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Extending the unconditional support in an Iwaniec–Luo–Sarnak family

Lucile Devin, Daniel Fiorilli and Anders Södergren

We study the harmonically weighted one-level density of low-lying zeros of L-functions in the family of holomorphic newforms of fixed even weight k and prime level N tending to infinity. For this family, Iwaniec, Luo and Sarnak proved that the Katz–Sarnak prediction for the one-level density holds unconditionally when the support of the Fourier transform of the implied test function is contained in $\left(-\frac{3}{2}, \frac{3}{2}\right)$. This result was improved by Ricotta–Royer, who increased the admissible support for $k \geq 4$ in a way that is asymptotically as good as the best known GRH result. We extend the admissible support for all $k \geq 2$ to $(-\Theta_k, \Theta_k)$, where $\Theta_2 = 1.866\ldots$ and Θ_k tends monotonically to 2 asymptotically five times faster than what was previously known. The main novelty in our analysis is the use of zero-density estimates for Dirichlet L-functions.

1. Introduction

Since the seminal paper of Iwaniec, Luo and Sarnak [Iwaniec et al. 2000] on low-lying zeros of families of L-functions attached to holomorphic modular forms, much work has been done towards the Katz–Sarnak heuristics [Katz and Sarnak 1999] (see for instance the discussion in [Sarnak et al. 2016]). In the family of cusp forms of fixed weight k and large level N, Iwaniec, Luo and Sarnak have proven that the Katz–Sarnak conjecture for the one-level density holds for test functions ϕ for which $\operatorname{supp}(\hat{\phi}) \subset \left(-\frac{3}{2}, \frac{3}{2}\right)$ unconditionally, and $\operatorname{supp}(\hat{\phi}) \subset (-2, 2)$ under the relevant Riemann hypothesis. Ricotta and Royer [2011] have on the one hand generalized this to higher symmetric power L-functions, and on the other hand improved the unconditional admissible support for $k \geq 4$. For the family we are interested in, they obtain the admissible support $\left(-2 + \frac{1}{k}, 2 - \frac{1}{k}\right)$, which is asymptotically as good as the conditional result mentioned above. These results have been refined in [Miller 2009; Miller and Montague 2011] to estimates containing lower-order terms. In the current paper, we extend the admissible support of the test function for the harmonically weighted one-level density: we show unconditionally that the Katz–Sarnak conjecture holds whenever $\operatorname{supp}(\hat{\phi}) \subset (-\Theta_k, \Theta_k)$, where $\Theta_2 = 1.866\ldots$, $\Theta_4 = 1.942\ldots$, and $2 - \Theta_k = \frac{2}{10k-5}$ for $k \geq 6$. This improves on the previous admissible values $\Theta_2 = 1.5$, $\Theta_4 = 1.75$, and $2 - \Theta_k = \frac{1}{k}$ for $k \geq 6$.

Before we state our main result, we need to introduce some notation. We fix a basis $B_k^*(N)$ of Hecke eigenforms of the space $H_k^*(N)$ of newforms of prime level N and weight k. We also renormalize so

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that for every $f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{(k-1)/2}e^{2\pi i n z}\in B_k^*(N)$, we have $\lambda_f(1)=1$. We use the harmonic weights defined as

$$\omega_f(N):=\frac{\Gamma(k-1)}{(4\pi)^{k-1}(f,f)_N},\qquad (f,f)_N:=\int_{\Gamma_0(N)\backslash\mathbb{H}} y^{k-2}|f(z)|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Note that by combining [Iwaniec et al. 2000, Lemma 2.5; Goldfeld et al. 1994; Iwaniec 1990], we have $(kN)^{-1-\varepsilon}\ll_\varepsilon \omega_f(N)\ll_\varepsilon (kN)^{-1+\varepsilon}$. Our main object of study is the one-level density, which is defined as

$$\mathcal{D}^*_{k,N}(\phi;X):=\frac{1}{\Omega_k(N)}\sum_{f\in B_k^*(N)}\omega_f(N)\sum_{\gamma_f}\phi\left(\gamma_f\frac{\log X}{2\pi}\right),$$

where $\rho_f=\frac{1}{2}+i\gamma_f$ runs through the nontrivial zeros of $L(s,f)$ (note that γ_f might be nonreal) and ϕ is an even Schwartz function whose Fourier transform is compactly supported. Here, it is important to note that we may holomorphically extend the definition of ϕ to the complex plane (see, e.g., [Rudin 1987, Section 19.1]) to account for possible zeros off the critical line, and we henceforth consider ϕ as an entire function. Moreover, the parameter $X=k^2N$ is the analytic conductor of f, and the total weight is given by

$$\Omega_k(N)=\sum_{f\in B_k^*(N)}\omega_f(N).$$

We have the estimate

$$\Omega_k(N)=1+O_k(N^{-1})$$

(see the second part of Lemma 2.4). We can now state our main result.

Theorem 1.1. *Let ϕ be an even Schwartz function for which* $\mathrm{supp}(\hat{\phi})\subset(-\Theta_k,\Theta_k)$, *where*

$$\Theta_k:=\begin{cases}1+\frac{\sqrt{3}}{2} & \text{if } k=2,\\ 2\left(1-\frac{1}{10k-5}\right) & \text{if } k\geq 4.\end{cases}\tag{1}$$

Then for N running through the set of prime numbers, we have the estimate

$$\mathcal{D}^*_{k,N}(\phi;X)=\int_{\mathbb{R}}W(O)(x)\phi(x)\,\mathrm{d}x+o_{N\to\infty}(1),\tag{2}$$

where $W(O)(x)=1+\frac{1}{2}\delta_0(x)$.

Remark 1.2. • In the literature on low-lying zeros, there are already several unconditional results with large admissible supports. We mention the result of Drappeau, Pratt and Radziwiłł [Drappeau et al. 2023] in the family of Dirichlet L-functions and that of Fouvry, Kowalski and Michel [Fouvry et al. 2014] in the family of symmetric square L-functions of holomorphic cusp forms of prime level.

• The use of zero-density estimates in order to circumvent the assumption of the Riemann hypothesis in the context of low-lying zeros is an idea which was first hinted by Brumer [1992], and subsequently successfully used by Kowalski and Michel [1999] in families of holomorphic cusp forms and Baier and Zhao [2008] in families of elliptic curve L-functions. The novelty in our approach is to use such estimates after an application of the Petersson formula, which relates low-lying zeros of cusp form L-functions to sums over zeros of Dirichlet L-functions, for which powerful zero-density estimates have been known for almost a century (see, e.g., [Linnik 1946; Montgomery 1971]).

- Our techniques also yield unconditional extensions of the admissible support in families of Maass forms L-functions in the level aspect. These results will appear in a forthcoming paper.

Here is a brief summary of the proof of Theorem 1.1. In Section 2, we apply the explicit formula and express the one-level density $\mathcal{D}^*_{k,N}(\phi; X)$ as a sum over eigenvalues of Hecke operators at prime power values. Averaging over the family of newforms of prime level, we apply the Petersson formula and turn this last expression into a sum of Kloosterman sums weighted by Bessel functions. In Section 3, we rewrite the Kloosterman sums in terms of Dirichlet characters and Gauss sums using orthogonality. This last expression allows us to apply Mellin inversion and use a variant of the explicit formula for Dirichlet L-functions. Finally, in Section 4 we complete the proof of Theorem 1.1 using zero-density estimates. It is the shape of these zero-density estimates which gives the exact restriction on the support which appears in Theorem 1.1. Note that if one assumes the grand density conjecture [Iwaniec and Kowalski 2004, p. 250], then one can prove that the Katz–Sarnak prediction holds in the full range $\operatorname{supp}(\hat{\phi}) \subset (-2, 2)$, independently of k (see Remark 4.2).

2. Prerequisites and first estimates

The goal of this section is to gather a few identities and estimates which will be central in our analysis, including the explicit and Petersson formulas. We then bound some of the terms in these formulas and reduce our problem to estimates on averages of Kloosterman sums.

We begin with the explicit formula.

Lemma 2.1 (Explicit formula). *Let ϕ be an even Schwartz function whose Fourier transform has compact support (which is canonically extended to an entire function). Then for $X > 1$ we have the formula*

$$\mathcal{D}^*_{k,N}(\phi; X) = \hat{\phi}(0)\frac{\log(N/\pi^2)}{\log X} + \frac{1}{\log X}\int_{\mathbb{R}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi i t}{\log X}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi i t}{\log X}\right)\right)\phi(t)\,dt - \frac{2}{\Omega_k(N)}\sum_{f\in B_k^*(N)}\omega_f(N)\sum_{p,\nu}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu/2}}\hat{\phi}\left(\frac{\nu\log p}{\log X}\right)\frac{\log p}{\log X}. \quad (3)$$

Here, $\alpha_f(p)$ and $\beta_f(p)$ are the local coefficients of the L-function

$$L(s, f) = \sum_{n\geq 1}\frac{\lambda_f(n)}{n^s} = \prod_p\left(1-\frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_f(p)}{p^s}\right)^{-1};$$

in particular, for $p \nmid N$ we have that $|\alpha_f(p)| = |\beta_f(p)| = 1$, and for $p \mid N$ we use the convention that $\beta_f(p) = 0$.

Proof. For $f \in B_k^*(N)$, the formula[1] [Iwaniec et al. 2000, (4.11)] reads

$$\sum_{\gamma_f} \phi\Big(\gamma_f \frac{\log X}{2\pi}\Big) = \hat{\phi}(0)\frac{\log(N/\pi^2)}{\log X} + \frac{1}{\log X}\int_{\mathbb{R}}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi i t}{\log X}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi i t}{\log X}\Big)\Big)\phi(t)\,dt - 2\sum_{p,\nu}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\nu/2}}\hat{\phi}\Big(\frac{\nu\log p}{\log X}\Big)\frac{\log p}{\log X}. \quad (4)$$

Next, we sum against the weight $\omega_f(N)$ to obtain the desired formula. □

We now introduce the Petersson formula, which involves the Kloosterman sum

$$S(m,n;c) := \sideset{}{^*}\sum_{x \bmod c} e\Big(\frac{mx+n\bar{x}}{c}\Big)$$

and the Bessel functions J_k.

Lemma 2.2. *Let $k \in \mathbb{N}$. We have the bound*

$$J_{k-1}(x) \ll \min\bigg(\frac{1}{(k-1)!}\Big(\frac{x}{2}\Big)^{k-1},\ x^{-\frac{1}{4}}\big(|x-k+1|+k^{\frac{1}{3}}\big)^{-\frac{1}{4}}\bigg).$$

Proof. This is [Iwaniec et al. 2000, (2.11)′, (2.11)″], which follows immediately from bounds in [Watson 1944; Krasikov 2006]; see [Devin et al. 2022, Lemma 3.2]. □

Lemma 2.3 (Petersson formula). *Assuming that N is prime, $(m,N)=1$, $(n,N^2)\mid N$, and k is an even integer, we have the formula*

$$\sum_{f\in B_k^*(N)} \omega_f(N)\lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^k \sum_{c\equiv 0 \bmod N} c^{-1}S(m,n;c)J_{k-1}(4\pi\sqrt{mn}/c) + O_\varepsilon\big((N\nu((n,N)))^{-1}(kmn)^\varepsilon\big), \quad (5)$$

where

$$\nu(t) := t\prod_{p\mid t}\Big(1+\frac{1}{p}\Big).$$

Proof. We combine [Iwaniec et al. 2000, Proposition 2.8] with [Iwaniec et al. 2000, Proposition 2.1], in the case where N is prime. In their notation, thanks to [Iwaniec et al. 2000, Lemma 2.5], we have

$$\sum_{f\in B_k^*(N)} \omega_f(N)\lambda_f(m)\lambda_f(n) = \frac{12}{(k-1)N}\Delta_{k,N}^*(m,n) = \sum_{LM=N}\frac{\mu(L)}{L\nu((n,L))}\sum_{\ell\mid L^\infty}\ell^{-1}\Delta_{k,M}(\ell^2 m,n)$$

$$= \delta(m,n) + 2\pi i^k \sum_{c\equiv 0 \bmod N} c^{-1}S(m,n;c)J_{k-1}(4\pi\sqrt{mn}/c) - \frac{1}{N\nu((n,N))}\sum_{\ell\mid N^\infty}\ell^{-1}\Delta_{k,1}(\ell^2 m,n). \quad (6)$$

[1] We fixed a minor typo in the arguments of the gamma factors.

The first equality above is a restatement of the definition of $\Delta^*_{k,N}(m,n)$ [Iwaniec et al. 2000, (2.53), (2.54)], and moreover we recall (see [Iwaniec et al. 2000, (2.7)]) that

$$\Delta_{k,N}(m,n) = \sum_{f\in B_k(N)} \omega_f(N)\lambda_f(m)\lambda_f(n)$$

denotes the sum over all forms in an orthogonal basis $B_k(N)$ of the space of cusp forms of level N and weight k (including the oldforms). To estimate the contribution of the last term on the right-hand side of (6), we use Deligne's bound $\lambda_f(n) \ll_\varepsilon n^\varepsilon$ (note that $B_k(1)$ can be chosen to include only newforms), which combined with the bounds $|B_k(1)| \ll k$ and $\omega_f(1) \ll_\varepsilon k^{-1+\varepsilon}$ yields the bound $\Delta_{k,1}(\ell^2 m,n) = O_\varepsilon((\ell mnk)^\varepsilon)$, resulting in the claimed estimate. □

We continue by estimating the sum over c in Lemma 2.3. This is similar to [Iwaniec et al. 2000, Corollary 2.10] along with the refinements in [Ricotta and Royer 2011, Section 4].

Lemma 2.4 (Estimated Petersson formula). *Let k be a fixed even integer. If N is prime, $N^2 \nmid n$ and $(m,N)=1$, we have*

$$\sum_{f\in B_k^*(N)} \omega_f(N)\lambda_f(m)\lambda_f(n) = \delta(m,n) + O_{k,\varepsilon}\big((n,N)^{-\frac{1}{2}}N^{-1+\varepsilon}(mn)^{\frac{1}{4}+\varepsilon}\big).$$

If in addition $mn \le N^2$, then we have the bound

$$\sum_{f\in B_k^*(N)} \omega_f(N)\lambda_f(m)\lambda_f(n) = \delta(m,n) + O_{k,\varepsilon}\big((N(n,N))^{-1}(mn)^\varepsilon + (n,N)^{-\frac{1}{2}}N^{-k+\frac{1}{2}+\varepsilon}(mn)^{\frac{k-1}{2}+\varepsilon}\big).$$

Proof. By Lemma 2.3, as N is prime, we have

$$\sum_{f\in B_k^*(N)} \omega_f(N)\lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^k \sum_{c\equiv 0 \bmod N} c^{-1}S(m,n;c)J_{k-1}(4\pi\sqrt{mn}/c) + O_{k,\varepsilon}\big((N(n,N))^{-1}(mn)^\varepsilon\big).$$

Using Weil's bound in the form [Iwaniec et al. 2000, (2.13)], together with Lemma 2.2, we have

$$\begin{aligned}\sum_{c\equiv 0 \bmod N} c^{-1}S(m,n;c)J_{k-1}\Big(4\pi\frac{\sqrt{mn}}{c}\Big) &\ll_{\varepsilon,k} \sum_{c\equiv 0 \bmod N} c^{-\frac{1}{2}+\varepsilon}\Big(\frac{\sqrt{mn}}{c}\Big)^{k-1}\frac{(m,n,c)}{(m,c)^{\frac{1}{2}}+(n,c)^{\frac{1}{2}}}\\ &\ll \frac{(m,n,N)}{(m,N)^{\frac{1}{2}}+(n,N)^{\frac{1}{2}}}N^{-k+\frac{1}{2}+\varepsilon}(mn)^{\frac{k-1}{2}}\sum_{b\ge 1}(m,n,b)^{\frac{1}{2}}b^{\frac{1}{2}-k+\varepsilon}\\ &\ll_\varepsilon \frac{(m,n,N)}{(m,N)^{\frac{1}{2}}+(n,N)^{\frac{1}{2}}}N^{-k+\frac{1}{2}+\varepsilon}(mn)^{\frac{k-1}{2}+\varepsilon},\end{aligned}$$

which concludes the proof in the case $mn \le N^2$. In case $N^2 < mn$, we cut the sum over c and apply Weil's bound:

$$\sum_{c\equiv 0 \bmod N} c^{-1} S(m,n;c) J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right)$$
$$\ll_{\varepsilon,k} \sum_{\substack{c\equiv 0 \bmod N\\ c>\sqrt{mn}}} c^{-\frac12+\varepsilon}\left(\frac{\sqrt{mn}}{c}\right)^{k-1} \frac{(m,n,c)}{(m,c)^{\frac12}+(n,c)^{\frac12}} + \sum_{\substack{c\equiv 0 \bmod N\\ c<\sqrt{mn}}} c^{-\frac12+\varepsilon}\left(\frac{\sqrt{mn}}{c}\right)^{-\frac12} \frac{(m,n,c)}{(m,c)^{\frac12}+(n,c)^{\frac12}}$$
$$\ll \frac{(m,n,N)}{(m,N)^{\frac12}+(n,N)^{\frac12}} N^{-1+\varepsilon}(mn)^{\frac14+\varepsilon},$$

which gives the desired bound. □

We now apply the Hecke relations and discard certain prime powers from the expression (3).

Lemma 2.5. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\operatorname{supp}(\hat\phi)) < \infty$. For N running through the set of prime numbers, we have that*

$$\begin{aligned}\mathcal{D}^*_{k,N}(\phi;X) &= \hat\phi(0)\frac{\log(N/\pi^2)}{\log X}\\ &\quad + \frac{1}{\log X}\int_{\mathbb{R}}\left(\frac{\Gamma'}{\Gamma}\left(\frac14+\frac{k+1}{4}+\frac{\pi i t}{\log X}\right)+\frac{\Gamma'}{\Gamma}\left(\frac14+\frac{k-1}{4}+\frac{\pi i t}{\log X}\right)\right)\phi(t)\,dt\\ &\quad + 2\sum_{p\nmid N}\frac1p\hat\phi\left(\frac{2\log p}{\log X}\right)\frac{\log p}{\log X}\\ &\quad - \frac{2}{\Omega_k(N)}\sum_{f\in B_k^*(N)}\omega_f(N)\sum_{\substack{p\nmid N\\ \nu\ge1}}\frac{\lambda_f(p^\nu)}{p^{\nu/2}}\hat\phi\left(\frac{\nu\log p}{\log X}\right)\frac{\log p}{\log X} + O_{k,\varepsilon}(N^{-1+\varepsilon}).\end{aligned} \tag{7}$$

Proof. Note that if $p=N$ then $\lambda_f(p^\nu)=\lambda_f(p)^\nu$, and moreover $\lambda_f(p) = O(p^{-1/2})$ [Miyake 1989, Theorem 4.6.17], so the contribution of this term to the prime sum in (3) is $O_k(N^{-1})$. Next, by the Hecke relations, the sum over prime numbers not dividing N in (3) is equal to

$$-\frac{2}{\Omega_k(N)}\sum_{f\in B_k^*(N)}\omega_f(N)\sum_{\substack{p,\nu\\ p\nmid N}}\frac{\lambda_f(p^\nu)}{p^{\nu/2}}\hat\phi\left(\frac{\nu\log p}{\log X}\right)\frac{\log p}{\log X} + \frac{2}{\Omega_k(N)}\sum_{f\in B_k^*(N)}\omega_f(N)\sum_{\substack{p,\nu\ge2\\ p\nmid N}}\frac{\lambda_f(p^{\nu-2})}{p^{\nu/2}}\hat\phi\left(\frac{\nu\log p}{\log X}\right)\frac{\log p}{\log X}.$$

The contribution of the summands with $\nu=2$ in the second term are given by

$$\frac{2}{\Omega_k(N)}\sum_{f\in B_k^*(N)}\omega_f(N)\sum_{p\nmid N}\frac{\lambda_f(1)}{p}\hat\phi\left(\frac{2\log p}{\log X}\right)\frac{\log p}{\log X} = 2\sum_{p\nmid N}\frac1p\hat\phi\left(\frac{2\log p}{\log X}\right)\frac{\log p}{\log X}.$$

The goal is now to estimate the contribution of the remaining terms, that is, the terms with $\nu \geq 3$ and $p \neq N$. From Lemma 2.4, we see that

$$\frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p \nmid N \\ \nu \geq 3}} \frac{\lambda_f(p^{\nu-2})}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X}$$
$$\ll_{k,\varepsilon} N^{-1+\varepsilon} \sum_{p \leq X^{\sigma/3}} \frac{\log p}{p^{\frac{1}{2}+2\varepsilon}} \sum_{3 \leq \nu \leq \sigma \log X/\log p} p^{(-\frac{1}{4}+\varepsilon)\nu}$$
$$\ll N^{-1+\varepsilon} \sum_{p \leq X^{\sigma/3}} \frac{\log p}{p^{\frac{5}{4}-\varepsilon}}$$
$$\ll N^{-1+\varepsilon},$$

which gives the claimed result. □

We are now ready to apply the Petersson formula.

Corollary 2.6. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\operatorname{supp}(\hat{\phi})) < 2$. Then, for N running through the set of prime numbers, we have that*

$$\mathcal{D}^*_{k,N}(\phi; X) = \hat{\phi}(0) + \frac{\phi(0)}{2} - \frac{4\pi i^k}{\Omega_k(N)} \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{1}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X}$$
$$\times \sum_{c \equiv 0 \bmod N} c^{-1} S(p^\nu, 1; c) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right) + O_k\left(\frac{1}{\log X}\right). \quad (8)$$

Proof. By Lemma 2.3, we have

$$-\frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{\lambda_f(p^\nu)}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X}$$
$$= -\frac{4\pi i^k}{\Omega_k(N)} \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{1}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} \sum_{c \equiv 0 \bmod N} c^{-1} S(p^\nu, 1; c) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right) + O_{k,\varepsilon}\left(N^{\frac{\sigma}{2}-1+\varepsilon}\right),$$

which, combined with Lemma 2.5 and [Iwaniec et al. 2000, (4.16), (4.20)] (see also [Devin et al. 2022, Lemmas 2.2 and 4.1]), gives the desired result. □

3. The prime sum in terms of zeros of Dirichlet L-functions

In the previous section we reduced the problem of estimating the one-level density to that of bounding sums over primes containing averages of Kloosterman sums. We now evaluate these averages using Dirichlet characters, which allow us to relate our problem to zeros of Dirichlet L-functions.

Lemma 3.1. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\operatorname{supp}(\hat{\phi})) < 2$. For N running through the set of prime numbers, we have that*

$$\mathcal{D}^*_{k,N}(\phi; X) = \hat{\phi}(0) + \frac{\phi(0)}{2} - \frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c\equiv 0 \bmod N \\ c<N^{1+1/(2k-3)}}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \bmod c \\ \chi\neq\chi_0}} \tau(\bar{\chi})^2 \\ \times \sum_{\substack{p\nmid c \\ \nu\geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\Big(\frac{\nu \log p}{\log X}\Big) \chi(p^\nu) J_{k-1}\Big(4\pi \frac{\sqrt{p^\nu}}{c}\Big) + O_k\Big(\frac{1}{\log X}\Big). \tag{9}$$

Proof. For $p \nmid c$, the Kloosterman sums satisfy

$$S(p^\nu, 1; c) = \frac{1}{\varphi(c)} \sum_{a \bmod c} S(a, 1; c) \sum_{\chi \bmod c} \bar{\chi}(a)\chi(p^\nu), \qquad \sum_{a \bmod c} \bar{\chi}(a) S(a, 1; c) = \tau(\bar{\chi})^2.$$

We substitute these expressions in Corollary 2.6. Let $C = N^{1+1/(2k-3)}$. Bounding the contribution of the primes dividing c using [Iwaniec et al. 2000, (2.13)] and Lemma 2.2, the sum on the right-hand side of (8) is equal to

$$-\frac{4\pi i^k}{\Omega_k(N)} \sum_{c\equiv 0 \bmod N} c^{-1} \sum_{\substack{p\nmid c \\ \nu\geq 1}} \frac{1}{p^{\nu/2}} \hat{\phi}\Big(\frac{\nu \log p}{\log X}\Big) \frac{\log p}{\log X} S(p^\nu, 1; c) J_{k-1}\Big(4\pi \frac{\sqrt{p^\nu}}{c}\Big) \\ + O_{k,\varepsilon}\big(N^{(\sigma-2)\frac{k-2}{2} - \frac{3}{2} + \varepsilon}\big)$$

$$= -\frac{4\pi i^k}{\Omega_k(N)} \sum_{\substack{c\equiv 0 \bmod N \\ c<C}} c^{-1} \sum_{\substack{p\nmid c \\ \nu\geq 1}} \frac{1}{p^{\nu/2}} \hat{\phi}\Big(\frac{\nu \log p}{\log X}\Big) \frac{\log p}{\log X} S(p^\nu, 1; c) J_{k-1}\Big(4\pi \frac{\sqrt{p^\nu}}{c}\Big) \\ + O_{k,\varepsilon}\big(N^{-\frac{3}{2}+\varepsilon} + N^{\frac{k}{2}(\sigma-2)+\varepsilon}\big)$$

$$= -\frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c\equiv 0 \bmod N \\ c<C}} \frac{1}{c\varphi(c)} \sum_{a \bmod c} \sum_{\chi \bmod c} \bar{\chi}(a) S(a, 1; c) \\ \times \sum_{\substack{p\nmid c \\ \nu\geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\Big(\frac{\nu \log p}{\log X}\Big) \chi(p^\nu) J_{k-1}\Big(4\pi \frac{\sqrt{p^\nu}}{c}\Big) + O_{k,\varepsilon}\big(N^{-\frac{3}{2}+\varepsilon} + N^{\frac{k}{2}(\sigma-2)+\varepsilon}\big)$$

$$= -\frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c\equiv 0 \bmod N \\ c<C}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \bmod c \\ \chi\neq\chi_0}} \tau(\bar{\chi})^2 \sum_{\substack{p\nmid c \\ \nu\geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\Big(\frac{\nu \log p}{\log X}\Big) \chi(p^\nu) J_{k-1}\Big(4\pi \frac{\sqrt{p^\nu}}{c}\Big) \\ + O_{k,\varepsilon}\big(N^{-\frac{3}{2}+\varepsilon} + N^{\frac{k}{2}(\sigma-2)+\varepsilon} + N^{k(\frac{\sigma}{2}-1)-1+\varepsilon}\big),$$

where the error terms account for the contributions of terms with $p \mid c$, terms with $c \geq C$, and the principal character. □

In the next lemma we express the third term on the right-hand side of (9) as a contour integral.

Lemma 3.2. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\mathrm{supp}(\hat{\phi})) < 2$. For N running through the set of prime numbers, we have that*

$$\mathcal{D}^*_{k,N}(\phi; X) = \hat{\phi}(0) + \frac{\phi(0)}{2} + O_k\left(\frac{1}{\log X}\right) + O_k\left(\sum_{\substack{c \equiv 0 \bmod N \\ c < N^{1+1/(2k-3)}}} \frac{1}{c\varphi(c)} \sum_{1 \neq d \mid c} d \sideset{}{^*}\sum_{\chi \bmod d} \left| \int_{(2)} \frac{L'\big(s+\frac{1}{2},\chi\big)}{L\big(s+\frac{1}{2},\chi\big)} \Psi_{\phi,X,c,k}(s)\,\mathrm{d}s \right| \right), \tag{10}$$

where the star over the sum means that we are summing over primitive characters, and where

$$\Psi_{\phi,X,c,k}(s) = \frac{1}{\log X} \int_0^\infty x^{s-1} \hat{\phi}\left(\frac{\log x}{\log X}\right) J_{k-1}\left(4\pi \frac{\sqrt{x}}{c}\right) \mathrm{d}x. \tag{11}$$

Proof. We once more use the shorthand $C = N^{1+1/(2k-3)}$. Note that the sum on the right-hand side of (9) is equal to

$$-\frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c \equiv 0 \bmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \bmod c \\ \chi \neq \chi_0}} \tau(\bar{\chi})^2 \sum_{\substack{p \nmid c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi^*(p^\nu) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right),$$

where χ^* is the primitive character modulo c^* inducing χ. Hence, this sum is

$$\begin{aligned} &\ll_k \frac{1}{\log X} \sum_{\substack{c \equiv 0 \bmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \bmod c \\ \chi \neq \chi_0}} c^* \left| \sum_{\substack{p \nmid c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi^*(p^\nu) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right) \right| \\ &= \frac{1}{\log X} \sum_{\substack{c \equiv 0 \bmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{1 \neq d \mid c} d \sideset{}{^*}\sum_{\chi \bmod d} \left| \sum_{\substack{p \nmid c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right) \right|. \end{aligned}$$

Here, the inner sum is equal to

$$\sum_{\substack{p \nmid d \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right) + O_\varepsilon\left(\frac{N^{\sigma(\frac{k}{2}-1)+\varepsilon}}{c^{k-1}}\right),$$

to which we may apply Mellin inversion. Doing so and interchanging the sum and integral thanks to absolute convergence (see Lemma 3.3 below), we obtain the identity

$$\frac{1}{\log X} \sum_{\substack{p \nmid c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \hat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}\left(4\pi \frac{\sqrt{p^\nu}}{c}\right) = -\frac{1}{2\pi i} \int_{(2)} \frac{L'\big(s+\frac{1}{2},\chi\big)}{L\big(s+\frac{1}{2},\chi\big)} \Psi_{\phi,X,c,k}(s)\,\mathrm{d}s + O_\varepsilon\left(\frac{N^{\sigma(\frac{k}{2}-1)+\varepsilon}}{c^{k-1}}\right).$$

The claimed estimate follows. □

We now establish analytic properties of the function $\Psi_{\phi,X,c,k}(s)$.

Lemma 3.3. *The function $\Psi_{\phi,X,c,k}(s)$ defined in (11) is entire and satisfies the bound*

$$\Psi_{\phi,X,c,k}(s) \ll_k \frac{X^{\sigma|\Re(s)+\frac{k-1}{2}|}}{(|s|+1)^2 c^{k-1}}.$$

Proof. The fact that $\hat{\phi}$ has compact support immediately implies that $\Psi_{\phi,X,c,k}(s)$ is entire. Assume now that $|s|>1$. We change variables $u=\log x/\log X$ in the definition of $\Psi_{\phi,X,c,k}(s)$ and then integrate by parts twice:

$$\begin{aligned}\Psi_{\phi,X,c,k}(s) &= \int_{\mathbb{R}} X^{us}\hat{\phi}(u) J_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)\,\mathrm{d}u\\ &= -\int_{\mathbb{R}} \frac{X^{us}}{s\log X}\Big(\hat{\phi}'(u)J_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)+\hat{\phi}(u)\frac{2\pi}{c}X^{\frac{u}{2}}\log X J'_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)\Big)\,\mathrm{d}u\\ &= \int_{\mathbb{R}} \frac{X^{us}}{(s\log X)^2}\Big(\hat{\phi}''(u)J_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)+\hat{\phi}'(u)\frac{4\pi}{c}X^{\frac{u}{2}}\log X J'_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)\\ &\qquad +\hat{\phi}(u)\frac{\pi}{c}X^{\frac{u}{2}}(\log X)^2 J'_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)+\hat{\phi}(u)\frac{4\pi^2}{c^2}X^{u}(\log X)^2 J''_{k-1}\Big(4\pi\frac{X^{\frac{u}{2}}}{c}\Big)\Big)\,\mathrm{d}u.\end{aligned}$$

By the identity

$$2J'_k(x) = J_{k-1}(x) - J_{k+1}(x)$$

(see [Watson 1944, equation (2), p. 17]) and Lemma 2.2, we deduce the bound[2]

$$\Psi_{\phi,X,c,k}(s) \ll_k \int_{-\sigma}^{\sigma} \frac{X^{u\Re(s)}}{|s|^2}\Big(\frac{X^{\frac{u}{2}}}{c}\Big)^{k-1}\,\mathrm{d}u \ll \frac{X^{\sigma|\Re(s)+(k-1)/2|}}{|s|^2 c^{k-1}}.$$

Finally, for $|s|\le 1$ the bound $\Psi_{\phi,X,c,k}(s)\ll X^{\sigma|\Re(s)+(k-1)/2|}/c^{k-1}$ follows directly from the definition. □

The next step is to move the contour of integration to the left in the integral appearing in (10).

Proposition 3.4. *Let ϕ be an even Schwartz test function for which $\sigma=\sup(\mathrm{supp}(\hat{\phi}))<2$. For $k\ge 2$ a fixed even integer, and for N running through the set of prime numbers, we have*

$$\begin{aligned}\mathcal{D}^*_{k,N}(\phi;X) &= \hat{\phi}(0)+\frac{\phi(0)}{2}+O_k\Big(\frac{1}{\log X}\Big)\\ &\quad + O_{k,\varepsilon}\Bigg(N^{\sigma(\frac{k}{2}-1)+\varepsilon}\sup_{N\le D<N^{1+1/(2k-3)}}\ \sup_{\frac{1}{2}\le\beta<1}\ \sup_{1\le T\le N^5}\frac{N^{\sigma\beta}}{D^kT^2}\sum_{\substack{d\sim D\\ d\equiv 0 \bmod N}}\ \sideset{}{^*}\sum_{\chi \bmod d}\ \sum_{\substack{\rho_\chi\\ \beta\le\Re(\rho_\chi)<\beta+1/\log N\\ T-1\le|\Im(\rho_\chi)|<2T}} 1\Bigg),\end{aligned} \tag{12}$$

where ρ_χ runs over the nontrivial zeros of the Dirichlet L-function $L(s,\chi)$ and $d\sim D$ means $\frac{D}{2}<d\le D$.

[2] Note that in the case $k=2$, we also use the identity $J_{-1}(x)=-J_1(x)$.

Proof. For an integer $d > 1$, we let $\chi \bmod d$ be a primitive character. We pull the contour of integration to the left in the integral

$$\frac{1}{2\pi i}\int_{(2)} \frac{L'(s+\frac{1}{2},\chi)}{L(s+\frac{1}{2},\chi)}\Psi_{\phi,X,c,k}(s)\,ds,$$

which appears in Lemma 3.2. Moreover, following [Davenport 2000, §19] and applying Lemma 3.3, this integral is easily shown to be equal to

$$\sum_{\rho_\chi}\Psi_{\phi,X,c,k}\big(\rho_\chi-\tfrac{1}{2}\big)+\Psi_{\phi,X,c,k}\big(-\tfrac{1}{2}\big)1_{\chi(-1)=1}+\frac{1}{2\pi i}\int_{(-1)}\frac{L'(s+\frac{1}{2},\chi)}{L(s+\frac{1}{2},\chi)}\Psi_{\phi,X,c,k}(s)\,ds.$$

Arguing as in [Davenport 2000, §19], we can show that on the line $\Re(s)=-1$ we have the bound

$$\frac{L'(s+\frac{1}{2},\chi)}{L(s+\frac{1}{2},\chi)} \ll \log(d|s|).$$

Combining this with Lemma 3.3, we deduce that

$$\frac{1}{2\pi i}\int_{(-1)}\frac{L'(s+\frac{1}{2},\chi)}{L(s+\frac{1}{2},\chi)}\Psi_{\phi,X,c,k}(s)\,ds \ll_k \int_{(-1)}\log(d|s|)\frac{X^{\sigma|\frac{k-3}{2}|}}{|s|^2c^{k-1}}\,ds \ll \frac{X^{\sigma|\frac{k-3}{2}|}}{c^{k-1}}\log d.$$

The total contribution of this term to the second error term in (10) is $\ll_{k,\varepsilon} N^{\sigma|(k-3)/2|-k+1+\varepsilon}$ (recall that $X=k^2N$), which is admissible for $\sigma<2$ and any even $k\geq 2$. Using Lemma 3.3, we see that the contribution of the terms $\Psi_{\phi,X,c,k}(-\frac{1}{2})$ is $O_{k,\varepsilon}(N^{\sigma(k/2-1)-k+1+\varepsilon})$, which is also admissible. We are left with the term involving nontrivial zeros of $L(s,\chi)$, whose contribution to the second error term in (10), setting $C=N^{1+1/(2k-3)}$, is

$$\begin{aligned}
&\ll_k \sum_{\substack{c\equiv 0 \bmod N\\ c<C}}\frac{1}{c\varphi(c)}\sum_{1\neq d|c} d \sideset{}{^*}\sum_{\chi \bmod d}\sum_{\rho_\chi}\big|\Psi_{\phi,X,c,k}\big(\rho_\chi-\tfrac{1}{2}\big)\big|\\
&\ll_k N^{\sigma(\frac{k}{2}-1)}\sum_{\substack{c\equiv 0 \bmod N\\ c<C}}\frac{1}{c^k\varphi(c)}\sum_{1\neq d|c} d \sideset{}{^*}\sum_{\chi \bmod d}\sum_{\rho_\chi}\frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2},
\end{aligned}$$

by Lemma 3.3. This expression is

$$\begin{aligned}
&\ll_\varepsilon N^{\sigma(\frac{k}{2}-1)-k-1+\varepsilon}\sum_{c'<C/N}\frac{1}{c'^{k+1}}\sum_{\substack{d|c'N\\ d\neq 1}} d \sideset{}{^*}\sum_{\chi \bmod d}\sum_{\rho_\chi}\frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2}\\
&\ll N^{\sigma(\frac{k}{2}-1)-k-1+\varepsilon}\sum_{1<d<C} d \sum_{\substack{c'<C/N\\ c'\equiv 0 \bmod d/(d,N)}}\frac{1}{c'^{k+1}}\sideset{}{^*}\sum_{\chi \bmod d}\sum_{\rho_\chi}\frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2}\\
&\ll N^{\sigma(\frac{k}{2}-1)-k-1+\varepsilon}\sum_{1<d<C} d\left(\frac{d}{(d,N)}\right)^{-k-1}\sideset{}{^*}\sum_{\chi \bmod d}\sum_{\rho_\chi}\frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2}\\
&\ll N^{\sigma(\frac{k}{2}-1)+\varepsilon}\sum_{\substack{d<C\\ d\equiv 0 \bmod N}} d^{-k}\sideset{}{^*}\sum_{\chi \bmod d}\sum_{\rho_\chi}\frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2}+O_\varepsilon\big(N^{\sigma\frac{k}{2}-k-1+\varepsilon}\big).
\end{aligned}$$

Truncating the sum over ρ_χ at $|\Im(\rho_\chi)| \le N^5$, we obtain the error term

$$O_\varepsilon\big(N^{k(\frac{\sigma}{2}-1)-4+\varepsilon}\big),$$

which is admissible. The final step is to decompose the sum over zeros into subsums in which $\Re(\rho_\chi)$ lies in a short vertical band of size $1/(\log N)$ (on which the function $N^{\sigma\Re(\rho_\chi)}$ is essentially constant), and in which $\Im(\rho_\chi)$ lies in a dyadic interval. We also cut the sum over d dyadically. This yields the subsums

$$\sum_{\substack{d\sim D\\ d\equiv 0 \bmod N}} d^{-k} \sideset{}{^*}\sum_{\chi \bmod d} \sum_{\substack{\rho_\chi\\ \beta\le\Re(\rho_\chi)<\beta+1/\log N\\ T-1\le|\Im(\rho_\chi)|<2T}} \frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2},$$

where the parameters D, β, T are such that $N \le D < C$, $0<\beta<1$, and $1 \le T \le N^5$. Note that by monotonicity and by symmetry of the zeros about the line $\Re(s) = \frac{1}{2}$, we may add the further restriction $\beta \ge \frac{1}{2}$. There are $\ll (\log N)^3$ such subsums, in each of which

$$d^{-k}\frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi|+1)^2} \ll \frac{N^{\sigma\beta}}{D^kT^2}.$$

The claimed result follows. □

4. Proof of Theorem 1.1

The object of this section is to state and apply a zero-density estimate on Dirichlet L-functions, which will be the last step in the proof of Theorem 1.1. For a Dirichlet character χ, we define

$$N(\beta,T,\chi) := \#\{\rho_\chi : \Re(\rho_\chi)\ge\beta, |\Im(\rho_\chi)|<T, L(\rho_\chi,\chi)=0\}.$$

Theorem 4.1. *Fix $\varepsilon>0$. In the range $\frac{1}{2}+\varepsilon\le\beta\le 1$ and for $h\in\mathbb{N}$, $Q\ge 1$, we have the bound*

$$\sum_{\substack{q\le Q\\ (q,h)=1}} \sideset{}{^*}\sum_{\psi \bmod q} \sum_{\xi \bmod h} N(\beta,T,\psi\xi) \ll_\varepsilon \big((hQT)^{(2+\varepsilon)(1-\beta)} + (hQ^2T)^{(1-\beta)\min(\frac{3}{2-\beta},\frac{3}{3\beta-1})}\big)(\log hQT)^{O_\varepsilon(1)}.$$

Proof. This statement is essentially contained in the proof of [Iwaniec and Kowalski 2004, Theorem 10.4]. The difference between our statement and theirs is that we require β to be at a positive distance from $\frac{1}{2}$. Looking at the second half of [Iwaniec and Kowalski 2004, p. 262], we note that the power of $\log hQT$ comes from a bound on the average of the divisor function and therefore depends only on the exponent to which the appropriate Dirichlet polynomial is raised. Moreover, near the end of the same page, it is noted that this exponent is bounded in terms of ε.

Therefore, going through the second paragraph of [Iwaniec and Kowalski 2004, p. 263], one can see that in the range $\frac{1}{2}+\varepsilon\le\beta\le\frac{3}{4}$, the implied constants in $\ll$ and in the power of $\log hQT$ in [Iwaniec and Kowalski 2004, equation (10.90)] depend only on ε. We conclude by noting that in the range $\frac{3}{4}\le\beta\le 1$, the implied constants in $\ll$ and in the power of $\log hQT$ are absolute: they depend on ε, which can, in this situation, be taken to be equal to $\frac{1}{4}$. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. From Proposition 3.4, we see that it is sufficient to show, for $C = N^{1+1/(2k-3)}$, that

$$N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N\le D<C} \sup_{\frac12\le\beta<1} \sup_{1\le T\le N^5} \frac{N^{\sigma\beta}}{D^kT^2} \sum_{\substack{d\sim D\\ d\equiv 0 \bmod N}} \sideset{}{^*}\sum_{\chi \bmod d} \sum_{\substack{\rho_\chi\\ \beta\le\Re(\rho_\chi)<\beta+1/\log N\\ T-1\le|\Im(\rho_\chi)|<2T}} 1 = o_{N\to\infty}(1). \tag{13}$$

Note that since $D < N^2$, N^2 does not divide d and thus we may write $d = fN$ with $(f, N) = 1$. Now, by the Chinese remainder theorem, every primitive character χ modulo d, where d is a multiple of N, can be decomposed as $\chi = \psi\xi$, where ψ is primitive modulo f and ξ is primitive modulo N. As a result, the left-hand side of (13) is

$$\ll_\varepsilon N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N\le D<C} \sup_{\frac12\le\beta<1} \sup_{1\le T\le N^5} \frac{N^{\sigma\beta}}{D^kT^2} \sum_{\substack{f\sim D/N\\ (f,N)=1}} \sideset{}{^*}\sum_{\psi \bmod f} \sideset{}{^*}\sum_{\xi \bmod N} N(\beta, 2T, \psi\xi).$$

We first apply the Riemann–von Mangoldt theorem in the range $\frac12 \le \beta < \frac12 + \varepsilon$ and deduce that

$$N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N\le D<C} \sup_{\frac12\le\beta<\frac12+\varepsilon} \sup_{1\le T\le N^5} \frac{N^{\sigma\beta}}{D^kT^2} \sum_{\substack{f\sim D/N\\ (f,N)=1}} \sideset{}{^*}\sum_{\psi \bmod f} \sideset{}{^*}\sum_{\xi \bmod N} N(\beta, 2T, \psi\xi) \ll N^{\sigma(\frac{k}{2}-\frac12+\varepsilon)+1-k+2\varepsilon},$$

which is admissible for $\sigma < 2 - 9\varepsilon$. Moreover, Theorem 4.1 implies that

$$\begin{aligned}
&N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N\le D<C} \sup_{\frac12+\varepsilon\le\beta<1} \sup_{1\le T\le N^5} \frac{N^{\sigma\beta}}{D^kT^2} \sum_{\substack{f\sim D/N\\ (f,N)=1}} \sideset{}{^*}\sum_{\psi \bmod f} \sideset{}{^*}\sum_{\xi \bmod N} N(\beta, 2T, \psi\xi)\\
&\quad\ll_\varepsilon N^{\sigma(\frac{k}{2}-1)+3\varepsilon} \sup_{N\le D<C} \sup_{\frac12+\varepsilon\le\beta<1} \sup_{1\le T\le N^5} \biggl(\frac{N^{\sigma\beta}}{D^kT^2}\Bigl((DT)^{2(1-\beta)}\\
&\qquad\qquad + (N^{-1}D^2T)^{(1-\beta)\min(\frac{3}{2-\beta},\frac{3}{3\beta-1})}\Bigr)\biggr).
\end{aligned}\tag{14}$$

As $k \ge 2$, the supremum over D is attained at $D = N$ and the supremum over T is attained at $T = 1$. Thus, the error term above is

$$\begin{aligned}
&\ll N^{\sigma(\frac{k}{2}-1)+3\varepsilon} \sup_{\frac12+\varepsilon\le\beta<1} N^{\sigma\beta-k}\bigl(N^{2(1-\beta)} + N^{(1-\beta)\min(\frac{3}{2-\beta},\frac{3}{3\beta-1})}\bigr)\\
&\le \sup_{\frac12\le\beta<1} \bigl(N^{\sigma(\beta+\frac{k}{2}-1)-k+2(1-\beta)+3\varepsilon} + N^{\sigma(\beta+\frac{k}{2}-1)-k+(1-\beta)\min(\frac{3}{2-\beta},\frac{3}{3\beta-1})+3\varepsilon}\bigr).
\end{aligned}$$

The first error term is admissible whenever

$$\sigma < \inf_{\frac12\le\beta<1} \frac{2\beta+k-2}{\beta+\frac{k}{2}-1} = 2.$$

As for the second, it is admissible whenever

$$\sigma < \inf_{\frac{1}{2}\le\beta<1} \frac{k-(1-\beta)\min\left(\frac{3}{2-\beta},\frac{3}{3\beta-1}\right)}{\beta+\frac{k}{2}-1} = \Theta_k,$$

where we recall that Θ_k is defined in (1). The last equality follows from optimizing the two cases in the minimum separately. Note that for $k=2$, the infimum is attained at $\beta=\sqrt{3}-1$, and for $k\ge 4$ it is attained at $\beta=\frac{3}{4}$. □

Remark 4.2. As mentioned in the introduction, the grand density conjecture

$$\sum_{\substack{q\le Q\\(q,k)=1}}\ \sideset{}{^*}\sum_{\psi \bmod q}\ \sum_{\xi \bmod k} N(\beta,T,\psi\xi) \ll (kQ^2T)^{2(1-\beta)}(\log kQT)^{O(1)}$$

(see [Iwaniec and Kowalski 2004, page 250]) implies the Katz–Sarnak conjecture in the full range $\operatorname{supp}(\hat{\phi})\subset(-2,2)$. Indeed, under this conjecture we can replace (14) by the expression

$$N^{\sigma(\frac{k}{2}-1)+2\varepsilon} \sup_{N\le D<N^{1+1/(2k-3)}}\ \sup_{\frac{1}{2}\le\beta<1}\ \sup_{1\le T\le N^5} \frac{N^{\sigma\beta}}{D^kT^2}(N^{-1}D^2T)^{2(1-\beta)} \ll \sup_{\frac{1}{2}\le\beta<1} N^{(\sigma-2)(\beta+\frac{k}{2}-1)+2\varepsilon},$$

which is clearly admissible when $\sigma<2$.

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