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Solvable and nonsolvable finite groups of the same order type

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We construct two groups of size $2^{365} \cdot 3^{105} \cdot 7^{104}$: a solvable group G and a nonsolvable group H such that for every integer n the groups have the same number of elements of order n. This answers a question posed in 1987 by John G. Thompson.

1. Introduction

The following metaquestion has been of considerable interest in the field of finite group theory.

Metaquestion. *Which properties of a group G can we discern from knowing only the orders of its elements*?

While some addressed this question by looking at just the *set* of orders of elements of G, see [Vasilev et al. 2009], we are interested in also using the *multiplicity* of these orders, which prompts the following definition.

Definition 1. The *order type* of a group G is the function

$$o_G : n \mapsto |\{g \in G \mid g \text{ has order } n\}|.$$

It is easy to give examples of an abelian group G and a nonabelian group H such that $o_G = o_H$. Indeed, take $G = C_3 \times C_3 \times C_3$ and H to be the Heisenberg group $(C_3 \times C_3) \rtimes C_3$, which is nonabelian of exponent 3.

On the other hand, the order type allows us to count the number of elements of order p^α for p a prime, so it can also tell us if a p-Sylow subgroup is normal. This implies that the order type can distinguish nilpotent groups from nonnilpotent ones. A similar argument can be given for supersolvable groups.

A problem posed in 1987 by John G. Thompson in a private communication with Wujie Shi (see [Shen et al. 2023; Shi 2025; Guralnick and Weiss 1993, Question 2.13; Khukhro and Mazurov 2014, Problem 12.37]) is the following.

Question. *Let G be a finite solvable group and H be any finite group such that $o_G = o_H$. Is H necessarily solvable*?

It was shown that the answer is positive if we assume that the prime graph of G is disconnected; see [Shen et al. 2023; Shen 2012] in the case when there are at most three different orders of elements. Some

MSC2020: 20D10.
Keywords: Thompson's problem, order type.

i	G_i	Id	m_i	H_i	Id	n_i
1	C_4	(4, 1)	9	C_2	(2, 1)	21
2	D_3	(6, 1)	6	C_3	(3, 1)	3
3	C_7	(7, 1)	1	Dic_3	(12, 1)	6
4	D_4	(8, 3)	9	A_4	(12, 3)	21
5	D_7	(14, 1)	18	SD_{16}	(16, 8)	3
6	$\mathrm{SL}(2, 3)$	(24, 3)	21	$C_7 \rtimes C_3$	(21, 1)	4
7	$C_{24} \rtimes C_2$	(48, 6)	3	D_{12}	(24, 6)	6
8	$C_7 \rtimes D_4$	(56, 7)	3	$C_3 \rtimes D_4$	(24, 8)	6
9	$C_7 \rtimes C_{12}$	(84, 1)	6	Dic_7	(28, 1)	15
10	Dic_{21}	(84, 5)	6	F_7	(42, 1)	18
11	$C_7 \rtimes A_4$	(84, 11)	21	D_{21}	(42, 5)	6
12	$C_7 \rtimes D_7$	(98, 4)	2	D_{24}	(48, 7)	3
13	$C_4 \rtimes F_7$	(168, 9)	21	D_{28}	(56, 5)	27
14	$C_{21} \rtimes D_4$	(168, 15)	9	$\mathrm{Dic}_7 \rtimes C_6$	(168, 11)	3
15	$C_7 \rtimes D_{12}$	(168, 17)	6	$C_{14}.A_4$	(168, 23)	21
16	$F_8 \rtimes C_3$	(168, 43)	3	$\mathrm{GL}(3, 2)$	(168, 42)	3
17	$D_8 \rtimes D_7$	(224, 106)	3	$C_7 \rtimes F_7$	(294, 10)	2
18	$C_7 \rtimes D_{24}$	(336, 31)	3	$D_{12}.D_7$	(336, 36)	3

Table 1. The groups G_i and H_i involved in Theorem A and their multiplicities m_i and n_i. The column labelled "Id" contains the Small Groups isomorphism type identifier (see [Besche et al. 2002]), while columns G_i and H_i give some idea of the group structure, but they do not necessarily identify the groups uniquely in the case of extensions.

works also showed results about solvability of groups (and supersolvability, nilpotence and cyclicity too) based on the average element order; see [Herzog et al. 2022; Lazorec and Tărnăuceanu 2023].

In this paper, we give a negative answer to Thompson's question. The example presented is not of minimal size; since the publication of the preprint of the present article a smaller example constructed in a similar manner was found by Peter Müller [2024].

Theorem A. *Let G_i and H_i be the collections of finite groups described in Table 1, and let m_i and n_i be the associated natural numbers from the tables. Let G and H be the direct products*

$$G = \prod G_i^{m_i}, \quad H = \prod H_i^{n_i}.$$

Then $o_G = o_H$, G is solvable, and H is not solvable.

The result has a corollary regarding permutation modules. The reader is referred to Question 2.12 in [Guralnick and Weiss 1993] for an explanation of an equivalent version of Thompson's problem stated in these terms.

Corollary B. *There exist subgroups G, H of S_n for $n = 2^{365} \cdot 3^{105} \cdot 7^{104}$ such that $\mathbb{Q}(S_n/G) \cong \mathbb{Q}(S_n/H)$, G is solvable and H is nonsolvable.*

2. Proof of Theorem A

Before presenting the proof, let us first rephrase the question slightly.

Definition 2. The *exponent type* of a group G is the function

$$e_G : n \mapsto |\{g \in G \mid g^n = 1\}|.$$

Lemma 3. *For groups G and H we have $o_G = o_H \iff e_G = e_H$.*

Proof. We have $g^n = 1$ if and only if $\mathrm{ord}(g)$ divides n. Thus $e_G(n) = \sum_{d \mid n} o_G(d)$, and $o_G(n) = \sum_{d \mid n} e_G(n/d) \cdot \mu(d)$, where μ is the Möbius function. □

Lemma 4. *Let G and H be any groups. Then $e_{G \times H} = e_G \cdot e_H$.*

Proof. For $(g, h) \in G \times H$ we have $(g, h)^n = (g^n, h^n)$, so $(g, h)^n$ is trivial if and only if g^n and h^n are trivial. □

Proof of Theorem A. With the help of Magma [Bosma et al. 1997] we check that—with the notable exception of H_{16}— all of the groups G_i and H_i are solvable. This means that G is solvable as a product of solvable groups, and H is nonsolvable as it contains a nonsolvable subgroup, namely $H_{16} \cong \mathrm{GL}(3, 2)$.

To prove $e_G = e_H$ we need to compute the exponent types e_{G_i} and e_{H_i}, and check that $\prod e_{G_i}^{m_i} = \prod e_{H_i}^{n_i}$.

The exponent types of groups G_i and H_i were computed with the help of Magma too (see the ancillary file `exponents_computation.m` available at arXiv [Piwek 2024]) and are listed in Tables 2 and 4. Only the columns labelled with divisors of 168 were included as $e_G(n) = e_G(\gcd(n, E_G))$ for E_G being the exponent of the group G. Since the products $\prod e_{G_i}(n)^{m_i}$ and $\prod e_{H_i}(n)^{n_i}$ become unmanageably large, we factorise them into their prime factors to compute the exponent types of G and H listed in Table 3. This computation is done in the ancillary Jupyter Notebook file `exponents_verification.ipynb`. □

3. How these groups were found

Finding groups G and H presented in Theorem A manually would be very difficult. Instead, we employed a computer-based search method outlined below.

Step 1. We used Magma to access the Small Groups database and to loop through all groups of size at most 2000, excluding groups of size divisible by 128, and to compute for each group

(1) whether it is solvable — using `IsSolvable` function,

(2) whether it is a nontrivial direct product — using a custom function `IsDirectProduct` (see the ancillary file `functions.m`), which searches for two normal subgroups with trivial intersection and whose sizes multiply to the size of the investigated group,

(3) its order type — using a simple custom function `OrderType`.

Step 2. We parsed the outputs into a `.csv` table and handled the remaining tasks using various Python libraries. Given our objective of constructing an example through the direct product of groups, we focussed solely on those groups that were not direct products themselves.

i	G_i	Id	m_i	E_{G_i}	1	2	3	4	6	7	8	12	14	21	24	28	42	56	84	168
1	C_4	(4, 1)	9	4	1	2	1	4	2	1	4	4	2	1	4	4	2	4	4	4
2	D_3	(6, 1)	6	6	1	4	3	4	6	1	4	6	4	3	6	4	6	4	6	6
3	C_7	(7, 1)	1	7	1	1	1	1	1	7	1	1	7	7	1	7	7	7	7	7
4	D_4	(8, 3)	9	4	1	6	1	8	6	1	8	8	6	1	8	8	6	8	8	8
5	D_7	(14, 1)	18	14	1	8	1	8	8	7	8	8	14	7	8	14	14	14	14	14
6	SL(2, 3)	(24, 3)	21	12	1	2	9	8	18	1	8	24	2	9	24	8	18	8	24	24
7	$C_{24} \rtimes C_2$	(48, 6)	3	24	1	14	3	28	18	1	32	36	14	3	48	28	18	32	36	48
8	$C_7 \rtimes D_4$	(56, 7)	3	28	1	18	1	32	18	7	32	32	42	7	32	56	42	56	56	56
9	$C_7 \rtimes C_{12}$	(84, 1)	6	84	1	2	15	16	30	7	16	72	14	21	72	28	42	28	84	84
10	Dic_{21}	(84, 5)	6	84	1	2	3	44	6	7	44	48	14	21	48	56	42	56	84	84
11	$C_7 \rtimes A_4$	(84, 11)	21	42	1	4	57	4	60	7	4	60	28	63	60	28	84	28	84	84
12	$C_7 \rtimes D_7$	(98, 4)	2	14	1	50	1	50	50	49	50	50	98	49	50	98	98	98	98	98
13	$C_4 \rtimes F_7$	(168, 9)	21	84	1	30	15	32	114	7	32	144	42	21	144	56	126	56	168	168
14	$C_{21} \rtimes D_4$	(168, 15)	9	84	1	22	3	64	54	7	64	96	70	21	96	112	126	112	168	168
15	$C_7 \rtimes D_{12}$	(168, 17)	6	84	1	50	3	64	54	7	64	96	98	21	96	112	126	112	168	168
16	$F_8 \rtimes C_3$	(168, 43)	3	42	1	8	57	8	120	49	8	120	56	105	120	56	168	56	168	168
17	$D_8 \rtimes D_7$	(224,106)	3	56	1	52	1	96	52	7	128	96	112	7	128	168	112	224	168	224
18	$C_7 \rtimes D_{24}$	(336, 31)	3	168	1	98	3	100	102	7	128	108	182	21	192	196	210	224	252	336

Table 2. The groups G_i, their multiplicities m_i, their exponents E_{G_i}, and their exponent types.

1	1	14	$2^{191}\cdot 3^{33}\cdot 5^{9}\cdot 7^{113}\cdot 13^{3}$
2	$2^{221}\cdot 3^{36}\cdot 5^{37}\cdot 7^{9}\cdot 11^{9}\cdot 13^{3}$	21	$3^{147}\cdot 5^{3}\cdot 7^{104}$
3	$3^{126}\cdot 5^{27}\cdot 19^{24}$	24	$2^{488}\cdot 3^{132}\cdot 5^{28}$
4	$2^{500}\cdot 3^{3}\cdot 5^{10}\cdot 7^{3}\cdot 11^{6}$	28	$2^{374}\cdot 3^{3}\cdot 7^{110}$
6	$2^{215}\cdot 3^{174}\cdot 5^{34}\cdot 13^{3}\cdot 17^{3}\cdot 19^{21}$	42	$2^{185}\cdot 3^{177}\cdot 5^{3}\cdot 7^{104}$
7	7^{107}	56	$2^{398}\cdot 7^{104}$
8	$2^{530}\cdot 5^{4}\cdot 11^{6}$	84	$2^{347}\cdot 3^{114}\cdot 7^{104}$
12	$2^{464}\cdot 3^{144}\cdot 5^{28}$	168	$2^{365}\cdot 3^{105}\cdot 7^{104}$

Table 3. The exponent type of both G and H.

Step 3. We rephrased the question as follows.

(1) First we converted the order types o_G to exponent types e_G as described in Section 2.

(2) Then we used multiplicative Möbius inversion on e_G defining the *revolved exponent type* $r_G : \mathbb{N} \to \mathbb{Q}$ as

$$r_G(n) := \prod_{d\,|\,n} e_G(n/d)^{\mu(d)}.$$

The advantage of doing so was that $r_G(n) = 1$ if $n \nmid E_G$. Indeed, for $k \mid E_G$ and $\gcd(m, E_G) = 1$ we get

$$r_G(mk) = \prod_{d\,|\,mk} e_G(mk/d)^{\mu(d)} = \prod_{e\,|\,m}\prod_{f\,|\,k} e_G(m/e\cdot k/f)^{\mu(ef)} = \prod_{e\,|\,m}\prod_{f\,|\,k} e_G(k/f)^{\mu(e)\mu(f)}$$
$$= \prod_{e\,|\,m}\Bigl(\prod_{f\,|\,k} e_G(k/f)^{\mu(f)}\Bigr)^{\mu(e)} = r_G(k)^{\sum_{e\,|\,m}\mu(e)} = \begin{cases} r_G(k) & \text{if } m = 1,\\ 1 & \text{otherwise.}\end{cases}$$

Additionally, we maintained the multiplicativity with respect to direct products: $r_{G_1\times G_2} = r_{G_1}\cdot r_{G_2}$. In order to operate with rational numbers we used a Python datatype `Fraction`.

(3) Finally, we factorised the revolved exponent types into their prime factors by defining $v_G(n, p)$ to be the exponent of a prime p in the factorisation of $r_G(n)$ into prime factors. Thus, for $G = \prod G_i^{m_i}$ we have

$$v_G(n, p) = \sum m_i \cdot v_{G_i}(n, p).$$

Additionally, for a fixed G, the values of $v_G(n, p)$ are zero for all but finitely many combinations (n, p).

At this point the question became: is there a positive rational linear combination $\sum k_i \cdot v_{N_i}$ for nonsolvable groups N_i in the rational span of v_{S_i} for S_i solvable groups?

Step 4. An even simpler, but inequivalent, question arises: is any v_N for a nonsolvable group N in the span of v_{S_i} for solvable groups S_i? This question is equivalent to determining whether the equation $Vx = v_N$ has a solution, where V is a matrix of $v_{S_i}(n, p)$ with rows indexed by pairs (n, p) and columns indexed by solvable groups S_i. A significant challenge is that the matrix V in our case is of size 9945×100972, albeit very sparse.

Step 5. We first attempted solving this numerically using least squares methods of the SciPy library [Virtanen et al. 2020], which gave two conclusions.

i	H_i	Id	n_i	E_{H_i}	1	2	3	4	6	7	8	12	14	21	24	28	42	56	84	168
1	C_2	(2, 1)	21	2	1	2	1	2	2	1	2	2	2	1	2	2	2	2	2	2
2	C_3	(3, 1)	3	3	1	1	3	1	3	1	1	3	1	3	3	1	3	1	3	3
3	Dic_3	(12, 1)	6	12	1	2	3	8	6	1	8	12	2	3	12	8	6	8	12	12
4	A_4	(12, 3)	21	6	1	4	9	4	12	1	4	12	4	9	12	4	12	4	12	12
5	SD_{16}	(16, 8)	3	8	1	6	1	12	6	1	16	12	6	1	16	12	6	16	12	16
6	$C_7 \rtimes C_3$	(21, 1)	4	21	1	1	15	1	15	7	1	15	7	21	15	7	21	7	21	21
7	D_{12}	(24, 6)	6	12	1	14	3	16	18	1	16	24	14	3	24	16	18	16	24	24
8	$C_3 \rtimes D_4$	(24, 8)	6	12	1	10	3	16	18	1	16	24	10	3	24	16	18	16	24	24
9	Dic_7	(28, 1)	15	28	1	2	1	16	2	7	16	16	14	7	16	28	14	28	28	28
10	F_7	(42, 1)	18	42	1	8	15	8	36	7	8	36	14	21	36	14	42	14	42	42
11	D_{21}	(42, 5)	6	42	1	22	3	22	24	7	22	24	28	21	24	28	42	28	42	42
12	D_{24}	(48, 7)	3	24	1	26	3	28	30	1	32	36	26	3	48	28	30	32	36	48
13	D_{28}	(56, 5)	27	28	1	30	1	32	30	7	32	32	42	7	32	56	42	56	56	56
14	$\mathrm{Dic}_7 \rtimes C_6$	(168, 11)	3	84	1	18	15	32	102	7	32	144	42	21	144	56	126	56	168	168
15	$C_{14}.A_4$	(168, 23)	21	84	1	2	57	8	114	7	8	120	14	63	120	56	126	56	168	168
16	$\mathrm{GL}(3, 2)$	(168, 42)	3	84	1	22	57	64	78	49	64	120	70	105	120	112	126	112	168	168
17	$C_7 \rtimes F_7$	(294, 10)	2	42	1	50	15	50	162	49	50	162	98	147	162	98	294	98	294	294
18	$D_{12}.D_7$	(336, 36)	3	168	1	14	3	100	18	7	128	108	98	21	192	196	126	224	252	336

Table 4. The groups H_i, their multiplicities n_i, their exponents E_{H_i}, and their exponent types.

(1) For many of the nonsolvable groups considered (e.g., $N = A_5$) the (numerical) projection of their v_N onto the column space of V was at distance about 1 from v_N, meaning that v_N was almost certainly not in the span.

(2) For some of the groups (e.g., the GL(3, 2)) the algorithm converged to a solution up to an error of 10^{-6}. This led us to search for an exact solution to $Vx = v_N$ for these groups.

Step 6. For $N = \mathrm{GL}(3, 2)$ we solved $Vx = v_N$ exactly using the symbolic computation library Sympy [Meurer et al. 2017]. A family of solutions was found; we chose one of them.

Step 7. We converted the solution back into the language of groups and verified it, cross-referencing the computed exponent types with the subgroup lattices available in the GroupNames database [Dokchitser 2020] and checking the computations on prime factorisations with the help of Python.

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