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Algebraic relations among hyperderivatives of periods and logarithms of Drinfeld modules

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We determine all algebraic relations among all hyperderivatives of the periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules defined over a separable closure of the rational function field. In particular, for periods and logarithms that are linearly independent over the endomorphism ring of the Drinfeld module, we prove the algebraic independence of their hyperderivatives and the hyperderivatives of the corresponding quasiperiods and quasilogarithms.

1. Introduction

The objects of study in the present paper are inspired by elliptic curves in the classical setting. Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$. The period conjecture states that the transcendence degree over $\overline{\mathbb{Q}}$ of the two periods $\{\omega_1, \omega_2\}$ and the two corresponding quasiperiods $\{\eta_1, \eta_2\}$ of E is 2 when E has complex multiplication (CM), and 4 otherwise. The CM case was confirmed to be true by Chudnovsky, while the non-CM case is still open. With regards to logarithms of E, one can expect logarithms of algebraic numbers that are linearly independent over $\mathrm{End}(E)$ to be algebraically independent over $\overline{\mathbb{Q}}$. Although linear independence over $\overline{\mathbb{Q}}$ of these logarithms is known due to Masser (for the CM case), Bertrand and Masser (for the non-CM case), and as a consequence of Wüstholz's analytic subgroup theorem, algebraic independence of these logarithms is still fully open. See [Baker and Wüstholz 2007; Waldschmidt 2008] for details.

In the function field setting, Drinfeld [1974] introduced "elliptic modules", now called Drinfeld modules, as an analogue of elliptic curves. Later, Anderson [1986] defined higher-dimensional generalizations of Drinfeld modules, called t-modules. One can ask analogous questions regarding algebraic independence

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of periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules and Anderson t-modules defined over algebraic function fields. Yu [1997] proved the sub-t-module theorem, a remarkable result regarding linear independence among logarithms of Anderson t-modules, which is an analogue of Wüstholz's analytic subgroup theorem, and proved the complete transcendence results concerning periods and logarithms of Drinfeld modules [Yu 1986; 1990]. Thiery [1992] proved algebraic independence results among periods and quasiperiods of rank-2 Drinfeld modules with complex multiplication. Chang and Papanikolas [2011; 2012] proved algebraic independence of periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules of arbitrary rank. The goal of the present paper is to generalize completely under separability hypothesis this work of Chang and Papanikolas [2011; 2012, Theorems 3.5.4 and 5.1.5, and Corollary 5.1.6] to include all hyperderivatives, which are defined below.

Let $\mathbb{F}_q$ be a finite field, where q is a positive power of a prime number p, and let θ be an indeterminate. For the rational function field $\mathbb{F}_q(\theta)$, the j-th hyperderivative $\partial_\theta^j : \mathbb{F}_q(\theta) \to \mathbb{F}_q(\theta)$ is defined by $\partial_\theta^j(\theta^m) := \binom{m}{j}\theta^{m-j}$, where $j \geq 0$. Taking the completion $\mathbb{F}_q((1/\theta))$ of $\mathbb{F}_q(\theta)$ with respect to its ∞-adic absolute value $|\cdot|_\infty$, $\partial_\theta^j(\cdot)$ extends uniquely to $\mathbb{F}_q((1/\theta))^{\mathrm{sep}}$. Note that hyperderivatives play a role analogous to that of formal derivatives in the classical case. Unlike in the classical setting of elliptic curves, one can take hyperderivatives of periods and logarithms of Anderson t-modules defined over $\mathbb{F}_q(\theta)^{\mathrm{sep}}$. Moreover, many interpretations of objects of interest in terms of logarithms of Anderson t-modules involve hyperderivatives. The entries of periods of the d-th tensor power $\mathfrak{C}^{\otimes d}$ of the Carlitz module $\mathfrak{C}$ (rank-1 Drinfeld module) are obtained using hyperderivatives [Maurischat 2018, Lemma 8.3] of Anderson–Thakur functions [1990, §2.5]. Also, Carlitz zeta values [Thakur 2004] appear in the last coordinate of a logarithm of $\mathfrak{C}^{\otimes d}$ [Anderson and Thakur 1990, Theorem 3.8.3]. Generalizing this, Chang, Green, and Mishiba [Chang et al. 2021] showed that multizeta values [Thakur 2004] also appear as coordinates of logarithms of a particular Anderson t-module and further showed that its periods and logarithms are obtained using hyperderivatives. There are also logarithmic interpretations of special values of Goss L-functions attached to Drinfeld modules in terms of logarithms of an Anderson t-module, where hyperderivatives play a crucial role [Gezmiş and Namoijam 2021]. These interpretations further motivate interest in determining algebraic independence of hyperderivatives of periods and logarithms of Anderson t-modules.

Algebraic independence among hyperderivatives of the fundamental period of the Carlitz module were proved by Denis [1993; 1995; 2000] and Maurischat [2018; 2022a]. Further work in this direction was also done in unpublished work by Brownawell and van der Poorten. Utilizing Yu's sub-t-module theorem, Brownawell and Denis [2000] and Brownawell [1999; 2001] investigated linear independence of hyperderivatives of logarithms and quasilogarithms of Drinfeld modules. In the present paper, we determine all algebraic independence results among all hyperderivatives of periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules of arbitrary rank under the hypothesis of separability.

1.1. ***Hyperderivatives of periods and logarithms.*** For a finite field $\mathbb{F}_q$, where q is a positive power of a prime number p, we set $A := \mathbb{F}_q[\theta]$, $k := \mathbb{F}_q(\theta)$ and $k_\infty := \mathbb{F}_q((1/\theta))$, the completion of k at its infinite place. We further set $\mathbb{K}$ to be the completion of an algebraic closure of k_∞, and let $\bar{k}$ and k^{sep} be the algebraic closure and the separable closure respectively of k inside $\mathbb{K}$. For a variable t independent from θ, we further define $\boldsymbol{A} := \mathbb{F}_q[t]$ and $\boldsymbol{k} := \mathbb{F}_q(t)$.

For $n \in \mathbb{Z}$, we define the *Frobenius twist* $\tau^n : \mathbb{K}(\!(t)\!) \to \mathbb{K}(\!(t)\!)$ by setting for $f = \sum_i a_i t^i$

$$\tau^n(f) := f^{(n)} = \sum_i a_i^{q^n} t^i. \tag{1.1.1}$$

For a field $K \subseteq \mathbb{K}$, we define the twisted power series ring $K[\![\tau]\!]$ subject to the condition $\tau c = c^q \tau$ for all $c \in K$. Then, we define the twisted polynomial ring $K[\tau]$ as the subring of $K[\![\tau]\!]$, where $K[\tau]$ is viewed as a subalgebra of the $\mathbb{F}_q$-linear endomorphisms of the additive group of K.

For a field $k \subseteq K \subseteq \mathbb{K}$, a *Drinfeld $\boldsymbol{A}$-module of rank r defined over K* is an $\mathbb{F}_q$-algebra homomorphism $\rho : \boldsymbol{A} \to K[\tau]$ uniquely determined by

$$\rho_t = \theta + \kappa_1 \tau + \cdots + \kappa_r \tau^r$$

such that $\kappa_r \neq 0$. The *exponential function* associated to ρ is given by

$$\operatorname{Exp}_\rho(z) = z + \sum_{h \geq 1} \alpha_h z^{q^h} \in K[\![z]\!]$$

and it satisfies the functional equation $\operatorname{Exp}_\rho(\theta z) = \rho_t(\operatorname{Exp}_\rho(z))$. The *period lattice* of ρ is the kernel Λ_ρ of Exp_ρ, which is a free discrete $\boldsymbol{A}$-submodule of rank r inside $\mathbb{K}$.

The de Rham cohomology theory for Drinfeld $\boldsymbol{A}$-modules was developed by Anderson, Deligne, Gekeler [1989] and Yu [1990]. A *ρ-biderivation* is an $\mathbb{F}_q$-linear map $\delta : \boldsymbol{A} \to \mathbb{K}[\tau]\tau$ satisfying, for all $a, b \in \boldsymbol{A}$,

$$\delta_{ab} = a(\theta)\delta_b + \delta_a \rho_b.$$

Let $u \in \mathbb{K}[\tau]$. Then, the ρ-biderivation $\delta^{(u)}$ defined by $\delta_a^{(u)} = u\rho_a - a(\theta)u$ for all $a \in \boldsymbol{A}$ is called an *inner biderivation*. If $u \in \mathbb{K}[\tau]\tau$, then $\delta^{(u)}$ is said to be *strictly inner*. The set of ρ-biderivations $\operatorname{Der}(\rho)$ forms a $\mathbb{K}$-vector space. The set of inner biderivations $\operatorname{Der}_{\text{in}}(\rho)$ and the set of strictly inner biderivations $\operatorname{Der}_{\text{si}}(\rho)$ are also $\mathbb{K}$-vector subspaces of $\operatorname{Der}(\rho)$. We define the *de Rham module for ρ* to be $\mathrm{H}^1_{\mathrm{DR}}(\rho) := \operatorname{Der}(\rho)/\operatorname{Der}_{\text{si}}(\rho)$, which is an r-dimensional $\mathbb{K}$-vector space. The de Rham module $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ parametrizes the extensions of ρ by $\mathbb{G}_a$.

For each $\delta \in \operatorname{Der}(\rho)$ there is a unique $\mathbb{F}_q$-linear and entire power series

$$\mathrm{F}_\delta(z) = \sum_{i \geq 1} c_i z^{(i)} \in \mathbb{K}[\![z]\!]$$

such that, for all $a \in \boldsymbol{A}$,

$$\mathrm{F}_\delta(a(\theta)z) = a(\theta)\mathrm{F}_\delta(z) + \delta_a(\operatorname{Exp}_\rho(z)). \tag{1.1.2}$$

We call F_δ the *quasiperiodic function* associated to δ. For $\lambda \in \Lambda_\rho$, the value $\mathrm{F}_\delta(\lambda)$ is called a *quasiperiod of ρ*. For $u \in \mathbb{K}$, the value $\mathrm{F}_\delta(u)$, which is a coordinate of logarithms of quasiperiodic extensions, is called a *quasilogarithm of ρ* (see [Brownawell and Papanikolas 2002; Namoijam and Papanikolas 2024]).

A $\mathbb{K}$-basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ is represented by $\{\delta_1, \ldots, \delta_r\}$, where δ_1 is the inner biderivation such that $(\delta_1)_t = \rho_t - \theta$, and $\delta_j(t) = \tau^{j-1}$ for $2 \leq j \leq r$. We see that $\mathrm{F}_{\delta_1}(z) = \operatorname{Exp}_\phi(z) - z$, and so $\mathrm{F}_{\delta_1}(\lambda) = -\lambda$ for all $\lambda \in \Lambda_\rho$. If we take $\{\lambda_1, \dots, \lambda_r\}$ to be an $\boldsymbol{A}$-basis of Λ_ρ and we set $\mathrm{F}_{\tau^{j-1}}(z) := \mathrm{F}_{\delta_j}(z)$ for $2 \le j \le r$, then we define the *period matrix of* ρ to be

$$\boldsymbol{P}_\rho := \begin{pmatrix} \lambda_1 & \mathrm{F}_\tau(\lambda_1) & \cdots & \mathrm{F}_{\tau^{r-1}}(\lambda_1) \\ \lambda_2 & \mathrm{F}_\tau(\lambda_2) & \cdots & \mathrm{F}_{\tau^{r-1}}(\lambda_2) \\ \vdots & \vdots & & \vdots \\ \lambda_r & \mathrm{F}_\tau(\lambda_r) & \cdots & \mathrm{F}_{\tau^{r-1}}(\lambda_r) \end{pmatrix},$$

which accounts for all periods and quasiperiods of ρ. The de Rham cohomology theory for Drinfeld $\boldsymbol{A}$-modules runs in parallel to the theory of elliptic functions such that the periods and quasiperiods summarized above play the role of periods and quasiperiods of the Weierstrass $\wp$-functions.

If the Drinfeld $\boldsymbol{A}$-module ρ is defined over k^{sep}, Denis [1995, p. 6] showed that, for a ρ-biderivation δ defined over k^{sep}, if $u \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u) \in k^{\mathrm{sep}}$, then $u \in k_\infty^{\mathrm{sep}}$ and $\mathrm{F}_\delta(u) \in k_\infty^{\mathrm{sep}}$ (see also [Namoijam and Papanikolas 2024, Lemma 4.22]). Therefore, for $n \ge 0$ we can consider $\partial_\theta^n(u)$ and $\partial_\theta^n(\mathrm{F}_\delta(u))$. Let $\partial_\theta^n(\boldsymbol{P}_\rho)$ be the matrix formed by entrywise action of $\partial_\theta^n(\,\cdot\,)$ on $\boldsymbol{P}_\rho$.

We define $\mathrm{End}(\rho) := \{x \in \mathbb{K} : x\Lambda_\rho \subseteq \Lambda_\rho\}$ and let K_ρ be its fraction field. Our first main result is as follows (restated as Theorem 4.5.1):

Theorem 1.1.3. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. If $\boldsymbol{s} = [K_\rho : k]$, then for $n \ge 1$ we have*

$$\mathrm{tr.deg}_{\bar{k}}\, \bar{k}(\boldsymbol{P}_\rho, \partial_\theta^1(\boldsymbol{P}_\rho), \dots, \partial_\theta^n(\boldsymbol{P}_\rho)) = (n+1)\cdot r^2/\boldsymbol{s}.$$

Building on Theorem 1.1.3, we prove algebraic independence among hyperderivatives of logarithms and quasilogarithms of Drinfeld $\boldsymbol{A}$-modules. Our second main result is as follows (restated as Theorem 5.4.4):

Theorem 1.1.4. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. Let $u_1, \dots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$ and suppose that $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w) = r/\boldsymbol{s} + w$, where $\boldsymbol{s} = [K_\rho : k]$. Then, for $n \ge 1$,*

$$\mathrm{tr.deg}_{\bar{k}}\, \bar{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(\mathrm{F}_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(\mathrm{F}_{\tau^i}(u_m))\}\right) = (n+1)(r^2/\boldsymbol{s} + rw).$$

For an arbitrary basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep}, we deduce the following corollary.

Corollary 1.1.5. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. Let $u_1, \dots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$. Let $\{\delta_1, \dots, \delta_r\}$ be a basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep}. If $u_1, \dots, u_w$ are linearly independent over K_ρ, then for $n \ge 1$ the $(n+1)rw$ quantities*

$$\left\{\bigcup_{s=0}^{n}\bigcup_{j=1}^{r}\big(\partial_\theta^s(\mathrm{F}_{\delta_j}(u_1)), \partial_\theta^s(\mathrm{F}_{\delta_j}(u_2)), \dots, \partial_\theta^s(\mathrm{F}_{\delta_j}(u_w))\big)\right\}$$

are algebraically independent over $\bar{k}$.

Combining Theorems 1.1.3, 1.1.4, and Corollary 1.1.5, the $\bar{k}$-linear relations among the periods and logarithms of ρ and their hyperderivatives induced by endomorphisms of ρ account for all the $\bar{k}$-algebraic relations among all hyperderivatives of the periods and logarithms as well as all hyperderivatives of the corresponding quasiperiods and quasilogarithms of ρ.

1.2. ***Remarks on structure of the paper.*** In [Namoijam and Papanikolas 2024], Papanikolas and the author showed that t-motives whose period matrices comprise the values of interest in Theorems 1.1.3 and 1.1.4 are constructed from the t-motive associated to prolongations [Maurischat 2018] of ρ, but did not prove any transcendence results about the values in question. Papanikolas's theorem [2008, Theorem 1.1.7] states that the transcendence degree of the period matrix of a t-motive is equal to the dimension of its Galois group. The primary hurdle, then, is determining the dimension of the associated Galois group of the t-motive.

The first goal of this paper is to explicitly determine the Galois group of the t-motive corresponding to the n-th prolongation t-module $\mathrm{P}_n\rho$ of ρ. To do this, we calculate the Zariski closure of the image of the Galois representation on the $\mathfrak{p}$-adic Tate module of $\mathrm{P}_n\rho$, for a nonzero prime $\mathfrak{p}$ of $\boldsymbol{A}$. Next, we immediately extend this result. We construct new t-motives whose period matrices are comprised of both periods and quasiperiods of $\mathrm{P}_n\rho$, and hyperderivatives of logarithms and quasilogarithms of ρ, and then determine their Galois groups. We construct a sequence of surjections between specific sub-t-motives using consecutive prolongations $\mathrm{P}_\ell\rho$ for $0 \le \ell \le n$. These surjections are crucial in establishing that algebraic independence over $\bar{k}$ of all hyperderivatives of the logarithms and quasilogarithms depends only on K_ρ-linear independence of the logarithms.

The paper is outlined as follows.

- In Section 2 we give necessary background concerning t-motives and their Galois groups. Next, we give a brief review of hyperderivatives and then discuss prolongations of dual t-motives introduced in [Maurischat 2018].

- In Section 3, we describe t-motives and rigid analytic trivializations corresponding to Drinfeld $\boldsymbol{A}$-modules and their prolongations; then we state Theorem 3.4.1. Based on Theorem 3.4.1 (see [Namoijam and Papanikolas 2024, §5.3] for a detailed account), to prove Theorem 1.1.3, for $n \ge 1$ we calculate the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ of the n-th prolongation $\mathrm{P}_n M_\rho$ of the t-motive M_ρ associated to ρ.

- We first make use of a direct connection $\Gamma_{\mathrm{P}_n M_\rho}$ has with Galois representations. For a nonzero prime $\mathfrak{p}$ of $\boldsymbol{A}$, let $\boldsymbol{A}_\mathfrak{p}$ be the completion of $\boldsymbol{A}$ and let $\boldsymbol{k}_\mathfrak{p}$ be its fraction field. For a Drinfeld $\boldsymbol{A}$-module ρ defined over K, where $k \subseteq K \subseteq \bar{k}$ with $[K:k] < \infty$, there is a representation $\varphi_\mathfrak{p} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ coming from the Galois action on the $\mathfrak{p}$-power torsion points $\rho[\mathfrak{p}^m] := \{x \in \mathbb{K} : \rho_{\mathfrak{p}^m}(x) = 0\}$. In Section 4, using Anderson generating functions and $\varphi_\mathfrak{p}$, we consider the Galois representation on the $\mathfrak{p}$-adic Tate module of the n-th prolongation t-module $\mathrm{P}_n\rho$ associated to ρ. The image of this Galois representation is determined using hyperderivatives of the image for the Drinfeld $\boldsymbol{A}$-module ρ and is naturally contained in the $\boldsymbol{k}_\mathfrak{p}$-valued points of $\Gamma_{\mathrm{P}_n M_\rho}$ (Theorem 4.1.6).

- For $n \ge 1$, $\mathrm{P}_{n-1}M_\rho$ is a sub-t-motive of $\mathrm{P}_n M_\rho$ and therefore, $\mathrm{P}_n M_\rho$ is not simple, which makes determining the Zariski closure of the aforementioned image a complicated task. To find the Zariski closure, we bring in differential algebraic geometry. We consider hyperdifferential polynomials (precise definition in Section A.1) to determine the above Zariski closure by first determining the defining differential ideal of the aforementioned image and then restricting to Zariski topology (Theorem 4.3.3). This allows us to prove Theorem 1.1.3 and compute the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ explicitly (Corollary 4.4.8).

• In Section 5, for $u_1, \dots, u_w \in \mathbb{K}$ satisfying $\operatorname{Exp}_\rho(u_i) \in k^{\mathrm{sep}}$ for each $1 \le i \le w$, we build on results of Section 4 to construct new t-motives $Y_{1,n}, \dots, Y_{w,n}$ such that the entries of the period matrix of $\bigoplus_{m=1}^{w} Y_{m,n}$ comprise $\bigcup_{s=0}^{n} \bigcup_{i=1}^{r-1} \bigcup_{m=1}^{w} \{\partial_\theta^s(u_m), \partial_\theta^s(\mathrm{F}_{\tau^i}(u_m))\}$. Let $\mathcal{T}$ denote the category of t-motives. In Lemma 5.4.3, we obtain a surjective map from certain sub-t-motives of $Y_{m,n}$ to corresponding sub-t-motives of $Y_{m,\ell}$ for $\ell \le n$ and $1 \le m \le w$. This map allows us to implement Theorem 5.2.2, which is based on an $\operatorname{End}_{\mathcal{T}}(M_\rho)$-linear independence result [Chang and Papanikolas 2012, Theorem 4.2.2], which enables us to prove Theorem 1.1.4.

• Finally, in the Appendix, we cover necessary background concerning differential algebraic geometry in positive characteristic. We explore various properties, especially a result on the determination of the Zariski closure of a set in a differential field (Lemma A.1.5).

2. Preliminaries

2.1. ***Notation.*** We continue with the notation introduced in Section 1.1. We also define the following.

Let $\mathbb{T}$ be the Tate algebra of the closed unit disk of $\mathbb{K}$,

$$\mathbb{T} := \left\{ \sum_{h=0}^{\infty} a_h t^h \in \mathbb{K}[\![t]\!] : \lim_{h\to\infty} |\, a_h \,|_\infty = 0 \right\},$$

and let $\mathbb{L}$ be its fraction field.

For $n \in \mathbb{Z}$, recall the Frobenius twist τ^n from (1.1.1). In some cases, we will write σ for τ^{-1}. For $M = (m_{ij}) \in \operatorname{Mat}_{e\times d}(\mathbb{K}(\!(t)\!))$, we define $M^{(n)}$ by setting $M^{(n)} := (m_{ij}^{(n)})$. Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be the Laurent polynomial ring over $\bar{k}(t)$ in σ subject to the relation

$$\sigma f = f^{(-1)}\sigma, \qquad f \in \bar{k}(t).$$

For a field $K \subseteq \mathbb{K}$, recall from Section 1.1 the twisted power series ring $K[\![\tau]\!]$ and the subring $K[\tau]$ given by $\tau f = f^{(1)}\tau$ for all $f \in K$. We also define $K[\![\sigma]\!]$ and $K[\sigma]$ when K is a perfect field. For $b = \sum c_i \tau^i \in \mathbb{K}[\tau]$, we define $b^* := \sum c_i^{(-i)} \sigma^i \in \mathbb{K}[\sigma]$. If $B = (b_{ij}) \in \operatorname{Mat}_{e\times d}(\mathbb{K}[\tau]) = \operatorname{Mat}_{e\times d}(\mathbb{K})[\tau]$, then we set $B^* := (b_{ji}^*)$. Thus, if $B \in \operatorname{Mat}_{e\times d}(\mathbb{K}[\tau])$ and $C \in \operatorname{Mat}_{d\times h}(\mathbb{K}[\tau])$, then $(BC)^* = C^* B^*$. Moreover, if $B = \beta_0 + \beta_1\tau + \cdots + \beta_\ell \tau^\ell$, then we set $\mathrm{d}B := \beta_0$.

2.2. ***Dual t-motives and t-motives.*** In this subsection, we briefly introduce the main tools used in Papanikolas's result. The reader is directed to [Papanikolas 2008] for further details. A *pre-t-motive M* is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module that is finite-dimensional over $\bar{k}(t)$. We denote by $\mathcal{P}$ the category of pre-t-motives whose morphisms are the left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module homomorphisms. Let $\boldsymbol{m} \in \operatorname{Mat}_{r\times 1}(M)$ be such that its entries form a $\bar{k}(t)$-basis of M. Then, there is a matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ such that

$$\sigma \boldsymbol{m} = \Phi \boldsymbol{m},$$

where the action of σ on $\boldsymbol{m}$ is entrywise. We say that M is *rigid analytically trivial* if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ such that

$$\Psi^{(-1)} = \Phi\Psi.$$

The matrix Ψ is called a *rigid analytic trivialization for* Φ. Set $M^\dagger := \mathbb{L} \otimes_{\bar{k}(t)} M$, where we give $M^\dagger$ a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module by letting σ act diagonally:

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m, \quad f \in \bar{k}(t),\ m \in M.$$

If we let

$$M^B := (M^\dagger)^\sigma := \{\mu \in M^\dagger : \sigma\mu = \mu\},$$

then M^B is a finite-dimensional vector space over $\boldsymbol{k}$, and $M \mapsto M^B$ is a covariant functor from $\mathcal{P}$ to the category of $\boldsymbol{k}$-vector spaces. The natural map $\mathbb{L} \otimes_{\bar{k}(t)} M^B \to M^\dagger$ is an isomorphism if and only if M is rigid analytically trivial [Papanikolas 2008, §3.3]. If Ψ is a rigid analytic trivialization of Φ, then the entries of $\Psi^{-1}\boldsymbol{m}$ form a $\boldsymbol{k}$-basis for M^B [loc. cit., Theorem 3.3.9(b)]. By [loc. cit., Theorem 3.3.15], the category $\mathcal{R}$ of *rigid analytically trivial pre-t-motives* forms a neutral Tannakian category over $\boldsymbol{k}$ with fiber functor $M \mapsto M^B$.

We now consider $\boldsymbol{A}$-finite dual t-motives, which were first introduced in [Anderson et al. 2004] (see also [Hartl and Juschka 2020; Namoijam and Papanikolas 2024]). A *dual t-motive* $\mathcal{M}$ is a left $\bar{k}[t, \sigma]$-module that is free and finitely generated as a left $\bar{k}[\sigma]$-module and such that $(t-\theta)^s\mathcal{M} \subseteq \sigma\mathcal{M}$ for $s \in \mathbb{N}$ sufficiently large. If, in addition, $\mathcal{M}$ is free and finitely generated as a left $\bar{k}[t]$-module, then $\mathcal{M}$ is said to be $\boldsymbol{A}$*-finite*. Thus, if the entries of $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(\mathcal{M})$ form a $\bar{k}[t]$-basis for $\mathcal{M}$, then there is a matrix $\Phi \in \mathrm{Mat}_r(\bar{k}[t])$ such that $\sigma\boldsymbol{m} = \Phi\boldsymbol{m}$ with $\det \Phi = c(t-\theta)^s$ for some $c \in \bar{k}^\times$, $s \geq 1$. We say that $\mathcal{M}$ is *rigid analytically trivial* if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$. In [Anderson et al. 2004], the term "dual t-motives" is used for $\boldsymbol{A}$-finite dual t-motives. We will consider both dual t-motives and $\boldsymbol{A}$-finite dual t-motives.

Given an $\boldsymbol{A}$-finite dual t-motive $\mathcal{M}$,

$$M := \bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{M}$$

is a pre-t-motive, where $\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m$. Then, $\mathcal{M} \mapsto M$ is a functor from the category of $\boldsymbol{A}$-finite dual t-motives to the category of pre-t-motives. We define the category $\mathcal{T}$ of t*-motives* to be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the essential image of rigid analytically trivial $\boldsymbol{A}$-finite dual t-motives under the assignment $\mathcal{M} \mapsto M$.

For a t-motive M, we let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{T}$ generated by M. As $\mathcal{T}_M$ is a neutral Tannakian category over $\boldsymbol{k}$, there is an affine group scheme Γ_M over $\boldsymbol{k}$, a subgroup of the $\boldsymbol{k}$-group scheme $\mathrm{GL}_r/\boldsymbol{k}$ of $r \times r$ invertible matrices, so that $\mathcal{T}_M$ is equivalent to the category of finite-dimensional representations of Γ_M over $\boldsymbol{k}$, i.e., $\mathcal{T}_M \approx \mathbf{Rep}(\Gamma_M, \boldsymbol{k})$ [Papanikolas 2008, §3.5]. We call Γ_M the *Galois group of M*.

2.3. ***The difference Galois group.*** We now present a brief summary of the construction of the Galois group of a t-motive as the Galois group of a system of difference equations. The reader is directed to [Papanikolas 2008] for further details. For a subfield $F \subset \mathbb{K}(\!(t)\!)$ invariant under the action of σ, let F^{σ} denote the elements of F fixed by σ. Note that the automorphism $\sigma : \mathbb{L} \to \mathbb{L}$ restricts to automorphisms of $\boldsymbol{k}$ and $\bar{k}(t)$, and $\boldsymbol{k} = \boldsymbol{k}^{\sigma} = \bar{k}(t)^{\sigma} = \mathbb{L}^{\sigma}$.

For a t-motive M, let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ denote the action of σ on a $\bar{k}(t)$-basis of M and let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ be the rigid analytic trivialization for Φ satisfying $\Psi^{(-1)} = \Phi\Psi$.

We define a $\bar{k}(t)$-algebra homomorphism $\nu : \bar{k}(t)[X, 1/\det X] \to \mathbb{L}$ by setting $\nu(X_{ij}) := \Psi_{ij}$, where $X = (X_{ij})$ is an $r \times r$ matrix of independent variables. We let $\mathfrak{p} := \ker \nu$ and $\Sigma := \operatorname{Im} \nu = \bar{k}(t)[\Psi, 1/\det \Psi] \subseteq \mathbb{L}$, and set $Z_{\Psi} = \operatorname{Spec} \Sigma$. Then, Z_{Ψ} is the smallest closed subscheme of $\mathrm{GL}_r / \bar{k}(t)$ such that $\Psi \in Z_{\Psi}(\mathbb{L})$.

Set $\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$ to be such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$, and let $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$. We define a $\boldsymbol{k}$-algebra homomorphism $\mu : \boldsymbol{k}[X, 1/\det X] \to \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}$ by setting $\mu(X_{ij}) := \widetilde{\Psi}_{ij}$. We let $\mathfrak{q} := \ker \mu$ and $\Delta := \operatorname{Im} \mu$, and set $\Gamma_{\Psi} = \operatorname{Spec} \Delta$. Then, Γ_{Ψ} is the smallest closed subscheme of $\mathrm{GL}_r / \boldsymbol{k}$ such that $\widetilde{\Psi} \in \Gamma_{\Psi}(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$. The following properties hold.

Theorem 2.3.1 [Papanikolas 2008, §4]. *Let M be a t-motive, and let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ represent multiplication by σ on a $\bar{k}(t)$-basis of M. Let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfy $\Psi^{(-1)} = \Phi\Psi$.*

(a) *The closed $\bar{k}(t)$-subscheme Z_{Ψ} is stable under right-multiplication by $\bar{k}(t) \times_{\boldsymbol{k}} \Gamma_{\Psi}$ and is a $\bar{k}(t) \times_{\boldsymbol{k}} \Gamma_{\Psi}$-torsor over $\bar{k}(t)$. In particular, $\Gamma_{\Psi}(\bar{\mathbb{L}}) = \Psi^{-1} Z_{\Psi}(\bar{\mathbb{L}})$.*

(b) *The $\boldsymbol{k}$-scheme Γ_{Ψ} is absolutely irreducible and smooth over $\bar{\boldsymbol{k}}$.*

(c) *$\Gamma_{\Psi} \cong \Gamma_M$ over $\boldsymbol{k}$.*

For the t-motive M, if $\Phi \in \mathrm{GL}_r(\bar{k}(t)) \cap \mathrm{Mat}_r(\bar{k}[t])$ and $\det \Phi = c(t-\theta)^s$ for some $c \in \bar{k}^{\times}$, $s \geq 1$, then we can pick Ψ to be in $\mathrm{GL}_r(\mathbb{T})$ [Papanikolas 2008, Proposition 3.3.9(c)]. Moreover, the entries of Ψ are regular at $t = \theta$ [Anderson et al. 2004, Proposition 3.1.3]. Let $\Psi|_{t=\theta}$ denote the specialization of the entries of Ψ at $t = \theta$ and let $\bar{k}(\Psi|_{t=\theta})$ be the field formed by adjoining the entries of $\Psi|_{t=\theta}$ to $\bar{k}$. The main theorem of [Papanikolas 2008] is as follows.

Theorem 2.3.2 [Papanikolas 2008, Theorem 1.1.7]. *Let M be a t-motive, and let Γ_M be its Galois group. Suppose that $\Phi \in \mathrm{GL}_r(\bar{k}(t)) \cap \mathrm{Mat}_r(\bar{k}[t])$ represents multiplication by σ on a $\bar{k}(t)$-basis of M and that* $\det \Phi = c(t-\theta)^s$, $c \in \bar{k}^{\times}$, $s \geq 1$. *Let $\Psi \in \mathrm{GL}_r(\mathbb{T})$ be a rigid analytic trivialization of Φ. Then,* $\operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi|_{t=\theta}) = \dim \Gamma_M$.

2.4. ***Hyperderivatives and hyperdifferential operators.*** For details beyond the review here, the reader may refer to [Brownawell 1999; Jeong 2011; Namoijam and Papanikolas 2024, §2.4]. For $m, j \geq 0$, let $\binom{m}{j} \in \mathbb{N}$ denote the usual binomial coefficient modulo p. Then, for F a field of characteristic $p > 0$ where θ is transcendental over F, the F-linear map $\partial_{\theta}^{j} : F[\theta] \to F[\theta]$ defined by setting

$$\partial_{\theta}^{j}(\theta^m) = \binom{m}{j}\theta^{m-j}$$

is called the *j-th hyperdifferential operator with respect to θ*. For each $f \in F[\theta]$, we call $\partial_\theta^j(f)$ the *j-th hyperderivative of f*. The definition of ∂_θ^j extends naturally to $\partial_\theta^j : F[\![\theta]\!] \to F[\![\theta]\!]$. The hyperdifferential operators satisfy various identities including the product rule

$$\partial_\theta^j(fg) = \sum_{i=0}^{j} \partial_\theta^i(f)\partial_\theta^{j-i}(g)$$

and the composition rule

$$\partial_\theta^i(\partial_\theta^j(f)) = \binom{i+j}{j}\partial_\theta^{i+j}(f).$$

The product rule extends ∂_θ^j to the Laurent series field $F(\!(\theta)\!)$, where as usual for $m > 0$ we have

$$\binom{-m}{j} = (-1)^j\binom{m+j-1}{j}.$$

For a place v of $F(\theta)$ there are unique extensions $\partial_\theta^j : F(\theta)_v \to F(\theta)_v$ and $\partial_\theta^j : F(\theta)_v^{\mathrm{sep}} \to F(\theta)_v^{\mathrm{sep}}$, where $F(\theta)_v^{\mathrm{sep}}$ is a separable closure of $F(\theta)_v$.

Proposition 2.4.1 (see [Brownawell 1999, §7; Jeong 2011, §2]). *Let F be a field of characteristic $p > 0$, and let v be a place of $F(\theta)$. Then, for $f \in F(\theta)_v^{\mathrm{sep}}$, $n \geqslant 0$, and $j \geq 1$, $\partial_\theta^j : F(\theta)_v^{\mathrm{sep}} \to F(\theta)_v^{\mathrm{sep}}$, $j \geqslant 0$, satisfies*

$$\partial_\theta^j(f^{p^n}) = \begin{cases} (\partial_\theta^e(f))^{p^n} & \text{if } j = ep^n, \\ 0 & \text{if } p^n \nmid j. \end{cases}$$

For $f \in F(\theta)_v^{\mathrm{sep}}$ and $n \geq 0$, we define the *d-matrix with respect to θ*, $d_{\theta,n}[f] \in \mathrm{Mat}_n(F(\theta)_v^{\mathrm{sep}})$ to be the upper-triangular $n \times n$ matrix

$$d_{\theta,n}[f] := \begin{pmatrix} f & \partial_\theta^1(f) & \cdots & \cdots & \partial_\theta^{n-1}(f) \\ & f & \partial_\theta^1(f) & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \partial_\theta^1(f) \\ & & & & f \end{pmatrix}. \tag{2.4.2}$$

Using the product rule, it is easy to see that $d_{\theta,n}[g] \cdot d_{\theta,n}[f] = d_{\theta,n}[gf]$. For a matrix $B := (b_{ij}) \in \mathrm{Mat}_{e_1 \times e_2}(F(\theta)_v^{\mathrm{sep}})$, we also define the d-matrix with respect to θ, $d_{\theta,n}[B] \in \mathrm{Mat}_{ne_1 \times ne_2}(F(\theta)_v^{\mathrm{sep}})$ as in (2.4.2), where we let $\partial_\theta^j(B) := (\partial_\theta^j(b_{ij})) \in \mathrm{Mat}_{e_1 \times e_2}(F(\theta)_v^{\mathrm{sep}})$.

We further define *partial hyperderivatives* for two independent variables θ and t to be the F-linear maps

$$\partial_\theta^j, \partial_t^j : F(\theta, t) \to F(\theta, t), \quad j \geq 0,$$

such that for $m \in \mathbb{Z}$ we have $\partial_\theta^j(\theta^m) = \binom{m}{j}\theta^{m-j}$, $\partial_t^j(t^m) = \binom{m}{j}t^{m-j}$, and $\partial_\theta^j(t^m) = \partial_t^j(\theta^m) = 0$. Thus, we have $\partial_\theta \circ \partial_t = \partial_t \circ \partial_\theta$. For $n \geq 0$, we define the d-matrices $d_{\theta,n}[\,\cdot\,]$ and $d_{t,n}[\,\cdot\,]$ with respect to each independent variable θ and t as in (2.4.2).

Note that ∂_t^j extends naturally to $\mathbb{T}$, and ∂_θ^j extends to $\mathbb{T} \cap k_\infty^{\mathrm{sep}}[\![t]\!]$.

2.5. ***Prolongations of dual t-motives.*** We review the construction of prolongations of dual t-motives, introduced in [Maurischat 2018]. For a left $\bar{k}[t,\sigma]$-module $\mathcal{M}$ and $n \geq 0$, we define the *n-th prolongation of $\mathcal{M}$* to be the left $\bar{k}[t,\sigma]$-module $\mathrm{P}_n\mathcal{M}$ generated by symbols $D_i m$, for $m \in \mathcal{M}$ and $0 \leq i \leq n$, subject to the relations

(a) $D_i(m_1 + m_2) = D_i m_1 + D_i m_2$,

(b) $D_i(a \cdot m) = \sum_{i=i_1+i_2} \partial_t^{i_1}(a) \cdot D_{i_2} m$,

(c) $\sigma(a \cdot D_i m) = a^{(-1)} \cdot D_i(\sigma m)$,

where $m, m_1, m_2 \in \mathcal{M}$ and $a \in \bar{k}[t]$.

If $\mathcal{M}$ is an $\boldsymbol{A}$-finite dual t-motive, then $\mathrm{P}_n\mathcal{M}$ is also an $\boldsymbol{A}$-finite dual t-motive [loc. cit., Theorem 3.4]. Thus, if the entries of $\boldsymbol{m} = [m_1, \dots, m_r]^\top \in \mathcal{M}^r$ form a $\bar{k}[t]$-basis of $\mathcal{M}$, then a $\bar{k}[t]$-basis of $\mathrm{P}_n\mathcal{M}$ is given by the entries of

$$\boldsymbol{D}_n\boldsymbol{m} := (D_n\boldsymbol{m}^\top, D_{n-1}\boldsymbol{m}^\top, \dots, D_0\boldsymbol{m}^\top)^\top \in (\mathrm{P}_n\mathcal{M})^{r(n+1)}, \tag{2.5.1}$$

where $D_i\boldsymbol{m} := (D_i m_1, \dots, D_i m_r)^\top \in (\mathrm{P}_n\mathcal{M})^r$ for each $0 \leq i \leq n$ [loc. cit., Proposition 4.2]. Also, if $\Phi \in \mathrm{GL}_r(\bar{k}[t])$ represents multiplication by σ on $\boldsymbol{m}$, then

$$\sigma(\boldsymbol{D}_n\boldsymbol{m}) = d_{t,n+1}[\Phi] \cdot \boldsymbol{D}_n\boldsymbol{m}. \tag{2.5.2}$$

If $\mathcal{M}$ is rigid analytically trivial with $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$, then since Frobenius twisting commutes with hyperdifferentiation with respect to t, we have

$$(d_{t,n+1}[\Psi])^{(-1)} = d_{t,n+1}[\Psi^{(-1)}] = d_{t,n+1}[\Phi\Psi] = d_{t,n+1}[\Phi] d_{t,n+1}[\Psi]. \tag{2.5.3}$$

Therefore, $\mathrm{P}_n\mathcal{M}$ is rigid analytically trivial.

Via $D_0 m \mapsto m$, we see that $\mathrm{P}_0\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$, and as in [loc. cit., Remark 3.2], for $0 \leq j \leq n-1$ we obtain a short exact sequence of dual t-motives

$$0 \to \mathrm{P}_j\mathcal{M} \to \mathrm{P}_n\mathcal{M} \xrightarrow{\mathbf{pr}_{n-j-1}} \mathrm{P}_{n-j-1}\mathcal{M} \to 0, \tag{2.5.4}$$

where $\mathbf{pr}_{n-j-1}(D_i m) := D_{i-j-1} m$ for $i > j$ and $\mathbf{pr}_{n-j-1}(D_i m) := 0$ for $i \leq j$ and $m \in \mathcal{M}$.

3. Rigid analytic trivializations and hyperderivatives

The goal of this section is to provide necessary background on Anderson t-modules for the purpose of studying Drinfeld $\boldsymbol{A}$-modules and their prolongations, and their connection to dual t-motives and rigid analytic trivializations via Anderson generating functions. Then, we state Theorem 3.4.1, which provides the connection between Taylor coefficients of series expansions of Anderson generating functions and hyperderivatives of periods, quasiperiods, logarithms, and quasilogarithms of a Drinfeld $\boldsymbol{A}$-module defined over k^{sep}.

3.1. ***Anderson t-modules, dual t-motives, and Anderson generating functions.*** For a field $K \subseteq \mathbb{K}$, an *Anderson t-module defined over K* is an $\mathbb{F}_q$-algebra homomorphism $\phi : \boldsymbol{A} \to \mathrm{Mat}_d(K[\tau])$ defined uniquely by

$$\phi_t = B_0 + B_1\tau + \cdots + B_\ell\tau^\ell,$$

where $B_i \in \mathrm{Mat}_d(K)$ for $0 \le i \le \ell$, and $\mathrm{d}\phi_t = B_0 = \theta I_d + N$ such that I_d is the $d \times d$ identity matrix and N is a nilpotent matrix. Then, ϕ defines an $\boldsymbol{A}$-module structure on $\mathbb{K}^d$ via

$$a \cdot \boldsymbol{x} = \phi_a(\boldsymbol{x}), \quad a \in \boldsymbol{A},\ \boldsymbol{x} \in \mathbb{K}^d. \tag{3.1.1}$$

We call d the *dimension of ϕ*. If $\phi_t = B_0 \in \mathrm{Mat}_d(K)$, then ϕ is said to be a *trivial* Anderson t-module. A nontrivial Anderson t-module of dimension 1 is called a *Drinfeld $\boldsymbol{A}$-module*.

There exists a unique power series $\mathrm{Exp}_\phi(\boldsymbol{z}) = \sum_{i=0}^{\infty} C_i \boldsymbol{z}^{(i)} \in \mathbb{K}[\![z_1, \dots, z_d]\!]^d$, $\boldsymbol{z} = [z_1, \dots, z_d]^{\mathsf{T}}$, so that $C_0 = I_d$ and satisfies

$$\mathrm{Exp}_\phi(\mathrm{d}\phi_a \boldsymbol{z}) = \phi_a(\mathrm{Exp}_\phi(\boldsymbol{z}))$$

for all $a \in \boldsymbol{A}$. Moreover, $\mathrm{Exp}_\phi(\boldsymbol{z})$ defines an entire function $\mathrm{Exp}_\phi : \mathbb{K}^d \to \mathbb{K}^d$. If Exp_ϕ is surjective, then we say that ϕ is *uniformizable*. The kernel $\Lambda_\phi \subseteq \mathbb{K}^d$ of Exp_ϕ is a free and finitely generated discrete $\boldsymbol{A}$-submodule of $\mathbb{K}^d$ through the action of $\mathrm{d}\phi(\boldsymbol{A})$ and it is called the *period lattice of ϕ*. If ϕ is uniformizable, then we have an isomorphism $\mathbb{K}^d/\Lambda_\phi \cong (\mathbb{K}^d, \phi)$ of $\boldsymbol{A}$-modules, where $(\mathbb{K}^d, \phi)$ denotes $\mathbb{K}^d$ together with the $\boldsymbol{A}$-module structure defined in (3.1.1) coming from ϕ. For more details about Anderson t-modules, see [Anderson 1986; Brownawell and Papanikolas 2020; Thakur 2004].

We define the dual t-motive $\mathcal{M}_\phi$ associated to a t-module ϕ defined over $K \subseteq \bar{k}$ in the following way. We let $\mathcal{M}_\phi := \mathrm{Mat}_{1\times d}(\bar{k}[\sigma])$. To give $\mathcal{M}_\phi$ the $\bar{k}[t,\sigma]$-module structure, set

$$a \cdot m = m\phi_a^*, \quad m \in \mathcal{M}_\phi,\ a \in \boldsymbol{A}, \tag{3.1.2}$$

where ϕ_a^* is defined as in Section 2.1. For each $m \in \mathcal{M}_\phi$, by straightforward computation we obtain $(t-\theta)^d \cdot m \in \sigma\mathcal{M}_\phi$. Thus, $\mathcal{M}_\phi$ defines a dual t-motive and (3.1.2) gives a unique correspondence between a t-module and its associated dual t-motive (see also [Brownawell and Papanikolas 2020, §4.4; Hartl and Juschka 2020; Namoijam and Papanikolas 2024, §2.3]). If $\mathcal{M}_\phi$ is $\boldsymbol{A}$-finite, then we say that ϕ *is $\boldsymbol{A}$-finite* and call the rank of $\mathcal{M}_\phi$ as a left $\bar{k}[t]$-module the *rank of ϕ*. The reader is directed to [Hartl and Juschka 2020; Namoijam and Papanikolas 2024, §2.3] for more information on dual t-motives associated to t-modules.

We conclude this subsection by introducing the Anderson generating functions associated to t-modules (see [Green 2022; Maurischat 2022c; Namoijam and Papanikolas 2024] for further details). For $\boldsymbol{y} \in \mathbb{K}^d$, we define the *Anderson generating function for ϕ* by the infinite series

$$\mathcal{G}_{\boldsymbol{y}}(t) := \sum_{m=0}^{\infty} \mathrm{Exp}_\phi(\mathrm{d}\phi_t^{-m-1}\boldsymbol{y})t^m \in \mathbb{T}^d. \tag{3.1.3}$$

We explore the properties we will use in Sections 3.3, 4.1, 5.1. For clarity, we will denote by $f_y(t)$ the Anderson generating function for a Drinfeld $\boldsymbol{A}$-module at $y \in \mathbb{K}$.

3.2. *Prolongations of Drinfeld A-modules and associated dual t-motives.* Let $\rho : \boldsymbol{A} \to K[\tau]$ be a Drinfeld $\boldsymbol{A}$-module defined over $K \subseteq \bar{k}$ such that

$$\rho_t = \theta + \kappa_1 \tau + \cdots + \kappa_r \tau^r,$$

where $\kappa_r \neq 0$. Drinfeld $\boldsymbol{A}$-modules are uniformizable and the rank of the period lattice Λ_ρ of ρ as an $\boldsymbol{A}$-module is r. As defined above for t-modules, we define the dual t-motive $\mathcal{M}_\rho := \bar{k}[\sigma]$. Then the set $\{m_1, m_2, \ldots, m_r\} = \{1, \sigma, \ldots, \sigma^{r-1}\}$ forms a $\bar{k}[t]$-basis for $\mathcal{M}_\rho$ [Chang and Papanikolas 2012, §3.3; Namoijam and Papanikolas 2024, Example 3.35], and with respect to this basis, multiplication by σ on $\mathcal{M}_\rho$ is represented by

$$\Phi_\rho := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\kappa_r^{(-r)} & -\kappa_1^{(-1)}/\kappa_r^{(-r)} & \cdots & -\kappa_{r-1}^{(-r+1)}/\kappa_r^{(-r)} \end{pmatrix}. \tag{3.2.1}$$

Thus, $\mathcal{M}_\rho$ is $\boldsymbol{A}$-finite. We let $M_\rho := \bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{M}_\rho$ be the pre-t-motive associated to $\mathcal{M}_\rho$.

For Drinfeld $\boldsymbol{A}$-modules ρ and ρ' defined over $K \subseteq \mathbb{K}$, a *morphism* $b : \rho \to \rho'$ is a twisted polynomial $b \in \mathbb{K}[\tau]$ such that $b\rho_a = \rho'_a b$ for all $a \in \boldsymbol{A}$. We say that b is defined over $L \subseteq \mathbb{K}$ if $b \in L[\tau]$. A morphism $b : \rho \to \rho'$ defined over $\bar{k}$ induces a morphism $B : \mathcal{M}_\rho \to \mathcal{M}_{\rho'}$ of $\boldsymbol{A}$-finite dual t-motives in the following way. If $b = \sum c_i \tau^i \in L[\tau]$, recall from Section 2.1 that $b^* = \sum c_i^{(-i)} \sigma^i$. Then, B is the $\bar{k}[\sigma]$-linear map such that $B(1) = b^*$ (see [Chang and Papanikolas 2011, Lemma 2.4.2]).

The map

$$\operatorname{End}(\rho) \to \{c \in \mathbb{K} : c\Lambda_\rho \subseteq \Lambda_\rho\}, \quad \sum c_i \tau^i \mapsto c_0, \tag{3.2.2}$$

is an isomorphism [Drinfeld 1974]. Throughout this paper, we identify $\operatorname{End}(\rho)$ with the image of this map and let K_ρ denote its fraction field. We state the following result due to Anderson.

Proposition 3.2.3 [Chang and Papanikolas 2012, Proposition 3.3.2, Corollary 3.3.3]. *The functor $\rho \to \mathcal{M}_\rho$ from the category of Drinfeld $\boldsymbol{A}$-modules defined over $K \subseteq \bar{k}$ to the category of $\boldsymbol{A}$-finite dual t-motives is fully faithful. Moreover,*

$$\operatorname{End}(\rho) \cong \operatorname{End}_{\bar{k}[t,\sigma]}(\mathcal{M}_\rho), \quad K_\rho \cong \operatorname{End}_{\mathcal{T}}(M_\rho),$$

and M_ρ is a simple left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module.

Remark 3.2.4. Let $i_t \in \operatorname{End}_{\bar{k}[t,\sigma]}(\mathcal{M}_\rho)$ be such that $i_t(1) = t \cdot 1 = \rho_t^*$. The isomorphism $\operatorname{End}(\rho) \cong \operatorname{End}_{\bar{k}[t,\sigma]}(\mathcal{M}_\rho)$ in Proposition 3.2.3 sends $\theta \mapsto i_t$ and so, it sends A to $\boldsymbol{A}$. Thus, $K_\rho \cong \operatorname{End}_{\mathcal{T}}(M_\rho)$ sends k to $\boldsymbol{k}$.

For $n \geq 0$, we define the *n-th prolongation t-module* $\mathrm{P}_n\rho$ *of* ρ to be the Anderson t-module associated to the n-th prolongation $\mathrm{P}_n\mathcal{M}_\rho$ of the $\boldsymbol{A}$-finite dual t-motive $\mathcal{M}_\rho$ (see for details [Maurischat 2018, §5; Namoijam and Papanikolas 2024, §5.2]). The Anderson t-module $\mathrm{P}_n\rho : \boldsymbol{A} \to \operatorname{Mat}_{n+1}(K[\tau])$ is of dimension $n+1$ and is defined by

$$(\mathrm{P}_n\rho)_t = \mathrm{d}(\mathrm{P}_n\rho)_t + \operatorname{diag}(\kappa_1)\tau + \cdots + \operatorname{diag}(\kappa_r)\tau^r,$$

where

$$\mathrm{d}(\mathrm{P}_n\rho)_t = \begin{pmatrix} \theta & & & & \\ -1 & \ddots & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & \theta \end{pmatrix}, \tag{3.2.5}$$

and $\mathrm{diag}(\kappa_i)$ is the $(n+1)\times(n+1)$ diagonal matrix with diagonal entries all equal to κ_i for each $1 \le i \le r$. If we set $\mathcal{M}_{\mathrm{P}_n\rho} := \mathrm{Mat}_{1\times(n+1)}(\bar{k}[\sigma])$ to be the dual t-motive associate to $\mathrm{P}_n\rho$ defined as in (3.1.2), then by [Namoijam and Papanikolas 2024, Proposition 5.22(b)] we have

$$\mathcal{M}_{\mathrm{P}_n\rho} = \mathrm{P}_n\mathcal{M}_\rho.$$

We define $\mathrm{P}_n M_\rho := \bar{k}(t) \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ to be the pre-t-motive associated to $\mathrm{P}_n\mathcal{M}_\rho$.

3.3. ***Rigid analytic trivializations.*** We fix our choice of Drinfeld $\boldsymbol{A}$-module ρ of rank r from Section 3.2 such that it is defined over $K = k^{\mathrm{sep}}$. In this subsection, we show that the $\boldsymbol{A}$-finite dual t-motive $\mathcal{M}_\rho$ associated to ρ is rigid analytically trivial by constructing the rigid analytic trivialization Ψ_ρ, and then extend to the prolongation t-module $\mathrm{P}_n\rho$. The details regarding Drinfeld $\boldsymbol{A}$-modules can be found in [Chang and Papanikolas 2012, §3.4; Namoijam and Papanikolas 2024, Example 4.1117].

For $u \in \mathbb{K}$, we let $f_u(t) \in \mathbb{T}$ denote the Anderson generating function of ρ given as in (3.1.3). For an $\boldsymbol{A}$-basis $\{\lambda_1, \dots, \lambda_r\}$ of Λ_ρ, we set $f_i(t) := f_{\lambda_i}(t)$ for each $1 \le i \le r$. Define the matrices

$$\Upsilon := \begin{pmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(r-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(r-1)} \\ \vdots & \vdots & & \vdots \\ f_r & f_r^{(1)} & \cdots & f_r^{(r-1)} \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} \kappa_1 & \kappa_2^{(-1)} & \cdots & \kappa_{r-1}^{(-r+2)} & \kappa_r^{(-r+1)} \\ \kappa_2 & \kappa_3^{(-1)} & \cdots & \kappa_r^{(-r+2)} & \\ \vdots & \vdots & & & \\ \kappa_{r-1} & \kappa_r^{(-1)} & & & \\ \kappa_r & & & & \end{pmatrix}. \tag{3.3.1}$$

By [Chang and Papanikolas 2012, §3.4] (see also [Namoijam and Papanikolas 2024, Lemma 4.47]), it follows that $\det \Upsilon \neq 0$. Set

$$\Psi_\rho := V^{-1}[\Upsilon^{(1)}]^{-1}. \tag{3.3.2}$$

Then $\Psi_\rho^{(-1)} = \Phi_\rho\Psi_\rho$. Thus, the pre-$t$-motive $M_\rho = \bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{M}_\rho$ is rigid analytically trivial and is in the category $\mathcal{T}$ of t-motives.

By (2.5.3), the n-th prolongation t-motive $\mathrm{P}_n M_\rho = \bar{k}(t) \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ is rigid analytically trivial with rigid analytic trivialization $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$. Thus,

$$\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[V]^{-1} d_{t,n+1}[\Upsilon^{(1)}]^{-1}. \tag{3.3.3}$$

3.4. ***Hyperderivatives of periods and logarithms.*** We continue with our choice of Drinfeld $\boldsymbol{A}$-module ρ of rank r defined over k^{sep}. Recall from Section 1.1 that a $\mathbb{K}$-basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ is represented by $\{\delta_1, \dots, \delta_r\}$, where δ_1 is the inner biderivation such that $(\delta_1)_t = \rho_t - \theta = \kappa\tau + \cdots + \kappa_r\tau^r$ and $\delta_j(t) = \tau^{j-1}$ for $2 \le j \le r$.

Let $\mathrm{F}_{\tau^{j-1}}(z)$ denote the quasiperiodic function associated to the biderivation $\delta_j : t \mapsto \tau^{j-1}$. Note that $\mathrm{F}_{\delta_1}(z) = \mathrm{Exp}_\rho(z) - z$. Then, we have the following result, which is a modified version for Drinfeld $\boldsymbol{A}$-modules, and its proof is due to Papanikolas and the author.

Theorem 3.4.1 (see [Namoijam and Papanikolas 2024, Theorem E]). *Let ρ be a Drinfeld $\boldsymbol{A}$-module defined over k^{sep} of rank r. Let $u \in \mathbb{K}^d$ satisfy $\mathrm{Exp}_\phi(u) \in (k^{\mathrm{sep}})^d$. Then, for $n \geqslant 0$,*

$$\mathrm{Span}_{\bar{k}}\Bigg(\{1\} \cup \bigcup_{s=0}^{n} \bigcup_{\ell=1}^{r} \{\partial_t^s(f_u^{(\ell)}(t))|_{t=\theta}\}\Bigg) = \mathrm{Span}_{\bar{k}}\Bigg(\{1\} \cup \bigcup_{s=0}^{n} \bigcup_{j=1}^{r-1} \{\partial_\theta^s(u),\, \partial_\theta^s(\mathrm{F}_{\tau^j}(u))\}\Bigg). \tag{3.4.2}$$

In particular, if $\{\lambda_1, \dots, \lambda_r\}$ is an $\boldsymbol{A}$-basis of the period lattice Λ_ρ, then

$$\mathrm{Span}_{\bar{k}}(d_{t,n+1}[\Psi_\rho]^{-1}|_{t=\theta}) = \mathrm{Span}_{\bar{k}}\Bigg(\bigcup_{s=0}^{n} \bigcup_{i=1}^{r} \bigcup_{j=1}^{r-1} \{\partial_\theta^s(\lambda_j),\, \partial_\theta^s(\mathrm{F}_{\tau^j}(\lambda_i))\}\Bigg). \tag{3.4.3}$$

By using Theorems 2.3.2 and 3.4.1, computing the dimension of the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ for $n \geq 1$ proves Theorem 1.1.3. Moreover, by (3.4.2) if we are able to construct appropriate t-motives whose periods span the hyperderivatives in question and determine the dimension of their associated Galois groups, then we can prove Theorem 1.1.4.

4. Hyperderivatives of periods and quasiperiods

Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep}. Let K_ρ be the fraction field of $\mathrm{End}(\rho)$ defined as in (3.2.2) and let $[K_\rho : k] = \boldsymbol{s}$. In this section, we prove Theorem 1.1.3 (restated as Theorem 4.5.1). To prove this theorem, we first show in Theorem 4.3.3 that $\dim \Gamma_{\mathrm{P}_n M_\rho} \geq (n+1)\cdot r^2/\boldsymbol{s}$, and in Theorem 4.4.6 that $\dim \Gamma_{\mathrm{P}_n M_\rho} \leq (n+1)\cdot r^2/\boldsymbol{s}$. Moreover, in Corollary 4.4.8 we explicitly compute the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ for all $n \geq 1$.

4.1. ***The $\mathfrak{p}$-adic Tate module and Anderson generation functions.*** Let ϕ be a uniformizable and $\boldsymbol{A}$-finite Anderson t-module of dimension d and rank r. For any $a \in \boldsymbol{A}$, the torsion $\boldsymbol{A}$-module $\phi[a] := \{\boldsymbol{x} \in \mathbb{K}^d \mid \phi_a(\boldsymbol{x}) = 0\}$ is isomorphic to $(\boldsymbol{A}/(a))^{\oplus r}$ (see [Anderson 1986; Thakur 2004, Theorem 7.2.1]). For a nonzero prime $\mathfrak{p}$ of $\boldsymbol{A}$, we define the $\mathfrak{p}$-*adic Tate module*

$$T_\mathfrak{p}(\phi) := \varprojlim_m \phi[\mathfrak{p}^m] \cong \boldsymbol{A}_\mathfrak{p}^{\oplus r},$$

where $\boldsymbol{A}_\mathfrak{p}$ is the completion of $\boldsymbol{A}$ at $\mathfrak{p}$. Now, we fix a Drinfeld $\boldsymbol{A}$-module ρ of rank r. If ρ is defined over K such that $k \subseteq K \subseteq \bar{k}$ and $[K : k] < \infty$, then note that every element of $\rho[\mathfrak{p}^m]$ is separable over K. Thus, the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of the separable closure of K inside $\bar{k}$ acts on $T_\mathfrak{p}(\rho)$, defining a representation

$$\varphi_\mathfrak{p} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_\mathfrak{p}(\rho)) \cong \mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p}).$$

Set $\mathrm{p} := \mathfrak{p}(\theta) \in A$. We fix an $\boldsymbol{A}$-basis $\{\lambda_1, \ldots, \lambda_r\}$ of Λ_ρ and define

$$\xi_{i,m} := \operatorname{Exp}_\rho\left(\frac{\lambda_i}{\mathrm{p}^{m+1}}\right) \in \rho[\mathfrak{p}^{m+1}]$$

for each $1 \le i \le r$ and $m \ge 0$. Then, $\{x_1, \ldots, x_r\}$ is an $\boldsymbol{A}_\mathfrak{p}$-basis of $T_\mathfrak{p}(\rho)$, where we set $x_i := (\xi_{i,0}, \xi_{i,1}, \ldots)$. Set $\boldsymbol{x} := [x_1, \ldots, x_r]^\mathsf{T}$. Then, for each $\epsilon \in \operatorname{Gal}(K^{\mathrm{sep}}/K)$ there exists $g_\epsilon \in \operatorname{GL}_r(\boldsymbol{A}_\mathfrak{p})$ such that

$$\varphi_\mathfrak{p}(\epsilon)\boldsymbol{x} = g_\epsilon \boldsymbol{x}. \tag{4.1.1}$$

Theorem 4.1.2 [Maurischat and Perkins 2022, Theorem 1.2]. *Let ρ be a Drinfeld $\boldsymbol{A}$-module defined over K such that $k \subseteq K \subseteq \bar{k}$ and $[K : k] < \infty$. Let $\boldsymbol{k}_\mathfrak{p}$ be the field of fractions of $\boldsymbol{A}_\mathfrak{p}$. For each $\epsilon \in \operatorname{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon \in \operatorname{GL}_r(\boldsymbol{A}_\mathfrak{p})$ be as in (4.1.1). Then, the assignment $\epsilon \mapsto g_\epsilon$ induces a group homomorphism*

$$\beta_0 : \operatorname{Gal}(K^{\mathrm{sep}}/K) \to \Gamma_{\Psi_\rho}(\boldsymbol{A}_\mathfrak{p}) := \operatorname{GL}_r(\boldsymbol{A}_\mathfrak{p}) \cap \Gamma_{\Psi_\rho}(\boldsymbol{k}_\mathfrak{p}).$$

Note that in the case of $\mathfrak{p} = t$, Theorem 4.1.2 was first proved by Chang and Papanikolas [2012, Theorem 3.5.1].

For the remainder of this subsection, we fix $n \ge 0$. By [Namoijam and Papanikolas 2024, Proposition 5.27], we have that, for $\boldsymbol{z} = [z_0, \ldots, z_n]^\mathsf{T}$,

$$\operatorname{Exp}_{\mathrm{P}_n\rho}(\boldsymbol{z}) = [\operatorname{Exp}_\rho(z_0), \ldots, \operatorname{Exp}_\rho(z_n)]^\mathsf{T}. \tag{4.1.3}$$

For $u \in \mathbb{K}$, set

$$(u)_j := [0, \ldots, 0, u, 0, \ldots, 0]^\mathsf{T} =\in \mathbb{K}^{n+1}, \tag{4.1.4}$$

where u is in the j-th entry and all other entries are 0. By (4.1.3), using the $\boldsymbol{A}$-basis $\{\lambda_1, \ldots, \lambda_r\}$ of Λ_ρ, an $\boldsymbol{A}$-basis of the period lattice $\Lambda_{\mathrm{P}_n\rho}$ of $\mathrm{P}_n\rho$ is

$$\{(\lambda_i)_j : 1 \le i \le r \text{ and } 1 \le j \le n+1\}.$$

We define

$$\chi_{i,m} := \operatorname{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right)$$

for each $1 \le i \le r$ and $m \ge 0$. By (3.1.3), the Anderson generating function $f_i(t) := f_{\lambda_i}(t)$ of ρ with respect to λ_i is

$$f_i(t) = \sum_{m=0}^{\infty} \operatorname{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right)t^m = \sum_{m=0}^{\infty} \chi_{i,m} t^m \in \mathbb{T} \cap K^{\mathrm{sep}}[[t]].$$

For each $1 \le i \le r$ and $1 \le j \le n+1$ we let $\mathcal{G}_{i,j}(t) := \mathcal{G}_{(\lambda_i)_j}(t)$ denote the Anderson generating function of $\mathrm{P}_n\rho$ with respect to $(\lambda_i)_j$. Then, by (3.1.3) we have

$$\mathcal{G}_{i,j}(t) = \sum_{m=0}^{\infty} \operatorname{Exp}_{\mathrm{P}_n\rho}\big((\mathrm{d}(\mathrm{P}_n\rho)_t)^{-m-1}(\lambda_i)_j\big)t^m \in \mathbb{T}^{n+1} \cap K^{\mathrm{sep}}[[t]]^{n+1}.$$

Observe that in (3.2.5), the subdiagonal entries of $\mathrm{d}(\mathrm{P}_n\rho)_t$ are $-\partial_\theta^1(\theta)$. Also, $(-1)^c\partial_\theta^c(\theta)=0$ for $c\geq 2$. Moreover, the e-th subdiagonal entries of $\mathrm{d}(\mathrm{P}_n\rho)_t^{-1}$ are $(-1)^e\partial_\theta^e(\theta^{-1})$ and so, by the product rule of hyperderivatives, for $h\in\mathbb{Z}$ we have

$$(\mathrm{d}(\mathrm{P}_n\rho)_t)^h=\begin{pmatrix} \theta^h & & & & & \\ -\partial_\theta^1(\theta^h) & \theta^h & & & & \\ \partial_\theta^2(\theta^h) & -\partial_\theta^1(\theta^h) & \theta^h & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ (-1)^n\partial_\theta^n(\theta^h) & \cdots & \cdots & \partial_\theta^2(\theta^h) & -\partial_\theta^1(\theta^h) & \theta^h \end{pmatrix}.$$

Note that for $m\geq 0$ and $c\geq 1$, we have $(-1)^c\partial_\theta^c(\theta^{-m-1})=\binom{m+c}{c}\theta^{-m-1-c}$. Then, it follows by using (4.1.3) that

$$\begin{aligned}\mathcal{G}_{i,j}(t)&=\sum_{m=0}^{\infty}\left(0,\dots,0,\,\chi_{i,m},\,\binom{m+1}{1}\chi_{i,m+1},\,\dots,\,\binom{m+(n+1-j)}{n+1-j}(\chi_{i,m+(n+1-j)})\right)^{\mathsf{T}}t^m\\ &=\left(0,\dots,0,\,\sum_{m=0}^{\infty}\chi_{i,m}t^m,\,\sum_{m=1}^{\infty}\chi_{i,m}\binom{m}{1}t^{m-1},\,\dots,\,\sum_{m=n+1-j}^{\infty}\chi_{i,m}\binom{m}{n+j-1}t^{m-(n+j-1)}\right)^{\mathsf{T}}.\end{aligned}$$

Thus,

$$\mathcal{G}_{i,j}(t)=\left(0,\dots,0,\,f_i,\,\partial_t^1(f_i),\,\dots,\,\partial_t^{n+1-j}(f_i)\right)^{\mathsf{T}}\in\mathbb{T}^{n+1}. \tag{4.1.5}$$

For our purpose, we consider the Galois group $\Gamma_{\Psi_{\mathrm{P}_n\rho}}$ and its principal homogeneous space $Z_{\Psi_{\mathrm{P}_n\rho}}$ as in Section 2.3, and we prove the following result.

Theorem 4.1.6. *Let ρ be a Drinfeld $\boldsymbol{A}$-module defined over K such that $k\subseteq K\subseteq\bar{k}$ and $[K:k]<\infty$, and for $n\geq 0$ let $\mathrm{P}_n\rho$ be its n-th prolongation t-module. Let $\boldsymbol{k}_\mathfrak{p}$ be the fraction field of $\boldsymbol{A}_\mathfrak{p}$. For each $\epsilon\in\mathrm{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon\in\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ be as in (4.1.1). Then, the assignment $\epsilon\mapsto d_{t,n+1}[g_\epsilon]$ induces a group homomorphism*

$$\beta_n:\mathrm{Gal}(K^{\mathrm{sep}}/K)\to\Gamma_{\Psi_{\mathrm{P}_n\rho}}(\boldsymbol{A}_\mathfrak{p}):=\mathrm{GL}_{(n+1)r}(\boldsymbol{A}_\mathfrak{p})\cap\Gamma_{\Psi_{\mathrm{P}_n\rho}}(\boldsymbol{k}_\mathfrak{p}).$$

We follow the methods used in [Maurischat and Perkins 2022]. Let $\overline{\mathbb{F}}_q$ denote an algebraic closure of $\mathbb{F}_q$ inside $\mathbb{K}$ and let $\zeta\in\overline{\mathbb{F}}_q$ be a root of $\mathfrak{p}$. We define the $\mathbb{K}$-algebra map $\mathcal{D}_\zeta:\mathbb{T}\to\mathbb{K}[\![X]\!]$ by

$$g\mapsto\sum_{m=0}^{\infty}\partial_t^m(g)|_{t=\zeta}X^m.$$

By [loc. cit., Lemma 2.2], the map $\mathcal{D}_\zeta:\boldsymbol{A}\to\mathbb{K}[\![X]\!]$ extends to an isomorphism $\mathcal{D}_\zeta:\boldsymbol{A}_\mathfrak{p}\to\mathbb{F}_q(\zeta)[\![X]\!]$.

The Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on $K^{\mathrm{sep}}[\![X]\!]$ by acting on each coefficient. We now consider the Galois action on Anderson generating functions of $\mathrm{P}_n\rho$ and their Frobenius twists.

Proposition 4.1.7 (cf. [Maurischat and Perkins 2022, Proposition 4.2]). *For each $\epsilon\in\mathrm{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon\in\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ be defined as in (4.1.1). Let $\mathcal{G}:=[\mathcal{G}_{1,1},\dots,\mathcal{G}_{r,1},\dots,\mathcal{G}_{1,n+1},\dots,\mathcal{G}_{r,n+1}]^{\mathsf{T}}\in\mathrm{Mat}_{r(n+1)\times n+1}(\mathbb{T})$. Then,*

$$\epsilon(\mathcal{D}_\zeta(\mathcal{G}))=\mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[g_\epsilon])\cdot\mathcal{D}_{\epsilon(\zeta)}(\mathcal{G}),$$

where

$$\epsilon(\mathcal{D}_\zeta(\mathcal{G})) = [\epsilon(\mathcal{D}_\zeta(\mathcal{G}_{1,1})), \dots, \epsilon(\mathcal{D}_\zeta(\mathcal{G}_{r,1})), \dots, \epsilon(\mathcal{D}_\zeta(\mathcal{G}_{1,n+1})), \dots, \epsilon(\mathcal{D}_\zeta(\mathcal{G}_{r,n+1}))]^{\mathsf{T}}.$$

Proof. Note that by (4.1.5), the j-th column of $\mathcal{G}$ for $1 \le j \le n+1$ is

$$[\partial_t^{j-1}(f_1), \dots, \partial_t^{j-1}(f_r), \partial_t^{j-2}(f_1), \dots, \partial_t^{j-2}(f_r), \dots, f_1, \dots, f_r, 0, \dots, 0]^{\mathsf{T}} \in \mathbb{T}^{r(n+1)}.$$

Then, for $m_1, m_2 \in \mathbb{N}$ and $1 \le i \le r$, since $\partial_t^{m_1}(\partial_t^{m_2}(f_i)) = \binom{m_1+m_2}{m_1}\partial_t^{m_1+m_2}(f_i)$, the result follows by using [Maurischat and Perkins 2022, Lemma 4.1]. □

Proposition 4.1.8 (cf. [Maurischat and Perkins 2022, Proposition 5.1]). *For $1 \le i, j \le r$, define $\Upsilon \in \mathrm{Mat}_r(\mathbb{T})$ so that $\Upsilon_{ij} := f_i^{(j-1)}(t)$ as in (3.3.1). Then, for any $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $g_\epsilon \in \mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ as in (4.1.1), we have*

$$\epsilon(\mathcal{D}_\zeta(d_{t,n+1}[\Upsilon]^{(1)})) = \mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[g_\epsilon]) \cdot \mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[\Upsilon]^{(1)})$$

and

$$\epsilon(\mathcal{D}_\zeta(\Psi_{\mathrm{P}_n\rho})) = \mathcal{D}_{\epsilon(\zeta)}(\Psi_{\mathrm{P}_n\rho}) \cdot \mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[g_\epsilon])^{-1}.$$

Proof. Since Frobenius twisting commutes with hyperdifferentiation with respect to t, we see by using (4.1.5) that for $1 \le j \le r$ and $0 \le \ell \le n$, the $(\ell r+j)$-th column of $d_{t,n+1}[\Upsilon^{(1)}]$ is given by the j-th Frobenius twist of the $(\ell+1)$-th column of $\mathcal{G}$. Moreover, by (3.3.3) we have $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[V]^{-1} d_{t,n+1}[\Upsilon^{(1)}]^{-1}$. Then by using Proposition 4.1.7, the results follow by a straightforward adaptation of the proof of [loc. cit., Proposition 5.1]. □

By an abuse of the notation $\mathcal{D}_\zeta$, we consider the homomorphism $\mathcal{D}_\zeta : \mathbb{T} \otimes_{\boldsymbol{A}} \boldsymbol{A}_\mathfrak{p} \to \mathbb{K}[\![X]\!]$ defined by

$$\sum_i g_i \otimes b_i \mapsto \sum_i \mathcal{D}_\zeta(g_i) \cdot \mathcal{D}_\zeta(b_i).$$

Note that $\mathcal{D}_\zeta$ is injective on $\mathbb{T}$, and so it extends to $\mathbb{L} \otimes_{\mathbb{T}} (\mathbb{T} \otimes_{\boldsymbol{A}} \boldsymbol{A}_\mathfrak{p}) \cong \mathbb{L} \otimes_{\boldsymbol{k}} \boldsymbol{k}_\mathfrak{p}$, that is, to a ring homomorphism

$$\widetilde{\mathcal{D}}_\zeta : \mathbb{L} \otimes_{\boldsymbol{k}} \boldsymbol{k}_\mathfrak{p} \to \mathbb{K}(\!(X)\!).$$

Proof of Theorem 4.1.6. Let $S \subseteq K^{\mathrm{per}}(t)[Y, 1/\det Y]$ denote a finite set of generators of the defining ideal of $Z_{\Psi_{\mathrm{P}_n\rho}}$, where K^{per} is the perfect closure of K in $\mathbb{K}$. Then, for any $h \in S$, we have $h(\Psi_{\mathrm{P}_n\rho}) = 0$. If $\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1} \in Z_{\Psi_{\mathrm{P}_n\rho}}(\mathbb{K}(\!(t)\!))$, then by Theorem 2.3.1 we have $d_{t,n+1}[g_\epsilon]^{-1} \in \Gamma_{\Psi_{\mathrm{P}_n\rho}}(\boldsymbol{k}_\mathfrak{p})$. Thus, to prove our result, we will show that $h(\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1}) = 0$ for every $h \in S$. The proof follows by a straightforward adaptation of the proof of [Maurischat and Perkins 2022, Theorem 1.2], but for completeness we provide a proof.

For $h \in S$, let $h_\zeta \in \mathbb{K}(\!(X)\!)[Y, 1/\det Y]$ denote its image after mapping its coefficients via the map $\widetilde{\mathcal{D}}_\zeta$. Then,

$$\begin{aligned}\widetilde{\mathcal{D}}_\zeta\big(h(\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1})\big) &= h_\zeta\big(\mathcal{D}_\zeta(\Psi_{\mathrm{P}_n\rho}) \cdot \mathcal{D}_\zeta(d_{t,n+1}[g_\epsilon]^{-1})\big)\\ &= h_\zeta\big(\epsilon(\mathcal{D}_{\epsilon^{-1}(\zeta)}(\Psi_{\mathrm{P}_n\rho}))\big) = \epsilon\big(h_{\epsilon^{-1}(\zeta)}(\mathcal{D}_{\epsilon^{-1}(\zeta)}(\Psi_{\mathrm{P}_n\rho}))\big)\\ &= \epsilon\big(\widetilde{\mathcal{D}}_{\epsilon^{-1}(\zeta)}(h(\Psi_{\mathrm{P}_n\rho}))\big) = 0, \qquad (4.1.9)\end{aligned}$$

where the second equality is by Proposition 4.1.8 and we obtain the third equality since the coefficients of h are in $K^{\mathrm{per}}(t)$. Since ζ is an arbitrary root of $\mathfrak{p}$, it follows by [Maurischat and Perkins 2022, Lemma 5.3] that $h(\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1}) = 0$. □

4.2. *Elements of* $\Gamma_{\mathrm{P}_n M_\rho}$. Let ρ a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep}, and consider the t-motive M_ρ associated to ρ (see Section 3.1). In this subsection, for $n \geq 1$ we study the structure of the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ of the n-th prolongation t-motive $\mathrm{P}_n M_\rho$. We let $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ denote the ring of endomorphisms of $\mathrm{P}_n M_\rho$ and set $\boldsymbol{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$. If the entries of $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ form a $\bar{k}(t)$-basis of M_ρ, then the entries of $\boldsymbol{D}_n\boldsymbol{m}$ form a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$ as in (2.5.1). Given $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$, let $\mathrm{H} \in \mathrm{Mat}_{r(n+1)}(\bar{k}(t))$ be such that $h(\boldsymbol{D}_n\boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n\boldsymbol{m}$. Since $h\sigma = \sigma h$ and $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, we have

$$d_{t,n+1}[\Phi_\rho]\mathrm{H} = \mathrm{H}^{(-1)} d_{t,n+1}[\Phi_\rho].$$

From this, we see σ fixes $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho]$, and thus $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho] \in \mathrm{Mat}_{r(n+1)}(\boldsymbol{k})$. We have thus defined the injective map

$$\begin{aligned} \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho) &\to \mathrm{End}((\mathrm{P}_n M_\rho)^B) = \mathrm{Mat}_{r(n+1)}(\boldsymbol{k}), \\ h &\mapsto h^B := d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho]. \end{aligned} \tag{4.2.1}$$

Since the tautological representation $\varpi_n : \Gamma_{\mathrm{P}_n M_\rho} \to \mathrm{GL}((\mathrm{P}_n M_\rho)^B)$ is functorial in $\mathrm{P}_n M_\rho$ [Papanikolas 2008, Theorem 4.5.3], for any $\boldsymbol{k}$-algebra R and $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$, it follows that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B & \xrightarrow{\varpi_n^{\mathrm{R}}(\mu)} & \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B \\ \big\downarrow{\scriptstyle 1\otimes h^B} & & \big\downarrow{\scriptstyle 1\otimes h^B} \\ \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B & \xrightarrow{\varpi_n^{\mathrm{R}}(\mu)} & \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B \end{array} \tag{4.2.2}$$

Proposition 4.2.3. *Given $f \in \boldsymbol{K}_\rho$, let $\mathrm{F} \in \mathrm{Mat}_r(\bar{k}(t))$ satisfy $f(\boldsymbol{m}) = \mathrm{F}\boldsymbol{m}$. Also, for $n \geq 1$ let $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ be such that $h(\boldsymbol{D}_n\boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n\boldsymbol{m}$, where $\mathrm{H} = (\mathrm{H}_{ij}) \in \mathrm{Mat}_{r(n+1)}(\bar{k}(t))$ and each H_{ij} is an $r \times r$ block for $1 \leq i, j \leq n+1$. Then*:

(a) *For $n \geq 1$ there exists $g \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ such that $g(\boldsymbol{D}_n\boldsymbol{m}) = d_{t,n+1}[\mathrm{F}]\boldsymbol{D}_n\boldsymbol{m}$.*

(b) *For $0 \leq j \leq n-1$, the matrix $\mathrm{H}_j := (\mathrm{H}_{uv}) \in \mathrm{Mat}_{r(j+1)}(\bar{k}(t))$, $j+1 \leq u \leq n+1$, $1 \leq v \leq j+1$, formed by the lower left $r(j+1) \times r(j+1)$ square of H represents an element of $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_j M_\rho)$.*

Proof. For part (a), since $f\sigma = \sigma f$, we have $\Phi_\rho \mathrm{F} = \mathrm{F}^{(-1)}\Phi_\rho$. Since multiplication by σ on $\mathrm{P}_n M_\rho$ is represented by $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, the proof of (a) follows from the observation that

$$d_{t,n+1}[\Phi_\rho]d_{t,n+1}[\mathrm{F}] = d_{t,n+1}[\mathrm{F}]^{(-1)} d_{t,n+1}[\Phi_\rho].$$

For part (b), using $d_{t,n+1}[\Phi_\rho]\mathrm{H} = \mathrm{H}^{(-1)} d_{t,n+1}[\Phi_\rho]$ and the definition of d-matrices, we see that, for $0 \leq j \leq n-1$,

$$d_{t,j+1}[\Phi_\rho]\mathrm{H}_j = \mathrm{H}_j^{(-1)} d_{t,j+1}[\Phi_\rho],$$

and the result follows. □

For any $n \geq 1$ and $0 \leq j \leq n-1$, since $\mathrm{P}_{n-j-1}M_\rho$ is a sub-t-motive of $\mathrm{P}_n M_\rho$, we have a surjective map of affine group schemes over $\boldsymbol{k}$,

$$\boldsymbol{\pi}_{n-j-1} : \Gamma_{\mathrm{P}_n M_\rho} \twoheadrightarrow \Gamma_{\mathrm{P}_{n-j-1} M_\rho}. \tag{4.2.4}$$

We are now ready to prove the main result of this subsection.

Theorem 4.2.5. *For each $n \geq 1$ and any $\boldsymbol{k}$-algebra* R, *an element of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form*

$$\mu_n = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} & \gamma_n \\ & \gamma_0 & \gamma_1 & \ddots & \gamma_{n-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \gamma_1 \\ & & & & \gamma_0 \end{pmatrix}, \tag{4.2.6}$$

where, for each $0 \leq i \leq n$, γ_i is an $r \times r$ block. Furthermore, for $0 \leq j \leq n-1$, the matrix μ_{n-j-1} formed by the upper left $r(n-j) \times r(n-j)$ square is an element of $\Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$. In particular, the map $\boldsymbol{\pi}^{(\mathrm{R})}_{n-j-1} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \twoheadrightarrow \Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$ maps an element μ_n of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ to the matrix μ_{n-j-1}.

Proof. Since the prolongation of an $\boldsymbol{A}$-finite dual t-motive is also an $\boldsymbol{A}$-finite dual t-motive, by (2.5.4) for any $n \geq 1$ and $0 \leq j \leq n-1$ we obtain a short exact sequence of t-motives

$$0 \to \mathrm{P}_j M_\rho \xrightarrow{\iota} \mathrm{P}_n M_\rho \xrightarrow{\mathbf{pr}_{n-j-1}} \mathrm{P}_{n-j-1} M_\rho \to 0, \tag{4.2.7}$$

where $\mathbf{pr}_{n-j-1}(D_i m) := D_{i-j-1} m$ for $i > j$ and $\mathbf{pr}_{n-j-1}(D_i m) := 0$ for $i \leq j$ and $m \in M_\rho$, and ι is the inclusion map. Note that $\mathrm{P}_0 M_\rho \cong M_\rho$ via $D_0 m \mapsto m$ for all $m \in M_\rho$.

For any $\boldsymbol{k}$-algebra R, we recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ from [Papanikolas 2008, §4.5]. Since $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$, the entries of $\boldsymbol{u}_n := d_{t,n+1}[\Psi_\rho]^{-1}\boldsymbol{D}_n\boldsymbol{m}$ form a $\boldsymbol{k}$-basis of $(\mathrm{P}_n M_\rho)^B$ [loc. cit., Proposition 3.3.9] and similarly, for $0 \leq j \leq n-1$, we have that the entries of $\boldsymbol{u}_{n-j-1} := d_{t,n-j}[\Psi_\rho]^{-1}\boldsymbol{D}_{n-j-1}\boldsymbol{m}$ form a $\boldsymbol{k}$-basis of $(\mathrm{P}_{n-j-1}M_\rho)^B$. For any $\mu_n \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $a_i \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \leq i \leq n$, the action of μ_n on $(a_0, \ldots, a_n) \cdot \boldsymbol{u}_n \in \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ is

$$(a_0, \ldots, a_n) \cdot d_{t,n+1}[\Psi_\rho]^{-1}\boldsymbol{D}_n\boldsymbol{m} \mapsto (a_0, \ldots, a_n) \cdot \mu_n^{-1} d_{t,n+1}[\Psi_\rho]^{-1}\boldsymbol{D}_n\boldsymbol{m}. \tag{4.2.8}$$

We first restrict the action of μ_n to $\mathrm{R}\otimes_{\boldsymbol{k}}(\mathrm{P}_j M_\rho)^B$ via the map ι in (4.2.7). So, we take $a_0, \ldots, a_{n-j-1} = 0$ and set $\mu_n^{-1} := (\mathrm{B}_{uw})$, $1 \leq u, w \leq n+1$, where each B_{uw} is an $r \times r$ block. By ι in (4.2.7), we see that μ_n leaves $(\mathrm{P}_j M_\rho)^B$ invariant and thus

$$\mathrm{B}_{n-j+v,1} = \mathrm{B}_{n-j+v,2} = \cdots = \mathrm{B}_{n-j+v,n-j} = \mathbf{0} \quad \text{for } 1 \leq v \leq j+1.$$

Moreover, since the nonzero a_i were chosen arbitrarily, we see that the matrix formed by the lower right $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$. Varying j from 0 to $n-1$, we see that μ_n^{-1} is a block upper triangular matrix and that the matrix formed by the lower right $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$ for each $0 \leq j \leq n-1$.

We return to arbitrary $a_i \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \le i \le n$. We restrict the action of μ_n to $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_{n-j-1}M_\rho)^B$ via the map $\mathbf{pr}_{n-j-1}$ in (4.2.7). Through $\mathbf{pr}_{n-j-1}$, we see that μ_n leaves $(\mathrm{P}_{n-j-1}M_\rho)^B$ invariant and so the matrix μ_{n-j-1} formed by the upper left $r(n-j)\times r(n-j)$ square of μ_n is an element of $\Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$. Varying j from 0 to $n-1$, we see that the matrices μ_{n-j-1} formed by the upper left $r(n-j)\times r(n-j)$ square of μ_n is an element of $\Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$ for each $0 \le j \le n-1$.

Now, we let $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ be such that for $\mathrm{H} \in \mathrm{Mat}_{r(n+1)}(\bar{k}(t))$ we have $h(\boldsymbol{D}_n\boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n\boldsymbol{m}$. Let $\mathrm{H} := (\mathrm{H}_{iw})$, where each (H_{iw}) is an $r\times r$ block. For $0 \le j \le n-1$, let $\mathrm{H}_j := (\mathrm{H}_{uv}) \in \mathrm{Mat}_{r(j+1)}(\bar{k}(t))$, $j+1 \le u \le n+1$, $1 \le v \le j+1$, be the matrix formed by the lower left $r(j+1)\times r(j+1)$ square of H. Using the definition of d-matrices, we see that the matrix formed by the lower left $r(j+1)\times r(j+1)$ square of $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}\, d_{t,n+1}[\Psi_\rho]$ is $d_{t,j+1}[\Psi_\rho]^{-1}\mathrm{H}_j\, d_{t,j+1}[\Psi_\rho]$. By Proposition 4.2.3(b), we have that $d_{t,j+1}[\Psi_\rho]^{-1}\mathrm{H}_j\, d_{t,j+1}[\Psi_\rho]$ is an element in the image of the natural embedding (4.2.1) for the j-th prolongation. Thus, by using the commutative diagram (4.2.2) for the n-th and the $(n-1)$-th prolongations, we see that since μ_n is upper triangular, the matrices formed by the lower right $rn\times rn$ square and the upper left $rn\times rn$ square of μ_n are equal. Comparing each $r\times r$ block in this equality, we get the required result. □

4.3. ***Lower bound on the dimension of*** $\boldsymbol{\Gamma_{\mathrm{P}_n M_\rho}}$**.** For this subsection, the reader is directed to the Appendix for details about differential algebra and differential algebraic geometry in characteristic $p > 0$. We note that the purpose of the Appendix is for use in this subsection to prove Theorem 4.3.3. For a nonzero prime $\mathfrak{p} \in \boldsymbol{A}$, let $\boldsymbol{A}_\mathfrak{p}$ denote the completion of $\boldsymbol{A}$ at $\mathfrak{p}$, and let $\boldsymbol{k}_\mathfrak{p}$ be the fraction field of $\boldsymbol{A}_\mathfrak{p}$. By the properties of hyperderivatives (see Section 2.4) we see that $(\boldsymbol{k}_\mathfrak{p}, \partial_t)$, where ∂_t represents hyperdifferentiation with respect to t, is a ∂_t-field. Using Theorem 4.2.5, by a slight abuse of notation, we make the choice to let the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$ be

$$\boldsymbol{X} := \begin{pmatrix} \boldsymbol{X}_0 & \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & \boldsymbol{X}_1 \\ & & & \boldsymbol{X}_0 \end{pmatrix}, \tag{4.3.1}$$

where $\boldsymbol{X}_h := ((X_h)_{i,j})$, an $r\times r$ matrix for $0 \le h \le n$. We set $\partial_t^\ell(\boldsymbol{X}_h) := (\partial_t^\ell((X_h)_{i,j}))$ and

$$\mathbf{vec}(\boldsymbol{X}_h) := [(X_h)_{1,1}, \dots, (X_h)_{r,1}, (X_h)_{1,2}, \dots, (X_h)_{r,2}, \dots, (X_h)_{1,r}, \dots, (X_h)_{r,r}]^\mathsf{T},$$

which consists of all entries of $\boldsymbol{X}_h$ lined up in a column vector.

Let $0 \le \alpha \le n$. As in Section A.1, we define $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha\}$ to be the ∂_t-polynomial ring over $\boldsymbol{k}_\mathfrak{p}$ with entries of each $\boldsymbol{X}_h$ for $0 \le h \le \alpha$ as ∂_t-indeterminates. We also define $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha, 1/\det \boldsymbol{X}_0\}$ to be the localization of $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha\}$ at $\det \boldsymbol{X}_0$. We define $\boldsymbol{k}_\mathfrak{p}[\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha]$ to be the usual polynomial ring over $\boldsymbol{k}_\mathfrak{p}$ with entries of each $\boldsymbol{X}_h$ for $h = 0, \dots, \alpha$ as indeterminates, and $\boldsymbol{k}_\mathfrak{p}[\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha, 1/\det \boldsymbol{X}_0]$ to be the localization of $\boldsymbol{k}_\mathfrak{p}[\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha]$ at $\det \boldsymbol{X}_0$.

We define the centralizer $\mathrm{Cent}_{\mathrm{GL}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ to be the algebraic group over $\boldsymbol{k}$ such that, for any $\boldsymbol{k}$-algebra R,

$$\mathrm{Cent}_{\mathrm{GL}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)(\mathrm{R}) := \{\gamma \in \mathrm{GL}_r(\mathrm{R}) : \gamma g = g\gamma \text{ for all } g \in \mathrm{R} \otimes_{\boldsymbol{k}} \boldsymbol{K}_\rho \subseteq \mathrm{Mat}_r(\mathrm{R})\}.$$

By [Pink 1997, Theorem 0.2] and [Pink and Rütsche 2009, Theorem 0.2], the image $\operatorname{Im}\beta_0$ of the homomorphism β_0 in Theorem 4.1.2 is equal to $\operatorname{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})}(\boldsymbol{K}_\rho)$ for all but finitely many primes of $\boldsymbol{A}$. Therefore, let $\mathfrak{p} \in \boldsymbol{A}$ be a nonzero prime such that $\operatorname{Im}\beta_0 = \operatorname{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})}(\boldsymbol{K}_\rho)$. Then, by [Chang and Papanikolas 2012, Theorem 3.5.4] we see that

$$\Gamma_{M_\rho}(\boldsymbol{A}_\mathfrak{p}) = \operatorname{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})}(\boldsymbol{K}_\rho) = \operatorname{Im}\beta_0. \tag{4.3.2}$$

Theorem 4.3.3. *Fix $n \geq 1$. Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let K_ρ be the fraction field of* $\operatorname{End}(\rho)$ *defined as in (3.2.2) and suppose that $[K_\rho : k] = \boldsymbol{s}$. Then,*

$$\dim \Gamma_{\mathrm{P}_n M_\rho} \geq (n+1)\frac{r^2}{\boldsymbol{s}}.$$

Proof. By Theorem 4.1.6, we see that the Zariski closure $\overline{\operatorname{Im}\beta_n}^Z$ of $\operatorname{Im}\beta_n$ is an algebraic subgroup of $\Gamma_{\mathrm{P}_n M_\rho}$. Therefore, our task is to prove that $\dim(\overline{\operatorname{Im}\beta_n}^Z) = (n+1)r^2/\boldsymbol{s}$. By [Chang and Papanikolas 2012, Theorem 3.5.4], we have $\Gamma_{M_\rho} = \operatorname{Cent}_{\mathrm{GL}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ and $\dim \Gamma_{M_\rho} = r^2/\boldsymbol{s}$. Since the defining polynomials of $\operatorname{Cent}_{\mathrm{Mat}_r(\boldsymbol{k})}(\boldsymbol{K}_\rho) = \operatorname{Lie}\Gamma_{M_\rho}$ are homogeneous degree-1 polynomials, let its defining equations be

$$\sum_{i,j=1}^{r} (b_u)_{i,j}(X_0)_{i,j} = 0, \quad (b_u)_{i,j} \in \boldsymbol{k},\ u = 1, \dots, r^2 - r^2/\boldsymbol{s}, \tag{4.3.4}$$

which can be written as

$$\boldsymbol{B} \cdot \mathbf{vec}(\boldsymbol{X}_0) = \mathbf{0}, \tag{4.3.5}$$

where we set $\boldsymbol{B}$ to be the $(r^2 - r^2/\boldsymbol{s}) \times r^2$ matrix of full rank with $(b_u)_{ij}$ as the $u \times ((j-1)r+i)$-th entry. We see that rank $\boldsymbol{B} = r^2 - \dim \Gamma_{M_\rho} = r^2 - r^2/\boldsymbol{s}$. Therefore, the defining ideal of Γ_{M_ρ} is the ideal generated by the entries of $\boldsymbol{B} \cdot \mathbf{vec}(\boldsymbol{X}_0)$ in $\boldsymbol{k}[\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0]$, the coordinate ring of $\mathrm{GL}_r/\boldsymbol{k}$.

For $0 \leq \alpha \leq n$, we define a monomial order on $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha\}$ and use the division algorithm [Iima and Yoshino 2009, Proposition 1.9] on it. We denote by $\mathbb{Z}_{\geq 0}^{(\infty)}$ the set of all sequences $(a_1, a_2, a_3, \dots)$ of nonnegative integers such that $a_i = 0$ for all but finitely many $i \geq 1$. Any monomial in $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha\}$ can be described uniquely as $\boldsymbol{X}^{\boldsymbol{b}} = \prod \partial_t^\ell((X_h)_{i,j})^{(b_{h,\ell})_{i,j}}$ for some

$$\boldsymbol{b} = (\boldsymbol{b}_{0,0}, \boldsymbol{b}_{0,1} \dots, \boldsymbol{b}_{1,0}, \boldsymbol{b}_{1,1}, \dots, \boldsymbol{b}_{\alpha,0}, \boldsymbol{b}_{\alpha,1}, \dots) \in \mathbb{Z}_{\geq 0}^{(\infty)},$$

where

$$\boldsymbol{b}_{h,\ell} = \mathbf{vec}(((b_{h,\ell})_{i,j}))^{\mathsf{T}} = [(b_{h,\ell})_{1,1}, \dots, (b_{h,\ell})_{r,1}, (b_{h,\ell})_{1,2}, \dots, (b_{h,\ell})_{r,2}, \dots, (b_{h,\ell})_{1,r}, \dots, (b_{h,\ell})_{r,r}]$$

for $0 \leq h \leq \alpha$ and $\ell \in \mathbb{Z}_{\geq 0}$ such that $((b_{h,\ell})_{i,j})$ is an $r \times r$ matrix and $(b_{h,\ell})_{i.j} = 0$ for all but a finite number of h, ℓ, i, j. We define a monomial order on $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_\alpha\}$ as in [loc. cit., Definition 1.1] in the following way:

- we set $\partial_t^\ell((X_h)_{1,1}) < \cdots < \partial_t^\ell((X_h)_{r,1}) < \cdots < \partial_t^\ell((X_h)_{1,r}) < \cdots < \partial_t^\ell((X_h)_{r,r})$,
- we set $\partial_t^\ell((X_h)_{i_1,j_1}) < \partial_t^{\ell+1}((X_h)_{i_2,j_2})$,

- we set $\partial_t^{\ell_1}((X_h)_{i_1,j_1}) < \partial_t^{\ell_2}((X_{h+1})_{i_2,j_2})$,
- we take the pure lexicographic order defined such that $X^{\boldsymbol{b}} < X^{\boldsymbol{c}}$ if the leftmost nonzero component of $\boldsymbol{b}-\boldsymbol{c}$ is negative,

where $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{(\infty)},\ \ell, \ell_1, \ell_2 \in \mathbb{Z}_{\geq 0},\ i, j, i_1, i_2, j_1, j_2 \in \{0, \dots, r\}$ and $0 \leq h \leq \alpha$.

Let $\mathfrak{I}(\operatorname{Im}\beta_0)$ denote the defining $\boldsymbol{k}_{\mathfrak{p}}$-$\partial_t$-ideal of $\operatorname{Im}\beta_0$ in $\boldsymbol{k}_{\mathfrak{p}}\{X_0, 1/\det X_0\}$, and let $\mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(X_0))$ denote the ∂_t-ideal in $\boldsymbol{k}_{\mathfrak{p}}\{X_0, 1/\det X_0\}$ generated by the homogeneous degree-1 polynomials given by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(X_0)$. Also, let $\mathfrak{R}(\boldsymbol{B}\cdot\mathbf{vec}(X_0))$ denote the radical ∂_t-ideal in $\boldsymbol{k}_{\mathfrak{p}}\{X_0, 1/\det X_0\}$ generated by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(X_0)$.

Claim 1. *We claim that* $\mathfrak{I}(\operatorname{Im}\beta_0) = \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(X_0))$.

Proof. By Proposition A.1.6, we have $\mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(X_0)) = \mathfrak{R}(\boldsymbol{B}\cdot\mathbf{vec}(X_0))$. Clearly, $\mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(X_0)) \subseteq \mathfrak{I}(\operatorname{Im}\beta_0)$. To show that $\mathfrak{I}(\operatorname{Im}\beta_0) \subseteq \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(X_0))$, let $P \in \mathfrak{I}(\operatorname{Im}\beta_0) \subseteq \boldsymbol{k}_{\mathfrak{p}}\{X_0, 1/\det X_0\}$. Let $(X_0)_{\vartheta_u,\omega_u}$ denote the leading variable of $\sum_{i,j=1}^r (b_u)_{i,j}(X_0)_{i,j}$ for $u = 1, \dots, r^2 - r^2/\boldsymbol{s}$ with respect to the monomial order above. This means that for $\ell > \vartheta_u,\ h > \omega_u$, the coefficients $(b_u)_{\ell h}$ are all 0. Moreover, by clearing denominators, we may assume that each $(b_u)_{i,j} \in \boldsymbol{A}$. Thus, the defining polynomials of $\operatorname{Im}\beta_0$ are now

$$\sum_{i=1}^{\vartheta_u}\sum_{j=1}^{\omega_u}(b_u)_{i,j}(X_0)_{i,j} = 0, \quad (b_u)_{i,j} \in \boldsymbol{A},\ u = 1, \dots, r^2 - r^2/\boldsymbol{s}, \tag{4.3.6}$$

Since the rank of $\boldsymbol{B}$ is full, we may pick $(b_u)_{i,j}$ so that for each $u = 1, \dots, r^2 - r^2/\boldsymbol{s} - 1$

$$(X_0)_{\vartheta_u,\omega_u} < (X_0)_{\vartheta_{u+1},\omega_{u+1}}.$$

By using the division algorithm [Iima and Yoshino 2009, Proposition 1.9], we can write

$$P = \sum_{u=1}^{r^2-r^2/\boldsymbol{s}}\ \sum_{\ell=0}^{\mu_u}\partial_t^{\ell}\left(\sum_{i=1}^{\vartheta_u}\sum_{j=1}^{\omega_u}(b_u)_{i,j}(X_0)_{i,j}\right)\cdot z_{\ell,u} + S,$$

where μ_u is the largest number such that $\partial_t^{\mu_u}((X_0)_{\vartheta_u,\omega_u})$ occurs as a variable in P, each $z_{\ell,u}$ is in $\boldsymbol{k}_{\mathfrak{p}}\{X_0, 1/\det X_0\}$, and the remainder S is an element of $\mathfrak{I}(\operatorname{Im}\beta_0) \setminus \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(X_0))$. Note that the variables $\partial_t^{\ell}((X_0)_{\vartheta_u,\omega_u})$ do not occur in S.

Suppose that $S \neq 0$. Then, note that there exist $\alpha \geq 0$ and $m \geq \alpha$ such that

$$S \in \boldsymbol{k}_{\mathfrak{p}}[\partial_t^{\alpha}((X_0)_{1,1}), \dots, \partial_t^{\alpha}((X_0)_{r,r}), \dots, \partial_t^{m}((X_0)_{1,1}), \dots, \partial_t^{m}((X_0)_{r,r})],$$

when S is regarded as a usual polynomial in the variables $\{\partial_t^{\ell}((X_0)_{i,j}) : \alpha \leq \ell \leq m,\ 1 \leq i, j, \leq r\}$ over $\boldsymbol{k}_{\mathfrak{p}}$. Suppose $\partial_t^{\alpha}((X_0)_{v_1,v_2})$ for some $1 \leq v_1, v_2 \leq r$ is the smallest, with respect to the above monomial order, among the variables $\partial_t^{\alpha}((X_0)_{i,j})$ occurring in S. We will show that the coefficients of S as a polynomial in the single variable $\partial_t^{\alpha}((X_0)_{v_1,v_2})$ over the ring

$$\boldsymbol{k}_{\mathfrak{p}}[\partial_t^{\alpha}((X_0)_{\gamma_1,\gamma_2}), \partial_t^{\ell}((X_0)_{i,j}) : v_1 < \gamma_1 \leq r,\ v_2 < \gamma_2 \leq r,\ 1 \leq i, j \leq r,\ \alpha < \ell \leq m]$$

are in $\mathfrak{I}(\operatorname{Im}\beta_0)$ as well.

We pick $\mathrm{v} > 1$ such that $q^{\mathrm{v}} > m$. Consider $\mathfrak{f} \in \boldsymbol{A}_{\mathfrak{p}}^{q^{\mathrm{v}}}$ of the form

$$\mathfrak{f} = \mathfrak{g} \cdot \prod_{u=1}^{r^2-r^2/s} (b_u)_{\vartheta_u,\omega_u}^{q^{\mathrm{v}}}, \tag{4.3.7}$$

where $\mathfrak{g} \in \boldsymbol{A}_{\mathfrak{p}}^{q^{\mathrm{v}}}$, $\mathfrak{g}|_{t=0} = 0$ and each $(b_u)_{\vartheta_u,\omega_u} \in \boldsymbol{A}$ is the coefficient of $(X_0)_{\vartheta_u,\omega_u}$ in (4.3.6). Note that $\mathfrak{f}|_{t=0} = 0$. Then, for $\alpha \le \ell \le m$ by using the product rule for hyperderivatives and Proposition 2.4.1 we have

$$\partial_t^{\ell}(t^{\alpha} \cdot \mathfrak{f}) = \partial_t^{\ell}(t^{\alpha}) \cdot \mathfrak{f} = \begin{cases} \mathfrak{f} & \text{for } \ell = \alpha, \\ 0 & \text{for } \alpha < \ell \le m. \end{cases}$$

For $\mathfrak{f}$ as in (4.3.7), consider $\mathfrak{G} = (\mathfrak{f}_{i,j}) \in \mathrm{Mat}_r(\boldsymbol{A}_{\mathfrak{p}})$, where we set $\mathfrak{f}_{v_1,v_2} = t^{\alpha} \cdot \mathfrak{f}$ for $(i, j) \ne (v_1, v_2), (\vartheta_u, \omega_u)$, $u = 1, \dots, r^2 - r^2/s$, we set $\mathfrak{f}_{i,j} = t^{\alpha-1} \cdot \mathfrak{f}$ (or $\mathfrak{f}_{i,j} = 0$ in the case $\alpha = 0$), and finally we pick the entries $\mathfrak{f}_{\vartheta_u,\omega_u} \in \boldsymbol{A}_{\mathfrak{p}}$ for each $u = 1, \dots, r^2 - r^2/s$ such that

$$(b_u)_{\vartheta_u,\omega_u} \cdot \mathfrak{f}_{\vartheta_u,\omega_u} = -\left(\sum_{i=1}^{\vartheta_u-1} \sum_{j=1}^{\omega_u-1} (b_u)_{i,j} \cdot \mathfrak{f}_{i,j} \right).$$

Note that each $\mathfrak{f}_{\vartheta_u,\omega_u}|_{t=0} = 0$. Then, $\mathfrak{G}$ satisfies (4.3.5), that is,

$$\boldsymbol{B} \cdot \mathbf{vec}(\mathfrak{G}) = 0.$$

Since $\mathfrak{f}_{i,j}|_{t=0} = 0$ for all $1 \le i, j \le r$, for any $\mathfrak{C} \in \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})}(\mathbf{K}_{\rho}) = \mathrm{Im}\,\beta_0$, we see that $\mathfrak{C} + \mathfrak{G} \in \mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})$. Moreover, $\mathfrak{C} + \mathfrak{G}$ satisfies $\boldsymbol{B} \cdot \mathbf{vec}(\mathfrak{C} + \mathfrak{G}) = 0$, and so

$$\mathfrak{C} + \mathfrak{G} = (\mathfrak{e}_{i,j}) \in \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})}(\mathbf{K}_{\rho}) = \mathrm{Im}\,\beta_0. \tag{4.3.8}$$

To prove $S = 0$, we adapt an argument of Maurischat [2022b, Corollary 6.4]. For any $\mathfrak{C} = (\mathfrak{c}_{i,j}) \in \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})}(\mathbf{K}_{\rho}) = \mathrm{Im}\,\beta_0$, consider the polynomial $W_{\mathfrak{C}}(Y) \in \boldsymbol{k}_{\mathfrak{p}}[Y]$ created from S by making the following assignments to the variables:

$$\begin{aligned} \partial_t^{\alpha}((X_0)_{v_1,v_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2}) + Y, \\ \partial_t^{\alpha}((X_0)_{\gamma_1,\gamma_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2}), \\ \partial_t^{\ell}((X_0)_{i,j}) &= \partial_t^{\ell}(\mathfrak{c}_{i,j}) \end{aligned}$$

for $v_1 < \gamma_1 \le r$, $v_2 < \gamma_2 \le r$, $1 \le i, j \le r$, and $\alpha < \ell \le m$. Note that for $\mathfrak{C} + \mathfrak{G}$ in (4.3.8)

$$\begin{aligned} \partial_t^{\alpha}(\mathfrak{e}_{v_1,v_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2} + t^{\alpha} \cdot \mathfrak{f}) = \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2}) + \mathfrak{f}, \\ \partial_t^{\alpha}(\mathfrak{e}_{\gamma_1,\gamma_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2} + t^{\alpha-1} \cdot \mathfrak{f}) = \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2}) \end{aligned}$$

for $v_1 < \gamma_1 \le r$, $v_2 < \gamma_2 \le r$, and

$$\partial_t^{\ell}(\mathfrak{e}_{i,j}) = \partial_t^{\ell}(\mathfrak{c}_{i,j} + t^{\alpha-1} \cdot \mathfrak{f}) = \partial_t^{\ell}(\mathfrak{c}_{i,j})$$

for $\alpha < \ell \le m$ and $1 \le i, j \le r$ such that $(i, j) \ne (\vartheta_u, \omega_u)$, where $u = 1, \dots, r^2 - r^2/s$. Thus, since the variables $\partial_t^{\ell}((X_0)_{\vartheta_u,\omega_u})$ do not occur in S, we see that $W_{\mathfrak{C}}(\mathfrak{f})$ is equal to the evaluation of S at the element $\mathfrak{C} + \mathfrak{G} \in \mathrm{Im}\,\beta_0$ and so,

$$W_{\mathfrak{C}}(\mathfrak{f}) = 0.$$

This implies that, for all $\mathfrak{C} \in \mathfrak{I}(\operatorname{Im}\beta_0)$, the single variable polynomial $W_{\mathfrak{C}}(Y)$ has infinitely many solutions $\mathfrak{f} \in \boldsymbol{A}_{\mathfrak{p}}^{q^\vee}$ of the form (4.3.7) and so $W_{\mathfrak{C}}(Y)$ is identically 0. Note that $W_{\mathfrak{C}}(\partial_t^{\alpha}((X_0)_{v_1,v_2}) - \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2}))$ is simply the polynomial in the variable $\partial_t^{\alpha}((X_0)_{v_1,v_2})$ obtained from S by letting

$$\partial_t^{\alpha}((X_0)_{\gamma_1,\gamma_2}) = \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2}), \quad \partial_t^{\ell}((X_0)_{i,j}) = \partial_t^{\ell}(\mathfrak{c}_{i,j})$$

for $v_1 < \gamma_1 \leq r,\ v_2 < \gamma_2 \leq r,\ 1 \leq i, j \leq r$, and $\alpha < \ell \leq m$. Since, for all $\mathfrak{C} \in \operatorname{Im}\beta_0$,

$$0 = W_{\mathfrak{C}}(\partial_t^{\alpha}((X_0)_{v_1,v_2}) - \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2})),$$

this implies that the coefficients of $\partial_t^{\alpha}((X_0)_{v_1,v_2})$ in the polynomial S also lie in $\mathfrak{I}(\operatorname{Im}\beta_0)$. If S' denotes such a coefficient and if $\partial_t^{\varkappa}((X_0)_{a_1,a_2})$ is the smallest variable with respect to the monomial order above occurring in S', then applying to S' the same process above, the coefficients of $\partial_t^{\varkappa}((X_0)_{a_1,a_2})$ in the polynomial S' also lie in $\mathfrak{I}(\operatorname{Im}\beta_0)$. Continuing like this, there is a nonzero element of $\boldsymbol{k}_{\mathfrak{p}}$ which is an element of $\mathfrak{I}(\operatorname{Im}\beta_0)$, which gives a contradiction to $\operatorname{Im}\beta_0 \neq \varnothing$. Thus, $S = 0$. □

Set

$$T := \mathfrak{R}\big(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0), \mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1)), \mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))\big)$$

to be the radical ∂_t-ideal in $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$ that is generated by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)$, $\mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1))$, $\mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))$, which are homogeneous degree-1 ∂_t-polynomials. Then, Proposition A.1.6 implies

$$T = \mathfrak{D}\big(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0), \mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1)), \mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))\big), \tag{4.3.9}$$

the ∂_t-ideal in $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$ generated by the set of homogeneous degree-1 ∂_t-polynomials given by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)$, $\mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1))$, $\mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))$.

Let $\mathfrak{I}(\operatorname{Im}\beta_n)$ denote the defining $\boldsymbol{k}_{\mathfrak{p}}$-$\partial_t$-ideal of $\operatorname{Im}\beta_n$ in $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$.

Claim 2. *We claim that $T = \mathfrak{I}(\operatorname{Im}\beta_n)$.*

Proof of Claim 2. By Theorem 4.1.6, clearly $T \subseteq \mathfrak{I}(\operatorname{Im}\beta_n)$. To show $\mathfrak{I}(\operatorname{Im}\beta_n) \subseteq T$, let $F \in \mathfrak{I}(\operatorname{Im}\beta_n) \subseteq \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$. Note that for $1 \leq h \leq n$, we have $\partial_t^{\ell}(\partial_t^h((X_0)_{i,j})) < \partial_t^{\ell}((X_h)_{i,j})$ and so the leading monomial of each $\partial_t^{\ell}(\partial_t^h((X_0)_{i,j}) - (X_h)_{i,j})$ is $\partial_t^{\ell}((X_h)_{i,j})$. Then, by using the division algorithm [Iima and Yoshino 2009, Proposition 1.9] we see that

$$F = \sum_{i,j=1}^{r}\sum_{h=1}^{n}\sum_{\ell=0}^{m_{h,i,j}} \partial_t^{\ell}\big(\partial_t^h((X_0)_{i,j}) - (X_h)_{i,j}\big)\cdot(w_{h,\ell})_{i,j} + H, \tag{4.3.10}$$

where $m_{h,i,j}$ is the largest number such that $\partial_t^{m_{h,i,j}}((X_h)_{i,j})$ occurs as a variable in F, each $(w_{h,\ell})_{i,j} \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$, and the remainder $H = H(\boldsymbol{X}_0)$ is an element of $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$. Note that for g_ϵ, $\operatorname{Im}\beta_n$, and $\operatorname{Im}\beta_0$ as in Theorem 4.1.6, there is a surjective map

$$\operatorname{Im}\beta_n \twoheadrightarrow \operatorname{Im}\beta_0$$

given by $d_{t,n+1}[g_\epsilon] \mapsto g_\epsilon$. Moreover we have $F(d_{t,n+1}[g_\epsilon]) = 0$. Since $T \subseteq \mathfrak{I}(\operatorname{Im}\beta_n)$ and

$$\sum_{i,j=1}^{r}\sum_{h=1}^{n}\sum_{\ell=0}^{m_{h,i,j}} \partial_t^{\ell}\big(\partial_t^{h}((X_0)_{i,j}) - (X_h)_{i,j}\big)\cdot (w_{h,\ell})_{i,j} \in T,$$

we obtain from (4.3.10) that $H(g_\epsilon) = 0$. Thus, $H(\boldsymbol{X}_0)$ is an element of $\mathfrak{I}(\operatorname{Im}\beta_0) = \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$. This proves our claim. Therefore, $\mathfrak{I}(\operatorname{Im}\beta_n) = T$. $\square$

We are now ready to compute $\overline{\operatorname{Im}\beta_n}^{Z}$.

Claim 3. *The defining equations of* $\overline{\operatorname{Im}\beta_n}^{Z}$ *are given by*

$$d_{t,n+1}[\boldsymbol{B}]\cdot\mathbf{vec}([\boldsymbol{X}_n,\dots,\boldsymbol{X}_0]^{\mathsf{T}}) = \mathbf{0}. \tag{4.3.11}$$

Proof of Claim 3. Based on Lemma A.1.5, we can find the defining equations of $\overline{\operatorname{Im}\beta_n}^{Z}$ if we determine

$$\boldsymbol{T} := T \cap \boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0, \boldsymbol{X}_1, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0]. \tag{4.3.12}$$

By the preceding arguments, an element of $F \in T = \mathfrak{I}(\overline{\operatorname{Im}\beta_n}^{\partial})$ is of the form (4.3.10), where $H \in \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)) \subseteq \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$. Suppose

$$H = \sum_{i,j=1}^{r}\sum_{u=1}^{r^2-r^2/s}\sum_{\varkappa=0}^{v_u} c_{u,\varkappa}\cdot\partial_t^{\varkappa}((b_u)_{i,j}(X_0)_{i,j}),$$

where $v_u \geq 0$ and $c_{u,\varkappa} \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$ for each $1 \leq u \leq r^2 - r^2/s$. By the product rule of hyperderivatives, we have $\partial_t^{\varkappa}((b_u)_{i,j}(X_0)_{i,j}) = \sum_{\alpha=0}^{\varkappa}\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^{\alpha}((X_0)_{i,j})$, and so rewriting F we have

$$F = \sum_{i,j=1}^{r}\Bigg(\sum_{h=1}^{n}\sum_{\ell=0}^{m_{h,i,j}}(w_{h,\ell})_{i,j}\cdot\partial_t^{\ell}\big(\partial_t^{h}((X_0)_{i,j})-(X_h)_{i,j}\big) + \sum_{u=1}^{r^2-r^2/s}\Bigg(\sum_{\varkappa=0}^{v_u}c_{u,\varkappa}\cdot\partial_t^{\varkappa}((b_u)_{i,j})\cdot(X_0)_{i,j}+\sum_{\varkappa=1}^{v_u}\sum_{\alpha=1}^{\varkappa}c_{u,\varkappa}\cdot\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^{\alpha}((X_0)_{i,j})\Bigg)\Bigg),$$

where $(w_{h,\ell})_{i,j} \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0,\dots,\boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$ and $m_{h,i,j} \in \mathbb{Z}_{\geq 0}$ for $1 \leq h \leq n$, $1 \leq i, j \leq r$.

Suppose that $F \in \boldsymbol{T} \subseteq \boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0,\dots,\boldsymbol{X}_n, 1/\det\boldsymbol{X}_0]$. Then, since $H \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$, we obtain

$$m_{h,i,j} = 0.$$

Additionally, for each $1 \leq i, j \leq r$ we have

$$\sum_{u=1}^{r^2-r^2/s}\sum_{\varkappa=1}^{v_u}\sum_{\alpha=1}^{\varkappa}c_{u,\varkappa}\cdot\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^{\alpha}((X_0)_{i,j}) + \sum_{h=1}^{n}(w_{h,0})_{i,j}\cdot\partial_t^{h}((X_0)_{i,j}) = 0.$$

From this, we see that $v_u \leq n$, and for $h > v_u$ we have $(w_{h,0})_{i,j} = 0$. Moreover, since

$$\sum_{\varkappa=1}^{v_u}\sum_{\alpha=1}^{\varkappa}c_{u,\varkappa}\cdot\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^{\alpha}((X_0)_{i,j}) = \sum_{h=1}^{v_u}\sum_{\gamma=h}^{v_u}c_{u,\gamma}\cdot\partial_t^{\gamma-h}((b_u)_{i,j})\cdot\partial_t^{h}((X_0)_{i,j}),$$

we have, for $1 \le h \le v_u$,

$$(w_{h,0})_{i,j} = -\sum_{u=1}^{r^2-r^2/s} \sum_{\gamma=h}^{v_u} c_{u,\gamma} \cdot \partial_t^{\gamma-h}((b_u)_{i,j}).$$

Thus, $F \in \boldsymbol{T}$ is of the form

$$\begin{aligned} F &= \sum_{i,j=1}^{r} \sum_{u=1}^{r^2-r^2/s} \left(\sum_{\varkappa=0}^{v_u} c_{u,\varkappa} \cdot \partial_t^{\varkappa}((b_u)_{i,j}) \cdot (X_0)_{i,j} + \sum_{h=1}^{v_u} \sum_{\gamma=h}^{v_u} c_{u,\gamma} \cdot \partial_t^{\gamma-h}((b_u)_{i,j}) \cdot (X_h)_{i,j} \right) \\ &= \sum_{u=1}^{r^2-r^2/s} \left(\sum_{\varkappa=0}^{v_u} c_{u,\varkappa} \cdot \partial_t^{\varkappa}(B_u) \cdot \mathbf{vec}(\boldsymbol{X}_0) + \sum_{h=1}^{v_u} \sum_{\gamma=h}^{v_u} c_{u,\gamma} \cdot \partial_t^{\gamma-h}(B_u) \cdot \mathbf{vec}(\boldsymbol{X}_h) \right), \end{aligned}$$

where B_u is the u-th row of $\boldsymbol{B}$ and each $c_{u,\varkappa} \in \boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0]$. Varying u from 1 to $r^2 - r^2/\boldsymbol{s}$ and varying each v_u from 0 to n, we see that the ideal in $\boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0]$ generated by (4.3.12) is the same as the ideal generated by

$$\left\{ \sum_{h=0}^{n} \partial_t^{n-h}(B_u) \cdot \mathbf{vec}(\boldsymbol{X}_h), \quad u = 1, \dots, r^2 - r^2/\boldsymbol{s} \right\},$$

which can be written as

$$d_{t,n+1}[\boldsymbol{B}] \cdot \mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}),$$

where we define $\mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}) := [(\mathbf{vec}\, \boldsymbol{X}_n)^{\mathsf{T}}, \dots, (\mathbf{vec}\, \boldsymbol{X}_0)^{\mathsf{T}}]^{\mathsf{T}}$. Since, by its definition, $d_{t,n+1}[\boldsymbol{B}]$ is a block upper triangular matrix with all diagonal blocks equal to $\boldsymbol{B}$, we have that

$$\operatorname{rank} d_{t,n+1}[\boldsymbol{B}] \ge (n+1) \cdot \operatorname{rank} \boldsymbol{B} = (n+1) \cdot (r^2 - r^2/\boldsymbol{s}).$$

Also, since $d_{t,n+1}[\boldsymbol{B}]$ is an $(n+1) \cdot (r^2 - r^2/\boldsymbol{s}) \times (n+1) \cdot r^2$ matrix, we have that $\operatorname{rank} d_{t,n+1}[\boldsymbol{B}] \le (n+1) \cdot (r^2 - r^2/\boldsymbol{s})$ and so $\operatorname{rank} d_{t,n+1}[\boldsymbol{B}] = (n+1) \cdot (r^2 - r^2/\boldsymbol{s})$. Since $\operatorname{rank} d_{t,n+1}[\boldsymbol{B}]$ is full, we see that

$$d_{t,n+1}[\boldsymbol{B}] \cdot \mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}) = \mathbf{0}$$

are the defining equations of $\overline{\operatorname{Im} \beta_n}^Z$. □

Since each $(b_u)_{ij}$ is an element of $\boldsymbol{k}$, we see that each entry of $d_{t,n+1}[\boldsymbol{B}]$ is an element of $\boldsymbol{k}$ and so, $\overline{\operatorname{Im} \beta_n}^Z$ is defined over $\boldsymbol{k}$. Moreover,

$$\dim \overline{\operatorname{Im} \beta_n}^Z = (n+1) \cdot r^2 - \operatorname{rank} d_{t,n+1}[\boldsymbol{B}] = (n+1) \cdot r^2 - (n+1) \cdot (r^2 - r^2/\boldsymbol{s}) = (n+1) \cdot r^2/\boldsymbol{s}, \quad (4.3.13)$$

which gives the desired result. □

4.4. ***Upper bound on the dimension of*** $\boldsymbol{\Gamma}_{\mathbf{P}_n M_\rho}$**.** Recall from Theorem 4.2.5 that for any $\boldsymbol{k}$-algebra R and $n \ge 1$, an element μ_n of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form as in (4.2.6).

Note that by (4.2.4), we have a short exact sequence of affine group schemes over $\boldsymbol{k}$,

$$1 \to Q_n \to \Gamma_{\mathrm{P}_n M_\rho} \xrightarrow{\pi_{n-1}} \Gamma_{\mathrm{P}_{n-1} M_\rho} \to 1, \quad (4.4.1)$$

where, by Theorem 4.2.5, $\boldsymbol{\pi}_{n-1}^{(\mathrm{R})} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \twoheadrightarrow \Gamma_{\mathrm{P}_{n-1} M_\rho}(\mathrm{R})$ maps μ_n to the matrix μ_{n-1} formed by the upper left $rn \times rn$ square. Consider

$$\nu = \begin{pmatrix} \mathrm{Id}_r & 0 & \cdots & 0 & \boldsymbol{v} \\ & \mathrm{Id}_r & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & \mathrm{Id}_r \end{pmatrix} \in \mathrm{GL}_{(n+1)r}(\mathrm{R}), \tag{4.4.2}$$

where $\boldsymbol{v} \in \mathrm{Mat}_r(\mathrm{R})$. Then, an element of $Q_n(\mathrm{R})$ is of the form (4.4.2). It can easily be checked that

$$\mu_n \nu \mu_n^{-1} = \begin{pmatrix} \mathrm{Id}_r & 0 & \cdots & 0 & \gamma_0 \boldsymbol{v} \gamma_0^{-1} \\ & \mathrm{Id}_r & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & \mathrm{Id}_r \end{pmatrix}. \tag{4.4.3}$$

Note that $\mathrm{P}_0 M_\rho$ is simply M_ρ via the map $D_0 m \mapsto m$ for all $m \in M_\rho$ and so M_ρ is a sub-t-motive of $\mathrm{P}_n M_\rho$. Thus, similarly, by (4.2.4) there is a surjective map of affine group schemes over $\boldsymbol{k}$,

$$\boldsymbol{\pi}_0 : \Gamma_{\mathrm{P}_n M_\rho} \twoheadrightarrow \Gamma_{M_\rho},$$

where, by Theorem 4.2.5, $\boldsymbol{\pi}_0^{(\mathrm{R})} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \to \Gamma_{M_\rho}(\mathrm{R})$ is the map given by $\mu_n \mapsto \gamma_0$. Thus, via conjugation there is a left action of Γ_{M_ρ} on Q_n given by (4.4.3).

Set $\boldsymbol{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$ and for a $\boldsymbol{k}$-algebra R define

$$\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)(\mathrm{R}) := \{\gamma \in \mathrm{Mat}_r(\mathrm{R}) : \gamma g = g\gamma \text{ for all } g \in \mathrm{R} \otimes_{\boldsymbol{k}} \boldsymbol{K}_\rho \subseteq \mathrm{Mat}_r(\mathrm{R})\}.$$

Lemma 4.4.4. *For $n \geq 1$, let $\nu \in Q_n(\mathrm{R})$ be as in (4.4.2). Then,*

$$\boldsymbol{v} \in \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)(\mathrm{R}).$$

Proof. The entries of $\boldsymbol{u}_n = d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}$ form a $\boldsymbol{k}$-basis of $(\mathrm{P}_n M_\rho)^B$ (see [Papanikolas 2008, Proposition 3.3.9]). Recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ from [loc. cit., §4.5] (see also (4.2.8)) as follows: for any $\mu_n \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $a_i \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \leq i \leq n$, the action of μ_n on $(a_0, \dots, a_n) \cdot \boldsymbol{u}_n \in \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ is

$$\varpi_n^{\mathrm{R}}(\mu_n) : (a_0, \dots, a_n) \cdot d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m} \mapsto (a_0, \dots, a_n) \cdot \mu_n^{-1} d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}. \tag{4.4.5}$$

Given $f \in \boldsymbol{K}_\rho$, let $\mathrm{F} \in \mathrm{Mat}_r(\bar{k}(t))$ satisfy $f(\boldsymbol{m}) = \mathrm{F}\boldsymbol{m}$. By Proposition 4.2.3(a), for $n \geq 1$ there exists $g \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ such that $g(\boldsymbol{D}_n \boldsymbol{m}) = d_{t,n+1}[\mathrm{F}] \boldsymbol{D}_n \boldsymbol{m}$ and so, $d_{t,n+1}[\Psi_\rho]^{-1} d_{t,n+1}[\mathrm{F}] d_{t,n+1}[\Psi_\rho] = d_{t,n+1}[\Psi_\rho^{-1}\mathrm{F}\Psi_\rho]$ is an element in the image of the natural embedding (4.2.1). Then by (4.4.5) and the commutative diagram (4.2.2), we have

$$d_{t,n+1}[\Psi_\rho^{-1}\mathrm{F}\Psi_\rho]\nu = \nu d_{t,n+1}[\Psi_\rho^{-1}\mathrm{F}\Psi_\rho].$$

This gives

$$\Psi_\rho^{-1}\mathrm{F}\Psi_\rho \boldsymbol{v} = \boldsymbol{v}\Psi_\rho^{-1}\mathrm{F}\Psi_\rho$$

and the desired result follows. □

Theorem 4.4.6. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and, for $n \geq 1$, let $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in (3.2.2) and suppose that $[K_\rho : k] = \boldsymbol{s}$. Then* $\dim \Gamma_{\mathrm{P}_n M_\rho} \leq (n+1)\cdot r^2/\boldsymbol{s}$.

Remark 4.4.7. The author thanks the referee for sharing the ideas of the following proof, which is an improvement on the ideas used in a previous proof the author obtained. The author's previous proof required a lemma proving smoothness of Q_n. This is no longer required and has been removed.

Proof. By Proposition 3.2.3 and Remark 3.2.4, we see that $[\boldsymbol{K}_\rho : \boldsymbol{k}] = \boldsymbol{s}$ and so, $\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ is an additive group scheme of dimension $r^2/\boldsymbol{s}$ over $\boldsymbol{k}$ [Farb and Dennis 1993, Theorem 3.15(3)].

As Q_n is defined as the kernel in (4.4.1), Q_n is a closed subgroup of $\Gamma_{\mathrm{P}_n M_\rho}$. Consider the closed immersion $\mathrm{Mat}_r/\boldsymbol{k} \hookrightarrow \mathrm{GL}_{(n+1)r}/\boldsymbol{k}$ defined by $\boldsymbol{v} \mapsto \nu$, where ν is of the form (4.4.2). Note that $Q_n \subseteq \mathrm{GL}_{(n+1)r}/\boldsymbol{k}$ is isomorphic to its preimage under this closed immersion. Thus, Q_n is closed in $\mathrm{Mat}_r/\boldsymbol{k}$, and hence closed in $\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ by Lemma 4.4.4. This implies that $\dim Q_n \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho) = r^2/\boldsymbol{s}$.

Now, by (4.4.1) our task is to prove that $\dim Q_n + \dim \Gamma_{\mathrm{P}_{n-1}M_\rho} \leq (n+1)\cdot r^2/\boldsymbol{s}$, which we show by induction. For the base case $n = 1$, since $\dim \Gamma_{M_\rho} = r^2/\boldsymbol{s}$ [Chang and Papanikolas 2012, Theorem 3.5.4] we see that $\dim Q_1 + \dim \Gamma_{M_\rho} \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho) + \dim \Gamma_{M_\rho} = 2\cdot r^2/\boldsymbol{s}$. Suppose we have shown that $\dim \Gamma_{\mathrm{P}_{n-1}M_\rho} \leq n\cdot r^2/\boldsymbol{s}$. By the same argument as in the base case, we obtain

$$\dim Q_n + \dim \Gamma_{\mathrm{P}_{n-1}M_\rho} \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho) + \dim \Gamma_{\mathrm{P}_{n-1}M_\rho} = (n+1)\cdot r^2/\boldsymbol{s}.$$ □

Corollary 4.4.8. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and, for $n \geq 1$, let $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let $\overline{\mathrm{Im}\,\beta_n}^Z$ be the Zariski closure of $\mathrm{Im}\,\beta_n$, where β_n is as in Theorem 4.1.6. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in (3.2.2) and suppose that $[K_\rho : k] = \boldsymbol{s}$. Then $\dim \Gamma_{\mathrm{P}_n M_\rho} = (n+1)\cdot r^2/\boldsymbol{s}$ and*

$$\overline{\mathrm{Im}\,\beta_n}^Z/\boldsymbol{k} = \Gamma_{\mathrm{P}_n M_\rho}.$$

Proof. We obtain $\dim \Gamma_{\mathrm{P}_n M_\rho} = (n+1)\cdot r^2/\boldsymbol{s}$ by combining Theorems 4.3.3 and 4.4.6. By (4.3.13) we see that $\dim \overline{\mathrm{Im}\,\beta_n}^Z = \dim \Gamma_{\mathrm{P}_n M_\rho}$. Then, since $\Gamma_{\mathrm{P}_n M_\rho}$ is connected and smooth by Theorem 2.3.1(b), we have $\overline{\mathrm{Im}\,\beta_n}^Z/\boldsymbol{k} = \Gamma_{\mathrm{P}_n M_\rho}$. □

Remark 4.4.9. By Corollary 4.4.8, we see $\dim Q_n = \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$. Since the defining polynomials of $\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ are degree-1 polynomials, it is connected and smooth. Thus, $Q_n = \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$.

4.5. *Algebraic independence of periods and quasiperiods.* The following result proves Theorem 1.1.3.

Theorem 4.5.1. *Fix $n \geq 1$. Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in (3.2.2) and* *suppose that* K_ρ *is separable over* k. *Let* M_ρ *and* $\mathrm{P}_n M_\rho$ *be the* t*-motives corresponding to* ρ *and* $\mathrm{P}_n\rho$ *respectively. Then,* $\operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi_{\mathrm{P}_n\rho}(\theta)) = (n+1)\cdot r^2/\boldsymbol{s}$, *where* $\boldsymbol{s} = [K_\rho : k]$. *In particular,*

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\Bigg(\bigcup_{s=1}^{n}\bigcup_{i=1}^{r-1}\bigcup_{j=1}^{r}\{\lambda_j,\, F_{\tau^i}(\lambda_j),\, \partial_\theta^s(\lambda_j),\, \partial_\theta^s(F_{\tau^i}(\lambda_j))\}\Bigg) = (n+1)\cdot r^2/\boldsymbol{s}.$$

Proof. By Theorem 2.3.2, we have $\dim \Gamma_{\mathrm{P}_n M_\rho} = \operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi_{\mathrm{P}_n\rho}|_{t=\theta})$. Since $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$, the result follows from Theorem 3.4.1 and Corollary 4.4.8. $\square$

5. Hyperderivatives of logarithms and quasilogarithms

In this section, we prove Theorem 1.1.4 (restated as Theorem 5.4.4) and Corollary 1.1.5. We fix a Drinfeld $\boldsymbol{A}$-module ρ of rank r defined over k^{sep} and an $\boldsymbol{A}$-basis $\{\lambda_1, \dots, \lambda_r\}$ of Λ_ρ as in Section 3.2. Let M_ρ be the t-motive associated to ρ along with a fixed $\bar{k}(t)$-basis $\{m_1, \dots, m_r\} \subseteq M_\rho$, multiplication by σ given by Φ_ρ as in (3.2.1), and rigid analytic trivialization Ψ_ρ as in (3.3.2). For each $n \ge 0$, let $\mathrm{P}_n M_\rho$ be the t-motive corresponding to the n-th prolongation $\mathrm{P}_n\rho$ of ρ as in Section 3.2. Note that $\mathrm{P}_0 M_\rho$ is simply M_ρ via the map $D_0 m \mapsto m$ for all $m \in M_\rho$. If $\boldsymbol{m} = (m_1, \dots, m_r)^{\mathsf{T}}$, then a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$ is given by the entries of $\boldsymbol{D}_n\boldsymbol{m} \in \mathrm{Mat}_{(n+1)r\times 1}(\mathrm{P}_n M_\rho)$ (see (2.5.1)) such that multiplication by σ is given by $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$ (see (2.5.2)) with rigid analytic trivialization $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$ (see (2.5.3)). We also set $\boldsymbol{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$ and let K_ρ denote the fraction field of $\mathrm{End}(\rho)$.

5.1. ***t-motives and quasilogarithms.*** Given $u \in \mathbb{K}$ such that $\operatorname{Exp}_\rho(u) = \alpha \in k^{\mathrm{sep}}$, let $f_u(t)$ be the Anderson generating function of ρ with respect to u given as in (3.1.3). Then, for $n \ge 1$, we see that the Anderson generating function of $\mathrm{P}_n\rho$ with respect to $\boldsymbol{u}_n := [u, 0, \dots, 0]^{\mathsf{T}} \in \mathbb{K}^{n+1}$ is $\mathcal{G}_{\boldsymbol{u}_n}(t) = [f_u(t), \partial_t^1(f_u(t)), \dots, \partial_t^n(f_u(t))]^{\mathsf{T}}$ (see (4.1.5)). Moreover, by (4.1.3),

$$\operatorname{Exp}_{\mathrm{P}_n\rho}(\boldsymbol{u}_n) = [\operatorname{Exp}_\rho(u), 0, \dots, 0]^{\mathsf{T}} = [\alpha, 0, \dots, 0]^{\mathsf{T}} \in (k^{\mathrm{sep}})^{n+1}.$$

We define

$$\boldsymbol{s}_\alpha := \begin{pmatrix} -(\kappa_1 f_u^{(1)}(t) + \cdots + \kappa_{r-1} f_u^{(r-1)}(t) + \kappa_r f_u^{(r)}(t)) \\ -(\kappa_2^{(-1)} f_u^{(1)}(t) + \cdots + \kappa_{r-1}^{(-1)} f_u^{(r-2)}(t) + \kappa_r^{(-1)} f_u^{(r-1)}(t)) \\ -(\kappa_3^{(-2)} f_u^{(1)}(t) + \cdots + \kappa_{r-1}^{(-2)} f_u^{(r-3)}(t) + \kappa_r^{(-2)} f_u^{(r-2)}(t)) \\ \vdots \\ -\kappa_r^{(-r+1)} f_u^{(1)}(t) \end{pmatrix}^{\mathsf{T}} \in \mathrm{Mat}_{1\times r}(\mathbb{T}),$$

and let $\boldsymbol{h}_{\alpha,n} := (\alpha, 0, \dots, 0) \in \mathrm{Mat}_{1\times(n+1)r}(k^{\mathrm{sep}})$. Let F_δ be the quasiperiodic function associated to ρ-biderivation δ, where $\delta_t = \kappa_1\tau + \cdots + \kappa_{r-1}\tau^{r-1} + \kappa_r\tau^r = \rho_t - \theta$. Then, by [Brownawell and Papanikolas 2002, Proposition 3.2.2] (see also [Namoijam and Papanikolas 2024, Proposition 4.3.5(a)]) we obtain

$$-u + \alpha = \mathrm{F}_\delta(u) = \kappa_1 f_u^{(1)}(\theta) + \cdots + \kappa_{r-1} f_u^{(r-1)}(\theta) + \kappa_r f_u^{(r)}(\theta). \tag{5.1.1}$$

We now define the pre-t-motive $Y_{\alpha,n}$ of dimension $(n+1)r+1$ over $\bar{k}(t)$ such that multiplication by σ is given by

$$\Phi_{\alpha,n} := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{h}_{\alpha,n} & 1 \end{pmatrix} \in \mathrm{Mat}_{(n+1)r+1}(\bar{k}[t]).$$

If we set $\boldsymbol{g}_{\alpha,n} := (\boldsymbol{s}_\alpha, \partial_t^1(\boldsymbol{s}_\alpha), \ldots, \partial_t^n(\boldsymbol{s}_\alpha))$, where the hyperderivatives are taken entrywise, then we have $\boldsymbol{g}_{\alpha,n}^{(-1)}\Phi_{\mathrm{P}_n\rho} = \boldsymbol{g}_{\alpha,n} + \boldsymbol{h}_{\alpha,n}$. We set

$$\Psi_{\alpha,n} := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{g}_{\alpha,n}\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix} \in \mathrm{Mat}_{(n+1)r+1}(\mathbb{T})$$

to obtain $\Psi_{\alpha,n}^{(-1)} = \Phi_{\alpha,n}\Psi_{\alpha,n}$. Thus, $Y_{\alpha,n}$ is rigid analytically trivial. The reader may consult [Namoijam and Papanikolas 2024, Lemma 5.65] for motivation behind the construction of $\boldsymbol{g}_{\alpha,n}$ and $\boldsymbol{h}_{\alpha,n}$.

Proposition 5.1.2 (cf. [Papanikolas 2008, Proposition 6.1.3]). *The pre-t-motive $Y_{\alpha,n}$ is a t-motive.*

Proof. Set $\mathcal{N} := \mathrm{Mat}_{1\times(n+1)r+1}(\bar{k}[t])$ and let $\boldsymbol{e} := [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{(n+1)r+1}]^{\mathsf{T}}$ be its standard $\bar{k}[t]$-basis. We give $\mathcal{N}$ a left $\bar{k}[t,\sigma]$-module structure by setting $\sigma\boldsymbol{e} = (t-\theta)\Phi_{\alpha,n}\boldsymbol{e}$. Let $\mathcal{C}$ be the $\boldsymbol{A}$-finite dual t-motive associated to the Carlitz module $\mathfrak{C}$ (rank-1 Drinfeld $\boldsymbol{A}$-module) given by $\mathfrak{C}_t = \theta + \tau$ and let $C := \bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{C}$ be the corresponding pre-t-motive. We obtain the following short exact sequence of $\bar{k}[t,\sigma]$-modules:

$$0 \to \mathcal{C} \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho \to \mathcal{N} \to \mathcal{C} \to 0. \tag{5.1.3}$$

Since $\mathcal{C}$ and $\mathcal{C} \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ are finitely generated left $\bar{k}[\sigma]$-modules, it follows from [Anderson et al. 2004, Proposition 4.3.2] that $\mathcal{N}$ is free and finitely generated as a left $\bar{k}[\sigma]$-module. Since $\mathcal{C} \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ is an $\boldsymbol{A}$-finite dual t-motive, we have

$$(t-\theta)^{v_1}(\mathcal{C} \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho) \subseteq \sigma(\mathcal{C} \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho)$$

for some $v_1 \in \mathbb{N}$. Moreover, $(t-\theta)\mathcal{C} = \sigma\mathcal{C}$ and so, by (5.1.3) we obtain $(t-\theta)^{v_2}\mathcal{N} \subseteq \sigma\mathcal{N}$ for $v_2 \in \mathbb{N}$ sufficiently large. Thus, we see that $\mathcal{N}$ is an $\boldsymbol{A}$-finite dual t-motive. Then, it follows from the discussion in [Papanikolas 2008, §3.4.10] that $Y_{\alpha,n}$ is a t-motive. □

5.2. *Nontriviality in* $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. We continue with the t-motive $Y_{\alpha,n}$ from the previous subsection. Let $\mathbf{1}$ denote the trivial object of the category $\mathcal{T}$ from Section 2.2. Note that $Y_{\alpha,n}$ represents a class in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. Suppose $e \in \mathrm{End}_{\mathcal{T}}(M_\rho)$ and let $\mathrm{E} \in \mathrm{Mat}_r(\bar{k}(t))$ be such that $e(\boldsymbol{m}) = \mathrm{E}\boldsymbol{m}$. If we set

$$\boldsymbol{E} := \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathrm{E} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{(n+1)r}(\bar{k}(t)), \tag{5.2.1}$$

then one checks easily that $\boldsymbol{E}$ represents an element $\boldsymbol{e}$ of $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$. For classes Y_1 and Y_2 in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$, if multiplication by σ on suitable $\bar{k}(t)$-bases are represented by

$$\begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_2 & 1 \end{pmatrix}$$

respectively, then their Baer sum in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$ is achieved by the matrix

$$\begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_1 + \boldsymbol{v}_2 & 1 \end{pmatrix}.$$

Moreover, we see that multiplication by σ on a $\bar{k}(t)$-basis of the pushout $\boldsymbol{e}_* Y_1$ is represented by

$$\begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_1 \boldsymbol{E} & 1 \end{pmatrix}.$$

Note that if $[K_\rho : k] = \boldsymbol{s}$, then $\{\lambda_1, \dots, \lambda_r\}$ span a K_ρ-vector space of dimension $r/\boldsymbol{s}$.

Theorem 5.2.2. *Suppose $u_1, \dots, u_w \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$ and $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w) = r/\boldsymbol{s} + w$, where $[K_\rho : k] = \boldsymbol{s}$. For $n \ge 1$, we let $Y_{i,n} := Y_{\alpha_i,n}$ be defined as in Section 5.1. Then, for $e_1, \dots, e_w \in \boldsymbol{K}_\rho$, not all zero, $S := \boldsymbol{e}_{1*} Y_{1,n} + \cdots + \boldsymbol{e}_{w*} Y_{w,n}$ is nontrivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$, where each $\boldsymbol{e}_i \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ corresponds to e_i as in (5.2.1).*

Proof. We adapt the ideas of the proof of [Chang and Papanikolas 2012, Theorem 4.2.2]. For each $1 \le i \le w$, we let $\boldsymbol{h}_{i,n} := \boldsymbol{h}_{\alpha_i,n}$ and $\boldsymbol{g}_{i,n} := \boldsymbol{g}_{\alpha_i,n}$. Fix $\mathrm{E}_i \in \mathrm{Mat}_r(\bar{k}(t))$ so that $e_i(\boldsymbol{m}) = \mathrm{E}_i \boldsymbol{m}$ for each $1 \le i \le w$. Then $\boldsymbol{e}_i(\boldsymbol{D}_n \boldsymbol{m}) = \boldsymbol{E}_i \cdot \boldsymbol{D}_n \boldsymbol{m}$, where $\boldsymbol{E}_i$ is as in (5.2.1) with $\mathrm{E}_i = \mathrm{E}$. By choosing an appropriate $\bar{k}(t)$-basis $\boldsymbol{s}$ for S, multiplication by σ on $\boldsymbol{s}$ is represented by

$$\Phi_S := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \sum_{i=1}^{w} \boldsymbol{h}_{i,n} \boldsymbol{E}_i & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\bar{k}(t)),$$

and a corresponding rigid analytic trivialization is represented by

$$\Psi_S := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \sum_{i=1}^{w} \boldsymbol{g}_{i,n} \boldsymbol{E}_i \Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\mathbb{L}).$$

Suppose on the contrary that S is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. Then, there exists another $\bar{k}(t)$-basis $\boldsymbol{s}'$ of S such that $\sigma \boldsymbol{s}' = (\Phi_{\mathrm{P}_n\rho} \oplus (1))\boldsymbol{s}'$, where $\Phi_{\mathrm{P}_n\rho} \oplus (1)$ is the block diagonal matrix with $\Phi_{\mathrm{P}_n\rho}$ and 1 in the diagonal blocks and all other entries are zero. If we let

$$\gamma = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & \mathbf{0} \\ \gamma_0 \cdots \gamma_n & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\bar{k}(t)),$$

where $\gamma_j := (\gamma_{j1}, \dots, \gamma_{jr})$ for each $0 \le j \le n$ be the matrix such that $\boldsymbol{s}' := \gamma \boldsymbol{s}$, then we obtain

$$\gamma^{(-1)} \Phi_S = (\Phi_{\mathrm{P}_n\rho} \oplus (1))\gamma. \tag{5.2.3}$$

Note from [Papanikolas 2008, Proof of Proposition 3.4.5] that all denominators of entries of γ are in $\boldsymbol{A}$ and so in particular, for each $0 \le j \le n$, the entries of γ_j are regular at $t = \theta, \theta^q, \theta^{q^2}, \dots$. Using $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, the $((n+1)r+1, (n-j)r+1)$-th entry of (5.2.3) for each $1 \le j \le n$ is

$$\sum_{h=0}^{n-j} \gamma_{h,r}^{(-1)} \partial_t^{n-j-h}((t-\theta)/\kappa_r^{(-r)}) = \gamma_{n-j,1},$$

and the $((n+1)r+1, nr+1)$-th entry is

$$\sum_{h=0}^{n} \gamma_{h,r}^{(-1)} \partial_t^{n-h}((t-\theta)/\kappa_r^{(-r)}) + \sum_{i=1}^{w} \alpha_i (\mathrm{E}_i)_{11} = \gamma_{n,1}.$$

For each $0 \le j \le n$, applying $(-1)^j \partial_t^j(\,\cdot\,)$ to the $((n+1)r+1, (n-j)r+1)$-th entry and then adding them (we also use the product rule of hyperderivatives and the property $\partial_t^v \partial_t^w(f(t)) = \binom{v+w}{v}\partial_t^{v+w}(f(t))$), we obtain

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,r})^{(-1)}(t-\theta)/\kappa_r^{(-r)} + \sum_{i=1}^{w} \alpha_i (\mathrm{E}_i)_{11} = \sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1}). \tag{5.2.4}$$

Specializing both sides of this equation at $t=\theta$, we obtain

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1})(\theta) = \sum_{i=1}^{w} \alpha_i (\mathrm{E}_i)_{11}(\theta). \tag{5.2.5}$$

By (5.2.3), we also have $(\gamma \Psi_S)^{(-1)} = (\Phi_{\mathrm{P}_n\rho} \oplus (1))(\gamma \Psi_S)$ and so by [Papanikolas 2008, §4.1.6], for some

$$\delta = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & \mathbf{0} \\ \delta_0 \cdots \delta_n & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\boldsymbol{k}),$$

where $\delta_j := (\delta_{j1}, \ldots, \delta_{jr})$ for each $0 \le j \le n$, we have

$$\gamma \Psi_S = (\Psi_{\mathrm{P}_n\rho} \oplus (1))\delta. \tag{5.2.6}$$

Since $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$, by applying to (5.2.6) the same methods applied on (5.2.3) to obtain (5.2.4), it follows that

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j}) + \sum_{i=1}^{w} \boldsymbol{s}_i \mathrm{E}_i = \sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j})\Psi_\rho^{-1}, \tag{5.2.7}$$

where for $\partial_t^j(\gamma_{n-j})$ and $\partial_t^j(\delta_{n-j})$, the hyperderivatives are taken entrywise. Since for each $1 \le i \le w$ the first entry of $\boldsymbol{s}_i(\theta)$ is $u_i - \alpha_i$ by (5.1.1), using [Chang and Papanikolas 2012, Proposition 4.1.1(b)] and specializing both sides of (5.2.7) at $t=\theta$, we see that

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1})(\theta) + \sum_{i=1}^{w}(u_i - \alpha_i)(\mathrm{E}_i)_{11}(\theta) = -\sum_{m=1}^{r}\sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j,m})(\theta)\lambda_m,$$

and so from (5.2.5) we have

$$\sum_{m=1}^{r}\sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j,m})(\theta)\lambda_m + \sum_{i=1}^{w}(\mathrm{E}_i)_{11}(\theta)u_i = 0.$$

Since $e_1, \ldots, e_w$ are not all zero, E_i is nonzero for some $1 \le i \le w$. Moreover, by Proposition 3.2.3 we see that $\boldsymbol{K}_\rho \cong K_\rho$ and so E_i is invertible. By [loc. cit., Proposition 4.1.1(b),(c)] we get $(\mathrm{E}_i)_{11}(\theta) \in K_\rho^\times$ and thus we get a contradiction to the assumption that $\{u_1, \ldots, u_w\}$ is K_ρ-linearly independent from each other and is K_ρ-linearly independent from $\{\lambda_1, \ldots, \lambda_r\}$. $\square$

5.3. ***Construction of the t-motives Y and N_n.*** In this subsection, we construct a t-motive that is suitable for the investigation of the hyperderivatives of logarithms and quasilogarithms of the Drinfeld $\boldsymbol{A}$-module ρ and the study of its Galois group. Suppose that we have $u_1, \dots, u_w \in \mathbb{K}$ with $\operatorname{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$. For $n \ge 0$, we let $\boldsymbol{h}_{\alpha_i} := \boldsymbol{h}_{\alpha_i,n}$, $\boldsymbol{g}_{\alpha_i} := \boldsymbol{g}_{\alpha_i,n}$, $Y_{i,n} := Y_{\alpha_i,n}$, $\Phi_{i,n} := \Phi_{\alpha_i,n}$ and $\Psi_{i,n} := \Psi_{\alpha_i,n}$ defined as in Section 5.1. The matrix $\Psi_n := \bigoplus_{i=1}^{w} \Psi_{i,n}$ is a rigid analytic trivialization for $Y_n := \bigoplus_{i=1}^{w} Y_{i,n}$.

Define the t-motive N_n such that multiplication by σ on a $\bar{k}(t)$-basis is given by $\Phi_{N_n} \in \mathrm{GL}_{(n+1)rw+1}(\bar{k}(t))$ along with rigid analytic trivialization $\Psi_{N_n} \in \mathrm{GL}_{(n+1)rw+1}(\mathbb{T})$ such that

$$\Phi_{N_n} := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & & & \\ & \ddots & & \\ & & \Phi_{\mathrm{P}_n\rho} & \\ \boldsymbol{h}_{\alpha_1} & \cdots & \boldsymbol{h}_{\alpha_w} & 1 \end{pmatrix} \quad \text{and} \quad \Psi_{N_n} := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & & & \\ & \ddots & & \\ & & \Psi_{\mathrm{P}_n\rho} & \\ \boldsymbol{g}_{\alpha_1}\Psi_{\mathrm{P}_n\rho} & \cdots & \boldsymbol{g}_{\alpha_w}\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix}. \tag{5.3.1}$$

Similar to $n = 0$ case [Chang and Papanikolas 2012, §5.1], N_n is an extension of $\mathbf{1}$ by $(\mathrm{P}_n M_\rho)^w$ which is a pullback of the surjective map $Y_n \twoheadrightarrow \mathbf{1}^w$ and the diagonal map $\mathbf{1} \to \mathbf{1}^w$. Thus, the two t-motives Y_n and N_n generate the same Tannakian subcategory of $\mathcal{T}$ and hence the Galois groups Γ_{Y_n} and Γ_{N_n} are isomorphic. For any $\boldsymbol{k}$-algebra R, an element of $\Gamma_{N_n}(\mathrm{R})$ is of the form

$$\nu = \begin{pmatrix} \mu & & & \\ & \ddots & & \\ & & \mu & \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_w & 1 \end{pmatrix}, \tag{5.3.2}$$

where $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and for each $1 \le i \le w$ we have $\boldsymbol{v}_i = (\boldsymbol{v}_{i,1}, \dots, \boldsymbol{v}_{i,n+1})$ such that $\boldsymbol{v}_{i,h} \in \mathbb{G}_a^r(\mathrm{R}) = \mathrm{Mat}_{1\times r}(\mathrm{R})$ for each $0 \le h \le n$. Since $(\mathrm{P}_n M_\rho)^w$ is a sub-t-motive of N_n, we have the following short exact sequence of affine group schemes over $\boldsymbol{k}$:

$$1 \to X_n \to \Gamma_{N_n} \xrightarrow{\pi_n} \Gamma_{\mathrm{P}_n M_\rho} \to 1, \tag{5.3.3}$$

where $\pi_n^{(\mathrm{R})} : \Gamma_{N_n}(\mathrm{R}) \to \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is the map $\nu \mapsto \mu$ (cf. [loc. cit., p. 138]). It can be checked directly that via conjugation (5.3.3) gives an action of any $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on

$$\boldsymbol{v} = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{(n+1)r} & \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_w & 1 \end{pmatrix} \in X_n(\mathrm{R})$$

given by

$$\nu\boldsymbol{v}\nu^{-1} = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{(n+1)r} & \\ \boldsymbol{u}_1\mu^{-1} & \cdots & \boldsymbol{u}_w\mu^{-1} & 1 \end{pmatrix}. \tag{5.3.4}$$

For $n \ge 0$, recall from (2.5.1) that if the entries of $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ form a $\bar{k}(t)$-basis of M_ρ, then the entries of $\boldsymbol{D}_n\boldsymbol{m}$ form a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$. Let $[\boldsymbol{D}_n\boldsymbol{m}^{\mathsf{T}}, y]^{\mathsf{T}}$ be a $\bar{k}(t)$-basis of N_n. Then, the entries of

$\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top$ form a $\mathbb{F}_q(t)$-basis of N_n^B [Papanikolas 2008, Proposition 3.3.9]. By construction, $\mathrm{P}_j M_\rho$ is a sub-t-motive of N_n for each $0 \le j \le n-1$ and we have a short exact sequence of t-motives

$$0 \to \mathrm{P}_j M_\rho \xrightarrow{\iota} N_n \xrightarrow{\mathbf{Pr}_{n-j-1}} N_{n-j-1} \to 0, \tag{5.3.5}$$

where $\mathbf{Pr}_{n-j-1}(D_h m) := D_{h-j-1}m$ for $h > j$, $\mathbf{Pr}_{n-j-1}(D_h m) := 0$ for $h \le j$ and $m \in M_\rho$, and $\mathbf{Pr}_{n-j-1}(x) := x$ for $x \in N_n/\mathrm{P}_n M_\rho$. Thus, as t-motives $N_n/\mathrm{P}_j M_\rho \cong N_{n-j-1}$ and so, N_{n-j-1} is an object in the Tannakian category $\mathcal{T}_{N_n}$. Therefore, we have a surjective map of affine group schemes $\Gamma_{N_n} \twoheadrightarrow \Gamma_{N_{n-j-1}}$. We now determine this surjective map. For any $\boldsymbol{k}$-algebra R, we recall the action of $\Gamma_{N_n}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} N_n^B$ from [Papanikolas 2008, §4.5] as follows: for any $\nu_n \in \Gamma_{N_n}(\mathrm{R})$, $b \in \mathrm{R}$ and $a_h \in \mathrm{Mat}_{1\times r}(\mathrm{R})$ where $0 \le h \le n$, the action of ν_n on $(a_0, \dots, a_n, b) \cdot \Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top \in R \otimes_{\boldsymbol{k}} N_n^B$ is

$$(a_0, \dots, a_n, b) \cdot \Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top \mapsto (a_0, \dots, a_n, b) \cdot \nu_n^{-1}\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top. \tag{5.3.6}$$

Note that $\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top = [(d_{t,n+1}[\Psi_\rho]^{-1}\boldsymbol{D}_n\boldsymbol{m})^\top, -\boldsymbol{g}_{\alpha_1} D_n\boldsymbol{m} + y]^\top$ by the definition of Ψ_{N_n} (see (5.3.1)). We restrict the action of ν_n to $\mathrm{R} \otimes_{\boldsymbol{k}} N_{n-j-1}^B$ via the map $\mathbf{Pr}_{n-j-1}$ in (5.3.5). Note that an element of $\Gamma_{N_n}(\mathrm{R})$ is of the form

$$\begin{pmatrix} \mu_n & \mathbf{0} \\ \boldsymbol{w}_n & 1 \end{pmatrix},$$

where $\mu_n \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and $\boldsymbol{w}_n = (w_0, \dots, w_n)$ such that each $w_h \in \mathbb{G}_a^r(\mathrm{R}) = \mathrm{Mat}_{1\times r}(\mathrm{R})$. Through $\mathbf{Pr}_{n-j-1}$, we see that ν_n leaves N_{n-j-1}^B invariant and so for

$$\nu_n = \begin{pmatrix} \mu_n & \mathbf{0} \\ \boldsymbol{w}_n & 1 \end{pmatrix} \in \Gamma_{N_n}(\mathrm{R}),$$

we obtain

$$\nu_{n-j-1} = \begin{pmatrix} \mu_{n-j-1} & \mathbf{0} \\ \boldsymbol{w}_{n-j-1} & 1 \end{pmatrix} \in \Gamma_{N_{n-j-1}}(\mathrm{R}), \tag{5.3.7}$$

where μ_{n-j-1} is the matrix formed by the upper left $r(n-j) \times r(n-j)$ square of μ_n and $\boldsymbol{w}_{n-j-1} = (w_0, \dots, w_{n-j-1})$. Note that by Theorem 4.2.5, we have $\mu_{n-j-1} \in \Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$. Thus, the surjective map $\Xi_{n-j-1} : \Gamma_{N_n} \to \Gamma_{N_{n-j-1}}$ is given by (cf. [Chang and Papanikolas 2011, Proposition 3.1.2])

$$\Xi_{n-j-1}^{(\mathrm{R})} : \nu_n \mapsto \nu_{n-j-1}. \tag{5.3.8}$$

Lemma 5.3.9. *Let $n \ge 1$. If K_ρ is separable over k, then X_n in (5.3.3) is $\boldsymbol{k}$-smooth.*

Proof. We adapt the ideas of the proof of [Chang and Papanikolas 2011, Proposition 4.1.2] and the proof of a lemma from a preliminary version of [Chang and Papanikolas 2012] (Lemma 5.1.3: arXiv:1005.5120v1). By [Springer 1998, Corollary 12.1.3] it suffices to show that for $n \ge 1$, the induced tangent map $d\pi_n$ at the identity is surjective onto $\mathrm{Lie}\,\Gamma_{\mathrm{P}_n M_\rho}$. We prove this for $w = 1$ as the argument used in this case can be applied in a straightforward manner to prove the arbitrary w case. We leave this task to the reader. Since $\boldsymbol{K}_\rho$ is separable over $\boldsymbol{k}$ (by hypothesis, Proposition 3.2.3, and Remark 3.2.4), we see from [Chang and Papanikolas 2012, Corollary 3.5.6] and [Waterhouse 1979, p. 61 Problem 14] that through conjugation by some $\mathrm{J} \in \mathrm{GL}_r(\boldsymbol{k}^{\mathrm{sep}})$, we have an isomorphism

$$\Gamma_{M_\rho} \times_{\boldsymbol{k}} \boldsymbol{K}_\rho \xrightarrow{\cong} \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho)_i,$$

where

$$\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho)_i := \left\{ \begin{pmatrix} \mathrm{GL}_{r/s} & & \\ & \ddots & \\ & & \mathrm{GL}_{r/s} \end{pmatrix} \right\},$$

and $(\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho)_i$ is the canonical embedding of $\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho$ into the i-th diagonal block matrix of $\mathrm{GL}_r / \boldsymbol{K}_\rho$. Making a change of basis, we obtain

$$\Gamma_{M_\rho} \times_{\boldsymbol{k}} \bar{\boldsymbol{k}} \xrightarrow{\cong} \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i.$$

For $n \geq 1$, it follows that via conjugation by $d_{t,n+1}[\mathrm{J}] \in \mathrm{GL}_{(n+1)r}(\boldsymbol{k}^{\mathrm{sep}})$ on $\Gamma_{\mathrm{P}_n M_\rho}$, we obtain $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$, an algebraic subgroup of $\mathrm{GL}_{(n+1)r} / \bar{\boldsymbol{k}}$, such that there is an isomorphism

$$\Gamma_{\mathrm{P}_n M_\rho} \times_{\boldsymbol{k}} \bar{\boldsymbol{k}} \xrightarrow{\cong} \bar{\Gamma}_{\mathrm{P}_n M_\rho}. \tag{5.3.10}$$

Let $\left(\bigoplus_{i=1}^{w} d_{t,n+1}[\mathrm{J}]\right) \oplus (1) \in \mathrm{GL}_{(n+1)rw+1}(\boldsymbol{k}^{\mathrm{sep}})$ be the block diagonal matrix with $d_{t,n+1}[\mathrm{J}]$ in the first w diagonal blocks and 1 in the last diagonal, and all other entries are zero. Then, via conjugation by $\left(\bigoplus_{i=1}^{w} d_{t,n+1}[\mathrm{J}]\right) \oplus (1)$ on Γ_{N_n} we obtain $\bar{\Gamma}_{N_n}$ such that we have an isomorphism $\Gamma_{N_n} \times_{\boldsymbol{k}} \bar{\boldsymbol{k}} \cong \bar{\Gamma}_{N_n}$. Moreover, $\bar{\Gamma}_{N_n}$ is an algebraic subgroup of $\mathrm{GL}_{(n+1)rw+1} / \bar{\boldsymbol{k}}$ such that $\bar{\pi}_n : \bar{\Gamma}_{N_n} \to \bar{\Gamma}_{\mathrm{P}_n M_\rho}$ induced by π_n in (5.3.3) is surjective. Thus, we are reduced to proving that the induced tangent map $\mathrm{d}\bar{\pi}_n : \mathrm{Lie}\, \bar{\Gamma}_{N_n} \to \mathrm{Lie}\, \bar{\Gamma}_{\mathrm{P}_n M_\rho}$ is surjective.

First we determine $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$. Recall $\boldsymbol{X}$, the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$ from (4.3.1). Since $d_{t,n+1}[\mathrm{J}]$ and its inverse are block upper triangular matrices, similar to $\boldsymbol{X}$ we make the choice to let the coordinates of $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$ be

$$\boldsymbol{Y} := \begin{pmatrix} \boldsymbol{Y}_0 & \boldsymbol{Y}_1 & \cdots & \boldsymbol{Y}_n \\ & \boldsymbol{Y}_0 & \ddots & \vdots \\ & & \ddots & \boldsymbol{Y}_1 \\ & & & \boldsymbol{Y}_0 \end{pmatrix},$$

where $\boldsymbol{Y}_h := ((Y_h)_{ij})$, an $r \times r$ matrix for $0 \leq h \leq n$. Then, by construction we have $\boldsymbol{X} = d_{t,n+1}[\mathrm{J}] \boldsymbol{Y} d_{t,n+1}[\mathrm{J}]^{-1}$ and so for each $0 \leq w \leq n$, we obtain

$$\boldsymbol{X}_w = \sum_{\substack{w_1+w_2=w \\ w_1,w_2 \geq 0}} \sum_{h=0}^{w_1} \partial_t^{w_1-h}(\mathrm{J}) \cdot \boldsymbol{Y}_{w_2} \cdot (\partial_t^h(\mathrm{J}))^{-1},$$

where the hyperderivatives are taken entrywise. Then, we have

$$\mathbf{vec}(\boldsymbol{X}_w) = \sum_{\substack{w_1+w_2=w \\ w_1,w_2 \geq 0}} \sum_{h=0}^{w_1} \left([(\partial_t^h(\mathrm{J}))^{-1}]^{\mathsf{T}} \otimes \partial_t^{w_1-h}(\mathrm{J}) \right) \cdot \mathbf{vec}(\boldsymbol{Y}_{w_2}) = \sum_{\substack{w_1+w_2=w \\ w_1,w_2 \geq 0}} \partial_t^{w_1}((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J}) \cdot \mathbf{vec}(\boldsymbol{Y}_{w_2}),$$

where we obtain the first equality by using properties of the Kronecker product and the second equality by further applying the product rule for hyperderivatives. This implies

$$\mathbf{vec}([\boldsymbol{X}_n, \ldots, \boldsymbol{X}_0]^{\mathsf{T}}) = d_{t,n+1}[(\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J}] \cdot \mathbf{vec}([\boldsymbol{Y}_n, \ldots, \boldsymbol{Y}_0]^{\mathsf{T}}), \tag{5.3.11}$$

where we set

$$\mathbf{vec}([\boldsymbol{X}_n, \ldots, \boldsymbol{X}_0]^{\mathsf{T}}) := [(\mathbf{vec}\,\boldsymbol{X}_n)^{\mathsf{T}}, \ldots, (\mathbf{vec}\,\boldsymbol{X}_0)^{\mathsf{T}}]^{\mathsf{T}},$$

and we further define $\mathbf{vec}([\boldsymbol{Y}_n, \ldots, \boldsymbol{Y}_0]^{\mathsf{T}})$ similarly. For $0 \le i \le n$, let $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_i, 1/\det \boldsymbol{Y}_0]$ denote the localization of $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_i]$ at $\det \boldsymbol{Y}_0$. Then, by (4.3.11), Corollary 4.4.8, and (5.3.11), the defining ideal of $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$ via the isomorphism (5.3.10) is the ideal in $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_n, 1/\det \boldsymbol{Y}_0]$ generated by the entries of

$$d_{t,n+1}[\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})] \cdot \mathbf{vec}([\boldsymbol{Y}_n, \ldots, \boldsymbol{Y}_0]^{\mathsf{T}}). \tag{5.3.12}$$

It is clear by observing $\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$ that, for $\boldsymbol{Y}_0 = ((Y_0)_{i,j})$, the defining ideal of $\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$ is the ideal in $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, 1/\det \boldsymbol{Y}_0]$ generated by

$$\{(Y_0)_{i,j} : (i, j) \neq (ur/s + v_1, ur/s + v_2),\ 0 \le u \le s-1 \text{ and } 1 \le v_1, v_2 \le r/s\}. \tag{5.3.13}$$

Moreover, by (4.3.5) and (5.3.11), the defining ideal of $\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$ is also generated by the entries of

$$(\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \mathbf{vec}(\boldsymbol{Y}_0). \tag{5.3.14}$$

By (5.3.13), in the defining ideal of $\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$, there are no linear relations among

$$\{(Y_0)_{i,j} : (i, j) = (ur/s + v_1, ur/s + v_2),\ 0 \le u \le s-1 \text{ and } 1 \le v_1, v_2 \le r/s\}. \tag{5.3.15}$$

Since (5.3.14) also generate the defining ideal of $\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$, we see that the entries of $\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})$ that give linear relations among the variables in (5.3.15) are all zero. Therefore, the hyperderivatives of these entries are also all zero. Using this and using (5.3.13), for $\gamma \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$ and for $0 \le \ell \le n$, we see that

$$\partial_t^{\ell} (\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \gamma = \mathbf{0}. \tag{5.3.16}$$

Moreover, by (5.3.14), for $1 \le h \le n$, the defining ideal of $\prod_{i=1}^{s} (\mathrm{Mat}_{r/s} / \bar{\boldsymbol{k}})_i$ is the ideal in $\bar{\boldsymbol{k}}[\boldsymbol{Y}_h]$ generated by the entries of

$$(\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \mathbf{vec}(\boldsymbol{Y}_h),$$

and similar to (5.3.16), for $\gamma' \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s} / \bar{\boldsymbol{k}})_i$ and for $0 \le \ell \le n$, we see that

$$\partial_t^{\ell} (\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \gamma' = \mathbf{0}.$$

Therefore, for all $\gamma_0 \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i$ and $\gamma_h \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s} / \bar{\boldsymbol{k}})_i$ where $1 \le h \le n$, we have

$$d_{t,n+1}[\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})] \cdot ([\gamma_n, \ldots, \gamma_0]^{\mathsf{T}}) = \mathbf{0}.$$

Thus, by (5.3.12) we have

$$\bar{\Gamma}_{\mathrm{P}_nM_\rho}=\left\{\left(\begin{array}{c|c|c|c}\gamma_0&\gamma_1&\cdots&\gamma_n\\ \hline &\gamma_0&\ddots&\vdots\\ \hline &&\ddots&\gamma_1\\ \hline &&&\gamma_0\end{array}\right):\gamma_0\in\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i,\ \gamma_h\in\prod_{i=1}^{s}(\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i,\ 1\le h\le n\right\},\tag{5.3.17}$$

where, for each i, $(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$ and $(\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i$ are the canonical embeddings of $\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}}$ and $\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}}$ respectively into the i-th diagonal block matrices of $\mathrm{GL}_r/\bar{\boldsymbol{k}}$ and $\mathrm{Mat}_r/\bar{\boldsymbol{k}}$.

We are now ready to prove that the induced tangent map $\mathrm{d}\bar{\pi}_n:\mathrm{Lie}\,\bar{\Gamma}_{N_n}\to\mathrm{Lie}\,\bar{\Gamma}_{\mathrm{P}_nM_\rho}$ is surjective. Let $w=1$ and consider the short exact sequence of linear algebraic groups

$$1\to\bar{X}_n\to\bar{\Gamma}_{N_n}\xrightarrow{\bar{\pi}_n}\bar{\Gamma}_{\mathrm{P}_nM_\rho}\to 1.\tag{5.3.18}$$

First suppose $n=1$. Then, by (5.3.17),

$$\bar{\Gamma}_{\mathrm{P}_1M_\rho}=\left\{\begin{pmatrix}\gamma_0&\gamma_1\\ \mathbf{0}&\gamma_0\end{pmatrix}:\gamma_0\in\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i,\ \gamma_1\in\prod_{i=1}^{s}(\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i\right\},\tag{5.3.19}$$

and by (5.3.2),

$$\bar{\Gamma}_{N_1}\subseteq\left\{\begin{pmatrix}\gamma_0&\gamma_1&\mathbf{0}\\ \mathbf{0}&\gamma_0&\mathbf{0}\\ z_0&z_1&1\end{pmatrix}:\begin{pmatrix}\gamma_0&\gamma_1\\ \mathbf{0}&\gamma_0\end{pmatrix}\in\bar{\Gamma}_{\mathrm{P}_1M_\rho},\ z_0,z_1\in\mathbb{G}_a^r\right\}.\tag{5.3.20}$$

From $\bar{\pi}_1$, we see that $\bar{X}_1$ is contained in the $2r$-dimensional additive group

$$G:=\left\{\begin{pmatrix}\mathrm{Id}_{r/s}&&&\\ &\ddots&&\\ &&\mathrm{Id}_{r/s}&\\ \boldsymbol{v}_1&\cdots&\boldsymbol{v}_{2s}&1\end{pmatrix}:\boldsymbol{v}_i\in\mathbb{G}_a^{r/s}\right\},$$

where we call $\boldsymbol{v}_1,\dots,\boldsymbol{v}_{2s}$ the coordinates of G. We see that via conjugation, $\bar{X}_1(\bar{\boldsymbol{k}})$ has a $\bar{\Gamma}_{\mathrm{P}_1M_\rho}(\bar{\boldsymbol{k}})$-module structure coming from (5.3.18) (see (5.3.4)). Using (5.3.19) and this module structure, one checks easily that there is a natural decomposition $\bar{X}_1(\bar{\boldsymbol{k}})=\prod_{i=1}^{2s}W_i$ such that each W_i is either zero or $\bar{\boldsymbol{k}}^{r/s}$. Fix any $1\le i\le s$. For any $\xi_i\in\mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$, we let

$$\bar{\xi}_i=\left(\begin{array}{ccccccccccc}\mathrm{Id}_{r/s}&&&&&&\mathbf{0}&&&&\\ &\ddots&&&&&&\ddots&&&\\ &&\xi_i&&&&&&\mathbf{0}&&\\ &&&\ddots&&&&&&\ddots&\\ &&&&\mathrm{Id}_{r/s}&&&&&\mathbf{0}&\\ \hline &&&&&\mathrm{Id}_{r/s}&&&&&\\ &&&&&&\ddots&&&&\\ &&&&&&&\xi_i&&&\\ &&&&&&&&\ddots&&\\ &&&&&&&&&\mathrm{Id}_{r/s}&\\ \boldsymbol{u}_1&\cdots&\boldsymbol{u}_i&\cdots&\boldsymbol{u}_s&\boldsymbol{u}_{s+1}&\cdots&\boldsymbol{u}_{s+i}&\cdots&\boldsymbol{u}_{2s}&1\end{array}\right)\in\bar{\Gamma}_{N_1}(\bar{\boldsymbol{k}})$$

be an arbitrary element, which by (5.3.18) for $n = 1$ and (5.3.19) is a preimage of the matrix formed by the upper left $2r \times 2r$ square of $\bar{\xi}_i$ under the map $\bar{\pi}_1$. For each $j \neq i$ with $1 \leq j \leq s$, we claim that if $\boldsymbol{u}_j \neq \boldsymbol{0}$ and $\boldsymbol{u}_{s+j} \neq \boldsymbol{0}$, then $W_j = W_{s+j} = \bar{\boldsymbol{k}}^{r/s}$. To prove this claim, assuming that $\boldsymbol{u}_j \neq \boldsymbol{0}$ and $\boldsymbol{u}_{s+j} \neq \boldsymbol{0}$ we pick $\delta_j \in \mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$ so that $\boldsymbol{u}_j\delta_j - \boldsymbol{u}_j \neq \boldsymbol{0}$ and $\boldsymbol{u}_{s+j}\delta_j - \boldsymbol{u}_{s+j} \neq \boldsymbol{0}$, and let $\bar{\delta}_j \in \overline{\Gamma}_{N_1}(\bar{\boldsymbol{k}})$ be such that

$$\bar{\pi}_1(\bar{\delta}_j) = \left(\begin{array}{ccccc|ccccc} \mathrm{Id}_{r/s} & & & & & & & \boldsymbol{0} & & \\ & \ddots & & & & & & & \ddots & \\ & & \delta_j & & & & & & \boldsymbol{0} & \\ & & & \ddots & & & & & & \ddots \\ & & & & \mathrm{Id}_{r/s} & & & & & \boldsymbol{0} \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \delta_j & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \mathrm{Id}_{r/s} \end{array}\right) \in \overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}}).$$

Then one checks directly that $\bar{\delta}_j^{-1}\bar{\xi}_i\bar{\delta}_j\bar{\xi}_i^{-1}$ is an element of $\overline{X}_1(\bar{\boldsymbol{k}})$ and its $\boldsymbol{v}_j$ and $\boldsymbol{v}_{s+j}$ coordinate vectors respectively are $\boldsymbol{u}_j\delta_j - \boldsymbol{u}_j$ and $\boldsymbol{u}_{s+j}\delta_j - \boldsymbol{u}_{s+j}$, and so it follows that $W_j = W_{s+j} = \bar{\boldsymbol{k}}^{r/s}$. Therefore, multiplying $\bar{\xi}_i$ by a suitable element of $\overline{X}_1(\bar{\boldsymbol{k}})$ we get an element of the form

$$\bar{\xi}_i' = \left(\begin{array}{ccccc|ccccc|c} \mathrm{Id}_{r/s} & & & & & & & \boldsymbol{0} & & & \\ & \ddots & & & & & & & \ddots & & \\ & & \xi_i & & & & & & \boldsymbol{0} & & \\ & & & \ddots & & & & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} & & & & & \boldsymbol{0} & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \xi_i & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & \\ \boldsymbol{0} & \cdots & \boldsymbol{u}_i & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{u}_{s+i} & \cdots & \boldsymbol{0} & 1 \end{array}\right) \in \overline{\Gamma}_{N_1}(\bar{\boldsymbol{k}}). \tag{5.3.21}$$

For any $\boldsymbol{b}_i \in \mathrm{Mat}_{r/s}(\bar{\boldsymbol{k}})$, by using a method similar to that above where we take an element of the form $\bar{\delta}_j$, we obtain an element of the form

$$\bar{\boldsymbol{b}}_i' = \left(\begin{array}{ccccc|ccccc|c} \mathrm{Id}_{r/s} & & & & & & & \boldsymbol{0} & & & \\ & \ddots & & & & & & & \ddots & & \\ & & \mathrm{Id}_{r/s} & & & & & & \boldsymbol{b}_i & & \\ & & & \ddots & & & & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} & & & & & \boldsymbol{0} & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \mathrm{Id}_{r/s} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & \\ \boldsymbol{0} & \cdots & \boldsymbol{w}_i & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{w}_{s+i} & \cdots & \boldsymbol{0} & 1 \end{array}\right) \in \overline{\Gamma}_{N_1}(\bar{\boldsymbol{k}}), \tag{5.3.22}$$

which is a preimage of the matrix formed by the upper left $2r \times 2r$ square of $\bar{\boldsymbol{b}}_i'$ under the map $\bar{\pi}_1$. Let $\overline{H}_{1,i}$ be the Zariski closure inside $\overline{\Gamma}_{N_1}$ of the subgroup generated by all $\bar{\xi}_i'$ with ξ_i running over all elements of $\mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$ and all $\bar{\boldsymbol{b}}_i'$ with $\boldsymbol{b}_i$ running over all elements of $\mathrm{Mat}_{r/s}(\bar{\boldsymbol{k}})$. For each $1 \le i \le \boldsymbol{s}$, let

$$(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i := \left\{ \begin{pmatrix} \gamma_0 & \gamma_1 \\ \mathbf{0} & \gamma_0 \end{pmatrix} : \gamma_0 \in (\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i,\ \gamma_1 \in (\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i \right\}. \tag{5.3.23}$$

Note that $\dim \overline{H}_{1,i} \le 2r^2/\boldsymbol{s}^2 + 2r/\boldsymbol{s}$.

First suppose that $\dim \overline{H}_{1,i} = 2r^2/\boldsymbol{s}^2 + 2r/\boldsymbol{s}$. Then, we could simply take $\bar{\xi}_i'$ and $\bar{\boldsymbol{b}}_i'$ so that $\boldsymbol{u}_i, \boldsymbol{u}_{\boldsymbol{s}+i}, \boldsymbol{w}_i$ and $\boldsymbol{w}_{\boldsymbol{s}+i}$ are zero. Taking the Zariski closure $\bar{S}_{1,i}$ inside $\overline{\Gamma}_{N_1}$ of the subgroup generated by all such $\bar{\xi}_i'$ and $\boldsymbol{b}_i'$ with ξ_i and $\boldsymbol{b}_i$ running over all elements of $\mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$ and $\mathrm{Mat}_{r/s}(\bar{\boldsymbol{k}})$ respectively, we obtain

$$\bar{S}_{1,i} = \left\{ \begin{pmatrix} \nu_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : \nu_i \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i \right\}. \tag{5.3.24}$$

Thus, $\overline{\Gamma}_{N_1}$ contains a copy of $(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ and so, restricting $\mathrm{d}\bar{\pi}_1$ to $\mathrm{Lie}\, \bar{S}_{1,i}$, we obtain a surjection onto $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$. As we vary all $1 \le i \le \boldsymbol{s}$, the surjection of $\mathrm{d}\bar{\pi}_1$ follows.

Next, suppose that $\dim \overline{H}_{1,i} < 2r^2/\boldsymbol{s}^2 + 2r/\boldsymbol{s}$. Then, via $\bar{\pi}_1$ we have a short exact sequence

$$1 \to \overline{Q}_{1,i} \to \overline{H}_{1,i} \xrightarrow{\bar{\pi}_{1,i}} (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i \to 1,$$

where $\dim \overline{Q}_{1,i} < 2r/\boldsymbol{s}$ and $\overline{Q}_{1,i}$ is contained in an additive subgroup of G whose $\boldsymbol{v}_j$ coordinate vector is zero for all $j \ne i, \boldsymbol{s}+i$, that is,

$$\overline{Q}_{1,i} \subseteq \left\{ \left(\begin{array}{ccccccccccccc} \mathrm{Id}_{r/s} & & & & & & & \mathbf{0} & & & & & \\ & \ddots & & & & & & & \ddots & & & & \\ & & \mathrm{Id}_{r/s} & & & & & & & \mathbf{0} & & & \\ & & & \ddots & & & & & & & \ddots & & \\ & & & & \mathrm{Id}_{r/s} & & & & & & & \mathbf{0} & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & & & \\ & & & & & & \ddots & & & & & & \\ & & & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & & & \\ \mathbf{0} & \cdots & \boldsymbol{v}_i & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{v}_{s+i} & \cdots & \mathbf{0} & 1 & & \end{array} \right) : \begin{array}{l} \boldsymbol{v}_i, \boldsymbol{v}_{s+i} \in \mathbb{G}_a^{r/s}, \\ \boldsymbol{v}_i = (\boldsymbol{v}_{i,1}, \ldots, \boldsymbol{v}_{i,r/s}), \\ \boldsymbol{v}_{s+i} = (\boldsymbol{v}_{s+i,1}, \ldots, \boldsymbol{v}_{s+i,r/s}) \end{array} \right\}. \tag{5.3.25}$$

Claim 4. *For $\overline{Q}_{1,i}$ if some entry of the $\boldsymbol{v}_i$ coordinate vector is nonzero or* $\dim \overline{Q}_{1,i} \ne r/\boldsymbol{s}$, *then* $\dim \overline{Q}_{1,i} = 0$.

Proof of Claim 4. We follow the argument of the proof of [Chang and Papanikolas 2011, Lemma 4.1.1]. Suppose $\dim \overline{Q}_{1,i} = m$, where $1 \le m < 2r/\boldsymbol{s}$. Note that $\overline{Q}_{1,i}$ is a vector group. If $\boldsymbol{v}_{i,j}, \boldsymbol{v}_{s+i,j} \ne 0$ for all $1 \le j \le r/\boldsymbol{s}$, let $\mu \in \overline{Q}_{1,i}(\bar{\boldsymbol{k}})$ such that all the entries of μ in the $\boldsymbol{v}_i$ coordinate vector are nonzero. For $a \in \bar{\boldsymbol{k}},\ a \ne 0, 1$ and $1 \le \ell \le r/\boldsymbol{s}$, pick $\eta_\ell, \varkappa_\ell \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ such that $\bar{\pi}_{1,i}(\eta_\ell), \bar{\pi}_{1,i}(\varkappa_\ell) \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}}))_i$, where

$$\bar{\pi}_{1,i}(\eta_\ell) = \left(\begin{array}{cccccc} \mathrm{Id}_{r/s} & & & \mathbf{0} & & \\ & \ddots & & & \ddots & \\ & & \mathfrak{a}_\ell & & & \mathbf{0} \\ & & & \ddots & & \\ & & & \mathrm{Id}_{r/s} & & \mathbf{0} \\ \hline & & & \mathrm{Id}_{r/s} & & \\ & & & & \ddots & \\ & & & & \mathfrak{a}_\ell & \\ & & & & & \ddots \\ & & & & & \mathrm{Id}_{r/s} \end{array}\right) \quad \text{for } \mathfrak{a}_\ell := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, \tag{5.3.26}$$

$$\bar{\pi}_{1,i}(\varkappa_\ell) = \left(\begin{array}{cccccc} \mathrm{Id}_{r/s} & & & \mathbf{0} & & \\ & \ddots & & & \ddots & \\ & & \mathrm{Id}_{r/s} & & \mathfrak{b}_\ell & \\ & & & \ddots & & \ddots \\ & & & \mathrm{Id}_{r/s} & & \mathbf{0} \\ \hline & & & \mathrm{Id}_{r/s} & & \\ & & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} & \\ & & & & & \ddots \\ & & & & & \mathrm{Id}_{r/s} \end{array}\right) \quad \text{for } \mathfrak{b}_\ell := \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & a & \\ & & & \ddots \\ & & & & 0 \end{pmatrix},$$

where $\mathfrak{a}_\ell$ and $\mathfrak{b}_\ell$ are $r/\boldsymbol{s} \times r/\boldsymbol{s}$, and a is in the ℓ-th diagonal entries. One checks directly that the $2r/\boldsymbol{s}$ vectors $\eta_\ell^{-1}\mu\eta_\ell$, $\varkappa_\ell^{-1}\mu\varkappa_\ell$, where $1 \le \ell \le m$, are $\bar{\boldsymbol{k}}$-linearly independent in $\overline{Q}_{1,i}(\bar{\boldsymbol{k}})$, which contradicts the assumption $\dim \overline{Q}_{1,i} = m < 2r/\boldsymbol{s}$. Thus, $\boldsymbol{v}_{i,u} = 0$ for some $1 \le u \le r/\boldsymbol{s}$. Since $m \neq 0$, at least one of $\boldsymbol{v}_{i,j}$, $\boldsymbol{v}_{\boldsymbol{s}+i,j}$ for some $1 \le j \le r/\boldsymbol{s}$ is nonzero, say $\boldsymbol{v}_{i,v}$ or $\boldsymbol{v}_{\boldsymbol{s}+i,v}$.

Let $\boldsymbol{P}_{u,v}$ be the permutation matrix obtained by switching the $((i-1)r/\boldsymbol{s}+u)$-th column and the $((i-1)r/\boldsymbol{s}+v)$-column of the $r \times r$ identity matrix. Pick $\gamma \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ such that

$$\bar{\pi}_{1,i}(\gamma) = \begin{pmatrix} \boldsymbol{P}_{u,v} & \\ & \boldsymbol{P}_{u,v} \end{pmatrix} \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}}))_i. \tag{5.3.27}$$

If $\boldsymbol{v}_{i,v}$ is nonzero, then since $\gamma^{-1}\overline{Q}_{1,i}\gamma \subseteq \overline{Q}_{1,i}$ we get a contradiction to $\boldsymbol{v}_{i,u} = 0$. Therefore, $\dim \overline{Q}_{1,i} = 0$.

Next suppose $\boldsymbol{v}_{\boldsymbol{s}+i,v}$ is nonzero but $\boldsymbol{v}_{i,j} = 0$ for all $1 \le j \le r/\boldsymbol{s}$. Then by hypothesis, $m < r/\boldsymbol{s}$. If $\boldsymbol{v}_{\boldsymbol{s}+i,j} \neq 0$ for all $1 \le j \le r/\boldsymbol{s}$, let $\vartheta \in \overline{Q}_{1,i}(\bar{\boldsymbol{k}})$ such that all the entries of ϑ in the $\boldsymbol{v}_{\boldsymbol{s}+i}$ coordinate vector are nonzero. Then, one checks directly that for η_ℓ as in (5.3.26), the $r/\boldsymbol{s}$ vectors $\eta_\ell^{-1}\vartheta\eta_\ell$ are $\bar{\boldsymbol{k}}$-linearly independent in $\overline{Q}_{1,i}(\bar{\boldsymbol{k}})$, which contradicts the assumption $\dim Q_{1,i} = m < r/\boldsymbol{s}$. Thus, $\boldsymbol{v}_{\boldsymbol{s}+i,u} = 0$ for some $1 \le u \le r/\boldsymbol{s}$. Then, since $\boldsymbol{v}_{\boldsymbol{s}+i,v}$ is nonzero and $\gamma^{-1}Q_{1,i}\gamma \subseteq \overline{Q}_{1,i}$ for γ as in (5.3.27), we get a contradiction to $\boldsymbol{v}_{\boldsymbol{s}+i,u} = 0$. Therefore, $\dim \overline{Q}_{1,i} = 0$. □

Claim 5. *If* $\dim \overline{Q}_{1,i} = 0$, *then* $\mathrm{d}\bar{\pi}_{1,i} : \mathrm{Lie}\, \overline{H}_{1,i} \to \mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ *is surjective.*

Proof of Claim 5. To prove that $d\bar{\pi}_{1,i}$ is surjective, we follow the argument of the proof of [Chang and Papanikolas 2011, Proposition 4.1.2]. We let the coordinates of $\overline{H}_{1,i}$ be as follows:

$$\boldsymbol{Z}_1 := \begin{pmatrix} \mathcal{Z}_0 & \mathcal{Z}_1 & \boldsymbol{0} \\ & \mathcal{Z}_0 & \boldsymbol{0} \\ \mathcal{W}_0 & \mathcal{W}_1 & 1 \end{pmatrix}, \tag{5.3.28}$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} \mathrm{Id}_{r/s} & & & \\ & \ddots & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{Z}_1 = \begin{pmatrix} \boldsymbol{0} & & & \\ & \ddots & & \\ & & (Z_1) & & \\ & & & \ddots & \\ & & & & \boldsymbol{0} \end{pmatrix},$$

such that (Z_0) and (Z_1) are the coordinates of $\mathrm{GL}_{r/s}$ and $\mathrm{Mat}_{r/s}$ respectively. For each $h = 0, 1$, we define (Z_h) to be the $r/s \times r/s$ block $((Z_h)_{a,b})$ for $1 \le a, b \le r/s$ and $\mathcal{W}_h := (0, \dots, 0, (W_h), 0, \dots, 0)$, where we set $(W_h) := (W_{h,1}, \dots, W_{h,r/s})$. For $1 \le u, v \le r/s$, we define the following one-dimensional subgroups of $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$:

$$T_{uv} := \left\{ \begin{pmatrix} \mathcal{B}_{uv} & \boldsymbol{0} \\ \boldsymbol{0} & \mathcal{B}_{uv} \end{pmatrix} \right\}, \quad U_{uv} := \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathcal{C}_{uv} \\ \boldsymbol{0} & \mathrm{Id}_r \end{pmatrix} \right\}, \tag{5.3.29}$$

where we set

$$\mathcal{B}_{uv} := \begin{pmatrix} \mathrm{Id}_{r/s} & & & \\ & \ddots & & \\ & & B_{uv} & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{C}_{uv} := \begin{pmatrix} \boldsymbol{0} & & & \\ & \ddots & & \\ & & C_{uv} & & \\ & & & \ddots & \\ & & & & \boldsymbol{0} \end{pmatrix} \tag{5.3.30}$$

such that

$$B_{vv} := \left\{ \begin{pmatrix} 1 & & \\ & \ddots & \\ & * & \ddots & \\ & & & 1 \end{pmatrix} \right\}, \quad B_{uv} := \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & * & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad C_{uv} := \left\{ \begin{pmatrix} 0 & & \\ & \ddots & * \\ & & \ddots \\ & & & 0 \end{pmatrix} \right\},$$

where $*$ in B_{uv} and C_{uv} are in the (u, v)-coordinates. Note that the Lie algebras of the $2 \cdot r^2/s^2$ algebraic groups T_{uv} and U_{uv} span $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$. In what follows, we construct one-dimensional algebraic subgroups T'_{uv} and U'_{uv} of $\overline{H}_{1,i}$ so that $T'_{uv} \cong T_{uv}$ and $U'_{uv} \cong U_{uv}$. Then, since $\mathrm{Lie}(\cdot)$ is a left exact functor, it follows that $\mathrm{Lie}\, T'_{uv} \cong \mathrm{Lie}\, T_{uv}$ and $\mathrm{Lie}\, U'_{uv} \cong \mathrm{Lie}\, U_{uv}$, and so $d\bar{\pi}_{1,i}$ is surjective. Since $\overline{Q}_{1,i}$ is a zero-dimensional vector group, $\bar{\pi}_{1,i}$ is injective on points and so it follows by checking directly that

- for $w \ne v$, all $W_{0,w}$ and $W_{1,w}$ coordinates of $\bar{\pi}_{1,i}^{-1}(T_{uv})$ are zero;
- all (W_0) coordinates of $\bar{\pi}_{1,i}^{-1}(U_{uv})$ are zero, and for $w \ne v$, all $W_{1,w}$ coordinates of $\bar{\pi}_{1,i}^{-1}(U_{uv})$ are zero.

To construct T'_{vv}, we let $a_v \in \bar{\boldsymbol{k}}^\times \setminus \overline{\mathbb{F}}_q^\times$ and pick an element $\gamma_{1,v} \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ so that

$$\bar{\pi}_{1,i}(\gamma_{1,v}) = \begin{pmatrix} \mathfrak{a}_v & \\ & \mathfrak{a}_v \end{pmatrix}, \quad \text{where } \mathfrak{a}_v = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a_v & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in (\mathrm{GL}_{r/s}(\bar{\boldsymbol{k}}))_i, \tag{5.3.31}$$

such that a_v is in the $(i \cdot r/s + v)$-th diagonal entry of $\mathfrak{a}_v$. For $1 \le v \le r/s$, we let $c_{0,v}$ and $c_{1,v}$ respectively be the $(2r+1, (i-1)\cdot r/s + v)$-th and the $(2r+1, (r+(i-1)\cdot r/s)+v)$-th the entry of $\gamma_{1,v}$. Let T'_{vv} be the Zariski closure of the subgroup of $\overline{H}_{1,i}$ generated by $\gamma_{1,v}$ for each $1 \le v \le r/s$. Then, one checks directly that the defining equations of the one-dimensional subgroup T'_{vv} of $\overline{H}_{1,i}$ can be written as

$$\begin{cases} (a_v - 1)W_{0,v} - c_{0,v}((Z_0)_{v,v} - 1) = 0, & 1 \le v \le r^2/s, \\ (Z_0)_{w,w} = 1, & w \neq v,\ 1 \le v \le r^2/s, \\ (Z_1)_{u,v} = 0, & 1 \le u,\ v \le r/s, \\ W_{h,w} = 0, & w \neq v,\ h = 0, 1,\ 1 \le v \le r^2/s, \\ W_{0,v} \cdot c_{1,v} - W_{1,v} \cdot c_{0,v} = 0, & 1 \le v \le r^2/s. \end{cases}$$

Then, we see that $T'_{vv} \cong T_{vv}$ via $\bar{\pi}_{1,i}$. To construct T'_{uv} when $u \neq v$, we let $b_{u,v} \in T_{uv}(\boldsymbol{k})$ be a $\boldsymbol{k}$-rational basis for the one-dimensional vector group T_{uv} and pick $b'_{u,v} \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ so that $\bar{\pi}_{1,i}(b'_{u,v}) = b_{u,v}$. We define T'_{uv} to be the one-dimensional vector group in $\overline{H}_{1,i}$ via the conjugations

$$\eta_v^{-1} b'_{uv} \eta_v \quad \text{for } \eta_v \in T'_{vv},\ v = 1, \dots, r/s.$$

Then, we have $T'_{uv} \cong T_{uv}$ via $\bar{\pi}_{1,i}$. Similarly, we use the methods used for T'_{vv} and conjugations as above to construct suitable one-dimensional U'_{uv} such that $U'_{uv} \cong U_{uv}$ for $1 \le u, v \le r/s$. The arguments are essentially the same as the ones used to construct T'_{vv} and T'_{uv}, and so we omit the details and leave it to the reader. This proves our claim. □

Claim 6. *For $\overline{Q}_{1,i}$ if all entries of the $\boldsymbol{v}_i$ coordinate vector are zero and $\dim \overline{Q}_{1,i} = r/s$, then $\mathrm{d}\bar{\pi}_{1,i} : \operatorname{Lie} \overline{H}_{1,i} \to \operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective.*

Proof of Claim 6. We have $\dim \overline{H}_{1,i} = 2r^2/s^2 + r/s$ and by (5.3.25),

$$\overline{Q}_{1,i} = \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{Id}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{z} & 1 \end{pmatrix} : \boldsymbol{z} = (\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{v}_{s+i}, \mathbf{0}, \dots, \mathbf{0}) \in \mathbb{G}_a^r \text{ where } \boldsymbol{v}_{s+i} \in \mathbb{G}_a^{r/s} \right\}.$$

Note that $\overline{\Gamma}_{N_0}$ is an algebraic subgroup of $\mathrm{GL}_{r+1}/\bar{\boldsymbol{k}}$ such that the surjective map $\overline{\Xi}_0 : \overline{\Gamma}_{N_1} \to \overline{\Gamma}_{N_0}$ induced by Ξ_0 in (5.3.8) is given by

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \mathbf{0} \\ \mathbf{0} & \gamma_0 & \mathbf{0} \\ z_0 & z_1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \gamma_0 & \mathbf{0} \\ z_0 & 1 \end{pmatrix}. \tag{5.3.32}$$

Then, the elements of $\operatorname{Ker}\Xi_0^{(\bar{k})} \subseteq \bar{\Gamma}_{N_1}(\bar{k})$ are of the form

$$\begin{pmatrix} \mathrm{Id}_r & \gamma_1 & \mathbf{0} \\ \mathbf{0} & \mathrm{Id}_r & \mathbf{0} \\ \mathbf{0} & z_1 & 1 \end{pmatrix}.$$

From this, we see that for any $\boldsymbol{b}_i \in \mathrm{Mat}_{r/s}(\bar{k})$, elements of the form $\bar{\boldsymbol{b}}_i'$ in (5.3.22) with $\boldsymbol{w}_i = \mathbf{0}$ are in $\bar{H}_{1,i}(\bar{k})$. Multiplying such $\bar{\boldsymbol{b}}_i'$ by suitable elements of $\bar{Q}_{1,i}(\bar{k})$, we have $\bar{\boldsymbol{b}}_i'$ of the form (5.3.22), where $\boldsymbol{w}_i = \boldsymbol{w}_{s+i} = \mathbf{0}$ in $\bar{H}_{1,i}(\bar{k})$. Let $\bar{P}_{1,i}$ be the Zariski closure inside $\bar{H}_{1,i}$ of the subgroup generated by all such $\bar{\boldsymbol{b}}_i'$ with $\boldsymbol{b}_i$ running over all elements of $\mathrm{Mat}_{r/s}(\bar{k})$. Then, clearly $\bar{P}_{1,i} \cong \mathrm{Mat}_{r/s}/\bar{k}$.

For any $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, multiplying the elements $\bar{\xi}_i' \in \bar{H}_{1,i}(\bar{k})$ of the form (5.3.21) by suitable elements of $Q_{1,i}(\bar{k})$, we obtain $\bar{\xi}_i'$, where $\boldsymbol{u}_{s+i} = \mathbf{0}$.

For all $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, if there is a $\bar{\xi}_i' \in \bar{H}_{1,i}(\bar{k})$ with $\boldsymbol{u}_i = \mathbf{0}$, then by using $\bar{P}_{1,i}$ and all such elements $\bar{\xi}_i'$ for all $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, we could simply construct $\bar{S}_{1,i}$ as in (5.3.24) and restrict $d\bar{\pi}_1$ to $\operatorname{Lie}\bar{S}_{1,i}$ to obtain a surjection onto $\operatorname{Lie}(\bar{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{k})_i$.

Next suppose $\boldsymbol{u}_i \neq \mathbf{0}$. Consider the short exact sequence of linear algebraic groups (see (5.3.18))

$$1 \to \bar{X}_0 \to \bar{\Gamma}_{N_0} \xrightarrow{\bar{\pi}_0} \bar{\Gamma}_{M_\rho} \to 1.$$

Consider the one-dimensional subgroups of $\bar{\Gamma}_{M_\rho}$ of the form $\mathcal{B}_{uv} \in (\mathrm{GL}_{r/s}/\bar{k})_i$ given in (5.3.30) for $1 \le u, v \le r/s$. The same methods used in Claim 5 to construct T_{uv}' can be applied in a straightforward manner to construct one-dimensional subgroups $\mathcal{B}_{uv}'$ of $\bar{\Gamma}_{N_0}$ so that $\mathcal{B}_{uv}' \cong \mathcal{B}_{uv}$. We leave this to the reader. For $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, consider $\bar{\xi}_i' \in \bar{H}_{1,i}(\bar{k})$ of the form (5.3.21) with $\boldsymbol{u}_{s+i} = \mathbf{0}$. Let $\bar{V}_{1,i}$ be the Zariski closure inside $\bar{H}_{1,i}$ of the subgroup generated by all such $\bar{\xi}_i'$ with ξ_i running over all elements of $\mathrm{GL}_{r/s}(\bar{k})$. Then, we can identify $\nu \in \bar{V}_{1,i}(\bar{k})$ with the image $\bar{\Xi}_0^{(\bar{k})}(\nu) \in \bar{\Gamma}_{N_0}(\bar{k})$ where $\bar{\Xi}_0$ is the surjective map (5.3.32). Via this identification, each $\mathcal{B}_{uv}'$ for $1 \le u, v \le r/s$ is a one-dimensional subgroup of $\bar{V}_{1,i}$. The Lie algebras of the r^2/s^2 subgroups $\mathcal{B}_{uv}$ span $\operatorname{Lie}\mathrm{GL}_{r/s}/\bar{k}$. Thus, since $\bar{P}_{1,i} \cong \mathrm{Mat}_{r/s}/\bar{k}$, the Lie algebras of each $\mathcal{B}_{uv}$ and $\operatorname{Lie}\bar{P}_{1,i}$ span $\operatorname{Lie}(\bar{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{k})_i$ by (5.3.23). Then, since $\operatorname{Lie}(\cdot)$ is a left exact, $d\bar{\pi}_{1,i}$ is surjective. □

As we vary all $1 \le i \le s$, the surjection of $d\bar{\pi}_1$ follows. Thus, for $n = 1$ the proof of the lemma is complete.

Now suppose $n > 1$. We follow the methods used for $n = 1$ to prove that the induced tangent map $d\bar{\pi}_n$ at the identity is surjective onto $\operatorname{Lie}\bar{\Gamma}_{\mathrm{P}_n M_\rho}$. Recall $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$ from (5.3.17). Let $w = 1$ and consider the short exact sequence (5.3.18) of linear algebraic groups. Fix $1 \le i \le s$. We follow the methods used for the construction of $\bar{H}_{1,i}$ above to construct the Zariski closure $\bar{H}_{n,i}$ inside $\bar{\Gamma}_{N_n}$ of the subgroup generated by suitably chosen elements of $\bar{\Gamma}_{N_n}$ such that $\bar{H}_{n,i}$ is contained in the $(n+1)r^2/s^2 + (n+1)r^2/s$-dimensional group

$$G_{n,i} := \left\{ \left(\begin{array}{cccc|c} \eta_0 & \eta_1 & \cdots & \eta_n & \mathbf{0} \\ & \eta_0 & \ddots & \vdots & \vdots \\ & & \ddots & \eta_1 & \vdots \\ & & & \eta_0 & \mathbf{0} \\ \hline \boldsymbol{s}_0 & \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_n & 1 \end{array} \right) : \begin{array}{l} \eta_0 \in (\mathrm{GL}_{r/s}/\bar{k})_i,\ \eta_j \in (\mathrm{Mat}_{r/s}/\bar{k})_i,\ 1 \le j \le n, \\ \boldsymbol{s}_h = (\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{s}_{h,i}, \mathbf{0}, \dots, \mathbf{0}),\ \boldsymbol{s}_{h,i} \in \mathbb{G}_a^{r/s} \text{ for each } 0 \le h \le n \end{array} \right\}.$$

Let

$$(\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i := \left\{ \left(\begin{array}{c|c|c|c} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \hline & \gamma_0 & \ddots & \vdots \\ \hline & & \ddots & \gamma_1 \\ \hline & & & \gamma_0 \end{array} \right) : \gamma_0 \in (\mathrm{GL}_{r/\boldsymbol{s}}/\bar{\boldsymbol{k}})_i,\ \gamma_j \in (\mathrm{Mat}_{r/\boldsymbol{s}}/\bar{\boldsymbol{k}})_i,\ \text{where } 1 \le j \le n \right\}.$$

If $\dim \bar{H}_{n,i} = (n+1)\cdot r^2/\boldsymbol{s}^2 + (n+1)\cdot r/\boldsymbol{s}$, similar to $\bar{S}_{1,i}$ in (5.3.24) we simply construct

$$\bar{S}_{n,i} = \left\{ \begin{pmatrix} \vartheta_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : \vartheta_i \in (\Gamma_{\mathrm{P}_n M_\rho}(\bar{\boldsymbol{k}}))_i \right\} \tag{5.3.33}$$

and restrict $\mathrm{d}\bar{\pi}_n$ to $\mathrm{Lie}\,\bar{S}_{n,i}$ to obtain a surjection onto $\mathrm{Lie}(\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$. As we vary all $1 \le i \le \boldsymbol{s}$, the surjection of $\mathrm{d}\bar{\pi}_n$ follows.

Next, suppose $\dim \bar{H}_{n,i} < (n+1)\cdot r^2/\boldsymbol{s}^2 + (n+1)\cdot r/\boldsymbol{s}$. Then, via $\bar{\pi}_n$ we have a short exact sequence

$$1 \to \bar{Q}_{n,i} \to \bar{H}_{n,i} \xrightarrow{\bar{\pi}_{n,i}} (\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i \to 1.$$

For $\bar{Q}_{n.i}$, if some entry of each $\boldsymbol{s}_{h,i}$ coordinate vector where $0 \le h \le n-\ell$ is nonzero or $\dim \bar{Q}_{n,i} \ne \ell r/\boldsymbol{s}$, then the methods used in Claim 4 to prove $\dim \bar{Q}_{1,i} = 0$ can be applied in a straightforward manner to prove $\dim \bar{Q}_{n,i} = 0$, which we leave to the reader.

Claim 7. *If* $\dim \bar{Q}_{n,i} = 0$, *then* $\mathrm{d}\bar{\pi}_{n,i} : \mathrm{Lie}\,\bar{H}_{n,i} \to \mathrm{Lie}(\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$ *is surjective.*

Proof of Claim 7. The proof follows the same line of argument as in the proof of Claim 5 ($n=1$ case) and so we include only a sketch. Similar to the coordinates $\boldsymbol{Z}_1$ of $\bar{H}_{1,i}$ in (5.3.28), we let the coordinates of $\bar{H}_{n,i}$ be

$$\boldsymbol{Z}_n = \begin{pmatrix} \mathcal{Z}_0 & \mathcal{Z}_1 & \cdots & \mathcal{Z}_n & \mathbf{0} \\ & \mathcal{Z}_0 & \ddots & \vdots & \vdots \\ & & \ddots & \mathcal{Z}_1 & \vdots \\ & & & \mathcal{Z}_0 & \mathbf{0} \\ \mathcal{W}_0 & \mathcal{W}_1 & \cdots & \mathcal{W}_n & 1 \end{pmatrix},$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} \mathrm{Id}_{r/\boldsymbol{s}} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/\boldsymbol{s}} \end{pmatrix}, \quad \mathcal{Z}_j = \begin{pmatrix} \mathbf{0} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathbf{0} \end{pmatrix}$$

for each $1 \le j \le n$ such that (Z_0) is as in (5.3.28) and (Z_j) is the $r/\boldsymbol{s} \times r/\boldsymbol{s}$ block $((Z_j)_{a,b})$ for $1 \le a, b \le r/\boldsymbol{s}$. Moreover, set $\mathcal{W}_h := (0, \ldots, 0, (W_h), 0, \ldots, 0)$, where we set $(W_h) := (W_{h,1}, \ldots, W_{h,r/\boldsymbol{s}})$ for each $0 \le h \le n$.

Now, we prove that $\mathrm{d}\bar{\pi}_{n,i} : \operatorname{Lie} \overline{H}_{n,i} \to \operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective. Similar to (5.3.29), we construct one-dimensional subgroups of $\overline{\Gamma}_{\mathrm{P}_n M_\rho}$:

$$T_{0,u,v} := \left\{ \begin{pmatrix} \mathcal{B}_{uv} & \mathbf{0} & \cdots & \mathbf{0} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \mathcal{B}_{uv} \end{pmatrix} \right\}, \quad U_{\ell,u,v} := \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathbf{0} & \cdots & \mathcal{C}_{uv} & \cdots & \mathbf{0} \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \mathcal{C}_{uv} \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \mathbf{0} \\ & & & & & \mathrm{Id}_r \end{pmatrix} \right\}$$

such that $\mathcal{B}_{uv}$ and $\mathcal{C}_{uv}$ are as in (5.3.30), and, for $1 \le \ell \le n$, $\mathcal{C}_{uv}$ is in the ℓ-th superdiagonal block of $U_{\ell,u,v}$. Similar to the $n=1$ case, note that the Lie algebras of the $(n+1)\cdot r^2/\boldsymbol{s}^2$ algebraic groups $T_{0,u,v}$ and $U_{\ell,u,v}$ span $\operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$. In what follows, we construct one-dimensional algebraic subgroups $T'_{0,u,v}$ and $U'_{\ell,u,v}$ of $\overline{H}_{n,i}$ so that $T'_{0,u,v} \cong T_{0,u,v}$ and $U'_{\ell,u,v} \cong U_{\ell,u,v}$. Then, since $\operatorname{Lie}(\,\cdot\,)$ is a left exact functor, it follows that $\operatorname{Lie} T'_{0,u,v} \cong \operatorname{Lie} T_{0,u,v}$ and $\operatorname{Lie} U'_{\ell,u,v} \cong \operatorname{Lie} U_{\ell,u,v}$, and so $\mathrm{d}\bar{\pi}_{n,i}$ is surjective. Since $\overline{Q}_{n,i}$ is a zero-dimensional vector group, $\bar{\pi}_{n,i}$ is injective on points and so it follows by checking directly that

- for $w \ne v$ and $0 \le h \le n$, all $W_{h,w}$ coordinates of $\bar{\pi}_{n,i}^{-1}(T_{0,u,v})$ are zero;
- all (W_0) coordinates of $\bar{\pi}_{n,i}^{-1}(U_{\ell,u,v})$ are zero, and for $w \ne v$ and $1 \le j \le n$, all $W_{j,w}$ coordinates of $\bar{\pi}_{n,i}^{-1}(U_{\ell,u,v})$ are zero.

To construct $T'_{0,v,v}$, we let $a_v \in \bar{\boldsymbol{k}}^\times \setminus \overline{\mathbb{F}}_q^\times$ and pick elements $\gamma_{n,v} \in \overline{H}_{n,i}(\bar{\boldsymbol{k}})$ so that

$$\bar{\pi}_{n,i}(\gamma_{n,v}) = \begin{pmatrix} \mathfrak{a}_v & & \\ & \ddots & \\ & & \mathfrak{a}_v \end{pmatrix},$$

where $\mathfrak{a}_v$ is as in (5.3.31). For $1 \le v \le r/\boldsymbol{s}$ and $0 \le h \le n$, we let $c_{h,v}$ be the $(nr+1, hr+(i-1)\cdot r/\boldsymbol{s}+v)$-th the entry of $\gamma_{n,v}$. Let $T'_{0,v,v}$ be the Zariski closure of the subgroup of $\overline{H}_{n,i}$ generated by $\gamma_{n,v}$. Then, one checks directly that the defining equations of the one-dimensional subgroup $T'_{0,v,v}$ of $\overline{H}_{n,i}$ can be written as

$$\begin{cases} (a_v - 1)W_{0,v} - c_{0,v}((Z_0)_{v,v} - 1) = 0, & 1 \le v \le r^2/\boldsymbol{s}, \\ (Z_0)_{w,w} = 1, & w \ne v,\ 1 \le v \le r^2/\boldsymbol{s}, \\ (Z_j)_{u,v} = 0, & 1 \le j \le \ell,\ 1 \le u, v \le r/\boldsymbol{s}, \\ W_{h,w} = 0, & w \ne v,\ 0 \le h \le \ell,\ 1 \le v \le r^2/\boldsymbol{s}, \\ W_{h_1,v} \cdot c_{h_2,v} - W_{h_2,v} \cdot c_{h_1,v} = 0, & 0 \le h_1, h_2 \le \ell,\ 1 \le v \le r^2/\boldsymbol{s}. \end{cases}$$

Then, we see that $T'_{0,v,v} \cong T_{0,v,v}$ via $\bar{\pi}_{n,i}$. Similarly, we use the methods used for $T'_{0,v,v}$ and conjugations as in the $n=1$ case to construct $U'_{\ell,u,v}$ such that $U'_{\ell,u,v} \cong U_{\ell,u,v}$ for all $1 \le u, v \le r/\boldsymbol{s},\ 1 \le \ell \le n$, and $T'_{0,u,v}$ such that $T'_{0,u,v} \cong T_{0,u,v}$ for all $1 \le u, v \le r/\boldsymbol{s},\ u \ne v$. The arguments are essentially the same as the arguments used to construct T'_{uv} and U'_{uv} in the $n=1$ case and $T'_{0,v,v}$ above, and so we omit the details and leave it to the reader. This proves our claim. ☐

For $Q_{n,i}$ if $\boldsymbol{s}_{h,i} = \mathbf{0}$ for all $0 \le h \le n-\ell$ and $\dim Q_{n,i} = \ell r/\boldsymbol{s}$, then the methods used in Claim 6 to prove that $\mathrm{d}\bar{\pi}_{1,i} : \mathrm{Lie}\, \overline{H}_{1,i} \to \mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective can be applied in a straightforward manner to prove that $\mathrm{d}\bar{\pi}_{n,i} : \mathrm{Lie}\, \overline{H}_{n,i} \to \mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective, which we leave to the reader.

As we vary all $1 \le i \le \boldsymbol{s}$ the surjection of $\mathrm{d}\bar{\pi}_n$ follows. Thus, for $n > 1$ the proof of the lemma is complete. □

5.4. *Algebraic independence of logarithms and quasilogarithms.* We now prove Theorem 1.1.4 (restated as Theorem 5.4.4) and Corollary 1.1.5. Recall the short exact sequence (5.3.3):

$$1 \to X_n \to \Gamma_{N_n} \xrightarrow{\pi_n} \Gamma_{\mathrm{P}_n M_\rho} \to 1.$$

We will first show that X_n can be identified with a $\Gamma_{\mathrm{P}_n M_\rho}$-submodule of $((\mathrm{P}_n M_\rho)^B)^w$. Let $\boldsymbol{n} \in \mathrm{Mat}_{((n+1)rw+1)\times 1}(N_n)$ be such that its entries form a $\bar{k}(t)$-basis of N_n and $\sigma \boldsymbol{n} = \Phi_{N_n}\boldsymbol{n}$. The entries of $\Psi_{N_n}^{-1}\boldsymbol{n}$ form a $\boldsymbol{k}$-basis of N_n^B [Papanikolas 2008, Proposition 3.3.9]. If we write $\boldsymbol{n} = [\boldsymbol{n}_1, \ldots, \boldsymbol{n}_w, y]^\intercal$, where each $\boldsymbol{n}_i \in \mathrm{Mat}_{(n+1)r\times 1}(N_n)$, then the entries of $[\boldsymbol{n}_1, \ldots, \boldsymbol{n}_w]^\intercal$ form a $\bar{k}(t)$-basis of $(\mathrm{P}_n M_\rho)^w$ and the entries of $\boldsymbol{u} := [\Psi_{\mathrm{P}_n M_\rho}^{-1}\boldsymbol{n}_1, \ldots, \Psi_{\mathrm{P}_n M_\rho}^{-1}\boldsymbol{n}_w]^\intercal$ form a $\boldsymbol{k}$-basis of $((\mathrm{P}_n M_\rho)^B)^w$. Given any $\boldsymbol{k}$-algebra R, we recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} ((\mathrm{P}_n M_\rho)^B)^w$ from [Papanikolas 2008, §4.5] (see also (4.2.8)) as follows: for any $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $\boldsymbol{v}_h \in \mathrm{Mat}_{1\times(n+1)r}(\mathrm{R})$, $0 \le h \le n$, the action of μ on $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_w) \cdot \boldsymbol{u} \in R \otimes_{\boldsymbol{k}} ((\mathrm{P}_n M_\rho)^B)^w$ is

$$(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_w) \cdot \boldsymbol{u} \mapsto (\boldsymbol{v}_1 \mu^{-1}, \ldots, \boldsymbol{v}_w \mu^{-1}) \cdot \boldsymbol{u}.$$

Thus, by (5.3.4) the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on $((\mathrm{P}_n M_\rho)^B)^w$ is compatible with the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on X_n. Then, when we regard $((\mathrm{P}_n M_\rho)^B)^w$ as a vector group over $\boldsymbol{k}$, by Lemma 5.3.9 we get the desired result.

Now, note that since X_n is a $\Gamma_{\mathrm{P}_n M_\rho}$-submodule of $((\mathrm{P}_n M_\rho)^w)^B$, by the equivalence of categories $\mathcal{T}_{\mathrm{P}_n M_\rho} \approx \mathbf{Rep}(\Gamma_{\mathrm{P}_n M_\rho}, \boldsymbol{k})$, there exists a sub-$t$-motive V_n of $(\mathrm{P}_n M_\rho)^w$ such that as $\Gamma_{\mathrm{P}_n M_\rho}$-modules

$$X_n \cong V_n^B. \tag{5.4.1}$$

By (4.2.7), we see that for any $n \ge 1$ and $0 \le j \le n-1$ we obtain a short exact sequence of t-motives

$$0 \to (\mathrm{P}_j M_\rho)^w \xrightarrow{\iota} (\mathrm{P}_n M_\rho)^w \xrightarrow{\mathbf{pr}_{w,n-j-1}} (\mathrm{P}_{n-j-1} M_\rho)^w \to 0. \tag{5.4.2}$$

Lemma 5.4.3. *For $n \ge 1$, let V_n be as in (5.4.1). Then, for $0 \le j \le n-1$ there is a surjective map of t-motives $\overline{\mathbf{pr}}_{w,n-j-1} : V_n \to V_{n-j-1}$ via the map $\mathbf{pr}_{w,n-j-1}$ in (5.4.2).*

Proof. We prove the result for $w = 1$. The following argument for $w = 1$ can be applied in a straightforward manner to prove the arbitrary w case, which we leave to the reader. Let $w = 1$. Recall from (5.3.7) that for any $\boldsymbol{k}$-algebra R if

$$\nu_n = \begin{pmatrix} \mu_n & \mathbf{0} \\ \boldsymbol{w}_n & 1 \end{pmatrix} \in \Gamma_{N_n}(\mathrm{R}),$$

then

$$\nu_{n-j-1} = \begin{pmatrix} \mu_{n-j-1} & \mathbf{0} \\ \boldsymbol{w}_{n-j-1} & 1 \end{pmatrix} \in \Gamma_{N_{n-j-1}}(\mathrm{R}),$$

where μ_{n-j-1} is the matrix formed by the $r(n-j)\times r(n-j)$ upper-left square of μ_n and $\boldsymbol{w}_{n-j-1}=(w_0,\dots,w_{n-j-1})$. Also recall from (5.3.8) that the surjective map of affine group schemes $\Gamma_{N_n}\twoheadrightarrow\Gamma_{N_{n-j-1}}$ is given by

$$\nu_n\mapsto\nu_{n-j-1}.$$

Since X_n and X_{n-j-1} are $\boldsymbol{k}$-smooth by Lemma 5.3.9, this map gives a surjective map of group schemes $X_n\to X_{n-j-1}$. By (5.4.1), this corresponds to a map of representations of $\Gamma_{\mathrm{P}_nM_\rho}$ over $\boldsymbol{k}$, $\overline{\mathbf{pr}}^B_{w,n-j-1}:V_n^B\to V_{n-j-1}^B$ via the map $\mathbf{pr}^B_{w,n-j-1}:((\mathrm{P}_nM_\rho)^w)^B\to((\mathrm{P}_{n-j-1}M_\rho)^w)^B$, where $\mathbf{pr}_{w,n-j-1}$ is as in (5.4.2). By the equivalence of categories $\mathcal{T}_{\mathrm{P}_nM_\rho}\approx\mathbf{Rep}(\Gamma_{\mathrm{P}_nM_\rho},\boldsymbol{k})$, we obtain the required conclusion. □

Theorem 5.4.4. *Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep}. Suppose that K_ρ is separable over k and $[K_\rho:k]=\mathbf{s}$. Let $u_1,\dots,u_w\in\mathbb{K}$ with $\mathrm{Exp}_\rho(u_i)=\alpha_i\in k^{\mathrm{sep}}$ for each $1\le i\le w$ and suppose that $\dim_{K_\rho}\mathrm{Span}_{K_\rho}(\lambda_1,\dots,\lambda_r,u_1,\dots,u_w)=r/\mathbf{s}+w$. For $n\ge1$, let N_n and Ψ_{N_n} be defined as in (5.3.1), and, for each $1\le i\le w$, let $Y_{i,n}:=Y_{u_i,n}$ be defined as in Section 5.2. Then, $\dim\Gamma_{N_n}=(n+1)\cdot r(r/\mathbf{s}+w)$. In particular,*

$$\operatorname{tr.deg}_{\bar{k}}\bar{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j),\partial_\theta^s(F_{\tau^i}(\lambda_j)),\partial_\theta^s(u_m),\partial_\theta^s(F_{\tau^i}(u_m))\}\right)=(n+1)(r^2/\mathbf{s}+rw).$$

Proof. From the construction of Ψ_{N_n}, by Theorem 3.4.1 we have

$$\bar{k}(\Psi_{N_n}|_{t=\theta})=\bar{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j),\partial_\theta^s(F_{\tau^i}(\lambda_j)),\partial_\theta^s(u_m),\partial_\theta^s(F_{\tau^i}(u_m))\}\right),$$

and by Theorems 2.3.2 and 4.5.1, we have

$$\dim\Gamma_{N_n}=\operatorname{tr.deg}_{\bar{k}}\bar{k}(\Psi_{N_n}|_{t=\theta})\le(n+1)\frac{r^2}{\mathbf{s}}+(n+1)rw.$$

Thus, we need to prove that $\dim X_n=(n+1)rw$, where X_n is as in (5.3.3). By (5.4.1) it suffices to show that $V_n^B\cong((\mathrm{P}_nM_\rho)^w)^B$. To prove this, we adapt the arguments of the proof of [Chang and Papanikolas 2012, Theorem 5.1.5] (see also [Hardouin 2011, Lemma 1.2]).

Note from (5.4.2) that for $n\ge1$ we have a short exact sequence of t-motives

$$0\to(\mathrm{P}_0M_\rho)^w\xrightarrow{\iota}(\mathrm{P}_nM_\rho)^w\xrightarrow{\mathbf{pr}_{w,n-1}}(\mathrm{P}_{n-1}M_\rho)^w\to0.$$

By Lemma 5.4.3, there is a surjective map $\overline{\mathbf{pr}}_{w,n-1}:V_n\to V_{n-1}$ via $\mathbf{pr}_{w,n-1}$. Then $\ker(\overline{\mathbf{pr}}_{w,n-1})$ is a sub-t-motive of M_ρ^w.

We claim that if $V_{n-1}\cong(\mathrm{P}_{n-1}M_\rho)^w$, then N_n/V_n is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1},\mathrm{P}_nM_\rho/V_n)$. Since $X_n\cong V_n^B$, we see that Γ_{N_n} acts on N_n^B/V_n^B through $\Gamma_{N_n}/X_n\cong\Gamma_{\mathrm{P}_nM_\rho}$ via (5.3.3). Since $\overline{\mathbf{pr}}_{w,n-1}$ is surjective onto $V_{n-1}\cong(\mathrm{P}_{n-1}M_\rho)^w$, by using (5.3.5) one finds that $N_n^B/V_n^B\cong N_0^B/(\ker\overline{\mathbf{pr}}_{w,n-1})^B$. Recall that for any $\boldsymbol{k}$-algebra R, an element of $\Gamma_{\mathrm{P}_nM_\rho}(\mathrm{R})$ is of the form (4.2.6) such that γ_0 is an element of $\Gamma_{M_\rho}(\mathrm{R})$. Then, (5.3.6) shows the action of $\Gamma_{\mathrm{P}_nM_\rho}$ on N_n^B/V_n^B is the same as the action of Γ_{M_ρ} on it. Thus, N_n^B/V_n^B is an extension of $\boldsymbol{k}$ by $((\mathrm{P}_n M_\rho)^w)^B / V_n^B$ in $\mathbf{Rep}(\Gamma_{M_\rho}, \boldsymbol{k})$. By [Chang and Papanikolas 2012, Corollary 3.5.7] and the equivalence of categories $\mathcal{T}_{M_\rho} \approx \mathbf{Rep}(\Gamma_{M_\rho}, \boldsymbol{k})$, we get the required conclusion of the claim.

Now, we prove the main result by induction. For $n = 1$ case, suppose to the contrary that $V_1^B \subsetneq ((\mathrm{P}_1 M_\rho)^w)^B$. From [loc. cit., Theorem 5.1.5], we have $M_\rho^w \cong V_0$ and so, since $M_\rho^w \cong (\mathrm{P}_0 M_\rho)^w$, we have $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subsetneq M_\rho^w$. Since M_ρ^w is completely reducible in $\mathcal{T}_{M_\rho}$ by [loc. cit., Corollary 3.3.3] and $\ker(\overline{\mathbf{pr}}_{w,n-1})$ is a sub-t-motive of M_ρ^w, there exists a nontrivial morphism $\phi_1 \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$ so that $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subseteq \ker \phi_1$. Moreover, the morphism ϕ_1 factors through the map $M_\rho^w / \ker(\overline{\mathbf{pr}}_{w,n-1}) \to M_\rho^w / (\ker \phi_1)$. Since $\phi_1 \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$, there exist $e_{i,1} \in \boldsymbol{K}_\rho$ not all zero such that $\phi_1(n_1, \ldots, n_w) = \sum_{i=1}^{w} e_{i,1}(n_i)$. For a $\bar{k}(t)$-basis $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ of M_ρ, suppose that $\mathrm{E}_{i,1} \in \mathrm{Mat}_r(\bar{k}(t))$ satisfies $e_{i,1}(\boldsymbol{m}) = \mathrm{E}_{i,1}\boldsymbol{m}$. Set

$$\boldsymbol{E}_{i,1} := \begin{pmatrix} \mathbf{0} & \mathrm{E}_{i,1} \\ & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{2r}(\bar{k}(t)).$$

Recall from Section 2.5 that $\boldsymbol{D}_1\boldsymbol{m}$ forms a $\bar{k}(t)$-basis of $\mathrm{P}_1 M_\rho$. By (5.2.1) there exists $\boldsymbol{e}_{i,1} \in \mathrm{End}_{\mathcal{T}}((\mathrm{P}_1 M)^w)$ such that $\boldsymbol{e}_{i,1}(\boldsymbol{D}_1\boldsymbol{m}) = \boldsymbol{E}_{i,1}\boldsymbol{D}_1\boldsymbol{m}$. Let $\psi_1 \in \mathrm{Hom}_{\mathcal{T}}((\mathrm{P}_1 M_\rho)^w, \mathrm{P}_1 M_\rho)$ such that $\psi_1(D_j n_1, \ldots, D_j n_w) = \sum_{i=1}^{w} \boldsymbol{e}_{i,1}(D_j n_i)$ for each $j = 0, 1$. We see $\ker \psi_1 / M_\rho^w \cong \ker \phi_1$ and $(\mathrm{P}_1 M_\rho)^w / \ker \psi_1 \cong M_\rho^w / \ker \phi_1 \cong M_\rho$. Then the pushout $\psi_{1*}N_1 := \boldsymbol{e}_{1,1*}Y_{1,1} + \cdots + \boldsymbol{e}_{w,1*}Y_{w,1}$ is a quotient of N_1/V_1. By using the claim above, it follows that $\psi_{1*}N_1$ is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_1 M_\rho)$. However, by Theorem 5.2.2, this is a contradiction.

Now suppose that we have shown the result for $n-1$, that is, $V_{n-1} \cong (\mathrm{P}_{n-1} M_\rho)^w$. Suppose that $V_n^B \subsetneq ((\mathrm{P}_n M_\rho)^w)^B$. Then, $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subsetneq M_\rho^w$. Since M_ρ^w is completely reducible in $\mathcal{T}_{M_\rho}$ by [Chang and Papanikolas 2012, Corollary 3.3.3] and $\ker(\overline{\mathbf{pr}}_{w,n-1})$ is a sub-t-motive of M_ρ^w, there exists a nontrivial morphism $\phi_n \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$ so that $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subseteq \ker \phi_n$. Moreover, the morphism ϕ_n factors through the map $M_\rho^w / \ker(\overline{\mathbf{pr}}_{w,n-1}) \to M_\rho^w / (\ker \phi_n)$. Since $\phi_n \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$, we can write $\phi_n(n_1, \ldots, n_w) = \sum_{i=1}^{w} e_{i,n}(n_i)$ for some $e_{1,n}, \ldots, e_{w,n} \in \boldsymbol{K}_\rho$ not all zero. Suppose that $e_{i,n}(\boldsymbol{m}) = \mathrm{E}_{i,n}\boldsymbol{m}$, where $\mathrm{E}_{i,n} \in \mathrm{Mat}_r(\bar{k}(t))$. Set

$$\boldsymbol{E}_{i,n} := \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathrm{E}_{i,n} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{(n+1)r}(\bar{k}(t)).$$

Recall also from Section 2.5 that $\boldsymbol{D}_n\boldsymbol{m}$ forms a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$. By (5.2.1) there exists $\boldsymbol{e}_{i,n} \in \mathrm{End}_{\mathcal{T}}((\mathrm{P}_n M)^w)$ such that $\boldsymbol{e}_{i,n}(\boldsymbol{D}_1\boldsymbol{m}) = \boldsymbol{E}_{i,n}\boldsymbol{D}_1\boldsymbol{m}$. Let $\psi_n \in \mathrm{Hom}_{\mathcal{T}}((\mathrm{P}_n M_\rho)^w, \mathrm{P}_n M_\rho)$ such that $\psi_1(D_j n_1, \ldots, D_j n_w) = \sum_{i=1}^{w} \boldsymbol{e}_{i,1}(D_j n_i)$ for each $0 \le j \le n$. Similar to the base case, we see that $\ker \psi_n / (\mathrm{P}_{n-1} M_\rho)^w \cong \ker \phi_n$ and $(\mathrm{P}_n M_\rho)^w / \ker \psi_n \cong M_\rho^w / \ker \phi_n \cong M_\rho$. Then the pushout $\psi_{n*}N_n := \boldsymbol{e}_{1,n*}Y_{1,n} + \cdots + \boldsymbol{e}_{w,n*}Y_{w,n}$ is a quotient of N_n/V_n. By using the claim above, it follows that $\psi_{n*}N_n$ is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. However, by Theorem 5.2.2, this is a contradiction. □

Proof of Corollary 1.1.5. Let $\{\eta_1, \ldots, \eta_\alpha\} \subseteq \{\lambda_1, \ldots, \lambda_r, u_1, \ldots, u_w\}$ be a maximal K_ρ-linearly independent set containing $\{u_1, \ldots, u_w\}$. Clearly, $r/s \le \alpha \le r/s + w$. Since the quasiperiodic functions F_δ are linear in δ and satisfy the difference equation (1.1.2), we have

$$\bar{k}\Bigl(\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\lambda_j, F_{\tau^i}(\lambda_j), u_m, F_{\tau^i}(u_m)\}\Bigr) = \bar{k}\Bigl(\bigcup_{j=1}^{r}\bigcup_{m=1}^{\alpha}\{F_{\delta_j}(\eta_m)\}\Bigr).$$

Moreover, for any $1 \le i_1, i_2 \le r,\ 1 \le j_1, j_2 \le \alpha,\ 0 \le s \le n$ and $v_1, v_2 \in K_\rho$, by the product rule of hyperderivatives we obtain

$$\partial_\theta^s(v_1 F_{\delta_{i_1}}(\eta_{j_1}) + v_2 F_{\delta_{i_2}}(\eta_{j_2})) = \sum_{h=0}^{s}\bigl(\partial_\theta^{s-h}(v_1)\partial_\theta^h(F_{\delta_{i_1}}(\eta_{j_1})) + \partial_\theta^{s-h}(v_2)\partial_\theta^h(F_{\delta_{i_1}}(\eta_{j_2}))\bigr).$$

Thus,

$$\bar{k}\Bigl(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}\Bigr) = \bar{k}\Bigl(\bigcup_{s=0}^{n}\bigcup_{j=1}^{r}\bigcup_{m=1}^{\alpha}\{\partial_\theta^s(F_{\delta_j}(\eta_m))\}\Bigr).$$

Then, the result follows by Theorem 5.4.4. □

Appendix: Differential algebraic geometry

We present a few topics from differential algebraic geometry in positive characteristic [Okugawa 1987] (see [Hardouin et al. 2016] for characteristic zero). For the most part, we follow the terminology of [Hardouin et al. 2016]. Even though the proofs of most of the results presented here are covered in [Okugawa 1987], we present them here nevertheless for completeness.

A.1. ***Differential algebraic geometry in positive characteristic.*** Let R be a commutative ring with unity of characteristic $p > 0$. A *differential ring* or *∂-ring* is a pair (R, ∂), where ∂ represents a sequence of additive maps $\partial^j : R \to R$ that satisfy

(1) $\partial^0(a) = a$,

(2) $\partial^j(a+b) = \partial^j(a) + \partial^j(b)$,

(3) $\partial^j(ab) = \sum_{i=0}^{j}\partial^i(a)\partial^{j-i}(b)$,

(4) $\partial^k\partial^j(a) = \binom{k+j}{j}\partial^{k+j}(a)$

for all $a, b \in R$ and $j, k \ge 0$. If R is a field, then we say that (R, ∂) is a *differential field* or a *∂-field*. When the context is clear, we shall write R instead of (R, ∂). Moreover, a *∂-morphism* between two ∂-rings R and S is a morphism of rings that commute with ∂. For a ∂-ring R, if we let $\mathfrak{I} \subseteq R$ be an ideal, then $\mathfrak{I}$ is called a *∂-ideal* if $\partial^j(\mathfrak{I}) \subseteq \mathfrak{I}$ for all $j \ge 1$. If, in addition, $\mathfrak{I}$ is a radical (respectively prime) ideal of the ∂-ring R regarded as a ring, then we say that $\mathfrak{I}$ is a *radical* (respectively *prime*) *∂-ideal* of the ∂-ring R. For a set $\Sigma \subseteq R$, the intersection of all ∂-ideals containing Σ is a ∂-ideal of R, which we denote by $\mathfrak{D}(\Sigma)$ and it is the smallest ∂-ideal of R containing Σ. We see that $\mathfrak{D}(\Sigma)$ is the ideal, generated $\{\partial^j(a) : j \ge 0,\ a \in \Sigma\}$, of the ∂-ring R regarded as a ring. We denote by $\mathfrak{R}(\mathfrak{D}(\Sigma))$ or $\mathfrak{R}(\Sigma)$ the radical of $\mathfrak{D}(\Sigma)$ in the ∂-ring R.

Proposition A.1.1 [Okugawa 1987, p. 45, Theorem 5]. *Let R be a ∂-ring of characteristic $p>0$ and let $\mathfrak{I}\subseteq R$ be a ∂-ideal of R. Then, the radical $\mathfrak{R}(\mathfrak{I})$ is a ∂-ideal of R.*

Proof. It suffices to prove that $\partial^j(\mathfrak{R}(\mathfrak{I}))\subseteq\mathfrak{R}(\mathfrak{I})$ for all $j\geq 1$. Let $a\in\mathfrak{R}(\mathfrak{I})$. Then $a^n\in\mathfrak{I}$ for some $n\geq 1$. For a sufficiently large $e\geq 1$, we see that $a^m\cdot a^n=a^{p^e}\in\mathfrak{I}$ for some $m\in\mathbb{N}$. Note that Proposition 2.4.1 applies here and so, for all $j\in\mathbb{N}$ we see that $\partial^{jp^e}(a^{p^e})=(\partial^j(a))^{p^e}$. Since $\mathfrak{I}$ is a ∂-ideal of R, we have $\partial^{jp^e}(a^{p^e})\in\mathfrak{I}$ for all $j\geq 1$. Thus, $(\partial^j(a))^{p^e}\in\mathfrak{I}$ and so $\partial^j(a)\in\mathfrak{R}(\mathfrak{I})$. □

Remark A.1.2. The proof of Proposition A.1.1 does not work in characteristic 0. See [Hardouin et al. 2016, Proposition 2.19] for the proof of the characteristic-0 case.

The *∂-polynomial ring* denoted by $R\{y_1,\dots,y_m\}$ in the ∂-variables $(y_1,\dots,y_m)$ is the polynomial ring over a ∂-ring R in the variables $\partial^j(y_i)$, $j\geq 0$, $1\leq i\leq m$, made into a ∂-ring by setting

(a) $\partial^j(a):=\partial^j(a)$ for $a\in R$,

(b) $\partial^k(\partial^j(y_i)):=\binom{k+j}{j}\partial^{k+j}(y_i)$, $k\geq 0$.

Here $y_1,\dots,y_m$ are called *∂-indeterminates*.

Let K be a ∂-field. A *∂-extension field of K* is a ∂-field L which is an extension field of the ∂-field K. Note that K and L are fields. Let $\overline{K}$ be an algebraic closure of the field K and K^{sep} be the separable closure of K in $\overline{K}$.

Proposition A.1.3. *There is a unique extension of $\partial^j: K\to K$ to $\partial^j: K^{\text{sep}}\to K^{\text{sep}}$ which satisfies all the rules of ∂.*

Proof. The proof follows the same argument as that for hyperderivatives [Conrad 2000, Theorem 5]. □

Let $a\in\overline{K}\setminus K^{\text{sep}}$. We say that ∂ can be *extended to a* if ∂ can be extended to some extension field of K^{sep} that contains a. The largest extension field $\overline{K}^{\partial}$ of K^{sep} in $\overline{K}$ that has an extension of ∂ is called the *∂-closure* of K in $\overline{K}$.

For a set $X\subseteq(\overline{K}^{\partial})^m$, if we set

$$\mathfrak{I}(X):=\{P\in K\{y_1,\dots,y_m\}: P(a_1,\dots,a_m)=0,\ (a_1,\dots,a_m)\in X\},$$

then $\mathfrak{I}(X)$ is a radical ∂-ideal in R, and we call it the *defining K-∂-ideal* of X.

Proposition A.1.4 [Hardouin et al. 2016, Proposition 3.8]. *Let $X_1, X_2\subseteq(\overline{K}^{\partial})^m$. Then:*

(1) *If $X_1\subseteq X_2$, then $\mathfrak{I}(X_2)\subseteq\mathfrak{I}(X_1)$.*

(2) $\mathfrak{I}(X_1\cup X_2)=\mathfrak{I}(X_1)\cap\mathfrak{I}(X_2)$.

Proof. The proofs follow the same line of argument as that for the Zariski topology. □

Given a set $X\subseteq((\overline{K}^{\partial})^m,\partial)$, we consider the Zariski closure $\overline{X}^Z\subseteq\overline{K}^m$ of X, the closure of X as a subset of $(\overline{K}^{\partial})^m$ equipped with the Zariski topology. Let $S\subseteq K[y_1,\dots,y_m]$ be a set of polynomials. The *zero set* of S is defined as

$$\mathcal{Z}(S):=\{(a_1,\dots,a_m)\in\overline{K}^m: f(a_1,\dots,a_m)=0,\ f\in S\}.$$

Lemma A.1.5 (cf. [Hardouin et al. 2016, Lemma 3.42]). *Let* $X \subseteq (\overline{K}^{\partial})^m$ *and let* $\mathfrak{I}(X) \subseteq K\{y_1, \dots, y_m\}$ *be its defining* K*-*∂*-ideal. Also, let* $K[y_1, \dots, y_m]$ *be the usual polynomial ring in the variables* $y_1, \dots, y_m$ *over the field* K*. Then its Zariski closure is the set*

$$\overline{X}^Z = \mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]),$$

where $\mathfrak{I}(X) \cap K[y_1, \dots, y_m] \subseteq K[y_1, \dots, y_m]$.

Proof. We follow the outline of the proof of [Hardouin et al. 2016, Lemma 3.42]. Since $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m])$ is Zariski closed, it is straightforward to see that

$$X \subseteq \overline{X}^Z \subseteq \mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]).$$

Conversely, if $S \subseteq K[y_1, \dots, y_m] \subseteq K\{y_1, \dots, y_m\}$ is such that $S \subseteq \mathfrak{I}(X)$, then clearly $\mathfrak{R}(S) \subseteq \mathfrak{I}(X)$. This implies $S \subseteq \mathfrak{R}(S) \cap K[y_1, \dots, y_m] \subseteq \mathfrak{I}(X) \cap K[y_1, \dots, y_m]$. Thus, $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]) \subseteq \mathcal{Z}(S)$. Since S was chosen arbitrarily, we see that $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]) \subseteq \overline{X}^Z$. □

If $f \in K\{y_1, \dots, y_m\}$ is a ∂-polynomial given by a linear combination over the ∂-field K of 1 and elements of the set $\{\partial^j(y_i) : j \geq 0,\ 1 \leq i \leq m\}$, then we say that f is a *degree*-1 ∂*-polynomial* in $K\{y_1, \dots, y_m\}$. Moreover if the coefficient of 1 is 0, then we say that such f is a *homogeneous degree*-1 ∂*-polynomial*.

Proposition A.1.6 [Okugawa 1987, p. 74, Theorem 5]. *Let* $S \subseteq K\{y_1, \dots, y_m\}$ *be a set of degree*-1 ∂*-polynomials. Then,* $\mathfrak{R}(S) = \mathfrak{D}(S)$.

Proof. It suffices to show that $\mathfrak{D}(S)$ is a prime ideal of the ∂-ring $K\{y_1, \dots, y_m\}$ regarded as a usual ring. By definition $\mathfrak{D}(S)$ is generated, as an ideal of the ring $K\{y_1, \dots, y_m\}$, by $\{\partial^j(L_i) : i, j \geq 0,\ L_i \in S\}$. Suppose that $f, g \notin \mathfrak{D}(S)$ such that $fg \in \mathfrak{D}(S)$. Then,

$$fg = \sum_{L_i \in S,\ j \geq 1} h_{i,\ell_j} \partial^{\ell_j}(L_i),$$

where $\ell_j \geq 0$, and $h_{i,\ell_j} \in K\{y_1, \dots, y_m\}$, and all but finitely many h_{i,ℓ_j} are zero. We see that fg is a polynomial in a finite subset of the variables $\{\partial^j(y_i) : j \geq 0,\ 1 \leq i \leq m\}$ over the ∂-field K regarded as a usual field. Let us denote this subset of variables by $\{x_1, \dots, x_n\}$ for some $n \geq 1$. Then, $L = (\{\partial^{\ell_j}(L_i)\})$ is an ideal in $K[x_1, \dots, x_n]$ such that $f, g \notin L$ and $fg \in L$ and so L is not a prime ideal. However, for a polynomial ring in finitely many indeterminates, ideals generated by degree-1 polynomials are prime ideals and thus, we obtain a contradiction. □

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