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Algebraic relations among hyperderivatives of periods and logarithms of Drinfeld modules

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We determine all algebraic relations among all hyperderivatives of the periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules defined over a separable closure of the rational function field. In particular, for periods and logarithms that are linearly independent over the endomorphism ring of the Drinfeld module, we prove the algebraic independence of their hyperderivatives and the hyperderivatives of the corresponding quasiperiods and quasilogarithms.

1. Introduction

The objects of study in the present paper are inspired by elliptic curves in the classical setting. Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$. The period conjecture states that the transcendence degree over $\overline{\mathbb{Q}}$ of the two periods $\{\omega_1, \omega_2\}$ and the two corresponding quasiperiods $\{\eta_1, \eta_2\}$ of E is 2 when E has complex multiplication (CM), and 4 otherwise. The CM case was confirmed to be true by Chudnovsky, while the non-CM case is still open. With regards to logarithms of E, one can expect logarithms of algebraic numbers that are linearly independent over $\operatorname{End}(E)$ to be algebraically independent over $\overline{\mathbb{Q}}$. Although linear independence over $\overline{\mathbb{Q}}$ of these logarithms is known due to Masser (for the CM case), Bertrand and Masser (for the non-CM case), and as a consequence of Wüstholz's analytic subgroup theorem, algebraic independence of these logarithms is still fully open. See [Baker and Wüstholz 2007; Waldschmidt 2008] for details.

In the function field setting, Drinfeld [1974] introduced "elliptic modules", now called Drinfeld modules, as an analogue of elliptic curves. Later, Anderson [1986] defined higher-dimensional generalizations of Drinfeld modules, called t-modules. One can ask analogous questions regarding algebraic independence

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of periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules and Anderson t-modules defined over algebraic function fields. Yu [1997] proved the sub-t-module theorem, a remarkable result regarding linear independence among logarithms of Anderson t-modules, which is an analogue of Wüstholz's analytic subgroup theorem, and proved the complete transcendence results concerning periods and logarithms of Drinfeld modules [Yu 1986; 1990]. Thiery [1992] proved algebraic independence results among periods and quasiperiods of rank-2 Drinfeld modules with complex multiplication. Chang and Papanikolas [2011; 2012] proved algebraic independence of periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules of arbitrary rank. The goal of the present paper is to generalize completely under separability hypothesis this work of Chang and Papanikolas [2011; 2012, Theorems 3.5.4 and 5.1.5, and Corollary 5.1.6] to include all hyperderivatives, which are defined below.

Let $\mathbb{F}_q$ be a finite field, where q is a positive power of a prime number p, and let θ be an indeterminate. For the rational function field $\mathbb{F}_q(\theta)$, the j-th hyperderivative $\partial_\theta^j : \mathbb{F}_q(\theta) \to \mathbb{F}_q(\theta)$ is defined by $\partial_\theta^j(\theta^m) := \binom{m}{j}\theta^{m-j}$, where $j \geq 0$. Taking the completion $\mathbb{F}_q(\!(1/\theta)\!)$ of $\mathbb{F}_q(\theta)$ with respect to its ∞-adic absolute value $|\cdot|_\infty$, $\partial_\theta^j(\cdot)$ extends uniquely to $\mathbb{F}_q(\!(1/\theta)\!)^{\mathrm{sep}}$. Note that hyperderivatives play a role analogous to that of formal derivatives in the classical case. Unlike in the classical setting of elliptic curves, one can take hyperderivatives of periods and logarithms of Anderson t-modules defined over $\mathbb{F}_q(\theta)^{\mathrm{sep}}$. Moreover, many interpretations of objects of interest in terms of logarithms of Anderson t-modules involve hyperderivatives. The entries of periods of the d-th tensor power $\mathfrak{C}^{\otimes d}$ of the Carlitz module $\mathfrak{C}$ (rank-1 Drinfeld module) are obtained using hyperderivatives [Maurischat 2018, Lemma 8.3] of Anderson–Thakur functions [1990, §2.5]. Also, Carlitz zeta values [Thakur 2004] appear in the last coordinate of a logarithm of $\mathfrak{C}^{\otimes d}$ [Anderson and Thakur 1990, Theorem 3.8.3]. Generalizing this, Chang, Green, and Mishiba [Chang et al. 2021] showed that multizeta values [Thakur 2004] also appear as coordinates of logarithms of a particular Anderson t-module and further showed that its periods and logarithms are obtained using hyperderivatives. There are also logarithmic interpretations of special values of Goss L-functions attached to Drinfeld modules in terms of logarithms of an Anderson t-module, where hyperderivatives play a crucial role [Gezmiş and Namoijam 2021]. These interpretations further motivate interest in determining algebraic independence of hyperderivatives of periods and logarithms of Anderson t-modules.

Algebraic independence among hyperderivatives of the fundamental period of the Carlitz module were proved by Denis [1993; 1995; 2000] and Maurischat [2018; 2022a]. Further work in this direction was also done in unpublished work by Brownawell and van der Poorten. Utilizing Yu's sub-t-module theorem, Brownawell and Denis [2000] and Brownawell [1999; 2001] investigated linear independence of hyperderivatives of logarithms and quasilogarithms of Drinfeld modules. In the present paper, we determine all algebraic independence results among all hyperderivatives of periods, quasiperiods, logarithms, and quasilogarithms of Drinfeld modules of arbitrary rank under the hypothesis of separability.

1.1. ***Hyperderivatives of periods and logarithms.*** For a finite field $\mathbb{F}_q$, where q is a positive power of a prime number p, we set $A := \mathbb{F}_q[\theta]$, $k := \mathbb{F}_q(\theta)$ and $k_\infty := \mathbb{F}_q(\!(1/\theta)\!)$, the completion of k at its infinite place. We further set $\mathbb{K}$ to be the completion of an algebraic closure of k_∞, and let $\bar{k}$ and k^{sep} be the algebraic closure and the separable closure respectively of k inside $\mathbb{K}$. For a variable t independent from θ, we further define $\boldsymbol{A} := \mathbb{F}_q[t]$ and $\boldsymbol{k} := \mathbb{F}_q(t)$.

For $n \in \mathbb{Z}$, we define the *Frobenius twist* $\tau^n : \mathbb{K}((t)) \to \mathbb{K}((t))$ by setting for $f = \sum_i a_i t^i$

$$\tau^n(f) := f^{(n)} = \sum_i a_i^{q^n} t^i. \tag{1.1.1}$$

For a field $K \subseteq \mathbb{K}$, we define the twisted power series ring $K[[\tau]]$ subject to the condition $\tau c = c^q \tau$ for all $c \in K$. Then, we define the twisted polynomial ring $K[\tau]$ as the subring of $K[[\tau]]$, where $K[\tau]$ is viewed as a subalgebra of the $\mathbb{F}_q$-linear endomorphisms of the additive group of K.

For a field $k \subseteq K \subseteq \mathbb{K}$, a *Drinfeld $\boldsymbol{A}$-module of rank r defined over K* is an $\mathbb{F}_q$-algebra homomorphism $\rho : \boldsymbol{A} \to K[\tau]$ uniquely determined by

$$\rho_t = \theta + \kappa_1 \tau + \cdots + \kappa_r \tau^r$$

such that $\kappa_r \neq 0$. The *exponential function* associated to ρ is given by

$$\operatorname{Exp}_\rho(z) = z + \sum_{h \geq 1} \alpha_h z^{q^h} \in K[[z]]$$

and it satisfies the functional equation $\operatorname{Exp}_\rho(\theta z) = \rho_t(\operatorname{Exp}_\rho(z))$. The *period lattice* of ρ is the kernel Λ_ρ of Exp_ρ, which is a free discrete $\boldsymbol{A}$-submodule of rank r inside $\mathbb{K}$.

The de Rham cohomology theory for Drinfeld $\boldsymbol{A}$-modules was developed by Anderson, Deligne, Gekeler [1989] and Yu [1990]. A *ρ-biderivation* is an $\mathbb{F}_q$-linear map $\delta : \boldsymbol{A} \to \mathbb{K}[\tau]\tau$ satisfying, for all $a, b \in \boldsymbol{A}$,

$$\delta_{ab} = a(\theta)\delta_b + \delta_a \rho_b.$$

Let $u \in \mathbb{K}[\tau]$. Then, the ρ-biderivation $\delta^{(u)}$ defined by $\delta_a^{(u)} = u\rho_a - a(\theta)u$ for all $a \in \boldsymbol{A}$ is called an *inner biderivation*. If $u \in \mathbb{K}[\tau]\tau$, then $\delta^{(u)}$ is said to be *strictly inner*. The set of ρ-biderivations $\operatorname{Der}(\rho)$ forms a $\mathbb{K}$-vector space. The set of inner biderivations $\operatorname{Der}_{\mathrm{in}}(\rho)$ and the set of strictly inner biderivations $\operatorname{Der}_{\mathrm{si}}(\rho)$ are also $\mathbb{K}$-vector subspaces of $\operatorname{Der}(\rho)$. We define the *de Rham module for ρ* to be $\mathrm{H}^1_{\mathrm{DR}}(\rho) := \operatorname{Der}(\rho)/\operatorname{Der}_{\mathrm{si}}(\rho)$, which is an r-dimensional $\mathbb{K}$-vector space. The de Rham module $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ parametrizes the extensions of ρ by $\mathbb{G}_a$.

For each $\delta \in \operatorname{Der}(\rho)$ there is a unique $\mathbb{F}_q$-linear and entire power series

$$\mathrm{F}_\delta(z) = \sum_{i \geq 1} c_i z^{(i)} \in \mathbb{K}[[z]]$$

such that, for all $a \in \boldsymbol{A}$,

$$\mathrm{F}_\delta(a(\theta)z) = a(\theta)\mathrm{F}_\delta(z) + \delta_a(\operatorname{Exp}_\rho(z)). \tag{1.1.2}$$

We call F_δ the *quasiperiodic function* associated to δ. For $\lambda \in \Lambda_\rho$, the value $\mathrm{F}_\delta(\lambda)$ is called a *quasiperiod of ρ*. For $u \in \mathbb{K}$, the value $\mathrm{F}_\delta(u)$, which is a coordinate of logarithms of quasiperiodic extensions, is called a *quasilogarithm of ρ* (see [Brownawell and Papanikolas 2002; Namoijam and Papanikolas 2024]).

A $\mathbb{K}$-basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ is represented by $\{\delta_1, \ldots, \delta_r\}$, where δ_1 is the inner biderivation such that $(\delta_1)_t = \rho_t - \theta$, and $\delta_j(t) = \tau^{j-1}$ for $2 \leq j \leq r$. We see that $\mathrm{F}_{\delta_1}(z) = \operatorname{Exp}_\phi(z) - z$, and so $\mathrm{F}_{\delta_1}(\lambda) = -\lambda$ for all $\lambda \in \Lambda_\rho$. If we take $\{\lambda_1, \dots, \lambda_r\}$ to be an $\boldsymbol{A}$-basis of Λ_ρ and we set $\mathrm{F}_{\tau^{j-1}}(z) := \mathrm{F}_{\delta_j}(z)$ for $2 \le j \le r$, then we define the *period matrix of* ρ to be

$$\boldsymbol{P}_\rho := \begin{pmatrix} \lambda_1 & \mathrm{F}_\tau(\lambda_1) & \cdots & \mathrm{F}_{\tau^{r-1}}(\lambda_1) \\ \lambda_2 & \mathrm{F}_\tau(\lambda_2) & \cdots & \mathrm{F}_{\tau^{r-1}}(\lambda_2) \\ \vdots & \vdots & & \vdots \\ \lambda_r & \mathrm{F}_\tau(\lambda_r) & \cdots & \mathrm{F}_{\tau^{r-1}}(\lambda_r) \end{pmatrix},$$

which accounts for all periods and quasiperiods of ρ. The de Rham cohomology theory for Drinfeld $\boldsymbol{A}$-modules runs in parallel to the theory of elliptic functions such that the periods and quasiperiods summarized above play the role of periods and quasiperiods of the Weierstrass $\wp$-functions.

If the Drinfeld $\boldsymbol{A}$-module ρ is defined over k^{sep}, Denis [1995, p. 6] showed that, for a ρ-biderivation δ defined over k^{sep}, if $u \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u) \in k^{\mathrm{sep}}$, then $u \in k_\infty^{\mathrm{sep}}$ and $\mathrm{F}_\delta(u) \in k_\infty^{\mathrm{sep}}$ (see also [Namoijam and Papanikolas 2024, Lemma 4.22]). Therefore, for $n \ge 0$ we can consider $\partial_\theta^n(u)$ and $\partial_\theta^n(\mathrm{F}_\delta(u))$. Let $\partial_\theta^n(\boldsymbol{P}_\rho)$ be the matrix formed by entrywise action of $\partial_\theta^n(\,\cdot\,)$ on $\boldsymbol{P}_\rho$.

We define $\mathrm{End}(\rho) := \{x \in \mathbb{K} : x\Lambda_\rho \subseteq \Lambda_\rho\}$ and let K_ρ be its fraction field. Our first main result is as follows (restated as Theorem 4.5.1):

Theorem 1.1.3. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. If $\boldsymbol{s} = [K_\rho : k]$, then for $n \ge 1$ we have*

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}(\boldsymbol{P}_\rho, \partial_\theta^1(\boldsymbol{P}_\rho), \dots, \partial_\theta^n(\boldsymbol{P}_\rho)) = (n+1)\cdot r^2/\boldsymbol{s}.$$

Building on Theorem 1.1.3, we prove algebraic independence among hyperderivatives of logarithms and quasilogarithms of Drinfeld $\boldsymbol{A}$-modules. Our second main result is as follows (restated as Theorem 5.4.4):

Theorem 1.1.4. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. Let $u_1, \dots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$ and suppose that $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w) = r/\boldsymbol{s} + w$, where $\boldsymbol{s} = [K_\rho : k]$. Then, for $n \ge 1$,*

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\Biggl(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(\mathrm{F}_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(\mathrm{F}_{\tau^i}(u_m))\}\Biggr) = (n+1)(r^2/\boldsymbol{s} + rw).$$

For an arbitrary basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep}, we deduce the following corollary.

Corollary 1.1.5. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. Let $u_1, \dots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$. Let $\{\delta_1, \dots, \delta_r\}$ be a basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep}. If $u_1, \dots, u_w$ are linearly independent over K_ρ, then for $n \ge 1$ the $(n+1)rw$ quantities*

$$\Biggl\{\bigcup_{s=0}^{n}\bigcup_{j=1}^{r}\bigl(\partial_\theta^s(\mathrm{F}_{\delta_j}(u_1)), \partial_\theta^s(\mathrm{F}_{\delta_j}(u_2)), \dots, \partial_\theta^s(\mathrm{F}_{\delta_j}(u_w))\bigr)\Biggr\}$$

are algebraically independent over $\bar{k}$.

Combining Theorems 1.1.3, 1.1.4, and Corollary 1.1.5, the $\bar{k}$-linear relations among the periods and logarithms of ρ and their hyperderivatives induced by endomorphisms of ρ account for all the $\bar{k}$-algebraic relations among all hyperderivatives of the periods and logarithms as well as all hyperderivatives of the corresponding quasiperiods and quasilogarithms of ρ.

1.2. ***Remarks on structure of the paper.*** In [Namoijam and Papanikolas 2024], Papanikolas and the author showed that t-motives whose period matrices comprise the values of interest in Theorems 1.1.3 and 1.1.4 are constructed from the t-motive associated to prolongations [Maurischat 2018] of ρ, but did not prove any transcendence results about the values in question. Papanikolas's theorem [2008, Theorem 1.1.7] states that the transcendence degree of the period matrix of a t-motive is equal to the dimension of its Galois group. The primary hurdle, then, is determining the dimension of the associated Galois group of the t-motive.

The first goal of this paper is to explicitly determine the Galois group of the t-motive corresponding to the n-th prolongation t-module $\mathrm{P}_n\rho$ of ρ. To do this, we calculate the Zariski closure of the image of the Galois representation on the $\mathfrak{p}$-adic Tate module of $\mathrm{P}_n\rho$, for a nonzero prime $\mathfrak{p}$ of $\boldsymbol{A}$. Next, we immediately extend this result. We construct new t-motives whose period matrices are comprised of both periods and quasiperiods of $\mathrm{P}_n\rho$, and hyperderivatives of logarithms and quasilogarithms of ρ, and then determine their Galois groups. We construct a sequence of surjections between specific sub-t-motives using consecutive prolongations $\mathrm{P}_\ell\rho$ for $0 \le \ell \le n$. These surjections are crucial in establishing that algebraic independence over $\bar{k}$ of all hyperderivatives of the logarithms and quasilogarithms depends only on K_ρ-linear independence of the logarithms.

The paper is outlined as follows.

- In Section 2 we give necessary background concerning t-motives and their Galois groups. Next, we give a brief review of hyperderivatives and then discuss prolongations of dual t-motives introduced in [Maurischat 2018].

- In Section 3, we describe t-motives and rigid analytic trivializations corresponding to Drinfeld $\boldsymbol{A}$-modules and their prolongations; then we state Theorem 3.4.1. Based on Theorem 3.4.1 (see [Namoijam and Papanikolas 2024, §5.3] for a detailed account), to prove Theorem 1.1.3, for $n \ge 1$ we calculate the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ of the n-th prolongation $\mathrm{P}_n M_\rho$ of the t-motive M_ρ associated to ρ.

- We first make use of a direct connection $\Gamma_{\mathrm{P}_n M_\rho}$ has with Galois representations. For a nonzero prime $\mathfrak{p}$ of $\boldsymbol{A}$, let $\boldsymbol{A}_\mathfrak{p}$ be the completion of $\boldsymbol{A}$ and let $\boldsymbol{k}_\mathfrak{p}$ be its fraction field. For a Drinfeld $\boldsymbol{A}$-module ρ defined over K, where $k \subseteq K \subseteq \bar{k}$ with $[K : k] < \infty$, there is a representation $\varphi_\mathfrak{p} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ coming from the Galois action on the $\mathfrak{p}$-power torsion points $\rho[\mathfrak{p}^m] := \{x \in \mathbb{K} : \rho_{\mathfrak{p}^m}(x) = 0\}$. In Section 4, using Anderson generating functions and $\varphi_\mathfrak{p}$, we consider the Galois representation on the $\mathfrak{p}$-adic Tate module of the n-th prolongation t-module $\mathrm{P}_n\rho$ associated to ρ. The image of this Galois representation is determined using hyperderivatives of the image for the Drinfeld $\boldsymbol{A}$-module ρ and is naturally contained in the $\boldsymbol{k}_\mathfrak{p}$-valued points of $\Gamma_{\mathrm{P}_n M_\rho}$ (Theorem 4.1.6).

- For $n \ge 1$, $\mathrm{P}_{n-1}M_\rho$ is a sub-t-motive of $\mathrm{P}_n M_\rho$ and therefore, $\mathrm{P}_n M_\rho$ is not simple, which makes determining the Zariski closure of the aforementioned image a complicated task. To find the Zariski closure, we bring in differential algebraic geometry. We consider hyperdifferential polynomials (precise definition in Section A.1) to determine the above Zariski closure by first determining the defining differential ideal of the aforementioned image and then restricting to Zariski topology (Theorem 4.3.3). This allows us to prove Theorem 1.1.3 and compute the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ explicitly (Corollary 4.4.8).

• In Section 5, for $u_1, \dots, u_w \in \mathbb{K}$ satisfying $\operatorname{Exp}_\rho(u_i) \in k^{\mathrm{sep}}$ for each $1 \le i \le w$, we build on results of Section 4 to construct new t-motives $Y_{1,n}, \dots, Y_{w,n}$ such that the entries of the period matrix of $\bigoplus_{m=1}^{w} Y_{m,n}$ comprise $\bigcup_{s=0}^{n} \bigcup_{i=1}^{r-1} \bigcup_{m=1}^{w} \{\partial_\theta^s(u_m), \partial_\theta^s(\mathrm{F}_{\tau^i}(u_m))\}$. Let $\mathcal{T}$ denote the category of t-motives. In Lemma 5.4.3, we obtain a surjective map from certain sub-t-motives of $Y_{m,n}$ to corresponding sub-t-motives of $Y_{m,\ell}$ for $\ell \le n$ and $1 \le m \le w$. This map allows us to implement Theorem 5.2.2, which is based on an $\operatorname{End}_{\mathcal{T}}(M_\rho)$-linear independence result [Chang and Papanikolas 2012, Theorem 4.2.2], which enables us to prove Theorem 1.1.4.

• Finally, in the Appendix, we cover necessary background concerning differential algebraic geometry in positive characteristic. We explore various properties, especially a result on the determination of the Zariski closure of a set in a differential field (Lemma A.1.5).

2. Preliminaries

2.1. ***Notation.*** We continue with the notation introduced in Section 1.1. We also define the following.

Let $\mathbb{T}$ be the Tate algebra of the closed unit disk of $\mathbb{K}$,

$$\mathbb{T} := \left\{ \sum_{h=0}^{\infty} a_h t^h \in \mathbb{K}[\![t]\!] : \lim_{h\to\infty} |\, a_h \,|_\infty = 0 \right\},$$

and let $\mathbb{L}$ be its fraction field.

For $n \in \mathbb{Z}$, recall the Frobenius twist τ^n from (1.1.1). In some cases, we will write σ for τ^{-1}. For $M = (m_{ij}) \in \operatorname{Mat}_{e\times d}(\mathbb{K}(\!(t)\!))$, we define $M^{(n)}$ by setting $M^{(n)} := (m_{ij}^{(n)})$. Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be the Laurent polynomial ring over $\bar{k}(t)$ in σ subject to the relation

$$\sigma f = f^{(-1)}\sigma, \quad f \in \bar{k}(t).$$

For a field $K \subseteq \mathbb{K}$, recall from Section 1.1 the twisted power series ring $K[\![\tau]\!]$ and the subring $K[\tau]$ given by $\tau f = f^{(1)}\tau$ for all $f \in K$. We also define $K[\![\sigma]\!]$ and $K[\sigma]$ when K is a perfect field. For $b = \sum c_i \tau^i \in \mathbb{K}[\tau]$, we define $b^* := \sum c_i^{(-i)} \sigma^i \in \mathbb{K}[\sigma]$. If $B = (b_{ij}) \in \operatorname{Mat}_{e\times d}(\mathbb{K}[\tau]) = \operatorname{Mat}_{e\times d}(\mathbb{K})[\tau]$, then we set $B^* := (b_{ji}^*)$. Thus, if $B \in \operatorname{Mat}_{e\times d}(\mathbb{K}[\tau])$ and $C \in \operatorname{Mat}_{d\times h}(\mathbb{K}[\tau])$, then $(BC)^* = C^* B^*$. Moreover, if $B = \beta_0 + \beta_1 \tau + \cdots + \beta_\ell \tau^\ell$, then we set $\mathrm{d}B := \beta_0$.

2.2. ***Dual t-motives and t-motives.*** In this subsection, we briefly introduce the main tools used in Papanikolas's result. The reader is directed to [Papanikolas 2008] for further details. A *pre-t-motive* M is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module that is finite-dimensional over $\bar{k}(t)$. We denote by $\mathcal{P}$ the category of pre-t-motives whose morphisms are the left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module homomorphisms. Let $\boldsymbol{m} \in \operatorname{Mat}_{r\times 1}(M)$ be such that its entries form a $\bar{k}(t)$-basis of M. Then, there is a matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ such that

$$\sigma \boldsymbol{m} = \Phi \boldsymbol{m},$$

where the action of σ on $\boldsymbol{m}$ is entrywise. We say that M is *rigid analytically trivial* if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ such that

$$\Psi^{(-1)} = \Phi\Psi.$$

The matrix Ψ is called a *rigid analytic trivialization for* Φ. Set $M^\dagger := \mathbb{L} \otimes_{\bar{k}(t)} M$, where we give $M^\dagger$ a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module by letting σ act diagonally:

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m, \quad f \in \bar{k}(t),\ m \in M.$$

If we let

$$M^B := (M^\dagger)^\sigma := \{\mu \in M^\dagger : \sigma\mu = \mu\},$$

then M^B is a finite-dimensional vector space over $\boldsymbol{k}$, and $M \mapsto M^B$ is a covariant functor from $\mathcal{P}$ to the category of $\boldsymbol{k}$-vector spaces. The natural map $\mathbb{L} \otimes_{\bar{k}(t)} M^B \to M^\dagger$ is an isomorphism if and only if M is rigid analytically trivial [Papanikolas 2008, §3.3]. If Ψ is a rigid analytic trivialization of Φ, then the entries of $\Psi^{-1}\boldsymbol{m}$ form a $\boldsymbol{k}$-basis for M^B [loc. cit., Theorem 3.3.9(b)]. By [loc. cit., Theorem 3.3.15], the category $\mathcal{R}$ of *rigid analytically trivial pre-t-motives* forms a neutral Tannakian category over $\boldsymbol{k}$ with fiber functor $M \mapsto M^B$.

We now consider $\boldsymbol{A}$-finite dual t-motives, which were first introduced in [Anderson et al. 2004] (see also [Hartl and Juschka 2020; Namoijam and Papanikolas 2024]). A *dual t-motive* $\mathcal{M}$ is a left $\bar{k}[t, \sigma]$-module that is free and finitely generated as a left $\bar{k}[\sigma]$-module and such that $(t-\theta)^s \mathcal{M} \subseteq \sigma\mathcal{M}$ for $s \in \mathbb{N}$ sufficiently large. If, in addition, $\mathcal{M}$ is free and finitely generated as a left $\bar{k}[t]$-module, then $\mathcal{M}$ is said to be $\boldsymbol{A}$*-finite*. Thus, if the entries of $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(\mathcal{M})$ form a $\bar{k}[t]$-basis for $\mathcal{M}$, then there is a matrix $\Phi \in \mathrm{Mat}_r(\bar{k}[t])$ such that $\sigma\boldsymbol{m} = \Phi\boldsymbol{m}$ with $\det \Phi = c(t-\theta)^s$ for some $c \in \bar{k}^\times$, $s \geq 1$. We say that $\mathcal{M}$ is *rigid analytically trivial* if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$. In [Anderson et al. 2004], the term “dual t-motives” is used for $\boldsymbol{A}$-finite dual t-motives. We will consider both dual t-motives and $\boldsymbol{A}$-finite dual t-motives.

Given an $\boldsymbol{A}$-finite dual t-motive $\mathcal{M}$,

$$M := \bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{M}$$

is a pre-t-motive, where $\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m$. Then, $\mathcal{M} \mapsto M$ is a functor from the category of $\boldsymbol{A}$-finite dual t-motives to the category of pre-t-motives. We define the category $\mathcal{T}$ of *t-motives* to be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the essential image of rigid analytically trivial $\boldsymbol{A}$-finite dual t-motives under the assignment $\mathcal{M} \mapsto M$.

For a t-motive M, we let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{T}$ generated by M. As $\mathcal{T}_M$ is a neutral Tannakian category over $\boldsymbol{k}$, there is an affine group scheme Γ_M over $\boldsymbol{k}$, a subgroup of the $\boldsymbol{k}$-group scheme $\mathrm{GL}_r/\boldsymbol{k}$ of $r \times r$ invertible matrices, so that $\mathcal{T}_M$ is equivalent to the category of finite-dimensional representations of Γ_M over $\boldsymbol{k}$, i.e., $\mathcal{T}_M \approx \mathbf{Rep}(\Gamma_M, \boldsymbol{k})$ [Papanikolas 2008, §3.5]. We call Γ_M the *Galois group of M*.

2.3. ***The difference Galois group.*** We now present a brief summary of the construction of the Galois group of a t-motive as the Galois group of a system of difference equations. The reader is directed to [Papanikolas 2008] for further details. For a subfield $F \subset \mathbb{K}((t))$ invariant under the action of σ, let F^σ denote the elements of F fixed by σ. Note that the automorphism $\sigma : \mathbb{L} \to \mathbb{L}$ restricts to automorphisms of $\boldsymbol{k}$ and $\bar{k}(t)$, and $\boldsymbol{k} = \boldsymbol{k}^\sigma = \bar{k}(t)^\sigma = \mathbb{L}^\sigma$.

For a t-motive M, let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ denote the action of σ on a $\bar{k}(t)$-basis of M and let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ be the rigid analytic trivialization for Φ satisfying $\Psi^{(-1)} = \Phi\Psi$.

We define a $\bar{k}(t)$-algebra homomorphism $\nu : \bar{k}(t)[X, 1/\det X] \to \mathbb{L}$ by setting $\nu(X_{ij}) := \Psi_{ij}$, where $X = (X_{ij})$ is an $r \times r$ matrix of independent variables. We let $\mathfrak{p} := \ker \nu$ and $\Sigma := \operatorname{Im} \nu = \bar{k}(t)[\Psi, 1/\det \Psi] \subseteq \mathbb{L}$, and set $Z_\Psi = \operatorname{Spec} \Sigma$. Then, Z_Ψ is the smallest closed subscheme of $\mathrm{GL}_r / \bar{k}(t)$ such that $\Psi \in Z_\Psi(\mathbb{L})$.

Set $\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$ to be such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$, and let $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$. We define a $\boldsymbol{k}$-algebra homomorphism $\mu : \boldsymbol{k}[X, 1/\det X] \to \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}$ by setting $\mu(X_{ij}) := \widetilde{\Psi}_{ij}$. We let $\mathfrak{q} := \ker \mu$ and $\Delta := \operatorname{Im} \mu$, and set $\Gamma_\Psi = \operatorname{Spec} \Delta$. Then, Γ_Ψ is the smallest closed subscheme of $\mathrm{GL}_r / \boldsymbol{k}$ such that $\widetilde{\Psi} \in \Gamma_\Psi(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$. The following properties hold.

Theorem 2.3.1 [Papanikolas 2008, §4]. *Let M be a t-motive, and let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$ represent multiplication by σ on a $\bar{k}(t)$-basis of M. Let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfy $\Psi^{(-1)} = \Phi\Psi$.*

(a) *The closed $\bar{k}(t)$-subscheme Z_Ψ is stable under right-multiplication by $\bar{k}(t) \times_{\boldsymbol{k}} \Gamma_\Psi$ and is a $\bar{k}(t) \times_{\boldsymbol{k}} \Gamma_\Psi$-torsor over $\bar{k}(t)$. In particular, $\Gamma_\Psi(\bar{\mathbb{L}}) = \Psi^{-1} Z_\Psi(\bar{\mathbb{L}})$.*

(b) *The $\boldsymbol{k}$-scheme Γ_Ψ is absolutely irreducible and smooth over $\bar{\boldsymbol{k}}$.*

(c) *$\Gamma_\Psi \cong \Gamma_M$ over $\boldsymbol{k}$.*

For the t-motive M, if $\Phi \in \mathrm{GL}_r(\bar{k}(t)) \cap \mathrm{Mat}_r(\bar{k}[t])$ and $\det \Phi = c(t-\theta)^s$ for some $c \in \bar{k}^\times$, $s \geq 1$, then we can pick Ψ to be in $\mathrm{GL}_r(\mathbb{T})$ [Papanikolas 2008, Proposition 3.3.9(c)]. Moreover, the entries of Ψ are regular at $t = \theta$ [Anderson et al. 2004, Proposition 3.1.3]. Let $\Psi|_{t=\theta}$ denote the specialization of the entries of Ψ at $t = \theta$ and let $\bar{k}(\Psi|_{t=\theta})$ be the field formed by adjoining the entries of $\Psi|_{t=\theta}$ to $\bar{k}$. The main theorem of [Papanikolas 2008] is as follows.

Theorem 2.3.2 [Papanikolas 2008, Theorem 1.1.7]. *Let M be a t-motive, and let Γ_M be its Galois group. Suppose that $\Phi \in \mathrm{GL}_r(\bar{k}(t)) \cap \mathrm{Mat}_r(\bar{k}[t])$ represents multiplication by σ on a $\bar{k}(t)$-basis of M and that* $\det \Phi = c(t-\theta)^s$, $c \in \bar{k}^\times$, $s \geq 1$. *Let $\Psi \in \mathrm{GL}_r(\mathbb{T})$ be a rigid analytic trivialization of Φ. Then,* $\operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi|_{t=\theta}) = \dim \Gamma_M$.

2.4. ***Hyperderivatives and hyperdifferential operators.*** For details beyond the review here, the reader may refer to [Brownawell 1999; Jeong 2011; Namoijam and Papanikolas 2024, §2.4]. For $m, j \geq 0$, let $\binom{m}{j} \in \mathbb{N}$ denote the usual binomial coefficient modulo p. Then, for F a field of characteristic $p > 0$ where θ is transcendental over F, the F-linear map $\partial_\theta^j : F[\theta] \to F[\theta]$ defined by setting

$$\partial_\theta^j(\theta^m) = \binom{m}{j} \theta^{m-j}$$

is called the *j-th hyperdifferential operator with respect to θ*. For each $f \in F[\theta]$, we call $\partial_\theta^j(f)$ the *j-th hyperderivative of f*. The definition of ∂_θ^j extends naturally to $\partial_\theta^j : F[\![\theta]\!] \to F[\![\theta]\!]$. The hyperdifferential operators satisfy various identities including the product rule

$$\partial_\theta^j(fg) = \sum_{i=0}^{j} \partial_\theta^i(f)\partial_\theta^{j-i}(g)$$

and the composition rule

$$\partial_\theta^i(\partial_\theta^j(f)) = \binom{i+j}{j}\partial_\theta^{i+j}(f).$$

The product rule extends ∂_θ^j to the Laurent series field $F(\!(\theta)\!)$, where as usual for $m > 0$ we have

$$\binom{-m}{j} = (-1)^j\binom{m+j-1}{j}.$$

For a place v of $F(\theta)$ there are unique extensions $\partial_\theta^j : F(\theta)_v \to F(\theta)_v$ and $\partial_\theta^j : F(\theta)_v^{\mathrm{sep}} \to F(\theta)_v^{\mathrm{sep}}$, where $F(\theta)_v^{\mathrm{sep}}$ is a separable closure of $F(\theta)_v$.

Proposition 2.4.1 (see [Brownawell 1999, §7; Jeong 2011, §2]). *Let F be a field of characteristic $p > 0$, and let v be a place of $F(\theta)$. Then, for $f \in F(\theta)_v^{\mathrm{sep}}$, $n \geqslant 0$, and $j \geq 1$, $\partial_\theta^j : F(\theta)_v^{\mathrm{sep}} \to F(\theta)_v^{\mathrm{sep}}$, $j \geqslant 0$, satisfies*

$$\partial_\theta^j(f^{p^n}) = \begin{cases} (\partial_\theta^e(f))^{p^n} & \text{if } j = ep^n, \\ 0 & \text{if } p^n \nmid j. \end{cases}$$

For $f \in F(\theta)_v^{\mathrm{sep}}$ and $n \geq 0$, we define the *d-matrix with respect to θ*, $d_{\theta,n}[f] \in \mathrm{Mat}_n(F(\theta)_v^{\mathrm{sep}})$ to be the upper-triangular $n \times n$ matrix

$$d_{\theta,n}[f] := \begin{pmatrix} f & \partial_\theta^1(f) & \cdots & \cdots & \partial_\theta^{n-1}(f) \\ & f & \partial_\theta^1(f) & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \partial_\theta^1(f) \\ & & & & f \end{pmatrix}. \tag{2.4.2}$$

Using the product rule, it is easy to see that $d_{\theta,n}[g] \cdot d_{\theta,n}[f] = d_{\theta,n}[gf]$. For a matrix $B := (b_{ij}) \in \mathrm{Mat}_{e_1 \times e_2}(F(\theta)_v^{\mathrm{sep}})$, we also define the d-matrix with respect to θ, $d_{\theta,n}[B] \in \mathrm{Mat}_{ne_1 \times ne_2}(F(\theta)_v^{\mathrm{sep}})$ as in (2.4.2), where we let $\partial_\theta^j(B) := (\partial_\theta^j(b_{ij})) \in \mathrm{Mat}_{e_1 \times e_2}(F(\theta)_v^{\mathrm{sep}})$.

We further define *partial hyperderivatives* for two independent variables θ and t to be the F-linear maps

$$\partial_\theta^j, \partial_t^j : F(\theta, t) \to F(\theta, t), \quad j \geq 0,$$

such that for $m \in \mathbb{Z}$ we have $\partial_\theta^j(\theta^m) = \binom{m}{j}\theta^{m-j}$, $\partial_t^j(t^m) = \binom{m}{j}t^{m-j}$, and $\partial_\theta^j(t^m) = \partial_t^j(\theta^m) = 0$. Thus, we have $\partial_\theta \circ \partial_t = \partial_t \circ \partial_\theta$. For $n \geq 0$, we define the d-matrices $d_{\theta,n}[\,\cdot\,]$ and $d_{t,n}[\,\cdot\,]$ with respect to each independent variable θ and t as in (2.4.2).

Note that ∂_t^j extends naturally to $\mathbb{T}$, and ∂_θ^j extends to $\mathbb{T} \cap k_\infty^{\mathrm{sep}}[\![t]\!]$.

2.5. ***Prolongations of dual t-motives.*** We review the construction of prolongations of dual t-motives, introduced in [Maurischat 2018]. For a left $\bar{k}[t,\sigma]$-module $\mathcal{M}$ and $n \geq 0$, we define the *n-th prolongation of $\mathcal{M}$* to be the left $\bar{k}[t,\sigma]$-module $\mathrm{P}_n\mathcal{M}$ generated by symbols $D_i m$, for $m \in \mathcal{M}$ and $0 \leq i \leq n$, subject to the relations

(a) $D_i(m_1 + m_2) = D_i m_1 + D_i m_2$,

(b) $D_i(a \cdot m) = \sum_{i=i_1+i_2} \partial_t^{i_1}(a) \cdot D_{i_2} m$,

(c) $\sigma(a \cdot D_i m) = a^{(-1)} \cdot D_i(\sigma m)$,

where $m, m_1, m_2 \in \mathcal{M}$ and $a \in \bar{k}[t]$.

If $\mathcal{M}$ is an $\boldsymbol{A}$-finite dual t-motive, then $\mathrm{P}_n\mathcal{M}$ is also an $\boldsymbol{A}$-finite dual t-motive [loc. cit., Theorem 3.4]. Thus, if the entries of $\boldsymbol{m} = [m_1, \dots, m_r]^\top \in \mathcal{M}^r$ form a $\bar{k}[t]$-basis of $\mathcal{M}$, then a $\bar{k}[t]$-basis of $\mathrm{P}_n\mathcal{M}$ is given by the entries of

$$\boldsymbol{D}_n\boldsymbol{m} := (D_n\boldsymbol{m}^\top, D_{n-1}\boldsymbol{m}^\top, \dots, D_0\boldsymbol{m}^\top)^\top \in (\mathrm{P}_n\mathcal{M})^{r(n+1)}, \tag{2.5.1}$$

where $D_i\boldsymbol{m} := (D_i m_1, \dots, D_i m_r)^\top \in (\mathrm{P}_n\mathcal{M})^r$ for each $0 \leq i \leq n$ [loc. cit., Proposition 4.2]. Also, if $\Phi \in \mathrm{GL}_r(\bar{k}[t])$ represents multiplication by σ on $\boldsymbol{m}$, then

$$\sigma(\boldsymbol{D}_n\boldsymbol{m}) = d_{t,n+1}[\Phi] \cdot \boldsymbol{D}_n\boldsymbol{m}. \tag{2.5.2}$$

If $\mathcal{M}$ is rigid analytically trivial with $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$, then since Frobenius twisting commutes with hyperdifferentiation with respect to t, we have

$$(d_{t,n+1}[\Psi])^{(-1)} = d_{t,n+1}[\Psi^{(-1)}] = d_{t,n+1}[\Phi\Psi] = d_{t,n+1}[\Phi]d_{t,n+1}[\Psi]. \tag{2.5.3}$$

Therefore, $\mathrm{P}_n\mathcal{M}$ is rigid analytically trivial.

Via $D_0 m \mapsto m$, we see that $\mathrm{P}_0\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$, and as in [loc. cit., Remark 3.2], for $0 \leq j \leq n-1$ we obtain a short exact sequence of dual t-motives

$$0 \to \mathrm{P}_j\mathcal{M} \to \mathrm{P}_n\mathcal{M} \xrightarrow{\mathbf{pr}_{n-j-1}} \mathrm{P}_{n-j-1}\mathcal{M} \to 0, \tag{2.5.4}$$

where $\mathbf{pr}_{n-j-1}(D_i m) := D_{i-j-1}m$ for $i > j$ and $\mathbf{pr}_{n-j-1}(D_i m) := 0$ for $i \leq j$ and $m \in \mathcal{M}$.

3. Rigid analytic trivializations and hyperderivatives

The goal of this section is to provide necessary background on Anderson t-modules for the purpose of studying Drinfeld $\boldsymbol{A}$-modules and their prolongations, and their connection to dual t-motives and rigid analytic trivializations via Anderson generating functions. Then, we state Theorem 3.4.1, which provides the connection between Taylor coefficients of series expansions of Anderson generating functions and hyperderivatives of periods, quasiperiods, logarithms, and quasilogarithms of a Drinfeld $\boldsymbol{A}$-module defined over k^{sep}.

3.1. ***Anderson t-modules, dual t-motives, and Anderson generating functions.*** For a field $K \subseteq \mathbb{K}$, an *Anderson t-module defined over K* is an $\mathbb{F}_q$-algebra homomorphism $\phi : \boldsymbol{A} \to \mathrm{Mat}_d(K[\tau])$ defined uniquely by

$$\phi_t = B_0 + B_1\tau + \cdots + B_\ell \tau^\ell,$$

where $B_i \in \mathrm{Mat}_d(K)$ for $0 \le i \le \ell$, and $\mathrm{d}\phi_t = B_0 = \theta I_d + N$ such that I_d is the $d \times d$ identity matrix and N is a nilpotent matrix. Then, ϕ defines an $\boldsymbol{A}$-module structure on $\mathbb{K}^d$ via

$$a \cdot \boldsymbol{x} = \phi_a(\boldsymbol{x}), \quad a \in \boldsymbol{A},\ \boldsymbol{x} \in \mathbb{K}^d. \tag{3.1.1}$$

We call d the *dimension of ϕ*. If $\phi_t = B_0 \in \mathrm{Mat}_d(K)$, then ϕ is said to be a *trivial* Anderson t-module. A nontrivial Anderson t-module of dimension 1 is called a *Drinfeld $\boldsymbol{A}$-module*.

There exists a unique power series $\mathrm{Exp}_\phi(\boldsymbol{z}) = \sum_{i=0}^\infty C_i \boldsymbol{z}^{(i)} \in \mathbb{K}[\![z_1, \ldots, z_d]\!]^d$, $\boldsymbol{z} = [z_1, \ldots, z_d]^\mathsf{T}$, so that $C_0 = I_d$ and satisfies

$$\mathrm{Exp}_\phi(\mathrm{d}\phi_a \boldsymbol{z}) = \phi_a(\mathrm{Exp}_\phi(\boldsymbol{z}))$$

for all $a \in \boldsymbol{A}$. Moreover, $\mathrm{Exp}_\phi(\boldsymbol{z})$ defines an entire function $\mathrm{Exp}_\phi : \mathbb{K}^d \to \mathbb{K}^d$. If Exp_ϕ is surjective, then we say that ϕ is *uniformizable*. The kernel $\Lambda_\phi \subseteq \mathbb{K}^d$ of Exp_ϕ is a free and finitely generated discrete $\boldsymbol{A}$-submodule of $\mathbb{K}^d$ through the action of $\mathrm{d}\phi(\boldsymbol{A})$ and it is called the *period lattice of ϕ*. If ϕ is uniformizable, then we have an isomorphism $\mathbb{K}^d/\Lambda_\phi \cong (\mathbb{K}^d, \phi)$ of $\boldsymbol{A}$-modules, where $(\mathbb{K}^d, \phi)$ denotes $\mathbb{K}^d$ together with the $\boldsymbol{A}$-module structure defined in (3.1.1) coming from ϕ. For more details about Anderson t-modules, see [Anderson 1986; Brownawell and Papanikolas 2020; Thakur 2004].

We define the dual t-motive $\mathcal{M}_\phi$ associated to a t-module ϕ defined over $K \subseteq \bar{k}$ in the following way. We let $\mathcal{M}_\phi := \mathrm{Mat}_{1\times d}(\bar{k}[\sigma])$. To give $\mathcal{M}_\phi$ the $\bar{k}[t, \sigma]$-module structure, set

$$a \cdot m = m\phi_a^*, \quad m \in \mathcal{M}_\phi,\ a \in \boldsymbol{A}, \tag{3.1.2}$$

where ϕ_a^* is defined as in Section 2.1. For each $m \in \mathcal{M}_\phi$, by straightforward computation we obtain $(t-\theta)^d \cdot m \in \sigma\mathcal{M}_\phi$. Thus, $\mathcal{M}_\phi$ defines a dual t-motive and (3.1.2) gives a unique correspondence between a t-module and its associated dual t-motive (see also [Brownawell and Papanikolas 2020, §4.4; Hartl and Juschka 2020; Namoijam and Papanikolas 2024, §2.3]). If $\mathcal{M}_\phi$ is $\boldsymbol{A}$-finite, then we say that *ϕ is $\boldsymbol{A}$-finite* and call the rank of $\mathcal{M}_\phi$ as a left $\bar{k}[t]$-module the *rank of ϕ*. The reader is directed to [Hartl and Juschka 2020; Namoijam and Papanikolas 2024, §2.3] for more information on dual t-motives associated to t-modules.

We conclude this subsection by introducing the Anderson generating functions associated to t-modules (see [Green 2022; Maurischat 2022c; Namoijam and Papanikolas 2024] for further details). For $\boldsymbol{y} \in \mathbb{K}^d$, we define the *Anderson generating function for ϕ* by the infinite series

$$\mathcal{G}_{\boldsymbol{y}}(t) := \sum_{m=0}^\infty \mathrm{Exp}_\phi(\mathrm{d}\phi_t^{-m-1}\boldsymbol{y})t^m \in \mathbb{T}^d. \tag{3.1.3}$$

We explore the properties we will use in Sections 3.3, 4.1, 5.1. For clarity, we will denote by $f_y(t)$ the Anderson generating function for a Drinfeld $\boldsymbol{A}$-module at $y \in \mathbb{K}$.

3.2. *Prolongations of Drinfeld A-modules and associated dual t-motives.* Let $\rho : \boldsymbol{A} \to K[\tau]$ be a Drinfeld $\boldsymbol{A}$-module defined over $K \subseteq \bar{k}$ such that

$$\rho_t = \theta + \kappa_1 \tau + \cdots + \kappa_r \tau^r,$$

where $\kappa_r \neq 0$. Drinfeld $\boldsymbol{A}$-modules are uniformizable and the rank of the period lattice Λ_ρ of ρ as an $\boldsymbol{A}$-module is r. As defined above for t-modules, we define the dual t-motive $\mathcal{M}_\rho := \bar{k}[\sigma]$. Then the set $\{m_1, m_2, \ldots, m_r\} = \{1, \sigma, \ldots, \sigma^{r-1}\}$ forms a $\bar{k}[t]$-basis for $\mathcal{M}_\rho$ [Chang and Papanikolas 2012, §3.3; Namoijam and Papanikolas 2024, Example 3.35], and with respect to this basis, multiplication by σ on $\mathcal{M}_\rho$ is represented by

$$\Phi_\rho := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\kappa_r^{(-r)} & -\kappa_1^{(-1)}/\kappa_r^{(-r)} & \cdots & -\kappa_{r-1}^{(-r+1)}/\kappa_r^{(-r)} \end{pmatrix}. \tag{3.2.1}$$

Thus, $\mathcal{M}_\rho$ is $\boldsymbol{A}$-finite. We let $M_\rho := \bar{k}(t) \otimes_{\bar{k}[t]} \mathcal{M}_\rho$ be the pre-t-motive associated to $\mathcal{M}_\rho$.

For Drinfeld $\boldsymbol{A}$-modules ρ and ρ' defined over $K \subseteq \mathbb{K}$, a *morphism* $b : \rho \to \rho'$ is a twisted polynomial $b \in \mathbb{K}[\tau]$ such that $b\rho_a = \rho'_a b$ for all $a \in \boldsymbol{A}$. We say that b is defined over $L \subseteq \mathbb{K}$ if $b \in L[\tau]$. A morphism $b : \rho \to \rho'$ defined over $\bar{k}$ induces a morphism $B : \mathcal{M}_\rho \to \mathcal{M}_{\rho'}$ of $\boldsymbol{A}$-finite dual t-motives in the following way. If $b = \sum c_i \tau^i \in L[\tau]$, recall from Section 2.1 that $b^* = \sum c_i^{(-i)} \sigma^i$. Then, B is the $\bar{k}[\sigma]$-linear map such that $B(1) = b^*$ (see [Chang and Papanikolas 2011, Lemma 2.4.2]).

The map

$$\operatorname{End}(\rho) \to \{c \in \mathbb{K} : c\Lambda_\rho \subseteq \Lambda_\rho\}, \quad \sum c_i \tau^i \mapsto c_0, \tag{3.2.2}$$

is an isomorphism [Drinfeld 1974]. Throughout this paper, we identify $\operatorname{End}(\rho)$ with the image of this map and let K_ρ denote its fraction field. We state the following result due to Anderson.

Proposition 3.2.3 [Chang and Papanikolas 2012, Proposition 3.3.2, Corollary 3.3.3]. *The functor $\rho \to \mathcal{M}_\rho$ from the category of Drinfeld $\boldsymbol{A}$-modules defined over $K \subseteq \bar{k}$ to the category of $\boldsymbol{A}$-finite dual t-motives is fully faithful. Moreover,*

$$\operatorname{End}(\rho) \cong \operatorname{End}_{\bar{k}[t,\sigma]}(\mathcal{M}_\rho), \qquad K_\rho \cong \operatorname{End}_{\mathcal{T}}(M_\rho),$$

and M_ρ is a simple left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module.

Remark 3.2.4. Let $i_t \in \operatorname{End}_{\bar{k}[t,\sigma]}(\mathcal{M}_\rho)$ be such that $i_t(1) = t \cdot 1 = \rho_t^*$. The isomorphism $\operatorname{End}(\rho) \cong \operatorname{End}_{\bar{k}[t,\sigma]}(\mathcal{M}_\rho)$ in Proposition 3.2.3 sends $\theta \mapsto i_t$ and so, it sends A to $\boldsymbol{A}$. Thus, $K_\rho \cong \operatorname{End}_{\mathcal{T}}(M_\rho)$ sends k to $\boldsymbol{k}$.

For $n \geq 0$, we define the *n-th prolongation t-module* $\mathrm{P}_n\rho$ *of* ρ to be the Anderson t-module associated to the n-th prolongation $\mathrm{P}_n\mathcal{M}_\rho$ of the $\boldsymbol{A}$-finite dual t-motive $\mathcal{M}_\rho$ (see for details [Maurischat 2018, §5; Namoijam and Papanikolas 2024, §5.2]). The Anderson t-module $\mathrm{P}_n\rho : \boldsymbol{A} \to \operatorname{Mat}_{n+1}(K[\tau])$ is of dimension $n+1$ and is defined by

$$(\mathrm{P}_n\rho)_t = \mathrm{d}(\mathrm{P}_n\rho)_t + \operatorname{diag}(\kappa_1)\tau + \cdots + \operatorname{diag}(\kappa_r)\tau^r,$$

where

$$\mathrm{d}(\mathrm{P}_n\rho)_t = \begin{pmatrix} \theta & & & & \\ -1 & \ddots & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & \theta \end{pmatrix}, \tag{3.2.5}$$

and $\mathrm{diag}(\kappa_i)$ is the $(n+1)\times(n+1)$ diagonal matrix with diagonal entries all equal to κ_i for each $1 \le i \le r$. If we set $\mathcal{M}_{\mathrm{P}_n\rho} := \mathrm{Mat}_{1\times(n+1)}(\bar{k}[\sigma])$ to be the dual t-motive associate to $\mathrm{P}_n\rho$ defined as in (3.1.2), then by [Namoijam and Papanikolas 2024, Proposition 5.22(b)] we have

$$\mathcal{M}_{\mathrm{P}_n\rho} = \mathrm{P}_n\mathcal{M}_\rho.$$

We define $\mathrm{P}_n M_\rho := \bar{k}(t)\otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ to be the pre-t-motive associated to $\mathrm{P}_n\mathcal{M}_\rho$.

3.3. *Rigid analytic trivializations.* We fix our choice of Drinfeld $\boldsymbol{A}$-module ρ of rank r from Section 3.2 such that it is defined over $K = k^{\mathrm{sep}}$. In this subsection, we show that the $\boldsymbol{A}$-finite dual t-motive $\mathcal{M}_\rho$ associated to ρ is rigid analytically trivial by constructing the rigid analytic trivialization Ψ_ρ, and then extend to the prolongation t-module $\mathrm{P}_n\rho$. The details regarding Drinfeld $\boldsymbol{A}$-modules can be found in [Chang and Papanikolas 2012, §3.4; Namoijam and Papanikolas 2024, Example 4.1117].

For $u \in \mathbb{K}$, we let $f_u(t) \in \mathbb{T}$ denote the Anderson generating function of ρ given as in (3.1.3). For an $\boldsymbol{A}$-basis $\{\lambda_1, \dots, \lambda_r\}$ of Λ_ρ, we set $f_i(t) := f_{\lambda_i}(t)$ for each $1 \le i \le r$. Define the matrices

$$\Upsilon := \begin{pmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(r-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(r-1)} \\ \vdots & \vdots & & \vdots \\ f_r & f_r^{(1)} & \cdots & f_r^{(r-1)} \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} \kappa_1 & \kappa_2^{(-1)} & \cdots & \kappa_{r-1}^{(-r+2)} & \kappa_r^{(-r+1)} \\ \kappa_2 & \kappa_3^{(-1)} & \cdots & \kappa_r^{(-r+2)} & \\ \vdots & \vdots & & & \\ \kappa_{r-1} & \kappa_r^{(-1)} & & & \\ \kappa_r & & & & \end{pmatrix}. \tag{3.3.1}$$

By [Chang and Papanikolas 2012, §3.4] (see also [Namoijam and Papanikolas 2024, Lemma 4.47]), it follows that $\det\Upsilon \neq 0$. Set

$$\Psi_\rho := V^{-1}[\Upsilon^{(1)}]^{-1}. \tag{3.3.2}$$

Then $\Psi_\rho^{(-1)} = \Phi_\rho\Psi_\rho$. Thus, the pre-$t$-motive $M_\rho = \bar{k}(t)\otimes_{\bar{k}[t]}\mathcal{M}_\rho$ is rigid analytically trivial and is in the category $\mathcal{T}$ of t-motives.

By (2.5.3), the n-th prolongation t-motive $\mathrm{P}_n M_\rho = \bar{k}(t)\otimes_{\bar{k}[t]}\mathrm{P}_n\mathcal{M}_\rho$ is rigid analytically trivial with rigid analytic trivialization $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$. Thus,

$$\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[V]^{-1} d_{t,n+1}[\Upsilon^{(1)}]^{-1}. \tag{3.3.3}$$

3.4. *Hyperderivatives of periods and logarithms.* We continue with our choice of Drinfeld $\boldsymbol{A}$-module ρ of rank r defined over k^{sep}. Recall from Section 1.1 that a $\mathbb{K}$-basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ is represented by $\{\delta_1, \dots, \delta_r\}$, where δ_1 is the inner biderivation such that $(\delta_1)_t = \rho_t - \theta = \kappa\tau + \cdots + \kappa_r\tau^r$ and $\delta_j(t) = \tau^{j-1}$ for $2 \le j \le r$.

Let $\mathrm{F}_{\tau^{j-1}}(z)$ denote the quasiperiodic function associated to the biderivation $\delta_j : t \mapsto \tau^{j-1}$. Note that $\mathrm{F}_{\delta_1}(z) = \mathrm{Exp}_\rho(z) - z$. Then, we have the following result, which is a modified version for Drinfeld $\boldsymbol{A}$-modules, and its proof is due to Papanikolas and the author.

Theorem 3.4.1 (see [Namoijam and Papanikolas 2024, Theorem E]). *Let ρ be a Drinfeld $\boldsymbol{A}$-module defined over k^{sep} of rank r. Let $u \in \mathbb{K}^d$ satisfy $\mathrm{Exp}_\phi(u) \in (k^{\mathrm{sep}})^d$. Then, for $n \geqslant 0$,*

$$\mathrm{Span}_{\bar{k}}\left(\{1\} \cup \bigcup_{s=0}^{n} \bigcup_{\ell=1}^{r} \{\partial_t^s(f_u^{(\ell)}(t))|_{t=\theta}\}\right) = \mathrm{Span}_{\bar{k}}\left(\{1\} \cup \bigcup_{s=0}^{n} \bigcup_{j=1}^{r-1} \{\partial_\theta^s(u), \partial_\theta^s(\mathrm{F}_{\tau^j}(u))\}\right). \tag{3.4.2}$$

In particular, if $\{\lambda_1, \dots, \lambda_r\}$ is an $\boldsymbol{A}$-basis of the period lattice Λ_ρ, then

$$\mathrm{Span}_{\bar{k}}(d_{t,n+1}[\Psi_\rho]^{-1}|_{t=\theta}) = \mathrm{Span}_{\bar{k}}\left(\bigcup_{s=0}^{n} \bigcup_{i=1}^{r} \bigcup_{j=1}^{r-1} \{\partial_\theta^s(\lambda_j), \partial_\theta^s(\mathrm{F}_{\tau^j}(\lambda_i))\}\right). \tag{3.4.3}$$

By using Theorems 2.3.2 and 3.4.1, computing the dimension of the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ for $n \geq 1$ proves Theorem 1.1.3. Moreover, by (3.4.2) if we are able to construct appropriate t-motives whose periods span the hyperderivatives in question and determine the dimension of their associated Galois groups, then we can prove Theorem 1.1.4.

4. Hyperderivatives of periods and quasiperiods

Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep}. Let K_ρ be the fraction field of $\mathrm{End}(\rho)$ defined as in (3.2.2) and let $[K_\rho : k] = \boldsymbol{s}$. In this section, we prove Theorem 1.1.3 (restated as Theorem 4.5.1). To prove this theorem, we first show in Theorem 4.3.3 that $\dim \Gamma_{\mathrm{P}_n M_\rho} \geq (n+1) \cdot r^2/\boldsymbol{s}$, and in Theorem 4.4.6 that $\dim \Gamma_{\mathrm{P}_n M_\rho} \leq (n+1) \cdot r^2/\boldsymbol{s}$. Moreover, in Corollary 4.4.8 we explicitly compute the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ for all $n \geq 1$.

4.1. ***The $\mathfrak{p}$-adic Tate module and Anderson generation functions.*** Let ϕ be a uniformizable and $\boldsymbol{A}$-finite Anderson t-module of dimension d and rank r. For any $a \in \boldsymbol{A}$, the torsion $\boldsymbol{A}$-module $\phi[a] := \{\boldsymbol{x} \in \mathbb{K}^d \mid \phi_a(\boldsymbol{x}) = 0\}$ is isomorphic to $(\boldsymbol{A}/(a))^{\oplus r}$ (see [Anderson 1986; Thakur 2004, Theorem 7.2.1]). For a nonzero prime $\mathfrak{p}$ of $\boldsymbol{A}$, we define the $\mathfrak{p}$-*adic Tate module*

$$T_\mathfrak{p}(\phi) := \varprojlim_m \phi[\mathfrak{p}^m] \cong \boldsymbol{A}_\mathfrak{p}^{\oplus r},$$

where $\boldsymbol{A}_\mathfrak{p}$ is the completion of $\boldsymbol{A}$ at $\mathfrak{p}$. Now, we fix a Drinfeld $\boldsymbol{A}$-module ρ of rank r. If ρ is defined over K such that $k \subseteq K \subseteq \bar{k}$ and $[K : k] < \infty$, then note that every element of $\rho[\mathfrak{p}^m]$ is separable over K. Thus, the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of the separable closure of K inside $\bar{k}$ acts on $T_\mathfrak{p}(\rho)$, defining a representation

$$\varphi_\mathfrak{p} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_\mathfrak{p}(\rho)) \cong \mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p}).$$

Set $\mathrm{p} := \mathfrak{p}(\theta) \in A$. We fix an $\boldsymbol{A}$-basis $\{\lambda_1, \dots, \lambda_r\}$ of Λ_ρ and define

$$\xi_{i,m} := \operatorname{Exp}_\rho\left(\frac{\lambda_i}{\mathrm{p}^{m+1}}\right) \in \rho[\mathfrak{p}^{m+1}]$$

for each $1 \le i \le r$ and $m \ge 0$. Then, $\{x_1, \dots, x_r\}$ is an $\boldsymbol{A}_\mathfrak{p}$-basis of $T_\mathfrak{p}(\rho)$, where we set $x_i := (\xi_{i,0}, \xi_{i,1}, \dots)$. Set $\boldsymbol{x} := [x_1, \dots, x_r]^\mathsf{T}$. Then, for each $\epsilon \in \operatorname{Gal}(K^{\mathrm{sep}}/K)$ there exists $g_\epsilon \in \operatorname{GL}_r(\boldsymbol{A}_\mathfrak{p})$ such that

$$\varphi_\mathfrak{p}(\epsilon)\boldsymbol{x} = g_\epsilon \boldsymbol{x}. \tag{4.1.1}$$

Theorem 4.1.2 [Maurischat and Perkins 2022, Theorem 1.2]. *Let ρ be a Drinfeld $\boldsymbol{A}$-module defined over K such that $k \subseteq K \subseteq \bar{k}$ and $[K : k] < \infty$. Let $\boldsymbol{k}_\mathfrak{p}$ be the field of fractions of $\boldsymbol{A}_\mathfrak{p}$. For each $\epsilon \in \operatorname{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon \in \operatorname{GL}_r(\boldsymbol{A}_\mathfrak{p})$ be as in (4.1.1). Then, the assignment $\epsilon \mapsto g_\epsilon$ induces a group homomorphism*

$$\beta_0 : \operatorname{Gal}(K^{\mathrm{sep}}/K) \to \Gamma_{\Psi_\rho}(\boldsymbol{A}_\mathfrak{p}) := \operatorname{GL}_r(\boldsymbol{A}_\mathfrak{p}) \cap \Gamma_{\Psi_\rho}(\boldsymbol{k}_\mathfrak{p}).$$

Note that in the case of $\mathfrak{p} = t$, Theorem 4.1.2 was first proved by Chang and Papanikolas [2012, Theorem 3.5.1].

For the remainder of this subsection, we fix $n \ge 0$. By [Namoijam and Papanikolas 2024, Proposition 5.27], we have that, for $\boldsymbol{z} = [z_0, \dots, z_n]^\mathsf{T}$,

$$\operatorname{Exp}_{\mathrm{P}_n\rho}(\boldsymbol{z}) = [\operatorname{Exp}_\rho(z_0), \dots, \operatorname{Exp}_\rho(z_n)]^\mathsf{T}. \tag{4.1.3}$$

For $u \in \mathbb{K}$, set

$$(u)_j := [0, \dots, 0, u, 0, \dots, 0]^\mathsf{T} =\in \mathbb{K}^{n+1}, \tag{4.1.4}$$

where u is in the j-th entry and all other entries are 0. By (4.1.3), using the $\boldsymbol{A}$-basis $\{\lambda_1, \dots, \lambda_r\}$ of Λ_ρ, an $\boldsymbol{A}$-basis of the period lattice $\Lambda_{\mathrm{P}_n\rho}$ of $\mathrm{P}_n\rho$ is

$$\{(\lambda_i)_j : 1 \le i \le r \text{ and } 1 \le j \le n+1\}.$$

We define

$$\chi_{i,m} := \operatorname{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right)$$

for each $1 \le i \le r$ and $m \ge 0$. By (3.1.3), the Anderson generating function $f_i(t) := f_{\lambda_i}(t)$ of ρ with respect to λ_i is

$$f_i(t) = \sum_{m=0}^{\infty} \operatorname{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right) t^m = \sum_{m=0}^{\infty} \chi_{i,m} t^m \in \mathbb{T} \cap K^{\mathrm{sep}}[\![t]\!].$$

For each $1 \le i \le r$ and $1 \le j \le n+1$ we let $\mathcal{G}_{i,j}(t) := \mathcal{G}_{(\lambda_i)_j}(t)$ denote the Anderson generating function of $\mathrm{P}_n\rho$ with respect to $(\lambda_i)_j$. Then, by (3.1.3) we have

$$\mathcal{G}_{i,j}(t) = \sum_{m=0}^{\infty} \operatorname{Exp}_{\mathrm{P}_n\rho}\big((\mathrm{d}(\mathrm{P}_n\rho)_t)^{-m-1}(\lambda_i)_j\big) t^m \in \mathbb{T}^{n+1} \cap K^{\mathrm{sep}}[\![t]\!]^{n+1}.$$

Observe that in (3.2.5), the subdiagonal entries of $\mathrm{d}(\mathrm{P}_n\rho)_t$ are $-\partial_\theta^1(\theta)$. Also, $(-1)^c\partial_\theta^c(\theta)=0$ for $c\geq 2$. Moreover, the e-th subdiagonal entries of $\mathrm{d}(\mathrm{P}_n\rho)_t^{-1}$ are $(-1)^e\partial_\theta^e(\theta^{-1})$ and so, by the product rule of hyperderivatives, for $h\in\mathbb{Z}$ we have

$$(\mathrm{d}(\mathrm{P}_n\rho)_t)^h=\begin{pmatrix}\theta^h &&&&& \\ -\partial_\theta^1(\theta^h) & \theta^h &&&& \\ \partial_\theta^2(\theta^h) & -\partial_\theta^1(\theta^h) & \theta^h &&& \\ \vdots && \ddots & \ddots && \\ \vdots &&& \ddots & \ddots & \\ (-1)^n\partial_\theta^n(\theta^h) & \cdots & \cdots & \partial_\theta^2(\theta^h) & -\partial_\theta^1(\theta^h) & \theta^h\end{pmatrix}.$$

Note that for $m\geq 0$ and $c\geq 1$, we have $(-1)^c\partial_\theta^c(\theta^{-m-1})=\binom{m+c}{c}\theta^{-m-1-c}$. Then, it follows by using (4.1.3) that

$$\begin{aligned}\mathcal{G}_{i,j}(t)&=\sum_{m=0}^{\infty}\left(0,\ldots,0,\,\chi_{i,m},\,\binom{m+1}{1}\chi_{i,m+1},\ldots,\binom{m+(n+1-j)}{n+1-j}(\chi_{i,m+(n+1-j)})\right)^{\mathsf{T}}t^m\\ &=\left(0,\ldots,0,\,\sum_{m=0}^{\infty}\chi_{i,m}t^m,\,\sum_{m=1}^{\infty}\chi_{i,m}\binom{m}{1}t^{m-1},\ldots,\sum_{m=n+1-j}^{\infty}\chi_{i,m}\binom{m}{n+j-1}t^{m-(n+j-1)}\right)^{\mathsf{T}}.\end{aligned}$$

Thus,

$$\mathcal{G}_{i,j}(t)=\left(0,\ldots,0,\,f_i,\,\partial_t^1(f_i),\ldots,\partial_t^{n+1-j}(f_i)\right)^{\mathsf{T}}\in\mathbb{T}^{n+1}. \tag{4.1.5}$$

For our purpose, we consider the Galois group $\Gamma_{\Psi_{\mathrm{P}_n\rho}}$ and its principal homogeneous space $Z_{\Psi_{\mathrm{P}_n\rho}}$ as in Section 2.3, and we prove the following result.

Theorem 4.1.6. *Let ρ be a Drinfeld $\boldsymbol{A}$-module defined over K such that $k\subseteq K\subseteq\bar{k}$ and $[K:k]<\infty$, and for $n\geq 0$ let $\mathrm{P}_n\rho$ be its n-th prolongation t-module. Let $\boldsymbol{k}_\mathfrak{p}$ be the fraction field of $\boldsymbol{A}_\mathfrak{p}$. For each $\epsilon\in\mathrm{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon\in\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ be as in (4.1.1). Then, the assignment $\epsilon\mapsto d_{t,n+1}[g_\epsilon]$ induces a group homomorphism*

$$\beta_n:\mathrm{Gal}(K^{\mathrm{sep}}/K)\to\Gamma_{\Psi_{\mathrm{P}_n\rho}}(\boldsymbol{A}_\mathfrak{p}):=\mathrm{GL}_{(n+1)r}(\boldsymbol{A}_\mathfrak{p})\cap\Gamma_{\Psi_{\mathrm{P}_n\rho}}(\boldsymbol{k}_\mathfrak{p}).$$

We follow the methods used in [Maurischat and Perkins 2022]. Let $\overline{\mathbb{F}}_q$ denote an algebraic closure of $\mathbb{F}_q$ inside $\mathbb{K}$ and let $\zeta\in\overline{\mathbb{F}}_q$ be a root of $\mathfrak{p}$. We define the $\mathbb{K}$-algebra map $\mathcal{D}_\zeta:\mathbb{T}\to\mathbb{K}[[X]]$ by

$$g\mapsto\sum_{m=0}^{\infty}\partial_t^m(g)|_{t=\zeta}X^m.$$

By [loc. cit., Lemma 2.2], the map $\mathcal{D}_\zeta:\boldsymbol{A}\to\mathbb{K}[[X]]$ extends to an isomorphism $\mathcal{D}_\zeta:\boldsymbol{A}_\mathfrak{p}\to\mathbb{F}_q(\zeta)[[X]]$.

The Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on $K^{\mathrm{sep}}[[X]]$ by acting on each coefficient. We now consider the Galois action on Anderson generating functions of $\mathrm{P}_n\rho$ and their Frobenius twists.

Proposition 4.1.7 (cf. [Maurischat and Perkins 2022, Proposition 4.2]). *For each $\epsilon\in\mathrm{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon\in\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})$ be defined as in (4.1.1). Let $\mathcal{G}:=[\mathcal{G}_{1,1},\ldots,\mathcal{G}_{r,1},\ldots,\mathcal{G}_{1,n+1},\ldots,\mathcal{G}_{r,n+1}]^{\mathsf{T}}\in\mathrm{Mat}_{r(n+1)\times n+1}(\mathbb{T})$. Then,*

$$\epsilon(\mathcal{D}_\zeta(\mathcal{G}))=\mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[g_\epsilon])\cdot\mathcal{D}_{\epsilon(\zeta)}(\mathcal{G}),$$

where

$$\epsilon(\mathcal{D}_\zeta(\mathcal{G})) = [\epsilon(\mathcal{D}_\zeta(\mathcal{G}_{1,1})), \dots, \epsilon(\mathcal{D}_\zeta(\mathcal{G}_{r,1})), \dots, \epsilon(\mathcal{D}_\zeta(\mathcal{G}_{1,n+1})), \dots, \epsilon(\mathcal{D}_\zeta(\mathcal{G}_{r,n+1}))]^{\mathsf{T}}.$$

Proof. Note that by (4.1.5), the j-th column of $\mathcal{G}$ for $1 \le j \le n+1$ is

$$[\partial_t^{j-1}(f_1), \dots, \partial_t^{j-1}(f_r), \partial_t^{j-2}(f_1), \dots, \partial_t^{j-2}(f_r), \dots, f_1, \dots, f_r, 0, \dots, 0]^{\mathsf{T}} \in \mathbb{T}^{r(n+1)}.$$

Then, for $m_1, m_2 \in \mathbb{N}$ and $1 \le i \le r$, since $\partial_t^{m_1}(\partial_t^{m_2}(f_i)) = \binom{m_1+m_2}{m_1}\partial_t^{m_1+m_2}(f_i)$, the result follows by using [Maurischat and Perkins 2022, Lemma 4.1]. □

Proposition 4.1.8 (cf. [Maurischat and Perkins 2022, Proposition 5.1]). *For $1 \le i, j \le r$, define $\Upsilon \in \mathrm{Mat}_r(\mathbb{T})$ so that $\Upsilon_{ij} := f_i^{(j-1)}(t)$ as in (3.3.1). Then, for any $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $g_\epsilon \in \mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})$ as in (4.1.1), we have*

$$\epsilon(\mathcal{D}_\zeta(d_{t,n+1}[\Upsilon]^{(1)})) = \mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[g_\epsilon]) \cdot \mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[\Upsilon]^{(1)})$$

and

$$\epsilon(\mathcal{D}_\zeta(\Psi_{\mathrm{P}_n\rho})) = \mathcal{D}_{\epsilon(\zeta)}(\Psi_{\mathrm{P}_n\rho}) \cdot \mathcal{D}_{\epsilon(\zeta)}(d_{t,n+1}[g_\epsilon])^{-1}.$$

Proof. Since Frobenius twisting commutes with hyperdifferentiation with respect to t, we see by using (4.1.5) that for $1 \le j \le r$ and $0 \le \ell \le n$, the $(\ell r+j)$-th column of $d_{t,n+1}[\Upsilon^{(1)}]$ is given by the j-th Frobenius twist of the $(\ell+1)$-th column of $\mathcal{G}$. Moreover, by (3.3.3) we have $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[V]^{-1} d_{t,n+1}[\Upsilon^{(1)}]^{-1}$. Then by using Proposition 4.1.7, the results follow by a straightforward adaptation of the proof of [loc. cit., Proposition 5.1]. □

By an abuse of the notation $\mathcal{D}_\zeta$, we consider the homomorphism $\mathcal{D}_\zeta : \mathbb{T} \otimes_{\boldsymbol{A}} \boldsymbol{A}_{\mathfrak{p}} \to \mathbb{K}[[X]]$ defined by

$$\sum_i g_i \otimes b_i \mapsto \sum_i \mathcal{D}_\zeta(g_i) \cdot \mathcal{D}_\zeta(b_i).$$

Note that $\mathcal{D}_\zeta$ is injective on $\mathbb{T}$, and so it extends to $\mathbb{L} \otimes_{\mathbb{T}} (\mathbb{T} \otimes_{\boldsymbol{A}} \boldsymbol{A}_{\mathfrak{p}}) \cong \mathbb{L} \otimes_{\boldsymbol{k}} \boldsymbol{k}_{\mathfrak{p}}$, that is, to a ring homomorphism

$$\widetilde{\mathcal{D}}_\zeta : \mathbb{L} \otimes_{\boldsymbol{k}} \boldsymbol{k}_{\mathfrak{p}} \to \mathbb{K}((X)).$$

Proof of Theorem 4.1.6. Let $S \subseteq K^{\mathrm{per}}(t)[Y, 1/\det Y]$ denote a finite set of generators of the defining ideal of $Z_{\Psi_{\mathrm{P}_n\rho}}$, where K^{per} is the perfect closure of K in $\mathbb{K}$. Then, for any $h \in S$, we have $h(\Psi_{\mathrm{P}_n\rho}) = 0$. If $\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1} \in Z_{\Psi_{\mathrm{P}_n\rho}}(\mathbb{K}((t)))$, then by Theorem 2.3.1 we have $d_{t,n+1}[g_\epsilon]^{-1} \in \Gamma_{\Psi_{\mathrm{P}_n\rho}}(\boldsymbol{k}_{\mathfrak{p}})$. Thus, to prove our result, we will show that $h(\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1}) = 0$ for every $h \in S$. The proof follows by a straightforward adaptation of the proof of [Maurischat and Perkins 2022, Theorem 1.2], but for completeness we provide a proof.

For $h \in S$, let $h_\zeta \in \mathbb{K}((X))[Y, 1/\det Y]$ denote its image after mapping its coefficients via the map $\widetilde{\mathcal{D}}_\zeta$. Then,

$$\begin{aligned}\widetilde{\mathcal{D}}_\zeta\big(h(\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1})\big) &= h_\zeta\big(\mathcal{D}_\zeta(\Psi_{\mathrm{P}_n\rho}) \cdot \mathcal{D}_\zeta(d_{t,n+1}[g_\epsilon]^{-1})\big)\\ &= h_\zeta\big(\epsilon(\mathcal{D}_{\epsilon^{-1}(\zeta)}(\Psi_{\mathrm{P}_n\rho}))\big) = \epsilon\big(h_{\epsilon^{-1}(\zeta)}(\mathcal{D}_{\epsilon^{-1}(\zeta)}(\Psi_{\mathrm{P}_n\rho}))\big)\\ &= \epsilon\big(\widetilde{\mathcal{D}}_{\epsilon^{-1}(\zeta)}(h(\Psi_{\mathrm{P}_n\rho}))\big) = 0, \end{aligned} \tag{4.1.9}$$

where the second equality is by Proposition 4.1.8 and we obtain the third equality since the coefficients of h are in $K^{\mathrm{per}}(t)$. Since ζ is an arbitrary root of $\mathfrak{p}$, it follows by [Maurischat and Perkins 2022, Lemma 5.3] that $h(\Psi_{\mathrm{P}_n\rho} \cdot d_{t,n+1}[g_\epsilon]^{-1}) = 0$. □

4.2. *Elements of* $\Gamma_{\mathrm{P}_n M_\rho}$. Let ρ a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep}, and consider the t-motive M_ρ associated to ρ (see Section 3.1). In this subsection, for $n \geq 1$ we study the structure of the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ of the n-th prolongation t-motive $\mathrm{P}_n M_\rho$. We let $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ denote the ring of endomorphisms of $\mathrm{P}_n M_\rho$ and set $\boldsymbol{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$. If the entries of $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ form a $\bar{k}(t)$-basis of M_ρ, then the entries of $\boldsymbol{D}_n\boldsymbol{m}$ form a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$ as in (2.5.1). Given $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$, let $\mathrm{H} \in \mathrm{Mat}_{r(n+1)}(\bar{k}(t))$ be such that $h(\boldsymbol{D}_n\boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n\boldsymbol{m}$. Since $h\sigma = \sigma h$ and $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, we have

$$d_{t,n+1}[\Phi_\rho]\mathrm{H} = \mathrm{H}^{(-1)} d_{t,n+1}[\Phi_\rho].$$

From this, we see σ fixes $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho]$, and thus $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho] \in \mathrm{Mat}_{r(n+1)}(\boldsymbol{k})$. We have thus defined the injective map

$$\begin{aligned} \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho) &\to \mathrm{End}((\mathrm{P}_n M_\rho)^B) = \mathrm{Mat}_{r(n+1)}(\boldsymbol{k}), \\ h &\mapsto h^B := d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho]. \end{aligned} \tag{4.2.1}$$

Since the tautological representation $\varpi_n : \Gamma_{\mathrm{P}_n M_\rho} \to \mathrm{GL}((\mathrm{P}_n M_\rho)^B)$ is functorial in $\mathrm{P}_n M_\rho$ [Papanikolas 2008, Theorem 4.5.3], for any $\boldsymbol{k}$-algebra R and $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$, it follows that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B & \xrightarrow{\varpi_n^{\mathrm{R}}(\mu)} & \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B \\ \downarrow{\scriptstyle 1\otimes h^B} & & \downarrow{\scriptstyle 1\otimes h^B} \\ \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B & \xrightarrow{\varpi_n^{\mathrm{R}}(\mu)} & \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B \end{array} \tag{4.2.2}$$

Proposition 4.2.3. *Given $f \in \boldsymbol{K}_\rho$, let $\mathrm{F} \in \mathrm{Mat}_r(\bar{k}(t))$ satisfy $f(\boldsymbol{m}) = \mathrm{F}\boldsymbol{m}$. Also, for $n \geq 1$ let $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ be such that $h(\boldsymbol{D}_n\boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n\boldsymbol{m}$, where $\mathrm{H} = (\mathrm{H}_{ij}) \in \mathrm{Mat}_{r(n+1)}(\bar{k}(t))$ and each H_{ij} is an $r \times r$ block for $1 \leq i, j \leq n+1$. Then*:

(a) *For $n \geq 1$ there exists $g \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ such that $g(\boldsymbol{D}_n\boldsymbol{m}) = d_{t,n+1}[\mathrm{F}]\boldsymbol{D}_n\boldsymbol{m}$.*

(b) *For $0 \leq j \leq n-1$, the matrix $\mathrm{H}_j := (\mathrm{H}_{uv}) \in \mathrm{Mat}_{r(j+1)}(\bar{k}(t))$, $j+1 \leq u \leq n+1$, $1 \leq v \leq j+1$, formed by the lower left $r(j+1) \times r(j+1)$ square of H represents an element of $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_j M_\rho)$.*

Proof. For part (a), since $f\sigma = \sigma f$, we have $\Phi_\rho \mathrm{F} = \mathrm{F}^{(-1)}\Phi_\rho$. Since multiplication by σ on $\mathrm{P}_n M_\rho$ is represented by $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, the proof of (a) follows from the observation that

$$d_{t,n+1}[\Phi_\rho]d_{t,n+1}[\mathrm{F}] = d_{t,n+1}[\mathrm{F}]^{(-1)} d_{t,n+1}[\Phi_\rho].$$

For part (b), using $d_{t,n+1}[\Phi_\rho]\mathrm{H} = \mathrm{H}^{(-1)} d_{t,n+1}[\Phi_\rho]$ and the definition of d-matrices, we see that, for $0 \leq j \leq n-1$,

$$d_{t,j+1}[\Phi_\rho]\mathrm{H}_j = \mathrm{H}_j^{(-1)} d_{t,j+1}[\Phi_\rho],$$

and the result follows. □

For any $n \geq 1$ and $0 \leq j \leq n-1$, since $\mathrm{P}_{n-j-1}M_\rho$ is a sub-t-motive of $\mathrm{P}_n M_\rho$, we have a surjective map of affine group schemes over $\boldsymbol{k}$,

$$\boldsymbol{\pi}_{n-j-1} : \Gamma_{\mathrm{P}_n M_\rho} \twoheadrightarrow \Gamma_{\mathrm{P}_{n-j-1} M_\rho}. \tag{4.2.4}$$

We are now ready to prove the main result of this subsection.

Theorem 4.2.5. *For each $n \geq 1$ and any $\boldsymbol{k}$-algebra* R, *an element of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form*

$$\mu_n = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} & \gamma_n \\ & \gamma_0 & \gamma_1 & \ddots & \gamma_{n-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \gamma_1 \\ & & & & \gamma_0 \end{pmatrix}, \tag{4.2.6}$$

where, for each $0 \leq i \leq n$, γ_i is an $r \times r$ block. Furthermore, for $0 \leq j \leq n-1$, the matrix μ_{n-j-1} formed by the upper left $r(n-j) \times r(n-j)$ square is an element of $\Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$. In particular, the map $\boldsymbol{\pi}^{(\mathrm{R})}_{n-j-1} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \twoheadrightarrow \Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$ maps an element μ_n of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ to the matrix μ_{n-j-1}.

Proof. Since the prolongation of an $\boldsymbol{A}$-finite dual t-motive is also an $\boldsymbol{A}$-finite dual t-motive, by (2.5.4) for any $n \geq 1$ and $0 \leq j \leq n-1$ we obtain a short exact sequence of t-motives

$$0 \to \mathrm{P}_j M_\rho \xrightarrow{\iota} \mathrm{P}_n M_\rho \xrightarrow{\mathbf{pr}_{n-j-1}} \mathrm{P}_{n-j-1} M_\rho \to 0, \tag{4.2.7}$$

where $\mathbf{pr}_{n-j-1}(D_i m) := D_{i-j-1} m$ for $i > j$ and $\mathbf{pr}_{n-j-1}(D_i m) := 0$ for $i \leq j$ and $m \in M_\rho$, and ι is the inclusion map. Note that $\mathrm{P}_0 M_\rho \cong M_\rho$ via $D_0 m \mapsto m$ for all $m \in M_\rho$.

For any $\boldsymbol{k}$-algebra R, we recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ from [Papanikolas 2008, §4.5]. Since $\Psi_{\mathrm{P}_n \rho} = d_{t,n+1}[\Psi_\rho]$, the entries of $\boldsymbol{u}_n := d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}$ form a $\boldsymbol{k}$-basis of $(\mathrm{P}_n M_\rho)^B$ [loc. cit., Proposition 3.3.9] and similarly, for $0 \leq j \leq n-1$, we have that the entries of $\boldsymbol{u}_{n-j-1} := d_{t,n-j}[\Psi_\rho]^{-1} \boldsymbol{D}_{n-j-1} \boldsymbol{m}$ form a $\boldsymbol{k}$-basis of $(\mathrm{P}_{n-j-1} M_\rho)^B$. For any $\mu_n \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $a_i \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \leq i \leq n$, the action of μ_n on $(a_0, \ldots, a_n) \cdot \boldsymbol{u}_n \in \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ is

$$(a_0, \ldots, a_n) \cdot d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m} \mapsto (a_0, \ldots, a_n) \cdot \mu_n^{-1} d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}. \tag{4.2.8}$$

We first restrict the action of μ_n to $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_j M_\rho)^B$ via the map ι in (4.2.7). So, we take $a_0, \ldots, a_{n-j-1} = 0$ and set $\mu_n^{-1} := (\mathrm{B}_{uw})$, $1 \leq u, w \leq n+1$, where each B_{uw} is an $r \times r$ block. By ι in (4.2.7), we see that μ_n leaves $(\mathrm{P}_j M_\rho)^B$ invariant and thus

$$\mathrm{B}_{n-j+v,1} = \mathrm{B}_{n-j+v,2} = \cdots = \mathrm{B}_{n-j+v,n-j} = \mathbf{0} \quad \text{for } 1 \leq v \leq j+1.$$

Moreover, since the nonzero a_i were chosen arbitrarily, we see that the matrix formed by the lower right $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$. Varying j from 0 to $n-1$, we see that μ_n^{-1} is a block upper triangular matrix and that the matrix formed by the lower right $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$ for each $0 \leq j \leq n-1$.

We return to arbitrary $a_i \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \le i \le n$. We restrict the action of μ_n to $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_{n-j-1} M_\rho)^B$ via the map $\mathbf{pr}_{n-j-1}$ in (4.2.7). Through $\mathbf{pr}_{n-j-1}$, we see that μ_n leaves $(\mathrm{P}_{n-j-1} M_\rho)^B$ invariant and so the matrix μ_{n-j-1} formed by the upper left $r(n-j) \times r(n-j)$ square of μ_n is an element of $\Gamma_{\mathrm{P}_{n-j-1} M_\rho}(\mathrm{R})$. Varying j from 0 to $n-1$, we see that the matrices μ_{n-j-1} formed by the upper left $r(n-j) \times r(n-j)$ square of μ_n is an element of $\Gamma_{\mathrm{P}_{n-j-1} M_\rho}(\mathrm{R})$ for each $0 \le j \le n-1$.

Now, we let $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ be such that for $\mathrm{H} \in \mathrm{Mat}_{r(n+1)}(\bar{k}(t))$ we have $h(\boldsymbol{D}_n \boldsymbol{m}) = \mathrm{H} \boldsymbol{D}_n \boldsymbol{m}$. Let $\mathrm{H} := (\mathrm{H}_{iw})$, where each (H_{iw}) is an $r \times r$ block. For $0 \le j \le n-1$, let $\mathrm{H}_j := (\mathrm{H}_{uv}) \in \mathrm{Mat}_{r(j+1)}(\bar{k}(t))$, $j+1 \le u \le n+1$, $1 \le v \le j+1$, be the matrix formed by the lower left $r(j+1) \times r(j+1)$ square of H. Using the definition of d-matrices, we see that the matrix formed by the lower left $r(j+1) \times r(j+1)$ square of $d_{t,n+1}[\Psi_\rho]^{-1} \mathrm{H}\, d_{t,n+1}[\Psi_\rho]$ is $d_{t,j+1}[\Psi_\rho]^{-1} \mathrm{H}_j\, d_{t,j+1}[\Psi_\rho]$. By Proposition 4.2.3(b), we have that $d_{t,j+1}[\Psi_\rho]^{-1} \mathrm{H}_j\, d_{t,j+1}[\Psi_\rho]$ is an element in the image of the natural embedding (4.2.1) for the j-th prolongation. Thus, by using the commutative diagram (4.2.2) for the n-th and the $(n-1)$-th prolongations, we see that since μ_n is upper triangular, the matrices formed by the lower right $rn \times rn$ square and the upper left $rn \times rn$ square of μ_n are equal. Comparing each $r \times r$ block in this equality, we get the required result. □

4.3. *Lower bound on the dimension of $\Gamma_{\mathrm{P}_n M_\rho}$.* For this subsection, the reader is directed to the Appendix for details about differential algebra and differential algebraic geometry in characteristic $p > 0$. We note that the purpose of the Appendix is for use in this subsection to prove Theorem 4.3.3. For a nonzero prime $\mathfrak{p} \in \boldsymbol{A}$, let $\boldsymbol{A}_\mathfrak{p}$ denote the completion of $\boldsymbol{A}$ at $\mathfrak{p}$, and let $\boldsymbol{k}_\mathfrak{p}$ be the fraction field of $\boldsymbol{A}_\mathfrak{p}$. By the properties of hyperderivatives (see Section 2.4) we see that $(\boldsymbol{k}_\mathfrak{p}, \partial_t)$, where ∂_t represents hyperdifferentiation with respect to t, is a ∂_t-field. Using Theorem 4.2.5, by a slight abuse of notation, we make the choice to let the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$ be

$$\boldsymbol{X} := \begin{pmatrix} \boldsymbol{X}_0 & \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & \boldsymbol{X}_1 \\ & & & \boldsymbol{X}_0 \end{pmatrix}, \tag{4.3.1}$$

where $\boldsymbol{X}_h := ((X_h)_{i,j})$, an $r \times r$ matrix for $0 \le h \le n$. We set $\partial_t^\ell(\boldsymbol{X}_h) := (\partial_t^\ell((X_h)_{i,j}))$ and

$$\mathbf{vec}(\boldsymbol{X}_h) := [(X_h)_{1,1}, \ldots, (X_h)_{r,1}, (X_h)_{1,2}, \ldots, (X_h)_{r,2}, \ldots, (X_h)_{1,r}, \ldots, (X_h)_{r,r}]^{\mathsf{T}},$$

which consists of all entries of $\boldsymbol{X}_h$ lined up in a column vector.

Let $0 \le \alpha \le n$. As in Section A.1, we define $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha\}$ to be the ∂_t-polynomial ring over $\boldsymbol{k}_\mathfrak{p}$ with entries of each $\boldsymbol{X}_h$ for $0 \le h \le \alpha$ as ∂_t-indeterminates. We also define $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha, 1/\det \boldsymbol{X}_0\}$ to be the localization of $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha\}$ at $\det \boldsymbol{X}_0$. We define $\boldsymbol{k}_\mathfrak{p}[\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha]$ to be the usual polynomial ring over $\boldsymbol{k}_\mathfrak{p}$ with entries of each $\boldsymbol{X}_h$ for $h = 0, \ldots, \alpha$ as indeterminates, and $\boldsymbol{k}_\mathfrak{p}[\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha, 1/\det \boldsymbol{X}_0]$ to be the localization of $\boldsymbol{k}_\mathfrak{p}[\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha]$ at $\det \boldsymbol{X}_0$.

We define the centralizer $\mathrm{Cent}_{\mathrm{GL}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ to be the algebraic group over $\boldsymbol{k}$ such that, for any $\boldsymbol{k}$-algebra R,

$$\mathrm{Cent}_{\mathrm{GL}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)(\mathrm{R}) := \{\gamma \in \mathrm{GL}_r(\mathrm{R}) : \gamma g = g\gamma \text{ for all } g \in \mathrm{R} \otimes_{\boldsymbol{k}} \boldsymbol{K}_\rho \subseteq \mathrm{Mat}_r(\mathrm{R})\}.$$

By [Pink 1997, Theorem 0.2] and [Pink and Rütsche 2009, Theorem 0.2], the image $\operatorname{Im}\beta_0$ of the homomorphism β_0 in Theorem 4.1.2 is equal to $\mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})}(\boldsymbol{K}_\rho)$ for all but finitely many primes of $\boldsymbol{A}$. Therefore, let $\mathfrak{p} \in \boldsymbol{A}$ be a nonzero prime such that $\operatorname{Im}\beta_0 = \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})}(\boldsymbol{K}_\rho)$. Then, by [Chang and Papanikolas 2012, Theorem 3.5.4] we see that

$$\Gamma_{M_\rho}(\boldsymbol{A}_\mathfrak{p}) = \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_\mathfrak{p})}(\boldsymbol{K}_\rho) = \operatorname{Im}\beta_0. \tag{4.3.2}$$

Theorem 4.3.3. *Fix $n \geq 1$. Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in* (3.2.2) *and suppose that* $[K_\rho : k] = \boldsymbol{s}$. *Then,*

$$\dim \Gamma_{\mathrm{P}_n M_\rho} \geq (n+1)\frac{r^2}{\boldsymbol{s}}.$$

Proof. By Theorem 4.1.6, we see that the Zariski closure $\overline{\operatorname{Im}\beta_n}^Z$ of $\operatorname{Im}\beta_n$ is an algebraic subgroup of $\Gamma_{\mathrm{P}_n M_\rho}$. Therefore, our task is to prove that $\dim(\overline{\operatorname{Im}\beta_n}^Z) = (n+1)r^2/\boldsymbol{s}$. By [Chang and Papanikolas 2012, Theorem 3.5.4], we have $\Gamma_{M_\rho} = \mathrm{Cent}_{\mathrm{GL}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ and $\dim \Gamma_{M_\rho} = r^2/\boldsymbol{s}$. Since the defining polynomials of $\mathrm{Cent}_{\mathrm{Mat}_r(\boldsymbol{k})}(\boldsymbol{K}_\rho) = \operatorname{Lie}\Gamma_{M_\rho}$ are homogeneous degree-1 polynomials, let its defining equations be

$$\sum_{i,j=1}^{r} (b_u)_{i,j}(X_0)_{i,j} = 0, \quad (b_u)_{i,j} \in \boldsymbol{k},\ u = 1, \ldots, r^2 - r^2/\boldsymbol{s}, \tag{4.3.4}$$

which can be written as

$$\boldsymbol{B} \cdot \mathbf{vec}(\boldsymbol{X}_0) = \mathbf{0}, \tag{4.3.5}$$

where we set $\boldsymbol{B}$ to be the $(r^2 - r^2/\boldsymbol{s}) \times r^2$ matrix of full rank with $(b_u)_{ij}$ as the $u \times ((j-1)r+i)$-th entry. We see that $\operatorname{rank}\boldsymbol{B} = r^2 - \dim\Gamma_{M_\rho} = r^2 - r^2/\boldsymbol{s}$. Therefore, the defining ideal of Γ_{M_ρ} is the ideal generated by the entries of $\boldsymbol{B} \cdot \mathbf{vec}(\boldsymbol{X}_0)$ in $\boldsymbol{k}[\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0]$, the coordinate ring of $\mathrm{GL}_r/\boldsymbol{k}$.

For $0 \leq \alpha \leq n$, we define a monomial order on $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha\}$ and use the division algorithm [Iima and Yoshino 2009, Proposition 1.9] on it. We denote by $\mathbb{Z}_{\geq 0}^{(\infty)}$ the set of all sequences $(a_1, a_2, a_3, \ldots)$ of nonnegative integers such that $a_i = 0$ for all but finitely many $i \geq 1$. Any monomial in $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha\}$ can be described uniquely as $\boldsymbol{X}^{\boldsymbol{b}} = \prod \partial_t^\ell((X_h)_{i,j})^{(b_{h,\ell})_{i,j}}$ for some

$$\boldsymbol{b} = (\boldsymbol{b}_{0,0}, \boldsymbol{b}_{0,1} \ldots, \boldsymbol{b}_{1,0}, \boldsymbol{b}_{1,1}, \ldots, \boldsymbol{b}_{\alpha,0}, \boldsymbol{b}_{\alpha,1}, \ldots) \in \mathbb{Z}_{\geq 0}^{(\infty)},$$

where

$$\boldsymbol{b}_{h,\ell} = \mathbf{vec}(((b_{h,\ell})_{i,j}))^{\mathsf{T}} = [(b_{h,\ell})_{1,1}, \ldots, (b_{h,\ell})_{r,1}, (b_{h,\ell})_{1,2}, \ldots, (b_{h,\ell})_{r,2}, \ldots, (b_{h,\ell})_{1,r}, \ldots, (b_{h,\ell})_{r,r}]$$

for $0 \leq h \leq \alpha$ and $\ell \in \mathbb{Z}_{\geq 0}$ such that $((b_{h,\ell})_{i,j})$ is an $r \times r$ matrix and $(b_{h,\ell})_{i.j} = 0$ for all but a finite number of h, ℓ, i, j. We define a monomial order on $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, \ldots, \boldsymbol{X}_\alpha\}$ as in [loc. cit., Definition 1.1] in the following way:

- we set $\partial_t^\ell((X_h)_{1,1}) < \cdots < \partial_t^\ell((X_h)_{r,1}) < \cdots < \partial_t^\ell((X_h)_{1,r}) < \cdots < \partial_t^\ell((X_h)_{r,r})$,
- we set $\partial_t^\ell((X_h)_{i_1,j_1}) < \partial_t^{\ell+1}((X_h)_{i_2,j_2})$,

- we set $\partial_t^{\ell_1}((X_h)_{i_1,j_1}) < \partial_t^{\ell_2}((X_{h+1})_{i_2,j_2})$,
- we take the pure lexicographic order defined such that $X^{\boldsymbol{b}} < X^{\boldsymbol{c}}$ if the leftmost nonzero component of $\boldsymbol{b}-\boldsymbol{c}$ is negative,

where $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{(\infty)}$, $\ell, \ell_1, \ell_2 \in \mathbb{Z}_{\geq 0}$, $i, j, i_1, i_2, j_1, j_2 \in \{0, \dots, r\}$ and $0 \leq h \leq \alpha$.

Let $\mathfrak{I}(\operatorname{Im}\beta_0)$ denote the defining $\boldsymbol{k}_\mathfrak{p}$-$\partial_t$-ideal of $\operatorname{Im}\beta_0$ in $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0\}$, and let $\mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$ denote the ∂_t-ideal in $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0\}$ generated by the homogeneous degree-1 polynomials given by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)$. Also, let $\mathfrak{R}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$ denote the radical ∂_t-ideal in $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0\}$ generated by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)$.

Claim 1. *We claim that* $\mathfrak{I}(\operatorname{Im}\beta_0) = \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$.

Proof. By Proposition A.1.6, we have $\mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)) = \mathfrak{R}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$. Clearly, $\mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)) \subseteq \mathfrak{I}(\operatorname{Im}\beta_0)$. To show that $\mathfrak{I}(\operatorname{Im}\beta_0) \subseteq \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$, let $P \in \mathfrak{I}(\operatorname{Im}\beta_0) \subseteq \boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0\}$. Let $(X_0)_{\vartheta_u,\omega_u}$ denote the leading variable of $\sum_{i,j=1}^r (b_u)_{i,j}(X_0)_{i,j}$ for $u = 1, \dots, r^2 - r^2/\boldsymbol{s}$ with respect to the monomial order above. This means that for $\ell > \vartheta_u$, $h > \omega_u$, the coefficients $(b_u)_{\ell h}$ are all 0. Moreover, by clearing denominators, we may assume that each $(b_u)_{i,j} \in \boldsymbol{A}$. Thus, the defining polynomials of $\operatorname{Im}\beta_0$ are now

$$\sum_{i=1}^{\vartheta_u}\sum_{j=1}^{\omega_u}(b_u)_{i,j}(X_0)_{i,j} = 0, \quad (b_u)_{i,j} \in \boldsymbol{A},\ u = 1, \dots, r^2 - r^2/\boldsymbol{s}, \tag{4.3.6}$$

Since the rank of $\boldsymbol{B}$ is full, we may pick $(b_u)_{i,j}$ so that for each $u = 1, \dots, r^2 - r^2/\boldsymbol{s} - 1$

$$(X_0)_{\vartheta_u,\omega_u} < (X_0)_{\vartheta_{u+1},\omega_{u+1}}.$$

By using the division algorithm [Iima and Yoshino 2009, Proposition 1.9], we can write

$$P = \sum_{u=1}^{r^2-r^2/\boldsymbol{s}}\sum_{\ell=0}^{\mu_u}\partial_t^\ell\left(\sum_{i=1}^{\vartheta_u}\sum_{j=1}^{\omega_u}(b_u)_{i,j}(X_0)_{i,j}\right)\cdot z_{\ell,u} + S,$$

where μ_u is the largest number such that $\partial_t^{\mu_u}((X_0)_{\vartheta_u,\omega_u})$ occurs as a variable in P, each $z_{\ell,u}$ is in $\boldsymbol{k}_\mathfrak{p}\{\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0\}$, and the remainder S is an element of $\mathfrak{I}(\operatorname{Im}\beta_0) \setminus \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$. Note that the variables $\partial_t^\ell((X_0)_{\vartheta_u,\omega_u})$ do not occur in S.

Suppose that $S \neq 0$. Then, note that there exist $\alpha \geq 0$ and $m \geq \alpha$ such that

$$S \in \boldsymbol{k}_\mathfrak{p}[\partial_t^\alpha((X_0)_{1,1}), \dots, \partial_t^\alpha((X_0)_{r,r}), \dots, \partial_t^m((X_0)_{1,1}), \dots, \partial_t^m((X_0)_{r,r})],$$

when S is regarded as a usual polynomial in the variables $\{\partial_t^\ell((X_0)_{i,j}) : \alpha \leq \ell \leq m,\ 1 \leq i, j, \leq r\}$ over $\boldsymbol{k}_\mathfrak{p}$. Suppose $\partial_t^\alpha((X_0)_{v_1,v_2})$ for some $1 \leq v_1, v_2 \leq r$ is the smallest, with respect to the above monomial order, among the variables $\partial_t^\alpha((X_0)_{i,j})$ occurring in S. We will show that the coefficients of S as a polynomial in the single variable $\partial_t^\alpha((X_0)_{v_1,v_2})$ over the ring

$$\boldsymbol{k}_\mathfrak{p}[\partial_t^\alpha((X_0)_{\gamma_1,\gamma_2}), \partial_t^\ell((X_0)_{i,j}) : v_1 < \gamma_1 \leq r,\ v_2 < \gamma_2 \leq r,\ 1 \leq i, j \leq r,\ \alpha < \ell \leq m]$$

are in $\mathfrak{I}(\operatorname{Im}\beta_0)$ as well.

We pick $\mathrm{v} > 1$ such that $q^{\mathrm{v}} > m$. Consider $\mathfrak{f} \in \boldsymbol{A}_{\mathfrak{p}}^{q^{\mathrm{v}}}$ of the form

$$\mathfrak{f} = \mathfrak{g} \cdot \prod_{u=1}^{r^2-r^2/s} (b_u)_{\vartheta_u,\omega_u}^{q^{\mathrm{v}}}, \tag{4.3.7}$$

where $\mathfrak{g} \in \boldsymbol{A}_{\mathfrak{p}}^{q^{\mathrm{v}}}$, $\mathfrak{g}|_{t=0} = 0$ and each $(b_u)_{\vartheta_u,\omega_u} \in \boldsymbol{A}$ is the coefficient of $(X_0)_{\vartheta_u,\omega_u}$ in (4.3.6). Note that $\mathfrak{f}|_{t=0} = 0$. Then, for $\alpha \le \ell \le m$ by using the product rule for hyperderivatives and Proposition 2.4.1 we have

$$\partial_t^{\ell}(t^{\alpha} \cdot \mathfrak{f}) = \partial_t^{\ell}(t^{\alpha}) \cdot \mathfrak{f} = \begin{cases} \mathfrak{f} & \text{for } \ell = \alpha, \\ 0 & \text{for } \alpha < \ell \le m. \end{cases}$$

For $\mathfrak{f}$ as in (4.3.7), consider $\mathfrak{G} = (\mathfrak{f}_{i,j}) \in \mathrm{Mat}_r(\boldsymbol{A}_{\mathfrak{p}})$, where we set $\mathfrak{f}_{v_1,v_2} = t^{\alpha} \cdot \mathfrak{f}$ for $(i, j) \ne (v_1, v_2), (\vartheta_u, \omega_u)$, $u = 1, \ldots, r^2 - r^2/s$, we set $\mathfrak{f}_{i,j} = t^{\alpha-1} \cdot \mathfrak{f}$ (or $\mathfrak{f}_{i,j} = 0$ in the case $\alpha = 0$), and finally we pick the entries $\mathfrak{f}_{\vartheta_u,\omega_u} \in \boldsymbol{A}_{\mathfrak{p}}$ for each $u = 1, \ldots, r^2 - r^2/s$ such that

$$(b_u)_{\vartheta_u,\omega_u} \cdot \mathfrak{f}_{\vartheta_u,\omega_u} = -\left(\sum_{i=1}^{\vartheta_u-1} \sum_{j=1}^{\omega_u-1} (b_u)_{i,j} \cdot \mathfrak{f}_{i,j} \right).$$

Note that each $\mathfrak{f}_{\vartheta_u,\omega_u}|_{t=0} = 0$. Then, $\mathfrak{G}$ satisfies (4.3.5), that is,

$$\boldsymbol{B} \cdot \mathbf{vec}(\mathfrak{G}) = 0.$$

Since $\mathfrak{f}_{i,j}|_{t=0} = 0$ for all $1 \le i, j \le r$, for any $\mathfrak{C} \in \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})}(\mathbf{K}_{\rho}) = \mathrm{Im}\,\beta_0$, we see that $\mathfrak{C} + \mathfrak{G} \in \mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})$. Moreover, $\mathfrak{C} + \mathfrak{G}$ satisfies $\boldsymbol{B} \cdot \mathbf{vec}(\mathfrak{C} + \mathfrak{G}) = 0$, and so

$$\mathfrak{C} + \mathfrak{G} = (\mathfrak{e}_{i,j}) \in \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})}(\mathbf{K}_{\rho}) = \mathrm{Im}\,\beta_0. \tag{4.3.8}$$

To prove $S = 0$, we adapt an argument of Maurischat [2022b, Corollary 6.4]. For any $\mathfrak{C} = (\mathfrak{c}_{i,j}) \in \mathrm{Cent}_{\mathrm{GL}_r(\boldsymbol{A}_{\mathfrak{p}})}(\mathbf{K}_{\rho}) = \mathrm{Im}\,\beta_0$, consider the polynomial $W_{\mathfrak{C}}(Y) \in \boldsymbol{k}_{\mathfrak{p}}[Y]$ created from S by making the following assignments to the variables:

$$\begin{aligned} \partial_t^{\alpha}((X_0)_{v_1,v_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2}) + Y, \\ \partial_t^{\alpha}((X_0)_{\gamma_1,\gamma_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2}), \\ \partial_t^{\ell}((X_0)_{i,j}) &= \partial_t^{\ell}(\mathfrak{c}_{i,j}) \end{aligned}$$

for $v_1 < \gamma_1 \le r$, $v_2 < \gamma_2 \le r$, $1 \le i, j \le r$, and $\alpha < \ell \le m$. Note that for $\mathfrak{C} + \mathfrak{G}$ in (4.3.8)

$$\begin{aligned} \partial_t^{\alpha}(\mathfrak{e}_{v_1,v_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2} + t^{\alpha} \cdot \mathfrak{f}) = \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2}) + \mathfrak{f}, \\ \partial_t^{\alpha}(\mathfrak{e}_{\gamma_1,\gamma_2}) &= \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2} + t^{\alpha-1} \cdot \mathfrak{f}) = \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2}) \end{aligned}$$

for $v_1 < \gamma_1 \le r$, $v_2 < \gamma_2 \le r$, and

$$\partial_t^{\ell}(\mathfrak{e}_{i,j}) = \partial_t^{\ell}(\mathfrak{c}_{i,j} + t^{\alpha-1} \cdot \mathfrak{f}) = \partial_t^{\ell}(\mathfrak{c}_{i,j})$$

for $\alpha < \ell \le m$ and $1 \le i, j \le r$ such that $(i, j) \ne (\vartheta_u, \omega_u)$, where $u = 1, \ldots, r^2 - r^2/s$. Thus, since the variables $\partial_t^{\ell}((X_0)_{\vartheta_u,\omega_u})$ do not occur in S, we see that $W_{\mathfrak{C}}(\mathfrak{f})$ is equal to the evaluation of S at the element $\mathfrak{C} + \mathfrak{G} \in \mathrm{Im}\,\beta_0$ and so,

$$W_{\mathfrak{C}}(\mathfrak{f}) = 0.$$

This implies that, for all $\mathfrak{C} \in \mathfrak{I}(\operatorname{Im}\beta_0)$, the single variable polynomial $W_{\mathfrak{C}}(Y)$ has infinitely many solutions $\mathfrak{f} \in \boldsymbol{A}_{\mathfrak{p}}^{q^{\vee}}$ of the form (4.3.7) and so $W_{\mathfrak{C}}(Y)$ is identically 0. Note that $W_{\mathfrak{C}}(\partial_t^{\alpha}((X_0)_{v_1,v_2}) - \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2}))$ is simply the polynomial in the variable $\partial_t^{\alpha}((X_0)_{v_1,v_2})$ obtained from S by letting

$$\partial_t^{\alpha}((X_0)_{\gamma_1,\gamma_2}) = \partial_t^{\alpha}(\mathfrak{c}_{\gamma_1,\gamma_2}), \quad \partial_t^{\ell}((X_0)_{i,j}) = \partial_t^{\ell}(\mathfrak{c}_{i,j})$$

for $v_1 < \gamma_1 \le r,\ v_2 < \gamma_2 \le r,\ 1 \le i, j \le r$, and $\alpha < \ell \le m$. Since, for all $\mathfrak{C} \in \operatorname{Im}\beta_0$,

$$0 = W_{\mathfrak{C}}(\partial_t^{\alpha}((X_0)_{v_1,v_2}) - \partial_t^{\alpha}(\mathfrak{c}_{v_1,v_2})),$$

this implies that the coefficients of $\partial_t^{\alpha}((X_0)_{v_1,v_2})$ in the polynomial S also lie in $\mathfrak{I}(\operatorname{Im}\beta_0)$. If S' denotes such a coefficient and if $\partial_t^{\varkappa}((X_0)_{a_1,a_2})$ is the smallest variable with respect to the monomial order above occurring in S', then applying to S' the same process above, the coefficients of $\partial_t^{\varkappa}((X_0)_{a_1,a_2})$ in the polynomial S' also lie in $\mathfrak{I}(\operatorname{Im}\beta_0)$. Continuing like this, there is a nonzero element of $\boldsymbol{k}_{\mathfrak{p}}$ which is an element of $\mathfrak{I}(\operatorname{Im}\beta_0)$, which gives a contradiction to $\operatorname{Im}\beta_0 \ne \varnothing$. Thus, $S = 0$. □

Set

$$T := \mathfrak{R}\big(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0), \mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1)), \mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))\big)$$

to be the radical ∂_t-ideal in $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0\}$ that is generated by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)$, $\mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1))$, $\mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))$, which are homogeneous degree-1 ∂_t-polynomials. Then, Proposition A.1.6 implies

$$T = \mathfrak{D}\big(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0), \mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1)), \mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))\big), \qquad (4.3.9)$$

the ∂_t-ideal in $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0\}$ generated by the set of homogeneous degree-1 ∂_t-polynomials given by the entries of $\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)$, $\mathbf{vec}(\partial_t^1(\boldsymbol{X}_0) - (\boldsymbol{X}_1))$, $\mathbf{vec}(\partial_t^2(\boldsymbol{X}_0) - (\boldsymbol{X}_2)), \dots, \mathbf{vec}(\partial_t^n(\boldsymbol{X}_0) - (\boldsymbol{X}_n))$.

Let $\mathfrak{I}(\operatorname{Im}\beta_n)$ denote the defining $\boldsymbol{k}_{\mathfrak{p}}$-$\partial_t$-ideal of $\operatorname{Im}\beta_n$ in $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0\}$.

Claim 2. *We claim that $T = \mathfrak{I}(\operatorname{Im}\beta_n)$.*

Proof of Claim 2. By Theorem 4.1.6, clearly $T \subseteq \mathfrak{I}(\operatorname{Im}\beta_n)$. To show $\mathfrak{I}(\operatorname{Im}\beta_n) \subseteq T$, let $F \in \mathfrak{I}(\operatorname{Im}\beta_n) \subseteq \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0\}$. Note that for $1 \le h \le n$, we have $\partial_t^{\ell}(\partial_t^h((X_0)_{i,j})) < \partial_t^{\ell}((X_h)_{i,j})$ and so the leading monomial of each $\partial_t^{\ell}(\partial_t^h((X_0)_{i,j}) - (X_h)_{i,j})$ is $\partial_t^{\ell}((X_h)_{i,j})$. Then, by using the division algorithm [Iima and Yoshino 2009, Proposition 1.9] we see that

$$F = \sum_{i,j=1}^{r} \sum_{h=1}^{n} \sum_{\ell=0}^{m_{h,i,j}} \partial_t^{\ell}\big(\partial_t^h((X_0)_{i,j}) - (X_h)_{i,j}\big)\cdot(w_{h,\ell})_{i,j} + H, \qquad (4.3.10)$$

where $m_{h,i,j}$ is the largest number such that $\partial_t^{m_{h,i,j}}((X_h)_{i,j})$ occurs as a variable in F, each $(w_{h,\ell})_{i,j} \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0\}$, and the remainder $H = H(\boldsymbol{X}_0)$ is an element of $\boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0\}$. Note that for g_{ϵ}, $\operatorname{Im}\beta_n$, and $\operatorname{Im}\beta_0$ as in Theorem 4.1.6, there is a surjective map

$$\operatorname{Im}\beta_n \twoheadrightarrow \operatorname{Im}\beta_0$$

given by $d_{t,n+1}[g_\epsilon] \mapsto g_\epsilon$. Moreover we have $F(d_{t,n+1}[g_\epsilon]) = 0$. Since $T \subseteq \mathfrak{I}(\operatorname{Im}\beta_n)$ and

$$\sum_{i,j=1}^{r}\sum_{h=1}^{n}\sum_{\ell=0}^{m_{h,i,j}} \partial_t^\ell\big(\partial_t^h((X_0)_{i,j}) - (X_h)_{i,j}\big)\cdot(w_{h,\ell})_{i,j} \in T,$$

we obtain from (4.3.10) that $H(g_\epsilon) = 0$. Thus, $H(\boldsymbol{X}_0)$ is an element of $\mathfrak{I}(\operatorname{Im}\beta_0) = \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0))$. This proves our claim. Therefore, $\mathfrak{I}(\operatorname{Im}\beta_n) = T$. □

We are now ready to compute $\overline{\operatorname{Im}\beta_n}^Z$.

Claim 3. *The defining equations of* $\overline{\operatorname{Im}\beta_n}^Z$ *are given by*

$$d_{t,n+1}[\boldsymbol{B}]\cdot\mathbf{vec}([\boldsymbol{X}_n,\dots,\boldsymbol{X}_0]^{\mathsf{T}}) = \mathbf{0}. \tag{4.3.11}$$

Proof of Claim 3. Based on Lemma A.1.5, we can find the defining equations of $\overline{\operatorname{Im}\beta_n}^Z$ if we determine

$$\boldsymbol{T} := T \cap \boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0, \boldsymbol{X}_1, \dots, \boldsymbol{X}_n, 1/\det\boldsymbol{X}_0]. \tag{4.3.12}$$

By the preceding arguments, an element of $F \in T = \mathfrak{I}(\overline{\operatorname{Im}\beta_n}^\partial)$ is of the form (4.3.10), where $H \in \mathfrak{D}(\boldsymbol{B}\cdot\mathbf{vec}(\boldsymbol{X}_0)) \subseteq \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$. Suppose

$$H = \sum_{i,j=1}^{r}\sum_{u=1}^{r^2-r^2/s}\sum_{\varkappa=0}^{v_u} c_{u,\varkappa}\cdot\partial_t^\varkappa((b_u)_{i,j}(X_0)_{i,j}),$$

where $v_u \ge 0$ and $c_{u,\varkappa} \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$ for each $1 \le u \le r^2 - r^2/s$. By the product rule of hyperderivatives, we have $\partial_t^\varkappa((b_u)_{i,j}(X_0)_{i,j}) = \sum_{\alpha=0}^{\varkappa}\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^\alpha((X_0)_{i,j})$, and so rewriting F we have

$$F = \sum_{i,j=1}^{r}\Bigg(\sum_{h=1}^{n}\sum_{\ell=0}^{m_{h,i,j}}(w_{h,\ell})_{i,j}\cdot\partial_t^\ell\big(\partial_t^h((X_0)_{i,j}) - (X_h)_{i,j}\big) \\ + \sum_{u=1}^{r^2-r^2/s}\Bigg(\sum_{\varkappa=0}^{v_u} c_{u,\varkappa}\cdot\partial_t^\varkappa((b_u)_{i,j})\cdot(X_0)_{i,j} + \sum_{\varkappa=1}^{v_u}\sum_{\alpha=1}^{\varkappa} c_{u,\varkappa}\cdot\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^\alpha((X_0)_{i,j})\Bigg)\Bigg),$$

where $(w_{h,\ell})_{i,j} \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0,\dots,\boldsymbol{X}_n, 1/\det\boldsymbol{X}_0\}$ and $m_{h,i,j} \in \mathbb{Z}_{\ge 0}$ for $1 \le h \le n,\ 1 \le i, j \le r$.

Suppose that $F \in \boldsymbol{T} \subseteq \boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0,\dots,\boldsymbol{X}_n, 1/\det\boldsymbol{X}_0]$. Then, since $H \in \boldsymbol{k}_{\mathfrak{p}}\{\boldsymbol{X}_0, 1/\det\boldsymbol{X}_0\}$, we obtain

$$m_{h,i,j} = 0.$$

Additionally, for each $1 \le i, j \le r$ we have

$$\sum_{u=1}^{r^2-r^2/s}\sum_{\varkappa=1}^{v_u}\sum_{\alpha=1}^{\varkappa} c_{u,\varkappa}\cdot\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^\alpha((X_0)_{i,j}) + \sum_{h=1}^{n}(w_{h,0})_{i,j}\cdot\partial_t^h((X_0)_{i,j}) = 0.$$

From this, we see that $v_u \le n$, and for $h > v_u$ we have $(w_{h,0})_{i,j} = 0$. Moreover, since

$$\sum_{\varkappa=1}^{v_u}\sum_{\alpha=1}^{\varkappa} c_{u,\varkappa}\cdot\partial_t^{\varkappa-\alpha}((b_u)_{i,j})\cdot\partial_t^\alpha((X_0)_{i,j}) = \sum_{h=1}^{v_u}\sum_{\gamma=h}^{v_u} c_{u,\gamma}\cdot\partial_t^{\gamma-h}((b_u)_{i,j})\cdot\partial_t^h((X_0)_{i,j}),$$

we have, for $1 \le h \le v_u$,

$$(w_{h,0})_{i,j} = -\sum_{u=1}^{r^2-r^2/s}\sum_{\gamma=h}^{v_u} c_{u,\gamma}\cdot \partial_t^{\gamma-h}((b_u)_{i,j}).$$

Thus, $F \in \boldsymbol{T}$ is of the form

$$\begin{aligned} F &= \sum_{i,j=1}^{r}\sum_{u=1}^{r^2-r^2/s}\left(\sum_{\varkappa=0}^{v_u} c_{u,\varkappa}\cdot \partial_t^{\varkappa}((b_u)_{i,j})\cdot (X_0)_{i,j} + \sum_{h=1}^{v_u}\sum_{\gamma=h}^{v_u} c_{u,\gamma}\cdot \partial_t^{\gamma-h}((b_u)_{i,j})\cdot (X_h)_{i,j}\right) \\ &= \sum_{u=1}^{r^2-r^2/s}\left(\sum_{\varkappa=0}^{v_u} c_{u,\varkappa}\cdot \partial_t^{\varkappa}(B_u)\cdot \mathbf{vec}(\boldsymbol{X}_0) + \sum_{h=1}^{v_u}\sum_{\gamma=h}^{v_u} c_{u,\gamma}\cdot \partial_t^{\gamma-h}(B_u)\cdot \mathbf{vec}(\boldsymbol{X}_h)\right), \end{aligned}$$

where B_u is the u-th row of $\boldsymbol{B}$ and each $c_{u,\varkappa} \in \boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0, 1/\det \boldsymbol{X}_0]$. Varying u from 1 to $r^2 - r^2/\boldsymbol{s}$ and varying each v_u from 0 to n, we see that the ideal in $\boldsymbol{k}_{\mathfrak{p}}[\boldsymbol{X}_0, \dots, \boldsymbol{X}_n, 1/\det \boldsymbol{X}_0]$ generated by (4.3.12) is the same as the ideal generated by

$$\left\{\sum_{h=0}^{n} \partial_t^{n-h}(B_u)\cdot \mathbf{vec}(\boldsymbol{X}_h), \quad u = 1, \dots, r^2 - r^2/\boldsymbol{s}\right\},$$

which can be written as

$$d_{t,n+1}[\boldsymbol{B}]\cdot \mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}),$$

where we define $\mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}) := [(\mathbf{vec}\,\boldsymbol{X}_n)^{\mathsf{T}}, \dots, (\mathbf{vec}\,\boldsymbol{X}_0)^{\mathsf{T}}]^{\mathsf{T}}$. Since, by its definition, $d_{t,n+1}[\boldsymbol{B}]$ is a block upper triangular matrix with all diagonal blocks equal to $\boldsymbol{B}$, we have that

$$\operatorname{rank} d_{t,n+1}[\boldsymbol{B}] \ge (n+1)\cdot \operatorname{rank} \boldsymbol{B} = (n+1)\cdot (r^2 - r^2/\boldsymbol{s}).$$

Also, since $d_{t,n+1}[\boldsymbol{B}]$ is an $(n+1)\cdot(r^2 - r^2/\boldsymbol{s}) \times (n+1)\cdot r^2$ matrix, we have that $\operatorname{rank} d_{t,n+1}[\boldsymbol{B}] \le (n+1)\cdot(r^2 - r^2/\boldsymbol{s})$ and so $\operatorname{rank} d_{t,n+1}[\boldsymbol{B}] = (n+1)\cdot(r^2 - r^2/\boldsymbol{s})$. Since $\operatorname{rank} d_{t,n+1}[\boldsymbol{B}]$ is full, we see that

$$d_{t,n+1}[\boldsymbol{B}]\cdot \mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}) = \mathbf{0}$$

are the defining equations of $\overline{\operatorname{Im}\beta_n}^Z$. □

Since each $(b_u)_{ij}$ is an element of $\boldsymbol{k}$, we see that each entry of $d_{t,n+1}[\boldsymbol{B}]$ is an element of $\boldsymbol{k}$ and so, $\overline{\operatorname{Im}\beta_n}^Z$ is defined over $\boldsymbol{k}$. Moreover,

$$\dim \overline{\operatorname{Im}\beta_n}^Z = (n+1)\cdot r^2 - \operatorname{rank} d_{t,n+1}[\boldsymbol{B}] = (n+1)\cdot r^2 - (n+1)\cdot(r^2 - r^2/\boldsymbol{s}) = (n+1)\cdot r^2/\boldsymbol{s}, \quad (4.3.13)$$

which gives the desired result. □

4.4. ***Upper bound on the dimension of*** $\boldsymbol{\Gamma_{\mathrm{P}_n M_\rho}}$. Recall from Theorem 4.2.5 that for any $\boldsymbol{k}$-algebra R and $n \ge 1$, an element μ_n of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form as in (4.2.6).

Note that by (4.2.4), we have a short exact sequence of affine group schemes over $\boldsymbol{k}$,

$$1 \to Q_n \to \Gamma_{\mathrm{P}_n M_\rho} \xrightarrow{\pi_{n-1}} \Gamma_{\mathrm{P}_{n-1} M_\rho} \to 1, \quad (4.4.1)$$

where, by Theorem 4.2.5, $\boldsymbol{\pi}_{n-1}^{(\mathrm{R})} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \twoheadrightarrow \Gamma_{\mathrm{P}_{n-1} M_\rho}(\mathrm{R})$ maps μ_n to the matrix μ_{n-1} formed by the upper left $rn \times rn$ square. Consider

$$\nu = \begin{pmatrix} \mathrm{Id}_r & 0 & \cdots & 0 & \boldsymbol{v} \\ & \mathrm{Id}_r & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & \mathrm{Id}_r \end{pmatrix} \in \mathrm{GL}_{(n+1)r}(\mathrm{R}), \tag{4.4.2}$$

where $\boldsymbol{v} \in \mathrm{Mat}_r(\mathrm{R})$. Then, an element of $Q_n(\mathrm{R})$ is of the form (4.4.2). It can easily be checked that

$$\mu_n \nu \mu_n^{-1} = \begin{pmatrix} \mathrm{Id}_r & 0 & \cdots & 0 & \gamma_0 \boldsymbol{v} \gamma_0^{-1} \\ & \mathrm{Id}_r & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & \mathrm{Id}_r \end{pmatrix}. \tag{4.4.3}$$

Note that $\mathrm{P}_0 M_\rho$ is simply M_ρ via the map $D_0 m \mapsto m$ for all $m \in M_\rho$ and so M_ρ is a sub-t-motive of $\mathrm{P}_n M_\rho$. Thus, similarly, by (4.2.4) there is a surjective map of affine group schemes over $\boldsymbol{k}$,

$$\boldsymbol{\pi}_0 : \Gamma_{\mathrm{P}_n M_\rho} \twoheadrightarrow \Gamma_{M_\rho},$$

where, by Theorem 4.2.5, $\boldsymbol{\pi}_0^{(\mathrm{R})} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \to \Gamma_{M_\rho}(\mathrm{R})$ is the map given by $\mu_n \mapsto \gamma_0$. Thus, via conjugation there is a left action of Γ_{M_ρ} on Q_n given by (4.4.3).

Set $\boldsymbol{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$ and for a $\boldsymbol{k}$-algebra R define

$$\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)(\mathrm{R}) := \{\gamma \in \mathrm{Mat}_r(\mathrm{R}) : \gamma g = g\gamma \text{ for all } g \in \mathrm{R} \otimes_{\boldsymbol{k}} \boldsymbol{K}_\rho \subseteq \mathrm{Mat}_r(\mathrm{R})\}.$$

Lemma 4.4.4. *For $n \geq 1$, let $\nu \in Q_n(\mathrm{R})$ be as in (4.4.2). Then,*

$$\boldsymbol{v} \in \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)(\mathrm{R}).$$

Proof. The entries of $\boldsymbol{u}_n = d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}$ form a $\boldsymbol{k}$-basis of $(\mathrm{P}_n M_\rho)^B$ (see [Papanikolas 2008, Proposition 3.3.9]). Recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ from [loc. cit., §4.5] (see also (4.2.8)) as follows: for any $\mu_n \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $a_i \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \leq i \leq n$, the action of μ_n on $(a_0, \dots, a_n) \cdot \boldsymbol{u}_n \in \mathrm{R} \otimes_{\boldsymbol{k}} (\mathrm{P}_n M_\rho)^B$ is

$$\varpi_n^{\mathrm{R}}(\mu_n) : (a_0, \dots, a_n) \cdot d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m} \mapsto (a_0, \dots, a_n) \cdot \mu_n^{-1} d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}. \tag{4.4.5}$$

Given $f \in \boldsymbol{K}_\rho$, let $\mathrm{F} \in \mathrm{Mat}_r(\bar{k}(t))$ satisfy $f(\boldsymbol{m}) = \mathrm{F}\boldsymbol{m}$. By Proposition 4.2.3(a), for $n \geq 1$ there exists $g \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ such that $g(\boldsymbol{D}_n \boldsymbol{m}) = d_{t,n+1}[\mathrm{F}] \boldsymbol{D}_n \boldsymbol{m}$ and so, $d_{t,n+1}[\Psi_\rho]^{-1} d_{t,n+1}[\mathrm{F}] d_{t,n+1}[\Psi_\rho] = d_{t,n+1}[\Psi_\rho^{-1} \mathrm{F} \Psi_\rho]$ is an element in the image of the natural embedding (4.2.1). Then by (4.4.5) and the commutative diagram (4.2.2), we have

$$d_{t,n+1}[\Psi_\rho^{-1} \mathrm{F} \Psi_\rho] \nu = \nu d_{t,n+1}[\Psi_\rho^{-1} \mathrm{F} \Psi_\rho].$$

This gives

$$\Psi_\rho^{-1}\mathrm{F}\Psi_\rho \boldsymbol{v} = \boldsymbol{v}\Psi_\rho^{-1}\mathrm{F}\Psi_\rho$$

and the desired result follows. □

Theorem 4.4.6. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and, for $n \geq 1$, let $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in (3.2.2) and suppose that $[K_\rho : k] = \boldsymbol{s}$. Then* $\dim \Gamma_{\mathrm{P}_n M_\rho} \leq (n+1)\cdot r^2/\boldsymbol{s}$.

Remark 4.4.7. The author thanks the referee for sharing the ideas of the following proof, which is an improvement on the ideas used in a previous proof the author obtained. The author's previous proof required a lemma proving smoothness of Q_n. This is no longer required and has been removed.

Proof. By Proposition 3.2.3 and Remark 3.2.4, we see that $[\boldsymbol{K}_\rho : \boldsymbol{k}] = \boldsymbol{s}$ and so, $\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ is an additive group scheme of dimension $r^2/\boldsymbol{s}$ over $\boldsymbol{k}$ [Farb and Dennis 1993, Theorem 3.15(3)].

As Q_n is defined as the kernel in (4.4.1), Q_n is a closed subgroup of $\Gamma_{\mathrm{P}_n M_\rho}$. Consider the closed immersion $\mathrm{Mat}_r/\boldsymbol{k} \hookrightarrow \mathrm{GL}_{(n+1)r}/\boldsymbol{k}$ defined by $\boldsymbol{v} \mapsto \nu$, where ν is of the form (4.4.2). Note that $Q_n \subseteq \mathrm{GL}_{(n+1)r}/\boldsymbol{k}$ is isomorphic to its preimage under this closed immersion. Thus, Q_n is closed in $\mathrm{Mat}_r/\boldsymbol{k}$, and hence closed in $\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ by Lemma 4.4.4. This implies that $\dim Q_n \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho) = r^2/\boldsymbol{s}$.

Now, by (4.4.1) our task is to prove that $\dim Q_n + \dim \Gamma_{\mathrm{P}_{n-1}M_\rho} \leq (n+1)\cdot r^2/\boldsymbol{s}$, which we show by induction. For the base case $n = 1$, since $\dim \Gamma_{M_\rho} = r^2/\boldsymbol{s}$ [Chang and Papanikolas 2012, Theorem 3.5.4] we see that $\dim Q_1 + \dim \Gamma_{M_\rho} \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho) + \dim \Gamma_{M_\rho} = 2\cdot r^2/\boldsymbol{s}$. Suppose we have shown that $\dim \Gamma_{\mathrm{P}_{n-1}M_\rho} \leq n\cdot r^2/\boldsymbol{s}$. By the same argument as in the base case, we obtain

$$\dim Q_n + \dim \Gamma_{\mathrm{P}_{n-1}M_\rho} \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho) + \dim \Gamma_{\mathrm{P}_{n-1}M_\rho} = (n+1)\cdot r^2/\boldsymbol{s}.$$ □

Corollary 4.4.8. *Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and, for $n \geq 1$, let $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let $\overline{\mathrm{Im}\,\beta_n}^Z$ be the Zariski closure of $\mathrm{Im}\,\beta_n$, where β_n is as in Theorem 4.1.6. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in (3.2.2) and suppose that $[K_\rho : k] = \boldsymbol{s}$. Then $\dim \Gamma_{\mathrm{P}_n M_\rho} = (n+1)\cdot r^2/\boldsymbol{s}$ and*

$$\overline{\mathrm{Im}\,\beta_n}^Z/\boldsymbol{k} = \Gamma_{\mathrm{P}_n M_\rho}.$$

Proof. We obtain $\dim \Gamma_{\mathrm{P}_n M_\rho} = (n+1)\cdot r^2/\boldsymbol{s}$ by combining Theorems 4.3.3 and 4.4.6. By (4.3.13) we see that $\dim \overline{\mathrm{Im}\,\beta_n}^Z = \dim \Gamma_{\mathrm{P}_n M_\rho}$. Then, since $\Gamma_{\mathrm{P}_n M_\rho}$ is connected and smooth by Theorem 2.3.1(b), we have $\overline{\mathrm{Im}\,\beta_n}^Z/\boldsymbol{k} = \Gamma_{\mathrm{P}_n M_\rho}$. □

Remark 4.4.9. By Corollary 4.4.8, we see $\dim Q_n = \dim \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$. Since the defining polynomials of $\mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$ are degree-1 polynomials, it is connected and smooth. Thus, $Q_n = \mathrm{Cent}_{\mathrm{Mat}_r/\boldsymbol{k}}(\boldsymbol{K}_\rho)$.

4.5. *Algebraic independence of periods and quasiperiods.* The following result proves Theorem 1.1.3.

Theorem 4.5.1. *Fix $n \geq 1$. Let ρ be a Drinfeld $\boldsymbol{A}$-module of rank r defined over k^{sep} and $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let K_ρ be the fraction field of* $\mathrm{End}(\rho)$ *defined as in (3.2.2) and* *suppose that K_ρ is separable over k. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Then*, $\operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi_{\mathrm{P}_n\rho}(\theta)) = (n+1)\cdot r^2/\boldsymbol{s}$, *where* $\boldsymbol{s} = [K_\rho : k]$. *In particular*,

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\Bigg(\bigcup_{s=1}^{n}\bigcup_{i=1}^{r-1}\bigcup_{j=1}^{r}\{\lambda_j,\, F_{\tau^i}(\lambda_j),\, \partial_\theta^s(\lambda_j),\, \partial_\theta^s(F_{\tau^i}(\lambda_j))\}\Bigg) = (n+1)\cdot r^2/\boldsymbol{s}.$$

Proof. By Theorem 2.3.2, we have $\dim \Gamma_{\mathrm{P}_n M_\rho} = \operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi_{\mathrm{P}_n\rho}|_{t=\theta})$. Since $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$, the result follows from Theorem 3.4.1 and Corollary 4.4.8. □

5. Hyperderivatives of logarithms and quasilogarithms

In this section, we prove Theorem 1.1.4 (restated as Theorem 5.4.4) and Corollary 1.1.5. We fix a Drinfeld $\boldsymbol{A}$-module ρ of rank r defined over k^{sep} and an $\boldsymbol{A}$-basis $\{\lambda_1, \dots, \lambda_r\}$ of Λ_ρ as in Section 3.2. Let M_ρ be the t-motive associated to ρ along with a fixed $\bar{k}(t)$-basis $\{m_1, \dots, m_r\} \subseteq M_\rho$, multiplication by σ given by Φ_ρ as in (3.2.1), and rigid analytic trivialization Ψ_ρ as in (3.3.2). For each $n \geq 0$, let $\mathrm{P}_n M_\rho$ be the t-motive corresponding to the n-th prolongation $\mathrm{P}_n\rho$ of ρ as in Section 3.2. Note that $\mathrm{P}_0 M_\rho$ is simply M_ρ via the map $D_0 m \mapsto m$ for all $m \in M_\rho$. If $\boldsymbol{m} = (m_1, \dots, m_r)^{\mathsf{T}}$, then a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$ is given by the entries of $\boldsymbol{D}_n\boldsymbol{m} \in \operatorname{Mat}_{(n+1)r\times 1}(\mathrm{P}_n M_\rho)$ (see (2.5.1)) such that multiplication by σ is given by $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$ (see (2.5.2)) with rigid analytic trivialization $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$ (see (2.5.3)). We also set $\boldsymbol{K}_\rho := \operatorname{End}_{\mathcal{T}}(M_\rho)$ and let K_ρ denote the fraction field of $\operatorname{End}(\rho)$.

5.1. ***t-motives and quasilogarithms.*** Given $u \in \mathbb{K}$ such that $\operatorname{Exp}_\rho(u) = \alpha \in k^{\mathrm{sep}}$, let $f_u(t)$ be the Anderson generating function of ρ with respect to u given as in (3.1.3). Then, for $n \geq 1$, we see that the Anderson generating function of $\mathrm{P}_n\rho$ with respect to $\boldsymbol{u}_n := [u, 0, \dots, 0]^{\mathsf{T}} \in \mathbb{K}^{n+1}$ is $\mathcal{G}_{\boldsymbol{u}_n}(t) = [f_u(t), \partial_t^1(f_u(t)), \dots, \partial_t^n(f_u(t))]^{\mathsf{T}}$ (see (4.1.5)). Moreover, by (4.1.3),

$$\operatorname{Exp}_{\mathrm{P}_n\rho}(\boldsymbol{u}_n) = [\operatorname{Exp}_\rho(u), 0, \dots, 0]^{\mathsf{T}} = [\alpha, 0, \dots, 0]^{\mathsf{T}} \in (k^{\mathrm{sep}})^{n+1}.$$

We define

$$\boldsymbol{s}_\alpha := \begin{pmatrix} -(\kappa_1 f_u^{(1)}(t) + \cdots + \kappa_{r-1} f_u^{(r-1)}(t) + \kappa_r f_u^{(r)}(t)) \\ -(\kappa_2^{(-1)} f_u^{(1)}(t) + \cdots + \kappa_{r-1}^{(-1)} f_u^{(r-2)}(t) + \kappa_r^{(-1)} f_u^{(r-1)}(t)) \\ -(\kappa_3^{(-2)} f_u^{(1)}(t) + \cdots + \kappa_{r-1}^{(-2)} f_u^{(r-3)}(t) + \kappa_r^{(-2)} f_u^{(r-2)}(t)) \\ \vdots \\ -\kappa_r^{(-r+1)} f_u^{(1)}(t) \end{pmatrix}^{\mathsf{T}} \in \operatorname{Mat}_{1\times r}(\mathbb{T}),$$

and let $\boldsymbol{h}_{\alpha,n} := (\alpha, 0, \dots, 0) \in \operatorname{Mat}_{1\times(n+1)r}(k^{\mathrm{sep}})$. Let F_δ be the quasiperiodic function associated to ρ-biderivation δ, where $\delta_t = \kappa_1\tau + \cdots + \kappa_{r-1}\tau^{r-1} + \kappa_r\tau^r = \rho_t - \theta$. Then, by [Brownawell and Papanikolas 2002, Proposition 3.2.2] (see also [Namoijam and Papanikolas 2024, Proposition 4.3.5(a)]) we obtain

$$-u + \alpha = \mathrm{F}_\delta(u) = \kappa_1 f_u^{(1)}(\theta) + \cdots + \kappa_{r-1} f_u^{(r-1)}(\theta) + \kappa_r f_u^{(r)}(\theta). \tag{5.1.1}$$

We now define the pre-t-motive $Y_{\alpha,n}$ of dimension $(n+1)r+1$ over $\bar{k}(t)$ such that multiplication by σ is given by

$$\Phi_{\alpha,n} := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{h}_{\alpha,n} & 1 \end{pmatrix} \in \mathrm{Mat}_{(n+1)r+1}(\bar{k}[t]).$$

If we set $\boldsymbol{g}_{\alpha,n} := (\boldsymbol{s}_\alpha, \partial_t^1(\boldsymbol{s}_\alpha), \dots, \partial_t^n(\boldsymbol{s}_\alpha))$, where the hyperderivatives are taken entrywise, then we have $\boldsymbol{g}_{\alpha,n}^{(-1)} \Phi_{\mathrm{P}_n\rho} = \boldsymbol{g}_{\alpha,n} + \boldsymbol{h}_{\alpha,n}$. We set

$$\Psi_{\alpha,n} := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{g}_{\alpha,n}\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix} \in \mathrm{Mat}_{(n+1)r+1}(\mathbb{T})$$

to obtain $\Psi_{\alpha,n}^{(-1)} = \Phi_{\alpha,n}\Psi_{\alpha,n}$. Thus, $Y_{\alpha,n}$ is rigid analytically trivial. The reader may consult [Namoijam and Papanikolas 2024, Lemma 5.65] for motivation behind the construction of $\boldsymbol{g}_{\alpha,n}$ and $\boldsymbol{h}_{\alpha,n}$.

Proposition 5.1.2 (cf. [Papanikolas 2008, Proposition 6.1.3]). *The pre-t-motive $Y_{\alpha,n}$ is a t-motive.*

Proof. Set $\mathcal{N} := \mathrm{Mat}_{1\times(n+1)r+1}(\bar{k}[t])$ and let $\boldsymbol{e} := [\boldsymbol{e}_1, \dots, \boldsymbol{e}_{(n+1)r+1}]^{\mathsf{T}}$ be its standard $\bar{k}[t]$-basis. We give $\mathcal{N}$ a left $\bar{k}[t,\sigma]$-module structure by setting $\sigma\boldsymbol{e} = (t-\theta)\Phi_{\alpha,n}\boldsymbol{e}$. Let $\mathcal{C}$ be the $\boldsymbol{A}$-finite dual t-motive associated to the Carlitz module $\mathfrak{C}$ (rank-1 Drinfeld $\boldsymbol{A}$-module) given by $\mathfrak{C}_t = \theta + \tau$ and let $C := \bar{k}(t)\otimes_{\bar{k}[t]}\mathcal{C}$ be the corresponding pre-t-motive. We obtain the following short exact sequence of $\bar{k}[t,\sigma]$-modules:

$$0 \to \mathcal{C} \otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho \to \mathcal{N} \to \mathcal{C} \to 0. \tag{5.1.3}$$

Since $\mathcal{C}$ and $\mathcal{C}\otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ are finitely generated left $\bar{k}[\sigma]$-modules, it follows from [Anderson et al. 2004, Proposition 4.3.2] that $\mathcal{N}$ is free and finitely generated as a left $\bar{k}[\sigma]$-module. Since $\mathcal{C}\otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ is an $\boldsymbol{A}$-finite dual t-motive, we have

$$(t-\theta)^{v_1}(\mathcal{C}\otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho) \subseteq \sigma(\mathcal{C}\otimes_{\bar{k}[t]} \mathrm{P}_n\mathcal{M}_\rho)$$

for some $v_1 \in \mathbb{N}$. Moreover, $(t-\theta)\mathcal{C} = \sigma\mathcal{C}$ and so, by (5.1.3) we obtain $(t-\theta)^{v_2}\mathcal{N} \subseteq \sigma\mathcal{N}$ for $v_2 \in \mathbb{N}$ sufficiently large. Thus, we see that $\mathcal{N}$ is an $\boldsymbol{A}$-finite dual t-motive. Then, it follows from the discussion in [Papanikolas 2008, §3.4.10] that $Y_{\alpha,n}$ is a t-motive. □

5.2. *Nontriviality in* $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. We continue with the t-motive $Y_{\alpha,n}$ from the previous subsection. Let $\mathbf{1}$ denote the trivial object of the category $\mathcal{T}$ from Section 2.2. Note that $Y_{\alpha,n}$ represents a class in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. Suppose $e \in \mathrm{End}_{\mathcal{T}}(M_\rho)$ and let $\mathrm{E} \in \mathrm{Mat}_r(\bar{k}(t))$ be such that $e(\boldsymbol{m}) = \mathrm{E}\boldsymbol{m}$. If we set

$$\boldsymbol{E} := \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathrm{E} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{(n+1)r}(\bar{k}(t)), \tag{5.2.1}$$

then one checks easily that $\boldsymbol{E}$ represents an element $\boldsymbol{e}$ of $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$. For classes Y_1 and Y_2 in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$, if multiplication by σ on suitable $\bar{k}(t)$-bases are represented by

$$\begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_2 & 1 \end{pmatrix}$$

respectively, then their Baer sum in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$ is achieved by the matrix

$$\begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_1 + \boldsymbol{v}_2 & 1 \end{pmatrix}.$$

Moreover, we see that multiplication by σ on a $\bar{k}(t)$-basis of the pushout $\boldsymbol{e}_* Y_1$ is represented by

$$\begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \boldsymbol{v}_1 \boldsymbol{E} & 1 \end{pmatrix}.$$

Note that if $[K_\rho : k] = \boldsymbol{s}$, then $\{\lambda_1, \dots, \lambda_r\}$ span a K_ρ-vector space of dimension $r/\boldsymbol{s}$.

Theorem 5.2.2. *Suppose $u_1, \dots, u_w \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$ and $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w) = r/\boldsymbol{s} + w$, where $[K_\rho : k] = \boldsymbol{s}$. For $n \ge 1$, we let $Y_{i,n} := Y_{\alpha_i,n}$ be defined as in Section 5.1. Then, for $e_1, \dots, e_w \in \boldsymbol{K}_\rho$, not all zero, $S := \boldsymbol{e}_{1*} Y_{1,n} + \cdots + \boldsymbol{e}_{w*} Y_{w,n}$ is nontrivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$, where each $\boldsymbol{e}_i \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ corresponds to e_i as in (5.2.1).*

Proof. We adapt the ideas of the proof of [Chang and Papanikolas 2012, Theorem 4.2.2]. For each $1 \le i \le w$, we let $\boldsymbol{h}_{i,n} := \boldsymbol{h}_{\alpha_i,n}$ and $\boldsymbol{g}_{i,n} := \boldsymbol{g}_{\alpha_i,n}$. Fix $\mathrm{E}_i \in \mathrm{Mat}_r(\bar{k}(t))$ so that $e_i(\boldsymbol{m}) = \mathrm{E}_i \boldsymbol{m}$ for each $1 \le i \le w$. Then $\boldsymbol{e}_i(\boldsymbol{D}_n \boldsymbol{m}) = \boldsymbol{E}_i \cdot \boldsymbol{D}_n \boldsymbol{m}$, where $\boldsymbol{E}_i$ is as in (5.2.1) with $\mathrm{E}_i = \mathrm{E}$. By choosing an appropriate $\bar{k}(t)$-basis $\boldsymbol{s}$ for S, multiplication by σ on $\boldsymbol{s}$ is represented by

$$\Phi_S := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \sum_{i=1}^w \boldsymbol{h}_{i,n} \boldsymbol{E}_i & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\bar{k}(t)),$$

and a corresponding rigid analytic trivialization is represented by

$$\Psi_S := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \sum_{i=1}^w \boldsymbol{g}_{i,n} \boldsymbol{E}_i \Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\mathbb{L}).$$

Suppose on the contrary that S is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. Then, there exists another $\bar{k}(t)$-basis $\boldsymbol{s}'$ of S such that $\sigma \boldsymbol{s}' = (\Phi_{\mathrm{P}_n\rho} \oplus (1))\boldsymbol{s}'$, where $\Phi_{\mathrm{P}_n\rho} \oplus (1)$ is the block diagonal matrix with $\Phi_{\mathrm{P}_n\rho}$ and 1 in the diagonal blocks and all other entries are zero. If we let

$$\gamma = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & \mathbf{0} \\ \gamma_0 \cdots \gamma_n & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\bar{k}(t)),$$

where $\gamma_j := (\gamma_{j1}, \dots, \gamma_{jr})$ for each $0 \le j \le n$ be the matrix such that $\boldsymbol{s}' := \gamma \boldsymbol{s}$, then we obtain

$$\gamma^{(-1)} \Phi_S = (\Phi_{\mathrm{P}_n\rho} \oplus (1))\gamma. \tag{5.2.3}$$

Note from [Papanikolas 2008, Proof of Proposition 3.4.5] that all denominators of entries of γ are in $\boldsymbol{A}$ and so in particular, for each $0 \le j \le n$, the entries of γ_j are regular at $t = \theta, \theta^q, \theta^{q^2}, \dots$. Using $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, the $((n+1)r+1, (n-j)r+1)$-th entry of (5.2.3) for each $1 \le j \le n$ is

$$\sum_{h=0}^{n-j} \gamma_{h,r}^{(-1)} \partial_t^{n-j-h}((t-\theta)/\kappa_r^{(-r)}) = \gamma_{n-j,1},$$

and the $((n+1)r+1, nr+1)$-th entry is

$$\sum_{h=0}^{n} \gamma_{h,r}^{(-1)} \partial_t^{n-h}((t-\theta)/\kappa_r^{(-r)}) + \sum_{i=1}^{w} \alpha_i (\mathrm{E}_i)_{11} = \gamma_{n,1}.$$

For each $0 \le j \le n$, applying $(-1)^j \partial_t^j(\,\cdot\,)$ to the $((n+1)r+1, (n-j)r+1)$-th entry and then adding them (we also use the product rule of hyperderivatives and the property $\partial_t^v \partial_t^w(f(t)) = \binom{v+w}{v}\partial_t^{v+w}(f(t))$), we obtain

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,r})^{(-1)}(t-\theta)/\kappa_r^{(-r)} + \sum_{i=1}^{w} \alpha_i (\mathrm{E}_i)_{11} = \sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1}). \tag{5.2.4}$$

Specializing both sides of this equation at $t=\theta$, we obtain

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1})(\theta) = \sum_{i=1}^{w} \alpha_i (\mathrm{E}_i)_{11}(\theta). \tag{5.2.5}$$

By (5.2.3), we also have $(\gamma\Psi_S)^{(-1)} = (\Phi_{\mathrm{P}_n\rho} \oplus (1))(\gamma\Psi_S)$ and so by [Papanikolas 2008, §4.1.6], for some

$$\delta = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & \mathbf{0} \\ \delta_0 \cdots \delta_n & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\boldsymbol{k}),$$

where $\delta_j := (\delta_{j1}, \dots, \delta_{jr})$ for each $0 \le j \le n$, we have

$$\gamma\Psi_S = (\Psi_{\mathrm{P}_n\rho} \oplus (1))\delta. \tag{5.2.6}$$

Since $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$, by applying to (5.2.6) the same methods applied on (5.2.3) to obtain (5.2.4), it follows that

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j}) + \sum_{i=1}^{w} \boldsymbol{s}_i \mathrm{E}_i = \sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j})\Psi_\rho^{-1}, \tag{5.2.7}$$

where for $\partial_t^j(\gamma_{n-j})$ and $\partial_t^j(\delta_{n-j})$, the hyperderivatives are taken entrywise. Since for each $1 \le i \le w$ the first entry of $\boldsymbol{s}_i(\theta)$ is $u_i - \alpha_i$ by (5.1.1), using [Chang and Papanikolas 2012, Proposition 4.1.1(b)] and specializing both sides of (5.2.7) at $t=\theta$, we see that

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1})(\theta) + \sum_{i=1}^{w}(u_i - \alpha_i)(\mathrm{E}_i)_{11}(\theta) = -\sum_{m=1}^{r}\sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j,m})(\theta)\lambda_m,$$

and so from (5.2.5) we have

$$\sum_{m=1}^{r}\sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j,m})(\theta)\lambda_m + \sum_{i=1}^{w}(\mathrm{E}_i)_{11}(\theta)u_i = 0.$$

Since $e_1, \dots, e_w$ are not all zero, E_i is nonzero for some $1 \le i \le w$. Moreover, by Proposition 3.2.3 we see that $\boldsymbol{K}_\rho \cong K_\rho$ and so E_i is invertible. By [loc. cit., Proposition 4.1.1(b),(c)] we get $(\mathrm{E}_i)_{11}(\theta) \in K_\rho^\times$ and thus we get a contradiction to the assumption that $\{u_1, \dots, u_w\}$ is K_ρ-linearly independent from each other and is K_ρ-linearly independent from $\{\lambda_1, \dots, \lambda_r\}$. □

5.3. ***Construction of the t-motives Y and N_n.*** In this subsection, we construct a t-motive that is suitable for the investigation of the hyperderivatives of logarithms and quasilogarithms of the Drinfeld $\boldsymbol{A}$-module ρ and the study of its Galois group. Suppose that we have $u_1, \dots, u_w \in \mathbb{K}$ with $\operatorname{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$. For $n \ge 0$, we let $\boldsymbol{h}_{\alpha_i} := \boldsymbol{h}_{\alpha_i,n}$, $\boldsymbol{g}_{\alpha_i} := \boldsymbol{g}_{\alpha_i,n}$, $Y_{i,n} := Y_{\alpha_i,n}$, $\Phi_{i,n} := \Phi_{\alpha_i,n}$ and $\Psi_{i,n} := \Psi_{\alpha_i,n}$ defined as in Section 5.1. The matrix $\Psi_n := \bigoplus_{i=1}^{w} \Psi_{i,n}$ is a rigid analytic trivialization for $Y_n := \bigoplus_{i=1}^{w} Y_{i,n}$.

Define the t-motive N_n such that multiplication by σ on a $\bar{k}(t)$-basis is given by $\Phi_{N_n} \in \mathrm{GL}_{(n+1)rw+1}(\bar{k}(t))$ along with rigid analytic trivialization $\Psi_{N_n} \in \mathrm{GL}_{(n+1)rw+1}(\mathbb{T})$ such that

$$\Phi_{N_n} := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & & & \\ & \ddots & & \\ & & \Phi_{\mathrm{P}_n\rho} & \\ \boldsymbol{h}_{\alpha_1} & \cdots & \boldsymbol{h}_{\alpha_w} & 1 \end{pmatrix} \quad \text{and} \quad \Psi_{N_n} := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & & & \\ & \ddots & & \\ & & \Psi_{\mathrm{P}_n\rho} & \\ \boldsymbol{g}_{\alpha_1}\Psi_{\mathrm{P}_n\rho} & \cdots & \boldsymbol{g}_{\alpha_w}\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix}. \tag{5.3.1}$$

Similar to $n = 0$ case [Chang and Papanikolas 2012, §5.1], N_n is an extension of $\mathbf{1}$ by $(\mathrm{P}_n M_\rho)^w$ which is a pullback of the surjective map $Y_n \twoheadrightarrow \mathbf{1}^w$ and the diagonal map $\mathbf{1} \to \mathbf{1}^w$. Thus, the two t-motives Y_n and N_n generate the same Tannakian subcategory of $\mathcal{T}$ and hence the Galois groups Γ_{Y_n} and Γ_{N_n} are isomorphic. For any $\boldsymbol{k}$-algebra R, an element of $\Gamma_{N_n}(\mathrm{R})$ is of the form

$$\nu = \begin{pmatrix} \mu & & & \\ & \ddots & & \\ & & \mu & \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_w & 1 \end{pmatrix}, \tag{5.3.2}$$

where $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and for each $1 \le i \le w$ we have $\boldsymbol{v}_i = (\boldsymbol{v}_{i,1}, \dots, \boldsymbol{v}_{i,n+1})$ such that $\boldsymbol{v}_{i,h} \in \mathbb{G}_a^r(\mathrm{R}) = \mathrm{Mat}_{1\times r}(\mathrm{R})$ for each $0 \le h \le n$. Since $(\mathrm{P}_n M_\rho)^w$ is a sub-t-motive of N_n, we have the following short exact sequence of affine group schemes over $\boldsymbol{k}$:

$$1 \to X_n \to \Gamma_{N_n} \xrightarrow{\pi_n} \Gamma_{\mathrm{P}_n M_\rho} \to 1, \tag{5.3.3}$$

where $\pi_n^{(\mathrm{R})} : \Gamma_{N_n}(\mathrm{R}) \to \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is the map $\nu \mapsto \mu$ (cf. [loc. cit., p. 138]). It can be checked directly that via conjugation (5.3.3) gives an action of any $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on

$$\boldsymbol{v} = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{(n+1)r} & \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_w & 1 \end{pmatrix} \in X_n(\mathrm{R})$$

given by

$$\nu \boldsymbol{v} \nu^{-1} = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{(n+1)r} & \\ \boldsymbol{u}_1\mu^{-1} & \cdots & \boldsymbol{u}_w\mu^{-1} & 1 \end{pmatrix}. \tag{5.3.4}$$

For $n \ge 0$, recall from (2.5.1) that if the entries of $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ form a $\bar{k}(t)$-basis of M_ρ, then the entries of $\boldsymbol{D}_n\boldsymbol{m}$ form a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$. Let $[\boldsymbol{D}_n\boldsymbol{m}^{\mathsf{T}}, y]^{\mathsf{T}}$ be a $\bar{k}(t)$-basis of N_n. Then, the entries of

$\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top$ form a $\mathbb{F}_q(t)$-basis of N_n^B [Papanikolas 2008, Proposition 3.3.9]. By construction, $\mathrm{P}_j M_\rho$ is a sub-t-motive of N_n for each $0 \le j \le n-1$ and we have a short exact sequence of t-motives

$$0 \to \mathrm{P}_j M_\rho \xrightarrow{\iota} N_n \xrightarrow{\mathbf{Pr}_{n-j-1}} N_{n-j-1} \to 0, \tag{5.3.5}$$

where $\mathbf{Pr}_{n-j-1}(D_h m) := D_{h-j-1}m$ for $h > j$, $\mathbf{Pr}_{n-j-1}(D_h m) := 0$ for $h \le j$ and $m \in M_\rho$, and $\mathbf{Pr}_{n-j-1}(x) := x$ for $x \in N_n/\mathrm{P}_n M_\rho$. Thus, as t-motives $N_n/\mathrm{P}_j M_\rho \cong N_{n-j-1}$ and so, N_{n-j-1} is an object in the Tannakian category $\mathcal{T}_{N_n}$. Therefore, we have a surjective map of affine group schemes $\Gamma_{N_n} \twoheadrightarrow \Gamma_{N_{n-j-1}}$. We now determine this surjective map. For any $\boldsymbol{k}$-algebra R, we recall the action of $\Gamma_{N_n}(\mathrm{R})$ on $\mathrm{R} \otimes_{\boldsymbol{k}} N_n^B$ from [Papanikolas 2008, §4.5] as follows: for any $\nu_n \in \Gamma_{N_n}(\mathrm{R})$, $b \in \mathrm{R}$ and $a_h \in \mathrm{Mat}_{1\times r}(\mathrm{R})$ where $0 \le h \le n$, the action of ν_n on $(a_0, \dots, a_n, b) \cdot \Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top \in R \otimes_{\boldsymbol{k}} N_n^B$ is

$$(a_0, \dots, a_n, b) \cdot \Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top \mapsto (a_0, \dots, a_n, b) \cdot \nu_n^{-1}\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top. \tag{5.3.6}$$

Note that $\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\top, y]^\top = [(d_{t,n+1}[\Psi_\rho]^{-1}\boldsymbol{D}_n\boldsymbol{m})^\top, -\boldsymbol{g}_{\alpha_1} D_n\boldsymbol{m} + y]^\top$ by the definition of Ψ_{N_n} (see (5.3.1)). We restrict the action of ν_n to $\mathrm{R} \otimes_{\boldsymbol{k}} N_{n-j-1}^B$ via the map $\mathbf{Pr}_{n-j-1}$ in (5.3.5). Note that an element of $\Gamma_{N_n}(\mathrm{R})$ is of the form

$$\begin{pmatrix} \mu_n & \mathbf{0} \\ \boldsymbol{w}_n & 1 \end{pmatrix},$$

where $\mu_n \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and $\boldsymbol{w}_n = (w_0, \dots, w_n)$ such that each $w_h \in \mathbb{G}_a^r(\mathrm{R}) = \mathrm{Mat}_{1\times r}(\mathrm{R})$. Through $\mathbf{Pr}_{n-j-1}$, we see that ν_n leaves N_{n-j-1}^B invariant and so for

$$\nu_n = \begin{pmatrix} \mu_n & \mathbf{0} \\ \boldsymbol{w}_n & 1 \end{pmatrix} \in \Gamma_{N_n}(\mathrm{R}),$$

we obtain

$$\nu_{n-j-1} = \begin{pmatrix} \mu_{n-j-1} & \mathbf{0} \\ \boldsymbol{w}_{n-j-1} & 1 \end{pmatrix} \in \Gamma_{N_{n-j-1}}(\mathrm{R}), \tag{5.3.7}$$

where μ_{n-j-1} is the matrix formed by the upper left $r(n-j) \times r(n-j)$ square of μ_n and $\boldsymbol{w}_{n-j-1} = (w_0, \dots, w_{n-j-1})$. Note that by Theorem 4.2.5, we have $\mu_{n-j-1} \in \Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$. Thus, the surjective map $\Xi_{n-j-1} : \Gamma_{N_n} \to \Gamma_{N_{n-j-1}}$ is given by (cf. [Chang and Papanikolas 2011, Proposition 3.1.2])

$$\Xi_{n-j-1}^{(\mathrm{R})} : \nu_n \mapsto \nu_{n-j-1}. \tag{5.3.8}$$

Lemma 5.3.9. *Let $n \ge 1$. If K_ρ is separable over k, then X_n in (5.3.3) is $\boldsymbol{k}$-smooth.*

Proof. We adapt the ideas of the proof of [Chang and Papanikolas 2011, Proposition 4.1.2] and the proof of a lemma from a preliminary version of [Chang and Papanikolas 2012] (Lemma 5.1.3: arXiv:1005.5120v1). By [Springer 1998, Corollary 12.1.3] it suffices to show that for $n \ge 1$, the induced tangent map $\mathrm{d}\pi_n$ at the identity is surjective onto $\mathrm{Lie}\,\Gamma_{\mathrm{P}_n M_\rho}$. We prove this for $w = 1$ as the argument used in this case can be applied in a straightforward manner to prove the arbitrary w case. We leave this task to the reader. Since $\boldsymbol{K}_\rho$ is separable over $\boldsymbol{k}$ (by hypothesis, Proposition 3.2.3, and Remark 3.2.4), we see from [Chang and Papanikolas 2012, Corollary 3.5.6] and [Waterhouse 1979, p. 61 Problem 14] that through conjugation by some $\mathrm{J} \in \mathrm{GL}_r(\boldsymbol{k}^{\mathrm{sep}})$, we have an isomorphism

$$\Gamma_{M_\rho} \times_{\boldsymbol{k}} \boldsymbol{K}_\rho \xrightarrow{\cong} \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho)_i,$$

where

$$\prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho)_i := \left\{ \begin{pmatrix} \mathrm{GL}_{r/s} & & \\ & \ddots & \\ & & \mathrm{GL}_{r/s} \end{pmatrix} \right\},$$

and $(\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho)_i$ is the canonical embedding of $\mathrm{GL}_{r/s} / \boldsymbol{K}_\rho$ into the i-th diagonal block matrix of $\mathrm{GL}_r / \boldsymbol{K}_\rho$. Making a change of basis, we obtain

$$\Gamma_{M_\rho} \times_{\boldsymbol{k}} \bar{\boldsymbol{k}} \xrightarrow{\cong} \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \bar{\boldsymbol{k}})_i.$$

For $n \geq 1$, it follows that via conjugation by $d_{t,n+1}[\mathrm{J}] \in \mathrm{GL}_{(n+1)r}(\boldsymbol{k}^{\mathrm{sep}})$ on $\Gamma_{\mathrm{P}_n M_\rho}$, we obtain $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$, an algebraic subgroup of $\mathrm{GL}_{(n+1)r} / \bar{\boldsymbol{k}}$, such that there is an isomorphism

$$\Gamma_{\mathrm{P}_n M_\rho} \times_{\boldsymbol{k}} \bar{\boldsymbol{k}} \xrightarrow{\cong} \bar{\Gamma}_{\mathrm{P}_n M_\rho}. \tag{5.3.10}$$

Let $\left(\bigoplus_{i=1}^{w} d_{t,n+1}[\mathrm{J}]\right) \oplus (1) \in \mathrm{GL}_{(n+1)rw+1}(\boldsymbol{k}^{\mathrm{sep}})$ be the block diagonal matrix with $d_{t,n+1}[\mathrm{J}]$ in the first w diagonal blocks and 1 in the last diagonal, and all other entries are zero. Then, via conjugation by $\left(\bigoplus_{i=1}^{w} d_{t,n+1}[\mathrm{J}]\right) \oplus (1)$ on Γ_{N_n} we obtain $\bar{\Gamma}_{N_n}$ such that we have an isomorphism $\Gamma_{N_n} \times_{\boldsymbol{k}} \bar{\boldsymbol{k}} \cong \bar{\Gamma}_{N_n}$. Moreover, $\bar{\Gamma}_{N_n}$ is an algebraic subgroup of $\mathrm{GL}_{(n+1)rw+1} / \bar{\boldsymbol{k}}$ such that $\bar{\pi}_n : \bar{\Gamma}_{N_n} \to \bar{\Gamma}_{\mathrm{P}_n M_\rho}$ induced by π_n in (5.3.3) is surjective. Thus, we are reduced to proving that the induced tangent map $\mathrm{d}\bar{\pi}_n : \mathrm{Lie}\, \bar{\Gamma}_{N_n} \to \mathrm{Lie}\, \bar{\Gamma}_{\mathrm{P}_n M_\rho}$ is surjective.

First we determine $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$. Recall $\boldsymbol{X}$, the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$ from (4.3.1). Since $d_{t,n+1}[\mathrm{J}]$ and its inverse are block upper triangular matrices, similar to $\boldsymbol{X}$ we make the choice to let the coordinates of $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$ be

$$\boldsymbol{Y} := \begin{pmatrix} \boldsymbol{Y}_0 & \boldsymbol{Y}_1 & \cdots & \boldsymbol{Y}_n \\ & \boldsymbol{Y}_0 & \ddots & \vdots \\ & & \ddots & \boldsymbol{Y}_1 \\ & & & \boldsymbol{Y}_0 \end{pmatrix},$$

where $\boldsymbol{Y}_h := ((Y_h)_{ij})$, an $r \times r$ matrix for $0 \leq h \leq n$. Then, by construction we have $\boldsymbol{X} = d_{t,n+1}[\mathrm{J}] \boldsymbol{Y} d_{t,n+1}[\mathrm{J}]^{-1}$ and so for each $0 \leq w \leq n$, we obtain

$$\boldsymbol{X}_w = \sum_{\substack{w_1 + w_2 = w \\ w_1, w_2 \geq 0}} \sum_{h=0}^{w_1} \partial_t^{w_1 - h}(\mathrm{J}) \cdot \boldsymbol{Y}_{w_2} \cdot (\partial_t^h(\mathrm{J}))^{-1},$$

where the hyperderivatives are taken entrywise. Then, we have

$$\mathbf{vec}(\boldsymbol{X}_w) = \sum_{\substack{w_1 + w_2 = w \\ w_1, w_2 \geq 0}} \sum_{h=0}^{w_1} \left([(\partial_t^h(\mathrm{J}))^{-1}]^{\mathsf{T}} \otimes \partial_t^{w_1 - h}(\mathrm{J}) \right) \cdot \mathbf{vec}(\boldsymbol{Y}_{w_2}) = \sum_{\substack{w_1 + w_2 = w \\ w_1, w_2 \geq 0}} \partial_t^{w_1}((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J}) \cdot \mathbf{vec}(\boldsymbol{Y}_{w_2}),$$

where we obtain the first equality by using properties of the Kronecker product and the second equality by further applying the product rule for hyperderivatives. This implies

$$\mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}) = d_{t,n+1}[(\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J}] \cdot \mathbf{vec}([\boldsymbol{Y}_n, \dots, \boldsymbol{Y}_0]^{\mathsf{T}}), \tag{5.3.11}$$

where we set

$$\mathbf{vec}([\boldsymbol{X}_n, \dots, \boldsymbol{X}_0]^{\mathsf{T}}) := [(\mathbf{vec}\,\boldsymbol{X}_n)^{\mathsf{T}}, \dots, (\mathbf{vec}\,\boldsymbol{X}_0)^{\mathsf{T}}]^{\mathsf{T}},$$

and we further define $\mathbf{vec}([\boldsymbol{Y}_n, \dots, \boldsymbol{Y}_0]^{\mathsf{T}})$ similarly. For $0 \le i \le n$, let $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, \dots, \boldsymbol{Y}_i, 1/\det \boldsymbol{Y}_0]$ denote the localization of $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, \dots, \boldsymbol{Y}_i]$ at $\det \boldsymbol{Y}_0$. Then, by (4.3.11), Corollary 4.4.8, and (5.3.11), the defining ideal of $\overline{\Gamma}_{\mathrm{P}_n M_\rho}$ via the isomorphism (5.3.10) is the ideal in $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, \dots, \boldsymbol{Y}_n, 1/\det \boldsymbol{Y}_0]$ generated by the entries of

$$d_{t,n+1}[\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})] \cdot \mathbf{vec}([\boldsymbol{Y}_n, \dots, \boldsymbol{Y}_0]^{\mathsf{T}}). \tag{5.3.12}$$

It is clear by observing $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$ that, for $\boldsymbol{Y}_0 = ((Y_0)_{i,j})$, the defining ideal of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$ is the ideal in $\bar{\boldsymbol{k}}[\boldsymbol{Y}_0, 1/\det \boldsymbol{Y}_0]$ generated by

$$\{(Y_0)_{i,j} : (i, j) \neq (ur/s + v_1, ur/s + v_2),\ 0 \le u \le s-1 \text{ and } 1 \le v_1, v_2 \le r/s\}. \tag{5.3.13}$$

Moreover, by (4.3.5) and (5.3.11), the defining ideal of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$ is also generated by the entries of

$$(\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \mathbf{vec}(\boldsymbol{Y}_0). \tag{5.3.14}$$

By (5.3.13), in the defining ideal of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$, there are no linear relations among

$$\{(Y_0)_{i,j} : (i, j) = (ur/s + v_1, ur/s + v_2),\ 0 \le u \le s-1 \text{ and } 1 \le v_1, v_2 \le r/s\}. \tag{5.3.15}$$

Since (5.3.14) also generate the defining ideal of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$, we see that the entries of $\boldsymbol{B}\cdot((\mathrm{J}^{-1})^{\mathsf{T}}\otimes\mathrm{J})$ that give linear relations among the variables in (5.3.15) are all zero. Therefore, the hyperderivatives of these entries are also all zero. Using this and using (5.3.13), for $\gamma \in \prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$ and for $0 \le \ell \le n$, we see that

$$\partial_t^{\ell}(\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \gamma = \mathbf{0}. \tag{5.3.16}$$

Moreover, by (5.3.14), for $1 \le h \le n$, the defining ideal of $\prod_{i=1}^{s}(\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i$ is the ideal in $\bar{\boldsymbol{k}}[\boldsymbol{Y}_h]$ generated by the entries of

$$(\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \mathbf{vec}(\boldsymbol{Y}_h),$$

and similar to (5.3.16), for $\gamma' \in \prod_{i=1}^{s}(\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i$ and for $0 \le \ell \le n$, we see that

$$\partial_t^{\ell}(\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})) \cdot \gamma' = \mathbf{0}.$$

Therefore, for all $\gamma_0 \in \prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\bar{\boldsymbol{k}})_i$ and $\gamma_h \in \prod_{i=1}^{s}(\mathrm{Mat}_{r/s}/\bar{\boldsymbol{k}})_i$ where $1 \le h \le n$, we have

$$d_{t,n+1}[\boldsymbol{B} \cdot ((\mathrm{J}^{-1})^{\mathsf{T}} \otimes \mathrm{J})] \cdot ([\gamma_n, \dots, \gamma_0]^{\mathsf{T}}) = \mathbf{0}.$$

Thus, by (5.3.12) we have

$$\bar{\Gamma}_{\mathrm{P}_n M_\rho} = \left\{ \left(\begin{array}{c|c|c|c} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \hline & \gamma_0 & \ddots & \vdots \\ \hline & & \ddots & \gamma_1 \\ \hline & & & \gamma_0 \end{array} \right) : \gamma_0 \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s} /\bar{\boldsymbol{k}})_i,\ \gamma_h \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s} /\bar{\boldsymbol{k}})_i,\ 1 \le h \le n \right\}, \tag{5.3.17}$$

where, for each i, $(\mathrm{GL}_{r/s} /\bar{\boldsymbol{k}})_i$ and $(\mathrm{Mat}_{r/s} /\bar{\boldsymbol{k}})_i$ are the canonical embeddings of $\mathrm{GL}_{r/s} /\bar{\boldsymbol{k}}$ and $\mathrm{Mat}_{r/s} /\bar{\boldsymbol{k}}$ respectively into the i-th diagonal block matrices of $\mathrm{GL}_r /\bar{\boldsymbol{k}}$ and $\mathrm{Mat}_r /\bar{\boldsymbol{k}}$.

We are now ready to prove that the induced tangent map $\mathrm{d}\bar{\pi}_n : \mathrm{Lie}\, \bar{\Gamma}_{N_n} \to \mathrm{Lie}\, \bar{\Gamma}_{\mathrm{P}_n M_\rho}$ is surjective. Let $w = 1$ and consider the short exact sequence of linear algebraic groups

$$1 \to \bar{X}_n \to \bar{\Gamma}_{N_n} \xrightarrow{\bar{\pi}_n} \bar{\Gamma}_{\mathrm{P}_n M_\rho} \to 1. \tag{5.3.18}$$

First suppose $n = 1$. Then, by (5.3.17),

$$\bar{\Gamma}_{\mathrm{P}_1 M_\rho} = \left\{ \begin{pmatrix} \gamma_0 & \gamma_1 \\ \mathbf{0} & \gamma_0 \end{pmatrix} : \gamma_0 \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s} /\bar{\boldsymbol{k}})_i,\ \gamma_1 \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s} /\bar{\boldsymbol{k}})_i \right\}, \tag{5.3.19}$$

and by (5.3.2),

$$\bar{\Gamma}_{N_1} \subseteq \left\{ \begin{pmatrix} \gamma_0 & \gamma_1 & \mathbf{0} \\ \mathbf{0} & \gamma_0 & \mathbf{0} \\ z_0 & z_1 & 1 \end{pmatrix} : \begin{pmatrix} \gamma_0 & \gamma_1 \\ \mathbf{0} & \gamma_0 \end{pmatrix} \in \bar{\Gamma}_{\mathrm{P}_1 M_\rho},\ z_0, z_1 \in \mathbb{G}_a^r \right\}. \tag{5.3.20}$$

From $\bar{\pi}_1$, we see that $\bar{X}_1$ is contained in the $2r$-dimensional additive group

$$G := \left\{ \begin{pmatrix} \mathrm{Id}_{r/s} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{r/s} & \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{2s} & 1 \end{pmatrix} : \boldsymbol{v}_i \in \mathbb{G}_a^{r/s} \right\},$$

where we call $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{2s}$ the coordinates of G. We see that via conjugation, $\bar{X}_1(\bar{\boldsymbol{k}})$ has a $\bar{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}})$-module structure coming from (5.3.18) (see (5.3.4)). Using (5.3.19) and this module structure, one checks easily that there is a natural decomposition $\bar{X}_1(\bar{\boldsymbol{k}}) = \prod_{i=1}^{2s} W_i$ such that each W_i is either zero or $\bar{\boldsymbol{k}}^{r/s}$. Fix any $1 \le i \le s$. For any $\xi_i \in \mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$, we let

$$\bar{\xi}_i = \left(\begin{array}{ccccccccccc} \mathrm{Id}_{r/s} & & & & & & \mathbf{0} & & & & \\ & \ddots & & & & & & \ddots & & & \\ & & \xi_i & & & & & & \mathbf{0} & & \\ & & & \ddots & & & & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \xi_i & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_i & \cdots & \boldsymbol{u}_s & \boldsymbol{u}_{s+1} & \cdots & \boldsymbol{u}_{s+i} & \cdots & \boldsymbol{u}_{2s} & 1 \end{array} \right) \in \bar{\Gamma}_{N_1}(\bar{\boldsymbol{k}})$$

be an arbitrary element, which by (5.3.18) for $n=1$ and (5.3.19) is a preimage of the matrix formed by the upper left $2r\times 2r$ square of $\bar{\xi}_i$ under the map $\bar{\pi}_1$. For each $j\neq i$ with $1\leq j\leq \mathbf{s}$, we claim that if $\boldsymbol{u}_j\neq \mathbf{0}$ and $\boldsymbol{u}_{s+j}\neq\mathbf{0}$, then $W_j=W_{s+j}=\bar{\boldsymbol{k}}^{r/s}$. To prove this claim, assuming that $\boldsymbol{u}_j\neq\mathbf{0}$ and $\boldsymbol{u}_{s+j}\neq\mathbf{0}$ we pick $\delta_j\in \mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$ so that $\boldsymbol{u}_j\delta_j-\boldsymbol{u}_j\neq\mathbf{0}$ and $\boldsymbol{u}_{s+j}\delta_j-\boldsymbol{u}_{s+j}\neq\mathbf{0}$, and let $\bar{\delta}_j\in\bar{\Gamma}_{N_1}(\bar{\boldsymbol{k}})$ be such that

$$\bar{\pi}_1(\bar{\delta}_j)=\left(\begin{array}{ccccc|ccccc}
\mathrm{Id}_{r/s} & & & & & & & \mathbf{0} & & \\
& \ddots & & & & & & & \ddots & \\
& & \delta_j & & & & & & \mathbf{0} & \\
& & & \ddots & & & & & & \ddots \\
& & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} \\
\hline
& & & & & \mathrm{Id}_{r/s} & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & \delta_j & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \mathrm{Id}_{r/s}
\end{array}\right)\in\bar{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}}).$$

Then one checks directly that $\bar{\delta}_j^{-1}\bar{\xi}_i\bar{\delta}_j\bar{\xi}_i^{-1}$ is an element of $\overline{X}_1(\bar{\boldsymbol{k}})$ and its $\boldsymbol{v}_j$ and $\boldsymbol{v}_{s+j}$ coordinate vectors respectively are $\boldsymbol{u}_j\delta_j-\boldsymbol{u}_j$ and $\boldsymbol{u}_{s+j}\delta_j-\boldsymbol{u}_{s+j}$, and so it follows that $W_j=W_{s+j}=\bar{\boldsymbol{k}}^{r/s}$. Therefore, multiplying $\bar{\xi}_i$ by a suitable element of $\overline{X}_1(\bar{\boldsymbol{k}})$ we get an element of the form

$$\bar{\xi}_i'=\left(\begin{array}{ccccc|cccccc}
\mathrm{Id}_{r/s} & & & & & & & \mathbf{0} & & & \\
& \ddots & & & & & & & \ddots & & \\
& & \xi_i & & & & & & \mathbf{0} & & \\
& & & \ddots & & & & & & \ddots & \\
& & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} & \\
\hline
& & & & & \mathrm{Id}_{r/s} & & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & & \xi_i & & & \\
& & & & & & & & \ddots & & \\
& & & & & & & & & \mathrm{Id}_{r/s} & \\
\mathbf{0} & \cdots & \boldsymbol{u}_i & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{u}_{s+i} & \cdots & \mathbf{0} & 1
\end{array}\right)\in\bar{\Gamma}_{N_1}(\bar{\boldsymbol{k}}). \tag{5.3.21}$$

For any $\boldsymbol{b}_i\in\mathrm{Mat}_{r/s}(\bar{\boldsymbol{k}})$, by using a method similar to that above where we take an element of the form $\bar{\delta}_j$, we obtain an element of the form

$$\bar{\boldsymbol{b}}_i'=\left(\begin{array}{ccccc|cccccc}
\mathrm{Id}_{r/s} & & & & & & & \mathbf{0} & & & \\
& \ddots & & & & & & & \ddots & & \\
& & \mathrm{Id}_{r/s} & & & & & & \boldsymbol{b}_i & & \\
& & & \ddots & & & & & & \ddots & \\
& & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} & \\
\hline
& & & & & \mathrm{Id}_{r/s} & & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & & \mathrm{Id}_{r/s} & & & \\
& & & & & & & & \ddots & & \\
& & & & & & & & & \mathrm{Id}_{r/s} & \\
\mathbf{0} & \cdots & \boldsymbol{w}_i & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{w}_{s+i} & \cdots & \mathbf{0} & 1
\end{array}\right)\in\bar{\Gamma}_{N_1}(\bar{\boldsymbol{k}}), \tag{5.3.22}$$

which is a preimage of the matrix formed by the upper left $2r \times 2r$ square of $\bar{\boldsymbol{b}}_i'$ under the map $\bar{\pi}_1$. Let $\overline{H}_{1,i}$ be the Zariski closure inside $\overline{\Gamma}_{N_1}$ of the subgroup generated by all $\bar{\xi}_i'$ with ξ_i running over all elements of $\mathrm{GL}_{r/s}(\bar{\boldsymbol{k}})$ and all $\bar{\boldsymbol{b}}_i'$ with $\boldsymbol{b}_i$ running over all elements of $\mathrm{Mat}_{r/s}(\bar{\boldsymbol{k}})$. For each $1 \le i \le \boldsymbol{s}$, let

$$(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i := \left\{ \begin{pmatrix} \gamma_0 & \gamma_1 \\ \mathbf{0} & \gamma_0 \end{pmatrix} : \gamma_0 \in (\mathrm{GL}_{r/\boldsymbol{s}}\,/\bar{\boldsymbol{k}})_i,\ \gamma_1 \in (\mathrm{Mat}_{r/\boldsymbol{s}}\,/\bar{\boldsymbol{k}})_i \right\}. \tag{5.3.23}$$

Note that $\dim \overline{H}_{1,i} \le 2r^2/\boldsymbol{s}^2 + 2r/\boldsymbol{s}$.

First suppose that $\dim \overline{H}_{1,i} = 2r^2/\boldsymbol{s}^2 + 2r/\boldsymbol{s}$. Then, we could simply take $\bar{\xi}_i'$ and $\bar{\boldsymbol{b}}_i'$ so that $\boldsymbol{u}_i$, $\boldsymbol{u}_{\boldsymbol{s}+i}$, $\boldsymbol{w}_i$ and $\boldsymbol{w}_{\boldsymbol{s}+i}$ are zero. Taking the Zariski closure $\bar{S}_{1,i}$ inside $\overline{\Gamma}_{N_1}$ of the subgroup generated by all such $\bar{\xi}_i'$ and $\boldsymbol{b}_i'$ with ξ_i and $\boldsymbol{b}_i$ running over all elements of $\mathrm{GL}_{r/\boldsymbol{s}}(\bar{\boldsymbol{k}})$ and $\mathrm{Mat}_{r/\boldsymbol{s}}(\bar{\boldsymbol{k}})$ respectively, we obtain

$$\bar{S}_{1,i} = \left\{ \begin{pmatrix} \nu_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : \nu_i \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i \right\}. \tag{5.3.24}$$

Thus, $\overline{\Gamma}_{N_1}$ contains a copy of $(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ and so, restricting $\mathrm{d}\bar{\pi}_1$ to $\mathrm{Lie}\,\bar{S}_{1,i}$, we obtain a surjection onto $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$. As we vary all $1 \le i \le \boldsymbol{s}$, the surjection of $\mathrm{d}\bar{\pi}_1$ follows.

Next, suppose that $\dim \overline{H}_{1,i} < 2r^2/\boldsymbol{s}^2 + 2r/\boldsymbol{s}$. Then, via $\bar{\pi}_1$ we have a short exact sequence

$$1 \to \overline{Q}_{1,i} \to \overline{H}_{1,i} \xrightarrow{\bar{\pi}_{1,i}} (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i \to 1,$$

where $\dim \overline{Q}_{1,i} < 2r/\boldsymbol{s}$ and $\overline{Q}_{1,i}$ is contained in an additive subgroup of G whose $\boldsymbol{v}_j$ coordinate vector is zero for all $j \ne i, \boldsymbol{s}+i$, that is,

$$\overline{Q}_{1,i} \subseteq \left\{ \left(\begin{array}{cccccccccccc} \mathrm{Id}_{r/s} & & & & & & \mathbf{0} & & & & & \\ & \ddots & & & & & & \ddots & & & & \\ & & \mathrm{Id}_{r/s} & & & & & & \mathbf{0} & & & \\ & & & \ddots & & & & & & \ddots & & \\ & & & & \mathrm{Id}_{r/s} & & & & & & \mathbf{0} & \\ \hline & & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & & \ddots & & & & \\ & & & & & & & & \mathrm{Id}_{r/s} & & & \\ & & & & & & & & & \ddots & & \\ & & & & & & & & & & \mathrm{Id}_{r/s} & \\ \mathbf{0} & \cdots & \boldsymbol{v}_i & \cdots & \mathbf{0} & & \mathbf{0} & \cdots & \boldsymbol{v}_{s+i} & \cdots & \mathbf{0} & 1 \end{array} \right) : \begin{array}{l} \boldsymbol{v}_i,\, \boldsymbol{v}_{\boldsymbol{s}+i} \in \mathbb{G}_a^{r/\boldsymbol{s}}, \\ \boldsymbol{v}_i = (\boldsymbol{v}_{i,1}, \ldots, \boldsymbol{v}_{i,r/\boldsymbol{s}}), \\ \boldsymbol{v}_{\boldsymbol{s}+i} = (\boldsymbol{v}_{\boldsymbol{s}+i,1}, \ldots, \boldsymbol{v}_{\boldsymbol{s}+i,r/\boldsymbol{s}}) \end{array} \right\}. \tag{5.3.25}$$

Claim 4. *For $\overline{Q}_{1,i}$ if some entry of the $\boldsymbol{v}_i$ coordinate vector is nonzero or* $\dim \overline{Q}_{1,i} \ne r/\boldsymbol{s}$, *then* $\dim \overline{Q}_{1,i} = 0$.

Proof of Claim 4. We follow the argument of the proof of [Chang and Papanikolas 2011, Lemma 4.1.1]. Suppose $\dim \overline{Q}_{1,i} = m$, where $1 \le m < 2r/\boldsymbol{s}$. Note that $\overline{Q}_{1,i}$ is a vector group. If $\boldsymbol{v}_{i,j}, \boldsymbol{v}_{\boldsymbol{s}+i,j} \ne 0$ for all $1 \le j \le r/\boldsymbol{s}$, let $\mu \in \overline{Q}_{1,i}(\bar{\boldsymbol{k}})$ such that all the entries of μ in the $\boldsymbol{v}_i$ coordinate vector are nonzero. For $a \in \bar{\boldsymbol{k}}$, $a \ne 0, 1$ and $1 \le \ell \le r/\boldsymbol{s}$, pick $\eta_\ell, \varkappa_\ell \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ such that $\bar{\pi}_{1,i}(\eta_\ell), \bar{\pi}_{1,i}(\varkappa_\ell) \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}}))_i$, where

$$\bar{\pi}_{1,i}(\eta_\ell) = \left(\begin{array}{ccccc|ccccc} \mathrm{Id}_{r/s} & & & & & \mathbf{0} & & & & \\ & \ddots & & & & & \ddots & & & \\ & & \mathfrak{a}_\ell & & & & & \mathbf{0} & & \\ & & & \ddots & & & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \mathfrak{a}_\ell & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \mathrm{Id}_{r/s} \end{array}\right) \quad \text{for } \mathfrak{a}_\ell := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tag{5.3.26}$$

$$\bar{\pi}_{1,i}(\varkappa_\ell) = \left(\begin{array}{ccccc|ccccc} \mathrm{Id}_{r/s} & & & & & \mathbf{0} & & & & \\ & \ddots & & & & & \ddots & & & \\ & & \mathrm{Id}_{r/s} & & & & & \mathfrak{b}_\ell & & \\ & & & \ddots & & & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \mathrm{Id}_{r/s} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \mathrm{Id}_{r/s} \end{array}\right) \quad \text{for } \mathfrak{b}_\ell := \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

where $\mathfrak{a}_\ell$ and $\mathfrak{b}_\ell$ are $r/\boldsymbol{s} \times r/\boldsymbol{s}$, and a is in the ℓ-th diagonal entries. One checks directly that the $2r/\boldsymbol{s}$ vectors $\eta_\ell^{-1}\mu\eta_\ell$, $\varkappa_\ell^{-1}\mu\varkappa_\ell$, where $1 \le \ell \le m$, are $\bar{\boldsymbol{k}}$-linearly independent in $\overline{Q}_{1,i}(\bar{\boldsymbol{k}})$, which contradicts the assumption $\dim \overline{Q}_{1,i} = m < 2r/\boldsymbol{s}$. Thus, $\boldsymbol{v}_{i,u} = 0$ for some $1 \le u \le r/\boldsymbol{s}$. Since $m \ne 0$, at least one of $\boldsymbol{v}_{i,j}$, $\boldsymbol{v}_{\boldsymbol{s}+i,j}$ for some $1 \le j \le r/\boldsymbol{s}$ is nonzero, say $\boldsymbol{v}_{i,v}$ or $\boldsymbol{v}_{\boldsymbol{s}+i,v}$.

Let $\boldsymbol{P}_{u,v}$ be the permutation matrix obtained by switching the $((i-1)r/\boldsymbol{s}+u)$-th column and the $((i-1)r/\boldsymbol{s}+v)$-column of the $r \times r$ identity matrix. Pick $\gamma \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ such that

$$\bar{\pi}_{1,i}(\gamma) = \begin{pmatrix} \boldsymbol{P}_{u,v} & \\ & \boldsymbol{P}_{u,v} \end{pmatrix} \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\bar{\boldsymbol{k}}))_i. \tag{5.3.27}$$

If $\boldsymbol{v}_{i,v}$ is nonzero, then since $\gamma^{-1}\overline{Q}_{1,i}\gamma \subseteq \overline{Q}_{1,i}$ we get a contradiction to $\boldsymbol{v}_{i,u} = 0$. Therefore, $\dim \overline{Q}_{1,i} = 0$.

Next suppose $\boldsymbol{v}_{\boldsymbol{s}+i,v}$ is nonzero but $\boldsymbol{v}_{i,j} = 0$ for all $1 \le j \le r/\boldsymbol{s}$. Then by hypothesis, $m < r/\boldsymbol{s}$. If $\boldsymbol{v}_{\boldsymbol{s}+i,j} \ne 0$ for all $1 \le j \le r/\boldsymbol{s}$, let $\vartheta \in \overline{Q}_{1,i}(\bar{\boldsymbol{k}})$ such that all the entries of ϑ in the $\boldsymbol{v}_{\boldsymbol{s}+i}$ coordinate vector are nonzero. Then, one checks directly that for η_ℓ as in (5.3.26), the $r/\boldsymbol{s}$ vectors $\eta_\ell^{-1}\vartheta\eta_\ell$ are $\bar{\boldsymbol{k}}$-linearly independent in $\overline{Q}_{1,i}(\bar{\boldsymbol{k}})$, which contradicts the assumption $\dim Q_{1,i} = m < r/\boldsymbol{s}$. Thus, $\boldsymbol{v}_{\boldsymbol{s}+i,u} = 0$ for some $1 \le u \le r/\boldsymbol{s}$. Then, since $\boldsymbol{v}_{\boldsymbol{s}+i,v}$ is nonzero and $\gamma^{-1}Q_{1,i}\gamma \subseteq \overline{Q}_{1,i}$ for γ as in (5.3.27), we get a contradiction to $\boldsymbol{v}_{\boldsymbol{s}+i,u} = 0$. Therefore, $\dim \overline{Q}_{1,i} = 0$. □

Claim 5. *If* $\dim \overline{Q}_{1,i} = 0$, *then* $\mathrm{d}\bar{\pi}_{1,i} : \mathrm{Lie}\, \overline{H}_{1,i} \to \mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ *is surjective.*

Proof of Claim 5. To prove that $d\bar{\pi}_{1,i}$ is surjective, we follow the argument of the proof of [Chang and Papanikolas 2011, Proposition 4.1.2]. We let the coordinates of $\overline{H}_{1,i}$ be as follows:

$$\boldsymbol{Z}_1 := \begin{pmatrix} \mathcal{Z}_0 & \mathcal{Z}_1 & \boldsymbol{0} \\ & \mathcal{Z}_0 & \boldsymbol{0} \\ \mathcal{W}_0 & \mathcal{W}_1 & 1 \end{pmatrix}, \tag{5.3.28}$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} \mathrm{Id}_{r/s} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{Z}_1 = \begin{pmatrix} \boldsymbol{0} & & & & \\ & \ddots & & & \\ & & (Z_1) & & \\ & & & \ddots & \\ & & & & \boldsymbol{0} \end{pmatrix},$$

such that (Z_0) and (Z_1) are the coordinates of $\mathrm{GL}_{r/s}$ and $\mathrm{Mat}_{r/s}$ respectively. For each $h = 0, 1$, we define (Z_h) to be the $r/s \times r/s$ block $((Z_h)_{a,b})$ for $1 \le a, b \le r/s$ and $\mathcal{W}_h := (0, \dots, 0, (W_h), 0, \dots, 0)$, where we set $(W_h) := (W_{h,1}, \dots, W_{h,r/s})$. For $1 \le u, v \le r/s$, we define the following one-dimensional subgroups of $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$:

$$T_{uv} := \left\{ \begin{pmatrix} \mathcal{B}_{uv} & \boldsymbol{0} \\ \boldsymbol{0} & \mathcal{B}_{uv} \end{pmatrix} \right\}, \quad U_{uv} := \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathcal{C}_{uv} \\ \boldsymbol{0} & \mathrm{Id}_r \end{pmatrix} \right\}, \tag{5.3.29}$$

where we set

$$\mathcal{B}_{uv} := \begin{pmatrix} \mathrm{Id}_{r/s} & & & & \\ & \ddots & & & \\ & & B_{uv} & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{C}_{uv} := \begin{pmatrix} \boldsymbol{0} & & & & \\ & \ddots & & & \\ & & C_{uv} & & \\ & & & \ddots & \\ & & & & \boldsymbol{0} \end{pmatrix} \tag{5.3.30}$$

such that

$$B_{vv} := \left\{ \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & * & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \right\}, \quad B_{uv} := \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & * & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad C_{uv} := \left\{ \begin{pmatrix} 0 & & \\ & \ddots & * \\ & & \ddots \\ & & & 0 \end{pmatrix} \right\},$$

where $*$ in B_{uv} and C_{uv} are in the (u, v)-coordinates. Note that the Lie algebras of the $2 \cdot r^2/s^2$ algebraic groups T_{uv} and U_{uv} span $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$. In what follows, we construct one-dimensional algebraic subgroups T'_{uv} and U'_{uv} of $\overline{H}_{1,i}$ so that $T'_{uv} \cong T_{uv}$ and $U'_{uv} \cong U_{uv}$. Then, since $\mathrm{Lie}(\,\cdot\,)$ is a left exact functor, it follows that $\mathrm{Lie}\, T'_{uv} \cong \mathrm{Lie}\, T_{uv}$ and $\mathrm{Lie}\, U'_{uv} \cong \mathrm{Lie}\, U_{uv}$, and so $d\bar{\pi}_{1,i}$ is surjective. Since $\overline{Q}_{1,i}$ is a zero-dimensional vector group, $\bar{\pi}_{1,i}$ is injective on points and so it follows by checking directly that

- for $w \ne v$, all $W_{0,w}$ and $W_{1,w}$ coordinates of $\bar{\pi}_{1,i}^{-1}(T_{uv})$ are zero;
- all (W_0) coordinates of $\bar{\pi}_{1,i}^{-1}(U_{uv})$ are zero, and for $w \ne v$, all $W_{1,w}$ coordinates of $\bar{\pi}_{1,i}^{-1}(U_{uv})$ are zero.

To construct T'_{vv}, we let $a_v \in \bar{\boldsymbol{k}}^\times \setminus \overline{\mathbb{F}}_q^\times$ and pick an element $\gamma_{1,v} \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ so that

$$\bar{\pi}_{1,i}(\gamma_{1,v}) = \begin{pmatrix} \mathfrak{a}_v & \\ & \mathfrak{a}_v \end{pmatrix}, \quad \text{where } \mathfrak{a}_v = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a_v & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in (\mathrm{GL}_{r/s}(\bar{\boldsymbol{k}}))_i, \tag{5.3.31}$$

such that a_v is in the $(i \cdot r/s + v)$-th diagonal entry of $\mathfrak{a}_v$. For $1 \le v \le r/s$, we let $c_{0,v}$ and $c_{1,v}$ respectively be the $(2r+1, (i-1)\cdot r/s + v)$-th and the $(2r+1, (r+(i-1)\cdot r/s)+v)$-th the entry of $\gamma_{1,v}$. Let T'_{vv} be the Zariski closure of the subgroup of $\overline{H}_{1,i}$ generated by $\gamma_{1,v}$ for each $1 \le v \le r/s$. Then, one checks directly that the defining equations of the one-dimensional subgroup T'_{vv} of $\overline{H}_{1,i}$ can be written as

$$\begin{cases} (a_v - 1)W_{0,v} - c_{0,v}((Z_0)_{v,v} - 1) = 0, & 1 \le v \le r^2/s, \\ (Z_0)_{w,w} = 1, & w \ne v,\ 1 \le v \le r^2/s, \\ (Z_1)_{u,v} = 0, & 1 \le u,\ v \le r/s, \\ W_{h,w} = 0, & w \ne v,\ h = 0, 1,\ 1 \le v \le r^2/s, \\ W_{0,v}\cdot c_{1,v} - W_{1,v}\cdot c_{0,v} = 0, & 1 \le v \le r^2/s. \end{cases}$$

Then, we see that $T'_{vv} \cong T_{vv}$ via $\bar{\pi}_{1,i}$. To construct T'_{uv} when $u \ne v$, we let $b_{u,v} \in T_{uv}(\boldsymbol{k})$ be a $\boldsymbol{k}$-rational basis for the one-dimensional vector group T_{uv} and pick $b'_{u,v} \in \overline{H}_{1,i}(\bar{\boldsymbol{k}})$ so that $\bar{\pi}_{1,i}(b'_{u,v}) = b_{u,v}$. We define T'_{uv} to be the one-dimensional vector group in $\overline{H}_{1,i}$ via the conjugations

$$\eta_v^{-1} b'_{uv} \eta_v \quad \text{for } \eta_v \in T'_{vv},\ v = 1, \dots, r/s.$$

Then, we have $T'_{uv} \cong T_{uv}$ via $\bar{\pi}_{1,i}$. Similarly, we use the methods used for T'_{vv} and conjugations as above to construct suitable one-dimensional U'_{uv} such that $U'_{uv} \cong U_{uv}$ for $1 \le u, v \le r/s$. The arguments are essentially the same as the ones used to construct T'_{vv} and T'_{uv}, and so we omit the details and leave it to the reader. This proves our claim. □

Claim 6. *For $\overline{Q}_{1,i}$ if all entries of the $\boldsymbol{v}_i$ coordinate vector are zero and $\dim \overline{Q}_{1,i} = r/s$, then $\mathrm{d}\bar{\pi}_{1,i} : \mathrm{Lie}\, \overline{H}_{1,i} \to \mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective.*

Proof of Claim 6. We have $\dim \overline{H}_{1,i} = 2r^2/s^2 + r/s$ and by (5.3.25),

$$\overline{Q}_{1,i} = \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{Id}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{z} & 1 \end{pmatrix} : \boldsymbol{z} = (\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{v}_{s+i}, \mathbf{0}, \dots, \mathbf{0}) \in \mathbb{G}_a^r \text{ where } \boldsymbol{v}_{s+i} \in \mathbb{G}_a^{r/s} \right\}.$$

Note that $\overline{\Gamma}_{N_0}$ is an algebraic subgroup of $\mathrm{GL}_{r+1}/\bar{\boldsymbol{k}}$ such that the surjective map $\overline{\Xi}_0 : \overline{\Gamma}_{N_1} \to \overline{\Gamma}_{N_0}$ induced by Ξ_0 in (5.3.8) is given by

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \mathbf{0} \\ \mathbf{0} & \gamma_0 & \mathbf{0} \\ z_0 & z_1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \gamma_0 & \mathbf{0} \\ z_0 & 1 \end{pmatrix}. \tag{5.3.32}$$

Then, the elements of $\operatorname{Ker}\Xi_0^{(\bar{k})} \subseteq \bar{\Gamma}_{N_1}(\bar{k})$ are of the form

$$\begin{pmatrix} \mathrm{Id}_r & \gamma_1 & \mathbf{0} \\ \mathbf{0} & \mathrm{Id}_r & \mathbf{0} \\ \mathbf{0} & z_1 & 1 \end{pmatrix}.$$

From this, we see that for any $\boldsymbol{b}_i \in \mathrm{Mat}_{r/s}(\bar{k})$, elements of the form $\bar{\boldsymbol{b}}_i'$ in (5.3.22) with $\boldsymbol{w}_i = \mathbf{0}$ are in $\bar{H}_{1,i}(\bar{k})$. Multiplying such $\bar{\boldsymbol{b}}_i'$ by suitable elements of $\bar{Q}_{1,i}(\bar{k})$, we have $\bar{\boldsymbol{b}}_i'$ of the form (5.3.22), where $\boldsymbol{w}_i = \boldsymbol{w}_{s+i} = \mathbf{0}$ in $\bar{H}_{1,i}(\bar{k})$. Let $\bar{P}_{1,i}$ be the Zariski closure inside $\bar{H}_{1,i}$ of the subgroup generated by all such $\bar{\boldsymbol{b}}_i'$ with $\boldsymbol{b}_i$ running over all elements of $\mathrm{Mat}_{r/s}(\bar{k})$. Then, clearly $\bar{P}_{1,i} \cong \mathrm{Mat}_{r/s}/\bar{k}$.

For any $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, multiplying the elements $\bar{\xi}_i' \in \bar{H}_{1,i}(\bar{k})$ of the form (5.3.21) by suitable elements of $Q_{1,i}(\bar{k})$, we obtain $\bar{\xi}_i'$, where $\boldsymbol{u}_{s+i} = \mathbf{0}$.

For all $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, if there is a $\bar{\xi}_i' \in \bar{H}_{1,i}(\bar{k})$ with $\boldsymbol{u}_i = \mathbf{0}$, then by using $\bar{P}_{1,i}$ and all such elements $\bar{\xi}_i'$ for all $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, we could simply construct $\bar{S}_{1,i}$ as in (5.3.24) and restrict $d\bar{\pi}_1$ to $\operatorname{Lie}\bar{S}_{1,i}$ to obtain a surjection onto $\operatorname{Lie}(\bar{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{k})_i$.

Next suppose $\boldsymbol{u}_i \neq \mathbf{0}$. Consider the short exact sequence of linear algebraic groups (see (5.3.18))

$$1 \to \bar{X}_0 \to \bar{\Gamma}_{N_0} \xrightarrow{\bar{\pi}_0} \bar{\Gamma}_{M_\rho} \to 1.$$

Consider the one-dimensional subgroups of $\bar{\Gamma}_{M_\rho}$ of the form $\mathcal{B}_{uv} \in (\mathrm{GL}_{r/s}/\bar{k})_i$ given in (5.3.30) for $1 \le u, v \le r/s$. The same methods used in Claim 5 to construct T'_{uv} can be applied in a straightforward manner to construct one-dimensional subgroups $\mathcal{B}'_{uv}$ of $\bar{\Gamma}_{N_0}$ so that $\mathcal{B}'_{uv} \cong \mathcal{B}_{uv}$. We leave this to the reader. For $\xi_i \in \mathrm{GL}_{r/s}(\bar{k})$, consider $\bar{\xi}_i' \in \bar{H}_{1,i}(\bar{k})$ of the form (5.3.21) with $\boldsymbol{u}_{s+i} = \mathbf{0}$. Let $\bar{V}_{1,i}$ be the Zariski closure inside $\bar{H}_{1,i}$ of the subgroup generated by all such $\bar{\xi}_i'$ with ξ_i running over all elements of $\mathrm{GL}_{r/s}(\bar{k})$. Then, we can identify $\nu \in \bar{V}_{1,i}(\bar{k})$ with the image $\bar{\Xi}_0^{(\bar{k})}(\nu) \in \bar{\Gamma}_{N_0}(\bar{k})$ where $\bar{\Xi}_0$ is the surjective map (5.3.32). Via this identification, each $\mathcal{B}'_{uv}$ for $1 \le u, v \le r/s$ is a one-dimensional subgroup of $\bar{V}_{1,i}$. The Lie algebras of the r^2/s^2 subgroups $\mathcal{B}_{uv}$ span $\operatorname{Lie}\mathrm{GL}_{r/s}/\bar{k}$. Thus, since $\bar{P}_{1,i} \cong \mathrm{Mat}_{r/s}/\bar{k}$, the Lie algebras of each $\mathcal{B}_{uv}$ and $\operatorname{Lie}\bar{P}_{1,i}$ span $\operatorname{Lie}(\bar{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{k})_i$ by (5.3.23). Then, since $\operatorname{Lie}(\cdot)$ is a left exact, $d\bar{\pi}_{1,i}$ is surjective. □

As we vary all $1 \le i \le s$, the surjection of $d\bar{\pi}_1$ follows. Thus, for $n = 1$ the proof of the lemma is complete.

Now suppose $n > 1$. We follow the methods used for $n = 1$ to prove that the induced tangent map $d\bar{\pi}_n$ at the identity is surjective onto $\operatorname{Lie}\bar{\Gamma}_{\mathrm{P}_n M_\rho}$. Recall $\bar{\Gamma}_{\mathrm{P}_n M_\rho}$ from (5.3.17). Let $w = 1$ and consider the short exact sequence (5.3.18) of linear algebraic groups. Fix $1 \le i \le s$. We follow the methods used for the construction of $\bar{H}_{1,i}$ above to construct the Zariski closure $\bar{H}_{n,i}$ inside $\bar{\Gamma}_{N_n}$ of the subgroup generated by suitably chosen elements of $\bar{\Gamma}_{N_n}$ such that $\bar{H}_{n,i}$ is contained in the $(n+1)r^2/s^2 + (n+1)r^2/s$-dimensional group

$$G_{n,i} := \left\{ \left(\begin{array}{c|c|c|c|c} \eta_0 & \eta_1 & \cdots & \eta_n & \mathbf{0} \\ \hline & \eta_0 & \ddots & \vdots & \vdots \\ \hline & & \ddots & \eta_1 & \vdots \\ \hline & & & \eta_0 & \mathbf{0} \\ \hline \boldsymbol{s}_0 & \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_n & 1 \end{array} \right) : \begin{array}{l} \eta_0 \in (\mathrm{GL}_{r/s}/\bar{k})_i,\ \eta_j \in (\mathrm{Mat}_{r/s}/\bar{k})_i,\ 1 \le j \le n, \\ \boldsymbol{s}_h = (\mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{s}_{h,i}, \mathbf{0}, \ldots, \mathbf{0}),\ \boldsymbol{s}_{h,i} \in \mathbb{G}_a^{r/s} \text{ for each } 0 \le h \le n \end{array} \right\}.$$

Let

$$(\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i := \left\{ \left(\begin{array}{c|c|c|c} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \hline & \gamma_0 & \ddots & \vdots \\ \hline & & \ddots & \gamma_1 \\ \hline & & & \gamma_0 \end{array} \right) : \gamma_0 \in (\mathrm{GL}_{r/\boldsymbol{s}}/\bar{\boldsymbol{k}})_i,\ \gamma_j \in (\mathrm{Mat}_{r/\boldsymbol{s}}/\bar{\boldsymbol{k}})_i,\ \text{where } 1 \le j \le n \right\}.$$

If $\dim \bar{H}_{n,i} = (n+1)\cdot r^2/\boldsymbol{s}^2 + (n+1)\cdot r/\boldsymbol{s}$, similar to $\bar{S}_{1,i}$ in (5.3.24) we simply construct

$$\bar{S}_{n,i} = \left\{ \begin{pmatrix} \vartheta_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : \vartheta_i \in (\Gamma_{\mathrm{P}_n M_\rho}(\bar{\boldsymbol{k}}))_i \right\} \tag{5.3.33}$$

and restrict $\mathrm{d}\bar{\pi}_n$ to Lie $\bar{S}_{n,i}$ to obtain a surjection onto $\mathrm{Lie}(\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$. As we vary all $1 \le i \le \boldsymbol{s}$, the surjection of $\mathrm{d}\bar{\pi}_n$ follows.

Next, suppose $\dim \bar{H}_{n,i} < (n+1)\cdot r^2/\boldsymbol{s}^2 + (n+1)\cdot r/\boldsymbol{s}$. Then, via $\bar{\pi}_n$ we have a short exact sequence

$$1 \to \bar{Q}_{n,i} \to \bar{H}_{n,i} \xrightarrow{\bar{\pi}_{n,i}} (\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i \to 1.$$

For $\bar{Q}_{n.i}$, if some entry of each $\boldsymbol{s}_{h,i}$ coordinate vector where $0 \le h \le n-\ell$ is nonzero or $\dim \bar{Q}_{n,i} \neq \ell r/\boldsymbol{s}$, then the methods used in Claim 4 to prove $\dim \bar{Q}_{1,i} = 0$ can be applied in a straightforward manner to prove $\dim \bar{Q}_{n,i} = 0$, which we leave to the reader.

Claim 7. *If* $\dim \bar{Q}_{n,i} = 0$, *then* $\mathrm{d}\bar{\pi}_{n,i} : \mathrm{Lie}\, \bar{H}_{n,i} \to \mathrm{Lie}(\bar{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$ *is surjective.*

Proof of Claim 7. The proof follows the same line of argument as in the proof of Claim 5 ($n=1$ case) and so we include only a sketch. Similar to the coordinates $\boldsymbol{Z}_1$ of $\bar{H}_{1,i}$ in (5.3.28), we let the coordinates of $\bar{H}_{n,i}$ be

$$\boldsymbol{Z}_n = \begin{pmatrix} \mathcal{Z}_0 & \mathcal{Z}_1 & \cdots & \mathcal{Z}_n & \mathbf{0} \\ & \mathcal{Z}_0 & \ddots & \vdots & \vdots \\ & & \ddots & \mathcal{Z}_1 & \vdots \\ & & & \mathcal{Z}_0 & \mathbf{0} \\ \mathcal{W}_0 & \mathcal{W}_1 & \cdots & \mathcal{W}_n & 1 \end{pmatrix},$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} \mathrm{Id}_{r/\boldsymbol{s}} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/\boldsymbol{s}} \end{pmatrix}, \quad \mathcal{Z}_j = \begin{pmatrix} \mathbf{0} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathbf{0} \end{pmatrix}$$

for each $1 \le j \le n$ such that (Z_0) is as in (5.3.28) and (Z_j) is the $r/\boldsymbol{s} \times r/\boldsymbol{s}$ block $((Z_j)_{a,b})$ for $1 \le a, b \le r/\boldsymbol{s}$. Moreover, set $\mathcal{W}_h := (0, \ldots, 0, (W_h), 0, \ldots, 0)$, where we set $(W_h) := (W_{h,1}, \ldots, W_{h,r/\boldsymbol{s}})$ for each $0 \le h \le n$.

Now, we prove that $d\bar{\pi}_{n,i} : \operatorname{Lie} \overline{H}_{n,i} \to \operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective. Similar to (5.3.29), we construct one-dimensional subgroups of $\overline{\Gamma}_{\mathrm{P}_n M_\rho}$:

$$T_{0,u,v} := \left\{ \begin{pmatrix} \mathcal{B}_{uv} & \mathbf{0} & \cdots & \mathbf{0} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \mathcal{B}_{uv} \end{pmatrix} \right\}, \quad U_{\ell,u,v} := \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathbf{0} & \cdots & \mathcal{C}_{uv} & \cdots & \mathbf{0} \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \mathcal{C}_{uv} \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \mathbf{0} \\ & & & & & \mathrm{Id}_r \end{pmatrix} \right\}$$

such that $\mathcal{B}_{uv}$ and $\mathcal{C}_{uv}$ are as in (5.3.30), and, for $1 \le \ell \le n$, $\mathcal{C}_{uv}$ is in the ℓ-th superdiagonal block of $U_{\ell,u,v}$. Similar to the $n = 1$ case, note that the Lie algebras of the $(n+1) \cdot r^2/\boldsymbol{s}^2$ algebraic groups $T_{0,u,v}$ and $U_{\ell,u,v}$ span $\operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$. In what follows, we construct one-dimensional algebraic subgroups $T'_{0,u,v}$ and $U'_{\ell,u,v}$ of $\overline{H}_{n,i}$ so that $T'_{0,u,v} \cong T_{0,u,v}$ and $U'_{\ell,u,v} \cong U_{\ell,u,v}$. Then, since $\operatorname{Lie}(\,\cdot\,)$ is a left exact functor, it follows that $\operatorname{Lie} T'_{0,u,v} \cong \operatorname{Lie} T_{0,u,v}$ and $\operatorname{Lie} U'_{\ell,u,v} \cong \operatorname{Lie} U_{\ell,u,v}$, and so $d\bar{\pi}_{n,i}$ is surjective. Since $\overline{Q}_{n,i}$ is a zero-dimensional vector group, $\bar{\pi}_{n,i}$ is injective on points and so it follows by checking directly that

- for $w \neq v$ and $0 \le h \le n$, all $W_{h,w}$ coordinates of $\bar{\pi}_{n,i}^{-1}(T_{0,u,v})$ are zero;
- all (W_0) coordinates of $\bar{\pi}_{n,i}^{-1}(U_{\ell,u,v})$ are zero, and for $w \neq v$ and $1 \le j \le n$, all $W_{j,w}$ coordinates of $\bar{\pi}_{n,i}^{-1}(U_{\ell,u,v})$ are zero.

To construct $T'_{0,v,v}$, we let $a_v \in \bar{\boldsymbol{k}}^\times \setminus \overline{\mathbb{F}}_q^\times$ and pick elements $\gamma_{n,v} \in \overline{H}_{n,i}(\bar{\boldsymbol{k}})$ so that

$$\bar{\pi}_{n,i}(\gamma_{n,v}) = \begin{pmatrix} \mathfrak{a}_v & & \\ & \ddots & \\ & & \mathfrak{a}_v \end{pmatrix},$$

where $\mathfrak{a}_v$ is as in (5.3.31). For $1 \le v \le r/\boldsymbol{s}$ and $0 \le h \le n$, we let $c_{h,v}$ be the $(nr+1, hr+(i-1)\cdot r/\boldsymbol{s}+v)$-th the entry of $\gamma_{n,v}$. Let $T'_{0,v,v}$ be the Zariski closure of the subgroup of $\overline{H}_{n,i}$ generated by $\gamma_{n,v}$. Then, one checks directly that the defining equations of the one-dimensional subgroup $T'_{0,v,v}$ of $\overline{H}_{n,i}$ can be written as

$$\begin{cases} (a_v - 1) W_{0,v} - c_{0,v}((Z_0)_{v,v} - 1) = 0, & 1 \le v \le r^2/\boldsymbol{s}, \\ (Z_0)_{w,w} = 1, & w \neq v,\ 1 \le v \le r^2/\boldsymbol{s}, \\ (Z_j)_{u,v} = 0, & 1 \le j \le \ell,\ 1 \le u, v \le r/\boldsymbol{s}, \\ W_{h,w} = 0, & w \neq v,\ 0 \le h \le \ell,\ 1 \le v \le r^2/\boldsymbol{s}, \\ W_{h_1,v} \cdot c_{h_2,v} - W_{h_2,v} \cdot c_{h_1,v} = 0, & 0 \le h_1, h_2 \le \ell,\ 1 \le v \le r^2/\boldsymbol{s}. \end{cases}$$

Then, we see that $T'_{0,v,v} \cong T_{0,v,v}$ via $\bar{\pi}_{n,i}$. Similarly, we use the methods used for $T'_{0,v,v}$ and conjugations as in the $n = 1$ case to construct $U'_{\ell,u,v}$ such that $U'_{\ell,u,v} \cong U_{\ell,u,v}$ for all $1 \le u, v \le r/\boldsymbol{s},\ 1 \le \ell \le n$, and $T'_{0,u,v}$ such that $T'_{0,u,v} \cong T_{0,u,v}$ for all $1 \le u, v \le r/\boldsymbol{s},\ u \neq v$. The arguments are essentially the same as the arguments used to construct T'_{uv} and U'_{uv} in the $n = 1$ case and $T'_{0,v,v}$ above, and so we omit the details and leave it to the reader. This proves our claim. □

For $Q_{n,i}$ if $\boldsymbol{s}_{h,i}=\mathbf{0}$ for all $0\le h\le n-\ell$ and $\dim Q_{n,i}=\ell r/\boldsymbol{s}$, then the methods used in Claim 6 to prove that $\mathrm{d}\bar{\pi}_{1,i}:\operatorname{Lie}\overline{H}_{1,i}\to\operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective can be applied in a straightforward manner to prove that $\mathrm{d}\bar{\pi}_{n,i}:\operatorname{Lie}\overline{H}_{n,i}\to\operatorname{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\bar{\boldsymbol{k}})_i$ is surjective, which we leave to the reader.

As we vary all $1\le i\le \boldsymbol{s}$ the surjection of $\mathrm{d}\bar{\pi}_n$ follows. Thus, for $n>1$ the proof of the lemma is complete. □

5.4. ***Algebraic independence of logarithms and quasilogarithms.*** We now prove Theorem 1.1.4 (restated as Theorem 5.4.4) and Corollary 1.1.5. Recall the short exact sequence (5.3.3):

$$1\to X_n\to\Gamma_{N_n}\xrightarrow{\pi_n}\Gamma_{\mathrm{P}_n M_\rho}\to 1.$$

We will first show that X_n can be identified with a $\Gamma_{\mathrm{P}_n M_\rho}$-submodule of $((\mathrm{P}_n M_\rho)^B)^w$. Let $\boldsymbol{n}\in\operatorname{Mat}_{((n+1)rw+1)\times 1}(N_n)$ be such that its entries form a $\bar{k}(t)$-basis of N_n and $\sigma\boldsymbol{n}=\Phi_{N_n}\boldsymbol{n}$. The entries of $\Psi_{N_n}^{-1}\boldsymbol{n}$ form a $\boldsymbol{k}$-basis of N_n^B [Papanikolas 2008, Proposition 3.3.9]. If we write $\boldsymbol{n}=[\boldsymbol{n}_1,\dots,\boldsymbol{n}_w,y]^{\mathsf{T}}$, where each $\boldsymbol{n}_i\in\operatorname{Mat}_{(n+1)r\times 1}(N_n)$, then the entries of $[\boldsymbol{n}_1,\dots,\boldsymbol{n}_w]^{\mathsf{T}}$ form a $\bar{k}(t)$-basis of $(\mathrm{P}_n M_\rho)^w$ and the entries of $\boldsymbol{u}:=[\Psi_{\mathrm{P}_n M_\rho}^{-1}\boldsymbol{n}_1,\dots,\Psi_{\mathrm{P}_n M_\rho}^{-1}\boldsymbol{n}_w]^{\mathsf{T}}$ form a $\boldsymbol{k}$-basis of $((\mathrm{P}_n M_\rho)^B)^w$. Given any $\boldsymbol{k}$-algebra R, we recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R}\otimes_{\boldsymbol{k}}((\mathrm{P}_n M_\rho)^B)^w$ from [Papanikolas 2008, §4.5] (see also (4.2.8)) as follows: for any $\mu\in\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $\boldsymbol{v}_h\in\operatorname{Mat}_{1\times(n+1)r}(\mathrm{R})$, $0\le h\le n$, the action of μ on $(\boldsymbol{v}_1,\dots,\boldsymbol{v}_w)\cdot\boldsymbol{u}\in R\otimes_{\boldsymbol{k}}((\mathrm{P}_n M_\rho)^B)^w$ is

$$(\boldsymbol{v}_1,\dots,\boldsymbol{v}_w)\cdot\boldsymbol{u}\mapsto(\boldsymbol{v}_1\mu^{-1},\dots,\boldsymbol{v}_w\mu^{-1})\cdot\boldsymbol{u}.$$

Thus, by (5.3.4) the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on $((\mathrm{P}_n M_\rho)^B)^w$ is compatible with the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on X_n. Then, when we regard $((\mathrm{P}_n M_\rho)^B)^w$ as a vector group over $\boldsymbol{k}$, by Lemma 5.3.9 we get the desired result.

Now, note that since X_n is a $\Gamma_{\mathrm{P}_n M_\rho}$-submodule of $((\mathrm{P}_n M_\rho)^w)^B$, by the equivalence of categories $\mathcal{T}_{\mathrm{P}_n M_\rho}\approx\mathbf{Rep}(\Gamma_{\mathrm{P}_n M_\rho},\boldsymbol{k})$, there exists a sub-$t$-motive V_n of $(\mathrm{P}_n M_\rho)^w$ such that as $\Gamma_{\mathrm{P}_n M_\rho}$-modules

$$X_n\cong V_n^B. \tag{5.4.1}$$

By (4.2.7), we see that for any $n\ge 1$ and $0\le j\le n-1$ we obtain a short exact sequence of t-motives

$$0\to(\mathrm{P}_j M_\rho)^w\xrightarrow{\iota}(\mathrm{P}_n M_\rho)^w\xrightarrow{\mathbf{pr}_{w,n-j-1}}(\mathrm{P}_{n-j-1}M_\rho)^w\to 0. \tag{5.4.2}$$

Lemma 5.4.3. *For $n\ge 1$, let V_n be as in (5.4.1). Then, for $0\le j\le n-1$ there is a surjective map of t-motives $\overline{\mathbf{pr}}_{w,n-j-1}:V_n\to V_{n-j-1}$ via the map $\mathbf{pr}_{w,n-j-1}$ in (5.4.2).*

Proof. We prove the result for $w=1$. The following argument for $w=1$ can be applied in a straightforward manner to prove the arbitrary w case, which we leave to the reader. Let $w=1$. Recall from (5.3.7) that for any $\boldsymbol{k}$-algebra R if

$$\nu_n=\begin{pmatrix}\mu_n & \mathbf{0}\\ \boldsymbol{w}_n & 1\end{pmatrix}\in\Gamma_{N_n}(\mathrm{R}),$$

then

$$\nu_{n-j-1}=\begin{pmatrix}\mu_{n-j-1} & \mathbf{0}\\ \boldsymbol{w}_{n-j-1} & 1\end{pmatrix}\in\Gamma_{N_{n-j-1}}(\mathrm{R}),$$

where μ_{n-j-1} is the matrix formed by the $r(n-j)\times r(n-j)$ upper-left square of μ_n and $\boldsymbol{w}_{n-j-1} = (w_0, \dots, w_{n-j-1})$. Also recall from (5.3.8) that the surjective map of affine group schemes $\Gamma_{N_n} \twoheadrightarrow \Gamma_{N_{n-j-1}}$ is given by

$$\nu_n \mapsto \nu_{n-j-1}.$$

Since X_n and X_{n-j-1} are $\boldsymbol{k}$-smooth by Lemma 5.3.9, this map gives a surjective map of group schemes $X_n \to X_{n-j-1}$. By (5.4.1), this corresponds to a map of representations of $\Gamma_{\mathrm{P}_n M_\rho}$ over $\boldsymbol{k}$, $\overline{\mathbf{pr}}^B_{w,n-j-1} : V_n^B \to V_{n-j-1}^B$ via the map $\mathbf{pr}^B_{w,n-j-1} : ((\mathrm{P}_n M_\rho)^w)^B \to ((\mathrm{P}_{n-j-1} M_\rho)^w)^B$, where $\mathbf{pr}_{w,n-j-1}$ is as in (5.4.2). By the equivalence of categories $\mathcal{T}_{\mathrm{P}_n M_\rho} \approx \mathbf{Rep}(\Gamma_{\mathrm{P}_n M_\rho}, \boldsymbol{k})$, we obtain the required conclusion. □

Theorem 5.4.4. *Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep}. Suppose that K_ρ is separable over k and $[K_\rho : k] = \boldsymbol{s}$. Let $u_1, \dots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $1 \le i \le w$ and suppose that $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w) = r/\boldsymbol{s} + w$. For $n \ge 1$, let N_n and Ψ_{N_n} be defined as in (5.3.1), and, for each $1 \le i \le w$, let $Y_{i,n} := Y_{u_i,n}$ be defined as in Section 5.2. Then, $\dim \Gamma_{N_n} = (n+1)\cdot r(r/\boldsymbol{s}+w)$. In particular,*

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}\right) = (n+1)(r^2/\boldsymbol{s} + rw).$$

Proof. From the construction of Ψ_{N_n}, by Theorem 3.4.1 we have

$$\bar{k}(\Psi_{N_n}|_{t=\theta}) = \bar{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}\right),$$

and by Theorems 2.3.2 and 4.5.1, we have

$$\dim \Gamma_{N_n} = \operatorname{tr.deg}_{\bar{k}} \bar{k}(\Psi_{N_n}|_{t=\theta}) \le (n+1)\frac{r^2}{\boldsymbol{s}} + (n+1)rw.$$

Thus, we need to prove that $\dim X_n = (n+1)rw$, where X_n is as in (5.3.3). By (5.4.1) it suffices to show that $V_n^B \cong ((\mathrm{P}_n M_\rho)^w)^B$. To prove this, we adapt the arguments of the proof of [Chang and Papanikolas 2012, Theorem 5.1.5] (see also [Hardouin 2011, Lemma 1.2]).

Note from (5.4.2) that for $n \ge 1$ we have a short exact sequence of t-motives

$$0 \to (\mathrm{P}_0 M_\rho)^w \xrightarrow{\iota} (\mathrm{P}_n M_\rho)^w \xrightarrow{\mathbf{pr}_{w,n-1}} (\mathrm{P}_{n-1} M_\rho)^w \to 0.$$

By Lemma 5.4.3, there is a surjective map $\overline{\mathbf{pr}}_{w,n-1} : V_n \to V_{n-1}$ via $\mathbf{pr}_{w,n-1}$. Then $\ker(\overline{\mathbf{pr}}_{w,n-1})$ is a sub-t-motive of M_ρ^w.

We claim that if $V_{n-1} \cong (\mathrm{P}_{n-1} M_\rho)^w$, then N_n/V_n is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho/V_n)$. Since $X_n \cong V_n^B$, we see that Γ_{N_n} acts on N_n^B/V_n^B through $\Gamma_{N_n}/X_n \cong \Gamma_{\mathrm{P}_n M_\rho}$ via (5.3.3). Since $\overline{\mathbf{pr}}_{w,n-1}$ is surjective onto $V_{n-1} \cong (\mathrm{P}_{n-1} M_\rho)^w$, by using (5.3.5) one finds that $N_n^B/V_n^B \cong N_0^B/(\ker \overline{\mathbf{pr}}_{w,n-1})^B$. Recall that for any $\boldsymbol{k}$-algebra R, an element of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form (4.2.6) such that γ_0 is an element of $\Gamma_{M_\rho}(\mathrm{R})$. Then, (5.3.6) shows the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on N_n^B/V_n^B is the same as the action of Γ_{M_ρ} on it. Thus, N_n^B/V_n^B is an extension of $\boldsymbol{k}$ by $((\mathrm{P}_n M_\rho)^w)^B / V_n^B$ in $\mathbf{Rep}(\Gamma_{M_\rho}, \boldsymbol{k})$. By [Chang and Papanikolas 2012, Corollary 3.5.7] and the equivalence of categories $\mathcal{T}_{M_\rho} \approx \mathbf{Rep}(\Gamma_{M_\rho}, \boldsymbol{k})$, we get the required conclusion of the claim.

Now, we prove the main result by induction. For $n = 1$ case, suppose to the contrary that $V_1^B \subsetneq ((\mathrm{P}_1 M_\rho)^w)^B$. From [loc. cit., Theorem 5.1.5], we have $M_\rho^w \cong V_0$ and so, since $M_\rho^w \cong (\mathrm{P}_0 M_\rho)^w$, we have $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subsetneq M_\rho^w$. Since M_ρ^w is completely reducible in $\mathcal{T}_{M_\rho}$ by [loc. cit., Corollary 3.3.3] and $\ker(\overline{\mathbf{pr}}_{w,n-1})$ is a sub-t-motive of M_ρ^w, there exists a nontrivial morphism $\phi_1 \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$ so that $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subseteq \ker \phi_1$. Moreover, the morphism ϕ_1 factors through the map $M_\rho^w / \ker(\overline{\mathbf{pr}}_{w,n-1}) \to M_\rho^w / (\ker \phi_1)$. Since $\phi_1 \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$, there exist $e_{i,1} \in \boldsymbol{K}_\rho$ not all zero such that $\phi_1(n_1, \dots, n_w) = \sum_{i=1}^w e_{i,1}(n_i)$. For a $\bar{k}(t)$-basis $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ of M_ρ, suppose that $\mathrm{E}_{i,1} \in \mathrm{Mat}_r(\bar{k}(t))$ satisfies $e_{i,1}(\boldsymbol{m}) = \mathrm{E}_{i,1}\boldsymbol{m}$. Set

$$\boldsymbol{E}_{i,1} := \begin{pmatrix} \mathbf{0} & \mathrm{E}_{i,1} \\ & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{2r}(\bar{k}(t)).$$

Recall from Section 2.5 that $\boldsymbol{D}_1\boldsymbol{m}$ forms a $\bar{k}(t)$-basis of $\mathrm{P}_1 M_\rho$. By (5.2.1) there exists $\boldsymbol{e}_{i,1} \in \mathrm{End}_{\mathcal{T}}((\mathrm{P}_1 M)^w)$ such that $\boldsymbol{e}_{i,1}(\boldsymbol{D}_1\boldsymbol{m}) = \boldsymbol{E}_{i,1}\boldsymbol{D}_1\boldsymbol{m}$. Let $\psi_1 \in \mathrm{Hom}_{\mathcal{T}}((\mathrm{P}_1 M_\rho)^w, \mathrm{P}_1 M_\rho)$ such that $\psi_1(D_j n_1, \dots, D_j n_w) = \sum_{i=1}^w \boldsymbol{e}_{i,1}(D_j n_i)$ for each $j = 0, 1$. We see $\ker \psi_1 / M_\rho^w \cong \ker \phi_1$ and $(\mathrm{P}_1 M_\rho)^w / \ker \psi_1 \cong M_\rho^w / \ker \phi_1 \cong M_\rho$. Then the pushout $\psi_{1*} N_1 := \boldsymbol{e}_{1,1*} Y_{1,1} + \cdots + \boldsymbol{e}_{w,1*} Y_{w,1}$ is a quotient of N_1 / V_1. By using the claim above, it follows that $\psi_{1*} N_1$ is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_1 M_\rho)$. However, by Theorem 5.2.2, this is a contradiction.

Now suppose that we have shown the result for $n-1$, that is, $V_{n-1} \cong (\mathrm{P}_{n-1} M_\rho)^w$. Suppose that $V_n^B \subsetneq ((\mathrm{P}_n M_\rho)^w)^B$. Then, $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subsetneq M_\rho^w$. Since M_ρ^w is completely reducible in $\mathcal{T}_{M_\rho}$ by [Chang and Papanikolas 2012, Corollary 3.3.3] and $\ker(\overline{\mathbf{pr}}_{w,n-1})$ is a sub-t-motive of M_ρ^w, there exists a nontrivial morphism $\phi_n \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$ so that $\ker(\overline{\mathbf{pr}}_{w,n-1}) \subseteq \ker \phi_n$. Moreover, the morphism ϕ_n factors through the map $M_\rho^w / \ker(\overline{\mathbf{pr}}_{w,n-1}) \to M_\rho^w / (\ker \phi_n)$. Since $\phi_n \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$, we can write $\phi_n(n_1, \dots, n_w) = \sum_{i=1}^w e_{i,n}(n_i)$ for some $e_{1,n}, \dots, e_{w,n} \in \boldsymbol{K}_\rho$ not all zero. Suppose that $e_{i,n}(\boldsymbol{m}) = \mathrm{E}_{i,n}\boldsymbol{m}$, where $\mathrm{E}_{i,n} \in \mathrm{Mat}_r(\bar{k}(t))$. Set

$$\boldsymbol{E}_{i,n} := \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathrm{E}_{i,n} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{(n+1)r}(\bar{k}(t)).$$

Recall also from Section 2.5 that $\boldsymbol{D}_n\boldsymbol{m}$ forms a $\bar{k}(t)$-basis of $\mathrm{P}_n M_\rho$. By (5.2.1) there exists $\boldsymbol{e}_{i,n} \in \mathrm{End}_{\mathcal{T}}((\mathrm{P}_n M)^w)$ such that $\boldsymbol{e}_{i,n}(\boldsymbol{D}_1\boldsymbol{m}) = \boldsymbol{E}_{i,n}\boldsymbol{D}_1\boldsymbol{m}$. Let $\psi_n \in \mathrm{Hom}_{\mathcal{T}}((\mathrm{P}_n M_\rho)^w, \mathrm{P}_n M_\rho)$ such that $\psi_1(D_j n_1, \dots, D_j n_w) = \sum_{i=1}^w \boldsymbol{e}_{i,1}(D_j n_i)$ for each $0 \le j \le n$. Similar to the base case, we see that $\ker \psi_n / (\mathrm{P}_{n-1} M_\rho)^w \cong \ker \phi_n$ and $(\mathrm{P}_n M_\rho)^w / \ker \psi_n \cong M_\rho^w / \ker \phi_n \cong M_\rho$. Then the pushout $\psi_{n*} N_n := \boldsymbol{e}_{1,n*} Y_{1,n} + \cdots + \boldsymbol{e}_{w,n*} Y_{w,n}$ is a quotient of N_n / V_n. By using the claim above, it follows that $\psi_{n*} N_n$ is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. However, by Theorem 5.2.2, this is a contradiction. □

Proof of Corollary 1.1.5. Let $\{\eta_1, \dots, \eta_\alpha\} \subseteq \{\lambda_1, \dots, \lambda_r, u_1, \dots, u_w\}$ be a maximal K_ρ-linearly independent set containing $\{u_1, \dots, u_w\}$. Clearly, $r/s \le \alpha \le r/s + w$. Since the quasiperiodic functions F_δ are linear in δ and satisfy the difference equation (1.1.2), we have

$$\bar{k}\Big(\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\lambda_j, F_{\tau^i}(\lambda_j), u_m, F_{\tau^i}(u_m)\}\Big) = \bar{k}\Big(\bigcup_{j=1}^{r}\bigcup_{m=1}^{\alpha}\{F_{\delta_j}(\eta_m)\}\Big).$$

Moreover, for any $1 \le i_1, i_2 \le r$, $1 \le j_1, j_2 \le \alpha$, $0 \le s \le n$ and $v_1, v_2 \in K_\rho$, by the product rule of hyperderivatives we obtain

$$\partial_\theta^s(v_1 F_{\delta_{i_1}}(\eta_{j_1}) + v_2 F_{\delta_{i_2}}(\eta_{j_2})) = \sum_{h=0}^{s}\big(\partial_\theta^{s-h}(v_1)\partial_\theta^h(F_{\delta_{i_1}}(\eta_{j_1})) + \partial_\theta^{s-h}(v_2)\partial_\theta^h(F_{\delta_{i_1}}(\eta_{j_2}))\big).$$

Thus,

$$\bar{k}\Big(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}\Big) = \bar{k}\Big(\bigcup_{s=0}^{n}\bigcup_{j=1}^{r}\bigcup_{m=1}^{\alpha}\{\partial_\theta^s(F_{\delta_j}(\eta_m))\}\Big).$$

Then, the result follows by Theorem 5.4.4. □

Appendix: Differential algebraic geometry

We present a few topics from differential algebraic geometry in positive characteristic [Okugawa 1987] (see [Hardouin et al. 2016] for characteristic zero). For the most part, we follow the terminology of [Hardouin et al. 2016]. Even though the proofs of most of the results presented here are covered in [Okugawa 1987], we present them here nevertheless for completeness.

A.1. ***Differential algebraic geometry in positive characteristic.*** Let R be a commutative ring with unity of characteristic $p > 0$. A *differential ring* or ∂-*ring* is a pair (R, ∂), where ∂ represents a sequence of additive maps $\partial^j : R \to R$ that satisfy

(1) $\partial^0(a) = a$,

(2) $\partial^j(a+b) = \partial^j(a) + \partial^j(b)$,

(3) $\partial^j(ab) = \sum_{i=0}^{j}\partial^i(a)\partial^{j-i}(b)$,

(4) $\partial^k\partial^j(a) = \binom{k+j}{j}\partial^{k+j}(a)$

for all $a, b \in R$ and $j, k \ge 0$. If R is a field, then we say that (R, ∂) is a *differential field* or a ∂-*field*. When the context is clear, we shall write R instead of (R, ∂). Moreover, a ∂-*morphism* between two ∂-rings R and S is a morphism of rings that commute with ∂. For a ∂-ring R, if we let $\mathfrak{I} \subseteq R$ be an ideal, then $\mathfrak{I}$ is called a ∂-*ideal* if $\partial^j(\mathfrak{I}) \subseteq \mathfrak{I}$ for all $j \ge 1$. If, in addition, $\mathfrak{I}$ is a radical (respectively prime) ideal of the ∂-ring R regarded as a ring, then we say that $\mathfrak{I}$ is a *radical* (respectively *prime*) ∂-*ideal* of the ∂-ring R. For a set $\Sigma \subseteq R$, the intersection of all ∂-ideals containing Σ is a ∂-ideal of R, which we denote by $\mathfrak{D}(\Sigma)$ and it is the smallest ∂-ideal of R containing Σ. We see that $\mathfrak{D}(\Sigma)$ is the ideal, generated $\{\partial^j(a) : j \ge 0,\ a \in \Sigma\}$, of the ∂-ring R regarded as a ring. We denote by $\mathfrak{R}(\mathfrak{D}(\Sigma))$ or $\mathfrak{R}(\Sigma)$ the radical of $\mathfrak{D}(\Sigma)$ in the ∂-ring R.

Proposition A.1.1 [Okugawa 1987, p. 45, Theorem 5]. *Let R be a ∂-ring of characteristic $p > 0$ and let $\mathfrak{I} \subseteq R$ be a ∂-ideal of R. Then, the radical $\mathfrak{R}(\mathfrak{I})$ is a ∂-ideal of R.*

Proof. It suffices to prove that $\partial^j(\mathfrak{R}(\mathfrak{I})) \subseteq \mathfrak{R}(\mathfrak{I})$ for all $j \geq 1$. Let $a \in \mathfrak{R}(\mathfrak{I})$. Then $a^n \in \mathfrak{I}$ for some $n \geq 1$. For a sufficiently large $e \geq 1$, we see that $a^m \cdot a^n = a^{p^e} \in \mathfrak{I}$ for some $m \in \mathbb{N}$. Note that Proposition 2.4.1 applies here and so, for all $j \in \mathbb{N}$ we see that $\partial^{jp^e}(a^{p^e}) = (\partial^j(a))^{p^e}$. Since $\mathfrak{I}$ is a ∂-ideal of R, we have $\partial^{jp^e}(a^{p^e}) \in \mathfrak{I}$ for all $j \geq 1$. Thus, $(\partial^j(a))^{p^e} \in \mathfrak{I}$ and so $\partial^j(a) \in \mathfrak{R}(\mathfrak{I})$. □

Remark A.1.2. The proof of Proposition A.1.1 does not work in characteristic 0. See [Hardouin et al. 2016, Proposition 2.19] for the proof of the characteristic-0 case.

The ∂-*polynomial ring* denoted by $R\{y_1, \ldots, y_m\}$ in the ∂-variables $(y_1, \ldots, y_m)$ is the polynomial ring over a ∂-ring R in the variables $\partial^j(y_i)$, $j \geq 0$, $1 \leq i \leq m$, made into a ∂-ring by setting

(a) $\partial^j(a) := \partial^j(a)$ for $a \in R$,

(b) $\partial^k(\partial^j(y_i)) := \binom{k+j}{j}\partial^{k+j}(y_i)$, $k \geq 0$.

Here $y_1, \ldots, y_m$ are called ∂-*indeterminates*.

Let K be a ∂-field. A ∂-*extension field of* K is a ∂-field L which is an extension field of the ∂-field K. Note that K and L are fields. Let $\overline{K}$ be an algebraic closure of the field K and K^{sep} be the separable closure of K in $\overline{K}$.

Proposition A.1.3. *There is a unique extension of $\partial^j : K \to K$ to $\partial^j : K^{\text{sep}} \to K^{\text{sep}}$ which satisfies all the rules of ∂.*

Proof. The proof follows the same argument as that for hyperderivatives [Conrad 2000, Theorem 5]. □

Let $a \in \overline{K} \setminus K^{\text{sep}}$. We say that ∂ can be *extended to a* if ∂ can be extended to some extension field of K^{sep} that contains a. The largest extension field $\overline{K}^{\partial}$ of K^{sep} in $\overline{K}$ that has an extension of ∂ is called the ∂-*closure* of K in $\overline{K}$.

For a set $X \subseteq (\overline{K}^{\partial})^m$, if we set

$$\mathfrak{I}(X) := \{P \in K\{y_1, \ldots, y_m\} : P(a_1, \ldots, a_m) = 0,\ (a_1, \ldots, a_m) \in X\},$$

then $\mathfrak{I}(X)$ is a radical ∂-ideal in R, and we call it the *defining K-∂-ideal* of X.

Proposition A.1.4 [Hardouin et al. 2016, Proposition 3.8]. *Let $X_1, X_2 \subseteq (\overline{K}^{\partial})^m$. Then*:

(1) *If $X_1 \subseteq X_2$, then $\mathfrak{I}(X_2) \subseteq \mathfrak{I}(X_1)$.*

(2) $\mathfrak{I}(X_1 \cup X_2) = \mathfrak{I}(X_1) \cap \mathfrak{I}(X_2)$.

Proof. The proofs follow the same line of argument as that for the Zariski topology. □

Given a set $X \subseteq ((\overline{K}^{\partial})^m, \partial)$, we consider the Zariski closure $\overline{X}^Z \subseteq \overline{K}^m$ of X, the closure of X as a subset of $(\overline{K}^{\partial})^m$ equipped with the Zariski topology. Let $S \subseteq K[y_1, \ldots, y_m]$ be a set of polynomials. The *zero set* of S is defined as

$$\mathcal{Z}(S) := \{(a_1, \ldots, a_m) \in \overline{K}^m : f(a_1, \ldots, a_m) = 0,\ f \in S\}.$$

Lemma A.1.5 (cf. [Hardouin et al. 2016, Lemma 3.42]). *Let $X \subseteq (\overline{K}^{\partial})^m$ and let $\mathfrak{I}(X) \subseteq K\{y_1, \dots, y_m\}$ be its defining K-∂-ideal. Also, let $K[y_1, \dots, y_m]$ be the usual polynomial ring in the variables $y_1, \dots, y_m$ over the field K. Then its Zariski closure is the set*

$$\overline{X}^Z = \mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]),$$

where $\mathfrak{I}(X) \cap K[y_1, \dots, y_m] \subseteq K[y_1, \dots, y_m]$.

Proof. We follow the outline of the proof of [Hardouin et al. 2016, Lemma 3.42]. Since $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m])$ is Zariski closed, it is straightforward to see that

$$X \subseteq \overline{X}^Z \subseteq \mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]).$$

Conversely, if $S \subseteq K[y_1, \dots, y_m] \subseteq K\{y_1, \dots, y_m\}$ is such that $S \subseteq \mathfrak{I}(X)$, then clearly $\mathfrak{R}(S) \subseteq \mathfrak{I}(X)$. This implies $S \subseteq \mathfrak{R}(S) \cap K[y_1, \dots, y_m] \subseteq \mathfrak{I}(X) \cap K[y_1, \dots, y_m]$. Thus, $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]) \subseteq \mathcal{Z}(S)$. Since S was chosen arbitrarily, we see that $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \dots, y_m]) \subseteq \overline{X}^Z$. □

If $f \in K\{y_1, \dots, y_m\}$ is a ∂-polynomial given by a linear combination over the ∂-field K of 1 and elements of the set $\{\partial^j(y_i) : j \geq 0,\ 1 \leq i \leq m\}$, then we say that f is a *degree*-1 ∂-*polynomial* in $K\{y_1, \dots, y_m\}$. Moreover if the coefficient of 1 is 0, then we say that such f is a *homogeneous degree*-1 ∂-*polynomial*.

Proposition A.1.6 [Okugawa 1987, p. 74, Theorem 5]. *Let $S \subseteq K\{y_1, \dots, y_m\}$ be a set of degree-*1 *∂-polynomials. Then, $\mathfrak{R}(S) = \mathfrak{D}(S)$.*

Proof. It suffices to show that $\mathfrak{D}(S)$ is a prime ideal of the ∂-ring $K\{y_1, \dots, y_m\}$ regarded as a usual ring. By definition $\mathfrak{D}(S)$ is generated, as an ideal of the ring $K\{y_1, \dots, y_m\}$, by $\{\partial^j(L_i) : i, j \geq 0,\ L_i \in S\}$. Suppose that $f, g \notin \mathfrak{D}(S)$ such that $fg \in \mathfrak{D}(S)$. Then,

$$fg = \sum_{L_i \in S,\ j \geq 1} h_{i,\ell_j} \partial^{\ell_j}(L_i),$$

where $\ell_j \geq 0$, and $h_{i,\ell_j} \in K\{y_1, \dots, y_m\}$, and all but finitely many h_{i,ℓ_j} are zero. We see that fg is a polynomial in a finite subset of the variables $\{\partial^j(y_i) : j \geq 0,\ 1 \leq i \leq m\}$ over the ∂-field K regarded as a usual field. Let us denote this subset of variables by $\{x_1, \dots, x_n\}$ for some $n \geq 1$. Then, $L = (\{\partial^{\ell_j}(L_i)\})$ is an ideal in $K[x_1, \dots, x_n]$ such that $f, g \notin L$ and $fg \in L$ and so L is not a prime ideal. However, for a polynomial ring in finitely many indeterminates, ideals generated by degree-1 polynomials are prime ideals and thus, we obtain a contradiction. □

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Mutation and torsion pairs

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Mutation of compact silting objects is a fundamental operation in the representation theory of finite-dimensional algebras due to its connections to cluster theory and to the lattice of torsion pairs in module or derived categories. We develop a theory of mutation in the broader framework of silting or cosilting t-structures in triangulated categories. We show that mutation of pure-injective cosilting objects encompasses the classical concept of mutation for compact silting complexes. As an application we prove that any minimal inclusion of torsion classes in the category of finitely generated modules over an artinian ring corresponds to an irreducible mutation. This generalises a well-known result for functorially finite torsion classes.

1. Introduction

The operation of mutation has a long history in representation theory and algebraic geometry. The aim of a mutation is to create a new object from an old one, changing a designated part of it and keeping the other part. This operation has been around for at least thirty years, for example in the study of exceptional collections of sheaves [Gorodentsev and Rudakov 1987] or in the combinatorial study of tilting modules [Riedtmann and Schofield 1991; Happel and Unger 2005]. Mutation also plays a central role in the foundations of cluster theory, which were set up by Fomin and Zelevinsky in the early 2000s. Their work, together with the categorification results of [Buan et al. 2006], reinforced the importance of mutation in contemporary representation theory.

As shown in [Aihara and Iyama 2012], the right framework to study mutation in derived categories of finite-dimensional algebras is provided by silting complexes, a significant generalisation of tilting

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modules which was introduced in [Keller and Vossieck 1988]. Indeed, the categorification of cluster algebras allows us to interpret clusters as silting complexes and cluster mutation as an operation that produces a new silting complex from a given one by exchanging a summand. It turns out that mutation of silting objects also provides a deep insight into the structure of the lattice tors(A) of torsion pairs in the category mod(A) of finite-dimensional modules over a finite-dimensional algebra A. In [Adachi et al. 2014] it was shown that minimal inclusions of functorially finite torsion classes correspond, in a suitable way, to a mutation operation on associated silting complexes. The following question was the initial motivation for this paper.

Question. *Do minimal inclusions of arbitrary torsion classes correspond to an operation of mutation*?

In order to answer this question, we are forced to leave the realm of finite-dimensional modules, but fortunately we can benefit from well-developed model-theoretic methods [Prest 2009; Laking 2020]. We approach the above question via cosilting theory, where small silting complexes are replaced by pure-injective cosilting objects. More recently, this concept has also appeared in the literature under the name derived injective (or just injective) object and turned out to be indispensable in various other contexts: in spectral algebraic geometry [Lurie 2016, Appendix C.5.7], deformation theory of dg categories [Genovese et al. 2021] or derived commutative algebra [Shaul 2018].

Our starting point here is an observation based on [Crawley-Boevey 1994; Breaz and Žemlička 2018; Zhang and Wei 2017]: every "small" torsion pair in mod(A) corresponds bijectively to a "large" torsion pair in the category Mod(A) of all A-modules, and the latter is determined by a large (i.e., not necessarily compact) pure-injective two-term cosilting complex σ. In other words, while two-term silting complexes in the sense of [Adachi et al. 2014] only detect functorially finite torsion pairs in mod(A), two-term cosilting complexes detect all torsion pairs in mod(A). In fact, pure-injective cosilting complexes give rise to a class of t-structures in the derived category that encompasses those associated to compact silting complexes.

Summary of main results. We first develop a general framework to study mutation of silting and cosilting objects in triangulated categories, without any compactness or pure-injectivity assumptions. Our Definitions 3.2 and 5.1 extend the concept of mutation in [Aihara and Iyama 2012] to the noncompact case, adopting the notion of (co)silting from [Psaroudakis and Vitória 2018; Nicolás et al. 2019].

Every cosilting object σ gives rise to a t-structure $\mathbb{T}_\sigma$ and to an abelian category $\mathcal{H}_\sigma$, the heart of $\mathbb{T}_\sigma$. Moreover, every subset $\mathscr{E}$ of Prod(σ) induces a set of injectives $H^0_\sigma(\mathscr{E})$ in $\mathcal{H}_\sigma$ and thus a hereditary torsion pair $(\mathcal{S}, \mathcal{R}) = ({}^{\perp_0}H^0_\sigma(\mathscr{E}), \operatorname{Cogen}(H^0_\sigma(\mathscr{E})))$. It turns out that such torsion pairs control the process of mutation via HRS-tilting.

Theorem A (Theorem 3.5). *Let $\mathcal{D}$ be a triangulated category with products, and let σ and σ' be two cosilting objects in $\mathcal{D}$. Let $\mathscr{E} = \operatorname{Prod}(\sigma) \cap \operatorname{Prod}(\sigma')$, and let $(\mathcal{S}, \mathcal{R})$ be the torsion pair in $\mathcal{H}_\sigma$ cogenerated by $H^0_\sigma(\mathscr{E})$.*

(1) *σ' is a right mutation of σ if and only if σ admits an $\mathscr{E}$-precover and $\mathbb{T}_{\sigma'}$ is the right HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair $(\mathcal{S}, \mathcal{R})$ in $\mathcal{H}_\sigma$.*

(2) *σ' is a left mutation of σ if and only if the class $\mathcal{S}$ is closed under products (hence a TTF-class) in $\mathcal{H}_\sigma$ and $\mathbb{T}_{\sigma'}$ is the left HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair $(\mathcal{T}, \mathcal{S})$ in $\mathcal{H}_\sigma$.*

An important difference with respect to the theory developed in [Aihara and Iyama 2012] is that, even for finite-dimensional algebras, mutation of large silting or cosilting objects is not always possible (Examples 4.10 and 9.15). Nevertheless, we show that cosilting mutation is, in itself, a generalisation of compact silting mutation. Indeed, any mutation of a compact silting object corresponds to a mutation of a pure-injective cosilting object (see Theorem 5.8). The class of pure-injective cosilting objects, which includes bounded cosilting complexes in the derived category of a ring (see Example 2.8), will play a distinguished role. The big advantage of working with a pure-injective cosilting object σ lies in the fact that the heart $\mathcal{H}_\sigma$ is a Grothendieck category [Angeleri Hügel et al. 2017] — and in this case we can characterise the *existence of mutations* as follows.

Theorem B (Theorem 4.9). *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$, and $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ a subcategory of $\mathrm{Prod}(\sigma)$. Let $(\mathcal{S}, \mathcal{R})$ be the torsion pair in $\mathcal{H}_\sigma$ cogenerated by $H^0_\sigma(\mathscr{E})$. The following statements are equivalent*:

(1) *σ admits a right mutation σ' with respect to $\mathscr{E}$.*

(2) *The torsion-free class $\mathcal{R} = \mathrm{Cogen}(H^0_\sigma(\mathscr{E}))$ in $\mathcal{H}_\sigma$ is closed under direct limits.*

(3) *The cosilting object σ admits an $\mathscr{E}$-cover.*

Dually, the following statements are equivalent:

(1) *σ admits a left mutation σ' with respect to $\mathscr{E}$.*

(2) *The torsion class $\mathcal{S} = {}^{\perp_0} H^0_\sigma(\mathscr{E})$ in $\mathcal{H}_\sigma$ is closed under products (that is, it is a TTF class).*

(3) *The object $\varepsilon_0 \oplus \varepsilon_1$ arising from an $\mathscr{E}$-envelope $\sigma \to \varepsilon_0$ and its cone ε_1 is a cosilting object.*

In both cases, if the equivalent conditions are satisfied, any mutation σ' as in (1) *is pure-injective.*

We also provide an interpretation of mutation of pure-injective cosilting objects in terms of localisation theory (see Section 6), which essentially states that the operation of mutation can be understood as a three-step process: first restrict the associated t-structures to certain subcategories; then shift one of the restricted t-structures; finally glue them back together.

The whole machinery of cosilting mutation in triangulated categories leads to an *answer to our motivating question* concerning the study of the lattice of torsion pairs. In [Demonet et al. 2023; Barnard et al. 2019], it was shown that minimal inclusions of torsion classes (not necessarily functorially finite) in $\mathrm{mod}(A)$, for a finite-dimensional algebra A, are parametrised by bricks. These bricks turn out to correspond to certain indecomposable summands of the associated (two-term) cosilting objects. We show that minimal inclusions of torsion classes then correspond to swapping precisely this indecomposable summand. We call this irreducible mutation. This result generalises the phenomenon that is well-understood for minimal inclusions of functorially finite torsion classes. While we prove the theorem below in a more general setting (see Setup 9.1 and Corollary 9.14), here we state it in the setting of artinian rings.

Theorem C (Corollary 9.14). *Let A be an artinian ring. Consider two cosilting torsion pairs $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ and $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ in* $\mathrm{mod}(A)$ *such that $\mathcal{U} \subseteq \mathcal{T}$. Let $\sigma_{\mathfrak{t}}$ and $\sigma_{\mathfrak{u}}$ be the two-term cosilting complexes associated to $\mathfrak{t}$ and $\mathfrak{u}$, respectively. Then the following statements are equivalent*:

(1) *$\sigma_{\mathfrak{t}}$ is an irreducible right mutation of $\sigma_{\mathfrak{u}}$.*

(2) *$\sigma_{\mathfrak{u}}$ is an irreducible left mutation of $\sigma_{\mathfrak{t}}$.*

(3) *The class $\mathcal{S} = \mathcal{T} \cap \mathcal{V}$ coincides with* $\mathrm{filt}(M)$ *for a brick M in* $\mathrm{mod}(A)$.

(4) *The inclusion $\mathcal{U} \subseteq \mathcal{T}$ is a minimal inclusion of torsion classes.*

In forthcoming work [Angeleri Hügel et al. 2024; 2025], we will specialise to the case of two-term cosilting complexes in the derived category of a finite-dimensional algebra A in order to obtain a more explicit description of the operation of mutation involving the approximation theory and the Ziegler spectrum of $\mathrm{Mod}(A)$.

Structure of the paper. The paper is organised as follows.

- In Section 2 we collect the necessary background on t-structures, HRS-tilts, silting and cosilting objects.
- In Section 3 we define cosilting mutation and interpret these operations in terms of HRS-tilts. In particular, we prove Theorem A above (Theorem 3.5).
- In Section 4 we specialise to pure-injective cosilting objects in compactly generated triangulated categories. In this setting the situation becomes more tractable, and we are able to provide necessary and sufficient conditions of the existence of cosilting mutation, proving Theorem B (Theorem 4.9).
- Section 5 is devoted to the dual situation of silting objects and we prove there that cosilting mutation generalises the notion of mutation for compact silting objects, as set up in [Aihara and Iyama 2012] (see Theorem 5.8).
- In Section 6, we interpret mutation from the point of view of localisation theory of triangulated categories and we observe that the operation of mutation can be broken into three parts: restriction, shifting and gluing (see Theorem 6.6).
- In Section 7 we consider general mutation of cosilting objects associated with torsion pairs in the heart of a pure-injective cosilting t-structure and we characterise this situation in terms of wide subcategories of the heart (Theorem 7.8).
- Section 8 clarifies the bijection between torsion pairs in "small" and "large" triangulated categories and describes mutation of "small" torsion pairs (Theorem 8.9).
- Finally in Section 9, we focus on mutation of torsion pairs in abelian length categories. It is in this framework that we prove, in particular, Theorem C (Corollary 9.14).

2. Preliminaries

2.1. ***Notation.*** All subcategories considered are strict and full. Given an object X in an additive category $\mathcal{A}$, we denote by $\mathrm{Add}(X)$ the subcategory whose objects are summands of existing coproducts of copies of X. Dually, we write $\mathrm{Prod}(X)$ for the subcategory whose objects are summands of existing products of copies of X. We denote the isomorphism classes of indecomposable objects in $\mathcal{A}$ by $\mathrm{Ind}(\mathcal{A})$.

Let now $\mathcal{A}$ be a complete and cocomplete abelian category. An object A in $\mathcal{A}$ is said to be *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ commutes with direct limits. Given a class of objects $\mathcal{X}$ in $\mathcal{A}$, we write $\mathrm{fp}\,\mathcal{X}$ for the collection of finitely presented objects in $\mathcal{X}$ and $\varinjlim \mathcal{X}$ for the subcategory of $\mathcal{A}$ formed by direct limits of objects in $\mathcal{X}$. We further denote by $\mathrm{Gen}(\mathcal{X})$ (respectively, $\mathrm{gen}(\mathcal{X})$) the subcategory formed by all epimorphic images of coproducts (respectively, of finite coproducts) of objects in $\mathcal{X}$, and by $\mathrm{Cogen}(\mathcal{X})$ the subcategory formed by all subobjects of products of objects in $\mathcal{X}$. Furthermore, we denote by $\mathrm{Filt}(\mathcal{X})$ the class of all objects M which admit an ascending chain $(M_\lambda, \lambda \leq \mu)$ of subobjects indexed over an ordinal number μ where $M_0 = 0$, all consecutive factors $M_{\lambda+1}/M_\lambda$ with $\lambda < \mu$ belong to $\mathcal{X}$, and $M = \bigcup_{\lambda \leq \mu} M_\lambda$. The class of objects with a finite filtration of this form is denoted by $\mathrm{filt}(\mathcal{X})$.

For a pair of full subcategories $\mathcal{M}$ and $\mathcal{N}$ of an abelian (respectively, triangulated) category $\mathcal{C}$, we use the notation $\mathcal{M} \star \mathcal{N}$ for the full subcategory of $\mathcal{C}$ consisting of objects X such that there exists a short exact sequence $0 \to M \to X \to N \to 0$ (respectively, a triangle $M \to X \to N \to M[1]$) with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

Let $\mathcal{M}$ be a class of objects in a triangulated category $\mathcal{D}$. Given a set of integers I (which is often expressed by symbols such as $> n$, $\neq n$, or just n), we write ${}^{\perp_I}\mathcal{M}$ for the orthogonal class given by the objects X satisfying $\mathrm{Hom}_{\mathcal{D}}(X, M[i]) = 0$ for all $M \in \mathcal{M}$ and $i \in I$, while $\mathcal{M}^{\perp_I}$ consists of the objects X such that $\mathrm{Hom}_{\mathcal{D}}(M, X[i]) = 0$ for all $M \in \mathcal{M}$ and $i \in I$. If $\mathcal{M}$ is a class of objects in an abelian category $\mathcal{A}$ and I is a set of natural numbers, we similarly denote by ${}^{\perp_I}\mathcal{M}$ the class given by the objects $X \in \mathcal{A}$ such that $\mathrm{Ext}^i_{\mathcal{A}}(X, M) = 0$ and by $\mathcal{M}^{\perp_I}$ the class of objects X satisfying $\mathrm{Ext}^i_{\mathcal{A}}(M, X) = 0$ for all $M \in \mathcal{M}$ and $i \in I$.

Finally, when R is a ring, $\mathrm{Mod}(R)$ denotes the category of all left R-modules and $\mathrm{D}(R)$ the unbounded derived category of $\mathrm{Mod}(R)$. If R is left coherent, then $\mathrm{mod}(R)$ denotes the abelian subcategory of finitely presented left R-modules and $\mathrm{D}^b(\mathrm{mod}(R))$ its bounded derived category.

2.2. ***Torsion pairs, t-structures and HRS-tilts.*** Recall that a *torsion pair* in an abelian (respectively, triangulated) category $\mathcal{C}$ is a pair of idempotent-complete subcategories $\mathrm{t} := (\mathcal{T}, \mathcal{F})$ such that $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all T in $\mathcal{T}$ and F in $\mathcal{F}$ and, furthermore, with the property that $\mathcal{C} = \mathcal{T} \star \mathcal{F}$. The subcategory $\mathcal{T}$ is often called a *torsion class*, while the subcategory $\mathcal{F}$ is often referred to as a *torsion-free class*. If a torsion-free class $\mathcal{F}$ is again a torsion class with respect to another torsion pair $(\mathcal{F}, \mathcal{G})$, then we say that $\mathcal{F}$ is a *torsion torsion-free class* (or TTF class, for short).

A torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian or triangulated category $\mathcal{C}$ is said to be

- *cogenerated by a subcategory* $\mathcal{S}$ if $\mathcal{T} = {}^{\perp_0}\mathcal{S}$, and
- *generated by a subcategory* $\mathcal{S}$ if $\mathcal{F} = \mathcal{S}^{\perp_0}$.

If $\mathcal{C}$ is abelian, we say that $(\mathcal{T}, \mathcal{F})$ is

- *hereditary* if $\mathcal{T}$ is closed under subobjects, and
- *cohereditary* if $\mathcal{F}$ is closed under quotient objects.

If $\mathcal{C}$ is furthermore AB5 (i.e., a cocomplete abelian category with exact direct limits), then $(\mathcal{T}, \mathcal{F})$ is

- *of finite type* if $\mathcal{F}$ is closed under direct limits in $\mathcal{C}$.

A subcategory $\mathcal{M}$ of an abelian (respectively, triangulated category) will be said to be *extension-closed* if $\mathcal{M} \star \mathcal{M} \subseteq \mathcal{M}$. An extension-closed subcategory $\mathcal{X}$ of a triangulated category $\mathcal{D}$ is said to be *suspended* (respectively, *cosuspended*) if $\mathcal{X}[1] \subseteq \mathcal{X}$ (respectively, if $\mathcal{X}[-1] \subseteq \mathcal{X}$).

A torsion pair $\mathbb{T} := (\mathcal{X}, \mathcal{Y})$ in a triangulated category for which $\mathcal{X}$ is suspended is called a *t-structure*. Then $\mathcal{X}$ is called the *aisle* and $\mathcal{Y}$ the *coaisle* of $\mathbb{T}$. Such torsion pairs give rise to an abelian subcategory of $\mathcal{D}$, the *heart* of the t-structure, which can be obtained as $\mathcal{H}_{\mathbb{T}} := \mathcal{X}[-1] \cap \mathcal{Y}$. Furthermore, there is a cohomological functor associated to $\mathbb{T}$, i.e., a functor $H^0_{\mathbb{T}} : \mathcal{D} \to \mathcal{H}_{\mathbb{T}}$ that sends triangles to long exact sequences, and we denote by $H^i_{\mathbb{T}} : \mathcal{D} \to \mathcal{H}_{\mathbb{T}}$ the functor given by $H^i_{\mathbb{T}}(X) = H^0_{\mathbb{T}}(X[i])$ for any i in $\mathbb{Z}$. The following useful lemma is a direct consequence of the construction of this cohomological functor.

Lemma 2.1. *Let $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ be a t-structure in a triangulated category $\mathcal{D}$. Let W be an object in the heart $\mathcal{H}$, and let Y be in $\mathcal{Y}$ and Z in $\mathcal{X}[-1]$. Then we have that* $\operatorname{Hom}_{\mathcal{D}}(W, Y) \cong \operatorname{Hom}_{\mathcal{H}}(W, H^0_{\mathbb{T}}(Y))$ *and* $\operatorname{Hom}_{\mathcal{D}}(Z, W) \cong \operatorname{Hom}_{\mathcal{H}}(H^0_{\mathbb{T}}(Z), W)$.

A t-structure $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ is said to be

- *nondegenerate* if $\bigcap_{n \in \mathbb{Z}} \mathcal{X}[n] = 0 = \bigcap_{n \in \mathbb{Z}} \mathcal{Y}[n]$.

It is easy to check that if $\mathbb{T}$ is nondegenerate, then the aisle $\mathcal{X}$ consists of the objects X with $H^k_{\mathbb{T}}(X) = 0$ for all $k \geq 0$, and the coaisle $\mathcal{Y}$ of those with $H^k_{\mathbb{T}}(X) = 0$ for all $k < 0$.

If $\mathcal{D}$ is a triangulated category with coproducts (respectively, products), then we say that $\mathbb{T}$ is

- *smashing* if the coaisle $\mathcal{Y}$ is closed under coproducts, and
- *cosmashing* if the aisle $\mathcal{X}$ is closed under products.

A nondegenerate t-structure $\mathbb{T}$ is smashing (respectively, cosmashing) if and only if the $H^0_{\mathbb{T}}$ preserves coproducts (respectively, products); see [Angeleri Hügel et al. 2017, Lemma 3.3]. For more details on t-structures we refer to [Beilinson et al. 1982].

For a torsion pair $\mathfrak{t} := (\mathcal{T}, \mathcal{F})$ in the heart $\mathcal{H}_{\mathbb{T}}$ of a t-structure $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$, we can build a new t-structure according to [Happel et al. 1996]. It is called the *left HRS-tilt* at $\mathfrak{t}$, and it is defined as

$$\mathbb{T}_{\mathfrak{t}^+} := (\mathcal{X}_{\mathfrak{t}^+} := \mathcal{X}[1] \star \mathcal{T}[1],\ \mathcal{Y}_{\mathfrak{t}^+} := \mathcal{F}[1] \star \mathcal{Y}).$$

The corresponding heart is then given by $\mathcal{H}_{\mathfrak{t}^+} = \mathcal{F}[1] \star \mathcal{T}$, it is equipped with a torsion pair $(\mathcal{F}[1], \mathcal{T})$ and the cohomological functor $H^0_{\mathfrak{t}^+} : \mathcal{D} \to \mathcal{H}_{\mathfrak{t}^+}$. Dually, we denote the *right HRS-tilt* at $\mathfrak{t}$ by

$$\mathbb{T}_{\mathfrak{t}^-} := (\mathcal{X}_{\mathfrak{t}^-} := \mathcal{X} \star \mathcal{T},\ \mathcal{Y}_{\mathfrak{t}^-} := \mathcal{F} \star (\mathcal{Y}[-1])).$$

The corresponding heart is then given by $\mathcal{H}_{\mathfrak{t}^-} = \mathcal{F} \star \mathcal{T}[-1]$, it is equipped with a torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ and the cohomological functor $H^0_{\mathfrak{t}^-} : \mathcal{D} \to \mathcal{H}_{\mathfrak{t}^-}$.

Remark 2.2. Note that $\mathbb{T}_{\mathfrak{t}^+} = \mathbb{T}_{\mathfrak{t}^-}[1]$, and $\mathbb{T} = (\mathbb{T}_{\mathfrak{t}^-})_{\mathfrak{s}^+}$ for the torsion pair $\mathfrak{s} = (\mathcal{F}, \mathcal{T}[-1])$ in $\mathcal{H}_{t^-}$. Furthermore, if $\mathbb{T}$ is nondegenerate, then the right HRS-tilt is given by

$$\mathcal{X}_{\mathfrak{t}^-} = \{X \in \mathcal{D} \mid H^0_{\mathbb{T}}(X) \in \mathcal{T} \text{ and } H^k_{\mathbb{T}}(X) = 0 \text{ for all } k > 0\},$$
$$\mathcal{Y}_{\mathfrak{t}^-} = \{X \in \mathcal{D} \mid H^0_{\mathbb{T}}(X) \in \mathcal{F} \text{ and } H^k_{\mathbb{T}}(X) = 0 \text{ for all } k < 0\},$$

and the corresponding statement holds true for the left HRS-tilt.

By construction, we have that $\mathcal{Y} \subseteq \mathcal{Y}_{\mathfrak{t}^+} \subseteq \mathcal{Y}[1]$ and $\mathcal{Y}[-1] \subseteq \mathcal{Y}_{\mathfrak{t}^-} \subseteq \mathcal{Y}$. These properties completely characterise $\mathbb{T}_{\mathfrak{t}^+}$ as a left HRS-tilt and $\mathbb{T}_{\mathfrak{t}^-}$ as a right HRS-tilt.

Proposition 2.3 [Polishchuk 2007, Lemma 1.1.2; Woolf 2010, Proposition 2.1]. *Let $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ be a t-structure in a triangulated category $\mathcal{D}$ with heart $\mathcal{H}$.*

(1) *The assignment $\mathfrak{t} \mapsto \mathbb{T}_{\mathfrak{t}^+}$ defines a bijection between torsion pairs in $\mathcal{H}$ and t-structures $\mathbb{T}' = (\mathcal{X}', \mathcal{Y}')$ in $\mathcal{D}$ with $\mathcal{Y} \subseteq \mathcal{Y}' \subseteq \mathcal{Y}[1]$. The inverse assignment takes a t-structure $\mathbb{T}'$ with heart $\mathcal{H}'$ to the torsion pair $\mathfrak{t} := (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ given by $\mathcal{T} = \mathcal{H} \cap \mathcal{H}'$ and $\mathcal{F} = \mathcal{H} \cap \mathcal{H}'[-1]$.*

(2) *The assignment $\mathfrak{t} \mapsto \mathbb{T}_{\mathfrak{t}^-}$ defines a bijection between torsion pairs in $\mathcal{H}$ and t-structures $\mathbb{T}' = (\mathcal{X}', \mathcal{Y}')$ in $\mathcal{D}$ with $\mathcal{Y}[-1] \subseteq \mathcal{Y}' \subseteq \mathcal{Y}$. The inverse assignment takes a t-structure $\mathbb{T}'$ with heart $\mathcal{H}'$ to the torsion pair $\mathfrak{t} := (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ given by $\mathcal{T} = \mathcal{H} \cap \mathcal{H}'[1]$ and $\mathcal{F} = \mathcal{H} \cap \mathcal{H}'$.*

2.3. ***Silting and cosilting t-structures.*** Recall that an object σ of a triangulated category $\mathcal{D}$ with (set-indexed) coproducts is said to be *silting* if the pair $(\sigma^{\perp_{\geq 0}}, \sigma^{\perp_{<0}})$ is a t-structure in $\mathcal{D}$. Dually, if $\mathcal{D}$ has (set-indexed) products, an object σ of $\mathcal{D}$ is said to be *cosilting* if $({}^{\perp_{\leq 0}}\sigma, {}^{\perp_{>0}}\sigma)$ is a t-structure in $\mathcal{D}$. In both cases we denote the associated (silting/cosilting) t-structure by $\mathbb{T}_\sigma$, its heart by $\mathcal{H}_\sigma$ and its associated cohomological functor by $H^0_\sigma : \mathcal{D} \to \mathcal{H}_\sigma$. Two (co)silting objects are said to be *equivalent* if they give rise to the same t-structure.

Remark 2.4. Note that it is our convention that the heart of a t-structure is contained in the coaisle, not in the aisle. This justifies the slight adaptation (by a shift) of the definition of the t-structure associated to a silting object presented above (compare with [Psaroudakis and Vitória 2018]).

We say that a subcategory $\mathcal{M}$ *generates* $\mathcal{D}$ if $\mathcal{M}^{\perp_{\mathbb{Z}}} = 0$ and $\mathcal{M}$ *cogenerates* $\mathcal{D}$ if ${}^{\perp_{\mathbb{Z}}}\mathcal{M} = 0$. If $\mathcal{M}$ consists of a single object M, we say that M (co)generates $\mathcal{D}$. It follows from the definition that every silting object generates $\mathcal{D}$, and every cosilting object cogenerates $\mathcal{D}$. We recall the following properties of (co)silting t-structures.

Proposition 2.5 [Psaroudakis and Vitória 2018, Proposition 4.3 and Lemma 4.5; Angeleri Hügel et al. 2017, Lemma 2.8, Theorem 3.5, and Corollary 3.8]. *Let σ be an object in a triangulated category $\mathcal{D}$.*

(1) *If $\mathcal{D}$ has coproducts and σ is a silting object, then the associated heart $\mathcal{H}_\sigma = \sigma^{\perp_{\neq 0}}$ is an abelian category with enough projectives, and the functor H^0_σ induces an equivalence of categories between* $\operatorname{Add}(\sigma)$ *and* $\operatorname{Proj}(\mathcal{H}_\sigma)$ *and a natural isomorphism* $\operatorname{Hom}_{\mathcal{D}}(\sigma, -) \cong \operatorname{Hom}_{\mathcal{D}}(H^0_\sigma(\sigma), H^0_\sigma(-))$. *Furthermore, if $\mathcal{X} = \sigma^{\perp_{>0}}$, then* $\operatorname{Add}(\sigma) = {}^{\perp_1}\mathcal{X} \cap \mathcal{X}$.

(2) *If* $\mathcal{D}$ *has products and* σ *is a cosilting object, then the associated heart* $\mathcal{H}_\sigma = {}^{\perp_{\neq 0}}\sigma$ *is an abelian category with enough injectives, and the functor* H^0_σ *induces an equivalence of categories between* $\mathrm{Prod}(\sigma)$ *and* $\mathrm{Inj}(\mathcal{H}_\sigma)$ *and a natural isomorphism* $\mathrm{Hom}_{\mathcal{D}}(-,\sigma) \cong \mathrm{Hom}_{\mathcal{D}}(H^0_\sigma(-), H^0_\sigma(\sigma))$. *Furthermore, if* $\mathcal{Y} = {}^{\perp_{>0}}\sigma$, *then* $\mathrm{Prod}(\sigma) = \mathcal{Y} \cap \mathcal{Y}^{\perp_1}$.

Remark 2.6. If $\mathcal{D}$ is a triangulated category with coproducts, then every heart is known to be cocomplete [Parra and Saorín 2015, Proposition 3.2]. Moreover, if a given t-structure is associated to a silting object, then the corresponding heart $\mathcal{H}$ has a projective generator by the proposition above. Then Freyd's adjoint functor theorem implies that $\mathcal{H}$ is also complete (see [Faith 1973, Proposition 6.4]) and products are necessarily exact. Moreover, $\mathcal{H}$ has the property of being *well-powered*, i.e., every object has a set of subobjects (see [Stenström 1975, Proposition IV.6.6]). Dually, if $\mathcal{D}$ is a triangulated category with products, then the heart $\mathcal{H}$ of any t-structure associated to a cosilting object is complete, cocomplete, with exact coproducts and well-powered.

It follows in both cases from [Dickson 1966] that torsion classes are precisely those closed under coproducts, extensions and quotients, and torsion-free classes are precisely those closed under products, extensions and subobjects. Furthermore, in any complete, cocomplete and well-powered abelian category $\mathcal{A}$, we then have that

- if $\mathscr{E}$ is a family of injective objects in $\mathcal{A}$, then $({}^{\perp_0}\mathscr{E}, \mathrm{Cogen}(\mathscr{E}))$ is a hereditary torsion pair in $\mathcal{A}$; and
- if $\mathscr{P}$ is a family of projective objects in $\mathcal{A}$, then $(\mathrm{Gen}(\mathscr{P}), \mathscr{P}^{\perp_0})$ is a cohereditary torsion pair in $\mathcal{A}$.

We close the section with a brief review of the notion of purity in triangulated categories. Assume now that $\mathcal{D}$ admits (set-indexed) coproducts. Recall that an object X in $\mathcal{D}$ is said to be *compact* if the functor $\mathrm{Hom}_{\mathcal{D}}(X,-)$ commutes with coproducts. If the subcategory $\mathcal{D}^c$ of compact objects is skeletally small and generates $\mathcal{D}$, then $\mathcal{D}$ is said to be *compactly generated*. It is well known that $\mathcal{D}$ then also admits products.

When $\mathcal{D}$ is a compactly generated triangulated category, the category of additive (contravariant) functors $(\mathcal{D}^c)^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$, denoted by $\mathrm{Mod}(\mathcal{D}^c)$, is a locally coherent Grothendieck category with enough projectives. Recall that a Grothendieck category $\mathcal{G}$ is said to be *locally coherent* if its subcategory of finitely presented objects $\mathrm{fp}\,\mathcal{G}$ is an abelian subcategory and generates $\mathcal{G}$.

Moreover, the functor $\boldsymbol{y} : \mathcal{D} \to \mathrm{Mod}(\mathcal{D}^c)$, defined by $\boldsymbol{y}X := \mathrm{Hom}_{\mathcal{D}}(-,X)_{|\mathcal{D}^c}$, sends triangles to long exact sequences. A *pure triangle* is a triangle in $\mathcal{D}$ of the form

$$\Delta :\; X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that $\boldsymbol{y}\Delta$ is a short exact sequence, i.e., $\boldsymbol{y}f$ is a monomorphism (and, hence, f is called a *pure monomorphism*), $\boldsymbol{y}g$ is an epimorphism (and, hence, g is called a *pure epimorphism*) and $\boldsymbol{y}h = 0$ (and, hence, h is called a *phantom map*). Note that, with this terminology in place, Δ is a pure triangle if and only if f is a pure monomorphism, and the latter is equivalent to g being a pure epimorphism and equivalent to h being a phantom map. We say that an object X of $\mathcal{D}$ is *pure-injective* (respectively, *pure-projective*) if $\boldsymbol{y}X$ is an injective (respectively, projective) object in $\mathrm{Mod}(\mathcal{D}^c)$. Equivalently, an object X of $\mathcal{D}$ is pure-injective (respectively, pure-projective) if and only if every pure triangle starting in X (respectively, ending in X) splits. It is well known that the pure-projective objects coincide precisely with $\mathrm{Add}(\mathcal{D}^c)$ [Beligiannis 2000, Lemma 8.1]. The following theorem explains the benefits of considering pure-injective/pure-projective cosilting/silting objects.

Theorem 2.7 [Angeleri Hügel et al. 2017; Nicolás et al. 2019]. *Let $\mathcal{D}$ be a compactly generated triangulated category.*

(1) *There is a bijection between equivalence classes of cosilting objects and smashing nondegenerate t-structures whose heart has an injective cogenerator. Furthermore, a cosilting object is pure-injective if and only if the heart of the associated t-structure is a Grothendieck category.*

(2) *There is a bijection between equivalence classes of silting objects and cosmashing nondegenerate t-structures whose heart has a projective generator. Furthermore, a silting object is pure-projective if and only if the associated t-structure is smashing and its heart is a Grothendieck category with a projective generator.*

Example 2.8 [Zhang and Wei 2017; Marks and Vitória 2018]. An object σ in the category $\mathsf{K}^b(\mathrm{Inj}(R))$ of bounded complexes of injective R-modules is cosilting in $\mathrm{D}(R)$ if and only if $\mathrm{Hom}_{\mathrm{D}(R)}(\sigma^I, \sigma[i]) = 0$ for all sets I and $i > 0$, and $\mathsf{K}^b(\mathrm{Inj}(R))$ is the smallest triangulated subcategory of $\mathrm{D}(R)$ containing $\mathrm{Prod}(\sigma)$. Such cosilting complexes are pure injective objects of $\mathrm{D}(R)$ and thus give rise to t-structures with Grothendieck heart.

3. The concept of mutation

Silting mutation was introduced in [Aihara and Iyama 2012]. In this section, we define mutation for large silting or cosilting objects, and in Theorem 5.8 we will show that it extends Aihara and Iyama's silting mutation. We prove that a (co)silting object σ' is a (left or right) mutation of another (co)silting object σ if and only if the associated (co)silting t-structures are related by a suitable (left or right) HRS-tilt. The torsion pair at which the HRS-tilt is performed is determined by the intersection $\mathcal{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$ in the cosilting case, and by $\mathcal{E} = \mathrm{Add}(\sigma) \cap \mathrm{Add}(\sigma')$ in the silting case.

We begin with a useful lemma. Recall that, given an object C in an additive category $\mathcal{C}$, we say that a morphism $g : M \to C$ with M in $\mathcal{M}$ is an *$\mathcal{M}$-precover* (or a *right $\mathcal{M}$-approximation*) if any other morphism $M' \to C$ with M' in $\mathcal{M}$ factors through g. If, in addition, any endomorphism $h : M \to M$ with $gh = g$ is an isomorphism, then g is called an *$\mathcal{M}$-cover*. Dually, one defines *$\mathcal{M}$-preenvelopes* (or *left $\mathcal{M}$-approximations*) and *$\mathcal{M}$-envelopes*.

Lemma 3.1. *Let σ be an object in a triangulated category $\mathcal{D}$.*

(1) *If $\mathcal{D}$ has products, σ is cosilting and $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ is a subcategory of $\mathrm{Prod}(\sigma)$, then*

  (a) *$\mathscr{E}$ is preenveloping in $\mathrm{Prod}(\sigma)$, and*

  (b) *$\mathscr{E}$ cogenerates $\mathcal{D}$ if and only if $\mathscr{E} = \mathrm{Prod}(\sigma)$.*

(2) *If $\mathcal{D}$ has coproducts, σ is silting and $\mathscr{P} = \mathrm{Add}(\mathscr{P})$ is a subcategory of $\mathrm{Add}(\sigma)$, then*
   (a) *$\mathscr{P}$ is precovering in $\mathrm{Add}(\sigma)$, and*
   (b) *$\mathscr{P}$ generates $\mathcal{D}$ if and only if $\mathscr{P} = \mathrm{Add}(\sigma)$.*

Proof. We prove the cosilting case; the silting case is analogous.

(1a): Consider the torsion pair $\mathfrak{s} := ({}^{\perp_0}H^0_\sigma(\mathscr{E}), \mathrm{Cogen}(H^0_\sigma(\mathscr{E})))$ and take $a : H^0_\sigma(\sigma) \to A$ to be the epimorphism to a torsion-free object A with a torsion kernel (with respect to $\mathfrak{s}$). Consider then a monomorphism $\pi : A \to H^0_\sigma(E)$ for some E in $\mathscr{E}$. It is easy to observe that $\pi \circ a$ is an $H^0_\sigma(\mathscr{E})$-preenvelope of $H^0_\sigma(\sigma)$, and by Proposition 2.5 the map $\Psi : \sigma \to E$ such that $H^0_\sigma(\Psi) = \pi \circ a$ is an $\mathscr{E}$-preenvelope of σ.

(1b): Let $\Psi : \sigma \to E$ be the preenvelope obtained above. If $\mathscr{E}$ cogenerates $\mathcal{D}$, then $H^0_\sigma(\mathscr{E})$ is a cogenerating class of injective objects in $\mathcal{H}_\sigma$. In particular, $H^0_\sigma(\Psi)$ must be a monomorphism and, thus, a split map. Therefore Ψ splits by Proposition 2.5, and we conclude that $\mathscr{E} = \mathrm{Prod}(\sigma)$. The converse holds by the definition of cosilting object. □

We will first discuss mutation of cosilting objects. Later we will restrict our attention to pure-injective cosilting objects, for which we can use some approximation-theoretic tools to simplify the definition below.

Definition 3.2. Let $\mathcal{D}$ be a triangulated category with products. Let σ and σ' be two cosilting objects in $\mathcal{D}$, and let $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$. We say that:

(1) σ' is a *left mutation* of σ if there is a triangle

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \sigma[1]$$

such that

- Φ is an $\mathscr{E}$-preenvelope of σ in $\mathcal{D}$, and
- $\varepsilon_0 \oplus \varepsilon_1$ is a cosilting object equivalent to σ'.

(2) σ' is a *right mutation* of σ if there is a triangle

$$\sigma[-1] \longrightarrow \gamma_1 \longrightarrow \gamma_0 \xrightarrow{\Phi} \sigma$$

such that

- Φ is an $\mathscr{E}$-precover of σ in $\mathcal{D}$, and
- $\gamma_0 \oplus \gamma_1$ is a cosilting object equivalent to σ'.

We will also say that σ' is a *left* (*or right*) *mutation* of σ *with respect to* $\mathscr{E}$.

Our first aim is to clarify the relation between mutations and HRS-tilts. This will allow us to see, in particular, that mutation is defined up to equivalence of cosilting objects. For that purpose, we will make use of the following lemma.

Lemma 3.3. *Let σ be a cosilting object in a triangulated category $\mathcal{D}$ with products, and let $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ be a subcategory of $\mathrm{Prod}(\sigma)$. Denote by*

$$\mathfrak{t} = (\mathcal{S}, \mathcal{R}) = ({}^{\perp_0}H^0_\sigma(\mathscr{E}), \mathrm{Cogen}(H^0_\sigma(\mathscr{E})))$$

the torsion pair in $\mathcal{H}_\sigma$ *cogenerated by the set of injective objects* $H^0_\sigma(\mathscr{E})$*, and let* $\mathbb{T} = (\mathcal{X}', \mathcal{Y}')$ *be the right HRS-tilt of* $\mathbb{T}_\sigma$ *at* $\mathfrak{t}$*. The following statements are equivalent for a map* $\Phi : E \to \sigma$ *with* E *in* $\mathscr{E}$*:*

(1) Φ *is an* $\mathscr{E}$*-precover of* σ *in* $\mathcal{D}$*.*

(2) $\phi := H^0_\sigma(\Phi)$ *is an* $\mathcal{R}$*-precover of* $H^0_\sigma(\sigma)$ *in* $\mathcal{H}_\sigma$*.*

(3) Φ *is a* $\mathcal{Y}'$*-precover of* σ *in* $\mathcal{D}$*.*

Proof. (1) ⇒ (2): The map ϕ is an $H^0_\sigma(\mathscr{E})$-precover because H^0_σ induces an equivalence of categories between $\mathrm{Prod}(\sigma)$ and $\mathrm{Inj}(\mathcal{H}_\sigma)$. Let $g : X \to H^0_\sigma(\sigma)$ be a map in $\mathcal{H}_\sigma$ with X in $\mathcal{R} = \mathrm{Cogen}(H^0_\sigma(\mathscr{E}))$. Since $H^0_\sigma(\sigma)$ is injective, g extends along any monomorphism $h : X \to H^0_\sigma(E')$ with E' in $\mathscr{E}$; that is, there is $t : H^0_\sigma(E') \to H^0_\sigma(\sigma)$ such that $t \circ h = g$. But then t must factor through ϕ, so there is $\psi : H^0_\sigma(E') \to H^0_\sigma(E)$ such that $\phi \circ \psi = t$. This shows that $g = \phi \circ \psi \circ h$, as wanted.

(2) ⇒ (3): Suppose that ϕ is an $\mathcal{R}$-precover of $H^0_\sigma(\sigma)$ in $\mathcal{H}_\sigma$ and let $f : Y \to \sigma$ be a map with Y in $\mathcal{Y}'$. Applying H^0_σ to the triangle induced by f we get an exact sequence

$$0 \longrightarrow H^0_\sigma(K) \longrightarrow H^0_\sigma(Y) \xrightarrow{H^0_\sigma(f)} H^0_\sigma(\sigma).$$

Now, since $\mathbb{T}_\sigma$ is nondegenerate, its right HRS-tilt $\mathbb{T}$ has the form described in Remark 2.2. In particular, $H^0_\sigma(Y)$ lies in $\mathcal{R}$, so the map $H^0_\sigma(f)$ factors through ϕ; that is, there is a map $\psi : H^0_\sigma(Y) \to H^0_\sigma(E)$ such that $H^0_\sigma(f) = \phi \circ \psi$. Since E lies in $\mathrm{Prod}(\sigma)$, we know from Proposition 2.5(2) that the functor H^0_σ induces an isomorphism $\mathrm{Hom}(Y, E) \cong \mathrm{Hom}(H^0_\sigma(Y), H^0_\sigma(E))$. Hence there is a unique map $\Psi : Y \to E$ such that $H^0_\sigma(\Psi) = \psi$. Clearly, it follows that $f = \Phi\Psi$ as wanted.

(3) ⇒ (1): This is clear from the fact that $\mathscr{E}$ is contained in $\mathcal{Y}'$ and E, by assumption, lies in $\mathscr{E}$. □

Remark 3.4. It is clear that the class $\mathcal{R}$ in the torsion pair $\mathfrak{t} = (\mathcal{S}, \mathcal{R})$ above is enveloping (as every torsion-free class is!), but without further assumptions on the cosilting object it may be hard to say whether it is precovering or not. We will discuss the approximation properties of such torsion pairs in Section 4 (compare also with the silting case in Remark 5.6).

We are now ready for our first theorem.

Theorem 3.5. *Let* $\mathcal{D}$ *be a triangulated category with products. Let* σ *and* σ' *be two cosilting objects in* $\mathcal{D}$*, and let* $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$*.*

(1) σ' *is a left mutation of* σ *if and only if* ${}^{\perp_0}H^0_\sigma(\mathscr{E})$ *is closed under products in* $\mathcal{H}_\sigma$ *and* $\mathbb{T}_{\sigma'}$ *is the left HRS-tilt of* $\mathbb{T}_\sigma$ *at the torsion pair* $\mathfrak{t} = (\mathcal{T}, {}^{\perp_0}H^0_\sigma(\mathscr{E}))$ *in* $\mathcal{H}_\sigma$*.*

(2) σ' *is a right mutation of* σ *if and only if* σ *admits an* $\mathscr{E}$*-precover and* $\mathbb{T}_{\sigma'}$ *is the right HRS-tilt of* $\mathbb{T}_\sigma$ *at the torsion pair* $\mathfrak{t} = ({}^{\perp_0}H^0_\sigma(\mathscr{E}), \mathrm{Cogen}(H^0_\sigma(\mathscr{E})))$ *in* $\mathcal{H}_\sigma$*.*

In both cases, the torsion pairs involved do not depend on the triangle in Definition 3.2.

Proof. Let $\mathbb{T} = \mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$ be the cosilting t-structure associated to σ with heart $\mathcal{H} = \mathcal{H}_\sigma$, and $\mathbb{T}' = \mathbb{T}_{\sigma'} = (\mathcal{X}', \mathcal{Y}')$ the cosilting t-structure associated to σ' with heart $\mathcal{H}' = \mathcal{H}_{\sigma'}$.

(1): Suppose first that σ' is a left mutation of σ. Let Φ be an $\mathscr{E}$-preenvelope of σ in $\mathcal{D}$ and consider the cosilting object $\tilde{\sigma} = \varepsilon_0 \oplus \varepsilon_1$, equivalent to σ', where ε_1 is defined via the triangle

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \sigma[1]. \qquad (\Delta_1)$$

Then $\mathbb{T}' = ({}^{\perp_{\leq 0}}\tilde{\sigma}, {}^{\perp_{>0}}\tilde{\sigma})$, and it is easy to see that $\mathcal{Y} \subseteq \mathcal{Y}' \subseteq \mathcal{Y}[1]$. From Proposition 2.3(1) we infer that $\mathbb{T}' = \mathbb{T}_{\mathfrak{t}^+}$ is the left HRS-tilt of $\mathbb{T}$ at the torsion pair $\mathfrak{t} := (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ given by $\mathcal{F} = \mathcal{H} \cap \mathcal{H}'[-1]$, and the latter coincides with $\mathcal{H} \cap \mathcal{Y}'[-1]$ since $\mathcal{X}[-1] \subseteq \mathcal{X}'[-2]$. Note that $\mathfrak{t}$ only depends on σ and σ'. It remains to verify that $\mathcal{F} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$.

In fact, we first observe that $\mathcal{F} = {}^{\perp_0}H^0_\sigma(\varepsilon_0)$. An object X of $\mathcal{H}$ lies in $\mathcal{F}$ if and only if $X[1]$ lies in $\mathcal{Y}'$. In particular, $\mathrm{Hom}_\mathcal{D}(X[1], \tilde{\sigma}[1]) = 0$ and, thus, $\mathrm{Hom}_\mathcal{D}(X, \tilde{\sigma}) = 0$. By Lemma 2.1 we conclude that $\mathrm{Hom}_\mathcal{H}(X, H^0_\sigma(\varepsilon_0)) \cong \mathrm{Hom}_\mathcal{D}(X, \varepsilon_0) = 0$. Conversely, if X belongs to ${}^{\perp_0}H^0_\sigma(\varepsilon_0)$, that is, $\mathrm{Hom}_\mathcal{D}(X, \varepsilon_0) = 0$, then applying $\mathrm{Hom}_\mathcal{D}(X[1], -)$ to the triangle above and keeping in mind that X lies in $\mathcal{Y} = {}^{\perp_{>0}}\sigma$ and ε_0 lies in $\mathrm{Prod}(\sigma)$, we see that $X[1]$ lies indeed in ${}^{\perp_{>0}}\tilde{\sigma} = \mathcal{Y}'$, which amounts to X lying in $\mathcal{F}$.

Finally, we show that ${}^{\perp_0}H^0_\sigma(\varepsilon_0) = {}^{\perp_0}H^0_\sigma(\mathscr{E})$, thus proving our claim. Note that, being left orthogonal to (classes of) injective objects in $\mathcal{H}$, both these classes are torsion (see Remark 2.6), and it is clear that ${}^{\perp_0}H^0_\sigma(\mathscr{E}) \subseteq {}^{\perp_0}H^0_\sigma(\varepsilon_0)$. So, in order to prove the desired equality, it is enough to show that the corresponding torsion-free classes satisfy $({}^{\perp_0}H^0_\sigma(\mathscr{E}))^{\perp_0} \subseteq ({}^{\perp_0}H^0_\sigma(\varepsilon_0))^{\perp_0}$, and for that it suffices to show that $H^0_\sigma(\mathscr{E})$ lies in $({}^{\perp_0}H^0_\sigma(\varepsilon_0))^{\perp_0} = \mathcal{F}^{\perp_0}$. Recall that $\mathcal{Y}' = \mathcal{F}[1] \star \mathcal{Y}$, and that $\mathrm{Prod}(\sigma) = \mathcal{Y} \cap (\mathcal{Y}[-1])^{\perp_0}$ by Proposition 2.5. Therefore, we have

$$\mathrm{Prod}(\sigma) \cap \mathcal{F}^{\perp_0} = \mathcal{Y} \cap (\mathcal{Y}[-1])^{\perp_0} \cap \mathcal{F}^{\perp_0} = \mathcal{Y} \cap (\mathcal{F} \star \mathcal{Y}[-1])^{\perp_0} = \mathcal{Y} \cap (\mathcal{Y}'[-1])^{\perp_0}.$$

But the latter class coincides with $\mathscr{E}$. Indeed, since $\mathcal{Y} \subseteq \mathcal{Y}'$ (and, thus, $(\mathcal{Y}[-1])^{\perp_0} \supseteq (\mathcal{Y}'[-1])^{\perp_0}$), we have

$$\mathcal{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma') = (\mathcal{Y} \cap (\mathcal{Y}[-1])^{\perp_0}) \cap (\mathcal{Y}' \cap (\mathcal{Y}'[-1])^{\perp_0}) = \mathcal{Y} \cap (\mathcal{Y}'[-1])^{\perp_0}.$$

So we have shown that, in fact, $\mathrm{Inj}(\mathcal{H}) \cap \mathcal{F}^{\perp_0} = H^0_\sigma(\mathscr{E})$.

Conversely, assume now that $\mathbb{T}'$ is the left HRS-tilt of $\mathbb{T}$ at the torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}_\sigma$, where $\mathcal{F} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$. Note here that since the hereditary torsion class ${}^{\perp_0}H^0_\sigma(\mathscr{E})$ is assumed to be closed under products, it is also a torsion-free class (see Remark 2.6), so the torsion pair $\mathfrak{t}$ is well-defined and the condition makes sense. Consider an $\mathscr{E}$-preenvelope $\Phi : \sigma \to \varepsilon_0$ of σ in $\mathcal{D}$ (see Lemma 3.1 for the existence of such a map). We complete Φ to a triangle of the form (Δ_1) and set $\tilde{\sigma} = \varepsilon_0 \oplus \varepsilon_1$. We will show that $\mathrm{Prod}(\sigma') = \mathrm{Prod}(\tilde{\sigma})$. Note that since σ cogenerates $\mathcal{D}$, so does $\tilde{\sigma}$. Hence, if we show that $\mathrm{Prod}(\tilde{\sigma})$ is contained in $\mathrm{Prod}(\sigma')$, Lemma 3.1(1b) guarantees the equality. For this purpose we show that $\tilde{\sigma}$ lies in $\mathcal{Y}' \cap \mathcal{Y}'[-1]^{\perp_0}$, which coincides with $\mathrm{Prod}(\sigma')$ by Proposition 2.5. Recall from Remark 2.2 that

$$\mathcal{Y}' = \mathcal{Y}_{\mathfrak{t}^+} = \mathcal{F}[1] \star \mathcal{Y} = \{X \in \mathcal{D} \mid H^{-1}_\sigma(X) \in \mathcal{F}, \text{ and } H^k_\sigma(X) = 0 \text{ for all } k < -1\}.$$

It is clear that ε_0 lies in $\mathcal{Y} \subset \mathcal{Y}'$. We claim that also ε_1 lies in $\mathcal{Y}'$. Indeed, we infer from the triangle above that $H^k_\sigma(\varepsilon_1) = 0$ for all $k < -1$, and moreover, since $H^0_\sigma(\Phi)$ is an $H^0_\sigma(\mathscr{E})$-preenvelope of $H^0_\sigma(\sigma)$, we also have that $H^{-1}_\sigma(\varepsilon_1)$ lies in ${}^{\perp_0}H^0_\sigma(\mathscr{E}) = \mathcal{F}$. Hence, $\tilde{\sigma}$ belongs to $\mathcal{Y}'$. Next, we pick an object Y

in $\mathcal{Y}'$. Then $H_\sigma^{-1}(Y)$ lies in $\mathcal{F}$ and, thus, $\mathrm{Hom}_\mathcal{D}(Y, \varepsilon_0[1]) \cong \mathrm{Hom}_\mathcal{D}(H_\sigma^{-1}(Y), H_\sigma^0(\varepsilon_0)) = 0$. Moreover, as $\mathcal{Y}' \subset \mathcal{Y}[1]$, we also have $\mathrm{Hom}_\mathcal{D}(Y, \sigma[2]) = 0$. Thus, we infer from a rotation of the triangle above that $\mathrm{Hom}_\mathcal{D}(Y, \varepsilon_1[1]) = 0$. This shows that $\tilde{\sigma}$ belongs to $\mathcal{Y}'^{\perp_1}$ and completes the proof.

(2): Suppose that σ' is a right mutation of σ. By definition, there is an $\mathscr{E}$-precover $\Phi : \gamma_0 \to \sigma$ in $\mathcal{D}$ and $\tilde{\sigma} = \gamma_0 \oplus \gamma_1$ is a cosilting object equivalent to σ', where γ_1 is defined by the triangle

$$\sigma[-1] \longrightarrow \gamma_1 \longrightarrow \gamma_0 \xrightarrow{\Phi} \sigma. \tag{Δ_2}$$

As in the previous part, one verifies that $\mathcal{Y}[-1] \subseteq \mathcal{Y}' \subseteq \mathcal{Y}$ and $\mathbb{T}' = \mathbb{T}_{\mathrm{t}^-}$ is the right HRS-tilt of $\mathbb{T}$ at a torsion pair $\mathrm{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ where $\mathcal{T} = {}^{\perp_0}H_\sigma^0(\gamma_0)$. Consequently, $\mathcal{F} = \mathrm{Cogen}(H_\sigma^0(\gamma_0))$ by Remark 2.6. It remains to check that $\mathcal{F} = \mathrm{Cogen}(H_\sigma^0(\mathscr{E}))$. However, we know from Lemma 3.3 that $H_\sigma^0(\Phi)$ is a $\mathrm{Cogen}(H_\sigma^0(\mathscr{E}))$-precover of $H_\sigma^0(\sigma)$. Since $H_\sigma^0(\sigma)$ is an injective cogenerator of $\mathcal{H}$, it follows that indeed $\mathcal{F} = \mathrm{Cogen}(H_\sigma^0(\mathscr{E}))$.

Conversely, suppose that σ admits an $\mathscr{E}$-precover $\Phi : \gamma_0 \to \sigma$ and that $\mathbb{T}_{\sigma'}$ is the right HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair t in $\mathcal{H}_\sigma$ given by $\mathrm{t} = ({}^{\perp_0}H_\sigma^0(\mathscr{E}), \mathrm{Cogen}(H_\sigma^0(\mathscr{E})))$. Complete Φ to a triangle of the form (Δ_2) and set $\tilde{\sigma} = \gamma_0 \oplus \gamma_1$. We have to show that $\mathrm{Prod}(\sigma') = \mathrm{Prod}(\tilde{\sigma})$. As in (1), it will suffice to show that $\tilde{\sigma}$ belongs to $\mathcal{Y}' \cap (\mathcal{Y}'[-1])^{\perp_0}$. Since γ_0 lies in $\mathcal{Y}$, there is a truncation triangle of the form

$$H_\sigma^0(\gamma_0) \longrightarrow \gamma_0 \longrightarrow Y[-1] \longrightarrow H_\sigma^0(\gamma_0)[1],$$

with Y in $\mathcal{Y}$ and, thus, γ_0 lies in $\mathcal{F} \star \mathcal{Y}[-1] = \mathcal{Y}_{\mathrm{t}^-} = \mathcal{Y}'$. Since γ_1 is an extension of $\sigma[-1]$ (which lies in $\mathcal{Y}[-1] \subseteq \mathcal{Y}'$) and γ_0, we conclude that also γ_1 (and, hence, $\tilde{\sigma}$) belongs to $\mathcal{Y}'$. Next, we pick an object Y in $\mathcal{Y}'$ and apply $\mathrm{Hom}_\mathcal{D}(Y, -)$ to the following rotation of the triangle (Δ_2):

$$\gamma_0 \xrightarrow{\Phi} \sigma \longrightarrow \gamma_1[1] \longrightarrow \gamma_0[1].$$

Since, from Lemma 3.3, Φ is a $\mathcal{Y}'$-precover, and since $\mathcal{Y}' \subseteq \mathcal{Y} = {}^{\perp_{>0}}\sigma$ and γ_0 lies in $\mathrm{Prod}(\sigma)$, we conclude that $\tilde{\sigma}$ belongs to $(\mathcal{Y}'[-1])^{\perp_0}$, completing the proof. □

Remark 3.6. Note that the HRS-tilts discussed above involve torsion pairs in $\mathcal{H}_\sigma$ of different flavours: in the case of left mutation, $\mathbb{T}_{\sigma'}$ is the HRS-tilt of $\mathbb{T}_\sigma$ at a torsion pair $(\mathcal{T}, \mathcal{F})$ for which $\mathcal{F}$ is a TTF class; in the case of right mutation, $\mathbb{T}_{\sigma'}$ is the HRS-tilt of $\mathbb{T}_\sigma$ at a hereditary torsion pair $(\mathcal{S}, \mathcal{R})$. Moreover, if there exist both a left and a right mutation with respect to $\mathscr{E}$, then $(\mathcal{T}, \mathcal{F})$ is left adjacent to $(\mathcal{S}, \mathcal{R})$, i.e., $\mathcal{S} = \mathcal{F}(= {}^{\perp_0}H_\sigma^0(\mathscr{E}))$.

Corollary 3.7. *Let $\mathcal{D}$ be a triangulated category with products. Then both left and right mutation of cosilting objects in $\mathcal{D}$ are well-defined up to equivalence. Moreover, the induced operations on equivalence classes of cosilting objects are inverse of each other.*

Proof. Theorem 3.5 characterises the fact that two cosilting objects σ and σ' are mutations of each other in terms of properties of the associated cosilting t-structures and of the class $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$. These t-structures and $\mathscr{E}$ depend exclusively on the equivalence class of the cosilting objects considered.

Moreover, using the fact that products of $\mathscr{E}$-precovers are $\mathscr{E}$-precovers, we note that if σ admits an $\mathscr{E}$-precover, then so does any cosilting object equivalent to σ.

For the second statement, we observe that σ' is a left mutation of σ if and only if σ is a right mutation of σ'. Indeed, if $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$, consider the triangle induced by an $\mathscr{E}$-preenvelope Φ

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \xrightarrow{\Psi} \varepsilon_1 \longrightarrow \sigma[1],$$

and the cosilting object $\tilde{\sigma} = \varepsilon_0 \oplus \varepsilon_1$, which, by assumption, is equivalent to σ'. It then follows that Ψ is an $\mathscr{E}$-precover of ε_1 (since $\mathrm{Hom}_{\mathcal{D}}(\mathscr{E}, \sigma[1]) = 0$), and $\theta = \Psi \oplus 1_{\varepsilon_0}$ is an $\mathscr{E}$-precover of $\tilde{\sigma}$. Therefore, the induced triangle

$$\sigma \longrightarrow \varepsilon_0 \oplus \varepsilon_0 \xrightarrow{\theta} \tilde{\sigma} \longrightarrow \sigma[1]$$

shows that σ is a right mutation of $\tilde{\sigma}$. The other implication is obtained by dual arguments. □

The torsion pairs appearing the theorem above are determined by the cosilting objects involved in the mutation process. In general it is difficult to characterise when a torsion pair arises in this way, although we will see later that this is possible in the setting of Section 4. Nevertheless, the following definition is helpful in characterising when mutation is possible.

Definition 3.8. Let $\mathcal{D}$ be a triangulated category with products and $\mathbb{T}$ a t-structure with heart $\mathcal{H}$. A torsion pair $\mathfrak{t} := (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ is said to be a *cosilting torsion pair* if and only if there is a cosilting object σ in $\mathcal{D}$ such that $\mathbb{T}_{\mathfrak{t}^-} = \mathbb{T}_\sigma$. We denote the collection of cosilting torsion pairs in $\mathcal{H}$ by $\mathrm{Cosilt}(\mathcal{H})$. If $\mathcal{D}$ is compactly generated, then we denote the set of cosilting torsion pairs in $\mathcal{H}$ arising from pure-injective cosilting objects by $\mathrm{Cosilt}_*(\mathcal{H})$.

Example 3.9. Let R be a ring. The modules $C = H^0(\sigma)$ arising as zero cohomologies of a cosilting complex $\sigma : I^0 \to I^1$ of length 2, concentrated in degrees 0 and 1, are precisely the *cosilting modules* introduced in [Breaz and Pop 2017]. The t-structure $\mathbb{T}_\sigma$ then coincides with the right HRS-tilt of the standard t-structure $\mathbb{T} = (\mathrm{D}^{\le -1}, \mathrm{D}^{\ge 0})$ of $\mathrm{D}(R)$ at the torsion pair $\mathfrak{t} := ({}^{\perp_0}C, \mathrm{Cogen}(C))$ in $\mathrm{Mod}(R)$ cogenerated by C. In other words, $\mathfrak{t}$ is contained in $\mathrm{Cosilt}(R) := \mathrm{Cosilt}(\mathrm{Mod}(R))$.

Furthermore, any cosilting torsion pair in $\mathrm{Mod}(R)$ is actually of this form. Indeed, let $\sigma \in \mathrm{D}(R)$ be an associated cosilting complex and $\mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$ the cosilting t-structure. Then $\sigma \in \mathcal{Y} \subseteq \mathrm{D}^{\ge 0}$ is isomorphic to a complex of injectives concentrated in nonnegative cohomological degrees. As on the other hand $\mathrm{D}^{\ge 1} \subseteq \mathcal{Y}$, we have

$$\sigma \in \mathcal{Y}^{\perp_1} \subseteq (\mathrm{D}^{\ge 1})^{\perp_1} = (\mathrm{D}^{\ge 2})^{\perp_0},$$

which implies that such a complex of injectives is homotopy equivalent to a 2-term complex concentrated in degrees 0 and 1 (see, e.g., [Parra et al. 2023a, Lemma 4.12]).

Recall further that all cosilting complexes of length 2 in $\mathrm{D}(R)$ are pure-injective (Example 2.8). Thus $\mathrm{Cosilt}(R) = \mathrm{Cosilt}_*(\mathrm{Mod}(R))$. In fact, it follows from [Breaz and Žemlička 2018; Zhang and Wei 2017] that a torsion pair in $\mathrm{Mod}(R)$ is cosilting if and only if it is of finite type. We are going to see in Proposition 4.5 that this is a special instance of a general phenomenon.

The following proposition explains the relation between cosilting torsion pairs and mutation, providing a criterion for the existence of mutation.

Proposition 3.10. *Let $\mathcal{D}$ be a triangulated category with products. Let σ be a cosilting object and $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ a subcategory of $\mathrm{Prod}(\sigma)$. Then we have*

(1) *σ admits a left mutation σ' with respect to $\mathscr{E}$ if and only if there is a cosilting torsion pair of the form $\mathfrak{t} := (\mathcal{T}, {}^{\perp_0}H^0_\sigma(\mathscr{E}))$ in $\mathcal{H}_\sigma$, and*

(2) *σ admits a right mutation σ' with respect to $\mathscr{E}$ if and only if σ admits an $\mathscr{E}$-precover and the torsion pair $\mathfrak{t} = (\mathcal{S}, \mathcal{R}) = ({}^{\perp_0}H^0_\sigma(\mathscr{E}), \mathrm{Cogen}(H^0_\sigma(\mathscr{E})))$ in $\mathcal{H}_\sigma$ is a cosilting torsion pair.*

Proof. (1): By Theorem 3.5, we only need to prove the "if" part. Suppose that there is a cosilting torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}_\sigma$ where $\mathcal{F} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$. By definition, there is a cosilting object σ' such that $\mathbb{T}_{\sigma'}$ coincides with the left HRS-tilt $\mathbb{T}_{\mathfrak{t}^+} = (\mathcal{X}', \mathcal{Y}')$ of the t-structure $\mathbb{T} = \mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$ associated to σ. As in the proof of Theorem 3.5, we see that $\mathrm{Prod}(\sigma')$ coincides with $\mathrm{Prod}(\tilde{\sigma})$, where $\tilde{\sigma} = \varepsilon_0 \oplus \varepsilon_1$ is obtained from a triangle

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \sigma[1],$$

where Φ is an $\mathscr{E}$-preenvelope. By Theorem 3.5, it thus only remains to show that $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$. For the inclusion $\mathscr{E} \subseteq \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$ we have to show that $\mathscr{E} \subseteq \mathrm{Prod}(\sigma') = \mathcal{Y}' \cap \mathcal{Y}'^{\perp_1}$. Of course, $\mathscr{E}$ is contained in $\mathcal{Y} \subseteq \mathcal{Y}'$. Now pick E in $\mathscr{E}$ and Y in $\mathcal{Y}'$, and consider a decomposition triangle with respect to the t-structure $\mathbb{T}$

$$A \longrightarrow Y \longrightarrow B \longrightarrow A[1],$$

where A in $\mathcal{X}$ and B in $\mathcal{Y} \subseteq \mathcal{Y}'$. In particular, we have that A lies in $\mathcal{X} \cap \mathcal{Y}' = \mathcal{F}[1]$, because $\mathcal{Y}' = \mathcal{Y}_{\mathfrak{t}^+} = \mathcal{F}[1] \star \mathcal{Y}$. Applying $\mathrm{Hom}_{\mathcal{D}}(-, E[1])$ to the triangle, and using that $A[-1] \in \mathcal{F} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$, we obtain

$$\mathrm{Hom}_{\mathcal{D}}(A, E[1]) \cong \mathrm{Hom}_{\mathcal{H}_\sigma}(A[-1], H^0_\sigma(E)) = 0.$$

Further, since E lies in $\mathrm{Prod}(\sigma)$, we have $\mathrm{Hom}_{\mathcal{D}}(B, E[1]) = 0$ and, thus, $\mathrm{Hom}_{\mathcal{D}}(Y, E[1]) = 0$. To prove the other inclusion, let X be an object in $\mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$. Clearly, $H^0_\sigma(X)$ is injective in $\mathcal{H}_\sigma$, because X lies in $\mathrm{Prod}(\sigma)$. Furthermore, if F is an object in $\mathcal{F}$, since $F[1]$ lies in $\mathcal{Y}' = {}^{\perp_{>0}}\sigma'$ and X lies in $\mathrm{Prod}(\sigma')$, we have

$$\mathrm{Hom}_{\mathcal{D}}(F, H^0_\sigma(X)) \cong \mathrm{Hom}_{\mathcal{D}}(F, X) \cong \mathrm{Hom}_{\mathcal{D}}(F[1], X[1]) = 0.$$

It follows that $H^0_\sigma(X)$ lies in $\mathcal{F}^{\perp_0} = \mathrm{Cogen}(H^0_\sigma(\mathscr{E}))$. We conclude that $H^0_\sigma(X)$ lies in $H^0_\sigma(\mathscr{E})$ and, thus, X lies in $\mathscr{E}$.

(2): By Theorem 3.5, we only need to prove the "if" part. Suppose that $\mathfrak{t} = ({}^{\perp_0}H^0_\sigma(\mathscr{E}), \mathrm{Cogen}(H^0(\mathscr{E})))$ is a cosilting torsion pair in $\mathcal{H}_\sigma$. Then there is a cosilting object σ' such that $\mathbb{T}_{\sigma'}$ coincides with the right HRS-tilt $\mathbb{T}_{\mathfrak{t}^-} = (\mathcal{X}', \mathcal{Y}')$ of the t-structure $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ associated to σ. By Theorem 3.5, it only remains to show that $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$. Now $\mathscr{E}$ is contained in $\mathrm{Prod}(\sigma)$, and also in $\mathcal{Y}'$, which consists of those objects X in $\mathcal{Y}$ for which $H^0_\sigma(X)$ lies in $\mathrm{Cogen}(H^0_\sigma(\mathscr{E}))$ by Remark 2.2. Moreover, given E in $\mathscr{E}$ and X in $\mathcal{Y}'$, we have $\mathrm{Hom}_{\mathcal{D}}(X, E[1]) = 0$ since X lies in $\mathcal{Y} = {}^{\perp_{>0}}\sigma$ and E lies in $\mathrm{Prod}(\sigma)$. We conclude that E is an object of $\mathcal{Y}' \cap \mathcal{Y}'^{\perp_1} = \operatorname{Prod}(\sigma')$. Conversely, take X in $\operatorname{Prod}(\sigma) \cap \operatorname{Prod}(\sigma')$. Then $H^0_\sigma(X)$ is injective in $\mathcal{H}_\sigma$, because X lies in $\operatorname{Prod}(\sigma)$. Furthermore, it lies in $\operatorname{Cogen}(H^0_\sigma(\mathscr{E}))$, because X lies in $\operatorname{Prod}(\sigma') \subseteq \mathcal{Y}'$. Hence $H^0_\sigma(X)$ lies in $H^0_\sigma(\mathscr{E})$ and, thus, X lies in $\mathscr{E}$. □

4. Mutation and purity

As recalled in Theorem 2.7, pure-injective cosilting objects or pure-projective silting objects give rise to Grothendieck hearts. In order to investigate mutations of these objects, we need to understand when HRS-tilts are Grothendieck categories. Recall that a cocomplete abelian category is said to be *AB*4 if coproducts are exact and it is said to be *AB*5 if all direct limits are exact.

4.1. *HRS-tilts with Grothendieck heart.* Our aim in this subsection is to generalise [Parra and Saorín 2016, Theorem 1.2], which characterises when the heart of an HRS-tilt is a Grothendieck category. This requires some preparation, however. Suppose that $\mathcal{H}$ is an AB4 abelian category, I a directed poset and $\mathcal{X} = (X_i, f_{ij} : X_i \to X_j \mid i, j \in I,\ i \geq j)$ a direct system in $\mathcal{H}$. This system gives rise to a complex

$$B\mathcal{X}_{\mathsf{aug}} : \cdots \longrightarrow \coprod_{i_0<i_1<i_2} X_{i_0} \xrightarrow{d_2} \coprod_{i_0<i_1} X_{i_0} \xrightarrow{d_1} \coprod_{i_0 \in I} X_{i_0} \xrightarrow{d_0} \varinjlim \mathcal{X} \longrightarrow 0$$

in $\mathcal{H}$, where d_0 is the canonical morphism and

$$d_n : \coprod_{i_0<\cdots<i_n} X_{i_0} \longrightarrow \coprod_{j_0<\cdots<j_{n-1}} X_{j_0}$$

for $n \geq 1$ are described as follows: if $(j_0 < \cdots < j_{n-1})$ is obtained from $(i_0 < \cdots < i_n)$ by removing i_k for some $0 \leq k \leq n$, then the component of d_n between the corresponding summands of the coproducts is equal to $(-1)^k f_{i_0 j_0} : X_{i_0} \to X_{j_0}$ (we stress that $f_{i_0 j_0} = \mathrm{id}_{X_{i_0}}$ if $k > 0$ as then $i_0 = j_0$). All the other components of d_n vanish by definition.

We denote by $B\mathcal{X}$ the complex obtained from $B\mathcal{X}_{\mathsf{aug}}$ by deleting the last term with the direct limit. Hence $B\mathcal{X}$ is a complex over $\mathcal{H}$ which consists just of coproducts of objects in $\mathcal{X}$ and we place the term $\coprod_{i_0 \in I} X_{i_0}$ in cohomological degree 0. Now we define for $n \geq 0$ functors

$$\varinjlim\nolimits_n : \mathcal{H}^I \to \mathcal{H}$$

by putting $\varinjlim_n \mathcal{X} := H^{-n}(B\mathcal{X})$. We summarise some well-known properties of these functors.

Lemma 4.1. *Given a short exact sequence* $0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$ *of* I*-shaped direct systems in an AB*4 *abelian category* $\mathcal{H}$, *there is a natural long exact sequence*

$$\cdots \to \varinjlim\nolimits_2 \mathcal{X} \to \varinjlim\nolimits_2 \mathcal{Y} \to \varinjlim\nolimits_2 \mathcal{Z} \to \varinjlim\nolimits_1 \mathcal{X} \to \varinjlim\nolimits_1 \mathcal{Y} \to \varinjlim\nolimits_1 \mathcal{Z} \to \varinjlim \mathcal{X} \to \varinjlim \mathcal{Y} \to \varinjlim \mathcal{Z} \to 0.$$

Moreover, the following statements are equivalent:

(1) $\mathcal{H}$ *is AB*5.

(2) $\varinjlim_1 \mathcal{X} = 0$ *for each directed poset* I *and an* I*-shaped direct system* $\mathcal{X}$.

(3) $\varinjlim_n \mathcal{X} = 0$ *for each directed poset* I, *an* I*-shaped direct system* $\mathcal{X}$ *and* $n > 0$.

Proof. The short exact sequence $0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$ yields a short exact sequence of complexes $0 \to B\mathcal{X} \to B\mathcal{Y} \to B\mathcal{Z} \to 0$ since we assume $\mathcal{H}$ to be AB4. The "moreover" part is standard: The implications (3) ⇒ (2) ⇒ (1) are straightforward. Regarding (1) ⇒ (3), one observes that $B\mathcal{X}_{\mathsf{aug}}$ is a direct limit of contractible complexes, and hence exact assuming (1). Indeed, I is easily seen to be a direct union of finite directed subsets $F \subseteq I$, and each such F has a unique maximal element $\max(F) \in F$ (which is an upper bound of all the elements in F). The corresponding complex for the restricted direct system $\mathcal{X}|_F$ is canonically identified with a subcomplex of $B\mathcal{X}_{\mathsf{aug}}$ of the form

$$B(\mathcal{X}|_F)_{\mathsf{aug}}: \cdots \longrightarrow \coprod_{i_0<i_1<i_2} X_{i_0} \xrightarrow{d_2} \coprod_{i_0<i_1} X_{i_0} \xrightarrow{d_1} \coprod_{i_0\in F} X_{i_0} \xrightarrow{d_0} X_{\max(F)} \longrightarrow 0$$

and is contractible. A particular nullhomotopy is given by $(s_n)_{n\geq 0}$, where

$$s_n: \coprod_{j_0<\cdots<j_{n-1}} X_{j_0} \longrightarrow \coprod_{i_0<\cdots<i_n} X_{i_0},$$

with the components $(-1)^n \mathrm{id}_{X_{j_0}}$ if $(i_0 < \cdots < i_{n-1} < i_n) = (j_0 < \cdots < j_{n-1} < \max(F))$, and zero maps otherwise. The special case of $s_0 : X_{\max(F)} \to \coprod_{i_0\in F} X_{i_0}$ is the obvious split inclusion. Finally, one observes that $B\mathcal{X}_{\mathsf{aug}} = \varinjlim_F B(\mathcal{X}|_F)_{\mathsf{aug}}$, where F runs over all finite directed subsets of I, ordered by inclusion. □

Now we focus on how the $\varinjlim_n$-functors interact with HRS tilting.

Lemma 4.2. *Let $\mathcal{D}$ be a triangulated category with coproducts, let $\mathbb{T}$ be a smashing t-structure with an AB5 heart $\mathcal{H}$, and let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{H}$. We recall that the heart $\mathcal{H}_{\mathfrak{t}^-}$ of the right HRS-tilt has a torsion pair $(\mathcal{F}, \mathcal{T}[-1])$. Then the following hold*:

(1) *If $\mathcal{X}$ is a direct system in $\mathcal{T}$ and $T = \varinjlim \mathcal{X}$ in $\mathcal{H}$, then $T[-1]$ is the direct limit of $\mathcal{X}[-1]$ in $\mathcal{H}_{\boldsymbol{t}^-}$ and $\varinjlim_n \mathcal{X}[-1] = 0$ in $\mathcal{H}_{\boldsymbol{t}^-}$ for each $n > 0$.*

(2) *Assume moreover that $(\mathcal{T}, \mathcal{F})$ is of finite type in $\mathcal{H}$. If $\mathcal{X}$ is a direct system in $\mathcal{F}$ and $F = \varinjlim \mathcal{X}$ in $\mathcal{H}$, then F is also the direct limit of $\mathcal{X}$ in $\mathcal{H}_{\boldsymbol{t}^-}$ and $\varinjlim_n \mathcal{X} = 0$ in $\mathcal{H}_{\boldsymbol{t}^-}$ for each $n > 0$.*

Proof. We prove (2) only, as the proof of (1) is completely analogous. Since the complex $B\mathcal{X}_{\mathsf{aug}}$ is acyclic in $\mathcal{H}$ by Lemma 4.1, we have induced triangles in $\mathcal{D}$,

$$Z_{n+1} \longrightarrow \coprod_{i_0<\cdots<i_n} X_{i_0} \longrightarrow Z_n \longrightarrow Z_{n+1}[1]$$

for all $n \geq 0$. Here, $Z_n = \mathrm{Im}(d_n)$ are the images in $\mathcal{H}$ and the coproducts can be taken equally well in $\mathcal{H}$ and in $\mathcal{D}$ as $\mathbb{T}$ is smashing. Since $\mathcal{F}$ is closed under coproducts ($\mathcal{H}$ is AB5) and subobjects, we have $Z_n \in \mathcal{F}$ for each $n > 1$. We also have $Z_0 \in \mathcal{F}$ since we assume $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ to be of finite type. Consequently, the triangles above induce short exact sequences in $\mathcal{H}_{\boldsymbol{t}^-}$. Note further that $\mathbb{T}_{\boldsymbol{t}^-}$ is a smashing t-structure and $\mathcal{H}_{\boldsymbol{t}^-}$ is closed under coproducts in $\mathcal{D}$. It follows that $\mathcal{H}_{\boldsymbol{t}^-}$ is AB4, and that the coproducts in the triangles above are also coproducts in $\mathcal{H}_{\boldsymbol{t}^-}$. Thus, the complexes $B\mathcal{X}_{\mathsf{aug}}$ taken in $\mathcal{H}$ and $\mathcal{H}_{\boldsymbol{t}^-}$ coincide and are both acyclic in the corresponding hearts. □

Now we can give the promised generalisation of [Parra and Saorín 2016, Theorem 1.2].

Theorem 4.3. *Let $\mathcal{D}$ be a triangulated category with coproducts, let $\mathbb{T}$ be a smashing t-structure with an AB5 heart $\mathcal{H}$, and let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{H}$. Then the torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ is of finite type in the right HRS-tilt $\mathcal{H}_{\mathfrak{t}^-}$ and the following statements are equivalent*:

(1) *$\mathcal{H}_{\mathfrak{t}^-}$ is AB5.*

(2) *The torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is of finite type in $\mathcal{H}$.*

If $\mathcal{D}$ is compactly generated, the conditions above are further equivalent to

(3) *$\mathcal{H}_{\mathfrak{t}^-}$ is a Grothendieck category.*

Proof. The torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ is of finite type in $\mathcal{H}_{\mathfrak{t}^-}$ thanks to Lemma 4.2(1). If $(\mathcal{T}, \mathcal{F})$ is of finite type in $\mathcal{H}$, then $\varinjlim_1$ vanishes in $\mathcal{H}_{\mathfrak{t}^-}$ on all direct systems in $\mathcal{F}$ or $\mathcal{T}[-1]$ by Lemma 4.2. If $\mathcal{X}$ is any direct system in $\mathcal{H}_{\mathfrak{t}^-}$, we obtain a short exact sequence of direct systems

$$0 \longrightarrow \mathcal{X}_{\mathcal{F}} \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}[-1]} \longrightarrow 0,$$

where $\mathcal{X}_{\mathcal{F}}$ and $\mathcal{X}_{\mathcal{T}[-1]}$ are the direct systems of torsion and torsion-free parts of $\mathcal{X}$ with respect to $(\mathcal{F}, \mathcal{T}[-1])$, respectively. If we take the direct limit, by Lemma 4.1, we obtain an exact sequence

$$0 = \varinjlim{}_1 \mathcal{X}_{\mathcal{F}} \longrightarrow \varinjlim{}_1 \mathcal{X} \longrightarrow \varinjlim{}_1 \mathcal{X}_{\mathcal{T}[-1]} = 0,$$

and the same lemma tells us (since, by [Parra and Saorín 2015, Proposition 3.3], $\mathcal{H}_{\mathfrak{t}^-}$ is AB4), that $\mathcal{H}_{\mathfrak{t}^-}$ is also AB5.

Suppose conversely that $\mathcal{H}_{\mathfrak{t}^-}$ is AB5. Then we can apply the previous arguments to $\mathcal{H}_{\mathfrak{t}^-}$ with torsion pair $(\mathcal{F}, \mathcal{T}[-1])$. As the left HRS-tilt is equivalent to $\mathcal{H}$ with $(\mathcal{T}, \mathcal{F})$, it follows that the latter torsion pair is of finite type.

Now assume that $\mathcal{D}$ is compactly generated, with $\mathcal{D}^c$ denoting its subcategory of compact objects. Suppose that (1) holds true. In order to prove that $\mathcal{H}_{\mathfrak{t}^-}$ is a Grothendieck category, it remains to exhibit a generator. We claim that $\coprod H^0_{\mathfrak{t}^-}(C)$, where the coproduct runs over all isoclasses of compact objects C in $\mathcal{D}$, is a generator of $\mathcal{H}_{\mathfrak{t}^-}$. Indeed, given any X in $\mathcal{H}_{\mathfrak{t}^-}$, the canonical map

$$\Phi : \coprod C^{(\mathrm{Hom}_{\mathcal{D}}(C,X))} \longrightarrow X$$

is a pure epimorphism. Note that $H^0_{\mathfrak{t}^-} : \mathcal{D} \to \mathcal{H}_{\mathfrak{t}^-}$ sends pure triangles in $\mathcal{D}$ to short exact sequences by [Krause 2000, Corollary 2.5]. Therefore, $H^0_{\mathfrak{t}^-}(\Phi)$ is an epimorphism, which proves the claim. □

4.2. *Mutation of pure-injective cosilting objects.* We are now ready to examine mutation of pure-injective cosilting objects. We will see that this setting is somewhat nicer than the general setting explored in Section 3.

In view of Proposition 3.10, we begin by investigating cosilting torsion pairs in Grothendieck hearts. As indicated in Example 3.9, when $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is a cosilting torsion pair in the heart $\mathcal{H}$ of a t-structure $\mathbb{T}$ in $\mathcal{D}$, and σ is a cosilting object such that $\mathbb{T}_{\mathfrak{t}^-} = \mathbb{T}_\sigma$, it is interesting to study properties of the object $C = H^0_{\mathbb{T}}(\sigma)$. As the following lemma says, this object in fact determines the torsion pair $\mathfrak{t}$ and hence the cosilting object σ up to equivalence. To this end, recall that an object C in a complete abelian category $\mathcal{H}$ is called *quasicotilting* if $\mathrm{Cogen}(C) = \mathrm{gen}(\mathrm{Cogen}(C)) \cap {}^{\perp_1}C$ (one can also define quasitilting objects and prove a dual analogous version of the following lemma, but it is not needed here).

Lemma 4.4. *Let $\mathcal{D}$ be a triangulated category with products and $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ be a t-structure with heart $\mathcal{H}$. Suppose that $\mathcal{H}$ is complete and that $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is a cosilting torsion pair in $\mathcal{H}$, and σ is a cosilting object such that $\mathbb{T}_{\mathfrak{t}^-} = \mathbb{T}_\sigma$. Then the object $C = H^0_{\mathbb{T}}(\sigma)$ is quasicotilting in $\mathcal{H}$ and $\mathcal{F} = \mathrm{Cogen}(C)$.*

Proof. We will adapt the argument for [Parra et al. 2023a, Theorem 4.1(3) ⇒ (2)]. Recall that there is a torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ in the heart $\mathcal{H}_\sigma$ of $\mathbb{T}_{\mathfrak{t}^-} = (\mathcal{X}_{\mathfrak{t}^-}, \mathcal{Y}_{\mathfrak{t}^-})$, and that the torsion part of $X \in \mathcal{H}_\sigma$ with respect to this torsion pair equals $H^0_{\mathbb{T}}(X)$ [Happel et al. 1996, Corollary 2.2]. Hence the functor $H^0_{\mathbb{T}}|_{\mathcal{H}_\sigma} : \mathcal{H}_\sigma \to \mathcal{H}$, being a composition of the right adjoints $\mathcal{H}_\sigma \to \mathcal{F}$ and $\mathrm{inc} : \mathcal{F} \to \mathcal{H}$, is itself a right adjoint. Since $E := H^0_\sigma(\sigma)$ is an injective cogenerator of $\mathcal{H}_\sigma$ by Proposition 2.5, any object $F \in \mathcal{F}$ admits a monomorphism $F \to E^I$ for some set I, which induces a monomorphism $F = H^0_{\mathbb{T}}(F) \to H^0_{\mathbb{T}}(E)^I$ in $\mathcal{H}$. Moreover, since $\sigma \in \mathcal{Y}_{\mathfrak{t}^-} \subseteq \mathcal{Y}$, we observe that E is an $\mathcal{X}_{\mathfrak{t}^-}[-1]$-coreflection of σ and C is an $\mathcal{X}[-1]$-coreflection of σ, which is the same as an $\mathcal{X}[-1]$-coreflection of $E \in \mathcal{Y}_{\mathfrak{t}^-}$. In particular, $C = H^0_{\mathbb{T}}(E)$, the object F embeds in C^I and, hence, $\mathcal{F} \subseteq \mathrm{Cogen}(C)$. Since clearly $C \in \mathcal{F}$ and $\mathcal{F}$ is closed under products and subobjects, we obtain the equality $\mathcal{F} = \mathrm{Cogen}(C)$.

In order to prove that C is quasicotilting, we first claim that $\mathrm{Ext}^1_{\mathcal{H}}(F, C) = \mathrm{Hom}_{\mathcal{D}}(F[-1], C) = 0$ for each F in $\mathcal{F} = \mathcal{X}[-1] \cap \mathcal{Y}_{\mathfrak{t}^-}$. To see this, recall from the discussion above that there is a decomposition triangle

$$Y[-2] \longrightarrow C \longrightarrow \sigma \longrightarrow Y[-1],$$

where $Y \in \mathcal{Y}$ (and $C \in \mathcal{X}[-1]$). Notice now that $\mathrm{Hom}_{\mathcal{D}}(F[-1], Y[-2]) = 0$ since $F[-1] \in \mathcal{X}[-2]$, and that $\mathrm{Hom}_{\mathcal{D}}(F[-1], \sigma) = 0$ as $F[-1] \in \mathcal{Y}_{\mathfrak{t}^-}[-1]$ and $\sigma \in (\mathcal{Y}_{\mathfrak{t}^-})^{\perp_1}$. It follows from the decomposition triangle that $\mathrm{Hom}_{\mathcal{D}}(F[-1], C) = 0$, proving the claim.

It remains to check that every object $X \in \mathrm{gen}(\mathrm{Cogen}(C)) \cap {}^{\perp_1}C$ lies in $\mathcal{F}$. Each such X is part of a short exact sequence in $\mathcal{H}$ of the form

$$0 \longrightarrow F' \longrightarrow F \longrightarrow X \longrightarrow 0,$$

with $F', F \in \mathcal{F}$. First note that since C is an $\mathcal{X}[-1]$-coreflection of σ, we have a natural isomorphism $\mathrm{Hom}_{\mathcal{H}}(-, C) \cong \mathrm{Hom}_{\mathcal{D}}(-, \sigma)|_{\mathcal{H}}$. Applying $\mathrm{Hom}_{\mathcal{H}}(-, C)$ to the above exact sequence and using this natural isomorphism, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{H}}(F, C) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}}(F', C) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{H}}(X, C) = 0 & & \\
\cong\downarrow & & \downarrow\cong & & & & \\
\mathrm{Hom}_{\mathcal{D}}(F, \sigma) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F', \sigma) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(X[-1], \sigma) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F[-1], \sigma) = 0
\end{array}$$

It follows that $\mathrm{Hom}_{\mathcal{H}_\sigma}(H^{-1}_\sigma(X), E) \cong \mathrm{Hom}_{\mathcal{D}}(X[-1], \sigma) = 0$, so $H^{-1}_\sigma(X) = 0$ and $X \in \mathcal{H} \cap \mathcal{H}_\sigma = \mathcal{F}$. □

We now characterise cosilting torsion pairs in Grothendieck hearts coming from pure-injective cosilting objects. A similar result is proved in [Parra et al. 2023a, Theorem A] (see also [Parra et al. 2023b, Proposition 6.20]) when the ambient category $\mathcal{D}$ is the derived category of $\mathcal{H}$.

Proposition 4.5. *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$. Let $\mathcal{H}$ be the heart of the associated t-structure and let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{H}$. The following statements are equivalent*:

(1) *The torsion pair $\mathfrak{t}$ is of finite type.*

(2) *The torsion-free class $\mathcal{F}$ is covering in $\mathcal{H}$.*

(3) *Any injective cogenerator of $\mathcal{H}$ admits an $\mathcal{F}$-cover.*

(4) *There is a quasicotilting object C in $\mathcal{H}$ such that $\mathcal{F} = \mathrm{Cogen}(C)$.*

(5) *The torsion pair $\mathfrak{t}$ is a cosilting torsion pair arising from a pure-injective cosilting object.*

(6) *The torsion pair $\mathfrak{t}$ is cosilting.*

Before we prove the proposition, we point out an immediate corollary, which goes back to [Bazzoni 2003, Theorem 2.8], later generalised in [Čoupek and Šťovíček 2020, Theorem 3.9].

Corollary 4.6. *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$ and let $\mathcal{H}$ be the heart of the associated t-structure. Then every cosilting torsion pair in $\mathcal{H}$ arises from a pure-injective cosilting object*, *i.e.*, $\mathrm{Cosilt}(\mathcal{H}) = \mathrm{Cosilt}_*(\mathcal{H})$.

Proof of Proposition 4.5. Let $\mathbb{T}$ denote the t-structure associated to σ. Recall from Theorem 2.7 that $\mathbb{T}$ is a smashing nondegenerate t-structure and that $\mathcal{H}$ is a Grothendieck category.

(1) $\Rightarrow$ (2): If $\mathcal{F}$ is closed under direct limits, it follows from [El Bashir 2006, Theorem 3.2] that $\mathcal{F}$ is covering.

(2) $\Rightarrow$ (3): This is trivial.

(4) $\Rightarrow$ (1): This follows from [Parra et al. 2023a, Theorem A].

(3) $\Rightarrow$ (4): Let E be an injective cogenerator of $\mathcal{H}$ and let $f : F \to E$ be an $\mathcal{F}$-cover. Consider the exact sequence

$$0 \longrightarrow K \xrightarrow{k} F \xrightarrow{f} E.$$

We claim that $C = F \oplus K$ is quasicotilting. First note that $\mathcal{F} = \mathrm{Cogen}(F) = \mathrm{Cogen}(C)$. Indeed $\mathrm{Cogen}(F)$ is clearly contained in $\mathcal{F}$ and, conversely, for any object X in $\mathcal{F}$, any given monomorphism $g : X \to E^I$ (which exists for some set I) will factor through the precover f^I via a monomorphism.

Next, we show the equality $\mathrm{Cogen}(C) = \mathrm{gen}(\mathrm{Cogen}(C)) \cap {}^{\perp_1}C$. Let us first verify that $\mathrm{Cogen}(C) \subseteq {}^{\perp_1}C$. If X lies in $\mathcal{F}$, consider an exact sequence

$$0 \longrightarrow F \xrightarrow{\alpha} Y \xrightarrow{\beta} X \longrightarrow 0.$$

Since $\mathcal{F} = \mathrm{Cogen}(F)$ is extension-closed, there is a monomorphism $\epsilon : Y \to F^J$ for some set J. Since E is injective, the map f extends along $\epsilon \circ \alpha$ to a map $g : F^J \to E$, i.e., $g \circ \epsilon \circ \alpha = f$. Since f is an $\mathcal{F}$-(pre)cover of E, there is $h : F^J \to F$ such that $f \circ h = g$. Finally, we observe that since f is right minimal, $f \circ h \circ \epsilon \circ \alpha = f$ implies that $h \circ \epsilon \circ \alpha$ is an isomorphism and, therefore, α splits. This shows that $\mathrm{Ext}^1_{\mathcal{H}}(X, F) = 0$. The condition $\mathrm{Ext}^1_{\mathcal{H}}(X, K) = 0$ follows from the fact that f is an $\mathcal{F}$-cover by a result known as Wakamatsu's lemma. This proves that $\mathcal{F} = \mathrm{Cogen}(C) \subseteq {}^{\perp_1}C$.

Let now X be an object in $\mathrm{gen}(\mathrm{Cogen}(F)) \cap {}^{\perp_1}K$ and consider a short exact sequence

$$0 \longrightarrow L \xrightarrow{\ a\ } Y \xrightarrow{\ b\ } X \longrightarrow 0,$$

with Y in $\mathrm{Cogen}(F)$. Let $\phi : X \to E$ be a nonzero map. Then $\phi \circ b$ factors through f since f is an $\mathcal{F}$-(pre)cover; i.e., there is map $d : Y \to F$ such that $f \circ d = \phi \circ b$. This induces a map $c : L \to K$ such that $k \circ c = d \circ a$. Since $\mathrm{Ext}^1_{\mathcal{H}}(X, K) = 0$, it follows that $\mathrm{Hom}_{\mathcal{H}}(a, K)$ is surjective and, thus, there is a map $g : Y \to K$ such that $g \circ a = c$. In summary, up to this moment, we have the solid part of the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\ a\ } & Y & \xrightarrow{\ b\ } & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle d} & \swarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & K & \xrightarrow{\ k\ } & F & \xrightarrow{\ f\ } & E & &
\end{array}$$

Now a standard argument (using for example a dual version of [Šťovíček et al. 2014, Lemma 2.8]) yields a map $\alpha : X \to F$, indicated by the dotted arrow above, such that $f \circ \alpha = \phi$. We have shown that $\mathrm{Hom}_{\mathcal{H}}(X, f)$ is a surjective map. As above, we infer that any given monomorphism $g : X \to E^I$ factors through f^I via a monomorphism, showing that X lies in $\mathrm{Cogen}(F)$, as wanted.

(1) ⇒ (5): If $\mathfrak{t}$ is of finite type, $\mathbb{T}_{\mathfrak{t}^-}$ is a smashing t-structure with a Grothendieck heart by Theorem 4.3, so it corresponds to a pure-injective cosilting object in $\mathcal{D}$ by Theorem 2.7.

(5) ⇒ (6): This is trivial.

(6) ⇒ (4): This is proved in Lemma 4.4. □

For pure-injective cosilting objects in compactly generated triangulated categories, we can now provide a more convenient characterisation for the existence of left or right mutation. We first need some preliminary results on the existence of approximations.

Lemma 4.7. *Let $\mathcal{A}$ be a (not necessarily additive) category, $\mathcal{R}$ a covering class and $\mathcal{I}$ an enveloping class. The class $\mathcal{R}$ is closed under $\mathcal{I}$-envelopes if and only if the class $\mathcal{I}$ is closed under $\mathcal{R}$-covers.*

Proof. Suppose that $\mathcal{R}$ is closed under $\mathcal{I}$-envelopes. First we show that $\mathcal{I}$ is closed under retracts. Suppose that we have an object J in $\mathcal{I}$ together with a retraction $\pi : J \to I$ and its right inverse $\iota : I \to J$. Consider an $\mathcal{I}$-envelope $f : I \to E_{\mathcal{I}}(I)$ of I. As f is an $\mathcal{I}$-envelope, there exists a factorisation $\iota = gf$ for some $g : E_{\mathcal{I}}(I) \to J$. Then $(\pi g) f = \pi\iota = \mathrm{id}$ and so f has a left inverse. Moreover, we have that $f(\pi g) f = f\pi\iota = f$ so, by the minimality of f, we have that $f(\pi g)$ is an isomorphism, so f has a right inverse. Thus, we conclude that f is an isomorphism and I lies in $\mathcal{I}$. Now, let $h : C_{\mathcal{R}}(I) \to I$ be an $\mathcal{R}$-cover of an object I in $\mathcal{I}$ and let $e : C_{\mathcal{R}}(I) \to E_{\mathcal{I}}(C_{\mathcal{R}}(I))$ be an $\mathcal{I}$-envelope of $C_{\mathcal{R}}(I)$. Since e is an $\mathcal{I}$-envelope and h is a morphism to $\mathcal{I}$, we have a factorisation $h = me$ for some $m : E_{\mathcal{I}}(C_{\mathcal{R}}(I)) \to I$. Moreover, since $E_{\mathcal{I}}(C_{\mathcal{R}}(I))$ lies in $\mathcal{R}$ and h is an $\mathcal{R}$-cover, there is a factorisation $m = hk$ for some $k : E_{\mathcal{I}}(C_{\mathcal{R}}(I)) \to C_{\mathcal{R}}(I)$. Then we have $hke = me = h$, and so ke is an isomorphism because h is minimal. We have shown that $C_{\mathcal{R}}(I)$ is a retract of $E_{\mathcal{I}}(C_{\mathcal{R}}(I))$ and so $C_{\mathcal{R}}(I)$ lies in $\mathcal{I}$. The converse statement is dual. □

Lemma 4.8. *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$, and $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ a subcategory of $\mathrm{Prod}(\sigma)$. Let $(\mathcal{S}, \mathcal{R})$ be the torsion pair in $\mathcal{H}_\sigma$ cogenerated by $H^0_\sigma(\mathscr{E})$.*

(1) *$\mathscr{E}$ is an enveloping class.*

(2) *If $(\mathcal{S}, \mathcal{R})$ is a cosilting torsion pair, then every object in $\mathrm{Prod}(\sigma)$ admits an $\mathscr{E}$-cover.*

Proof. (1) Recall from Lemma 3.1 that $\mathscr{E}$ is preenveloping. Since σ is pure-injective and $\mathcal{H}_\sigma$ is a Grothendieck category, it follows that, in fact, $\mathscr{E}$ is enveloping. Indeed, this is the same as proving that $H^0_\sigma(\mathscr{E})$ is enveloping in $\mathcal{H}_\sigma$, and such envelopes can be constructed as injective envelopes of the torsion-free part with respect to the hereditary torsion pair $(\mathcal{S}, \mathcal{R})$.

(2) By Proposition 4.5, the torsion-free class $\mathcal{R}$ is a covering class in $\mathcal{H}_\sigma$. Hence, if γ is in $\mathrm{Prod}(\sigma)$, the injective object $H^0_\sigma(\gamma)$ of $\mathcal{H}_\sigma$ admits an $\mathcal{R}$-cover. Moreover, since $(\mathcal{S}, \mathcal{R})$ is a hereditary torsion pair, $\mathcal{R}$ is closed under injective envelopes and, thus, by Lemma 4.7, injectives are closed under $\mathcal{R}$-covers. The assertion then follows by Lemma 3.3. □

The following theorem refines Proposition 3.10 in the case the ambient category is compactly generated and the cosilting object is pure-injective. It also says that mutation of such objects is automatically again pure-injective.

Theorem 4.9. *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$, and $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ a subcategory of $\mathrm{Prod}(\sigma)$. Let $(\mathcal{S}, \mathcal{R})$ be the torsion pair in $\mathcal{H}_\sigma$ cogenerated by $H^0_\sigma(\mathscr{E})$.*

(1) *The following statements are equivalent*:

   (a) *σ admits a left mutation σ' with respect to $\mathscr{E}$.*

   (b) *The torsion class $\mathcal{S} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$ in $\mathcal{H}_\sigma$ is closed under products (that is, it is a TTF class).*

   (c) *The object $\varepsilon_0 \oplus \varepsilon_1$ arising from an $\mathscr{E}$-envelope $\sigma \to \varepsilon_0$ and its cone ε_1 is a cosilting object.*

(2) *The following statements are equivalent*:

   (a) *σ admits a right mutation σ' with respect to $\mathscr{E}$.*

   (b) *The torsion-free class $\mathcal{R} = \mathrm{Cogen}(H^0_\sigma(\mathscr{E}))$ in $\mathcal{H}_\sigma$ is closed under direct limits.*

   (c) *The cosilting object σ admits an $\mathscr{E}$-cover.*

In both cases, if the equivalent conditions are satisfied, any mutation σ' as in (a) *is pure-injective.*

Proof. The last assertion of the theorem is a consequence of Corollary 4.6.

(1): (a) ⇒ (c): By definition, there are a cosilting object σ' and a triangle

$$\sigma \xrightarrow{\Phi'} \varepsilon_0' \longrightarrow \varepsilon_1' \longrightarrow \sigma[1]$$

such that Φ' is an $\mathscr{E}$-preenvelope of σ and $\sigma' = \varepsilon_0' \oplus \varepsilon_1'$ up to equivalence. Consider now the triangle

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \sigma[1]$$

arising from an $\mathscr{E}$-envelope of σ and the object $\sigma'' = \varepsilon_0 \oplus \varepsilon_1$. By well-known properties of envelopes, we have that ε_i is isomorphic to a direct summand of ε_i' for $i = 0$ and $i = 1$. This implies that $\mathrm{Prod}(\sigma'') \subseteq \mathrm{Prod}(\sigma')$. Moreover, σ'' cogenerates $\mathcal{D}$ since so does σ. We infer from Lemma 3.1 that $\mathrm{Prod}(\sigma'') = \mathrm{Prod}(\sigma')$, thus proving that σ'' is a cosilting object (equivalent to σ').

The implications (a) ⇒ (b) and (c) ⇒ (b) follow immediately from Theorem 3.5(1).

(b) ⇒ (a): Since ${}^{\perp_0}H^0_\sigma(\mathscr{E})$ is a torsion class, $\mathfrak{t} := (\mathcal{T}, {}^{\perp_0}H^0_\sigma(\mathscr{E}))$ is a torsion pair of finite type in the Grothendieck category $\mathcal{H}_\sigma$, and hence a cosilting torsion pair associated to a pure-injective cosilting object by Proposition 4.5. It follows from Proposition 3.10(1) that σ admits a left mutation σ' with respect to $\mathscr{E}$.

(2): Recall from Theorem 2.7(1) that $\mathcal{H}_\sigma$ is a Grothendieck category. We apply Proposition 4.5 to the torsion pair $(\mathcal{S}, \mathcal{R})$ in $\mathcal{H}_\sigma$.

(a) ⇔ (b): By Proposition 4.5, a torsion pair in $\mathcal{H}_\sigma$ is cosilting if and only if it is of finite type, and the associated cosilting object must be pure-injective in this case. It follows from Proposition 3.10 and Lemma 4.8 that (b) amounts to the existence of a right mutation σ' of σ with respect to $\mathscr{E}$.

(b) ⇔ (c): This follows combining Proposition 4.5 with Lemmas 4.8 and 3.3. □

Example 4.10. Let A be the path algebra over an algebraically closed field k of the Kronecker quiver

$$\bullet \rightrightarrows \bullet.$$

Recall that the finite-dimensional indecomposable regular modules form a tubular family $(\boldsymbol{t}_x)_{x\in\mathbb{X}}$ indexed by the projective line $\mathbb{X} = \mathbb{P}^1(k)$. We pick a subset $P \subseteq \mathbb{X}$, denote by $\bar{P} = \mathbb{X} \setminus P$ its complement, and consider the torsion pair $(\mathcal{T}_P, \mathcal{F}_P)$ in $\mathrm{Mod}(A)$ generated by $\boldsymbol{t}_P = \bigcup_{x\in P} \boldsymbol{t}_x$. If $P = \varnothing$, we take the torsion pair generated by the preinjective modules (whose torsion class is indeed contained in $\mathcal{T}_Q$ for all $Q \neq \varnothing$). It is a cosilting torsion pair cogenerated by the cosilting (in fact, even cotilting) module

$$C_P = G \oplus \prod\{S[-\infty] \mid S \in \boldsymbol{t}_P\} \oplus \coprod\{S[\infty] \mid S \in \boldsymbol{t}_{\bar{P}}\},$$

where $S[-\infty]$ and $S[\infty]$ denote the adic and the Prüfer module corresponding to the simple regular module S, respectively, and G is the generic module. Notice that the corresponding cosilting complex σ_P is quasi-isomorphic to C_P and lies in the heart $\mathcal{H}_{\sigma_P} = \sigma_P^{\perp_{\neq 0}}$.

Let us look at the two extreme cases

$$C_\varnothing = G \oplus \coprod\{S[\infty] \mid S \in \boldsymbol{t}_\mathbb{X}\} \quad \text{and} \quad C_\mathbb{X} = G \oplus \prod\{S[-\infty] \mid S \in \boldsymbol{t}_\mathbb{X}\}.$$

For any $P \subseteq \mathbb{X}$, we have that σ_P is a right mutation of $\sigma_\varnothing$ at the set

$$\mathscr{E} = \mathrm{Add}\left(G \oplus \coprod\{S[\infty] \mid S \in \boldsymbol{t}_{\overline{P}}\}\right)$$

(if $P \neq \mathbb{X}$, we can also express $\mathscr{E}$ as $\mathrm{Prod}(\{S[\infty] \mid S \in \boldsymbol{t}_{\overline{P}}\})$). Indeed, we can construct an $\mathscr{E}$-cover of $\sigma_\varnothing$ from the canonical sequences

$$0 \longrightarrow S[-\infty] \longrightarrow G^{(I)} \longrightarrow S[\infty] \longrightarrow 0, \quad S \in \boldsymbol{t}_P,$$

as in [Buan and Krause 2003, Lemma 2.4], which are easily seen to be $\mathscr{E}$-covers since $\mathrm{Ext}^1_A(G, S[-\infty]) = 0$ and $S[-\infty]$ is indecomposable. When taking a product of these short exact sequences for all $S \in P$ together with the trivial short exact sequences $0 \to 0 \to S[\infty] \to S[\infty] \to 0$ for all $S \in \overline{P}$, we obtain a short exact sequence of the form

$$0 \longrightarrow \prod\{S[-\infty] \mid S \in \boldsymbol{t}_P\} \longrightarrow G^{(J)} \oplus \prod\{S[\infty] \mid S \in \boldsymbol{t}_{\overline{P}}\} \xrightarrow{\ \phi\ } \prod_{S\in\mathbb{X}} S[\infty] \longrightarrow 0.$$

Since the middle term lies in $\mathscr{E}$, the map ϕ is an $\mathscr{E}$-precover. Moreover, the right-hand side term is a cotilting module equivalent to $C_\varnothing$ (the generic module is a summand of the term by [Ringel 1998, Proposition 4]), while the sum of the left-hand and the middle terms is a cotilting module equivalent to C_P, so the last short exact sequence yields an approximation triangle witnessing that σ_P is a right mutation of $\sigma_\varnothing$.

On the other hand, $\sigma_\mathbb{X}$ does *not* admit right mutation at $\mathscr{E} = \mathrm{Prod}(\{S[-\infty] \mid S \in \boldsymbol{t}_P\})$ for any nonempty subset $P \subset \mathbb{X}$. In fact, condition (2b) in Theorem 4.9 fails, due to the fact that the generic module G is not contained in the torsion-free class $\mathrm{Cogen}(\mathscr{E})$ in $\mathcal{H}_{\sigma_\mathbb{X}}$, although it can be realised as a direct summand of a direct limit of a direct system $S[-\infty] \to S[-\infty] \to \cdots$ for any simple regular S by [Ringel 1998, Proposition 4].

Similarly, $\sigma_\varnothing$ does not admit a left mutation at $\mathscr{E} = \mathrm{Prod}(\{S[\infty] \mid S \in \boldsymbol{t}_P\})$ for any proper subset $P \subset \mathbb{X}$, because condition (1b) in Theorem 4.9 fails. Namely, ${}^{\perp_0}H^0_{\sigma_\varnothing}(\mathscr{E})$ contains any $S[\infty]$ with $S \in \boldsymbol{t}_{\overline{P}}$, but the generic module G is not contained in ${}^{\perp_0}H^0_{\sigma_\varnothing}(\mathscr{E})$, although it can be realised as a direct summand of a direct product of $S[\infty]$ for any simple regular S by [Ringel 1998, Proposition 4].

Example 4.11. Let $\mathcal{D}$ be a compactly generated triangulated category and let σ be a pure-injective cosilting object in $\mathcal{D}$. Recall from Theorems 2.7 and 4.9 that the associated t-structure has a Grothendieck heart $\mathcal{H}$, and right mutations of σ bijectively correspond to hereditary torsion pair of finite type in $\mathcal{H}$.

In this example, we consider a commutative noetherian ring R and $\mathcal{D} = \mathrm{D}(R)$. In this setting, a combination of [Alonso Tarrío et al. 2010, Theorem 3.10 and Theorem 3.11] and [Hrbek and Nakamura 2021, Corollary 2.14] (see [Angeleri Hügel and Hrbek 2021, Theorem 3.8] for details) yields a bijection between

- equivalence classes of pure-injective cosilting objects in $\mathcal{D}$, and
- *nondegenerate sp-filtrations of* $\mathsf{Spec}(R)$, i.e., functions ϕ from $\mathbb{Z}$ to the power set $\mathscr{P}(\mathsf{Spec}(R))$ of $\mathsf{Spec}(R)$ such that
  - for any i in $\mathbb{Z}$, $\phi(i)$ is *specialisation-closed*, i.e., for any $\mathfrak{p}$ in $\phi(i)$ and for any $\mathfrak{p} \subseteq \mathfrak{q}$, $\mathfrak{q}$ is in $\phi(i)$.
  - ϕ is decreasing, i.e., $\phi(i) \supseteq \phi(i+1)$.
  - the intersection over $\mathbb{Z}$ of all sets $\phi(i)$ is the empty set, and their union is $\mathsf{Spec}(R)$.

Moreover this bijection restricts to a bijection between equivalence classes of cosilting complexes in $\mathcal{D}$ that lie in $\mathsf{K}^b(\mathsf{Inj}(R))$ (which are automatically pure-injective, see Example 2.8) and the sp-filtrations ϕ of $\mathsf{Spec}(R)$ for which there are integers $a \leq b$ such that $\phi(a) = \mathsf{Spec}(R)$ and $\phi(b) = \varnothing$. Denote by ϕ_σ the sp-filtration associated to a pure-injective cosilting object σ.

Recall that a cosilting object σ is said to be cotilting if $\mathsf{Prod}(\sigma)$ is contained in the heart $\mathcal{H}_\sigma$ of the t-structure associated to σ. We review some examples in the literature of mutations of cotilting objects in $\mathcal{D}$ which lie in $\mathsf{K}^b(\mathsf{Inj}(R))$.

(1) It follows from [Angeleri Hügel and Hrbek 2017, Theorem 5.1] that every two-term cosilting complex in $\mathcal{D}$ is a right mutation of the injective minimal cogenerator of Mod(R).

(2) All cotilting modules over a commutative noetherian ring, and more generally, all cotilting modules of cofinite type over an arbitrary commutative ring can be constructed as iterated right mutations of an injective cogenerator of Mod(R); see [Šťovíček et al. 2014, §4] and [Hrbek and Šťovíček 2020, §8].

(3) In some cases, we know how to translate right mutation of pure-injective cosilting objects into an operation on sp-filtrations. It follows from [Pavon and Vitória 2021, Theorem 4.5(1b)] that, given a specialisation-closed subset W of $\mathsf{Spec}(R)$, there is a hereditary torsion pair of finite type $\boldsymbol{t}_W := (\mathcal{T}_W, \mathcal{F}_W)$ in $\mathcal{H}_\sigma$ in which $\mathcal{T}_W$ coincides with the class of objects of $\mathcal{H}_\sigma$ supported in W. Hence, it makes sense to consider the right mutation of σ associated with W, which we may denote by σ_W: it is the cosilting object (unique up to equivalence) corresponding the right HRS-tilt of the t-structure associated to σ with respect to the torsion pair $\boldsymbol{t}_W$. It then follows from [loc. cit., Proposition 4.7(2)] that the sp-filtration ϕ_{σ_W} associated to σ_W is given by

$$\phi_{\sigma_W}(i) = (W \cap \phi_\sigma(i-1)) \cup \phi_\sigma(i).$$

Indeed, ϕ_{σ_W} describes a compactly generated t-structure that is a right HRS-tilt[1] of the t-structure associated to ϕ_σ, and [loc. cit., Proposition 4.7(2)] states that the torsion class that gives rise to this HRS-tilt must be the one supported on the union over $\mathbb{Z}$ of $\phi_{\sigma_W}(i) \setminus \phi_\sigma(i)$. This union is clearly W and, thus, the torsion pair whose right HRS-tilt corresponds to ϕ_{σ_W} is $\boldsymbol{t}_W$.

Example 4.12. When $\mathcal{D} = \mathrm{D}(\mathrm{Qcoh}(\mathbb{X}))$ for a noetherian scheme $\mathbb{X}$, a class of mutations of the injective cogenerator of Qcoh($\mathbb{X}$) was studied in [Čoupek and Šťovíček 2020, §6] (albeit not in this terminology). However, in contrast to the previous example (where $\mathbb{X}$ was an affine scheme), in this case there can be more 2-term cosilting complexes than hereditary torsion pairs of finite type (see [loc. cit., Example 6.14], which is closely related to Example 4.10 as $\mathrm{D}(\mathrm{Qcoh}(\mathbb{P}^1_k)) \simeq \mathrm{D}(A)$). This means that not every 2-term cosilting complex in $\mathcal{D}$ can be obtained by a right mutation of the injective cogenerator of Qcoh($\mathbb{X}$).

In the following proposition, we will use the notation Ind(σ) for the collection of isoclasses of indecomposable objects in Prod(σ), where σ is a pure-injective cosilting object. Observe that, by Proposition 2.5, we have that Ind(σ) is a set because the isomorphism classes of the indecomposable injective objects in the Grothendieck category $\mathcal{H}_\sigma$ form a set. We will show that if pure-injective cosilting

[1]Note that, in the reference, left HRS-tilts are used for this description; since we are discussing right mutations, we use right HRS-tilts. These differ from the left HRS-tilts by a $[-1]$-shift (see Remark 2.2).

objects σ and σ' are related by a mutation, there is a natural bijection between $\mathrm{Ind}(\sigma)$ and $\mathrm{Ind}(\sigma')$. While in the case of silting mutation of compact silting objects over a finite-dimensional algebra this was implicit from the very beginning (see for example [Aihara and Iyama 2012, Corollary 2.28]), in the case of cosilting mutation in our context, this phenomenon was noticed for derived categories of modules over commutative noetherian rings [Šťovíček et al. 2014, Theorem 5.4] and quasicoherent sheaves on noetherian schemes [Čoupek and Šťovíček 2020, Remark 6.5].

Proposition 4.13. *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$ and $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ a subcategory of $\mathrm{Prod}(\sigma)$ such that there is a right mutation of σ at $\mathscr{E}$, say σ'. For each object α in $\mathrm{Ind}(\sigma) \setminus \mathrm{Ind}(\mathscr{E})$, consider the triangle induced by an $\mathscr{E}$-cover Φ of α*

$$\alpha' \xrightarrow{\ \Omega\ } e_0 \xrightarrow{\ \Phi\ } \alpha \longrightarrow \alpha'[1]. \qquad (\Delta_3)$$

Then, the assignment $\alpha \mapsto \alpha'$ defines a bijection between $\mathrm{Ind}(\sigma)\setminus\mathrm{Ind}(\mathscr{E})$ and $\mathrm{Ind}(\sigma')\setminus\mathrm{Ind}(\mathscr{E})$ and the map Ω in each such triangle is an $\mathscr{E}$-envelope. As a consequence, there is a bijection between $\mathrm{Ind}(\sigma)$ and $\mathrm{Ind}(\sigma')$.

Note that by Corollary 3.7 a result analogous to the one above is available for left mutations.

Example 4.14. While reading the proof of the proposition, it is instructive to keep in mind that it also covers the case of trivial right mutations where $\mathscr{E} = 0$ and $\sigma' = \sigma[-1]$.

Proof of Proposition 4.13. We first show that the assignment is well-defined. Let α be an indecomposable object in $\mathrm{Prod}(\sigma) \setminus \mathscr{E}$; this implies that $\alpha' \neq 0$. Note also since Φ is an $\mathscr{E}$-cover of an indecomposable object, Φ is an indecomposable object in the category of morphisms in $\mathcal{D}$. To see that, if $\Phi = \Phi_0 \oplus \Phi_1$, then one of the Φ_i must be of the form $e_{0,i} \to 0$, which contradicts the fact that Φ is a cover. Since the completion of a direct sum of maps to a triangle is isomorphic to the direct sum of the two triangles completing the summands, it follows that (Δ_3) is an indecomposable triangle (this makes sense since triangles in $\mathcal{D}$ themselves form an additive category). Now it quickly follows that Ω is an $\mathscr{E}$-envelope. Indeed, an $\mathscr{E}$-envelope exists (Lemma 4.8) and is a summand of Ω. However, Ω must be indecomposable in the category of morphisms, or else (Δ_3) could not be indecomposable in the category of triangles. By the same token, α' is indecomposable, since otherwise that $\mathscr{E}$-envelope Ω could be expressed as a direct sum of $\mathscr{E}$-envelopes of summands of α'. Finally, α' cannot lie in $\mathscr{E}$, as otherwise Ω had to be an isomorphism and α the zero object. This completes a proof of the fact that the assignment from the statement of the proposition is well-defined.

Further observe that, in particular, we have shown that α is determined up to isomorphism from α', and the assignment is injective. Regarding the surjectivity, suppose that α' is an indecomposable object in $\mathrm{Prod}(\sigma') \setminus \mathscr{E}$. Since σ' is (up to equivalence) the right mutation of σ at $\mathscr{E}$, we have σ is (up to equivalence) the left mutation of σ at $\mathscr{E}$ (see Corollary 3.7). As every object in $\mathrm{Prod}(\sigma')$ admits an $\mathscr{E}$-envelope (see Lemma 4.8) we can take $\Omega : \alpha' \to e_0$ to be this envelope and let α be its cone. We therefore obtain a triangle as in the statement of the proposition, and dual arguments to the ones presented above can be used to show that Φ is an $\mathscr{E}$-cover (which we know to exist, see Lemma 4.8) and that α is indecomposable in $\mathrm{Prod}(\sigma) \setminus \mathscr{E}$, thus finishing the proof. □

5. Silting mutation

In this section, we state the dual results for silting objects. Then we establish a compatibility between silting and cosilting mutation, showing that cosilting mutation encompasses mutation between compact silting objects.

Definition 5.1. Let $\mathcal{D}$ be a triangulated category with coproducts. Let σ and σ' be two silting objects in $\mathcal{D}$, and let $\mathscr{P} = \mathrm{Add}(\sigma) \cap \mathrm{Add}(\sigma')$. We say that:

(1) σ' is a *left mutation* of σ if there is a triangle

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \sigma[1]$$

such that

- Φ is a $\mathscr{P}$-preenvelope of σ in $\mathcal{D}$, and
- $\varepsilon_0 \oplus \varepsilon_1$ is a silting object equivalent to σ'.

(2) σ' is a *right mutation* of σ if there is a triangle

$$\sigma[-1] \longrightarrow \gamma_1 \longrightarrow \gamma_0 \xrightarrow{\Phi} \sigma$$

such that

- Φ is a $\mathscr{P}$-precover of σ in $\mathcal{D}$, and
- $\gamma_0 \oplus \gamma_1$ is a silting object equivalent to σ'.

We will also say that σ' is a *left* (*or right*) *mutation* of σ *with respect to* $\mathscr{P}$.

Silting mutation can also be expressed in terms of HRS-tilts. The proof of the following theorem is dual to the one of Theorem 3.5.

Theorem 5.2. *Let $\mathcal{D}$ be a triangulated category with coproducts, let σ and σ' be two silting objects in $\mathcal{D}$, and let $\mathscr{P} = \mathrm{Add}(\sigma) \cap \mathrm{Add}(\sigma')$. Then we have that*:

(1) *σ' is a left mutation of σ if and only if σ admits a $\mathscr{P}$-preenvelope and $\mathbb{T}_{\sigma'}$ is the left HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair $\mathfrak{t} = (\mathrm{Gen}(H^0_\sigma(\mathscr{P})), H^0_\sigma(\mathscr{P})^{\perp_0})$ in $\mathcal{H}_\sigma$.*

(2) *σ' is a right mutation of σ if and only if $H^0_\sigma(\mathscr{P})^{\perp_0}$ is closed under coproducts in $\mathcal{H}_\sigma$ and $\mathbb{T}_{\sigma'}$ is the right HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair $\mathfrak{t} = (H^0_\sigma(\mathscr{P})^{\perp_0}, \mathcal{F})$ in $\mathcal{H}_\sigma$.*

In both cases, the torsion pairs involved do not depend on the choice of the triangle in Definition 5.1.

Again, as noted for cosilting mutation in Remark 3.6, the torsion pairs involved have different flavours: left mutation will yield an HRS-tilt at a cohereditary torsion pair, while right mutation will give rise to an HRS-tilt at a torsion pair $(\mathcal{T}, \mathcal{F})$ for which $\mathcal{T}$ is a TTF class.

Definition 5.3. Let $\mathcal{D}$ be a triangulated category with coproducts and $\mathbb{T}$ a t-structure with heart $\mathcal{H}$. A torsion pair $\mathfrak{t} := (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ is said to be a *silting torsion pair* if and only if there is a silting object σ in $\mathcal{D}$ such that $\mathbb{T}_{\mathfrak{t}^+} = \mathbb{T}_\sigma$.

Example 5.4. Let R be a ring. The modules $T = H^0(\sigma)$ arising as zero cohomologies of a silting complex $\sigma : P_1 \to P_0$ of length 2 concentrated in cohomological degrees -1 and 0 are precisely the *silting modules* introduced in [Angeleri Hügel et al. 2016]. The t-structure $\mathbb{T}_\sigma$ then coincides with the left HRS-tilt of the standard t-structure of $\mathrm{D}(R)$ at the torsion pair $\mathfrak{t} := (\mathrm{Gen}(T), T^{\perp_0})$ in $\mathrm{Mod}(R)$ generated by T. In other words, $\mathfrak{t}$ is a silting torsion pair in the sense of the definition above, and as in Example 3.9, one can show that all silting torsion pairs in the sense of the definition are of this form.

Again, we can extract from the theorem above a criterion for the existence of a mutation with respect to a given subset $\mathscr{P}$. Once again, we omit the proof as it is dual to the proof of Proposition 3.10.

Proposition 5.5. *Let $\mathcal{D}$ be a triangulated category with coproducts. Let σ be a silting object and $\mathscr{P} = \mathrm{Add}(\mathscr{P})$ a subcategory of $\mathrm{Add}(\sigma)$. Then we have that*:

(1) *σ admits a left mutation σ' with respect to $\mathscr{P}$ if and only if σ admits a $\mathscr{P}$-preenvelope and the pair $(\mathrm{Gen}(H^0_\sigma(\mathscr{P})), H^0_\sigma(\mathscr{P})^{\perp_0})$ in $\mathcal{H}_\sigma$ is a silting torsion pair.*

(2) *σ admits a right mutation σ' with respect to $\mathscr{P}$ if and only if the torsion class $H^0_\sigma(\mathscr{P})^{\perp_0}$ defines a silting torsion pair $\mathfrak{t} := (H^0_\sigma(\mathscr{P})^{\perp_0}, \mathcal{F})$ in $\mathcal{H}_\sigma$.*

Remark 5.6. In the case where $\mathcal{D}$ is a compactly generated triangulated category, the condition that σ admits a $\mathscr{P}$-preenvelope is redundant.

Indeed, let $\mathbb{T} = \mathbb{T}_\sigma$ be a t-structure associated to a silting object σ, let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a silting torsion pair in the heart $\mathcal{H}_\sigma$ and let γ be a silting object in $\mathcal{D}$ such that $\mathbb{T}_{\mathfrak{t}^+} = \mathbb{T}_\gamma$. We know from [Angeleri Hügel et al. 2020, Proposition 3.8] (see also [Bondarko 2016, Theorem 3.2.4]) that $\gamma^{\perp_{>0}}$ is a TTF class. Let $\Phi : \sigma \to B$ denote a $\gamma^{\perp_{>0}}$-preenvelope of σ. In particular, $H^0_\sigma(B)$ lies in $\mathcal{T}$. We claim that $\phi := H^0_\sigma(\Phi)$ is a $\mathcal{T}$-preenvelope of the projective generator $H^0_\sigma(\sigma)$. To that end, suppose that $f : H^0_\sigma(\sigma) \to T$ is a morphism in $\mathcal{H}_\sigma$ with T in $\mathcal{T}$. Then the composition $f \circ \pi : \sigma \to T$, where $\pi : \sigma \to H^0_\sigma(\sigma)$ is the natural truncation map, factors through Φ (since T lies in $\gamma^{\perp_{>0}}$). In other words, there is $\alpha : B \to T$ such that $f \circ \pi = \alpha \circ \Phi$. If we apply H^0_σ to this equality we get $f = H^0_\sigma(\alpha) \circ \phi$, as wanted.

Finally, if $\mathcal{T} = \mathrm{Gen}(H^0_\sigma(\mathscr{P}))$ as in Proposition 5.5(1), there is an epimorphism $p : H^0_\sigma(P) \to H^0_\sigma(B)$ in $\mathcal{H}_\sigma$ with P in $\mathscr{P}$ and the preenvelope $\phi : H^0_\sigma(\sigma) \to H^0_\sigma(B)$ factors through p as $H^0_\sigma(\sigma)$ is projective in $\mathcal{H}_\sigma$. Clearly, the resulting map $\phi' : H^0_\sigma(\sigma) \to H^0_\sigma(P)$ is a $\mathcal{T}$-preenvelope as well and there is a map $\Phi' : \sigma \to P$ such that $\phi' = H^0_\sigma(\Phi')$ by Proposition 2.5(1). Finally, Φ' is a $\mathscr{P}$-preenvelope by arguments dual to those in Lemma 3.3.

Let us now delve into the connection between silting mutation and cosilting mutation. For this purpose we restrict ourselves to the setting of compactly generated triangulated categories. Recall that, in a compactly generated triangulated category $\mathcal{D}$, for any compact object K there is an object $\mathbb{BC}(K)$, called the *Brown–Comenetz dual of* K, such that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathcal{D}}(K, -), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathbb{BC}(K)).$$

Note that, since K is compact, $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathcal{D}}(K,-),\mathbb{Q}/\mathbb{Z})$ sends pure triangles to short exact sequences and, thus, $\mathbb{BC}(K)$ is pure-injective by [Krause 2000, Corollary 2.5]. The following lemma is an easy observation.

Lemma 5.7. *Let $\mathcal{D}$ be a compactly generated triangulated category and σ a compact silting object. Then $\mathbb{BC}(\sigma)$ is a pure-injective cosilting object such that $\mathbb{T}_\sigma = \mathbb{T}_{\mathbb{BC}(\sigma)}$.*

Recall that in [Aihara and Iyama 2012, Section 4] the authors consider compact silting objects in compactly generated triangulated categories. To see that, for a compact object, the definition of silting given in [loc. cit., Definition 4.1] is equivalent to the definition of silting given is Section 2.3, we refer the reader to [loc. cit., Corollary 4.7]. The following theorem shows that the operation of mutation of compact silting objects defined in [loc. cit.] is a special case of the operation of mutation of both silting objects and pure-injective cosilting objects.

Theorem 5.8. *Let $\mathcal{D}$ be a compactly generated triangulated category. Let σ be a compact silting object in $\mathcal{D}$ and $\mathfrak{p} = \mathrm{add}(\mathfrak{p})$ a subcategory of $\mathrm{add}(\sigma)$ and define $\mathscr{P} = \mathrm{Add}(\mathfrak{p})$.*

(1) *Any $\mathfrak{p}$-preenvelope of σ is a $\mathscr{P}$-preenvelope and any $\mathfrak{p}$-precover of σ is a $\mathscr{P}$-precover.*

(2) *If σ admits a $\mathfrak{p}$-preenvelope (respectively, a $\mathfrak{p}$-precover), then it has a compact left (respectively, right) mutation σ' with respect to $\mathscr{P}$. Moreover, the t-structure $\mathbb{T}_{\sigma'}$ is the cosilting t-structure associated to a pure-injective left (respectively, right) mutation of the cosilting object $\mathbb{BC}(\sigma)$.*

Proof. (1): Let $\Phi : \sigma \to \varepsilon_0$ be a $\mathfrak{p}$-preenvelope of σ and let $f : \sigma \to P$ be a map to an object P in $\mathscr{P}$. Without loss of generality we may assume that P is a coproduct of objects in $\mathfrak{p}$. Since σ is compact, f factors through a finite subsum of objects in $\mathfrak{p}$ and hence through Φ. Dually, let $\Psi : \gamma_0 \to \sigma$ be a $\mathfrak{p}$-precover and $g : P \to \sigma$ a map from an object P in $\mathscr{P}$. Again, without loss of generality assume that P is a coproduct of objects in $\mathfrak{p}$. Since a factorisation through Ψ exists for each summand of P, the universal property of the coproduct yields a factorisation for g.

(2): If σ admits a $\mathfrak{p}$-preenvelope $\Phi : \sigma \to \varepsilon_0$, the triangle

$$\sigma \xrightarrow{\Phi} \varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \sigma[1]$$

yields a compact silting object $\sigma' = \varepsilon_0 \oplus \varepsilon_1$ by [Aihara and Iyama 2012, Theorem 2.31]. By (1) this is the left mutation of σ with respect to $\mathscr{P}$ in the sense of Definition 5.1.

By (the proof of) Theorem 5.2, the t-structure $\mathbb{T}_{\sigma'}$ is the left HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair in $\mathcal{H}_\sigma$ with torsion-free class $\mathcal{F} = H^0_\sigma(\mathscr{P})^{\perp_0} = H^0_\sigma(\varepsilon_0)^{\perp_0}$ in $\mathcal{H}_\sigma$. By Proposition 2.5, $\mathcal{F} = \varepsilon_0^{\perp_0} \cap \mathcal{H}_\sigma$ in $\mathcal{D}$. Consider now the Brown–Comenetz dual of ε_0 and define $\mathscr{E} := \mathrm{Prod}(\mathbb{BC}(\varepsilon_0))$. Note that $\mathcal{F} = {}^{\perp_0}\mathbb{BC}(\varepsilon_0) \cap \mathcal{H}_\sigma$ in $\mathcal{D}$ since $\mathrm{Hom}_{\mathcal{D}}(\varepsilon_0, X) = 0$ if and only if $\mathrm{Hom}_{\mathcal{D}}(X, \mathbb{BC}(\varepsilon_0)) = 0$ and, thanks to Proposition 2.5, we infer that $\mathcal{F} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$ in $\mathcal{H}_\sigma$. Recall from Lemma 5.7 that $\mathbb{T}_\sigma = \mathbb{T}_{\mathbb{BC}(\sigma)}$ and $\mathbb{T}_{\sigma'} = \mathbb{T}_{\mathbb{BC}(\sigma')}$. By Theorem 3.5(1) the t-structure $\mathbb{T}_{\sigma'}$ then coincides with the one associated to a left mutation of $\mathbb{BC}(\sigma)$ with respect to $\mathscr{E}$.

Now we turn to the dual case. If $\gamma_0 \to \sigma$ is a $\mathfrak{p}$-precover, then we see with analogous arguments that there is a compact silting object σ' such that $\mathbb{T}_{\sigma'}$ is the right HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair in $\mathcal{H}_\sigma$ with torsion class $\mathcal{T} = H^0_\sigma(\gamma_0)^{\perp_0} = {}^{\perp_0}\mathbb{BC}(\gamma_0)$, and Lemma 5.7 yields again that $\mathbb{T}_\sigma = \mathbb{T}_{\mathbb{BC}(\sigma)}$ and $\mathbb{T}_{\sigma'} = \mathbb{T}_{\mathbb{BC}(\sigma')}$. Note that $\mathcal{T} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$, where $\mathscr{E} = \mathrm{Prod}(\mathbb{BC}(\gamma_0)) \subseteq \mathrm{Prod}(\mathbb{BC}(\sigma))$. Hence $\mathbb{T}_{\sigma'}$ is the right HRS-tilt of $\mathbb{T}_\sigma$ at the cosilting torsion pair cogenerated by $H^0_\sigma(\mathscr{E})$. Moreover, $\mathbb{BC}(\sigma)$ has an $\mathscr{E}$-cover by Lemma 4.8. So we infer from Theorem 3.5(2) that the t-structure $\mathbb{T}_{\sigma'}$ coincides with the one associated to a right mutation of $\mathbb{BC}(\sigma)$ with respect to $\mathscr{E}$. □

6. Mutation and localisation

In this section we will show that, in nice enough contexts, mutation can be understood as three-step process: first restrict the t-structures (to certain subcategories); then shift one of the restricted t-structures; finally glue them back together. In order to prove this, we need to review some ideas concerning restricting and gluing.

6.1. ***Restricting and gluing along (co)localising sequences.*** A sequence of exact functors between triangulated (respectively, abelian categories)

$$\mathcal{B} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$$

is said to be a *short exact sequence* if F is fully faithful, the Verdier quotient (respectively, the Serre quotient) $\mathcal{D}/\mathsf{Im}(F)$ is well-defined (where $\mathsf{Im}(F)$ denotes the essential image of the functor F), there are an equivalence $L : \mathcal{D}/\mathsf{Im}(F) \to \mathcal{C}$ and a natural isomorphism $\theta : G \to L \circ q$, where $q : \mathcal{D} \to \mathcal{D}/\mathsf{Im}(F)$.

A short exact sequence of triangulated (respectively, abelian) categories as above is said to be a *localising sequence* if both F and G admit right adjoints. Dually, it is said to be a *colocalising sequence* if both F and G admit left adjoints. A short exact sequence is said to be a *recollement* if it is both a localising and a colocalising sequence. Note that a localising sequence of triangulated categories

$$\mathcal{B} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{D} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{C}$$

can be transformed into a colocalising sequence

$$\mathcal{C} \underset{j^*}{\overset{j_*}{\rightleftarrows}} \mathcal{D} \underset{i_*}{\overset{i^!}{\rightleftarrows}} \mathcal{B}.$$

However, the same observation does not hold for abelian categories. This is due to the fact that, in the triangulated setting, adjoints of exact functors are exact, while this is not the case in the abelian setting. Recall that, for abelian categories, quotient functors by *Serre subcategories* (i.e., subcategories closed under subobjects, quotient objects and extensions) are always exact.

Definition 6.1. Given a short exact sequence of triangulated categories

$$\mathcal{B} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C},$$

we say that a t-structure $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}$ *restricts along the exact sequence* if $(\mathcal{X} \cap \mathsf{Im}(F), \mathcal{Y} \cap \mathsf{Im}(F))$ is a t-structure in $\mathsf{Im}(F)$ and $(G(\mathcal{X}), G(\mathcal{Y}))$ is a t-structure in $\mathcal{C}$.

Proposition 6.2. *Let $\mathbb{T}$ be a t-structure in a triangulated category $\mathcal{D}$ and suppose there is a short exact sequence of triangulated categories*

$$\mathcal{B} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}.$$

(1) [Chuang and Rouquier 2017, Lemma 3.3] *The following statements are equivalent for a t-structure $\mathbb{T}$:*

(a) *$\mathbb{T}$ restricts along the exact sequence.*

(b) *$G(\mathbb{T}) := (G(\mathcal{X}), G(\mathcal{Y}))$ is a t-structure in $\mathcal{C}$.*

(c) *$\mathbb{T} \cap \mathsf{Im}(F) := (\mathcal{X} \cap \mathsf{Im}(F), \mathcal{Y} \cap \mathsf{Im}(F))$ is a t-structure in $\mathsf{Im}(F)$, and the heart of this t-structure is a Serre subcategory of the heart of $\mathbb{T}$.*

(2) *If $\mathbb{T}$ restricts along the exact sequence, then there is an induced short exact sequence of abelian categories formed by the associated hearts. If the exact sequence of triangulated categories is a localising, respectively colocalising, sequence, then so is the corresponding sequence of hearts.*

Proof. For simplicity, since F is fully faithful, we identify $\mathcal{B}$ with $\mathsf{Im}(F)$ and assume, without loss of generality, that F is the inclusion functor. Denote by $\mathcal{H}$ the heart of $\mathbb{T}$ and by $\mathcal{H}_\mathcal{B}$ and $\mathcal{H}_\mathcal{C}$ the hearts of $G(\mathbb{T})$ and $\mathbb{T} \cap \mathsf{Im}(F)$ respectively. Define $\bar{F} : \mathcal{H}_\mathcal{B} \to \mathcal{H}$ to be the restriction of F to $\mathcal{H}_\mathcal{B}$ and $\bar{G} : \mathcal{H} \to \mathcal{H}_\mathcal{C}$ to be the restriction of G to $\mathcal{H}$. Note that these functors are well-defined by the construction of the t-structures $G(\mathbb{T})$ and $\mathbb{T} \cap \mathsf{Im}(F)$. Under the equivalent conditions of (1), it follows as in [Chuang and Rouquier 2017, Lemma 3.9] (see also [Beilinson et al. 1982, Section 1.4; Beligiannis and Reiten 2007, Proposition 2.5]) that there is a short exact sequence of abelian categories

$$\mathcal{H}_\mathcal{B} \xrightarrow{\bar{F}} \mathcal{H} \xrightarrow{\bar{G}} \mathcal{H}_\mathcal{C}.$$

Next we observe that if F has a right adjoint, then so does $\bar{F}$. Indeed, if $R : \mathcal{D} \to \mathcal{B}$ is a right adjoint to F, then X is an object of $\mathcal{B}$, and D is an object of $\mathcal{D}$, then we have a canonical isomorphism

$$\mathrm{Hom}_\mathcal{D}(F(X), D) \cong \mathrm{Hom}_\mathcal{B}(X, R(D)).$$

In particular, $\mathrm{Hom}_\mathcal{B}(X, R(D)) = 0$ for each $X \in \mathcal{X} \cap \mathsf{Im}(F)$ whenever $D \in \mathcal{Y}$. It immediately follows that $R(\mathcal{Y}) \subseteq \mathcal{Y} \cap \mathsf{Im}(F)$ and that we have canonical isomorphisms

$$\mathrm{Hom}_\mathcal{H}(\bar{F}(X), D) = \mathrm{Hom}_\mathcal{D}(F(X), D) \cong \mathrm{Hom}_\mathcal{B}(X, R(D)) \cong \mathrm{Hom}_{\mathcal{H}_\mathcal{B}}(X, H^0_{\mathbb{T}_\mathcal{B}}(R(D)))$$

for any $X \in \mathcal{H}_\mathcal{B}$ and $D \in \mathcal{H}$. Therefore the following composition is a right adjoint to $\bar{F}$:

$$\mathcal{H} \xrightarrow{\mathsf{inc}} \mathcal{D} \xrightarrow{R} \mathcal{B} \xrightarrow{H^0_{\mathbb{T}_\mathcal{B}}} \mathcal{H}_\mathcal{B}.$$

An analogous argument shows that if $S : \mathcal{C} \to \mathcal{D}$ is a right adjoint to G, then the composition

$$\mathcal{H}_\mathcal{C} \xrightarrow{\mathsf{inc}} \mathcal{C} \xrightarrow{S} \mathcal{B} \xrightarrow{H^0_{\mathbb{T}}} \mathcal{H}$$

is a right adjoint to $\bar{G} : \mathcal{H} \to \mathcal{H}_\mathcal{C}$. Indeed, if $C \in G(\mathcal{Y})$, then $\mathrm{Hom}_\mathcal{D}(D, S(C)) \cong \mathrm{Hom}_\mathcal{C}(G(D), C) = 0$ for any $D \in \mathcal{X}$. Thus, $S(G(\mathcal{Y})) \subseteq \mathcal{Y}$ and, for any D in $\mathcal{H}$ and C in $\mathcal{H}_\mathcal{C}$, we have canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{H}_\mathcal{C}}(\bar{G}(D), C) = \mathrm{Hom}_\mathcal{C}(G(D), C) \cong \mathrm{Hom}_\mathcal{D}(D, S(C)) \cong \mathrm{Hom}_\mathcal{H}(D, H^0_\mathbb{T}(S(C))).$$

Finally, the assertion for left adjoints follows in similar fashion. □

Remark 6.3. Let $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ be a t-structure in a triangulated category $\mathcal{D}$ which restricts along a short exact sequence of triangulated categories

$$\Delta:\ \mathcal{B} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{C}.$$

Then it follows from [Beilinson et al. 1982] that:

(1) If Δ is a localising sequence, then $\mathcal{Y} = (\mathcal{Y} \cap \mathsf{Im}(i_*)) \star j_* j^*(\mathcal{Y})$, where j_* is the right adjoint of j^*.

(2) If Δ is a colocalising sequence, then $\mathcal{X} = j_! j^* \mathcal{X} \star (\mathcal{X} \cap \mathsf{Im}(i_*))$, where $j_!$ is the left adjoint of j^*.

6.2. *Mutation and localising sequences.* We are now going to explore the relation between mutation of pure-injective cosilting (respectively, pure-projective silting) objects and categorical localisations. First of all, given a cosilting object σ, we show that sets of pure-injective objects in $\mathrm{Prod}(\sigma)$ induce localising sequences both at triangulated and abelian level.

Proposition 6.4. *Let σ be a cosilting object in a compactly generated triangulated category $\mathcal{D}$. Let $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ be a subcategory of $\mathrm{Prod}(\sigma)$ in which every object is pure-injective. Then there is a torsion pair $({}^{\perp_{\mathbb{Z}}}\mathscr{E}, \mathcal{C}_{\mathscr{E}})$ in $\mathcal{D}$ and, thus, a localising sequence*

$$\begin{array}{c} {}^{\perp_{\mathbb{Z}}}\mathscr{E} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{C}_{\mathscr{E}}, \\ \text{with right adjoints } i^!: \mathcal{D} \to {}^{\perp_{\mathbb{Z}}}\mathscr{E},\ j_*: \mathcal{C}_{\mathscr{E}} \to \mathcal{D} \end{array}$$

where i_* and j_* are the inclusion functors. Moreover, the cosilting t-structure $\mathbb{T}_\sigma$ restricts along the sequence, thus giving rise to a localising sequence of abelian categories

$$\begin{array}{c} \mathcal{S} \xrightarrow{\bar{i}_*} \mathcal{H}_\sigma \xrightarrow{\bar{j}^*} \mathcal{S}^{\perp_{0,1}}, \\ \text{with right adjoints } \bar{i}^!: \mathcal{H}_\sigma \to \mathcal{S},\ \bar{j}_*: \mathcal{S}^{\perp_{0,1}} \to \mathcal{H}_\sigma \end{array}$$

where $\bar{i}_*$ and $\bar{j}_*$ are the inclusion functors, $\mathcal{S} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$ is a hereditary torsion class, and $\mathcal{S}^{\perp_{0,1}}$ is the associated Giraud subcategory in $\mathcal{H}_\sigma$.

Proof. First, we observe that since $\mathscr{E}$ is made of pure-injective objects, it follows from [Saorín and Šťovíček 2023, Proposition 6.9] that there is a torsion pair of the form $({}^{\perp_{\mathbb{Z}}}\mathscr{E}, \mathcal{C}_{\mathscr{E}})$ in $\mathcal{D}$. This is equivalent to the existence of the claimed localisation sequence. We now show that $\mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$ restricts along this exact sequence using Proposition 6.2. Indeed, if D lies in ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$, consider its truncation triangle for $\mathbb{T}_\sigma$

$$x(D) \longrightarrow D \longrightarrow y(D) \longrightarrow x(X)[1].$$

Clearly ${}^{\perp_{\mathbb{Z}}}\mathscr{E} = {}^{\perp_{>0}}\mathscr{E} \cap {}^{\perp_{\leq 0}}\mathscr{E}$ and, since $\mathscr{E}$ is contained in $\mathrm{Prod}(\sigma)$, we have that $\mathcal{X} \subseteq {}^{\perp_{\leq 0}}\mathscr{E}$ and $\mathcal{Y} \subseteq {}^{\perp_{>0}}\mathscr{E}$ and, thus, $x(D)$ lies in ${}^{\perp_{\leq 0}}\mathscr{E}$ and $y(D)$ lies in ${}^{\perp_{>0}}\mathscr{E}$. Moreover, since ${}^{\perp_{\leq 0}}\mathscr{E}$ is suspended, $y(D)$ also lies in ${}^{\perp_{\leq 0}}\mathscr{E}$, and since ${}^{\perp_{>0}}\mathscr{E}$ is cosuspended, $x(D)$ lies in ${}^{\perp_{>0}}\mathscr{E}$. This shows that the truncation triangle restricts to ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$. Furthermore, note that $\mathcal{H}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}$ is the subcategory of objects X in $\mathcal{H}_\sigma$ such that $\mathrm{Hom}_{\mathcal{H}}(X, H^0_\sigma(E)) = 0$ for all E in $\mathscr{E}$. Recall that $H^0_\sigma(E)$ is injective for all E in $\mathscr{E}$ and, therefore, $\mathcal{H}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E} = {}^{\perp_0}H^0_\sigma(\mathscr{E}) = \mathcal{S}$ is a hereditary torsion class in $\mathcal{H}_\sigma$. Our claim then follows from Proposition 6.2 and from the fact that the right adjoint of the localising functor in a short exact sequence of abelian categories identifies the quotient category with the Giraud subcategory $\mathcal{S}^{\perp_{0,1}}$ associated to $\mathcal{S}$. $\square$

Next, as promised earlier, we prove that if σ and σ' are pure-injective cosilting objects which are mutations of each other, the associated t-structures can be described in terms of some operations along the localising sequence induced by $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$.

Lemma 6.5. *Let σ and σ' be pure-injective cosilting objects in a compactly generated triangulated category $\mathcal{D}$. Let $\mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$ and $\mathbb{T}_{\sigma'} = (\mathcal{X}', \mathcal{Y}')$ be the associated cosilting t-structures and let*

$$^{\perp_{\mathbb{Z}}}\mathscr{E} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{C}_{\mathscr{E}}$$

be the localising sequence induced by $\mathscr{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$.

(1) *If σ' is a right mutation of σ, then we have the following equalities for the restricted t-structures*:

(a) $\mathbb{T}_{\sigma'} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E} = (\mathbb{T}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[-1]$ *in* ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$.

(b) $j^*(\mathbb{T}_{\sigma'}) = j^*(\mathbb{T}_\sigma)$ *in* $\mathcal{C}_{\mathscr{E}}$.

(2) *If σ' is a left mutation of σ, then we have the following equalities for the restricted t-structures*:

(a) $\mathbb{T}_{\sigma'} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E} = (\mathbb{T}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[1]$ *in* ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$.

(b) $j^*(\mathbb{T}_{\sigma'}) = j^*(\mathbb{T}_\sigma)$ *in* $\mathcal{C}_{\mathscr{E}}$.

Proof. We prove (1). The assertion (2) follows analogously, taking into account Corollary 3.7(3).

By assumption we know that $\mathbb{T}_{\sigma'}$ is the right HRS-tilt of $\mathbb{T}_\sigma$ at the torsion pair $(\mathcal{S}, \mathcal{R})$ in $\mathcal{H}_\sigma$. Recall from Proposition 6.4 that the t-structures $\mathbb{T}_\sigma$ and $\mathbb{T}_{\sigma'}$ restrict along the localising sequence determined by $\mathscr{E}$, and the heart of $\mathbb{T}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}$ is $\mathcal{H}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E} = {}^{\perp_0}H^0_\sigma(\mathscr{E}) = \mathcal{S}$. We show that the heart of $\mathbb{T}_{\sigma'} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}$ coincides with $\mathcal{S}[-1]$. To this end, we consider the torsion pair $(\mathcal{R}, \mathcal{S}[-1])$ in $\mathcal{H}_{\sigma'}$. It is clear that $\mathcal{S}[-1] \subseteq \mathcal{H}_{\sigma'} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}$. For the converse, we pick an object X in $\mathcal{H}_{\sigma'} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}$ with torsion decomposition

$$0 \longrightarrow r(X) \longrightarrow X \longrightarrow X/r(X) \longrightarrow 0,$$

where $r(X)$ is in $\mathcal{R}$, and $X/r(X)$ is in $\mathcal{S}[-1]$ and thus in ${}^{\perp_{\mathbb{Z}}}\mathscr{E} \cap \mathcal{H}_{\sigma'}$. Since this exact sequence yields a triangle in $\mathcal{D}$ and since ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$ is triangulated, it follows that $r(X)$ lies in ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$. But then $r(X)$ lies in $\mathcal{R}$ and in $\mathcal{H}_\sigma \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E} = \mathcal{S}$. We conclude that $r(X) = 0$, as wanted.

Now, combining the fact that the heart of $\mathbb{T}_{\sigma'} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}$ lies in $\mathcal{Y}[-1]$ with the inclusions $\mathcal{Y}[-1] \subseteq \mathcal{Y}' \subseteq \mathcal{Y}$, we easily obtain the equality in (1a). Next, we check the equality in (1b). Since the right adjoint j_* of the quotient functor j^* is fully faithful, this amounts to verifying $j_*j^*(\mathcal{Y}) = j_*j^*(\mathcal{Y}')$. Observe that $\mathcal{Y}' \subseteq \mathcal{Y}$ implies $j_*j^*(\mathcal{Y}') \subseteq j_*j^*(\mathcal{Y})$. For the reverse inclusion, we pick an object X in $j_*j^*(\mathcal{Y})$ and consider a truncation triangle with respect to the t-structure $\mathbb{T}_{\sigma'}$,

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1],$$

with A in $\mathcal{X}'$ and B in $\mathcal{Y}'$. Again, since $j_*j^*(\mathcal{Y}') \subseteq \mathcal{Y}' \subseteq \mathcal{Y}$ and since the latter is cosuspended, we have that A lies in $\mathcal{X}' \cap \mathcal{Y}$, which coincides with the torsion class $\mathcal{S} = {}^{\perp_0}H^0_\sigma(\mathscr{E})$ of $\mathcal{H}_\sigma$. But this means that A lies in ${}^{\perp_{\mathbb{Z}}}\mathscr{E}$ and, in particular, it has no maps to objects in the essential image of j_*, where X lies. Hence X is isomorphic to B, proving the desired equality. □

Now we can give a precise description of mutated cosilting t-structures associated to pure-injective cosilting objects in terms of localisation.

Theorem 6.6. *Let σ be a pure-injective cosilting object in a compactly generated triangulated category $\mathcal{D}$ with associated t-structure $\mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$. Let $\mathscr{E} = \mathrm{Prod}(\mathscr{E})$ be a subcategory of $\mathrm{Prod}(\sigma)$, and consider the localising sequence induced by $\mathscr{E}$*

$$^{\perp_{\mathbb{Z}}}\mathscr{E} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{C}_{\mathscr{E}}.$$

(1) *σ admits a right mutation with respect to $\mathscr{E}$ if and only if there is a cosilting object σ' with associated coaisle $\mathcal{Y}' = (\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[-1] \star j_* j^*(\mathcal{Y})$.*

(2) *σ admits a left mutation with respect to $\mathscr{E}$ if and only if there is a cosilting object σ' with associated coaisle $\mathcal{Y}' = (\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[1] \star j_* j^*(\mathcal{Y})$.*

Proof. Recall that a mutation of a pure-injective cosilting object is again pure-injective by Theorem 4.9. The "only if" part in (1) and (2) follows directly from Lemma 6.5 and Remark 6.3. For the "if" part, let us consider the hereditary torsion pair $(\mathcal{S}, \mathcal{R})$ in $\mathcal{H}_\sigma$ cogenerated by $H^0_\sigma(\mathscr{E})$ and verify the conditions (1b) and (2b) in Theorem 4.9, respectively. In fact, we show in both cases that σ' is the corresponding mutation.

(1): We have to show that the torsion-free class $\mathcal{R}$ is closed under direct limits. By Proposition 4.5 this amounts to proving that $(\mathcal{S}, \mathcal{R})$ is a cosilting torsion pair. Indeed, we claim that $\mathbb{T}_{\sigma'}$ is a right HRS-tilt of $\mathbb{T}_\sigma$ at $(\mathcal{S}, \mathcal{R})$. For a proof, we apply Proposition 2.3 and verify that $\mathcal{Y}[-1] \subseteq \mathcal{Y}' \subseteq \mathcal{Y}$ and $\mathcal{R} = \mathcal{H}_\sigma \cap \mathcal{H}_{\sigma'}$.

Observe first that $\mathcal{Y}' \subseteq \mathcal{Y}$ since $j_* j^*(\mathcal{Y}) \subseteq \mathcal{Y}$ by Proposition 6.4 and Remark 6.3. On the other hand, let Y be an object in $\mathcal{Y}$ and consider the triangle associated to $Y[-1]$ given by the localising sequence induced by $\mathscr{E}$,

$$A \longrightarrow Y[-1] \xrightarrow{\alpha} j_* j^* Y[-1] \longrightarrow A[1]. \qquad (\Delta_4)$$

As $j_* j^* Y[-1] \in \mathcal{Y}[-1] \subseteq \mathcal{Y}$ and, consequently, $A \in (\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[-1]$, we have $\mathcal{Y}[-1] \subseteq \mathcal{Y}'$.

We now finish the proof of part (1) by showing that $\mathcal{H}_\sigma \cap \mathcal{Y}' = \mathcal{H}_\sigma \cap \mathcal{H}_{\sigma'} = \mathcal{R}$. To that end, let Y be an object in $\mathcal{H}_\sigma$ and consider the triangle (Δ_4) associated to Y given by the localising sequence induced by $\mathscr{E}$. By assumption, Y lies in $\mathcal{Y}'$ if and only if A lies in $(\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[-1]$. Now, applying the functor H^0_σ to the triangle, and using that $j_* j^*(\mathcal{Y}) \subseteq \mathcal{Y}$, we see that A lies in $\mathcal{Y}$, and we get an exact sequence

$$0 \longrightarrow H^0_\sigma(A) \longrightarrow Y \xrightarrow{H^0_\sigma(\alpha)} H^0_\sigma(j_* j^* Y).$$

Observe that A lies in $(\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[-1]$ if and only if $H^0_\sigma(A) = 0$, which means that $H^0_\sigma(\alpha)$ is a monomorphism. But $H^0_\sigma(\alpha)$ is the reflection of Y in the Giraud subcategory $\mathcal{S}^{\perp_{0,1}}$ (see Proposition 6.4) and, hence, it is a monomorphism if and only if Y lies in the torsion-free class $\mathcal{R}$.

(2): We have to show that the torsion class $\mathcal{S}$ is closed under direct products. For this purpose, it suffices to show that $\mathcal{S} = H^{-1}_\sigma(\mathcal{Y}')$. Indeed, $\mathcal{Y}'$ is contained in $\mathcal{Y}[1]$, and by the definition of products in the heart $\mathcal{H}_\sigma$, the functor H^{-1}_σ sends products in $\mathcal{Y}[1]$ to products in $\mathcal{H}_\sigma$. Hence $H^{-1}_\sigma(\mathcal{Y}')$ is closed under products, because so is $\mathcal{Y}'$ being the coaisle of a t-structure.

Now, as $j_*j^*(\mathcal{Y})$ is contained in $\mathcal{Y}$, we have by assumption

$$H_\sigma^{-1}(\mathcal{Y}') = H_\sigma^{-1}((\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[1] \star j_*j^*(\mathcal{Y})) = H_\sigma^{-1}((\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E})[1]) = H_\sigma^0(\mathcal{Y} \cap {}^{\perp_{\mathbb{Z}}}\mathscr{E}),$$

which clearly consists of the objects X in $\mathcal{H}_\sigma$ such that $\mathrm{Hom}_{\mathcal{D}}(X, \mathscr{E}) = 0$. Since $\mathscr{E}$ is contained in $\mathrm{Prod}(\sigma)$, the latter are precisely the objects of $\mathcal{S} = {}^{\perp_0}H_\sigma^0(\mathscr{E})$, as wanted. □

Example 6.7. We continue Example 4.10 over the Kronecker algebra A. Let $P \subset \mathbb{X}$, and consider the right mutation σ_P of $\sigma_\varnothing$ at the set $\mathscr{E} = \mathrm{Prod}(\{S[\infty] \mid S \in \boldsymbol{t}_{\overline{P}}\})$. Notice that the heart $\mathcal{H}$ associated with $\sigma_\varnothing$ is equivalent to the category of quasicoherent sheaves over the projective line $\mathbb{X}$, where the simple sheaves are in bijection with the simple regular A-modules. Moreover, the hereditary torsion pair $(\mathcal{S}, \mathcal{R})$ in $\mathcal{H}$ cogenerated by $\mathscr{E}$ corresponds to the torsion pair generated by the simple sheaves which are determined by the family of tubes $\boldsymbol{t}_P = \bigcup_{x \in P} \boldsymbol{t}_x$. Combining Proposition 6.4 with [Angeleri Hügel and Kussin 2017, Corollary 5.8] we obtain a localising sequence of hearts

$$\mathcal{S} = \varinjlim\nolimits_{\mathcal{H}} \boldsymbol{t}_P \longrightarrow \mathcal{H} \longrightarrow \mathrm{Mod}(A_P),$$

where A_P is a hereditary ring obtained as the universal localisation of A at the modules from $\boldsymbol{t}_P$. On the other hand, the heart $\mathcal{H}'$ associated with the mutation σ_P is a locally coherent Grothendieck category which is neither hereditary nor locally noetherian if $P \neq \varnothing$.

Finally, we turn to the dual case. We first need a technical lemma.

Lemma 6.8. *Let $\mathcal{D}$ be a compactly generated triangulated category and $P \in \mathcal{D}$ a pure-projective object. Then there exists a set of maps $\mathcal{I}$ between compact objects such that*

$$P^{\perp_0} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(f, X) = 0 \textit{ for each } f \in \mathcal{I}\}.$$

Proof. Recall from Section 2.3 that the category of pure-projective objects is equivalent to that of projective $\mathcal{D}^c$-modules via $\boldsymbol{y} : \mathcal{D} \to \mathrm{Mod}(\mathcal{D}^c)$. By a theorem of Kaplansky, every projective module is a direct sum of countably generated ones [Mitchell 1972, Lemma 36.3], so we can without loss of generality assume that $\boldsymbol{y}P$ is countably generated. In particular, there is a sequence of compact objects of $\mathcal{D}$,

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} \cdots$$

such that $\boldsymbol{y}P = \varinjlim \boldsymbol{y}C_n$ and, given any $X \in \mathcal{D}$, we have isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(P, X) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}^c)}(\boldsymbol{y}P, \boldsymbol{y}X) \cong \varprojlim \mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}^c)}(\boldsymbol{y}C_n, \boldsymbol{y}X) \cong \varprojlim \mathrm{Hom}_{\mathcal{D}}(C_n, X)$$

(the outer isomorphisms follow by the Yoneda lemma and the fact that $\boldsymbol{y}$ preserves coproducts and summands). Moreover, as one can trace back to [Whitehead 1980, Theorem 1.9] and as is explained in [Herbera and Příhoda 2014, §1], up to passing to a cofinal subsystem, one can assume that there exist morphisms

$$C_2 \xleftarrow{g_2} C_3 \xleftarrow{g_3} C_4 \xleftarrow{g_4} \cdots$$

such that $g_{n+1} f_{n+1} f_n = f_n$ for each $n > 0$.

We claim that, given any $X \in \mathcal{D}$, we have $\mathrm{Hom}_{\mathcal{D}}(P, X) = 0$ if and only if $\mathrm{Hom}_{\mathcal{D}}(f_n, X) = 0$ for each $n > 0$. The "if" part being clear, we focus on the "only if" part. Applying $\mathrm{Hom}_{\mathcal{D}}(-, X)$ to the direct system above, we obtain an inverse system of abelian groups

$$\mathrm{Hom}_{\mathcal{D}}(C_1, X) \xleftarrow{f_1^*} \mathrm{Hom}_{\mathcal{D}}(C_2, X) \xleftarrow{f_2^*} \mathrm{Hom}_{\mathcal{D}}(C_3, X) \xleftarrow{f_3^*} \cdots$$

and one readily verifies that $\mathrm{Im}(f_n^*) = \mathrm{Im}(f_n^* f_{n+1}^*)$ for each $n > 0$, as $f_n^*(h) = hf_n = hg_{n+1} f_{n+1} f_n = f_n^* f_{n+1}^*(hg_{n+1})$ for any $h : C_{n+1} \to X$. Using the same argument as in [Šaroch and Šťovíček 2008, Lemma 4.5], we deduce that also the image of the limit map

$$\mathrm{Hom}_{\mathcal{D}}(C_n, X) \longleftarrow \varprojlim \mathrm{Hom}_{\mathcal{D}}(C_n, X) \cong \mathrm{Hom}_{\mathcal{D}}(P, X)$$

equals $\mathrm{Im}(f_n^*)$ for each $n > 0$. Hence, if $\mathrm{Hom}_{\mathcal{D}}(P, X) = 0$, then $\mathrm{Im}(f_n^*) = 0$ for each $n > 0$, or in other words $\mathrm{Hom}_{\mathcal{D}}(f_n, X) = 0$ for each $n > 0$. □

Now we see that pure-projective silting objects even induce recollements of triangulated categories and of the associated hearts.

Proposition 6.9. *Let σ be a silting object in a compactly generated triangulated category $\mathcal{D}$. Let $\mathscr{P} = \mathrm{Add}(\mathscr{P})$ be a subcategory of $\mathrm{Add}(\sigma)$ in which every object is pure-projective. Then there is a TTF triple $(\mathcal{S}_{\mathscr{P}}, \mathscr{P}^{\perp_{\mathbb{Z}}}, \mathcal{C}_{\mathscr{P}})$ in $\mathcal{D}$ and, thus, a recollement*

$$\mathscr{P}^{\perp_{\mathbb{Z}}} \underset{\curvearrowleft\, i^!}{\overset{\curvearrowleft\, i^*}{\xrightarrow{\;i_*\;}}} \mathcal{D} \underset{\curvearrowleft\, j_*}{\overset{\curvearrowleft\, j_!}{\xrightarrow{\;j^*\;}}} \mathcal{S}_{\mathscr{P}},$$

where i_* and $j_!$ are the inclusion functors. Moreover, the silting t-structure $\mathbb{T}_\sigma$ restricts along the sequence, thus giving rise to a recollement of abelian categories

$$\mathcal{T} \underset{\curvearrowleft\, \bar{i}^!}{\overset{\curvearrowleft\, \bar{i}^*}{\xrightarrow{\;\bar{i}_*\;}}} \mathcal{H}_\sigma \underset{\curvearrowleft\, \bar{j}_*}{\overset{\curvearrowleft\, \bar{j}_!}{\xrightarrow{\;\bar{j}^*\;}}} \mathcal{T}^{\perp_{0,1}},$$

where $\bar{i}_*$ and $\bar{j}_*$ are the inclusion functors, $\mathcal{T} = H^0_\sigma(\mathscr{P})^{\perp_0}$ is a TTF class, and $\mathcal{T}^{\perp_{0,1}}$ is the associated Giraud subcategory in $\mathcal{H}_\sigma$.

If σ and σ' are pure-projective silting objects with associated t-structures $\mathbb{T}_\sigma = (\mathcal{X}, \mathcal{Y})$ and $\mathbb{T}_{\sigma'} = (\mathcal{X}', \mathcal{Y}')$, and σ' is a left mutation of σ at $\mathscr{P}$, then we have $\mathcal{X}' = j_! j^*(\mathcal{X}) \star (\mathcal{X} \cap \mathscr{P}^{\perp_{\mathbb{Z}}})[1]$, while in the case of right mutation we have $\mathcal{X}' = j_! j^*(\mathcal{X}) \star (\mathcal{X} \cap \mathscr{P}^{\perp_{\mathbb{Z}}})[-1]$.

Proof. The class $\mathscr{P}^{\perp_{\mathbb{Z}}}$ is preenveloping by Lemma 6.8 and [Krause 2000, Proposition 3.11], and the inclusion $i_* : \mathscr{P}^{\perp_{\mathbb{Z}}} \to \mathscr{D}$ has a left adjoint i^* by [Neeman 2010, Proposition 5.1]. In fact, the latter tells us that there is colocalising sequence

$$\mathscr{P}^{\perp_{\mathbb{Z}}} \overset{\curvearrowleft\, i^*}{\xrightarrow{\;i_*\;}} \mathcal{D} \overset{\curvearrowleft\, j_!}{\xrightarrow{\;j^*\;}} \mathcal{S}_{\mathscr{P}}.$$

It follows from that $\mathscr{P}^{\perp_{\mathbb{Z}}}$ is itself a compactly generated triangulated category and that the sequence is also localising; see [Krause 2000, Proposition 2.6 and Lemma 4.1] (see also [Laking and Vitória 2020, Proposition 6.3]). Hence, we get the recollement. The proof that the t-structure restricts along the recollement is analogous to Proposition 6.4. From [Beilinson et al. 1982] it follows that there is such a recollement of hearts. Finally, the last statement is shown with arguments similar to those in the proof of Lemma 6.5. □

7. Mutation of torsion pairs

We now wish to consider mutations of cosilting torsion pairs. We will make use of the notation set up in Definition 3.8. We will also freely use the fact that a cosilting object σ' is a right mutation of a cosilting object σ if and only if σ is a left mutation of σ'; see Corollary 3.7.

Definition 7.1. Let $\mathcal{D}$ be a triangulated category with products and coproducts and $\mathbb{T}$ a t-structure with heart $\mathcal{H}$. Furthermore, let $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be in $\mathrm{Cosilt}(\mathcal{H})$ with associated cosilting objects $\sigma_{\mathfrak{u}}$ and $\sigma_{\mathfrak{t}}$ in $\mathcal{D}$. If $\sigma_{\mathfrak{t}}$ is a right mutation of $\sigma_{\mathfrak{u}}$, then we will say that $\mathfrak{t}$ is a *right mutation* of $\mathfrak{u}$ and $\mathfrak{u}$ is a *left mutation* of $\mathfrak{t}$.

By Corollary 3.7, the definition is independent of the cosilting objects we choose to represent $\mathfrak{u}$ and $\mathfrak{t}$.

7.1. ***Inclusions of torsion pairs and filtration triples.*** If a cosilting torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is a right mutation of some cosilting torsion pair $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$, then, by Theorem 3.5, we have that $\mathbb{T}_{\mathfrak{t}^-}$ is a right HRS-tilt of $\mathbb{T}_{\mathfrak{u}^-}$. By Proposition 2.3, we have that $\mathcal{X}_{\mathfrak{u}^-} \subseteq \mathcal{X}_{\mathfrak{t}^-}$ and so $\mathcal{U} \subseteq \mathcal{T}$. Consequently, we begin by studying such nested torsion pairs, which are known to give rise to filtrations (see [Baumann et al. 2014]).

Definition 7.2. Let $\mathcal{U}, \mathcal{S}, \mathcal{F}$ be full subcategories of an abelian category $\mathcal{A}$. We will call $(\mathcal{U}, \mathcal{S}, \mathcal{F})$ a *filtration triple* if $\mathrm{Hom}_{\mathcal{A}}(\mathcal{U}, \mathcal{S}) = 0$, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{U}, \mathcal{F}) = 0$, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{S}, \mathcal{F}) = 0$ and, for every object X in $\mathcal{A}$, there exists a filtration

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 = X$$

such that $X_1/X_0 = X_1$ lies in $\mathcal{U}$, X_2/X_1 lies in $\mathcal{S}$ and $X_3/X_2 = X/X_2$ lies in $\mathcal{F}$.

Proposition 7.3. *Let $\mathcal{A}$ be an abelian category. There is a bijection between*

(1) *pairs of torsion pairs $(\mathcal{U}, \mathcal{V})$, $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ with $\mathcal{U} \subseteq \mathcal{T}$, and*

(2) *filtration triples $(\mathcal{U}, \mathcal{S}, \mathcal{F})$ in $\mathcal{A}$.*

The mutually inverse bijections are given by

$$(\mathcal{U}, \mathcal{V}), (\mathcal{T}, \mathcal{F}) \mapsto (\mathcal{U}, \mathcal{V} \cap \mathcal{T}, \mathcal{F}) \quad \textit{and} \quad (\mathcal{U}, \mathcal{S}, \mathcal{F}) \mapsto (\mathcal{U}, \mathcal{S} \star \mathcal{F}), (\mathcal{U} \star \mathcal{S}, \mathcal{F}).$$

Proof. Let $(\mathcal{U}, \mathcal{V})$, $(\mathcal{T}, \mathcal{F})$ be a pair of torsion pairs in $\mathcal{A}$ such that $\mathcal{U} \subseteq \mathcal{T}$. We show that $(\mathcal{U}, \mathcal{V} \cap \mathcal{T}, \mathcal{F})$ is a filtration triple in $\mathcal{A}$. Let $\mathcal{S} := \mathcal{V} \cap \mathcal{T}$. Firstly, the Hom-orthogonality conditions are clear because $\mathrm{Hom}_{\mathcal{A}}(\mathcal{U}, \mathcal{V}) = 0 = \mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F})$. Let X be an arbitrary object in $\mathcal{A}$ with torsion decompositions

$$0 \longrightarrow X_1 \longrightarrow X \longrightarrow X/X_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X_2 \longrightarrow X \longrightarrow X/X_2 \longrightarrow 0,$$

where X_1 is contained in $\mathcal{U}$, X/X_1 is in $\mathcal{V}$, X_2 is in $\mathcal{T}$ and X/X_2 is in $\mathcal{F}$. Then X_2/X_1 is the torsion-free part of X_2 with respect to $(\mathcal{U},\mathcal{V})$ and is a quotient of X_2, and so X_2/X_1 lies in $\mathcal{S}$. Hence $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq X$ is the desired filtration of X.

Let $(\mathcal{U},\mathcal{S},\mathcal{F})$ be a filtration triple in $\mathcal{A}$. We show that $(\mathcal{U}\star\mathcal{S},\mathcal{F})$ is a torsion pair in $\mathcal{A}$; the proof that $(\mathcal{U},\mathcal{S}\star\mathcal{F})$ is a torsion pair is similar. If we consider objects U in $\mathcal{U}$, S in $\mathcal{S}$ and an exact sequence

$$0 \longrightarrow U \longrightarrow X \longrightarrow S \longrightarrow 0$$

and we apply $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{F})$, then we obtain that $\mathrm{Hom}_{\mathcal{A}}(X,\mathcal{F}) = 0$ and hence $\mathrm{Hom}_{\mathcal{A}}(\mathcal{U}\star\mathcal{S},\mathcal{F}) = 0$. Moreover, by the definition of filtration triple, for each object A in $\mathcal{A}$, we have a short exact sequence

$$0 \longrightarrow A_2 \longrightarrow A \longrightarrow A/A_2 \longrightarrow 0,$$

with A/A_2 in $\mathcal{F}$, A_2 in $\mathcal{U}\star\mathcal{S}$ because there is a short exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_1/A_2 \longrightarrow 0,$$

where A_1 lies in $\mathcal{U}$ and A_1/A_2 lies in $\mathcal{S}$. Thus $(\mathcal{U}\star\mathcal{S},\mathcal{F})$ is a torsion pair. □

Proposition 7.4. *Let $\mathcal{H}$ be the heart of a t-structure in a triangulated category $\mathcal{D}$ and let $(\mathcal{U},\mathcal{S},\mathcal{F})$ be a filtration triple in $\mathcal{H}$. The following statements hold*:

(1) $(\mathcal{F},\mathcal{U}[-1],\mathcal{S}[-1])$ *is a filtration triple in $\mathcal{H}_{\mathfrak{t}^-}$, where $\mathfrak{t} = (\mathcal{U}\star\mathcal{S},\mathcal{F})$.*

(2) $(\mathcal{S},\mathcal{F},\mathcal{U}[-1])$ *is a filtration triple in $\mathcal{H}_{\mathfrak{u}^-}$, where $\mathfrak{u} = (\mathcal{U},\mathcal{S}\star\mathcal{F})$.*

Proof. We prove statement (1); the proof of (2) is similar. We show that $(\mathcal{F},\mathcal{U}[-1],\mathcal{S}[-1])$ is a filtration triple in $\mathcal{H}_{\mathfrak{t}^-}$. First we prove the Hom-orthogonality conditions

$$\mathrm{Hom}_{\mathcal{H}_{\mathfrak{t}^-}}(\mathcal{U}[-1],\mathcal{S}[-1]) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}[-1],\mathcal{S}[-1]) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{U},\mathcal{S}) = \mathrm{Hom}_{\mathcal{H}}(\mathcal{U},\mathcal{S}) = 0.$$

Also, since $\mathcal{U},\mathcal{S} \subset \mathcal{T} := \mathcal{U}\star\mathcal{S}$ and $(\mathcal{F},\mathcal{T}[-1])$ is a torsion pair in $\mathcal{H}_{\mathfrak{t}^-}$, it follows that

$$\mathrm{Hom}_{\mathcal{H}_{\mathfrak{t}^-}}(\mathcal{F},\mathcal{U}[-1]) = 0 = \mathrm{Hom}_{\mathcal{H}_{\mathfrak{t}^-}}(\mathcal{F},\mathcal{S}[-1]).$$

Next we prove the existence of a filtration by $(\mathcal{F},\mathcal{U}[-1],\mathcal{S}[-1])$ for an arbitrary object $Y \in \mathcal{H}_{\mathfrak{t}^-}$. The torsion pair $(\mathcal{F},\mathcal{T}[-1])$ with $\mathcal{T}[-1] = \mathcal{U}[-1]\star\mathcal{S}[-1]$ induces the following commutative diagram in $\mathcal{H}_{\mathfrak{t}^-}$ with exact rows and columns, where the object X is obtained as a pullback:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & U[-1] & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & Y & \longrightarrow & T[-1] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S[-1] & = & S[-1] & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then, the sequence $0 = Y_0 \subseteq F \subseteq X \subseteq Y$ is a $(\mathcal{F},\mathcal{U}[-1],\mathcal{S}[-1])$-filtration of Y. □

Let us summarise the situation in Proposition 7.4 as follows:

$$\begin{array}{ccc} (\mathcal{U},\mathcal{S},\mathcal{F}) \text{ in } \mathcal{H} & \overset{\mathfrak{t}^-}{\rightsquigarrow} & (\mathcal{F},\mathcal{U}[-1],\mathcal{S}[-1]) \text{ in } \mathcal{H}_{\mathfrak{t}^-} \\ {\scriptstyle \mathfrak{u}^-}\wr\!\downarrow & & \\ (\mathcal{S},\mathcal{F},\mathcal{U}[-1]) \text{ in } \mathcal{H}_{\mathfrak{u}^-} & & \end{array}$$

We are now going to see that this diagram can be completed to a commutative triangle. Recall that the collection of torsion pairs in an abelian category $\mathcal{H}$ has a partial order: we say that $\mathfrak{u}$ is less than $\mathfrak{t}$ if $\mathcal{U} \subseteq \mathcal{T}$. This gives rise to a poset which we denote by $\operatorname{tors}(\mathcal{H})$.

Proposition 7.5. *Let $\mathcal{H}$ be the heart of a t-structure $\mathbb{T} = (\mathcal{X},\mathcal{Y})$ in a triangulated category $\mathcal{D}$, and let $\mathfrak{u} = (\mathcal{U},\mathcal{V})$ be a torsion pair in $\mathcal{H}$. The assignment taking $\mathfrak{t} = (\mathcal{T},\mathcal{F})$ to $\mathfrak{s} = (\mathcal{T}\cap\mathcal{V}, \mathcal{F}\star\mathcal{U}[-1])$ induces an order-preserving bijection between*

(1) *torsion pairs $\mathfrak{t} = (\mathcal{T},\mathcal{F})$ in $\mathcal{H}$ with $\mathcal{U} \subseteq \mathcal{T}$, and*

(2) *torsion pairs $\mathfrak{s} = (\mathcal{S},\mathcal{R})$ in $\mathcal{H}_{\mathfrak{u}^-}$ with $\mathcal{S} \subseteq \mathcal{V}$.*

Moreover, if $\mathfrak{t}$ and $\mathfrak{s}$ correspond to each other under this bijection, then $(\mathbb{T}_{\mathfrak{u}^-})_{\mathfrak{s}^-} = \mathbb{T}_{\mathfrak{t}^-}$. That is, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T} & \overset{\mathfrak{t}^-}{\rightsquigarrow} & \mathbb{T}_{\mathfrak{t}^-} \\ {\scriptstyle \mathfrak{u}^-}\wr\!\downarrow & \overset{\mathfrak{s}^-}{\nearrow} & \\ \mathbb{T}_{\mathfrak{u}^-} & & \end{array}$$

Proof. In view of Proposition 7.3, the statement can be rephrased in terms of a bijection between filtration triples $(\mathcal{U},\mathcal{S},\mathcal{F})$ in $\mathcal{H}$ with $\mathcal{S}\star\mathcal{F} = \mathcal{V}$ and filtration triples $(\mathcal{S},\mathcal{F},\mathcal{U}[-1])$ in $\mathcal{H}_{\mathfrak{u}^-}$ with the same property. By Proposition 7.4, every filtration triple $(\mathcal{U},\mathcal{S},\mathcal{F})$ in $\mathcal{H}$ induces a filtration triple $(\mathcal{S},\mathcal{F},\mathcal{U}[-1])$ in $\mathcal{H}_{\mathfrak{u}^-}$, which in turn induces a filtration triple $(\mathcal{U}[-1],\mathcal{S}[-1],\mathcal{F}[-1])$ in $(\mathcal{H}_{\mathfrak{u}^-})_{\boldsymbol{v}^-}$, where $\boldsymbol{v} = (\mathcal{V},\mathcal{U}[-1])$ is the tilted torsion pair in $\mathcal{H}_{\mathfrak{u}^-}$. By Remark 2.2 we have $(\mathcal{H}_{\mathfrak{u}^-})_{\boldsymbol{v}^-} = \mathcal{H}[-1]$, and the latter filtration triple corresponds to the filtration triple $(\mathcal{U},\mathcal{S},\mathcal{F})$ in $\mathcal{H}$. This establishes the desired bijection, which is order-preserving by construction.

Next, we prove the stated equality of t-structures by comparing the coaisles. We have $\mathcal{Y}_{\mathfrak{t}^-} = \mathcal{F}\star\mathcal{Y}[-1]$ and $\mathcal{Y}_{\mathfrak{u}^-} = \mathcal{V}\star\mathcal{Y}[-1]$. Then, keeping in mind that $\mathcal{R} = \mathcal{F}\star\mathcal{U}[-1]$, we obtain

$$(\mathcal{Y}_{\mathfrak{u}^-})_{\mathfrak{s}^-} = \mathcal{R}\star\mathcal{Y}_{\mathfrak{u}^-}[-1] = \mathcal{F}\star\mathcal{U}[-1]\star\mathcal{V}[-1]\star\mathcal{Y}[-2] = \mathcal{F}\star\mathcal{H}[-1]\star\mathcal{Y}[-2] = \mathcal{F}\star\mathcal{Y}[-1] = \mathcal{Y}_{\mathfrak{t}^-}. \qquad \square$$

7.2. ***Mutation of cosilting torsion pairs.*** In this subsection, we apply Proposition 7.5 to cosilting torsion pairs and determine when they are related by mutation.

We will see that this can be expressed in terms of the notion of a wide subcategory. Recall that a full additive subcategory $\mathcal{W}$ of an abelian category $\mathcal{H}$ is an *exact abelian subcategory* if it is closed under kernels and cokernels. The subcategory $\mathcal{W}$ is *wide* if it is an exact abelian subcategory that is closed under extensions. We will need the following lemma.

Lemma 7.6. *Let $\mathcal{D}$ be an arbitrary triangulated category and let $\mathcal{H}$ and $\mathcal{H}'$ be hearts of t-structures $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{X}', \mathcal{Y}')$ respectively. Then the following statements hold*:

(1) *Let $\mathcal{W}$ be a full subcategory of $\mathcal{H}$. Then $\mathcal{W}$ is an exact abelian subcategory of $\mathcal{H}$ if and only if $\mathcal{W} \star \mathcal{W}[1] \subseteq \mathcal{W}[1] \star \mathcal{W}$ (equivalently, the cone of every morphism f in $\mathcal{W}$ is contained in $\mathcal{W}[1] \star \mathcal{W}$).*

(2) *Let $\mathcal{W}$ be a full subcategory contained in $\mathcal{H} \cap \mathcal{H}'$. Then $\mathcal{W}$ is an exact abelian subcategory of $\mathcal{H}$ if and only if $\mathcal{W}$ is an exact abelian subcategory of $\mathcal{H}'$.*

(3) *Let $\mathcal{W}$ be a full subcategory contained in $\mathcal{H} \cap \mathcal{H}'$. Then $\mathcal{W}$ is a wide subcategory of $\mathcal{H}$ if and only if $\mathcal{W}$ is a wide subcategory of $\mathcal{H}'$.*

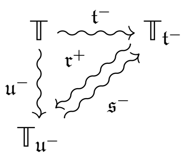
Proof. Statements (2) and (3) follow immediately from the first statement because then the required closure conditions depend only on the ambient triangulated category and not on the specific t-structures. We therefore prove statement (1). Let f be a morphism in $\mathcal{W}$ and let $L := \operatorname{cone}(f)$. Consider the triangle $L_{\mathcal{X}} \to L \to L_{\mathcal{Y}} \to L_{\mathcal{X}}[1]$ corresponding to the t-structure $(\mathcal{X}, \mathcal{Y})$. Then $\operatorname{Ker}(f) = L_{\mathcal{X}}[-1]$ and $\operatorname{Coker}(f) = L_{\mathcal{Y}}$, so the statement follows. □

Let us now consider the following situation.

Setup 7.7. Let $\mathcal{D}$ be a compactly generated triangulated category with a t-structure $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ and let $\mathcal{H}$ be the heart of $\mathbb{T}$. Let further $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be torsion pairs in $\operatorname{Cosilt}_*(\mathcal{H})$ such that $\mathcal{U} \subseteq \mathcal{T}$. Let us fix the following notation:

- $\mathfrak{s} = (\mathcal{S}, \mathcal{R}) = (\mathcal{T} \cap \mathcal{V}, \mathcal{F} \star \mathcal{U}[-1])$ is the torsion pair in $\mathcal{H}_{\mathfrak{u}^-}$ uniquely determined by $\mathfrak{u}$ and $\mathfrak{t}$ according to Proposition 7.5.
- $\mathfrak{r} = (\mathcal{R}, \mathcal{S}[-1])$ is the tilted torsion pair of $\mathfrak{s}$ in $(\mathbb{T}_{\mathfrak{u}^-})_{\mathfrak{s}^-} = \mathbb{T}_{\mathfrak{t}^-}$.

We can visualise the setup in the following commutative diagram:



Theorem 7.8. *Suppose we are in Setup 7.7. Then $\mathfrak{s}$ is in $\operatorname{Cosilt}_*(\mathcal{H}_{\mathfrak{u}^-})$ and $\mathfrak{r}$ is in $\operatorname{Cosilt}_*(\mathcal{H}_{\mathfrak{t}^-})$. Moreover, the following statements are equivalent*:

(1) *$\mathfrak{t}$ is a right mutation of $\mathfrak{u}$.*

(2) *$\mathcal{S}$ is a wide subcategory of $\mathcal{H}$.*

(3) *$\mathfrak{s} = (\mathcal{S}, \mathcal{R})$ is a hereditary torsion pair in $\mathcal{H}_{\mathfrak{u}^-}$.*

(4) *$\mathcal{S}[-1]$ is a TTF class in $\mathcal{H}_{\mathfrak{t}^-}$.*

Proof. Let $\sigma_{\mathfrak{t}}$ and $\sigma_{\mathfrak{u}}$ denote the pure-injective cosilting objects in $\mathcal{D}$ associated to $\mathfrak{u}$ and $\mathfrak{t}$ respectively. The fact that $\mathfrak{s}$ is a cosilting torsion pair follows immediately from the fact that $(\mathbb{T}_{\mathfrak{u}^-})_{\mathfrak{s}^-} = \mathbb{T}_{\mathfrak{t}^-} = \mathbb{T}_{\sigma_{\mathfrak{t}}}$ by

Proposition 7.5. Similarly, we have that $(\mathbb{T}_{\mathrm{t}^-})_{\mathfrak{r}^-} = \mathbb{T}_{\mathrm{u}^-}[-1] = \mathbb{T}_{\sigma_{\mathrm{u}}}[-1]$ by Remark 2.2. From Corollary 4.6 we infer that $\mathfrak{s}$ is in $\mathrm{Cosilt}_*(\mathcal{H}_{\mathrm{u}^-})$ and $\mathfrak{r}$ is in $\mathrm{Cosilt}_*(\mathcal{H}_{\mathrm{t}^-})$.

(1) ⇔ (3): Since σ_{t} and σ_{u} are pure-injective, the hearts $\mathcal{H}_{\mathrm{u}^-} = \mathcal{H}_{\sigma_{\mathrm{u}}}$ and $\mathcal{H}_{\mathrm{t}^-} = \mathcal{H}_{\sigma_{\mathrm{t}}}$ are Grothendieck categories by Theorem 2.7. It is well known that, in a Grothendieck category, a torsion pair is hereditary if and only if it is cogenerated by a class of injective objects.

Now, if σ_{t} is a right mutation of σ_{u}, then we know from Theorem 3.5 that the t-structure $\mathbb{T}_{\sigma_{\mathrm{t}}} = \mathbb{T}_{\mathrm{t}^-}$ is the right HRS-tilt of $\mathbb{T}_{\sigma_{\mathrm{u}}} = \mathbb{T}_{\mathrm{u}^-}$ at the torsion pair $({}^{\perp_0}H^0_{\mathrm{u}^-}(\mathscr{E}), \mathrm{Cogen}(H^0_{\mathrm{u}^-}(\mathscr{E})))$ in $\mathcal{H}_{\mathrm{u}^-}$ induced by $\mathscr{E} = \mathrm{Prod}(\sigma_{\mathrm{u}}) \cap \mathrm{Prod}(\sigma_{\mathrm{t}})$. But $\mathbb{T}_{\mathrm{t}^-}$ is also the right HRS-tilt of $\mathbb{T}_{\mathrm{u}^-}$ at the torsion pair $\mathfrak{s} = (\mathcal{S}, \mathcal{R})$. It follows that $\mathfrak{s} = ({}^{\perp_0}H^0_{\mathrm{u}^-}(\mathscr{E}), \mathrm{Cogen}(H^0_{\mathrm{u}^-}(\mathscr{E})))$. We know that $H^0_{\mathrm{u}^-}(\mathscr{E})$ is a set of injective objects in $\mathcal{H}_{\mathrm{u}^-}$ (see Proposition 2.5) and so we have shown that (3) holds.

Conversely, suppose that $\mathcal{S} = {}^{\perp_0}\mathcal{I}$ for some class $\mathcal{I} = \mathrm{Prod}(\mathcal{I})$ of injective objects in $\mathcal{H}_{\mathrm{u}^-}$. Then there is a class $\mathscr{E} = \mathrm{Prod}(\mathscr{E}) \subseteq \mathrm{Prod}(\sigma_{\mathrm{u}})$ such that $\mathcal{I} = H^0_{\mathrm{u}^-}(\mathscr{E})$ (see Proposition 2.5 again) and, moreover, $\mathfrak{s} = ({}^{\perp_0}H^0_{\mathrm{u}^-}(\mathscr{E}), \mathrm{Cogen}(H^0_{\mathrm{u}^-}(\mathscr{E})))$ is a cosilting torsion pair in $\mathcal{H}_{\mathrm{u}^-}$ because $(\mathbb{T}_{\mathrm{u}^-})_{\mathfrak{s}^-} = \mathbb{T}_{\mathrm{t}^-} = \mathbb{T}_{\sigma_{\mathrm{t}}}$ by Proposition 7.5. It follows from Proposition 4.5 and Theorem 4.9 that σ_{u} admits a right mutation $\tilde{\sigma}$ with respect to $\mathscr{E}$, which is equivalent to σ_{t} because $\mathbb{T}_{\tilde{\sigma}}$ is the right HRS-tilt of $\mathbb{T}_{\mathrm{u}^-}$ at $\mathfrak{s}$.

(2) ⇔ (3): Note that $\mathcal{S}$ is a full subcategory of $\mathcal{D}$ that is contained in $\mathcal{H} \cap \mathcal{H}_{\mathrm{u}^-}$ so, by Lemma 7.6(3), we have that $\mathcal{S}$ is a wide subcategory of $\mathcal{H}$ if and only if $\mathcal{S}$ is a wide subcategory of $\mathcal{H}_{\mathrm{u}^-}$. Since $\mathcal{S}$ is a torsion class in $\mathcal{H}_{\mathrm{u}^-}$, it is closed under extensions and quotients. The equivalence of (2) and (3) then follows from the fact that $\mathfrak{s}$ is hereditary if and only if $\mathcal{S}$ is closed under subobjects in $\mathcal{H}_{\mathrm{u}^-}$ if and only if $\mathcal{S}$ is closed under kernels in $\mathcal{H}_{\mathrm{u}^-}$ if and only if $\mathcal{S}$ is a wide subcategory of $\mathcal{H}_{\mathrm{u}^-}$.

(2) ⇔ (4): By analogous arguments as in the previous paragraph, we observe that $\mathcal{S}$ is wide in $\mathcal{H}$ if and only if $\mathfrak{r} = (\mathcal{R}, \mathcal{S}[-1])$ is a cohereditary torsion pair in $\mathcal{H}_{\mathrm{t}^-}$, that is, if and only if $\mathcal{S}[-1]$ is closed under quotient objects in $\mathcal{H}_{\mathrm{t}^-}$. Since a torsion-free class in a Grothendieck category is always closed under coproducts, this happens if and only if $\mathcal{S}[-1]$ is a torsion class in $\mathcal{H}_{\mathrm{t}^-}$. □

Inspired by Theorem 7.8(2), we conclude the section with a useful criterion for when the intersection of a torsion and a torsion-free class is a wide subcategory. It is closely related to a construction of [Ingalls and Thomas 2009, §2.3], which was generalised in [Marks and Šťovíček 2017, §3].

Proposition 7.9. *Let $\mathcal{H}$ be an abelian category and $\mathrm{u} = (\mathcal{U}, \mathcal{V})$ and $\mathrm{t} = (\mathcal{T}, \mathcal{F})$ be torsion pairs in $\mathcal{H}$ such that $\mathcal{U} \subseteq \mathcal{T}$. Then the following are equivalent*:

(1) *$\mathcal{T} \cap \mathcal{V}$ is a wide subcategory of $\mathcal{H}$.*

(2) *If $g : T \to V$ is a map in $\mathcal{H}$ with T in $\mathcal{T}$ and V in $\mathcal{V}$, then* $\mathrm{Ker}(g)$ *lies in $\mathcal{T}$ and* $\mathrm{Coker}(g)$ *lies in $\mathcal{V}$.*

Proof. Let us write $\mathcal{W} := \mathcal{T} \cap \mathcal{V}$.

(1) ⇒ (2): Let $g : T \to V$ be a morphism in $\mathcal{H}$ with T in $\mathcal{T}$ and V in $\mathcal{V}$. By Proposition 7.3, we have that T lies in $\mathcal{U} \star \mathcal{W}$ and V lies in $\mathcal{W} \star \mathcal{F}$, so there are short exact sequences

$$0 \longrightarrow U \longrightarrow T \xrightarrow{b} S_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow S_2 \xrightarrow{a} V \longrightarrow F \longrightarrow 0,$$

with S_1 and S_2 in $\mathcal{W}$, U in $\mathcal{U}$ and F in $\mathcal{F}$. Since $\mathrm{Hom}_A(U, V) = 0$ and $\mathrm{Hom}_A(S_1, F) = 0$, we may use the kernel/cokernel properties to obtain a morphism $f : S_1 \to S_2$ such that $g = afb$. By assumption we have that both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ lie in $\mathcal{W}$. By taking the pullback of the canonical embedding $\mathrm{Ker}(f) \to S_1$ along b, we obtain a short exact sequence

$$0 \longrightarrow U \longrightarrow K \longrightarrow \mathrm{Ker}(f) \longrightarrow 0.$$

By checking the universal property, it is straightforward to show that $K \cong \mathrm{Ker}(g)$ and, hence, $\mathrm{Ker}(g)$ lies in $\mathcal{U} \star \mathcal{W} = \mathcal{T}$. A dual argument yields that there is a short exact sequence

$$0 \longrightarrow \mathrm{Coker}(f) \longrightarrow \mathrm{Coker}(g) \longrightarrow F \longrightarrow 0,$$

and hence $\mathrm{Coker}(g)$ lies in $\mathcal{W} \star \mathcal{F} = \mathcal{V}$.

(2) $\Rightarrow$ (1): Let $g : T \to V$ be a map with T and V in $\mathcal{W}$. Then $\mathrm{Ker}(g)$ lies in $\mathcal{T}$ by assumption and $\mathrm{Ker}(g)$ lies in $\mathcal{V}$ because $\mathcal{V}$ is a torsion-free class in $\mathcal{H}$. Therefore, $\mathrm{Ker}(g)$ lies in $\mathcal{W}$. Similarly, we have that $\mathrm{Coker}(g)$ lies in $\mathcal{W}$. □

8. Mutations of torsion pairs in $\mathrm{D}^b(\mathrm{mod}(R))$

In this section we will assume that $\mathcal{D}$ is the derived category $\mathrm{D}(R)$ of a left coherent ring R. We will use the techniques developed in the previous sections to study torsion pairs in hearts of t-structures in $\mathrm{D}^b(\mathrm{mod}(R))$. In order to do that, we must first lift these t-structures to the whole derived category $\mathcal{D}$ and then extend the torsion pairs from the original heart to the lifted heart. Let us begin with the process of extending torsion pairs within a locally coherent Grothendieck category (of which $\mathrm{Mod}(R)$ is, by assumption on R, an example).

Proposition 8.1 [Crawley-Boevey 1994, Lemma 4.4]. *Let $\mathcal{A}$ be a locally coherent Grothendieck category and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathrm{fp}\,\mathcal{A}$. Then*:

(1) *The pair $\vec{\mathfrak{t}} = (\vec{\mathcal{T}}, \vec{\mathcal{F}}) := ({}^{\perp}(\mathcal{T}^{\perp}), \mathcal{T}^{\perp})$ in $\mathcal{A}$ is a torsion pair, called the lift of* $\mathfrak{t}$.

(2) *The assignment of a torsion pair in $\mathrm{fp}\,\mathcal{A}$ to its lift in $\mathcal{A}$ induces a bijection between*
   (a) *torsion pairs in $\mathrm{fp}\,\mathcal{A}$,*
   (b) *torsion pairs $(\mathcal{X}, \mathcal{Y})$ of finite type in $\mathcal{A}$ such that $\mathcal{X} \cap \mathrm{fp}\,\mathcal{A}$ is a torsion class in $\mathrm{fp}\,\mathcal{A}$.*

If $\mathcal{A}$ is locally noetherian (i.e., if $\mathcal{A}$ is a Grothendieck category with a set of noetherian generators), then the assignment above establishes a bijection between torsion pairs in $\mathrm{fp}\,\mathcal{A}$ and torsion pairs of finite type in $\mathcal{A}$.

Note that torsion pairs of finite type in $\mathrm{Mod}(R)$, for any ring R, are precisely the ones in $\mathrm{Cosilt}_*(\mathrm{Mod}(R))$. This also holds for a more general class of hearts in $\mathrm{D}(R)$; see Proposition 4.5.

Let us now consider an analogous result for certain t-structures in $\mathrm{D}^b(\mathrm{mod}(R))$. Recall that a t-structure $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ in $\mathrm{D}(R)$ or in $\mathrm{D}^b(\mathrm{mod}(R))$ is called *intermediate* if there are integers $m \geq n$ such that

$$\mathrm{D}^{\geq 0}[m] \subseteq \mathcal{Y} \subseteq \mathrm{D}^{\geq 0}[n],$$

where $\mathrm{D}^{\geq 0}$ is the standard coaisle in $\mathrm{D}(R)$ or in $\mathrm{D}^b(\mathrm{mod}(R))$, respectively.

Proposition 8.2 [Marks and Zvonareva 2023, Lemma 3.1 and Corollary 4.2]. *Let R be a left coherent ring and let $\mathbb{T} := (\mathcal{U}, \mathcal{V})$ be an intermediate t-structure in $\mathrm{D}^b(\mathrm{mod}(R))$. Then*:

(1) *The pair $\vec{\mathbb{T}} = (\vec{\mathcal{U}}, \vec{\mathcal{V}}) := ({}^{\perp}(\mathcal{U}^{\perp}), \mathcal{U}^{\perp})$ in $\mathrm{D}(R)$ is a t-structure, called the lift of $\mathbb{T}$.*

(2) *The assignment of a t-structure in $\mathrm{D}^b(\mathrm{mod}(R))$ to its lift in $\mathrm{D}(R)$ induces a bijection between*
   (a) *intermediate t-structures in $\mathrm{D}^b(\mathrm{mod}(R))$,*
   (b) *intermediate, compactly generated t-structures in $\mathrm{D}(R)$ with a locally coherent heart $\vec{\mathcal{H}}$ such that $\mathrm{fp}\,\vec{\mathcal{H}} = \vec{\mathcal{H}} \cap \mathrm{D}^b(\mathrm{mod}(R))$.*

Note that the t-structures in $\mathrm{D}(R)$ obtained as lifts of t-structures in $\mathrm{D}^b(\mathrm{mod}(R))$ correspond to pure-injective cosilting objects (see [Angeleri Hügel et al. 2017, Theorem 4.9]); that is, for every intermediate t-structure $\mathbb{T}$ in $\mathrm{D}^b(\mathrm{mod}(R))$ with heart $\mathcal{H}$, there is a pure-injective cosilting object σ such that $\vec{\mathbb{T}} = \mathbb{T}_\sigma$. Note that, since $\mathbb{T}$ is intermediate, it follows that σ is in fact a complex in $\mathsf{K}^b(\mathsf{Inj}(R))$ (see, for example, [Psaroudakis and Vitória 2018, Proposition 4.16]). The t-structure $\vec{\mathbb{T}} = \mathbb{T}_\sigma$ has a locally coherent Grothendieck heart $\mathcal{H}_\sigma$ with $\mathrm{fp}\,\mathcal{H}_\sigma = \mathcal{H}$. We will often denote this heart by $\vec{\mathcal{H}}$; see the remark below.

Remark 8.3. Let us briefly justify the notation $(\vec{-})$ used in the assignments discussed in the two theorems. In fact, if $\mathcal{H}$ is a locally coherent Grothendieck category, it is shown in [Crawley-Boevey 1994] that, for a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{fp}\,\mathcal{H}$, we have that ${}^{\perp}(\mathcal{T}^{\perp})$ and $\mathcal{T}^{\perp}$ are the closure under direct limits of $\mathcal{T}$ and $\mathcal{F}$, respectively, inside $\mathcal{H}$. Similarly, if $(\mathcal{U}, \mathcal{V})$ is an intermediate t-structure in $\mathrm{D}^b(\mathrm{mod}(R))$ for a left coherent ring R, it is shown in [Marks and Zvonareva 2023] that ${}^{\perp}(\mathcal{U}^{\perp})$, $\mathcal{U}^{\perp}$ and the heart ${}^{\perp}(\mathcal{U}^{\perp})[-1] \cap \mathcal{U}^{\perp}$ are the closure under directed homotopy colimits of $\mathcal{U}$, $\mathcal{V}$ and of the heart $\mathcal{U}[-1] \cap \mathcal{V}$, respectively, inside $\mathrm{D}(R)$. Recall that in the derived category of a ring, we may consider directed homotopy colimits as the derived functor of the direct limit functor.

Finally, the following proposition relates the two lifts enunciated in the theorems above via HRS-tilting. This result is essentially contained in [Saorín 2017, Proposition 5.1] and [Marks and Zvonareva 2023, Proposition 5.1]. We include a proof since the formulations therein are slightly different.

Proposition 8.4. *Let R be a left coherent ring and $\mathbb{T} = (\mathcal{X}, \mathcal{Y})$ an intermediate t-structure in $\mathrm{D}^b(\mathrm{mod}(R))$ with heart $\mathcal{H}$. Consider a torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ and a torsion pair $\mathfrak{p} = (\mathcal{P}, \mathcal{Q})$ in the heart $\vec{\mathcal{H}}$ of $\vec{\mathbb{T}}$ in $\mathrm{D}(R)$. Then $\mathfrak{p}$ is the lift of $\mathfrak{t}$ to $\vec{\mathcal{H}}$ if and only if $\vec{\mathbb{T}}_{\mathfrak{p}^-}$ is the lift of $\mathbb{T}_{\mathfrak{t}^-}$ to $\mathrm{D}(R)$, i.e.,*

$$\mathfrak{p} = \vec{\mathfrak{t}} \iff \vec{\mathbb{T}}_{\mathfrak{p}^-} = \vec{\mathbb{T}}_{\mathfrak{t}^-}.$$

In particular, for any torsion pair $\mathfrak{t}$ in $\mathcal{H}$, the heart of $\vec{\mathbb{T}}_{\vec{\mathfrak{t}}^-}$ is a locally coherent Grothendieck category with $\mathcal{H}_{\mathfrak{t}^-}$ as its subcategory of finitely presented objects.

Proof. We make use of the description of $\vec{\mathbb{T}}$ given in Remark 8.3. Suppose that $\vec{\mathbb{T}}_{\mathfrak{p}^-} = \vec{\mathbb{T}}_{\mathfrak{t}^-}$. Since $H^0_{\vec{\mathbb{T}}}$ sends directed homotopy colimits in $\mathrm{D}(R)$ to direct limits in $\vec{\mathcal{H}}$ (see [Saorín et al. 2023, Lemma 5.7]), it follows that an object X of $\mathrm{D}(R)$ lies in $\vec{\mathcal{T}}$ if and only if it lies in $\vec{\mathcal{H}} \cap \vec{\mathcal{X}}_{\mathfrak{t}^-}$. This latter intersection coincides by assumption with $\vec{\mathcal{X}}_{\mathfrak{p}^-} \cap \vec{\mathcal{H}}$, which is precisely $\mathcal{P}$, thus proving the desired equality.

Conversely, suppose that $\mathfrak{p} = \vec{\mathfrak{t}}$. Let X be an object in $\vec{\mathcal{X}}_{\mathfrak{t}^-}$. Then, there is a directed coherent diagram (i.e., an object of the derived category $\mathrm{D}(\mathrm{Mod}(R)^I)$ of I-shaped diagrams of R-modules) that gives rise to $(X_i)_{i\in I}$ in $\mathcal{X}^I_{\mathfrak{t}^-}$ such that $\mathbb{L}\varinjlim_{i\in I} X_i = X$. Again since $H^0_{\vec{\mathbb{T}}}$ sends directed homotopy colimits in $\mathrm{D}(R)$ to direct limits in $\vec{\mathcal{H}}$, we have that $H^0_{\vec{\mathbb{T}}}(X)$ lies in $\vec{\mathcal{T}}$, and this latter class coincides with $\mathcal{P}$ by assumption. As a consequence, since $\vec{\mathbb{T}}$ is intermediate and, thus, nondegenerate, we have that $\vec{\mathcal{X}}_{\mathfrak{t}^-} \subseteq \vec{\mathcal{X}}_{\mathfrak{p}^-}$ (see also Remark 2.2). Conversely, since both $\vec{\mathcal{T}}$ and $\vec{\mathcal{X}}$ are contained in $\vec{\mathcal{X}}_{\mathfrak{t}^-}$, it follows that $\vec{\mathcal{X}}_{\mathfrak{p}^-} = \vec{\mathcal{X}} \star \mathcal{P} = \vec{\mathcal{X}} \star \vec{\mathcal{T}}$ is contained in $\vec{\mathcal{X}}_{\mathfrak{t}^-}$. The final statement follows from Proposition 8.2. □

Informally, one of the implications of the statement above tells us that the tilt at the lifted torsion pair coincides with the lift of the tilted t-structure. In other words, the operations *lift* and *tilt*, when correctly interpreted, commute.

Example 8.5. Let us go back to the setting of Example 4.11, i.e., let R be a commutative noetherian ring and $\mathcal{D} = \mathrm{D}(R)$. From Proposition 8.2, every intermediate t-structure $\mathbb{T}$ in $\mathrm{D}^b(\mathrm{mod}(R))$ gives rise to a cosilting object σ in $\mathcal{D}$, lying in $\mathsf{K}^b(\mathsf{Inj}(R))$, such that $\vec{\mathbb{T}} = \mathbb{T}_\sigma$. It is shown in [Pavon and Vitória 2021, Corollaries 6.17 and 6.18] that the cosilting complexes σ obtained in this way are actually cotilting, and that there is a bijection between hereditary torsion pairs of finite type in $\mathcal{H}_\sigma$ and specialisation-closed subsets of $\mathrm{Spec}(R)$. In particular, if σ is associated to an sp-filtration ϕ_σ, then the right mutations of σ are precisely the cosilting objects associated to the sp-filtrations of the form ϕ_{σ_W} described in Example 4.11, for W a specialisation closed subset of $\mathrm{Spec}(R)$.

Moreover, it follows from [Pavon and Vitória 2021, Proposition 6.10 and Corollary 6.15] that every cosilting object σ obtained from a lift of an intermediate t-structure in $\mathrm{D}^b(\mathrm{mod}(R))$ (and, thus, cotilting) is an iterated right mutation of a shift of the injective cogenerator. A key to this observation is in the spirit of the proposition above: lifting and then tilting yields the same result as tilting first and then lifting.

We are now ready to establish the setup with which we will work in this section.

Setup 8.6. Let R be a left coherent ring and $\mathbb{T}$ an intermediate t-structure in $\mathrm{D}^b(\mathrm{mod}(R))$ with heart $\mathcal{H}$. Consider two torsion pairs $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ with $\mathcal{U} \subseteq \mathcal{T}$ and let us fix the notation:

- σ denotes a pure-injective cosilting object in $\mathrm{D}(R)$ such that $\vec{\mathbb{T}} = \mathbb{T}_\sigma$.
- $\mathfrak{s} = (\mathcal{S}, \mathcal{R}) = (\mathcal{T} \cap \mathcal{V}, \mathcal{F} \star \mathcal{U}[-1])$ denotes the torsion pair in $\mathcal{H}_{\mathfrak{u}^-}$ determined by $\mathfrak{u}$ and $\mathfrak{t}$ by Proposition 7.5.
- $\mathfrak{r} = (\mathcal{R}, \mathcal{S}[-1])$ denotes the tilted torsion pair of $\mathfrak{s}$ in $(\mathbb{T}_{\mathfrak{u}^-})_{\mathfrak{s}^-} = \mathbb{T}_{\mathfrak{t}^-}$.

An important example of Setup 8.6 is given by taking $\mathbb{T}$ to be the standard t-structure in $\mathrm{D}(R)$ and $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ any torsion pairs in $\mathrm{mod}(R)$ with $\mathcal{U} \subseteq \mathcal{T}$.

The following lemma makes it clear that, in our setup, Proposition 7.5 is compatible with the operations of lifting of torsion pairs and t-structures. This will be useful for us later on.

Lemma 8.7. *Suppose we are in Setup 8.6 and consider the torsion pair $\vec{\mathfrak{t}}$ and $\vec{\mathfrak{u}}$ in $\vec{\mathcal{H}}$. Then the torsion pairs $(\vec{\mathcal{T}} \cap \vec{\mathcal{V}}, \vec{\mathcal{F}} \star \vec{\mathcal{U}}[-1])$ in $\vec{\mathcal{H}}_{\vec{\mathfrak{u}}^-}$ and $(\vec{\mathcal{F}} \star \vec{\mathcal{U}}[-1], (\vec{\mathcal{T}} \cap \vec{\mathcal{V}})[-1])$ in $\vec{\mathcal{H}}_{\vec{\mathfrak{t}}^-}$ given by Proposition 7.5 coincide with $\vec{\mathfrak{s}}$ and $\vec{\mathfrak{r}}$, respectively.*

Proof. Note that, by Proposition 8.4, we have $\vec{\mathcal{H}}_{\vec{\mathfrak{u}}^-} = \vec{\mathcal{H}}_{\mathfrak{u}^-}$ and $\vec{\mathcal{H}}_{\vec{\mathfrak{t}}^-} = \vec{\mathcal{H}}_{\mathfrak{t}^-}$. By Proposition 7.5, the torsion pair for which a right HRS-tilt allows us to pass from $\vec{\mathcal{H}}_{\mathfrak{u}^-}$ to $\vec{\mathcal{H}}_{\mathfrak{t}^-}$ is uniquely determined as the torsion pair $\mathfrak{p} := (\vec{\mathcal{T}} \cap \vec{\mathcal{V}}, \vec{\mathcal{F}} \star \vec{\mathcal{U}}[-1])$. On the other hand, it follows from Proposition 8.4 that this torsion pair must be $\vec{\mathfrak{s}}$. An analogous argument holds for the equality $(\vec{\mathcal{F}} \star \vec{\mathcal{U}}[-1], (\vec{\mathcal{T}} \cap \vec{\mathcal{V}})[-1]) = \vec{\mathfrak{r}}$ in $\vec{\mathbb{T}}_{t^-}$. □

In the context of the lemma above, when $\vec{\mathfrak{t}}$ is a right mutation of $\vec{\mathfrak{u}}$, the torsion pair $\vec{\mathfrak{s}}$ is a hereditary torsion pair of finite type, as shown in Theorem 7.8 and Proposition 4.5. We will make use of the close relationship between such torsion pairs and the spectrum of locally coherent Grothendieck categories, which we summarise in the next theorem.

Theorem 8.8 [Krause 1997; Herzog 1997]. *Let $\mathcal{A}$ be a locally coherent Grothendieck category $\mathcal{A}$. The (isoclasses of) indecomposable injective objects form a topological space*, $\operatorname{Spec}(\mathcal{A})$, *with a basis of open subsets given by sets of the form*

$$\mathcal{O}(C) = \{E \in \operatorname{Spec}(\mathcal{A}) \mid \operatorname{Hom}_{\mathcal{H}}(C, E) \neq 0\}, \quad C \in \operatorname{fp}\mathcal{A}.$$

There are bijections between

(a) *hereditary torsion pairs of finite type in $\mathcal{A}$,*

(b) *Serre subcategories of* $\operatorname{fp}\mathcal{A}$, *and*

(c) *open subsets of* $\operatorname{Spec}(\mathcal{A})$.

The bijection between (a) *and* (b) *is given by the assignments $(\mathcal{S}, \mathcal{R}) \mapsto \mathcal{S} \cap \operatorname{fp}\mathcal{A}$ and $\mathcal{L} \mapsto ({}^{\perp_0}(\mathcal{L}^{\perp_0}), \mathcal{L}^{\perp_0})$. The assignment* (b) → (c) *takes a Serre subcategory $\mathcal{L}$ to $\mathcal{O} = \{E \in \operatorname{Spec}(\mathcal{A}) \mid E \notin \mathcal{L}^{\perp_0}\}$. The assignment* (c) → (a) *maps an open set $\mathcal{O}$ to the hereditary torsion pair $(\mathcal{S}, \mathcal{R})$ cogenerated by the complement $\mathcal{O}^c$.*

Let us come back to Setup 8.6. We are now in a position to show that, if the lifted torsion pairs $\vec{\mathfrak{t}}$ and $\vec{\mathfrak{u}}$ in $\vec{\mathcal{H}}$ are related by mutation, then this mutation is controlled by objects of $\mathcal{H} = \operatorname{fp}\vec{\mathcal{H}}$.

Theorem 8.9. *Suppose we are in Setup 8.6. The following statements are equivalent*:

(1) $\vec{\mathfrak{t}}$ *is a right mutation of* $\vec{\mathfrak{u}}$.

(2) *$\mathcal{S}$ is a wide subcategory of $\mathcal{H}$.*

(3) *If $g : T \to V$ is a map in $\mathcal{H}$ with T in $\mathcal{T}$ and V in $\mathcal{V}$, then* $\operatorname{Ker}(g)$ *lies in $\mathcal{T}$ and* $\operatorname{Coker}(g)$ *lies in $\mathcal{V}$.*

Proof. (1) ⇒ (2): From Theorem 7.8 and Lemma 8.7, the class $\vec{\mathcal{S}}$ is a wide subcategory in $\vec{\mathcal{H}}$. We prove that $\mathcal{S} = \vec{\mathcal{S}} \cap \mathcal{H}$, thus showing that $\mathcal{S}$ is a wide subcategory of $\mathcal{H}$. We have $\mathcal{S} = \mathcal{V} \cap \mathcal{T} = \vec{\mathcal{V}} \cap \vec{\mathcal{T}} \cap \mathcal{H}$. The latter class coincides with $\vec{\mathcal{V}} \cap \vec{\mathcal{T}} \cap \mathrm{D}^b(\operatorname{mod}(R))$ by Proposition 8.2 and, moreover, from Lemma 8.7, we have that it equals $\vec{\mathcal{S}} \cap \mathrm{D}^b(\operatorname{mod}(R))$. Using Proposition 8.2 again, we conclude our claim.

(2) ⇒ (1): Both hearts $\mathcal{H}$ and $\mathcal{H}_{\mathfrak{u}^-}$ in $\mathrm{D}^b(\operatorname{mod}(R))$ contain $\mathcal{W}$. By Lemma 7.6(3), our assumption implies that $\mathcal{W} = \mathcal{S} \cap \mathcal{H}_{\mathfrak{u}^-}$ is a wide subcategory of $\mathcal{H}_{\mathfrak{u}^-}$. In fact, it is even a Serre subcategory: it is closed under quotients in $\mathcal{H}_{\mathfrak{u}^-}$ because $\mathcal{S}$ is a torsion class, and so it is also closed under subobjects. By Theorem 8.8 we then have that $\vec{\mathfrak{s}}$ is a hereditary torsion pair of finite type and, therefore, by Lemma 8.7 and Theorem 7.8, we conclude that $\vec{\mathfrak{t}}$ is a right mutation of $\vec{\mathfrak{u}}$.

(2) ⇔ (3): This is an immediate consequence of Proposition 7.9. □

9. Mutation and simple objects

In this section we specialise the results of Section 8 to the case where the heart $\mathcal{H}$ (of an intermediate t-structure in $D^b(\text{mod}(R))$, R left coherent) is a length category. These hearts are known to occur frequently when R is an artinian ring. In this setting we will show that mutation is controlled by simple objects.

9.1. ***Abelian length categories.*** Recall that an abelian category $\mathcal{A}$ is called a *length category* if $\text{filt}(\mathcal{S}) = \mathcal{A}$, where $\mathcal{S}$ is the set of simple objects in $\mathcal{A}$. A Grothendieck category $\mathcal{G}$ is called *locally finite* if it has a set of finite-length generators. Recall that an object is of finite length if and only if it is both noetherian and artinian. By [Popescu 1973, Proposition 8.2] we have that $\mathcal{G}$ is locally finite if and only if $\text{fp}\,\mathcal{G}$ is a length category if and only if $\mathcal{G}$ is locally noetherian and $\text{Filt}(\Omega) = \mathcal{G}$, where Ω is the set of simple objects in $\mathcal{G}$. In particular, if in Setup 8.6 $\mathcal{H}$ is a length category, Proposition 8.1 tells us that every torsion pair of finite type in $\vec{\mathcal{H}}$ is of the form $\vec{\mathfrak{v}}$ for a torsion pair $\mathfrak{v}$ in $\mathcal{H}$. We will replace Setup 8.6 with the following.

Setup 9.1. (= Setup 8.6 + $\mathcal{H}$ length category) Let R be a left coherent ring and $\mathbb{T}$ an intermediate t-structure in $D^b(\text{mod}(R))$ whose heart $\mathcal{H}$ is a length category. Consider two torsion pairs $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$ with $\mathcal{U} \subseteq \mathcal{T}$ and let us fix the notation:

- σ denotes a pure-injective cosilting object in $D(R)$ such that $\vec{\mathbb{T}} = \mathbb{T}_\sigma$.
- $\mathfrak{s} = (\mathcal{S}, \mathcal{R}) = (\mathcal{T} \cap \mathcal{V}, \mathcal{F} \star \mathcal{U}[-1])$ denotes the torsion pair in $\mathcal{H}_{\mathfrak{u}^-}$ determined by $\mathfrak{u}$ and $\mathfrak{t}$ by Proposition 7.5.
- $\mathfrak{r} = (\mathcal{R}, \mathcal{S}[-1])$ denotes the tilted torsion pair of $\mathfrak{s}$ in $(\mathbb{T}_{\mathfrak{u}^-})_{\mathfrak{s}^-} = \mathbb{T}_{\mathfrak{t}^-}$.

An important example of Setup 9.1 is given by taking $\mathbb{T}$ to be the standard t-structure in $D^b(\text{mod}(R))$, with R being artinian, and any pair of torsion pairs $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\text{mod}(R)$ with $\mathcal{U} \subseteq \mathcal{T}$.

Remark 9.2. In view of Proposition 8.1, mutation of torsion pairs in $\text{Cosilt}_*(\vec{\mathcal{H}})$ admits an interpretation inside the lattice $\text{tors}(\mathcal{H})$ of torsion classes in $\mathcal{H}$ with partial order given by inclusion. We refer to [Demonet et al. 2023; Asai 2020; Barnard et al. 2019; Asai and Pfeifer 2022] for details about the lattice structure of $\text{tors}(\mathcal{H})$.

Following [Asai and Pfeifer 2022, Section 6], we will say that $\mathfrak{t}$ *is a right mutation of* $\mathfrak{u}$ (and $\mathfrak{u}$ a left mutation of $\mathfrak{t}$) when we are in the situation of Theorem 8.9. Note also that condition (2) in that theorem, in the terminology of [loc. cit.], states that the interval $[\mathcal{U}, \mathcal{T}]$ is a wide interval of $\text{tors}(\mathcal{H})$.

The following well-known theorem due to Ringel tells us that every object in a wide subcategory $\mathcal{W}$ of $\mathcal{H}$ admits a finite filtration by simple objects of $\mathcal{W}$. This is the point of view from which wide intervals are studied in [loc. cit.]. An object X in $\mathcal{H}$ is called a *brick* if $\text{End}_{\mathcal{H}}(X)$ is a skew-field. A collection of bricks Ω in $\mathcal{H}$ is called a *semibrick* if $\text{Hom}_{\mathcal{H}}(S, S') = 0$ whenever S and S' are in Ω and $S \neq S'$.

Theorem 9.3 [Ringel 1976]. *Let $\mathcal{A}$ be a length category. If we assign to a wide subcategory $\mathcal{W}$ of $\mathcal{A}$ the set $\mathcal{M}$ of its simple objects, we obtain a semibrick $\mathcal{M}$ such that $\mathcal{W} = \text{filt}(\mathcal{M})$. This yields a one-one correspondence between wide subcategories and semibricks in $\mathcal{A}$.*

When a wide subcategory arises as in Theorem 8.9(2), it is possible to characterise its simple objects more precisely, and we do so in Lemma 9.6. First we need the following definition.

Definition 9.4 [Angeleri Hügel et al. 2024]. Let $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ be torsion pair in an abelian category $\mathcal{H}$. We say that a nonzero object M in $\mathcal{V}$ is *almost torsion* (*for* $\mathfrak{u}$) if the following conditions are satisfied:

(i) All proper quotients of M are in $\mathcal{U}$.

(ii) For all short exact sequences $0 \to M \to Y \to Z \to 0$, with Y in $\mathcal{V}$, we have that Z lies in $\mathcal{V}$.

Almost torsion-free objects for $\mathfrak{u}$ are defined dually.

Remark 9.5. (1) If $\mathcal{H}$ is a Grothendieck category and $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ is a hereditary torsion pair in $\mathcal{H}$, it is well known that the right adjoint of the localisation functor $\mathcal{H} \to \mathcal{H}/\mathcal{U}$ establishes an equivalence between the Serre quotient and the subcategory $\mathcal{U}^{\perp_{0,1}}$ of $\mathcal{H}$. It follows that the torsion-free almost torsion objects in $\mathcal{H}$ (for $\mathfrak{u}$) are precisely the simple objects in $\mathcal{U}^{\perp_{0,1}}$ (see [Angeleri Hügel et al. 2024, Example 3.3], where this is written for module categories; the same proof holds for Grothendieck categories).

(2) [Angeleri Hügel et al. 2024; Rapa 2019, Theorem 2.3.6] If $\mathcal{H}$ is the heart of a t-structure in an arbitrary triangulated category, then an object M in $\mathcal{V}$ is almost torsion if and only if M becomes a (torsion) simple object in the tilted heart $\mathcal{H}_{\mathfrak{u}^-} = \mathcal{V} \star \mathcal{U}[-1]$. The almost torsion-free objects for $\mathfrak{u}$ are precisely those N in $\mathcal{U}$ for which $N[-1]$ becomes a simple object in $\mathcal{H}_{\mathfrak{u}^-}$.

(3) [Sentieri 2023, Section 2] Let A be a finite-dimensional algebra and $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$ a torsion pair in $\mathrm{mod}(A)$ with lifted torsion pair $\vec{\mathfrak{u}}$ in $\mathrm{Mod}(A)$. The finite-dimensional torsion-free, almost torsion modules for $\vec{\mathfrak{u}}$ coincide with the torsion-free, almost torsion modules for $\mathfrak{u}$ and are precisely the minimal extending modules defined in [Barnard et al. 2019]. Moreover, all torsion, almost torsion-free modules for $\vec{\mathfrak{u}}$ are finite-dimensional and coincide with the torsion, almost torsion-free modules for $\mathfrak{u}$, that is, with the minimal coextending modules from [Barnard et al. 2019].

(4) It is easy to check that the arguments in [Sentieri 2023] yield the same results for locally finite categories. In particular, in the situation of Setup 9.1, every object M which is torsion-free, almost torsion for $\mathfrak{u}$ becomes a simple object in $\vec{\mathcal{H}}_{\mathfrak{u}^-}$, and every object N which is torsion, almost torsion-free for $\mathfrak{t}$ gives rise to a simple object $N[-1]$ in $\vec{\mathcal{H}}_{\mathfrak{t}^-}$.

Lemma 9.6. *Suppose we are in Setup 9.1. If* $\mathfrak{t}$ *is a right mutation of* $\mathfrak{u}$, *then the following statements are equivalent for an object* B *in* $\mathcal{H}$:

(1) *B is contained in the semibrick associated to the wide subcategory $\mathcal{S}$ of $\mathcal{H}$.*

(2) *B is a torsion-free, almost torsion object for* $\mathfrak{u}$ *that belongs to* $\mathcal{T}$.

(3) *B is a torsion, almost torsion-free object for* $\mathfrak{t}$ *that belongs to* $\mathcal{V}$.

Proof. Let $\mathcal{M}$ denote the semibrick associated to $\mathcal{S}$, that is, the set of simple objects of $\mathcal{S}$. We show the equivalence of (1) and (2). The equivalence of (1) and (3) uses a dual argument.

We begin by showing that every B in $\mathcal{M}$ is torsion-free, almost torsion for $\mathfrak{u}$. Let $N \cong B/K$ be a proper factor of B; we wish to show that N lies in $\mathcal{U}$ and hence condition (i) from Definition 9.4 holds.

Let U_N and V_N be objects in $\mathcal{U}$ and $\mathcal{V}$ such that there is a short exact sequence $0 \to U_N \to N \to V_N \to 0$. First suppose that $U_N \neq 0$ and consider the pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & U_N & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{h} & B & \xrightarrow{l} & N & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & \downarrow & & \\
 & & & & V_N & = & V_N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By condition (3) of Theorem 8.9 applied to f, we have that X lies in $\mathcal{S}$. Since B is a simple object in $\mathcal{S}$, the morphism g is an isomorphism and so $V_N = 0$. That is, we have $N = U_N$, which lies in $\mathcal{U}$. Now suppose that $U_N = 0$ and so N lies in $\mathcal{V}$. We may apply condition (3) of Theorem 8.9 to l and so we have that K lies in $\mathcal{S}$. Then h is an isomorphism; that is, we have $N = 0$. Condition (ii) in Definition 9.4 follows immediately from statement (3) of Theorem 8.9.

Conversely, if B in $\mathcal{H}$ is torsion-free, almost torsion for $\mathfrak{u}$ and belongs to $\mathcal{T}$, then certainly B lies in $\mathcal{S}$, and every nonzero subobject K in $\mathcal{S}$ of B satisfies that B/K lies in $\mathcal{S} \subseteq \mathcal{V}$, but also in $\mathcal{U}$ by condition (i) in Definition 9.4; hence $K = B$. This shows that B is a simple object of $\mathcal{S}$; hence it belongs to $\mathcal{M}$. □

Let $\mathcal{A}$ be a Grothendieck category, and let Ω be the set of isoclasses of simple objects in $\mathcal{A}$. Given a subset Ω' of Ω, we consider the torsion pair generated by Ω'. It has the shape $(\mathrm{Filt}(\Omega'), (\Omega')^{\perp_0})$; see [Stenström 1975, Proposition VIII.3.2]. Moreover, it is hereditary because, for every simple S in Ω and every object M in $\mathcal{G}$, there exists a nonzero map $S \to E(M)$ if and only if S embeds in M, and so $(\Omega')^{\perp_0}$ is closed under injective envelopes. By [loc. cit., Lemma VIII.2.4], the torsion pairs generated by subsets of Ω are precisely the hereditary torsion pairs of the form $(\mathcal{S}, \mathcal{R})$ with $\mathcal{S} \subseteq \mathrm{Filt}(\Omega)$; we call such a pair a *simple torsion pair*. Observe that an object M is contained in $\mathrm{Filt}(\Omega)$ if and only if every nonzero quotient of M has a nonzero socle; see [loc. cit., Proposition VII.2.5]. For the sake of the next result, we will say that a TTF class $\mathcal{F}$ is a *simple TTF class* if $\mathcal{F} = \mathrm{Filt}(\Omega')$ for some set $\Omega' \subseteq \Omega$.

Theorem 9.7. *Suppose we are in Setup 9.1. Let $\mathcal{M}$ be the set of all torsion-free, almost torsion objects for $\mathfrak{u}$ which belong to $\mathcal{T}$. Let $\mathcal{N}$ be the set of all torsion, almost torsion-free objects for $\mathfrak{t}$ which belong to $\mathcal{V}$. The following statements are equivalent*:

(1) $\mathfrak{t}$ *is a right mutation of* $\mathfrak{u}$.

(2) $\mathcal{S} = \mathrm{filt}(\mathcal{M})$ *in* $\mathcal{H}$.

(3) $\mathcal{S} = \mathrm{filt}(\mathcal{N})$ *in* $\mathcal{H}$.

(4) *The pair* $\vec{\mathfrak{s}}$ *is a simple (hereditary) torsion pair in* $\vec{\mathcal{H}}_{\mathfrak{u}^-}$.

(5) *The class* $\vec{\mathcal{S}}[-1]$ *is a simple TTF class in* $\vec{\mathcal{H}}_{\mathfrak{t}^-}$.

Proof. The equivalence of the first three statements follows immediately from Theorem 8.9, Theorem 9.3 and Lemma 9.6.

(2) $\Rightarrow$ (4): First observe that, since $\mathcal{S}$ is extension-closed, condition (2) implies that $\mathcal{S}$ also coincides with $\mathrm{filt}(\mathcal{M})$ in $\mathcal{H}_{\mathfrak{u}^-}$. Hence, we have that $\vec{\mathcal{R}} = \mathcal{S}^{\perp_0} = \mathcal{M}^{\perp_0}$ in $\vec{\mathcal{H}}_{\mathfrak{u}^-}$, and the latter is the torsion-free class in a simple hereditary torsion pair in $\vec{\mathcal{H}}_{\mathfrak{u}^-}$ by Remark 9.5. Thus, $\vec{\mathfrak{s}}$ is indeed a simple torsion pair in $\vec{\mathcal{H}}_{\mathfrak{u}^-}$.

(4) $\Rightarrow$ (5): We know from Theorem 7.8 that $\vec{\mathcal{S}}[-1]$ is a TTF class in $\vec{\mathcal{H}}_{\mathfrak{t}^-}$. By assumption, there exists a set of simple objects Ω' in $\vec{\mathcal{H}}_{\mathfrak{u}^-}$ such that $\vec{\mathcal{S}} = \mathrm{Filt}(\Omega')$. It follows easily from the definitions that the torsion, almost torsion-free objects for a hereditary torsion pair coincide with the torsion simple objects. Thus, the objects in $\vec{\mathcal{S}}$ that are almost torsion-free coincide with $\mathcal{M}$. By Remark 9.5, we have that the objects $\Omega'[-1]$ are simple in $\vec{\mathcal{H}}_{\mathfrak{t}^-} = (\vec{\mathcal{H}}_{\mathfrak{u}^-})_{\vec{\mathfrak{s}}^-}$. Since, considering the subcategory $\mathrm{Filt}(\Omega')$ of $\vec{\mathcal{H}}_{\mathfrak{u}^-}$ and the subcategory $\mathrm{Filt}(\Omega'[-1])$ of $\vec{\mathcal{H}}_{\mathfrak{t}^-}$, we have

$$\vec{\mathcal{S}}[-1] = (\mathrm{Filt}(\Omega'))[-1] = \mathrm{Filt}(\Omega'[-1])$$

and, thus, $\vec{\mathcal{S}}[-1]$ is a simple TTF class in $\vec{\mathcal{H}}_{\mathfrak{t}^-}$.

(5) $\Rightarrow$ (1): This implication is immediate by Theorem 7.8. □

Generalising results from [Ingalls and Thomas 2009] again, we obtain certain "distinguished" mutations of a torsion pair in a length category.

Lemma 9.8. *Suppose $\mathcal{H}$ is a length category with a torsion pair $\mathfrak{v} = (\mathcal{X}, \mathcal{Y})$.*

(1) *There is a torsion pair $\check{\mathfrak{v}} = (\check{\mathcal{X}}, \check{\mathcal{Y}})$ in $\mathcal{H}$ such that*

$$\check{\mathcal{X}} = \{X \in \mathcal{H} \mid \textit{every } f \in \mathrm{Hom}_{\mathcal{H}}(X, Y) \textit{ with } Y \in \mathcal{Y} \textit{ has } \mathrm{Coker}(f) \in \mathcal{Y}\}.$$

(2) *There is a torsion pair $\hat{\mathfrak{v}} = (\widehat{\mathcal{X}}, \widehat{\mathcal{Y}})$ in $\mathcal{H}$ such that*

$$\widehat{\mathcal{Y}} = \{Y \in \mathcal{H} \mid \textit{every } f \in \mathrm{Hom}_{\mathcal{H}}(X, Y) \textit{ with } X \in \mathcal{X} \textit{ has } \mathrm{Ker}(f) \in \mathcal{X}\}.$$

Moreover, we have $\widehat{\mathcal{X}} \subseteq \mathcal{X} \subseteq \check{\mathcal{X}}$ and both $\check{\mathcal{X}} \cap \mathcal{Y}$ and $\mathcal{X} \cap \widehat{\mathcal{Y}}$ are wide subcategories of $\mathcal{H}$.

Proof. We prove only (1) and one half of the final statements. The others follow by a dual analogous argument.

It is easy to check that $\mathcal{X} \subseteq \check{\mathcal{X}}$ and $\check{\mathcal{X}}$ is closed under quotients. To see that $\check{\mathcal{X}}$ is closed under extensions, consider an exact sequence $0 \to X_1 \to X \to X_2 \to 0$ in $\mathcal{H}$ such that $X_1, X_2 \in \check{\mathcal{X}}$ and consider further a homomorphism $f : X \to Y$ with $Y \in \mathcal{Y}$. If we denote by Y_1 the image of the composition $X_1 \to X \to Y$ and by Y_2 the cokernel of the same map, we obtain a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & X & \longrightarrow & X_2 & \longrightarrow & 0\\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y & \longrightarrow & Y_2 & \longrightarrow & 0\\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{Coker}(f) & \longrightarrow & C_2 & \longrightarrow & 0
\end{array}$$

Now $Y_2 \in \mathcal{Y}$ since $X_1 \in \check{\mathcal{X}}$, and $C_2 \in \mathcal{Y}$ since $X_2 \in \check{\mathcal{X}}$. This shows that $\mathrm{Coker}(f) \in \mathcal{Y}$ and, hence, $X \in \check{\mathcal{X}}$. As $\check{\mathcal{X}} \subseteq \mathcal{H}$ is closed under quotients and extensions in the length category $\mathcal{H}$, it is a torsion class in $\mathcal{H}$. Finally, $\check{\mathcal{X}} \cap \mathcal{Y}$ is a wide subcategory of $\mathcal{H}$ by the argument for [Ingalls and Thomas 2009, Proposition 2.12]. □

Combining the last two results with those from [Asai and Pfeifer 2022] allows us to describe all right or left mutations of a torsion pair in terms of almost torsion objects or almost torsion-free objects, respectively. Moreover, we identify the "distinguished" mutations from Lemma 9.8 as "extremal" mutations of the given torsion pair.

Corollary 9.9. *Suppose we are in Setup 9.1.*

(1) *Let $\mathcal{M}$ be a representative set of isomorphism classes of torsion-free, almost torsion objects for $\mathfrak{u}$. Then the right mutations of $\mathfrak{u}$ bijectively correspond to subsets of $\mathcal{M}$. In particular, $\mathfrak{u}$ admits a proper right mutation if and only if there are torsion-free, almost torsion objects for $\mathfrak{u}$. Moreover, if $\mathfrak{t}$ is a right mutation of $\mathfrak{u}$, then $\mathcal{U} \subseteq \mathcal{T} \subseteq \check{\mathcal{U}}$.*

(2) *Let $\mathcal{N}$ be a representative set of isomorphism classes of torsion, almost torsion-free objects for $\mathfrak{t}$. Then the left mutations of $\mathfrak{t}$ bijectively correspond to subsets of $\mathcal{N}$. In particular, $\mathfrak{t}$ admits a proper left mutation if and only if there are torsion, almost torsion-free objects for $\mathfrak{t}$. Moreover, if $\mathfrak{u}$ is a left mutation of $\mathfrak{t}$, then $\widehat{\mathcal{T}} \subseteq \mathcal{U} \subseteq \mathcal{T}$.*

Proof. We prove (1); the argument for (2) is dual. To start with, note that $\mathcal{M}$ is a semibrick by Remark 9.5(1) (as it is a set of pairwise nonisomorphic simple objects in $\mathcal{H}_{\mathfrak{u}^-}$), and so is any subset of $\mathcal{M}$.

The assignment between right mutations and subsets of $\mathcal{M}$ can be described as follows. Given a right mutation $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ of $\mathfrak{u} = (\mathcal{U}, \mathcal{V})$, the intersection $\mathcal{S} = \mathcal{T} \cap \mathcal{V}$ is a wide subcategory of $\mathcal{H}$ by Theorem 8.9, and so is of the form $\mathcal{S} = \mathrm{filt}(\mathcal{M}')$ for a unique subset $\mathcal{M}' \subseteq \mathcal{M}$ by Theorem 9.7 (recall that simply $\mathcal{M}' = \mathcal{M} \cap \mathcal{S}$ by Theorem 9.3). This assignment is injective since one can recover $\mathcal{T}$ from $\mathcal{M}'$ as $\mathcal{T} = \mathcal{U} \star \mathrm{filt}(\mathcal{M}')$; see Proposition 7.3.

On the other hand, if $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{V}$ is any subset, then $\mathcal{M}' \subseteq \check{\mathcal{U}}$ by the very definition of almost torsion objects for $\mathfrak{u}$. Following [Asai and Pfeifer 2022, §6], we denote by $\mathcal{W}_r(\mathcal{V}) := \check{\mathcal{U}} \cap \mathcal{V}$ the wide subcategory of $\mathcal{H}$ obtained by applying Lemma 9.8 to $\mathfrak{u}$. Note that $\mathcal{W}_r(\mathcal{V})$ is also a wide subcategory of $\mathcal{H}_{\mathfrak{u}^-}$ by Lemma 7.6; hence $\mathcal{M}'$ becomes a set of simple objects in $\mathcal{W}_r(\mathcal{V})$. As $\mathcal{W}_r(\mathcal{V})$ is necessarily an abelian length category, $\mathcal{S} := \mathrm{filt}(\mathcal{M}')$ is torsion class of a hereditary torsion pair in $\mathcal{W}_r(\mathcal{V})$. Now it follows from [loc. cit., Theorems 4.2 and 6.6] that $\mathcal{T} := \mathcal{U} \star \mathcal{S}$ is a torsion class in $\mathcal{H}$ such that $\mathcal{V} \cap \mathcal{T} = \mathcal{S} = \mathrm{filt}(\mathcal{M}')$. Hence, the assignment from the previous paragraph is also surjective. □

9.2. ***Irreducible mutations.*** In this final subsection we consider irreducible mutations of torsion pairs. The notation $\mathrm{Ind}(\sigma)$ and $\mathrm{Ind}(\sigma')$ is used for the isoclasses of indecomposable objects in $\mathrm{Prod}(\sigma)$ and $\mathrm{Prod}(\sigma')$ respectively.

Definition 9.10. Suppose σ and σ' are pure-injective cosilting objects in a compactly generated triangulated category $\mathcal{D}$ and let $\mathscr{E}$ be the class $\mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$. Suppose σ' is a right mutation of σ or, equivalently, that σ is a left mutation of σ' (see Corollary 3.7). We will say that σ' is an *irreducible right mutation* of σ if $|\mathrm{Ind}(\sigma) \setminus \mathrm{Ind}(\mathscr{E})| = 1$. We will say that σ is an *irreducible left mutation* of σ' if $|\mathrm{Ind}(\sigma') \setminus \mathrm{Ind}(\mathscr{E})| = 1$.

Recall that, by Proposition 4.13, there is a bijection between $\mathrm{Ind}(\sigma) \setminus \mathrm{Ind}(\mathscr{E})$ and $\mathrm{Ind}(\sigma') \setminus \mathrm{Ind}(\mathscr{E})$ and so σ' is an irreducible right mutation of σ if and only if σ is an irreducible left mutation of σ'.

Notation. Within Setup 9.1, we fix some further notation that we use in the remainder of the section.

- Denote by $\sigma_{\mathfrak{u}}$ and $\sigma_{\mathfrak{t}}$ the cosilting objects in $\mathrm{D}(R)$ such that $\mathcal{H}_{\sigma_{\mathfrak{u}}} = \vec{\mathcal{H}}_{\mathfrak{u}^-}$ and $\mathcal{H}_{\sigma_{\mathfrak{t}}} = \vec{\mathcal{H}}_{\mathfrak{t}^-}$.
- In the case where $\mathfrak{t}$ is a right mutation of $\mathfrak{u}$, let $\mathcal{M}$ be the semibrick associated to $\mathcal{S}$. For each M in $\mathcal{M}$, we know from Remark 9.5 and Lemma 9.6 that
  - M is a simple object in $\vec{\mathcal{H}}_{\mathfrak{u}^-}$; we denote by σ_M the object in $\mathrm{Ind}(\sigma_{\mathfrak{u}})$ such that $H^0_{\sigma_{\mathfrak{u}}}(\sigma_M)$ is the injective envelope of M in the locally coherent Grothendieck category $\mathcal{H}_{\sigma_{\mathfrak{u}}}$.
  - $M[-1]$ is a simple object in $\vec{\mathcal{H}}_{\mathfrak{t}^-}$ and we similarly denote by $\sigma_{M[-1]}$ the object in $\mathrm{Ind}(\sigma_{\mathfrak{t}})$ such that $H^0_{\mathfrak{t}^-}(\sigma_{M[-1]})$ is the injective envelope of $M[-1]$ in the locally coherent Grothendieck category $\mathcal{H}_{\sigma_{\mathfrak{t}}}$.

Lemma 9.11. *Suppose we are in Setup 9.1 and that $\mathfrak{t}$ is a right mutation of $\mathfrak{u}$. Let $\sigma_{\mathfrak{u}}$ and $\sigma_{\mathfrak{t}}$ be a cosilting object associated to $\mathfrak{u}$ and $\mathfrak{t}$ respectively and consider $\mathscr{E} = \mathrm{Prod}(\sigma_{\mathfrak{u}}) \cap \mathrm{Prod}(\sigma_{\mathfrak{t}})$. Then*

(1) *the set $\mathrm{Ind}(\sigma_{\mathfrak{u}}) \setminus \mathrm{Ind}(\mathscr{E})$ coincides with $\{\sigma_M \mid M \in \mathcal{M}\}$, and*

(2) *the set $\mathrm{Ind}(\sigma_{\mathfrak{t}}) \setminus \mathrm{Ind}(\mathscr{E})$ coincides with $\{\sigma_{M[-1]} \mid M \in \mathcal{M}\}$.*

Proof. (1) By Theorems 3.5(2) and 9.7, we have that

$$(\vec{\mathcal{S}}, \vec{\mathcal{R}}) = ({}^{\perp_0} H^0_{\sigma_{\mathfrak{u}}}(\mathscr{E}), \mathrm{Cogen}(H^0_{\sigma_{\mathfrak{u}}}(\mathscr{E}))) = (\mathrm{Filt}(\mathcal{M}), \mathcal{M}^{\perp_0})$$

is a hereditary torsion pair of finite type in $\mathcal{H}_{\sigma_{\mathfrak{u}}}$. By Theorem 8.8

$$\mathcal{O} := \{E \in \mathrm{Spec}(\mathcal{H}_{\sigma_{\mathfrak{u}}}) \mid E \notin \mathcal{M}^{\perp_0}\}$$

is the associated open set in $\mathrm{Spec}(\mathcal{H}_{\sigma_{\mathfrak{u}}})$, and it clearly consists of the injective envelopes of the simple objects from $\mathcal{M}$. In other words, we have that $\mathcal{O} = \{H^0_{\sigma_{\mathfrak{u}}}(\sigma_M) \mid M \in \mathcal{M}\}$. Since $H^0_{\sigma_{\mathfrak{u}}}(\mathscr{E})$ is the class of torsion-free injective objects of $\mathcal{H}_{\sigma_{\mathfrak{u}}}$, it follows that $H^0_{\sigma_{\mathfrak{u}}}$ induces a bijection between $\mathrm{Ind}(\sigma_{\mathfrak{u}}) \setminus \mathrm{Ind}(\mathscr{E})$ and $\mathcal{O}$, and the claim is proven.

(2) It follows from the proof of Theorem 9.7, Remark 9.5 and the fact that direct limits in both $\vec{\mathcal{H}}_{\mathfrak{t}^-}$ and $\vec{\mathcal{H}}_{\mathfrak{u}^-}$ are directed homotopy colimits (see [Saorín et al. 2023, Corollary 5.8]) that $\overrightarrow{\mathcal{S}[-1]} = \vec{\mathcal{S}}[-1] = \mathrm{Filt}(\mathcal{N}[-1])$ in $\mathcal{H}_{\mathfrak{t}^-}$. Recall that, by Lemma 8.7, $\vec{\mathbb{T}}_{\mathfrak{u}^-}$ is the left HRS-tilt of $\vec{\mathbb{T}}_{\mathfrak{t}^-}$ at the torsion pair $\vec{\mathfrak{r}} = (\vec{\mathcal{R}}, \overrightarrow{\mathcal{S}[-1]})$. Since $\mathfrak{u}$ is a left mutation of $\mathfrak{t}$, it follows from Theorem 3.5(1) that $\overrightarrow{\mathcal{S}[-1]} = {}^{\perp_0} H^0_{\sigma_{\mathfrak{t}}}(\mathscr{E})$. The same arguments as (1) yield that $\mathrm{Ind}(\sigma_{\mathfrak{t}}) \setminus \mathrm{Ind}(\mathscr{E})$ coincides with $\{\sigma_{M[-1]} \mid M \in \mathcal{M}\}$. □

Remark 9.12. It follows from Lemma 9.11 that right mutation of a torsion pair $\mathfrak{u}$ within Setup 9.1 consists of removing indecomposable summands of an associated cosilting object $\sigma_{\mathfrak{u}}$ in $\mathrm{D}(R)$ and replacing them with new ones. Indeed, since σ_M corresponds to the injective envelope of a simple object in $\mathcal{H}_{\sigma_{\mathfrak{u}}}$ for every M in $\mathcal{M}$, it follows that σ_M is a direct summand of every cosilting object equivalent to $\sigma_{\mathfrak{u}}$. Similarly, for every M in $\mathcal{M}$, the indecomposable object $\sigma_{M[-1]}$ is a direct summand of every cosilting object corresponding to $\mathfrak{t}$.

As a corollary of the results above we are able to characterise minimal inclusions of torsion classes in length hearts $\mathcal{H}$ (as in Setup 9.1) in terms of irreducible mutations of the associated cosilting objects. Recall that if the inclusion $\mathcal{U} \subseteq \mathcal{T}$ is proper, it is said to be a *minimal inclusion of torsion classes* if for any other torsion class $\mathcal{X}$ of $\mathcal{H}$, if $\mathcal{U} \subseteq \mathcal{X} \subseteq \mathcal{T}$ then either $\mathcal{U} = \mathcal{X}$ or $\mathcal{T} = \mathcal{X}$.

Remark 9.13. If in Setup 9.1 we have $\mathcal{H} = \mathrm{mod}(A)$ for a finite-dimensional algebra A, it is shown in [Barnard et al. 2019, Theorem 2.8] that if $\mathcal{U} \subseteq \mathcal{T}$ is a minimal inclusion of torsion classes, then $\mathcal{T}$ can be built by adjoining to $\mathcal{U}$ an indecomposable module satisfying certain properties. This module turns out to be precisely the unique torsion-free, almost torsion module for the torsion pair $\mathfrak{u}$, as shown in [Sentieri 2023]. It can easily be checked that these arguments hold also for an arbitrary length category $\mathcal{H}$, by replacing the notion of dimension by length where necessary.

Corollary 9.14. *Suppose we are in Setup 9.1. The following statements are equivalent*:

(1) $\sigma_{\mathfrak{t}}$ *is an irreducible right mutation of* $\sigma_{\mathfrak{u}}$.

(2) *The class* $\mathcal{S}$ *coincides with* $\mathrm{filt}(M)$ *for a brick* M *in* $\mathcal{H}$.

(3) *The inclusion* $\mathcal{U} \subseteq \mathcal{T}$ *is a minimal inclusion of torsion classes.*

Proof. (2) $\Rightarrow$ (1): By Theorem 9.3, we have that $\mathcal{S}$ is a wide subcategory of $\mathcal{H}$ and so, by Theorem 8.9, we have that $\sigma_{\mathfrak{t}}$ is a right mutation of $\sigma_{\mathfrak{u}}$. It follows from Lemma 9.6 that $\mathcal{M} = \{M\}$. By Lemma 9.11, we have that $\sigma_{\mathfrak{t}}$ is an irreducible right mutation of $\sigma_{\mathfrak{u}}$.

(3) $\Rightarrow$ (2): By Remark 9.13, there is a torsion-free, almost torsion object for $\mathfrak{u}$ in $\mathcal{S}$. By Remark 9.5, it follows that M is a simple object in $\mathcal{H}_{\mathfrak{u}^-}$. Thus $\mathcal{S}' := \mathrm{filt}(M)$ is a nontrivial torsion class in $\mathcal{H}_{\mathfrak{u}^-}$ that is contained in $\mathcal{S}$, and in particular, in $\mathcal{V}$. By Proposition 7.5, there exists a torsion class $\mathcal{U} \subsetneq \mathcal{T}' \subseteq \mathcal{T}$ in $\mathcal{H}$. By assumption, we have that $\mathcal{T} = \mathcal{T}'$ and so, by another application of Proposition 7.5, we conclude that $\mathcal{S} = \mathcal{S}'$.

(1) $\Rightarrow$ (3): By assumption and Lemma 9.11, the semibrick $\mathcal{M}$ associated to $\mathcal{S}$ consists of the unique (up to isomorphism) simple object S in $\mathcal{S}$, and $\mathcal{S} = \mathrm{filt}(S)$. Consider a torsion class $\mathcal{X}$ in $\mathcal{H}$ such that $\mathcal{U} \subseteq \mathcal{X} \subseteq \mathcal{T}$. We must show that $\mathcal{U} = \mathcal{X}$ or $\mathcal{T} = \mathcal{X}$. By Proposition 7.5, there exists a torsion class $\mathcal{S}' \subseteq \mathcal{S} \subseteq \mathcal{V}$ in $\mathcal{H}_{\mathfrak{u}^-}$ (given by $\mathcal{X} \cap \mathcal{V}$). If $\mathcal{S}'$ is trivial, then by Proposition 7.5, $\mathcal{U} = \mathcal{X}$. Suppose that $\mathcal{S}'$ is nontrivial and let X be a nonzero object of $\mathcal{S}'$. Then X is contained in $\mathcal{S} = \mathrm{filt}(S)$; thus S is a quotient of X and lies in $\mathcal{S}'$. This shows that $\mathcal{S}' = \mathcal{S}$ and, thus, that $\mathcal{T} = \mathcal{X}$, again by Proposition 7.5. □

Example 9.15. We revisit Example 4.10. This time we consider the indecomposable preprojective modules P_n, $n \in \mathbb{N}$, over the Kronecker algebra A and the torsion pairs $\mathfrak{t}_n = ({}^{\perp_0} P_n, \mathrm{Cogen}(P_n))$ cogenerated by them. It is well known that $\mathfrak{t}_n$ is an irreducible mutation of $\mathfrak{t}_{n+1}$; the corresponding wide subcategory is ${}^{\perp_0}P_n \cap \mathrm{Cogen}(P_{n+1}) \cap \mathrm{mod}(A) = \mathrm{add}(P_{n+1})$. Notice that ${}^{\perp_0}P_n \cap \mathrm{Cogen}(P_{n+2}) \cap \mathrm{mod}(A) = \mathrm{add}(P_{n+1} \oplus P_{n+2})$ is not wide, so $\mathfrak{t}_n$ is not a mutation of $\mathfrak{t}_{n+2}$. This shows that a sequence of irreducible mutations is not a mutation in general.

We can also rediscover the fact that $\sigma_{\mathbb{X}}$ does not admit right mutation. Indeed, $\sigma_{\mathbb{X}}$ is associated with the torsion pair $\mathfrak{u} = (\mathrm{Gen}(\boldsymbol{t}), \mathcal{V})$ generated by all finite-dimensional indecomposable regular modules, so any torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $\mathrm{Cosilt}(A)$ lying above $\mathfrak{u}$ has the form $\mathfrak{t} = \mathfrak{t}_n$ for some n, and $\mathcal{T} \cap \mathcal{V} \cap \mathrm{mod}(A) = \mathrm{add}(P_{n+1} \oplus P_{n+2} \oplus \cdots)$ is clearly not wide.

Finally, we remark that the set $\mathscr{E}$ in Lemma 9.11 may differ from $\mathrm{Prod}(\{\sigma_M \mid M \in \Omega \setminus \mathcal{M}\})$, where Ω is the set of isoclasses of simple objects in $\mathcal{H}_{\mathfrak{u}^-}$. To this end, we consider σ_P with $P = \mathbb{X} \setminus \{x\}$ for some $x \in \mathbb{X}$. Here $\mathcal{M}$ only contains the simple regular module S corresponding to the tube $\boldsymbol{t}_x$, and the set Ω consists of the Prüfer module S_∞ and the adic modules corresponding to $\boldsymbol{t}_P = \bigcup_{y \neq x} \boldsymbol{t}_y$. Thus the generic module G belongs to the set $\mathscr{E}$, but not to $\mathrm{Prod}(\{\sigma_M \mid M \in \Omega \setminus \mathcal{M}\})$.

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Elliptic KZB connections via universal vector extensions

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Using the formalism of bar complexes and their relative versions, we give a new, purely algebraic, construction of the so-called *universal elliptic KZB connection* in arbitrary level. We compute explicit analytic formulae, and we compare our results with previous approaches to elliptic KZB equations and multiple elliptic polylogarithms in the literature.

Our approach is based on a number of results concerning logarithmic differential forms on universal vector extensions of elliptic curves. Let S be a scheme of characteristic 0, $E \to S$ be an elliptic curve, $f : E^\natural \to S$ be its universal vector extension, and $\pi : E^\natural \to E$ be the natural projection. Given a finite subset of torsion sections $Z \subset E(S)$, we study the dg-algebra over $\mathcal{O}_S$ of relative logarithmic differentials $\mathcal{A} = f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)$. In particular, we prove that the residue exact sequence in degree 1 splits canonically, and we derive the formality of $\mathcal{A}$. When S is smooth over a field k of characteristic 0, we also prove that sections of $\mathcal{A}^1$ admit canonical lifts to absolute logarithmic differentials in $f_*\Omega^1_{E^\natural/k}(\log \pi^{-1}Z)$, which extends a well-known property for regular differentials given by the "crystalline nature" of universal vector extensions.

1. Introduction

The main goal of this paper is to give a purely algebraic construction of the so-called universal elliptic Knizhnik–Zamolodchikov–Bernard (KZB) connection in arbitrary level [Bernard 1988; Levin and Racinet 2007; Calaque et al. 2009; Calaque and Gonzalez 2020] (see also [Hain 2020; Luo 2019; Hopper 2024]) in terms of universal vector extensions of elliptic curves. In doing so, we establish a number of new results concerning logarithmic differential forms on universal vector extensions. As an application, we

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shall also obtain new algebraic formulas for elliptic KZB connections, which in particular yield an explicit solution to some rationality questions concerning these equations.

Our principal motivation is to provide an algebraic approach to multiple elliptic polylogarithms [Beilinson and Levin 1994; Levin 1997; Levin and Racinet 2007; Brown and Levin 2011] and their closely related notions, such as elliptic multiple zeta values [Enriquez 2016]. In the literature, these objects are usually defined and studied using analytic versions of the elliptic KZB connection, and algebraicity is only shown a posteriori. This obscures the essentially algebraic nature of the elliptic KZB connection, and makes the relation to arithmetic algebraic geometry more indirect — e.g., it is not clear how to write special values of multiple elliptic polylogarithms in terms of periods, in the sense of Kontsevich and Zagier (see [Fonseca and Matthes 2020]). In this work, we use the universal vector extension of an elliptic curve to give a purely algebraic definition of the elliptic KZB connection and we also show how to retrieve the various versions found in the literature via "analytification". Our theory is in complete analogy to the genus-0 case, the Knizhnik–Zamolodchikov (KZ) connection [1984], which is most naturally defined using algebraic formulas.

1.1. ***The elliptic KZB connection over*** $\mathbb{C}$**.** The elliptic KZB connection on a complex elliptic curve (E, O) can be defined as a proalgebraic connection with logarithmic singularities at O

$$\nabla_E : \mathcal{V}_E \longrightarrow \Omega^1_E(\log O) \mathbin{\hat{\otimes}} \mathcal{V}_E$$

satisfying the following universal property: given a base point $b \in E \setminus O$, there is a vector v_E in the fibre $\mathcal{V}_E(b)$ such that, for every *unipotent* connection $\nabla : \mathcal{V} \to \Omega^1_E(\log O) \otimes \mathcal{V}$ equipped with $v \in \mathcal{V}(b)$, there is a unique morphism $(\mathcal{V}_E, \nabla_E) \to (\mathcal{V}, \nabla)$ sending v_E to v (see [Kim 2009, Section 1]). Recall that "unipotent" means that $(\mathcal{V}, \nabla)$ can be written as a finite iterated extension of the trivial connection $(\mathcal{O}_E, d)$.

Alternatively, by Serre's GAGA and the Riemann–Hilbert correspondence, $(\mathcal{V}_E, \nabla_E)$ can be characterised by its prolocal system of horizontal sections $\mathbb{V}_E$, whose stalk at $x \in E \setminus O$ is the prounipotent completion over $\mathbb{C}$ of the fundamental torsor of paths $\pi_1(E \setminus O; b, x)$:

$$\mathbb{V}_{E,x} = \pi_1^{\mathrm{un}}(E \setminus O; b, x).$$

Concretely, local sections of $\mathbb{V}_E$ are described by holomorphic functions, possibly multivalued, given by homotopy-invariant linear combinations of iterated integrals à la Chen

$$x \longmapsto \int_b^x \omega_1 \cdots \omega_n$$

of 1-forms ω_i on the once-punctured elliptic curve $E \setminus O$. Thus, the elliptic KZB connection can be thought of more concretely as the differential equations these iterated integrals satisfy. Note that the equations themselves do not depend on the choice of base point b.

The above description in terms of local systems immediately generalises to a family of elliptic curves

$$E \overset{O}{\underset{\curvearrowleft}{\longrightarrow}} S,$$

where S is a complex manifold. Here, working locally over S, one can take b to be a base section of $E \to S$ and consider the local system $\mathbb{V}_E$ whose stalk at a point $x \in E \setminus O$ above $s \in S$ is

$$\mathbb{V}_{E,x} = \pi_1^{\text{un}}(E_s \setminus O(s); b(s), x).$$

In particular, the elliptic KZB connection of the family is an integrable connection $(\mathcal{V}_E, \nabla_E)$ on the total space of the family minus the identity section $E \setminus O$, which restricts to the previously defined $(\mathcal{V}_{E_s}, \nabla_{E_s})$ on every fibre $E_s \setminus O(s)$. In other words, it can be regarded as an *isomonodromic deformation*, parametrised by S, of the elliptic KZB connection at one fibre. It is also possible to characterise $(\mathcal{V}_E, \nabla_E)$ directly by a relative version of the universal property recalled in the first paragraph above (see [Chiarellotto et al. 2023, Section 3]).

The (level-1) *universal* elliptic KZB connection corresponds to the universal family $\mathcal{E} \to \mathcal{M}_{1,1}$. Higher-level elliptic KZB connections are defined analogously, with logarithmic singularities on torsion points $E[N]$; see [Calaque and Gonzalez 2020; Hopper 2024]. For the purposes of this introduction, we focus on the level-1 case, although all of our results work more generally in arbitrary level.

1.2. ***Construction of KZB over the universal vector extension.*** We state some of our results in simplified form. Let S be a scheme of characteristic 0, $(E/S, O)$ be an elliptic curve over S, and

$$\pi : E^\natural \longrightarrow E$$

be its universal vector extension. Formally, $E^\natural$ is given as an extension of E by a certain vector group of rank 1, in the category of commutative S-group schemes, satisfying a suitable universal property (see Section 2.1 below for a precise definition), and π is the natural projection. In this paper, the key property of $\pi : E^\natural \to E$ is that it is a principal $\mathbb{G}_a$-bundle over which every S-unipotent vector bundle trivialises.

We shall directly construct a connection on $E^\natural$ with logarithmic singularities along the vertical divisor $\pi^{-1}O$ which can be shown a posteriori to be the pullback of the elliptic KZB connection on E by π. Our first result describes global relative differential forms on $E^\natural/S$ with logarithmic singularities along $\pi^{-1}O$. For simplicity, assume that

$$S = \operatorname{Spec} R$$

is affine and small enough so that $\pi^{-1}O \cong \mathbb{G}_{a,S} = \operatorname{Spec} R[t]$.

Theorem 1.1. *There exists a* $\nu \in \Gamma(E^\natural, \Omega^1_{E^\natural/S})$ *such that* $\nu|_{\pi^{-1}O} = dt$. *Given such a* ν, *there is a unique family* $(\omega^{(n)})_{n \geq 0}$ *in* $\Gamma(E^\natural, \Omega^1_{E^\natural/S}(\log \pi^{-1}O))$ *such that*

$$\Gamma(E^\natural, \Omega^1_{E^\natural/S}) = R\nu \oplus R\omega^{(0)}$$

and, for $n \geq 1$,

(a) $\operatorname{Res}(\omega^{(n)}) = t^{n-1}/(n-1)!$,

(b) $\omega^{(n)} \wedge \omega^{(0)} = 0$,

(c) $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$.

Moreover,

$$\Gamma(E^\natural, \Omega^1_{E^\natural/S}(\log \pi^{-1}O)) = R\nu \oplus \bigoplus_{n\geq 0} R\omega^{(n)},$$
$$\Gamma(E^\natural, \Omega^2_{E^\natural/S}(\log \pi^{-1}O)) = \bigoplus_{n\geq 0} R\nu \wedge \omega^{(n)}.$$

We call $\omega^{(0)}, \omega^{(1)}, \ldots$ *Kronecker differentials*, as they are purely algebraic variants of classical elliptic functions considered by Kronecker (see Section 1.3 below).

Under the above hypotheses, we can explicitly construct a *relative KZB connection* on $E^\natural/S$ by setting

$$\nabla_{E^\natural/S} : \mathcal{O}_{E^\natural} \hat{\otimes} R\langle\langle a, b\rangle\rangle \longrightarrow \Omega^1_{E^\natural/S}(\log \pi^{-1}O) \hat{\otimes} R\langle\langle a, b\rangle\rangle, \quad \nabla_{E^\natural/S} = d + \omega_{E^\natural/S},$$

with

$$\omega_{E^\natural/S} = -\nu \otimes a - \sum_{n\geq 0} \omega^{(n)} \otimes \mathrm{ad}_a^n b.$$

Here, $R\langle\langle a, b\rangle\rangle$ denotes a ring of noncommutative power series with the (a, b)-adic topology, and ad_a is the operator $x \mapsto ax - xa$. In the above formula for $\omega_{E^\natural/S}$, an element of $R\langle\langle a, b\rangle\rangle$ acts on $R\langle\langle a, b\rangle\rangle$ by left multiplication. The integrability of $\nabla_{E^\natural/S}$, which amounts to the equation

$$d\omega_{E^\natural/S} + \omega_{E^\natural/S} \wedge \omega_{E^\natural/S} = 0,$$

follows from Theorem 1.1(b)–(c), together with the fact that ν and $\omega^{(0)}$ are closed 1-forms (Proposition 2.5).

Remark 1.2. The above explicit formula for the relative elliptic KZB connection is actually derived from a natural construction involving the bar complex of the dg-algebra $\Gamma(E^\natural, \Omega^\bullet_{E^\natural/S}(\log \pi^{-1}O))$, which holds for arbitrary S of characteristic 0 (see Section 1.6). This construction also commutes with arbitrary base change in S. In Proposition 5.12, we characterise it by a universal property as in Section 1.1.

From now on, assume moreover that S is smooth over a field k of characteristic 0. The next step is to lift the relative KZB connection to an absolute integrable k-connection, the "isomonodromic deformation":

$$\nabla_{E^\natural/S/k} : \mathcal{O}_{E^\natural} \hat{\otimes} R\langle\langle a, b\rangle\rangle \longrightarrow \Omega^1_{E^\natural/k}(\log \pi^{-1}O) \hat{\otimes} R\langle\langle a, b\rangle\rangle, \quad \nabla_{E^\natural/S/k} = d + \omega_{E^\natural/S/k}.$$

In this simplified situation, this amounts to the construction of the absolute connection form $\omega_{E^\natural/S/k}$, which is a suitable lift of the relative connection form $\omega_{E^\natural/S}$ satisfying the integrability equation

$$d\omega_{E^\natural/S/k} + \omega_{E^\natural/S/k} \wedge \omega_{E^\natural/S/k} = 0.$$

Our next result shows that relative logarithmic differentials on $E^\natural/S$ admit canonical lifts to absolute differentials. This extends a well-known property for regular differentials on universal vector extensions reflecting their "crystalline nature" (see [Bost 2013, Section 6; Fonseca and Matthes 2024]).

Theorem 1.3. *The relative differentials* $\nu, \omega^{(0)}, \omega^{(1)}, \dots \in \Gamma(E^\natural, \Omega^1_{E^\natural/S}(\log \pi^{-1}O))$ *lift uniquely to absolute differentials* $\tilde{\nu}, \tilde{\omega}^{(0)}, \tilde{\omega}^{(1)}, \dots \in \Gamma(E^\natural, \Omega^1_{E^\natural/k}(\log \pi^{-1}O))$ *such that*

$$e^*\tilde{\nu} = e^*\tilde{\omega}^{(0)} = 0,$$

where $e \in E^\natural(S)$ *denotes the identity section, and, for* $n \geq 1$,

$$\tilde{\omega}^{(n)} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv n\alpha_{21} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(n+1)} \bmod \Omega^2_{R/k} \wedge \Gamma(E^\natural, \Omega^1_{E^\natural/k}),$$

where $\alpha_{21} \in \Omega^1_{R/k}$ *is a coefficient of the Gauss–Manin connection matrix* (see [Fonseca and Matthes 2024, Remark 3.7])

$$d\tilde{\omega}^{(0)} = \alpha_{11} \wedge \tilde{\omega}^{(0)} + \alpha_{21} \wedge \tilde{\nu},$$
$$d\tilde{\nu} = \alpha_{12} \wedge \tilde{\omega}^{(0)} + \alpha_{22} \wedge \tilde{\nu}.$$

Now it is natural to consider the canonical lift of the relative KZB connection:

$$\tilde{\nabla}_{E^\natural/S} : \mathcal{O}_{E^\natural} \hat{\otimes} R\langle\langle a, b\rangle\rangle \longrightarrow \Omega^1_{E^\natural/k}(\log \pi^{-1} O) \hat{\otimes} R\langle\langle a, b\rangle\rangle, \quad \tilde{\nabla}_{E^\natural/S} = d + \tilde{\omega}_{E^\natural/S},$$

with

$$\tilde{\omega}_{E^\natural/S} = -\tilde{\nu} \otimes a - \sum_{n \geq 0} \tilde{\omega}^{(n)} \otimes \operatorname{ad}_a^n b.$$

Crucially, this k-connection is *not* integrable. The next result computes its curvature.

Theorem 1.4. *There is a unique* 1*-form over* S *with coefficients in* k*-derivations of* $R\langle\langle a, b\rangle\rangle$

$$\Phi \in \Omega^1_{R/k} \hat{\otimes} \operatorname{Der}_k R\langle\langle a, b\rangle\rangle$$

such that

$$d\tilde{\omega}_{E^\natural/S} + \tilde{\omega}_{E^\natural/S} \wedge \tilde{\omega}_{E^\natural/S} + \Phi(\tilde{\omega}_{E^\natural/S}) = 0$$

in $\Gamma(E^\natural, \Omega^2_{E^\natural/k}(\log \pi^{-1} O)) \hat{\otimes} R\langle\langle a, b\rangle\rangle$*; here,* $\Phi(\tilde{\omega}_{E^\natural/S})$ *is the* 2*-form with coefficients in* $R\langle\langle a, b\rangle\rangle$ *obtained by "evaluating"* Φ *at* $\tilde{\omega}_{E^\natural/S}$ (see Section 6.4). *In particular, the connection on* $\mathcal{O}_{E^\natural} \hat{\otimes} R\langle\langle a, b\rangle\rangle$ *defined by*

$$\nabla_{E^\natural/S/k} = d + \omega_{E^\natural/S/k}, \quad \omega_{E^\natural/S/k} = \tilde{\omega}_{E^\natural/S} + \Phi,$$

is integrable.

In short, the elliptic KZB connection of the family is obtained by "correcting" the canonical lift of the relative elliptic KZB connection by Φ. We actually retrieve Φ as the connection form of the dual of the Gauss–Manin connection on the relative fundamental Hopf algebra of $E \setminus O$; see Section 1.6.

1.3. ***Analytic formulae.*** All of the above can be explicitly computed on a given family. In order to compare our results with the traditional analytic approach in the literature, we work out in detail the case of the universal framed elliptic curve over the upper half-plane $\mathcal{E} \to \mathfrak{H}$, whose fibre at $\tau \in \mathfrak{H}$ is

$$\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

In this analytic situation, the universal vector extension can be uniformised as follows:

$$\mathcal{E}^\natural_\tau = \mathbb{C}^2/L_\tau, \quad L_\tau = \{(m + n\tau, 2\pi i n) \in \mathbb{C}^2 : m, n \in \mathbb{Z}\}.$$

Let $\theta_\tau(z)$ be Jacobi's odd theta function[1], and consider the so-called *Kronecker theta function* (see [Levin and Racinet 2007, Section 2; Brown and Levin 2011, Section 3.4])

$$F_\tau(z, x) = \frac{\theta'_\tau(0)\theta_\tau(z + x)}{\theta_\tau(z)\theta_\tau(x)}.$$

[1] Our normalisation is that of Proposition 7.1 below.

If (z, w) denotes the coordinates on $\mathbb{C}^2$, then the Kronecker differentials associated to

$$\nu = dw \in \Gamma(\mathcal{E}_\tau^\natural, \Omega^1_{\mathcal{E}_\tau^\natural})$$

by Theorem 1.1 are given by

$$\omega^{(n)} = \varphi_\tau^{(n)}(z, w)\, dz \in \Gamma(\mathcal{E}_\tau^\natural, \Omega^1_{\mathcal{E}_\tau^\natural}(\log \pi^{-1} O)),$$

where $\varphi_\tau^{(n)}(z, w)$ are complex-analytic functions on $\mathcal{E}_\tau^\natural$ defined by the generating series

$$e^{wx} F_\tau(z, x) = \sum_{n\geq 0} \varphi_\tau^{(n)}(z, w) x^{n-1}.$$

Note that $\omega^{(0)} = dz$. Thus, the relative KZB connection form is

$$\omega_{\mathcal{E}^\natural/\mathfrak{H}} = -dw \otimes a - \sum_{n\geq 0} \varphi_\tau^{(n)}(z, w)\, dz \otimes \mathrm{ad}_a^n\, b = -dw \otimes a - dz \otimes \mathrm{ad}_a\, e^{w\, \mathrm{ad}_a} F_\tau(z, \mathrm{ad}_a) b.$$

The canonical lifts of the above relative differentials, characterised by the properties of Theorem 1.3, are explicitly given by

$$\tilde{\nu} = dw, \quad \tilde{\omega}^{(n)} = \varphi_\tau^{(n)}(z, w)\Big(dz - w\, \frac{d\tau}{2\pi i}\Big) + n\varphi_\tau^{(n+1)}(z, w)\, \frac{d\tau}{2\pi i}.$$

By direct computation of the curvature $d\tilde{\omega}_{\mathcal{E}^\natural/\mathfrak{H}} + \tilde{\omega}_{\mathcal{E}^\natural/\mathfrak{H}} \wedge \tilde{\omega}_{\mathcal{E}^\natural/\mathfrak{H}}$, we obtain

$$\Phi = -\frac{d\tau}{2\pi i} \otimes D_\tau, \quad D_\tau = b\frac{\partial}{\partial a} + \frac{1}{2} \sum_{n\geq 2} (2n-1) G_{2n}(\tau) \sum_{\substack{j+k=2n-1 \\ j,k>0}} [(-\mathrm{ad}_a)^j b, \mathrm{ad}_a^k\, b]\, \frac{\partial}{\partial b}, \tag{1}$$

where $G_{2n}(\tau) = \sum_{(r,s)\neq(0,0)} (r + s\tau)^{-2n}$ are the classical Eisenstein series. The final expression for the KZB connection form then becomes

$$\omega_{\mathcal{E}^\natural/\mathfrak{H}/\mathbb{C}} = -dw \otimes a - dz \otimes \mathrm{ad}_a\, e^{w\, \mathrm{ad}_a} F_\tau(z, \mathrm{ad}_a) b - \frac{d\tau}{2\pi i} \otimes (\mathrm{ad}_a\, F_\tau'(z, w, \mathrm{ad}_a) b + D_\tau),$$

where

$$F_\tau'(z, w, x) = e^{wx} \frac{\partial}{\partial x} F_\tau(z, x) + \frac{1}{x^2}.$$

1.4. ***Relation to the literature.*** The elliptic KZB connection [Bernard 1988] arose in conformal field theory as a genus-1 version of the KZ connection [Knizhnik and Zamolodchikov 1984], which in its simplest guise is the proalgebraic connection

$$\nabla_{\mathrm{KZ}} : \mathcal{V}_{\mathrm{KZ}} \longrightarrow \Omega^1_{\mathbb{P}^1}(\log\{0, 1, \infty\})\, \hat{\otimes}\, \mathcal{V}_{\mathrm{KZ}}, \quad \nabla_{\mathrm{KZ}} = d - \frac{dz}{z} \otimes x_0 - \frac{dz}{1-z} \otimes x_1, \tag{2}$$

where $\mathcal{V}_{\mathrm{KZ}}$ is the trivial provector bundle over $\mathbb{P}^1$ with fibre the algebra of noncommutative power series $\mathbb{C}\langle\langle x_0, x_1\rangle\rangle$. The KZ connection encodes quantities of deep arithmetic interest, obtained as iterated integrals of the differential 1-forms $\frac{dz}{z}$, $\frac{dz}{1-z}$. Namely, flat sections of ∇_{KZ} are described by multiple polylogarithms, and their monodromy by multiple zeta values; see for instance [Brown 2013, Section 4].

Motivated by an elliptic analogue of the theory of multiple polylogarithms, Levin and Racinet [2007] were led to consider elliptic KZB connections as defined in Section 1.1. In contrast to the genus-0 case, however, the provector bundle $\mathcal{V}_E$ is *not* trivial, since the condition

$$H^1(E, \mathcal{O}_E) \neq 0 \tag{3}$$

amounts to the existence of nontrivial unipotent vector bundles on E. To obtain a formula as explicit as (2), they compute the pullback of the elliptic KZB connection on $\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ by the (analytic) uniformisation map $\mathbb{C} \to \mathcal{E}_\tau$:

$$\nabla_\tau : \mathcal{O}_{\mathbb{C}} \mathbin{\hat{\otimes}} \mathbb{C}\langle\!\langle a, b\rangle\!\rangle \longrightarrow \Omega^1_{\mathbb{C}}(\log(\mathbb{Z}+\tau\mathbb{Z})) \mathbin{\hat{\otimes}} \mathbb{C}\langle\!\langle a, b\rangle\!\rangle, \quad \nabla_\tau = d - dz \otimes \mathrm{ad}_a\, F_\tau(z, \mathrm{ad}_a)b,$$

with corresponding action of $\mathbb{Z}+\tau\mathbb{Z}$ on an element $f(a,b)$ of $\mathbb{C}\langle\!\langle a, b\rangle\!\rangle$ given by $(m+n\tau)\cdot f(a,b) = e^{-2\pi i n a} f(a,b)$.

By considering the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathcal{E}^\natural_\tau \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{C} & \longrightarrow & \mathcal{E}_\tau \end{array}$$

where horizontal arrows are the natural uniformisation maps and the left vertical arrow is the projection $(z, w) \mapsto z$, one can readily check that $f(a,b) \mapsto e^{-wa} f(a,b)$ induces an isomorphism between the pullbacks to $\mathbb{C}^2$ of our $(\mathcal{O}_{\mathcal{E}^\natural_\tau} \mathbin{\hat{\otimes}} \mathbb{C}\langle\!\langle a, b\rangle\!\rangle, \nabla_{\mathcal{E}^\natural_\tau})$, as given in Section 1.3, and Levin and Racinet's $(\mathcal{O}_{\mathbb{C}} \mathbin{\hat{\otimes}} \mathbb{C}\langle\!\langle a, b\rangle\!\rangle, \nabla_\tau)$.

At this point, it is also instructive to compare our construction with Brown and Levin's theory [2011] of multiple elliptic polylogarithms, which rely on real-analytic logarithmic 1-forms $\nu_{\mathrm{BL}}, \omega^{(0)}_{\mathrm{BL}}, \omega^{(1)}_{\mathrm{BL}}, \ldots$ defined by

$$\nu_{\mathrm{BL}} = 2\pi i\, dr, \qquad e^{2\pi i r x} F_\tau(z, x)\, dz = \sum_{n\geq 0} \omega^{(n)}_{\mathrm{BL}} x^{n-1},$$

where $r(z) = \mathrm{Im}(z)/\mathrm{Im}(\tau)$. The presence of r in their construction is justified by the transformation property

$$r(z+m+n\tau) = r(z) + n,$$

which, together with the modularity properties of Kronecker's function, implies that the differentials $\nu_{\mathrm{BL}}, \omega^{(n)}_{\mathrm{BL}}$ descend to $\mathcal{E}_\tau$. In this sense, the nonalgebraicity in Brown and Levin's work is also related to the same cohomological obstruction (3), which turns out to be equivalent to the nonexistence of a holomorphic function on $\mathcal{E}_\tau$ which transforms in the same way as r.

Our Kronecker differentials $\nu, \omega^{(n)}$ should be regarded as algebraic avatars of Brown and Levin's differentials $\nu_{\mathrm{BL}}, \omega^{(n)}_{\mathrm{BL}}$. Indeed, the projection $\pi : \mathcal{E}^\natural_\tau \to \mathcal{E}_\tau$ admits a real-analytic section $\sigma : \mathcal{E}_\tau \to \mathcal{E}^\natural_\tau$ induced by $z \mapsto (z, 2\pi i r(z))$, and we have

$$\nu_{\mathrm{BL}} = \sigma^*\nu, \qquad \omega^{(n)}_{\mathrm{BL}} = \sigma^*\omega^{(n)}.$$

With this point of view, the universal vector extension $\mathcal{E}^\natural$ can be naively thought of as a space over the elliptic curve $\mathcal{E}_\tau$ obtained by adjoining a formal variable w which transforms as $2\pi i r$ under the action of $\mathbb{Z}+\tau\mathbb{Z}$.

The *universal* elliptic KZB connection was first considered in explicit form by Calaque, Enriquez, and Etingof [Calaque et al. 2009], in relation to the representation theory of braid monodromy groups. It is defined as an integrable connection on $\mathcal{V}_{\mathrm{KZB}}$, the trivial infinite-rank vector bundle over $\mathfrak{H}\times\mathbb{C}$ with fibre $\mathbb{C}\langle\langle a,b\rangle\rangle$, given by

$$\nabla_{\mathrm{KZB}} = d - dz\otimes \mathrm{ad}_a\, F_\tau(z,\mathrm{ad}_a)b - \frac{d\tau}{2\pi i}\otimes(\mathrm{ad}_a\, G_\tau(z,\mathrm{ad}_a)b + D_\tau),$$

where

$$G_\tau(z,x) = \frac{\partial}{\partial x}F_\tau(z,x) + \frac{1}{x^2},$$

and D_τ is as in (1). By considering a suitable action of $\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, the connection $(\mathcal{V}_{\mathrm{KZB}},\nabla_{\mathrm{KZB}})$ is then proved to descend to the universal elliptic curve, seen as the orbifold quotient $\mathcal{E} = (\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2)\backslash\backslash(\mathfrak{H}\times\mathbb{C})$.

The comparison with our connection on the universal vector extension is done via a universal analogue of the previous commutative diagram:

$$\begin{array}{ccc} \mathfrak{H}\times\mathbb{C}^2 & \longrightarrow & \mathcal{E}^\natural \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ \mathfrak{H}\times\mathbb{C} & \longrightarrow & \mathcal{E}. \end{array}$$

It follows from the explicit expressions in Section 1.3 that the pullback to $\mathfrak{H}\times\mathbb{C}^2$ of $(\mathcal{O}_{\mathcal{E}^\natural}\hat{\otimes}\mathbb{C}\langle\langle a,b\rangle\rangle,\nabla_{\mathcal{E}^\natural/\mathfrak{H}/\mathbb{C}})$ is isomorphic to the pullback of $(\mathcal{V}_{\mathrm{KZB}},\nabla_{\mathrm{KZB}})$. There are also similar formulae for higher-level universal elliptic KZB connections due to Calaque and Gonzalez [2020] (see [Hopper 2024]), and a comparison in full generality is worked out in Section 7 below.

1.5. ***Towards motivic multiple elliptic polylogarithms.*** The present work has been originally motivated by the development of a motivic theory of multiple elliptic polylogarithms, in the framework of Brown's *motivic periods* [2014; 2017] (which have been applied with great success to arithmetic questions concerning multiple zeta values). Recall that motivic periods involve Betti and algebraic de Rham realisations; this paper is purely devoted to the algebraic de Rham aspects of the theory (see [Fonseca and Matthes 2020]).

More precisely, we are concerned here with *unipotent algebraic de Rham fundamental groups*. In the Tannakian formalism, this amounts to the study of unipotent vector bundles with connection over punctured elliptic curves (see the Appendix). Our use of the universal vector extension is motivated by the cohomological properties (over a field k of characteristic 0)

$$H^0(E^\natural,\mathcal{O}_{E^\natural}) = k,\qquad H^1(E^\natural,\mathcal{O}_{E^\natural}) = 0,$$

which imply that every unipotent vector bundle over $E^\natural$ is canonically trivial. The importance of the universal vector extension in the algebraic de Rham fundamental group theory of a punctured elliptic curve has been previously advocated by Deligne (personal communications with P. Etingof and R. Hain, 2015), and some of its Tannakian implications have already been explored by Enriquez and Etingof [2018] (see Section A5 below).

The above discussion is also connected to a number of natural algebraicity questions that have been raised in the literature concerning elliptic KZB equations, as the usual approach relies on analytic uniformisation maps. The algebraicity over $\mathbb{Q}$ of the universal elliptic KZB connection in level 1 was proved by Luo [2019] (see [Levin and Racinet 2007, Section 5]) by making essential use of the moduli space $\mathcal{M}_{1,\vec{1}}$ classifying elliptic curves with a nonzero tangent vector at the identity. Here, the map $\mathcal{M}_{1,\vec{1}} \to \mathcal{M}_{1,1}$, or the corresponding map on universal elliptic curves, can be thought of as a particular $\mathbb{G}_m$-bundle over which the algebraicity question becomes computationally tractable. In this sense, our approach, which is based on the construction of the elliptic KZB connection on the universal vector extension of a family of elliptic curves, is not far in spirit from that of Luo, with the difference that we use a $\mathbb{G}_a$-bundle instead of a $\mathbb{G}_m$-bundle.

Algebraicity problems were also considered in the literature concerning elliptic polylogarithm sheaves (in the sense of Beilinson and Levin [1994]), usually motivated by arithmetic questions concerning p-adic realisations of elliptic polylogarithm functions [Bannai et al. 2010; Sprang 2020]. Note that Sprang's approach [2020] also relies on universal vector extensions, and it would be interesting to compare it with the methods of this paper.

1.6. ***What we do.*** Let k be a field of characteristic 0, S be a smooth k-scheme, and $(p : E \to S, O)$ be an elliptic curve over S. Consider its universal vector extension $f : E^\natural \to S$, which fits into a short exact sequence of commutative S-group schemes

$$0 \longrightarrow \mathbb{V}(R^1 p_* \mathcal{O}_E) \longrightarrow E^\natural \xrightarrow{\pi} E \longrightarrow 0,$$

and is universal for extensions of E by an S-vector group (see Section 2.1). Let $Z \subset E$ be a subscheme given by a finite union of torsion sections of p. For simplicity, we also assume Z contains the identity section O.

We shall construct the elliptic KZB connection over $E^\natural$ with logarithmic singularities along $\pi^{-1}Z$. Our approach is based on the *bar construction* formalism. We refer to the recent work of Chiarellotto, Di Proietto, and Shiho [Chiarellotto et al. 2023] for general comparison statements of different approaches to unipotent fundamental groups. Regarding our particular framework, we have included in the Appendix precise comparison results with the Tannakian formalism over a field.

1.6.1. *Relative differentials.* Consider the dg-algebra over $\mathcal{O}_S$ of relative logarithmic differential forms

$$\mathcal{A} := f_* \Omega^\bullet_{E^\natural/S}(\log \pi^{-1} Z).$$

It follows from a result due independently to Coleman [1998] and Laumon [1996] (Theorem 2.3) that $\mathcal{A}$ is a model for the de Rham cohomology of $E \setminus Z$ over S (Proposition 2.7):

$$H^\bullet(\mathcal{A}) \cong H^\bullet_{\mathrm{dR}}((E \setminus Z)/S).$$

Our main results in Section 3 concern the structure of $\mathcal{A}$ as a dg-algebra over $\mathcal{O}_S$. In particular, we obtain in Theorem 3.7 a decomposition

$$\mathcal{A}^1 = f_* \Omega^1_{E^\natural/S} \oplus \bigoplus_{n \geq 1} \mathcal{K}^{(n)},$$

where the $\mathcal{O}_S$-submodules $\mathcal{K}^{(n)}$ are characterised by conditions involving the residue along $\pi^{-1}Z$ and the dg-algebra structure of $\mathcal{A}$. Concretely, locally over S, we have

$$f_*\Omega^1_{E^\natural/S} = \mathcal{O}_S\nu \oplus \mathcal{O}_S\omega^{(0)}, \quad \mathcal{K}^{(n)} = \bigoplus_{P\in Z(S)} \mathcal{O}_S\omega_P^{(n)},$$

where $\omega_P^{(n)}$ are Kronecker differentials with logarithmic singularities along $\pi^{-1}P$.

This also allows us to prove the formality of the dg-algebra $\mathcal{A}$. More precisely, we obtain a dg-quasi-isomorphism (Theorem 3.9)

$$\mathcal{A} \longrightarrow H^\bullet(\mathcal{A})$$

which plays a key role in the rest of the paper (see [Brown and Levin 2011, Theorem 19]).

1.6.2. *Relative KZB.* With $\mathcal{A}$ as above (note that $\mathcal{A}$ is a connected dg-algebra over $\mathcal{O}_S$), we consider the bar complex

$$0 \longrightarrow B^0(\mathcal{A}) \xrightarrow{d_B} B^1(\mathcal{A}) \longrightarrow \cdots .$$

We shall only need the first two terms, which are explicitly given by $B^0(\mathcal{A}) = \bigoplus_{n\geq 0}(\mathcal{A}^1)^{\otimes n}$ and $B^1(\mathcal{A}) = \bigoplus_{n\geq 1}\bigoplus_{1\leq i\leq n}(\mathcal{A}^1)^{\otimes i-1}\otimes\mathcal{A}^2\otimes(\mathcal{A}^1)^{\otimes n-i}$, where tensor products are over $\mathcal{O}_S$. Decomposable tensors are denoted by $a_1\otimes\cdots\otimes a_n = [a_1\mid\cdots\mid a_n]$, and the differential in degree 0 is explicitly given by

$$d_B : B^0(\mathcal{A}) \longrightarrow B^1(\mathcal{A}),$$
$$[a_1\mid\cdots\mid a_n] \longmapsto -\sum_{i=1}^{n}[a_1\mid\cdots\mid a_{i-1}\mid da_i\mid a_{i+1}\mid\cdots\mid a_n] - \sum_{i=1}^{n-1}[a_1\mid\cdots\mid a_{i-1}\mid a_i\wedge a_{i+1}\mid a_{i+2}\mid\cdots\mid a_n].$$

Finally, we consider the $\mathcal{O}_S$-module

$$\mathcal{H}_{E/S,Z} := H^0(B(\mathcal{A})) = \ker(d_B : B^0(\mathcal{A}) \longrightarrow B^1(\mathcal{A})).$$

It comes with a natural commutative Hopf algebra structure, given by the deconcatenation coproduct and the shuffle product, and a natural filtration $L_n\mathcal{H}_{E/S,Z}$ by length (see Section 5.1). Note that $\mathcal{H}_{E/S,Z}$ can be thought of as the Hopf algebra corresponding to the relative de Rham unipotent fundamental group of $E\setminus Z$ over S at certain "canonical base point" (see Section A5 for a discussion in the case where S is the spectrum of a field).

In Section 5, the *relative elliptic KZB connection* is naturally defined on the pullback of the continuous dual

$$\mathcal{H}^\vee_{E/S,Z} := \varprojlim_{n\geq 0} \mathcal{H}om_{\mathcal{O}_S}(L_n\mathcal{H}_{E/S,Z}, \mathcal{O}_S)$$

by

$$\nabla_{E^\natural/S,Z} : f^*\mathcal{H}^\vee_{E/S,Z} \longrightarrow \Omega^1_{E^\natural/S}(\log\pi^{-1}Z)\,\hat{\otimes}\, f^*\mathcal{H}^\vee_{E/S,Z}, \quad \nabla_{E^\natural/S,Z} = d + \omega_{E^\natural/S,Z},$$

where the *KZB form* $\omega_{E^\natural/S,Z} \in \Gamma(S, \mathcal{A}^1 \,\hat{\otimes}\, \mathcal{H}^\vee_{E/S,Z})$ is the length-1 component of the element in $\Gamma(S, \mathcal{H}_{E/S,Z} \,\hat{\otimes}\, \mathcal{H}^\vee_{E/S,Z})$ induced by the Hopf algebra antipode, and acts on $\mathcal{H}^\vee_{E/S,Z}$ by left multiplication (see Section 5.3). In Proposition 5.12, we prove the integrability of $(f^*\mathcal{H}^\vee_{E/S,Z}, \nabla_{E^\natural/S,Z})$ and we characterise it via a universal property.

Building on the results of Section 3, we show that $\mathcal{H}_{E/S,Z}$ is canonically isomorphic to the tensor coalgebra $T^c H^1_{\mathrm{dR}}((E\setminus Z)/S)$ (Theorem 5.8). In particular, locally over S, we show in Theorem 5.15 that the continuous dual $\mathcal{H}^\vee_{E/S,Z}$ is isomorphic to the algebra of noncommutative power series

$$\mathcal{H}^\vee_{E/S,Z} \cong \frac{\mathcal{O}_S\langle\langle a,b,c_P : P\in Z(S)\rangle\rangle}{\left\langle \sum_{P\in Z(S)} c_P - [a,b]\right\rangle},$$

and, under the above isomorphism, the KZB form is given by

$$\omega_{E^\natural/S,Z} = -\nu\otimes a - \omega^{(0)}\otimes b - \sum_{n\geq 1}\sum_{P\in Z(S)} \omega_P^{(n)}\otimes \mathrm{ad}_a^{n-1} c_P.$$

When $Z=O$ (level 1), we recover the expressions in Section 1.2.

1.6.3. *Canonical lifts.* Our next results concern the sheaves of *absolute* logarithmic differentials $f_*\Omega^\bullet_{E^\natural/k}(\log \pi^{-1}Z)$. It is known that $E^\natural/S$ admits a natural horizontal foliation — formally, a D-group scheme structure — which amounts to a splitting of

$$0 \longrightarrow f^*\Omega^1_{S/k} \longrightarrow \Omega^1_{E^\natural/k} \longrightarrow \Omega^1_{E^\natural/S} \longrightarrow 0$$

satisfying a certain integrability condition and a compatibility with the group scheme structure [Bost 2013, Section 6.4]. We show in [Fonseca and Matthes 2024] that this splitting actually comes from the splitting of

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k} \longrightarrow f_*\Omega^1_{E^\natural/S} \longrightarrow 0$$

induced by the retraction $f_*\Omega^1_{E^\natural/k}\to\Omega^1_{S/k}$ given by restriction to the identity section $e\in E^\natural(S)$:

$$f_*\Omega^1_{E^\natural/k} = \Omega^1_{S/k}\oplus\mathcal{N},\quad \mathcal{N}=\ker(e^*).$$

In Theorem 4.9, we extend this result to logarithmic differentials by showing that there is a unique splitting of

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k}(\log\pi^{-1}Z) \longrightarrow f_*\Omega^1_{E^\natural/S}(\log\pi^{-1}Z) \longrightarrow 0$$

of the form

$$f_*\Omega^1_{E^\natural/k}(\log\pi^{-1}Z) = \Omega^1_{S/k}\oplus\mathcal{N}\oplus\bigoplus_{n\geq 1}\mathcal{L}^{(n)},$$

where each $\mathcal{L}^{(n)}$ maps isomorphically onto $\mathcal{K}^{(n)}$, and for every $n\geq 1$

$$\mathcal{L}^{(n)}\wedge\mathcal{N}\wedge\mathcal{N} \equiv d\mathcal{N}\wedge\mathcal{L}^{(n+1)} \mod f_*\mathcal{F}^2.$$

Here, $f_*\mathcal{F}^2$ is the second step of the (direct image) Koszul filtration on $f_*\Omega^\bullet_{E^\natural/k}(\log\pi^{-1}Z)$, given by the ideal generated by $\Omega^2_{S/k}$ (see Section 4.2).

Locally over S, the above result implies that the Kronecker differentials admit canonical lifts to sections of $f_*\Omega^1_{E^\natural/k}(\log\pi^{-1}Z)$, which we denote by $\tilde{\nu}$, $\tilde{\omega}^{(0)}$, and $\tilde{\omega}_P^{(n)}$ for $n\geq 1$ and $P\in Z(S)$.

1.6.4. *Absolute KZB.* Finally, in Section 6, the *elliptic KZB connection* associated to $E/S/k$ punctured at Z is constructed as a suitable lift of the S-connection $\nabla_{E^\natural/S,Z}$ to an integrable k-connection

$$\nabla_{E^\natural/S/k,Z} : f^*\mathcal{H}^\vee_{E/S,Z} \longrightarrow \Omega^1_{E^\natural/k}(\log \pi^{-1}Z) \hat{\otimes} f^*\mathcal{H}^\vee_{E/S,Z}.$$

As explained in Section 1.2, the key idea is to obtain $\nabla_{E^\natural/S/k,Z}$ as a correction of the canonical lift of the relative connection $\tilde{\nabla}_{E^\natural/S/k,Z}$ by a certain k-connection on S, which is dual to a *Gauss–Manin connection*

$$\delta : \mathcal{H}_{E/S,Z} \longrightarrow \Omega^1_{S/k} \otimes \mathcal{H}_{E/S,Z}.$$

Our construction of δ is a bar complex variant of the usual Katz–Oda procedure [1968], and relies on the use of *relative bar complexes*. It depends crucially on Coleman and Laumon's result on the cohomology of universal vector extensions. We have also drawn inspiration from an analogous approach developed by Brown and Levin [2012]. In the C^∞ context, there is a similar construction due to Hain and Zucker [1987, Proposition 4.11].

Consider the dg-algebra $\mathcal{C} := f_*\Omega^\bullet_{E^\natural/k}(\log \pi^{-1}Z)$, which contains $\Omega := \Omega^\bullet_{S/k}$ as a dg-subalgebra, and form the relative bar complex

$$0 \longrightarrow B^0_\Omega(\mathcal{C}) \xrightarrow{d_B} B^1_\Omega(\mathcal{C}) \longrightarrow \cdots,$$

the definition of which is similar to the usual bar complex, but tensor products are now taken over the noncommutative ring Ω (see Section 6.1). Then, the Koszul filtration on $\mathcal{C}$ induces a decreasing filtration

$$B_\Omega(\mathcal{C}) = F^0B_\Omega(\mathcal{C}) \supset F^1B_\Omega(\mathcal{C}) \supset F^2B_\Omega(\mathcal{C}) \supset \cdots$$

satisfying $B_\Omega(\mathcal{C})/F^1B_\Omega(\mathcal{C}) \cong B(\mathcal{A})$. We construct a projection

$$\pi : F^1B_\Omega(\mathcal{C}) \longrightarrow \Omega^1_{S/k} \otimes B(\mathcal{A})[-1]$$

which factors through $F^1B_\Omega(\mathcal{C})/F^2B_\Omega(\mathcal{C})$, and we define

$$\delta(\xi) = -\pi(d_B\tilde{\xi}),$$

where $\tilde{\xi}$ is the canonical lift of the section ξ of $\mathcal{H}_{E/S,Z} = H^0(B(\mathcal{A}))$.

We prove in Theorems 6.6 and 6.10 that the above definition yields an integrable k-connection on $\mathcal{H}_{E/S,Z}$ which preserves the length filtration and restricts to the tensor power of the Gauss–Manin connection on the graded quotients $L_n\mathcal{H}_{E/S,Z}/L_{n-1}\mathcal{H}_{E/S,Z} \cong H^1_{\mathrm{dR}}((E \setminus Z)/S)^{\otimes n}$. Moreover, its continuous dual

$$\delta^\vee : \mathcal{H}^\vee_{E/S,Z} \longrightarrow \Omega^1_{S/k} \hat{\otimes} \mathcal{H}^\vee_{E/S,Z}$$

is a derivation of $\mathcal{H}^\vee_{E/S,Z}$ with coefficients in $\Omega^1_{S/k}$, and the k-connection

$$\nabla_{E^\natural/S/k,Z} : f^*\mathcal{H}^\vee_{E/S,Z} \longrightarrow \Omega^1_{E^\natural/k}(\log \pi^{-1}Z) \hat{\otimes} f^*\mathcal{H}^\vee_{E/S,Z}, \quad \nabla_{E^\natural/S/k,Z} = f^*\delta^\vee + \tilde{\omega}_{E^\natural/S,Z},$$

is an integrable lift of the relative KZB connection $\nabla_{E^\natural/S,Z}$.

2. The universal vector extension of an elliptic curve

2.1. *Definition.* Let S be a scheme. The *S-vector group* associated to a quasicoherent $\mathcal{O}_S$-module $\mathcal{F}$ is the S-group scheme $\mathbb{V}(\mathcal{F}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}\,\mathcal{F})$. If $\mathcal{F}$ is locally free of rank r, then $\mathbb{V}(\mathcal{F})$ is locally isomorphic to $(\mathbb{G}_{a,S})^r$.

Let $p : E \to S$ be an elliptic curve. The *universal vector extension* of E/S is a commutative S-group scheme $f : E^\natural \to S$ with a morphism of S-group schemes $\pi : E^\natural \to E$ fitting into an exact sequence (of abelian fppf sheaves over S)

$$0 \longrightarrow \mathbb{V}(R^1 p_*\mathcal{O}_E) \longrightarrow E^\natural \xrightarrow{\ \pi\ } E \longrightarrow 0, \tag{4}$$

which is universal for extensions of E by an S-vector group. Namely,

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, R^1 p_*\mathcal{O}_E) \longrightarrow \mathrm{Ext}_{S_{\mathrm{fppf}}}(E, \mathbb{V}(\mathcal{F})),$$
$$\varphi \longmapsto \text{the class of the pushout of (4) along } \mathbb{V}(\varphi) : \mathbb{V}(R^1 p_*\mathcal{O}_E) \to \mathbb{V}(\mathcal{F}),$$

is an isomorphism for any vector bundle $\mathcal{F}$ over S [Mazur and Messing 1974, Proposition 1.10] (see [Fonseca and Matthes 2024, Section 2.2]). Since $R^1 p_*\mathcal{O}_E$ is a line bundle over S, it follows from (4) that $\pi : E^\natural \to E$ is a $\mathbb{G}_a$-bundle. In particular, $f : E^\natural \to S$ is smooth, of finite presentation, and of relative dimension 2. Moreover, the formation of the universal vector extension is compatible with every base change $S' \to S$, meaning that there is a natural S'-isomorphism $E^\natural \times_S S' \cong (E \times_S S')^\natural$.

Example 2.1. Assume that $S = \mathrm{Spec}\, R$ is affine, with 6 invertible in R, and that E/S admits a Weierstrass equation of the form

$$E : y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3, \quad g_2, g_3 \in R,\ g_2^3 - 27 g_3^2 \in R^\times.$$

Let $q(x) = 4x^3 - g_2 x - g_3 \in R[x]$, and set

$$U_1 = \mathrm{Spec}\, \frac{R[x, y, t]}{(y^2 - q(x))}, \quad U_2 = \mathrm{Spec}\, \frac{R[x, z, t]}{(z - z^3 q(x/z))}.$$

Let U_{12} be the open subscheme of U_1 given by $y \neq 0$, and U_{21} be the open subscheme of U_2 given by $z \neq 0$. Then, $E^\natural$ is isomorphic to the gluing of U_1 and U_2 along the isomorphism $U_{12} \xrightarrow{\sim} U_{21}$ given on the corresponding R-algebras by

$$\frac{R[x, z^{\pm 1}, t]}{(z - z^3 q(x/z))} \xrightarrow{\sim} \frac{R[x, y^{\pm 1}, t]}{(y^2 - q(x))}, \quad (x, z, t) \longmapsto \left(\frac{x}{y}, \frac{1}{y}, t - \frac{q'(x)}{6y}\right),$$

where $q'(x) = 12x^2 - g_2$. This follows from the interpretation of $E^\natural$ as a moduli space of line bundles on E equipped with an S-connection [Katz 1977, C.1–C.2]. For instance, if R' is an R-algebra, a point $(a, b, c) \in U_1(R')$ corresponds to the isomorphism class of

$$(\mathcal{O}(P) \otimes \mathcal{O}(O)^{-1}, d + \omega_{P,c}), \quad \omega_{P,c} = \left(\frac{1}{2}\frac{y+b}{x-a} + c\right)\frac{dx}{y},$$

where $O = (0 : 1 : 0) \in E(R')$ is the identity, and $P = (a : b : 1) \in E(R')$.

In the category of complex analytic spaces, the universal vector extension of an elliptic curve is more conveniently described in terms of its uniformisation.

Example 2.2. Assume that S is locally of finite type over $\mathbb{C}$. Then the analytification $E^{\natural,\mathrm{an}}$ is uniformised by the rank-2 holomorphic vector bundle $V^{\mathrm{an}} := \mathbb{V}(H^1_{\mathrm{dR}}(E/S))^{\mathrm{an}}$ over S^{an}: there is an exact sequence of relative complex Lie groups over S^{an}

$$0 \longrightarrow L \longrightarrow V^{\mathrm{an}} \xrightarrow{\exp} E^{\natural,\mathrm{an}} \longrightarrow 0,$$

where L is the space over S^{an} corresponding to the locally constant sheaf $(R^1 p^{\mathrm{an}}_* \mathbb{Z})^\vee$. The map $L \to V^{\mathrm{an}}$ is induced by the morphism $(R^1 p^{\mathrm{an}}_* \mathbb{Z})^\vee \to (H^1_{\mathrm{dR}}(E/S)^{\mathrm{an}})^\vee$, which sends a locally constant family of topological 1-cycles γ to the functional $\alpha \mapsto \int_\gamma \alpha$. We refer to [Mazur and Messing 1974, I.4.4] for a proof. In the particular case where $S = \operatorname{Spec}\mathbb{C}$, let (ω, η) be a basis of $H^1_{\mathrm{dR}}(E/\mathbb{C})$. Then,

$$E^{\natural,\mathrm{an}} \cong \mathbb{C}^2/L, \quad L = \{(\textstyle\int_\gamma \omega, \int_\gamma \eta) \in \mathbb{C}^2 : \gamma \in H_1(E^{\mathrm{an}}, \mathbb{Z})\}.$$

2.2. *Coherent and de Rham cohomology.*

We keep the above notation.

Theorem 2.3 (Coleman, Laumon). *If S is of characteristic* 0, *then*

$$R^i f_* \mathcal{O}_{E^\natural} = \begin{cases} \mathcal{O}_S, & i = 0, \\ 0, & i \geq 1. \end{cases}$$

For a proof of the more general statement concerning universal vector extensions of abelian schemes, we refer to [Laumon 1996, Théorème 2.4.1] or to [Coleman 1998, Corollary 2.7]. For elliptic curves, one may also give an elementary proof by computing the Čech cohomology of the affine cover given in Example 2.1.

Remark 2.4 (algebraic versus analytic functions). Let $S = \operatorname{Spec}\mathbb{C}$. The above theorem implies that $\Gamma(E^\natural, \mathcal{O}_{E^\natural}) = \mathbb{C}$, i.e., that every global regular function on $E^\natural$ is constant. The analogous statement in the analytic category is *false*. In other words, $E^{\natural,\mathrm{an}}$ admits nonconstant holomorphic functions. In fact, it follows from Example 2.2 that $E^{\natural,\mathrm{an}}$ is isomorphic to $\mathbb{C}^2/\mathbb{Z}^2 \cong \mathbb{C}^\times \times \mathbb{C}^\times$.

From now on, we assume that S is of characteristic 0. Let $e \in E^\natural(S)$ be the identity section. Since $E^\natural/S$ is a smooth group scheme, we have for every $n \geq 0$ an isomorphism

$$f^* e^* \Omega^n_{E^\natural/S} \xrightarrow{\sim} \Omega^n_{E^\natural/S},$$

obtained by extending cotangent vectors at the identity to invariant differential forms via the group law. Then, it follows from the projection formula and Theorem 2.3 that

$$R^i f_* \Omega^n_{E^\natural/S} \cong \begin{cases} e^* \Omega^n_{E^\natural/S}, & i = 0, \\ 0, & i \geq 1. \end{cases} \tag{5}$$

The above equation for $i = 0$ means that every section of $f_* \Omega^n_{E^\natural/S}$ is an invariant differential n-form on $E^\natural/S$.

Proposition 2.5. *Every section of $f_*\Omega^n_{E^\natural/S}$ is a closed differential form. The induced map*

$$f_*\Omega^n_{E^\natural/S} \longrightarrow H^n_{\mathrm{dR}}(E^\natural/S)$$

is an isomorphism of $\mathcal{O}_S$-modules.

This statement is contained in [Fonseca and Matthes 2024, Propositions 2.4 and 2.7] (see also [Coleman 1998, Theorem 2.2]). We reproduce a short proof for completeness.

Proof. We have seen that sections of $f_*\Omega^n_{E^\natural/S}$ are invariant. That every invariant differential form is closed is a general property of smooth commutative group schemes; it is a consequence of the Maurer–Cartan equation. Now, that the natural maps $f_*\Omega^n_{E^\natural/S} \to H^n_{\mathrm{dR}}(E^\natural/S)$ are isomorphisms follows from the f_*-acyclicity of $\Omega^n_{E^\natural/S}$ (see (5)). □

In fact, the above isomorphism is also compatible with the natural product structures, yielding an isomorphism of dg-algebras over $\mathcal{O}_S$:

$$f_*\Omega^\bullet_{E^\natural/S} \cong H^\bullet_{\mathrm{dR}}(E^\natural/S).$$

Since $\pi : E^\natural \to E$ is a $\mathbb{G}_a$-bundle, it follows from the Künneth formula that $\pi^* : H^\bullet_{\mathrm{dR}}(E/S) \to H^\bullet_{\mathrm{dR}}(E^\natural/S)$ is also an isomorphism of dg-algebras over $\mathcal{O}_S$. Thus, we obtain an isomorphism

$$f_*\Omega^\bullet_{E^\natural/S} \cong H^\bullet_{\mathrm{dR}}(E/S). \tag{6}$$

Remark 2.6. By [Mazur and Messing 1974, I.4], applying the functor Lie_S to the exact sequence (4) gives rise to a short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow (R^1 p_*\mathcal{O}_E)^\vee \longrightarrow \mathrm{Lie}_S\, E^\natural \longrightarrow \mathrm{Lie}_S\, E \longrightarrow 0,$$

isomorphic to the dual of the Hodge–de Rham short exact sequence

$$0 \longrightarrow p_*\Omega^1_{E/S} \longrightarrow H^1_{\mathrm{dR}}(E/S) \longrightarrow R^1 p_*\mathcal{O}_E \longrightarrow 0. \tag{7}$$

The composition of isomorphisms $f_*\Omega^1_{E^\natural/S} \cong (\mathrm{Lie}_S\, E^\natural)^\vee \cong H^1_{\mathrm{dR}}(E/S)$ coincides with the isomorphism (6).

The fact that the dg-algebra $f_*\Omega^\bullet_{E^\natural/S}$ is a model for the de Rham cohomology of E/S can be generalised to punctured elliptic curves as follows.

Proposition 2.7. *Let $Z \subsetneq E$ be a smooth closed S-subscheme. For every $n \geq 0$, the natural maps*

$$H^n(f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)) \longrightarrow H^n_{\mathrm{dR}}(\pi^{-1}(E \setminus Z)/S) \xleftarrow{\ \pi^*\ } H^n_{\mathrm{dR}}((E \setminus Z)/S)$$

are isomorphisms.

Proof. That $\pi^* : H^n_{\mathrm{dR}}((E \setminus Z)/S) \to H^n_{\mathrm{dR}}(\pi^{-1}(E \setminus Z)/S)$ is an isomorphism follows from the Künneth formula and from the fact that $\pi : \pi^{-1}(E \setminus Z) \to E \setminus Z$ is a $\mathbb{G}_a$-bundle.

By Deligne's theorem [1970, Chapter II, Corollaire 3.15, Remarque 3.16], there is a canonical isomorphism

$$\mathbb{R}^n f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z) \xrightarrow{\sim} H^n_{\mathrm{dR}}(\pi^{-1}(E \setminus Z)/S).$$

We are left to prove that $\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)$ is a complex of f_*-acyclic sheaves. For this, let $i : \pi^{-1}Z \to E^\natural$ be the inclusion, and consider the Poincaré residue exact sequence

$$0 \longrightarrow \Omega^\bullet_{E^\natural/S} \longrightarrow \Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z) \xrightarrow{\text{Res}} i_*\Omega^\bullet_{\pi^{-1}Z/S}[-1] \longrightarrow 0.$$

As each $\Omega^n_{E^\natural/S}$ is f_*-acyclic by (5), and each $i_*\Omega^{n-1}_{\pi^{-1}Z/S}$ is f_*-acyclic by the fact that both i and $f\circ i$ are affine, we conclude from long exact sequence in cohomology associated to the above short exact sequence that each $\Omega^n_{E^\natural/S}(\log \pi^{-1}Z)$ is f_*-acyclic. □

3. Relative logarithmic differentials on the universal vector extension

3.1. ***Kronecker differentials.*** Recall that the sheaf of invariant differential 1-forms on the vector group $\mathbb{V}(R^1p_*\mathcal{O}_E)/S$ is canonically isomorphic to $R^1p_*\mathcal{O}_E$: a local section t of $R^1p_*\mathcal{O}_E$ corresponds to the relative 1-form dt. Thus, under the identification (6), the Hodge–de Rham exact sequence (7) corresponds to the short exact sequence of sheaves of invariant differential forms

$$0 \longrightarrow p_*\Omega^1_{E/S} \longrightarrow f_*\Omega^1_{E^\natural/S} \longrightarrow R^1p_*\mathcal{O}_E \longrightarrow 0, \tag{8}$$

where the left arrow is given by pullback by π, and the right arrow is given by restriction to $\mathbb{V}(R^1p_*\mathcal{O}_E) \cong \pi^{-1}O \hookrightarrow E^\natural$.

Assume that S is affine and that $R^1p_*\mathcal{O}_E$ is free with trivialisation t. Under the identification

$$f_*\mathcal{O}_{\pi^{-1}O} \cong \bigoplus_{n\geq 0}(R^1p_*\mathcal{O}_E)^{\otimes n} \cong \mathcal{O}_S[t],$$

we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\Omega^1_{E^\natural/S} & \longrightarrow & f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O) & \xrightarrow{\text{Res}} & \mathcal{O}_S[t] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & f_*\Omega^2_{E^\natural/S} & \longrightarrow & f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}O) & \xrightarrow{\text{Res}} & \mathcal{O}_S[t]dt & \longrightarrow & 0 \end{array} \tag{9}$$

where the rows are Poincaré residue exact sequences (see the proof of Proposition 2.7).

Lemma 3.1. *Let α be a global section of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O)$ satisfying* $\text{Res}(\alpha) = 1$. *Then d restricts to an isomorphism*

$$d : \mathcal{O}_S\alpha \xrightarrow{\sim} f_*\Omega^2_{E^\natural/S}.$$

Proof. Since $\text{Res}(d\alpha) = d\,\text{Res}(\alpha) = 0$, it follows from the residue exact sequence in degree 2 that $d\alpha$ is a section of the line bundle $f_*\Omega^2_{E^\natural/S}$, so that the map in the statement is well-defined. To verify that it is an isomorphism, it suffices to prove that it is surjective. Since $H^2(f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}O)) = 0$ by Proposition 2.7, for any section β of $f_*\Omega^2_{E^\natural/S}$ there is a section α' of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O)$ satisfying $d\alpha' = \beta$. By the commutativity of (9), $\text{Res}(\alpha')$ is in $\mathcal{O}_S$, so that $\text{Res}(\alpha')\alpha - \alpha'$ is in $f_*\Omega^1_{E^\natural/S}$. This shows that $d(\text{Res}(\alpha')\alpha) = \beta$. □

Remark 3.2. Note that α in the above statement is unique up to a section of $f_*\Omega^1_{E^\natural/S}$, so that $d\alpha$ is independent of any choice; it gives a canonical trivialisation of $f_*\Omega^2_{E^\natural/S}$. In particular, we obtain a symplectic pairing $\langle\cdot,\cdot\rangle : f_*\Omega^1_{E^\natural/S} \otimes f_*\Omega^1_{E^\natural/S} \to \mathcal{O}_S$ defined by $\langle\omega_1,\omega_2\rangle\, d\alpha = \omega_1 \wedge \omega_2$, which induces by (8) the classical isomorphism $p_*\Omega^1_{E/S} \cong (R^1 p_*\mathcal{O}_E)^\vee$. Under the identification (6), the pairing $\langle\cdot,\cdot\rangle$ is the usual de Rham pairing on $H^1_{\mathrm{dR}}(E/S)$ (see [Coleman 1998]).

Theorem 3.3. *Assume that S is affine, $R^1 p_*\mathcal{O}_E = \mathcal{O}_S t$ is free, and let ν be a global section of $f_*\Omega^1_{E^\natural/S}$ satisfying*

$$\nu|_{\pi^{-1}O} = dt.$$

Then, there exists a unique family $\{\omega^{(n)}\}_{n\geq 0}$ of global sections of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O)$ such that $\omega^{(0)}$ trivialises $p_*\Omega^1_{E/S}$ and, for every $n \geq 1$,

(i) $\mathrm{Res}(\omega^{(n)}) = t^{n-1}/(n-1)!$,

(ii) $\omega^{(n)} \wedge \omega^{(0)} = 0$, *and*

(iii) $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$.

Moreover, we have

$$f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O) = f_*\Omega^1_{E^\natural/S} \oplus \bigoplus_{n\geq 1} \mathcal{O}_S\omega^{(n)}, \qquad f_*\Omega^1_{E^\natural/S} = \mathcal{O}_S\nu \oplus \mathcal{O}_S\omega^{(0)}, \tag{10}$$

$$f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}O) = f_*\Omega^2_{E^\natural/S} \oplus \bigoplus_{n\geq 1} \mathcal{O}_S\nu\wedge\omega^{(n)}, \quad f_*\Omega^2_{E^\natural/S} = \mathcal{O}_S\nu \wedge \omega^{(0)}. \tag{11}$$

Proof. Let α_0 be any trivialisation of $p_*\Omega^1_{E/S}$ (note that $p_*\Omega^1_{E/S}$ is free by Remark 3.2), so that

$$f_*\Omega^1_{E^\natural/S} = \mathcal{O}_S\nu \oplus \mathcal{O}_S\alpha_0, \quad f_*\Omega^2_{E^\natural/S} = \mathcal{O}_S\nu\wedge\alpha_0. \tag{12}$$

It follows from the residue exact sequence (9) in degree 1 that, for every $n \geq 1$, there is a global section α_n of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O)$ such that

$$\mathrm{Res}(\alpha_n) = \frac{t^{n-1}}{(n-1)!}.$$

Our goal is to modify each α_n by a section of $f_*\Omega^1_{E^\natural/S}$ so that properties (ii) and (iii) are also satisfied.

Since the restriction of α_0 to $\pi^{-1}O$ vanishes by (8), the 2-form $\alpha_n \wedge \alpha_0$ is in $f_*\Omega^2_{E^\natural/S}$. Thus, by (12), up to adding a multiple of ν to α_n, we can assume that

$$\alpha_n \wedge \alpha_0 = 0.$$

Let $n \geq 1$. Since

$$\mathrm{Res}(d\alpha_n) = d\,\mathrm{Res}(\alpha_n) = d\left(\frac{t^{n-1}}{(n-1)!}\right) = \frac{t^{n-2}\,dt}{(n-2)!} = \mathrm{Res}(\nu\wedge\alpha_{n-1}),$$

it follows from the residue exact sequence (9) in degree 2 and from (12) that there exists a global section r_{n-1} of $\mathcal{O}_S$ such that

$$d\alpha_n = \nu\wedge\alpha_{n-1} + r_{n-1}\nu\wedge\alpha_0.$$

For every $n \geq 0$, we set

$$\omega^{(n)} := \alpha_n + r_n \alpha_0.$$

Thus, $\omega^{(0)}$ is a global section of $p_*\Omega^1_{E/S}$ and $\omega^{(n)}$ are global sections of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O)$ satisfying (i), (ii), and (iii). Furthermore, since $d\omega^{(1)} = \nu \wedge \omega^{(0)}$, it follows from Lemma 3.1 that $\nu \wedge \omega^{(0)}$ is a trivialisation of $f_*\Omega^2_{E^\natural/S}$, so that $\{\nu, \omega^{(0)}\}$ is a trivialisation of $f_*\Omega^1_{E^\natural/S}$. Then, (10) and (11) follow from the residue exact sequence (9).

To prove uniqueness, let $\{\lambda_n\}_{n\geq 0}$ be a family of global sections of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}O)$ such that λ_0 trivialises $p_*\Omega^1_{E/S}$ and λ_n satisfies (i), (ii), and (iii) for $n \geq 1$. Then, we can write $\lambda_0 = u\omega^{(0)}$ for some $u \in \Gamma(S, \mathcal{O}_S^\times)$, and, by (i), for every $n \geq 1$,

$$\lambda_n = \omega^{(n)} + a_n \omega^{(0)} + b_n \nu$$

for some $a_n, b_n \in \Gamma(S, \mathcal{O}_S)$. From (ii), we conclude that $b_n = 0$. Finally, property (iii) applied to the family $\{\lambda_n\}_{n\geq 0}$ yields

$$\nu \wedge \omega^{(n-1)} = \begin{cases} u\nu \wedge \omega^{(0)}, & n = 1, \\ \nu \wedge \omega^{(n-1)} + a_{n-1}\nu \wedge \omega^{(0)}, & n \geq 2, \end{cases}$$

which implies that $u = 1$ and $a_n = 0$ for every $n \geq 1$. □

Remark 3.4. It follows from uniqueness and from Theorem 3.3(i), (ii), and (iii), that a change in ν to $\nu' = u\nu + v\omega^{(0)}$, with $u \in \Gamma(S, \mathcal{O}_S^\times)$ and $v \in \Gamma(S, \mathcal{O}_S)$, changes $\omega^{(n)}$ to $\omega^{(n)\prime} = u^{n-1}\omega^{(n)}$.

Corollary 3.5. *With notation as in Theorem 3.3, we have $\omega^{(n)} \wedge \omega^{(m)} = 0$ for every $n, m \geq 0$.*

Proof. Since $E^\natural/S$ is smooth of relative dimension 2, in a formal neighbourhood of any point in the smooth relative effective Cartier divisor $\pi^{-1}O$, we can find S-coordinates (x, t) such that $dx = \omega^{(0)}$, $dt = \nu$, so that $\pi^{-1}O$ is given by $x = 0$. Since $\omega^{(n)}$ has logarithmic singularities along $\pi^{-1}O$, there are power series $F_n(x, t)$, $G_n(x, t)$ with coefficients in $\mathcal{O}_S$ such that $\omega^{(n)} = F_n(x, t)(dx/x) + G_n(x, t)\, dt$. By equation (ii) of Theorem 3.3, we have $G_n(x, t) = 0$, so that $\omega^{(n)} = F_n(x, t)(dx/x)$. The statement follows immediately. □

3.2. ***Kronecker subbundles.*** Let $Z \subsetneq E$ be a closed subscheme of E given by the union of *torsion* sections $P \in E(S)$. Assume moreover that Z contains the identity section: $O \in Z(S)$.

Example 3.6. We can always take $Z = O$. If E/S admits a full level-N structure, we can also consider $Z = E[N]$, or any other subscheme thereof which is flat over S and contains the identity.

By pulling Z back by the projection $\pi : E^\natural \to E$, we obtain a divisor $\pi^{-1}Z$ on $E^\natural$, which can be written as a disjoint union:

$$\pi^{-1}Z = \bigsqcup_{P \in Z(S)} \pi^{-1}P.$$

Since $P \in Z(S)$ is torsion and $\pi : E^\natural \to E$ is a $\mathbb{G}_a$-bundle, P admits a unique lift to a torsion section $P^\natural \in E^\natural(S)$ [Katz 1977, Lemma C.1.1]. Thus, we obtain a trivialisation

$$\pi^{-1}Z \xrightarrow{\sim} Z \times_S \mathbb{V}(R^1 p_*\mathcal{O}_E), \tag{13}$$

given on the component $\pi^{-1}P$ by $x \mapsto (P, x - P^\natural)$. This yields an isomorphism

$$f_*\mathcal{O}_{\pi^{-1}Z} \cong p_*\mathcal{O}_Z \otimes \bigoplus_{n\geq 0}(R^1 p_*\mathcal{O}_E)^{\otimes n} \cong \bigoplus_{n\geq 0} p_*\mathcal{O}_Z \otimes (R^1 p_*\mathcal{O}_E)^{\otimes n}, \tag{14}$$

where tensor products are taken over $\mathcal{O}_S$.

Theorem 3.7. *There is a decomposition*

$$f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z) = f_*\Omega^1_{E^\natural/S} \oplus \bigoplus_{n\geq 1} \mathcal{K}^{(n)},$$

where $\mathcal{K}^{(n)}$ are subbundles of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z)$ uniquely determined by the following properties:

(i) *Under the identification* (14), *the residue map*

$$\mathrm{Res} : f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \longrightarrow f_*\mathcal{O}_{\pi^{-1}Z}$$

restricts to an isomorphism between $\mathcal{K}^{(n)}$ and $p_*\mathcal{O}_Z \otimes (R^1 p_*\mathcal{O}_E)^{\otimes(n-1)}$.

(ii) $\mathcal{K}^{(n)} \wedge \mathcal{K}^{(0)} = 0$ *in* $f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}Z)$, *where*

$$\mathcal{K}^{(0)} := p_*\Omega^1_{E/S}$$

is seen as a rank-1 subbundle of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z)$ via pullback by π.

(iii) $d\mathcal{K}^{(n)} = f_*\Omega^1_{E^\natural/S} \wedge \mathcal{K}^{(n-1)}$ *in* $f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}Z)$.

Moreover, we have

$$f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}Z) = \bigoplus_{n\geq 0} f_*\Omega^1_{E^\natural/S} \wedge \mathcal{K}^{(n)}.$$

Proof. It suffices to prove existence and uniqueness locally over S. Thus, we may assume that S is affine and that $R^1 p_*\mathcal{O}_E = \mathcal{O}_S t$ is free. Let $\nu, \omega^{(0)}, \omega^{(1)}, \ldots$ be as in Theorem 3.3. For any $P \in Z(S)$, let $\tau_{-P^\natural} : E^\natural \to E^\natural$ be the translation by $-P^\natural$, and set, for every $n \geq 0$

$$\omega_P^{(n)} := \tau^*_{-P^\natural}\omega^{(n)} \in \Gamma(S, f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}P)). \tag{15}$$

Let us also denote by $t_P = \tau^*_{-P^\natural}t$ the coordinate on $\pi^{-1}P$ induced by t under the isomorphism (13). Since $\nu, \omega^{(0)}$ are invariant — in particular, $\omega_P^{(0)} = \omega^{(0)}$ — it follows from Theorem 3.3 that, for every $n \geq 1$,

(a) $\mathrm{Res}(\omega_P^{(n)}) = t_P^{n-1}/(n-1)!$,

(b) $\omega_P^{(n)} \wedge \omega^{(0)} = 0$, and

(c) $d\omega_P^{(n)} = \nu \wedge \omega_P^{(n-1)}$.

For $n \geq 1$, set

$$\mathcal{K}^{(n)} := \bigoplus_{P \in Z(S)} \mathcal{K}_P^{(n)}, \quad \mathcal{K}_P^{(n)} := \mathcal{O}_S \omega_P^{(n)}.$$

Then, the residue exact sequence, together with (a), (b), and (c), show that $\mathcal{K}^{(n)}$ so defined satisfies all the properties in the statement.

To prove uniqueness, let $\{\mathcal{L}_n\}_{n \geq 1}$ be another family of subbundles of $f_* \Omega^1_{E^\natural/S}(\log \pi^{-1} Z)$ satisfying (i), (ii), and (iii). By (i), $\mathcal{L}_n$ is trivialised by 1-forms $\lambda_{n,P}$, $P \in Z(S)$, satisfying (a). Then, by the residue exact sequence, there are global sections $a_{n,P}$, $b_{n,P}$ of $\mathcal{O}_S$ such that

$$\lambda_{n,P} = \omega_P^{(n)} + a_{n,P}\omega^{(0)} + b_{n,P}\nu.$$

It follows from (ii) that $\lambda_{n,P}$ satisfies (b), so that $b_{n,P} = 0$. By (iii), we have

$$d\lambda_{n,P} = c_{n,P}\nu \wedge \lambda_{n-1,P}$$

for some global section $c_{n,P}$ of $\mathcal{O}_S$, where we set $\lambda_{0,P} := \omega^{(0)}$. Thus, also using (c) for $\omega_P^{(n)}$, we obtain, for $n = 1$,

$$c_{1,P}\nu \wedge \omega^{(0)} = \nu \wedge \omega^{(0)},$$

and, for $n \geq 2$,

$$c_{n,P}\nu \wedge \omega_P^{(n-1)} + c_{n,P}a_{n-1,P}\nu \wedge \omega^{(0)} = \nu \wedge \omega_P^{(n-1)}.$$

Thus, $c_{1,P} = 1$, and, by taking residues along $\pi^{-1}P$, we see that $c_{n,P} = 1$ for $n \geq 2$. Then, it follows from the same equation that $a_{n-1,P} = 0$. We conclude that $\lambda_{n,P} = \omega_P^{(n)}$, which yields $\mathcal{L}_n = \mathcal{K}^{(n)}$. □

We shall always denote by

$$\mathcal{K}^{(n)} = \bigoplus_{P \in Z(S)} \mathcal{K}_P^{(n)}$$

the decomposition induced by part (i) of Theorem 3.7 and $p_*\mathcal{O}_Z \cong \bigoplus_{P \in Z(S)} \mathcal{O}_S$. When $R^1 p_* \mathcal{O}_E$ is free, a choice of ν as in Theorem 3.3 induces trivialisations $\omega_P^{(n)}$ of $\mathcal{K}_P^{(n)}$ as in the proof of the above theorem, so that

$$f_*\Omega^1_{E^\natural/S}(\log \pi^{-1} Z) = \mathcal{O}_S \nu \oplus \mathcal{O}_S \omega^{(0)} \oplus \bigoplus_{\substack{n \geq 1 \\ P \in Z(S)}} \mathcal{O}_S \omega_P^{(n)}, \tag{16}$$

$$f_*\Omega^2_{E^\natural/S}(\log \pi^{-1} Z) = \mathcal{O}_S \nu \wedge \omega^{(0)} \oplus \bigoplus_{\substack{n \geq 1 \\ P \in Z(S)}} \mathcal{O}_S \nu \wedge \omega_P^{(n)}. \tag{17}$$

3.3. ***The dg-algebra of relative logarithmic differentials.*** With same hypotheses and notation as in the last sections, consider the dg-algebra over $\mathcal{O}_S$

$$\mathcal{A} := f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1} Z),$$

and consider the submodule of $\mathcal{A}^1$

$$\mathcal{K} = \bigoplus_{n \geq 0} \mathcal{K}^{(n)}.$$

It follows from Theorem 3.7 that the dg-algebra $\mathcal{A}$ is generated by its degree-1 sections. More precisely, we have the following structure result.

Corollary 3.8. *$\mathcal{A}$ is locally free as an $\mathcal{O}_S$-module, and the wedge product induces an isomorphism of graded $\mathcal{O}_S$-algebras*

$$\bigwedge^\bullet \mathcal{A}^1/\langle \bigwedge^2 \mathcal{K}\rangle \xrightarrow{\sim} \mathcal{A}.$$

Proof. That $\mathcal{A}$ is locally free as an $\mathcal{O}_S$-module follows from Theorem 3.7 and from the fact that $R^1 p_* \mathcal{O}_E$ is locally free (see (16) and (17)).

It follows from Theorem 3.7 (see also Corollary 3.5) that the wedge product gives a well-defined morphism of graded $\mathcal{O}_S$-algebras $\bigwedge^\bullet \mathcal{A}^1/\langle \bigwedge^2 \mathcal{K}\rangle \to \mathcal{A}$. This is an isomorphism in degree 0 by Theorem 2.3, and is trivially an isomorphism in degree 1. To see that it is also an isomorphism in higher degrees, we may argue locally over S. Thus, we may assume that $\mathcal{A}^1/\mathcal{K} \cong R^1 p_* \mathcal{O}_E$ is free (of rank 1), and that the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^1/\mathcal{K} \longrightarrow 0$$

splits

$$\mathcal{A}^1 \cong (\mathcal{A}^1/\mathcal{K}) \oplus \mathcal{K}.$$

This implies that

$$\bigwedge^2 \mathcal{A}^1 \cong ((\mathcal{A}^1/\mathcal{K}) \otimes \mathcal{K}) \oplus \bigwedge^2 \mathcal{K} \cong \mathcal{A}^2 \oplus \bigwedge^2 \mathcal{K},$$

and, for $n \geq 3$,

$$\bigwedge^n \mathcal{A}^1 \cong \big((\mathcal{A}^1/\mathcal{K}) \otimes \bigwedge^{n-1} \mathcal{K}\big) \oplus \bigwedge^n \mathcal{K} \subset \mathrm{im}\big(\bigwedge^{n-2} \mathcal{A}^1 \otimes \bigwedge^2 \mathcal{K} \longrightarrow \bigwedge^n \mathcal{A}^1\big). \qquad \square$$

Our next result concerns the cohomology of $\mathcal{A}$. It follows from $\mathcal{A}^0 = \mathcal{O}_S$ (Theorem 2.3) that $H^0(\mathcal{A}) = \mathcal{O}_S$ and that

$$H^1(\mathcal{A}) = \ker(d : \mathcal{A}^1 \longrightarrow \mathcal{A}^2) \subset \mathcal{A}^1.$$

Moreover, it follows from Proposition 2.7 or from Theorem 3.7 that $H^n(\mathcal{A}) = 0$ for $n \geq 2$.

Theorem 3.9. *We have a decomposition of $\mathcal{O}_S$-modules*

$$\mathcal{A}^1 = H^1(\mathcal{A}) \oplus \mathcal{K}_O^{(1)} \oplus \bigoplus_{n\geq 2} \mathcal{K}^{(n)}.$$

Let $\rho^1 : \mathcal{A}^1 \to H^1(\mathcal{A})$ be the projector induced by the above decomposition, and

$$\rho : \mathcal{A} \longrightarrow H^\bullet(\mathcal{A})$$

be the induced morphism of graded $\mathcal{O}_S$-algebras ($\rho^0 = \mathrm{id}$, and $\rho^n = 0$ for $n \geq 2$). Then, ρ is a dg-quasi-isomorphism.

Proof. The second assertion follows immediately from the (see Proposition 2.7). To prove the decomposition, we may work locally over S. Let $\nu, \omega^{(0)}, \omega_P^{(1)}, \omega_P^{(2)}, \ldots$ be as in (16). It follows from the equations

$$d\omega^{(0)} = d\nu = 0, \quad d\omega_P^{(1)} = \nu \wedge \omega^{(0)}, \quad d\omega_P^{(n)} = \nu \wedge \omega_P^{(n-1)} \quad (n \geq 2),$$

that

$$H^1(\mathcal{A}) = \mathcal{O}_S\nu \oplus \mathcal{O}_S\omega^{(0)} \oplus \bigoplus_{P\in Z(S)\setminus\{O\}} \mathcal{O}_S(\omega_P^{(1)} - \omega_O^{(1)}).$$

To conclude, we simply remark that

$$\mathcal{K}^{(1)} = \bigoplus_{P\in Z(S)} \mathcal{O}_S\omega_P^{(1)} = \mathcal{O}_S\omega_O^{(1)} \oplus \bigoplus_{P\in Z(S)\setminus\{O\}} \mathcal{O}_S(\omega_P^{(1)} - \omega_O^{(1)}).$$ □

4. Canonical lifts of relative logarithmic differentials

4.1. *Canonical lifts of regular differentials.* Keep the same notation and hypotheses of Section 3.2, and assume moreover that S is smooth over a field k of characteristic 0.

We review some of the results of [Fonseca and Matthes 2024]. It follows from the smoothness of $f : E^\natural \to S$ that we have an exact sequence of $\mathcal{O}_{E^\natural}$-modules

$$0 \longrightarrow f^*\Omega^1_{S/k} \longrightarrow \Omega^1_{E^\natural/k} \longrightarrow \Omega^1_{E^\natural/S} \longrightarrow 0.$$

By Theorem 2.3, we obtain an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k} \longrightarrow f_*\Omega^1_{E^\natural/S} \longrightarrow 0,$$

which admits a canonical splitting, as follows.

Proposition 4.1. *The pullback by the identity section $e \in E^\natural(S)$ induces a retraction $e^* : f_*\Omega^1_{E^\natural/k} \to \Omega^1_{S/k}$. In particular, if $\mathcal{N} := \ker(e^*)$, we obtain a decomposition*

$$f_*\Omega^1_{E^\natural/k} = \Omega^1_{S/k} \oplus \mathcal{N}, \quad \mathcal{N} \cong f_*\Omega^1_{E^\natural/S}.$$

Proof. See [Fonseca and Matthes 2024, Theorem 3.2]. □

It follows that any section ω of $f_*\Omega^1_{E^\natural/S}$ lifts canonically to a section of $\mathcal{N} \subset f_*\Omega^1_{E^\natural/k}$, which we denote by $\tilde{\omega}$. For the next result, recall that $H^1_{\mathrm{dR}}(E/S)$ is endowed with an integrable k-connection, the *Gauss–Manin connection*. Under the identification (6), we get an integrable k-connection

$$\nabla : f_*\Omega^1_{E^\natural/S} \longrightarrow \Omega^1_{S/k} \otimes f_*\Omega^1_{E^\natural/S}.$$

Proposition 4.2. *There is a canonical isomorphism $\Omega^1_{S/k} \otimes f_*\Omega^1_{E^\natural/S} \cong f_*\Omega^2_{E^\natural/k}/\Omega^2_{S/k}$, under which we have*

$$\nabla\omega = d\tilde{\omega} \bmod \Omega^2_{S/k}$$

for any section ω of $f_*\Omega^1_{E^\natural/S}$.

Proof. See [Fonseca and Matthes 2024, Propositions 2.6 and 3.5]. □

Example 4.3. If $\omega^{(0)}$, ν is a trivialisation of $f_*\Omega^1_{E^\natural/S}$ as in Section 3.1, and $\alpha_{ij} \in \Gamma(S, \Omega^1_{S/k})$ are defined by

$$\nabla\omega^{(0)} = \alpha_{11} \otimes \omega^{(0)} + \alpha_{21} \otimes \nu,$$

$$\nabla\nu = \alpha_{12} \otimes \omega^{(0)} + \alpha_{22} \otimes \nu,$$

then the canonical lifts satisfy (see [Fonseca and Matthes 2024, Remark 3.7])

$$\begin{aligned} d\tilde{\omega}^{(0)} &= \alpha_{11}\wedge\tilde{\omega}^{(0)} + \alpha_{21}\wedge\tilde{\nu}, \\ d\tilde{\nu} &= \alpha_{12}\wedge\tilde{\omega}^{(0)} + \alpha_{22}\wedge\tilde{\nu}. \end{aligned} \tag{18}$$

Note that the above equations hold "on the nose", and not only modulo $\Omega^2_{S/k}$.

For later reference, we consider the following auxiliary results.

Lemma 4.4. *Assume that S is affine, that $R^1p_*\mathcal{O}_E$ is trivial, and let $\omega^{(0)}$, ν, and t be as in Theorem 3.3. Then*

$$\tilde{\omega}^{(0)}|_{\pi^{-1}O} = -\alpha_{21}t, \quad \tilde{\nu}|_{\pi^{-1}O} = dt - \alpha_{22}t.$$

Proof. As $\pi^{-1}O$ is isomorphic to $\mathbb{G}_{a,S}$ via the coordinate t, we have

$$\Gamma(S, f_*\Omega^1_{\pi^{-1}O/k}) \cong \Gamma(S,\Omega^1_{S/k})[t] + \Gamma(S,\mathcal{O}_S)[t]\,dt.$$

Thus, since $\omega^{(0)}|_{\pi^{-1}O} = 0$ and $\nu|_{\pi^{-1}O} = dt$, we can write

$$\tilde{\omega}^{(0)}|_{\pi^{-1}O} = \sum_{n\geq 0}\gamma_n t^n, \quad \tilde{\nu}|_{\pi^{-1}O} = dt + \sum_{n\geq 0}\delta_n t^n \tag{19}$$

for unique $\gamma_n, \delta_n \in \Gamma(S, \Omega^1_{S/k})$, with $\gamma_n = \delta_n = 0$ for $n \gg 0$. The condition $e^*\tilde{\omega}^{(0)} = e^*\tilde{\nu} = 0$ implies that $\gamma_0 = \delta_0 = 0$. Plugging (19) into (18), a short calculation shows that $\gamma_n = \delta_n = 0$ for $n \geq 2$, and that $\gamma_1 = -\alpha_{21}$ and $\delta_1 = -\alpha_{22}$. □

Lemma 4.5. *Let $\tilde{\omega} \in \Gamma(S,\mathcal{N})$ and $Q \in E^\natural(S)$ be a torsion section. If $\tau_Q : E^\natural \to E^\natural$ denotes the translation by Q, then $\tau_Q^*\tilde{\omega} = \tilde{\omega}$. In particular, global sections of $\mathcal{N}$ are invariant under translation by $P^\natural \in E^\natural(S)$ for every $P \in Z(S)$.*

Proof. Since every section of $f_*\Omega^1_{E^\natural/S}$ is invariant (see Section 2.2), there exists $\beta \in \Gamma(S, \Omega^1_{S/k})$ such that $\tau_Q^*\tilde{\omega} = \tilde{\omega} + \beta$. By induction, for any $n \geq 1$, we obtain

$$\tau_{nQ}^*\tilde{\omega} = \tilde{\omega} + n\beta.$$

By pulling back the above equation by e, we get

$$(nQ)^*\tilde{\omega} = e^*\tilde{\omega} + n\beta = n\beta,$$

where we have also used that $e^*\tilde{\omega} = 0$. Since Q is torsion and S is of characteristic 0, we conclude that $\beta = 0$. □

4.2. *Canonical lifts of Kronecker differentials.* Define the *Koszul filtration* $\mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots$ on the complex $\Omega^\bullet_{E^\natural/k}(\log \pi^{-1}Z)$ by

$$\mathcal{F}^p := \operatorname{im}\bigl(f^*\Omega^p_{S/k} \otimes \Omega^\bullet_{E^\natural/k}(\log\pi^{-1}Z)[-p] \longrightarrow \Omega^\bullet_{E^\natural/k}(\log\pi^{-1}Z)\bigr),$$

where the above map is given by the wedge product. This gives rise to a filtration $f_*\mathcal{F}^p$ on $f_*\Omega^\bullet_{E^\natural/k}(\log\pi^{-1}Z)$ by direct image on each component.

Proposition 4.6. *For every $p \geq 0$, there is a canonical isomorphism of complexes of $\mathcal{O}_S$-modules*

$$f_*\mathcal{F}^p/f_*\mathcal{F}^{p+1} \cong \Omega^p_{S/k} \otimes f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)[-p].$$

Proof. Using that

$$0 \longrightarrow f^*\Omega^1_{S/k} \longrightarrow \Omega^1_{E^\natural/k}(\log \pi^{-1}Z) \longrightarrow \Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \longrightarrow 0$$

is locally split, and that $\Omega^\bullet_{E^\natural/k}(\log \pi^{-1}Z) = \bigwedge^\bullet \Omega^1_{E^\natural/k}(\log \pi^{-1}Z)$, we see that the Koszul filtration satisfies

$$\mathcal{F}^p/\mathcal{F}^{p+1} \cong f^*\Omega^p_{S/k} \otimes \Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)[-p] \quad \text{for every } p \geq 0. \tag{20}$$

Thus, by the projection formula, it suffices to check that each $\mathcal{F}^p$ is a complex of f_*-acyclic $\mathcal{O}_{E^\natural}$-modules. This statement is local on S, so we may assume that $S \to \operatorname{Spec} k$ is finitely presented, which implies in particular that $\mathcal{F}^N = 0$ for some $N \geq 0$. We prove the desired assertion, which is trivially true for $p \geq N$, by descending induction on p. By (20), we have a short exact sequence

$$0 \longrightarrow \mathcal{F}^{p+1} \longrightarrow \mathcal{F}^p \longrightarrow f^*\Omega^p_{S/k} \otimes \Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)[-p] \longrightarrow 0.$$

Note that $f^*\Omega^p_{S/k} \otimes \Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)[-p]$ is a complex of f_*-acyclic $\mathcal{O}_{E^\natural}$-modules by the same argument in the proof of Proposition 2.7. Thus, if $\mathcal{F}^{p+1}$ is also a complex of f_*-acyclic $\mathcal{O}_{E^\natural}$-modules, so is $\mathcal{F}^p$ by the long exact sequence in cohomology. □

Theorem 4.7. *Assume that S is affine, that $R^1 p_*\mathcal{O}_E$ is free, and let $\nu, \omega^{(0)}, \omega^{(1)}, \ldots$ be as in Theorem 3.3. There is a unique family $\{\tilde{\omega}^{(n)}\}_{n\geq 1}$ of global sections of $f_*\Omega^1_{E^\natural/k}(\log \pi^{-1}O)$ such that, for every $n \geq 1$, $\tilde{\omega}^{(n)}$ is a lift of $\omega^{(n)}$ satisfying*

$$\tilde{\omega}^{(n)} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv n\alpha_{21} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(n+1)} \mod f_*\mathcal{F}^2,$$

where $\alpha_{21} \in \Gamma(S, \Omega^1_{S/k})$ is a coefficient of the Gauss–Manin connection as in Example 4.3.

Proof. We first prove uniqueness. Any other lift of $\omega^{(n)}$ would be of the form $\tilde{\omega}^{(n)} + \beta_n$ for some $\beta_n \in \Gamma(S, \Omega^1_{S/k})$. By the condition in the statement, we deduce that

$$\beta_n \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv 0 \mod f_*\mathcal{F}^2.$$

In other words, the image of $\beta_n \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)}$ in $f_*\mathcal{F}^1/f_*\mathcal{F}^2$ vanishes. By Proposition 4.6, this means that $\beta_n \otimes \nu \wedge \omega^{(0)} = 0$ in $\Omega^1_{S/k} \otimes f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}O)$, which implies that $\beta_n = 0$, since $\nu \wedge \omega^{(0)}, \nu \wedge \omega^{(1)}, \ldots$ is a trivialisation of $f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}O)$ by Theorem 3.3.

For the existence, let $\varphi_n \in \Gamma(S, f_*\Omega^1_{E^\natural/k}(\log \pi^{-1}O))$ be any lift of $\omega^{(n)}$. By Proposition 4.6 and the fact that $f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}O)$ is trivialised by $\nu \wedge \omega^{(0)}, \nu \wedge \omega^{(1)}, \ldots$, there exist unique $\gamma_{n,i} \in \Gamma(S, \Omega^1_{S/k})$ such that

$$\varphi_n \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv \sum_{i\geq 0} \gamma_{n,i} \wedge \tilde{\nu} \wedge \varphi_i \mod f_*\mathcal{F}^2.$$

We take residues along $\pi^{-1}O$ on both sides of the above equation. By Theorem 3.3 and Lemma 4.4, we get on the one hand

$$\mathrm{Res}(\varphi_n \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)}) = \frac{t^n}{(n-1)!}\alpha_{21} \wedge dt + \frac{t^{n+1}}{(n-1)!}\alpha_{22} \wedge \alpha_{21},$$

and, on the other hand,

$$\mathrm{Res}(\gamma_{n,i} \wedge \tilde{\nu} \wedge \varphi_i) = \begin{cases} 0, & i = 0, \\ \dfrac{t^{i-1}}{(i-1)!}\gamma_{n,i} \wedge dt + \dfrac{t^i}{(i-1)!}\alpha_{22} \wedge \gamma_{n,i}, & i \geq 1. \end{cases}$$

In particular, we get

$$\frac{t^n}{(n-1)!}\alpha_{21} \wedge dt = \sum_{i \geq 1} \frac{t^{i-1}}{(i-1)!}\gamma_{n,i} \wedge dt,$$

so that $\gamma_{n,n+1} = n\alpha_{21}$ and $\gamma_{n,i} = 0$ for $i \notin \{0, n+1\}$. We conclude that

$$\varphi_n \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv \gamma_{n,0} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} + n\alpha_{21} \wedge \tilde{\nu} \wedge \varphi_{n+1} \mod f_*\mathcal{F}^2.$$

Thus, $\tilde{\omega}^{(n)} := \varphi_n - \gamma_{n,0}$ are lifts of $\omega^{(n)}$ satisfying the equation in the statement. □

Remark 4.8. If we consider $\nu' = u\nu + v\omega^{(0)}$ as in Remark 3.4, then, by uniqueness, the canonical lifts of the corresponding Kronecker differentials are $\tilde{\omega}^{(n)\prime} = u^{n-1}\tilde{\omega}^{(n)}$.

Theorem 4.9. *Let $\mathcal{K}^{(n)}$ be the Kronecker subbundles of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z)$ defined in Theorem 3.7. For every $n \geq 1$, there is a unique lift of $\mathcal{K}^{(n)}$ to a subbundle $\mathcal{L}^{(n)}$ of $f_*\Omega^1_{E^\natural/k}(\log \pi^{-1}Z)$ such that*

$$\mathcal{L}^{(n)} \wedge \mathcal{N} \wedge \mathcal{N} \equiv d\mathcal{N} \wedge \mathcal{L}^{(n+1)} \mod f_*\mathcal{F}^2. \tag{21}$$

Moreover, we have

$$f_*\Omega^1_{E^\natural/k}(\log \pi^{-1}Z) = \Omega^1_{S/k} \oplus \mathcal{N} \oplus \bigoplus_{n \geq 1} \mathcal{L}^{(n)}.$$

Proof. We may work locally over S, so that the hypotheses of Theorem 4.7 are satisfied. The proof is similar to that of Theorem 3.7: for every $P \in Z(S)$, we set

$$\tilde{\omega}_P^{(n)} := \tau^*_{-P^\natural}\tilde{\omega}^{(n)}. \tag{22}$$

Using that $\tilde{\nu}$ and $\tilde{\omega}^{(0)}$ are invariant under translation by $P^\natural$ (Lemma 4.5), we obtain

$$d\tilde{\omega}_P^{(n)} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv n\alpha_{21} \wedge \tilde{\nu} \wedge \tilde{\omega}_P^{(n+1)} \mod f_*\mathcal{F}^2.$$

Thus, the subbundles

$$\mathcal{L}^{(n)} := \bigoplus_{P \in Z(S)} \mathcal{L}_P^{(n)}, \qquad \mathcal{L}_P^{(n)} = \mathcal{O}_S\tilde{\omega}_P^{(n)},$$

satisfy (21). The second assertion follows immediately from the exactness of

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k}(\log \pi^{-1}Z) \longrightarrow f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \longrightarrow 0$$

and from Theorem 3.7. □

For uniqueness, let $\mathcal{L}^{(n)\prime}$ be another family of subbundles of $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z)$ satisfying (21). Since $\mathcal{L}^{(n)\prime}$ is isomorphic to $\mathcal{K}^{(n)} = \bigoplus_{P\in Z(S)} \mathcal{K}^{(n)}_P$, we have a decomposition $\mathcal{L}^{(n)\prime} = \bigoplus_{P\in Z(S)} \mathcal{L}^{(n)\prime}_P$. Let $\tilde{\omega}^{(n)\prime}_P$ be the trivialisation of $\mathcal{L}^{(n)\prime}_P$ corresponding to the trivialisation $\omega^{(n)}_P$ of $\mathcal{K}^{(n)}_P$. Let $\beta^n_P \in \Gamma(S, \Omega^1_{S/k})$ be such that

$$\tilde{\omega}^{(n)\prime}_P = \tilde{\omega}^{(n)}_P + \beta^n_P.$$

Since $\pi^{-1}Z = \bigsqcup_{P\in Z(S)} \pi^{-1}P$, and $\tilde{\omega}^{(n)\prime}_P$ has singularities only along the component $\pi^{-1}P$, equation (21) implies that, for every $P \in Z(S)$, we have

$$\mathcal{O}_S\tilde{\omega}^{(n)\prime}_P \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} \equiv \mathcal{O}_S\alpha_{21} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(n+1)\prime}_P \mod f_*\mathcal{F}^2.$$

Under the identification $f_*\mathcal{F}^1/f_*\mathcal{F}^2 \cong \Omega^1_{S/k} \otimes f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z)$ of Proposition 4.6, we deduce that

$$\beta^n_P \otimes \nu \wedge \omega^{(0)} \in \mathcal{O}_S\alpha_{21} \otimes \nu \wedge \omega^{(n+1)}_P.$$

This is only possible if $\beta^n_P = 0$, since $\nu \wedge \omega^{(0)}, \nu \wedge \omega^{(1)}_P, \dots$ trivialise $f_*\Omega^2_{E^\natural/S}(\log \pi^{-1}Z)$. □

5. Relative elliptic KZB connections

5.1. ***Reminders on the bar construction.*** We work in the category of $\mathcal{O}_S$-modules for some scheme S. All tensor products are taken over $\mathcal{O}_S$.

Let $\mathcal{A}$ be a graded-commutative dg-algebra over $\mathcal{O}_S$, and assume that $\mathcal{A}$ is *connected*: $\mathcal{A}^0 = \mathcal{O}_S$. We denote by

$$\mathcal{I} := \bigoplus_{n\geq 1} \mathcal{A}^n$$

the kernel of the augmentation $\mathcal{A} \to \mathcal{O}_S$ given by projection onto the component of degree 0. Local sections of a tensor power $\mathcal{I}^{\otimes n}$ will be written in "bar notation"

$$a_1 \otimes \cdots \otimes a_n =: [a_1 \mid \cdots \mid a_n].$$

The *bar construction* associated to $\mathcal{A}$ is the total complex of $\mathcal{O}_S$-modules $(B^\bullet(\mathcal{A}), d_B)$ associated to the double complex $(B^{\bullet,\bullet}(\mathcal{A}), d_1, d_2)$ given by

$$B^{-s,t}(\mathcal{A}) := (\mathcal{I}^{\otimes s})^t, \quad s,t \geq 0,$$

$$d_1 : B^{-s,t}(\mathcal{A}) \longrightarrow B^{-s,t+1}(\mathcal{A}), \quad [a_1 \mid \cdots \mid a_n] \longmapsto \sum_{i=1}^{n} (-1)^i [Ja_1 \mid \cdots \mid Ja_{i-1} \mid da_i \mid a_{i+1} \mid \cdots \mid a_n],$$

$$d_2 : B^{-s,t}(\mathcal{A}) \longrightarrow B^{-s+1,t}(\mathcal{A}), \quad [a_1 \mid \cdots \mid a_n] \longmapsto \sum_{i=1}^{n-1} (-1)^{i-1} [Ja_1 \mid \cdots \mid Ja_{i-1} \mid Ja_i \wedge a_{i+1} \mid a_{i+2} \mid \cdots \mid a_n],$$

where $J : \mathcal{I} \to \mathcal{I}$ is the involution acting by $(-1)^j$ in degree j. Note that $B^{-s,t}(\mathcal{A})$ has total degree $t - s$ and the bar differential (i.e., the total differential) d_B has total degree 1.

The *length filtration* on $B(\mathcal{A})$ is the increasing exhaustive filtration by the $\mathcal{O}_S$-submodules

$$L_n B(\mathcal{A}) := \bigoplus_{m=0}^{n} \mathcal{I}^{\otimes m}, \quad n \geq 0.$$

Note that d_1 sends $L_n B(\mathcal{A})$ to itself, while d_2 sends it to $L_{n-1}B(\mathcal{A})$. In particular, the bar differential $d_B = d_1 + d_2$ preserves the length filtration.

The bar construction is functorial: if $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is a morphism of connected graded-commutative dg-algebras, then

$$B(\varphi) : B(\mathcal{A}_1) \longrightarrow B(\mathcal{A}_2), \quad B(\varphi) := \bigoplus_{n \geq 0} \bar{\varphi}^{\otimes n},$$

is a morphism of complexes of $\mathcal{O}_S$-modules, where $\bar{\varphi} : \mathcal{I}_1 \to \mathcal{I}_2$ is obtained from φ by restriction to the kernel of the augmentation. The next result shows that, under a suitable Künneth-type condition, the functor B also preserves quasi-isomorphisms.

Lemma 5.1. *Let $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ be a dg-quasi-isomorphism between connected graded-commutative dg-algebras. Assume that for all $m, n \geq 0$ the natural maps*

$$\bigoplus_{i_1+\cdots+i_n=m} H^{i_1}(\mathcal{I}_j) \otimes \cdots \otimes H^{i_n}(\mathcal{I}_j) \longrightarrow H^m(\mathcal{I}_j^{\otimes n}) \tag{23}$$

are isomorphisms for $j = 1, 2$. Then, the induced map $B(\varphi) : B(\mathcal{A}_1) \to B(\mathcal{A}_2)$ is a quasi-isomorphism.

Proof. It suffices to prove that the induced map $L_n B(\mathcal{A}_1) \to L_n B(\mathcal{A}_2)$ is a quasi-isomorphism for every $n \geq 0$. We proceed by induction on n, the base case $n = 0$ being trivial. Now fix $n \geq 1$ and consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_{n-1}B(\mathcal{A}_1) & \longrightarrow & L_n B(\mathcal{A}_1) & \longrightarrow & L_n B(\mathcal{A}_1)/L_{n-1}B(\mathcal{A}_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{n-1}B(\mathcal{A}_2) & \longrightarrow & L_n B(\mathcal{A}_2) & \longrightarrow & L_n B(\mathcal{A}_2)/L_{n-1}B(\mathcal{A}_2) & \longrightarrow & 0
\end{array}$$

The vertical arrow on the left is a quasi-isomorphism by induction hypothesis. That the vertical arrow on the right is a quasi-isomorphism follows from the isomorphisms (23). We conclude by an application of the five lemma. □

Remark 5.2. For the hypotheses of Lemma 5.1 to be satisfied, it is sufficient that all of $\mathcal{I}_j^n$, $d\mathcal{I}_j^n$, and $H^n(\mathcal{I}_j)$ are flat $\mathcal{O}_S$-modules ($n \geq 0$, $j = 1, 2$); see [Weibel 1994, Theorem 3.6.3].

We state without proof the following standard result.

Proposition 5.3. *The cohomology in degree* 0 *of the bar construction*

$$H^0(B(\mathcal{A})) = \ker(d_B : B^0(\mathcal{A}) \longrightarrow B^1(\mathcal{A}))$$

is naturally equipped with the structure of a filtered commutative Hopf algebra over $\mathcal{O}_S$, given by:

• *A commutative multiplication* $\sqcup\!\sqcup : H^0(B(\mathcal{A})) \otimes H^0(B(\mathcal{A})) \to H^0(B(\mathcal{A}))$, *the* ***shuffle product***, *which is defined on local sections by*

$$[a_1 \mid \cdots \mid a_m] \sqcup\!\sqcup [a_{m+1} \mid \cdots \mid a_{m+n}] = \sum_{\sigma} [a_{\sigma(1)} \mid \cdots \mid a_{\sigma(m+n)}],$$

where the sum ranges over all permutations σ *of* $\{1, \ldots, m+n\}$ *such that* σ^{-1} *is strictly increasing on* $\{1, \ldots, m\}$ *and* $\{m+1, \ldots, m+n\}$. *The unit for* $\sqcup\!\sqcup$ *is* $1 \in B^{0,0}(\mathcal{A}) = \mathcal{O}_S$.

• *A comultiplication* $\Delta : H^0(B(\mathcal{A})) \to H^0(B(\mathcal{A})) \otimes H^0(B(\mathcal{A}))$, *the* ***deconcatenation coproduct***, *which is defined on local sections by*

$$\Delta([a_1 \mid \cdots \mid a_n]) = [a_1 \mid \cdots \mid a_n] \otimes 1 + 1 \otimes [a_1 \mid \cdots \mid a_n] + \sum_{r=1}^{n-1} [a_1 \mid \cdots \mid a_r] \otimes [a_{r+1} \mid \cdots \mid a_n].$$

The counit for Δ *is the augmentation map* $\varepsilon : H^0(B(\mathcal{A})) \to \mathcal{O}_S$.

• *An antipode* $\sigma : H^0(B(\mathcal{A})) \to H^0(B(\mathcal{A}))$, *given on local sections by*

$$\sigma([a_1 \mid \cdots \mid a_n]) = (-1)^n [a_n \mid \cdots \mid a_1].$$

• *A length filtration* $L_n H^0(B(\mathcal{A})) := L_n B(\mathcal{A}) \cap H^0(B(\mathcal{A}))$.

Example 5.4. If $\mathcal{A}^n = 0$ for $n \geq 2$, then $d_B = 0$, so that

$$H^0(B(\mathcal{A})) = B^0(\mathcal{A}) = T^c \mathcal{A}^1.$$

Here, $T^c\mathcal{A}^1 = \bigoplus_{n \geq 0} (\mathcal{A}^1)^{\otimes n}$ denotes the *tensor coalgebra* on $\mathcal{A}^1$, with the above structure of filtered Hopf algebra over $\mathcal{O}_S$. If moreover $\mathcal{A}^1 = \mathcal{O}_S\alpha_1 \oplus \cdots \oplus \mathcal{O}_S\alpha_r$, then we can identify

$$H^0(B(\mathcal{A})) \cong \mathcal{O}_S\langle \alpha_1, \ldots, \alpha_r \rangle,$$

where the length filtration L_n is spanned by words in $\alpha_1, \ldots, \alpha_r$ of length $\leq n$.

In general, denote by $\mathrm{pr}_n : H^0(B(\mathcal{A})) \to (\mathcal{A}^1)^{\otimes n}$ the natural projection onto the component of pure length n. Comodules for the Hopf algebra $H^0(B(\mathcal{A}))$ can be characterised as follows.

Proposition 5.5. *Let* $\mathcal{E}$ *be a vector bundle over* S, *and* $\rho : \mathcal{E} \to H^0(B(\mathcal{A})) \otimes \mathcal{E}$ *be an* $\mathcal{O}_S$*-morphism. Write* $\rho = \sum_{n \geq 0} \rho_n$, *where* $\rho_n = (\mathrm{pr}_n \otimes \mathrm{id}) \circ \rho : \mathcal{E} \to (\mathcal{A}^1)^{\otimes n} \otimes \mathcal{E}$. *Then,* ρ *is a comodule structure if and only if*

(i) $\rho_0 = \mathrm{id}_{\mathcal{E}}$, *and*

(ii) ρ_n *is the* n*-fold composition*

$$[\rho_1]^n := (\mathrm{id}_{(\mathcal{A}^1)^{\otimes n-1}} \otimes \rho_1) \circ \cdots \circ (\mathrm{id}_{\mathcal{A}^1} \otimes \rho_1) \circ \rho_1 : \mathcal{E} \longrightarrow (\mathcal{A}^1)^{\otimes n} \otimes \mathcal{E}$$

for every $n \geq 1$.

Moreover, given an $\mathcal{O}_S$*-linear map* $\omega : \mathcal{E} \to \mathcal{A}^1 \otimes \mathcal{E}$, *there exists a comodule structure* $\rho : \mathcal{E} \to H^0(B(\mathcal{A})) \otimes \mathcal{E}$ *satisfying* $\rho_1 = \omega$ *if and only if* ω *is* ***locally nilpotent*** (*i.e., locally over* S, *we have* $[\omega]^n = 0$ *for* $n \gg 0$), *and*

$$d\omega + \omega \wedge \omega = 0$$

in $\mathcal{A}^2 \otimes \mathcal{E}nd(\mathcal{E})$.

Proof. Since the counit ε of $H^0(B(\mathcal{A}))$ is equal to pr_0, statement (i) is equivalent to the comodule axiom $(\varepsilon \otimes \mathrm{id}) \circ \rho = \mathrm{id}$. Let

$$\delta_{i,j} : (\mathcal{A}^1)^{\otimes i+j} \xrightarrow{\sim} (\mathcal{A}^1)^{\otimes i} \otimes (\mathcal{A}^1)^{\otimes j}, \quad [a_1 \mid \cdots \mid a_{i+j}] \longmapsto [a_1 \mid \cdots \mid a_i] \otimes [a_{i+1} \mid \cdots \mid a_{i+j}],$$

be the "deconcatenation isomorphisms". Using the expression $\rho = \sum_{n \geq 0} \rho_n$ and the definition of the deconcatenation coproduct Δ of $H^0(B(\mathcal{A}))$, we obtain

$$(\Delta \otimes \mathrm{id}) \circ \rho = \sum_{n \geq 0} (\Delta \otimes \mathrm{id}) \circ \rho_n = \sum_{n \geq 0} \sum_{i+j=n} (\delta_{i,j} \otimes \mathrm{id}) \circ \rho_n = \sum_{i,j \geq 0} (\delta_{i,j} \otimes \mathrm{id}) \circ \rho_{i+j}$$

and

$$(\mathrm{id} \otimes \rho) \circ \rho = \Big(\mathrm{id} \otimes \sum_{i \geq 0} \rho_i \Big) \circ \sum_{j \geq 0} \rho_j = \sum_{i,j \geq 0} (\mathrm{id} \otimes \rho_i) \circ \rho_j .$$

By induction, this shows that the comodule axiom $(\Delta \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \rho$ is equivalent to (ii).

For the last assertion, note that any $\mathcal{O}_S$-linear map $\omega : \mathcal{E} \to \mathcal{A}^1 \otimes \mathcal{E}$ defines an $\mathcal{O}_S$-linear map

$$\rho := ([\omega]^n)_{\geq 0} : \mathcal{E} \longrightarrow \prod_{n \geq 0} (\mathcal{A}^1)^{\otimes n} \otimes \mathcal{E}.$$

We regard $T^c \mathcal{A}^1 = \bigoplus_{n \geq 0} (\mathcal{A}^1)^{\otimes n}$ as a submodule of $\prod_{n \geq 0} (\mathcal{A}^1)^{\otimes n}$. Since $\mathcal{E}$ is an $\mathcal{O}_S$-module of finite type, we see that ρ factors through $T^c \mathcal{A}^1 \otimes \mathcal{E}$ if and only if, locally over S, we have $[\omega]^n = 0$ for every sufficiently large n. Finally, since $H^0(B(\mathcal{A})) = \ker(d_B)$ and $\mathcal{E}$ is flat, the image of ρ is contained in $H^0(B(\mathcal{A})) \otimes \mathcal{E}$ if and only if $(d_B \otimes \mathrm{id}) \circ \rho = 0$. In bar notation, we have

$$\begin{aligned}(d_B \otimes \mathrm{id}) \circ \rho &= \sum_{n \geq 1} d_B \underbrace{[\omega \mid \cdots \mid \omega]}_{\text{length } n} = -\sum_{n \geq 1} \Bigg(\sum_{i=1}^{n} \underbrace{[\omega \mid \cdots \mid \overbrace{d\omega}^{i\text{-th position}} \mid \cdots \mid \omega]}_{\text{length } n} + \sum_{i=1}^{n-1} \underbrace{[\omega \mid \cdots \mid \overbrace{\omega \wedge \omega}^{i\text{-th position}} \mid \cdots \mid \omega]}_{\text{length } n-1} \Bigg) \\ &= -\sum_{n \geq 1} \sum_{i=1}^{n} \underbrace{[\omega \mid \cdots \mid \overbrace{d\omega + \omega \wedge \omega}^{i\text{-th position}} \mid \cdots \mid \omega]}_{\text{length } n},\end{aligned}$$

so that the identity $(d_B \otimes \mathrm{id}) \circ \rho = 0$ is equivalent to $d\omega + \omega \wedge \omega = 0$. □

5.2. *Fundamental Hopf algebra.* Let S be a scheme of characteristic 0, $p : E \to S$ be an elliptic curve, and Z be a divisor on E/S as in Section 3.2. Let $f : E^\natural \to S$ be the universal vector extension of E/S and $\pi : E^\natural \to E$ be the natural projection.

Definition 5.6. The *de Rham fundamental Hopf algebra* of E/S punctured at Z is the filtered Hopf algebra over $\mathcal{O}_S$ defined by

$$\mathcal{H}_{E/S,Z} := H^0(B(f_* \Omega^\bullet_{E^\natural/S}(\log \pi^{-1} Z))).$$

The affine group scheme over S corresponding to $\mathcal{H}_{E/S,Z}$ can be regarded as a base-point-free version of the "relative unipotent de Rham fundamental group" of $E \setminus Z$ over S (see [Chiarellotto et al. 2023]). We refer to the Appendix for precise comparison theorems in the case where S is the spectrum of a field.

Example 5.7. Assume that S is affine, that $R^1 p_* \mathcal{O}_E$ is trivial, and let ν, $\omega^{(0)}$, $\omega_P^{(1)}$, $\omega_P^{(2)}$, ... be as in (16). Then, a section ξ of $\mathcal{H}_{E/S,Z}$ is an $\mathcal{O}_S$-linear combination of words in these 1-forms, satisfying $d_B\xi = 0$. For instance,

$$[\omega^{(0)} \mid \omega^{(0)}], \quad [\omega^{(0)} \mid \nu] + [\omega_P^{(1)}], \quad [\omega_P^{(1)} - \omega_O^{(1)} \mid \nu] + [\omega^{(2)} - \omega_O^{(2)}]$$

are sections of length 2, and

$$[\omega^{(0)} \mid \nu \mid \omega^{(0)} \mid \nu \mid \nu] + [\omega_P^{(1)} \mid \omega^{(0)} \mid \nu \mid \nu] - [\omega^{(0)} \mid \omega_P^{(1)} \mid \nu \mid \nu] + [\omega^{(0)} \mid \nu \mid \omega_P^{(1)} \mid \nu] + 2[\omega_P^{(1)} \mid \omega^{(1)} \mid \nu] \\ - 2[\omega^{(0)} \mid \omega_P^{(2)} \mid \nu] + [\omega^{(0)} \mid \nu \mid \omega_P^{(2)}] + 3[\omega_P^{(1)} \mid \omega_P^{(2)}] - 3[\omega^{(0)} \mid \omega_P^{(3)}]$$

is a section of length 5.

Theorem 5.8. *The projector ρ from Theorem 3.9 induces an isomorphism of filtered Hopf algebras over $\mathcal{O}_S$*

$$\mathcal{H}_{E/S,Z} \xrightarrow{\sim} T^c H^1_{\mathrm{dR}}((E \setminus Z)/S).$$

Proof. This follows immediately from the fact that ρ is a dg-quasi-isomorphism (Theorem 3.9) and from Lemma 5.1. To verify the Künneth-type hypotheses of Lemma 5.1 (see Remark 5.2), we apply Theorems 3.7 and 3.9. □

Example 5.9. Locally over S, the isomorphism of the above theorem is given explicitly by writing a section of $\mathcal{H}_{E/S,Z}$ as an $\mathcal{O}_S$-linear combination of words in $\omega^{(0)}$, ν, $\omega_P^{(1)} - \omega_O^{(1)}$, $\omega_O^{(1)}$, $\omega_P^{(n)}$ (where $n \geq 2$ and $P \in Z(S)$), and by sending $\omega_O^{(1)}$, $\omega_P^{(n)}$ to zero. For instance, the length-2 section

$$[\omega^{(0)} \mid \nu] + [\omega_P^{(1)}] = [\omega^{(0)} \mid \nu] + [\omega_P^{(1)} - \omega_O^{(1)}] + [\omega_O^{(1)}]$$

of $\mathcal{H}_{E/S,Z}$ is sent to the length-2 section $[\omega^{(0)} \mid \nu] + [\omega_P^{(1)} - \omega_O^{(1)}]$ of $T^c H^1_{\mathrm{dR}}((E \setminus Z)/S)$.

We also deduce from the above theorem that the formation of $\mathcal{H}_{E/S,Z}$ commutes with every base change in S.

Corollary 5.10. *For any morphism of schemes $\varphi : S' \to S$, the natural map*

$$\varphi^* \mathcal{H}_{E/S,Z} \longrightarrow \mathcal{H}_{(E \times_S S')/S, Z \times_S S'}$$

is an isomorphism of filtered Hopf algebras over $\mathcal{O}_{S'}$.

Proof. It is well known that, for proper and smooth schemes, the formation of the de Rham cohomology commutes with arbitrary base change. Using the residue exact sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(E/S) \longrightarrow H^1_{\mathrm{dR}}((E \setminus Z)/S) \xrightarrow{\mathrm{Res}} H^0_{\mathrm{dR}}(Z/S) \longrightarrow H^2_{\mathrm{dR}}(E/S) \longrightarrow 0$$

we deduce that the formation of $H^1_{\mathrm{dR}}((E \setminus Z)/S)$ commutes with arbitrary base change. To conclude we apply Theorem 5.8 and the fact that the formation of the universal vector extension also commutes with arbitrary base change (see Section 2.1). □

5.3. *Elliptic KZB connection.* We keep the notation of Section 5.2. It follows from Theorem 5.8 that there are canonical isomorphisms

$$L_n\mathcal{H}_{E/S,Z}/L_{n-1}\mathcal{H}_{E/S,Z} \cong H^1_{\mathrm{dR}}((E\setminus Z)/S)^{\otimes n}. \tag{24}$$

In particular, each $L_n\mathcal{H}_{E/S,Z}$ is a vector bundle over S. The *continuous dual* of $\mathcal{H}_{E/S,Z}$ is the $\mathcal{O}_S$-module

$$\mathcal{H}^\vee_{E/S,Z} := \varprojlim_n (L_n\mathcal{H}_{E/S,Z})^\vee, \quad (L_n\mathcal{H}_{E/S,Z})^\vee = \mathcal{H}om_{\mathcal{O}_S}(L_n\mathcal{H}_{E/S,Z}, \mathcal{O}_S),$$

with the dual structure of a completed Hopf algebra over $\mathcal{O}_S$ (see [Burgos Gil and Fresán, Section 3.2.6]). For instance, its multiplication is given by

$$\Delta^\vee : \mathcal{H}^\vee_{E/S,Z} \hat{\otimes} \mathcal{H}^\vee_{E/S,Z} \longrightarrow \mathcal{H}^\vee_{E/S,Z}, \quad \Delta^\vee := \varprojlim_n (\Delta|_{L_n\mathcal{H}_{E/S,Z}})^\vee.$$

Regarding the restriction of the antipode σ to $L_n\mathcal{H}_{E/S,Z}$ as a global section of $L_n\mathcal{H}_{E/S,Z}\otimes(L_n\mathcal{H}_{E/S,Z})^\vee$, we get a global section

$$\hat{\sigma} = \varprojlim_n \sigma|_{L_n\mathcal{H}_{E/S,Z}} \in \Gamma(S, \mathcal{H}_{E/S,Z} \hat{\otimes} \mathcal{H}^\vee_{E/S,Z}).$$

Definition 5.11. The *KZB form* of E/S punctured at Z is the global section

$$\omega_{E^\natural/S,Z} := (\mathrm{pr}_1 \otimes \mathrm{id})(\hat{\sigma}) \in \Gamma(S, f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \hat{\otimes} \mathcal{H}^\vee_{E/S,Z}),$$

where $\mathrm{pr}_1 : \mathcal{H}_{E/S,Z} \to f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z)$ is the projection onto the component of length 1.

By letting $\omega_{E^\natural/S,Z}$ act on $\mathcal{H}^\vee_{E/S,Z}$ by left multiplication via $\Delta^\vee$, we can also regard it as an $\mathcal{O}_S$-linear map

$$\omega_{E^\natural/S,Z} : \mathcal{H}^\vee_{E/S,Z} \longrightarrow f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \hat{\otimes} \mathcal{H}^\vee_{E/S,Z}.$$

Proposition 5.12. *We have*

$$d\omega_{E^\natural/S,Z} + \omega_{E^\natural/S,Z} \wedge \omega_{E^\natural/S,Z} = 0.$$

Moreover, if we let

$$1 := \varprojlim_n (\varepsilon|_{L_n\mathcal{H}_{E/S,Z}})^\vee \in \Gamma(S, \mathcal{H}^\vee_{E/S,Z}),$$

then the triple $(\mathcal{H}^\vee_{E/S,Z}, \omega_{E^\natural/S,Z}, 1)$ satisfies the following universal property: for every triple $(\mathcal{E}, \omega, e)$, where $\mathcal{E}$ is a vector bundle over S, $\omega : \mathcal{E} \to f_*\Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \otimes \mathcal{E}$ is a locally nilpotent $\mathcal{O}_S$-linear map satisfying $d\omega + \omega \wedge \omega = 0$, and $e \in \Gamma(S, \mathcal{E})$, there is a unique $\mathcal{O}_S$-linear map $\varphi : \mathcal{H}^\vee_{E/S,Z} \to \mathcal{E}$ such that $(\mathrm{id} \otimes \varphi) \circ \omega_{E^\natural/S,Z} = \omega \circ \varphi$ and $\varphi(1) = e$.

Proof. Let $\rho : \mathcal{E} \to \mathcal{H}_{E/S,Z} \otimes \mathcal{E}$ be the comodule structure corresponding to ω as in Proposition 5.5. Working locally over S, we may assume that there is some $n_0 \geq 0$ such that $\rho(\mathcal{E}) \subset L_{n_0}\mathcal{H}_{E/S,Z} \otimes \mathcal{E}$. For every $n \geq n_0$, define

$$\varphi_n : (L_n\mathcal{H}_{E/S,Z})^\vee \longrightarrow \mathcal{E}, \quad \lambda \longmapsto (\lambda \otimes \mathrm{id}) \circ (\sigma \otimes \mathrm{id}) \circ \rho(e). \tag{25}$$

Then, a straightforward computation shows that $\varphi := \varprojlim_n \varphi_n : \mathcal{H}^\vee_{E/S,Z} \to \mathcal{E}$ is an $\mathcal{O}_S$-linear map satisfying the properties in the statement. To prove uniqueness, let $\rho_{E/S,Z} : \mathcal{H}^\vee_{E/S,Z} \to \mathcal{H}_{E/S,Z} \hat{\otimes} \mathcal{H}^\vee_{E/S,Z}$ be the completed comodule structure corresponding to $\omega_{E^\natural/S,Z}$ via Proposition 5.5. It is given by left multiplication by $\hat{\sigma}$. If $\varphi' : \mathcal{H}^\vee_{E/S,Z} \to \mathcal{E}$ is an $\mathcal{O}_S$-morphism satisfying $\varphi'(1) = e$ and $(\mathrm{id} \otimes \varphi') \circ \omega_{E^\natural/S,Z} = \omega \circ \varphi'$, then it follows from Proposition 5.5 that this last equation can be lifted to

$$(\mathrm{id} \otimes \varphi') \circ \rho_{E/S,Z} = \rho \circ \varphi'.$$

Thus,

$$\rho(e) = (\mathrm{id} \otimes \varphi') \circ \rho_{E/S,Z}(1) = (\mathrm{id} \otimes \varphi')(\hat{\sigma}).$$

Let $n \geq n_0$ be as above. For any section λ of $(L_n\mathcal{H}_{E/S,Z})^\vee$, we have

$$(\lambda \otimes \mathrm{id}) \circ \rho(e) = (\lambda \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi')(\hat{\sigma}) = \varphi' \circ (\lambda \otimes \mathrm{id})(\hat{\sigma}) = \varphi'_n(\lambda \circ \sigma|_{L_n\mathcal{H}_{E/S,Z}}),$$

where in the last equality we regard $\lambda \circ \sigma|_{L_n\mathcal{H}_{E/S,Z}}$ as a section of $(L_n\mathcal{H}_{E/S,Z})^\vee$. Finally, using that σ is an involution, we conclude that φ'_n is given by the same formula as φ_n in (25). □

Definition 5.13. Consider the completed pullback $f^*\mathcal{H}^\vee_{E/S,Z} := \varprojlim_n f^*(L_n\mathcal{H}_{E/S,Z})^\vee$. The *relative elliptic KZB connection* of E/S punctured at Z is the S-connection

$$\nabla_{E^\natural/S,Z} : f^*\mathcal{H}^\vee_{E/S,Z} \longrightarrow \Omega^1_{E^\natural/S}(\log \pi^{-1}Z) \hat{\otimes} f^*\mathcal{H}^\vee_{E/S,Z}, \quad \nabla_{E^\natural/S,Z} = d + \omega_{E^\natural/S,Z}.$$

Proposition 5.14. *The formation of $\nabla_{E^\natural/S,Z}$ is compatible with arbitrary base change in S.*

Proof. This follows from Corollary 5.10 and from the fact that $\omega_{E^\natural/S,Z}$ is induced by the antipode of the Hopf algebra $\mathcal{H}_{E/S,Z}$. □

We shall now give explicit formulas for the relative elliptic KZB connection. Assuming that S is affine and that $R^1p_*\mathcal{O}_E$ is trivial, let $\nu, \omega^{(0)}, \omega^{(1)}_P, \omega^{(2)}_P, \ldots$ be as in (16). Recall that

$$\{\nu, \omega^{(0)}\} \cup \{\omega^{(1)}_P - \omega^{(1)}_O : P \in Z(S) \setminus \{O\}\}$$

trivialises the vector bundle $H^1(f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z))$ over S (see the proof of Theorem 3.9). We denote by

$$\{a, b\} \cup \{b_P : P \in Z(S) \setminus \{O\}\}$$

the dual trivialisation of $H^1(f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1}Z))^\vee$. By Theorem 5.8 (see Example 5.4), we have

$$\mathcal{H}^\vee_{U/S} \cong \mathcal{O}_S\langle\langle a, b, b_P : P \in Z(S) \setminus \{O\}\rangle\rangle.$$

Note that Hopf algebra unit 1 as defined in Proposition 5.12 gets identified with the constant 1. It will be more convenient to write

$$\mathcal{O}_S\langle\langle a, b, b_P : P \in Z(S) \setminus \{O\}\rangle\rangle \cong \frac{\mathcal{O}_S\langle\langle a, b, c_P : P \in Z(S)\rangle\rangle}{\left\langle \sum_{P \in Z(S)} c_P - [a, b]\right\rangle},$$

where the isomorphism is given by sending b_P to c_P for every $P \in Z(S) \setminus \{O\}$.

Theorem 5.15. *With the above notation, we have*

$$\omega_{E^\natural/S,Z} = -\nu \otimes a - \omega^{(0)} \otimes b - \sum_{n \geq 1} \sum_{P \in Z(S)} \omega^{(n)}_P \otimes \mathrm{ad}^{n-1}_a c_P, \tag{26}$$

where $\mathrm{ad}_a x = [a, x] = ax - xa$.

Proof. Call ω' the right-hand side of (26). We show that ω' satisfies the universal property for the KZB form. For this, let $(\mathcal{E}, \omega, e)$ be a triple as in Proposition 5.12. As $\nu, \omega^{(0)}, \omega_P^{(n)}$ $(n \geq 1,\ P \in Z(S))$ trivialise $f_*\Omega^1_{E^\natural/S}(\log \pi^{-1} Z)$, we can uniquely write

$$\omega = -\nu \otimes A - \omega^{(0)} \otimes B - \sum_{n\geq 1} \sum_{P \in Z(S)} \omega_P^{(n)} \otimes C_P^{(n)},$$

where $A, B, C_P^{(n)}$ are nilpotent endomorphisms of $\mathcal{E}$ (with $C_P^{(n)} = 0$ for $n \gg 0$). Since

$$0 = d\omega + \omega \wedge \omega = \nu \wedge \omega^{(0)} \otimes \Bigl([A, B] - \sum_{P \in Z(S)} C_P^{(1)}\Bigr) + \sum_{n\geq 1} \nu \wedge \omega_P^{(n)} \otimes \bigl([A, C_P^{(n)}] - C_P^{(n+1)}\bigr),$$

we conclude that

$$\sum_{P \in Z(S)} C_P^{(1)} = [A, B], \quad C_P^{(n)} = \mathrm{ad}_A^{n-1} C_P^{(1)}.$$

Thus, there is a unique $\mathcal{O}_S$-morphism $\varphi : \mathcal{H}^\vee_{E/S,Z} \to \mathcal{E}$ satisfying $\varphi(1) = e$, $\varphi(a) = Ae$, $\varphi(b) = Be$, and $\varphi(c_P) = C_P^{(1)} e$ for every $P \in Z(S)$. To finish the proof, it suffices to remark that these conditions are equivalent to $\varphi(1) = e$ and $(\mathrm{id} \otimes \varphi) \circ \omega' = \omega \circ \varphi$. □

Example 5.16. When $Z = O$, we have $\mathcal{H}^\vee_{E/S,O} \cong \mathcal{O}_S\langle\langle a, b\rangle\rangle$. Since $c_O = [a, b] = \mathrm{ad}_a b$, the formula for $\omega_{E^\natural/S,Z}$ becomes

$$\omega_{E^\natural/S,O} = -\nu \otimes a - \sum_{n\geq 0} \omega^{(n)} \otimes \mathrm{ad}_a^n b,$$

where $\omega^{(n)} = \omega_O^{(n)}$.

6. Absolute elliptic KZB connections

6.1. ***Bar construction relative to a dg-algebra.*** Let S be a smooth scheme over a field k of characteristic 0. In this subsection, we consider a variant of the bar construction relative to the dg-algebra $\Omega := \Omega^\bullet_{S/k}$.

To lighten the notation, tensor products without subscripts are over $\mathcal{O}_S$. If $\mathcal{F}$ and $\mathcal{G}$ are bimodules over the (noncommutative) $\mathcal{O}_S$-algebra Ω, we define

$$\mathcal{F} \otimes_\Omega \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})/\mathcal{R},$$

where $\mathcal{R}$ is the submodule generated by $(f \wedge \omega) \otimes g - f \otimes (J\omega \wedge g)$, with f, g, ω sections of $\mathcal{F}, \mathcal{G}, \Omega$ respectively.

Let $\mathcal{C} = \bigoplus_{n\geq 0} \mathcal{C}^n$ be a graded $\mathcal{O}_S$-algebra equipped with a k-linear differential $d : \mathcal{C} \to \mathcal{C}[1]$ making it a commutative dg-algebra over k. Assume that $\mathcal{C}^0 = \mathcal{O}_S$ and that $\mathcal{C}$ contains Ω both as a graded subalgebra over $\mathcal{O}_S$ and as a dg-subalgebra over k. The *relative bar construction* on $\mathcal{C}$ is defined as follows. Let $\mathcal{J} = \bigoplus_{n\geq 1} \mathcal{C}^n$ be the kernel of the projection on the component of degree 0 $\mathcal{C} \to \mathcal{O}_S$, and set

$$B_\Omega^{-s,t}(\mathcal{C}) := (\mathcal{J}^{\otimes_\Omega s})^t, \quad s, t \geq 0.$$

Here, we also denote local sections of a tensor power $\mathcal{J}^{\otimes_\Omega n}$ in "bar notation":

$$c_1 \otimes \cdots \otimes c_n =: [c_1 \mid \cdots \mid c_n].$$

With the above notion of tensor product, one can readily check that

$$d_1 : B_\Omega^{-s,t}(\mathcal{C}) \longrightarrow B_\Omega^{-s,t+1}(\mathcal{C}), \quad [c_1 \mid \cdots \mid c_n] \longmapsto \sum_{i=1}^{n} (-1)^i [Jc_1 \mid \cdots \mid Jc_{i-1} \mid dc_i \mid c_{i+1} \mid \cdots \mid c_n],$$

$$d_2 : B_\Omega^{-s,t}(\mathcal{C}) \longrightarrow B_\Omega^{-s+1,t}(\mathcal{C}), \quad [c_1 \mid \cdots \mid c_n] \longmapsto \sum_{i=1}^{n-1} (-1)^{i-1} [Jc_1 \mid \cdots \mid Jc_{i-1} \mid Jc_i \wedge c_{i+1} \mid c_{i+2} \mid \cdots \mid c_n],$$

are well-defined (even though d is only k-linear), so that we obtain a double complex $(B_\Omega^{\bullet,\bullet}(\mathcal{C}), d_1, d_2)$. The relative bar construction is the associated total complex $(B_\Omega^\bullet(\mathcal{C}), d_B)$.

Remark 6.1. Similarly to the usual bar construction, as an $\mathcal{O}_S$-module, $B_\Omega(\mathcal{C})$ is simply the tensor module

$$B_\Omega(\mathcal{C}) = \bigoplus_{n \geq 0} \mathcal{J}^{\otimes_\Omega n},$$

with a shift on the grading: $\deg([c_1 \mid \cdots \mid c_n]) = \sum_{i=1}^n (\deg(c_i) - 1)$.

6.2. *Koszul filtration on the relative bar construction.*

We take

$$\mathcal{C} := f_* \Omega^\bullet_{E^\natural/k}(\log \pi^{-1} Z),$$

with hypotheses and notation as in Section 5.2. The Koszul filtration by $\mathcal{O}_{E^\natural}$-submodules $\mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots$ on $\Omega^\bullet_{E^\natural/k}(\log \pi^{-1} Z)$ (see Section 4.2) induces by direct image a filtration by $\mathcal{O}_S$-submodules $f_*\mathcal{F}^0 \supset f_*\mathcal{F}^1 \supset \cdots$ on $\mathcal{C}$. Note that each $f_*\mathcal{F}^p$ is actually a Ω-submodule of $\mathcal{C}$. Let $F^\bullet \mathcal{J}$ be the induced filtration on $\mathcal{J} \subset \mathcal{C}$, so that

$$F^p \mathcal{J} = \begin{cases} \mathcal{J}, & p = 0, \\ f_*\mathcal{F}^p, & p \geq 1. \end{cases}$$

This induces a filtration on $B_\Omega(\mathcal{C})$:

$$F^p B_\Omega(C) = \bigoplus_{n \geq 1} \sum_{p_1 + \cdots + p_n = p} \operatorname{im}(F^{p_1}\mathcal{J} \otimes_\Omega \cdots \otimes_\Omega F^{p_n}\mathcal{J} \longrightarrow \mathcal{J}^{\otimes_\Omega n}).$$

In what follows, let $\mathcal{A} = f_*\Omega^\bullet_{E^\natural/S}(\log \pi^{-1} Z)$ and $\mathcal{I} = \bigoplus_{n \geq 1} \mathcal{A}^n$ (see Section 5.1). By Proposition 4.6, we have $\mathcal{J}/F^1\mathcal{J} \cong \mathcal{I}$. Let

$$B_\Omega(\mathcal{C}) \longrightarrow B(\mathcal{A})$$

be the natural map induced by the quotient map $\mathcal{J} \to \mathcal{I}$.

Lemma 6.2. *The sequence*

$$0 \longrightarrow F^1 B_\Omega(\mathcal{C}) \longrightarrow B_\Omega(\mathcal{C}) \longrightarrow B(\mathcal{A}) \longrightarrow 0$$

is exact.

Proof. By flatness of $\mathcal{J}$, $F^1\mathcal{J}$, and $\mathcal{I}$, we have, for every $n \geq 1$, an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \bigoplus_{i=1}^n \mathcal{J}^{\otimes i-1} \otimes F^1\mathcal{J} \otimes \mathcal{J}^{\otimes n-i} \longrightarrow \mathcal{J}^{\otimes n} \longrightarrow \mathcal{I}^{\otimes n} \longrightarrow 0.$$

Since the map $\mathcal{J}^{\otimes n} \to \mathcal{I}^{\otimes n}$ factors through $\mathcal{J}^{\otimes_\Omega n} \to \mathcal{I}^{\otimes n}$, we obtain an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \bigoplus_{i=1}^n \mathcal{J}^{\otimes_\Omega i-1} \otimes_\Omega F^1\mathcal{J} \otimes_\Omega \mathcal{J}^{\otimes_\Omega n-i} \longrightarrow \mathcal{J}^{\otimes_\Omega n} \longrightarrow \mathcal{I}^{\otimes n} \longrightarrow 0.$$

This proves that the sequence in the statement is an exact sequence of $\mathcal{O}_S$-modules. □

Let

$$\sigma_{n,i} : \mathcal{I}^{\otimes i-1} \otimes (\Omega^1[1] \otimes \mathcal{I}) \otimes \mathcal{I}^{\otimes n-i} \xrightarrow{\sim} \Omega^1[1] \otimes \mathcal{I}^{\otimes n}$$

be the "Koszul sign rule" isomorphism given by

$$[a_1 \mid \cdots \mid a_{i-1} \mid \omega \otimes a_i \mid a_{i+1} \mid \cdots \mid a_n] \longmapsto (-1)^{i-1} \omega \otimes [Ja_1 \mid \cdots \mid Ja_{i-1} \mid a_i \mid \cdots \mid a_n].$$

Lemma 6.3. *For $n \geq 1$ and $1 \leq i \leq n$, let*

$$\pi_{n,i} : \mathcal{J}^{\otimes i-1} \otimes F^1\mathcal{J} \otimes \mathcal{J}^{\otimes n-i} \longrightarrow \Omega^1[1] \otimes \mathcal{I}^{\otimes n}$$

be the projection given by $F^1\mathcal{J} \to F^1\mathcal{J}/F^2\mathcal{J} \cong \Omega^1[1] \otimes \mathcal{A} \to \Omega^1[1] \otimes \mathcal{I}$ (see Proposition 4.6) on the i-th factor and by $\mathcal{J} \to \mathcal{J}/F^1\mathcal{J} \cong \mathcal{I}$ on the other factors, composed with $\sigma_{n,i}$. Then:

(i) *The sum*

$$\pi_n = \sum_{1 \leq i \leq n} \pi_{n,i} : \bigoplus_{i=1}^n \mathcal{J}^{\otimes i-1} \otimes F^1\mathcal{J} \otimes \mathcal{J}^{\otimes n-i} \longrightarrow \Omega^1[1] \otimes \mathcal{I}^{\otimes n}$$

factors through an $\mathcal{O}_S$-linear map

$$\pi_n : \sum_{i=1}^n \mathrm{im}(\mathcal{J}^{\otimes_\Omega i-1} \otimes_\Omega F^1\mathcal{J} \otimes_\Omega \mathcal{J}^{\otimes_\Omega n-i} \longrightarrow \mathcal{J}^{\otimes_\Omega n}) \longrightarrow \Omega^1[1] \otimes \mathcal{I}^{\otimes n}.$$

(ii) *The induced $\mathcal{O}_S$-linear map*

$$\pi : F^1 B_\Omega(\mathcal{C}) \longrightarrow \Omega^1[1] \otimes B(\mathcal{A}) \cong \Omega^1 \otimes B(\mathcal{A})[-1]$$

is a morphism of complexes over k.

Proof. For clarity, we only give details for the case $n = 2$; the general case is similar.

To prove (i), we first remark that, from the flatness of $F^1\mathcal{J}$, $\mathcal{J}$, and $\mathcal{J}/F^1\mathcal{J} \cong \mathcal{I}$, we obtain an exact sequence

$$0 \longrightarrow (F^1\mathcal{J})^{\otimes 2} \longrightarrow (F^1\mathcal{J} \otimes \mathcal{J}) \oplus (\mathcal{J} \otimes F^1\mathcal{J}) \longrightarrow \mathcal{J}^{\otimes 2},$$

where the injection is given by $x \mapsto (x, -x)$. For $i = 1, 2$, we have $\pi_{2,i}((F^1\mathcal{J})^{\otimes 2}) = 0$, since there is always a projection $\mathcal{J} \to \mathcal{J}/F^1\mathcal{J}$ in one of the factors. This shows that π_2 factors through

$$\pi_2 : \mathrm{im}((F^1\mathcal{J} \otimes \mathcal{J}) \oplus (\mathcal{J} \otimes F^1\mathcal{J}) \longrightarrow \mathcal{J}^{\otimes 2}) = F^1\mathcal{J} \otimes \mathcal{J} + \mathcal{J} \otimes F^1\mathcal{J} \longrightarrow \Omega^1[1] \otimes \mathcal{I}^{\otimes 2}.$$

We are left to show that π_2 factors through the image of $\mathcal{F}^1\mathcal{J}\otimes\mathcal{J}+\mathcal{J}\otimes F^1\mathcal{J}$ in $\mathcal{J}^{\otimes_\Omega 2}$. By definition, the kernel of $\mathcal{J}^{\otimes 2}\to\mathcal{J}^{\otimes_\Omega 2}$ is generated by sections of the form

$$x=(c_1\wedge\omega)\otimes c_2-c_1\otimes(J\omega\wedge c_2),$$

with c_1, c_2 sections of $\mathcal{J}$ and ω a section of Ω^n with $n\geq 1$. Every such element is in $\mathcal{F}^1\mathcal{J}\otimes\mathcal{J}+\mathcal{J}\otimes F^1\mathcal{J}$, so we only need to prove that $\pi_2(x)=0$. Using that $\Omega^i=\bigwedge^i\Omega^1$, we can reduce to the case where ω is a section of Ω^1: if $\omega=\omega'\wedge\omega''$, then we can write

$$x=((c_1\wedge\omega')\wedge\omega'')\otimes c_2-(c_1\wedge\omega')\otimes(J\omega''\wedge c_2)+(c_1\wedge\omega')\otimes(J\omega''\wedge c_2)-c_1\otimes(J\omega'\wedge(J\omega''\wedge c_2)).$$

Finally, assuming that ω is a section of Ω^1, we have $x=(c_1\wedge\omega)\otimes c_2+c_1\otimes(\omega\wedge c_2)$, so that

$$\begin{aligned}\pi(x)&=\pi_{2,1}((c_1\wedge\omega)\otimes c_2)+\pi_{2,2}(c_1\otimes(\omega\wedge c_2))\\&=\pi_{2,1}((\omega\wedge Jc_1)\otimes c_2)+\pi_{2,2}(c_1\otimes(\omega\wedge c_2))\\&=\omega\otimes(Jc_1\otimes c_2)-\omega\otimes(Jc_1\otimes c_2)=0.\end{aligned}$$

This ends the proof of (i).

To prove (ii), we simply compute

$$\begin{aligned}d_B([\omega\wedge c_1\,|\,c_2\,|\,\cdots\,|\,c_n])=&-[d\omega\wedge c_1-\omega\wedge dc_1\,|\,c_2\,|\,\cdots\,|\,c_n]\\&-\sum_{i=2}^{n}(-1)^i[\omega\wedge Jc_1\,|\,\cdots\,|\,Jc_{i-1}\,|\,dc_i\,|\,c_{i+1}\,|\,\cdots\,|\,c_n]\\&-\sum_{i=1}^{n}(-1)^{i-1}[\omega\wedge Jc_1\,|\,\cdots\,|\,Jc_{i-1}\,|\,Jc_i\wedge c_{i+1}\,|\,c_{i+2}\,|\,\cdots\,|\,c_n]\end{aligned}$$

Thus,

$$\pi_2(d_B([\omega\wedge c_1\,|\,c_2\,|\,\cdots\,|\,c_n]))=-(\mathrm{id}\otimes d_B)(\pi_2([\omega\wedge c_1\,|\,c_2\,|\,\cdots\,|\,c_n])).$$

This finishes the proof, since every section of $F^1B_\Omega(\mathcal{C})$ is a combination of sections of the form $[\omega\wedge c_1\,|\,c_2\,|\,\cdots\,|\,c_n]$. □

6.3. ***Gauss–Manin connection on the fundamental Hopf algebra.*** Consider the splitting

$$0\longrightarrow\Omega^1\longrightarrow\mathcal{C}^1\overset{\curvearrowleft}{\longrightarrow}\mathcal{A}^1\longrightarrow 0,$$

induced by the canonical lift of Kronecker differentials, as in Section 4. Locally,

$$\nu\longmapsto\tilde{\nu},\quad \omega_P^{(n)}\longmapsto\tilde{\omega}_P^{(n)},\quad n\geq 0,\ P\in Z(S).$$

It induces a splitting

$$0\longrightarrow F^1B^0_\Omega(\mathcal{C})\longrightarrow B^0_\Omega(\mathcal{C})\overset{s}{\longrightarrow}B^0(\mathcal{A})\longrightarrow 0.$$

given by

$$s[a_1\,|\,\cdots\,|\,a_n]=[\tilde{a}_1\,|\,\cdots\,|\,\tilde{a}_n].$$

Lemma 6.4. *If ξ is a section of $H^0(B(\mathcal{A}))$, then $d_B\circ s(\xi)\in F^1B^1_\Omega(\mathcal{C})$.*

Proof. This follows immediately from the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1 B^0_\Omega(\mathcal{C}) & \longrightarrow & B^0_\Omega(\mathcal{C}) & \underset{\overset{s}{\curvearrowleft}}{\longrightarrow} & B^0(\mathcal{A}) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle d_B} & & \big\downarrow{\scriptstyle d_B} & & \\ 0 & \longrightarrow & F^1 B^1_\Omega(\mathcal{C}) & \longrightarrow & B^1_\Omega(\mathcal{C}) & \longrightarrow & B^1(\mathcal{A}) & \longrightarrow & 0 \end{array}$$

given by Lemma 6.2. □

Thus, we can define a k-linear map

$$\delta : H^0(B(\mathcal{A})) \longrightarrow \Omega^1 \otimes B^0(\mathcal{A}), \quad \delta = -\pi \circ d_B \circ s.$$

Lemma 6.5. *The image of δ is contained in $\Omega^1 \otimes H^0(B(\mathcal{A}))$.*

Proof. By definition, $(\mathrm{id} \otimes d_B) \circ \delta = -(\mathrm{id} \otimes d_B) \circ \pi \circ d_B \circ s$. Then, using that π is a morphism of k-complexes (part (ii) of Lemma 6.3), we get $-(\mathrm{id} \otimes d_B) \circ \pi \circ d_B \circ s = \pi \circ d_B \circ d_B \circ s = 0$. □

In the notation of Definition 5.6, we obtain a k-linear map

$$\delta : \mathcal{H}_{E/S,Z} \longrightarrow \Omega^1_{S/k} \otimes \mathcal{H}_{E/S,Z}.$$

Theorem 6.6. *The above-defined map δ is an integrable k-connection on the $\mathcal{O}_S$-module $\mathcal{H}_{E/S,Z}$. Moreover:*

(i) *The connection δ preserves the length filtration.*

(ii) *For every $n \geq 1$, the induced connection on $L_n\mathcal{H}_{E/S,Z}/L_{n-1}\mathcal{H}_{E/S,Z}$ gets identified with the n-th tensor power of the Gauss–Manin connection on $H^1_{\mathrm{dR}}((E \setminus Z)/S)$ under the isomorphism (24). In particular, δ is regular singular at infinity.*

(iii) *The deconcatenation coproduct Δ and the antipode σ are horizontal for δ, namely,*

$$(\mathrm{id} \otimes \Delta) \circ \delta = (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta) \circ \Delta, \quad \delta \circ \sigma = (\mathrm{id} \otimes \sigma) \circ \delta.$$

(iv) *The formation of δ is compatible with base change of the form $S' \to S$, where S' is a smooth k-scheme, and with extension of scalars $k \subset k'$.*

Proof. To show that δ is a connection, let $\gamma = \sum_j [c^j_1 \mid \cdots \mid c^j_{n_j}]$ be a section of $B^0_\Omega(\mathcal{C})$ such that $d_B\gamma$ lies in $F^1 B^1_\Omega(\mathcal{C})$. For a section r of $\mathcal{O}_S$, we have

$$d_B(r\gamma) = \sum_j d_B[rc^j_1 \mid \cdots \mid c^j_{n_j}] = -\sum_j [dr \wedge c^j_1 \mid \cdots \mid c^j_{n_j}] + r d_B \gamma.$$

In particular, $d_B(r\gamma)$ is also a section of $F^1 B^1_\Omega(\mathcal{C})$, and

$$\pi \circ d_B(r\gamma) = -dr \otimes \bar{\gamma} + r\pi \circ d_B(\gamma),$$

where $\bar{\gamma}$ denotes the image of γ in $B^0(\mathcal{A})$. If $\gamma = s(\xi)$ for some section ξ of $\mathcal{H}_{E/S,Z} = H^0(B(\mathcal{A}))$, this shows that

$$\delta(r\xi) = dr \otimes \xi + r\delta(\xi).$$

Thus, δ is a k-connection.

Note that the definition of δ only involves the canonical splitting s, which is compatible with base change, and natural operations on the bar construction. Therefore, (iv) follows from Corollary 5.10. The compatibility with base change immediately implies the integrability of δ, since the moduli stack of elliptic curves is a 1-dimensional smooth Deligne–Mumford stack, and every connection on a smooth curve is integrable.

Properties (i) and (iii) are straightforward to verify. We are left to prove (ii). Recall that, by Katz and Oda's construction [1968], the Gauss–Manin connection ∇ on $H^1_{\mathrm{dR}}((E\setminus Z)/S)$ can be explicitly described as follows. Under the isomorphism

$$H^1_{\mathrm{dR}}((E\setminus Z)/S)\cong H^1(\mathcal{A})=\ker(d:\mathcal{A}^1\longrightarrow\mathcal{A}^2)$$

of Proposition 2.7 (see Section 3.3), for a cohomology class given by a closed differential form ω in $\mathcal{A}^1$, let ω' be any lift of ω to an absolute differential form in $\mathcal{C}^1$; then

$$\nabla\omega=d\omega' \bmod f_*\mathcal{F}^2=\sum_{i=1}^{n}\alpha_i\otimes\omega_i \tag{27}$$

for unique α_i in Ω^1 and ω_i in $\mathcal{A}^1$ closed (see Proposition 4.6).

Now, given $[\omega_1\mid\cdots\mid\omega_n]$ in $H^1(\mathcal{A})^{\otimes n}$, let ξ be a section of $L_n\mathcal{H}_{E/S,Z}$ mapping to $\omega_1\otimes\cdots\otimes\omega_n$ under the isomorphism (24). Since $\mathcal{H}_{E/S,Z}\subset B^0(\mathcal{A})=T^c\mathcal{A}^1$, we can write $\xi=\xi_n+\xi_{n-1}+\cdots+\xi_0$, where each ξ_i is of pure length i, and $\xi_n=[\omega_1\mid\cdots\mid\omega_n]$. We have

$$(-d_B\circ s)(\xi)=\sum_{i=1}^{n}[\tilde\omega_1\mid\cdots\mid d\tilde\omega_i\mid\cdots\mid\tilde\omega_n]+\text{lower length}.$$

Then, since each $\tilde\omega_i$ is a lift of ω_i to an absolute differential in $\mathcal{C}^1$, it follows from (27) and from the definition of π that

$$\delta(\xi)=\nabla^{\otimes n}([\omega_1\mid\cdots\mid\omega_n])+\text{lower length}.$$

The last statement of (ii) follows from the regularity at infinity of the Gauss–Manin connection [Deligne 1970, II, §7] and from the fact that regularity is preserved by extensions [loc. cit., II, Proposition 4.6(i)]. □

6.4. *Lifting the relative elliptic KZB connection.* The absolute elliptic KZB connection will be given by combining the relative connection $\nabla_{E^\natural/S,Z}$ (Definition 5.13) with the dual of δ:

$$\delta^\vee:\mathcal{H}^\vee_{E/S,Z}\longrightarrow\Omega^1_{S/k}\mathbin{\hat\otimes}\mathcal{H}^\vee_{E/S,Z}.$$

For this, let

$$\tilde\omega_{E^\natural/S,Z}=(s\otimes\mathrm{id})(\omega_{E^\natural/S,Z})\in\Gamma(S,\ f_*\Omega^1_{E^\natural/k}(\log\pi^{-1}Z)\mathbin{\hat\otimes}\mathcal{H}^\vee_{E/S,Z})$$

be the canonical lift of the KZB form, and let it act on $\mathcal{H}^\vee_{E/S,Z}$ by left multiplication.

Definition 6.7. The *absolute elliptic KZB connection* of $E/S/k$ punctured at Z is the k-connection

$$\nabla_{E^\natural/S/k,Z}:f^*\mathcal{H}^\vee_{E/S,Z}\longrightarrow\Omega^1_{E^\natural/k}(\log\pi^{-1}Z)\mathbin{\hat\otimes}f^*\mathcal{H}^\vee_{E/S,Z},\qquad\nabla_{E^\natural/S/k,Z}=f^*\delta^\vee+\tilde\omega_{E^\natural/S,Z}.$$

Proposition 6.8. *The formation of $\nabla_{E^\natural/S/k,Z}$ is compatible with every base change of the form $S' \to S$, where S' is a smooth k-scheme, and with extension of scalars $k \subset k'$.*

Proof. This follows immediately from Proposition 5.14 and Theorem 6.6. □

To prove integrability, consider the following lemma.

Lemma 6.9. *Let A be a k-algebra (not necessarily commutative) and $(\Omega, d, \wedge)$ be a dg-algebra over k. Let $\varphi \in \Omega^1 \otimes \mathrm{Der}_k(A)$ and $\alpha \in \Omega^1 \otimes A$. We identify A as a subspace of $\mathrm{End}_k(A)$ by left multiplication. Then, the following equation holds in $\Omega^2 \otimes \mathrm{End}_k(A)$:*

$$d(\varphi+\alpha)+(\varphi+\alpha)\wedge(\varphi+\alpha)=(d\varphi+\varphi\wedge\varphi)+(d\alpha+\alpha\wedge\alpha)+\varphi(\alpha),$$

where $\varphi(\alpha)$ is the element of $\Omega^2 \otimes A$ given by "evaluating" φ at α.

Proof. It suffices to prove that $\varphi\wedge\alpha+\alpha\wedge\varphi=\varphi(\alpha)$. Since this equation is linear in φ and in α, we can assume that $\varphi=\omega\otimes\partial$ and $\alpha=\eta\otimes a$. Then,

$$\begin{aligned}\varphi\wedge\alpha+\alpha\wedge\varphi&=\omega\wedge\eta\otimes(\partial\circ a)+\eta\wedge\omega\otimes(a\circ\partial)\\&=\omega\wedge\eta\otimes(\partial\circ a-a\circ\partial)=\omega\wedge\eta\otimes\partial(a)=\varphi(\alpha),\end{aligned}$$

where we used that ∂ is a derivation in the penultimate equality above. □

Theorem 6.10. *The k-connection $\nabla_{E^\natural/S/k,Z}$ is integrable.*

Proof. We may work locally over S and assume that $R^1p_*\mathcal{O}_E$ is trivial. Moreover, by Proposition 6.8, we can also assume that $\dim S=1$ (see the proof of Theorem 6.6). Then, with notation as in (16) and Section 5.3, we identify

$$\mathcal{H}^\vee_{E/S,Z}\cong\mathcal{O}_S\hat{\otimes}A,\qquad A=\frac{k\langle\langle a,b,c_P : P\in Z(S)\rangle\rangle}{\langle\sum_{P\in Z(S)}c_P-[a,b]\rangle}$$

and we write

$$\delta^\vee=d+\Phi,\qquad \Phi\in\Gamma(S,\Omega^1_{S/k})\otimes\mathrm{End}_k(A),$$

so that $\nabla_{E^\natural/S/k,Z}$ is a k-connection on $\mathcal{O}_{E^\natural}\hat{\otimes}A$, given by

$$\nabla_{E^\natural/S/k,Z}=d+\Phi+\tilde{\omega}_{E^\natural/S,Z}.$$

Since the multiplication in $\mathcal{H}^\vee_{E/S,Z}$ is given by the dual of the deconcatenation coproduct Δ, it follows from Theorem 6.6 that $\Phi\in\Gamma(S,\Omega^1_{S/k})\otimes\mathrm{Der}_k(A)$. Moreover, since $\delta^\vee$ is integrable, we conclude from Lemma 6.9 that the integrability of $\nabla_{E^\natural/S/k,Z}$ is equivalent to the equation

$$d\tilde{\omega}_{E^\natural/S,Z}+\tilde{\omega}_{E^\natural/S,Z}\wedge\tilde{\omega}_{E^\natural/S,Z}+\Phi(\tilde{\omega}_{E^\natural/S,Z})=0. \tag{28}$$

Recall from Section 5.3 that $\hat{\sigma}$ is the element of $\Gamma(S,\mathcal{H}_{E/S,Z})\hat{\otimes}A\cong\Gamma(S,\mathcal{H}_{E/S,Z}\hat{\otimes}\mathcal{H}^\vee_{E/S,Z})$ corresponding to the antipode σ of $\mathcal{H}_{E/S,Z}$, and that $\omega_{E^\natural/S,Z}=(\mathrm{pr}_1\otimes\mathrm{id})(\hat{\sigma})$. Since left multiplication by $\hat{\sigma}$

defines a (completed) $\mathcal{H}_{E/S,Z}$-comodule structure on $\mathcal{H}^\vee_{E/S,Z}$, it follows from Proposition 5.5 that

$$\hat{\sigma} = \sum_{n\geq 0} \underbrace{[\omega_{E^\natural/S,Z} \mid \cdots \mid \omega_{E^\natural/S,Z}]}_{\text{length } n}.$$

Since $\sigma : \mathcal{H}_{E/S,Z} \to \mathcal{H}_{E/S,Z}$ is horizontal for the connection δ by Theorem 6.6, and since $\delta^\vee$ is defined as the dual of δ, we have

$$(\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta^\vee)(\hat{\sigma}) = 0. \tag{29}$$

On the one hand, using the definition of $\delta = -\pi \circ d_B \circ s$, we get

$$(\delta \otimes \mathrm{id})(\hat{\sigma}) = \sum_{n\geq 1}\sum_{i=1}^{n} (\pi \otimes \mathrm{id})(\underbrace{[\tilde{\omega}_{E^\natural/S,Z} \mid \cdots \mid \overbrace{d\tilde{\omega}_{E^\natural/S,Z} + \tilde{\omega}_{E^\natural/S,Z} \wedge \tilde{\omega}_{E^\natural/S,Z}}^{i\text{-th position}} \mid \cdots \mid \tilde{\omega}_{E^\natural/S,Z}]}_{\text{length } n}).$$

On the other hand, using that Φ has coefficients in k-derivations of A, we obtain

$$(\mathrm{id} \otimes \delta^\vee)(\hat{\sigma}) = \sum_{n\geq 1}\sum_{i=1}^{n} \underbrace{[\omega_{E^\natural/S,Z} \mid \cdots \mid \overbrace{\Phi(\omega_{E^\natural/S,Z})}^{i\text{-th position}} \mid \cdots \mid \omega_{E^\natural/S,Z}]}_{\text{length } n}.$$

Thus, (29) is equivalent to

$$(\pi \otimes \mathrm{id})(d\tilde{\omega}_{E^\natural/S,Z} + \tilde{\omega}_{E^\natural/S,Z} \wedge \tilde{\omega}_{E^\natural/S,Z} + \Phi(\tilde{\omega}_{E^\natural/S,Z})) = 0.$$

To conclude, we simply remark that the hypothesis $\dim S = 1$ implies that $F^2\mathcal{J} = f_*\mathcal{F}^2 = 0$, so that π is injective on $F^1\mathcal{J} = f_*\mathcal{F}^1$ (see Proposition 4.6), which yields (28). ☐

7. Analytic formulae

7.1. *Kronecker differentials.* Let E be an elliptic curve over $\mathbb{C}$ and let $\tau \in \mathfrak{H}$ be such that $E^{\mathrm{an}} \cong \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, so that $H_1(E^{\mathrm{an}}, \mathbb{Z}) \cong \mathbb{Z} + \tau\mathbb{Z}$. Consider the basis (ω, η) of $H^1_{\mathrm{dR}}(E/\mathbb{C})$ given by

$$\omega = dz, \quad \eta = (\wp_\tau(z) + G_2(\tau))\, dz, \tag{30}$$

where $\wp_\tau(z) = z^{-2} + \sum'_{m,n}((z - m - n\tau)^{-2} - (m + n\tau)^{-2})$ is the Weierstrass $\wp$-function associated to the lattice $\mathbb{Z} + \tau\mathbb{Z}$, and $G_2(\tau) = \sum_n \sum'_m (m + n\tau)^{-2}$ is the Eisenstein series of weight 2 and level 1. It follows from Example 2.2 and [Katz 1973, Lemma A1.3.9] that

$$E^{\natural,\mathrm{an}} \cong \mathbb{C}^2/L_\tau, \quad L_\tau = \{(m + n\tau, 2\pi i n) \in \mathbb{C}^2 : m, n \in \mathbb{Z}\}. \tag{31}$$

If z, w denote the coordinates on $\mathbb{C}^2$, then the basis (ω, η) of $H^1_{\mathrm{dR}}(E/\mathbb{C})$ corresponds under the isomorphism (6) to the basis

$$\omega^{(0)} := dz, \quad \nu := dw$$

of $\Gamma(E^\natural, \Omega^1_{E^\natural/\mathbb{C}})$.

Proposition 7.1. *Let*

$$F_\tau(z,x) := \frac{\theta'_\tau(0)\theta_\tau(z+x)}{\theta_\tau(z)\theta_\tau(x)},$$

where

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})z}, \quad q = e^{2\pi i\tau},$$

denotes Jacobi's odd theta function. Consider the functions $\varphi_\tau^{(n)}(z,w)$ defined by the generating series

$$e^{wx} F_\tau(z,x) = \sum_{n\geq 0} \varphi_\tau^{(n)}(z,w)x^{n-1}.$$

Then the Kronecker differentials associated to $\nu = dw$ (Theorem 3.3) are given by

$$\omega^{(n)} = \varphi_\tau^{(n)}(z,w)\,dz, \quad n \geq 0.$$

Proof. It follows from the transformation property

$$F_\tau(z+m+n\tau, x) = e^{-2\pi i n x} F_\tau(z,x) \tag{32}$$

of the Kronecker function (see [Levin and Racinet 2007, equations (10) and (11)]) that $\varphi_\tau^{(n)}(z,w)\,dz$ are well-defined 1-forms on $E^\natural$ with logarithmic poles along the divisor $\pi^{-1}O$ (which is explicitly given by the equation $z=0$ under (31)). A straightforward computation shows that they satisfy properties (i), (ii), and (iii) of Theorem 3.3. By uniqueness, we conclude that $\omega^{(n)} = \varphi_\tau^{(n)}(z,w)\,dz$. □

Corollary 7.2. *Let $P \in E(\mathbb{C})$ be a torsion point represented by $\alpha + \beta\tau$, for some $\alpha, \beta \in \mathbb{Q}$, under $E^{\mathrm{an}} \cong \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Then, for every $n \geq 0$,*

$$\omega_P^{(n)} = \varphi_\tau^{(n)}(z-\alpha-\beta\tau, w-2\pi i\beta)\,dz.$$

Proof. It suffices to use (15) and to note that $P^\natural \in E^\natural(\mathbb{C})$ is represented by $(\alpha+\beta\tau, 2\pi i\beta)$ under (31). □

7.2. ***Analytic canonical lifts.*** We work in the category of complex analytic spaces. Let $p : \mathcal{E}_{\mathfrak{H}} \to \mathfrak{H}$ be the universal framed elliptic curve over the upper half-plane $\mathfrak{H}$, with fibre at $\tau \in \mathfrak{H}$ given by $p^{-1}(\tau) = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) =: \mathcal{E}_\tau$, and let $\mathcal{E}^\natural$ be its universal vector extension. Explicitly, we can write

$$\mathcal{E}_{\mathfrak{H}}^\natural \cong (\mathbb{C}^2 \times \mathfrak{H})/L,$$

where $L \to \mathfrak{H}$ is the "relative lattice" with fibre at $\tau \in \mathfrak{H}$ given by L_τ as in (31). We denote by $f : \mathcal{E}_{\mathfrak{H}}^\natural \to \mathfrak{H}$ the structure map, and by $\pi : \mathcal{E}_{\mathfrak{H}}^\natural \to \mathcal{E}_{\mathfrak{H}}$ the projection, which in this case is induced by $(z,w) \mapsto z$. Note that the divisor $\pi^{-1}O$ is given by $z=0$, and the identity section $e \in \mathcal{E}_{\mathfrak{H}}^\natural(\mathfrak{H})$ is given by $z=w=0$.

In Section 7.1, we have described the differentials

$$\nu = dw, \quad \omega^{(0)} = dz, \quad \omega^{(1)} = \varphi_\tau^{(1)}(z,w)\,dz, \quad \ldots, \tag{33}$$

which we now regard as global sections of $f_*\Omega^1_{\mathcal{E}_{\mathfrak{H}}^\natural/\mathfrak{H}}(\log \pi^{-1}O)$. Next, we describe their canonical lifts to "absolute differentials" on $\mathcal{E}_{\mathfrak{H}}^\natural$, i.e., global sections of $f_*\Omega^1_{\mathcal{E}_{\mathfrak{H}}^\natural}(\log \pi^{-1}O)$. As our algebraic results do not immediately apply to the analytic category due to the failure of GAGA (see Remark 2.4), we shall first define these canonical lifts by explicit formulas, and then characterise them via uniqueness statements which mimic their algebraic counterparts. Set

$$\tilde{\nu} := dw, \quad \tilde{\omega}^{(0)} := dz - w\frac{d\tau}{2\pi i}.$$

Proposition 7.3. *The above formulas yield well-defined absolute* 1*-forms* $\tilde{\nu}, \tilde{\omega}^{(0)} \in \Gamma(\mathcal{E}^{\natural}_{\mathfrak{H}}, \Omega^1_{\mathcal{E}^{\natural}_{\mathfrak{H}}})$ *lifting the relative* 1*-forms* $\nu, \omega^{(0)} \in \Gamma(\mathcal{E}^{\natural}_{\mathfrak{H}}, \Omega^1_{\mathcal{E}^{\natural}_{\mathfrak{H}}/\mathfrak{H}})$*. Moreover, we have*

$$e^*\tilde{\omega}^{(0)} = e^*\tilde{\nu} = 0, \tag{34}$$

$$d\tilde{\omega}^{(0)} = \frac{d\tau}{2\pi i} \wedge \tilde{\nu}, \quad d\tilde{\nu} = 0, \tag{35}$$

and these properties characterise $\tilde{\nu}, \tilde{\omega}^{(0)}$ *uniquely among lifts of* $\nu, \omega^{(0)}$*.*

Proof. For the first claim, it suffices to check that the above formulas for $\tilde{\nu}$ and $\tilde{\omega}^{(0)}$ are invariant under the action of the lattice L. For instance

$$d(z+m+n\tau) - (w+2\pi i n)\frac{d\tau}{2\pi i} = dz - w\frac{d\tau}{2\pi i}.$$

Equations (34) and (35) are straightforward from the explicit formulas. The last claim follows from the following fact: if γ is a global section of $\Omega^1_{\mathcal{E}^{\natural}}$ of the form

$$\gamma = f(z,w,\tau)d\tau, \quad f \in \Gamma(\mathcal{E}^{\natural}_{\mathfrak{H}}, \mathcal{O}_{\mathcal{E}^{\natural}_{\mathfrak{H}}}),$$

satisfying

$$0 = e^*\gamma = f(0,0,\tau)\,d\tau$$

and

$$0 = d\gamma = \frac{\partial f}{\partial z}(z,w,\tau)\,dz \wedge d\tau + \frac{\partial f}{\partial w}(z,w,\tau)\,dw \wedge d\tau$$

then $\gamma = 0$. □

Remark 7.4. Note that $(\omega^{(0)}, \nu)$ corresponds to the frame $(\omega, \eta) = (dz, (\wp_\tau(z) + G_2(\tau))\,dz)$ of the analytic de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{E}_{\mathfrak{H}}/\mathfrak{H})$. The Gauss–Manin connection in this frame is given by (see [Katz 1973, Section A1])

$$\nabla\omega = \frac{d\tau}{2\pi i} \otimes \eta, \quad \nabla\eta = 0,$$

so that (35) are the analytic versions of (18).

Define lifts of $\omega^{(n)}$ $(n \geq 1)$ to absolute logarithmic 1-forms by

$$\tilde{\omega}^{(n)} := \varphi^{(n)}_\tau(z,w)\Big(dz - w\frac{d\tau}{2\pi i}\Big) + n\varphi^{(n+1)}_\tau(z,w)\frac{d\tau}{2\pi i},$$

where $\varphi^{(n)}_\tau(z,w)$ is as in Proposition 7.1.

Proposition 7.5. *For every* $n \geq 1$*, the above formula yields a well-defined absolute logarithmic* 1*-form* $\tilde{\omega}^{(n)} \in \Gamma(\mathcal{E}^{\natural}_{\mathfrak{H}}, \Omega^1_{\mathcal{E}^{\natural}_{\mathfrak{H}}}(\log \pi^{-1}O))$ *which lifts the relative logarithmic* 1*-form* $\omega^{(n)}$*. Moreover, we have*

$$\tilde{\omega}^{(n)} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} = n\frac{d\tau}{2\pi i} \wedge \tilde{\nu} \wedge \tilde{\omega}^{(n+1)} \tag{36}$$

and this property characterises $\tilde{\omega}^{(n)}$ *uniquely among lifts of* $\omega^{(n)}$*.*

Proof. The first claim follows from Proposition 7.3 and (32). Equation (36) follows immediately from the explicit formulas. To show that it characterises $\tilde{\omega}^{(n)}$ uniquely among lifts of $\omega^{(n)}$, we let

$$\tilde{\omega}^{(n)\prime} = \tilde{\omega}^{(n)} + f_n(z, w, \tau)\, d\tau, \qquad f_n(z, w, \tau) \in \Gamma(\mathcal{E}^{\natural}_{\mathfrak{H}}, \mathcal{O}_{\mathcal{E}^{\natural}_{\mathfrak{H}}}),$$

be other lifts of $\omega^{(n)}$ satisfying (36). Then

$$(\tilde{\omega}^{(n)} + f_n(z, w, \tau)\, d\tau) \wedge \tilde{\nu} \wedge \tilde{\omega}^{(0)} = n \frac{d\tau}{2\pi i} \wedge \tilde{\nu} \wedge (\tilde{\omega}^{(n+1)} + f_{n+1}(z, w, \tau)\, d\tau)$$

if and only if

$$f_n(z, w, \tau) d\tau \wedge dw \wedge dz = 0,$$

so that $f_n = 0$ for every $n \geq 1$. □

As an application, we can use the above uniqueness statements to prove an algebraicity result.

Proposition 7.6. *Consider the pullback diagram*

$$\begin{array}{ccc} \mathcal{E}_{\mathfrak{H}} & \xrightarrow{\ \psi\ } & E^{\mathrm{an}} \\ \downarrow & \square & \downarrow \\ \mathfrak{H} & \xrightarrow[\ s\ ]{} & S^{\mathrm{an}} \end{array} \tag{37}$$

where

- $S = \operatorname{Spec} \mathbb{C}[g_2, g_3, (g_2^3 - 27g_3^2)^{-1}]$, *and E/S is the universal Weierstrass elliptic curve given by the equation* $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$,
- *the map* $s : \mathfrak{H} \to S^{\mathrm{an}}$ *is given by* $s(\tau) = (g_2(\tau), g_3(\tau))$, *where* $g_2(\tau) = 60 \sum'_{m,n} (m + n\tau)^{-4}$ *and* $g_3(\tau) = 140 \sum'_{m,n} (m + n\tau)^{-6}$,
- *the map* $\psi : \mathcal{E}_{\mathfrak{H}} \to E^{\mathrm{an}}$ *is given on each fibre by*

$$\psi_\tau : \mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \xrightarrow{\sim} E^{\mathrm{an}}_{s(\tau)}, \qquad [z] \longmapsto \begin{cases} (\wp_\tau(z) : \wp'_\tau(z) : 1), & [z] \neq 0, \\ (0 : 1 : 0), & [z] = 0. \end{cases}$$

Consider the frame $(\omega_{\mathrm{alg}}, \eta_{\mathrm{alg}}) = (dx/y, x\, dx/y)$ *of the algebraic de Rham cohomology* $H^1_{\mathrm{dR}}(E/S)$, *and let* $\omega^{(n)}_{\mathrm{alg}} \in \Gamma(E^\natural, \Omega^1_{E^\natural/S}(\log \pi^{-1} O))$ *and* $\tilde{\omega}^{(n)}_{\mathrm{alg}} \in \Gamma(E^\natural, \Omega^1_{E^\natural/\mathbb{C}}(\log \pi^{-1} O))$ $(n \geq 1)$ *be the corresponding Kronecker differentials and their canonical lifts* (see Remark 3.2). *Then,*

$$\psi^* \omega^{(n)}_{\mathrm{alg}} = \omega^{(n)}, \qquad \psi^* \tilde{\omega}^{(n)}_{\mathrm{alg}} = \tilde{\omega}^{(n)}$$

for every $n \geq 1$.

Proof. Let $(\omega^{(0)}_{\mathrm{alg}}, \nu_{\mathrm{alg}})$ be the frame of $f_* \Omega^1_{E^\natural/S}$ corresponding to $(\omega_{\mathrm{alg}}, \eta_{\mathrm{alg}})$. Since the frame $(\omega^{(0)}, \nu)$ of $f_* \Omega^1_{\mathcal{E}_{\mathfrak{H}}/\mathfrak{H}}$ corresponds to (ω, η), which is given by the formula (30), we obtain

$$\psi^* \omega^{(0)}_{\mathrm{alg}} = dz = \omega^{(0)}, \qquad \psi^* \nu_{\mathrm{alg}} = dw - G_2\, dz = \nu - G_2 \omega^{(0)}.$$

It follows from Remark 3.4 and from the uniqueness part of Theorem 3.3 that $\psi^* \omega^{(n)}_{\mathrm{alg}}$ and $\omega^{(n)}$ agree in every fibre of $f : \mathcal{E}^{\natural}_{\mathfrak{H}} \to \mathfrak{H}$, so that

$$\psi^* \omega^{(n)}_{\mathrm{alg}} = \omega^{(n)}$$

for every $n \geq 1$.

Let $(\alpha_{ij})_{1\leq i,j\leq 2}$ be the matrix of the Gauss–Manin connection ∇ on $H^1_{\mathrm{dR}}(E/S)$ in the frame $(\omega_{\mathrm{alg}}, \eta_{\mathrm{alg}})$, so that

$$\nabla\omega_{\mathrm{alg}} = \alpha_{11}\otimes\omega_{\mathrm{alg}} + \alpha_{21}\otimes\eta_{\mathrm{alg}},$$
$$\nabla\eta_{\mathrm{alg}} = \alpha_{12}\otimes\omega_{\mathrm{alg}} + \alpha_{22}\otimes\eta_{\mathrm{alg}}.$$

It follows from Remark 7.4 that the Gauss–Manin connection ∇ on $H^1_{\mathrm{dR}}(\mathcal{E}_{\mathfrak{H}}/\mathfrak{H})$ satisfies

$$\nabla\psi^*\omega_{\mathrm{alg}} = \nabla\omega = \frac{d\tau}{2\pi i}\otimes\eta = \frac{d\tau}{2\pi i}\otimes(\psi^*\eta_{\mathrm{alg}}+G_2\psi^*\omega_{\mathrm{alg}}) = G_2\frac{d\tau}{2\pi i}\otimes\psi^*\omega_{\mathrm{alg}}+\frac{d\tau}{2\pi i}\otimes\psi^*\eta_{\mathrm{alg}},$$
$$\nabla\psi^*\eta_{\mathrm{alg}} = \nabla(\eta-G_2\omega) = -dG_2\otimes\omega-G_2\frac{d\tau}{2\pi i}\otimes\eta = \Big(-dG_2-G_2^2\frac{d\tau}{2\pi i}\Big)\otimes\psi^*\omega_{\mathrm{alg}}-G_2\frac{d\tau}{2\pi i}\otimes\psi^*\eta_{\mathrm{alg}}.$$

Since the formation of the Gauss–Manin connection commutes with base change, we conclude from the above equations that

$$\begin{pmatrix} s^*\alpha_{11} & s^*\alpha_{12} \\ s^*\alpha_{21} & s^*\alpha_{22}\end{pmatrix} = \begin{pmatrix} G_2\frac{d\tau}{2\pi i} & -dG_2 - G_2^2\frac{d\tau}{2\pi i} \\ \frac{d\tau}{2\pi i} & -G_2\frac{d\tau}{2\pi i}\end{pmatrix}.$$

Then, using the equations of Example 4.3, one can check that

$$d(\psi^*\tilde{\omega}^{(0)}_{\mathrm{alg}}) = \frac{d\tau}{2\pi i}\wedge(\psi^*\tilde{\nu}_{\mathrm{alg}} + G_2\tilde{\omega}^{(0)}), \quad d(\psi^*\tilde{\nu}_{\mathrm{alg}} + G_2\tilde{\omega}^{(0)}) = 0,$$

and we conclude from Proposition 7.3 that

$$\psi^*\tilde{\omega}^{(0)}_{\mathrm{alg}} = \tilde{\omega}^{(0)}, \quad \psi^*\tilde{\nu}_{\mathrm{alg}} = \tilde{\nu} - G_2\tilde{\omega}^{(0)}.$$

By pulling back the defining equation for $\tilde{\omega}^{(n)}_{\mathrm{alg}}$ in Theorem 4.7, we obtain

$$\psi^*\tilde{\omega}^{(n)}_{\mathrm{alg}}\wedge(\tilde{\nu} - G_2\tilde{\omega}^{(0)})\wedge\tilde{\omega}^{(0)} = n\frac{d\tau}{2\pi i}\wedge(\tilde{\nu} - G_2\tilde{\omega}^{(0)})\wedge\psi^*\tilde{\omega}^{(n+1)}_{\mathrm{alg}}$$

(note that $f_*\mathcal{F}^2 = 0$ since $\mathfrak{H}$ is 1-dimensional). Clearly,

$$\psi^*\tilde{\omega}^{(n)}_{\mathrm{alg}}\wedge(\tilde{\nu} - G_2\tilde{\omega}^{(0)})\wedge\tilde{\omega}^{(0)} = \psi^*\tilde{\omega}^{(n)}_{\mathrm{alg}}\wedge\tilde{\nu}\wedge\tilde{\omega}^{(0)}.$$

Since $\psi^*\tilde{\omega}^{(n+1)}_{\mathrm{alg}}$ differs from $\tilde{\omega}^{(n+1)}$ by an element of $\Gamma(\mathcal{E}^\natural_{\mathfrak{H}}, \mathcal{O}_{\mathcal{E}^\natural_{\mathfrak{H}}})\,d\tau$ and $\tilde{\omega}^{(0)}\wedge\tilde{\omega}^{(n+1)}$ is a multiple of $d\tau$, we obtain

$$n\frac{d\tau}{2\pi i}\wedge(\tilde{\nu} - G_2\tilde{\omega}^{(0)})\wedge\psi^*\tilde{\omega}^{(n+1)}_{\mathrm{alg}} = n\frac{d\tau}{2\pi i}\wedge\tilde{\nu}\wedge\psi^*\tilde{\omega}^{(n+1)}_{\mathrm{alg}}.$$

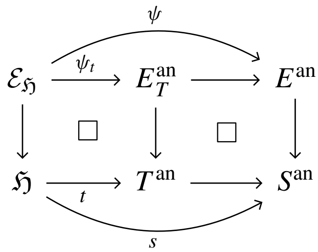
Thus, by Proposition 7.5, we conclude that $\psi^*\tilde{\omega}^{(n)}_{\mathrm{alg}} = \tilde{\omega}^{(n)}$ for every $n \geq 1$. □

Let $P \in \mathcal{E}_{\mathfrak{H}}(\mathfrak{H})$ be a torsion section. Then, there are $\alpha, \beta \in \mathbb{Q}$ such that $P(\tau)$ is represented by $\alpha + \beta\tau$ under $\mathcal{E}_\tau \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for every $\tau \in \mathfrak{H}$. The next result gives the canonical lift of $\omega^{(n)}_P$, defined in Corollary 7.2.

Corollary 7.7. *Set*

$$\tilde{\omega}^{(n)}_P := \varphi^{(n)}_\tau(z-\alpha-\beta\tau, w-2\pi i\beta)\Big(dz - w\frac{d\tau}{2\pi i}\Big) + n\varphi^{(n+1)}_\tau(z-\alpha-\beta\tau, w-2\pi i\beta)\frac{d\tau}{2\pi i}.$$

Let T be an S-scheme and assume that the diagram (37) factors as



Let $P_{\mathrm{alg}} \in E_T(T)$ be a torsion section and $\tilde{\omega}^{(n)}_{\mathrm{alg},P_{\mathrm{alg}}}$ be the corresponding Kronecker differentials on E_T with logarithmic singularities along $\pi^{-1}P_{\mathrm{alg}}$. If $P = t^* P_{\mathrm{alg}}$, then $\psi_t^* \tilde{\omega}^{(n)}_{\mathrm{alg},P_{\mathrm{alg}}} = \tilde{\omega}^{(n)}_P$.

Proof. The unique lift to a torsion section $P^\natural \in \mathcal{E}^\natural_{\mathfrak{H}}(\mathfrak{H})$ is such that $P^\natural(\tau)$ is represented by $(\alpha+\beta\tau, 2\pi i\beta)$ under $\mathcal{E}^\natural_\tau \cong \mathbb{C}^2/L_\tau$ for every $\tau \in \mathfrak{H}$. One can check that $\tilde{\omega}^{(0)}$ and $\tilde{\nu}$ are invariant under translation by $-P^\natural$ (see Lemma 4.5), so that $\tilde{\omega}^{(n)}_P = \tau^*_{-P^\natural}\tilde{\omega}^{(n)}$. Then, the statement follows immediately from (22) and from the previous theorem. □

7.3. *Level-N elliptic KZB connection.* Let $N \geq 1$ be an integer and denote by $\Gamma_N := \mathcal{E}_{\mathfrak{H}}[N](\mathfrak{H})$ the group of N-torsion sections of $p : \mathcal{E}_{\mathfrak{H}} \to \mathfrak{H}$. Note that $\Gamma_N \cong (N^{-1}\mathbb{Z})^2/\mathbb{Z}^2$.

Consider the completed Hopf algebra over $\mathcal{O}_{\mathfrak{H}}$ given by

$$\mathcal{A}_N := \frac{\mathcal{O}_{\mathfrak{H}}\langle\langle a, b, c_P : P \in \Gamma_N \rangle\rangle}{\left\langle \sum_{P\in\Gamma_N} c_P - [a,b] \right\rangle}.$$

If E is an (algebraic) complex elliptic curve such that $E^{\mathrm{an}} \cong \mathcal{E}_\tau$, then it follows from the discussion in Section 5.3 that the fibre of $\mathcal{A}_N$ at τ is isomorphic to the dual of the fundamental Hopf algebra of $E/\mathbb{C}$ punctured at $Z = E[N]$:

$$\mathcal{A}_{N,\tau} \cong \mathcal{H}^\vee_{E/\mathbb{C},E[N]}.$$

Then, the relative elliptic KZB connection is given by

$$\overline{\nabla}_N : f^*\mathcal{A}_N \longrightarrow \Omega^1_{\mathcal{E}^\natural_{\mathfrak{H}}/\mathfrak{H}}(\log \pi^{-1}\mathcal{E}_{\mathfrak{H}}[N]) \hat{\otimes} f^*\mathcal{A}_N, \quad \overline{\nabla}_N = d + \omega_N,$$

where

$$\omega_N = -\nu \otimes a - \omega^{(0)} \otimes b - \sum_{n\geq 1}\sum_{P\in\Gamma_N} \omega_P^{(n)} \otimes \mathrm{ad}_a^{n-1} c_P,$$

with ν, $\omega^{(0)}$, and $\omega_P^{(n)}$ as in the above paragraphs. If $P(\tau)$ is represented by $\alpha+\beta\tau$, let us define

$$k_P(z,w,x) := e^{(w-2\pi i\beta)x} F_\tau(z-\alpha-\beta\tau, x) - \frac{1}{x},$$

the dependence on τ being omitted in the notation for simplicity. Then, it follows from the results of Section 7.1 that we can rewrite

$$\omega_N = -dw \otimes a - dz \otimes \left(b + \sum_{P\in\Gamma_N} k_P(z,w,\mathrm{ad}_a)c_P\right).$$

According to Section 6.4, the relative connection $\overline{\nabla}_N$ lifts to an absolute elliptic KZB connection, which we shall denote more simply as

$$\nabla_N : f^*\mathcal{A}_N \longrightarrow \Omega^1_{\mathcal{E}^\natural_{\mathfrak{H}}}(\log \pi^{-1}\mathcal{E}_{\mathfrak{H}}[N]) \hat{\otimes} f^*\mathcal{A}_N,$$

and which is an integrable connection explicitly given by (see the proof of Theorem 6.10)

$$\nabla_N = d + \tilde{\omega}_N + \Phi_N,$$

where $\tilde{\omega}_N$ is the canonical lift of the KZB form ω_N, and Φ_N is the matrix of the dual Gauss–Manin connection $\delta^\vee$ (see Section 6.3). Next, we determine ∇_N explicitly.

Lemma 7.8. *For every $P \in \Gamma_N$, set*

$$g_P(z, w, x) := \frac{\partial}{\partial x} k_P(z, w, x) - w k_P(z, w, x).$$

We have

$$\tilde{\omega}_N = -dw \otimes a - dz \otimes \Bigl(b + \sum_{P\in\Gamma_N} k_P(z, w, \mathrm{ad}_a) c_P\Bigr) - \frac{d\tau}{2\pi i} \otimes \Bigl(-wb + \sum_{P\in\Gamma_N} g_P(z, w, \mathrm{ad}_a) c_P\Bigr).$$

Proof. By definition,

$$\tilde{\omega}_N = -\tilde{\nu} \otimes a - \tilde{\omega}^{(0)} \otimes b - \sum_{n\geq 1} \sum_{P\in\Gamma_N} \tilde{\omega}^{(n)}_P \otimes \mathrm{ad}^{n-1}_a c_P.$$

We put together the explicit expressions for the canonical lifts: $\tilde{\nu} = dw$, $\tilde{\omega}^{(0)} = dz - w\frac{d\tau}{2\pi i}$, and, by Corollary 7.7,

$$\begin{aligned}\sum_{n\geq 1} \tilde{\omega}^{(n)}_P x^{n-1} &= \sum_{n\geq 1} \varphi^{(n)}_\tau(z-\alpha-\beta\tau, w-2\pi i\beta) x^{n-1}\, dz \\ &\qquad + \sum_{n\geq 1} \bigl(n\varphi^{(n+1)}_\tau(z-\alpha-\beta\tau, w-2\pi i\beta)x^{n-1} - w\varphi^{(n)}_\tau(z-\alpha-\beta\tau, w-2\pi i\beta)x^{n-1}\bigr)\frac{d\tau}{2\pi i} \\ &= k_P(z, w, x)\, dz + g_P(z, w, x)\frac{d\tau}{2\pi i}.\end{aligned}$$

□

For $Q \in \Gamma_N$, let us define $A_{m,Q}(\tau)$ by the generating series

$$g_{-Q}(0, 0, x) = k'_{-Q}(0, 0, x) = \sum_{m\geq 0} A_{m,Q}(\tau) x^m,$$

where $k'_{-Q}(z, w, x) = (\partial/\partial x) k_{-Q}(z, w, x)$. When $Q = O$, we have

$$g_O(0, 0, x) = \frac{\partial}{\partial x}\Bigl(\frac{\partial}{\partial x} \log \theta_\tau(x)\Bigr) + \frac{1}{x^2} = -\Bigl(\wp_\tau(x) - \frac{1}{x^2}\Bigr) = \sum_{k\geq 2} (2k-1) G_{2k}(\tau) x^{2k-2},$$

so that $A_{m,O}$ are level-1 Eisenstein series. For general Q, the functions $A_{m,Q}$ are Eisenstein series of level N; see [Hopper 2024, Proposition 10.1].

Theorem 7.9. *We have*

$$\Phi_N = -\frac{d\tau}{2\pi i} \otimes \Big(b\frac{\partial}{\partial a} + \frac{1}{2} \sum_{Q\in\Gamma_N} \sum_{m\geq 0} A_{m,Q}(\tau)\delta_{m,Q} \Big),$$

where

$$\delta_{m,Q} := \sum_{\substack{i,j\geq 0 \\ i+j=m-1}} \sum_{P\in\Gamma_N} [(-\operatorname{ad}_a)^i c_P, \operatorname{ad}_a^j c_{P-Q}]\frac{\partial}{\partial b} + \sum_{P\in\Gamma_N} [c_P, \operatorname{ad}_a^m c_{P-Q} + (-\operatorname{ad}_a)^m c_{P+Q}]\frac{\partial}{\partial c_P}.$$

The proof is based on the following lemma.

Lemma 7.10. *Let* $A_N = \mathbb{C}\langle\!\langle a, b, c_P : P \in \Gamma_N \rangle\!\rangle / \big\langle \sum_{P\in\Gamma_N} c_P - [a,b] \big\rangle$, *so that* $\mathcal{A}_N = \mathcal{O}_{\mathfrak{H}} \hat{\otimes} A_N$, *and let* $\Psi \in \Gamma(\mathfrak{H}, \Omega^1_{\mathfrak{H}}) \otimes \operatorname{Der}_{\mathbb{C}}(A_N)$. *The following are equivalent*:

(i) *The connection* $d + \tilde{\omega}_N + \Psi$ *is integrable.*

(ii) $d\tilde{\omega}_N + \tilde{\omega}_N \wedge \tilde{\omega}_N + \Psi(\tilde{\omega}_N) = 0$.

(iii) $\Psi = \Phi_N$.

Proof. The equivalence between (i) and (ii) follows immediately from Lemma 6.9. Since $\nabla_N = d + \tilde{\omega}_N + \Phi_N$ is integrable, (ii) is equivalent to

$$\Psi(\tilde{\omega}_N) = \Phi_N(\tilde{\omega}_N). \tag{38}$$

To finish, it is enough to verify that (38) implies (iii). Let us write $\Phi_N = -\frac{d\tau}{2\pi i} \otimes \partial_N$ and $\Psi = -\frac{d\tau}{2\pi i} \otimes D$. Then, (38) is equivalent to

$$\begin{aligned} &\frac{d\tau}{2\pi i} \wedge \tilde{\nu} \otimes Da + \frac{d\tau}{2\pi i} \wedge \tilde{\omega}^{(0)} \otimes Db + \sum_{n\geq 1} \sum_{P\in\Gamma_N} \frac{d\tau}{2\pi i} \wedge \tilde{\omega}_P^{(n)} \otimes D \operatorname{ad}_a^{n-1} c_P \\ &\qquad = \frac{d\tau}{2\pi i} \wedge \tilde{\nu} \otimes \partial_N a + \frac{d\tau}{2\pi i} \wedge \tilde{\omega}^{(0)} \otimes \partial_N b + \sum_{n\geq 1} \sum_{P\in\Gamma_N} \frac{d\tau}{2\pi i} \wedge \tilde{\omega}_P^{(n)} \otimes \partial_N \operatorname{ad}_a^{n-1} c_P. \end{aligned} \tag{39}$$

Since

$$\frac{d\tau}{2\pi i} \wedge \tilde{\nu}, \quad \frac{d\tau}{2\pi i} \wedge \tilde{\omega}^{(0)}, \quad \frac{d\tau}{2\pi i} \wedge \tilde{\omega}_P^{(n)}, \quad P \in \Gamma_N,\ n \geq 1,$$

trivialise $\mathcal{F}^{1,2} \cong f^*\Omega^1_{\mathfrak{H}} \otimes \Omega^1_{\mathcal{E}_{\mathfrak{H}}{}^\natural/\mathfrak{H}}(\log \pi^{-1}\mathcal{E}_{\mathfrak{H}}[N])$ (see (20)), it follows from (39) that $Da = \partial_N a$, $Db = \partial_N b$, and $Dc_P = \partial_N c_P$ for all $P \in \Gamma_N$; thus $\Psi = \Phi_N$. □

Proof of Theorem 7.9. The proof is a long computation; we merely indicate the main steps. Let

$$\Psi := -\frac{d\tau}{2\pi i} \otimes D, \quad D := \Big(b\frac{\partial}{\partial a} + \frac{1}{2} \sum_{Q\in\Gamma_N} \sum_{m\geq 0} A_{m,Q}(\tau)\delta_{m,Q} \Big).$$

By Lemma 7.10, it suffices to prove that

$$d\tilde{\omega}_N + \tilde{\omega}_N \wedge \tilde{\omega}_N + \Psi(\tilde{\omega}_N) = 0.$$

It follows from the "mixed heat equation" [Brown and Levin 2011, Proposition 5(ii)] that

$$2\pi i \frac{\partial}{\partial \tau} k_P(z, w, x) = \frac{\partial}{\partial z} g_P(z, w, x),$$

so that

$$\begin{aligned} d\tilde{\omega}_N &= -dw\wedge dz\otimes\Bigg(\sum_{P\in\Gamma_N}\frac{\partial}{\partial w}k_P(z,w,\mathrm{ad}_a)c_P\Bigg)-dw\wedge\frac{d\tau}{2\pi i}\otimes\Bigg(-b+\sum_{P\in\Gamma_N}\frac{\partial}{\partial w}g_P(z,w,\mathrm{ad}_a)c_P\Bigg)\\ &= -dw\wedge dz\otimes\Bigg(\mathrm{ad}_a\, b+\sum_{P\in\Gamma_N}\mathrm{ad}_a\, k_P(z,w,\mathrm{ad}_a)c_P\Bigg)\\ &\qquad -dw\wedge\frac{d\tau}{2\pi i}\otimes\Bigg(-b-w\,\mathrm{ad}_a\, b+\sum_{P\in\Gamma_N}\mathrm{ad}_a\, g_P(z,w,\mathrm{ad}_a)c_P\Bigg), \end{aligned}\tag{40}$$

where in the last equality we used the equations

$$\frac{\partial}{\partial w}k_P(z,w,x)=xk_P(z,w,x)+1,\qquad \frac{\partial}{\partial x}g_P(z,w,x)=xg_P(z,w,x)-w,\qquad \sum_{P\in\Gamma_N}c_P=\mathrm{ad}_a\, b.$$

Now, we have

$$\begin{aligned} \tilde{\omega}_N\wedge\tilde{\omega}_N &= dw\wedge dz\otimes\Bigg(\mathrm{ad}_a\, b+\sum_{P\in\Gamma_N}\mathrm{ad}_a\, k_P(z,w,\mathrm{ad}_a)c_P\Bigg)\\ &\quad +dw\wedge\frac{d\tau}{2\pi i}\otimes\Bigg(-w\,\mathrm{ad}_a\, b+\sum_{P\in\Gamma_N}\mathrm{ad}_a\, g_P(z,w,\mathrm{ad}_a)c_P\Bigg)\\ &\quad +dz\wedge\frac{d\tau}{2\pi i}\otimes\Bigg[b+\sum_{P\in\Gamma_N}k_P(z,w,\mathrm{ad}_a)c_P,\ -wb+\sum_{Q\in\Gamma_N}g_Q(z,w,\mathrm{ad}_a)c_Q\Bigg]. \end{aligned}\tag{41}$$

By putting (40) and (41) together, we get

$$\begin{aligned} d\tilde{\omega}_N+\tilde{\omega}_N\wedge\tilde{\omega}_N &= -\frac{d\tau}{2\pi i}\wedge dw\otimes b\\ &\quad -\frac{d\tau}{2\pi i}\wedge dz\otimes\Bigg[b+\sum_{P\in\Gamma_N}k_P(z,w,\mathrm{ad}_a)c_P,\ -wb+\sum_{Q\in\Gamma_N}g_Q(z,w,\mathrm{ad}_a)c_Q\Bigg]. \end{aligned}\tag{42}$$

We borrow the following notation from [Levin and Racinet 2007]: for $f(x,y)=\sum_{i,j\geq 0}f_{i,j}x^iy^j\in\mathbb{C}[\![x,y]\!]$ and $t,r,s\in A_N$, we set

$$f(x,y)[\![r,s]\!]_t:=\sum_{i,j\geq 0}f_{i,j}[\mathrm{ad}_t^i\, r,\mathrm{ad}_t^j\, s].$$

Note that

$$f(x,y)[\![r,s]\!]_t=-f(y,x)[\![s,r]\!]_t.\tag{43}$$

To finish the proof, we are left to show that the right-hand side of (42) is equal to $-\Psi(\tilde{\omega}_N)$, or equivalently that

$$\begin{aligned} D\Bigg(b+\sum_{P\in\Gamma_N}k_P(z,w,\mathrm{ad}_a)c_P\Bigg) &= \Bigg[b+\sum_{P\in\Gamma_N}k_P(z,w,\mathrm{ad}_a)c_P,\ -wb+\sum_{Q\in\Gamma_N}g_Q(z,w,\mathrm{ad}_a)c_Q\Bigg]\\ &= \mathrm{ad}_b\sum_{P\in\Gamma_N}k'_P(z,w,\mathrm{ad}_a)c_P+\frac{1}{2}\sum_{P,Q\in\Gamma_N}f_{P,Q}(x,y)[\![c_Q,c_P]\!]_a, \end{aligned}\tag{44}$$

where $k'_P(z,w,x)=\frac{\partial}{\partial x}k_P(z,w,x)$ and

$$f_{P,Q}(x,y)=k_Q(z,w,x)k'_P(z,w,y)-k_P(z,w,y)k'_Q(z,w,x).$$

We now compute the left-hand side of (44). Using the symmetry $g_{-Q}(0,0,x) = g_Q(0,0,-x)$, we get by direct computation

$$D(b) = \frac{1}{2} \sum_{P,Q\in\Gamma_N} \frac{g_{P-Q}(0,0,y) - g_{Q-P}(0,0,x)}{x+y} [\![c_Q, c_P]\!]_a.$$

Applications of [Levin and Racinet 2007, Lemma 3.1.4] and (43) show that

$$D\Bigl(\sum_{P\in\Gamma_N} k_P(z,w,\mathrm{ad}_a)c_P\Bigr) = \mathrm{ad}_b \sum_{P\in\Gamma_N} k'_P(z,w,\mathrm{ad}_a)c_P + \frac{1}{2}\sum_{P,Q\in\Gamma_N} h_{P,Q}(x,y)[\![c_Q,c_P]\!]_a,$$

where

$$\begin{aligned} h_{P,Q}(x,y) = {}& \frac{k_P(z,w,x+y) - k_P(z,w,y) - xk'_P(z,w,y)}{x^2} \\ & - \frac{k_Q(z,w,x+y) - k_Q(z,w,x) - yk'_Q(z,w,x)}{y^2} \\ & + k_Q(z,w,x+y)g_{P-Q}(0,0,y) - k_P(z,w,x+y)g_{Q-P}(0,0,x). \end{aligned}$$

Thus, to establish (44), it suffices to prove that, for every $P, Q \in \Gamma_N$, we have

$$f_{P,Q}(x,y) - h_{P,Q}(x,y) - \frac{g_{P-Q}(0,0,y) - g_{Q-P}(0,0,x)}{x+y} = 0,$$

which is equivalent to

$$\begin{aligned} &\Bigl(k'_{P-Q}(0,0,y) - \frac{1}{y^2}\Bigr)\Bigl(k_Q(z,w,x+y) + \frac{1}{x+y}\Bigr) - \Bigl(k'_{Q-P}(0,0,x) - \frac{1}{x^2}\Bigr)\Bigl(k_P(z,w,x+y) + \frac{1}{x+y}\Bigr) \\ &\quad + \Bigl(k'_Q(z,w,x) - \frac{1}{x^2}\Bigr)\Bigl(k_P(z,w,y) + \frac{1}{y}\Bigr) - \Bigl(k'_P(z,w,y) - \frac{1}{y^2}\Bigr)\Bigl(k_Q(z,w,x) + \frac{1}{x}\Bigr) = 0. \end{aligned}$$

This, in turn, follows immediately from the formula [Brown and Levin 2011, equation (3.3)] in the case $P = Q$, and from the Fay identity [Brown and Levin 2011, Proposition 5(iii)] in the case $P \neq Q$. □

Appendix: Unipotent connections and Tannakian theory

We fix a base field k of characteristic 0, which will be implicit throughout this appendix.

A1. *Unipotent connections.* Let X be a smooth k-scheme and D be a normal crossings divisor in X. Recall that a *vector bundle with integrable connection over X with logarithmic singularities along D* is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a vector bundle over X and $\nabla : \mathcal{E} \to \Omega^1_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{E}$ is an integrable logarithmic k-connection on $\mathcal{E}$. A morphism $(\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$ is a horizontal $\mathcal{O}_X$-linear map $f : \mathcal{E} \to \mathcal{E}'$, meaning that $\nabla' \circ f = (\mathrm{id} \otimes f) \circ \nabla$. We thus obtain a category which we denote by $\mathrm{VIC}(X, \log D)$. When D is empty, it is denoted by $\mathrm{VIC}(X)$.

Example A.1. If $\mathcal{E} = \mathcal{O}_X \otimes_k F$ for some k-vector space F, then we can write $\nabla = d + \omega$ for a unique $\omega \in \Gamma(X, \Omega^1_{X/k}(\log D)) \otimes_k \mathrm{End}_k(F)$. Integrability amounts to the equation $d\omega + \omega \wedge \omega = 0$ in $\Gamma(X, \Omega^2_{X/k}(\log D)) \otimes_k \mathrm{End}_k(F)$. In general, a *frame* of a connection $(\mathcal{E}, \nabla)$ is an isomorphism $(\mathcal{O}_X^{\oplus n}, d + \omega) \cong (\mathcal{E}, \nabla)$, and $\omega \in M_{n\times n}(\Gamma(X, \Omega^1_{X/k}(\log D)))$ is the *matrix of* ∇ in this frame.

Recall that the category $\mathrm{VIC}(X, \log D)$ is k-linear and admits the usual multilinear operations, such as duals and tensor products. A *horizontal section* of $(\mathcal{E}, \nabla)$ is some $s \in \Gamma(X, \mathcal{E})$ satisfying $\nabla s = 0$; it can also be thought as a morphism $(\mathcal{O}_X, d) \to (\mathcal{E}, \nabla)$. In particular, a morphism $(\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$ is the same as a horizontal section of $(\mathcal{E}, \nabla)^\vee \otimes (\mathcal{E}', \nabla')$.

Definition A.2. We say that an object $(\mathcal{E}, \nabla)$ of $\mathrm{VIC}(X, \log D)$ is *unipotent* if it admits a finite filtration

$$0 = (\mathcal{E}_0, \nabla_0) \subseteq (\mathcal{E}_1, \nabla_1) \subseteq \cdots \subseteq (\mathcal{E}_n, \nabla_n) = (\mathcal{E}, \nabla)$$

such that, for every $1 \leq i \leq n$, the quotient $(\mathcal{E}_i, \nabla_i)/(\mathcal{E}_{i-1}, \nabla_{i-1})$ is isomorphic to $(\mathcal{O}_X \otimes F_i, d \otimes \mathrm{id})$ for some k-vector space F_i. The full subcategory of $\mathrm{VIC}(X, \log D)$ given by unipotent objects is denoted by $\mathrm{UVIC}(X, \log D)$ (or $\mathrm{UVIC}(X)$, when D is empty). The smallest n for which such a filtration exists is the *index of unipotency of* $(\mathcal{E}, \nabla)$.

A2. *Local form.* In this subsection, we give a local characterisation of unipotent connections.

Lemma A.3. *Let A be a commutative k-algebra and $\Omega^\bullet \hookrightarrow \Omega^\bullet_{A/k}$ be a subcomplex of k-modules such that*

$$H^n(\Omega^\bullet) \xrightarrow{\sim} H^n(\Omega^\bullet_{A/k}) \text{ is an isomorphism and } H^{n+1}(\Omega^\bullet) \hookrightarrow H^{n+1}(\Omega^\bullet_{A/k}) \text{ is injective} \qquad (\mathrm{C}_n)$$

for some $n \geq 0$. Given $\omega \in \Omega^n_{A/k}$, the following are equivalent:

(1) $d\omega \in \Omega^{n+1}$.

(2) *There exists $\nu \in \Omega^{n-1}_{A/k}$ such that $\omega + d\nu \in \Omega^n$.*

In practice, we only use this lemma when $\Omega^\bullet \hookrightarrow \Omega^\bullet_{A/k}$ is a quasi-isomorphism, so that the conditions (C_n) are satisfied for every $n \geq 0$.

Proof. Only the direction (1) ⇒ (2) is nontrivial. The form $d\omega \in \Omega^{n+1}$ is closed and defines a cohomology class in $H^{n+1}(\Omega^\bullet)$. As $H^{n+1}(\Omega^\bullet) \hookrightarrow H^{n+1}(\Omega^\bullet_{A/k})$ is injective, and $d\omega$ is exact in $\Omega^\bullet_{A/k}$, it must also be exact in $\Omega^\bullet$. Thus, there exists $\eta \in \Omega^n$ such that $d\eta = d\omega$. Since $\omega - \eta \in \Omega^n_{A/k}$ is closed, it defines a cohomology class in $H^n(\Omega^\bullet_{A/k})$. Finally, from the isomorphism $H^n(\Omega^\bullet) \xrightarrow{\sim} H^n(\Omega^\bullet_{A/k})$, we obtain $\nu \in \Omega^{n-1}_{A/k}$ such that $\omega - \eta + d\nu \in \Omega^n$; as $\eta \in \Omega^n$, we conclude that $\omega + d\nu \in \Omega^n$. □

Theorem A.4. *Let $X = \operatorname{Spec} A$ be a smooth affine k-scheme, $(\mathcal{E}, \nabla)$ be an object of $\mathrm{UVIC}(X)$, and $\Omega^\bullet \hookrightarrow \Omega^\bullet_{A/k}$ be a subcomplex of k-modules satisfying condition (C_1). Then $\mathcal{E}$ is trivial and there exists a frame*

$$(\mathcal{O}_X^{\oplus n}, d + \omega) \cong (\mathcal{E}, \nabla)$$

in which the matrix ω is strictly upper triangular and has all of its entries in Ω^1.

Proof. We proceed by induction on the rank n of $\mathcal{E}$. The base case $n = 1$ is trivial, since $(\mathcal{E}, \nabla)$ must be isomorphic to $(\mathcal{O}_X, d)$. Assume that the statement holds in rank $\leq n - 1$. By the unipotency of $(\mathcal{E}, \nabla)$, there is an exact sequence

$$0 \longrightarrow (\mathcal{E}', \nabla') \longrightarrow (\mathcal{E}, \nabla) \longrightarrow (\mathcal{O}_X, d) \longrightarrow 0,$$

where $(\mathcal{E}', \nabla')$ is an object of $\mathrm{UVIC}(X)$, with $\mathcal{E}'$ of rank $n-1$. By the induction hypothesis, $\mathcal{E}'$ is trivial and admits a frame $e' : (\mathcal{O}_X^{\oplus n-1}, d+\omega') \xrightarrow{\sim} (\mathcal{E}', \nabla\,|_{\mathcal{E}'})$ in which ω' is strictly upper-triangular and has all of its entries in Ω^1.

As X is affine, there is a splitting

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \underset{\curvearrowleft e_n}{\longrightarrow} \mathcal{O}_X \longrightarrow 0,$$

so that $\mathcal{E}$ is trivial and admits the frame $(e', e_n) : (\mathcal{O}_X^{\oplus n}, d+\omega) \xrightarrow{\sim} (\mathcal{E}, \nabla)$, in which

$$\omega = \left(\begin{array}{ccc|c} & & & \omega_{1\,n} \\ & \omega'_{ij} & & \vdots \\ & & & \omega_{n-1\,n} \\ \hline 0 & \cdots & 0 & 0 \end{array}\right),$$

where $\omega_{ij} = \omega'_{ij} \in \Omega^1$ whenever $j < n$.

To finish, we explain how to inductively modify e_n so that $\omega_{in} \in \Omega^1$ for all $1 \le i \le n$. Note that $\omega_{nn} = 0 \in \Omega^1$. By descending induction in i, assume that $\omega_{kn} \in \Omega^1$ for all $i+1 \le k \le n$. The integrability equation $d\omega + \omega \wedge \omega = 0$ at the entry (i, n) means that

$$d\omega_{in} + \sum_{k=i+1}^{n-1} \omega_{ik} \wedge \omega_{kn} = 0.$$

Since, for every $i+1 \le k \le n-1$, both ω_{ik} and ω_{kn} belong to Ω^1, we have $\sum_{k=i+1}^{n-1} \omega_{ik} \wedge \omega_{kn} \in \Omega^2$. By Lemma A.3, there exists $g \in A$ such that $\omega_{in} + dg \in \Omega^1$. Thus

$$\nabla(e_n + g e'_i) = \sum_{k=1}^{i-1} (\omega_{kn} + g\omega_{ki}) \otimes e'_k + (\omega_{in} + dg) \otimes e'_i + \sum_{k=i+1}^{n-1} \omega_{kn} \otimes e'_k$$

and we conclude that the matrix $\tilde{\omega}$ of ∇ in the frame $(e', e_n + g e'_i)$ satisfies $\tilde{\omega}_{kn} \in \Omega^1$ for all $i \le k \le n$. □

A3. *Canonical extension.* Let X be a smooth k-scheme and D be a normal crossings divisor in X. We say that an object $(\mathcal{E}, \nabla)$ of $\mathrm{VIC}(X, \log D)$ (resp. $\mathrm{VIC}(X \setminus D)$) is *locally unipotent along* D if, for every $x \in D$, there exists an open neighbourhood V of x such that the restriction $(\mathcal{E}, \nabla)|_V$ (resp. $(\mathcal{E}, \nabla)|_{V \setminus D}$) is unipotent. For simplicity, let us denote the corresponding full subcategories of $\mathrm{VIC}(X, \log D)$ and $\mathrm{VIC}(X \setminus D)$ by L_1 and L_2.

Theorem A.5 (see [Deligne 1970, Proposition 5.2]). *If $j : X \setminus D \to X$ denotes the inclusion, then the restriction functor*

$$j^* : L_1 \longrightarrow L_2, \quad (\mathcal{E}, \nabla) \longmapsto (\mathcal{E}, \nabla)|_{X \setminus D},$$

is an equivalence of tensor categories.

Proof. Since

$$\mathrm{Hom}((\mathcal{E}', \nabla'), (\mathcal{E}, \nabla)) \cong \mathrm{Hom}((\mathcal{O}_X, d), (\mathcal{E}', \nabla')^\vee \otimes (\mathcal{E}, \nabla))$$

and since $\mathrm{Hom}((\mathcal{O}_X, d), (\mathcal{E}, \nabla))$ is canonically isomorphic to $\Gamma(X, \mathcal{E}^\nabla)$, where $\mathcal{E}^\nabla$ denotes the subsheaf of horizontal sections of $\mathcal{E}$, to show that $j^* : L_1 \to L_2$ is fully faithful, it is enough to prove that

$$\Gamma(X, \mathcal{E}^\nabla) \longrightarrow \Gamma(X \setminus D, \mathcal{E}^\nabla), \quad s \longmapsto s|_{X \setminus D}, \tag{45}$$

is bijective for every $(\mathcal{E}, \nabla)$ in L_1.

As $\mathcal{E}$ is locally free, the injectivity of (45) follows from the fact that D is locally defined by a torsion-free section of $\mathcal{O}_X$. Granted the injectivity, we can prove the surjectivity of (45) locally. We can thus assume that $(\mathcal{E}, \nabla)$ admits a frame $e : (\mathcal{O}_X^{\oplus n}, d + \omega) \xrightarrow{\sim} (\mathcal{E}, \nabla)$ in which ω is strictly upper triangular (Theorem A.4). We must prove that a horizontal section $s \in \Gamma(X \setminus D, \mathcal{E}^\nabla)$ extends to X. By writing $s = \sum_{j=1}^n s_j \otimes e_j$, with $s_j \in \Gamma(X \setminus D, \mathcal{O}_X)$, we get

$$0 = \nabla(s) = \sum_{i=1}^n \left(ds_i + \sum_{j=i+1}^n s_j \omega_{ij} \right) \otimes e_i.$$

Since $ds_n = 0$, s_n extends to X. By descending induction on i, it follows from the above equation that ds_i has at most logarithmic singularities along D, so that s_i extends to X for every $1 \le i \le n$. This finishes the proof that (45) is bijective.

We are left to prove that $j^* : L_1 \to L_2$ is essentially surjective; we use that it is fully faithful to reduce it to a local statement. We can thus assume that X is affine and that $(\mathcal{E}, \nabla)$ is in $\mathrm{UVIC}(X \setminus D)$. Since we are in characteristic 0, the injection $\Omega^\bullet_{X/k}(\log D) \hookrightarrow j_* \Omega^\bullet_{(X \setminus D)/k}$ is a quasi-isomorphism (see [Deligne 1970, Corollaire 3.14, Remarque 3.16]), so that we can apply Theorem A.4 to find a frame $e : (\mathcal{O}_{X \setminus D}^{\oplus n}, d + \omega) \xrightarrow{\sim} (\mathcal{E}, \nabla)$ in which $\omega_{ij} \in \Gamma(X, \Omega^1_{X/k}(\log D))$ for every $1 \le i, j \le n$. Then, $(\mathcal{O}_X^{\oplus n}, d + \omega)$ is an extension of $(\mathcal{E}, \nabla)$. □

Given an object $(\mathcal{E}, \nabla)$ of L_2, the unique object $(\bar{\mathcal{E}}, \bar{\nabla})$ of L_1 such that $(\bar{\mathcal{E}}, \bar{\nabla})|_{X \setminus D} = (\mathcal{E}, \nabla)$ is called the *canonical extension* of $(\mathcal{E}, \nabla)$. This yields a quasi-inverse to the restriction $j^* : L_1 \to L_2$, which can be checked to be additive, exact, and tensor.

Corollary A.6. *With the above notation, the restriction functor*

$$\mathrm{UVIC}(X, \log D) \longrightarrow \mathrm{UVIC}(X \setminus D), \quad (\mathcal{E}, \nabla) \longmapsto (\mathcal{E}, \nabla)|_{X \setminus D}, \tag{46}$$

is an equivalence of tensor categories.

Proof. Since $\mathrm{UVIC}(X, \log D)$ (resp. $\mathrm{UVIC}(X \setminus D)$) is a full subcategory of L_1 (resp. L_2), it follows immediately from Theorem A.5 that (46) is fully faithful. To see that it is essentially surjective, we must check that, for any object $(\mathcal{E}, \nabla)$ of $\mathrm{UVIC}(X \setminus D)$, its canonical extension $(\bar{\mathcal{E}}, \bar{\nabla})$ is in $\mathrm{UVIC}(X, \log D)$. If $(\mathcal{E}, \nabla) = (\mathcal{O}_{X \setminus D} \otimes F, d \otimes \mathrm{id})$ for some k-vector space F, then it follows from the uniqueness of the canonical extension that $(\bar{\mathcal{E}}, \bar{\nabla}) = (\mathcal{O}_X \otimes F, d \otimes \mathrm{id})$. Since the canonical extension functor is exact, the general case follows by induction on the index of unipotency of $(\mathcal{E}, \nabla)$. □

A4. $\mathbb{A}^1$-*invariance.* By an *affine bundle* we mean a morphism of k-schemes $Y \to X$ which is, locally over X, of the form $\mathbb{A}^m_U \to U$.

Theorem A.7. *If X is a smooth k-scheme and $\pi : Y \to X$ is an affine bundle, then the pullback functor*

$$\pi^* : \mathrm{UVIC}(X) \longrightarrow \mathrm{UVIC}(Y), \quad (\mathcal{E}, \nabla) \longmapsto (\pi^*\mathcal{E}, \pi^*\nabla),$$

is an equivalence of tensor categories.

Proof. By the same initial argument of the proof of Theorem A.5, to verify that π^* is fully faithful, it is enough to check that, for every object $(\mathcal{E}, \nabla)$ of $\mathrm{UVIC}(X)$, the pullback map on horizontal sections

$$\Gamma(X, \mathcal{E}^{\nabla}) \longrightarrow \Gamma(Y, (\pi^*\mathcal{E})^{\pi^*\nabla}) \tag{47}$$

is bijective.

The injectivity of (47) immediately follows from the faithful flatness of π. As in Theorem A.5, granted the injectivity, we can reduce the proof of surjectivity to a local statement. Thus, we can assume that $X = \mathrm{Spec}\, B$ is affine, and that $Y = \mathbb{A}^m_X$. By induction on m, we can further assume that $m = 1$, so that $Y = \mathrm{Spec}\, A$, with $A = B[t]$. Set $M = \Gamma(X, \mathcal{E})$; the map (47) then becomes the inclusion of k-vector spaces

$$M^{\nabla} \to M[t]^{\pi^*\nabla}.$$

An element of $M[t]$ is of the form $q = \sum_{n\geq 0} x_n t^n$ for some $x_n \in M$. If q is horizontal for $\pi^*\nabla$, then

$$0 = \pi^*\nabla(q) = \sum_{n\geq 0} t^n \nabla(x_n) + \sum_{n\geq 1} dt \otimes n t^{n-1} x_n,$$

and we must have $\sum_{n\geq 1} dt \otimes n t^{n-1} x_n = 0$. Since k is of characteristic 0, we get $x_n = 0$ for every $n \geq 1$. Thus, $q = x_0$ is in the image of (47).

We first prove that $\pi^* : \mathrm{UVIC}(X) \to \mathrm{UVIC}(Y)$ is essentially surjective locally on X. Let $(\mathcal{E}, \nabla)$ be an object of $\mathrm{UVIC}(Y)$. We use the notation from the last paragraph: $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$, with $A = B[t]$. Since $\Omega^\bullet_{B/k} \hookrightarrow \Omega^\bullet_{A/k}$ is a quasi-isomorphism, we can apply Theorem A.4 to find a frame $(\mathcal{O}_Y^{\oplus n}, d + \omega) \xrightarrow{\sim} (\mathcal{E}, \nabla)$ in which ω has all of its entries in $\Omega^1_{B/k}$. Thus, $\pi^*(\mathcal{O}_X^{\oplus n}, d + \omega) \cong (\mathcal{E}, \nabla)$.

In general, let $(\mathcal{E}, \nabla)$ be an object of $\mathrm{UVIC}(Y)$. Since we already know that π^* is fully faithful, the above local constructions glue, yielding a (locally unipotent) vector bundle with integrable k-connection $(\mathcal{E}', \nabla')$ on X satisfying $\pi^*(\mathcal{E}', \nabla') \cong (\mathcal{E}, \nabla)$. We are left to check that $(\mathcal{E}', \nabla')$ is unipotent. If $(\mathcal{E}, \nabla) = (\mathcal{O}_X \otimes F, d \otimes \mathrm{id})$ for some k-vector space F, then $(\mathcal{E}', \nabla') \cong (\mathcal{O}_Y \otimes F, d \otimes \mathrm{id})$ by the fully faithfulness of π^*. Since π is faithfully flat, the pullback π^* is an exact functor, so that the general case follows by induction on the index of relative unipotency of $(\mathcal{E}, \nabla)$. □

The above statement also admits a logarithmic version. We keep the above notation and let D be a normal crossings divisor in X.

Theorem A.8. *With the above notation, the pullback functor*

$$\pi^* : \mathrm{UVIC}(X, \log D) \longrightarrow \mathrm{UVIC}(Y, \log \pi^{-1} D), \quad (\mathcal{E}, \nabla) \longmapsto (\pi^*\mathcal{E}, \pi^*\nabla),$$

is an equivalence of tensor categories.

Proof. One can prove it directly, as in the proof of Theorem A.7, or derive it as a corollary of Theorems A.5 and A.7. Indeed, let $j : X \setminus D \to X$ be the inclusion. Then, the diagram of pullback functors

$$\begin{array}{ccc} \mathrm{UVIC}(X, \log D) & \xrightarrow{j^*} & \mathrm{UVIC}(X \setminus D) \\ \pi^* \downarrow & & \downarrow \\ \mathrm{UVIC}(Y, \log \pi^{-1} D) & \longrightarrow & \mathrm{UVIC}(Y \setminus \pi^{-1} D) \end{array}$$

commutes, so that π^* is fully faithful and essentially surjective because all other arrows are. ☐

A5. *De Rham fundamental group of a punctured elliptic curve.* Recall that, if X is a smooth and geometrically connected k-scheme, $\mathrm{UVIC}(X)$ is a neutral Tannakian category over k (see [Deligne 1989, §10.26]). Given a fibre functor $b : \mathrm{UVIC}(X) \to \mathrm{Vect}_k$, the *unipotent de Rham fundamental group* of X at b is the Tannakian fundamental group

$$\pi_1^{\mathrm{dR}}(X, b) := \underline{\mathrm{Aut}}^{\otimes}_{\mathrm{UVIC}(X)}(b).$$

It is a prounipotent affine group scheme over k.

Let E be an elliptic curve over k, $Z \subset E$ be a divisor as in Section 3.2, and $\pi : E^\natural \to E$ be the canonical projection from the universal vector extension. The following two results can be attributed to Deligne (see [Enriquez and Etingof 2018]).

Lemma A.9. *If $\mathcal{V}$ is a unipotent vector bundle over $E^\natural$, then the natural map $\Gamma(E^\natural, \mathcal{V}) \otimes_k \mathcal{O}_{E^\natural} \to \mathcal{V}$ is an isomorphism.*

Proof. This follows, as in [Deligne 1989, Proposition 12.3], by an inductive argument on the rank of $\mathcal{V}$, using that $H^0(E^\natural, \mathcal{O}_{E^\natural}) = k$ and $H^1(E^\natural, \mathcal{O}_{E^\natural}) = \mathrm{Ext}^1(\mathcal{O}_{E^\natural}, \mathcal{O}_{E^\natural}) = 0$ (Theorem 2.3). ☐

Proposition A.10. *The functor*

$$b_{\mathrm{can}} : \mathrm{UVIC}(E \setminus Z) \longrightarrow \mathrm{Vect}_k, \quad (\mathcal{E}, \nabla) \longmapsto \Gamma(E^\natural, \pi^* \overline{\mathcal{E}}),$$

is a fibre functor over k.

Proof. Let $\mathrm{UV}(E^\natural)$ be the category of unipotent vector bundles on $E^\natural$. The functor b_{can} is the composition

$$\mathrm{UVIC}(E \setminus Z) \longrightarrow \mathrm{UVIC}(E, \log Z) \xrightarrow{\pi^*} \mathrm{UVIC}(E^\natural, \log \pi^{-1} Z) \longrightarrow \mathrm{UV}(E^\natural) \xrightarrow{\Gamma(E^\natural, -)} \mathrm{Vect}_k,$$

where the first arrow is the canonical extension and the third arrow is the forgetful functor $(\mathcal{V}, \nabla) \mapsto \mathcal{V}$. By Corollary A.6 and Theorem A.8, the first two arrows are k-linear equivalences of tensor categories. The third is trivially a k-linear tensor faithful functor. Finally, the last arrow is k-linear, tensor, and faithful by Lemma A.9. ☐

Our next goal is to relate the fundamental group $\pi_1^{\mathrm{dR}}(E \setminus Z, b_{\mathrm{can}})$ with the Hopf algebra $\mathcal{H}_{E/k,Z}$ constructed in Section 5.2. Let $(\mathcal{E}, \nabla)$ be an object of $\mathrm{UVIC}(E \setminus Z)$ and write $V := b_{\mathrm{can}}(\mathcal{E}, \nabla)$. It follows from Lemma A.9 that

$$(\pi^* \overline{\mathcal{E}}, \pi^* \overline{\nabla}) \cong (\mathcal{O}_{E^\natural} \otimes V, d + \omega)$$

for a unique nilpotent (in the sense of Proposition 5.5) $\omega \in \Gamma(E^\natural, \Omega^1_{E^\natural/k}(\log \pi^{-1}Z)) \otimes \mathrm{End}(V)$ satisfying $d\omega + \omega\wedge\omega = 0$. Thus, it defines a $\mathcal{H}_{E/k,Z}$-comodule structure $\rho = \sum_{n\geq 0}[\omega]^n$ on V. These constructions are natural, so that we obtain a functor

$$\mathrm{UVIC}(E\setminus Z) \longrightarrow \mathrm{Comod}(\mathcal{H}_{E/k,Z}), \quad (\mathcal{E}, \nabla) \longmapsto (V, \rho), \tag{48}$$

extending b_{can}.

Theorem A.11. *The functor (48) is an equivalence of tensor categories over k. In particular, it induces an isomorphism of affine group schemes over k:*

$$\pi_1^{\mathrm{dR}}(E\setminus Z, b_{\mathrm{can}}) \cong \mathrm{Spec}\,\mathcal{H}_{E/k,Z}.$$

Proof. That (48) is a k-linear equivalence of categories is an immediate consequence of Corollary A.6, Theorem A.8, and Proposition 5.5. We are left to show that (48) is a tensor functor.

We already know that b_{can} is tensor by Proposition A.10. Now, the tensor structure on the category $\mathrm{Comod}(\mathcal{H}_{E/k,Z})$ is induced by the shuffle product: given comodules (V, ρ), (V, ρ'), the tensor comodule structure $\rho \sqcup\!\sqcup\, \rho'$ on $V \otimes V'$ is given by

$$V\otimes V' \xrightarrow{\rho\otimes\rho'} (\mathcal{H}_{E/k,Z}\otimes V)\otimes(\mathcal{H}_{E/k,Z}\otimes V') \cong (\mathcal{H}_{E/k,Z}\otimes\mathcal{H}_{E/k,Z})\otimes(V\otimes V') \xrightarrow{\sqcup\!\sqcup\otimes \mathrm{id}} \mathcal{H}_{E/k,Z}\otimes(V\otimes V'),$$

where all of the above tensor products are over k. By Proposition 5.5, if $\rho = \sum_{i\geq 0}[\omega]^i$ and $\rho' = \sum_{j\geq 0}[\omega']^j$, then

$$\rho \sqcup\!\sqcup\, \rho' = \sum_{i,j\geq 0}[\omega]^i \sqcup\!\sqcup\, [\omega']^j = \sum_{n\geq 0}[\omega\otimes \mathrm{id} + \mathrm{id}\otimes\omega']^n.$$

To finish, we simply remark that the tensor structure of $\mathrm{UVIC}(E^\natural, \log \pi^{-1}Z)$ is given by

$$(\mathcal{O}_{E^\natural}\otimes V, d+\omega)\otimes(\mathcal{O}_{E^\natural}\otimes V', d+\omega') \cong (\mathcal{O}_{E^\natural}\otimes V\otimes V', d+\omega\otimes\mathrm{id}+\mathrm{id}\otimes\omega'). \qquad \square$$

Corollary A.12. *There is a canonical isomorphism $\pi_1^{\mathrm{dR}}(E\setminus Z, b_{\mathrm{can}}) \cong \mathrm{Spec}\, T^c H^1_{\mathrm{dR}}((E\setminus Z)/k)$.*

Proof. This is an immediate consequence of Theorems A.11 and 3.9. $\square$

Note that $\mathcal{H}^\vee_{E/k,Z}$ is a projective limit of finite-dimensional k-vector spaces, and the pullback $f^*\mathcal{H}^\vee_{E/k,Z}$ is simply the base change $\mathcal{O}_{E^\natural}\hat{\otimes}\mathcal{H}^\vee_{E/k,Z}$. Let

$$\nabla_{E^\natural/k,Z} : \mathcal{O}_{E^\natural}\hat{\otimes}\mathcal{H}^\vee_{E/k,Z} \longrightarrow \Omega^1_{E^\natural/k}(\log\pi^{-1}Z)\hat{\otimes}\mathcal{H}^\vee_{E/k,Z}, \quad \nabla_{E^\natural/k,Z} = d + \omega_{E^\natural/k,Z},$$

be the elliptic KZB connection of E/k punctured at Z constructed in Section 5.3. It is a pro-object of $\mathrm{UVIC}(E^\natural, \log\pi^{-1}Z)$. By Theorem A.8, it corresponds to a pro-object $(\mathcal{V}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}})$ of $\mathrm{UVIC}(E, \log Z)$. Note that $\Gamma(E^\natural, \pi^*\mathcal{V}_{\mathrm{KZB}})$ is the complete Hopf algebra $\mathcal{H}^\vee_{E/k,Z}$, and we denote by $1\in\mathcal{H}^\vee_{E/k,Z}$ its unit.

Proposition A.13. *The provector bundle with logarithmic connection $(\mathcal{V}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}})$ (resp. its restriction $(\mathcal{V}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}})|_{E\setminus Z}$) satisfies the following universal property: for every triple $(\mathcal{V}, \nabla, v)$, where $(\mathcal{V}, \nabla)$ is* *an object of* $\mathrm{UVIC}(E, \log Z)$ (*resp.* $\mathrm{UVIC}(E \setminus Z)$), *and* $v \in \Gamma(E^\natural, \pi^*\mathcal{V})$ (*resp.* $v \in \Gamma(E^\natural, \pi^*\overline{\mathcal{V}})$), *there is a unique horizontal map*

$$\varphi : (\mathcal{V}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}}) \longrightarrow (\mathcal{V}, \nabla) \quad (\textit{resp. } \varphi : (\mathcal{V}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}})|_{E\setminus Z} \longrightarrow (\mathcal{V}, \nabla))$$

satisfying $\varphi(1) = v$.

Proof. It suffices to combine Proposition 5.12 with the equivalence of Theorem A.11. □

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Mean values of long Dirichlet polynomials with divisor coefficients

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We prove an asymptotic formula for the mean value of long smoothed Dirichlet polynomials with divisor coefficients. Our result has a main term that includes all lower-order terms and a power saving error term. This is derived from a more general theorem on mean values of long smoothed Dirichlet polynomials that was previously established by the second and third authors (*Adv. Math.* **410**:B (2022)). We thus establish a stronger form of a conjecture of Conrey and Gonek (*Duke Math. J.* **107**:3 (2001), Conjecture 4) in the case of divisor functions.

1. Introduction

Mean values of Dirichlet polynomials play an important role in analytic number theory. They have important applications to zero-density estimates, primes in short intervals, gaps between primes and mean values of L-functions. Although we will describe some elements of the theory, one may consult [Iwaniec and Kowalski 2004, Chapters 9, 10; Montgomery 1994, Chapter 7] for a comprehensive discussion on mean values of Dirichlet polynomials.

For a sequence of complex numbers $(a(n))$, an associated Dirichlet polynomial is a partial sum in the form

$$\sum_{n\le K}\frac{a(n)}{n^s}.$$

By [Montgomery and Vaughan 1974, Corollary 3], this has the approximate behavior

$$\frac{1}{T}\int_0^T\biggl|\sum_{n\le K}a(n)n^{-\sigma-it}\biggr|^2\,dt\asymp\sum_{n\le K}|a(n)|^2n^{-2\sigma}\quad\text{as } K\to\infty, \tag{1}$$

provided that $K=O(T)$. If $K=o(T)$, then $\asymp$ can be replaced by $\sim$ and thus one has an asymptotic formula.

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Note that the integral on the left-hand side of (1) is called a mean value of the Dirichlet polynomial. If K is not $O(T)$, then this integral is referred to as a mean value of a long Dirichlet polynomial and it is considerably more difficult to evaluate. Observe that when the left-hand side of (1) is expanded out via the identity $|z|^2 = z\bar{z}$, one encounters correlation sums in the form

$$\sum_{n\le x} a(n)\overline{a(n+r)} \quad \text{for } r \in \mathbb{Z}^+, \tag{2}$$

which are viewed as part of the off-diagonal contribution. In this case, the integral in (1) depends, in a crucial way, on the asymptotic behavior of such correlation sums. Goldston and Gonek [1998] provided very precise formulae for mean values of this type under some conditions on the behavior of $(a(n))$. Indeed, their work can lead to asymptotic formulae for mean values of general Dirichlet polynomials in the case that $T \le K \le T^{1+\eta}$ for some $\eta < 1$ if there is square-root cancellation in the error term of their formula for (2). The reader is referred to Theorems 1–3 and their corollaries in [Goldston and Gonek 1998].

In their work on the sixth and eighth moments of the Riemann zeta function, Conrey and Gonek [2001] conjectured an asymptotic formula for the mean values of long Dirichlet polynomials when $a(n) = \tau_k(n)$ and $K = T^{1+\eta}$ with $0 < \eta < 1$. Here for $k \in \mathbb{N}$, τ_k denotes the k-th divisor function, which is defined as

$$\tau_k(n) = \#\{(n_1, \ldots, n_k) \in \mathbb{N}^k \mid n_1 \cdots n_k = n\} \quad \text{for } n \in \mathbb{N}.$$

For example, for $k = 2$, $\tau_2(n)$ is the ordinary divisor function $d(n)$.

Conjecture 1.1 [Conrey and Gonek 2001, Conjecture 4]. *Let T be sufficiently large and $K = T^{1+\eta}$ with $\eta \in (0, 1)$. Then*

$$\int_T^{2T} \left| \sum_{n\le K} \frac{\tau_k(n)}{n^{1/2+it}} \right|^2 dt \sim \frac{a_k}{\Gamma(k^2+1)}\, w_k\!\left(\frac{\log K}{\log T}\right) T(\log T)^{k^2},$$

where

$$a_k = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{\alpha=0}^{\infty} \frac{\tau_k^2(p^\alpha)}{p^\alpha} \right\},$$

$$w_k(x) = x^{k^2} \left\{ 1 - \sum_{n=0}^{k^2-1} \binom{k^2}{n+1} \gamma_k(n)(-1)^n (1 - x^{-n-1}) \right\},$$

$$\gamma_k(n) = \sum_{i=1}^{k}\sum_{j=1}^{k} \binom{k}{i}\binom{k}{j}\binom{n-1}{i+j-2}\binom{i+j-2}{j-1} \quad \textit{for } n \in \mathbb{Z}^+ \textit{ and } \gamma_k(0) = k.$$

The case $k = 2$ of Conjecture 1.1 was established by Bettin and Conrey [2021] for all $\eta > 0$. In this article we prove a stronger form of the conjecture in the same case, but for $0 < \eta < \frac{1}{3}$ and for smoothed Dirichlet polynomials. To be precise, we obtain all lower-order terms with a power savings error term. We note that both the error term and the range for η in our theorem below depend directly on bounds for the error term in the binary additive divisor problem. We discuss this in more detail in Remark 1.4 below.

Before presenting our result, we need to set some notation. Let $(a(n))$ and $(b(n))$ be sequences and φ be some real-valued smooth function. We will specify the properties that φ is required to have in Section 2.1. We define the smoothed Dirichlet polynomials

$$\mathbb{A}_{a,\varphi}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \varphi\left(\frac{n}{K}\right) \quad \text{and} \quad \mathbb{B}_{b,\varphi}(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \varphi\left(\frac{n}{K}\right).$$

We then consider the mean value

$$\mathscr{D}_{a,b;\omega}(K) = \int_{\mathbb{R}} \omega(t) \mathbb{A}_{a,\varphi}\left(\tfrac{1}{2}+it\right) \mathbb{B}_{b,\varphi}\left(\tfrac{1}{2}-it\right) dt, \tag{3}$$

where ω is a complex-valued smooth function that satisfies the conditions

$$\omega \text{ is smooth}, \tag{4}$$

$$\text{the support of } \omega \text{ lies in } [c_1 T, c_2 T], \text{ where } 0 < c_1 < c_2, \tag{5}$$

$$\text{for some positive absolute constant } \nu, \text{ there exists } T_0 \geq T^{\nu} \text{ such that } T_0 \ll T \text{ and } \omega^{(j)}(t) \ll T_0^{-j}. \tag{6}$$

The Fourier transform of ω is

$$\hat{\omega}(u) = \int_{\mathbb{R}} \omega(t) e^{-2\pi i u t}\, dt. \tag{7}$$

It satisfies the following property:

$$\text{If } |u| \gg T_0^{-1+\varepsilon}, \text{ then } |\hat{\omega}(u)| \ll T^{-A} \text{ for any } A > 0. \tag{8}$$

Since throughout the paper we will only study the case where $a(n) = \tau_k(n)$ and $b(n) = \tau_\ell(n)$ for some positive integers k, ℓ, in order to simplify our notation, we set

$$\mathscr{D}_{k,\ell;\omega}(K) := \mathscr{D}_{\tau_k,\tau_\ell;\omega}(K).$$

We also need to introduce some real sequences (g_j) and (δ_j). These are defined as coefficients in the Taylor series

$$f(s) := s\,\zeta(1+s) = \sum_{j=0}^{\infty} g_j s^j, \quad h(s) := \frac{1}{\zeta(2+s)} = \sum_{j=0}^{\infty} \delta_j s^j.$$

Another sequence (c_j), which depends on the smoothing function φ, is defined as follows. Let

$$G(s) := -2 \int_0^{\infty} \varphi(t) \varphi'(t) t^s\, dt.$$

Observe that $G(s)$ is entire. We then write its Taylor series expansion as

$$G(s) = \sum_{j=0}^{\infty} c_j s^j.$$

With these definitions in hand, we can state our main result.

Theorem 1.2. *Let $K = T^{1+\eta}$ with $0 < \eta < \frac{1}{3}$. Suppose that a weight function ω satisfies conditions (4), (5) and (6) with $\nu > \frac{1}{9}(5+3(\eta+1))$, while φ is a function satisfying the conditions in (11). Then for*

$$\mathscr{D}_{2,2;\omega}(K) = \int_{\mathbb{R}} \omega(t) \left| \sum_{n=1}^{\infty} \frac{\tau_2(n)}{n^{1/2+it}} \varphi\left(\frac{n}{K}\right) \right|^2 dt,$$

we have

$$\mathscr{D}_{2,2;\omega}(K) = \sum_{j=0}^{4} \int_{-\infty}^{\infty} \omega(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt + O\left(T^{3(1+\eta)/4+\varepsilon}\left(\frac{T}{T_0}\right)^{9/4} + T^{1-\eta/2}\right),$$

where each $Q_j(x, y) \in \mathbb{R}[x, y]$ is a polynomial of degree j given by

$$\begin{aligned}
Q_4(x, y) &= \frac{1}{4!\,\zeta(2)}(-x^4 + 8x^3y - 24x^2y^2 + 32xy^3 - 14y^4),\\
Q_3(x, y) &= \left(\frac{2\delta_0 g_1}{3} + \frac{\delta_1}{3} - \frac{c_1\delta_0}{6}\right)x^3 + (-4\delta_0 g_1 - 2\delta_1 + c_1\delta_0)x^2y\\
&\quad + (8\delta_0 g_1 + 4\delta_1 - 2c_1\delta_0)xy^2 + \left(-4\delta_0 g_1 - 2\delta_1 + \frac{4c_1\delta_0}{3}\right)y^3,\\
Q_2(x, y) &= \left(-2\delta_0 g_2 - 3\delta_0 g_1^2 - 4\delta_1 g_1 - 2\delta_2 + 2c_1\delta_0 g_1 + c_1\delta_1 - \frac{c_2\delta_0}{2}\right)x^2\\
&\quad + \big(8\delta_0 g_2 + 12\delta_0 g_1^2 + 16\delta_1 g_1 + 8\delta_2 - 8c_1\delta_0 g_1 - 4c_1\delta_1 + 2c_2\delta_0\big)xy\\
&\qquad + \big(-5\delta_0 g_1^2 - 4\delta_2 - 6\delta_0 g_2 - 8\delta_1 g_1 + 8c_1\delta_0 g_1 + 4c_1\delta_1 - 2c_2\delta_0\big)y^2,\\
Q_1(x, y) &= \big(4\delta_0 g_3 + 12\delta_0 g_1 g_2 + 4\delta_0 g_1^3 + 8\delta_1 g_2 + 12\delta_1 g_1^2 + 16\delta_2 g_1 + 8\delta_3 - 4c_1\delta_0 g_2 - 6c_1\delta_0 g_1^2\\
&\qquad - 8c_1\delta_1 g_1 - 4c_1\delta_2 + 4c_2\delta_0 g_1 + 2c_2\delta_1 - c_3\delta_0\big)x\\
&\quad + \big(-12\delta_0 g_3 - 4\delta_0 g_1 g_2 - 8\delta_1 g_2 + 4\delta_1 g_1^2 + 4\delta_0 g_1^3 + 8c_1\delta_0 g_2 + 12c_1\delta_0 g_1^2 + 16c_1\delta_1 g_1\\
&\qquad + 8c_1\delta_2 - 8c_2\delta_0 g_1 - 4c_2\delta_1 + 2c_3\delta_0\big)y,\\
Q_0(x, y) &= 16\delta_4 - 16\delta_1 g_3 + 32\delta_3 g_1 + 32g_1^2\delta_2 - 24\delta_0 g_4 + 8g_2^2\delta_0 + 5\delta_0 g_1^4 + 16\delta_1 g_1^3 - 8\delta_0 g_1 g_3\\
&\quad + 16\delta_1 g_1 g_2 + 12\delta_0 g_1^2 g_2 + 12g_1^2\delta_1 c_1 + 12\delta_0 g_1 g_2 c_1 + 8\delta_3 c_1 + 4\delta_0 g_1^3 c_1 + 4\delta_0 g_3 c_1\\
&\qquad + 8\delta_1 g_2 c_1 + 16 g_1\delta_2 c_1 - 4\delta_2 c_2 - 6g_1^2\delta_0 c_2 - 4\delta_0 g_2 c_2 - 8g_1\delta_1 c_2 + 4g_1\delta_0 c_3 + 2\delta_1 c_3 - \delta_0 c_4.
\end{aligned}$$

In Appendix A, we show how to remove the smooth function ω and derive the following result.

Corollary 1.3. *Let $K = T^{1+\eta}$ with $0 < \eta < \frac{1}{3}$, and let φ be a function satisfying the conditions in (11). Then, as $T \to \infty$, we have*

$$\int_T^{2T} \left| \sum_{n=1}^{\infty} \frac{\tau_2(n)}{n^{1/2+it}} \varphi\left(\frac{n}{K}\right) \right|^2 dt = \sum_{j=0}^{4} \int_T^{2T} Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt + O(T^{\max\{(12+3\eta)/13,\, 1-\eta/2\}}),$$

where the polynomials $Q_j(x, y)$ are as given in Theorem 1.2.

Observe that asymptotically, this result has the same leading term as the one in the conjecture of Conrey and Gonek in the case $k = 2$ for $0 < \eta < \frac{1}{3}$.

Our theorem depends on a result of Hughes and Young [2010, Theorem 5.1 and (74)], who applied Duke, Friedlander and Iwaniec's version of the δ-method [Duke et al. 1994]. Their work only makes use of the Weil bound for Kloosterman sums. Using ideas from Aryan [2017] and Topacogullari [2017], the main theorem in [Duke et al. 1994] can be improved by applying the spectral theory of automorphic forms and bounds for sums of Kloosterman sums. The spectral theory of automorphic forms was first applied to the classical additive divisor sum $D(x,r)=\sum_{n\le x} d(n)d(n+r)$ in the case $r=1$ in [Deshouillers and Iwaniec 1982]. Their ideas were extended in a wide-ranging way by Motohashi [1994], who derived an exact formula for this sum. From this formula he derived extremely strong uniform estimates for $D(x,r)$ that were uniform in r. His results were later improved by Meurman [2001] in some ranges of r. The works of Aryan [2017] and Topacogullari [2017] rely heavily on ideas from these aforementioned articles.

Our main result in this paper follows from [Hamieh and Ng 2022, Theorem 1.1], which requires an asymptotic formula for additive divisor sums involving the shifted divisor function rather than the ordinary divisor function (see Section 3 below for more details). Therefore, the aforementioned articles of Aryan and Topacogullari cannot be applied directly as they prove correlation estimates for the ordinary divisor functions of the type

$$\sum_{m-n=r} d(m)d(n)f(m,n),$$

where $f(m,n)$ are certain smoothing functions. Instead, one would need to replace $d(m)$ and $d(n)$ by the shifted divisor functions

$$\sigma_{a_1,a_2}(m)=\sum_{d_1d_2=m} d_1^{-a_1}d_2^{-a_2} \quad \text{and} \quad \sigma_{b_1,b_2}(n)=\sum_{d_1d_2=n} d_1^{-b_1}d_2^{-b_2},$$

where $a_1,a_2,b_1,b_2\in\mathbb{C}$. A second issue is that the smoothing functions in [Aryan 2017; Topacogullari 2017] are not general enough. For instance, in [Aryan 2017, p. 1458, equation (0.8)] $f(x,y)$ is supported on a box of the shape $[X,2X]\times[X,2X]$ and satisfies the bound

$$\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} f(x,y)\ll X^{-i-j}.$$

We would have to consider smoothing functions on more general boxes $[X,2X]\times[Y,2Y]$, which satisfy the bound $(P/X)^i(P/Y)^j$ for some parameter $P\ge 1$. In applications, it is important to have this extra parameter P. In [Topacogullari 2017], while the smoothing function satisfies a bound of the desired shape (see [Topacogullari 2017, p. 155]), it is also restricted to be of the form $f(x,y)=w_1(x/X)w_2(y/Y)$, where w_1 and w_2 are smooth compactly supported functions. The smoothing function required for the application of [Hamieh and Ng 2022, Theorem 1.1] is not of this form.

By applying the advanced techniques employed in [Aryan 2017; Topacogullari 2017] to the setting of shifted divisor functions while incorporating a more general smoothing to the correlation sum, it is likely that one could improve [Hughes and Young 2010, Theorem 5.1 and (74)]. This would result in an improvement of both the error term and the range of η in our Theorem 1.2.

Remark 1.4. If the binary divisor conjecture $\mathcal{AD}_{2,2}(\vartheta_{2,2}, C_{2,2}, \beta_{2,2})$ holds for a triple $(\vartheta_{2,2}, C_{2,2}, \beta_{2,2}) \in \left[\frac{1}{2}, 1\right) \times [0, \infty) \times (0, 1]$ (see Conjecture 3.1 below for notation), then Theorem 1.2 holds for

$$\eta < \frac{1}{\vartheta_{2,2}} - 1 \quad \text{and} \quad \nu > \frac{C_{2,2} + (\vartheta_{2,2} + \varepsilon)(\eta + 1)}{1 + C_{2,2}}$$

with an error term

$$O\left(T^{\vartheta_{2,2}(1+\eta)+\varepsilon}\left(\frac{T}{T_0}\right)^{1+C_{2,2}} + T^{1-\eta/2}\right).$$

One expects that the methods of [Aryan 2017; Topacogullari 2017] would lead to $\mathcal{AD}_{2,2}(\vartheta_{2,2}, C_{2,2}, \beta_{2,2})$ with $\beta_{2,2} = 1 - \epsilon$ and $\vartheta_{2,2} = \frac{1}{2} + \theta$, where θ is the current best bound for the Ramanujan conjecture. We are not able to predict the improved value of $C_{2,2}$ without going through certain technical aspects of the proof. In particular, if the Ramanujan conjecture is true so that $\vartheta_{2,2} = \frac{1}{2}$, then Theorem 1.2 will hold for Dirichlet polynomials with length $K = T^c$ for any $c < 2$.

Our approach in proving Theorem 1.2 is slightly different from that in [Goldston and Gonek 1998; Conrey and Gonek 2001]. In both works, one of the key steps is to express the mean value in (1) in terms of the correlation sums in (2) via partial summation. Whereas in the work of the second and the third authors [Hamieh and Ng 2022], the starting point is to split the sum into sums over dyadic intervals via a smooth partition of unity. Furthermore, they also work with shifted divisor functions. Conditionally on the additive divisor conjecture [Hamieh and Ng 2022, Conjecture 4], they compute the mean value

$$\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K),$$

where

$$\sigma_{\mathcal{I}}(n) = \sum_{d_1 \cdots d_k = n} d_1^{-a_1} \cdots d_k^{-a_k} \quad \text{and} \quad \sigma_{\mathcal{J}}(n) = \sum_{d_1 \cdots d_\ell = n} d_1^{-b_1} \cdots d_k^{-b_\ell}$$

are shifted divisor functions associated to sets of complex numbers $\mathcal{I} = \{a_1, \ldots, a_k\}$ and $\mathcal{J} = \{b_1, \ldots, b_\ell\}$. Then $\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K)$ is evaluated by using a smooth partition of unity. Thus, instead of the correlation sums as in (2), the authors work with the smoothed correlation sums

$$\sum_{\substack{m,n\in\mathbb{Z} \\ m-n=r}} \sigma_{\mathcal{I}}(n)\sigma_{\mathcal{J}}(n)F(m,n), \tag{9}$$

where F is a smooth function defined on a box $[M, 2M] \times [N, 2N]$. The main term for $\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K)$ is expressed in terms of a *diagonal* contribution and an *off-diagonal* contribution. The diagonal contribution equals a contour integral involving the Dirichlet series

$$Z_{\mathcal{I},\mathcal{J}}(s) = \sum_{m=1}^{\infty} \frac{\sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(m)}{m^{1+s}}.$$

These contour integrals can be evaluated similarly to integrals that one encounters in standard applications of Perron's formula.

The most difficult part is the computation of the off-diagonal terms. They may be expressed as a certain average of sums of type (9). On the additive divisor conjecture, conjectural main terms for sums of this type are inserted and a formula for $\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K)$ is obtained. This idea of considering smoothed sums originated in [Duke et al. 1994] and was employed in a similar context in [Hughes and Young 2010; Ng 2021; Ng et al. 2025]. Once the main terms from the additive divisor conjecture are inserted, there is still a lengthy calculation that needs to be done. One encounters Dirichlet series of the shape

$$H_{\mathcal{I},\mathcal{J};\{a_{i_1}\},\{b_{i_2}\}}(s)=\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{c_q(r)G_{\mathcal{I}}(1-a_{i_1},q)G_{\mathcal{J}}(1-b_{i_2},q)}{q^{2-a_{i_1}-b_{i_2}}r^{a_{i_1}+b_{i_2}+s}},\tag{10}$$

where $i_1\in\{1,\ldots,k\}$, $i_2\in\{1,\ldots,\ell\}$, $c_q(r)$ is the Ramanujan sum, and $G_{\mathcal{I}}(1-a_{i_1},q)$ and $G_{\mathcal{J}}(1-b_{i_2},q)$ are multiplicative functions that arise from the additive divisor conjecture (see (27) and (28) below). Indeed, in some approximate way,

$$G_{\mathcal{I}}(1-a_{i_1},q)\approx\sigma_{\mathcal{I}\setminus\{a_{i_1}\}}(q)\quad\text{ and }\quad G_{\mathcal{J}}(1-b_{i_2},q)\approx\sigma_{\mathcal{J}\setminus\{b_{i_2}\}}(q).$$

One requires a meromorphic continuation of the Dirichlet series $H_{\mathcal{I},\mathcal{J};\{a_{i_1}\},\{b_{i_2}\}}(s)$ to the region $\Re(s)\geq -1$. Furthermore, numerous facts about the gamma function are used; including the beta function identity and various versions of Stirling's formula. At the end, the off-diagonal contribution can be expressed as a sum of contour integrals of the functions $H_{\mathcal{I},\mathcal{J};\{a_{i_1}\},\{b_{i_2}\}}(s)$. From these expressions, the integrals corresponding to the diagonal and off-diagonal contributions can be evaluated by a contour shift and the residue theorem.

In order to prove Theorem 1.2, firstly, we will apply the main theorem of [Hamieh and Ng 2022] to our special case. The theorem provides a general asymptotic formula in the form

$$\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K)\sim\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)+\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$$

as $K\to\infty$, where the terms on the right-hand side are as in (35) and (36). We will prove in Lemma 3.3 that both $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ and $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ are holomorphic as functions of elements of the sets $\mathcal{I}=\{a_1,\ldots,a_k\}$ and $\mathcal{J}=\{b_1,\ldots,b_\ell\}$. Note that if $k=\ell=2$ and $a_j=b_j=0$ for $j=1,2$, then $\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K)$ becomes $\mathscr{D}_{2,2;\omega}(K)$. Upon explicit computations, each of the main terms $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ and $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ will be expressed as a sum of polar terms in a, b, $a-b$ or $a+b$ in the setting $\mathcal{I}=\{a,0\}$ and $\mathcal{J}=\{b,0\}$. We will then carefully analyze all the terms, and show that these polar terms cancel each other while the remaining terms match the ones in our main theorem.

This idea of working with the shifted divisor functions $\sigma_{\mathcal{I}}(n)$ and $\sigma_{\mathcal{J}}(n)$ and then setting the shifts equal to zero originated in [Ingham 1927]. An advantage of this approach is that when computing the residues one only deals with simple poles. Still, it is quite technical to find a formula for the mean value in terms of the shifts and show that the polar terms are indeed canceled out. On the other hand, it is also possible to compute $\mathscr{D}_{2,2;\omega}(K)$ directly, that is, without using any shifts. In that case, one must deal with poles of higher order, so the residue calculations will be more complicated.

We now comment on the use of the smooth weight functions w and φ in our definition of $\mathscr{D}_{k,\ell;\omega}(K)$. Note that the function φ appears in the definitions of $\mathbb{A}_{a,\varphi}(s)$ and $\mathbb{B}_{b,\varphi}(s)$. Classical forms of the approximate functional equation do not have smooth weights and they have much weaker error terms. In comparison, weighted approximate equations have much smaller error terms (see [Titchmarsh 1986, (4.20.1), (4.20.2); Iwaniec and Kowalski 2004, Theorem 5.3]). By introducing the function φ, one is able to make use of the Mellin transform instead of Perron's formula. This has the advantage of providing much better decay rates in the resulting complex integrals. The other weight function ω can be thought of as a smooth approximation to the indicator function $\mathbb{1}_{[T,2T]}(t)$. The purpose of weighing the mean value with such a function is to improve the estimation of the off-diagonal terms. As in [Hamieh and Ng 2022, (4.17) and (4.18)], for example, employing the bound in (8) for $\hat{\omega}$ allows one to dispense of many error terms.

On another note, mean values of weighted long Dirichlet polynomials with the von Mangoldt function $\Lambda(n)$ as their coefficients have been computed. Based on the results of Goldston and Gonek [1998], Chan [2004] computed asymptotically such mean values, assuming a version of the twin prime conjecture involving correlations of $\Lambda(n)$. Heap [2022] proved upper and lowers bounds for these types of mean values assuming the Riemann hypothesis. His work circumvents the estimation of correlation sums by writing the Dirichlet polynomial as an integral of the logarithm of the zeta function. On the critical line, the logarithm of the zeta function can be approximated by a short Dirichlet polynomial on average, so the problem then reduces to estimating moments of the short Dirichlet polynomial. His work is more closely related to the articles of Soundararajan [2009] and Harper [2013] on upper bounds of the zeta function and it also employs techniques related to the pair correlation of zeros of the zeta function as in [Montgomery 1973].

We remind the reader that mean values of long Dirichlet polynomials are known to be closely related to the moments of the Riemann zeta function (see [Conrey and Gonek 2001; Ivić 1997a; 1997b]). The $2k$-th moment is defined as

$$I_k(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt.$$

For $I_2(T)$, the fourth moment of the Riemann zeta function, Heath-Brown [1979] was the first to show that it is asymptotic to $T\mathcal{P}_4(\log T)$ for a certain polynomial $\mathcal{P}_4$ of degree four as $T \to \infty$. However, he did not compute all coefficients of this polynomial. Conrey [1996] gave several formulae for the coefficients of this polynomial. Conrey et al. [2008] provided numerical values for all coefficients of $\mathcal{P}_4$. Now, by the formulae in Conjecture 1.1, it is proposed that the first few polynomials for the asymptotics of the moments of Dirichlet polynomials are

$$w_2(x) = -x^4+8x^3-24x^2+32x-14,$$
$$w_3(x) = -2x^9+27x^8-324x^7+2268x^6-8694x^5+19278x^4-25452x^3+19764x^2-8343x+1479.$$

As it turns out, the polynomials $w_3(x)$ and $w_4(x)$ are intimately related to the sixth and the eighth moments of the Riemann zeta-function, respectively. Indeed, the identities

$$w_3(x)+w_3(3-x) = 42 \quad \text{and} \quad 2w_4(2) = w_4(2)+w_4(2) = 24024$$

led to Conrey and Gonek's conjectures [2001]

$$I_3(T) \sim \frac{42a_3}{9!} T\log^9 T \quad \text{and} \quad I_4(T) \sim \frac{24024a_4}{16!} T\log^{16} T.$$

Their work also provided a heuristic argument showing that $I_k(T)$ could be expressed as a sum of two mean values of long Dirichlet polynomials of k-divisor functions for $k = 3, 4$ as in Conjecture 1.1.

Finally, we note that with the same approach as in this article, it is likely that one could establish an asymptotic formula for $\mathscr{D}_{k,2;\omega}(K)$ for each other integer $k \geq 3$ for some $K = T^{1+\eta_k}$, where $0 < \eta_k < 1$, by building on the ideas in [Drappeau 2017; Topacogullari 2017; 2018]. This is current work in progress. However, this approach would not allow one to estimate $\mathscr{D}_{k,k;\omega}(K)$ for $K \geq T^2$. Conrey and Keating [2016; 2019] introduced a method with new divisor sums to estimate $\mathscr{D}_{k,k;\omega}(K)$ for such K. This created a new branch in this area of research, which is active at the present time.

Conventions and notation. Given two functions $f(x)$ and $g(x)$, we shall interchangeably use the notation $f(x) = O(g(x))$, $f(x) \ll g(x)$ and $g(x) \gg f(x)$ to mean that there exists $M > 0$ such that $|f(x)| \leq M|g(x)|$ for all sufficiently large x. The statement $f(x) \asymp g(x)$ means that the estimates $f(x) \ll g(x)$ and $g(x) \ll f(x)$ simultaneously hold.

Per our notation, ε denotes an arbitrarily small positive constant which may vary from instance to instance. The letter p will always be used to denote a prime number. We also adopt the usual notation that for $s \in \mathbb{C}$, its real part is $\sigma = \Re(s)$. The integral notation

$$\int_{(c)} f(s)\, ds =: \int_{c-i\infty}^{c+i\infty} f(s)\, ds$$

for a complex function $f(s)$ and real number c will be used frequently.

Give two sequences $(a(n))$, $(b(n))$, we define their additive convolution $((a \star b)(n))$ by

$$(a \star b)(n) = \sum_{\substack{u,v \geq 0 \\ u+v=n}} a(u)b(v).$$

This is so that

$$\left(\sum_{n=0}^{\infty} a(n)X^n\right)\left(\sum_{n=0}^{\infty} b(n)X^n\right) = \sum_{n=0}^{\infty} (a \star b)(n)X^n$$

for a variable X. We will also use the notation $(-1)^{\bullet}$ to denote the sequence $((-1)^n)_{n=0}^{\infty}$.

Organization. The plan of our paper is as follows. In Section 2 we define some special functions and fix the notation that will be used throughout the paper. In Section 3, we recall the main theorem in [Hamieh and Ng 2022], which provides an asymptotic formula for $\mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K)$. We prove that the main terms $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ and $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ in this formula are holomorphic functions of the elements of $\mathcal{I}$ and $\mathcal{J}$. Then in Section 4, we prove Theorem 1.2 by computing $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K) + \mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ explicitly in a special case of $|\mathcal{I}| = |\mathcal{J}| = 2$. In Appendix A, we show that Theorem 1.2 will still hold if the weight function w in the mean value is replaced by $1_{[T,2T]}$ and thus prove Corollary 1.3. Finally in Appendix B, we rewrite the expressions for $Q_0(x, y)$, $Q_1(x, y)$, $Q_2(x, y)$, and $Q_3(x, y)$ that appear in Theorem 1.2 in terms of the γ_j and $\zeta^{(j)}(2)$ for suitable j.

2. Setting and preliminaries

2.1. ***Properties of φ.*** For a fixed number $\mu \in \left(0, \frac{1}{2}\right)$, let φ be a smooth, nonnegative function defined on $\mathbb{R}_{\geq 0}$ such that

$$\varphi(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } t \geq 1+\mu, \end{cases} \qquad \varphi^{(j)}(t) \ll \mu^{-j} \quad \text{for all } j \geq 0. \tag{11}$$

Its Mellin transform is

$$\Phi(s) = \int_0^\infty \varphi(t) t^{s-1}\, dt, \tag{12}$$

which converges absolutely for $\Re(s) > 0$. The function Φ has an analytic continuation to the entire complex plane with the exception of a simple pole at $s = 0$ with residue 1.

For $c > 0$ and $\Re(s) > c$, we define

$$\Phi_2(s) = \frac{1}{2\pi i} \int_{(c)} \Phi(s_1)\Phi(s-s_1)\, ds_1. \tag{13}$$

Observe that

$$\Phi_2(s) = \int_0^\infty \varphi(t)^2 t^{s-1}\, dt \quad \text{and} \quad \varphi(t)^2 = \frac{1}{2\pi i} \int_{(c)} \Phi_2(s) t^{-s}\, ds \quad \text{for } c > 0. \tag{14}$$

Note that $\Phi_2(s)$ has a simple pole at $s = 0$. It also satisfies the bound

$$|\Phi_2(s)| \ll_m \frac{\mu^{1-m}(1+\mu)^{\sigma+m-1}}{|s(s+1)\cdots(s+m-1)|} \tag{15}$$

for $m \geq 1$ and $s \in \mathbb{C} \setminus \{0, -1, \ldots, -(m-1)\}$.

2.2. ***Taylor expansions of some functions.*** First, we recall the definitions of the functions f, h and G:

$$\begin{aligned} f(s) &= s\,\zeta(1+s) = \sum_{j=0}^\infty g_j s^j, \\ h(s) &= \frac{1}{\zeta(2+s)} = \sum_{j=0}^\infty \delta_j s^j, \\ G(s) &= -2\int_0^\infty \varphi(t)\varphi'(t) t^s\, dt = \sum_{j=0}^\infty c_j s^j. \end{aligned} \tag{16}$$

We will provide precise formulae for these coefficients, g_j, δ_j, and c_j. Recall that

$$\zeta(s) = \frac{1}{s-1} + \sum_{j=0}^\infty \frac{(-1)^j \gamma_j}{j!}(s-1)^j, \tag{17}$$

where, for $j \geq 0$,

$$\gamma_j = \lim_{m\to\infty}\left(\sum_{k=1}^{m} \frac{(\log k)^j}{k} - \frac{(\log m)^{j+1}}{j+1}\right).$$

It follows that the function $f(s) = s\zeta(1+s)$ is entire with the Taylor series expansion

$$f(s) = \sum_{j=0}^{\infty} g_j s^j,$$

where

$$g_j = \begin{cases} 1 & \text{for } j = 0, \\ \dfrac{(-1)^{j-1}\gamma_{j-1}}{(j-1)!} & \text{for } j \geq 1. \end{cases} \tag{18}$$

Observe that $g_1 = \gamma_0 = \gamma$ is Euler's constant.

We also have

$$\frac{1}{\zeta(2+s)} = \sum_{j=0}^{\infty} \delta_j s^j, \tag{19}$$

where

$$\delta_0 = \frac{1}{\zeta(2)}, \quad \delta_1 = -\frac{\zeta'(2)}{\zeta(2)^2}, \quad \delta_2 = \frac{2(\zeta'(2))^2 - \zeta(2)\zeta''(2)}{\zeta(2)^3},$$

$$\delta_3 = \frac{-6(\zeta'(2))^3 - \zeta'''(2)\zeta(2)^2 + 6\zeta(2)\zeta'(2)\zeta''(2)}{\zeta(2)^4}, \tag{20}$$

$$\delta_4 = \frac{24(\zeta'(2))^4 - \zeta^{(4)}(2)\zeta(2)^3 + 6\zeta(2)^2(\zeta''(2))^2 + 8\zeta'''(2)\zeta(2)^2\zeta'(2) - 36\zeta(2)(\zeta'(2))^2\zeta''(2)}{\zeta(2)^5},$$

and in general δ_j lies in the field generated by the $\zeta^{(j)}(2)$ with $j \in \mathbb{N}$.

Now, since

$$G(s) = -2\int_0^{\infty} \varphi(t)\varphi'(t)t^s\,dt = s\Phi_2(s)$$

for $\Phi_2(s)$ as defined in (14), it follows that $G(s)$ is an entire function and the coefficients of its Taylor series are given by

$$c_j = \frac{(-2)}{j!}\int_0^{\infty} \varphi(t)\varphi'(t)(\log t)^j\,dt \quad \text{for } j = 0, 1, \ldots.$$

In particular, we have

$$c_0 = -2\int_0^{\infty} \varphi(t)\varphi'(t)\,dt = \varphi(1)^2 = 1.$$

Furthermore, we find that

$$|c_j| \ll \mu^{-1}\int_1^{1+\mu} \log^j(t)\,dt \ll \mu^{-1}\cdot\mu\cdot\max_{t\in[1,1+\mu]} \log^j(t) = \log^j(1+\mu) \ll \mu^j$$

as long as $\mu \in [0, 1]$.

We also introduce the following entire functions and their Taylor expansions, which will appear in our calculations in Section 4:

$$F(s) = sf'(s) - f(s) = s^2\zeta'(1+s) = \sum_{j=0}^{\infty} g'_j s^j, \qquad \text{where } g'_j = (j-1)g_j,$$

$$H(s) = -\frac{\zeta'}{\zeta^2}(2+s) = \sum_{j=0}^{\infty} \delta'_j s^j, \qquad \text{where } \delta'_j = (j+1)\delta_{j+1}, \tag{21}$$

$$L = \log\frac{t}{2\pi}, \quad Y = \log K, \quad X = Y - L, \; E_1(s) = e^{sL} = \sum_{j=0}^{\infty} \alpha_j s^j, \quad E_2(s) = e^{sY} = \sum_{j=0}^{\infty} \beta_j s^j.$$

3. A mean value theorem under an additive divisor conjecture

We now recall the main theorem in [Hamieh and Ng 2022], in which an asymptotic formula is established for mean values of long Dirichlet polynomials with higher-order shifted divisor functions, assuming a smoothed additive divisor conjecture for higher-order shifted divisor functions. Before we state this result, we shall introduce some necessary notation and recall the statement of the additive divisor conjecture.

We set

$$\mathcal{K} = \{1, \dots, k\} \quad \text{and} \quad \mathcal{L} = \{1, \dots, \ell\}.$$

Throughout this section, $\mathcal{I}$ and $\mathcal{J}$ are multisets of complex numbers indexed by $\mathcal{K}$ and $\mathcal{L}$ respectively and are given by

$$\mathcal{I} = \{a_1, \dots, a_k\} \quad \text{and} \quad \mathcal{J} = \{b_1, \dots, b_\ell\}$$

such that

$$|a_i|, |b_j| \ll \frac{1}{\log T} \quad \text{for } i \in \mathcal{K} \text{ and } j \in \mathcal{L} \tag{22}$$

and

$$|a_{i_1} - a_{i_2}| \gg \frac{1}{\log T} \quad \text{and} \quad |b_{j_1} - b_{j_2}| \gg \frac{1}{\log T} \quad \text{for } i_1 \neq i_2 \in \mathcal{K} \text{ and } j_1 \neq j_2 \in \mathcal{L}, \tag{23}$$

for some parameter $T \geq 2$.

Also, for a set of shifts $\mathcal{I} = \{a_1, \dots, a_k\}$ as before, we define a shifted divisor function as

$$\sigma_{\mathcal{I}}(n) = \sum_{d_1 \cdots d_k = n} d_1^{-a_1} \cdots d_k^{-a_k}.$$

Observe that if $\mathcal{I} = \{0, \dots, 0\}$, then $\sigma_{\mathcal{I}}(n) = \tau_k(n)$.

3.1. ***The additive divisor conjecture.*** We define the shifted convolution sum

$$D_{F;\mathcal{I},\mathcal{J}}(r) = \sum_{\substack{m,n \geq 1 \\ m-n=r}} \sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(n)F(m,n). \tag{24}$$

Here we assume that, for some X, Y and $P \geq 1$,

$$\text{support}(F) \subset [X, 2X] \times [Y, 2Y] \tag{25}$$

and that

$$x^m y^n F^{(m,n)}(x, y) \ll_{m,n} P^{m+n}. \tag{26}$$

For a finite multiset of complex numbers $A = \{a_1, \ldots, a_m\}$ and $s \in \mathbb{C}$, we define two multiplicative functions $n \mapsto g_A(s, n)$ and $n \mapsto G_A(s, n)$ by

$$g_A(s, n) = \prod_{p^e || n} \biggl(\sum_{j=0}^{\infty} \frac{\sigma_A(p^{j+e})}{p^{js}}\biggr) \bigg/ \biggl(\sum_{j=0}^{\infty} \frac{\sigma_A(p^j)}{p^{js}}\biggr) \tag{27}$$

and

$$G_A(s, n) = \sum_{d|n} \frac{\mu(d) d^s}{\phi(d)} \sum_{e|d} \frac{\mu(e)}{e^s} g_A\Bigl(s, \frac{ne}{d}\Bigr). \tag{28}$$

Notice that for $n \in \mathbb{N}$ we have

$$\sum_{j=1}^{\infty} \frac{\sigma_A(jn)}{j^s} = g_A(s, n) \prod_{a \in A} \zeta(s + a).$$

We are now prepared to state a conjectural asymptotic formula for $D_{F;\mathcal{I},\mathcal{J}}(r)$.

Conjecture 3.1 (k-ℓ additive divisor conjecture). *There exists a triple*

$$(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \in \bigl[\tfrac{1}{2}, 1\bigr) \times [0, \infty) \times (0, 1]$$

for which the following holds (henceforth to be referred to as $\mathcal{AD}_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell})$ conjecture).

Let ε be a positive absolute constant, $P > 1$, and $X, Y > \frac{1}{2}$ satisfy $Y \asymp X$. Let F be a smooth function satisfying the conditions (25) and (26), and suppose that $\mathcal{I} = \{a_1, a_2, \ldots, a_k\}$ and $\mathcal{J} = \{b_1, \ldots, b_\ell\}$ are sets of distinct complex numbers such that $|a_i|, |b_j| \ll 1/\log X$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$. Then for $D_{F;\mathcal{I},\mathcal{J}}(r)$ as defined in (24), in the cases where X is sufficiently large (in absolute terms), one has

$$D_{F;\mathcal{I},\mathcal{J}}(r) = \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \prod_{j_1 \neq i_1} \zeta(1 - a_{i_1} + a_{j_1}) \prod_{j_2 \neq i_2} \zeta(1 - b_{i_2} + b_{j_2}) \sum_{q=1}^{\infty} \frac{c_q(r) G_{\mathcal{I}}(1 - a_{i_1}, q) G_{\mathcal{J}}(1 - b_{i_2}, q)}{q^{2 - a_{i_1} - b_{i_2}}}$$
$$\times \int_{\max(0,r)}^{\infty} f(x, x - r) x^{-a_{i_1}} (x - r)^{-b_{i_2}}\, dx + O(P^{C_{k,\ell}} X^{\vartheta_{k,\ell} + \varepsilon})$$

uniformly for $1 \leq |r| \ll X^{\beta_{k,\ell}}$.

We say that $\mathcal{AD}_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell})$ holds if the k-ℓ additive divisor conjecture holds for a triple $(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \in \bigl[\frac{1}{2}, 1\bigr) \times [0, \infty) \times (0, 1]$. It is important to note that in the case $|\mathcal{I}| = |\mathcal{J}| = 2$, Hughes and Young [2010, p. 218] proved that $\mathcal{AD}_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell})$ holds for $\vartheta_{2,2} = \frac{3}{4}$, $C_{2,2} = \frac{5}{4}$ and $\beta_{2,2} = 1$ by using Duke, Friedlander and Iwaniec's δ-method [Duke et al. 1994].

3.2. ***Mean values of long Dirichlet polynomials with shifted divisor functions as coefficients.*** We now consider the mean value of long Dirichlet polynomials associated with the shifted divisor functions $\sigma_{\mathcal{I}}$ and $\sigma_{\mathcal{J}}$ as defined in (3). For simplicity, we set

$$\mathscr{D}_{\mathcal{I},\mathcal{J};\omega}(K) = \mathscr{D}_{\sigma_{\mathcal{I}},\sigma_{\mathcal{J}};\omega}(K).$$

Definition. Let $\mathcal{I}$, $\mathcal{J}$ be finite multisets of complex numbers. We define $\mathcal{B}(\mathcal{I},\mathcal{J})$ as the series

$$\mathcal{B}(\mathcal{I},\mathcal{J}) = \sum_{n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(n)\sigma_{\mathcal{J}}(n)}{n}, \tag{29}$$

provided that the series converges (for example, when $\Re(a), \Re(b) > 0$ for all $a \in \mathcal{I}$ and $b \in \mathcal{J}$), and by analytic continuation elsewhere.

Observe that when the series (29) converges, we use the multiplicativity of $\sigma_{\mathcal{I}}\sigma_{\mathcal{J}}$ to write

$$\mathcal{B}(\mathcal{I},\mathcal{J}) = \prod_p \sum_{u=0}^{\infty} \frac{\sigma_{\mathcal{I}}(p^u)\sigma_{\mathcal{J}}(p^u)}{p^u}.$$

Upon factoring out $\prod_p \prod_{\substack{i\in\mathcal{K}\\ j\in\mathcal{L}}} (1-p^{-1-a_i-b_j})^{-1}$ from the right-hand side of this, we obtain

$$\mathcal{B}(\mathcal{I},\mathcal{J}) = \prod_p \prod_{\substack{i\in\mathcal{K}\\ j\in\mathcal{L}}} (1-p^{-1-a_i-b_j})^{-1} \prod_p \prod_{\substack{i\in\mathcal{K}\\ j\in\mathcal{L}}} (1-p^{-1-a_i-b_j}) \sum_{u=0}^{\infty} \frac{\sigma_{\mathcal{I}}(p^u)\sigma_{\mathcal{J}}(p^u)}{p^u}.$$

Definition. For a prime p and $s \in \mathbb{C}$, we set $z_p(s) = (1-p^{-s})^{-1}$. From the local factors $z_p(s)$, we define

$$\mathcal{Z}(\mathcal{I},\mathcal{J}) = \prod_p \prod_{\substack{i\in\mathcal{K}\\ j\in\mathcal{L}}} z_p(1+a_i+b_j), \tag{30}$$

$$\mathcal{A}(\mathcal{I},\mathcal{J}) = \prod_p \prod_{\substack{i\in\mathcal{K}\\ j\in\mathcal{L}}} z_p^{-1}(1+a_i+b_j) \sum_{u=0}^{\infty} \frac{\sigma_{\mathcal{I}}(p^u)\sigma_{\mathcal{J}}(p^u)}{p^u}. \tag{31}$$

Observe that we have

$$\mathcal{Z}(\mathcal{I},\mathcal{J}) = \prod_{i\in\mathcal{K},\, j\in\mathcal{L}} \zeta(1+a_i+b_j)$$

and

$$\mathcal{B}(\mathcal{I},\mathcal{J}) = \mathcal{A}(\mathcal{I},\mathcal{J})\mathcal{Z}(\mathcal{I},\mathcal{J}). \tag{32}$$

Next, we require some notation regarding set operations. Given a multiset $U = \{\alpha_1, \dots, \alpha_n\}$ and $\xi \in \mathbb{C}$, we define $U+\xi = \{\alpha_1+\xi, \dots, \alpha_n+\xi\}$. We also set $-U = \{-\alpha_1, \dots, -\alpha_n\}$. With this notation, the identity

$$\sigma_{U+\xi}(n) = n^{-\xi}\sigma_U(n) \tag{33}$$

holds.

We are now ready to state [Hamieh and Ng 2022, Theorem 1.1].

Theorem 3.2. *Let $|\mathcal{I}| = k$ and $|\mathcal{J}| = \ell$ with $k, \ell \geq 2$, and suppose that elements of both $\mathcal{I}$ and $\mathcal{J}$ satisfy (22) and (23). Assume that $\mathcal{AD}_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell})$ holds for some triple $(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \in \big[\frac{1}{2}, 1\big) \times [0, \infty) \times (0, 1]$. Let $K = T^{1+\eta}$ with $\eta > 0$, and let ω satisfy (4), (5), and (6) with*

$$\nu > \frac{(1-\beta_{k,\ell})(1+\eta)}{1-\epsilon} \quad \text{and} \quad 0 < \epsilon < 1.$$

Then we have

$$\mathscr{D}_{\mathcal{I},\mathcal{J};\omega}(K) = \mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K) + \mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K) + O\bigg(K^{\vartheta_{k,\ell}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C_{k,\ell}}\bigg), \tag{34}$$

where

$$\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K) = \frac{\hat{\omega}(0)}{2\pi i}\int_{(c)} K^s \Phi_2(s) \mathcal{B}(\mathcal{I}+s, \mathcal{J})\, ds \tag{35}$$

and

$$\begin{aligned}\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K) = \int_0^\infty \omega(t) \sum_{i\in\mathcal{K}, j\in\mathcal{L}} \Big(\frac{t}{2\pi}\Big)^{-a_i-b_j} \mathcal{Z}(\mathcal{I}\setminus\{a_i\}, \{-a_i\})\mathcal{Z}(\{-b_j\}, \mathcal{J}\setminus\{b_j\}) \\ \times \frac{1}{2\pi i}\int_{(c)} \Phi_2(s)\Big(\frac{2\pi K}{t}\Big)^s \mathcal{Z}\big((\mathcal{I}\setminus\{a_i\})+s, \mathcal{J}\setminus\{b_j\}\big)\zeta(1-a_i-b_j-s) \\ \times \mathcal{A}\big((\mathcal{I}\setminus\{a_i\})\cup\{-b_j-s\}, ((\mathcal{J}\setminus\{b_j\})+s)\cup\{-a_i\}\big)\, ds\, dt.\end{aligned} \tag{36}$$

Here $c > 0$ is fixed and $\Phi_2(s)$ is as defined in (13).

3.3. ***Holomorphy of $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ and $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$.*** We will now prove that the main term in the asymptotic formula (34) is holomorphic as a function of the shifts $a_1, \ldots, a_k, b_1, \ldots, b_\ell$. As a consequence of this, in another lemma we will prove that Theorem 3.2 holds without the restrictions in (23).

Lemma 3.3. *Under the hypothesis of Theorem 3.2 and with the same definitions, the terms $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ and $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$, which are written in (35) and (36), respectively, are both holomorphic as functions of the variables $a_1, \ldots, a_k, b_1, \ldots, b_\ell$.*

Proof. We follow the argument that was employed in [Baluyot and Turnage-Butterbaugh 2025, Section 6]. Recall that $a_i, b_j \ll 1/\log T$ for all $i \in \mathcal{K}$ and $j \in \mathcal{L}$.

First, we consider $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$. We repeat (35) here:

$$\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K) = \frac{\hat{\omega}(0)}{2\pi i}\int_{(c)} K^s \Phi_2(s) \mathcal{B}(\mathcal{I}+s, \mathcal{J})\, ds,$$

where, as in (32) , we have

$$\mathcal{B}(\mathcal{I}+s, \mathcal{J}) = \mathcal{A}(\mathcal{I}+s, \mathcal{J})\mathcal{Z}(\mathcal{I}+s, \mathcal{J})$$

with

$$\mathcal{A}(\mathcal{I}+s, \mathcal{J}) = \prod_p \prod_{x\in\mathcal{I}+s, y\in\mathcal{J}} \Big(1-\frac{1}{p^{1+x+y}}\Big)\sum_{u=0}^\infty \frac{\sigma_{\mathcal{I}+s}(p^u)\sigma_{\mathcal{J}}(p^u)}{p^u}, \quad \mathcal{Z}(\mathcal{I}+s, \mathcal{J}) = \prod_{x\in\mathcal{I}+s, y\in\mathcal{J}} \zeta(1+x+y).$$

From (29) and (33), we see that $\mathcal{B}(\mathcal{I}+s,\mathcal{J})$ is holomorphic as a function of the variables a_i and b_j due to the restriction on the size of the a_i and b_j. Thus $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ is also holomorphic in the a_i and the b_j.

We proceed with $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$, which, by (36), is given as

$$\begin{aligned}\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K) = \int_0^\infty \omega(t) \sum_{i_0\in\mathcal{K},\, j_0\in\mathcal{L}} \left(\frac{t}{2\pi}\right)^{-a_{i_0}-b_{j_0}} \mathcal{Z}(\mathcal{I}\setminus\{a_{i_0}\},\{-a_{i_0}\})\mathcal{Z}(\{-b_{j_0}\},\mathcal{J}\setminus\{b_{j_0}\})\\ \times \frac{1}{2\pi i}\int_{(c)} \Phi_2(s)\left(\frac{2\pi K}{t}\right)^s \mathcal{Z}\big((\mathcal{I}\setminus\{a_{i_0}\})+s,\mathcal{J}\setminus\{b_{j_0}\}\big)\zeta(1-a_{i_0}-b_{j_0}-s)\\ \times \mathcal{A}\big((\mathcal{I}\setminus\{a_{i_0}\})\cup\{-b_{j_0}-s\},((\mathcal{J}\setminus\{b_{j_0}\})+s)\cup\{-a_{i_0}\}\big)\,ds\,dt.\end{aligned}$$

For now, we assume that both sets $\mathcal{I}$ and $\mathcal{J}$ have distinct elements and that their intersection is empty. We expand each $\mathcal{Z}$-term and $\mathcal{A}$-term in the above. By definition,

$$\mathcal{Z}(\mathcal{I}\setminus\{a_{i_0}\},\{-a_{i_0}\}) = \prod_{\substack{x\in\mathcal{I}\setminus\{a_{i_0}\},\\ y\in\{-a_{i_0}\}}} \zeta(1+x+y) = \prod_{i\neq i_0}\zeta(1+a_i-a_{i_0}), \tag{37}$$

$$\mathcal{Z}(\{-b_{j_0}\},\mathcal{J}\setminus\{b_{j_0}\}) = \prod_{\substack{x\in\mathcal{J}\setminus\{b_{j_0}\},\\ y\in\{-b_{j_0}\}}} \zeta(1+x+y) = \prod_{j\neq j_0}\zeta(1+b_j-b_{j_0}). \tag{38}$$

Also, by an argument of inclusion-exclusion we have

$$\begin{aligned}\mathcal{Z}(\{-b_{j_0}\},\mathcal{J}\setminus\{b_{j_0}\}) &= \prod_{\substack{i\in\mathcal{K},\, j\in\mathcal{L},\\ i\neq i_0,\, j\neq j_0}} \zeta(1+a_i+s+b_j)\\ &= \frac{\prod_{i\in\mathcal{K},\, j\in\mathcal{L}}\zeta(1+a_i+s+b_j)}{\prod_{i\in\mathcal{K}}\zeta(1+a_i+s+b_{j_0})\prod_{j\in\mathcal{L}}\zeta(1+a_{i_0}+s+b_j)}\zeta(1+a_{i_0}+s+b_{j_0}).\end{aligned} \tag{39}$$

For the $\mathcal{A}$-term as defined via (31), we note the following for its Euler product part:

$$\begin{aligned}\prod_{\substack{x\in(\mathcal{I}\setminus\{a_{i_0}\})\cup\{-b_{j_0}-s\},\\ y\in((\mathcal{J}\setminus\{b_{j_0}\})+s)\cup\{-a_{i_0}\}}} \left(1-\frac{1}{p^{1+x+y}}\right) = \prod_{x\in\mathcal{I}\setminus\{a_{i_0}\},\, y\in(\mathcal{J}\setminus\{b_{j_0}\})+s} \left(1-\frac{1}{p^{1+x+y}}\right) \prod_{\substack{x\in\mathcal{I}\setminus\{a_{i_0}\},\\ y\in\{-a_{i_0}\}}} \left(1-\frac{1}{p^{1+x+y}}\right)\\ \times \prod_{\substack{x\in\{-b_{j_0}-s\},\\ y\in(\mathcal{J}\setminus\{b_{j_0}\})+s}} \left(1-\frac{1}{p^{1+x+y}}\right) \prod_{\substack{x\in\{-b_{j_0}-s\},\\ y\in\{-a_{i_0}\}}} \left(1-\frac{1}{p^{1+x+y}}\right).\end{aligned}$$

Again by inclusion-exclusion, this can also be written as

$$\begin{aligned}&\prod_{\substack{x\in(\mathcal{I}\setminus\{a_{i_0}\})\cup\{-b_{j_0}-s\},\\ y\in((\mathcal{J}\setminus\{b_{j_0}\})+s)\cup\{-a_{i_0}\}}} \left(1-\frac{1}{p^{1+x+y}}\right)\\ &= \prod_{i\in\mathcal{K},\, j\in\mathcal{L}} \left(1-\frac{1}{p^{1+a_i+b_j+s}}\right) \prod_{i\in\mathcal{K}} \left(1-\frac{1}{p^{1+a_i+b_{j_0}+s}}\right)^{-1} \prod_{j\in\mathcal{L}} \left(1-\frac{1}{p^{1+a_{i_0}+b_j+s}}\right)^{-1} \left(1-\frac{1}{p}\right)^{-2}\\ &\quad\times \prod_{i\in\mathcal{K}} \left(1-\frac{1}{p^{1+a_i-a_{i_0}}}\right) \prod_{j\in\mathcal{L}} \left(1-\frac{1}{p^{1+b_j-b_{j_0}}}\right) \left(1-\frac{1}{p^{1+a_{i_0}+b_{j_0}+s}}\right) \left(1-\frac{1}{p^{1-a_{i_0}-b_{j_0}-s}}\right).\end{aligned} \tag{40}$$

In view of these expressions, it will be useful to define for each prime p

$$\begin{aligned}\mathcal{P}(p) &= \mathcal{P}(z_1, z_2, s;\, p)\\ &:= \prod_{i\in\mathcal{K}, j\in\mathcal{L}}\left(1-\frac{1}{p^{1+a_i+b_j+s}}\right)\prod_{i\in\mathcal{K}}\left(1-\frac{1}{p^{1+a_i-z_2+s}}\right)^{-1}\prod_{j\in\mathcal{L}}\left(1-\frac{1}{p^{1+b_j-z_1+s}}\right)^{-1}\\ &\quad\times\left(1-\frac{1}{p}\right)^{-2}\prod_{i\in\mathcal{K}}\left(1-\frac{1}{p^{1+a_i+z_1}}\right)\prod_{j\in\mathcal{L}}\left(1-\frac{1}{p^{1+b_j+z_2}}\right)\left(1-\frac{1}{p^{1-z_1-z_2+s}}\right)\left(1-\frac{1}{p^{1+z_1+z_2-s}}\right).\end{aligned}$$

By using Cauchy's theorem, we can now write $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ as a sum of residues and thus as an integral. By (37), (38) and (39) we have

$$\begin{aligned}&\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)\\ &= \int_0^\infty \omega(t)\sum_{i\in\mathcal{K}, j\in\mathcal{L}}\left(\frac{t}{2\pi}\right)^{z_1+z_2}\frac{1}{2\pi i}\int_{(c)}\Phi_2(s)\left(\frac{2\pi K}{t}\right)^s\\ &\quad\times\frac{1}{(2\pi i)^2}\int_{|z_1|=c/4}\int_{|z_2|=c/4}\prod_{i\in\mathcal{K}}\zeta(1+z_1+a_i)\prod_{j\in\mathcal{L}}\zeta(1+z_2+b_j)\zeta(1+z_1+z_2-s)\zeta(1-z_1-z_2+s)\\ &\quad\times\frac{\prod_{i\in\mathcal{K}, j\in\mathcal{L}}\zeta(1+a_i+b_j+s)}{\prod_{i\in\mathcal{K}}\zeta(1+a_i-z_2+s)\prod_{j\in\mathcal{L}}\zeta(1-z_1+b_j+s)}\\ &\quad\times\prod_p\mathcal{P}(p)\sum_{u=0}^\infty\frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p^u)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p^u)}{p^u}\,dz_1\,dz_2\,ds\,dt. \qquad (41)\end{aligned}$$

This is because the pairs $z_1=-a_i$ and $z_2=-b_j$ for $i\in\mathcal{K}$, $j\in\mathcal{L}$ are the only poles of the above integrand, all of which are simple.

Moreover, the integrand is holomorphic as a function of the a_i, b_j whenever they are distinct as per our assumption. This is clear to see for the part of the integrand that involves ζ-values. It thus remains to show that the Euler product in the above converges absolutely. For this, note that by (40) we have

$$\begin{aligned}&\mathcal{P}(p)\sum_{u=0}^\infty\frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p^u)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p^u)}{p^u}\\ &= \prod_{\substack{x\in(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\},\\ y\in((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}}\left(1-\frac{1}{p^{1+x+y}}\right)\sum_{u=0}^\infty\frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p^u)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p^u)}{p^u}\\ &= \left(1+\frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p)}{p}\right)\prod_{\substack{x\in(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\},\\ y\in((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}}\left(1-\frac{1}{p^{1+x+y}}\right)+O_\varepsilon\left(\frac{1}{p^{2-8c+\varepsilon}}\right). \qquad (42)\end{aligned}$$

In the last step, we used the estimate

$$\sum_{u=2}^\infty\frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p^u)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p^u)}{p^u}\ll_\varepsilon\frac{1}{p^{2-8c+\varepsilon}}\qquad\text{for suitable } \varepsilon>0. \qquad (43)$$

This estimate follows from the fact that, for any $\varepsilon > 0$,

$$\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p^u) \ll_\varepsilon p^{u(-\min_{v\in(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}\{\Re v\}+\varepsilon)} \ll p^{u(2c+\varepsilon)},$$

since

$$\Re(v) \gg -c - \frac{c}{4} \geq -2c \quad \text{for } v \in (\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\},$$

and from the similar estimate

$$\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p^u) \ll_\varepsilon p^{u(2c+\varepsilon)}.$$

On the other hand, as in [Baluyot and Turnage-Butterbaugh 2025, Lemma 4.1], we see that, for suitable $\varepsilon > 0$,

$$\prod_{\substack{x\in(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\},\\ y\in((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}} \left(1 - \frac{1}{p^{1+x+y}}\right) = 1 - \frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p)}{p} + O\left(\frac{1}{p^{1+\varepsilon}}\right).$$

By combining this with (43), we obtain

$$\left(1 + \frac{\sigma_{(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\}}(p)\sigma_{((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}(p)}{p}\right) \prod_{\substack{x\in(\mathcal{I}\setminus\{-z_1\})\cup\{z_2-s\},\\ y\in((\mathcal{J}\setminus\{-z_2\})+s)\cup\{z_1\}}} \left(1 - \frac{1}{p^{1+x+y}}\right) = 1 + O\left(\frac{1}{p^{1+\varepsilon}}\right) + O_\varepsilon\left(\frac{1}{p^{2-8c+\varepsilon}}\right)$$

for $\varepsilon > 0$ sufficiently small. Finally by (42) and by choosing $c > 0$ suitably, we deduce that the Euler product in (41) converges absolutely, hence it is holomorphic in the a_i and b_j. Therefore, we have shown that if both $\mathcal{I}$ and $\mathcal{J}$ have no repeated elements and that they don't have any elements in common, then the right-hand side of (41) is a holomorphic function of the a_i and b_j. By analytic continuation, the same expression, and thus $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$, is a holomorphic function of the shifts $a_1, \dots, a_k, b_1, \dots, b_\ell$ that satisfy the condition $a_i, b_j \ll 1/\log T$ for all i, j. □

Lemma 3.4. *Theorem 3.2 holds without assuming the size restriction in (23).*

Proof. We follow the argument that was employed in [Ng 2021, Section 5]. We set $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_\ell)$. We also let $L(\boldsymbol{a}, \boldsymbol{b}) = \mathscr{D}_{\mathcal{I},\mathcal{J};\omega}(K)$ and $R(\boldsymbol{a}, \boldsymbol{b}) = \mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K) + \mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ for convenience. By Theorem 3.2, we know that

$$L(\boldsymbol{a}, \boldsymbol{b}) - R(\boldsymbol{a}, \boldsymbol{b}) = O\left(K^{\vartheta_{k,\ell}+\varepsilon}\left(\frac{T}{T_0}\right)^{1+C_{k,\ell}}\right), \tag{44}$$

provided that coordinates of $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the conditions (22) and (23). By Lemma 3.3, we also know that $L(\boldsymbol{a}, \boldsymbol{b}) - R(\boldsymbol{a}, \boldsymbol{b})$ is holomorphic as a function of the variables $a_1, \dots, a_k, b_1, \dots, b_\ell$.

Suppose that $a_1, \dots, a_k, b_1, \dots, b_\ell$ are complex numbers satisfying $|a_j|, |b_j| \leq C_0/\log T$ for some positive constant C_0. Consider the polydisc $D \subset \mathbb{C}^{k+\ell}$ given by

$$D = \prod_{j=1}^{k} D_j \prod_{j=1}^{\ell} \widetilde{D}_j,$$

where
$$D_j=\{z\in\mathbb{C}:|z-a_j|\le r_j\},\quad \widetilde{D}_j=\{z\in\mathbb{C}:|z-b_j|\le r_j\}\quad\text{and}\quad r_j=\frac{2^{j+1}C_0}{\log T}.$$

Let ∂D_j and $\partial\widetilde{D}_j$ be the boundaries of the discs D_j and $\widetilde{D}_j$ respectively. By Cauchy's integral formula, we have
$$L(\boldsymbol{a},\boldsymbol{b})-R(\boldsymbol{a},\boldsymbol{b})=\frac{1}{(2\pi i)^{k+\ell}}\int_{\partial D_1}\cdots\int_{\partial D_k}\int_{\partial\widetilde{D}_1}\cdots\int_{\partial\widetilde{D}_\ell}\frac{L(\boldsymbol{z},\boldsymbol{w})-R(\boldsymbol{z},\boldsymbol{w})}{(\boldsymbol{z}-\boldsymbol{a})(\boldsymbol{w}-\boldsymbol{b})}\,\boldsymbol{dz}\,\boldsymbol{dw},\tag{45}$$
where
$$\boldsymbol{dz}=dz_1\cdots dz_k,\quad \boldsymbol{dw}=dw_1\cdots dw_\ell,\quad \boldsymbol{z}-\boldsymbol{a}=\prod_{j=1}^{k}(z_j-a_j)\quad\text{and}\quad \boldsymbol{w}-\boldsymbol{b}=\prod_{j=1}^{\ell}(w_j-b_j).$$

Observe that for $1\le j_2<j_1\le k$ we have
$$|z_{j_1}-z_{j_2}|\ge|z_{j_1}-a_{j_1}|-|z_{j_2}-a_{j_2}|-|a_{j_1}|-|a_{j_2}|\ge\frac{2C_0}{\log T},$$
$$|w_{j_1}-w_{j_2}|\ge|w_{j_1}-b_{j_1}|-|w_{j_2}-b_{j_2}|-|b_{j_1}|-|b_{j_2}|\ge\frac{2C_0}{\log T}.$$

Hence z_j and w_j satisfy the conditions (22) and (23). In particular, (44) holds for $(z_1,\dots,z_k)\in\prod_{j=1}^{k}\partial D_j$ and $(w_1,\dots,w_\ell)\in\prod_{j=1}^{\ell}\partial\widetilde{D}_j$. More precisely, we have
$$L(\boldsymbol{z},\boldsymbol{w})-R(\boldsymbol{z},\boldsymbol{w})=O\biggl(K^{\vartheta_{k,\ell}+\varepsilon}\Bigl(\frac{T}{T_0}\Bigr)^{1+C_{k,\ell}}\biggr).$$

By using this bound in (45), we obtain
$$\begin{aligned}L(\boldsymbol{a},\boldsymbol{b})-R(\boldsymbol{a},\boldsymbol{b})&\ll K^{\vartheta_{k,\ell}+\varepsilon}\Bigl(\frac{T}{T_0}\Bigr)^{1+C_{k,\ell}}\prod_{j=1}^{k}\frac{\operatorname{length}(\partial D_j)}{r_j}\prod_{j=1}^{\ell}\frac{\operatorname{length}(\partial\widetilde{D}_j)}{r_j}\\&\ll K^{\vartheta_{k,\ell}+\varepsilon}\Bigl(\frac{T}{T_0}\Bigr)^{1+C_{k,\ell}},\end{aligned}$$
as desired. □

4. Proof of Theorem 1.2

As a first step in proving Theorem 1.2, we shall apply Theorem 3.2 with $\mathcal{I}=\{a,0\}$ and $\mathcal{J}=\{b,0\}$. In the case $|\mathcal{I}|=|\mathcal{J}|=2$, we know that $\mathcal{AD}_{k,\ell}(\vartheta_{k,\ell},C_{k,\ell},\beta_{k,\ell})$ holds with $\vartheta_{2,2}=\frac{3}{4}$, $C_{2,2}=\frac{5}{4}$, and $\beta_{2,2}=1$ [Hughes and Young 2010, p. 218]. Hence, Theorem 3.2 holds unconditionally for any $\eta<\frac{1}{3}$.

In order to compute $\mathscr{D}_{2,2;\omega}(K)$, we will simplify the expressions for $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ and $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ that were provided by Theorem 3.2. We will move the contours of integration to the left, and then the residues that are obtained will be part of the main term in our formula for $\mathscr{D}_{2,2;\omega}(K)$. Once we obtain the whole main term in terms of a and b, we will first let b tend to a, and then let a tend to 0. The resulting limit will provide us with the result of Theorem 1.2.

Note that we will frequently refer to the special functions that were defined in (16) and (21).

4.1. *Computing* $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$.

Proposition 4.1. *Let* $\mathcal{I}=\{a,0\}$ *and* $\mathcal{J}=\{b,0\}$, *and let* $\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)$ *be defined by* (35). *Then we have*

$$\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)=\hat{\omega}(0)(\mathcal{R}_1(a,b)+\mathcal{R}_1'(a,b))+O(TK^{-1/2+2\delta}),$$

where

$$\begin{aligned}\mathcal{R}_1(a,b)=(Y+c_1+\gamma_0)&\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)+2\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}H(a+b)\\&+\left(\frac{f'(a+b)}{a+b}-\frac{f(a+b)}{(a+b)^2}\right)\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)+\frac{f(a+b)}{a+b}\left(\frac{f'(a)}{a}-\frac{f(a)}{a^2}\right)\frac{f(b)}{b}h(a+b)\\&+\frac{f(a+b)}{a+b}\frac{f(a)}{a}\left(\frac{f'(b)}{b}-\frac{f(b)}{b^2}\right)h(a+b),\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_1'(a,b)=a^{-2}G(-a)K^{-a}\frac{f(b)}{b}\frac{f(b-a)}{b-a}f(-a)h(b-a)+b^{-2}G(-b)K^{-b}\frac{f(a)}{a}\frac{f(a-b)}{a-b}f(-b)h(a-b)\\+(a+b)^{-2}G(-a-b)K^{-a-b}\frac{f(-b)}{b}\frac{f(-a)}{a}f(-a-b)h(-a-b).\end{aligned}$$

Proof. By (1.31) and then by (1.28) in [Hamieh and Ng 2022], we can write

$$\begin{aligned}\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)&=\frac{\hat{\omega}(0)}{2\pi i}\int_{(2c)}K^s\Phi_2(s)\mathcal{B}(\mathcal{I}_s,\mathcal{J})\,ds\,ds\\&=\frac{\hat{\omega}(0)}{2\pi i}\int_{(2c)}K^s\Phi_2(s)\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)\zeta(1+s)}{\zeta(2+2s+a+b)}\,ds.\end{aligned}$$

We move the line of integration to $\Re(s)=-\frac{1}{2}+2\delta$ capturing the residue of the integrand at $s=0$ in addition to the residues at $s=-a,-b,-a-b$. This gives

$$\begin{aligned}\mathcal{M}_{0,\mathcal{I},\mathcal{J};\omega}(K)=\hat{\omega}(0)\operatorname{Res}_{s=0}&\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right)\\&+\hat{\omega}(0)\operatorname{Res}_{s=-a}\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right)\\&+\hat{\omega}(0)\operatorname{Res}_{s=-b}\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right)\\&+\hat{\omega}(0)\operatorname{Res}_{s=-a-b}\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right)\\&+\frac{\hat{\omega}(0)}{2\pi i}\int_{(-1/2+2\delta)}\Phi_2(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)\zeta(1+s)}{\zeta(2+2s+a+b)}\,ds.\end{aligned}$$

It follows from (15) that

$$\int_{(-1/2+2\delta)}\Phi_2(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)\zeta(1+s)}{\zeta(2+2s+a+b)}\,ds\ll K^{-1/2+2\delta}.$$

Let us now compute the residue of

$$s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}$$

at $s=0$. This is

$$\begin{aligned}&Y\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)+c_1\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)\\&\quad+\left(\frac{f'(a+b)}{a+b}-\frac{f(a+b)}{(a+b)^2}\right)\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)+\frac{f(a+b)}{a+b}\left(\frac{f'(a)}{a}-\frac{f(a)}{a^2}\right)\frac{f(b)}{b}h(a+b)\\&\quad+\frac{f(a+b)}{a+b}\frac{f(a)}{a}\left(\frac{f'(b)}{b}-\frac{f(b)}{b^2}\right)h(a+b)+\gamma_0\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)\\&\quad+2\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}H(a+b).\end{aligned}$$

Further, this is equal to

$$\begin{aligned}&(Y+c_1+\gamma_0)\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)+2\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}H(a+b)\\&\quad+\left(\frac{f'(a+b)}{a+b}-\frac{f(a+b)}{(a+b)^2}\right)\frac{f(a)}{a}\frac{f(b)}{b}h(a+b)+\frac{f(a+b)}{a+b}\left(\frac{f'(a)}{a}-\frac{f(a)}{a^2}\right)\frac{f(b)}{b}h(a+b)\\&\quad+\frac{f(a+b)}{a+b}\frac{f(a)}{a}\left(\frac{f'(b)}{b}-\frac{f(b)}{b^2}\right)h(a+b),\end{aligned}$$

which is $\mathcal{R}_1(a,b)$. The desired result is obtained by simply observing that

$$\begin{aligned}\mathcal{R}'_1(a,b)&=\operatorname{Res}_{s=-a}\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right)\\&\quad+\operatorname{Res}_{s=-b}\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right)\\&\quad+\operatorname{Res}_{s=-a-b}\left(s^{-2}G(s)K^s\frac{\zeta(1+a+b+s)\zeta(1+a+s)\zeta(1+b+s)f(s)}{\zeta(2+2s+a+b)}\right).\end{aligned}$$ □

We will now rewrite $\mathcal{R}_1(a,b)$ whereby we simplify its expression. For this, we introduce some notation:

$$\begin{aligned}\mathcal{L}_0&:=Y+c_1+g_1,\\ \kappa_{11}(a,b)&:=f(a)f(b)f(a+b)h(a+b),\\ \tilde{\kappa}_{11}(a,b)&:=f(a)f(b)f(a+b)H(a+b),\\ \kappa_{12}(a,b)&:=f(a)f(b)h(a+b)\big((a+b)f'(a+b)-f(a+b)\big)=f(a)f(b)F(a+b)h(a+b),\\ \kappa_{13}(a,b)&:=f(b)f(a+b)h(a+b)(af'(a)-f(a))=F(a)f(b)f(a+b)h(a+b),\\ \kappa_{14}(a,b)&:=f(a)f(a+b)h(a+b)(bf'(b)-f(b))=f(a)F(b)f(a+b)h(a+b).\end{aligned}$$

Observe that we can now write

$$\mathcal{R}_1(a,b)=(\mathcal{R}_{11}+\mathcal{R}_{12}+\mathcal{R}_{13}+\mathcal{R}_{14})(a,b),$$

where we set

$$\begin{aligned}
\mathcal{R}_{11}(a,b) &= \frac{1}{ab(a+b)}\mathcal{L}_0\kappa_{11}(a,b) + 2\frac{1}{ab(a+b)}\tilde{\kappa}_{11}(a,b),\\
\mathcal{R}_{12}(a,b) &= \frac{1}{ab(a+b)^2}\kappa_{12}(a,b),\\
\mathcal{R}_{13}(a,b) &= \frac{1}{a^2b(a+b)}\kappa_{13}(a,b),\\
\mathcal{R}_{14}(a,b) &= \frac{1}{ab^2(a+b)}\kappa_{14}(a,b).
\end{aligned} \tag{46}$$

4.2. ***Computing*** $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$**.** First, we observe that by (36) we have

$$\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K) = \sum_{i_1\in\mathcal{K},i_2\in\mathcal{L}} \frac{c_{i_1,i_2}}{2\pi i}\int_{-\infty}^{\infty}\omega(t)\int_{\Re(s)=2\epsilon} I_{i_1i_2}(s,t)\,ds\,dt$$

for sufficiently small $\epsilon > 0$, where

$$c_{i_1,i_2} = \mathcal{Z}(\mathcal{I}\backslash\{a_{i_1}\},\{-a_{i_1}\})\mathcal{Z}(\{-b_{i_2}\},\mathcal{J}\backslash\{-b_{i_2}\}) = \prod_{j_1\in\mathcal{K}\backslash\{i_1\}}\zeta(1-a_{i_1}+a_{j_1})\prod_{j_2\in\mathcal{L}\backslash\{i_2\}}\zeta(1-b_{i_2}+b_{j_2}),$$

$$I_{i_1i_2}(s,t) = \Phi_2(s)K^s\left(\frac{2\pi}{t}\right)^{a_{i_1}+b_{i_2}+s}\zeta(1-a_{i_1}-b_{i_2}-s)\prod_{\substack{j_1\in\mathcal{K}\backslash\{i_1\}\\ j_2\in\mathcal{L}\backslash\{i_2\}}}\zeta(1+a_{j_1}+b_{j_2}+s)$$
$$\times\mathcal{A}\big((\mathcal{I}\setminus\{a_{i_1}\})\cup\{-b_{i_2}-s\},((\mathcal{J}\setminus\{b_{i_2}\})+s)\cup\{-a_{i_1}\}\big).$$

Since we chose $\mathcal{I}=\{a,0\}$ and $\mathcal{J}=\{b,0\}$, the terms c_{i_1,i_2} and $I_{i_1,i_2}(s,t)$ that appear in $\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)$ can be written more explicitly. We find that

$$\begin{aligned}
I_{11}(s,t) &= \Phi_2(s)K^s\left(\frac{2\pi}{t}\right)^{a+b+s}\zeta(1-a-b-s)\zeta(1+s)\mathcal{A}\big(\{0,-b-s\},\{s,-a\}\big)\\
&= \frac{G(s)}{s}K^s\left(\frac{2\pi}{t}\right)^{a+b+s}\zeta(1-a-b-s)\zeta(1+s)\frac{1}{\zeta(2-a-b)},\\
I_{12}(s,t) &= \Phi_2(s)K^s\left(\frac{2\pi}{t}\right)^{a+s}\zeta(1-a-s)\zeta(1+b+s)\mathcal{A}\big(\{0,-s\},\{b+s,-a\}\big)\\
&= \frac{G(s)}{s}K^s\left(\frac{2\pi}{t}\right)^{a+s}\zeta(1-a-s)\zeta(1+b+s)\frac{1}{\zeta(2+b-a)},\\
I_{21}(s,t) &= \Phi_2(s)K^s\left(\frac{2\pi}{t}\right)^{b+s}\zeta(1-b-s)\zeta(1+a+s)\mathcal{A}\big(\{a,-b-s\},\{s,0\}\big)\\
&= \frac{G(s)}{s}K^s\left(\frac{2\pi}{t}\right)^{b+s}\zeta(1-b-s)\zeta(1+a+s)\frac{1}{\zeta(2+a-b)},
\end{aligned}$$

$$I_{22}(s,t) = \Phi_2(s) K^s \left(\frac{2\pi}{t}\right)^s \zeta(1-s)\zeta(1+a+b+s)\mathcal{A}\big(\{a,-s\},\{b+s,0\}\big)$$
$$= \frac{G(s)}{s} K^s \left(\frac{2\pi}{t}\right)^s \zeta(1-s)\zeta(1+a+b+s)\frac{1}{\zeta(2+a+b)}.$$

One can compute c_{i_1,i_2} in a straightforward manner. We collect the results in Table 1.

Hence we can write

$$\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\omega(t)\int_{\Re(s)=2\epsilon}\big(\zeta(1-a)\zeta(1-b)I_{11}(s,t)+\zeta(1-a)\zeta(1+b)I_{12}(s,t) +\zeta(1+a)\zeta(1-b)I_{21}(s,t)+\zeta(1+a)\zeta(1+b)I_{22}(s,t)\big)\,ds\,dt. \quad (47)$$

Proposition 4.2. *Let $K = T^{1+\eta}$ with $0 < \eta < \frac{1}{3}$, and suppose that a weight function ω satisfies (4), (5), and (6) with $\nu > (5+3(\eta+1))/9$. Let $\mathcal{I} = \{a, 0\}$ and $\mathcal{J} = \{b, 0\}$ satisfy (22) and (23). In particular, assume that $|a|, |b| \le \delta$ with $\delta < \eta/2(2+3\eta)$. Then we have*

$$\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K) = \int_{-\infty}^{\infty}\omega(t)\cdot(-\mathcal{R}_1'(a,b)+\mathcal{R}_2(a,b))\,dt + O(K^{-1/2+3\delta}T^{3/2-\delta}),$$

where $\mathcal{R}_1'(a,b)$ is as given in Proposition 4.1 and

$$\begin{aligned}\mathcal{R}_2(a,b) = &-\left(\frac{2\pi}{t}\right)^{a+b} h(-a-b)\frac{f(-a)}{a}\frac{f(-b)}{b}\left(\frac{F(-a-b)}{(a+b)^2}+\frac{f(-a-b)}{a+b}(X+g_1+c_1)\right)\\ &+\left(\frac{2\pi}{t}\right)^{a} h(b-a)\frac{f^2(-a)}{a^2}\frac{f^2(b)}{b^2}+\left(\frac{2\pi}{t}\right)^{b} h(a-b)\frac{f^2(-b)}{b^2}\frac{f^2(a)}{a^2}\\ &-h(a+b)\frac{f(a)}{a}\frac{f(b)}{b}\left(\frac{F(a+b)}{(a+b)^2}+\frac{f(a+b)}{a+b}(X-g_1+c_1)\right)\\ &+K^{-b}\left(\frac{2\pi}{t}\right)^{a-b} h(b-a)\frac{G(-b)}{b}\frac{f(b-a)}{b-a}\frac{f(-a)}{a}\frac{f(b)}{b}\\ &+K^{-a}\left(\frac{2\pi}{t}\right)^{b-a} h(a-b)\frac{G(-a)}{a}\frac{f(a-b)}{a-b}\frac{f(a)}{a}\frac{f(-b)}{b}\\ &-K^{-a-b}\left(\frac{2\pi}{t}\right)^{-a-b} h(a+b)\frac{G(-a-b)}{a+b}\frac{f(a+b)}{a+b}\frac{f(a)}{a}\frac{f(b)}{b}.\end{aligned} \quad (48)$$

(i_1,i_2)	c_{i_1,i_2}	$I_{i_1 i_2}(s,t)$
$(1,1)$	$\zeta(1-a)\zeta(1-b)$	$\frac{1}{s^2}G(s)K^s\left(\frac{2\pi}{t}\right)^{a+b+s}\zeta(1-a-b-s)f(s)\frac{1}{\zeta(2-a-b)}$
$(1,2)$	$\zeta(1-a)\zeta(1+b)$	$\frac{1}{s}G(s)K^s\left(\frac{2\pi}{t}\right)^{a+s}\zeta(1-a-s)\zeta(1+b+s)\frac{1}{\zeta(2+b-a)}$
$(2,1)$	$\zeta(1+a)\zeta(1-b)$	$\frac{1}{s}G(s)K^s\left(\frac{2\pi}{t}\right)^{b+s}\zeta(1-b-s)\zeta(1+a+s)\frac{1}{\zeta(2+a-b)}$
$(2,2)$	$\zeta(1+a)\zeta(1+b)$	$-\frac{1}{s^2}G(s)K^s\left(\frac{2\pi}{t}\right)^{s}\zeta(1+a+b+s)f(-s)\frac{1}{\zeta(2+a+b)}$

Table 1. The terms c_{i_1,i_2} and $I_{i_1,i_2}(s,t)$.

Proof. Observe that by Table 1, each of $I_{11}(s,t)$ and $I_{22}(s,t)$ in (47) has

- a double pole at $s=0$,
- a simple pole at $s=-(a+b)$,

whereas $I_{12}(s,t)$ and $I_{21}(s,t)$ in (47) each has

- a simple pole at $s=0$,
- a simple pole at $s=-a$,
- a simple pole at $s=-b$.

We denote by $R_{i_1i_2}(a,b)$ the sum of the residues of $I_{i_1i_2}(s,t)$ at these poles. Moving the contour of integration in (47) to the line $\Re(s)=-\frac{1}{2}+3\delta$ gives

$$\begin{aligned}\mathcal{M}_{1,\mathcal{I},\mathcal{J};\omega}(K)=\int_{-\infty}^{\infty}\omega(t)\big(&\zeta(1-a)\zeta(1-b)R_{11}(a,b)+\zeta(1-a)\zeta(1+b)R_{12}(a,b)\\ &+\zeta(1+a)\zeta(1-b)R_{21}(a,b)+\zeta(1+a)\zeta(1+b)R_{22}(a,b)\big)\,dt\\ &+\sum_{i_1\in\mathcal{K},i_2\in\mathcal{L}}\frac{c_{i_1,i_2}}{2\pi i}\int_{-\infty}^{\infty}\omega(t)\int_{(-1/2+3\delta)}I_{i_1i_2}(s,t)\,ds\,dt. \qquad (49)\end{aligned}$$

We first estimate the second term on the right-hand side, which is equal to

$$\begin{aligned}\sum_{i_1\in\mathcal{K},i_2\in\mathcal{L}}c_{i_1,i_2}\frac{1}{2\pi i}\int_{(-1/2+3\delta)}&\mathcal{A}((\mathcal{I}\setminus\{a_{i_1}\})\cup\{-b_{i_2}-s\},((\mathcal{J}\setminus\{b_{i_2}\})+s)\cup\{-a_{i_1}\})K^s\Phi_2(s)\\ &\times\zeta(1-a_{i_1}-b_{i_2}-s)\prod_{\substack{j_1\in\mathcal{K}\setminus\{i_1\}\\ j_2\in\mathcal{L}\setminus\{i_2\}}}\zeta(1+a_{j_1}+b_{j_2}+s)\int_{-\infty}^{\infty}\left(\frac{t}{2\pi}\right)^{-s-a_{i_1}-b_{i_2}}\omega(t)\,dt\,ds.\end{aligned}$$

By using $|\zeta(\sigma+it)|\ll t^{(1-\sigma)/2}\log t$ for $\sigma\in(0,1)$ and $|\zeta(\sigma+it)|\ll 1$ for $\sigma\in[1.01,2]$, we observe that for $s=-\frac{1}{2}+3\delta+iu$, we have

$$\zeta(1-a_{i_1}-b_{i_2}-s)\prod_{\substack{j_1\in\mathcal{K}\setminus\{i_1\}\\ j_2\in\mathcal{L}\setminus\{i_2\}}}\zeta(1+a_{j_1}+b_{j_2}+s)\ll\big((|u|+1)^{1/4-\delta/2}\log(2+|u|)\big)^{(k-1)(\ell-1)}.$$

We also know by [Hamieh and Ng 2022, Proposition 5.2] that

$$\mathcal{A}((\mathcal{I}\setminus\{a_{i_1}\})\cup\{-b_{i_2}-s\},((\mathcal{J}\setminus\{b_{i_2}\})+s)\cup\{-a_{i_1}\})=O(1)$$

when $\Re(s)\geq -1+2\delta+\epsilon$. It follows that

$$\begin{aligned}\int_{(-1/2+3\delta)}&\mathcal{A}((\mathcal{I}\setminus\{a_{i_1}\})\cup\{-b_{i_2}-s\},((\mathcal{J}\setminus\{b_{i_2}\})+s)\cup\{-a_{i_1}\})K^s\Phi_2(s)\\ &\times\zeta(1-a_{i_1}-b_{i_2}-s)\prod_{\substack{j_1\in\mathcal{K}\setminus\{i_1\}\\ j_2\in\mathcal{L}\setminus\{i_2\}}}\zeta(1+a_{j_1}+b_{j_2}+s)\left(\frac{t}{2\pi}\right)^{-s}ds\ll K^{-1/2+3\delta}t^{1/2-3\delta}.\end{aligned}$$

Therefore,

$$\sum_{i_1\in\mathcal{K},i_2\in\mathcal{L}} c_{i_1,i_2}\frac{1}{2\pi i}\int_{(-1/2+3\delta)} \mathcal{A}((\mathcal{I}\setminus\{a_{i_1}\})\cup\{-b_{i_2}-s\},((\mathcal{J}\setminus\{b_{i_2}\})+s)\cup\{-a_{i_1}\})K^s\Phi_2(s)$$
$$\times\zeta(1-a_{i_1}-b_{i_2}-s)\prod_{\substack{j_1\in\mathcal{K}\setminus\{i_1\}\\ j_2\in\mathcal{L}\setminus\{i_2\}}}\zeta(1+a_{j_1}+b_{j_2}+s)\int_{-\infty}^{\infty}\left(\frac{t}{2\pi}\right)^{-s-a_{i_1}-b_{i_2}} ds\,\omega(t)\,dt$$
$$\ll K^{-1/2+3\delta}\int_{-\infty}^{\infty}\omega(t)t^{1/2-\delta}\,dt \ll K^{-1/2+3\delta}T^{3/2-\delta}. \quad (50)$$

Note that since $K=T^{1+\eta}$, we require $\delta<\eta/2(2+3\eta)$.

Next, we compute the terms $R_{11}(a,b)$, $R_{12}(a,b)$, $R_{21}(a,b)$ and $R_{22}(a,b)$ in (49). We have

$$R_{11}(a,b)=\mathrm{Res}_{s=0}(I_{11}(s))+\mathrm{Res}_{s=-a-b}(I_{11}(s)).$$

For the first residue, we have

$$\mathrm{Res}_{s=0}(I_{11}(s))=\mathrm{Res}_{s=0}\left(\frac{U(s)}{s^2}\right)=U'(0),$$

where

$$U(s)=\left(\frac{2\pi}{t}\right)^{a+b}\frac{1}{\zeta(2-a-b)}\left(\frac{K}{\frac{t}{2\pi}}\right)^{s}\zeta(1-a-b-s)f(s)G(s).$$

Since $X=\log\bigl(K/\frac{t}{2\pi}\bigr)$, we have

$$U'(0)=\frac{\left(\frac{2\pi}{t}\right)^{a+b}}{\zeta(2-a-b)}\bigl(X\zeta(1-a-b)f(0)G(0)-\zeta'(1-a-b)f(0)G(0)+\zeta(1-a-b)f'(0)G(0)$$
$$+\zeta(1-a-b)f(0)G'(0)\bigr).$$

It follows that

$$\mathrm{Res}_{s=0}(I_{11}(s))=U'(0)=\frac{\left(\frac{2\pi}{t}\right)^{a+b}}{\zeta(2-a-b)}\bigl(-\zeta'(1-a-b)+\zeta(1-a-b)(X+g_1+c_1)\bigr).$$

Since $s=-(a+b)$ is a simple pole, we have

$$\mathrm{Res}_{s=-a-b}(I_{11}(s))=-\Phi_2(-a-b)K^{-a-b}\zeta(1-a-b)\frac{1}{\zeta(2-a-b)}.$$

Thus we obtain

$$R_{11}(a,b)=\frac{\left(\frac{2\pi}{t}\right)^{a+b}}{\zeta(2-a-b)}\bigl(-\zeta'(1-a-b)+\zeta(1-a-b)(X+g_1+c_1)\bigr)$$
$$-\Phi_2(-a-b)K^{-a-b}\zeta(1-a-b)\frac{1}{\zeta(2-a-b)}$$
$$=-\left(\frac{2\pi}{t}\right)^{a+b}h(-a-b)\left(\frac{F(-a-b)}{(a+b)^2}+\frac{f(-a-b)}{a+b}(X+g_1+c_1)\right)$$
$$-K^{-a-b}h(-a-b)\frac{G(-a-b)}{a+b}\frac{f(-a-b)}{a+b}. \quad (51)$$

Next for R_{22}, we note that

$$R_{22}(a,b) = \mathrm{Res}_{s=0}(I_{22}(s)) + \mathrm{Res}_{s=-a-b}(I_{22}(s)).$$

Here

$$\mathrm{Res}_{s=0}(I_{22}(s)) = \mathrm{Res}_{s=0}\left(\frac{V(s)}{s^2}\right) = V'(0),$$

where

$$V(s) = -\frac{1}{\zeta(2+a+b)}\left(\frac{K}{\frac{t}{2\pi}}\right)^s \zeta(1+a+b+s) f(-s) G(s).$$

We compute

$$V'(0) = -\frac{1}{\zeta(2+a+b)}\big(X\zeta(1+a+b) f(0)G(0) + \zeta'(1+a+b) f(0)G(0) \\ + \zeta(1+a+b)(-1) f'(0)G(0) + \zeta(1+a+b) f(0)G'(0)\big).$$

It then follows that

$$\mathrm{Res}_{s=0}(I_{22}(s)) = V'(0) = -\frac{1}{\zeta(2+a+b)}\big(\zeta'(1+a+b) + \zeta(1+a+b)(X - g_1 + c_1)\big).$$

For the other residue, since $s = -(a+b)$ is a simple pole we have

$$\mathrm{Res}_{s=-a-b}(I_{22}(s)) = \Phi_2(-a-b)K^{-a-b}\left(\frac{2\pi}{t}\right)^{-a-b}\zeta(1+a+b)\frac{1}{\zeta(2+a+b)}.$$

Hence

$$\begin{aligned} R_{22}(a,b) &= -\frac{1}{\zeta(2+a+b)}\big(\zeta'(1+a+b) + \zeta(1+a+b)(X - g_1 + c_1)\big) \\ &\qquad + \Phi_2(-a-b)K^{-a-b}\left(\frac{2\pi}{t}\right)^{-a-b}\zeta(1+a+b)\frac{1}{\zeta(2+a+b)} \\ &= -h(a+b)\left(\frac{F(a+b)}{(a+b)^2} + \frac{f(a+b)}{a+b}(X - g_1 + c_1)\right) \\ &\qquad + K^{-a-b}\left(\frac{2\pi}{t}\right)^{-a-b} h(a+b)\frac{G(-a-b)}{-a-b}\frac{f(a+b)}{a+b}. \end{aligned} \tag{52}$$

It remains to compute R_{12} and R_{21}. We have

$$R_{12}(a,b) = \mathrm{Res}_{s=0}(I_{12}(s)) + \mathrm{Res}_{s=-a}(I_{12}(s)) + \mathrm{Res}_{s=-b}(I_{12}(s)),$$
$$R_{21}(a,b) = \mathrm{Res}_{s=0}(I_{21}(s)) + \mathrm{Res}_{s=-a}(I_{21}(s)) + \mathrm{Res}_{s=-b}(I_{21}(s)).$$

For $R_{12}(a,b)$, we note that

$$\mathrm{Res}_{s=0}(I_{12}(s)) = \left(\frac{2\pi}{t}\right)^a \zeta(1-a)\zeta(1+b)\frac{1}{\zeta(2+b-a)},$$
$$\mathrm{Res}_{s=-a}(I_{12}(s)) = -\Phi_2(-a)K^{-a}\zeta(1+b-a)\frac{1}{\zeta(2+b-a)},$$
$$\mathrm{Res}_{s=-b}(I_{12}(s)) = \Phi_2(-b)K^{-b}\left(\frac{2\pi}{t}\right)^{a-b}\zeta(1+b-a)\frac{1}{\zeta(2+b-a)},$$

so

$$R_{12}(a,b) = -\left(\frac{2\pi}{t}\right)^{a} h(b-a)\frac{f(-a)}{a}\frac{f(b)}{b} + K^{-a}h(b-a)\frac{G(-a)}{a}\frac{f(b-a)}{b-a} - K^{-b}\left(\frac{2\pi}{t}\right)^{a-b} h(b-a)\frac{G(-b)}{b}\frac{f(b-a)}{b-a}. \quad (53)$$

For $R_{21}(a,b)$, we will use

$$\mathrm{Res}_{s=0}(I_{21}(s)) = \left(\frac{2\pi}{t}\right)^{b}\zeta(1-b)\zeta(1+a)\frac{1}{\zeta(2+a-b)},$$

$$\mathrm{Res}_{s=-a}(I_{21}(s)) = -\Phi_2(-b)K^{-b}\zeta(1+a-b)\frac{1}{\zeta(2+a-b)},$$

$$\mathrm{Res}_{s=-b}(I_{21}(s)) = \Phi_2(-a)K^{-a}\left(\frac{2\pi}{t}\right)^{b-a}\zeta(1-b+a)\frac{1}{\zeta(2+a-b)},$$

and find that

$$R_{21}(a,b) = -\left(\frac{2\pi}{t}\right)^{b} h(a-b)\frac{f(-b)}{b}\frac{f(a)}{a} + K^{-b}h(a-b)\frac{G(-b)}{b}\frac{f(a-b)}{a-b} - K^{-a}\left(\frac{2\pi}{t}\right)^{b-a} h(a-b)\frac{G(-a)}{a}\frac{f(a-b)}{a-b}. \quad (54)$$

Inserting (50)–(54) into (49) yields the desired result. □

Now we will rewrite $\mathcal{R}_2(a,b)$ and simplify its expression. We set

$$\mathcal{L}' = X + g_1 + c_1 \quad \text{and} \quad \mathcal{L}'' = X - g_1 + c_1,$$

and also

$$\begin{aligned}
\kappa_{25}(a,b) &= -E_1(-a-b)h(-a-b)f(-a)f(-b)F(-a-b),\\
\tilde{\kappa}_{25}(a,b) &= E_1(-a-b)h(-a-b)f(-a)f(-b)f(-a-b),\\
\kappa_{26}(a,b) &= E_1(-a)h(b-a)f(-a)^2 f(b)^2,\\
\kappa_{27}(a,b) &= E_1(-b)h(a-b)f(a)^2 f(-b)^2,\\
\kappa_{28}(a,b) &= h(a+b)f(a)f(b)((a+b)f'(a+b) - f(a+b)),\\
\tilde{\kappa}_{28}(a,b) &= h(a+b)f(a)f(b)f(a+b),\\
\kappa_{29}(a,b) &= E_2(-b)E_1(b-a)h(b-a)G(-b)f(-a)f(b)f(b-a),\\
\kappa_{210}(a,b) &= E_2(-a)E_1(a-b)h(a-b)G(-a)f(a)f(-b)f(a-b),\\
\kappa_{211}(a,b) &= E_2(-a-b)E_1(a+b)h(a+b)f(a)f(b)G(-a-b)f(a+b).
\end{aligned} \quad (55)$$

With this notation and by (48), we can write

$$\mathcal{R}_2(a,b) = \big(\mathcal{R}_{25} + \mathcal{R}_{26} + \mathcal{R}_{27} + \mathcal{R}_{28} + \mathcal{R}_{29} + \mathcal{R}_{210} + \mathcal{R}_{211}\big)(a,b),$$

where we set

$$
\begin{aligned}
\mathcal{R}_{25}(a,b) &= \frac{1}{ab(a+b)^2}\kappa_{25}(a,b) - \frac{1}{ab(a+b)}\tilde{\kappa}_{25}(a,b)\mathcal{L}', \\
\mathcal{R}_{26}(a,b) &= \frac{1}{(ab)^2}\kappa_{26}(a,b), \\
\mathcal{R}_{27}(a,b) &= \frac{1}{(ab)^2}\kappa_{27}(a,b), \\
\mathcal{R}_{28}(a,b) &= -\frac{1}{ab(a+b)^2}\kappa_{28}(a,b) - \frac{1}{ab(a+b)}\tilde{\kappa}_{28}(a,b)\mathcal{L}'', \\
\mathcal{R}_{29}(a,b) &= \frac{1}{ab^2(b-a)}\kappa_{29}(a,b), \\
\mathcal{R}_{210}(a,b) &= \frac{1}{a^2b(a-b)}\kappa_{210}(a,b), \\
\mathcal{R}_{211}(a,b) &= -\frac{1}{ab(a+b)^2}\kappa_{211}(a,b).
\end{aligned}
\tag{56}
$$

By Theorem 3.2 and Propositions 4.1 and 4.2 we arrive at the following proposition.

Proposition 4.3. *Let* $\mathcal{I} = \{a, 0\}$ *and* $\mathcal{J} = \{b, 0\}$. *Then*

$$
\mathscr{D}_{\mathcal{I},\mathcal{J};\omega}(K) = \int_{-\infty}^{\infty} \omega(t)\cdot \mathcal{R}(a,b)\,dt + O\left(T^{3(1+\eta)/4+\varepsilon}\left(\frac{T}{T_0}\right)^{9/4} + T^{1-\eta/2}\right),
$$

where

$$
\begin{aligned}
\mathcal{R}(a,b) = \frac{1}{ab}\Bigg(&\frac{1}{(a+b)}(L+2g_1)\kappa_{11}(a,b) + 2\frac{1}{(a+b)}\tilde{\kappa}_{11}(a,b) + \frac{1}{a(a+b)}\kappa_{13}(a,b) \\
&+ \frac{1}{b(a+b)}\kappa_{14}(a,b) + \frac{1}{(a+b)^2}\kappa_{25}(a,b) - \frac{1}{(a+b)}\tilde{\kappa}_{25}(a,b)\mathcal{L}' + \frac{1}{ab}\kappa_{26}(a,b) \\
&+ \frac{1}{ab}\kappa_{27}(a,b) + \frac{1}{b(b-a)}\kappa_{29}(a,b) + \frac{1}{a(a-b)}\kappa_{210}(a,b) - \frac{1}{(a+b)^2}\kappa_{211}(a,b)\Bigg).
\end{aligned}
$$

Proof. We have $\mathcal{R}(a,b) = \mathcal{R}_1(a,b) + \mathcal{R}_2(a,b)$. The result follows from (46), (56), and the observations that

$$
\kappa_{12}(a,b) = \kappa_{28}(a,b), \quad \kappa_{11}(a,b) = \tilde{\kappa}_{28}(a,b) \quad \text{and} \quad \mathcal{L}_0 - \mathcal{L}'' = \log\frac{t}{2\pi} + 2g_1 = L + 2g_1. \qquad \square
$$

4.3. *Computing* $\lim_{a,b\to 0}\mathcal{R}(a,b)$. Our goal is now reduced to computing the limit of $\mathcal{R}(a,b)$ as $a, b \to 0$. To this end, we write down the Taylor series expansions of the entire functions κ_{1*}, κ_{2*} and $\tilde{\kappa}_{2*}$ using (21) and (55), and then we combine the terms with similar coefficients to obtain the expression

$$
\mathcal{R}(a,b) = \big(A_1 + \tilde{A}_1 + A_2 + A_3 + A_4 + A_5 + A_6\big)(a,b), \tag{57}
$$

where the functions A_1, $\tilde{A}_1$, A_2, A_3, A_4, A_5, and A_6 are given as follows:

$$\begin{aligned}
A_1(a,b) &= \frac{1}{ab(a+b)}(L+2g_1)\kappa_{11}(a,b) + \frac{1}{a^2b(a+b)}\kappa_{13}(a,b) + \frac{1}{ab^2(a+b)}\kappa_{14}(a,b) \\
&= \frac{1}{ab(a+b)} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (g\star\delta)_{j_3} (L+2g_1) a^{j_1} b^{j_2} (a+b)^{j_3} \\
&\quad + \frac{1}{a^2b(a+b)} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (g\star\delta)_{j_3} (j_1-1) a^{j_1} b^{j_2} (a+b)^{j_3} \\
&\quad + \frac{1}{ab^2(a+b)} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (g\star\delta)_{j_3} (j_2-1) a^{j_1} b^{j_2} (a+b)^{j_3}, \\
\tilde{A}_1(a,b) &= \frac{2}{ab(a+b)}\tilde{\kappa}_{11}(a,b) = \frac{2}{ab(a+b)} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (g\star\delta')_{j_3} a^{j_1} b^{j_2} (a+b)^{j_3}, \\
A_2(a,b) &= \frac{1}{(ab)^2}\kappa_{26}(a,b) + \frac{1}{(ab)^2}\kappa_{27}(a,b) \\
&= \frac{1}{(ab)^2} \sum_{j_1,j_2,j_3} (-1)^{j_1} (\alpha * g * g)_{j_1} (g*g)_{j_2} \delta_{j_3} \{a^{j_1} b^{j_2} + (-1)^{j_3} a^{j_2} b^{j_1}\} (b-a)^{j_3}, \\
A_3(a,b) &= \frac{1}{ab^2(b-a)}\kappa_{29}(a,b) + \frac{1}{a^2b(a-b)}\kappa_{210}(a,b) \\
&= \frac{1}{ab^2(b-a)} \sum_{j_1,j_2,j_3} g_{j_1} (c\star\beta\star(-1)^{\bullet} g)_{j_2} (g\star\alpha\star\delta)_{j_3} (-1)^{j_1+j_2} a^{j_1} b^{j_2} (b-a)^{j_3} \\
&\quad - \frac{1}{a^2b(b-a)} \sum_{j_1,j_2,j_3} g_{j_1} (c\star\beta\star(-1)^{\bullet} g)_{j_2} (g\star\alpha\star\delta)_{j_3} (-1)^{j_1+j_2} (-1)^{j_3} a^{j_2} b^{j_1} (b-a)^{j_3}, \\
A_4(a,b) &= -\frac{1}{ab(a+b)^2}\kappa_{211}(a,b) \\
&= -\frac{1}{ab(a+b)^2} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (g\star\delta\star\alpha\star(-1)^{\bullet} c\star(-1)^{\bullet}\beta)_{j_3} a^{j_1} b^{j_2} (a+b)^{j_3}, \\
A_5(a,b) &= \frac{1}{ab(a+b)^2}\kappa_{25}(a,b) \\
&= -\frac{1}{ab(a+b)^2} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (\alpha\star g'\star\delta)_{j_3} (-1)^{j_1+j_2+j_3} a^{j_1} b^{j_2} (a+b)^{j_3}, \\
A_6(a,b) &= -\frac{\mathcal{L}'}{ab(a+b)}\tilde{\kappa}_{25}(a,b) \\
&= -\frac{\mathcal{L}'}{ab(a+b)} \sum_{j_1,j_2,j_3} g_{j_1} g_{j_2} (g\star\alpha\star\delta)_{j_3} (-1)^{j_1+j_2+j_3} a^{j_1} b^{j_2} (a+b)^{j_3}.
\end{aligned}$$

We will first compute $\lim_{b\to a}\mathcal{R}(a,b)$ and then use Maple to find $\lim_{a\to 0}(\lim_{b\to a}\mathcal{R}(a,b))$. It is straightforward to see that

$$\begin{aligned}
\lim_{b\to a} A_1(a,b) &= \frac{1}{2a^3}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(g\star\delta)_{j_3}(L+2g_1)2^{j_3}a^{j_1+j_2+j_3}\\
&\quad+\frac{1}{2a^4}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(g\star\delta)_{j_3}(j_1-1)2^{j_3}a^{j_1+j_2+j_3}\\
&\quad+\frac{1}{2a^4}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(g\star\delta)_{j_3}(j_2-1)2^{j_3}a^{j_1+j_2+j_3},\\
\lim_{b\to a}\tilde{A}_1(a,b) &= \frac{1}{a^3}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(g\star\delta')_{j_3}2^{j_3}a^{j_1+j_2+j_3},\\
\lim_{b\to a} A_2(a,b) &= \frac{2\delta_0}{a^4}\sum_{j_1,j_2}(-1)^{j_1}(\alpha*g*g)_{j_1}(g*g)_{j_2}a^{j_1+j_2},\\
\lim_{b\to a} A_4(a,b) &= -\frac{1}{4a^4}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(g\star\delta\star\alpha\star(-1)^{\bullet}c\star(-1)^{\bullet}\beta)_{j_3}2^{j_3}a^{j_1+j_2+j_3},\\
\lim_{b\to a} A_5(a,b) &= -\frac{1}{4a^4}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(\alpha\star g'\star\delta)_{j_3}(-1)^{j_1+j_2+j_3}2^{j_3}a^{j_1+j_2+j_3},\\
\lim_{b\to a} A_6(a,b) &= -\frac{\mathcal{L}'}{2a^3}\sum_{j_1,j_2,j_3} g_{j_1}g_{j_2}(g\star\alpha\star\delta)_{j_3}(-1)^{j_1+j_2+j_3}2^{j_3}a^{j_1+j_2+j_3}.
\end{aligned} \tag{58}$$

It remains to compute $\lim_{b\to a}A_3(a,b)$. We have

$$\begin{aligned}
A_3(a,b) &= \frac{1}{ab^2(b-a)}\sum_{j_1,j_2} g_{j_1}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_1+j_2}a^{j_1}b^{j_2}\\
&\qquad\times\Big((g\star\alpha\star\delta)_0+(g\star\alpha\star\delta)_1(b-a)+\sum_{j_3\geq 2}(g\star\alpha\star\delta)_{j_3}(b-a)^{j_3}\Big)\\
&\quad-\frac{1}{a^2b(b-a)}\sum_{j_1,j_2} g_{j_1}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_1+j_2}a^{j_2}b^{j_1}\\
&\qquad\times\Big((g\star\alpha\star\delta)_0-(g\star\alpha\star\delta)_1(b-a)+\sum_{j_3\geq 2}(g\star\alpha\star\delta)_{j_3}(-1)^{j_3}(b-a)^{j_3}\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{b\to a}A_3(a,b) &= (g\star\alpha\star\delta)_0\lim_{b\to a}\Big\{\frac{a}{a^2b^2(b-a)}\sum_{j_1} g_{j_1}(-1)^{j_1}a^{j_1}\sum_{j_2}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_2}b^{j_2}\\
&\qquad-\frac{b}{a^2b^2(b-a)}\sum_{j_1} g_{j_1}(-1)^{j_1}b^{j_1}\sum_{j_2}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_2}a^{j_2}\Big\}\\
&\quad+\frac{(g\star\alpha\star\delta)_1}{a^3}\sum_{j_1} g_{j_1}(-1)^{j_1}a^{j_1}\sum_{j_2}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_2}a^{j_2}.
\end{aligned} \tag{59}$$

At this point, we need the following lemma to simplify the limit on the right-hand side of (59).

Lemma 4.4. *Let f_1 and f_2 be entire functions. Consider*

$$F(z_1, z_2) := \frac{f_1(z_1)f_2(z_2) - f_1(z_2)f_2(z_1)}{z_1 - z_2}.$$

Then

$$\lim_{b\to a} F(a, b) = f_1'(a)f_2(a) - f_1(a)f_2'(a).$$

Proof. Note that if $a \neq b$, then

$$F(a, b) = \frac{(f_1(a) - f_1(b))f_2(a)}{a - b} - \frac{f_1(a)(f_2(a) - f_2(b))}{a - b}.$$

As $b \to a$, we obtain $F(a, b) \to f_1'(a)f_2(a) - f_1(a)f_2'(a)$. □

We may apply this lemma for

$$f_1(z) = z\sum_{j_1} g_{j_1}(-1)^{j_1}z^{j_1} \quad \text{and} \quad f_2(z) = \sum_{j_2}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_2}z^{j_2}.$$

These are both entire functions since $f_1(z) = -z^2\zeta(1-z)$ and $f_2(z) = zG(-z)e^{-zY}\zeta(1+z)$. When we apply the lemma in this setting, (59) becomes

$$\begin{aligned}
\lim_{b\to a} A_3 &= (g\star\alpha\star\delta)_0 \lim_{b\to a}\left(\frac{1}{a^2b^2(b-a)}f_1(a)f_2(b) - \frac{1}{a^2b^2(b-a)}f_1(b)f_2(a)\right) + \frac{(g\star\alpha\star\delta)_1}{a^4}f_1(a)f_2(a) \\
&= \frac{(g\star\alpha\star\delta)_0}{a^4}\big(f_1'(a)f_2(a) - f_1(a)f_2'(a)\big) + \frac{(g\star\alpha\star\delta)_1}{a^4}f_1(a)f_2(a) \\
&= \frac{(g\star\alpha\star\delta)_0}{a^4}\sum_{j_1,j_2} g_{j_1}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(j_2 - j_1 - 1)(-1)^{j_1+j_2}a^{j_1+j_2} \\
&\qquad + \frac{2(g\star\alpha\star\delta)_1}{a^3}\sum_{j_1,j_2} g_{j_1}(c\star\beta\star(-1)^{\bullet}g)_{j_2}(-1)^{j_1+j_2}a^{j_1+j_2}.
\end{aligned}$$

Then upon adding the right-hand sides of (58) to the right-hand side of the last equation above, we obtain

$$\begin{aligned}
\mathcal{R}(a, a) &= \lim_{b\to a}\mathcal{R}(a, b) \\
&= \frac{1}{a^4}\left(\sum_{j_1,j_2,j_3} C_1(j_1, j_2, j_3)a^{j_1+j_2+j_3} + \sum_{j_1,j_2} C_2(j_1, j_2)a^{j_1+j_2}\right) \\
&\qquad + \frac{1}{a^3}\left(\sum_{j_1,j_2,j_3} D_1(j_1, j_2, j_3)a^{j_1+j_2+j_3} + \sum_{j_1,j_2} D_2(j_1, j_2)a^{j_1+j_2}\right),
\end{aligned}$$

where

$$\begin{aligned}
C_1(j_1, j_2, j_3) &= \tfrac{1}{2}(j_1 + j_2 - 2)2^{j_3}g_{j_1}g_{j_2}(g\star\delta)_{j_3} - \tfrac{1}{4}2^{j_3}g_{j_1}g_{j_2}(g\star\delta\star\alpha\star(-1)^{\bullet}c\star(-1)^{\bullet}\beta)_{j_3} \\
&\qquad - \tfrac{1}{4}(-1)^{j_1+j_2+j_3}2^{j_3}g_{j_1}g_{j_2}(\alpha\star g'\star\delta)_{j_3},
\end{aligned}$$

$$C_2(j_1,j_2)=2\delta_0(-1)^{j_1}(\alpha*g*g)_{j_1}(g*g)_{j_2}+(-1)^{j_1+j_2}(j_2-j_1-1)(g\star\alpha\star\delta)_0 g_{j_1}(c\star\beta\star(-1)^{\bullet}g)_{j_2},$$
$$D_1(j_1,j_2,j_3)=\tfrac{1}{2}(L+2g_1)2^{j_3}g_{j_1}g_{j_2}(g\star\delta)_{j_3}+2^{j_3}g_{j_1}g_{j_2}(g\star\delta')_{j_3}-\tfrac{1}{2}\mathcal{L}'(-1)^{j_1+j_2+j_3}2^{j_3}g_{j_1}g_{j_2}(g\star\alpha\star\delta)_{j_3},$$
$$D_2(j_1,j_2)=2(-1)^{j_1+j_2}(g\star\alpha\star\delta)_1 g_{j_1}(c\star\beta\star(-1)^{\bullet}g)_{j_2}.$$

Hence
$$\mathcal{R}(a,a)=\frac{1}{a^4}\sum_{j=0}^{\infty}C(j)a^j,$$
where
$$C(0)=C_1(0,0,0)+C_2(0,0),$$
and for $j\in\mathbb{N}$ we have
$$C(j)=\sum_{\substack{j_1,j_2,j_3\\ j_1+j_2+j_3=j-1}}D_1(j_1,j_2,j_3)+\sum_{\substack{j_1,j_2\\ j_1+j_2=j-1}}D_2(j_1,j_2)+\sum_{\substack{j_1,j_2,j_3\\ j_1+j_2+j_3=j}}C_1(j_1,j_2,j_3)+\sum_{\substack{j_1,j_2\\ j_1+j_2=j}}C_2(j_1,j_2).$$
Using Maple we show that $C(j)=0$ for $j=0,1,2,3$, and we compute

$$\begin{aligned}
\mathcal{R}(0,0)&=C(4)\\
&=-\tfrac{7}{12}\delta_0L^4-\delta_0L^2Y^2+\tfrac{4}{3}\delta_0L^3Y+\tfrac{1}{3}\delta_0LY^3-\tfrac{1}{24}\delta_0Y^4+\big(-2\delta_1+\tfrac{4}{3}\delta_0c_1-4g_1\delta_0\big)L^3\\
&\quad+\big(\tfrac{1}{3}\delta_1-\tfrac{1}{6}\delta_0c_1+\tfrac{2}{3}g_1\delta_0\big)Y^3+(-2\delta_0c_1+4\delta_1+8g_1\delta_0)L^2Y+(-2\delta_1-4g_1\delta_0+\delta_0c_1)LY^2\\
&\quad+\big(8g_1c_1\delta_0-8g_1\delta_1-4\delta_2-5g_1^2\delta_0+4c_1\delta_1-2\delta_0c_2-6g_2\delta_0\big)L^2\\
&\quad+\big(-8g_1c_1\delta_0+12g_1^2\delta_0-4c_1\delta_1+2\delta_0c_2+8\delta_0g_2+16g_1\delta_1+8\delta_2\big)LY\\
&\quad+\big(\delta_1c_1+2g_1c_1\delta_0-2\delta_2-3g_1^2\delta_0-\tfrac{1}{2}\delta_0c_2-2\delta_0g_2-4g_1\delta_1\big)Y^2\\
&\quad+\big(12g_1^2\delta_0c_1+8c_1\delta_0g_2+16g_1c_1\delta_1-8g_1c_2\delta_0-4\delta_0g_1g_2-8\delta_1g_2-12\delta_0g_3-4c_2\delta_1\\
&\qquad\qquad+2\delta_0c_3+8\delta_2c_1+4\delta_0g_1^3+4g_1^2\delta_1\big)L\\
&\quad+\big(-\delta_0c_3-6g_1^2\delta_0c_1-4c_1\delta_0g_2-8g_1c_1\delta_1+4g_1\delta_0c_2+12\delta_0g_1g_2+8\delta_3+12g_1^2\delta_1\\
&\qquad\qquad+4\delta_0g_1^3-4c_1\delta_2+2\delta_1c_2+4\delta_0g_3+16g_1\delta_2+8\delta_1g_2\big)Y\\
&\quad+16\delta_4-16\delta_1g_3+32\delta_3g_1+32g_1^2\delta_2-24\delta_0g_4+8g_2^2\delta_0+5\delta_0g_1^4+16\delta_1g_1^3-8\delta_0g_1g_3\\
&\quad+16\delta_1g_1g_2+12\delta_0g_1^2g_2+12g_1^2\delta_1c_1+12\delta_0g_1g_2c_1+8\delta_3c_1+4\delta_0g_1^3c_1+4\delta_0g_3c_1\\
&\quad+8\delta_1g_2c_1+16g_1\delta_2c_1-4\delta_2c_2-6g_1^2\delta_0c_2-4\delta_0g_2c_2-8g_1\delta_1c_2+4g_1\delta_0c_3+2\delta_1c_3-\delta_0c_4.
\end{aligned}$$

Note that the above expression is a polynomial in Y and L. By collecting terms of the same degree in $\mathbb{R}[Y,L]$ and noting from (21) that $Y=\log K$ and $L=\log\frac{t}{2\pi}$, we find that
$$\mathcal{R}(0,0)=\sum_{j=0}^{4}Q_j(Y,L),$$
where the polynomials Q_j are defined within the statement of Theorem 1.2.

Appendix A: Proof of Corollary 1.3

Let $r(t) = \mathbb{1}_{[T,2T]}(t)$ and choose smooth functions $\omega^+(t)$ and $\omega^-(t)$ which satisfy

$$\omega^-(t) \le r(t) \le \omega^+(t),$$

where

$$\omega^+(t) = \begin{cases} 0 & \text{if } t < T - T_0 \text{ or } t > 2T + T_0, \\ 1 & \text{if } T + T_0 \le t \le 2T - T_0, \end{cases}$$

and also

$$(\omega^\pm)^{(j)} \ll T_0^{-j}.$$

Note that

$$\mathscr{D}_{2,2;\omega^-}(K) \le \mathscr{D}_{2,2;r}(K) \le \mathscr{D}_{2,2;\omega^+}(K), \tag{60}$$

where we let

$$\mathscr{D}_{2,2;\omega^\pm}(K) = \sum_{j=0}^{4} \int_{-\infty}^{\infty} \omega_\pm(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt + O\left(T^{3(1+\eta)/4+\varepsilon}\left(\frac{T}{T_0}\right)^{9/4}\right) + O(T^{1-\eta/2}).$$

It follows from the above that

$$\sum_{j=0}^{4}\left\{\int_{-\infty}^{\infty} \omega_+(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt - \int_{-\infty}^{\infty} r(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt\right\}$$
$$= \sum_{j=0}^{4}\left\{\int_{T-T_0}^{T} + \int_{2T}^{2T+T_0}\right\} \omega_+(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt \ll T_0 (\log T)^4.$$

Note that a similar argument establishes the same bound when ω^+ is replaced by ω^-. Thus by (60) we have

$$\mathscr{D}_{2,2;r}(K) = \sum_{j=0}^{4} \int_{-\infty}^{\infty} r(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt + O\left(T^{3(1+\eta)/4+\varepsilon}\left(\frac{T}{T_0}\right)^{9/4}\right)$$
$$+ O(T^{1-\eta/2}) + O(T_0(\log T)^4).$$

We then select $T_0 = T^{(12+3\eta)/13}$ so that the first and the third error terms are equal, and obtain

$$\mathscr{D}_{2,2;r}(K) = \sum_{j=0}^{4} \int_{-\infty}^{\infty} r(t) Q_j\left(\log K, \log \frac{t}{2\pi}\right) dt + O(T^{\max\{(12+3\eta)/13, 1-\eta/2\}}).$$

Appendix B: Computation of the coefficients in Theorem 1.2

In this section, we rewrite the expressions for $Q_0(x, y)$, $Q_1(x, y)$, $Q_2(x, y)$, and $Q_3(x, y)$ that appear in Theorem 1.2 by using the definitions of g_j and δ_j in terms of γ_{j-1} and $\zeta^{(j)}(2)$ as described in (18) and (20). Note that $c_0 = 1$ and the rest of the coefficients c_j that appear in Theorem 1.2 depend on the smoothing function φ.

Using Maple we compute the following expressions for Q_3, Q_2, Q_1 and Q_0:

$$Q_3(x,y)=\left(\frac{4\gamma}{\pi^2}-\frac{12\zeta'(2)}{\pi^4}-\frac{c_1}{\pi^2}\right)x^3+\left(\frac{6c_1}{\pi^2}-\frac{24\gamma}{\pi^2}+\frac{72\zeta'(2)}{\pi^4}\right)x^2y+\left(-\frac{12c_1}{\pi^2}+\frac{48\gamma}{\pi^2}-\frac{144\zeta'(2)}{\pi^4}\right)x\,y^2 +\left(-\frac{24\gamma}{\pi^2}+\frac{72\zeta'(2)}{\pi^4}+\frac{8c_1}{\pi^2}\right)y^3,$$

$$\begin{aligned}Q_2(x,y)=&\left(\frac{12\gamma_1}{\pi^2}-\frac{18\gamma^2}{\pi^2}+\frac{144\zeta'(2)\gamma}{\pi^4}-\frac{432\zeta'(2)^2}{\pi^6}+\frac{36\zeta''(2)}{\pi^4}+\frac{12c_1\gamma}{\pi^2}-\frac{36\zeta'(2)c_1}{\pi^4}-\frac{3c_2}{\pi^2}\right)x^2\\&+\left(-\frac{48c_1\gamma}{\pi^2}+\frac{72\gamma^2}{\pi^2}+\frac{144\zeta'(2)c_1}{\pi^4}+\frac{12c_2}{\pi^2}-\frac{48\gamma_1}{\pi^2}-\frac{576\zeta'(2)\gamma}{\pi^4}+\frac{1728\zeta'(2)^2}{\pi^6}-\frac{144\zeta''(2)}{\pi^4}\right)xy\\&+\left(\frac{48c_1\gamma}{\pi^2}-\frac{30\gamma^2}{\pi^2}-\frac{144\zeta'(2)c_1}{\pi^4}-\frac{12c_2}{\pi^2}+\frac{36\gamma_1}{\pi^2}+\frac{288\zeta'(2)\gamma}{\pi^4}-\frac{864\zeta'(2)^2}{\pi^6}+\frac{72\zeta''(2)}{\pi^4}\right)y^2,\end{aligned}$$

$$\begin{aligned}Q_1(x,y)=&\left(-\frac{36c_1\gamma^2}{\pi^2}+\frac{24\gamma^3}{\pi^2}+\frac{24c_1\gamma_1}{\pi^2}+\frac{288c_1\zeta'(2)\gamma}{\pi^4}+\frac{24c_2\gamma}{\pi^2}-\frac{72\gamma\gamma_1}{\pi^2}-\frac{432\zeta'(2)\gamma^2}{\pi^4}\right.\\&\qquad-4c_1\left(\frac{216\zeta'(2)^2}{\pi^6}-\frac{18\zeta''(2)}{\pi^4}\right)-\frac{72c_2\zeta'(2)}{\pi^4}-\frac{6c_3}{\pi^2}+\frac{12\gamma_2}{\pi^2}+\frac{288\zeta'(2)\gamma_1}{\pi^4}\\&\qquad\left.+16\left(\frac{216\zeta'(2)^2}{\pi^6}-\frac{18\zeta''(2)}{\pi^4}\right)\gamma-\frac{10368\zeta'(2)^3}{\pi^8}+\frac{1728\zeta'(2)\zeta''(2)}{\pi^6}-\frac{48\zeta'''(2)}{\pi^4}\right)x\\&+\left(\frac{72c_1\gamma^2}{\pi^2}+\frac{24\gamma^3}{\pi^2}-\frac{48c_1\gamma_1}{\pi^2}-\frac{576c_1\zeta'(2)\gamma}{\pi^4}-\frac{48c_2\gamma}{\pi^2}+\frac{24\gamma\gamma_1}{\pi^2}-\frac{144\zeta'(2)\gamma^2}{\pi^4}\right.\\&\qquad\left.+8c_1\left(\frac{216\zeta'(2)^2}{\pi^6}-\frac{18\zeta''(2)}{\pi^4}\right)+\frac{144c_2\zeta'(2)}{\pi^4}+\frac{12c_3}{\pi^2}-\frac{36\gamma_2}{\pi^2}-\frac{288\zeta'(2)\gamma_1}{\pi^4}\right)y,\end{aligned}$$

$$\begin{aligned}Q_0(x,y)=&-\frac{31104\zeta'(2)^2\zeta''(2)}{\pi^8}+\frac{1152\zeta'(2)\zeta'''(2)}{\pi^6}-\frac{72\zeta'(2)c_3}{\pi^4}-\frac{6c_4}{\pi^2}\\&+8\left(-\frac{1296\zeta'(2)^3}{\pi^8}+\frac{216\zeta'(2)\zeta''(2)}{\pi^6}-\frac{6\zeta'''(2)}{\pi^4}\right)c_1-4\left(\frac{216\zeta'(2)^2}{\pi^6}-\frac{18\zeta''(2)}{\pi^4}\right)c_2-\frac{24\zeta^{(4)}(2)}{\pi^4}\\&+\frac{864\zeta''(2)^2}{\pi^6}+\frac{124416\zeta'(2)^4}{\pi^{10}}+\frac{24\gamma_3}{\pi^2}+\frac{30\gamma^4}{\pi^2}+\frac{48\gamma_1^2}{\pi^2}+\frac{24\gamma_1c_2}{\pi^2}-\frac{36\gamma^2c_2}{\pi^2}-\frac{72\gamma^2\gamma_1}{\pi^2}\\&+32\left(-\frac{1296\zeta'(2)^3}{\pi^8}+\frac{216\zeta'(2)\zeta''(2)}{\pi^6}-\frac{6\zeta'''(2)}{\pi^4}\right)\gamma+32\gamma^2\left(\frac{216\zeta'(2)^2}{\pi^6}-\frac{18\zeta''(2)}{\pi^4}\right)\\&+16\gamma\left(\frac{216\zeta'(2)^2}{\pi^6}-\frac{18\zeta''(2)}{\pi^4}\right)c_1+\frac{288\zeta'(2)\gamma_2}{\pi^4}+\frac{12\gamma_2c_1}{\pi^2}+\frac{24\gamma c_3}{\pi^2}+\frac{24\gamma^3c_1}{\pi^2}\\&-\frac{24\gamma\gamma_2}{\pi^2}-\frac{576\zeta'(2)\gamma^3}{\pi^4}+\frac{288\zeta'(2)\gamma_1c_1}{\pi^4}+\frac{288\gamma\zeta'(2)c_2}{\pi^4}+\frac{576\zeta'(2)\gamma\gamma_1}{\pi^4}\\&-\frac{72\gamma\gamma_1c_1}{\pi^2}-\frac{432\gamma^2\zeta'(2)c_1}{\pi^4}.\end{aligned}$$

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