

Algebra & Number Theory

Volume 19
2025
No. 8

**Rational points of rigid-analytic sets:
a Pila–Wilkie-type theorem**

Gal Binyamini and Fumiharu Kato



Rational points of rigid-analytic sets: a Pila–Wilkie-type theorem

Gal Binyamini and Fumiharu Kato

We establish a rigid-analytic analog of the Pila–Wilkie counting theorem, giving subpolynomial upper bounds for the number of rational points in the transcendental part of a \mathbb{Q}_p -analytic set and the number of rational functions in a $\mathbb{F}_q((t))$ -analytic set. For $\mathbb{Z}((t))$ -analytic sets, we prove such bounds uniformly for the specialization to every nonarchimedean local field.

1. Introduction

Our goal in this paper is to introduce a nonarchimedean version of the Pila–Wilkie counting theorem which applies uniformly to analytic sets over local fields in the mixed-characteristic case $F = \mathbb{Q}_p$ and equicharacteristic case $F = \mathbb{F}_q((t))$. We begin by recalling the classical statement of the Pila–Wilkie counting theorem.

1.1. Classical Pila–Wilkie. For $X \subset \mathbb{R}^n$, we define the *algebraic part* X^{alg} to be the union of all connected, positive-dimensional semialgebraic sets contained in X . We set $X^{\text{tran}} := X \setminus X^{\text{alg}}$. For $x \in \mathbb{Q}^n$, we define $H(x) := \max_i H(x_i)$, where $H(x_i)$ is the height of x_i . We set

$$X(\mathbb{Q}, H) := \{x \in X \cap \mathbb{Q}^n \mid H(x) \leq H\}.$$

With this notation, Pila and Wilkie [2006] proved the following counting theorem.

Theorem 1.1.1. *Let X be definable in an o-minimal structure. Then, for every $\varepsilon > 0$, there exists a constant $C(X, \varepsilon)$ such that*

$$\#X^{\text{tran}}(\mathbb{Q}, H) \leq C(X, \varepsilon) \cdot H^\varepsilon.$$

We refer the unfamiliar reader to [Scanlon 2012] for an introduction to the notion of o-minimal structures. Theorem 1.1.1 has found numerous applications in an area of arithmetic geometry known as “unlikely intersection problems” following a general strategy of Pila and Zannier. We also refer the reader to [Scanlon 2012] for a survey in this direction.

MSC2020: 11G50, 14G05, 14G22.

Keywords: rigid-analytic spaces, Pila–Wilkie theorem, rational points, point counting.

1.2. Our main result. We consider a rigid-analytic analog of the Pila–Wilkie theorem, where definable sets are replaced by the spectra of *affinoid* algebras. For the precise definitions see Section 2. For the purposes of this introduction, we first introduce two types of affinoid algebras over local fields.

Set either $F = \mathbb{Q}_p$ and $V = \mathbb{Z}_p$, or $F = \mathbb{F}_q((t))$ and $V = \mathbb{F}_q[[t]]$. An affinoid F -algebra is an F -algebra of the form $\mathcal{B} := B[1/p]$ (resp. $\mathcal{B} := B[1/t]$), where B is a quotient $V\langle\langle x_1, \dots, x_n \rangle\rangle/\mathfrak{b}$ of a formal power series ring (converging in the p -adic (resp. t -adic) topologies). The set of F -rational points of the rigid-analytic spectrum of such an affinoid algebra can naturally be interpreted as a subset of the unit polydisc V^n in F^n . We will be concerned with counting *rational* points in such a spectrum. In particular, for $F = \mathbb{Q}_p$, we set

$$(\mathrm{Sp} \mathcal{B})(\mathbb{Q}, H) := \{x \in \mathrm{Sp} \mathcal{B} \cap \mathbb{Q}^n \mid \max_i H(x_i) \leq H\}$$

as before, and, for $F = \mathbb{F}_q((t))$, we set

$$(\mathrm{Sp} \mathcal{B})(\mathbb{F}_q(t), H) := \{x \in \mathrm{Sp} \mathcal{B} \cap \mathbb{F}_q(t)^n \mid \max_i \deg_t(x_i) \leq \log_q H\}.$$

We say that an irreducible closed subset of $\mathrm{Sp} F\langle\langle x_1, \dots, x_n \rangle\rangle$ is *algebraic* if it is an irreducible component of a closed set defined by a *polynomial ideal* $I \subset F\langle\langle x_1, \dots, x_n \rangle\rangle$, i.e., an ideal obtained by extension from $F[x_1, \dots, x_n]$. More generally we say that a closed subset of $\mathrm{Sp} F\langle\langle x_1, \dots, x_n \rangle\rangle$ is algebraic if its irreducible components are algebraic. Finally we define $(\mathrm{Sp} \mathcal{B})^{\mathrm{alg}}$ to be the union of all irreducible positive-dimensional closed algebraic subsets of $\mathrm{Sp} \mathcal{B}$, and $(\mathrm{Sp} \mathcal{B})^{\mathrm{tran}}$ to be its complement in $\mathrm{Sp} \mathcal{B}$. We refer the reader to Sections 2.11–2.13 for the accurate definition of irreducibility, dimension, and algebraic and transcendental parts, respectively. Note however that while we develop these notions in somewhat greater generality in the present paper, for the definition of algebraic and transcendental parts, it is sufficient to define them over complete nonarchimedean fields, and the corresponding definition may in fact be found in standard treatments such as [Bosch et al. 1984, p. 300].

Our first analog of the Pila–Wilkie theorem is as follows.

Theorem 1.2.1. *Let \mathcal{B} be an affinoid F -algebra as above. Then, for every $\varepsilon > 0$, there exists a constant $C(\mathcal{B}, \varepsilon)$ such that*

$$\#(\mathrm{Sp} \mathcal{B})^{\mathrm{tran}}(F_0, H) \leq C(\mathcal{B}, \varepsilon) \cdot H^\varepsilon, \quad F_0 = \begin{cases} \mathbb{Q} & (F = \mathbb{Q}_p), \\ \mathbb{F}_q(t) & (F = \mathbb{F}_q((t))). \end{cases}$$

In the \mathbb{Q}_p case, our result is a special case of the result in [Cluckers et al. 2015], where a similar result was proved more generally for *subanalytic* \mathbb{Q}_p sets. In the $\mathbb{F}_q((t))$ case the result is new.

Remark 1.2.2. Demangos [2015; 2016] claimed a proof of Theorem 1.2.1 in both the \mathbb{Q}_p and the $\mathbb{F}_q((t))$ cases. However it has been known to the experts for several years that the proof contains a substantial gap in both cases.

Our main result is a uniform version of Theorem 1.2.1 where we allow the field F to vary. Toward this end, we let \mathcal{O} denote the ring of integers of an algebraic number field and replace F by $\mathcal{O}((t))$ and V by $\mathcal{O}[[t]]$. We develop an analogous notion of affinoid algebras in this context. For the purpose of this

introduction, the reader may consider an affinoid algebra to be an $\mathcal{O}((t))$ -algebra $\mathcal{B} := B[1/t]$, where B is a quotient $B = \mathcal{O}[[t]]\langle\langle x_1, \dots, x_n \rangle\rangle/\mathfrak{b}$ of a power series ring converging in the t -adic topology. Consider some local nonarchimedean field F_α with valuation ring V_α and an adic homomorphism $\alpha : \mathcal{O}[[t]] \rightarrow V_\alpha$. Let π be a uniformizer of V_α , $q_\alpha = p^f$ be the number of elements of the residue field $V_\alpha/(\pi)$, and $r_\alpha \geq 1$ be the number given by $\alpha(t) = u\pi^{r_\alpha}$, where $u \in V_\alpha^\times$. Then the fiber $(\mathrm{Sp} \mathcal{B})_\alpha$ is an F_α -affinoid space. We set

$$\sigma_\alpha = \begin{cases} [F_\alpha : \mathbb{Q}_p] & (F_\alpha \supset \mathbb{Q}_p), \\ r_\alpha & (F_\alpha \supset \mathbb{F}_q((t))). \end{cases}$$

Our main theorem is as follows.

Theorem 1.2.3. *Let \mathcal{B} be an affinoid $\mathcal{O}((t))$ -algebra. Then, for every $\varepsilon > 0$, there exists a constant $C(\mathcal{B}, \varepsilon, \sigma_\alpha)$ such that, for every fiber $(\mathrm{Sp} \mathcal{B})_\alpha$ as above,*

$$\#(\mathrm{Sp} \mathcal{B})_\alpha^{\mathrm{tran}}(F_{\alpha,0}, H) \leq C(\mathcal{B}, \varepsilon, \sigma_\alpha) \cdot q_\alpha^n \cdot H^\varepsilon, \quad F_{\alpha,0} = \begin{cases} \mathbb{Q} & (F_\alpha \supset \mathbb{Q}_p), \\ \mathbb{F}_q(t) & (F_\alpha \supset \mathbb{F}_q((t))). \end{cases}$$

Theorem 1.2.1 follows immediately from Theorem 1.2.3. A similar result, which holds more generally for $\mathcal{O}[[t]]$ -subanalytic sets, was established in [Cluckers et al. 2020, Theorem B]. However, that similar result holds for local fields with residue characteristic larger than some $N = N(X, \varepsilon)$.

More generally, consider a morphism of affinoid algebras $\mathcal{A} \rightarrow \mathcal{B}$, and view $\mathrm{Sp} \mathcal{B} \rightarrow \mathrm{Sp} \mathcal{A}$ as a family of rigid-analytic spaces. Then any adic homomorphism $\alpha : A \rightarrow V_\alpha$ from a formal model A of \mathcal{A} corresponds to a point of $\mathrm{Sp} \mathcal{A}$, and we have the following.

Theorem 1.2.4. *Let \mathcal{A} and \mathcal{B} be affinoid $\mathcal{O}((t))$ -algebras. Then, for any $\varepsilon > 0$ and any positive integer σ , there exists a constant $C(\mathcal{A}, \mathcal{B}, \varepsilon, \sigma)$ such that, for every fiber $(\mathrm{Sp} \mathcal{B})_\alpha$ as above,*

$$\#(\mathrm{Sp} \mathcal{B})_\alpha^{\mathrm{tran}}(F_{\alpha,0}, H) \leq C(\mathcal{A}, \mathcal{B}, \varepsilon, \sigma) \cdot q_\alpha^n \cdot H^\varepsilon, \quad F_{\alpha,0} = \begin{cases} \mathbb{Q} & (F_\alpha \supset \mathbb{Q}_p), \\ \mathbb{F}_q(t) & (F_\alpha \supset \mathbb{F}_q((t))). \end{cases}$$

Theorem 1.2.4 restricts to Theorem 1.2.3 when $\mathcal{A} = \mathcal{O}((t))$.

1.3. Overconvergent versions. Most of the technical work in the present paper is carried out with the flavor of “overconvergence”. That is, in the notation of the previous section, we consider a formal model B and affinoid algebra $\mathcal{B} := B[1/t]$, but restrict our attention to the part of $\mathrm{Sp} \mathcal{B}$ that belongs to the polydisc of radius $|t^\delta|$ for some $\delta \in \mathbb{Q}$ with $\delta > 0$. Denote this by $\mathrm{Sp} \mathcal{B}^{\delta}$. For this overconvergent part, we have the following more uniform version of Theorem 1.2.3.

Theorem 1.3.1. *Let \mathcal{B} be an affinoid $\mathcal{O}((t))$ -algebra. Then, for any $\varepsilon > 0$ and any positive integer σ , there exists a constant $C(\mathcal{B}, \varepsilon, \delta, \sigma)$ such that, for every fiber $(\mathrm{Sp} \mathcal{B})_\alpha$ as in Theorem 1.2.3,*

$$\#(\mathrm{Sp} \mathcal{B}^{\delta})_\alpha^{\mathrm{tran}}(F_{\alpha,0}, H) \leq C(\mathcal{B}, \varepsilon, \delta, \sigma) \cdot H^\varepsilon, \quad F_{\alpha,0} = \begin{cases} \mathbb{Q} & (F_\alpha \supset \mathbb{Q}_p), \\ \mathbb{F}_q(t) & (F_\alpha \supset \mathbb{F}_q((t))). \end{cases}$$

Note that here we avoid the extra q_α^n -term, making the result truly uniform over all \mathbb{Q}_p - and $\mathbb{F}_q((t))$ -points. The proof of Theorem 1.3.1 is given in the first part of Section 5.2. Theorem 1.2.3 is obtained

from Theorem 1.3.1 by covering $\mathrm{Sp} \mathcal{B}$ by q_α^n polydiscs of radius $|t|$ and applying Theorem 1.3.1 to each of them. This is carried out in the second part of Section 5.2.

Theorem 1.2.3 is a special case of Theorem 5.1.3, which establishes a similar result where the rigid-analytic space $\mathrm{Sp} \mathcal{B}$ is also allowed to vary in a rigid-analytic family and the constant $C(\mathcal{B}, \varepsilon, \delta, \sigma)$ is uniformly bounded over the family. Similar statements have been obtained in the original Pila–Wilkie setting [Pila and Wilkie 2006] as well as in [Cluckers et al. 2015; 2020].

1.4. Comparison with earlier work. The work of [Cluckers et al. 2015] on Pila–Wilkie counting for \mathbb{Q}_p -subanalytic sets is roughly analogous to the proof in the classical case. The key difficulty is to find a suitable replacement for the reparametrization lemma (proved by Yomdin [1987] and Gromov [1987] in the algebraic setting and extended to the general o-minimal structure by Pila and Wilkie [2006]). This is accomplished in [Cluckers et al. 2015] through a systematic study of Lipschitz continuous cell decompositions. In [Cluckers et al. 2020], this is extended to $\mathbb{Z}[[t]]$. The model-theoretic machinery then allows one to specialize uniformly to every nonarchimedean local field of sufficiently high characteristic.

Our proofs follow a different approach to the Pila–Wilkie theorem introduced in [Binyamini and Novikov 2017a]. This approach avoids the use of the reparametrization theorem and replaces it by an argument in the spirit of Weierstrass preparation: instead of covering an analytic set by smooth charts as in the reparametrization lemma, one covers it by *Weierstrass polydiscs* where analytic functions can be Weierstrass prepared. It turns out, perhaps unsurprisingly, that Noether normalization provides a very direct analog of the Weierstrass polydisc construction in the rigid-analytic setting. Our main technical result in this direction, Theorem 3.3.2, shows that Noether normalization can be performed uniformly in families — giving a suitable replacement of the reparametrization lemma. The main advantage of this approach is that Noether normalization is much easier to carry out in positive characteristics than the reparametrization approach of [Cluckers et al. 2015; 2020], and it is this feature that allows us to carry out our proofs over $\mathbb{F}_q((t))$ and indeed uniformly over all characteristics.

Remark 1.4.1. In this paper we restrict our attention to the setting of affinoid spaces, which is less general than the subanalytic spaces considered in [Cluckers et al. 2015; 2020]. In the archimedean setting, the approach of [Binyamini and Novikov 2017a] was also used with relatively little effort to recover the subanalytic case from the analytic case. It seems possible that an analogous approach would yield a “subanalytic” version of our result in the rigid-analytic setting as well. In fact, the contributions of van den Dries [1992], Lipshitz and Robinson [2005], and Cluckers and Lipshitz [2011], together with the overconvergent approach by Martin [2016], may be suitable for our purposes.

1.5. Toward polylogarithmic counting theorems. The Weierstrass polydisc construction developed in [Binyamini and Novikov 2017a] has played the central role in many further developments around the Pila–Wilkie theorem concerning questions of effectivity [Binyamini 2019] as well as in the direction of the Wilkie conjecture, i.e., the improvement of the asymptotic $O(H^\varepsilon)$ to a polylogarithmic $(\log H)^\kappa$ [Binyamini and Novikov 2017b; Binyamini 2022]. In the nonarchimedean context, the same idea was used in [Binyamini et al. 2022] to prove a polylogarithmic counting result for germs of analytic varieties

defined by Pfaffian or Noetherian functions. Namely, the number of $\mathbb{C}(t)$ -rational curves in such a germ grows polynomially as a function of the degree.

Since our approach here establishes a very direct analog of the Weierstrass polydisc construction in rigid-analytic geometry, it seems likely that it could lead to similar improvements in the rigid-analytic setting. We intend to pursue these applications in forthcoming work. To illustrate the potential of this approach, we prove the following polylogarithmic interpolation result, which can be seen as a first step toward polylogarithmic counting.

Theorem 1.5.1. *Let \mathcal{B} be an affinoid $\mathcal{O}((t))$ -algebra. Then there exists a constant*

$$C = \begin{cases} C(\mathcal{B}, \delta, [F : \mathbb{Q}_p]/r_\alpha) & (F_\alpha \supset \mathbb{Q}_p), \\ C(\mathcal{B}, \delta) & (F_\alpha \supset \mathbb{F}_q((t))) \end{cases}$$

such that, for every fiber $(\mathrm{Sp} \mathcal{B})_\alpha$ as in Theorem 1.2.3, the set $(\mathrm{Sp} \mathcal{B}^{\delta})_\alpha^{\mathrm{tran}}(F_{\alpha,0}, H)$ is contained in an algebraic hypersurface of degree $C(\log_{q_\alpha} H)^d$.

Similar results have been obtained in [Cluckers et al. 2015] in the \mathbb{Q}_p setting and in [Cluckers et al. 2020] uniformly for sufficiently large primes. However, experience from the classical \mathbb{R} context [Binyamini and Novikov 2017b] and the $\mathbb{C}((t))$ context [Binyamini et al. 2022] shows that the approach using Weierstrass polydiscs makes it easier to extend such results into a full-fledged polylogarithmic counting theorem.

1.6. Organization of this paper. This paper is organized as follows. In Section 2 we develop some background material on rigid-analytic geometry over $\mathcal{O}((t))$, where \mathcal{O} is an integer ring of a number field. In Section 3 we develop a uniform version of the Noether normalization theorem in this context, which forms our rigid-analytic analog of the “Weierstrass polydisc” construction of [Binyamini and Novikov 2017a]. In Section 4 we prove a result on interpolation of rational points in rigid spaces using algebraic hypersurfaces, assuming that the space is Noether normalized. Finally in Section 5 we prove the point counting theorems.

1.7. Conventions.

- We denote by \mathbb{N} the set of all nonnegative integers.
- For a local ring R , we denote by \mathfrak{m}_R its maximal ideal.
- For $v = a/b \in \mathbb{Q}$ a reduced fraction, we define $H(v) = \max\{|a|, |b|\}$. For $v = a(t)/b(t) \in \mathbb{F}_q((t))$ a reduced fraction, we define $H(v) = \max\{q^{\deg a}, q^{\deg b}\}$.

2. Rigid geometry

2.1. Admissible algebras. Let F_0 be a number field fixed once for all, and $\mathcal{O} = \mathcal{O}_{F_0}$ be the integer ring of F_0 . We consider the formal power series ring $\mathcal{O}[[t]]$ with the t -adic topology. We denote by $\mathcal{O}[[t]]\langle\langle \mathbf{x} \rangle\rangle$ (where $\mathbf{x} = (x_1, \dots, x_n)$) the *restricted power series ring over $\mathcal{O}[[t]]$* , i.e., the t -adic completion of the polynomial ring $\mathcal{O}[[t]][\mathbf{x}]$.

Definition 2.1.1. (1) A *topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra* is an $\mathcal{O}[[t]]$ -algebra A that is isomorphic to an $\mathcal{O}[[t]]$ -algebra of the form $\mathcal{O}[[t]]\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}$, where $\mathfrak{a} \subset \mathcal{O}[[t]]\langle\langle \mathbf{x} \rangle\rangle$ is an ideal. By a *morphism of topologically-of-finite-type $\mathcal{O}[[t]]$ -algebras*, we mean an $\mathcal{O}[[t]]$ -algebra homomorphism.

(2) A topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra is said to be *admissible* if it is t -torsion free.

Note that topologically-of-finite-type $\mathcal{O}[[t]]$ -algebras are Noetherian and t -adically complete, and every morphism between them is adic. We denote by $\mathbf{Ad}_{\mathcal{O}[[t]]}$ the category of admissible $\mathcal{O}[[t]]$ -algebras and $\mathcal{O}[[t]]$ -algebra homomorphisms.

The notion of topologically-of-finite-type $\mathcal{O}[[t]]$ -algebras gives rise to the notion of finite-type formal schemes over $\mathcal{O}[[t]]$. Note that if X is a finite-type formal scheme over $\mathcal{O}[[t]]$, then X_0 (the closed fiber by $t = 0$) is a finite-type \mathcal{O} -scheme and hence is Jacobson.

2.1.2. Restricted power series ring over A . For any topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra A , we denote by $A\langle\langle \mathbf{x} \rangle\rangle$ the t -adic completion of the polynomial ring $A[\mathbf{x}]$, or, what amounts to the same,

$$A\langle\langle \mathbf{x} \rangle\rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \mathbf{x}^\nu \in A[[\mathbf{x}]] \mid \begin{array}{l} \text{for any } m \in \mathbb{Z}_{>0}, \text{ there exists } M \in \mathbb{Z}_{>0} \\ \text{such that } |\nu| \geq M \Rightarrow a_\nu \in t^m A \end{array} \right\}.$$

Lemma 2.1.3. *Let A be a topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra and $N \subset A$ its t -torsion part, i.e., the ideal consisting of the t -torsion elements. Then the t -torsion part of $A\langle\langle \mathbf{x} \rangle\rangle$ is the ideal*

$$\left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \mathbf{x}^\nu \mid a_\nu \in N \text{ for any } \nu \in \mathbb{N}^n \right\}. \tag{1}$$

Thus we have

$$A\langle\langle \mathbf{x} \rangle\rangle / (t\text{-torsion}) \cong (A / (t\text{-torsion}))\langle\langle \mathbf{x} \rangle\rangle.$$

In particular, if A is admissible, then so is $A\langle\langle \mathbf{x} \rangle\rangle$.

Proof. Let M be the t -torsion part of $A\langle\langle \mathbf{x} \rangle\rangle$. It is clear that M is contained in the ideal (1). Since A is Noetherian, there exists sufficiently large $m > 0$ such that $t^m N = 0$. If $F \in A\langle\langle \mathbf{x} \rangle\rangle$ belongs to the ideal (1), then we have $t^m F = 0$, and hence $F \in M$. □

2.2. Classical points. We consider rigid spaces of finite type over $\mathcal{S} = (\mathrm{Spf} \mathcal{O}[[t]])^{\mathrm{rig}}$. A finite-type affinoid over \mathcal{S} , for example, is a rigid space isomorphic to a rigid space of the form $(\mathrm{Spf} A)^{\mathrm{rig}}$ given by a topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra A .

Let \mathcal{X} be a rigid space of finite type over \mathcal{S} . Simply by a *point* of \mathcal{X} , we usually mean a point of the associated Zariski–Riemann space $\langle \mathcal{X} \rangle$. A *classical point* of \mathcal{X} is a retro-compact point-like rigid subspace of \mathcal{X} [Fujiwara and Kato 2018, II.8.2.8]. Note that any classical point of \mathcal{X} is closed [Fujiwara and Kato 2018, II.8.2.9]. As usual, the set of all classical points of \mathcal{X} is denoted by $\langle \mathcal{X} \rangle^{\mathrm{cl}}$. If $\mathcal{X} = (\mathrm{Spf} A)^{\mathrm{rig}}$, where A is an admissible $\mathcal{O}[[t]]$ -algebra, then classical points are in canonical one-to-one correspondence with closed points of the Noetherian scheme $\mathrm{Spec} \mathcal{A}$, where $\mathcal{A} = A[1/t]$ [Fujiwara and Kato 2018, II.8.2.11].

Remark 2.2.1. The classical points in the situation of classical rigid geometry are often referred to as *rig-points*; see [Bosch 2014, §8.3].

Lemma 2.2.2. *Any classical point of a rigid space of finite type over \mathcal{S} is isomorphic to $(\mathrm{Spf} V)^{\mathrm{rig}}$, where V is a complete discrete valuation ring with finite residue field.*

Proof. By [Fujiwara and Kato 2018, II.8.2.6], the classical points are of the form $(\mathrm{Spf} V)^{\mathrm{rig}}$ for a complete discrete valuation ring V . Since V is topologically of finite type over $\mathcal{O}[[t]]$, V/tV is of finite type over \mathcal{O} . As the residue field of V is of finite type over \mathcal{O} , it is a finite field. \square

Corollary 2.2.3. *If A is a topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra, and \mathfrak{m} is a maximal ideal of $A = A[1/t]$, then the residue field A/\mathfrak{m} is a nonarchimedean local field, i.e., a complete discrete valuation field with finite residue field.*

Example 2.2.4. Let us consider the case $\mathcal{O} = \mathbb{Z}$, and let p be a prime number.

(1) A classical point $(\mathrm{Spf} \mathbb{Z}_p)^{\mathrm{rig}} \hookrightarrow (\mathrm{Spf} \mathbb{Z}[[t]])^{\mathrm{rig}}$ is given by the surjective homomorphism $\mathbb{Z}[[t]] \rightarrow \mathbb{Z}_p$ that maps t to p .

(2) A classical point $(\mathrm{Spf} \mathbb{F}_p[[t]])^{\mathrm{rig}} \hookrightarrow (\mathrm{Spf} \mathbb{Z}[[t]])^{\mathrm{rig}}$ is given by the canonical homomorphism

$$\mathbb{Z}[[t]] \rightarrow \mathbb{F}_p[[t]] \cong \mathbb{Z}[[t]]/p\mathbb{Z}[[t]].$$

Example 2.2.5. Consider the homomorphism

$$\mathbb{Z}[[t]] \rightarrow R = \mathbb{Z}[[t]]/(p^3 - t^2).$$

The normalization \tilde{R} of R is a ramified quadratic extension of \mathbb{Z}_p . In fact, \tilde{R} can be obtained from the strict transform of the admissible blow-up $X \rightarrow \mathrm{Spf} \mathbb{Z}[[t]]$ along the admissible ideal $J = (p, t)$; i.e.,

$$X = \mathrm{Spf} \mathbb{Z}[[t]]\langle\langle t/p \rangle\rangle \cup \mathrm{Spf} \mathbb{Z}[[t]]\langle\langle p/t \rangle\rangle \rightarrow \mathrm{Spf} \mathbb{Z}[[t]],$$

and the strict transform on $\mathrm{Spf} R$ is $\mathrm{Spf} \tilde{R} \rightarrow \mathrm{Spf} R$, where $\tilde{R} \cong \mathbb{Z}[[t]]\langle\langle t/p \rangle\rangle/(p - (t/p)^2)$. In particular, the closed immersion $\mathrm{Spf} \tilde{R} \hookrightarrow X$ gives rise to a classical point of $(\mathrm{Spf} \mathbb{Z}[[t]])^{\mathrm{rig}}$.

Lemma 2.2.6. *Let $\phi : \mathcal{O}[[t]] \rightarrow V$, where V is as in Lemma 2.2.2, be an adic morphism that arises from a classical point of a rigid space of finite type over \mathcal{S} , and set $\mathfrak{q} = \phi^{-1}(\mathfrak{m}_V)$ and $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}$.*

(a) \mathfrak{q} (resp. \mathfrak{p}) is a maximal ideal of $\mathcal{O}[[t]]$ (resp. \mathcal{O}), and $\mathfrak{q} = \mathfrak{p}\mathcal{O}[[t]] + (t)$;

(b) V is finite over $\mathcal{O}[[t]]$.

Proof. Since the residue field $k = V/\mathfrak{m}_V$ is finite, \mathfrak{q} (resp. \mathfrak{p}) is a maximal ideal of $\mathcal{O}[[t]]$ (resp. \mathcal{O}). Since $\phi(t) \in \mathfrak{m}_V$, we have $\mathfrak{p}\mathcal{O}[[t]] + (t) \subset \mathfrak{q}$. But, since $\mathcal{O}[[t]]/(\mathfrak{p}\mathcal{O}[[t]] + (t)) \cong \mathcal{O}/\mathfrak{p}$, we have that $\mathfrak{p}\mathcal{O}[[t]] + (t)$ is maximal, and hence we have $\mathfrak{p}\mathcal{O}[[t]] + (t) = \mathfrak{q}$. Since V/tV is a finite ring, V/tV is finite over $\mathcal{O}[[t]]/t\mathcal{O}[[t]]$. Then, by [Matsumura 1989, 8.4], V is finite over $\mathcal{O}[[t]]$. \square

Lemma 2.2.7 (functoriality of classical points). *Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of rigid spaces of finite type over S and $(\mathrm{Spf} V)^{\mathrm{rig}} \hookrightarrow \mathcal{X}$ be a classical point. Then there exists uniquely a commutative diagram*

$$\begin{array}{ccc} (\mathrm{Spf} V)^{\mathrm{rig}} & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ (\mathrm{Spf} W)^{\mathrm{rig}} & \hookrightarrow & \mathcal{Y} \end{array}$$

where $W \rightarrow V$ is finite and $(\mathrm{Spf} W)^{\mathrm{rig}} \hookrightarrow \mathcal{Y}$ is a classical point. Thus we have the map $\langle \mathcal{X} \rangle^{\mathrm{cl}} \rightarrow \langle \mathcal{Y} \rangle^{\mathrm{cl}}$ between the sets of classical points.

Proof. We may assume that $\mathcal{X} = (\mathrm{Spf} A)^{\mathrm{rig}}$ and $\mathcal{Y} = (\mathrm{Spf} B)^{\mathrm{rig}}$, and that $\mathcal{X} \rightarrow \mathcal{Y}$ is induced from a morphism $B \rightarrow A$ of admissible $\mathcal{O}[[t]]$ -algebras. Then any classical point of \mathcal{X} corresponds to an adic morphism $A \rightarrow V$ to a complete discrete valuation ring V with finite residue field. Since classical points in our situation are closed, we may assume that $A \rightarrow V$ is surjective.

Let $\phi : \mathcal{O}[[t]] \rightarrow V$ be given by the composition $\mathcal{O}[[t]] \rightarrow B \rightarrow A \rightarrow V$, and set $\mathfrak{q} = \phi^{-1}(\mathfrak{m}_V)$ and $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}$. Note that, since $\sum_{v \geq 0} a_v t^v \in \mathcal{O}[[t]]$ does not belong to \mathfrak{q} if and only if $a_0 \notin \mathfrak{p}$, we have $\widehat{\mathcal{O}[[t]]}_{\mathfrak{q}} = \mathcal{O}_{\mathfrak{p}}[[t]]$. Let R be the image of $\widehat{\mathcal{O}[[t]]}_{\mathfrak{q}}$ in V , which is a t -adically complete local subring of V , and let \widetilde{R} be the normalization of R in V , which is a t -adically complete discrete valuation subring of V . Let $J \subset R$ be the admissible ideal such that \widetilde{R} is the admissible blow-up along J .

Consider the base change $B_R = B \widehat{\otimes}_{\mathcal{O}[[t]]} R \rightarrow A_R = A \widehat{\otimes}_{\mathcal{O}[[t]]} R$ and the strict transform $B_{\widetilde{R}} \rightarrow A_{\widetilde{R}}$ by $R \rightarrow \widetilde{R}$. Since $A_{\widetilde{R}} \rightarrow V$ is surjective, it defines a classical point of $\mathcal{X}_{\widetilde{R}} = (\mathrm{Spf} A_{\widetilde{R}})^{\mathrm{rig}}$. By the functoriality of classical points over valuation rings of height 1 [Fujiwara and Kato 2018, II.8.2.14], one has a commutative diagram of the form

$$\begin{array}{ccc} \mathrm{Spf} V & \hookrightarrow & \mathrm{Spf} A_{\widetilde{R}} \\ \downarrow & & \downarrow \\ \mathrm{Spf} W & \xrightarrow{i} & Z \\ & & \downarrow \pi \\ & & \mathrm{Spf} B_{\widetilde{R}} \end{array}$$

where W is a complete discrete valuation ring with finite residue field, $W \rightarrow V$ is finite, i is a closed immersion, and π is an admissible blow-up.

Let $\pi' : Z \rightarrow \mathrm{Spf} B_R$ be the composition $Z \xrightarrow{\pi} \mathrm{Spf} B_{\widetilde{R}} \rightarrow \mathrm{Spf} B_R$ of two admissible blow-ups, which is again an admissible blow-up [Fujiwara and Kato 2018, II.1.1.10], and let $J' \subset B_R$ be the corresponding admissible ideal. Let $J_1 \subset B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t]]_{\mathfrak{q}}$ be the pull-back of J' by the surjection $B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t]]_{\mathfrak{q}} \rightarrow B_R$. Then J_1 is an admissible ideal of $B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t]]_{\mathfrak{q}}$, and the corresponding admissible blow-up $\pi_1 : Z_1 \rightarrow \mathrm{Spf} B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t]]_{\mathfrak{q}}$ gives rise to π' by passage to the strict transform. Note that $Z \hookrightarrow Z_1$ is a closed immersion. Suppose $t^N \in J_1$. Then J_1 corresponds to a finitely generated ideal \bar{J}_1 of $B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t]]_{\mathfrak{q}}/(t^N) = B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}_{\mathfrak{p}}[[t]]/(t^N)$, which extends to a finitely generated ideal \bar{J}_2

of $B \otimes_{\mathcal{O}[[t]]} \mathcal{O}[1/a][t]/(t^N)$ for some $a \in \mathcal{O} \setminus \mathfrak{p}$. Then the pull-back $J_2 \subset B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[1/a][[t]]$ of \bar{J}_2 gives rise to the admissible blow-up $\pi_2 : Z_2 \rightarrow \mathrm{Spf} B \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[1/a][[t]]$ that extends π_1 . Now, due to [Fujiwara and Kato 2018, II.1.1.9], we can further extend π_2 to an admissible blow-up $\pi_3 : Y \rightarrow \mathrm{Spf} B$. Thus we have a locally closed immersion $\mathrm{Spf} W \rightarrow Y$, which defines a classical point of $\mathcal{Y} = Y^{\mathrm{rig}}$ (see [Fujiwara and Kato 2018, II.8.2.9 (2)]) and fits in with the following commutative diagram:

$$\begin{array}{ccccc}
 \mathrm{Spf} V & \hookrightarrow & \mathrm{Spf} A_R & \hookrightarrow & \mathrm{Spf} A \\
 \downarrow & & & & \downarrow \\
 \mathrm{Spf} W & \xrightarrow{i} & Z & \hookrightarrow & Y \\
 & & \downarrow \pi & & \downarrow \\
 & & \mathrm{Spf} B_R & \hookrightarrow & \mathrm{Spf} B
 \end{array}$$

The uniqueness follows from [Fujiwara and Kato 2018, II.8.2.11]. □

Corollary 2.2.8. *Let $A \rightarrow B$ be a morphism of topologically-of-finite-type $\mathcal{O}[[t]]$ -algebras. Then $\mathrm{Spec} B[1/t] \rightarrow \mathrm{Spec} A[1/t]$ maps closed points to closed points.*

2.3. Affinoid algebras. Set $\mathcal{O}((t)) = \mathcal{O}[[t]][1/t]$.

Definition 2.3.1. An *affinoid $\mathcal{O}((t))$ -algebra* is an $\mathcal{O}((t))$ -algebra that is isomorphic to an $\mathcal{O}((t))$ -algebra of the form $A[1/t]$ by a topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra A . By a *morphism of affinoid $\mathcal{O}((t))$ -algebras*, we mean an $\mathcal{O}((t))$ -algebra homomorphism.

Definition 2.3.2. (1) A *formal model* of an affinoid $\mathcal{O}((t))$ -algebra \mathcal{A} is a topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra A with an isomorphism $A[1/t] \xrightarrow{\sim} \mathcal{A}$ of $\mathcal{O}((t))$ -algebras.

(2) A formal model is said to be *admissible* if it is given by an admissible $\mathcal{O}[[t]]$ -algebra.

If A is a formal model of \mathcal{A} , then $A/(t\text{-torsion})$ gives an admissible formal model. In particular, any affinoid $\mathcal{O}((t))$ -algebra \mathcal{A} admits an admissible formal model. Note that a formal model A of \mathcal{A} is admissible if and only if the map $A \rightarrow \mathcal{A}$ is injective.

We denote by $\mathbf{Af}_{\mathcal{O}((t))}$ the category of affinoid $\mathcal{O}((t))$ -algebras and $\mathcal{O}((t))$ -algebra homomorphisms. There is a functor

$$(\cdot)[1/t] : \mathbf{Ad}_{\mathcal{O}[[t]]} \rightarrow \mathbf{Af}_{\mathcal{O}((t))}, \quad A \mapsto A[1/t].$$

Proposition 2.3.3. *Any affinoid $\mathcal{O}((t))$ -algebra is Jacobson.*

Proof. It suffices to show that, for any affinoid $\mathcal{O}((t))$ -domain \mathcal{A} , the intersection of all maximal ideals of \mathcal{A} is zero. Let \mathfrak{p} be the kernel of $\mathcal{O} \rightarrow \mathcal{A}$, which is a prime ideal of \mathcal{O} . If $\mathfrak{p} \neq (0)$, then \mathfrak{p} is a maximal ideal of \mathcal{O} , and \mathcal{A} is an affinoid algebra over the complete discrete valuation field $(\mathcal{O}/\mathfrak{p})((t))$. Hence \mathcal{A} is known to be Jacobson. Suppose $\mathfrak{p} = (0)$. For any nonzero $f \in \mathcal{A}$, consider the nonempty open subset $D(f) = \mathrm{Spec} \mathcal{A}_f$ of $\mathrm{Spec} \mathcal{A}$. We need to show that $D(f)$ contains a closed point of $\mathrm{Spec} \mathcal{A}$. Since the image of $\phi : \mathrm{Spec} \mathcal{A} \rightarrow \mathrm{Spec} \mathcal{O}$ is dense and \mathcal{O} is Jacobson, there exists a closed point x of $\mathrm{Spec} \mathcal{O}$ such

that $\phi^{-1}(x) \cap D(f) \neq \emptyset$. The fiber $\phi^{-1}(x)$ is given by $\text{Spec } \mathcal{A}/\mathfrak{q}\mathcal{A}$, where \mathfrak{q} is the maximal ideal of \mathcal{O} corresponding to x and $\mathcal{A}/\mathfrak{q}\mathcal{A}$ is an affinoid algebra over the complete discrete valuation field $(\mathcal{O}/\mathfrak{q})((t))$. Since $\mathcal{A}/\mathfrak{q}\mathcal{A}$ is known to be Jacobson, there exists a closed point of the closed subset $\phi^{-1}(x) \subset \text{Spec } \mathcal{A}$ in $\phi^{-1}(x) \cap D(f)$. □

2.3.4. Tate algebra. Let A be an admissible $\mathcal{O}[[t]]$ -algebra, and let $\mathcal{A} = A[1/t]$ be the associated affinoid $\mathcal{O}((t))$ -algebra. We set

$$A\langle\langle \mathbf{x} \rangle\rangle := A\langle\langle \mathbf{x} \rangle\rangle[1/t] = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \mathbf{x}^\nu \in A[[\mathbf{x}]] \mid \begin{array}{l} \text{for any } m \in \mathbb{Z}_{>0}, \text{ there exists } M \in \mathbb{Z}_{>0} \\ \text{such that } |\nu| \geq M \Rightarrow a_\nu \in t^m A \end{array} \right\},$$

which is an affinoid $\mathcal{O}((t))$ -algebra, called the *Tate algebra* over A .

2.4. Norms on affinoid algebras. The ring $\mathcal{O}((t))$ is equipped with a nonarchimedean norm

$$|\cdot| : \mathcal{O}((t)) \rightarrow \mathbb{R}_{\geq 0}$$

defined as follows: for $a(t) = \sum_{k \geq n} a_k t^k$ with $a_n \neq 0$, we set $|a(t)| = e^{-n}$, where $e > 1$ is a real number fixed once for all, and $|0| = 0$. Note that $\mathcal{O}((t))$ is complete with respect to this norm.

For any maximal ideal \mathfrak{m} of $\mathcal{O}((t))$, the residue field $K = \mathcal{O}((t))/\mathfrak{m}$ is a nonarchimedean local field (Corollary 2.2.3), and has the unique norm $|\cdot|_K$ such that $|\bar{t}|_K = e^{-1}$, where \bar{t} is the image of t in K . Note that, for any $a(t) \in \mathcal{O}((t))$, we have

$$|\bar{a}(\bar{t})|_K \leq |a(t)|,$$

where $\bar{a}(\bar{t})$ is the image of $a(t)$ in K .

2.4.1. Gauss norm. The Tate algebra $\mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle$ (where $\mathbf{x} = (x_1, \dots, x_n)$) is equipped with the *Gauss norm* $\|\cdot\|$, which is defined, as usual, as follows:

$$\left\| \sum_{\nu} a_\nu \mathbf{x}^\nu \right\| = \max_{\nu} |a_\nu|.$$

As usual, one can show that the Gauss norm is multiplicative, i.e., $\|FG\| = \|F\| \cdot \|G\|$, and that the Tate algebra $\mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle$ is complete with respect to the Gauss norm.

For any maximal ideal \mathfrak{m} of $\mathcal{O}((t))$, the Tate algebra $\mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle$ is specialized to the usual Tate algebra $K\langle\langle \mathbf{x} \rangle\rangle$ over $K = \mathcal{O}((t))/\mathfrak{m}$, and if $\|\cdot\|_K$ denotes the usual Gauss norm on $K\langle\langle \mathbf{x} \rangle\rangle$ relative to the norm $|\cdot|_K$ on K as above, we have

$$\|\bar{F}(\mathbf{x})\|_K \leq \|F(\mathbf{x})\|$$

for any $F(\mathbf{x}) \in \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle$, where $\bar{F}(\mathbf{x})$ denotes the image of $F(\mathbf{x})$ in $K\langle\langle \mathbf{x} \rangle\rangle$.

2.4.2. Residue norm. For an affinoid $\mathcal{O}((t))$ -algebra \mathcal{A} with a presentation $\mathcal{A} \cong \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}$, one has the *residue norm* $\|\cdot\|_{\mathcal{A}}$ defined by

$$\|f\|_{\mathcal{A}} = \inf\{\|F\| \mid F \in \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle \text{ and } F \bmod \mathfrak{a} = f\}$$

for $f \in \mathcal{A}$.

For any maximal ideal \mathfrak{m} of $\mathcal{O}((t))$, the presentation $\mathcal{A} \cong \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}$ is specialized to a presentation $\mathcal{A}_K \cong K\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}K\langle\langle \mathbf{x} \rangle\rangle$ of the usual affinoid algebra \mathcal{A}_K over $K = \mathcal{O}((t))/\mathfrak{m}$ and thus gives rise to a residue norm $\|\cdot\|_{\mathcal{A}_K}$ on \mathcal{A}_K (induced from the Gauss norm on $K\langle\langle \mathbf{x} \rangle\rangle$ defined as above). We have

$$\|\bar{f}\|_{\mathcal{A}_K} \leq \|f\|_{\mathcal{A}} \tag{2}$$

for any $f \in \mathcal{A}$, where \bar{f} is the image of f in \mathcal{A} .

2.5. Topology on affinoid algebras. Any residue norm on an affinoid algebra \mathcal{A} defines a complete topology on \mathcal{A} . We now show that this topology does not depend on the choice of the residue norm.

Lemma 2.5.1. *Let \mathcal{A} and \mathcal{B} be affinoid $\mathcal{O}((t))$ -algebras considered with arbitrary residue norms. Then any $\mathcal{O}((t))$ -algebra homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is continuous.*

Proof. Due to closed graph theorem for metrically complete topological groups [Pettis 1974], we only need to show the following: if a sequence $\{f_n\}$ in \mathcal{A} converges to 0 and $\{\varphi(f_n)\}$ converges to an element g in \mathcal{B} , then $g = 0$. Take any maximal ideal $\mathfrak{m} \subset \mathcal{B}$, and let $\mathfrak{n} \subset \mathcal{O}((t))$ be the image of \mathfrak{m} by $\text{Spec } \mathcal{B} \rightarrow \text{Spec } \mathcal{O}((t))$, which is a maximal ideal of $\mathcal{O}((t))$ due to Corollary 2.2.8. Set $K = \mathcal{O}((t))/\mathfrak{n}$ (see Corollary 2.2.3). For any $l \geq 1$, $\varphi_l : \mathcal{A} \rightarrow \mathcal{B}/\mathfrak{m}^l$ is continuous, since $\mathcal{A}/\ker(\varphi_l) \hookrightarrow \mathcal{B}/\mathfrak{m}^l$ is a mapping between finite dimensional K -linear spaces, and the topology of $\mathcal{A}/\ker(\varphi_l)$ is the induced one from that of \mathcal{A} . Hence we have $g \in \mathfrak{m}^l$ for any $l \geq 1$. Then by Krull’s theorem [Bourbaki 1972, III, §3.2, Corollary], the ideal $\text{Ann}(g) = \{x \in \mathcal{B} \mid xg = 0\}$ is not contained in any maximal ideal of \mathcal{B} . Hence $1 \in \text{Ann}(g)$, i.e., $g = 0$. \square

Corollary 2.5.2. *The residue norm on an affinoid algebra \mathcal{A} gives a well-defined topology on \mathcal{A} ; i.e., the induced topology does not depend on the choice of a presentation $\mathcal{A} \cong \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}$.*

Hence, in the sequel, we will always consider affinoid $\mathcal{O}((t))$ -algebras with the *canonical* topology as above. Any $\mathcal{O}((t))$ -algebra homomorphism between affinoid algebras is continuous with respect to the canonical topology.

Lemma 2.5.3. *Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra and A an admissible formal model of \mathcal{A} . Then the restriction to A of the canonical topology on \mathcal{A} coincides with the t -adic topology.*

Proof. The assertion is clear if $\mathcal{A} = \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle$ and $A = \mathcal{O}[[t]]\langle\langle \mathbf{x} \rangle\rangle$, i.e., the topology by the Gauss norm on $\mathcal{O}[[t]]\langle\langle \mathbf{x} \rangle\rangle$ coincides with the t -adic topology. In general, take a presentation $\mathcal{A} \cong \mathcal{O}[[t]]\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}$ and the induced presentation $\mathcal{A} \cong \mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}\mathcal{O}((t))\langle\langle \mathbf{x} \rangle\rangle$. Then the topology on A , which is the restriction of the topology by the induced residue norm, is the quotient topology of the t -adic topology and hence is the t -adic topology. \square

Corollary 2.5.4. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of affinoid $\mathcal{O}((t))$ -algebras and A an admissible formal model of \mathcal{A} . Then there exists an admissible formal model B of \mathcal{B} and a morphism $\phi : A \rightarrow B$ such that $\varphi = \phi[1/t]$.*

Proof. Take an admissible formal model B of \mathcal{B} , and let B' be the image of the continuous morphism $A \widehat{\otimes}_{\mathcal{O}[[t]]} B \rightarrow B$. Since B' contains B , B' is a formal model of \mathcal{B} , which admits $A \rightarrow B'$, as desired. \square

Corollary 2.5.5. *Let $\mathcal{A} \rightarrow \mathcal{B}$ be a morphism of affinoid $\mathcal{O}((t))$ -algebras. Then $\mathrm{Spec} \mathcal{B} \rightarrow \mathrm{Spec} \mathcal{A}$ maps closed points to closed points.*

Proof. This is immediate from Corollaries 2.5.4 and 2.2.8. □

2.6. Reduction map. Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra and $A \subset \mathcal{A}$ an admissible formal model of \mathcal{A} . Set $A_0 = A/tA$. Then we have the so-called *reduction map*

$$\mathrm{red}_A : \mathrm{Spm} \mathcal{A} \rightarrow \mathrm{Spm} A_0$$

from the maximal spectrum of \mathcal{A} to the maximal spectrum of A_0 , defined as follows. Any closed point $x \in \mathrm{Spec} \mathcal{A}$ corresponds to a classical point $(\mathrm{Spf} V)^{\mathrm{rig}} \rightarrow (\mathrm{Spf} A)^{\mathrm{rig}}$, where V is a complete discrete valuation ring with finite residue field, whence a finite morphism $\mathrm{Spf} V \rightarrow \mathrm{Spf} A$ of t -adic formal schemes. The last morphism induces a finite morphism $\mathrm{Spec} k \rightarrow \mathrm{Spec} A_0$, where k is the residue field of V , and hence a closed point of $\mathrm{Spec} A_0$, which we define to be $\mathrm{red}_A(x)$.

The reduction map can be defined in a slightly more general situation. Let $X' \rightarrow X = \mathrm{Spf} \mathcal{A}$ be an admissible blow-up. Then the finite adic map $\mathrm{Spf} V \rightarrow \mathrm{Spf} \mathcal{A}$ has a unique lift $\mathrm{Spf} V \rightarrow X'$, which gives rise to a finite morphism $\mathrm{Spec} k \rightarrow X'_0$, where X'_0 is the closed subscheme of X' defined by $t = 0$, which is a finite-type \mathcal{O} -scheme. We thus have a reduction map

$$\mathrm{red}_{X'} : \mathrm{Spm} \mathcal{A} \rightarrow (X'_0)^{\mathrm{cl}},$$

which maps to the set of closed points of X'_0 .

Lemma 2.6.1. *The reduction map $\mathrm{red}_{X'}$ is surjective.*

Proof. First, note that there exists a map $(X'_0)^{\mathrm{cl}} \rightarrow \mathrm{Spm} \mathcal{O}$ since X'_0 is of finite type over \mathcal{O} , which is Jacobson. Hence $(X'_0)^{\mathrm{cl}}$ is the disjoint union of the fibers of this map. The fiber over $x \in \mathrm{Spm} \mathcal{O}$ of the reduction map $\mathrm{red}_{X'}$ is the reduction map $\mathrm{Spm} \mathcal{A}/\mathfrak{p}_x \mathcal{A} \rightarrow (X'_{0,x})^{\mathrm{cl}}$ in classical rigid geometry, where $\mathfrak{p}_x \subset \mathcal{O}$ is the maximal ideal corresponding to x ; note that $\mathcal{A}/\mathfrak{p}_x \mathcal{A}$ is an affinoid algebra over the complete discrete valuation field $(\mathcal{O}/\mathfrak{p}_x)((t))$ and the reduction map is considered with respect to the formal model $A/\mathfrak{p}_x A$ of $\mathcal{A}/\mathfrak{p}_x \mathcal{A}$. As the reduction map in the classical situation is known to be surjective (e.g., [Bosch 2014, §8.3, Proposition 8]), the assertion follows. □

2.7. Power-bounded elements. Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra. For any closed point $x \in \mathrm{Spm} \mathcal{A}$, the residue field K_x at x is a nonarchimedean local field (Corollary 2.2.3) and has the unique nonarchimedean norm $|\cdot|_x$ defined as in Section 2.4. One has a map

$$\mathcal{A} \rightarrow \mathbb{R}_{\geq 0}, \quad f \mapsto |f(x)| := |f|_x,$$

and, for any $f \in \mathcal{A}$, the so-called *spectral seminorm*

$$|f|_{\mathrm{sp}} = \sup\{|f(x)| \mid x \in \mathrm{Spm} \mathcal{A}\},$$

which is power-multiplicative, i.e., $|f^n|_{\mathrm{sp}} = (|f|_{\mathrm{sp}})^n$ for any $n \geq 0$.

Lemma 2.7.1. *The spectral seminorm is bounded by any residue norm, i.e., $|f|_{\text{sp}} \leq \|f\|_{\mathcal{A}}$ for any $f \in \mathcal{A}$, where $\|\cdot\|_{\mathcal{A}}$ is a residue seminorm relative to an arbitrary presentation $\mathcal{A} \cong \mathcal{O}(\!(t)\!)\langle\langle \mathbf{x} \rangle\rangle/\mathfrak{a}$.*

Proof. For any closed point $x \in \text{Spm } \mathcal{A}$, let $y \in \text{Spm } \mathcal{O}(\!(t)\!)$ be its image (see Corollary 2.5.5) and K the residue field at y . Then $|f(x)| = |\bar{f}(x)|$, where \bar{f} is the image of f in \mathcal{A}_K . It is known in the classical rigid geometry that $|\bar{f}(x)| \leq \|\bar{f}\|_{\mathcal{A}_K}$, and, by (2) in Section 2.4, we have $|f(x)| \leq \|f\|_{\mathcal{A}}$. \square

Proposition 2.7.2. *Let \mathcal{A} be an affinoid $\mathcal{O}(\!(t)\!)$ -algebra. Then the following conditions for $f \in \mathcal{A}$ are equivalent:*

- (a) f is **power-bounded**, i.e., the set $\{f^n\}$ is bounded by a residue norm $|\cdot|_{\mathcal{A}}$ on \mathcal{A} (note that the boundedness does not depend on the choice of $|\cdot|_{\mathcal{A}}$).
- (b) $|f|_{\text{sp}} \leq 1$.
- (c) f is integral over any admissible formal model A of \mathcal{A} .
- (d) There exists an admissible formal model A of \mathcal{A} that contains f .

Proof. The spirit of the proof is just the same as that of the corresponding theorem in the classical rigid geometry. However, we include the proof here for the reader’s convenience.

Let us first show (a) \Rightarrow (b). Suppose that f is power-bounded and $|f|_{\text{sp}} > 1$. Since $|\cdot|_{\text{sp}}$ is power-multiplicative, $\{|f^n|_{\text{sp}}\}$ is not bounded, and hence $\{|f^n|_{\mathcal{A}}\}$ for any residue norm $|\cdot|_{\mathcal{A}}$ is not bounded, either.

Second we show (c) \Rightarrow (d). Suppose f is integral over an admissible formal model A of \mathcal{A} . Then $A[f]$ is finite over A and hence is topologically of finite type, which gives an admissible formal model of \mathcal{A} .

Next we show (d) \Rightarrow (a). Take an admissible formal model A such that $f \in A$ and a presentation $A \cong \mathcal{O}[\![t]\!] \langle\langle \mathbf{x} \rangle\rangle / \mathfrak{a}$. Then we have $\mathcal{A} = \mathcal{O}(\!(t)\!)\langle\langle \mathbf{x} \rangle\rangle / \mathfrak{a} \mathcal{O}(\!(t)\!)\langle\langle \mathbf{x} \rangle\rangle$, and one has the residue norm $|\cdot|_{\mathcal{A}}$ on \mathcal{A} with respect to this presentation. Since $|g|_{\mathcal{A}} \leq 1$ whenever $g \in A$, we have (a).

Finally, let us show (b) \Rightarrow (c). Let A be an admissible formal model of \mathcal{A} . Take $N \geq 0$ sufficiently large such that $g = t^N f \in A$, and consider the admissible ideal $J = (t^N, g)$ of A . Let $X' \rightarrow X = \text{Spf } A$ be the admissible blow-up along J . We have $X = U^+ \cup U^-$, where

$$U^+ = \text{Spf } A \langle\langle g/t^N \rangle\rangle / (t\text{-torsion}),$$

$$U^- = \text{Spf } A \langle\langle t^N/g \rangle\rangle / (g\text{-torsion}).$$

We are going to show $X' = U^+$. What we want to show is that the closed subscheme Z_0 of $Z = X' \setminus U^+$ defined by $t = 0$ is empty. First note that Z_0 is a finite-type scheme over \mathcal{O} . By our assumption (b), for any $x \in \text{Spm } \mathcal{A}$, the image $f(x)$ of f in the residue field K_x at x belongs to the valuation ring V_x . Now the map $\text{Spf } V_x \rightarrow \text{Spf } A$ lifts to $\text{Spf } V_x \rightarrow X'$, whose image lies in U^+ since $g = t^N f$. This means that the image of the reduction map $\text{Spm } \mathcal{A} \rightarrow X'_0$ lies in U_0^+ . Since the reduction map $\text{Spm } \mathcal{A} \rightarrow (X'_0)^{\text{cl}}$ is surjective by Lemma 2.6.1, we deduce that Z_0 has no closed point of X'_0 . Since X'_0 is of finite type over \mathcal{O} and hence is Jacobson, this means $Z_0 = \emptyset$.

Now, since $\pi : X' \rightarrow X = \text{Spf } A$ is proper and affine, it is finite. In particular, $\Gamma(X, \pi_* \mathcal{O}_{X'}) = A[f]$ is finite over A , as desired. \square

Corollary 2.7.3. *Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra, and let $f_1, \dots, f_n \in \mathcal{A}$ be power-bounded elements. Then there exists an admissible formal model A of \mathcal{A} that contains f_1, \dots, f_n .*

Corollary 2.7.4. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of affinoid $\mathcal{O}((t))$ -algebras, and let $g_1, \dots, g_n \in \mathcal{B}$ be power-bounded elements of \mathcal{B} . Then φ lifts uniquely to a morphism $A\langle\langle x_1, \dots, x_n \rangle\rangle \rightarrow \mathcal{B}$ such that $x_i \mapsto g_i$ for $i = 1, \dots, n$.*

Corollary 2.7.5. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of affinoid $\mathcal{O}((t))$ -algebras, and let A (resp. B) be an admissible formal model of \mathcal{A} (resp. \mathcal{B}). Then there exists an admissible formal model B' of \mathcal{B} such that:*

- (a) B' admits a homomorphism $\phi : A \rightarrow B'$ of admissible $\mathcal{O}[[t]]$ -algebras such that $\phi[1/t] = \varphi$.
- (b) $B \subset B'$, and B' is finite over B .
- (c) $\text{Spf } B' \rightarrow \text{Spf } B$ is an admissible blow-up.

Proof. Take a presentation $A = \mathcal{O}[[t]]\langle\langle x_1, \dots, x_n \rangle\rangle/\mathfrak{a}$, and set $f_i = x_i \bmod \mathfrak{a}$ for $i = 1, \dots, n$. Set $g_i = \varphi(f_i)$ for $i = 1, \dots, n$. Then, since $|f_i|_{\text{sp}} \leq 1$, we have $|g_i|_{\text{sp}} \leq 1$, and thus $B' = B[g_1, \dots, g_n]$ is an admissible formal model of \mathcal{B} , which is finite over B .

We need to show that $\text{Spf } B' \rightarrow \text{Spf } B$ is an admissible blow-up. To this end, take $N \geq 0$ sufficiently large such that $t^N g_i \in B$ for any $i = 1, \dots, n$, and consider the admissible blow-up $X'' \rightarrow \text{Spf } B$ along the admissible ideal $J = (t^N, t^N g_1, \dots, t^N g_n)$. Then X'' is covered by $\text{Spf } B_i$ for $i = 0, 1, \dots, n$, where

$$B_0 = B \left\langle\left\langle \frac{t^N g_1}{t^N}, \dots, \frac{t^N g_n}{t^N} \right\rangle\right\rangle / (t\text{-torsion}),$$

which is nothing but B' , and, for $i = 1, \dots, n$,

$$B_i = B \left\langle\left\langle \frac{t^N}{t^N g_i}, \frac{t^N g_1}{t^N g_i}, \dots, \frac{t^N g_n}{t^N g_i} \right\rangle\right\rangle / (t^N g_i\text{-torsion}),$$

which is isomorphic to the complete localization $B'_{\{g_i\}}$ of B' . Hence we have $X'' = \text{Spf } B'$, and thus we have shown that $\text{Spf } B' \rightarrow \text{Spf } B$ is an admissible blow-up, as desired. \square

Corollary 2.7.6. *Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra, and let A and A' be admissible formal models of \mathcal{A} such that $A \subset A'$. Then A' is finite over A . Moreover, $\text{Spf } A' \rightarrow \text{Spf } A$ is an admissible blow-up.*

Proof. Since A' is topologically of finite type over A , one can write $A' = A\langle\langle f_1, \dots, f_n \rangle\rangle$ for some $f_1, \dots, f_n \in A'$. By Proposition 2.7.2, each f_i is integral over A , and hence $A' = A[f_1, \dots, f_n]$, which is finite over A . The other part can be shown similarly to the proof of Corollary 2.7.5. \square

Corollary 2.7.7. *Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra, and let A and A' be admissible formal models of \mathcal{A} . Then there exists an admissible formal model A'' of \mathcal{A} such that:*

- (a) A'' contains both A and A' , and A'' is finite over A and A' .
- (b) $\text{Spf } A'' \rightarrow \text{Spf } A$ and $\text{Spf } A'' \rightarrow \text{Spf } A'$ are admissible blow-ups.

Proof. Take a presentation $A = \mathcal{O}[[t]]\langle\langle x_1, \dots, x_n \rangle\rangle/\mathfrak{a}$ (resp. $A' = \mathcal{O}[[t]]\langle\langle x'_1, \dots, x'_m \rangle\rangle/\mathfrak{a}'$), and set $f_i = x_i \bmod \mathfrak{a}$ for $i = 1, \dots, n$ (resp. $f'_j = x'_j \bmod \mathfrak{a}'$ for $j = 1, \dots, m$). Then f_i and f'_j are power-bounded elements, and set $A'' = A[f'_1, \dots, f'_m] = A'[f_1, \dots, f_n]$. Then this A'' satisfies condition (a). Condition (b) follows from Corollary 2.7.6. \square

Remark 2.7.8. What we have seen in Corollaries 2.7.5 and 2.7.7 can be summed up in the language of [Fujiwara and Kato 2018, II.A.4(g)] to say that affinoid $\mathcal{O}((t))$ -algebras, like classical affinoid algebras, have the so-called *canonical rigidification*.

2.8. Rigid analytic geometry over $\mathcal{O}((t))$. By Corollary 2.7.7, for any $\mathcal{O}((t))$ -affinoid algebra \mathcal{A} , the affinoid space $(\mathrm{Spf} A)^{\mathrm{rig}}$ does not depend on the choice of the admissible formal model A of \mathcal{A} . Hence we can write, as in the classical rigid geometry,

$$\mathrm{Sp} \mathcal{A} := (\mathrm{Spf} A)^{\mathrm{rig}}.$$

Remark 2.8.1. Note that this definition of $\mathrm{Sp} \mathcal{A}$ differs (harmlessly) from the one in the classical situation, where affinoid spaces in the classical setting are defined as the analytic spaces supported on the set of all maximal ideals (i.e., classical points) of \mathcal{A} .

By Corollary 2.7.5, any $\mathcal{O}((t))$ -algebra homomorphism $\mathcal{A} \rightarrow \mathcal{B}$ between affinoid $\mathcal{O}((t))$ -algebras induces a well-defined morphism $\mathrm{Sp} \mathcal{B} \rightarrow \mathrm{Sp} \mathcal{A}$ of rigid spaces over $\mathcal{O}((t))$.

2.8.2. K -valued points. Let $\mathcal{O}[[t]] \rightarrow K$ be a finite $\mathcal{O}((t))$ -algebra, where K is a nonarchimedean local field. Then, for any affinoid space $\mathrm{Sp} \mathcal{A}$, one can consider a K -valued point $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$ as a morphism between rigid spaces over $\mathcal{O}((t))$. By Corollary 2.5.5, α factors as

$$\begin{array}{ccc} \mathrm{Sp} K & \xrightarrow{\alpha} & \mathrm{Sp} \mathcal{A} \\ & \searrow & \nearrow \beta \\ & \mathrm{Sp} L & \end{array}$$

where K/L is a finite extension of nonarchimedean local fields and β is a classical point.

2.8.3. Specialization to the classical rigid geometry. Consider a K -valued point $\mathrm{Sp} K \hookrightarrow \mathrm{Sp} \mathcal{O}[[t]]$, where K is a nonarchimedean local field. Then any affinoid space $\mathrm{Sp} \mathcal{A}$ over $\mathcal{O}((t))$ can be specialized to the affinoid space $\mathrm{Sp} \mathcal{A}_K$ over K in the sense of classical rigid geometry, where $\mathcal{A}_K = \mathcal{A} \otimes_{\mathcal{O}((t))} K$.

More generally, for any finite-type rigid space \mathcal{X} over $\mathcal{O}((t))$, one has the base change \mathcal{X}_K , which is a finite-type rigid space over K .

2.9. Extension by roots of t . For a positive integer N , one has the finite extension $\mathcal{O}[[t]] \rightarrow \mathcal{O}[[t]][t^{1/N}] = \mathcal{O}[[t, s]]/(s^N - t)$. Since the latter ring is t -adically complete, it coincides with its completion $\widehat{\mathcal{O}[[t^{1/N}]]}$. Note that $\widehat{\mathcal{O}[[t^{1/N}]]} = \mathcal{O}[[t]][[t^{1/N}]]$ is faithfully flat over $\mathcal{O}[[t]]$, and that $\mathcal{O}[[t^{1/N}]] [1/t] = \mathcal{O}((t^{1/N}))$. Note also that, if A is t -adically complete, then $A \otimes_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]] = A \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]]$, since the left-hand side is already t -adically complete.

Let \mathcal{A} be an affinoid integral domain over $\mathcal{O}((t))$. Consider the simple field extension $\text{Frac}(\mathcal{A})(t^{1/N})$ of $\text{Frac}(\mathcal{A})$ generated by an N -th root of t . We denote by

$$\mathcal{A}[t^{1/N}]$$

the subring of $\text{Frac}(\mathcal{A})(t^{1/N})$ generated by \mathcal{A} and $t^{1/N}$. This is an affinoid $\mathcal{O}((t^{1/N}))$ -algebra since there exists a surjection $\mathcal{A} \otimes_{\mathcal{O}((t))} \mathcal{O}((t^{1/N})) \rightarrow \mathcal{A}[t^{1/N}]$ and $\mathcal{A} \otimes_{\mathcal{O}((t))} \mathcal{O}((t^{1/N})) = \mathcal{A} \widehat{\otimes}_{\mathcal{O}((t))} \mathcal{O}((t^{1/N}))$ is an affinoid $\mathcal{O}((t^{1/N}))$ -algebra.

If A is an admissible formal model of \mathcal{A} , one can similarly define

$$A[t^{1/N}]$$

to be the subring of $\text{Frac}(A)(t^{1/N})$ generated by A and $t^{1/N}$, which is the surjective image of the canonical map $A \otimes_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]] = A \widehat{\otimes}_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]] \rightarrow A[t^{1/N}]$. Clearly, $A[t^{1/N}]$ is an admissible formal model of $\mathcal{A}[t^{1/N}]$.

2.10. Affinoid algebra with a convergence condition. In the sequel, by the *denominator* of a rational number δ , we mean the smallest positive integer $d(\delta)$ such that $d(\delta) \cdot \delta \in \mathbb{Z}$. Similarly, the denominator of a vector $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{Q}^n$ of rational numbers is the smallest positive integer $d(\delta)$ such that $d(\delta) \cdot \delta \in \mathbb{Z}^n$. We write

$$t^\delta = (t^{\delta_1}, \dots, t^{\delta_n}) \quad \text{and} \quad t^\delta \mathbf{x} = (t^{\delta_1} x_1, \dots, t^{\delta_n} x_n),$$

and, for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$,

$$\delta \cdot \nu = \delta_1 \nu_1 + \dots + \delta_n \nu_n.$$

2.10.1. Affinoid algebra with a convergence condition. For $\delta = (\delta_1, \dots, \delta_n) \in (\mathbb{Q}_{\geq 0})^n$ and an affinoid algebra $\mathcal{A} = A[1/t]$, where A is an admissible $\mathcal{O}[[t]]$ -algebra, we set

$$\mathcal{A}\langle\langle \mathbf{x}; \delta \rangle\rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \mathbf{x}^\nu \in \mathcal{A}[[\mathbf{x}]] \mid |t|^{-\delta \cdot \nu} |a_\nu| \rightarrow 0 \text{ as } |\nu| \rightarrow \infty \right\}$$

(not depending on the choice of a residue norm $|\cdot|$ on \mathcal{A}) and

$$A\langle\langle \mathbf{x}; \delta \rangle\rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \mathbf{x}^\nu \in \mathcal{A}\langle\langle \mathbf{x}; \delta \rangle\rangle \mid |a_\nu| \leq |t|^{\delta \cdot \nu} \right\},$$

which is an A -algebra. Note that the condition $|a_\nu| \leq |t|^{\delta \cdot \nu}$ is equivalent to $a_\nu^{d(\delta)} \in t^{d(\delta)(\delta \cdot \nu)} A$ and hence is independent of choice of the residue norm $|\cdot|$.

If $\delta_i = m_i/d_i$ is the irreducible fraction with $d_i > 0$ for $i = 1, \dots, n$, then

$$A\langle\langle \mathbf{x}; \delta \rangle\rangle \cong A\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle / \mathfrak{a},$$

where $\mathbf{y} = (y_1, \dots, y_n)$ and \mathfrak{a} is the ideal generated by $t^{m_i} x_i^{d_i} - y_i^{d_i}$ for $i = 1, \dots, n$. In particular, $A\langle\langle \mathbf{x}; \delta \rangle\rangle$ is an admissible $\mathcal{O}[[t]]$ -algebra, and hence $\mathcal{A}\langle\langle \mathbf{x}; \delta \rangle\rangle = A\langle\langle \mathbf{x}; \delta \rangle\rangle[1/t]$ is an affinoid $\mathcal{O}((t))$ -algebra.

Moreover, it is easy to see that if $\mathbf{x}' = (x'_1, \dots, x'_m)$ and $\boldsymbol{\delta}' = (\delta'_1, \dots, \delta'_m) \in (\mathbb{Q}_{\geq 0})^m$, then

$$\begin{aligned} \mathcal{A}\langle\langle \mathbf{x}, \mathbf{x}'; (\boldsymbol{\delta}, \boldsymbol{\delta}') \rangle\rangle &\cong \mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle \widehat{\otimes}_{\mathcal{A}} \mathcal{A}\langle\langle \mathbf{x}'; \boldsymbol{\delta}' \rangle\rangle, \\ A\langle\langle \mathbf{x}, \mathbf{x}'; (\boldsymbol{\delta}, \boldsymbol{\delta}') \rangle\rangle &\cong A\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle \widehat{\otimes}_A A\langle\langle \mathbf{x}'; \boldsymbol{\delta}' \rangle\rangle, \end{aligned}$$

where $(\boldsymbol{\delta}, \boldsymbol{\delta}') = (\delta_1, \dots, \delta_n, \delta'_1, \dots, \delta'_m)$.

We note that $\mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle$ is an affinoid algebra corresponding to the polydisk of polyradius $|t|^{-\boldsymbol{\delta}} = (|t|^{-\delta_1}, \dots, |t|^{-\delta_n})$, i.e.,

$$\begin{aligned} A\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle \otimes_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]] &\cong (A \otimes_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]])\langle\langle t^{\boldsymbol{\delta}} \mathbf{x} \rangle\rangle, \\ \mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle \otimes_{\mathcal{O}((t))} \mathcal{O}((t^{1/N})) &\cong (\mathcal{A} \otimes_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]])\langle\langle t^{\boldsymbol{\delta}} \mathbf{x} \rangle\rangle, \end{aligned}$$

and, if \mathcal{A} is an integral domain,

$$A\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle[t^{1/N}] \cong A[t^{1/N}]\langle\langle t^{\boldsymbol{\delta}} \mathbf{x} \rangle\rangle, \quad \mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle[t^{1/N}] \cong \mathcal{A}[t^{1/N}]\langle\langle t^{\boldsymbol{\delta}} \mathbf{x} \rangle\rangle,$$

where N is a multiple of $d(\boldsymbol{\delta})$.

Lemma 2.10.2. *Let $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$, $\boldsymbol{\delta}' = (\delta'_1, \dots, \delta'_n) \in (\mathbb{Q}_{>0})^n$, and suppose $\boldsymbol{\delta}' \leq \boldsymbol{\delta}$, i.e., $\delta'_i \leq \delta_i$ for $i = 1, \dots, n$. Then $\mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle \rightarrow \mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta}' \rangle\rangle$ is flat.*

Proof. We may assume that $n = 1$, i.e., it suffices to show that $\mathcal{A}\langle\langle x; \boldsymbol{\delta} \rangle\rangle \rightarrow \mathcal{A}\langle\langle x; \boldsymbol{\delta}' \rangle\rangle$ for $\boldsymbol{\delta}' \leq \boldsymbol{\delta}$ is flat. Since the extension $\mathcal{O}[[t]] \rightarrow \mathcal{O}[[t^{1/N}]]$ is faithfully flat, we may replace \mathcal{A} by $\mathcal{A} \otimes_{\mathcal{O}[[t]]} \mathcal{O}[[t^{1/N}]]$, and thus we may assume that $\boldsymbol{\delta}, \boldsymbol{\delta}' \in \mathbb{Z}$. Then $\mathrm{Sp} \mathcal{A}\langle\langle x; \boldsymbol{\delta}' \rangle\rangle \rightarrow \mathrm{Sp} \mathcal{A}\langle\langle x; \boldsymbol{\delta} \rangle\rangle$ is an open immersion since $\mathrm{Spf} A\langle\langle x; \boldsymbol{\delta}' \rangle\rangle \rightarrow \mathrm{Spf} A\langle\langle x; \boldsymbol{\delta} \rangle\rangle$ is one of the affine patches of the admissible blow-up of $\mathrm{Spf} A\langle\langle x; \boldsymbol{\delta} \rangle\rangle$ along the admissible ideal $(t^{\boldsymbol{\delta}-\boldsymbol{\delta}'}, x)$. Then the desired flatness follows from [Fujiwara and Kato 2018, II.6.6.1 (1)]. \square

2.10.3. Unit-polydisk part. If $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$, $\boldsymbol{\delta}' = (\delta'_1, \dots, \delta'_n) \in (\mathbb{Q}_{>0})^n$ with $\boldsymbol{\delta}' \leq \boldsymbol{\delta}$, then $\mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle$ is contained in $\mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta}' \rangle\rangle$. In this situation, if \mathcal{B} is an affinoid algebra over $\mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle$, then we define

$$\mathcal{B}^{\natural\boldsymbol{\delta}'} = \mathcal{B} \widehat{\otimes}_{\mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle} \mathcal{A}\langle\langle \mathbf{x}; \boldsymbol{\delta}' \rangle\rangle.$$

Specifically, $\mathcal{B}^{\natural 0}$ will be called the *unit-polydisk part* of \mathcal{B} , which we denote by \mathcal{B}^{\natural} .

2.11. Irreducible decomposition.

Lemma 2.11.1. *Let \mathcal{X} be a rigid space of finite type over $\mathcal{O}((t))$, and let $\mathcal{N}_{\mathcal{X}}$ be the ideal sheaf of $\mathcal{O}_{\mathcal{X}}$ consisting of local nilpotent sections. Then $\mathcal{N}_{\mathcal{X}}$ is a coherent ideal of $\mathcal{O}_{\mathcal{X}}$.*

Proof. This can be shown similarly to [Fujiwara and Kato 2018, II.8.2.9], using the fact that any topologically-of-finite-type $\mathcal{O}[[t]]$ -algebra is excellent (due to Gabber; see [Kurano and Shimomoto 2021]). \square

With the above lemma, one finds that all the rest of [Fujiwara and Kato 2018, II.8.3 (b)] are valid for rigid spaces of finite type over $\mathcal{O}((t))$. In particular, an affinoid $\mathrm{Sp} \mathcal{A} = (\mathrm{Spf} A)^{\mathrm{rig}}$, where \mathcal{A} is an affinoid $\mathcal{O}((t))$ -algebra and A is an admissible formal model of \mathcal{A} , is reduced (resp. irreducible) if and only if

the affine scheme $\text{Spec } \mathcal{A}$ also is (see [Fujiwara and Kato 2018, II.8.3.5]). Thus one has the irreducible decomposition of $\mathcal{X} = \text{Sp } \mathcal{A}$; if

$$\text{Spec } \mathcal{A} = \bigcup_{i=1}^m X_i$$

is the irreducible decomposition of the Noetherian scheme $\text{Spec } \mathcal{A}$, then

$$\mathcal{X} = \bigcup_{i=1}^m \mathcal{X}_i,$$

where

$$\mathcal{X}_i = \text{Sp } \mathcal{A}/\mathfrak{q}_i \cong (\text{Spf } A/\mathfrak{q}_i \cap A)^{\text{rig}}$$

and \mathfrak{q}_i is the minimal prime ideal of \mathcal{A} corresponding to X_i , gives the irreducible decomposition of \mathcal{X} .

2.12. Dimension. We refer to [Fujiwara and Kato 2018, II.10] for the general notion of dimension of rigid spaces. First of all, for a rigid space \mathcal{X} and a point $x \in \langle \mathcal{X} \rangle$, the dimension of \mathcal{X} at x , denoted by $\dim_x(\mathcal{X})$, is the Krull dimension of the local ring $\mathcal{O}_{\mathcal{X},x}$. The dimension $\dim(\mathcal{X})$ of \mathcal{X} is then defined to be the supremum of $\dim_x(\mathcal{X})$ for all $x \in \langle \mathcal{X} \rangle$. Note that if \mathcal{X} is a rigid space of finite type over $\mathcal{O}(\!(t)\!)$, then $\dim(\mathcal{X})$ is the supremum of $\dim_x(\mathcal{X})$ for all classical points x of \mathcal{X} (see [Fujiwara and Kato 2018, II.10.1.9]). Moreover, if $\mathcal{X} = \text{Sp } \mathcal{A}$, where \mathcal{A} is an affinoid $\mathcal{O}(\!(t)\!)$ -algebra, then $\dim_x(\mathcal{X})$ at any classical point x coincides with $\dim_{s(x)}(\text{Spec } \mathcal{A})$, where $s(x)$ is the closed point of $\text{Spec } \mathcal{A}$ corresponding to x . In particular, we have

$$\dim(\mathcal{X}) = \dim(\mathcal{A}).$$

Lemma 2.12.1 (fiber dimension theorem). *Let $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism between rigid spaces of finite type over $\mathcal{O}(\!(t)\!)$, $y = \varphi(x) \in \langle \mathcal{Y} \rangle^{\text{cl}}$ a classical point, and $x \in \langle \mathcal{X} \rangle^{\text{cl}}$ a classical point over y , i.e., $y = \varphi(x)$. Then we have*

$$\dim_x(\mathcal{X}) - \dim_y(\mathcal{Y}) \leq \dim_x(\mathcal{X}_y).$$

Proof. What we need to show is

$$\dim \mathcal{O}_{\mathcal{X},x} - \dim \mathcal{O}_{\mathcal{Y},y} \leq \dim \mathcal{O}_{\mathcal{X},x}/\mathfrak{m}_y \mathcal{O}_{\mathcal{X},x},$$

where \mathfrak{m}_y is the maximal ideal of $\mathcal{O}_{\mathcal{Y},y}$. Since the local rings $\mathcal{O}_{\mathcal{X},x}$ and $\mathcal{O}_{\mathcal{Y},y}$ are Noetherian, this follows from [Matsumura 1989, 15.1]. □

The following lemma will be used later.

Lemma 2.12.2. *Let $\phi : \mathcal{A}\langle\langle y_1, \dots, y_d; \delta_1, \dots, \delta_d \rangle\rangle \rightarrow \mathcal{B}$ be a finite morphism of affinoid $\mathcal{O}(\!(t)\!)$ -algebras with nilpotent kernel. Then, for any morphism $\mathcal{A} \rightarrow \mathcal{R}$ of affinoid $\mathcal{O}(\!(t)\!)$ -algebras, the induced map*

$$\phi_{\mathcal{R}} : \mathcal{R}\langle\langle y_1, \dots, y_d; \delta_1, \dots, \delta_d \rangle\rangle \rightarrow \mathcal{B}_{\mathcal{R}} = \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{R}$$

is finite and $\ker(\phi_{\mathcal{R}})$ is contained in the nilpotent radical.

Proof. First, since the extension of the form $\mathcal{O}[[t]] \rightarrow \mathcal{O}[[t^{1/N}]]$ is faithfully flat, one can first reduce to the case where $\delta_1, \dots, \delta_d \in \mathbb{Z}$ by faithfully flat descent, and then to the case $\delta_1 = \dots = \delta_d = 0$ by replacing each y_i by $t^{\delta_i} y_i$ for $i = 1, \dots, d$.

Second, as the map in question is clearly finite, we only need to show that $\phi_{\mathcal{R}}$ consists of nilpotent elements. Suppose not, and take a nonnilpotent $f \in \ker(\phi_{\mathcal{R}})$. By Proposition 2.3.3, there exists a classical point $x = \mathfrak{m}_x$ of $\mathrm{Sp} \mathcal{R} \langle\langle y_1, \dots, y_d \rangle\rangle$ (\mathfrak{m}_x is the corresponding maximal ideal of $\mathcal{R} \langle\langle y_1, \dots, y_d \rangle\rangle$) such that $f \notin \mathfrak{m}_x$. Let $y = \mathfrak{m}_y$ be the image of x in $\mathrm{Sp} \mathcal{R}$, which is a classical point of $\mathrm{Sp} \mathcal{R}$ (Lemma 2.2.7). Let $K = \mathcal{R}/\mathfrak{m}_y$, and consider the induced diagram

$$K \rightarrow K \langle\langle y_1, \dots, y_d \rangle\rangle \xrightarrow{\phi_K} \mathcal{B}_K,$$

where ϕ_K is, by our construction, finite but not injective. It is clear that

$$\dim \mathrm{Sp} \mathcal{B}_K = \dim K \langle\langle y_1, \dots, y_d \rangle\rangle / \ker(\phi_K) < d.$$

On the other hand, by Lemma 2.12.1,

$$\dim \mathrm{Sp} \mathcal{B}_K \geq \dim_x \mathrm{Sp} \mathcal{B} - \dim_y \mathrm{Sp} \mathcal{A} = d,$$

which is absurd. □

2.13. Algebraic and transcendental parts. Having formally introduced the notions of irreducibility and dimension in our context, we now give the general form of the definition of algebraic and transcendental parts.

Let F be a local nonarchimedean field and $\mathrm{Sp} \mathcal{B}$ a closed subset of $\mathrm{Sp} F((t)) \langle\langle \mathbf{x} \rangle\rangle$. We say that $\mathrm{Sp} \mathcal{B}$ is *algebraic* if each of its irreducible components is an irreducible component of a closed set defined by a *polynomial ideal* $I \subset F \langle\langle x_1, \dots, x_n \rangle\rangle$, i.e., an ideal obtained by extension from $F[x_1, \dots, x_n]$. Note that this is a property of $\mathrm{Sp} \mathcal{B}$ as a subset of $\mathrm{Sp} F((t)) \langle\langle \mathbf{x} \rangle\rangle$, not an intrinsic property of $\mathrm{Sp} \mathcal{B}$.

We define $(\mathrm{Sp} \mathcal{B})^{\mathrm{alg}}$ to be the union of all irreducible positive-dimensional closed algebraic subsets of $\mathrm{Sp} \mathcal{B}$ and $(\mathrm{Sp} \mathcal{B})^{\mathrm{tran}}$ to be the its complement in $\mathrm{Sp} \mathcal{B}$.

2.14. Formal blow-up of affinoids. Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra and A an admissible formal model of \mathcal{A} . Let $J = (f_0, \dots, f_r)$ be an ideal of A . We want to discuss the *formal blow-up* of $\mathrm{Sp} \mathcal{A}$ along the ideal J . This includes the notion of admissible blow-ups in the case where J is an admissible ideal.

Consider the algebraic blow-up $Y = \mathrm{Proj} \bigoplus_{k=0}^{\infty} J^k \rightarrow \mathrm{Spec} A$ along J . Then

$$Y = \bigcup_{i=0}^r \mathrm{Spec} A_i, \quad A_i = A \left[\frac{f_0}{f_i}, \dots, \frac{f_r}{f_i} \right] / f_i\text{-torsion}.$$

Taking formal completion,

$$\widehat{Y} = \bigcup_{i=0}^r \mathrm{Spf} \widehat{A}_i, \quad \widehat{A}_i = A \left\langle\left\langle \frac{f_0}{f_i}, \dots, \frac{f_r}{f_i} \right\rangle\right\rangle / f_i\text{-torsion},$$

and the associated rigid spaces, we have the desired formal blow-up $\mathcal{Y} \rightarrow \mathrm{Sp} \mathcal{A}$, where

$$\mathcal{Y} = \widehat{Y}^{\mathrm{rig}} = \bigcup_{i=0}^r \mathrm{Sp} \mathcal{A}_i, \quad \mathcal{A}_i = \widehat{A}_i[1/t] = \mathcal{A} \left\langle \left\langle \frac{f_0}{f_i}, \dots, \frac{f_r}{f_i} \right\rangle \right\rangle / f_i\text{-torsion}.$$

2.14.1. Essential center. Let us say that the ideal $\mathcal{J} = J\mathcal{A}$ is the *essential center* of the formal blow-up $\mathcal{Y} = \bigcup_{i=0}^r \mathrm{Sp} \mathcal{A}_i \rightarrow \mathrm{Sp} \mathcal{A}$. Note that if J is an admissible ideal, then $\mathcal{J} = \mathcal{A}$, i.e., the essential center is “empty” in the affinoid space $\mathrm{Sp} \mathcal{A}$.

If $\mathrm{Sp} \mathcal{B} \rightarrow \mathrm{Sp} \mathcal{A}$ is a morphism of finite type, take an admissible formal model $A \rightarrow B$ of $\mathcal{A} \rightarrow \mathcal{B}$. Then $\mathcal{X} = \widehat{X}^{\mathrm{rig}}$, where X is the formal blow-up of $\mathrm{Spec} B$ along JB , and the map $X \rightarrow Y$ from the universality of blow-ups gives rise to the morphism $\mathcal{X} \rightarrow \mathcal{Y}$, which we call the *strict transform* of the formal blow-up $\mathcal{Y} \rightarrow \mathrm{Sp} \mathcal{A}$ by $\mathrm{Sp} \mathcal{B} \rightarrow \mathrm{Sp} \mathcal{A}$.

Lemma 2.14.2. *Let $\mathrm{Sp} \mathcal{A}$ be an irreducible affinoid, and let $\mathcal{Y} \rightarrow \mathrm{Sp} \mathcal{A}$ be the formal blow-up along $J = (f_0, \dots, f_r) \subset \mathcal{A}$. Suppose J is not nilpotent.*

(1) *The morphism $\mathcal{Y} \rightarrow \mathrm{Sp} \mathcal{A}$ is surjective. Moreover, any classical point $\mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$, where K is a nonarchimedean local field, lifts to a classical point $\mathrm{Sp} K \rightarrow \mathcal{Y}$ of \mathcal{Y} .*

(2) *If \mathcal{A} is an integral domain, then so is \mathcal{A}_i for $i = 0, \dots, r$.*

Proof. (1) For any rigid point $\alpha : \mathrm{Spf} W \rightarrow \mathrm{Spf} A$, where W is a t -adically complete valuation ring, the extension ideal JW is invertible (since J is finitely generated), and so there exists a factoring map $\mathrm{Spec} W \rightarrow Y$ which gives, by formal completion, a rigid point above α . The second assertion can be shown similarly.

(2) If \mathcal{A} is an integral domain, then so is A , since A is t -torsion free. Then A_i ($i = 0, \dots, r$) are integral domains [EGA II 1961, II, (8.1.4)]. Since A_i are excellent, \widehat{A}_i are integral domains, and so are \mathcal{A}_i for $i = 0, \dots, r$. □

3. Noether normalization for rigid spaces over $\mathcal{O}((t))$

3.1. Diagram modeled on a rooted tree.

3.1.1. Rooted tree. In what follows, by a rooted tree we simply mean a finite directed rooted tree with the orientation away from the root. As usual, we will often regard a rooted tree T as a category with the unique minimal object. The set of all vertices is denoted by V_T , and, for any $v \in V_T$, the set of all edges outgoing from v is denoted by $E_T^+(v)$. If $v \rightarrow u$ is a directed edge of T , we say v is a *parent* of u , and u is a *child* of v . Note that the set $E_T^+(v)$ can be identified with the set of all children of v . We say a vertex w is a *descendant* of v , and v is an *ancestor* of w , if there exists a directed path from v to w . A maximal vertex, i.e., a vertex with no children, will be called a *leaf*.

A *basic tree* is a height-1 rooted tree, i.e., a rooted tree consisting of a root and its children. Any rooted tree can be decomposed into the union of basic trees.

3.1.2. Diagrams. We will consider functors of the form $T \rightarrow \mathbf{Ad}_{\mathcal{O}[[t]]}$ or $T \rightarrow \mathbf{Af}_{\mathcal{O}((t))}$ from a rooted tree, which we shall simply call *diagrams*. A diagram of admissible algebras $A_* : T \rightarrow \mathbf{Ad}_{\mathcal{O}[[t]]}$ induces by composition with the functor $(\cdot)[1/t] : \mathbf{Ad}_{\mathcal{O}[[t]]} \rightarrow \mathbf{Af}_{\mathcal{O}((t))}$ (see Section 2.3) the diagram $T \rightarrow \mathbf{Af}_{\mathcal{O}((t))}$ of affinoid algebras, which we denote by $A_*[1/t]$. *Basic diagrams* are diagrams modeled on a basic tree.

Let A_* be a diagram modeled on a rooted tree. An *ideal* \mathfrak{a}_* of A_* consists of ideals $\mathfrak{a}_v \subset A_v$ for any vertex $v \in V_T$ such that, for any edge $v \rightarrow u$ of T , we have $\mathfrak{a}_v A_u \subset \mathfrak{a}_u$. In this situation, one can construct the diagram A_*/\mathfrak{a}_* given by A_v/\mathfrak{a}_v for $v \in V_T$.

Example 3.1.3. We will often use diagrams of the following form, denoted by

$$\mathcal{O}[[t^{1/N_*}]] \quad \text{or} \quad \mathcal{O}((t^{1/N_*})) (= \mathcal{O}[[t^{1/N_*}]] [1/t])$$

and defined as follows. The diagram $\mathcal{O}[[t^{1/N_*}]]$ modeled on a rooted tree T consists of $\mathcal{O}[[t^{1/N_v}]]$ for vertices v , where N_v are positive integers, such that, for any edge $v \rightarrow u$ of T , N_u is a multiple of N_v and

$$\mathcal{O}[[t^{1/N_v}]] \hookrightarrow \mathcal{O}[[t^{1/N_u}]]$$

is the unique morphism determined by $t^{1/N_v} \mapsto (t^{1/N_u})^{N_u/N_v}$.

Definition 3.1.4. A *transformation diagram modeled on a rooted tree T* is a chain of morphisms

$$\mathcal{O}[[t^{1/N_*}]] \rightarrow A_* \rightarrow A_* \langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle \rightarrow B_* = A_* \langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle / \mathfrak{a}_* \tag{3}$$

of diagrams of admissible $\mathcal{O}[[t]]$ -algebras modeled on T satisfying the following conditions:

- (a) For $v \in V_T$, we have $\mathbf{x}_v = (x_{v,1}, \dots, x_{v,n})$ and $\boldsymbol{\delta}_v = (\delta_{v,1}, \dots, \delta_{v,n}) \in (\mathbb{Q}_{>0})^n$, where the number of variables n is fixed and does not depend on v .
- (b) For any directed edge $v \rightarrow u$ of T , we have $\boldsymbol{\delta}_v \geq \boldsymbol{\delta}_u$, i.e., $\delta_{v,i} \geq \delta_{u,i}$ for $i = 1, \dots, n$.
- (c) For any directed edge $v \rightarrow u$, we have $\mathfrak{a}_v A_u \langle \mathbf{x}_u; \boldsymbol{\delta}_u \rangle \subset \mathfrak{a}_u$.

In the following, we will write $\mathcal{A}_* = A_*[1/t]$ and $\mathcal{B}_* = B_*[1/t]$. The induced diagram

$$\mathcal{O}((t^{1/N_*})) \rightarrow \mathcal{A}_* \rightarrow \mathcal{A}_* \langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle \rightarrow \mathcal{B}_* = \mathcal{A}_* \langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle / \mathfrak{a}_* \mathcal{A}_* \langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle \tag{4}$$

will be called the *transformation diagram of affinoid $\mathcal{O}((t))$ -algebras modeled on T* .

3.2. Rational diagrams. Let us say that a coordinate $\mathbf{y} = (y_1, \dots, y_n)$ is a \mathbb{Z} -rational coordinate transform of $\mathbf{x} = (x_1, \dots, x_n)$ if the coordinate transformation is given by polynomials over \mathbb{Z} , i.e., $y_i = F_i(\mathbf{x})$ is given by $F_i \in \mathbb{Z}[\mathbf{x}]$ and $x_j = G_j(\mathbf{y})$ is given by $G_j \in \mathbb{Z}[\mathbf{y}]$ ($i, j = 1, \dots, n$). Note that \mathbb{Z} -rational coordinate transformation preserves the unit-polydisk since any polynomial mapping maps points in the unit-polydisk to a point in the unit-polydisk.

Definition 3.2.1. (1) A *rational diagram modeled on a rooted tree T* is a transformation diagram of admissible $\mathcal{O}[[t]]$ -integral domains modeled on T as in (3) of Definition 3.1.4 satisfying the following conditions:

- (a) Every vertex v is equipped with an ideal $J_v = (c_0, \dots, c_{r_v})$ of A_v with fixed generators, and the set $E_T^+(v)$ of edges outgoing from v consists of exactly $r_v + 1$ elements $v \rightarrow u_i$ ($i = 0, \dots, r_v$) such that each A_{u_i} is of the form $A_{u_i} = A_i[t^{1/d_i}]$, where d_i is a positive integer and

$$A_i = A_v \left\langle \left\langle \frac{c_0}{c_i}, \dots, \frac{c_{r_v}}{c_i} \right\rangle \right\rangle / c_i\text{-torsion},$$

i.e., A_{u_i} is the i -th affine patch of the formal blow-up (see Lemma 2.14.2 (2)) of A with respect to $J_v = (c_0, \dots, c_{r_v})$ with a base extension by a root of t , and N_{u_i} is the least common multiple of N_v and d_i .

- (b) For any directed edge $v \rightarrow u$ of T , we have that $\mathbf{x}_u = (x_{u,1}, \dots, x_{u,n})$ is a \mathbb{Z} -rational coordinate transform of $\mathbf{x}_v = (x_{v,1}, \dots, x_{v,n})$.

- (c) Each B_{u_i} is of the form $B_{u_i} = B_i[t^{1/d_i}]$, where d_i is as in (a) and B_i is the strict transform

$$B_i = (B_v \widehat{\otimes}_{A_v \langle \langle \mathbf{x}_v; \delta_v \rangle \rangle} A_i \langle \langle \mathbf{x}_{u_i}; \delta_{u_i} \rangle \rangle) / c_i\text{-torsion}.$$

(2) A stratified rational diagram modeled on a rooted tree T is a transformation diagram of admissible $\mathcal{O}[[t]]$ -algebras modeled on T as in (3) of Definition 3.1.4 that satisfies the following conditions:

- (d) For any $v \in V_T$, the set $E_T^+(v)$ of edges outgoing from v is partitioned into two parts,

$$E_T^+(v) = E_T^{+,1}(v) \amalg E_T^{+,2}(v),$$

which we call *type 1* and *type 2* edges, respectively.

- (e) If $E_T^{+,1}(v) \neq \emptyset$, then A_v and B_v are integral domains, and the conditions (a) \sim (c) in (1) with $E_T^+(v)$ replaced by $E_T^{+,1}(v)$ is satisfied; i.e., $E_T^{+,1}(v)$ consists of exactly $r_v + 1$ elements $v \rightarrow u_i$ ($i = 0, \dots, r_v$), and $A_v \rightarrow A_{u_i}$ are of the form as in (a) constructed from a formal blow-up of A_v along an ideal J_v .

- (f) The type 2 edges are of the form $E_T^{+,2}(v) = \{v \rightarrow w_{ij} \mid i = 1, \dots, l, j = 1, \dots, m_i\}$ such that

- (f1) $N_{w_{ij}} = N_v$, $\mathbf{x}_{w_{ij}} = \mathbf{x}_v$, and $\delta_{w_{ij}} = \delta_v$ for all (i, j) ;
- (f2) for any $i = 1, \dots, l$, all the $A_{w_{ij}}$ are equal to an A_i ;
- (f3) $\mathcal{A}_i = A_i[1/t]$ and $\mathcal{B}_{w_{ij}} = B_{w_{ij}}[1/t]$ are given by the irreducible decompositions (Section 2.11)

$$\mathrm{Sp} \mathcal{A}_v / J_v = \bigcup_{i=1}^l \mathrm{Sp} \mathcal{A}_i, \quad \mathrm{Sp} \mathcal{B}_v \otimes_{\mathcal{A}_v} \mathcal{A}_i = \bigcup_{j=1}^{m_i} \mathrm{Sp} \mathcal{B}_{w_{ij}};$$

if \mathfrak{p}_i is the kernel of $\mathcal{A}_v \rightarrow \mathcal{A}_i$, then $A_i = A_v / \mathfrak{p}_i \cap A_v$; if \mathfrak{q}_{ij} is the kernel of $\mathcal{A}_i \langle \langle \mathbf{x}_v; \delta_v \rangle \rangle \rightarrow \mathcal{B}_v \otimes_{\mathcal{A}_v} \mathcal{A}_i \rightarrow \mathcal{B}_{w_{ij}}$, then $\mathfrak{a}_{w_{ij}} = \mathfrak{q}_{ij} \cap A_i \langle \langle \mathbf{x}_v; \delta_v \rangle \rangle$ and $B_{w_{ij}} = A_i \langle \langle \mathbf{x}_v; \delta_v \rangle \rangle / \mathfrak{a}_{w_{ij}}$.

In other words, the type 2 edges comprise the basic diagram by irreducible decomposition of the induced family over the essential blow-up center.

Remark 3.2.2. Note that if A_v and B_v are integral domains at a vertex v of T , then by Lemma 2.14.2 (2) A_w and B_w for any descendants w of v are integral domains.

Remark 3.2.3. If $\mathcal{O}[[t^{1/N_*}]] \rightarrow A_* \rightarrow A_*\langle\mathbf{x}_*; \boldsymbol{\delta}_*\rangle \rightarrow B_* = A_*\langle\mathbf{x}_*; \boldsymbol{\delta}_*\rangle/\mathfrak{a}_*$ is a stratified rational diagram (resp. rational diagram) modeled on a rooted tree T and $S \subset T$ is a subtree consisting of descendants of a fixed vertex v , then the restriction

$$\mathcal{O}[[t^{1/N_*}]]|_S \rightarrow A_*|_S \rightarrow A_*\langle\mathbf{x}_*; \boldsymbol{\delta}_*\rangle|_S \rightarrow B_*|_S = A_*\langle\mathbf{x}_*; \boldsymbol{\delta}_*\rangle/\mathfrak{a}_*|_S$$

on S is a stratified rational diagram (resp. rational diagram) modeled on S .

Proposition 3.2.4. *Let N be a multiple of all the N_v .*

(1) *Consider the morphism*

$$\coprod_{\lambda} \mathrm{Sp} \mathcal{A}_{\lambda}[t^{1/N}] \rightarrow \mathrm{Sp} \mathcal{A}_{v_0}[t^{1/N}],$$

where λ runs through all leaves of T . Then any classical point $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}_{v_0}[t^{1/N}]$ lifts to a classical point of the left-hand side.

(2) *Consider the morphism*

$$\coprod_{\lambda} \mathrm{Sp}(\mathcal{B}_{\lambda}[t^{1/N}])^{\natural} \rightarrow \mathrm{Sp}(\mathcal{B}_{v_0}[t^{1/N}])^{\natural},$$

where λ runs through all leaves of T and $(\cdot)^{\natural}$ denotes the unit-polydisk part (see page 1597). Then any classical point $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp}(\mathcal{B}_{v_0}[t^{1/N}])^{\natural}$ valued in a subfield $F \subset K$ lifts to a classical point of the left-hand side valued in F .

Proof. (1) It suffices to show that the morphism

$$\coprod_{v \rightarrow w} \mathrm{Sp} \mathcal{A}_w[t^{1/N}] \rightarrow \mathrm{Sp} \mathcal{A}_v[t^{1/N}],$$

where v is a vertex of T and w runs through all children of v , has a similar property. If the image of a classical point $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}_v[t^{1/N}]$ lies outside of the essential center, then the assertion follows from Lemma 2.14.2 (1). If not, it lifts to the component corresponding to a type 2 edge.

(2) The existence of the lifting can be shown similarly to (1). Note that if w is a child of v , since the coordinate $\mathbf{x}_w = (x_{w,1}, \dots, x_{w,n})$ is a \mathbb{Z} -rational coordinate change of $\mathbf{x}_v = (x_{v,1}, \dots, x_{v,n})$, both “unit-polydisk part” and “valued in F ” are preserved by the morphism $\mathrm{Sp} \mathcal{B}_w[t^{1/N}] \rightarrow \mathrm{Sp} \mathcal{B}_v[t^{1/N}]$. \square

3.3. Noether normalization.

Proposition 3.3.1. *Let $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$, $\boldsymbol{\delta}' = (\delta'_1, \dots, \delta'_n) \in \mathbb{Q}^n$ be vectors of rational numbers such that $0 < \boldsymbol{\delta}' < \boldsymbol{\delta}$ (i.e., $0 < \delta'_i < \delta_i$ for $i = 1, \dots, n$). Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -integral domain and $\mathcal{B} = \mathcal{A}\langle\mathbf{x}; \boldsymbol{\delta}\rangle/\mathfrak{a}$, where $\mathbf{x} = (x_1, \dots, x_n)$ and \mathfrak{a} is a prime ideal of $\mathcal{A}\langle\mathbf{x}; \boldsymbol{\delta}\rangle$. Let A be an admissible formal model of \mathcal{A} , and set $B = A\langle\mathbf{x}; \boldsymbol{\delta}\rangle/\underline{\mathfrak{a}}$, where $\underline{\mathfrak{a}} = \mathfrak{a} \cap A\langle\mathbf{x}; \boldsymbol{\delta}\rangle$, which is an admissible formal model of \mathcal{B} . Then there exists a rational diagram*

$$\mathcal{O}[[t^{1/N_*}]] \rightarrow A_* \rightarrow A_*\langle\mathbf{x}_*; \boldsymbol{\delta}_*\rangle \rightarrow B_* = A_*\langle\mathbf{x}_*; \boldsymbol{\delta}_*\rangle/\underline{\mathfrak{a}}_*$$

modeled on a rooted tree T such that the following conditions are satisfied:

- (a) Let $v_0 \in V_T$ be the root of T . Then $A_{v_0} = A$, $\mathbf{x}_{v_0} = \mathbf{x}$, $\delta_{v_0} = \delta$, and $\underline{\mathfrak{a}}_{v_0} = \underline{\mathfrak{a}}$ (hence $B_{v_0} = B$).
- (b) For any leaf λ of T , we have $\delta' < \delta_\lambda < \delta$, and the map $A_\lambda \rightarrow B_\lambda$ factors as

$$A_\lambda \rightarrow A_\lambda \langle\langle x_{\lambda,1}, \dots, x_{\lambda,d_\lambda}; \delta_{\lambda,1}, \dots, \delta_{\lambda,d_\lambda} \rangle\rangle \hookrightarrow B_\lambda,$$

where $d_\lambda \geq 0$ is a nonnegative integer and the right-hand map is injective and finite. Moreover, one can take δ_λ such that $\delta_{\lambda,1} = \dots = \delta_{\lambda,d_\lambda}$.

Proof. The proposition follows by induction from the following:

(*) If $\mathfrak{a} \neq (0)$, then, for any $\delta'' \in (\mathbb{Q}_{>0})^n$ such that $\delta'' < \delta$, there exists a rational diagram as above satisfying the condition (a) and

- (b') for any leaf λ of T , we have $\delta'' < \delta_\lambda < \delta$ for all $i = 1, \dots, n$, and there exists a finite injection of the form

$$A_\lambda \langle\langle x_{\lambda,1}, \dots, x_{\lambda,n-1}; \delta_{\lambda,1}, \dots, \delta_{\lambda,n-1} \rangle\rangle / \mathfrak{a}'_\lambda \hookrightarrow B_\lambda,$$

where $\delta_{\lambda,1} = \dots = \delta_{\lambda,n-1}$; moreover, we have $\dim A_\lambda = A$ and $\dim B_\lambda = \dim B$.

Indeed, the proposition follows by repeated application of (*), each time replacing A by A_λ and B by $A_\lambda \langle\langle y_1, \dots, y_{n-1} \rangle\rangle / \mathfrak{a}'_\lambda$, as long as $\mathfrak{a}'_\lambda \neq (0)$.

Consider the *reversed lexicographical ordering* on \mathbb{N}^n , i.e., for $\nu = (\nu_1, \dots, \nu_n)$, $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, we have $\nu < \mu$ if and only if $\nu \neq \mu$ and $\nu_k < \mu_k$, where $k = \max\{i \mid \nu_i \neq \mu_i\}$.

Suppose $\mathfrak{a} \neq (0)$. Take a nonzero $F \in \underline{\mathfrak{a}} = \mathfrak{a} \cap A \langle\langle \mathbf{x}; \delta \rangle\rangle$ and write

$$F(\mathbf{x}) = \sum_{\nu \in \mathbb{N}^n} c_\nu \mathbf{x}^\nu, \quad c_\nu \in A.$$

Consider the content ideal of F :

$$\text{Cont}(F) = (c_\nu \mid \nu \in \mathbb{N}^n).$$

This is a nonzero ideal of a Noetherian ring $A \langle\langle \mathbf{x}; \delta \rangle\rangle$ and is of the form $(c_{\nu_0}, \dots, c_{\nu_s})$, where we suppose that $\nu_0 < \dots < \nu_s$ and that the set of generators has been chosen such that ν_s is the smallest possible. Let us call such an ν_s the *maximal content degree* of F , denoted by $\text{mcd}(F)$. The proof of (*) will be done by induction on $\text{mcd}(F)$; suppose that the claim is already proved for smaller $\text{mcd}(F)$.

Step 1. We consider the formal blow-up $\{A \rightarrow A_i \mid i = 0, \dots, s\}$ along the content ideal, i.e.,

$$A_i = A \left\langle\left\langle \frac{c_{\nu_0}}{c_{\nu_i}}, \dots, \frac{c_{\nu_r}}{c_{\nu_i}} \right\rangle\right\rangle / c_{\nu_i}\text{-torsion}.$$

For each $i = 0, \dots, s$, we have $F(\mathbf{x})$ in $A_i \langle\langle \mathbf{x}; \delta \rangle\rangle$ is of the form $c_{\nu_i} F_i$ for $F_i \in A_i \langle\langle \mathbf{x}; \delta \rangle\rangle$. We have $c_{\nu_i} F_i = 0$ in B_i , the strict transform of B by $A \rightarrow A_i$, and, since B_i is c_{ν_i} -torsion free, we have $F_i = 0$ in B_i , and so $F_i \in \underline{\mathfrak{a}} A_i \langle\langle \mathbf{x}; \delta \rangle\rangle$. For $i < s$, $F_i(\mathbf{x})$ has a strictly smaller maximal content degree since $c_{\nu_s} = (c_{\nu_s}/c_{\nu_i})c_{\nu_i}$, and hence the claim (*) for A_i follows by induction. The last patch A_s will be treated in the next step (with A_s replaced by A).

Step 2. We discuss the case where $\text{mcd}(F) = \nu_0$ and $c_{\nu_0} = 1$. Take a natural number M such that

- M is not divisible by the characteristic of A ,
- $M > |\nu_0| + 1$ and $M^{M-1} > n|\nu_0|$,
- if we set

$$\boldsymbol{\varepsilon} = (1/M^{M^{n-1}}, \dots, 1/M^{M^2}, 1/M^M, 1/M),$$

then $\delta'' < \delta - \boldsymbol{\varepsilon}$.

Set $N = M^{M^{n-1}}$ and replace A by $A[t^{1/N}]$. Note that:

- (i) If $\nu_0 < \mu$, then $\boldsymbol{\varepsilon} \cdot \nu_0 < \boldsymbol{\varepsilon} \cdot \mu$.

Indeed, if $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and $\nu_0 = (\nu_{0,1}, \dots, \nu_{0,n})$ with $\mu_n = \nu_{0,n}, \dots, \mu_{k+1} = \nu_{0,k+1}$ and $\mu_k > \nu_{0,k}$ for $1 \leq k \leq n$, then

$$\begin{aligned} \boldsymbol{\varepsilon} \cdot (\boldsymbol{\mu} - \nu_0) &= M^{-M^{n-k}} (\mu_k - \nu_{0,k}) + M^{-M^{n-k+1}} (\mu_{k+1} - \nu_{0,k+1}) + \dots + M^{-M^{n-1}} (\mu_1 - \nu_{0,1}) \\ &\geq M^{-M^{n-k}} - M^{-M^{n-k+1}} |\nu_0| - \dots - M^{-M^{n-1}} |\nu_0| \\ &\geq M^{-M^{n-k}} \left\{ 1 - \frac{(k-1)|\nu_0|}{M^{M-1}} \right\} \geq M^{-M^{n-k}} \left\{ 1 - \frac{n|\nu_0|}{M^{M-1}} \right\} > 0. \end{aligned}$$

We need to consider the scaling of the coordinate $\tilde{\boldsymbol{x}} = t^{-\boldsymbol{\varepsilon}} \boldsymbol{x}$. Note that $A\langle\langle \tilde{\boldsymbol{x}}; \boldsymbol{\delta} \rangle\rangle = A\langle\langle \boldsymbol{x}; \boldsymbol{\delta} - \boldsymbol{\varepsilon} \rangle\rangle$ ($\leftrightarrow A\langle\langle \boldsymbol{x}; \boldsymbol{\delta} \rangle\rangle$). If we set $\tilde{F}(\tilde{\boldsymbol{x}}) = F(\boldsymbol{x})$, we have

$$\tilde{F}(\tilde{\boldsymbol{x}}) = \sum_{\nu \in \mathbb{N}^n} \tilde{c}_\nu \tilde{\boldsymbol{x}}^\nu, \quad \tilde{c}_\nu = t^{\boldsymbol{\varepsilon} \cdot \nu} c_\nu.$$

By (i) above, we have:

- (ii) If $\nu_0 < \mu$, then $\tilde{c}_{\nu_0} = t^{\boldsymbol{\varepsilon} \cdot \nu_0} |t^{\boldsymbol{\varepsilon} \cdot \nu_0} c_{\mu}| |t^{\boldsymbol{\varepsilon} \cdot \mu} c_{\mu}| = \tilde{c}_{\mu}$; a fortiori, $\tilde{c}_{\mu}/\tilde{c}_{\nu_0}$ is divisible by $t^{1/N}$.

Indeed, $\tilde{c}_{\mu}/\tilde{c}_{\nu_0}$ is divisible by $t^{\boldsymbol{\varepsilon} \cdot (\mu - \nu_0)}$, where $\boldsymbol{\varepsilon} \cdot (\mu - \nu_0) > 0$ by (i) and, in the notation as in the proof of (i) above,

$$\boldsymbol{\varepsilon} \cdot (\boldsymbol{\mu} - \nu_0) = M^{-M^{n-1}} \{M^{M^{n-1}-M^{n-k}} (\mu_k - \nu_{0,k}) + \dots + (\mu_1 - \nu_{0,1})\} = \frac{1}{N} \cdot (\text{positive integer}).$$

In particular, we have $\text{mcd}(\tilde{F}) \leq \nu_0 = \text{mcd}(F)$. Let

$$\text{Cont}(\tilde{F}(\tilde{\boldsymbol{x}})) = (\tilde{c}_{\mu_0}, \dots, \tilde{c}_{\mu_r})$$

$(\mu_0 < \dots < \mu_r)$ be the content ideal of \tilde{F} in A' , where $\mu_r = \text{mcd}(\tilde{F})$. By (ii) we have $\mu_r \leq \nu_0$. If $\mu_r < \nu_0$, then we can finish by induction. So we assume $\mu_r = \nu_0$. We perform the formal blow-up along the content ideal of $\tilde{F}(\tilde{\boldsymbol{x}})$, obtaining affine patches \tilde{A}_j for $j = 0, \dots, r$ as before. Similarly as before, we may finish by induction for the patches \tilde{A}_j for $j < r$.

On the patch \tilde{A}_r , the strict transform \tilde{F}_r of \tilde{F} is of the form

$$\tilde{F}_r(\tilde{\boldsymbol{x}}) = f_r(\tilde{\boldsymbol{x}}) + t^{1/N} G(\tilde{\boldsymbol{x}}),$$

where $f_r(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^{\nu_0} + \dots$ is a monic polynomial and all terms in $G(\tilde{\boldsymbol{x}})$ are of degree strictly larger than ν_0 .

Now we consider the famous coordinate change

$$\begin{cases} x_n = y_n, \\ x_k = y_n^{M^{n-k}} + y_k \quad (k = 1, \dots, n-1). \end{cases}$$

By our choice of ϵ , the above coordinate change gives rise to a coordinate change

$$\begin{cases} \tilde{x}_n = \tilde{y}_n, \\ \tilde{x}_k = \tilde{y}_n^{M^{n-k}} + \tilde{y}_k \quad (k = 1, \dots, n-1) \end{cases}$$

from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}} = t^{-\epsilon} \mathbf{y}$, and we have

$$\tilde{F}_r(\tilde{\mathbf{y}}) = f_r(\tilde{\mathbf{y}}) + t^{1/N} G(\tilde{\mathbf{y}}),$$

where

(iii) $f_r(\tilde{\mathbf{y}}) = u \tilde{y}_n^L + (\text{lower})$, where u is a unit and $L = M^{n-1} \nu_{0,1} + M^{n-2} \nu_{0,2} + \dots + \nu_{0,n}$;

(iv) all terms in $G(\tilde{\mathbf{y}})$ are of degree strictly larger than ν_0 .

Now let B'_r be the strict transform of B by $A \rightarrow \tilde{A}_r$, and set $\tilde{B}_r = B'_r \widehat{\otimes}_{\tilde{A}_r \langle \tilde{\mathbf{y}}; \delta \rangle} \tilde{A}_r \langle \tilde{\mathbf{y}}; \delta \rangle$. Consider the surjection

$$\tilde{A}_r \langle \tilde{\mathbf{y}}; \delta \rangle \rightarrow \tilde{B}_r$$

whose kernel is $\mathfrak{a}_r := \mathfrak{a} \tilde{A}_r \langle \tilde{\mathbf{y}}; \delta \rangle$. Let \tilde{B}'_r be the image of

$$\tilde{A}_r \langle \tilde{\mathbf{y}}'; \tilde{\delta}' \rangle := \tilde{A}_r \langle \tilde{y}_1, \dots, \tilde{y}_{n-1}; \delta_1, \dots, \delta_{n-1} \rangle \rightarrow \tilde{B}_r.$$

We have $\tilde{B}_r = \tilde{B}'_r \langle \tilde{y}_n; \delta_n \rangle$. Note that

$$\tilde{A}_r \langle \tilde{\mathbf{y}}; \delta \rangle = \tilde{A}_r \langle \mathbf{y}; \delta - \epsilon \rangle \quad \text{and} \quad \tilde{A}_r \langle \tilde{\mathbf{y}}'; \tilde{\delta}' \rangle = \tilde{A}_r \langle \mathbf{y}'; \delta' - \epsilon' \rangle,$$

where $\epsilon' = (\epsilon_1, \dots, \epsilon_{n-1})$, and hence $\tilde{B}_r = \tilde{B}'_r \langle y_n; \delta_n - \epsilon_n \rangle$.

We need to show that \tilde{B}_r is finite over \tilde{B}'_r . It suffices to show that $(\tilde{B}_r)_0 = \tilde{B}_r / t^{1/N} \tilde{B}_r$ is finite over $(\tilde{B}'_r)_0 = \tilde{B}'_r / t^{1/N} \tilde{B}'_r$ [Matsumura 1989, 8.4]. But this follows from (iii) and (iv).

To conclude the proof, let λ be a leaf, and let $A_\lambda = A_r$, $\mathbf{x}_\lambda = \mathbf{y}$, $\delta_\lambda = \delta - \epsilon$, and $B_\lambda = \tilde{B}_r$. Moreover, we claim that we can put $\delta_{\lambda,1} = \dots = \delta_{\lambda,n-1}$. To this end, define $\delta = \min\{\delta_{\lambda,i} \mid i = 1, \dots, n-1\}$, and let $\mathbf{x}'_\lambda = (x_{\lambda,1}, \dots, x_{\lambda,n-1})$ and $\delta'_\lambda = (\delta_{\lambda,1}, \dots, \delta_{\lambda,n-1})$. We take the base change of the finite injective map

$$A_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle / \mathfrak{a}'_\lambda \hookrightarrow B_\lambda = A_\lambda \langle \mathbf{x}_\lambda; \delta_\lambda \rangle / \mathfrak{a}_\lambda \cong (A_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle \widehat{\otimes}_{A_\lambda} A_\lambda \langle x_n; \delta_{\lambda,n} \rangle) / \mathfrak{a}_\lambda \quad (5)$$

by $A_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle \rightarrow A_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle$, followed by killing t -torsions. Here, by an abuse of notation, δ in the right-hand side stands for the vector (δ, \dots, δ) of size $n-1$. We claim that the resulting map

$$A_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle / \mathfrak{a}''_\lambda \rightarrow B'_\lambda \cong A_\lambda \langle \mathbf{x}'_\lambda, x_n; \delta, \delta_{\lambda,n} \rangle / \mathfrak{a}'''_\lambda$$

is again finite and injective. Finiteness is obvious. To show it is injective, it suffices to show that its localization by $1/t$,

$$A_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle / \mathfrak{a}''_\lambda \cdot A_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle \rightarrow B'_\lambda, \quad (6)$$

is injective. Consider the base change diagram

$$\begin{array}{ccc}
 \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle / \mathfrak{a}'_\lambda \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle & \longrightarrow & \mathcal{B}_\lambda \otimes_{\mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle} \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle \\
 \uparrow & & \uparrow \\
 \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle / \mathfrak{a}'_\lambda \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle & \longleftarrow & \mathcal{B}_\lambda
 \end{array}$$

by $\mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle \rightarrow \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle$, where the lower horizontal arrow is the localization by $1/t$ of (5), which is finite and injective. The upper horizontal arrow is injective, since $\mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle \rightarrow \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle$ is flat due to Lemma 2.10.2. Moreover, by finiteness of the horizontal arrows, $\mathcal{B}_\lambda \otimes_{\mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta'_\lambda \rangle} \mathcal{A}_\lambda \langle \mathbf{x}'_\lambda; \delta \rangle$ is already complete and hence coincides with \mathcal{B}'_λ , i.e., the upper horizontal arrow coincides with (6), whence the desired injectivity. Now, we finish the proof by replacing $\mathcal{A}_\lambda \langle \mathbf{x}_\lambda; \delta_\lambda \rangle$ by $\mathcal{A}_\lambda \langle \mathbf{x}'_\lambda, x_{\lambda,n}; \delta, \delta_{\lambda,n} \rangle$ and \mathcal{B}_λ by \mathcal{B}'_λ . \square

Let us call the rational diagram

$$\mathcal{O}[[t^{1/N_*}]] \rightarrow A_* \rightarrow A_* \langle \mathbf{x}_*; \delta_* \rangle \rightarrow B_* = A_* \langle \mathbf{x}_*; \delta_* \rangle / \mathfrak{a}_*$$

modeled on T as in Proposition 3.3.1 a *normalization diagram*. Note that if $S \subset T$ is a subtree of descendants of a fixed vertex, then $\mathcal{O}[[t^{1/N_*}]]|_S \rightarrow A_* \langle \mathbf{x}_*; \delta_* \rangle|_S \rightarrow B_*|_S = A_* \langle \mathbf{x}_*; \delta_* \rangle / \mathfrak{a}_*|_S$ is again a normalization diagram.

Theorem 3.3.2 (stratified Noether normalization). *Let $\delta = (\delta_1, \dots, \delta_n), \delta' = (\delta'_1, \dots, \delta'_n) \in \mathbb{Q}^n$ be vectors of rational numbers such that $0 < \delta' < \delta$ (i.e., $0 < \delta'_i < \delta_i$ for $i = 1, \dots, n$). Let \mathcal{A} be an affinoid $\mathcal{O}((t))$ -algebra and $\mathcal{B} = \mathcal{A} \langle \mathbf{x}; \delta \rangle / \mathfrak{a}$, where $\mathbf{x} = (x_1, \dots, x_n)$ and \mathfrak{a} is an ideal of $\mathcal{A} \langle \mathbf{x}; \delta \rangle$. Let A be an admissible formal model of \mathcal{A} , and set $B = \mathcal{A} \langle \mathbf{x}; \delta \rangle / \underline{\mathfrak{a}}$, where $\underline{\mathfrak{a}} = \mathfrak{a} \cap \mathcal{A} \langle \mathbf{x}; \delta \rangle$, which is an admissible formal model of \mathcal{B} . Then there exists a stratified rational diagram*

$$\mathcal{O}[[t^{1/N_*}]] \rightarrow A_* \rightarrow A_* \langle \mathbf{x}_*; \delta_* \rangle \rightarrow B_* = A_* \langle \mathbf{x}_*; \delta_* \rangle / \mathfrak{a}_*$$

modeled on a rooted tree T such that the conditions (a) and (b) in Proposition 3.3.1 are satisfied.

Proof. First perform the formal blow-up of A along (0), which gives the empty set of type 1 edges, and the type 2 edges correspond to the irreducible decomposition of $\mathrm{Sp} \mathcal{B}$. Then one can construct the desired stratified normalization diagram by the normalization diagrams as in Proposition 3.3.1 starting from the irreducible components, joining at each vertex the normalization diagrams from irreducible components of the induced families over the essential center of the formal blow-ups. \square

The stratified rational diagram as above will be called the *stratified normalization diagram*. Note that if $S \subset T$ is a subtree of descendants of a fixed vertex, then

$$A_*|_S \rightarrow A_* \langle \mathbf{x}_*; \delta_* \rangle|_S \rightarrow B_*|_S = A_* \langle \mathbf{x}_*; \delta_* \rangle / \mathfrak{a}_*|_S$$

is again a stratified normalization diagram.

4. Interpolation of algebraic points

4.1. Interpolation determinants.

4.1.1. Situation. In this section, we fix $\delta > 0$, an admissible $\mathcal{O}[[t]]$ -algebra A , and $B = A\langle\langle x; \delta \rangle\rangle/\mathfrak{a}$, where $x = (x_1, \dots, x_n)$ and δ stands for the row vector (δ, \dots, δ) . Moreover, we assume that there exists a factoring map

$$A \rightarrow A\langle\langle x_1, \dots, x_d; \delta \rangle\rangle \xrightarrow{\phi} B \quad (7)$$

such that

- $d \leq n$,
- ϕ is finite with nilpotent kernel.

We set $\mathcal{A} = A[1/t]$ and $\mathcal{B} = B[1/t]$ and fix a minimal set of generators v_1, \dots, v_E of B as an $A\langle\langle x_1, \dots, x_d; \delta \rangle\rangle$ -module.

We moreover fix a K -valued point $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$, where K is a nonarchimedean local field finite over $\mathcal{O}((t))$, and let $\mathrm{Sp} \mathcal{B}_\alpha \rightarrow \mathrm{Sp} K$ be the fiber of $\mathrm{Sp} \mathcal{B} \rightarrow \mathrm{Sp} \mathcal{A}$ over α , i.e., $\mathcal{B}_\alpha = \mathcal{B} \widehat{\otimes}_{\mathcal{A}} K (= \mathcal{B} \otimes_{\mathcal{A}} K)$, which is an affinoid algebra over K . Note that \mathcal{B}_α has $B_\alpha = B \widehat{\otimes}_A V_K / (t\text{-torsion}) (= B \otimes_A V_K / (t\text{-torsion}))$ as an admissible formal model, where V_K is the valuation ring of K . We have the base change of (7)

$$V_K \rightarrow V_K\langle\langle x_1, \dots, x_d; \delta \rangle\rangle \xrightarrow{\phi_\alpha} B_\alpha, \quad (8)$$

where ϕ_α is finite with nilpotent kernel due to Lemma 2.12.2.

Note that K has the unique norm $|\cdot|$ such that $|t| = e^{-1}$, where $e > 1$ is the real number fixed in Section 2.4.

4.1.2. Interpolation determinants. Let $\alpha_i : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{B}_\alpha$ for $i = 1, \dots, \mu$ be K -valued points over $\mathrm{Sp} K$, i.e., sections of $\mathrm{Sp} \mathcal{B}_\alpha \rightarrow \mathrm{Sp} K$, and $p_i : B_\alpha \rightarrow V_K$ the V_K -algebra homomorphisms corresponding to α_i .

Set $\mathbf{p} = (p_1, \dots, p_\mu)$. For $\mathbf{f} = (f_1, \dots, f_\mu) \in B_\alpha^\mu$, we consider the *interpolation determinant*

$$\Delta(\mathbf{f}, \mathbf{p}) = \det[p_j(f_i)] \mid i, j = 1, \dots, \mu].$$

We want to estimate the norm of $\Delta(\mathbf{f}, \mathbf{p})$ in terms of

$$\rho = \rho(\mathbf{p}) = |t^\delta| \cdot \max_{\substack{j, j' = 1, \dots, \mu \\ i = 1, \dots, d}} |p_j(x_i) - p_{j'}(x_i)|,$$

i.e., the radius of the smallest ball containing the projections of the points $\alpha_1, \dots, \alpha_\mu$ to the y -coordinates, normalized by a factor of $|t^\delta|$. The following estimate of $|\Delta(\mathbf{f}, \mathbf{p})|$ is the key to our approach, which goes back to an idea by Bombieri and Pila [1989]. Cluckers, Comte and Loeser [Cluckers et al. 2015] have used a similar nonarchimedean analog in their work using smooth parametrizations, but we require a somewhat different statement as we replace parametrizing maps by finite modules. In the complex analytic context, a similar argument using finite modules was used in Binyamini and Novikov [2017a].

Proposition 4.1.3. *We have*

$$|\Delta(\mathbf{f}, \mathbf{p})| \leq \rho^{C_d E^{-1/d} \mu^{1+1/d}},$$

where C_d is a positive constant depending only on d . (Recall that E is the number of minimal generators of B as an $A\langle\langle x_1, \dots, x_d; \delta \rangle\rangle$ -module; see the situation given in Section 4.1).

Proof. We first claim that we may assume, without loss of generality, that

$$\rho = |t^\delta| \cdot \max_{\substack{j=1, \dots, \mu \\ i=1, \dots, d}} |p_j(x_i)|. \tag{9}$$

To do this, we consider an automorphism (parallel translation) Φ of $V_K\langle\langle x_1, \dots, x_d; \delta \rangle\rangle$ of the form $\Phi(x_i) = x_i - s_i$ for $i = 1, \dots, d$, where the $s_i \in V_K$ are chosen such that $(s_i =) p_1(s_i) = p_1(x_i)$. Pulling back by Φ , we have a similar situation, where B_α is replaced by an isomorphic copy

$$\Phi^* B_\alpha = B_\alpha \otimes_{V_K\langle\langle x_1, \dots, x_d; \delta \rangle\rangle, \Phi} V_K\langle\langle x_1, \dots, x_d; \delta \rangle\rangle.$$

Note that, on $\Phi^* B_\alpha$, we have $p_1(x_i) = 0$ for $i = 1, \dots, d$. Then the desired inequality is equivalent to the corresponding one on $\Phi^* B_\alpha$ in which the ρ is given as in (9).

For $i = 1, \dots, \mu$, write

$$f_i = \sum_{k=1}^E f_{ik} v_k, \quad f_{ik} \in V_K\langle\langle x_1, \dots, x_d; \delta \rangle\rangle,$$

and further expand

$$f_{ik} = \sum_{v \in \mathbb{N}^d} c_{i,k,v} \mathbf{y}^v, \quad c_{i,k,v} \in t^{v \cdot \delta} V_K,$$

where $\mathbf{y} = (x_1, \dots, x_d)$. We now expand $\Delta(\mathbf{f}, \mathbf{p})$ by multilinearity with respect to each column, as a sum of interpolation determinants of the form $\Delta(\mathbf{f}', \mathbf{p})$, where $f'_i = c_{i,k_i,v_i} \mathbf{y}^{v_i} v_{k_i}$ for some choice of k_i and v_i for each $i = 1, \dots, \mu$. It will be enough to estimate each such $\Delta(\mathbf{f}', \mathbf{p})$ separately.

Now comes the main idea of Bombieri and Pila: if, for two different columns corresponding to f'_q and f'_r , one makes the same choice $k = k_q = k_r$ and $v = v_q = v_r$, then

$$\mathbf{p}(f'_q) = c_{q,k,v} \cdot \mathbf{p}(\mathbf{y}^v v_k), \quad \mathbf{p}(f'_r) = c_{r,k,v} \cdot \mathbf{p}(\mathbf{y}^v v_k),$$

and the corresponding columns thus agree up to a constant from K , and hence the determinant vanishes.

So, in order for the determinant $\Delta(\mathbf{f}', \mathbf{p})$ to be nonzero, one has to choose, for each pair (k_i, v_i) , at most one $c_{i,k_i,v_i} \mathbf{y}^{v_i} v_{k_i}$. For any such choice and any p_j , we have

$$|p_j(c_{i,k_i,v_i} \mathbf{y}^{v_i} v_{k_i})| \leq |t|^{\delta v_i} \cdot |p_j(\mathbf{y}^{v_i})| \leq \rho^{|v_i|},$$

and therefore

$$|\Delta(\mathbf{f}', \mathbf{p})| \leq \rho^{\sum_{i=1, \dots, \mu} |v_i|}. \tag{10}$$

It is clear that the largest value for the right-hand side will be obtained if we choose as many monomials \mathbf{y}^{v_i} as possible with $|v_i| = 0$, then as many as possible with $|v_i| = 1$, etc. There are at most $\Theta_d(k^{d-1})$

monomials \mathbf{y}^{v_i} with $|v_i| = k$, and each can appear at most E times for different choices of $k_i = 1, \dots, E$. Since the number of summands is μ , we solve

$$\sum_{k=1}^N k^{d-1} E = \mu \quad \Rightarrow \quad N = \Omega_d \left(\frac{\mu}{E} \right)^{1/d}$$

and deduce that the largest possible $|\Delta(\mathbf{f}', \mathbf{p})|$ will include all possible summands up to order $|v| = \lfloor N \rfloor$. In this case, we have

$$\sum_{i=1}^{\mu} |v_i| \geq \sum_{k=1}^N k^d E = \Omega_d(E \mu^{d+1}) = \Omega_d(E^{-1/d} \mu^{1+1/d}),$$

which, combined with (10), yields our assertion. □

4.2. Polynomial interpolation determinants.

4.2.1. Situation. We continue working in the situation in Section 4.1, and we further make the following setup.

Assume that the K -valued point $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$ lies over a fixed F -valued point $\beta : \mathrm{Sp} F \rightarrow \mathrm{Sp} \mathcal{O}((t))$, i.e., F is a nonarchimedean local subfield of K such that K/F is a finite extension, and

$$\begin{array}{ccc} \mathrm{Sp} K & \xrightarrow{\alpha} & \mathrm{Sp} \mathcal{A} \\ \downarrow & & \downarrow \\ \mathrm{Sp} F & \xrightarrow{\beta} & \mathrm{Sp} \mathcal{O}((t)) \end{array}$$

is commutative. We denote by $\pi \in V_F$ the uniformizer of F and by $q = p^f$ the number of elements of the residue field $V_F/(\pi)$. Note that F has the unique norm $|\cdot|$ such that $|t| = e^{-1}$, where $e > 1$ is the real number fixed in Section 2.4, and this norm coincides with the restriction of the norm on K as in Section 4.1. We have $t = u\pi^r$ by $u \in V_F^\times$ and $r \geq 1$. In particular, we have $|t| \leq |\pi|$.

Note that, even though we are interested in counting F -valued points of the affinoid spaces, we need to consider a larger field K to capture them, since we occasionally take base change by $\mathcal{O}((t)) \rightarrow \mathcal{O}((t^{1/N}))$.

Let $\mu = \mu(D)$ be the dimension of the space of polynomials of degree at most $D \in \mathbb{N}$ in $d + 1$ variables. We fix $d + 1$ functions $\mathbf{f} = (f_1, \dots, f_{d+1}) \in B_\alpha^{d+1}$ and a μ -tuple of K -valued points $\mathbf{p} = (p_1, \dots, p_\mu)$ (with $p_i : B_\alpha \rightarrow V_K$), as before. We assume

- $p_j(f_i) \in F$ for $j = 1, \dots, \mu$ and $i = 1, \dots, d + 1$,
- $p_j(x_i) \in F$ for $j = 1, \dots, \mu$ and $i = 1, \dots, d$.

We define the *polynomial interpolation determinant* $\Delta^D(\mathbf{f}, \mathbf{p})$ to be

$$\Delta^D(\mathbf{f}, \mathbf{p}) = \Delta(\mathbf{g}, \mathbf{p}), \quad \mathbf{g} = (\mathbf{f}^v \mid v \in \mathbb{N}^{d+1}, |v| \leq D).$$

In the following, we will denote by $|\cdot|_\infty$ the archimedean norm in order to distinguish it from the norm $|\cdot|$. The following lemma follows by elementary linear algebra.

Lemma 4.2.2. *Let P be a collection of adic morphisms $p : B_\alpha \rightarrow V_K$ over V_K such that $p(f_i) \in F$ for $i = 1, \dots, d+1$, and suppose that $\Delta^D(\mathbf{f}, \mathbf{p})$ vanishes for every μ -tuple \mathbf{p} of points from P . Then the set of points*

$$\{p(\mathbf{f}) \mid p \in P\} \subset F^{d+1}$$

is contained in a hypersurface defined by a polynomial $Q \in F[x_1, \dots, x_{d+1}]$ of degree at most D .

Now, let $H \in \mathbb{N}$ be an arbitrary natural number, and set $h = \log_q H$.

Proposition 4.2.3. *Suppose $H(p_i(f_j)) \leq H$ for $i = 1, \dots, \mu$ and $j = 1, \dots, d+1$. Then either $\Delta^D(\mathbf{f}, \mathbf{p}) = 0$ or*

$$|\Delta^D(\mathbf{f}, \mathbf{p})| \geq \begin{cases} |t|^{([F:\mathbb{Q}_p]/r)\{\log_q(\mu!)+(d+2)D\mu h\}} & (F \supset \mathbb{Q}_p), \\ |t|^{(d+2)D\mu h} & (F \supset \mathbb{F}_q((t))). \end{cases}$$

Proof. Suppose that $\Delta^D(\mathbf{f}, \mathbf{p}) \neq 0$, and let us first treat the case where F is a finite extension of \mathbb{Q}_p , so that $p_j(f_i) = a_{ij}/b_{ij}$ are rational numbers and $|a_{ij}|_\infty, |b_{ij}|_\infty \leq H$. Set $p = v\pi^s$ for $v \in V_F^\times$ and $s \geq 1$, so that

$$|q|^r = |t|^{fs} = |t|^{[F:\mathbb{Q}_p]}.$$

All numbers in the column of $\Delta^D(\mathbf{f}, \mathbf{p})$ corresponding to p_j have denominators dividing $b_j = \prod_i b_{ij}^D$, and setting $b = \prod_j b_j$ we have $|b|_\infty \leq H^{(d+1)D\mu}$. Since every entry in the matrix defining $\Delta^D(\mathbf{f}, \mathbf{p})$ is bounded in the classical absolute value by H^D , we also have

$$|\Delta^D(\mathbf{f}, \mathbf{p})|_\infty \leq \mu! H^{D\mu}.$$

Then $b\Delta^D(\mathbf{f}, \mathbf{p}) \in \mathbb{Z}$ and

$$|b\Delta^D(\mathbf{f}, \mathbf{p})|_\infty \leq \mu! H^{(d+2)D\mu},$$

and therefore

$$|\Delta^D(\mathbf{f}, \mathbf{p})| \geq |b\Delta^D(\mathbf{f}, \mathbf{p})| \geq |q|^{\log_q |b\Delta^D(\mathbf{f}, \mathbf{p})|_\infty} = |t|^{(fs/r)\{\log_q(\mu!)+(d+2)D\mu h\}}.$$

Similarly, in the $F = \mathbb{F}_q((\pi))$ case, all rational functions $p_j(f_i) = a_{ij}/b_{ij} \in \mathbb{F}_q(t)$ in the column of $\Delta^D(\mathbf{f}, \mathbf{p})$ corresponding to p_j have denominators dividing b_j as above, and b has degree bounded by $(d+1)Dh$. Since every entry in the matrix defining $\Delta^D(\mathbf{f}, \mathbf{p})$ is a rational function of degree at most Dh , we also have

$$\deg_t \Delta^D(\mathbf{f}, \mathbf{p}) \leq D\mu h.$$

Then $b\Delta^D(\mathbf{f}, \mathbf{p}) \in \mathbb{F}_q[t]$ and

$$\deg_t b\Delta^D(\mathbf{f}, \mathbf{p}) \leq (d+2)D\mu h,$$

and therefore

$$|\Delta^D(\mathbf{f}, \mathbf{p})| \geq |b\Delta^D(\mathbf{f}, \mathbf{p})| \geq \deg_t b\Delta^D(\mathbf{f}, \mathbf{p}) \geq |t|^{(d+2)D\mu h},$$

as claimed. \square

4.3. Interpolation by a hypersurface. Let P be as in Lemma 4.2.2, and let

$$\rho = \rho(P) = |t|^\delta \cdot \max_{\substack{p \neq p' \in P \\ i=1, \dots, d}} |p(x_i) - p'(x_i)|.$$

Comparing Propositions 4.1.3 and 4.2.3 we obtain the following corollary.

Corollary 4.3.1. *Suppose $H(p(f_i)) \leq H$ for $i = 1, \dots, d+1$ and $p \in P$. Let $\varepsilon > 0$. There exist a constant*

$$D = \begin{cases} D(d, \varepsilon, E, [F : \mathbb{Q}_p]) & (F \supset \mathbb{Q}_p), \\ D(d, \varepsilon, E, r) & (F \supset \mathbb{F}_q((t))), \end{cases}$$

which does not depend on $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$, such that if $\rho < |\pi|^{h\varepsilon}$, then the set of points

$$\{p(\mathbf{f}) \mid p \in P\} \subset F^{d+1}$$

is contained in a hypersurface defined by a polynomial $Q \in F[X_1, \dots, X_{d+1}]$ of degree at most D .

Proof. By Lemma 4.2.2, it is enough to prove that $\Delta^D(\mathbf{f}, \mathbf{p}) = 0$ for every μ -tuple $\mathbf{p} \subset P$. Assume the contrary. In the $F \supset \mathbb{Q}_p$ case, we have by Proposition 4.2.3 a lower bound

$$|\Delta^D(\mathbf{f}, \mathbf{p})| \geq |t|^{([F:\mathbb{Q}_p]/r)\{\log_q(\mu!)+(d+2)D\mu h\}}.$$

On the other hand, by Proposition 4.1.3, we have an upper bound

$$|\Delta(\mathbf{f}, \mathbf{p})| \leq \rho^{C_d E^{-1/d} \mu^{1+1/d}}.$$

Thus

$$\rho^{C_d E^{-1/d} \mu^{1+1/d}} \geq |t|^{([F:\mathbb{Q}_p]/r)\{\log_q(\mu!)+(d+2)D\mu h\}}.$$

Taking $\mu \sim_d D^{d+1}$ and $\rho < |\pi|^{\varepsilon h} = |t|^{\varepsilon h/r}$ into account, we have a contradiction as soon as

$$C'_d h \varepsilon E^{-1/d} D^{d+2+1/d} \geq [F : \mathbb{Q}_p] \{(d+1) \log_q D + (d+2) C''_d D^{d+2} h\}$$

with suitable constants C'_d, C''_d . Thus we obtain a contradiction, with a suitable $D \sim_d E \cdot ([F : \mathbb{Q}_p]/\varepsilon)^d$, for instance.

In the $F \supset \mathbb{F}_q((t))$ case, we have

$$\rho^{C_d E^{-1/d} \mu^{1+1/d}} \geq |t|^{(d+2)D\mu h}.$$

Similarly to the previous case, we have a contradiction as soon as

$$C'_d \varepsilon E^{-1/d} D^{d+2+1/d} \geq (d+2) C''_d D^{d+2} r$$

with suitable constants C'_d and C''_d . Thus we obtain a contradiction, with a suitable $D \sim_d E \cdot (r/\varepsilon)^d$, for instance. \square

Proposition 4.3.2. *Let $H \in \mathbb{N}$ and $\varepsilon > 0$. There exist a constant*

$$D' = \begin{cases} D'(d, \varepsilon, E, \delta, [F : \mathbb{Q}_p]) & (F \supset \mathbb{Q}_p), \\ D'(d, \varepsilon, E, \delta, r) & (F \supset \mathbb{F}_q((t))), \end{cases}$$

not depending on $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$, and a collection of H^ε hypersurfaces in F^{d+1} , each of degree at most D' , such that the following holds: for every K -valued point $p : \mathcal{B}_\alpha \rightarrow K$ over α such that $p(f_i) \in F$ and $H(p(f_i)) \leq H$ for $i = 1, \dots, d + 1$, the point $p(\mathbf{f})$ belongs to one of the hypersurfaces.

Proof. Suppose first that $h\varepsilon/(2d) \geq 1$. Apply Corollary 4.3.1 with $\varepsilon/(2d)$ in place of ε , and denote by D' the resulting degree. It will suffice to subdivide the collection of points $p : \mathcal{B}_\alpha \rightarrow V_K$ as above into $H^\varepsilon = q^{h\varepsilon}$ subcollections P_q satisfying $\rho(P_q) \leq |\pi|^{h\varepsilon/(2d)}$. For each j the residues $p(x_j) \in V_F/(\pi)^{\lceil h\varepsilon/(2d) \rceil}$ take at most $q^{\lceil h\varepsilon/(2d) \rceil}$ different values. Defining each collection to consist of points that give the same residue for each j , we obtain

$$q^{d\lceil h\varepsilon/(2d) \rceil} < q^{d \cdot h\varepsilon/d} \leq H^\varepsilon$$

such collections, as required.

Now suppose $h\varepsilon/(2d) < 1$. Apply Corollary 4.3.1 with $\delta\varepsilon/(2d)$ in place of ε , and denote by D' the resulting degree. Then the collection of all points $p : \mathcal{B}_\alpha \rightarrow V_K$ as above satisfies

$$\rho(P) \leq |t|^\delta \leq |\pi|^{h\delta\varepsilon/(2d)},$$

so a single hypersurface of degree D' suffices to interpolate all points in P . □

We also prove a polylogarithmic interpolation result, as follows.

Proposition 4.3.3. *Let $H \in \mathbb{N}$ and $\varepsilon > 0$. There exist a constant*

$$C = \begin{cases} C(d, E, \delta, [F : \mathbb{Q}_p]/r) & (F \supset \mathbb{Q}_p), \\ C(d, E, \delta) & (F \supset \mathbb{F}_q((t))), \end{cases}$$

not depending on $\alpha : \mathrm{Sp} K \rightarrow \mathrm{Sp} \mathcal{A}$, and a hypersurface $V \subset K^{d+1}$ of degree at most $C \cdot h^d$ such that the following holds: for every K -valued point $p : \mathcal{B}_\alpha \rightarrow K$ over α such that $p(f_j) \in F$ and $H(p(f_j)) \leq H$ for $j = 1, \dots, d + 1$, we have $p(\mathbf{f}) \in V$.

Proof. By Lemma 4.2.2 it is enough to prove that $\Delta^D(\mathbf{f}, \mathbf{p}) = 0$ for every μ -tuple \mathbf{p} of points as above. Assume the contrary. In the $F \supset \mathbb{Q}_p$ case, we have by Proposition 4.2.3 a lower bound

$$|\Delta^D(\mathbf{f}, \mathbf{p})| \geq |t|^{\lfloor [F:\mathbb{Q}_p]/r \rfloor \{ \log_q(\mu!) + (d+2)D\mu h \}}.$$

On the other hand, by Proposition 4.1.3, we have an upper bound

$$|\Delta(\mathbf{f}, \mathbf{p})| \leq \rho^{C_d E^{-1/d} \mu^{1+1/d}} \leq |t|^{\delta C_d E^{-1/d} \mu^{1+1/d}}.$$

Thus

$$|t|^{\delta C_d E^{-1/d} \mu^{1+1/d}} \geq |t|^{\lfloor [F:\mathbb{Q}_p]/r \rfloor \{ \log_q(\mu!) + (d+2)D\mu h \}}.$$

Recalling that $\mu \sim_d D^{d+1}$, we have a contradiction as soon as

$$C'_d r \delta E^{-1/d} D^{d+2+1/d} \geq [F : \mathbb{Q}_p] \{ (d + 1) \log_q D + (d + 2) C''_d D^{d+2} h \}$$

with suitable constants C'_d, C''_d . Thus we obtain a contradiction, with a suitable $D \sim_d E \cdot (h \cdot [F : \mathbb{Q}_p] / \varepsilon r)^d$, for instance.

In the $F \supset \mathbb{F}_q((t))$ case, we have

$$|t|^{\delta C_d E^{-1/d} \mu^{1+1/d}} \geq |t|^{(d+2)D\mu h}.$$

Similarly to the previous case, we have a contradiction as soon as

$$C'_d \delta E^{-1/d} D^{d+2+1/d} \geq (d+2)C''_d D^{d+2} h$$

with suitable constants C'_d and C''_d , for instance with a suitable $D \sim_d E(h/\varepsilon)^d$. □

4.4. Families of hypersurfaces.

4.4.1. The space \mathcal{H}_n . As usual, we identify \mathbb{N}^n with the set of all monomials in n variables $X = (X_1, \dots, X_n)$ by $v = (v_1, \dots, v_n) \mapsto X^v = X_1^{v_1} \cdots X_n^{v_n}$, and let

$$L(D, n) = \{v \in \mathbb{N}^n \mid |v| \leq D\}$$

denote the set of monomials of degree at most D . Let $\mathcal{O}((t))^{L(D,n)}$ be the $\mathcal{O}((t))$ -module of all maps of the form $L(D, n) \rightarrow \mathcal{O}((t))$, and consider the $\mathcal{O}((t))$ -scheme

$$P_{D,n} = \text{Proj}(\text{Sym } \mathcal{O}((t))^{L(D,n)}),$$

which is the parameter space that parametrizes all hypersurfaces of degree at most D in n variables over $\mathcal{O}((t))$. Let

$$\begin{array}{ccc} H_{D,n} & \hookrightarrow & P_{D,n} \times_{\mathcal{O}((t))} \text{Spec } \mathcal{O}((t))[X_1, \dots, X_n] \\ \downarrow & & \swarrow \text{pr}_1 \\ P_{D,n} & \longleftarrow & \end{array}$$

be the universal family.

Similarly, we consider the analytic parameter space $\mathcal{P}_{D,n} = P_{D,n}^{\text{an}}$ and the universal family

$$\begin{array}{ccc} \mathcal{H}_{D,n} & \hookrightarrow & \mathcal{P}_{D,n} \times_{\mathcal{O}((t))} \text{Sp } \mathcal{O}((t))\langle\langle X_1, \dots, X_n \rangle\rangle \\ \downarrow & & \swarrow \text{pr}_1 \\ \mathcal{P}_{D,n} & \longleftarrow & \end{array}$$

Note that the fiber of $\mathcal{H}_{D,n} \rightarrow \mathcal{P}_{D,n}$ over a K -valued point $\text{Sp } K \rightarrow \mathcal{P}_{D,n}$ is an affinoid, which is the intersection of the fiber $(H_{D,n} \times_{P_{D,n}} \text{Spec } K)^{\text{an}}$ and the closed unit-disc $\text{Sp } K\langle\langle X_1, \dots, X_n \rangle\rangle$.

4.4.2. The space $\mathcal{H}_{D,n,d}$. Let $d < n$. For any subset $I = \{i_1, \dots, i_{d+1}\} \subset \{1, \dots, n\}$ of $d+1$ elements, let $P_{D,d+1}^I$ (resp. $\mathcal{P}_{D,d+1}^I$) be the copy of $P_{D,d+1}$ (resp. $\mathcal{P}_{D,d+1}$) with the coordinates $X_I = (X_{i_1}, \dots, X_{i_{d+1}})$. We set

$$P_{D,n,d} = \prod_I P_{D,n,d}^I \quad \left(\text{resp. } \mathcal{P}_{D,n,d} = \prod_I \mathcal{P}_{D,n,d}^I \right),$$

where the product is taken over all subsets $I \subset \{1, \dots, n\}$ of $d + 1$ elements. Similarly to the above constructions, we have the universal families

$$H_{D,n,d} = \prod_I H_{D,d+1}^I \rightarrow P_{D,n,d}, \quad \mathcal{H}_{D,n,d} = \prod_I \mathcal{H}_{D,d+1}^I \rightarrow \mathcal{P}_{D,n,d}.$$

Lemma 4.4.3. *The fibers of $\mathcal{H}_{D,n,d} \rightarrow \mathcal{P}_{D,n,d}$ over K -valued points are of dimension at most d .*

Proof. It suffices to show that the fibers of $H_{D,n,d} \rightarrow P_{D,n,d}$ over closed points of the form $\text{Spec } K \rightarrow P_{D,n,d}$, where K is a field, are of dimension at most d . This is then completely a question in classical algebraic geometry. Let \mathcal{F} be such a fiber over K , and suppose that the dimension of \mathcal{F} is $k > d$. Then, at a generic point, \mathcal{F} is smooth of dimension k , and in particular one can choose a subset $I \subset \{1, \dots, n\}$ of size $d + 1$ such that the projection $\mathbf{X} = (X_1, \dots, X_n) \mapsto \mathbf{X}_I = (X_{i_1}, \dots, X_{i_{d+1}})$ restricted on \mathcal{F} is smooth on a Zariski open subset. But this contradicts the fact that the image of this projection is contained in a hypersurface defined by a nonzero polynomial of degree at most D . \square

5. Proof of the counting theorems

In this section we prove the point-counting theorems. In Section 5.1 we formulate and prove a version of the counting theorem which is uniform in families for the overconvergent setting. In Section 5.2 we deduce from this general result the various point-counting theorems formulated in the introduction.

5.1. The general counting theorem. Let $\delta = (\delta_1, \dots, \delta_n) \in (\mathbb{Q}_{>0})^n$ be a vector of positive rational numbers, \mathcal{A} an affinoid $\mathcal{O}((t))$ -algebra, and $\mathcal{B} = \mathcal{A}\langle\langle \mathbf{x}; \delta \rangle\rangle/\mathfrak{a}$, where $\mathbf{x} = (x_1, \dots, x_n)$ and \mathfrak{a} is an ideal of $\mathcal{A}\langle\langle \mathbf{x}; \delta \rangle\rangle$. Let A be an admissible formal model of \mathcal{A} , and set $B = A\langle\langle \mathbf{x}; \delta \rangle\rangle/\mathfrak{a}$, where $\mathfrak{a} = \mathfrak{a} \cap A\langle\langle \mathbf{x}; \delta \rangle\rangle$, which is an admissible formal model of \mathcal{B} . Thus we have a family

$$\varphi : \mathcal{X} = \text{Sp } \mathcal{B} \rightarrow \mathcal{Y} = \text{Sp } \mathcal{A}$$

of analytic subspaces in the polydisk of polyradius $|t|^{-\delta} = (|t|^{-\delta_1}, \dots, |t|^{-\delta_n})$.

Fix $f_1, \dots, f_n \in \mathcal{B}$ which are algebraic over $\mathcal{A}[x]$ and generate \mathcal{B} over \mathcal{A} , and set $\mathbf{f} := (f_1, \dots, f_n)$.

For a classical point $\beta : \text{Sp } F \rightarrow \text{Sp } \mathcal{O}((t))$ and a classical point $\alpha : \text{Sp } K \rightarrow \mathcal{Y}$ over β , we denote by $\mathcal{X}_\alpha = \text{Sp } \mathcal{B}_\alpha$ the fiber of φ over α . For any classical point $\alpha' : \text{Sp } K' \rightarrow \mathcal{X}_\alpha$ with the corresponding homomorphism $q : \mathcal{B}_\alpha \rightarrow K'$, where K' is a finite extension of F containing K , such that $q(f_i), q(x_i) \in F$ for $i = 1, \dots, n$, the height $H(\alpha'; \mathbf{f})$ is the maximum of the heights of $q(f_i) \in F$ ($i = 1, \dots, n$). Denote by $\mathcal{X}_\alpha(V_F, H; \mathbf{f})$ the set of all such points α' of the unit-polydisk part $(\mathcal{X}_\alpha)^\natural$ such that $H(\alpha'; \mathbf{f}) \leq H$, i.e.,

$$\mathcal{X}_\alpha(V_F, H; \mathbf{f}) = \left\{ \alpha' : \text{Sp } K' \rightarrow (\mathcal{X}_\alpha)^\natural \mid \begin{array}{l} q(f_i) \text{ and } q(x_i) \text{ lies in } F \text{ for } i = 1, \dots, n \\ \text{and } H(\alpha'; \mathbf{f}) \leq H \end{array} \right\}.$$

Finally, let σ_β be the constant

$$\sigma_\beta = \begin{cases} [F : \mathbb{Q}_p] & (F \supset \mathbb{Q}_p), \\ r_\beta & (F \supset \mathbb{F}_q((t))), \end{cases}$$

where r_β is the positive integer such that $t = u\pi^{r_\beta}$, with π the uniformizer of V_F and $u \in V_F^\times$.

Definition 5.1.1. Let K be a nonarchimedean local field.

- (1) An irreducible analytic subspace $\mathcal{Z} \hookrightarrow \mathrm{Sp} K \langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle$ is said to be *algebraic* if there exists an algebraic subvariety $Z \hookrightarrow \mathrm{Spec} K[\mathbf{x}]$ such that $\mathcal{Z}^{\mathrm{cl}} \subset Z$ and $\dim \mathcal{Z} = \dim Z$.
- (2) An analytic subspace $\mathcal{Z} \hookrightarrow \mathrm{Sp} K \langle\langle \mathbf{x}; \boldsymbol{\delta} \rangle\rangle$ is said to be *algebraic* if each of its irreducible components is algebraic.

Definition 5.1.2. Let $\varphi : \mathcal{X} = \mathrm{Sp} \mathcal{B} \rightarrow \mathcal{Y} = \mathrm{Sp} \mathcal{A}$ be a morphism of affinoids over $\mathcal{O}(\!(t)\!)$ as above.

- (1) The family φ is said to have *algebraic fibers* if every fiber \mathcal{X}_α over a classical point of \mathcal{Y} is algebraic in the sense of Definition 5.1.1 (2).
- (2) The *maximal fiber dimension* of the family φ is the maximum over the dimension of the fibers \mathcal{X}_α , where α ranges over the classical points of \mathcal{Y} .

Note that under our assumptions on \mathbf{f} , being algebraic over the \mathbf{f} coordinates is equivalent to being algebraic over the \mathbf{x} coordinates.

Theorem 5.1.3. Let $\varepsilon > 0$. There exist a transformation diagram of the form

$$\mathcal{O}[\![t^{1/N_*}]\!] \rightarrow A_* \rightarrow A_* \langle\langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle\rangle \rightarrow B_* = A_* \langle\langle \mathbf{x}_*; \boldsymbol{\delta}_* \rangle\rangle / \mathfrak{a}_* \tag{11}$$

of admissible $\mathcal{O}[\![t]\!]$ -algebras modeled on a rooted tree T and, for any positive integer σ , a constant $C(B, \varepsilon, \sigma)$ satisfying the following conditions:

- (a) $A_{v_0} \rightarrow B_{v_0}$ at the root v_0 coincides with $A \rightarrow B$.
- (b) For any leaf w , we have that $\varphi_w : \mathcal{X}_w = \mathrm{Sp} \mathcal{B}_w \rightarrow \mathcal{Y}_w = \mathrm{Sp} \mathcal{A}_w$ has equidimensional algebraic fibers.
- (c) For any classical point $\alpha : \mathrm{Sp} K \rightarrow \mathcal{Y}$ above $\beta : \mathrm{Sp} F \rightarrow \mathrm{Sp} \mathcal{O}(\!(t)\!)$, we have

$$\mathcal{X}_\alpha(V_F, H; \mathbf{f}) \subset \bigcup_{(w_j, \alpha_j)} (\mathcal{X}_{w_j})_{\alpha_j}(V_F, H; \mathbf{f}),$$

where $\{(w_j, \alpha_j)\}$ is a collection of $C(B, \varepsilon, \sigma) \cdot H^\varepsilon$ pairs, where w_j is a leaf of T and α_j is a classical point of \mathcal{Y}_{w_j} lying over α . Here \mathbf{f} on the right-hand side denotes the image of \mathbf{f} in $(B_{w_j})_{\alpha_j}$.

Remark 5.1.4. In fact, the families $\varphi_w : \mathcal{X}_w = \mathrm{Sp} \mathcal{B}_w \rightarrow \mathcal{Y}_w = \mathrm{Sp} \mathcal{A}_w$ as in (b) that we will obtain in the following proof are not only of algebraic fibers, but are algebraic over \mathcal{Y}_w .

To show the theorem, we need the following proposition, which is the main inductive step in the proof of the theorem.

Proposition 5.1.5. Let $\varepsilon > 0$. There exists a transformation diagram of the form (11) as above satisfying the conditions (a), (c), and the following (b)' instead of (b):

- (b') For any leaf w , either $\varphi_w : \mathcal{X}_w = \mathrm{Sp} \mathcal{B}_w \rightarrow \mathcal{Y}_w = \mathrm{Sp} \mathcal{A}_w$ has equidimensional algebraic fibers, or the maximal fiber dimension of φ_w is strictly smaller than that of φ .

Proof. We begin constructing T by forming the stratified Noether normalization tree starting from $A \rightarrow B$. According to Proposition 3.2.4, it will suffice to construct a transformation diagram from each of the leaves of the stratified Noether normalization tree. Hence we may assume without loss of generality that $A \rightarrow B$ is already normalized, i.e., there exists a factoring map

$$A \rightarrow A\langle\langle x_1, \dots, x_d; \delta \rangle\rangle \hookrightarrow B$$

which is finite and injective.

Let $I = \{i_1, \dots, i_{d+1}\}$ be a subset of size $d + 1$ of $\{1, \dots, n\}$, and write $\mathbf{f}_I = (f_{i_1}, \dots, f_{i_{d+1}})$. We apply Proposition 4.3.2 with ε/N and \mathbf{f}_I , where N is the number of possible $I \subset \{1, \dots, n\}$, and, repeating this for all $I \subset \{1, \dots, n\}$ of size $d + 1$, we conclude that the set $\mathcal{X}_\alpha(V_F, H)$ is contained in at most H^ε \mathbf{f} -pullbacks of fibers of $\mathcal{H}_{D,n,d} \rightarrow \mathcal{P}_{D,n,d}$, where D is the number given in Proposition 4.3.2.

More formally, one can replace $\mathcal{P}_{D,n,d}$ by an affinoid open subspace that contains all the points over which the fibers of $\mathcal{H}_{D,n,d}$ are those we are considering, and so we may assume that both $\mathcal{H}_{D,n,d}$ and $\mathcal{P}_{D,n,d}$ are affinoids. Let \mathcal{A}' be the affinoid algebra such that $(\mathcal{Y}' :=) \operatorname{Sp} \mathcal{A}' = \operatorname{Sp} \mathcal{A} \times_{\mathcal{O}((t))} \mathcal{P}_{D,n,d}$, and define $\mathcal{X}' = \operatorname{Sp} \mathcal{B}'$ by the Cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \hookrightarrow & \mathcal{P}_{D,n,d} \times_{\mathcal{O}((t))} \operatorname{Sp} \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{H}_{D,n,d} & \hookrightarrow & \mathcal{P}_{D,n,d} \times_{\mathcal{O}((t))} \operatorname{Sp} \mathcal{O}((t))\langle\langle X_1, \dots, X_n \rangle\rangle \end{array}$$

where the right vertical arrow is induced by $X_i \mapsto f_i$ for $i = 1, \dots, n$. By what we have seen above, we have

$$\mathcal{X}_\alpha(V_F, H; \mathbf{f}) \subset \bigcup_j \mathcal{X}'_{\alpha_j}(V_F, H; \mathbf{f}), \tag{12}$$

where $\{\alpha_j\}$ is a collection of $C_n \cdot H^\varepsilon$ points of $\mathcal{Y}' = \operatorname{Sp} \mathcal{A}'$ lying over α , which correspond to suitable choices of the hypersurfaces of the form $Q(\mathbf{f}_I) = 0$ for each I . Here C_n is some constant depending only on n .

Now we augment our tree by adding a child v of the root, and set $A_v = A'$ and $B_v = B'$, where A' and B' are suitable admissible formal models of \mathcal{A}' and \mathcal{B}' , respectively. We then perform another stratified Noether normalization of $A_v \rightarrow B_v$, appending the resulting tree to the root v .

Proposition 3.2.4 and (12) ensure that the condition (c) in the statement holds. It remains to verify the condition (b)'. Having performed stratified Noether normalization, we know that each leaf $\varphi_w : \mathcal{X}_w \rightarrow \mathcal{Y}_w$ has constant fiber dimension $d = d_w$. If $d_w < d$, we are done. Otherwise, the fibers of \mathcal{X}_w over classical points of \mathcal{Y}_w are d -equidimensional and contained in a fiber of $\mathcal{H}_{D,n,d}$, which is of dimension d due to Lemma 4.4.3. Hence $\varphi_w : \mathcal{X}_w \rightarrow \mathcal{Y}_w$ has algebraic fibers, as desired. \square

Proof of Theorem 5.1.3. This is proved by applying Proposition 5.1.5 to the initial family $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ and then applying it repeatedly to each of the leaves that have nonalgebraic fibers. Since the maximal fiber dimension of the fibers drops at each step, this process terminates after at most n repetitions. \square

5.2. Proofs of the statements from the Introduction.

5.2.1. Proof of Theorem 1.3.1. Let B denote a formal model for \mathcal{B} , and recall that

$$B = \mathcal{O}[[t]]\langle\langle x_1, \dots, x_n \rangle\rangle/\mathfrak{b}.$$

Set $B' = \mathcal{O}[[t]]\langle\langle tx_1, \dots, tx_n \rangle\rangle/\mathfrak{b}'$, where \mathfrak{b}' is the image of \mathfrak{b} by the map $x_i \rightarrow tx_i$. Then $\mathrm{Sp} B^{\natural 1}$ corresponds to $(\mathrm{Sp} B')^{\natural 1}$ under this rescaling. We will now apply Theorem 5.1.3 to B' , where we take $A = \mathcal{O}[[t]]$ and $\mathbf{f} = (tx_1, \dots, tx_n)$. In the resulting tree T , whenever a leaf w has positive-dimensional fibers, it corresponds by definition to the algebraic part of $\mathrm{Sp} B$. The remaining fibers are zero-dimensional, and in particular their cardinality is uniformly bounded by some constant depending only on B (for instance by Remark 5.1.4). The conclusion of Theorem 1.3.1 therefore follows from the conclusion of Theorem 5.1.3.

5.2.2. Proof of Theorem 1.2.4. Let B denote a formal model for \mathcal{B} and recall that $B = A\langle\langle x_1, \dots, x_n \rangle\rangle/\mathfrak{b}$. Set $A' = A\langle\langle y_1, \dots, y_n \rangle\rangle$ and $B' = A'\langle\langle tz_1, \dots, tz_n \rangle\rangle/\mathfrak{b}'$, where \mathfrak{b}' is the image of \mathfrak{b} by the map $x_i = y_i + tz_i$.

Fix α and F_α as in the statement of Theorem 1.2.4, and let V_α be the valuation ring of F_α . Let p be the morphism $p : A \rightarrow V_\alpha$ corresponding to α . Recall that q_α is the cardinality of the residue field of F_α . Then we may choose a collection of q_α^n points $\alpha_j \in V_\alpha^n$ corresponding to morphisms $p_j : A' \rightarrow V_\alpha$ with $p_j|_A = p$ such that the union of polydiscs of radius $|t|$ around each α_j covers V_α^n .

Now apply Theorem 5.1.3 to B' with $f_i = y_i + tz_i$. Then the points in $\mathcal{X}_{\alpha_j}(V_\alpha, H; \mathbf{f})$ are in bijection with the points of $(\mathrm{Sp} B)_\alpha(F_{\alpha,0}, H)$ belonging to the ball of radius $|t|$ around α_j , and in particular the union of q_α^n such sets is in bijection with $(\mathrm{Sp} B)_\alpha(F_{\alpha,0}, H)$. The conclusion of Theorem 1.2.4 thus follows from Theorem 5.1.3.

Acknowledgements

We would like to thank Raf Cluckers for several corrections and many important comments on the first version of this manuscript.

Binyamini was supported by the Israel Science Foundation (grant no. 1167/17). This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement no. 802107). Kato was supported by JSPS KAKENHI grant number 17H02832.

References

- [Binyamini 2019] G. Binyamini, “Density of algebraic points on Noetherian varieties”, *Geom. Funct. Anal.* **29**:1 (2019), 72–118. MR Zbl
- [Binyamini 2022] G. Binyamini, “Point counting for foliations over number fields”, *Forum Math. Pi* **10** (2022), art.id. e6. MR Zbl
- [Binyamini and Novikov 2017a] G. Binyamini and D. Novikov, “The Pila–Wilkie theorem for subanalytic families: a complex analytic approach”, *Compos. Math.* **153**:10 (2017), 2171–2194. MR Zbl
- [Binyamini and Novikov 2017b] G. Binyamini and D. Novikov, “Wilkie’s conjecture for restricted elementary functions”, *Ann. of Math. (2)* **186**:1 (2017), 237–275. MR Zbl

- [Binyamini et al. 2022] G. Binyamini, R. Cluckers, and D. Novikov, “Point counting and Wilkie’s conjecture for non-Archimedean Pfaffian and Noetherian functions”, *Duke Math. J.* **171**:9 (2022), 1823–1842. MR Zbl
- [Bombieri and Pila 1989] E. Bombieri and J. Pila, “The number of integral points on arcs and ovals”, *Duke Math. J.* **59**:2 (1989), 337–357. MR Zbl
- [Bosch 2014] S. Bosch, *Lectures on formal and rigid geometry*, Lecture Notes in Mathematics **2105**, Springer, 2014. MR Zbl
- [Bosch et al. 1984] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis: a systematic approach to rigid analytic geometry*, Grundle. Math. Wissen. **261**, Springer, 1984. MR Zbl
- [Bourbaki 1972] N. Bourbaki, *Elements of mathematics: commutative algebra*, Hermann, Paris, 1972. MR Zbl
- [Cluckers and Lipshitz 2011] R. Cluckers and L. Lipshitz, “Fields with analytic structure”, *J. Eur. Math. Soc.* **13**:4 (2011), 1147–1223. MR Zbl
- [Cluckers et al. 2015] R. Cluckers, G. Comte, and F. Loeser, “Non-Archimedean Yomdin–Gromov parametrizations and points of bounded height”, *Forum Math. Pi* **3** (2015), art. id. e5. MR Zbl
- [Cluckers et al. 2020] R. Cluckers, A. Forey, and F. Loeser, “Uniform Yomdin–Gromov parametrizations and points of bounded height in valued fields”, *Algebra Number Theory* **14**:6 (2020), 1423–1456. MR Zbl
- [Demangos 2015] L. Demangos, “ T -modules and Pila–Wilkie estimates”, *J. Number Theory* **154** (2015), 201–277. MR Zbl
- [Demangos 2016] L. Demangos, “A few remarks on a Manin–Mumford conjecture in function field arithmetic and generalized Pila–Wilkie estimates”, *Riv. Math. Univ. Parma (N.S.)* **7**:1 (2016), 205–216. MR Zbl
- [van den Dries 1992] L. van den Dries, “Analytic Ax–Kochen–Ersov theorems”, pp. 379–398 in *Proceedings of the International Conference on Algebra, Part 3* (Novosibirsk, 1989), edited by L. A. Bokut’ et al., Contemp. Math. **131**, Amer. Math. Soc., Providence, RI, 1992. MR Zbl
- [EGA II 1961] A. Grothendieck, “Éléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes”, *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 5–222. MR Zbl
- [Fujiwara and Kato 2018] K. Fujiwara and F. Kato, *Foundations of rigid geometry, I*, Eur. Math. Soc., Zürich, 2018. MR Zbl
- [Gromov 1987] M. Gromov, “Entropy, homology and semialgebraic geometry”, exposé 663, pp. 225–240 in *Séminaire Bourbaki, 1985/86*, Astérisque **145-146**, Soc. Math. France, Paris, 1987. MR Zbl
- [Kurano and Shimomoto 2021] K. Kurano and K. Shimomoto, “Ideal-adic completion of quasi-excellent rings (after Gabber)”, *Kyoto J. Math.* **61**:3 (2021), 707–722. MR Zbl
- [Lipshitz and Robinson 2005] L. Lipshitz and Z. Robinson, “Uniform properties of rigid subanalytic sets”, *Trans. Amer. Math. Soc.* **357**:11 (2005), 4349–4377. MR Zbl
- [Martin 2016] F. Martin, “Overconvergent subanalytic subsets in the framework of Berkovich spaces”, *J. Eur. Math. Soc.* **18**:10 (2016), 2405–2457. MR Zbl
- [Matsumura 1989] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1989. MR Zbl
- [Pettis 1974] B. J. Pettis, “Closed graph and open mapping theorems in certain topologically complete spaces”, *Bull. London Math. Soc.* **6** (1974), 37–41. MR Zbl
- [Pila and Wilkie 2006] J. Pila and A. J. Wilkie, “The rational points of a definable set”, *Duke Math. J.* **133**:3 (2006), 591–616. MR Zbl
- [Scanlon 2012] T. Scanlon, “A proof of the André–Oort conjecture via mathematical logic (after Pila, Wilkie and Zannier)”, exposé 1037, pp. 299–315 in *Séminaire Bourbaki: 2010/2011*, Astérisque **348**, Soc. Math. France, Paris, 2012. MR Zbl
- [Yomdin 1987] Y. Yomdin, “Volume growth and entropy”, *Israel J. Math.* **57**:3 (1987), 285–300. MR Zbl

Communicated by Anand Pillay

Received 2023-07-12 Revised 2024-05-17 Accepted 2024-09-03

gal.binyamini@weizmann.ac.il

Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel

bungen@math.titech.ac.jp

Department of Mathematics, Tokyo Institute of Technology, Tokyo, Japan

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR
Antoine Chambert-Loir
Université Paris-Diderot
France

EDITORIAL BOARD CHAIR
David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Jason P. Bell	University of Waterloo, Canada	Philippe Michel	École Polytechnique Fédérale de Lausanne
Bhargav Bhatt	University of Michigan, USA	Martin Olsson	University of California, Berkeley, USA
Frank Calegari	University of Chicago, USA	Irena Peeva	Cornell University, USA
J.-L. Colliot-Thélène	CNRS, Université Paris-Saclay, France	Jonathan Pila	University of Oxford, UK
Brian D. Conrad	Stanford University, USA	Anand Pillay	University of Notre Dame, USA
Samit Dasgupta	Duke University, USA	Bjorn Poonen	Massachusetts Institute of Technology, USA
Hélène Esnault	Freie Universität Berlin, Germany	Victor Reiner	University of Minnesota, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Peter Sarnak	Princeton University, USA
Sergey Fomin	University of Michigan, USA	Michael Singer	North Carolina State University, USA
Edward Frenkel	University of California, Berkeley, USA	Vasudevan Srinivas	SUNY Buffalo, USA
Wee Teck Gan	National University of Singapore	Shunsuke Takagi	University of Tokyo, Japan
Andrew Granville	Université de Montréal, Canada	Pham Huu Tiep	Rutgers University, USA
Ben J. Green	University of Oxford, UK	Ravi Vakil	Stanford University, USA
Christopher Hacon	University of Utah, USA	Akshay Venkatesh	Institute for Advanced Study, USA
Roger Heath-Brown	Oxford University, UK	Melanie Matchett Wood	Harvard University, USA
János Kollár	Princeton University, USA	Shou-Wu Zhang	Princeton University, USA
Michael J. Larsen	Indiana University Bloomington, USA		

PRODUCTION

production@msp.org
Silvio Levy, Scientific Editor


See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2025 is US \$565/year for the electronic version, and \$820/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2025 Mathematical Sciences Publishers

Algebra & Number Theory

Volume 19 No. 8 2025

The core of monomial ideals	1463
LOUIZA FOULI, JONATHAN MONTAÑO, CLAUDIA POLINI and BERND ULRICH	
Pullback formulas for arithmetic cycles on orthogonal Shimura varieties	1495
BENJAMIN HOWARD	
Weyl sums with multiplicative coefficients and joint equidistribution	1549
MATTEO BORDIGNON, CYNTHIA BORTOLOTTI and BRYCE KERR	
Rational points of rigid-analytic sets: a Pila–Wilkie-type theorem	1581
GAL BINYAMINI and FUMIHARU KATO	
Extending the unconditional support in an Iwaniec–Luo–Sarnak family	1621
LUCILE DEVIN, DANIEL FIORILLI and ANDERS SÖDERGREN	
On the maximum gonality of a curve over a finite field	1637
XANDER FABER, JON GRANTHAM and EVERETT W. HOWE	
Solvable and nonsolvable finite groups of the same order type	1663
PAWEŁ PIWEK	