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Weyl sums with multiplicative coefficients and joint equidistribution

Matteo Bordignon, Cynthia Bortolotto and Bryce Kerr

We generalise a result of Montgomery and Vaughan regarding exponential sums with multiplicative coefficients to the setting of Weyl sums. As applications, we establish a joint equidistribution result for roots of polynomial congruences and polynomial values which generalises a result of Hooley. We also obtain some new results for mixed character sums.

1. Introduction

Let $A \geqslant 1$, and let f be a multiplicative function satisfying $|f(p)| \leqslant A$ for any prime p and also $\sum_{n \le N} |f(n)|^2 \leqslant A^2 N$ for all natural numbers N. For $\alpha \in \mathbb{R}$, set

$$S(\alpha) := \sum_{1 \leqslant n \leqslant N} f(n) e(\alpha n),$$

where $e(x) = \exp(2\pi i x)$.

These sums appear to be considered first by Daboussi [1975], who showed that if $|\alpha - a/q| \leqslant 1/q^2$, $(a, q) = 1$ and $3 \leqslant q \leqslant (N/\log N)^{1/2}$, then

$$S(\alpha) \ll \frac{N}{(\log\log q)^{1/2}},$$

with the implied constant depending only on A.

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This result was improved by Montgomery and Vaughan [1977, Corollary 1] who show that, assuming $|\alpha - a/q| \leqslant 1/q^2$, $(a,q)=1$ and $2 \leqslant R \leqslant q \leqslant N/R$, we have

$$S(\alpha) \ll \frac{N}{\log N} + \frac{N(\log R)^{3/2}}{R^{1/2}}. \tag{1}$$

We refer the reader to [Montgomery and Vaughan 1977, Section 7] for a demonstration that the term $N/\log N$ is sharp.

The optimal dependence on R in (1) is an open problem and has been the subject of several works (see for example [Bachman 2003; de la Bretèche 1998]), and it is expected the estimate (1) may be improved to

$$S(\alpha) \ll \frac{N}{\log N} + \frac{N}{R^{1/2}}. \tag{2}$$

Recently, de la Bretèche and Granville [2022] have studied in detail the sums $S(\alpha)$ on major arcs. Their estimates suggest the following conjecture (see [de la Bretèche and Granville 2022, Equation 1.4]):

$$S(\alpha) \ll \frac{N}{\log N} + \frac{N}{q^{1/2}(1+|\beta|x)}, \quad \alpha = \frac{a}{q} + \beta, \quad (a,q)=1.$$

We also note that [de la Bretèche and Granville 2022] contains some nice applications to circle method-type problems.

The estimate (1) has important applications to Dirichlet L-functions. Montgomery and Vaughan [1977] have shown how it may be combined with the generalised Riemann hypothesis (GRH) to obtain a sharp upper bound for Dirichlet L-functions at the point $s=1$. One may also combine (1) with estimates for short character sums to obtain unconditional variants of Montgomery and Vaughan's result. We refer the reader to [Granville and Soundararajan 2007; Hildebrand 1988b; 1988a] for progress in this direction.

Since the work of Montgomery and Vaughan, exponential sums with multiplicative coefficients have appeared in a number of different contexts, and a variety of techniques have been developed to facilitate the reduction to bilinear forms. Some examples include work of Karatsuba [2010] on short Kloosterman sums, which has been refined by Korolëv; see for example [Korolëv 2018]. Kátai [1986] and Indlekofer and Kátai [1989] have considered more general functions in the exponential factor. Bourgain, Sarnak and Ziegler [Bourgain et al. 2013] have established a finite version of Vinogradov's bilinear sum inequality. Gong and Jia [2015] have considered shifted character sums with multiplicative coefficients and Korolëv and Shparlinski [2020] dealt with sums over trace functions with multiplicative coefficients.

In this paper, we revisit the approach of Montgomery and Vaughan and generalise it to the setting of sums of the form

$$\sum_{n \leqslant N} f(n) e(F(n)), \tag{3}$$

where F is a polynomial with real coefficients and f is a multiplicative function satisfying

$$f(p) = O(1), \quad \sum_{n \leqslant N} |f(n)| = O(N), \quad \sum_{n \leqslant N} |f(n)|^2 = O(N(\log N)^A)$$

for any $A \geq 0$.

Problems of this sort have previously been considered by Jiang, Lü and Wang [Jiang et al. 2021], who showed that one may replace an assumption on the ℓ_2 norm,

$$\sum_{n \leqslant N} |f(n)|^2 \ll N,$$

with an assumption of the form

$$\sum_{\substack{p \leqslant N \\ p \text{ prime}}} |f(p)||f(p+h)| \ll \frac{h}{\phi(h)} \frac{N}{(\log N)^2}.$$

Such a relaxation is significant in the context of GL_m L-functions in the absence of progress towards the Ramanujan conjectures.

Matthiesen [2018] considered sums of the form (3) over polynomial nilsequences with slightly weaker conditions on f than Montgomery and Vaughan. These results were later applied to linear correlations of multiplicative functions [Matthiesen 2020]. We also mention [Matomäki et al. 2023], which considers exponential sums with multiplicative functions over nilsequences on average over short intervals.

2. Main results

Theorem 1. *Let $A > 0$ and $c > 0$ be real numbers and f be a multiplicative function satisfying*

$$|f(p)| \leqslant C \quad \text{for each prime } p, \tag{4}$$

$$\sum_{n \leqslant N} |f(n)| = O(N), \tag{5}$$

$$\sum_{n \leqslant N} |f(n)|^2 = O(N(\log N)^A). \tag{6}$$

Let F be a polynomial of degree $d \geqslant 1$ with real coefficients given by

$$F(x) = \alpha_d x^d + \cdots + \alpha_1 x.$$

Let $R \geqslant 1$, and suppose there exist integers ℓ, a and q, with $1 \leqslant q \leqslant R$, $1 \leqslant \ell \leqslant d$, $(a,q) = 1$ and

$$\left|\alpha_\ell - \frac{a}{q}\right| \leqslant \frac{1}{Rq}.$$

Write $C = A/(2r)$. Then, for any $r > d(d+1)$, we have

$$\sum_{1 \leqslant n \leqslant N} f(n) e(F(n)) \ll N\left(\frac{1}{(\log N)^{1-C}} + (\log N)^C \left(\frac{q}{N^\ell} + \frac{1}{q}\right)^{1/(4r^2)}\right) + (N R^{1/\ell})^{1/2},$$

where the implied constant depends on A and r. In particular, if we suppose that

$$(\log N)^{4r^2} \leqslant q \leqslant \frac{N^\ell}{(\log N)^{4r^2}},$$

then

$$\sum_{1 \leqslant n \leqslant N} f(n) e(F(n)) \ll \frac{N}{(\log N)^{1-C}}. \tag{7}$$

To demonstrate the precision of the above estimate, in Section 10 we prove that, for any polynomial F and integer N, there exists $f = f_{F,N}$ such that

$$\left|\sum_{1\leqslant n\leqslant N} f(n)e(F(n))\right| \geqslant \frac{1}{10}\frac{N}{\log N}.$$

The proof of Theorem 1 follows the outline of [Montgomery and Vaughan 1977] and starts with a combinatorial decomposition of multiplicative functions based on Möbius inversion. This reduces the problem to estimating bilinear forms over polynomials with summation restricted to points under the hyperbola. We then use Montgomery and Vaughan's partition of the parabola into disjoint rectangles to which techniques related to the Vinogradov mean value theorem may be applied. It will be fundamental to develop a version of the Vinogradov mean value theorem for primes in large translated intervals; this is Lemma 10. We should note that we introduced the general condition (6) with the aim of using Theorem 1 in the proof of Theorem 2.

3. Applications

3.1. ***Joint equidistribution.*** As an application of Theorem 1, we prove a joint equidistribution result. Throughout this section, we let $p \in \mathbb{Z}[x]$ be irreducible over $\mathbb{Q}$ of degree $e \geqslant 2$, and we consider the ratios v/n, where the v are the roots of the polynomial p modulo n:

$$p(v) \equiv 0 \bmod n.$$

Consider the sequence $(g_k)_{k\geqslant 1}$ of these ratios, defined in such a way that the corresponding denominators are in ascending order. Hooley [1964, Theorem 2] proved that this sequence is equidistributed in $\mathbb{R}/\mathbb{Z}$.

We now let

$$F(x) = \alpha_1 x + \cdots + \alpha_d x^d \in \mathbb{R}[x] \quad \text{with } d \geqslant 1$$

have an irrational coefficient and define

$$A(F, p)_k = (g_k, F(n))_{k\geqslant 1},$$

where g_k is as above and n is taken so that

$$g_k = v/n, \qquad p(v) \equiv 0 \bmod n.$$

We prove the following result.

Theorem 2. *The sequence $A(F, p)_k$ is equidistributed in $(\mathbb{R}/\mathbb{Z})^2$.*

This indicates that the sequence $(F(n))_{n\geqslant 1}$ is somehow not correlated with the sequence $(g_n)_{n\geqslant 1}$.

3.2. ***Mixed character sums.*** We next explain how Theorem 1 is related to sums considered in [Enflo 1995; Chang 2010; Heath-Brown and Pierce 2015]. Theorem 1 implies large exponential sums must correspond to pretentious multiplicative functions.

Corollary 3. *With the notation and conditions as in Theorem 1, suppose further that f is completely multiplicative. Let $r \geqslant d(d+1)$, and suppose that*

$$\left|\sum_{1\leqslant n\leqslant N} f(n)e(F(n))\right| \gg \frac{N}{(\log N)^{1-A/(2r)}}.$$

There exists an integer

$$\ell \leqslant (\log N)^{d(4r^2+4rA)}$$

and a multiplicative character $\psi \bmod \ell$ such that

$$\sum_{1\leqslant n\leqslant N} f(n)e(F(n)) \ll (\log N)^{4r^2+4rA} \max_{u\leqslant N}\left|\sum_{n\leqslant u} \psi(n)f(n)\right|.$$

Corollary 3 implies one may bound character sums mixed by polynomials by reducing to pure character sums; we refer the reader to Section 8 for more precise results. It is worth mentioning that, by combining Corollary 3 with Haláz's theorem, it's possible to show that sums satisfying the conditions of Corollary 3 must be pretentious in the sense of [Granville and Soundararajan 2007].

Corollary 4. *Let $F(n)$ be a polynomial of degree d with real coefficients and χ a primitive character modulo q. Suppose that δ and ε satisfy*

$$\max_{\substack{\ell\ll(\log q)^{100d^3}\\ \psi \bmod \ell\\ u\leqslant N}}\left|\sum_{n\leqslant u} \psi(n)\chi(n)\right| \leqslant Nq^{-\varepsilon} \quad \textit{provided } N \geqslant q^{\delta}.$$

Then we have

$$\sum_{n\leqslant N} \chi(n)e(F(n)) \ll \frac{N}{(\log N)^{1-1/(d(d+1))}} \quad \textit{provided } N \geqslant q^{\delta}. \tag{8}$$

In particular, estimate (8) holds under the following conditions:

- for arbitrarily small δ assuming the generalised Riemann hypothesis,
- for $\delta = \frac{1}{3}$ and an arbitrary integer q, which follows from the Burgess bound; see for example [Iwaniec and Kowalski 2004].

Let $\varepsilon > 0$ be small. Enflo [1995] has previously established that, if q is prime,

$$\sum_{n\leqslant N} \chi(n)e(F(n)) \ll N^{1-\delta}, \quad \text{provided } N \geqslant q^{1/4+\varepsilon},$$

and we refer the reader to [Chang 2010; Heath-Brown and Pierce 2015] for quantitative improvements on Enflo's result. Chang [2014] has shown

$$\sum_{n\leqslant N} \chi(n)e(F(n)) \ll N^{1-\delta}, \quad \text{provided } N \geqslant q^{\varepsilon},$$

provided q is suitably smooth/powerful.

Corollary 4 provides some new instances where one may bound mixed character sums nontrivially. We refer the reader to Section 8 for more details.

4. Preliminary results

4.1. ***Reduction to bilinear forms.*** We proceed in a similar fashion to [Montgomery and Vaughan 1977, Section 2], which reduces the proof of Theorem 1 to bounding bilinear forms under the hyperbola.

Lemma 5. *Let f be a multiplicative function satisfying the conditions of Theorem 1, F be any real-valued function and $\epsilon > 0$. Then, for any integer N, we have*

$$\left|\sum_{1\leqslant n\leqslant N} f(n)e(F(n))\right| \ll \frac{N}{(\log N)^{1-\epsilon}} + \frac{1}{\log N}\left|\sum_{1\leqslant np\leqslant N} f(n)f(p)(\log p)e(F(np))\right|. \tag{9}$$

Proof. We follow the argument from [Montgomery and Vaughan 1977, Section 2] with some modifications to deal with the condition

$$\sum_{n\leqslant N}|f(n)|^2 \ll N(\log N)^A.$$

Consider

$$S = \sum_{n\leqslant N} f(n)e(F(n))\log(N/n).$$

Since

$$S = \log N\sum_{n\leqslant N} f(n)e(F(n)) - \sum_{n\leqslant N} f(n)(\log n)e(F(n)),$$

it is sufficient to show

$$S \ll N(\log N)^{\epsilon} \tag{10}$$

and

$$\sum_{n\leqslant N} f(n)(\log n)e(F(n)) \ll \left|\sum_{1\leqslant np\leqslant N} f(n)f(p)(\log p)e(F(np))\right| + N(\log N)^{\epsilon}. \tag{11}$$

Let r be a large real number, and apply Hölder's inequality, (5) and (6) to get

$$|S|^{2r} \ll \left(\sum_{n\leqslant N}(\log(N/n))^{2r}\right)\left(\sum_{n\leqslant N}|f(n)|^2\right)\left(\sum_{n\leqslant N}|f(n)|\right)^{2r-2} \ll \left(\sum_{n\leqslant N}(\log(N/n))^{2r}\right)N^{2r-1}(\log N)^A.$$

Since

$$\sum_{n\leqslant N}(\log(N/n))^{2r} \ll \sum_{\substack{j\\ 1\leqslant 2^j\leqslant N}} j^{2r}\sum_{N/2^{j+1}\leqslant n\leqslant N/2^{j-1}} 1 \ll N\sum_{j\geqslant 1}\frac{j^{2r}}{2^j} \ll N,$$

we obtain (10) after taking r sufficiently large. Since

$$\log n = \sum_{d\mid n}\Lambda(d),$$

we have

$$\begin{aligned}\sum_{n\leqslant N} f(n)(\log n)e(F(n)) &= \sum_{nm\leqslant N}\Lambda(m)f(nm)e(F(nm))\\ &= \sum_{mn\leqslant N}\Lambda(m)f(n)f(m)e(F(nm)) + O\left(\sum_{mn\leqslant N}\Lambda(m)|f(nm)-f(n)f(m)|\right).\end{aligned}$$

Using (4) and (5),

$$\sum_{mn\leqslant N}\Lambda(m)f(n)f(m)e(F(nm))=\sum_{pn\leqslant N}(\log p)f(p)f(n)e(F(np))+O\biggl(N\sum_{k\geqslant 2}\sum_{p^k\leqslant N}\frac{|f(p^k)|}{p^k}\biggr)$$
$$=\sum_{pn\leqslant N}(\log p)f(p)f(n)e(F(np))+O(N),$$

where we have used the Cauchy–Schwarz inequality and (6) to estimate

$$\sum_{k\geqslant 2}\sum_{p^k\leqslant N}\frac{|f(p^k)|}{p^k}\ll\biggl(\sum_{k\geqslant 2}\sum_{p}\frac{1}{p^{0.8k}}\biggr)\biggl(\sum_{n}\frac{|f(n)|^2}{n^{1.2}}\biggr)=O(1).$$

Hence it remains to show

$$\sum_{mn\leqslant N}\Lambda(m)|f(nm)-f(n)f(m)|=O(N(\log N)^{\epsilon}).\tag{12}$$

From (5),

$$\sum_{mn\leqslant N}\Lambda(m)|f(nm)-f(n)f(m)|\ll\sum_{k\geqslant 1}\sum_{p^k\leqslant N}\sum_{\substack{n\leqslant N/p^k\\ p\mid n}}|f(p^k)||f(n)|+|f(p^kn)|$$
$$\ll\sum_{k,j\geqslant 1}\sum_{p^{k+j}\leqslant N}\sum_{\substack{n\leqslant N/p^{k+j}\\ (n,p)=1}}(|f(p^{k+j})|+|f(p^k)||f(p^j)|)|f(n)|$$
$$\ll N\sum_{k,j\geqslant 1}\sum_{p}\frac{|f(p^k)||f(p^j)|+|f(p^{k+j})|}{p^{k+j}},$$

and after applying the Cauchy–Schwarz inequality, (6) and partial summation over p, k and j as above, we establish (12) and complete the proof. □

We require a generalisation of [Montgomery and Vaughan 1977, Section 3] for multiplicative functions f satisfying (6).

Lemma 6. *Let the notation and conditions be as in Theorem 1. Suppose s is a parameter and, for each $0\leqslant i\leqslant\log_2 N$, write*

$$J_i=\min\bigl(i+1,\ \lfloor\log_2 N\rfloor-i+1,\ \bigl\lfloor\log_2\tfrac{1}{2}(64N/s)\bigr\rfloor\bigr).\tag{13}$$

We partition the hyperbola into rectangles

$$\mathcal{R}_i=(0,2^i]\times\biggl(\frac{N}{2^{i+1}},\frac{N}{2^i}\biggr],\qquad 0\leqslant i\leqslant\log_2 N,\tag{14}$$

and

$$\mathcal{R}_{i,j,k}=\biggl(\frac{2^{i+j}}{k},\frac{2^{i+j+1}}{2k-1}\biggr]\times\biggl(\frac{(k-1)N}{2^{i+j}},\frac{(2k-1)N}{2^{i+j+1}}\biggr],\qquad 0\leqslant i\leqslant\log_2 N,\ \ 1\leqslant j\leqslant J_i,\ \ 2^{j-1}\leqslant k\leqslant 2^j.\tag{15}$$

Then each $\mathcal{R}_{i,j,k}$ is a rectangle of the form $(P,P']\times(N,N']$, with P,P',N,N' satisfying

$$P'-P\geqslant\tfrac{1}{4},\qquad N'-N\geqslant\tfrac{1}{4},\qquad (P'-P)(N'-N)\gg s.$$

Let $\mathcal{E}$ denote the set of points (p, n) satisfying $1 \leqslant pn \leqslant N$ with p prime which do not lie in any of the rectangles (14) or (15). Then, for any fixed $\epsilon > 0$, we have

$$\sum_{(p,n)\in\mathcal{E}} f(p)(\log p) f(n) e(F(np)) \ll (\log N)^{\epsilon}(N + (Ns)^{1/2}\log(2N/s)\log s).$$

Proof. Our proof is similar to that of [Montgomery and Vaughan 1977, Section 3] with some minor modifications. Recall that $\mathcal{E}$ denotes the set of points (p, n) satisfying $1 \leqslant pn \leqslant N$ with p prime which do not lie in any of the rectangles (14) or (15). We write the partition

$$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3,$$

where

$$\mathcal{E}_1 = \left\{(p, n) \in \mathcal{E} : pn \leqslant N,\ \frac{N}{2^{i+1}} < n \leqslant \frac{N}{2^i},\ J_i > i+1\right\},$$
$$\mathcal{E}_2 = \left\{(p, n) \in \mathcal{E} : pn \leqslant N,\ \frac{N}{2^{i+1}} < n \leqslant \frac{N}{2^i},\ J_i = \lfloor \log_2 N \rfloor - i + 1\right\},$$
$$\mathcal{E}_3 = \left\{(p, n) \in \mathcal{E} : pn \leqslant N,\ \frac{N}{2^{i+1}} < n \leqslant \frac{N}{2^i},\ J_i = \left\lfloor \tfrac{1}{2}\log_2(64N/s)\right\rfloor\right\},$$

which allows us to bound

$$\sum_{(p,n)\in\mathcal{E}} f(p)(\log p) f(n) e(F(np)) \ll \sum_{(p,n)\in\mathcal{E}_1} |f(p)||f(n)|\log p + \sum_{(p,n)\in\mathcal{E}_2} |f(p)||f(n)|\log p + \sum_{(p,n)\in\mathcal{E}_3} |f(p)||f(n)|\log p.$$

Consider first summation over $\mathcal{E}_1$. We apply Hölder's inequality twice; first with the exponents $(2r, 2r/(2r-1))$ and second with the exponents $(2r-1, (2r-2)/(2r-1))$. Together with (4), this gives

$$\left(\sum_{(p,n)\in\mathcal{E}_1} |f(p)||f(n)|\log p\right)^{2r} \ll \left(\sum_{(p,n)\in\mathcal{E}_1} |f(n)|\right)^{2r-2}\left(\sum_{(p,n)\in\mathcal{E}_1} |f(n)|^2\right)\left(\sum_{(p,n)\in\mathcal{E}_1} (\log p)^{2r}\right).$$

For each prime p,

$$\#\{n : (p, n) \in \mathcal{E}_1\} \ll \frac{N}{p^2},$$

so that

$$\sum_{(p,n)\in\mathcal{E}_1} (\log p)^{2r} \ll N \sum_{p} \frac{(\log p)^{2r}}{p^2} \ll N.$$

For each $n \in \mathbb{N}$,

$$\#\{p : (p, n) \in \mathcal{E}_1\} \ll 1,$$

and hence, by (5) and (6),

$$\sum_{(p,n)\in\mathcal{E}_1} |f(n)| \ll \sum_{n\leqslant N} |f(n)| \ll N, \quad \sum_{(p,n)\in\mathcal{E}_1} |f(n)|^2 \ll \sum_{n\leqslant N} |f(n)|^2 \ll N(\log N)^A.$$

Taking r sufficiently large, the above estimates combine to give

$$\sum_{(p,n)\in\mathcal{E}_1} |f(p)||f(n)|\log p \ll N(\log N)^{\epsilon}.$$

Consider next $\mathcal{E}_2$. By the Cauchy–Schwarz inequality,

$$\Bigg(\sum_{(p,n)\in\mathcal{E}_2} |f(p)||f(n)|\log p\Bigg)^2 \ll \sum_{(p,n)\in\mathcal{E}_2} |f(n)|^2 \sum_{(p,n)\in\mathcal{E}_2} (\log p)^2.$$

If $(p,n)\in\mathcal{E}_2$ then $n \ll N^{1/2}$ and, for fixed n, there exists some H such that

$$\{p : (p,n)\in\mathcal{E}_2\} \subseteq [H, H+N/n^2].$$

Hence, by Hölder's inequality and the Brun–Titchmarsh theorem, we get

$$\sum_{(p,n)\in\mathcal{E}_2} |f(p)||f(n)|\log p \ll \Bigg(N\sum_{n\leqslant N^{1/2}} \frac{|f(n)|^2}{n^2\log(4N/n^2)}\Bigg)^{1/2}\Bigg(\sum_{(p,n)\in\mathcal{E}_2}(\log p)^2\Bigg)^{1/2}.$$

Note by (6) and partial summation,

$$\sum_{n\leqslant N^{1/2}} \frac{|f(n)|^2}{n^2\log(4N/n^2)} \ll \int_1^{N^{1/2}}\Bigg(\sum_{n\leqslant t}|f(n)|^2\Bigg)\frac{1}{t^3(\log(4N/t))^2}\,dt \ll \int_1^{N^{1/2}} \frac{(\log t+2)^A}{t^2(\log(4N/t))}\,dt \ll \frac{1}{(\log N)}.$$

For each p,

$$\#\{n : (p,n)\in\mathcal{E}_2\} \ll 1,$$

so that

$$\sum_{(p,n)\in\mathcal{E}_2}(\log p)^2 \ll \sum_{p\ll N}(\log p)^2 \ll N\log N.$$

The above estimates combine to give

$$\sum_{(p,n)\in\mathcal{E}_2} |f(p)||f(n)|\log p \ll N.$$

Finally consider $\mathcal{E}_3$. By Hölder's inequality,

$$\Bigg(\sum_{(p,n)\in\mathcal{E}_3} |f(p)||f(n)|\log p\Bigg)^{2r} \ll \Bigg(\sum_{(N/q)^{1/2}\leqslant n\leqslant (Nq)^{1/2}} |f(n)|^{2r/(2r+1)}\log p\Bigg)^{2r-1}\Bigg(\sum_{(N/q)^{1/2}\leqslant p\leqslant (Nq)^{1/2}}\frac{\log p}{p}\Bigg).$$

If $(p,n)\in\mathcal{E}_3$, then

$$\left(\frac{N}{s}\right)^{1/2} \leqslant p \leqslant (Ns)^{1/2},$$

and, for each p, there exists some H such that

$$\{n : (p,n)\in\mathcal{E}_3\} \subseteq [H, H+O((Ns)^{1/2}/p)]$$

and, for each n, there exists some H such that

$$\{p : (p, n) \in \mathcal{E}_3\} \subseteq [H, H + O((Ns)^{1/2}/n)].$$

Using the Brun–Titchmarsh inequality, we see that

$$\biggl(\sum_{(p,n)\in\mathcal{E}_3} |f(p)||f(n)| \log p\biggr)^{2r} \ll (Ns)^{1/2}\biggl(\sum_{(N/s)^{1/2}\leqslant n\leqslant (Ns)^{1/2}} |f(n)|^{2r/(2r+1)} \frac{\log (2N/n)}{n \log (2Ns/n^2)}\biggr)^{2r-1}\biggl(\sum_{(N/s)^{1/2}\leqslant p\leqslant (Ns)^{1/2}} \frac{\log p}{p}\biggr).$$

We have

$$\sum_{(N/s)^{1/2}\leqslant p\leqslant (Ns)^{1/2}} \frac{\log p}{p} \ll \log s,$$

and Hölder's inequality combined with (5) and (6) gives

$$\begin{aligned}\biggl(\sum_{(N/s)^{1/2}\leqslant n\leqslant (Ns)^{1/2}} |f(n)|^{2r/(2r+1)} \frac{\log (2N/n)}{n \log (2Ns/n^2)}\biggr)^{2r-1} &\ll (\log(2N/s))^{2r-1}\biggl(\sum_{(N/s)^{1/2}\leqslant n\leqslant (Ns)^{1/2}} \frac{|f(n)|}{n}\biggr)^{2r-2}\biggl(\sum_{(N/s)^{1/2}\leqslant n\leqslant (Ns)^{1/2}} \frac{|f(n)|^2}{n}\biggr)\\ &\ll (\log(2N/s))^{2r-1}(\log N)^{A}(\log s)^{2r-1},\end{aligned}$$

which, after taking r sufficiently large, results in

$$\sum_{(p,n)\in\mathcal{E}_3} |f(p)||f(n)| \log p \ll (Ns)^{1/2}(\log N)^{\varepsilon}(\log s) \log (2N/s). \qquad \square$$

5. Sums over bilinear forms

5.1. ***The Vinogradov mean value theorem.*** Given integers r, d and V, we let $J_{r,d}(V)$ count the number of solutions to the system of equations

$$v_1^j + \cdots + v_r^j - v_{r+1}^j - \cdots - v_{2r}^j = 0, \quad 1 \leqslant j \leqslant d,$$

with variables satisfying

$$1 \leqslant v_1, \ldots, v_{2r} \leqslant V.$$

We will use a consequence of Bourgain, Demeter and Guth's work on the Vinogradov mean value theorem; see [Bourgain et al. 2016, Section 5].

Lemma 7. *Assume $d \geqslant 2$ and $r > d(d+1)$. Then we have*

$$J_{r,d}(V) \ll V^{2r-d(d+1)/2}.$$

Combining Lemma 7 with the fact that the Vinogradov system is translation invariant, we obtain the following.

Corollary 8. *Let $d \geqslant 2$, $r > d(d+1)$ be integers and $\mathcal{M}(k)$, $1 \leqslant k \leqslant K$, be disjoint intervals satisfying*

$$\mathcal{M}(k) = (M'(k), M''(k)], \quad M''(k) - M'(k) \leqslant Y.$$

Then the number of solutions to the system of equations

$$n_1^j + \cdots - n_{2r}^j = 0, \quad n_i \in \mathcal{M}(k), \ 1 \leqslant k \leqslant K, \tag{16}$$

is bounded by

$$O(KY^{2r-d(d+1)/2}).$$

We will also require an estimate for the number of solutions to the Vinogradov system with prime variables in translated intervals; here it will be fundamental that the intervals will not be too short. To obtain such a result we need the following intermediary lemma that follows directly from [Hua 1965, Theorem 10]. Here it appears clear why it is important that the intervals we work with are quite large compared to their starting point. For convenience we set $L = \log P$.

Lemma 9. *Let $0 < Q \leqslant c_1(k)L^{\sigma_1}$, $X \gg 1$ and*

$$S(X, P) = \sum_{\substack{X < p \leqslant X+P \\ p \equiv t \bmod Q}} e(f(p)),$$

in which

$$P \gg \frac{X}{(\log X)^M}$$

for any $M \gg 1$ and

$$f(x) = \frac{h}{q}x^k + \alpha_1 x^{k-1} + \cdots + \alpha_k, \quad (h, q) = 1,$$

the numbers $\alpha_1, \ldots, \alpha_{d-1}$ being real. Suppose that $L^\sigma < q \leqslant P^k L^{-\sigma}$. For arbitrary $\sigma_0 > 0$, when $\sigma \leqslant 2^{6k}(\sigma_0 + \sigma_1 + 1)$, we always have

$$|S(X, P)| \leqslant c_2(k)\frac{P}{QL^{\sigma_0 - M}}.$$

This lemma allows us to prove the following estimate for the number of solutions to the Vinogradov system with prime variables in large translated intervals.

Lemma 10. *Let $d \geqslant 2$ be an integer and $X, Y \gg 1$. If $r > d(d+1)$, the number of solutions to the equation*

$$p_1^j + \cdots - p_{2r}^j = 0, \quad 1 \leqslant j \leqslant d, \ \ Y \leqslant p_i \leqslant Y + X, \ \ X \gg \frac{Y}{(\log Y)^C}, \ \ p_i \text{ prime},$$

for any $C \gg 1$ is bounded by

$$O\left(\frac{X^{2r-d(d+1)/2}}{(\log X)^{2r}}\right),$$

with the implied constant depending on r and C.

Proof. The result follows in a straightforward way from [Hua 1965, Theorem 16] redefining in part (1)

$$S(\alpha_k,\dots,\alpha_1)=\sum_{Y\leqslant p\leqslant Y+X} e(f(p)),\qquad f(x)=\alpha_d x^d+\dots+\alpha_1 x,$$

and introducing the two following changes which account for the translation in the set of primes and optimising the range of r in Hua's result: In part (3) of the proof we use our Lemma 9 instead of [Hua 1965, Theorem 10]; this is possible as s_1 in [Hua 1965, Lemma 10.8] is of arbitrary size. While in part (5) of the proof we need to substitute [Hua 1965, Theorem 15] with [Bourgain et al. 2016, Theorem 1] for $d\geqslant 4$ and [Wooley 2016] for $d=3$. Doing this we need to be careful to isolate, in [Hua 1965, p. 144], $|S(\alpha_k,\dots,\alpha_1)|^{d(d+1)-\epsilon}$ instead of $|S(\alpha_k,\dots,\alpha_1)|^{s_0-1}$ and, in [Hua 1965, p. 145], $|S(\alpha_k,\dots,\alpha_1)|^{s-\epsilon}$ instead of $|S(\alpha_k,\dots,\alpha_1)|^{s-1}$. Here $\epsilon>0$ is chosen so that $s=d(d+1)+2\epsilon$. □

5.2. *Bounding bilinear forms.* It is well known that one may use Lemma 7 to estimate bilinear forms with Weyl sums over rectangles. We next show how one may obtain sharper results by averaging bilinear forms over a family of disjoint rectangles.

Lemma 11. *Let $F\in\mathbb{R}[X]$ be a polynomial of degree $d\geqslant 2$ of the form*

$$F(x)=\alpha_d x^d+\dots+\alpha_1 x.$$

Let M and N be integers and $\alpha(n)$ and $\beta(m)$ two sequences of complex numbers satisfying

$$|\beta(m)|\leqslant 1$$

and

$$\sum_{n\leqslant N}|\alpha(n)|\ll N,\qquad \sum_{n\leqslant N}|\alpha(n)|^2\ll N(\log N)^A \tag{17}$$

for $A\geqslant 0$. For $1\leqslant k\leqslant K$, let

$$\mathcal{R}(k)=\mathcal{L}(k)\times\mathcal{M}(k) \tag{18}$$

be a rectangle of the form

$$\mathcal{L}(k)=(Q'(k),Q''(k)],\qquad \mathcal{M}(k)=(M'(k)\times M''(k)]$$

such that $\mathcal{L}(k)\subseteq(0,Q)$, with $Q\gg 1$, are disjoint and satisfy

$$Q''(k)-Q'(k)\leqslant X \tag{19}$$

or

$$\frac{Q'(k)}{(\log Q'(k))^C}\ll Q''(k)-Q'(k)\ll X, \tag{20}$$

and $\mathcal{M}(k)\subseteq(0,M]$ are disjoint and satisfy

$$M''(k)-M'(k)\leqslant Y,\qquad M''(k)\leqslant 2M'(k)$$

and $K\ll M$. Let I denote the sum

$$I=\sum_{k=1}^{K}\sum_{(p,n)\in\mathcal{R}(k)}\alpha(n)\beta(p)e(F(np)). \tag{21}$$

Let $R \geqslant 1$, and suppose there exists $q \leqslant R$ and $1 \leqslant \ell \leqslant d$ such that

$$\left|\alpha_\ell - \frac{a}{q}\right| \leqslant \frac{1}{qR} \tag{22}$$

for some $(a, q) = 1$. For $r > d(d+1)$, we have

$$I^{4r^2} \ll m(X)(\log M)^{2rA} M^{4r^2} \left(\frac{X}{\log X}\right)^{4r^2} \left(\frac{M}{Y}\right)^{d(d-1)/2} \left(\frac{Q}{X}\right)^{d(d-1)/2} \times \left(\frac{q}{XQ^{\ell-1}YM^{\ell-1}} + \frac{1}{XQ^{\ell-1}} + \frac{1}{YM^{\ell-1}} + \frac{1}{q}\right),$$

where

$$m(X) = \begin{cases} (\log X)^{4r} & \text{when (19) holds,} \\ 1 & \text{when (20) holds,} \end{cases}$$

and the implied constant depends on C.

Proof. Recall

$$F(x) = \alpha_d x^d + \cdots + \alpha_1 x,$$

and define

$$S_k = \sum_{(p,n)\in\mathcal{R}(k)} \alpha(n)\beta(p)e(F(np)), \tag{23}$$

so that

$$I = \sum_{k=1}^{K} S_k. \tag{24}$$

Fix some $1 \leqslant k \leqslant K$ and consider (23). Recalling (18), by Hölder's inequality, for any integer $r \geqslant 1$,

$$S_k \leqslant \left(\sum_{n\in\mathcal{M}(k)} |\alpha(n)|^{2r/(2r-1)}\right)^{1-1/(2r)} \left(\sum_{n\in\mathcal{M}(k)} \left|\sum_{p\in\mathcal{L}(k)} \beta(p)e(F(np))\right|^{2r}\right)^{1/(2r)}.$$

After another application of Hölder's inequality,

$$I^{2r} \leqslant \left(\sum_{1\leqslant k\leqslant K} \sum_{n\in\mathcal{M}(k)} |\alpha(n)|^{2r/(2r-1)}\right)^{2r-1} \sum_{1\leqslant k\leqslant K} \sum_{n\in\mathcal{M}(k)} \left|\sum_{p\in\mathcal{L}(k)} \beta(p)e(F(np))\right|^{2r}.$$

Hence, from assumptions on the $\mathcal{M}(k)$,

$$I^{2r} \ll \left(\sum_{n\leqslant M} |\alpha(n)|^{2r/(2r-1)}\right)^{2r-1} \sum_{1\leqslant k\leqslant K} \sum_{n\in\mathcal{M}(k)} \left|\sum_{p\in\mathcal{L}(k)} \beta(p)e(F(np))\right|^{2r},$$

which combined with (17) implies that

$$I^{2r} \ll M^{2r-1}(\log M)^A \sum_{1\leqslant k\leqslant K} \sum_{n\in\mathcal{M}(k)} \left|\sum_{p\in\mathcal{L}(k)} \beta(p)e(F(np))\right|^{2r}.$$

Write $\lambda = (\lambda_1, \ldots, \lambda_d)$, and let $J_k(\lambda)$ count the number of solutions to

$$p_1^j + \cdots - p_{2r}^j = \lambda_j, \quad 1 \leqslant j \leqslant d, \quad p_i \in \mathcal{L}(k), \quad p_i \text{ prime}.$$

Note by assumptions on $\mathcal{L}(k)$, if the $p_1, \ldots, p_{2r} \in \mathcal{L}(k)$ satisfy

$$p_1^j + \cdots - p_{2r}^j = \lambda_j,$$

then

$$\lambda_j \ll X Q^{j-1}.$$

Expanding the $2r$-th power and interchanging summation gives

$$I^{2r} \ll M^{2r-1} (\log M)^A \sum_{1 \leqslant k \leqslant K} \sum_{\substack{\lambda \\ \lambda_j \ll X Q^{j-1} \\ 1 \leqslant j \leqslant d}} J_k(\lambda) \left| \sum_{n \in \mathcal{M}(k)} e(\alpha_d \lambda_d n^d + \cdots + \alpha_1 \lambda_1 n) \right|.$$

After another application of Hölder's inequality, we get

$$I^{4r^2} \ll M^{4r^2-2r} (\log M)^{2rA} \left(\sum_{1 \leqslant k \leqslant K} \sum_{\lambda} J_k(\lambda) \right)^{2r-2} \left(\sum_{1 \leqslant k \leqslant K} \sum_{\lambda} J_k(\lambda)^2 \right) \times \sum_{1 \leqslant k \leqslant K} \sum_{\substack{\lambda_j \ll X Q^{j-1} \\ 1 \leqslant j \leqslant d}} \left| \sum_{n \in \mathcal{M}(k)} e(\alpha_d \lambda_d n^d + \cdots + \alpha_1 \lambda_1 n) \right|^{2r}.$$

We have

$$\sum_{1 \leqslant k \leqslant K} \sum_{\lambda} J_k(\lambda) \leqslant \sum_{1 \leqslant k \leqslant K} |\{p_1, \ldots, p_{2r} \in \mathcal{L}(k)\}|.$$

Since, for each $1 \leqslant k \leqslant K$, $\mathcal{L}(k)$ is an interval of length at most X, the Brun–Titchmarsh theorem implies

$$\sum_{1 \leqslant k \leqslant K} \sum_{\lambda} J_k(\lambda) \ll \frac{K X^{2r}}{(\log X)^{2r}}.$$

The term

$$\sum_{1 \leqslant k \leqslant K} \sum_{\lambda} J_k(\lambda)^2$$

counts the number of solutions to the system of equations

$$p_1^j + \cdots - p_{4r}^j = 0, \quad 1 \leqslant j \leqslant d, \quad p_i \in \mathcal{L}(k), \quad p_i \text{ prime}, \quad 1 \leqslant k \leqslant K. \tag{25}$$

By Corollary 8, ignoring the condition that p_i is prime if (19) holds and using Lemma 10 when (20) holds, we get

$$\sum_{1 \leqslant k \leqslant K} \sum_{\lambda} J_k(\lambda)^2 \ll K m_1(X),$$

where

$$m_1(X) = \begin{cases} X^{4r-d(d+1)/2} & \text{when (19) holds,} \\ \dfrac{X^{4r-d(d+1)/2}}{(\log X)^{4r}} & \text{when (20) holds} \end{cases}$$

since $r > d(d+1)$. Combined with the observations above, this implies

$$I^{4r^2} \ll m(X) M^{4r^2-2r} (\log M)^{2rA} K^{2r-1} \left(\frac{X}{\log X}\right)^{4r^2} \frac{I_0}{X^{d(d+1)/2}}, \tag{26}$$

where

$$I_0 = \sum_{1 \leqslant k \leqslant K} \sum_{\substack{\lambda_j \ll X Q^{j-1} \\ 1 \leqslant j \leqslant d}} \left| \sum_{n \in \mathcal{M}(k)} e(\alpha_d \lambda_d n^d + \cdots + \alpha_1 \lambda_1 n) \right|^{2r}.$$

Let

$$S(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2,$$

so that

$$\hat{S}(x) = \max\{0, 1 - |x|\} \tag{27}$$

and

$$S(x) \gg 1 \quad \text{if } |x| \leqslant \tfrac{1}{4}.$$

There exists a constant c_0 such that

$$I_0 \ll \sum_{1 \leqslant k \leqslant K} \sum_{\substack{\lambda_j \ll X Q^{j-1} \\ 1 \leqslant j \leqslant d \\ j \neq \ell}} \sum_{\lambda_\ell \in \mathbb{Z}} S\left(\frac{c_0 \lambda_\ell}{X Q^{\ell-1}}\right) \left| \sum_{n \in \mathcal{M}(k)} e(\alpha_d \lambda_d n^d + \cdots + \alpha_1 \lambda_1 n) \right|^{2r}.$$

Expanding the $2r$-th power and interchanging summation,

$$I_0 \ll \sum_{\substack{1 \leqslant k \leqslant K \\ \mu_j \ll Y N^{j-1}}} L_k(\mu) \prod_{\substack{j=1 \\ j \neq \ell}}^{d} \left| \sum_{\lambda \ll X Q^{j-1}} e(\alpha_j \lambda \mu_j) \right| \left| \sum_{\lambda \in \mathbb{Z}} S\left(\frac{c_0 \lambda_\ell}{X Q^{\ell-1}}\right) e(\alpha_\ell \lambda \mu_\ell) \right|,$$

where $\mu = (\mu_1, \ldots, \mu_d)$ and $L_k(\mu)$ counts the number of solutions to

$$n_1^j + \cdots - n_{2r}^j = \mu_j, \quad n_i \in \mathcal{M}(k), \quad 1 \leqslant k \leqslant K, \quad 1 \leqslant j \leqslant d.$$

Using that

$$L_k(\mu) \leqslant L_k(0)$$

and applying Corollary 8, we obtain

$$I_0 \ll K Y^{2r-d(d+1)/2} \sum_{\substack{\mu \\ \mu_j \ll Y M^{j-1}}} \prod_{\substack{j=1 \\ j \neq \ell}}^{d} \left| \sum_{\lambda \ll X Q^{j-1}} e(\alpha_j \lambda \mu_j) \right| \left| \sum_{\lambda \in \mathbb{Z}} S\left(\frac{c_0 \lambda}{X q^{\ell-1}}\right) e(\alpha_\ell \lambda \mu_\ell) \right|.$$

Combining the inequality above with (26), we have

$$I^{4r^2} \ll m(X) (\log M)^{2rA} M^{4r^2} \left(\frac{X}{\log X}\right)^{4r^2} \frac{I_1}{(XY)^{d(d+1)/2}}, \tag{28}$$

where

$$I_1 = \sum_{\mu_j \ll Y M^{j-1}} \prod_{\substack{j=1 \\ j \neq \ell}}^{d} \left| \sum_{\lambda \ll X Q^{j-1}} e(\alpha_j \lambda \mu_j) \right| \left| \sum_{\lambda \in \mathbb{Z}} S\left(\frac{c_0 \lambda}{X q^{\ell-1}}\right) e(\alpha_\ell \lambda \mu_\ell) \right|.$$

In I_1, we bound every term trivially except the one with index ℓ to obtain

$$I_1 \ll \left(\frac{Y}{M}\right)^d \left(\frac{X}{Q}\right)^d (QM)^{d(d+1)/2} \frac{1}{XQ^{\ell-1}} \frac{1}{YM^{\ell-1}} \sum_{\mu \ll YM^{\ell-1}} \left| \sum_{\lambda\in\mathbb{Z}} S\left(\frac{c_0\lambda}{Xq^{\ell-1}}\right) e(\alpha_\ell \lambda \mu_\ell) \right|. \tag{29}$$

By Poisson summation,

$$\sum_{\lambda\in\mathbb{Z}} S\left(\frac{c_0\lambda_\ell}{XQ^{\ell-1}}\right) e(\alpha_\ell \mu \lambda) = \frac{XQ^{\ell-1}}{c_0} \sum_{\lambda\in\mathbb{Z}} \hat{S}\left(\frac{XQ^{\ell-1}(\lambda-\alpha_\ell\mu)}{c_0}\right),$$

and hence from (27)

$$\begin{aligned}\sum_{\mu \ll YM^{\ell-1}} \left| \sum_{\lambda\in\mathbb{Z}} S\left(\frac{c_0\lambda_\ell}{XQ^{\ell-1}}\right) e(\alpha_\ell \mu \lambda) \right| &\ll XQ^{\ell-1} \sum_{\mu \ll YM^{\ell-1}} \sum_{\lambda\in\mathbb{Z}} \max\left\{0, 1 - \left| \frac{XQ^{\ell-1}(\lambda-\alpha_\ell\mu)}{c_0} \right| \right\} \\ &\ll XQ^{\ell-1} \left| \left\{ \mu \ll YM^{\ell-1} : \|\alpha_\ell\mu\| \leqslant \frac{100c_0}{XQ^{\ell-1}} \right\} \right|.\end{aligned}$$

There exists some real number β such that

$$\left| \left\{ \mu \ll YM^{\ell-1} : \|\alpha_\ell\mu\| \leqslant \frac{100c_0}{XQ^{\ell-1}} \right\} \right| \ll \left(1+\frac{YM^{\ell-1}}{q}\right) \left| \left\{ 0 \leqslant \mu \leqslant q : \|\alpha_\ell\mu+\beta\| \leqslant \frac{100c_0}{XQ^{\ell-1}} \right\} \right|.$$

Recalling (22),

$$\left| \left\{ \mu \ll YM^{\ell-1} : \|\alpha_\ell\mu\| \leqslant \frac{100c_0}{XQ^{\ell-1}} \right\} \right| \ll \left(1+\frac{YM^{\ell-1}}{q}\right) \left| \left\{ 0 \leqslant \mu \leqslant q : \left\|\frac{a\mu}{q}+\beta\right\| \leqslant \frac{100c_0}{XQ^{\ell-1}} + \frac{1}{R} \right\} \right|,$$

which implies that

$$\left| \left\{ \mu \ll YM^{\ell-1} : \|\alpha_\ell\mu\| \leqslant \frac{100c_0}{XQ^{\ell-1}} \right\} \right| \ll \left(1+\frac{YM^{\ell-1}}{q}\right)\left(1+\frac{q}{XQ^{\ell-1}}\right),$$

and hence

$$\sum_{\mu \ll YM^{\ell-1}} \left| \sum_{\lambda\in\mathbb{Z}} S\left(\frac{c_0\lambda}{Xq^{\ell-1}}\right) e(\alpha_\ell \lambda \mu_\ell) \right| \ll XQ^{\ell-1}YM^{\ell-1}\left(\frac{1}{YM^{\ell-1}}+\frac{1}{q}\right)\left(1+\frac{q}{XQ^{\ell-1}}\right).$$

Combined with (28) and (29), this gives

$$\begin{aligned}I^{4r^2} \ll m(X)(\log M)^{2rA} M^{4r^2} \left(\frac{X}{\log X}\right)^{4r^2} \left(\frac{M}{Y}\right)^{d(d-1)/2} \left(\frac{Q}{X}\right)^{d(d-1)/2} \\ \times \left(\frac{q}{XQ^{\ell-1}YM^{\ell-1}} + \frac{1}{XQ^{\ell-1}} + \frac{1}{YM^{\ell-1}} + \frac{1}{q}\right). \quad \square\end{aligned}$$

6. Proof of Theorem 1

We apply Lemma 5 and Lemma 6 with

$$s = \frac{q^{1/\ell}}{(\log N)^4}$$

to get

$$\sum_{1\leqslant n\leqslant N} f(n)e(F(n)) \ll \frac{N}{(\log N)^{1-\varepsilon}} + (Nq^{1/\ell})^{1/2} + \sum_{0\leqslant i\leqslant \log_2 N} \frac{i}{\log N} S_i + \sum_{0\leqslant i\leqslant \log_2 N} \frac{i}{\log N} \sum_{1\leqslant j\leqslant J_i} S_{i,j},$$

where

$$S_i = \sum_{(p,n)\in\mathcal{R}_i} \frac{\log p}{i} f(p) f(n) e(F(pn))$$

and

$$S_{i,j} = \sum_{2^{j-1}\leqslant k\leqslant 2^j} \sum_{(p,n)\in\mathcal{R}_{i,j,k}} \frac{\log p}{i} f(p) f(n) e(F(pn)).$$

Note if $(p,n)\in\mathcal{R}_i$ or $(p,n)\in\mathcal{R}_{i,j,k}$ then

$$\frac{\log p}{i} \ll 1.$$

By Lemma 11, observing that in this case condition (20) holds, we have

$$S_i \ll (\log N)^{A/(2r)} \frac{N}{i}\left(\frac{q}{N^\ell} + \frac{1}{2^{\ell i}} + \frac{2^{\ell i}}{N} + \frac{1}{q}\right)^{1/(4r^2)},$$

which implies that

$$\sum_{0\leqslant i\leqslant \log_2 N} \frac{i}{\log N} S_i \ll \frac{N}{(\log N)^{1-A/(2r)}}\left(1 + \log N\left(\frac{q}{N^\ell} + \frac{1}{q}\right)^{1/(4r^2)}\right). \tag{30}$$

Consider next the sums $S_{i,j}$. We apply Lemma 11 with parameters

$$K = 2^{j-1}, \quad M = \frac{N}{2^i}, \quad Q = 2^{i+1}, \quad X = 2^{i-j+1}, \quad Y = 32N2^{-i-j}.$$

We first focus on the case when $j \geqslant \log i^c$ for a fixed $c \gg 1$. Here we use Lemma 11 with condition (19), which easily gives

$$S_{i,j} \ll \frac{N}{i^{c_1} 2^{j/8}}\left(\frac{q}{N^\ell} + \frac{1}{2^{\ell i}} + \left(\frac{2^i}{N}\right)^\ell + \frac{1}{q}\right)^{1/(4r^2)}$$

for some $c_1 \gg 1$. For fixed $i \leqslant \log_2 N$, recalling that J_i is given by (13), we have

$$\sum_{\log i^c\leqslant j\leqslant J_i} S_{i,j} \ll \frac{N}{i^{c_1}}\left(\frac{q}{N^\ell} + \frac{1}{2^{\ell i}} + \left(\frac{2^i}{N}\right)^\ell + \frac{1}{q}\right)^{1/(4r^2)},$$

and hence

$$\begin{aligned}\sum_{0\leqslant i\leqslant \log_2 N} \frac{i}{\log N} \sum_{\log i^c\leqslant j\leqslant J_i} S_{i,j} &\ll \frac{N}{\log N} \sum_{i\leqslant \log_2 N} \frac{1}{i^{c_1-1}}\left(\frac{q}{N^\ell} + \frac{1}{2^{\ell i}} + \left(\frac{2^i}{N}\right)^\ell + \frac{1}{q}\right)^{1/(4r^2)} \\ &\ll N\left(\frac{1}{\log N} + \frac{q}{N^\ell} + \frac{1}{q}\right).\end{aligned}$$

Write $C = A/(2r)$. We then focus on the case when $j \leqslant \log i^c$. Here we use Lemma 11 with condition (20), which easily gives

$$S_{i,j} \ll \frac{1}{(i-j+1)^{1-C}} \frac{N}{2^{j(1-d(d-1)/(4r^2))}}\left(\frac{2^{2j}q}{N^\ell} + \frac{2^j}{2^{\ell i}} + 2^j\left(\frac{2^i}{N}\right)^\ell + \frac{1}{q}\right)^{1/(4r^2)},$$

with the implied constant depending on the fixed real number c. For $i \leqslant \log_2 N$, we have

$$\sum_{1\leqslant j\leqslant \log i^c} S_{i,j} \ll \frac{N}{i^{1-C}}\left(\frac{q}{N^\ell}+\frac{1}{2^{\ell i}}+\left(\frac{2^i}{N}\right)^\ell+\frac{1}{q}\right)^{1/(4r^2)},$$

and hence

$$\sum_{0\leqslant i\leqslant \log_2 N} \frac{i}{\log N} \sum_{1\leqslant j\leqslant \log i^c} S_{i,j} \ll \frac{N}{\log N} \sum_{i\leqslant \log_2 N} i^C\left(\frac{q}{N^\ell}+\frac{1}{2^{\ell i}}+\left(\frac{2^i}{N}\right)^\ell+\frac{1}{q}\right)^{1/(4r^2)}$$
$$\ll N\left(\frac{1}{(\log N)^{1-C}}+\frac{q(\log N)^C}{N^\ell}+\frac{(\log N)^C}{q}\right).$$

Combining the above estimates, we complete the proof. □

7. Proof of Corollary 3

Suppose

$$F(x)=\alpha_1 x+\cdots+\alpha_d x^d.$$

By Dirichlet's theorem, for each $1\leqslant \ell\leqslant d$, there exists integers r_ℓ and s_ℓ, with $(r_\ell, s_\ell)=1$ and

$$r_\ell \leqslant \frac{N^\ell}{(\log N)^{4r^2+4rA}},$$

such that

$$\left|\alpha_\ell-\frac{r_\ell}{s_\ell}\right| \leqslant \frac{(\log N)^{4r^2+4rA}}{q_\ell N^\ell}.$$

By Theorem 1, we may suppose, for each $1\leqslant \ell\leqslant d$,

$$s_\ell \leqslant (\log N)^{4r^2+4rA}.$$

By partial summation, we have

$$\left|\sum_{n\leqslant N} f(n)e(F(n))\right| \ll (\log N)^{4r^2+4rA} \max_{u\leqslant N} T(u),$$

with

$$T(u):=\sum_{n\leqslant u} f(n)e(F_1(n))$$

and

$$F_1(x):=\frac{r_d}{s_d}x^d+\cdots+\frac{r_1}{s_1}x.$$

Defining $k:=\operatorname{lcm}(s_d,\ldots,s_1)$, we have

$$T(u)=\sum_{a\leqslant k} e(F_1(a))S(a),$$

with

$$S(a):=\sum_{\substack{n\leqslant u\\ n\equiv a \bmod k}} f(n).$$

Let $e = (a, k)$, and write

$$a' = \frac{a}{e}, \quad k' = \frac{k}{e},$$

so that

$$S(a) = \sum_{\substack{n \leqslant u \\ n \equiv a \bmod k \\ n \equiv 0 \bmod e}} f(n) = \sum_{\substack{n \leqslant u/e \\ n \equiv a' \bmod k'}} f(en),$$

which after expanding into multiplicative characters gives

$$S(a) = \sum_{\substack{n \leqslant u \\ n \equiv a \bmod k \\ n \equiv 0 \bmod e}} f(n) = \frac{f(e)}{\phi(k')} \sum_{\psi \bmod k'} \bar{\psi}(a') \sum_{n \leqslant u/e} \psi(n) f(n),$$

where we have used the assumption that f is completely multiplicative. Hence

$$|S(a)| \leqslant \frac{1}{\phi(k')} \sum_{\psi \bmod k'} \left| \sum_{n \leqslant u/e} \psi(n) f(n) \right|.$$

Thus, observing that

$$k = \operatorname{lcm}(s_d, \dots, s_1) \leqslant \prod_{1 \leqslant i \leqslant d} s_i \leqslant (\log N)^{d(4r^2 + 4rA)},$$

we conclude the proof. □

8. Short mixed character sums

We next use Corollary 3 to show how one may estimate short mixed character sums assuming GRH.

Corollary 12. *Assume the GRH, and let $\varepsilon > 0$. Uniformly over all primitive characters $\chi \bmod q$ and polynomials F of degree d with real coefficients, if N satisfies*

$$\log N \geqslant B \log\log q \tag{31}$$

for a suitable fixed constant B depending only on ε, then

$$\left| \sum_{n \leqslant N} \chi(n) e(F(n)) \right| \ll \frac{N}{(\log N)^{1-\varepsilon}}.$$

We need the following lemma on convolution of Dirichlet characters, which follows from Theorem 2 in [Granville and Soundararajan 2001] and Corollary 1.3 in [Hildebrand and Tenenbaum 1993], remembering that there they have $u = \log x / \log y$.

Lemma 13. *Let χ be a nonprincipal character modulo q. Assuming the GRH, for any x such that*

$$\frac{\log x}{\log\log q} \geqslant B$$

for a sufficiently large constant B, we have

$$\left| \sum_{n \leqslant x} \chi(n) \right| \ll x (\log q)^{-(1+o(1)) \frac{\log x \log\left(\frac{\log x}{2 \log\log q}\right)}{2(\log\log q)^2}}.$$

We can thus prove Corollary 12 in a similar fashion to [Hildebrand 1988b, Lemma 3] and [Kerr 2020, Lemma 7].

Proof of Corollary 12. Clearly we may suppose

$$\left|\sum_{n\leqslant N}\chi(n)e(F(n))\right| \gg \frac{N}{(\log N)^{1-\varepsilon}}.$$

Let k be as in Corollary 3.

For any ψ with modulus smaller than or equal to k, by Lemma 13, we have

$$\max_{u\leqslant N}\left|\sum_{n\leqslant u}\psi(n)\chi(n)\right| \ll N(\log q)^{-(1+o(1))\frac{\log N\log\left(\frac{\log N}{2\log\log q}\right)}{2(\log\log q)^2}}.$$

Therefore, by Corollary 3, taking r large enough in terms of ε,

$$\left|\sum_{n\leqslant N}\chi(n)e(F(n))\right| \ll N(\log x)^{d^5}(\log q)^{-(1+o(1))\frac{\log N\log\left(\frac{\log N}{2\log\log q}\right)}{2(\log\log q)^2}} \ll \frac{N}{(\log N)^{1-\varepsilon}}. \qquad \square$$

9. On the correlation between roots of polynomial congruences and polynomial values

We now prove Theorem 2. Let $p\in\mathbb{Z}[x]$ be irreducible over $\mathbb{Q}$ of degree $e\geqslant 2$, and consider the ratios v/n, where v are the roots of the polynomial p modulo n. In particular, v and n satisfy

$$p(v)\equiv 0 \bmod n.$$

Define the sequence $(g_k)_{k\geqslant 1}$ of these ratios in such a way that the corresponding denominators are in ascending order.

Hooley [1964, Theorem 1] proved that $(g_k)_{k\geqslant 1}$ is equidistributed in $\mathbb{R}/\mathbb{Z}$. Even stronger than that, he showed

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}\left|\sum_{\substack{v\in\mathbb{Z}/n\mathbb{Z}\\ p(v)\equiv 0 \bmod n}} e\left(\frac{hv}{n}\right)\right| = 0 \tag{32}$$

for every $h\in\mathbb{Z}\backslash\{0\}$. To prove Theorem 2, by the Weyl equidistribution criteria, it suffices to prove that, for any $(h_1,h_2)\neq 0$, we have

$$\sum_{n\leqslant N}e(h_1F(n))\sum_{\substack{v\in\mathbb{Z}/n\mathbb{Z}\\ p(v)\equiv 0 \bmod n}} e\left(\frac{h_2v}{n}\right) = o(N).$$

If $h_2\neq 0$ this is true by (32). Thus, the only cases that need to be considered are $h_2=0$ and $h_1\neq 0$. If we let

$$\varrho(n)=|\{v\in\mathbb{Z}/n\mathbb{Z} : p(v)\equiv 0 \bmod n\}|, \tag{33}$$

then the problem is therefore to show that

$$\sum_{n\leqslant N}\varrho(n)e(F(n)) = o(N).$$

We first prove some properties of ϱ.

Lemma 14. *The function ϱ is multiplicative and satisfies the following*:

(1) *For all* $\epsilon > 0$, $\varrho(n) \leqslant_\epsilon n^\epsilon$ *for* $n \to \infty$.

(2) *There exists a constant* $\lambda > 0$ *such that*

$$\sum_{n=1}^{N} \varrho(n) \sim \lambda N.$$

(3) *There exists a constant* $A \geqslant 0$ *such that*

$$\sum_{n=1}^{N} \varrho(n)^2 \ll N(\log N)^A$$

for $N \geqslant 2$.

(4) *There exists a constant* $D \geqslant 1$ *such that*

$$\varrho(mn) \leqslant D^{\mathrm{disc}(f)} \varrho(m)\varrho(n)$$

for all $m, n \in \mathbb{Z}_{>0}$.

Proof. The Chinese remainder theorem implies that ϱ is a multiplicative function. From Wirsing's theorem [1961, Satz 1] we conclude the second item. The proof of the third item is standard (see, e.g., [Crişan and Pollack 2020, Lemma 2.7]).

To conclude the fourth item, we observe that $\varrho(p^\alpha) = \varrho(p)$ for all primes p, $p \nmid \mathrm{disc}(f)$ and $\alpha \in \mathbb{Z}_{\geqslant 1}$, and $\varrho(p^\alpha) \leqslant D$ for all primes p and a constant $D > 0$ (for a proof, see, e.g., [Crişan and Pollack 2020, Lemma 2.4]). We also note that if $p \mid \mathrm{disc}(f)$ and $\varrho(p^\alpha) \neq 0$ then $\varrho(p^\beta) \neq 0$ for all $\beta \leqslant \alpha$, and we can conclude the result by factoring m and n in primes. Note this also implies the first item, since if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ then

$$\varrho(n) \ll D^k \ll D^{\log n/\log\log n}. \qquad \square$$

From now on, we fix $r^* = 2\max(d(d+1), A)$, $B = 4{r^*}^2 + 4rA + 1$ and the constants λ, D and A as in Lemma 14.

From Dirichlet's theorem, we can find a_i and $1 \leqslant q_i \leqslant N^i/(\log N)^B$ coprime integers satisfying

$$\left|\alpha_i - \frac{a_i}{q_i}\right| \leqslant \frac{(\log N)^B}{q_i N^i} \tag{34}$$

for $1 \leqslant i \leqslant d$. We also let

$$q = \mathrm{lcm}(q_1, \ldots, q_d).$$

Proposition 15. *If there exists an* $l \in \{1, \ldots, d\}$ *such that* q_l *satisfies* $q_l \geqslant (\log N)^B$ *then*

$$\sum_{n=1}^{N} \varrho(n) e(F(n)) \ll \frac{N}{(\log N)^{1/2}}.$$

Proof. This follows directly from Theorem 1 applied to $f = \rho$, $r = r^*$ and $R = N^\ell/(\log N)^B$. $\square$

We next focus on establishing the following result.

Proposition 16. *Let* $\varepsilon > 0$, *and suppose that* $q_i < (\log N)^B$ *for all* $1 \leqslant i \leqslant d$. *Then we*

$$\sum_{n=1}^{N} \varrho(n)e(F(n)) \ll \frac{N}{q^{1/(2d(d+1))-\epsilon}},$$

where the implied constant depends on ϵ.

Combining Proposition 16 with Proposition 15, one may deduce Theorem 2. Indeed, since at least one of the α_i is irrational, as $N \to \infty$, the integers q_i in (34) must satisfy $q_i \to \infty$, which implies

$$\sum_{n=1} \varrho(n)e(F(n)) = o(N).$$

To prove Proposition 16 we will proceed with an algebraic approach. We split the task into four subsections: First we make some reductions and provide a proof of the proposition, conditional to further analysis of a different sum. This new sum will be analysed using a Dirichlet series. In the next subsection, we write the Dirichlet series that we want to analyse in terms of better understood L-functions and in the third subsection we state some of its properties. At last, we gather all the results and conclude the proof.

9.1. ***First reductions.*** We assume $q_i < (\log N)^B$ for all $1 \leqslant i \leqslant d$, and we set, for $r \,|\, q$ and $Q \in \mathbb{Z}[x]$,

$$S_r(N, F) = \sum_{\substack{n=1 \\ (n,q)=r}}^{N} \varrho(n)e(F(n));$$

we omit r from the notation when the sum is over all integers $1 \leqslant n \leqslant N$.

We can split $S(N, F)$ into two sums as follows:

$$S(N, F) = \sum_{r\,|\,q} S_r(N, F) = \sum_{\substack{r\,|\,q \\ r<q^{1/(d(d+1))}}} S_r(N, F) + \sum_{\substack{r\,|\,q \\ r\geqslant q^{1/(d(d+1))}}} S_r(N, F). \tag{35}$$

We can bound trivially the second sum of the right-hand side of (35) using Lemma 14. Indeed,

$$\Bigg|\sum_{rqr\geqslant q^{1/(d(d+1))}} S_r(N, F)\Bigg| \leqslant \sum_{\substack{r\,|\,q \\ r\geqslant q^{1/(d(d+1))}}} \sum_{\substack{n=1 \\ (n,q)=r}}^{N} \varrho(n) = \sum_{\substack{r\,|\,q \\ r\geqslant q^{1/(d(d+1))}}} \sum_{\substack{n=1 \\ (n,q/r)=1}}^{N/r} \varrho(rn) \ll_\epsilon \frac{N}{q^{1/(d(d+1))-\epsilon}}. \tag{36}$$

We turn our attention to $S_r(N, F)$ when $r \leqslant 1/(d(d+1))$. Let

$$\widetilde{F}(x) = \sum_{j=1}^{d} (a_j/q_j)x^j \quad \text{and} \quad Q(x) = F(x) - \widetilde{F}(x).$$

Observe that integration by parts yields

$$S_r(N, F) = e(Q(N))S_r(N, \widetilde{F}) - 2\pi i \sum_{j=1}^{d} j\beta_j \int_1^N u^{j-1} e(Q(u))S_r(u, \widetilde{F})\, du, \tag{37}$$

where β_j is defined by

$$\alpha_j = \frac{a_j}{q_j} + \beta_j. \tag{38}$$

In particular, from (34),

$$|\beta_j| \leqslant \frac{(\log N)^B}{q_j N^j}. \tag{39}$$

The above reduces to analysing $S_r(N, \widetilde{F})$ instead of $S_r(N, F)$.

Observe that, since we are summing over multiples of r, we can rewrite $e(\widetilde{F}(n))$ as $e(\widetilde{F}_r(n))$, where $\widetilde{F}_r(n) := \widetilde{F}(rn)$, so that the latter is a periodic function modulo q/r. We let $q/r = r'$, and we decompose $S_r(u, \widetilde{F}_r)$ using Dirichlet characters modulo r' as follows:

$$S_r(N, \widetilde{F}) = \sum_{x \bmod r'} e(\widetilde{F}_r(x)) \sum_{\substack{m \leqslant N/r \\ (m,r')=1 \\ m \equiv x \bmod r'}} \varrho(rm) = \frac{1}{\varphi(r')} \sum_{x \bmod r'} e(\widetilde{F}_r(x)) \sum_{\chi \bmod r'} \overline{\chi(x)} \sum_{m \leqslant N/r} \chi(m)\varrho(rm).$$

We will prove the following.

Proposition 17. *Let χ be a Dirichlet character mod r'. There exist constants $c > 0$ and $0 \leqslant \delta(\chi) \leqslant \lambda r$ such that*

$$\sum_{m \leqslant N/r} \chi(m)\varrho(rm) = \delta(\chi)\frac{N}{r} + O(Ne^{-\frac{1}{c}\sqrt{\log N}}). \tag{40}$$

Moreover, if $\delta(\chi) \neq 0$ then χ has conductor $h \,|\, \mathrm{disc}(p)$.

We next show that Proposition 17 implies we can conclude the proof of Proposition 16. Indeed, we obtain

$$S_r(N, \widetilde{F}) = \frac{N}{r\varphi(r')} \sum_{x \bmod r'} e(\widetilde{F}_r(x)) \sum_{\chi \bmod r'} \overline{\chi(x)}\delta(\chi) + O(Ne^{-\frac{1}{c}\sqrt{\log N}}),$$

which substituted into (37) and using (39) implies

$$S_r(N, F) = e(Q(N))S_r(N, \widetilde{F}) - C\int_1^N u\Big(2\pi i \sum_{j=1}^d j\beta_j u^{j-1}\Big) e(Q(u))\,du + O(Ne^{-\frac{1}{c}\sqrt{\log N}}),$$

with

$$C = \frac{1}{r\varphi(r')} \sum_{x \bmod r'} e(\widetilde{F}_r(x)) \sum_{\chi \bmod r'} \overline{\chi(x)}\delta(\chi).$$

Note that integrating by parts yields

$$\int_1^N u\Big(2\pi i \sum_{j=1}^d j\beta_j u^{j-1}\Big) e(Q(u))\,du \ll N,$$

and since if $\delta(\chi) \neq 0$ then χ has conductor $h \,|\, \mathrm{disc}(p)$, the above implies

$$S_r(N, F) \ll \frac{N}{r}\frac{1}{\phi(r')} \sum_{x \bmod r'} \chi(x) e(\widetilde{F}_r(x)) + O(Ne^{-\frac{1}{c}\sqrt{\log N}}) \tag{41}$$

for some χ with conductor $h \,|\, \mathrm{disc}(p)$. Recall that

$$\widetilde{F}_r(x) = \widetilde{F}(rx) = \sum_{j=1}^d \frac{a_j r^j}{q_j} x^j = \frac{\sum_{j=1}^d (q a_j r^j / q_j) x^j}{q}.$$

Let q' denote the smallest divisor of q such that the polynomial

$$P(x) = \sum_{j=1}^{d} (q a_j r^j / q_j) x^j$$

is constant mod q/q'. Note that, for each $1 \leqslant j \leqslant d$,

$$\frac{q a_j r^j}{q_j} \equiv 0 \bmod q/q'.$$

Since $(a_j, q_j) = 1$, this implies

$$q' > \frac{q_j}{r^j} \geqslant \left(\prod_{j=1}^{d} \frac{q_j}{r^j}\right)^{1/d} \geqslant \frac{q^{1/d}}{r^{(d+1)/2}} \geqslant q^{1/(2d)},$$

provided $r \leqslant q^{1/(d(d+1))}$.

It follows from [Cochrane and Zheng 1999, Corollary 1.1] and the Chinese remainder theorem that

$$\sum_{x \bmod r'} \chi(x) e(\widetilde{F}_r(x)) \ll \frac{r'}{q'} (q')^{1-1/d} \ll \frac{r'}{q^{1/(2d(d+1))}}.$$

Combining the above with (41) gives

$$S_r(N, F) \leqslant \frac{N}{q^{1/(2d(d+1))}} \quad \text{provided } r \leqslant q^{1/(d(d+1))}.$$

Summing over $r < q^{1/(d(d+1))}$, we get

$$\sum_{\substack{r \mid q \\ r < q^{1/(d(d+1))}}} S_r(N, \widetilde{F}) \leqslant \frac{N}{q^{1/(2d(d+1))+o(1)}}.$$

Together with (35) and (36) this concludes the proof of Proposition 16. We will now focus on the proof of Proposition 17. In order to understand the sum

$$\sum_{n \leqslant N/r} \varrho(rn) \chi(n), \tag{42}$$

we study the Dirichlet series for $\chi \bmod r'$:

$$D_r(s, \chi) = \sum_{n=1}^{\infty} \varrho(rn) \chi(n) n^{-s}.$$

9.2. *Decomposing $D_r(s, \chi)$.* Obverse that $D_r(s, \chi)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Our next goal is to extend $D_r(s, \chi)$ to the left of $\mathrm{Re}(s) > 1$ so that we can use contour integration to estimate the sum (42). We will do this by decomposing D_r in terms of well-known Artin L-functions. With this in mind, we fix some notation.

Denote by K_f the splitting field of f in $\mathbb{C}$, by G the Galois group of K_f over $\mathbb{Q}$ and by $\mathbb{Q}(e(1/r'))$ the cyclotomic field generated by r'-roots of unity. Observe that the latter is a Galois extension of $\mathbb{Q}$, with Galois group $C_{r'}$ isomorphic to $(\mathbb{Z}/r'\mathbb{Z})^\times$.

We also consider the compositum of K_f and $\mathbb{Q}(e(1/r'))$ and denote it by $K_{f,r'}$ and its Galois group over $\mathbb{Q}$ by $G_{r'}$. Observe that there is a natural injection

$$G_{r'} \to G \times C_{r'} \simeq G \times (\mathbb{Z}/r'\mathbb{Z})^{\times}.$$

Moreover, observe that, from the extension theorem for field automorphism, it follows that the projections $p_1 : G_{r'} \to G$ and $p_2 : G_{r'} \to (\mathbb{Z}/r'\mathbb{Z})^{\times}$ are surjective.

We denote by $\hat{G}$ the finite set of isomorphism classes of complex irreducible representations of G and, for $\pi \in \hat{G}$, we write χ_π for the character of π.

Note that χ can be viewed as a character of $(\mathbb{Z}/r'\mathbb{Z})^{\times}$ and consequently as a character $\eta : C_{r'} \to \mathbb{C}^{\times}$. It is known that $\eta(\sigma_p) = \chi(p)$ for $p \nmid r'$ and σ_p the Frobenius automorphism at p.

So, we consider the representations $p_1^*\pi = \pi \circ p_1$ and $p_2^*\eta = \eta \circ p_2$ of $G_{r'}$ and observe that their tensor product satisfies

$$\operatorname{tr}(p_1^*\pi \otimes p_2^*\eta(\sigma_p)) = \chi_\pi(\sigma_p)\chi(p) \tag{43}$$

for $p \nmid \operatorname{disc}(f)$ and $p \nmid r'$.

Proposition 18. *$D_r(s, \chi)$ has an expression*

$$D_r(s, \chi) = F_r(s, \chi) \prod_{\pi \in \hat{G}} E_{r,\pi}(s, \chi)(L(s, K_{f,r'}/\mathbb{Q}, p_1^*\pi \otimes p_2^*\eta))^{m_\pi}$$

for $m_\pi \geqslant 0$ an integer and, for any $\epsilon > 0$, $F_r \ll_\epsilon q^\epsilon$ and $E_{r,\pi} \ll q^\epsilon$ for $\operatorname{Re}(s) \geqslant \frac{3}{4}$.

To prove the proposition we will need the following lemma.

Lemma 19. *Denote by π_f the permutation representation of G acting on the set of roots of f in $\mathbb{C}$, and let*

$$\pi_f = \bigoplus_{\pi \in \hat{G}} m_\pi \pi$$

be its decomposition in irreducible representations, where $m_\pi \geqslant 0$ are integers. For all $p \nmid \operatorname{disc}(f)$, we have

$$\varrho(p) = \sum_{\pi} m_\pi \cdot \chi_\pi(\sigma_p) \tag{44}$$

for $\sigma_p \in G$ the Frobenius automorphism at p.

Proof. The proof follows by noting that $\varrho(p)$ is the number of fixed points of the Frobenius automorphism σ_p at p, which is also the character at σ_p of the permutation representation. □

Proof of Proposition 18. First we observe that D_r can be decomposed as

$$D_r(s, \chi) = \sum_{e|r^\infty} \sum_{(k,r)=1} \chi(ek)\varrho(erk)(ek)^{-s} = \sum_{e|r^\infty} \varrho(er)\chi(e)e^{-s} \sum_{(k,r)=1} \chi(k)\varrho(k)k^{-s},$$

where the notation $e|r^\infty$ means that e runs over all possible products of powers of the primes that divide r.

We define

$$E_r(s,\chi)=\sum_{e\mid r^\infty}\varrho(er)\chi(e)e^{-s}$$

$$\widetilde{D}_r(s,\chi)=\sum_{(k,r)=1}\chi(k)\varrho(k)k^{-s},$$

and, using Lemma 14 and the property $\varrho(p^n)\leqslant C$ for all $n\geqslant 1$ and p prime, we observe that $E_r(s,\chi)\ll r^\epsilon$ uniformly for $\mathrm{Re}(s)\geqslant\frac{3}{4}$ and that $\widetilde{D}_r(s,\chi)$ converges absolutely for $\mathrm{Re}(s)>1$.

To be able to explore the properties of ϱ, we factor the primes that divide the discriminant of f in $\widetilde{D}_r(s,\chi)$ and, since $r\mid q$, we also factor the remaining primes that divide q. This yields

$$\widetilde{D}_r(s,\chi)=E_{1,r}(s,\chi)\prod_{\substack{p\nmid \mathrm{disc}(f)\\ p\nmid q}}\sum_{k\geqslant 0}\varrho(p^k)\chi(p)^k p^{-ks},$$

where $E_{1,r}(s,\chi)$ is entire and bounded for $\mathrm{Re}(s)\geqslant\frac{3}{4}$ by $C(\epsilon)q^\epsilon$.

We proceed in a similar manner and factor out the prime powers with exponent larger than 2, obtaining

$$\widetilde{D}_r(s,\chi)=E_{1,r}(s,\chi)E_2(s,\chi)\prod_{\substack{p\nmid \mathrm{disc}(f)\\ p\nmid q}}(1+\varrho(p)\chi(p)p^{-s}) \tag{45}$$

for $E_2(s,\chi)$ holomorphic and bounded in vertical strips by $C(\epsilon)q^\epsilon$ for $\mathrm{Re}(s)\geqslant\frac{3}{4}$.

Now we can use Lemma 19 and a further decomposition to write

$$\widetilde{D}_r(s,\chi)=E_{1,r}(s,\chi)E_2(s,\chi)E_{3,r}(s,\chi)\prod_{\pi\in\hat{G}}\prod_p(1+\chi_\pi(\sigma_p)\chi(p)p^{-s})^{m_\pi}$$

for $E_{3,r}$ holomorphic and uniformly bounded in vertical strips by $C(\epsilon,f)q^\epsilon$ for $\mathrm{Re}(s)\geqslant\frac{3}{4}$. We now set $F_r=E_{1,r}E_2E_{3,r}$. To conclude, we use equality (43) and yet another similar decomposition as before and obtain

$$\prod_p(1+\chi_\pi(\sigma_p)\chi(p)p^{-s})=E_{r,\pi}(s,\chi)L(s,K_{f,q}/\mathbb{Q},p_1^*\pi\otimes p_2^*\eta).$$ □

Observe that now we are dealing with Artin L-functions, which are better understood than the original Dirichlet series. Since p_1 is surjective and π is irreducible, it follows that $p_1^*\pi$ is an irreducible representation. Furthermore, since $p_2^*\eta$ is of dimension 1, we can conclude that $p_1^*\pi\otimes p_2^*\eta$ is also an irreducible representation. Artin's conjecture states that this L-function is entire except if the representation is trivial — which can occur if $p_1^*\pi$ is the inverse of $p_2^*\eta$.

To avoid assuming Artin's conjecture, we will use the Brauer induction theorem instead (see, e.g., [Serre 1977, Theorem 19]). It states that, for every subgroup H of G and one-dimensional character $\beta_H: H\to\mathbb{C}^\times$, there exist integers n_{π,β_H} such that

$$\chi_\pi=\sum_H\sum_{\beta_H}n_{\pi,\beta_H}\cdot\mathrm{Ind}_H^G\beta_H.$$

Observe that this implies that $L(s, K_{f,q}/\mathbb{Q}, p_1^*\pi)$ can be represented as

$$L(s, K_{f,q}/\mathbb{Q}, p_1^*\pi) = L(s, K_f/\mathbb{Q}, \pi) = \prod_H \prod_{\beta_H} L(s, K_f/\mathbb{Q}, \mathrm{Ind}_H^G \beta_H)^{n_{\pi,\beta_H}}$$
$$= \prod_H \prod_{\beta_H} L(s, K_f/K_H, \beta_H)^{n_{\pi,\beta_H}},$$

where $K_H \subset K_f$ is the subfield fixed by H — which implies that H is the Galois group of K_f over K_H.

We can now introduce the twist by the Dirichlet character. We write $H' = p_1^{-1}(H)$ and observe that we can write

$$p_1^*\pi = \bigoplus_H n_{\pi,\beta_H} \mathrm{Ind}_{H'}^{G_{r'}} (p_1^*\beta_H),$$

and thus

$$p_1^*\pi \otimes p_2^*\eta = \bigoplus_H \bigoplus_{\beta_H} n_{\pi,\beta_H} \mathrm{Ind}_{H'}^{G_{r'}} (p_1^*\beta_H \otimes \mathrm{Res}_{H'}^{G_{r'}} p_2^*\eta).$$

So, defining $\theta_{\beta_H} = p_1^*\beta_H \otimes \mathrm{Res}_{H'}^{G_{r'}} p_2^*\eta$, we can finally write

$$L(s, K_{f,q}/\mathbb{Q}, p_1^*\chi \otimes p_2^*\eta) = \prod_H \prod_{\beta_H} L(s, K_{f,q}/K_H, \theta_{\beta_H})^{n_{\pi,\beta_H}},$$

and thus D_r has the representation

$$D_r(s, \chi) = F_r(s, \chi) \prod_{\pi \in \hat{G}} E_{r,\pi}(s, \chi) \prod_H \prod_{\beta_H} L(s, K_{f,q}/K_H, \theta_{\beta_H})^{n_{\pi,\beta_H}}. \tag{46}$$

Now the $L(s, K_{f,q}/K_H, \theta_{\beta_H})^{n_{\pi,\beta_H}}$ are Artin L-functions of dimension 1, which we know are entire except for a pole at $s = 1$ if θ_{β_H} is the trivial one-dimensional character.

9.3. ***Bounds on the Artin L-functions near s = 1.*** We denote by $q_{\pi,i}$ the conductor of $L(s, K_{f,q}/K_H, \theta_{\beta_H})$ and observe that we can bound it as follows (see, e.g., [Bushnell and Henniart 1997]):

$$q_{\pi,\beta_H} \leqslant \mathrm{cond}(p_1^*\beta_H) \cdot \mathrm{cond}(\mathrm{Res}_{H'}^{G_q} p_2^*\eta) \leqslant Mq,$$

where $M = \sup_{\pi,\beta_H} \mathrm{cond}(p_1^*\beta_H)$. It is well known that $L(s, K_{f,q}/K_H, \theta_{\beta_H})$ can have at most one real zero β in the region

$$\sigma \geqslant 1 - \frac{c}{\log(q_{\pi,\beta_H}(|t|+3))} \tag{47}$$

for $s = \sigma + it \in \mathbb{C}$ and c a universal constant; see, e.g., [Iwaniec and Kowalski 2004, Theorem 5.35]. Note that this zero can only exist if θ_i is a quadratic character.

To simplify the notation, in what follows we let $g = \theta_{\beta_H}$ and we omit the field $K_{f,q}/K_H$. Denote by β the possible exception in region (47) and let $\boldsymbol{q}$ be the conductor of g. Also, we set $r_1 = 1$ if g has an exceptional zero β, and $r_1 = 0$ otherwise. Analogously, $r_2 = 1$ if g is the trivial character and consequently has a pole at $s = 1$, and $r_2 = 0$ if not.

Proposition 20. *Let $s=\sigma+it\in\mathbb{C}$ satisfy the conditions*

$$\sigma \geqslant 1-\frac{c}{10\log(Mq(|t|+3))}, \tag{48}$$

$$|s-\beta| \geqslant \frac{1}{20\log(3Mq)}, \tag{49}$$

$$|s-1| \geqslant \frac{1}{20\log(3Mq)}. \tag{50}$$

We have that

$$L(s,g) \ll \log(Mq(|t|+3)), \qquad \frac{1}{L(s,g)} \ll \log(Mq(|t|+3)),$$

where the implied constant does not depend on the parameters.

If g is not trivial, condition (50) can be dropped. Condition (49) can be dropped if g has no exceptional zero.

Proof. We set $\gamma(s,g)=\pi^{-s/2}\gamma\big(\frac{1}{2}(s+\kappa_g)\big)$, where $\kappa_g=0$ or 1, and let $\Lambda(s,g)=\boldsymbol{q}^{s/2}\gamma(s,g)L(s,g)$ be the extended L-function. It is a known fact that $\Lambda(s,g)$ is a meromorphic function of order at most 1. Thus, we can proceed as in [Montgomery and Vaughan 2007, Theorem 11.4] to obtain the proposition. □

9.4. *Finishing the proof.*

Proof of Proposition 16. To bound the sum $\sum_{n\leqslant N/r}\varrho(rn)\chi(n)$ we first smooth it:

$$\sum_{\substack{n=1\\(n,q)=r}}^{N/r}\varrho(n)\chi(n)=\sum_{\substack{n=1\\(n,q)=r}}^{\infty}\varrho(n)\chi(n)\phi(n)+O(\widetilde{N}),$$

where ϕ is defined as

$$\phi(x)=\min\left(x,\,1,\,1+\frac{N/r-x}{\widetilde{N}}\right)$$

for $0\leqslant x\leqslant N/r+\widetilde{N}$, and $\phi(x)=0$ for $x>N/r+\widetilde{N}$, with $\widetilde{N}>0$ to be chosen later. Observe that Mellin inversion implies that

$$\begin{aligned}\sum_{\substack{n=1\\(n,q)=r}}^{\infty}\varrho(n)\chi(n)\phi(n)&=\frac{1}{2\pi i}\int_{(3)}D_r(s,\chi)\hat{\phi}(s)\,ds\\&=\frac{1}{2\pi i}\int_{(3)}F_r(s,\chi)\prod_{\pi\in\hat{G}}E_{r,\pi}(s,\chi)(L(s,K_{f,r'}/\mathbb{Q},p_1^*\pi\otimes p_2^*\eta))^{m_\pi}\hat{\phi}(s)\,ds\\&=\frac{1}{2\pi i}\int_{(3)}F_r(s,\chi)\prod_{\pi\in\hat{G}}E_{r,\pi}(s,\chi)\prod_{H}\prod_{\beta_H}L(s,K_{f,q}/K'_H,\theta_{\beta_H})^{n_{\pi,\beta_H}}\hat{\phi}(s)\,ds,\end{aligned} \tag{51}$$

where (3) denotes the contour obtained by traversing the straight line $\{s=3+it,\,t\in\mathbb{R}\}$ from $t=-\infty$ to $t=+\infty$.

Our goal is to shift the contour of integration and use the information on the L-functions

$$L(s, K_{f,q}/K'_H, \theta_{\beta_H})$$

to obtain good bounds for $S_r(s, \widetilde{F}_r)$. Observe that we can find $\frac{1}{2}c \leqslant Q \leqslant \frac{1}{10}c$ such that all the conditions of Proposition 20 are satisfied for

$$s \in \mathcal{Z} = \left\{ s = \sigma + it, \sigma = 1 - \frac{Q}{\log (Mq(|t|+3))} \right\}.$$

Moving the contour of integration in (51) to $\mathcal{Z}$, we get

$$\sum_{\substack{n=1\\(n,q)=r}}^{\infty} \varrho(n)\chi(n)\phi(n) = \frac{1}{2\pi i}\int_{\mathcal{Z}} D_r(s,\chi)\hat{\phi}(s)\,ds + \alpha\,\mathrm{Res}(D_r(s,\chi),1)\cdot\frac{N}{r} + \tilde{\alpha}\,\mathrm{Res}(D_r(s,\chi),\beta)\cdot N^{\beta}, \tag{52}$$

where $\alpha = 1$ if there is $\pi \in \hat{G}$ for which $p_1^*\pi \otimes p_2^*\eta$ is the principal character in $K_{f,r'}/\mathbb{Q}$ with $m_\pi = 1$, and $\alpha = 0$ otherwise. Likewise, $\tilde{\alpha} = 1$ if $L(s, K_{f,q}/K'_H, \theta_{\beta_H})$ has an exceptional zero β and $n_{\pi,\beta_H} = -1$, and $\tilde{\alpha} = 0$ otherwise. Note from Proposition 18 that there is at most one trivial character $p_1^*\pi \otimes p_2^*\eta$ and one quadratic character θ_{β_H}. Moreover, if $p_1^*\pi \otimes p_2^*\eta$ is the trivial character, then we observe that χ must have a conductor with modulus h that divides $\mathrm{disc}(f)$.

We bound each term of the right-hand side of (52) separately. To bound the first term, we note that the Mellin transform of ϕ satisfies

$$\hat{\phi}(s) = \int_0^{N+\widetilde{N}} \phi(z)z^{s-1}\,dz \ll \frac{N^{\sigma}}{|s|}\min\left(1, \frac{N}{|s|\widetilde{N}}\right),$$

where $\sigma = \mathrm{Re}(s)$; see [Iwaniec and Kowalski 2004, Theorem 5.12]. Thus,

$$\int_{\mathcal{Z}} D_r(s,\chi)\hat{\phi}(s)\,ds \ll \int_{\mathcal{Z}} |D_r(s,\chi)|\frac{N^{\sigma}}{|s|}\min\left(1,\frac{N}{|s|\widetilde{N}}\right)|ds|.$$

We let $T = N/(r\widetilde{N})$ and $\sigma(T) = 1 - Q/\log(Mq(T+3))$, and we note that the observation above and Propositions 18 and 20 imply that

$$\int_{\mathcal{Z}} D_r(s,\chi)\hat{\phi}(s)\,ds \ll N^{\sigma(T)}q^{Y\epsilon}$$

for a constant $Y > 0$ that only depends on f.

We now deal with the last term $\tilde{\alpha}\,\mathrm{Res}(D_r(s,\chi),\beta)\cdot N^{\beta}$. Siegel proved (see, e.g., [Iwaniec and Kowalski 2004, Theorem 5.28]) that if the exceptional zero β exists then, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that

$$\beta \leqslant 1 - \frac{c(\epsilon)}{\mathrm{cond}(\theta_i)^{\epsilon}} \leqslant 1 - \frac{c(\epsilon)}{Mq^{\epsilon}}. \tag{53}$$

To bound the residue at β we notice that, if θ_{π,β_H} is not trivial, we can use Proposition 20 and obtain

$$|L(s, K_{f,q}/K'_H, \theta_{\beta_H})|^{n_{\pi,i}} \ll |\log 3Mq|^{|n_{\pi,\beta_H}|}.$$

If θ_{β_H} is trivial then

$$|L(s, K_{f,q}/K'_H, \theta_{\beta_H})|^{n_{\pi,\beta_H}} \ll_\epsilon q^{\epsilon|n_{\pi,\beta_H}|},$$

and, for the residue given by the L-function with quadratic character, we have

$$\operatorname{Res}\left(\frac{1}{L(s, K_{f,q}/K'_H, \theta_{\beta_H})}, \beta\right) = \frac{1}{L(s, K_{f,q}/K'_H, \theta_{\beta_H})}.$$

We can deduce that $L'(s, K_{f,q}/K'_H, \theta_{\beta_H}) \gg 1$ from the last part of Theorem 11.4 in [Montgomery and Vaughan 2007]. Putting everything together, we obtain

$$\operatorname{Res}(D_r(s, \chi), \beta) \cdot N^\beta \leqslant q^\epsilon N^{1-C(\epsilon)/q^\epsilon}.$$

Pick $T = \exp\left(\frac{1}{3}\sqrt{\log N}\right)$. Since $q \leqslant (\log N)^B$, the considerations above imply that

$$S_r(N, \widetilde{F}_r) = \alpha \operatorname{Res}(D_r(s, \chi), 1) \cdot \frac{N}{r} + O(Ne^{-\frac{1}{c(\epsilon)}\sqrt{\log N}}) \tag{54}$$

for a constant $c(\epsilon)$ and $\epsilon > 0$ sufficiently small. Since

$$\sum_{n \leqslant N} \varrho(n) \leqslant \lambda N$$

and the error term in (54) is $o(N)$, we conclude that

$$\operatorname{Res}(D_r(s, \chi), 1) \leqslant \lambda r. \qquad \square$$

10. Sharpness of Theorem 1

Following the approach of [Montgomery and Vaughan 1977], we will construct a completely multiplicative function $f = f_{N,F}$, with $|f(n)| \leqslant 1$, such that

$$\left|\sum_{n \leqslant N} f(n)e(F(n))\right| \geqslant \frac{1}{10}\frac{N}{\log N}. \tag{55}$$

We first observe that the function $G : \mathbb{C} \to \mathbb{C}$ given by

$$G(z) = \sum_{n \leqslant N} z^{\Omega(n)} e(F(n)) + \sum_{N/2 < p \leqslant N} (1 - ze(F(p)))$$

is entire and its value at zero satisfies

$$G(0) = \sum_{N/2 < p \leqslant N} 1 \geqslant \frac{1}{10}\frac{N}{\log N}$$

for N sufficiently large. Thus, by the maximum modulus principle, there exists a $z_0 \in \mathbb{C}$, $|z_0| = 1$, such that $|G(z_0)| \geqslant |G(0)|$.

Define the completely multiplicative function f by $f(p) = z_0$ for $p \leqslant \frac{1}{2}N$, and $f(p) = e(-F(p))$ for $p > \frac{1}{2}N$. To conclude that (55) is satisfied, we just observe that

$$\sum_{n \leqslant N} f(n)e(F(n)) = \sum_{n \leqslant N} z_0^{\Omega(n)} e(F(n)) + \sum_{N/2 < p \leqslant N} (1 - z_0 e(F(p))) = G(z_0) \gg \frac{N}{\log N}.$$

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