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Prismatic G-displays and descent theory

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For a smooth affine group scheme G over the ring of p-adic integers $\mathbb{Z}_p$ and a cocharacter μ of G, we study G-μ-displays over the prismatic site of Bhatt and Scholze. In particular, we obtain several descent results for them. If $G = \mathrm{GL}_n$, then our G-μ-displays can be thought of as Breuil–Kisin modules with some additional conditions. The relation between our G-μ-displays and prismatic F-gauges introduced by Drinfeld and Bhatt–Lurie is also discussed.

In fact, our main results are formulated and proved for smooth affine group schemes over the ring of integers $\mathcal{O}_E$ of any finite extension E of $\mathbb{Q}_p$ by using $\mathcal{O}_E$-prisms, which are $\mathcal{O}_E$-analogues of prisms.

1. Introduction

Bhatt and Scholze [2022] introduced the theory of prisms. The category of (bounded) prisms with the flat topology is called the absolute prismatic site. It has been observed that *prismatic F-crystals* on the absolute prismatic site introduced in [Bhatt and Scholze 2023] play significant roles in various aspects of arithmetic geometry. For a smooth affine group scheme G over the ring of p-adic integers $\mathbb{Z}_p$, we provide a systematic study of prismatic F-crystals with certain G-actions, which we call *prismatic G-μ-displays*. The results obtained here will be used to study the deformation theory of prismatic G-μ-displays in [Ito 2025]. We also discuss the relation between prismatic G-μ-displays and *prismatic F-gauges* introduced in [Drinfeld 2024; Bhatt and Lurie 2022a; 2022b; Bhatt 2022].

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1.1. ***Prismatic Dieudonné crystals.*** Anschütz and Le Bras [2023] introduced *prismatic Dieudonné crystals*, which are prismatic F-crystals with additional conditions, and showed that prismatic Dieudonné crystals can be used to classify p-divisible groups in mixed characteristic. The notion of prismatic G-μ-displays can be seen as a generalization of that of prismatic Dieudonné crystals. Before discussing prismatic G-μ-displays, let us state our main result for prismatic Dieudonné crystals.

Let k be a perfect field of characteristic $p > 0$ and let $W(k)$ be the ring of p-typical Witt vectors of k. Let R be a complete regular local ring over $W(k)$ with residue field k. There exists a pair

$$(A, I) = (W(k)[[t_1, \dots, t_n]], (\mathcal{E})),$$

with an isomorphism $R \simeq A/I$ over $W(k)$, where $\mathcal{E} \in W(k)[[t_1, \dots, t_n]]$ is a formal power series whose constant term is p. Here A admits a Frobenius endomorphism $\phi : A \to A$ such that it acts on $W(k)$ as the usual Frobenius and sends t_i to t_i^p for each i. The pair (A, I) is a typical example of a prism. Let $(R)_{\Delta}$ be the absolute prismatic site of R (where R is equipped with the p-adic topology). We regard (A, I) as an object of $(R)_{\Delta}$. We will prove (and generalize) the following result.

Theorem 1.1.1 (Proposition 7.1.1). *The category of prismatic Dieudonné crystals on $(R)_{\Delta}$ is equivalent to the category of minuscule Breuil–Kisin modules over (A, I).*

Anschütz and Le Bras [2023, Theorem 5.12] proved Theorem 1.1.1 when the dimension of R is ≤ 1 (or equivalently $n \leq 1$) and stated that their result should be generalized to R of arbitrary dimension.

Remark 1.1.2. A *minuscule Breuil–Kisin module* over (A, I) is a free A-module M of finite rank equipped with an A-linear homomorphism

$$F_M : \phi^* M := A \otimes_{\phi, A} M \to M$$

whose cokernel is killed by I. For a prismatic Dieudonné crystal $\mathcal{M}$ on $(R)_{\Delta}$, the value $\mathcal{M}(A, I)$ at $(A, I) \in (R)_{\Delta}$ is by definition a minuscule Breuil–Kisin module over (A, I), and the construction $\mathcal{M} \mapsto \mathcal{M}(A, I)$ induces an equivalence between the two categories in Theorem 1.1.1.

Anschütz and Le Bras [2023, Theorem 4.74] showed that the category of prismatic Dieudonné crystals on $(R)_{\Delta}$ is equivalent to the category of p-divisible groups over R. In fact, such an equivalence is obtained not only for R but also for any quasisyntomic ring (in the sense of [Bhatt et al. 2019, Definition 4.10]), where we need to replace prismatic Dieudonné crystals by *admissible* prismatic Dieudonné crystals [Anschütz and Le Bras 2023, Definition 4.5].

Although (admissible) prismatic Dieudonné crystals are theoretically important, it is difficult to describe them explicitly in general. Theorem 1.1.1 provides a practical way to study prismatic Dieudonné crystals on $(R)_{\Delta}$. For example, this shows that giving a prismatic Dieudonné crystal on $(R)_{\Delta}$ is equivalent to giving a minuscule Breuil–Kisin module over (A, I). The latter can be carried out in a much simpler way than the former.

1.2. *Prismatic G-μ-displays.* Let G be a smooth affine group scheme over $\mathbb{Z}_p$ and $\mu : \mathbb{G}_m \to G_{W(k)}$ a cocharacter defined over $W(k)$, where $G_{W(k)} := G \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} W(k)$. We will generalize Theorem 1.1.1 to prismatic G-μ-displays, or equivalently G-Breuil–Kisin modules of type μ, as explained below.

Let (A, I) be a bounded prism in the sense of [Bhatt and Scholze 2022] such that A is a $W(k)$-algebra. A G-Breuil–Kisin module over (A, I) is a G-torsor $\mathcal{P}$ over Spec A with an isomorphism

$$F_{\mathcal{P}} : (\phi^* \mathcal{P})[1/I] \xrightarrow{\sim} \mathcal{P}[1/I]$$

of G-torsors over Spec $A[1/I]$, where $\phi^*\mathcal{P}$ is the base change of $\mathcal{P}$ along the Frobenius $\phi : A \to A$. We say that $\mathcal{P}$ is *of type* μ if, (p, I)-completely étale locally on A, there exists some trivialization $\mathcal{P} \simeq G_A$ under which the isomorphism $F_{\mathcal{P}}$ is given by $g \mapsto Xg$ for an element X in the double coset

$$G(A)\mu(d)G(A) \subset G(A[1/I]),$$

where $d \in I$ is a generator. The notion of G-Breuil–Kisin modules of type μ is important in the study of integral models of (local) Shimura varieties; see Section 1.3.

We will study G-Breuil–Kisin modules of type μ via the theory of higher frames and G-μ-displays developed in [Lau 2021]. More precisely, we introduce and study the groupoid

$$G\text{-}\mathrm{Disp}_\mu(A, I)$$

of *G-μ-displays over* (A, I). It turns out that $G\text{-}\mathrm{Disp}_\mu(A, I)$ is equivalent to the groupoid of G-Breuil–Kisin modules of type μ over (A, I) (Proposition 5.3.8). For a p-adically complete ring R, the groupoid of *prismatic G-μ-displays over* R is defined to be

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta}) := 2 - \varprojlim_{(A,I)\in(R)_{\Delta}} G\text{-}\mathrm{Disp}_\mu(A, I).$$

The main result of this paper is as follows. Let R be a complete regular local ring over $W(k)$ with residue field k. As in Section 1.1, there exists a prism $(W(k)[[t_1, \dots, t_n]], (\mathcal{E}))$ with an isomorphism $R \simeq W(k)[[t_1, \dots, t_n]]/\mathcal{E}$ over $W(k)$.

Theorem 1.2.1 (Theorem 6.1.3). *Assume that the cocharacter μ is* 1*-bounded. Then the following natural functor is an equivalence*:

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta}) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(W(k)[[t_1, \dots, t_n]], (\mathcal{E})).$$

See Definition 4.2.3 for the definition of 1-bounded cocharacters. If G is reductive, then μ is 1-bounded if and only if μ is minuscule. A minuscule Breuil–Kisin module of rank N over $(W(k)[[t_1, \dots, t_n]], (\mathcal{E}))$ can be regarded as a GL_N-Breuil–Kisin module of type μ over $(W(k)[[t_1, \dots, t_n]], (\mathcal{E}))$ for a minuscule cocharacter μ. Theorem 1.1.1 is a special case of Theorem 1.2.1; see Section 7 for details.

We make a few comments on the proof of Theorem 1.2.1. To simplify the notation, we set $(A, I) := (W(k)[[t_1, \dots, t_n]], (\mathcal{E}))$. As in the proof of [Anschütz and Le Bras 2023, Theorem 5.12], the key part of the proof is to show that every G-μ-display $\mathcal{Q}$ over (A, I) admits a unique descent datum. More precisely, let $(A^{(2)}, I^{(2)})$ be the coproduct of two copies of (A, I) in $(R)_{\Delta}$ and let $p_1, p_2 : (A, I) \to (A^{(2)}, I^{(2)})$ be the associated morphisms. Then we will prove that there exists a unique isomorphism

$$\epsilon : p_1^* \mathcal{Q} \xrightarrow{\sim} p_2^* \mathcal{Q}$$

of G-μ-displays over $(A^{(2)}, I^{(2)})$ satisfying the usual cocycle condition over the coproduct $(A^{(3)}, I^{(3)})$ of three copies of (A, I). In the case where $G = \mathrm{GL}_N$, the proof of this claim goes along the same lines as that of [Anschütz and Le Bras 2023], but it requires some additional arguments when $n \geq 2$. For general G, we will use some techniques from the proof of [Lau 2021, Proposition 7.1.5].

We also give some basic definitions and results on prismatic G-μ-displays. In particular, we establish several descent results for prismatic G-μ-displays, such as flat descent (Proposition 5.2.8) and p-complete arc-descent (Corollary 5.6.10). We also introduce the *Hodge filtration* and the *underlying G-ϕ-module* of a prismatic G-μ-display. These notions will be needed in the Grothendieck–Messing deformation theory for prismatic G-μ-displays studied in [Ito 2025].

Remark 1.2.2. In fact, Theorem 1.2.1 will be formulated and proved for a smooth affine group scheme G over the ring of integers $\mathcal{O}_E$ of any finite extension E of $\mathbb{Q}_p$. For this, we will use $\mathcal{O}_E$-analogues of prisms, called *$\mathcal{O}_E$-prisms*. This notion was already introduced in [Marks 2023] (in which these objects are called E-typical prisms). Section 2 is devoted to discussing results analogous to those of [Bhatt and Scholze 2022, Sections 2 and 3] for $\mathcal{O}_E$-prisms. We will define G-μ-displays for bounded $\mathcal{O}_E$-prisms in the same way, and prove the above results for them. As explained in Remark 1.3.3 below, it will be convenient to establish our results in this generality, but the reader (who is familiar with the theory of prisms) may assume that $\mathcal{O}_E = \mathbb{Z}_p$ and skip Section 2 on a first reading. The arguments for general $\mathcal{O}_E$ are the same as for the case where $\mathcal{O}_E = \mathbb{Z}_p$.

Remark 1.2.3. G-Breuil–Kisin modules of type μ may be more familiar to readers than prismatic G-μ-displays. However, in order to prove Theorems 1.1.1 and 1.2.1, and other descent results (e.g., Corollaries 5.3.9 and 5.6.10), it is essential to work with prismatic G-μ-displays.

Remark 1.2.4. We briefly discuss how our results are related to the theory of G-objects in crystalline $\mathbb{Z}_p$-local systems. Here we follow the terminology of [Imai et al. 2024]. Let $R = \mathcal{O}_K$ be the ring of integers of a finite totally ramified extension K of $W(k)[1/p]$. Bhatt and Scholze [2023] proved that the category of prismatic F-crystals on $(\mathcal{O}_K)_{\Delta}$ is equivalent to the category $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(K)$ of free $\mathbb{Z}_p$-modules T of finite rank with a continuous $\mathrm{Gal}(\overline{K}/K)$-action such that $T[1/p]$ is crystalline. (Here $\mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K.) Using this result, together with [Imai et al. 2024], one can prove that there is an equivalence of groupoids

$$G\text{-}\mathrm{Disp}_{\mu}((\mathcal{O}_K)_{\Delta}) \xrightarrow{\sim} G\text{-}\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,\mu}(K) \tag{1-1}$$

if G is reductive, where $G\text{-}\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,\mu}(K)$ is the groupoid of G-objects in $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(K)$ having cocharacter μ in the sense of [Imai et al. 2024]. More specifically, this follows from [loc. cit., Theorem 2, Proposition 3.13] and [Ito 2025, Proposition 5.1.16]. Let $\mathcal{E} \in W(k)[[t]]$ be the Eisenstein polynomial of a uniformizer

$\varpi \in \mathcal{O}_K$. Then (1-1) and Theorem 1.2.1 give an equivalence

$$G\text{-}\mathrm{Disp}_\mu(W(k)[[t]], (\mathcal{E})) \xrightarrow{\sim} G\text{-}\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,\mu}(K).$$

A similar result was previously obtained in [Levin 2015, Corollary 4.3.8] by a completely different method. (The result of [Bhatt and Scholze 2023] was generalized to higher-dimensional smooth p-adic formal schemes over $\mathcal{O}_K$; see [Du et al. 2024; Guo and Reinecke 2024]. However, since a higher-dimensional complete regular local ring R as in Theorem 1.2.1 is in general not topologically of finite type over $W(k)$ with respect to the *p-adic* topology, we cannot directly apply those results to obtain an analogue of (1-1) for R. We do not pursue this issue here.)

We will also discuss (in the case where $\mathcal{O}_E = \mathbb{Z}_p$) the relation between prismatic G-μ-displays and prismatic F-gauges in vector bundles introduced in [Drinfeld 2024; Bhatt and Lurie 2022a; 2022b; Bhatt 2022]. In particular, for a quasisyntomic ring S over $W(k)$, we introduce[1] the groupoid

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(S)$$

of *prismatic G-F-gauges of type μ over S* and construct a fully faithful functor

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(S) \to G\text{-}\mathrm{Disp}_\mu((S)_{\Delta}).$$

See Section 8 for details. This functor can be thought of as a generalization of the fully faithful functor from the category of admissible prismatic Dieudonné crystals on $(S)_{\Delta}$ to the category of prismatic Dieudonné crystals on $(S)_{\Delta}$ (see Example 8.1.15). If S is a complete regular local ring over $W(k)$ with residue field k, then the above functor is an equivalence (Corollary 8.2.12). Thus, we can rephrase Theorem 1.2.1 as follows:

Corollary 1.2.5 (Theorem 6.1.3, Corollary 8.2.12). *Let the notation be as in Theorem 1.2.1. Assume that μ is 1-bounded. Then we have a natural equivalence*

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(W(k)[[t_1, \dots, t_n]], (\mathcal{E})).$$

1.3. ***Motivation.*** The primary motivation behind this work is to understand some classification results on p-divisible groups and the local structure of integral local Shimura varieties defined in [Scholze and Weinstein 2020], by using the theory of prisms. In the following, we explain this in more detail with a brief review of previous studies.

We first explain the motivation for $G = \mathrm{GL}_N$. Let $\mathcal{O}_K$ be the ring of integers of a finite totally ramified extension K of $W(k)[1/p]$. Let $\mathcal{E} \in W(k)[[t]]$ be the Eisenstein polynomial of a uniformizer $\varpi \in \mathcal{O}_K$.

Remark 1.3.1. Anschütz and Le Bras [2023, Theorem 5.12] obtained the equivalence of categories

$$\left\{ \begin{matrix} p\text{-divisible groups} \\ \text{over } \mathcal{O}_K \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{minuscule Breuil–Kisin modules} \\ \text{over } (W(k)[[t]], (\mathcal{E})) \end{matrix} \right\} \tag{1-2}$$

[1]After this work was completed, Gardner and Madapusi [2024] announced that they defined (certain objects which are essentially equivalent to) prismatic G-F-gauges of type μ for more general p-adically complete rings, using the stacky approach of Drinfeld and Bhatt–Lurie. See also [Imai et al. 2023] for the relation between our prismatic G-F-gauges of type μ and those defined in [Gardner and Madapusi 2024].

by combining the classification theorem [Anschütz and Le Bras 2023, Theorem 4.74] with Theorem 1.1.1 for $R = \mathcal{O}_K$. This result was conjectured in [Breuil 1998], proved in [Kisin 2006; 2009] when $p \geq 3$, and proved in [Kim 2012; Liu 2013; Lau 2014] for all $p > 0$.

We consider the pair $(W(k)[\![t]\!]/t^n, (\mathcal{E}))$, which is naturally a bounded prism for every $n \geq 1$. Lau [2014] obtained the equivalence of categories

$$\left\{ \begin{matrix} p\text{-divisible groups} \\ \text{over } \mathcal{O}_K/\varpi^n \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{minuscule Breuil–Kisin modules} \\ \text{over } (W(k)[\![t]\!]/t^n, (\mathcal{E})) \end{matrix} \right\} \tag{1-3}$$

by a deformation-theoretic argument and then proved (1-2) by taking the limit over n; see [Lau 2014, Corollary 5.4, Theorem 6.6]. His proof uses the theory of *displays*, which was initiated by Zink and developed by many authors, including Zink [2001; 2002] and Lau [2008; 2014]. This classification result over $\mathcal{O}_K/\varpi^n$ is important in its own right. For example, this is a key ingredient in the construction of integral canonical models of Shimura varieties of abelian type with hyperspecial level structure in characteristic $p = 2$; see [Kim and Madapusi Pera 2016] for details.

Contrary to Lau's approach, it is not clear whether the results in [Anschütz and Le Bras 2023] imply (1-3) since the Grothendieck–Messing deformation theory does not apply directly to prismatic Dieudonné crystals. This point is discussed in [Ito 2025], where we develop the deformation theory for prismatic Dieudonné crystals, or more generally for prismatic G-μ-displays when μ is 1-bounded. In [loc. cit.], we construct universal deformations of prismatic G-μ-displays over k as certain prismatic G-μ-displays over complete regular local rings of higher dimension, where Theorem 1.2.1 plays a crucial role. As a result, we can give an alternative proof of the equivalences (1-2) and (1-3).

Remark 1.3.2. In the proof of [Anschütz and Le Bras 2023, Theorem 4.74] (and hence in the proof of the equivalence (1-2) of that work), they use [Scholze and Weinstein 2013, Theorem B], which says that for an algebraically closed complete extension C of $\mathbb{Q}_p$, the category of p-divisible groups over $\mathcal{O}_C$ is equivalent to the category of free $\mathbb{Z}_p$-modules T of finite rank together with a C-subspace of $T \otimes_{\mathbb{Z}_p} C$. In [Ito 2025], we also give an alternative proof of this result.

We now explain our motivation for general G. The notion of G-μ-displays ("displays with G-μ-structures") was first introduced in [Bültel 2008; Bültel and Pappas 2020] to study Rapoport–Zink spaces and integral models of Shimura varieties (where G is reductive and μ is minuscule). The theory of G-μ-displays has been developed in various settings; see for example [Langer and Zink 2019; Pappas 2023; Lau 2021; Daniels 2021; Bartling 2022]. In fact, the notion of G-μ-displays over *perfect* prisms was already used in [Bartling 2022]. Bartling used G-μ-displays over perfect prisms to prove the local representability and the formal smoothness of integral local Shimura varieties with hyperspecial level structure, under a certain nilpotence assumption (introduced in [Bültel and Pappas 2020, Definition 3.4.2]). In [Ito 2025], we prove the same assertion without any nilpotence assumptions, by using the universal deformations of prismatic G-μ-displays over k.

Remark 1.3.3. In [Ito 2025], we establish the above results not only when G is defined over $\mathbb{Z}_p$ but also when G is defined over $\mathcal{O}_E$, where E is any finite extension of $\mathbb{Q}_p$. For this, it will be convenient to work with $\mathcal{O}_E$-prisms.

The theory of G-μ-displays also has applications to K3 surfaces and related varieties; see [Langer and Zink 2019; Lau 2021; Inoue 2024]. In a future work, we plan to employ prismatic G-μ-displays to investigate the deformation theory for these varieties.

1.4. ***Outline of this paper.*** This paper is organized as follows. In Section 2, we collect some basic definitions and facts about $\mathcal{O}_E$-prisms. In Section 3, we discuss the notion of displayed Breuil–Kisin modules (of type μ), which will serve as examples of prismatic G-μ-displays. In Section 4, we introduce and study the *display group* $G_\mu(A, I)$, which is used in the definition of prismatic G-μ-displays. The structural results about $G_\mu(A, I)$ obtained there play crucial roles in the study of prismatic G-μ-displays.

Sections 5 and 6 are the main parts of this paper. In Section 5, we introduce prismatic G-μ-displays, give some basic definitions (e.g., Hodge filtrations and underlying G-ϕ-modules), and establish several descent results. In Section 6, we prove our main result (Theorem 1.2.1).

In Section 7, we make a few remarks on prismatic Dieudonné crystals, and prove Theorem 1.1.1. Finally, in Section 8, we provide a comparison between prismatic G-μ-displays and prismatic F-gauges. In particular, we introduce the notion of prismatic G-F-gauges of type μ for quasisyntomic rings over $W(k)$.

Notation. In this paper, all rings are commutative and unital. For a module M over a ring R and a ring homomorphism $f : R \to R'$, the tensor product $M \otimes_R R'$ is denoted by $M_{R'}$ or f^*M. For a scheme X over R, the base change $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is denoted by $X_{R'}$ or f^*X. We use similar notation for the base change of group schemes, p-divisible groups, etc. Moreover, all actions of groups will be right actions, unless otherwise stated.

2. Preliminaries on $\mathcal{O}_E$-prisms

Throughout this paper, we fix a prime number p. Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}_q$. Here $\mathbb{F}_q$ is a finite field with q elements. We fix a uniformizer $\pi \in \mathcal{O}_E$.

In this section, we study an "$\mathcal{O}_E$-analogue" of the notion of prisms. Such objects are called $\mathcal{O}_E$-prisms in this paper. This notion was also introduced in [Marks 2023] (in which $\mathcal{O}_E$-prisms are called *E-typical prisms*). We discuss some properties of $\mathcal{O}_E$-prisms which we need in the sequel. We hope that this section will also help readers unfamiliar with [Bhatt and Scholze 2022] to understand some basic facts about prisms.

2.1. ***Prisms.*** We first recall the definition of bounded prisms.

Let A be a $\mathbb{Z}_{(p)}$-algebra. A *δ-structure* on A is a map $\delta : A \to A$ of sets with the following properties:

(1) $\delta(1) = 0$.

(2) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.

(3) $\delta(x + y) = \delta(x) + \delta(y) + (x^p + y^p - (x + y)^p)/p$.

A *δ-ring* is a pair (A, δ) of a $\mathbb{Z}_{(p)}$-algebra A and a δ-structure $\delta : A \to A$. The above equalities imply that

$$\phi : A \to A, \quad x \mapsto x^p + p\delta(x),$$

is a ring homomorphism which is a lift of the Frobenius $A/p \to A/p$, $x \mapsto x^p$.

In the following, for a ring A and an ideal $I \subset A$, we say that an A-module M is I-adically complete (or x-adically complete if I is generated by an element $x \in I$) if the natural homomorphism

$$M \to \widehat{M} := \varprojlim_n M/I^n M$$

is bijective.

Definition 2.1.1 [Bhatt and Scholze 2022]. A *bounded prism* is a pair (A, I) of a δ-ring A and a Cartier divisor $I \subset A$ with the following properties:

(1) A is (p, I)-adically complete.

(2) A/I has bounded p-torsion, i.e., $(A/I)[p^\infty] = (A/I)[p^n]$ for some integer $n > 0$.

(3) We have $p \in (I, \phi(I))$.

We say that a bounded prism (A, I) is *orientable* if I is principal.

Remark 2.1.2. Under the condition that A/I has bounded p^∞-torsion, the requirement that A is (p, I)-adically complete is equivalent to saying that A is *derived* (p, I)-adically complete; see [Bhatt and Scholze 2022, Lemma 3.7]. We refer to [loc. cit., Section 1.2] and [Stacks 2005–, Tag 091N] for the notion of derived complete modules (or complexes). For a ring A and a finitely generated ideal $I \subset A$, if an A-module M is I-adically complete, then M is derived I-adically complete; see [Stacks 2005–, Tag 091T] or [Positselski 2023, Lemma 2.3].

2.2. *δ_E-rings.* In this subsection, we recall the notion of δ_E-rings, which is an "$\mathcal{O}_E$-analogue" of the notion of a δ-ring. We define

$$\delta_{\mathcal{O}_E,\pi} : \mathcal{O}_E \to \mathcal{O}_E, \quad x \mapsto (x - x^q)/\pi.$$

Definition 2.2.1 [Marks 2023, Definition 2.2]. (1) Let A be an $\mathcal{O}_E$-algebra. A *δ_E-structure* on A is a map $\delta_E : A \to A$ of sets with the following properties:

(a) $\delta_E(xy) = x^q\delta_E(y) + y^q\delta_E(x) + \pi\delta_E(x)\delta_E(y)$.

(b) $\delta_E(x + y) = \delta_E(x) + \delta_E(y) + (x^q + y^q - (x + y)^q)/\pi$.

(c) $\delta_E : A \to A$ is compatible with $\delta_{\mathcal{O}_E,\pi}$, i.e., we have $\delta_E(x) = \delta_{\mathcal{O}_E,\pi}(x)$ for any $x \in \mathcal{O}_E$.

A *δ_E-ring* is a pair (A, δ_E) of an $\mathcal{O}_E$-algebra A and a δ_E-structure $\delta_E : A \to A$.

(2) A homomorphism $f : (A, \delta_E) \to (A', \delta'_E)$ of δ_E-rings is a homomorphism $f : A \to A'$ of $\mathcal{O}_E$-algebras such that $f \circ \delta_E = \delta'_E \circ f$.

The term $(x^q + y^q - (x+y)^q)/\pi$ in (b) makes sense since we can write it as

$$(x^q + y^q - (x+y)^q)/\pi = -\sum_{0<i<q} \left(\binom{q}{i}/\pi\right) x^i y^{q-i}.$$

We usually denote a δ_E-ring (A, δ_E) simply by A.

Remark 2.2.2. The notion of δ_E-rings also appeared in [Borger 2011, Remark 1.19; Li 2022] for example. In the end of the latter work, Li suggests using δ_E-structures for the study of prismatic sites of higher level over ramified bases.

Remark 2.2.3. The notion of δ_E-rings is essentially independent of the choice of π. More precisely, let $\pi' \in \mathcal{O}_E$ be another uniformizer. We write $\pi = u\pi'$ for a unique unit $u \in \mathcal{O}_E^\times$. If an $\mathcal{O}_E$-algebra A is equipped with a δ_E-structure $\delta_E : A \to A$ with respect to π, then it also admits a δ_E-structure with respect to π', defined by $x \mapsto u\delta_E(x)$.

For a δ_E-ring A, we define

$$\phi_A : A \to A, \quad x \mapsto x^q + \pi\delta_E(x).$$

We see that ϕ_A is a homomorphism of $\mathcal{O}_E$-algebras and is a lift of the q-th power Frobenius $A/\pi \to A/\pi$, $x \mapsto x^q$. The homomorphism ϕ_A is called the *Frobenius* of the δ_E-ring A. When there is no ambiguity, we omit the subscript and simply write $\phi = \phi_A$.

Remark 2.2.4. If A is a π-torsion-free $\mathcal{O}_E$-algebra, then the construction $\delta_E \mapsto \phi$ gives a bijection between the set of δ_E-structures on A and the set of homomorphisms $\phi : A \to A$ over $\mathcal{O}_E$ that are lifts of $A/\pi \to A/\pi$, $x \mapsto x^q$.

Example 2.2.5 (free δ_E-rings). We define an endomorphism ϕ of the polynomial ring $\mathcal{O}_E[X_0, X_1, X_2, \dots]$ by $X_i \mapsto X_i^q + \pi X_{i+1}$ $(i \geq 0)$. By Remark 2.2.4, we get the corresponding δ_E-structure on the ring $\mathcal{O}_E[X_0, X_1, X_2, \dots]$, which sends X_i to X_{i+1}. We write

$$\mathcal{O}_E\{X\}$$

for the resulting δ_E-ring. As in the proof of [Bhatt and Scholze 2022, Lemma 2.11], one can check that $\mathcal{O}_E\{X\}$ has the following property: For a δ_E-ring A and an element $x \in A$, there exists a unique homomorphism $f : \mathcal{O}_E\{X\} \to A$ of δ_E-rings with $f(X_0) = x$. In other words, the δ_E-ring $\mathcal{O}_E\{X\}$ is a free object with basis $X := X_0$ in the category of δ_E-rings.

The same argument as in the proof of [loc. cit., Lemma 2.11] also shows that the Frobenius $\phi : \mathcal{O}_E\{X\} \to \mathcal{O}_E\{X\}$ is faithfully flat; this fact will be used in Section 2.6.

Lemma 2.2.6. *For a δ_E-ring A, the Frobenius $\phi : A \to A$ is a homomorphism of δ_E-rings.*

Proof. Let $x \in A$ be an element. We have to show that $\phi(\delta_E(x)) = \delta_E(\phi(x))$. Since there exists a (unique) homomorphism $f : \mathcal{O}_E\{X\} \to A$ of δ_E-rings with $f(X) = x$, it suffices to prove the assertion for $A = \mathcal{O}_E\{X\}$, which is clear since A is π-torsion-free and $\phi : A \to A$ is ϕ-equivariant. □

Following [loc. cit., Remark 2.4], we shall give a characterization of δ_E-rings in terms of *ramified* Witt vectors. For an $\mathcal{O}_E$-algebra A, let

$$W_{\mathcal{O}_E,\pi,2}(A)$$

denote the *ring of π-typical Witt vectors of length* 2: the underlying set of $W_{\mathcal{O}_E,\pi,2}(A)$ is $A \times A$, and for $(x_0, x_1), (y_0, y_1) \in W_{\mathcal{O}_E,\pi,2}(A)$, we have

$$(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1 + (x_0^q + y_0^q - (x_0 + y_0)^q)/\pi),$$
$$(x_0, x_1) \cdot (y_0, y_1) = (x_0 y_0, x_0^q y_1 + y_0^q x_1 + \pi x_1 y_1).$$

If $\mathcal{O}_E = \mathbb{Z}_p$ and $\pi = p$, then $W_{\mathcal{O}_E,\pi,2}(A)$ is the ring $W_2(A)$ of p-typical Witt vectors of length 2. For a detailed treatment of the rings of π-typical Witt vectors (of any length), we refer to [Schneider 2017, Section 1.1; Borger 2011].

Remark 2.2.7 (cf. [Bhatt and Scholze 2022, Remark 2.4]). The map

$$\mathcal{O}_E \to W_{\mathcal{O}_E,\pi,2}(A), \quad x \mapsto (x, \delta_{\mathcal{O}_E,\pi}(x)),$$

is a ring homomorphism for any $\mathcal{O}_E$-algebra A. We regard $W_{\mathcal{O}_E,\pi,2}(A)$ as an $\mathcal{O}_E$-algebra by this homomorphism. Let

$$\epsilon : W_{\mathcal{O}_E,\pi,2}(A) \to A, \quad (x_0, x_1) \mapsto x_0,$$

denote the projection map, which is a homomorphism of $\mathcal{O}_E$-algebras. For a δ_E-structure $\delta_E : A \to A$, the map $s : A \mapsto W_{\mathcal{O}_E,\pi,2}(A)$ defined by $x \mapsto (x, \delta_E(x))$ is a homomorphism of $\mathcal{O}_E$-algebras such that $\epsilon \circ s = \mathrm{id}_A$. By this procedure, we obtain a bijection between the set of δ_E-structures on A and the set of homomorphisms $s : A \to W_{\mathcal{O}_E,\pi,2}(A)$ of $\mathcal{O}_E$-algebras satisfying $\epsilon \circ s = \mathrm{id}_A$.

Remark 2.2.8 (cf. [Bhatt and Scholze 2022, Remark 2.7]). It follows from Remark 2.2.7 that the category of δ_E-rings admits all limits and colimits, and they are preserved by the forgetful functor from the category of δ_E-rings to the category of $\mathcal{O}_E$-algebras.

The following two lemmas will be used frequently in the sequel.

Lemma 2.2.9. *Let $A = (A, \delta_E)$ be a δ_E-ring and $I \subset A$ an ideal. Then I is stable under δ_E if and only if A/I admits a δ_E-structure that is compatible with the one on A. If such a δ_E-structure on A/I exists, then it is unique.*

Proof. This follows immediately from the definition of δ_E-structures (see the proof of [Bhatt and Scholze 2022, Lemma 2.9]). □

Lemma 2.2.10. *Let A be a δ_E-ring and let $I \subset A$ be a finitely generated ideal containing π. Then, for any integer $n \geq 1$, there exists an integer $m \geq 1$ such that, for any $x \in A$, we have $\delta_E(x + I^m) \subset \delta_E(x) + I^n$. In particular, the I-adic completion of A admits a unique δ_E-structure that is compatible with the one on A.*

Proof. The proof is the same as that of [Bhatt and Scholze 2022, Lemma 2.17]. □

We shall discuss some properties of perfect δ_E-rings, which are defined as follows.

Definition 2.2.11. We say that a δ_E-ring A is *perfect* if the Frobenius $\phi : A \to A$ is bijective.

Lemma 2.2.12 [Marks 2023, Lemma 2.11]. *A perfect δ_E-ring A is π-torsion-free.*

Proof. This is proved in [Marks 2023, Lemma 2.11], and follows from the same argument as in the proof of [Bhatt and Scholze 2022, Lemma 2.28]. □

Example 2.2.13. Let R be an $\mathbb{F}_q$-algebra. Assume that R is perfect (i.e., $R \to R$, $x \mapsto x^p$, is bijective). Let $W(R)$ be the ring of p-typical Witt vectors and we define

$$W_{\mathcal{O}_E}(R) := W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E.$$

Let $\phi : W_{\mathcal{O}_E}(R) \to W_{\mathcal{O}_E}(R)$ denote the base change of the q-th power Frobenius of $W(R)$. This is a lift of the q-th power Frobenius of $W_{\mathcal{O}_E}(R)/\pi = R$. Since $W_{\mathcal{O}_E}(R)$ is π-torsion-free, we obtain the corresponding δ_E-structure on $W_{\mathcal{O}_E}(R)$. It is clear that $W_{\mathcal{O}_E}(R)$ is a perfect δ_E-ring.

Lemma 2.2.14. *The functor $R \mapsto W_{\mathcal{O}_E}(R)$ from the category of perfect $\mathbb{F}_q$-algebras to the category of π-adically complete $\mathcal{O}_E$-algebras admits a right adjoint given by $A \mapsto \varprojlim_{x \mapsto x^p} A/\pi A$.*

Proof. This is well known in the case where $\mathcal{O}_E = \mathbb{Z}_p$ (see [Szamuely and Zábrádi 2018, Proposition 3.12] for example). The general case follows from this special case. □

Corollary 2.2.15 [Marks 2023, Proposition 2.13]. *The following categories are equivalent*:

- *The category $\mathcal{C}_1$ of π-adically complete perfect δ_E-rings (A, δ_E).*
- *The category $\mathcal{C}_2$ of π-adically complete and π-torsion-free $\mathcal{O}_E$-algebras A such that $A/\pi A$ is perfect.*
- *The category $\mathcal{C}_3$ of perfect $\mathbb{F}_q$-algebras R.*

More precisely, the functors $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3 \to \mathcal{C}_1$, defined by $(A, \delta_E) \mapsto A$, $A \mapsto A/\pi$, $R \mapsto W_{\mathcal{O}_E}(R)$, respectively, are equivalences.

Proof. Using Lemma 2.2.14, one can prove the assertion in exactly the same way as [Bhatt and Scholze 2022, Corollary 2.31]. □

Corollary 2.2.16. *Let A be a perfect δ_E-ring and B a π-adically complete δ_E-ring. Then any homomorphism $A \to B$ of $\mathcal{O}_E$-algebras is a homomorphism of δ_E-rings.*

Proof. We may assume that A is π-adically complete. It then follows from Lemma 2.2.14 and Corollary 2.2.15 that $A \to B$ is ϕ-equivariant. We recall that $\phi : B \to B$ is a homomorphism of δ_E-rings by Lemma 2.2.6. Let B^{perf} be a limit of the diagram

$$B \xleftarrow{\phi} B \xleftarrow{\phi} B \longleftarrow \cdots$$

in the category of δ_E-rings, which is a perfect δ_E-ring. Since A is perfect, $A \to B$ factors through a ϕ-equivariant homomorphism $A \to B^{\mathrm{perf}}$ of $\mathcal{O}_E$-algebras. It follows from the π-torsion-freeness of B^{perf} (see Lemma 2.2.12) that $A \to B^{\mathrm{perf}}$ is a homomorphism of δ_E-rings. Since $A \to B$ is the composition $A \to B^{\mathrm{perf}} \to B$, the assertion follows. □

2.3. ***$\mathcal{O}_E$-prisms.*** We now introduce $\mathcal{O}_E$-prisms.

Definition 2.3.1 [Marks 2023, Definition 3.1]. (1) An *$\mathcal{O}_E$-prism* is a pair (A, I) of a δ_E-ring A and a Cartier divisor $I \subset A$ such that A is derived (π, I)-adically complete and $\pi \in I + \phi(I)A$.

(2) We say that an $\mathcal{O}_E$-prism (A, I) is *bounded* if A/I has bounded p^∞-torsion.

(3) We say that an $\mathcal{O}_E$-prism (A, I) is *orientable* if I is principal.

(4) An *oriented and bounded* $\mathcal{O}_E$-prism (A, d) is an orientable and bounded $\mathcal{O}_E$-prism (A, I) with a choice of a generator $d \in I$.

(5) A map $f : (A, I) \to (A', I')$ of $\mathcal{O}_E$-prisms is a homomorphism $f : A \to A'$ of δ_E-rings such that $f(I) \subset I'$.

If $\mathcal{O}_E = \mathbb{Z}_p$, then bounded $\mathcal{O}_E$-prisms are nothing but bounded prisms.

Remark 2.3.2. Let (A, I) be a bounded $\mathcal{O}_E$-prism. By [Tian 2023, Proposition 2.5(1)] (see also Lemma 2.5.1 below), we see that A is (π, I)-adically complete. Moreover, since A/I is derived π-adically complete and has bounded p^∞-torsion, it follows that A/I is π-adically complete (see [Bhatt et al. 2019, Lemma 4.7] for example).

Let A be a δ_E-ring. Following [Bhatt and Scholze 2022, Definition 2.19], we say that an element $d \in A$ is *distinguished* if $\delta_E(d) \in A^\times$, i.e., $\delta_E(d)$ is a unit. Since $\delta_{\mathcal{O}_E,\pi}(\pi) = 1 - \pi^{q-1} \in \mathcal{O}_E^\times$, we see that $\pi \in A$ is distinguished.

Lemma 2.3.3. *Let A be a δ_E-ring and $d \in A$ an element. Assume that π is contained in the Jacobson radical* $\mathrm{rad}(A)$ *of A.*

(1) *Assume that $d = fh$ for some elements $f, h \in A$ with $f \in \mathrm{rad}(A)$. If d is distinguished, then f is distinguished and $h \in A^\times$.*

(2) *Assume that $d \in \mathrm{rad}(A)$. Then d is distinguished if and only if $\pi \in (d, \phi(d))$.*

Proof. This can be proved exactly in the same way as [Bhatt and Scholze 2022, Lemmas 2.24 and 2.25]. See also [Marks 2023, Lemma 2.9]. □

The following rigidity property plays a fundamental role in the theory of $\mathcal{O}_E$-prisms.

Lemma 2.3.4 (cf. [Bhatt and Scholze 2022, Lemma 3.5]). *Let $(A, I) \to (A', I')$ be a map of $\mathcal{O}_E$-prisms. Then the natural homomorphism $I \otimes_A A' \to IA'$ is an isomorphism and $IA' = I'$.*

Proof. By using [Marks 2023, Lemma 3.4], this follows from the same argument as in the proof of [Bhatt and Scholze 2022, Lemma 3.5]. We recall the argument in the case where both (A, I) and (A', I') are orientable. It follows from Lemma 2.3.3(2) that any generator $d \in I$ is distinguished. Let $d' \in I'$ be a generator. Then Lemma 2.3.3(1) implies that d is mapped to ud' for some $u \in A'^\times$. In particular, the image of d in A' is a nonzerodivisor, and we obtain $I \otimes_A A' \xrightarrow{\sim} IA'$ and $IA' = I'$. □

The following lemma will be used several times in this paper.

Lemma 2.3.5 (cf. the proof of [Bhatt and Scholze 2022, Lemma 4.8]). *Let A be a perfect δ_E-ring and (B, I) a bounded $\mathcal{O}_E$-prism. Then any homomorphism $A \to B/I$ of $\mathcal{O}_E$-algebras lifts uniquely to a homomorphism $A \to B$ of δ_E-rings.*

Proof. By Corollary 2.2.16, it is enough to check that the homomorphism $A \to B/I$ lifts uniquely to a homomorphism $A \to B$ of $\mathcal{O}_E$-algebras. We may assume that A is π-adically complete, and then $A \simeq W_{\mathcal{O}_E}(R)$ for some perfect $\mathbb{F}_q$-algebra R by Corollary 2.2.15. Since B is (π, I)-adically complete and B/I is π-adically complete, it suffices to prove that, for every integer $n \geq 1$, any homomorphism $W_n(R) \to B/(\pi^n, I)$ of $W_n(\mathbb{F}_q)$-algebras lifts uniquely to a homomorphism $W_n(R) \to B/(\pi^n, I^n)$ of $W_n(\mathbb{F}_q)$-algebras (here $W_n(R) = W(R)/p^n$). This follows from the fact that the cotangent complex $L_{W_n(R)/W_n(\mathbb{F}_q)}$ is acyclic [Szamuely and Zábrádi 2018, Lemma 3.27(1)]. □

We give some examples of $\mathcal{O}_E$-prisms.

Example 2.3.6 (cf. [Bhatt and Scholze 2022, Example 1.3(1)]). Let A be a π-adically complete and π-torsion-free $\mathcal{O}_E$-algebra. Let $\phi : A \to A$ be a homomorphism over $\mathcal{O}_E$ which is a lift of the q-th power Frobenius of A/π. This homomorphism induces a δ_E-structure on A, and the pair $(A, (\pi))$ is a bounded $\mathcal{O}_E$-prism.

Definition 2.3.7 ($\mathcal{O}_E$-prism over $\mathcal{O}$). Let k be a perfect field containing $\mathbb{F}_q$. We will write

$$\mathcal{O} := W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$$

instead of $W_{\mathcal{O}_E}(k)$. An $\mathcal{O}_E$-prism *over* $\mathcal{O}$ is an $\mathcal{O}_E$-prism (A, I) with a homomorphism $\mathcal{O} \to A$ of δ_E-rings.

Let $\mathcal{O} = W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ be as in Definition 2.3.7. Let

$$\mathfrak{S}_{\mathcal{O}} := \mathcal{O}[[t_1, \dots, t_n]]$$

(where $n \geq 0$) and let $\phi : \mathfrak{S}_{\mathcal{O}} \to \mathfrak{S}_{\mathcal{O}}$ be the homomorphism such that $\phi(t_i) = t_i^q$ $(1 \leq i \leq n)$ and its restriction to $\mathcal{O}$ agrees with the Frobenius of $\mathcal{O}$. Since $\mathfrak{S}_{\mathcal{O}}$ is π-torsion-free, ϕ gives rise to a δ_E-structure on $\mathfrak{S}_{\mathcal{O}}$.

Proposition 2.3.8 (cf. [Bhatt and Scholze 2022, Example 1.3(3)]). *Let $\mathcal{E} \in \mathfrak{S}_{\mathcal{O}}$ be a formal power series whose constant term is a uniformizer of $\mathcal{O}$. Then the pair $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ is a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.*

Proof. We shall show that $\pi \in (\mathcal{E}, \phi(\mathcal{E}))$; the other required conditions are clearly satisfied. It is enough to check that $\delta_E(\mathcal{E}) \in \mathfrak{S}_{\mathcal{O}}^{\times}$. For this, it suffices to show that the image of $\delta_E(\mathcal{E})$ in $\mathfrak{S}_{\mathcal{O}}/(t_1, \dots, t_n) = \mathcal{O}$ is a unit, which is clear since the constant term of $\mathcal{E}$ is a uniformizer of $\mathcal{O}$. □

We call $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ an $\mathcal{O}_E$-prism *of Breuil–Kisin type* in this paper. Here n could be any nonnegative integer. Such a pair is also considered in [Cheng 2018].

2.4. ***Perfectoid rings and $\mathcal{O}_E$-prisms.*** The notion of (integral) perfectoid rings in the sense of [Bhatt et al. 2018, Definition 3.5] plays a central role in the theory of prismatic G-μ-displays. We refer to [loc. cit., Section 3] and [Česnavičius and Scholze 2024, Section 2] for basic properties of perfectoid rings. We recall the definition of perfectoid rings and some notation from [Bhatt et al. 2018, Section 3].

A ring R is a *perfectoid ring* if there exists an element $\varpi \in R$ such that $p \in (\varpi)^p$ and R is ϖ-adically complete, the Frobenius $R/p \to R/p$, $x \mapsto x^p$ is surjective, and the kernel of $\theta : W(R^\flat) \to R$ is principal. Here

$$R^\flat := \varprojlim_{x \mapsto x^p} R/p$$

and $\theta : W(R^\flat) \to R$ is the unique homomorphism whose reduction modulo p is the projection $R^\flat \to R/p$, $(x_0, x_1, \dots) \mapsto x_0$. The homomorphism θ is the counit of the adjunction given in Lemma 2.2.14 (in the case where $\mathcal{O}_E = \mathbb{Z}_p$). By [Bhatt et al. 2018, Lemma 3.9], there is an element $\varpi^\flat \in R^\flat$ such that $\theta([\varpi^\flat])$ is a unit multiple of ϖ, where $[-]$ denotes the Teichmüller lift.

Example 2.4.1. (1) An $\mathbb{F}_p$-algebra R is a perfectoid ring if and only if it is perfect; see [Bhatt et al. 2018, Example 3.15].

(2) Let V be a p-adically complete valuation ring with algebraically closed fraction field. Then V is a perfectoid ring. This follows from [loc. cit., Lemma 3.10].

Let $\mathcal{O}$ be as in Definition 2.3.7. If R is a perfectoid ring over $\mathcal{O}$ (i.e., R is a perfectoid ring with a ring homomorphism $\mathcal{O} \to R$), then $R^\flat$ is naturally a k-algebra, and $W_{\mathcal{O}_E}(R^\flat) = W(R^\flat) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ is an $\mathcal{O}$-algebra. Let

$$\theta_{\mathcal{O}_E} : W_{\mathcal{O}_E}(R^\flat) \to R$$

be the homomorphism induced from θ.

Lemma 2.4.2 (cf. [Fargues and Scholze 2021, Proposition II.1.4]). *The kernel* $\operatorname{Ker}\theta_{\mathcal{O}_E}$ *of* $\theta_{\mathcal{O}_E}$ *is a principal ideal. Moreover, any generator of* $\operatorname{Ker}\theta_{\mathcal{O}_E}$ *is a nonzerodivisor in* $W_{\mathcal{O}_E}(R^\flat)$.

Proof. Let $\varpi \in R$ be an element such that R is ϖ-adically complete and $p \in (\varpi)^p$. By [Bhatt et al. 2018, Lemma 3.10(i)], after replacing ϖ by $\theta([(\varpi^\flat)^{1/p^n}])$ for some integer $n > 0$, we have $\pi \in (\varpi)$. Then we can write $\pi = \theta([\varpi^\flat]x)$ for some element $x \in W(R^\flat)$ since θ is surjective. We shall show that $\pi - [\varpi^\flat]x$ generates $\operatorname{Ker}\theta_{\mathcal{O}_E}$.

Let $\mathcal{E}(T) \in W(k)[T]$ be the (monic) Eisenstein polynomial of $\pi \in \mathcal{O}$ so that we have $W(k)[T]/\mathcal{E}(T) \xrightarrow{\sim} \mathcal{O}$, $T \mapsto \pi$. We see that

$$\begin{aligned} W_{\mathcal{O}_E}(R^\flat)/(\pi - [\varpi^\flat]x) &\simeq W(R^\flat)[T]/(\mathcal{E}(T), T - [\varpi^\flat]x) \\ &\simeq W(R^\flat)/\mathcal{E}([\varpi^\flat]x). \end{aligned}$$

It thus suffices to show that $\mathcal{E}([\varpi^\flat]x)$ is a generator of the kernel $\operatorname{Ker}\theta$ of θ. It is clear that $\mathcal{E}([\varpi^\flat]x) \in \operatorname{Ker}\theta$. Since $\mathcal{E}([\varpi^\flat]x)$ is a unit multiple of an element of the form $p + [\varpi^\flat]y$, the proof of [Bhatt et al. 2018, Lemma 3.10] shows that $\mathcal{E}([\varpi^\flat]x) \in \operatorname{Ker}\theta$ is a generator.

It remains to prove that any generator $\xi \in \operatorname{Ker}\theta_{\mathcal{O}_E}$ is a nonzerodivisor. We recall that $\operatorname{Ker}\theta$ is generated by a nonzerodivisor $\xi' \in W(R^\flat)$. Since $W(\mathbb{F}_q) \to \mathcal{O}_E$ is flat, the element ξ' is a nonzerodivisor in $W_{\mathcal{O}_E}(R^\flat)$. This implies that ξ is a nonzerodivisor since we have $\xi' \in (\xi) = \operatorname{Ker}\theta_{\mathcal{O}_E}$. □

Proposition 2.4.3 (cf. [Bhatt and Scholze 2022, Example 1.3(2)]). *Let R be a perfectoid ring over $\mathcal{O}$ and we write $I_R := \operatorname{Ker}\theta_{\mathcal{O}_E}$. Then the pair*

$$(W_{\mathcal{O}_E}(R^\flat), I_R)$$

is an orientable and bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Proof. By the proof of Lemma 2.4.2, we know that I_R is generated by a nonzerodivisor of the form $\xi = \pi - [(\varpi')^\flat]b$, where $\varpi' \in R$ is such that R is ϖ'-adically complete and $p \in (\varpi')^p$. In order to show that $W_{\mathcal{O}_E}(R^\flat)$ is (π, ξ)-adically complete, it suffices to show that $W(R^\flat)$ is $(p, [(\varpi')^\flat])$-adically complete, which is easy to check (see also the proof of [Česnavičius and Scholze 2024, Proposition 2.1.11(b)]). Moreover $W_{\mathcal{O}_E}(R^\flat)/\xi = R$ has bounded p^∞-torsion by [Bhatt and Scholze 2022, Lemma 3.8].

It remains to show that $\pi \in (\xi, \phi(\xi))$. It suffices to prove that $\delta_E(\xi) \in W_{\mathcal{O}_E}(R^\flat)^\times$. The image of $\delta_E(\xi)$ in $W_{\mathcal{O}_E}(R^\flat)/[(\varpi')^\flat]$ is equal to $1 - \pi^{q-1}$ (we note that $W_{\mathcal{O}_E}(R^\flat)/[(\varpi')^\flat]$ is π-torsion-free) and hence is a unit, which in turn implies that $\delta_E(\xi) \in W_{\mathcal{O}_E}(R^\flat)^\times$. □

A homomorphism $R \to S$ of perfectoid rings over $\mathcal{O}$ induces a map $(W_{\mathcal{O}_E}(R^\flat), I_R) \to (W_{\mathcal{O}_E}(S^\flat), I_S)$ of $\mathcal{O}_E$-prisms over $\mathcal{O}$.

Remark 2.4.4. We say that an $\mathcal{O}_E$-prism (A, I) is *perfect* if the δ_E-ring A is perfect. By [Marks 2023, Lemma 3.10], a perfect $\mathcal{O}_E$-prism (A, I) is bounded and orientable. Moreover, in [loc. cit., Theorem 3.18], it is proved that A/I is a perfectoid ring. These facts, together with Lemma 2.3.5 and Proposition 2.4.3, imply that the functor $(A, I) \mapsto A/I$ from the category of perfect $\mathcal{O}_E$-prisms to that of perfectoid rings over $\mathcal{O}_E$ is an equivalence, whose inverse is given by $R \mapsto (W_{\mathcal{O}_E}(R^\flat), I_R)$. This is an analogue of [Bhatt and Scholze 2022, Theorem 3.10].

2.5. *Prismatic sites.* For a ring A, let $D(A)$ denote the derived category of A-modules. Let $I \subset A$ be a finitely generated ideal. We say that a complex $M \in D(A)$ is *I-completely flat* (resp. *I-completely faithfully flat*) if $M \otimes^{\mathbb{L}}_A A/I$ is concentrated in degree 0 and it is a flat (resp. faithfully flat) A/I-module. One can easily check that this definition is equivalent to the one introduced in [Bhatt and Scholze 2022, Section 1.2].

Lemma 2.5.1. *Let (A, I) be a bounded $\mathcal{O}_E$-prism.*

(1) *For a complex $M \in D(A)$, the derived (π, I)-adic completion of M is isomorphic to*

$$R\varprojlim_n (M \otimes^{\mathbb{L}}_A A/(\pi, I)^n).$$

In particular, if M is (π, I)-completely flat, then the derived (π, I)-adic completion of M is concentrated in degree 0.

(2) *Let M be an A-module. Assume that M is (π, I)-completely flat and derived (π, I)-adically complete. Then M is (π, I)-adically complete. Moreover the natural homomorphism $M \otimes_A I \to M$ is injective and $M/I^n M$ has bounded p^∞-torsion for any n.*

Proof. (1) The assertion follows from [Tian 2023, Proposition 2.5(1)] or the proof of [Bhatt and Scholze 2022, Lemma 3.7(1)]. This can also be deduced from the results discussed in [Yekutieli 2021]; by

[Yekutieli 2021, Corollary 3.5, Theorem 3.11], it suffices to prove that the ideal $(\pi, I) \subset A$ is weakly proregular in the sense of [loc. cit., Definition 3.2], which follows from the same argument as in the proof of [loc. cit., Theorem 7.3].

(2) It follows from (1) that M is (π, I)-adically complete. The second statement can be proved in the same way as [Bhatt and Scholze 2022, Lemma 3.7(2)]. (In [loc. cit.], we should assume that M is derived (p, I)-adically complete.) □

We say that a map $f : (A, I) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms is a *(faithfully) flat map* if $A \to A'$ is (π, I)-completely (faithfully) flat. If f is a faithfully flat map, then we say that (A', I') is a flat covering of (A, I).

Remark 2.5.2. For a map $f : (A, I) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms, we have $A' \otimes^{\mathbb{L}}_A A/I \simeq A'/I'$ by Lemma 2.3.4, which in turn implies that

$$A' \otimes^{\mathbb{L}}_A A/(\pi, I) \simeq A'/I' \otimes^{\mathbb{L}}_{A/I} A/(\pi, I). \tag{2-1}$$

In particular f is a (faithfully) flat map if and only if $A/I \to A'/I'$ is π-completely (faithfully) flat.

Definition 2.5.3. Let R be a π-adically complete $\mathcal{O}_E$-algebra. Let

$$(R)_{\mathbb{\Delta}, \mathcal{O}_E}$$

denote the category of bounded $\mathcal{O}_E$-prisms (A, I) together with a homomorphism $R \to A/I$ of $\mathcal{O}_E$-algebras. The morphisms $f : (A, I) \to (A', I')$ in $(R)_{\mathbb{\Delta}, \mathcal{O}_E}$ are the maps of $\mathcal{O}_E$-prisms such that $A/I \to A'/I'$ is a homomorphism of R-algebras. We endow the opposite category $(R)^{\mathrm{op}}_{\mathbb{\Delta}, \mathcal{O}_E}$ with the topology generated by the faithfully flat maps. This topology is called the *flat topology*.

We note that $(\mathcal{O}_E)_{\mathbb{\Delta}, \mathcal{O}_E}$ is just the category of bounded $\mathcal{O}_E$-prisms. If $\mathcal{O}_E = \mathbb{Z}_p$, then $(R)_{\mathbb{\Delta}, \mathcal{O}_E}$ is the category $(R)_{\mathbb{\Delta}}$ introduced in [Bhatt and Scholze 2022, Remark 4.7]. The category $(R)_{\mathbb{\Delta}}$ (or its opposite) is called the *absolute prismatic site of R*.

Remark 2.5.4. A diagram

$$(A_2, I_2) \xleftarrow{g} (A_1, I_1) \xrightarrow{f} (A_3, I_3)$$

in $(R)_{\mathbb{\Delta}, \mathcal{O}_E}$ such that g is a flat map, admits a colimit (i.e., a pushout). Indeed, by Lemma 2.5.1(1), the (π, I_3)-adic completion $A := (A_2 \otimes_{A_1} A_3)^\wedge$ is isomorphic to the derived (π, I_3)-adic completion of $A_2 \otimes^{\mathbb{L}}_{A_1} A_3$. In particular A is (π, I_3)-completely flat over A_3. (Here we use that J-complete flatness is preserved under base change and taking derived J-adic completions.) It follows from Remark 2.2.8 and Lemma 2.2.10 that A admits a unique δ_E-structure that is compatible with the δ_E-structures on A_2 and A_3. By Lemma 2.5.1(2), we see that $(A, I_3 A)$ is a bounded $\mathcal{O}_E$-prism. By construction, $(A, I_3 A)$ is a colimit of the above diagram. As a result, it follows that $(R)^{\mathrm{op}}_{\mathbb{\Delta}, \mathcal{O}_E}$ is indeed a site.

Remark 2.5.5 (cf. [Bhatt and Scholze 2022, Corollary 3.12]). A faithfully flat map $(A, I) \to (A', I')$ induces faithfully flat homomorphisms $A/(\pi, I)^n \to A'/(\pi, I')^n$ and $A/(\pi^n, I) \to A'/(\pi^n, I')$ for any n.

It follows that the functors

$$\mathcal{O}_{\Delta} : (R)_{\Delta,\mathcal{O}_E} \to \mathrm{Set}, \quad (A, I) \mapsto A,$$
$$\mathcal{O}_{\bar{\Delta}} : (R)_{\Delta,\mathcal{O}_E} \to \mathrm{Set}, \quad (A, I) \mapsto A/I,$$

form sheaves with respect to the flat topology. Here Set is the category of sets.

More generally, we have the following descent result.

Proposition 2.5.6. *The fibered category over $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to each $(A, I) \in (\mathcal{O}_E)_{\Delta,\mathcal{O}_E}$ the category of finite projective A-modules satisfies descent with respect to the flat topology. The same holds for finite projective A/I-modules.*

Proof. For a ring B and an ideal $J \subset B$ such that B is J-adically complete, the natural functor

$$\{\text{finite projective } B\text{-modules}\} \longrightarrow 2-\varprojlim_n \{\text{finite projective } B/J^n\text{-modules}\}$$

is an equivalence; see for example [Bhatt 2016, Lemma 4.11]. The assertions of the proposition follow from this fact and faithfully flat descent for finite projective modules over $A/(\pi, I)^n$ and $A/(\pi^n, I)$, respectively. See also [Anschütz and Le Bras 2023, Lemma A.1, Proposition A.3]. □

Definition 2.5.7. For a bounded $\mathcal{O}_E$-prism (A, I), let

$$(A, I)_{\Delta}$$

be the category of bounded $\mathcal{O}_E$-prisms (B, J) with a map $(A, I) \to (B, J)$. We endow $(A, I)^{\mathrm{op}}_{\Delta}$ with the flat topology induced from $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$.

Example 2.5.8. (1) Let $\mathcal{O}$ be as in Definition 2.3.7. The category $(\mathcal{O})_{\Delta,\mathcal{O}_E}$ is the same as the category of bounded $\mathcal{O}_E$-prisms over $\mathcal{O}$ by Lemma 2.3.5.

(2) Let R be a perfectoid ring over $\mathcal{O}_E$. It follows from Lemma 2.3.5 that $(R)_{\Delta,\mathcal{O}_E}$ is the same as the category $(W_{\mathcal{O}_E}(R^\flat), I_R)_{\Delta}$.

Let $A \to B$ be a ring homomorphism and $I \subset A$ a finitely generated ideal. We say that $A \to B$ is *I-completely étale* if B is derived I-adically complete, $B \otimes^{\mathbb{L}}_A A/I$ is concentrated in degree 0, and B/IB is étale over A/I. We write $A_{I\text{-ét}}$ for the category of I-completely étale A-algebras. If $I = 0$, then $A_{I\text{-ét}}$ is just the category $A_{\text{ét}}$ of étale A-algebras.

Lemma 2.5.9. *Let (A, I) be a bounded $\mathcal{O}_E$-prism.*

(1) *A ring homomorphism $A/I \to C$ is π-completely étale if and only if C is π-adically complete and C/π^n is étale over $A/(\pi^n, I)$ for every integer $n \geq 1$. If this is the case, then C has bounded p^∞-torsion.*

(2) *A ring homomorphism $A \to B$ is (π, I)-completely étale if and only if B is (π, I)-adically complete and $B/(\pi, I)^n$ is étale over $A/(\pi, I)^n$ for every $n \geq 1$. If this is the case, then $B \otimes^{\mathbb{L}}_A A/I \xrightarrow{\sim} B/IB$ and $A/I \to B/IB$ is π-completely étale.*

(3) *The functors*

$$A_{(\pi,I)\text{-ét}} \to (A/I)_{\pi\text{-ét}} \to (A/(\pi,I))_{\text{ét}},$$

where the first one is defined by $B \mapsto B/IB$ *and the second one is defined by* $C \mapsto C/\pi$, *are equivalences.*

Proof. This result is well known to specialists, but we include a proof for the convenience of the reader.

(1) Assume that $A/I \to C$ is π-completely étale. Then, since A/I has bounded p^∞-torsion, [Bhatt et al. 2019, Lemma 4.7] implies that C is π-adically complete and has bounded p^∞-torsion. Since C/π^n is flat over $A/(\pi^n, I)$ and C/π is étale over $A/(\pi, I)$, it follows that C/π^n is étale over $A/(\pi^n, I)$ for any n.

We next prove the "if" direction, so we assume that C is π-adically complete and C/π^n is étale over $A/(\pi^n, I)$ for any n. We want to show that $C \otimes^{\mathbb{L}}_{A/I} A/(\pi, I)$ is concentrated in degree 0. There exists an étale A/I-algebra C_0 such that $C_0/\pi \simeq C/\pi$ over $A/(\pi, I)$; see for example [Arabia 2001, Section 1.1] or [Stacks 2005–, Tag 04D1] (this is known as a special case of Elkik's result [1973]). One easily sees that the derived π-adic completion of C_0, the π-adic completion of C_0, and C are isomorphic to each other. Then we obtain

$$C \otimes^{\mathbb{L}}_{A/I} A/(\pi, I) \simeq C_0 \otimes^{\mathbb{L}}_{A/I} A/(\pi, I) \simeq C_0/\pi.$$

This proves the assertion.

(2) Assume that $A \to B$ is (π, I)-completely étale. We easily see that $B/(\pi, I)^n$ is étale over $A/(\pi, I)^n$. It follows from Lemma 2.5.1 that B is (π, I)-adically complete and we have $B \otimes^{\mathbb{L}}_A A/I \xrightarrow{\sim} B/IB$. It is then clear that $A/I \to B/IB$ is π-completely étale.

The "if" direction can be proved by the same argument as in (1). Suppose that B is (π, I)-adically complete and $B/(\pi, I)^n$ is étale over $A/(\pi, I)^n$ for any n. As above, there exists an étale A-algebra B_0 such that the (π, I)-adic completion of B_0 is isomorphic to B. It follows from Lemma 2.5.1(1) that B is isomorphic to the derived (π, I)-adic completion of B_0, which in turn implies that B is (π, I)-completely étale over A.

(3) This follows from the proofs of (1) and (2). □

Lemma 2.5.10 (cf. [Bhatt and Scholze 2022, Lemma 2.18]). *Let* (A, I) *be a bounded* $\mathcal{O}_E$*-prism and* $A \to B$ *a* (π, I)*-completely étale homomorphism. Then* B *admits a unique* δ_E*-structure compatible with that on* A*. Moreover, the pair* (B, IB) *is a bounded* $\mathcal{O}_E$*-prism.*

Proof. It suffices to prove the first statement by Lemma 2.5.1. For this, we proceed as in the proof of [Bhatt and Scholze 2022, Lemma 2.18].

We regard $W_{\mathcal{O}_E,\pi,2}(B)$ as an A-algebra via the composition $A \to W_{\mathcal{O}_E,\pi,2}(A) \to W_{\mathcal{O}_E,\pi,2}(B)$, where $A \to W_{\mathcal{O}_E,\pi,2}(A)$ is the homomorphism corresponding to the δ_E-structure on A (Remark 2.2.7). Then $W_{\mathcal{O}_E,\pi,2}(B)$ is (π, I)-adically complete. Indeed, we have an exact sequence of A-modules

$$0 \longrightarrow \phi_* B \xrightarrow{V} W_{\mathcal{O}_E,\pi,2}(B) \xrightarrow{\epsilon} B \longrightarrow 0,$$

where we write ϕ_*B for B regarded as an A-algebra via the composition $A \xrightarrow{\phi} A \to B$, and $V : \phi_*B \to W_{\mathcal{O}_E,\pi,2}(B)$ is defined by $x \mapsto (0, x)$. Since $B \otimes^{\mathbb{L}}_A A/(\pi, I)^n$ is concentrated in degree 0 and both ϕ_*B and B are (π, I)-adically complete, we can conclude that $W_{\mathcal{O}_E,\pi,2}(B)$ is (π, I)-adically complete.

As in the proof of Lemma 2.5.9, there exists an étale A-algebra B_0 such that the (π, I)-adic completion of B_0 is isomorphic to B. Since $W_{\mathcal{O}_E,\pi,2}(B)$ is $(\operatorname{Ker}\epsilon)$-adically complete and $A \to B_0$ is étale, there exists a unique homomorphism $s_0 : B_0 \to W_{\mathcal{O}_E,\pi,2}(B)$ of A-algebras such that $\epsilon \circ s_0$ coincides with $B_0 \to B$. Then, since $W_{\mathcal{O}_E,\pi,2}(B)$ is (π, I)-adically complete, we see that $s_0 : B_0 \to W_{\mathcal{O}_E,\pi,2}(B)$ extends to a unique homomorphism $s : B \to W_{\mathcal{O}_E,\pi,2}(B)$ of A-algebras such that $\epsilon \circ s = \mathrm{id}_B$, which corresponds to a unique δ_E-structure on B compatible with that on A by virtue of Remark 2.2.7. □

Example 2.5.11. Let R be a perfectoid ring over $\mathcal{O}_E$ and let $R \to S$ be a π-completely étale (or equivalently, p-completely étale) homomorphism. By [Anschütz and Le Bras 2023, Corollary 2.10] or [Lau 2018, Lemma 8.11], we see that S is a perfectoid ring. Moreover, the isomorphism (2-1) implies that $W_{\mathcal{O}_E}(R^\flat) \to W_{\mathcal{O}_E}(S^\flat)$ is (π, I_R)-completely étale.

Let (A, I) be a bounded $\mathcal{O}_E$-prism. We say that a homomorphism $B \to B'$ of (π, I)-completely étale A-algebras is a (π, I)-*completely étale covering* if

$$\operatorname{Spec} B'/(\pi, I) \to \operatorname{Spec} B/(\pi, I)$$

is surjective, or equivalently, the homomorphism $B \to B'$ is (π, I)-completely faithfully flat. We note that $B \to B'$ is automatically (π, I)-completely étale.

Definition 2.5.12. We write

$$(A, I)_{\text{ét}}$$

for the category of (π, I)-completely étale A-algebras instead of $A_{(\pi,I)\text{-ét}}$. We endow the opposite category $(A, I)_{\text{ét}}^{\text{op}}$ with the topology generated by the (π, I)-completely étale coverings, which is called the (π, I)-*completely étale topology*.

The category $(A, I)_{\text{ét}}^{\text{op}}$ admits fiber products. Indeed, a colimit of the diagram $C \leftarrow B \to D$ in $(A, I)_{\text{ét}}$ is given by the (π, I)-adic completion of $C \otimes_B D$; see Remark 2.5.4. It follows that $(A, I)_{\text{ét}}^{\text{op}}$ is a site.

Remark 2.5.13. Recall that, for a (π, I)-completely étale A-algebra $B \in (A, I)_{\text{ét}}$, the pair (B, IB) is naturally a bounded $\mathcal{O}_E$-prism by Lemma 2.5.10. We can regard $(A, I)_{\text{ét}}$ as a full subcategory of the category $(A, I)_{\Delta}$. The (π, I)-completely étale topology on $(A, I)_{\text{ét}}^{\text{op}}$ coincides with the one induced from the flat topology.

Remark 2.5.14. Any bounded $\mathcal{O}_E$-prism (A, I) admits a (π, I)-completely étale covering $A \to B$ such that (B, IB) is orientable. Indeed, there exists an étale and faithfully flat homomorphism $A \to A'$ such that IA' is principal. The (π, I)-adic completion B of A' is a (π, I)-completely étale covering of A (by Lemma 2.5.1). Since IA' is principal, the bounded $\mathcal{O}_E$-prism (B, IB) is orientable.

2.6. ***Prismatic envelopes for regular sequences.*** The existence and the flatness of prismatic envelopes for regular sequences are proved in [Bhatt and Scholze 2022, Proposition 3.13]. In this subsection, we give an analogous result for $\mathcal{O}_E$-prisms. We will freely use the formalism of *animated rings* here. For the definition and properties of animated rings, see for example [Česnavičius and Scholze 2024, Section 5] and [Bhatt and Lurie 2022a, Appendix A]. (See also [Lurie 2016, Chapter 25], where animated rings are called simplicial rings.)

We recall some terminology from [Česnavičius and Scholze 2024; Bhatt and Scholze 2022]. To an animated ring A, we can attach a graded ring of homotopy groups $\bigoplus_{n\geq 0}\pi_n(A)$. For an animated ring A, the *derived quotient* of A with respect to a sequence $x_1,\dots,x_n\in\pi_0(A)$ is defined by

$$A/^{\mathbb{L}}(x_1,\dots,x_n) := A\otimes^{\mathbb{L}}_{\mathbb{Z}[X_1,\dots,X_n]}\mathbb{Z}[X_1,\dots,X_n]/(X_1,\dots,X_n).$$

Here $\mathbb{Z}[X_1,\dots,X_n]\to A$ is a morphism such that the induced ring homomorphism $\mathbb{Z}[X_1,\dots,X_n]\to\pi_0(A)$ is given by $X_i\mapsto x_i$. In [Bhatt and Scholze 2022], the derived quotient is denoted by $\mathrm{Kos}(A;x_1,\dots,x_n)$. We say that a morphism $A\to B$ of animated rings is *flat* (resp. *faithfully flat*) if $\pi_0(B)$ is flat (resp. faithfully flat) over $\pi_0(A)$ and we have $\pi_n(A)\otimes_{\pi_0(A)}\pi_0(B)\xrightarrow{\sim}\pi_n(B)$ for any $n\geq 0$.

Before stating the result, let us quickly recall the definition of an $\mathcal{O}_E$-PD structure, and its relation to δ_E-structures.

Definition 2.6.1 [Hopkins and Gross 1994, Section 10; Faltings 2002, Definition 14]. Let A be an $\mathcal{O}_E$-algebra and $I\subset A$ an ideal. An *$\mathcal{O}_E$-PD structure* on I is a map $\gamma_\pi: I\to I$ of sets with the following properties:

(1) $\pi\gamma_\pi(x)=x^q$.

(2) $\gamma_\pi(ax)=a^q\gamma_\pi(x)$, where $a\in A$.

(3) $\gamma_\pi(x+y)=\gamma_\pi(x)+\gamma_\pi(y)+(x^q+y^q-(x+y)^q)/\pi$.

Example 2.6.2 [Faltings 2002, Section 7]. Let $n\geq 0$ be an integer and let $\mathcal{O}_E[(X_{i,j})]$ be the polynomial ring with variables $X_{i,j}$ indexed by integers i,j with $1\leq i\leq n$ and $j\geq 0$. We write

$$\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$$

for the quotient of $\mathcal{O}_E[(X_{i,j})]$ by the ideal generated by the elements $X_{i,j}^q-\pi X_{i,j+1}$ for all i,j. The image of $X_{i,0}$ in $\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$ is denoted by X_i. We see that $\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$ is canonically isomorphic to the $\mathcal{O}_E$-subalgebra of $E[X_1,\dots,X_n]$ generated by $X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}}$ ($1\leq i\leq n$ and $j\geq 0$). The ideal $I^{\mathrm{PD}}\subset\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$ generated by the elements $X_{i,j}$ admits an $\mathcal{O}_E$-PD structure γ_π such that $\gamma_\pi(X_{i,j})=X_{i,j+1}$. In fact, the pair $(\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}, I^{\mathrm{PD}})$ is the $\mathcal{O}_E$-PD envelope (in the usual sense) of the polynomial ring $\mathcal{O}_E[X_1,\dots,X_n]$ with respect to the ideal $(X_1,\dots,X_n)$.

Lemma 2.6.3 (cf. [Bhatt and Scholze 2022, Lemma 2.38]). *Let B be a π-torsion-free $\mathcal{O}_E$-algebra. Let $x_1,\dots,x_n\in B$ be a sequence such that $(B/\pi)/^{\mathbb{L}}(\bar{x}_1,\dots,\bar{x}_n)$ is concentrated in degree* 0, *where*

$\bar{x}_1, \dots, \bar{x}_n \in B/\pi$ are the images of $x_1, \dots, x_n \in B$. We set

$$C := B \otimes^{\mathbb{L}}_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}},$$

where $\mathcal{O}_E[X_1, \dots, X_n] \to B$ is defined by $X_i \mapsto x_i$. Then C is concentrated in degree 0 and $\pi_0(C)$ is π-torsion-free. Moreover the pair $(\pi_0(C), I^{\mathrm{PD}}\pi_0(C))$ is the $\mathcal{O}_E$-PD envelope of B with respect to the ideal $(x_1, \dots, x_n)$.

In the following, we will write

$$D_{(x_1,\dots,x_n)}(B) := \pi_0(C) = B \otimes_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}}.$$

Proof. The proof is identical to that of [Bhatt and Scholze 2022, Lemma 2.38]. □

We also need the following construction. Let B be a δ_E-ring. Let $d \in B$ be an element and be $x_1, \dots, x_n \in B$ a sequence. We set $B\{X\} := B \otimes_{\mathcal{O}_E} \mathcal{O}_E\{X\}$ and let

$$B\{X_1, \dots, X_n\}$$

be the n-th tensor power of $B\{X\}$ over B. We consider the diagram of δ_E-rings

$$B \xleftarrow{f} B\{X_1, \dots, X_n\} \xrightarrow{g} B\{Y_1, \dots, Y_n\},$$

where f is defined by $X_i \mapsto x_i$ and g is defined by $X_i \mapsto dY_i$. Let

$$B\{x_1/d, \dots, x_n/d\}$$

denote the pushout of this diagram, which is a δ_E-ring over B with the following property: For a homomorphism $B \to C$ of δ_E-rings such that the image of d is a nonzerodivisor in C and $x_i \in dC$ for all i, there exists a unique homomorphism $B\{x_1/d, \dots, x_n/d\} \to C$ of δ_E-rings over B. We let $x_i/d \in B\{x_1/d, \dots, x_n/d\}$ denote the image of $Y_i \in B\{Y_1, \dots, Y_n\}$.

Using this construction, we can relate δ_E-structures to $\mathcal{O}_E$-PD structures.

Lemma 2.6.4 (cf. [Bhatt and Scholze 2022, Lemma 2.36]). *We have a natural isomorphism*

$$(\mathcal{O}_E\{X_1, \dots, X_n\})\{\phi(X_1)/\pi, \dots, \phi(X_n)/\pi\} \simeq D_{(X_1,\dots,X_n)}(\mathcal{O}_E\{X_1, \dots, X_n\})$$

of $\mathcal{O}_E\{X_1, \dots, X_n\}$-algebras.

Proof. This can be proved in the same way as [Bhatt and Scholze 2022, Lemma 2.36]. We include a sketch of the proof.

We set $B := \mathcal{O}_E\{X_1, \dots, X_n\}$. Since the Frobenius $\phi : B \to B$ is faithfully flat by Example 2.2.5, it follows that $C := B\{\phi(X_1)/\pi, \dots, \phi(X_n)/\pi\}$ is π-torsion-free. Since $\phi(X_i)/\pi = X_i^q/\pi + \delta_E(X_i)$, C can be regarded as the smallest δ_E-subring of $B[1/\pi]$ which contains B and X_i^q/π $(1 \leq i \leq n)$. On the other hand, since $D := D_{(X_1,\dots,X_n)}(B)$ is π-torsion-free by Lemma 2.6.3, we see that D is the $\mathcal{O}_E$-subalgebra of $B[1/\pi]$ generated by B and $X_i^{q^j}/\pi^{1+q+\dots+q^{j-1}}$ $(1 \leq i \leq n$ and $j \geq 1)$.

We shall prove that $C = D$ in $B[1/\pi]$. Let us first show that $D \subset C$. For this, it suffices to prove that, for every $1 \le i \le n$, we have $X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}} \in C$ for all $j \ge 1$. We proceed by induction on j. The assertion is clear for $j = 1$. Assume that the assertion is true for some $j \ge 1$. Then we have

$$\begin{aligned}\delta_E(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}}) &= \phi(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}})/\pi - (X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}})^q/\pi \\ &= \phi(X_i)^{q^j}/\pi^{2+q+\cdots+q^{j-1}} - X_i^{q^{j+1}}/\pi^{1+q+\cdots+q^j} \in C.\end{aligned}$$

To prove that $X_i^{q^{j+1}}/\pi^{1+q+\cdots+q^j} \in C$, it is enough to show that $\phi(X_i)^{q^j}/\pi^{2+q+\cdots+q^{j-1}} \in C$. Since $\phi(X_i)/\pi = X_i^q/\pi + \delta_E(X_i)$ is contained in C, the assertion now follows from the inequality $q^j \ge 2+q+\cdots+q^{j-1}$.

It remains to prove that $C \subset D$. Since $\phi(X_i)/\pi = X_i^q/\pi + \delta_E(X_i)$ is contained in D, the inequality $q^j \ge 2+q+\cdots+q^{j-1}$ for $j \ge 1$ implies that

$$\phi(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}}) = \phi(X_i)^{q^j}/\pi^{1+q+\cdots+q^{j-1}} \in \pi D$$

for $j \ge 1$. Since

$$(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}})^q = \pi(X_i^{q^{j+1}}/\pi^{1+q+\cdots+q^j}) \in \pi D,$$

we see that ϕ preserves D and that the reduction modulo π of $\phi : D \to D$ is the q-th power Frobenius. This implies that $C \subset D$. □

Corollary 2.6.5 (cf. [Bhatt and Scholze 2022, Corollary 2.39]). *Let B be a π-torsion-free δ_E-ring. Let $x_1, \dots, x_n \in B$ be a sequence such that $(B/\pi)/^{\mathbb{L}}(\bar{x}_1, \dots, \bar{x}_n)$ is concentrated in degree 0. We set $D := B \otimes^{\mathbb{L}}_{\mathcal{O}_E\{X_1,\dots,X_n\}} \mathcal{O}_E\{Y_1, \dots, Y_n\}$ where $\mathcal{O}_E\{X_1, \dots, X_n\} \to B$ is defined by $X_i \mapsto \phi(x_i)$ and $\mathcal{O}_E\{X_1, \dots, X_n\} \to \mathcal{O}_E\{Y_1, \dots, Y_n\}$ is defined by $X_i \mapsto \pi Y_i$. Then D is concentrated in degree 0. Moreover*

$$\pi_0(D) = B\{\phi(x_1)/\pi, \dots, \phi(x_n)/\pi\}$$

is π-torsion-free, and is isomorphic to $D_{(x_1,\dots,x_n)}(B)$ as a B-algebra.

Proof. Since $\phi : \mathcal{O}_E\{X_1, \dots, X_n\} \to \mathcal{O}_E\{X_1, \dots, X_n\}$ is faithfully flat by Example 2.2.5, we have an identification

$$D = B \otimes^{\mathbb{L}}_{\mathcal{O}_E\{X_1,\dots,X_n\}} (\mathcal{O}_E\{X_1, \dots, X_n\})\{\phi(X_1)/\pi, \dots, \phi(X_n)/\pi\},$$

where $\mathcal{O}_E\{X_1, \dots, X_n\} \to B$ is defined by $X_i \mapsto x_i$. Then Lemma 2.6.4 implies that

$$D \simeq B \otimes^{\mathbb{L}}_{\mathcal{O}_E\{X_1,\dots,X_n\}} D_{(X_1,\dots,X_n)}(\mathcal{O}_E\{X_1, \dots, X_n\}).$$

By Lemma 2.6.3, we have

$$D_{(X_1,\dots,X_n)}(\mathcal{O}_E\{X_1, \dots, X_n\}) \simeq \mathcal{O}_E\{X_1, \dots, X_n\} \otimes^{\mathbb{L}}_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}}$$

and thus $D \simeq B \otimes^{\mathbb{L}}_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}}$. The assertion then follows by applying Lemma 2.6.3 again. □

Now we can state the desired result:

Proposition 2.6.6 (cf. [Bhatt and Scholze 2022, Proposition 3.13]). *Assume that (A, I) is an orientable and bounded $\mathcal{O}_E$-prism. Let $d \in I$ be a generator. Let B be a δ_E-ring over A. Let $x_1, \dots, x_n \in B$ be a sequence such that the induced morphism*

$$A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n) \tag{2-2}$$

of animated rings is flat. (In other words, the sequence $x_1, \dots, x_n \in B$ is (π, I)-completely regular relative to A in the sense of [Bhatt and Scholze 2022, Definition 2.42].) *We set $J := (d, x_1, \dots, x_n) \subset B$. Then the following assertions hold*:

(1) *The (π, I)-adic completion $B\{J/I\}^\wedge$ of $B\{x_1/d, \dots, x_n/d\}$ is (π, I)-completely flat over A. In particular, the pair*

$$(B\{J/I\}^\wedge, IB\{J/I\}^\wedge)$$

is an orientable and bounded $\mathcal{O}_E$-prism. Moreover $B\{J/I\}^\wedge$ is (π, I)-completely faithfully flat over A if the morphism (2-2) is faithfully flat.

(2) *For a bounded $\mathcal{O}_E$-prism (D, ID) over (A, I) and a homomorphism $B \to D$ of δ_E-rings over A such that $JD \subset ID$, there exists a unique map of $\mathcal{O}_E$-prisms*

$$(B\{J/I\}^\wedge, IB\{J/I\}^\wedge) \to (D, ID)$$

over B. Moreover, the formation of $B\{J/I\}^\wedge$ commutes with base change along any map $(A, I) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms, and also commutes with base change along any (π, I)-completely flat homomorphism $B \to B'$ of δ_E-rings.

See [Bhatt and Scholze 2022, Proposition 3.13(3)] for the precise meaning of the last statement.

Proof. Let $C := B \otimes^{\mathbb{L}}_{B\{X_1,\dots,X_n\}} B\{Y_1, \dots, Y_n\}$ be the pushout of the diagram $B \leftarrow B\{X_1, \dots, X_n\} \to B\{Y_1, \dots, Y_n\}$ in the ∞-category of animated rings, where the first map is defined by $X_i \mapsto x_i$ and the second one is defined by $X_i \mapsto dY_i$. It suffices to prove that if the morphism (2-2) is flat (resp. faithfully flat), then C is (π, I)-completely flat (resp. (π, I)-completely faithfully flat) over A. Indeed, if this is true, then the derived (π, I)-adic completion of C is isomorphic to $B\{J/I\}^\wedge$, and in particular $B\{J/I\}^\wedge$ is (π, I)-completely flat (resp. (π, I)-completely faithfully flat) over A. It is then easy to see that $B\{J/I\}^\wedge$ has the desired properties.

In order to prove that C is (π, I)-completely flat (resp. (π, I)-completely faithfully flat) over A, one can argue as in the proof of [loc. cit., Proposition 3.13]. (The faithful flatness is not discussed in [loc. cit.], but the same argument works.) The only difference is that we have to use $\mathcal{O}_E$-PD structures, instead of usual PD structures. Here we need the results established above (e.g., Corollary 2.6.5). The details are left to the reader. □

The bounded $\mathcal{O}_E$-prism $(B\{J/I\}^\wedge, IB\{J/I\}^\wedge)$ is called the *prismatic envelope* of B over (A, I) with respect to the ideal J.

Remark 2.6.7. As in [Bhatt and Scholze 2022, Proposition 3.13], we need to use animated δ_E-rings in the proof of Proposition 2.6.6. (For example, in the proof of [loc. cit., Proposition 3.13], the notion of animated δ_E-rings is used to obtain the description of the bottom right vertex C'' of the diagram appearing there.) One can define the notion of animated δ_E-rings in the same way as in [Mao 2024, Section 5] (i.e., by animating δ_E-rings). Alternatively, we can follow the approach employed in [Bhatt and Lurie 2022b, Appendix A].

3. Displayed Breuil–Kisin modules

In this section, we study Breuil–Kisin modules for bounded $\mathcal{O}_E$-prisms. We introduce the notions of a displayed Breuil–Kisin module and of a minuscule Breuil–Kisin module. These objects serve as examples of prismatic G-μ-displays introduced in Section 5.

3.1. *Displayed Breuil–Kisin modules.* We use the following notation. Let A be a ring. For A-modules M and N, the set of A-linear homomorphisms $M \to N$ is denoted by $\mathrm{Hom}_A(M, N)$. Let $I \subset A$ be a Cartier divisor. For an integer $n \geq 1$, we define $I^{-n} := \mathrm{Hom}_A(I^n, A)$. We have a natural injection $I^{-n} \hookrightarrow I^{-n-1}$ for any integer n. We then define

$$A[1/I] := \varinjlim_n I^{-n},$$

which is an A-algebra. For an A-module M, we set $M[1/I] := M \otimes_A A[1/I]$. If I is generated by a nonzerodivisor d, then we have $A[1/I] = A[1/d]$ and $I^{-n} = d^{-n}A$.

Lemma 3.1.1. *Let M, N be finite projective A-modules and let $F : N[1/I] \xrightarrow{\sim} M[1/I]$ be an $A[1/I]$-linear isomorphism. For an integer i, we set*

$$\mathrm{Fil}^i(N) := \{x \in N \mid F(x) \in M \otimes_A I^i\},$$

where we view $M \otimes_A I^i$ as a subset of $M[1/I]$. Let m be an integer. Then the following are equivalent:

(1) $\mathrm{Fil}^{m+1}(N) \subset IN$.

(2) $M \otimes_A I^m \subset F(N)$.

If these equivalent conditions are satisfied, then F restricts to an isomorphism $\mathrm{Fil}^m(N) \xrightarrow{\sim} M \otimes_A I^m$, and in particular $\mathrm{Fil}^m(N)$ is a finite projective A-module.

Proof. The final statement clearly follows from (2). We shall prove that (1) and (2) are equivalent. For this, we can reduce to the case where $I = (d)$ is principal.

Assume that (1) holds. Let $x \in M$. We want to show that $d^m x \in F(N)$. For a large enough integer n, we have $d^n x \in F(N)$. Let $y \in N$ be an element such that $F(y) = d^n x$. If $n \geq m+1$, then we have $y \in \mathrm{Fil}^{m+1}(N) \subset IN$, which in turn implies that $d^{n-1}x \in F(N)$. From this observation, we can conclude that $d^m x \in F(N)$.

Assume that (2) holds. Let $y \in \mathrm{Fil}^{m+1}(N)$. There exists an element $x \in M$ such that $F(y) = d^{m+1}x$. The condition (2) implies that $d^m x = F(z)$ for some $z \in N$. It then follows that $y = dz \in IN$. □

Let (A, I) be a bounded $\mathcal{O}_E$-prism.

Definition 3.1.2. A *Breuil–Kisin module* over (A, I) is a pair (M, F_M) consisting of a finite projective A-module M and an $A[1/I]$-linear isomorphism

$$F_M : (\phi^* M)[1/I] \xrightarrow{\sim} M[1/I],$$

where $\phi^* M := A \otimes_{\phi, A} M$. When there is no possibility of confusion, we simply write M instead of (M, F_M). For an integer i, we set

$$\mathrm{Fil}^i(\phi^* M) := \{x \in \phi^* M \mid F_M(x) \in M \otimes_A I^i\}.$$

Let $P^i \subset (\phi^* M)/I(\phi^* M)$ be the image of $\mathrm{Fil}^i(\phi^* M)$. We often write

$$M_{\mathrm{dR}} := (\phi^* M)/I(\phi^* M).$$

Remark 3.1.3. If $F_M(\phi^* M) \subset M$, then we say that M is *effective*. In this case, the induced homomorphism $\phi^* M \to M$ is again denoted by F_M. The cokernel of $F_M : \phi^* M \to M$ is killed by some power of I.

Conversely, for a finite projective A-module M and a homomorphism $F_M : \phi^* M \to M$ of A-modules whose cokernel is killed by some power of I, the induced homomorphism $(\phi^* M)[1/I] \to M[1/I]$ is an isomorphism. Indeed, it is clear that $(\phi^* M)[1/I] \to M[1/I]$ is surjective, which in turn implies that it is an isomorphism since (the vector bundles on Spec A associated with) $\phi^* M$ and M have the same rank. In particular, it follows that $F_M : \phi^* M \to M$ is injective.

Remark 3.1.4. For any integer i, we have $I\,\mathrm{Fil}^{i-1}(\phi^* M) = \mathrm{Fil}^i(\phi^* M) \cap I(\phi^* M)$. In other words, the natural homomorphism $\mathrm{Fil}^i(\phi^* M)/I\,\mathrm{Fil}^{i-1}(\phi^* M) \to P^i$ is bijective. We have $P^i = M_{\mathrm{dR}}$ for small enough i and $P^i = 0$ for large enough i.

Definition 3.1.5. Let M be a Breuil–Kisin module over (A, I). We say that M is *displayed* if the A/I-submodule $P^i \subset M_{\mathrm{dR}}$ is a direct summand for every i. In this case, the filtration $\{P^i\}_{i \in \mathbb{Z}}$ is called the *Hodge filtration*. We say that M is *minuscule* if it is displayed, and if we have $P^i = M_{\mathrm{dR}}$ for any $i \leq 0$ and $P^i = 0$ for any $i \geq 2$.

The following proposition, which is basically a consequence of [Anschütz and Le Bras 2023, Remark 4.25], shows that the definition of a minuscule Breuil–Kisin module given in Definition 3.1.5 agrees with the usual one employed in the literature (for example in [Kisin 2006, Section 2.2] and [Anschütz and Le Bras 2023, Definition 4.24]).

Proposition 3.1.6. *Let M be a Breuil–Kisin module over (A, I). The following statements are equivalent*:

(1) *M is minuscule.*

(2) *M is effective, and the cokernel* Coker F_M *of $F_M : \phi^* M \to M$ is killed by I.*

Proof. Assume that (1) holds. It follows from $P^0 = M_{\mathrm{dR}}$ and Nakayama's lemma that $\mathrm{Fil}^0(\phi^* M) = \phi^* M$. Moreover, we have $IM \subset F_M(\phi^* M)$ by Lemma 3.1.1. This proves that (1) implies (2).

Assume that (2) holds. It follows from Lemma 3.1.1 that $\mathrm{Fil}^2(\phi^*M) \subset I(\phi^*M)$, and hence $P^i = 0$ for any $i \geq 2$. Since M is effective, we have $P^i = M_{\mathrm{dR}}$ for any $i \leq 0$. It remains to prove that P^1 is a direct summand of M_{dR}. For this, it suffices to show that $(\phi^*M)/\mathrm{Fil}^1(\phi^*M)$ is projective as an A/I-module. Since we have the exact sequence of A/I-modules

$$0 \to (\phi^*M)/\mathrm{Fil}^1(\phi^*M) \to M/IM \to \mathrm{Coker}\, F_M \to 0,$$

it suffices to prove that $\mathrm{Coker}\, F_M$ is a projective A/I-module. With Lemma 3.1.7 below, this follows from the same argument as in [Anschütz and Le Bras 2023, Remark 4.25]. ☐

Lemma 3.1.7. *Let (A, I) be an $\mathcal{O}_E$-prism. For a perfect field k containing $\mathbb{F}_q$ and a homomorphism $g : A/I \to k$ of $\mathcal{O}_E$-algebras, there exists a map $(A, I) \to (\mathcal{O}, (\pi))$ of $\mathcal{O}_E$-prisms which induces g, where $\mathcal{O} := W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.*

Proof. Let $A_{\mathrm{perf}} := \varinjlim_\phi A$ be a colimit of the diagram $A \xrightarrow{\phi} A \xrightarrow{\phi} A \to \cdots$, which is a perfect δ_E-ring. Since k is perfect, the homomorphism $A/\pi \to k$ induced by the composition $A \to A/I \to k$ factors through a homomorphism $A_{\mathrm{perf}}/\pi \to k$. This homomorphism lifts uniquely to a homomorphism $A_{\mathrm{perf}} \to \mathcal{O}$ of δ_E-rings by Lemma 2.3.5. The composition $A \to A_{\mathrm{perf}} \to \mathcal{O}$ gives a map $(A, I) \to (\mathcal{O}, (\pi))$ which induces g, as desired. ☐

We shortly discuss the relation between the notion of minuscule Breuil–Kisin modules and that of windows introduced by Zink and Lau. We recall the notion of windows, adapted to our context. Let (A, d) be an oriented and bounded $\mathcal{O}_E$-prism.

Definition 3.1.8. A *window* over (A, d) is a quadruple

$$\underline{N} = (N, \mathrm{Fil}^1(N), \Phi, \Phi_1),$$

where N is a finite projective A-module, $\mathrm{Fil}^1(N) \subset N$ is an A-submodule, $\Phi : N \to N$ and $\Phi_1 : \mathrm{Fil}^1(N) \to N$ are ϕ-linear homomorphisms, such that the following conditions hold:

(1) We have $dN \subset \mathrm{Fil}^1(N)$, and $\Phi(x) = \Phi_1(dx)$ for every $x \in N$.

(2) The image $P^1 \subset N/dN$ of $\mathrm{Fil}^1(N)$ is a direct summand of N/dN.

(3) The linearization $1 \otimes \Phi_1 : \phi^* \mathrm{Fil}^1(N) \to N$ of Φ_1 is an isomorphism.

Proposition 3.1.9 (cf. [Cais and Lau 2017, Lemma 2.1.16; Anschütz and Le Bras 2023, Proposition 4.26]). *For a window $\underline{N}$ over (A, d), the pair $(\mathrm{Fil}^1(N), F)$, where $F : \phi^* \mathrm{Fil}^1(N) \to \mathrm{Fil}^1(N)$ is defined by $F = d(1 \otimes \Phi_1)$, is a minuscule Breuil–Kisin module over $(A, (d))$. This construction gives an equivalence between the category of windows over (A, d) and the category of minuscule Breuil–Kisin modules over $(A, (d))$.*

Proof. By virtue of Proposition 3.1.6, we can use the same argument as in the proof of [Cais and Lau 2017, Lemma 2.1.16]. ☐

We study the structure of displayed Breuil–Kisin modules. For this, we introduce the following definition. Let (A, I) be a bounded $\mathcal{O}_E$-prism.

Definition 3.1.10. Let M be a Breuil–Kisin module over (A, I). A decomposition $\phi^*M = \bigoplus_{j\in\mathbb{Z}} L_j$ is called a *normal decomposition* if the isomorphism $F_M : (\phi^*M)[1/I] \xrightarrow{\sim} M[1/I]$ restricts to an isomorphism

$$\bigoplus_{j\in\mathbb{Z}} (L_j \otimes_A I^{-j}) \xrightarrow{\sim} M$$

of A-modules.

Remark 3.1.11. If $\phi^*M = \bigoplus_{j\in\mathbb{Z}} L_j$ is a normal decomposition, then we have

$$\mathrm{Fil}^i(\phi^*M) = \Big(\bigoplus_{j\geq i} L_j\Big) \oplus \Big(\bigoplus_{j<i} I^{i-j} L_j\Big)$$

for every $i \in \mathbb{Z}$. In particular, a Breuil–Kisin module over (A, I) which admits a normal decomposition is displayed. In the next lemma, we shall prove that the converse is also true.

Lemma 3.1.12. *Let M be a displayed Breuil–Kisin module over (A, I). Then there exists a normal decomposition $\phi^*M = \bigoplus_{j\in\mathbb{Z}} L_j$.*

Proof. We choose a decomposition $(\phi^*M)/I(\phi^*M) = \bigoplus_{j\in\mathbb{Z}} K_j$ such that $P^i = \bigoplus_{j\geq i} K_j$ for every i. Since K_j is a finite projective A/I-module and A is I-adically complete, there exists a finite projective A-module L_j such that $L_j/IL_j \simeq K_j$ for every j; see [Stacks 2005–, Tag 0D4A] or [Greco 1968, Theorem 5.1] for example. Moreover we have $L_j = 0$ for all but finitely many j. Since $\mathrm{Fil}^i(\phi^*M) \to P^i$ is surjective, there exists a homomorphism $L_i \to \mathrm{Fil}^i(\phi^*M)$ which fits into the following commutative diagram:

$$\begin{array}{ccc} L_i & \longrightarrow & K_i \\ \downarrow & & \downarrow \\ \mathrm{Fil}^i(\phi^*M) & \longrightarrow & P^i \end{array}$$

The induced homomorphism $\bigoplus_{j\in\mathbb{Z}} L_j \to \phi^*M$ is an isomorphism since it is a lift of the isomorphism $\bigoplus_{j\in\mathbb{Z}} K_j \xrightarrow{\sim} (\phi^*M)/I(\phi^*M)$. We shall prove that, under this isomorphism, $\mathrm{Fil}^i(\phi^*M)$ coincides with $\big(\bigoplus_{j\geq i} L_j\big) \oplus \big(\bigoplus_{j<i} I^{i-j} L_j\big)$ for any $i \in \mathbb{Z}$. This implies that $\bigoplus_{j\in\mathbb{Z}} L_j$ is a normal decomposition.

We proceed by induction on i. The assertion clearly holds for small enough i. Let us assume that the assertion holds for an integer i. Since

$$\Big(\bigoplus_{j\geq i} IL_j\Big) \oplus \Big(\bigoplus_{j<i} I^{i+1-j} L_j\Big) = I\,\mathrm{Fil}^i(\phi^*M) \subset \mathrm{Fil}^{i+1}(\phi^*M)$$

and $\bigoplus_{j\geq i+1} L_j \subset \mathrm{Fil}^{i+1}(\phi^*M)$ by construction, we obtain

$$\Big(\bigoplus_{j\geq i+1} L_j\Big) \oplus \Big(\bigoplus_{j<i+1} I^{i+1-j} L_j\Big) \subset \mathrm{Fil}^{i+1}(\phi^*M).$$

The left-hand side contains $I\,\mathrm{Fil}^i(\phi^*M)$ and the quotient by $I\,\mathrm{Fil}^i(\phi^*M)$ is equal to P^{i+1}. The same holds for the right-hand side by Remark 3.1.4. Therefore, this inclusion is actually an equality. □

Let $f:(A,I)\to(A',I')$ be a map of bounded $\mathcal{O}_E$-prisms. For a Breuil–Kisin module (M,F_M) over (A,I), let $F_{M_{A'}}:(\phi^*(M_{A'}))[1/I']\xrightarrow{\sim} M_{A'}[1/I']$ be the base change of F_M, where $M_{A'}:=M\otimes_A A'$. We also write f^*M for $(M_{A'},F_{M_{A'}})$.

Proposition 3.1.13. *Let (M,F_M) be a Breuil–Kisin module over (A,I).*

(1) *Assume that (M,F_M) is displayed. Then $(M_{A'},F_{M_{A'}})$ is a displayed Breuil–Kisin module over (A',I'), and we have $\mathrm{Fil}^i(\phi^*M)\otimes_A A'\xrightarrow{\sim}\mathrm{Fil}^i(\phi^*(M_{A'}))$ for any integer i.*

(2) *Assume that $(M_{A'},F_{M_{A'}})$ is displayed and $f:(A,I)\to(A',I')$ is a faithfully flat map of $\mathcal{O}_E$-prisms. Then (M,F_M) is displayed.*

Proof. (1) This follows from Remark 3.1.11, Lemma 3.1.12, and the fact that normal decompositions are preserved under base change.

(2) We note that $\mathrm{Fil}^i(\phi^*(M_{A'}))$ is a finite projective A'-module for any i by Lemma 3.1.12 and Remark 3.1.11. Since $\mathrm{Fil}^i(\phi^*(M_{A'}))$ is stable under the natural descent datum of $\phi^*(M_{A'})$ (with respect to the flat covering $(A,I)\to(A',I')$) by (1), it follows from Proposition 2.5.6 that there is a descending filtration $\{\mathrm{Fil}^i\}_{i\in\mathbb{Z}}$ of ϕ^*M by finite projective A-submodules such that $\mathrm{Fil}^i\otimes_A A'\to\phi^*(M_{A'})$ induces an isomorphism $\mathrm{Fil}^i\otimes_A A'\xrightarrow{\sim}\mathrm{Fil}^i(\phi^*(M_{A'}))$ for any i.

Let m be an integer such that $M\otimes_A I^m\subset F_M(\phi^*M)$. Then $\mathrm{Fil}^m=\mathrm{Fil}^m(\phi^*M)$ (see Lemma 3.1.1). Moreover, we have $I\,\mathrm{Fil}^{i-1}\subset\mathrm{Fil}^i$ for any i, and $I\,\mathrm{Fil}^{i-1}=\mathrm{Fil}^i$ for $i\geq m+1$. In particular, we obtain $\mathrm{Fil}^i=\mathrm{Fil}^i(\phi^*M)$ for $i\geq m$.

Let i be any integer. We claim that the natural homomorphism of A/I-modules

$$\iota:\mathrm{Fil}^i/I\,\mathrm{Fil}^{i-1}\to(\phi^*M)/I(\phi^*M)$$

is injective and its cokernel is a finite projective A/I-module. Indeed, it suffices to show that for every closed point $x\in\operatorname{Spec}A/I$, the base change of ι to the residue field $k(x)$ is injective. Since x is contained in $\operatorname{Spec}A/(\pi,I)$ and $\operatorname{Spec}A'/(\pi,I')\to\operatorname{Spec}A/(\pi,I)$ is surjective, it is enough to prove that the base change of ι along $A/I\to A'/I'$ is injective and its cokernel is a finite projective A'/I'-module. This follows from the assumption that $(M_{A'},F_{M_{A'}})$ is displayed.

It follows from the claim that $I\,\mathrm{Fil}^{i-1}=I(\phi^*M)\cap\mathrm{Fil}^i$, or equivalently, $\mathrm{Fil}^{i-1}=\phi^*M\cap(\mathrm{Fil}^i\otimes_A I^{-1})$. Since $\mathrm{Fil}^i=\mathrm{Fil}^i(\phi^*M)$ for $i\geq m$, we can conclude that $\mathrm{Fil}^i=\mathrm{Fil}^i(\phi^*M)$ for any i. This, together with the claim, shows that (M,F_M) is displayed. □

Remark 3.1.14. The functor

$$(\mathcal{O}_E)_{\Delta,\mathcal{O}_E}\to\mathrm{Set},\quad (B,J)\mapsto B[1/J],$$

forms a sheaf (with respect to the flat topology) by Lemma 2.3.4 and Remark 2.5.5. Thus for finite projective A-modules M,M', the functor $(A,I)_\Delta\to\mathrm{Set}$ which associates to each $(B,J)\in(A,I)_\Delta$ the set of isomorphisms $M_B[1/J]\xrightarrow{\sim} M'_B[1/J]$ forms a sheaf. This fact, together with Proposition 2.5.6, implies that the fibered category over $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to a bounded $\mathcal{O}_E$-prism (A,I) the category of Breuil–Kisin modules over (A,I) satisfies descent with respect to the flat topology.

Corollary 3.1.15. *The fibered category over $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to a bounded $\mathcal{O}_E$-prism (A, I) the category of displayed Breuil–Kisin modules over (A, I) satisfies descent with respect to the flat topology.*

Proof. This follows from Proposition 3.1.13 and Remark 3.1.14. □

We finish this subsection by giving an example of a Breuil–Kisin module which is not displayed.

Example 3.1.16. Let (A, I) be an orientable and bounded $\mathcal{O}_E$-prism. We assume that A/I is π-torsion-free and $A/I \neq 0$. We set $M := A^2$ and let $F_M : \phi^* M \to M$ be the homomorphism defined by the matrix $\begin{pmatrix} \pi & d \\ d & d^2 \end{pmatrix}$ for a generator $d \in I$. The pair (M, F_M) is a Breuil–Kisin module over (A, I). We claim that P^1/P^2 is not π-torsion-free, and thus (M, F_M) is not displayed. Indeed, since $(d, 1) \in \mathrm{Fil}^1(\phi^* M)$, we have $(0, 1) \in P^1 \subset M_{\mathrm{dR}}$. One can check that the image of $(0, 1)$ in P^1/P^2 is not zero and is killed by π.

3.2. ***Breuil–Kisin modules of type μ.*** Here we introduce the notion of Breuil–Kisin modules *of type* μ. Let

$$\mu : \mathbb{G}_m \to \mathrm{GL}_{n,\mathcal{O}}$$

be a cocharacter defined over $\mathcal{O}$, where $\mathcal{O} = W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ is as in Definition 2.3.7. There is a unique tuple $(m_1, \dots, m_n)$ of integers $m_1 \geq \cdots \geq m_n$ such that μ is conjugate to the cocharacter defined by $t \mapsto \mathrm{diag}\,(t^{m_1}, \dots, t^{m_n})$. By abuse of notation, the tuple $(m_1, \dots, m_n)$ is also denoted by μ. Let $r_i \in \mathbb{Z}_{\geq 0}$ be the number of occurrences of i in $(m_1, \dots, m_n)$. We set $L := \mathcal{O}_E^n$ and $L_{\mathcal{O}} := L \otimes_{\mathcal{O}_E} \mathcal{O}$. The cocharacter μ induces an action of $\mathbb{G}_m$ on $L_{\mathcal{O}}$. We have the weight decomposition

$$L_{\mathcal{O}} = \bigoplus_{j \in \mathbb{Z}} L_{\mu,j},$$

where an element $t \in \mathbb{G}_m(\mathcal{O}) = \mathcal{O}^\times$ acts on $L_{\mu,j}$ by $x \mapsto t^j x$ for every $j \in \mathbb{Z}$. (See for example [Conrad et al. 2015, Lemma A.8.8] for the existence of the weight decomposition over a ring.) The rank of $L_{\mu,j}$ is equal to r_j.

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Definition 3.2.1. Let M be a displayed Breuil–Kisin module over (A, I). We say that M is *of type* μ if, for the Hodge filtration $\{P^i\}_{i\in\mathbb{Z}}$, the successive quotient P^i/P^{i+1} is of rank r_i (i.e., the corresponding vector bundle on $\mathrm{Spec}\, A/I$ has constant rank r_i) for every i. We say that M is *banal* if all successive quotients P^i/P^{i+1} are free A/I-modules.

We write $\mathrm{BK}_\mu(A, I)$ (resp. $\mathrm{BK}_\mu(A, I)_{\mathrm{banal}}$) for the category of Breuil–Kisin modules over (A, I) of type μ (resp. banal Breuil–Kisin modules over (A, I) of type μ).

Remark 3.2.2. We set

$$\mathrm{Fil}^i_\mu := \Big(\bigoplus_{j \geq i} (L_{\mu,j})_A\Big) \oplus \Big(\bigoplus_{j<i} I^{i-j}(L_{\mu,j})_A\Big) \subset A^n,$$

where $(L_{\mu,j})_A := L_{\mu,j} \otimes_{\mathcal{O}} A$. Let $M \in \mathrm{BK}_\mu(A, I)_{\mathrm{banal}}$. Let $\phi^* M = \bigoplus_{j\in\mathbb{Z}} L_j$ be a normal decomposition. Then, each L_j is a free A-module of rank r_j. Thus there is an isomorphism $A^n \simeq \phi^* M$ such that the filtration $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ coincides with $\{\mathrm{Fil}^i(\phi^* M)\}_{i\in\mathbb{Z}}$.

Remark 3.2.3. Let M be a displayed Breuil–Kisin module over (A, I). Then there exists a (π, I)-completely étale covering $A \to A_1 \times \cdots \times A_m$ such that for each $1 \le j \le m$, the base change of M to the bounded $\mathcal{O}_E$-prism (A_j, IA_j) (see Lemma 2.5.10) is of type μ for some μ and banal. Indeed, by Lemma 2.5.9, it suffices to prove that there exists an étale and faithfully flat homomorphism $A/(\pi, I) \to B_1 \times \cdots \times B_m$ such that, for all $1 \le j \le m$ and i, the base change $P^i/P^{i+1} \otimes_{A/I} B_j$ is free over B_j, which is clear.

4. Display group

Let G be a smooth affine group scheme over $\mathcal{O}_E$. Let k be a perfect field containing $\mathbb{F}_q$ and we set $\mathcal{O} := W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. Let

$$\mu : \mathbb{G}_m \to G_{\mathcal{O}} := G \times_{\operatorname{Spec} \mathcal{O}_E} \operatorname{Spec} \mathcal{O}$$

be a cocharacter defined over $\mathcal{O}$. In this section, we introduce the display group $G_\mu(A, I)$ for an orientable and bounded $\mathcal{O}_E$-prism (A, I) over $\mathcal{O}$. The display group will be used in the definition of prismatic G-μ-displays.

4.1. *Definition of the display group.* Let A be an $\mathcal{O}$-algebra with an ideal $I \subset A$ which is generated by a nonzerodivisor $d \in A$.

Definition 4.1.1. We define

$$G_\mu(A, I) := \{g \in G(A) \mid \mu(d) g \mu(d)^{-1} \text{ lies in } G(A) \subset G(A[1/I])\}.$$

The group $G_\mu(A, I)$ is called the *display group*. We note that $G_\mu(A, I)$ does not depend on the choice of d.

Remark 4.1.2. The definition of the display group given here is a translation of the one given in [Lau 2021] to our setting; see Remark 5.2.3 for details. If G is reductive and μ is minuscule, such a group was also considered in [Bültel and Pappas 2020].

For the cocharacter $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$, we endow $G_{\mathcal{O}}$ with the action of $\mathbb{G}_m$ defined by

$$G_{\mathcal{O}}(R) \times \mathbb{G}_m(R) \to G_{\mathcal{O}}(R), \quad (g, t) \mapsto \mu(t)^{-1} g \mu(t), \tag{4-1}$$

for every $\mathcal{O}$-algebra R. We note that this action is the inverse of the one used in Definition 4.1.1. We write $G = \operatorname{Spec} A'_G$ and $A_G := A'_G \otimes_{\mathcal{O}_E} \mathcal{O}$, so that $G_{\mathcal{O}} = \operatorname{Spec} A_G$. Let

$$A_G = \bigoplus_{i \in \mathbb{Z}} A_{G,i}$$

be the weight decomposition with respect to the action of $\mathbb{G}_m$. An element $t \in \mathbb{G}_m(R) = R^\times$ acts on $A_{G,i} \otimes_{\mathcal{O}} R$ by $x \mapsto t^i x$.

Remark 4.1.3. Let R be an $\mathcal{O}$-algebra. For any $t \in \mathbb{G}_m(R)$ and any $g \in G_{\mathcal{O}}(R)$ with corresponding homomorphism $g^* : A_G \to R$, the homomorphism

$$(\mu(t)^{-1} g \mu(t))^* : A_G \to R$$

corresponding to $\mu(t)^{-1} g \mu(t) \in G_{\mathcal{O}}(R)$ sends an element $x \in A_{G,i}$ to $t^i g^*(x) \in R$.

Lemma 4.1.4. *Let $g \in G(A)$ be an element. Then $g \in G_\mu(A, I)$ if and only if $g^*(x) \in I^i A$ for every $i > 0$ and every $x \in A_{G,i}$.*

Proof. This follows from Remark 4.1.3. □

Example 4.1.5. Assume that $G = \mathrm{GL}_n$. Let $\mu : \mathbb{G}_m \to \mathrm{GL}_{n,\mathcal{O}}$ be a cocharacter and let $(m_1, \dots, m_n)$ be the corresponding tuple of integers $m_1 \geq \cdots \geq m_n$ as in Section 3.2. Let $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ be the filtration of $M := A^n$ defined as in Remark 3.2.2. Then we have

$$(\mathrm{GL}_n)_\mu(A, I) = \{g \in \mathrm{GL}_n(A) \mid g(\mathrm{Fil}^i_\mu) = \mathrm{Fil}^i_\mu \text{ for every } i \in \mathbb{Z}\}.$$

Let $d \in I$ be a generator. For any $g \in (\mathrm{GL}_n)_\mu(A, I)$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Fil}^{m_1}_\mu & \xrightarrow{\quad g \quad} & \mathrm{Fil}^{m_1}_\mu \\ \simeq\downarrow & & \downarrow\simeq \\ M & \xrightarrow{\mu(d) g \mu(d)^{-1}} & M \end{array}$$

where $\mathrm{Fil}^{m_1}_\mu \xrightarrow{\sim} M$ is defined by $d^{-m_1}\mu(d)$.

4.2. *Properties of the display group.* For the cocharacter $\mu : \mathbb{G}_m \to G_\mathcal{O}$, we consider the closed subgroup schemes $P_\mu, U^-_\mu \subset G_\mathcal{O}$ over $\mathcal{O}$ defined by, for every $\mathcal{O}$-algebra R,

$$P_\mu(R) = \bigl\{g \in G(R) \mid \lim_{t\to 0} \mu(t) g \mu(t)^{-1} \text{ exists}\bigr\},$$
$$U^-_\mu(R) = \bigl\{g \in G(R) \mid \lim_{t\to 0} \mu(t)^{-1} g \mu(t) = 1\bigr\}.$$

(We refer to [Conrad et al. 2015, Lemma 2.1.4] for the definition of $\lim_{t\to 0} \mu(t) g \mu(t)^{-1}$.) We see that P_μ and U^-_μ are stable under the action of $\mathbb{G}_m$ on $G_\mathcal{O}$ given by (4-1). The group schemes P_μ and U^-_μ are smooth over $\mathcal{O}$. Moreover, the multiplication map

$$U^-_\mu \times_{\operatorname{Spec} \mathcal{O}} P_\mu \to G_\mathcal{O}$$

is an open immersion. See [Conrad et al. 2015, Section 2.1], especially Proposition 2.1.8 of that work, for details.

Remark 4.2.1. We have employed slightly different notation than in [Lau 2021]. For example, in that work, the subgroup P_μ (resp. U^-_μ) is denoted by P^- (resp. U^+).

Lemma 4.2.2. (1) *Let R be an $\mathcal{O}$-algebra and $g \in G_\mathcal{O}(R)$ an element. Then $g \in P_\mu(R)$ if and only if $g^*(x) = 0$ for every $i > 0$ and every $x \in A_{G,i}$.*

(2) *We have $P_\mu(A) \subset G_\mu(A, I)$, and the image of $G_\mu(A, I)$ in $G(A/I)$ under the projection $G(A) \to G(A/I)$ is contained in $P_\mu(A/I)$. Moreover $\mu(d) P_\mu(A) \mu(d)^{-1}$ is contained in $P_\mu(A)$.*

Proof. Remark 4.1.3 immediately implies (1). Assertion (2) follows from (1) and Lemma 4.1.4. □

Definition 4.2.3 [Lau 2021, Definition 6.3.1]. The action of $\mathbb{G}_m$ on $G_{\mathcal{O}}$ given in (4-1) induces an action of $\mathbb{G}_m$ on the Lie algebra $\mathrm{Lie}(G_{\mathcal{O}})$. Let

$$\mathrm{Lie}(G_{\mathcal{O}}) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

be the weight decomposition with respect to the action of $\mathbb{G}_m$. We say that the cocharacter $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ is 1-*bounded* if $\mathfrak{g}_i = 0$ for $i \geq 2$.

In general, the Lie algebra $\mathrm{Lie}(U_\mu^-)$ of U_μ^- coincides with $\bigoplus_{i \geq 1} \mathfrak{g}_i$. (We also note that $\mathrm{Lie}(P_\mu) = \bigoplus_{i \leq 0} \mathfrak{g}_i$.) Thus μ is 1-bounded if and only if $\mathrm{Lie}(U_\mu^-) = \mathfrak{g}_1$.

Remark 4.2.4. If G is a reductive group scheme over $\mathcal{O}_E$, then μ is 1-bounded if and only if μ is minuscule, that is, the equality $\mathrm{Lie}(G_{\mathcal{O}}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ holds.

Example 4.2.5. Assume that $G = \mathrm{GL}_n$. The cocharacter $\mathbb{G}_m \to \mathrm{GL}_{n,\mathcal{O}}$ defined by

$$t \mapsto \mathrm{diag}\,(\underbrace{t^m, \dots, t^m}_{s}, \underbrace{t^{m-1}, \dots, t^{m-1}}_{n-s})$$

for some integers m and s $(0 \leq s \leq n)$ is 1-bounded. In fact, any 1-bounded cocharacter of $\mathrm{GL}_{n,\mathcal{O}}$ is conjugate to a cocharacter of this form.

For a free $\mathcal{O}$-module M of finite rank, we let $V(M)$ denote the group scheme over $\mathcal{O}$ defined by $R \mapsto M \otimes_{\mathcal{O}} R$ for every $\mathcal{O}$-algebra R.

Lemma 4.2.6. *There exists a $\mathbb{G}_m$-equivariant isomorphism*

$$\log : U_\mu^- \xrightarrow{\sim} V(\mathrm{Lie}(U_\mu^-))$$

of schemes over $\mathcal{O}$ which induces the identity on the Lie algebras. If μ is 1*-bounded, then the isomorphism* log *is unique, and it is an isomorphism of group schemes over $\mathcal{O}$.*

Proof. The same arguments as in the proofs of [Lau 2021, Lemmas 6.1.1 and 6.3.2] work here. □

Remark 4.2.7. (1) An isomorphism $\log : U_\mu^- \xrightarrow{\sim} V(\mathrm{Lie}(U_\mu^-))$ as in Lemma 4.2.6 induces a bijection

$$U_\mu^-(A) \cap G_\mu(A, I) \xrightarrow{\sim} \bigoplus_{i \geq 1} I^i(\mathfrak{g}_i \otimes_{\mathcal{O}} A).$$

(2) If μ is 1-bounded, then we identify U_μ^- with $V(\mathrm{Lie}(U_\mu^-))$ by the unique isomorphism log. In particular, we view $\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A$ as a subgroup of $G(A)$. We then obtain

$$I(\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A) = (\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A) \cap G_\mu(A, I).$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} I(\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A) & \hookrightarrow & G_\mu(A, I) \\ \downarrow {\scriptstyle dv \mapsto v} & & \downarrow {\scriptstyle g \mapsto \mu(d) g \mu(d)^{-1}} \\ \mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A & \hookrightarrow & G(A) \end{array}$$

Proposition 4.2.8 (cf. [Lau 2021, Lemma 6.2.2]). *Assume that A is I-adically complete. Then the multiplication map*

$$(U_\mu^-(A) \cap G_\mu(A, I)) \times P_\mu(A) \to G_\mu(A, I) \tag{4-2}$$

is bijective.

Proof. Since $P_\mu(A) \subset G_\mu(A, I)$ by Lemma 4.2.2, the map (4-2) is well-defined. Since the map $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu \to G_\mathcal{O}$ is an open immersion, the map (4-2) is injective.

We shall show that the map (4-2) is surjective. Let $g \in G_\mu(A, I)$ be an element. By Lemma 4.2.2, the image of g in $G(A/I)$ is contained in $P_\mu(A/I)$. Since P_μ is smooth and A is I-adically complete, there exists an element $t \in P_\mu(A)$ whose image in $P_\mu(A/I)$ coincides with the image of g. The restriction of the morphism $gt^{-1} : \operatorname{Spec} A \to G_\mathcal{O}$ to $\operatorname{Spec} A/I$ factors through the open subscheme $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu$. Since $I \subset \operatorname{rad}(A)$, it follows that $gt^{-1} : \operatorname{Spec} A \to G_\mathcal{O}$ itself factors through $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu$. In other words, there are elements $u \in U_\mu^-(A)$ and $t' \in P_\mu(A)$ such that $g = ut't$. We note that $u \in G_\mu(A, I)$. In conclusion, we have shown that g is contained in the image of the map (4-2). □

Proposition 4.2.9. *Assume that $\mu : \mathbb{G}_m \to G_\mathcal{O}$ is 1-bounded. Assume further that A is I-adically complete. Then the multiplication map*

$$I(\operatorname{Lie}(U_\mu^-) \otimes_\mathcal{O} A) \times P_\mu(A) \to G_\mu(A, I) \tag{4-3}$$

is bijective. Moreover $G_\mu(A, I)$ coincides with the inverse image of $P_\mu(A/I)$ in $G(A)$ under the projection $G(A) \to G(A/I)$, and we have the bijection

$$G(A)/G_\mu(A, I) \xrightarrow{\sim} G(A/I)/P_\mu(A/I). \tag{4-4}$$

Proof. It follows from Remark 4.2.7 and Proposition 4.2.8 that the map (4-3) is bijective. Let $G'_\mu \subset G(A)$ be the inverse image of $P_\mu(A/I)$. We have $G_\mu(A, I) \subset G'_\mu$. By the same argument as in the proof of Proposition 4.2.8, one can show that $G'_\mu \subset I(\operatorname{Lie}(U_\mu^-) \otimes_\mathcal{O} A) \times P_\mu(A)$. Thus, we obtain $G_\mu(A, I) = G'_\mu$.

It remains to prove that the map (4-4) is bijective. Since G is smooth and A is I-adically complete, the projection $G(A) \to G(A/I)$ is surjective, which in turn implies the surjectivity of (4-4). The injectivity follows from the equality $G_\mu(A, I) = G'_\mu$. □

For an integer $m \geq 0$, let $G^{\geq m}(A)$ be the kernel of $G(A) \to G(A/I^m)$. We set

$$G_\mu^{\geq m}(A, I) := G_\mu(A, I) \cap G^{\geq m}(A).$$

We record a structural result about the quotient $G_\mu^{\geq m}(A, I)/G_\mu^{\geq m+1}(A, I)$.

Lemma 4.2.10. *Assume that A is I-adically complete. Then we have the isomorphisms of groups*

$$G^{\geq m}(A)/G^{\geq m+1}(A) \simeq \begin{cases} G(A/I) & (m = 0), \\ \operatorname{Lie}(G_\mathcal{O}) \otimes_\mathcal{O} I^m/I^{m+1} & (m \geq 1), \end{cases}$$

$$G_\mu^{\geq m}(A, I)/G_\mu^{\geq m+1}(A, I) \simeq \begin{cases} P_\mu(A/I) & (m = 0), \\ \left(\bigoplus_{i \leq m} \mathfrak{g}_i\right) \otimes_\mathcal{O} I^m/I^{m+1} & (m \geq 1). \end{cases}$$

Proof. Since A is I-adically complete and G is smooth, the map $G(A) \to G(A/I^m)$ is surjective. It follows that $G^{\geq m}(A)/G^{\geq m+1}(A)$ is isomorphic to the kernel $\mathrm{Ker}(G(A/I^{m+1}) \to G(A/I^m))$ of $G(A/I^{m+1}) \to G(A/I^m)$. This is equal to $G(A/I)$ when $m = 0$. If $m \geq 1$, then we have a canonical identification

$$\mathrm{Ker}(G(A/I^{m+1}) \to G(A/I^m)) = \mathrm{Lie}(G_{\mathcal{O}}) \otimes_{\mathcal{O}} I^m/I^{m+1}$$

since $I^m/I^{m+1} \subset A/I^{m+1}$ is a square zero ideal. This proves the first assertion.

Since $P_\mu(A) \to P_\mu(A/I)$ is surjective (as A is I-adically complete and P_μ is smooth), it follows from Lemma 4.2.2 that $G_\mu(A, I)/G_\mu^{\geq 1}(A, I) \simeq P_\mu(A/I)$. To prove the second assertion, it then suffices to show that the image of the natural homomorphism

$$G_\mu^{\geq m}(A, I) \to \mathrm{Ker}(G(A/I^{m+1}) \to G(A/I^m)) = \mathrm{Lie}(G_{\mathcal{O}}) \otimes_{\mathcal{O}} I^m/I^{m+1}$$

is $\left(\bigoplus_{i \leq m} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m/I^{m+1}$ for any $m \geq 1$. By Proposition 4.2.8, we may identify $G_\mu^{\geq m}(A, I)$ with

$$(U_\mu^-(A) \cap G_\mu^{\geq m}(A, I)) \times P_\mu^{\geq m}(A),$$

where $P_\mu^{\geq m}(A) := P_\mu(A) \cap G^{\geq m}(A)$. By the same argument as above, we have

$$P_\mu^{\geq m}(A)/P_\mu^{\geq m+1}(A) \simeq \mathrm{Lie}(P_\mu) \otimes_{\mathcal{O}} I^m/I^{m+1} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m/I^{m+1}.$$

It now suffices to prove that the image of the natural homomorphism

$$U_\mu^-(A) \cap G_\mu^{\geq m}(A, I) \to \mathrm{Ker}(U_\mu^-(A/I^{m+1}) \to U_\mu^-(A/I^m)) = \mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} I^m/I^{m+1} \tag{4-5}$$

is $\left(\bigoplus_{1 \leq i \leq m} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m/I^{m+1}$. For this, we fix an isomorphism $\log : U_\mu^- \xrightarrow{\sim} V(\mathrm{Lie}(U_\mu^-))$ as in Lemma 4.2.6. Since log induces the identity on the Lie algebras, the isomorphism

$$\mathrm{Ker}(U_\mu^-(A/I^{m+1}) \to U_\mu^-(A/I^m)) \xrightarrow{\sim} \mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} I^m/I^{m+1}$$

induced by log is the same as the one in (4-5). Since log induces

$$U_\mu^-(A) \cap G_\mu^{\geq m}(A, I) \xrightarrow{\sim} \left(\bigoplus_{1 \leq i \leq m} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m \oplus \left(\bigoplus_{i \geq m+1} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^i,$$

by Remark 4.2.7(1), the result follows. □

4.3. ***Display groups on prismatic sites.*** In this subsection, for a bounded $\mathcal{O}_E$-prism (A, I) over $\mathcal{O}$, we define the display group sheaf $G_{\mu,A,I}$ on the site $(A, I)_{\text{ét}}^{\text{op}}$ and discuss some basic results on $G_{\mu,A,I}$-torsors.

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. We begin with a comparison result between torsors over $\mathrm{Spec}\, A$ (or $\mathrm{Spec}\, A/I$) with respect to the usual étale topology, and torsors on the sites $(A, I)_{\text{ét}}^{\text{op}}$ and $(A, I)_{\Delta}^{\text{op}}$ from Section 2.5. To an affine scheme X over $\mathcal{O}$ (or A), we attach a functor

$$X_{\Delta,A} : (A, I)_{\Delta} \to \mathrm{Set}, \quad (B, J) \mapsto X(B).$$

This forms a sheaf (with respect to the flat topology) by Remark 2.5.5 since $X(B)$ can be regarded as the set of homomorphisms $R \to B$ of $\mathcal{O}$-algebras (or A-algebras) where $X = \mathrm{Spec}\, R$. Similarly, to an affine scheme X over $\mathcal{O}$ (or A/I), we attach a sheaf

$$X_{\bar{\Delta},A} : (A, I)_{\Delta} \to \mathrm{Set}, \quad (B, J) \mapsto X(B/J).$$

The restrictions of these sheaves to $(A, I)_{\text{ét}}$ are denoted by the same notation (see also Remark 2.5.13).

Proposition 4.3.1. *Let H be a smooth affine group scheme over $\mathcal{O}$.*

(1) *For an $H_{A/I}$-torsor $\mathcal{P}$ over* Spec A/I *with respect to the étale topology, which is an affine scheme over A/I, the sheaf $\mathcal{P}_{\bar{\Delta},A}$ on $(A, I)^{\mathrm{op}}_{\Delta}$ is an $H_{\bar{\Delta},A}$-torsor with respect to the flat topology. The functor*

$$\mathcal{P} \mapsto \mathcal{P}_{\bar{\Delta},A}$$

is an equivalence from the groupoid of $H_{A/I}$-torsors over Spec A/I *to the groupoid of $H_{\bar{\Delta},A}$-torsors on $(A, I)^{\mathrm{op}}_{\Delta}$. The same holds if we replace $(A, I)^{\mathrm{op}}_{\Delta}$ by $(A, I)^{\mathrm{op}}_{\text{ét}}$.*

(2) *The construction*

$$\mathcal{P} \mapsto \mathcal{P}_{\Delta,A}$$

gives an equivalence from the groupoid of H_A-torsors over Spec A *to the groupoid of $H_{\Delta,A}$-torsors on $(A, I)^{\mathrm{op}}_{\Delta}$. The same holds if we replace $(A, I)^{\mathrm{op}}_{\Delta}$ by $(A, I)^{\mathrm{op}}_{\text{ét}}$.*

Proof. (1) It follows from Lemma 2.5.9 that $\mathcal{P}_{\bar{\Delta},A}$ is trivialized by a (π, I)-completely étale covering of A. Thus $\mathcal{P}_{\bar{\Delta},A}$ is an $H_{\bar{\Delta},A}$-torsor on both $(A, I)^{\mathrm{op}}_{\Delta}$ and $(A, I)^{\mathrm{op}}_{\text{ét}}$. It then suffices to prove that the fibered category over $(A, I)^{\mathrm{op}}_{\Delta}$ which associates to each $(B, J) \in (A, I)_{\Delta}$ the groupoid of $H_{B/J}$-torsors over Spec B/J is a stack with respect to the flat topology.

It is known that, for any $\mathcal{O}$-algebra R, the groupoid of H_R-torsors over Spec R is equivalent to the groupoid of exact tensor functors $\mathrm{Rep}_{\mathcal{O}}(H) \to \mathrm{Vect}(R)$, where $\mathrm{Rep}_{\mathcal{O}}(H)$ is the category of algebraic representations of H on free $\mathcal{O}$-modules of finite rank, and $\mathrm{Vect}(R)$ is the category of finite projective R-modules; see [Scholze and Weinstein 2020, Theorem 19.5.1] and [Broshi 2013, Theorem 1.2]. (Although this result is stated only for the case where $\mathcal{O} = \mathbb{Z}_p$ in [Scholze and Weinstein 2020, Theorem 19.5.1], the proof also works for general $\mathcal{O}$.) Using this Tannakian perspective, the desired claim follows from Proposition 2.5.6 and the following fact: For a π-completely faithfully flat homomorphism $C \to C'$ of π-adically complete $\mathcal{O}$-algebras, a complex

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of finite projective C-modules is exact if the base change

$$0 \to M_1 \otimes_C C' \to M_2 \otimes_C C' \to M_3 \otimes_C C' \to 0$$

is exact. (This fact follows from the following criterion: A complex $0 \to M_1 \to M_2 \to M_3 \to 0$ of finite projective modules over a ring C is exact if for every closed point $x \in \operatorname{Spec} C$, its base change to the residue field $k(x)$ is exact.)

(2) This can be proved in the same way as (1). □

Definition 4.3.2. Let $(A, I)_{\Delta,\mathrm{ori}}$ be the category of *orientable* and bounded $\mathcal{O}_E$-prisms (B, J) with a map $(A, I) \to (B, J)$. We endow $(A, I)^{\mathrm{op}}_{\Delta,\mathrm{ori}}$ with the flat topology. If (A, I) is orientable, then we have $(A, I)_{\Delta,\mathrm{ori}} = (A, I)_{\Delta}$.

Remark 4.3.3. By Remark 2.5.14, the objects in $(A, I)^{\mathrm{op}}_{\Delta,\mathrm{ori}}$ form a basis for $(A, I)^{\mathrm{op}}_{\Delta}$. We may identify sheaves on $(A, I)^{\mathrm{op}}_{\Delta,\mathrm{ori}}$ with sheaves on $(A, I)^{\mathrm{op}}_{\Delta}$.

Definition 4.3.4. We define the functor

$$G_{\Delta,A} : (A, I)_{\Delta} \to \mathrm{Set}, \quad (B, J) \mapsto G(B).$$

As explained above, the functor $G_{\Delta,A}$ forms a group sheaf. We also define the functor

$$G_{\mu,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G_{\mu}(B, J).$$

Since the functor $(A, I)_{\Delta} \to \mathrm{Set}$, $(B, J) \mapsto G(B[1/J])$ forms a group sheaf (Remark 3.1.14), it follows that $G_{\mu,A,I}$ forms a group sheaf. We regard $G_{\mu,A,I}$ as a group sheaf on $(A, I)^{\mathrm{op}}_{\Delta}$. The restrictions of $G_{\Delta,A}$ and $G_{\mu,A,I}$ to $(A, I)_{\text{ét}}$ will be denoted by the same notation.

We remark that Proposition 4.3.1 cannot be applied directly to $G_{\mu,A,I}$-torsors. However, it is still useful for analyzing $G_{\mu,A,I}$-torsors in several places below, since we have the following lemma. For the notation used below, see Lemma 4.2.10.

Lemma 4.3.5. (1) *For an integer $m \geq 0$, the functor*

$$G^{=m}_{\mu,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G^{\geq m}_{\mu}(B, J)/G^{\geq m+1}_{\mu}(B, J),$$

forms a group sheaf, and it is isomorphic to $(P_{\mu})_{\bar{\Delta},A}$ (resp. $V\big(\bigoplus_{i\leq m} \mathfrak{g}_i\big)_{\bar{\Delta},A}$) if $m = 0$ (resp. $m \geq 1$). Moreover, the functor

$$G^{<m}_{\mu,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G_{\mu}(B, J)/G^{\geq m}_{\mu}(B, J),$$

forms a group sheaf.

(2) *For a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\Delta}$, we write $\mathcal{Q}^{<m}$ for the pushout of $\mathcal{Q}$ along $G_{\mu,A,I} \to G^{<m}_{\mu,A,I}$* (see Remark 4.3.6 below). *Then we have $\mathcal{Q} \xrightarrow{\sim} \varprojlim_m \mathcal{Q}^{<m}$. The same holds for $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\text{ét}}$.*

Proof. (1) The statement about $G^{=m}_{\mu,A,I}$ follows from Lemma 4.2.10. Using the exact sequence

$$1 \mapsto G^{=m}_{\mu,A,I}(B) \to G^{<m+1}_{\mu,A,I}(B) \to G^{<m}_{\mu,A,I}(B) \to 1,$$

the statement about $G^{<m}_{\mu,A,I}$ then follows by induction on m.

(2) We may assume that $\mathcal{Q}$ is a trivial $G_{\mu,A,I}$-torsor and (A, I) is orientable. Then it is enough to prove that $G_{\mu,A,I} \xrightarrow{\sim} \varprojlim_m G^{<m}_{\mu,A,I}$ on $(A, I)^{\mathrm{op}}_{\Delta}$. By Proposition 4.2.8, the multiplication map

$$(U^{-}_{\mu}(A) \cap G_{\mu}(A, I)) \times P_{\mu}(A) \to G_{\mu}(A, I)$$

is bijective. Note that $G_{\mu}(A, I)/G^{\geq m}_{\mu}(A, I)$ can be identified with the image of $G_{\mu}(A, I)$ in $G(A/I^m)$. Let $U^{<m}$ be the image of $U^{-}_{\mu}(A) \cap G_{\mu}(A, I)$ in $U^{-}_{\mu}(A/I^m)$. Then the multiplication map induces a

bijection

$$U^{<m} \times P_\mu(A/I^m) \xrightarrow{\sim} G_\mu(A, I)/G_\mu^{\geq m}(A, I).$$

We have $P_\mu(A) \xrightarrow{\sim} \varprojlim P_\mu(A/I^m)$. Moreover, using Lemma 4.2.6, one can check that

$$U_\mu^-(A) \cap G_\mu(A, I) \xrightarrow{\sim} \varprojlim U^{<m}.$$

Thus, we obtain $G_\mu(A, I) \xrightarrow{\sim} \varprojlim G_\mu(A, I)/G_\mu^{\geq m}(A, I)$. The same assertion holds for any $(B, J) \in (A, I)_\Delta$, and hence $G_{\mu,A,I} \xrightarrow{\sim} \varprojlim G_{\mu,A,I}^{<m}$. □

Remark 4.3.6. Let $f : H \to H'$ be a homomorphism of groups and Q a set with an H-action. We can attach to Q a set Q^f with an H'-action and an H-equivariant map $Q \to Q^f$ with the following universal property: for any set Q' with an H'-action and any H-equivariant map $Q \to Q'$, the map $Q \to Q'$ factors through a unique H'-equivariant map $Q^f \to Q'$. Explicitly, we can define Q^f as the contracted product

$$Q^f = (Q \times H')/H,$$

where the action of an $h \in H$ on $Q \times H'$ is defined by $(x, h') \mapsto (xh, f(h)^{-1}h')$. We call Q^f the pushout of Q along $f : H \to H'$.

Similarly, for a homomorphism $f : H \to H'$ of group sheaves on a site and a sheaf Q with an action of H, we can form the pushout Q^f with the same properties as above. If Q is an H-torsor, then Q^f is an H'-torsor.

We will use the following notation. Let us denote the inclusion $G_{\mu,A,I} \hookrightarrow G_{\Delta,A}$ by τ. The composition of τ with the projection map $G_{\Delta,A} \to G_{\bar{\Delta},A}$ is denoted by $\bar{\tau}$. (Here $G_{\bar{\Delta},A} := (G_{\mathcal{O}})_{\bar{\Delta},A}$.) By Lemma 4.2.2, the homomorphism $\bar{\tau}$ factors through a homomorphism $\bar{\tau}_P : G_{\mu,A,I} \to (P_\mu)_{\bar{\Delta},A}$. In summary, we have the following commutative diagram of group sheaves on $(A, I)_\Delta^{\mathrm{op}}$ (or on $(A, I)_{\text{ét}}^{\mathrm{op}}$):

$$\begin{array}{ccc} G_{\mu,A,I} & \xrightarrow{\tau} & G_{\Delta,A} \\ {\scriptstyle \bar{\tau}_P}\downarrow & \searrow^{\bar{\tau}} & \downarrow \\ (P_\mu)_{\bar{\Delta},A} & \longrightarrow & G_{\bar{\Delta},A} \end{array} \tag{4-6}$$

Corollary 4.3.7. *A $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)_\Delta^{\mathrm{op}}$ is trivial if the pushout of $\mathcal{Q}$ along $\bar{\tau}_P : G_{\mu,A,I} \to (P_\mu)_{\bar{\Delta},A}$ is trivial as a $(P_\mu)_{\bar{\Delta},A}$-torsor on $(A, I)_\Delta^{\mathrm{op}}$. The same holds if we replace $(A, I)_\Delta^{\mathrm{op}}$ by $(A, I)_{\text{ét}}^{\mathrm{op}}$.*

Proof. We prove the assertion for $(A, I)_\Delta^{\mathrm{op}}$; the argument for $(A, I)_{\text{ét}}^{\mathrm{op}}$ is similar. By Lemma 4.3.5(2), it suffices to show that $\mathcal{Q}^{<m}$ is trivial as a $G_{\mu,A,I}^{<m}$-torsor for any m. We proceed by induction on m. The assertion is true for $m = 1$ by our assumption. We assume that $\mathcal{Q}^{<m}$ is trivial for an integer $m \geq 1$, so that there exists an element $x \in \mathcal{Q}^{<m}(A)$. The fiber of the morphism $\mathcal{Q}^{<m+1} \to \mathcal{Q}^{<m}$ at x is a $G_{\mu,A,I}^{=m}$-torsor. Lemma 4.3.5(1) shows that $G_{\mu,A,I}^{=m} \simeq V\big(\bigoplus_{i \leq m} \mathfrak{g}_i\big)_{\bar{\Delta},A}$. By Proposition 4.3.1, the fiber arises from a $V\big(\bigoplus_{i \leq m} \mathfrak{g}_i\big)_{A/I}$-torsor over $\operatorname{Spec} A/I$, which is trivial since $\operatorname{Spec} A/I$ is affine. This implies that the $G_{\mu,A,I}^{<m+1}$-torsor $\mathcal{Q}^{<m+1}$ is trivial, as desired. □

Corollary 4.3.8. *A $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\Delta}$ is trivialized by a (π, I)-completely étale covering $A \to B$, i.e., the restriction of $\mathcal{Q}$ to $(B, IB)^{\mathrm{op}}_{\Delta}$ is trivial. Moreover, the restriction functor induces an equivalence from the groupoid of $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\Delta}$ to the groupoid of $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\text{ét}}$.*

Proof. The first assertion follows from Corollary 4.3.7 since any $(P_\mu)_{\bar{\Delta},A}$-torsor $\mathcal{P}$ on $(A, I)^{\mathrm{op}}_{\Delta}$ arises from a $(P_\mu)_{A/I}$-torsor over Spec A/I with respect to the étale topology, which in turn implies that $\mathcal{P}$ is trivialized by a (π, I)-completely étale covering $A \to B$ (see also Lemma 2.5.9). The second assertion is a formal consequence of the first one. □

Remark 4.3.9. To a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\Delta}$, we can associate the $G_{\Delta,A}$-torsor $\mathcal{Q}^{\tau}$ and the $(P_\mu)_{\bar{\Delta},A}$-torsor $\mathcal{Q}^{\bar{\tau}_P}$ on $(A, I)^{\mathrm{op}}_{\Delta}$, and there is a canonical isomorphism between the $G_{\bar{\Delta},A}$-torsors associated with $\mathcal{Q}^{\tau}$ and $\mathcal{Q}^{\bar{\tau}_P}$. We assume that μ is 1-bounded. Then, by Proposition 4.2.9, this construction induces an equivalence from the groupoid of $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\Delta}$ to the groupoid of triples consisting of a $G_{\Delta,A}$-torsor, a $(P_\mu)_{\bar{\Delta},A}$-torsor, and an isomorphism between the $G_{\bar{\Delta},A}$-torsors associated with them. The same holds if we replace $(A, I)^{\mathrm{op}}_{\Delta}$ by $(A, I)^{\mathrm{op}}_{\text{ét}}$. Corollaries 4.3.7 and 4.3.8 also follow from this fact and Proposition 4.3.1 when μ is 1-bounded.

5. Prismatic G-μ-displays

In this section, we come to the heart of this paper, namely prismatic G-μ-displays. We first discuss the notion of G-Breuil–Kisin modules of type μ in Section 5.1. Then we introduce and study prismatic G-μ-displays in Sections 5.2–5.7. Our prismatic G-μ-displays are essentially equivalent to G-Breuil–Kisin modules of type μ, and the latter may be more familiar to readers. Nevertheless, in many cases, such as the proof of the main result (Theorem 6.1.3) of this paper, it will be crucial to work with prismatic G-μ-displays.

We retain the notation of Section 4. Recall that G is a smooth affine group scheme over $\mathcal{O}_E$ and $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ is a cocharacter defined over $\mathcal{O} = W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.

5.1. *G-Breuil–Kisin modules of type μ.*

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Definition 5.1.1. A *G-Breuil–Kisin module* over (A, I) is a pair $(\mathcal{P}, F_{\mathcal{P}})$ consisting of a G_A-torsor $\mathcal{P}$ over Spec A (with respect to the étale topology) and an isomorphism

$$F_{\mathcal{P}} : (\phi^*\mathcal{P})[1/I] \xrightarrow{\sim} \mathcal{P}[1/I]$$

of $G_{A[1/I]}$-torsors over Spec $A[1/I]$.

Here, for a G_A-torsor $\mathcal{P}$ over Spec A, we let $\phi^*\mathcal{P}$ denote the base change of $\mathcal{P}$ along the Frobenius $\phi : A \to A$. Since ϕ is $\mathcal{O}_E$-linear and G is defined over $\mathcal{O}_E$, we have $\phi^* G_A = G_A$, and hence $\phi^*\mathcal{P}$ is a G_A-torsor over Spec A. Moreover, we write $\mathcal{P}[1/I] := \mathcal{P} \times_{\operatorname{Spec} A} \operatorname{Spec} A[1/I]$. When there is no ambiguity, we simply write $\mathcal{P} = (\mathcal{P}, F_{\mathcal{P}})$.

Example 5.1.2. Assume that $G = \mathrm{GL}_n$. Let (M, F_M) be a Breuil–Kisin module of rank n over (A, I). Let

$$\mathcal{P}(M) := \underline{\mathrm{Isom}}(A^n, M)$$

be the $\mathrm{GL}_{n,A}$-torsor over Spec A defined by sending an A-algebra B to the set of isomorphisms $B^n \simeq M_B$. Together with the isomorphism $(\phi^*\mathcal{P}(M))[1/I] \xrightarrow{\sim} \mathcal{P}(M)[1/I]$ induced by F_M, we regard $\mathcal{P}(M)$ as a GL_n-Breuil–Kisin module over (A, I). This construction $M \mapsto \mathcal{P}(M)$ induces an equivalence between the groupoid of Breuil–Kisin modules of rank n over (A, I) and the groupoid of GL_n-Breuil–Kisin modules over (A, I).

Remark 5.1.3. Let $\mathcal{P}$ and $\mathcal{P}'$ be G_A-torsors over Spec A. Using that the functor $(\mathcal{O}_E)_{\Delta,\mathcal{O}_E} \to \mathrm{Set}$, $(B, J) \mapsto B[1/J]$, forms a sheaf (see Remark 3.1.14) and that $\mathcal{P}, \mathcal{P}'$ are affine and flat over Spec A, one can show that the functor $(A, I)_\Delta \to \mathrm{Set}$ which associates to each $(B, J) \in (A, I)_\Delta$ the set of isomorphisms $\mathcal{P}_B[1/J] \xrightarrow{\sim} \mathcal{P}'_B[1/J]$ of $G_{B[1/J]}$-torsors forms a sheaf. This, together with Proposition 4.3.1, implies that the fibered category over $(A, I)_\Delta^{\mathrm{op}}$ which associates to each $(B, J) \in (A, I)_\Delta$ the groupoid of G-Breuil–Kisin modules over (B, J) is a stack with respect to the flat topology.

We introduce G-Breuil–Kisin modules of type μ. Recall that for a (π, I)-completely étale A-algebra $B \in (A, I)_{\text{ét}}$, the pair (B, IB) is naturally a bounded $\mathcal{O}_E$-prism; see Lemma 2.5.10.

Definition 5.1.4 (G-Breuil–Kisin module of type μ). We say that a G-Breuil–Kisin module $(\mathcal{P}, F_\mathcal{P})$ over (A, I) is *of type* μ if there exists a (π, I)-completely étale covering $A \to B$ such that (B, IB) is orientable, the base change $\mathcal{P}_B$ is a trivial G_B-torsor, and via some (and hence any) trivialization $\mathcal{P}_B \simeq G_B$, the isomorphism $F_\mathcal{P}$ is given by $g \mapsto Yg$ for an element Y in the double coset

$$G(B)\mu(d)G(B) \subset G(B[1/IB]),$$

where $d \in IB$ is a generator. If these conditions are satisfied for $B = A$, then we say that $(\mathcal{P}, F_\mathcal{P})$ is *banal*.

We write

$$G\text{-BK}_\mu(A, I) \quad \text{and} \quad G\text{-BK}_\mu(A, I)_{\text{banal}}$$

for the groupoid of G-Breuil–Kisin modules of type μ over (A, I) and the groupoid of banal G-Breuil–Kisin modules of type μ over (A, I) (when (A, I) is orientable), respectively.

Remark 5.1.5. By Remark 5.1.3, the fibered category over $(A, I)_{\text{ét}}^{\mathrm{op}}$ which associates to each $B \in (A, I)_{\text{ét}}$ the groupoid of G-Breuil–Kisin modules of type μ over (B, IB) is a stack with respect to the (π, I)-completely étale topology. We will prove that the same result holds for the flat topology in Corollary 5.3.9 below, using G-μ-displays introduced in the next subsection.

Example 5.1.6. Let M be a Breuil–Kisin module of rank n over (A, I) and let $\mathcal{P}(M)$ be the associated GL_n-Breuil–Kisin module over (A, I) (see Example 5.1.2). If $\mathcal{P}(M)$ is of type μ, then M is of type μ in the sense of Definition 3.2.1 by Proposition 3.1.13. We will prove that the converse is also true in Example 5.3.10.

5.2. *G-μ-displays.* We now introduce prismatic G-μ-displays. To an orientable and bounded $\mathcal{O}_E$-prism (A, I) over $\mathcal{O}$, we attach the display group $G_\mu(A, I)$ as in Definition 4.1.1. Since G is defined over $\mathcal{O}_E$, the Frobenius ϕ of A induces a homomorphism $\phi : G(A) \to G(A)$. For each generator $d \in I$, we define the homomorphism

$$\sigma_{\mu,d} : G_\mu(A, I) \to G(A), \quad g \mapsto \phi(\mu(d)g\mu(d)^{-1}). \tag{5-1}$$

We endow $G(A)$ with the following action of $G_\mu(A, I)$:

$$G(A) \times G_\mu(A, I) \to G(A), \quad (X, g) \mapsto X \cdot g := g^{-1} X \sigma_{\mu,d}(g). \tag{5-2}$$

We write $G(A) = G(A)_d$ when we regard $G(A)$ as a set with this action of $G_\mu(A, I)$. For another generator $d' \in I$, we have $d = ud'$ for a unique element $u \in A^\times$. The map $G(A)_d \to G(A)_{d'}$ defined by $X \mapsto X\phi(\mu(u))$ is $G_\mu(A, I)$-equivariant. Then we define the set

$$G(A)_I := \varprojlim_d G(A)_d$$

equipped with a natural action of $G_\mu(A, I)$, where d runs over the set of generators $d \in I$. The projection map $G(A)_I \to G(A)_d$ is an isomorphism. For an element $X \in G(A)_I$, let

$$X_d \in G(A)_d$$

denote the image of X. Although $G(A)_I$ depends on the cocharacter μ, we omit it from the notation. We hope that this will not cause any confusion.

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. We recall the category $(A, I)_{\Delta,\mathrm{ori}}$ from Definition 4.3.2. We define the functor

$$G_{\Delta,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G(B)_J.$$

This forms a sheaf. We regard $G_{\Delta,A,I}$ as a sheaf on $(A, I)_{\Delta}^{\mathrm{op}}$ (see Remark 4.3.3). The sheaf $G_{\Delta,A,I}$ is equipped with a natural action of the group sheaf $G_{\mu,A,I}$ on $(A, I)_{\Delta}^{\mathrm{op}}$ defined in Definition 4.3.4.

The restriction of $G_{\Delta,A,I}$ to $(A, I)_{\text{ét}}$ is denoted by the same notation. We define prismatic G-μ-displays, using the (π, I)-completely étale topology, as follows.

Definition 5.2.1 (G-μ-display). Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

(1) A *G-μ-display* over (A, I) is a pair

$$(\mathcal{Q}, \alpha_{\mathcal{Q}}),$$

where $\mathcal{Q}$ is a $G_{\mu,A,I}$-torsor on $(A, I)_{\text{ét}}^{\mathrm{op}}$ and $\alpha_{\mathcal{Q}} : \mathcal{Q} \to G_{\Delta,A,I}$ is a $G_{\mu,A,I}$-equivariant map of sheaves. The $G_{\mu,A,I}$-torsor $\mathcal{Q}$ is called the *underlying $G_{\mu,A,I}$-torsor* of $(\mathcal{Q}, \alpha_{\mathcal{Q}})$. We say that $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ is *banal* if $\mathcal{Q}$ is trivial as a $G_{\mu,A,I}$-torsor. When there is no possibility of confusion, we write $\mathcal{Q}$ instead of $(\mathcal{Q}, \alpha_{\mathcal{Q}})$.

(2) An isomorphism $g : (\mathcal{Q}, \alpha_{\mathcal{Q}}) \to (\mathcal{R}, \alpha_{\mathcal{R}})$ of G-μ-displays over (A, I) is an isomorphism $g : \mathcal{Q} \xrightarrow{\sim} \mathcal{R}$ of $G_{\mu,A,I}$-torsors such that $\alpha_{\mathcal{R}} \circ g = \alpha_{\mathcal{Q}}$.

We write

$$G\text{-}\mathrm{Disp}_\mu(A, I) \quad \text{and} \quad G\text{-}\mathrm{Disp}_\mu(A, I)_{\mathrm{banal}}$$

for the groupoid of G-μ-displays over (A, I) and the groupoid of banal G-μ-displays over (A, I), respectively.

Remark 5.2.2. The notion of G-μ-displays was originally introduced in [Bültel 2008; Bültel and Pappas 2020; Lau 2021] in different settings. The definition given here is an adaptation of Lau's approach to the context of ($\mathcal{O}_E$-)prisms; see also Remark 5.2.3 below. If $\mathcal{O}_E = \mathbb{Z}_p$ and μ is 1-bounded, the notion of G-μ-displays for an oriented perfect prism has already appeared in [Bartling 2022]. He also claimed that the same construction should work for more general oriented prisms in [loc. cit., Remark 14].

Remark 5.2.3. Assume that (A, I) is orientable. We consider the graded ring

$$\mathrm{Rees}(I^\bullet) := \Big(\bigoplus_{i\geq 0} I^i t^{-i}\Big) \oplus \Big(\bigoplus_{i<0} A t^{-i}\Big) \subset A[t, t^{-1}],$$

where the degree of t is -1. Let $\tau : \mathrm{Rees}(I^\bullet) \to A$ be the homomorphism of A-algebras defined by $t \mapsto 1$. For a generator $d \in I$, let $\sigma_d : \mathrm{Rees}(I^\bullet) \to A$ be the homomorphism defined by $a_i t^{-i} \mapsto \phi(a_i d^{-i})$ for any $i \in \mathbb{Z}$. The triple

$$(\mathrm{Rees}(I^\bullet), \sigma_d, \tau)$$

can be viewed as an analogue of a higher frame introduced in [Lau 2021, Definition 2.0.1]. We note that by Lemma 4.1.4, the homomorphism τ induces an isomorphism between the display group $G_\mu(A, I)$ and the subgroup

$$G(\mathrm{Rees}(I^\bullet))^0 \subset G(\mathrm{Rees}(I^\bullet))$$

consisting of homomorphisms $g^* : A_G \to \mathrm{Rees}(I^\bullet)$ of graded $\mathcal{O}$-algebras. Under this isomorphism, the homomorphism $\sigma_{\mu,d}$ agrees with the one $G(\mathrm{Rees}(I^\bullet))^0 \to G(A)$ induced by σ_d. Therefore, the action (5-2) is consistent with the one considered in [loc. cit., (5-2)].

Remark 5.2.4. Let $\tilde{k}$ be a perfect field containing k. We set $\widetilde{\mathcal{O}} := W(\tilde{k}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. Let $\tilde{\mu} : \mathbb{G}_m \to G_{\widetilde{\mathcal{O}}}$ be the base change of μ. Then, for a bounded $\mathcal{O}_E$-prism (A, I) over $\widetilde{\mathcal{O}}$, a G-$\tilde{\mu}$-display over (A, I) is the same as a G-μ-display over (A, I).

We have the following alternative description of banal G-μ-displays, which we will use frequently in the sequel.

Remark 5.2.5. Assume that (A, I) is orientable. Let

$$[G(A)_I / G_\mu(A, I)]$$

denote the groupoid whose objects are the elements $X \in G(A)_I$ and whose morphisms are defined by $\mathrm{Hom}(X, X') = \{g \in G_\mu(A, I) \mid X' \cdot g = X\}$. Here $(-) \cdot g$ denotes the action of $g \in G_\mu(A, I)$ on $G(A)_I$. To each $X \in G(A)_I$, we attach a banal G-μ-display

$$\mathcal{Q}_X := (G_{\mu,A,I}, \alpha_X)$$

over (A, I), where $\alpha_X : G_{\mu,A,I} \to G_{\Delta,A,I}$ is given by $1 \mapsto X$. We obtain an equivalence

$$[G(A)_I / G_\mu(A, I)] \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(A, I)_{\mathrm{banal}}, \quad X \mapsto \mathcal{Q}_X,$$

of groupoids.

We discuss the notion of base change for G-μ-displays. Let $f : (A, I) \to (A', I')$ be a map of orientable and bounded $\mathcal{O}_E$-prisms over $\mathcal{O}$. We have natural homomorphisms $f : G(A) \to G(A')$ and

$f : G_\mu(A, I) \to G_\mu(A', I')$. Let $d \in I$ and $d' \in I'$ be generators and let $u \in A'^\times$ be the unique element satisfying $f(d) = ud'$. Then the composition of $G_\mu(A, I)$-equivariant maps

$$G(A)_I \simeq G(A)_d \to G(A')_{d'} \simeq G(A')_{I'},$$

where the second map is defined by $X \mapsto f(X)\phi(\mu(u))$, is independent of the choices of d and d', and is also denoted by f.

We now consider a map $f : (A, I) \to (A', I')$ of (not necessarily orientable) bounded $\mathcal{O}_E$-prisms over $\mathcal{O}$. The functor $(A, I)^{\mathrm{op}}_{\text{ét}} \to (A', I')^{\mathrm{op}}_{\text{ét}}$ sending $B \in (A, I)^{\mathrm{op}}_{\text{ét}}$ to the (π, I)-adic completion B' of $B \otimes_A A'$ induces a morphism of the associated topoi

$$f : ((A', I')^{\mathrm{op}}_{\text{ét}})^\sim \to ((A, I)^{\mathrm{op}}_{\text{ét}})^\sim$$

(since it sends (π, I)-completely étale coverings to (π, I')-completely étale coverings, sends final objects to final objects, and commutes with fiber products). We have a natural homomorphism $f : f^{-1}G_{\mu,A,I} \to G_{\mu,A',I'}$ of group sheaves. Moreover, the maps $G(A)_I \to G(A')_{I'}$ defined in the orientable case glue together to a morphism $f : f^{-1}G_{\Delta,A,I} \to G_{\Delta,A',I'}$ of sheaves.

Definition 5.2.6. Let $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ be a G-μ-display over (A, I). Let $f^*\mathcal{Q}$ be the pushout of the $f^{-1}G_{\mu,A,I}$-torsor $f^{-1}\mathcal{Q}$ along $f : f^{-1}G_{\mu,A,I} \to G_{\mu,A',I'}$. By the universal property of $f^*\mathcal{Q}$, the composition

$$f^{-1}\mathcal{Q} \xrightarrow{f^{-1}(\alpha_{\mathcal{Q}})} f^{-1}G_{\Delta,A,I} \to G_{\Delta,A',I'}$$

factors through a unique $G_{\mu,A',I'}$-equivariant map $f^*(\alpha_{\mathcal{Q}}) : f^*\mathcal{Q} \to G_{\Delta,A',I'}$. The base change of $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ along $f : (A, I) \to (A', I')$ is defined to be $(f^*\mathcal{Q}, f^*(\alpha_{\mathcal{Q}}))$.

Example 5.2.7. Assume that (A, I) is orientable. For the banal G-μ-display $\mathcal{Q}_X$ associated with an element $X \in G(A)_I$ (see Remark 5.2.5), we have $f^*(\mathcal{Q}_X) = \mathcal{Q}_{f(X)}$.

By definition, it is clear that G-μ-displays form a stack with respect to the (π, I)-completely étale topology. In fact, we can prove the following flat descent result, which is an analogue of [Lau 2021, Lemma 5.4.2].

Proposition 5.2.8 (flat descent). *The fibered category over $(\mathcal{O})^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to each $(A, I) \in (\mathcal{O})_{\Delta,\mathcal{O}_E}$ the groupoid $G\text{-}\mathrm{Disp}_\mu(A, I)$ is a stack with respect to the flat topology.*

Proof. It suffices to prove that $G\text{-}\mathrm{Disp}_\mu(A, I)$ is equivalent to the groupoid of pairs $(\mathcal{Q}, \alpha_{\mathcal{Q}})$, where $\mathcal{Q}$ is a $G_{\mu,A,I}$-torsor on $(A, I)^{\mathrm{op}}_{\Delta}$ (with respect to the flat topology) and $\alpha_{\mathcal{Q}} : \mathcal{Q} \to G_{\Delta,A,I}$ is a $G_{\mu,A,I}$-equivariant map of sheaves on $(A, I)^{\mathrm{op}}_{\Delta}$. This follows from Corollary 4.3.8. □

5.3. *G-μ-displays and G-Breuil–Kisin modules of type μ.* Here we shall show that G-μ-displays are essentially equivalent to G-Breuil–Kisin modules of type μ. Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Definition 5.3.1. To a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$, we attach a G_A-torsor $\mathcal{Q}_{\mathrm{BK}}$ over $\operatorname{Spec} A$ as follows. We first assume that (A, I) is orientable. Let $d \in I$ be a generator. Let $\mathcal{Q}_{\mathrm{BK},d}$ be the pushout of $\mathcal{Q}$ along the homomorphism

$$G_{\mu,A,I} \hookrightarrow G_{\Delta,A}, \quad g \mapsto \mu(d)g\mu(d)^{-1}.$$

Let $d' \in I$ be another generator and let $u \in A^\times$ be the unique element such that $d = ud'$. We define $\mathrm{ad}(\mu(u)) : G_{\Delta,A} \xrightarrow{\sim} G_{\Delta,A}$ by $g \mapsto \mu(u)g\mu(u)^{-1}$. The pushout $(\mathcal{Q}_{\mathrm{BK},d'})^{\mathrm{ad}(\mu(u))}$ can be identified with $\mathcal{Q}_{\mathrm{BK},d}$. The composition

$$\mathcal{Q}_{\mathrm{BK},d'} \to (\mathcal{Q}_{\mathrm{BK},d'})^{\mathrm{ad}(\mu(u))} = \mathcal{Q}_{\mathrm{BK},d} \xrightarrow{x \mapsto x \cdot \mu(u)} \mathcal{Q}_{\mathrm{BK},d} \tag{5-3}$$

is an isomorphism of $G_{\Delta,A}$-torsors. (See Remark 4.3.6 for the first map.) Then we define

$$\mathcal{Q}_{\mathrm{BK}} := \varprojlim_{d} \mathcal{Q}_{\mathrm{BK},d},$$

where d runs over the set of generators $d \in I$.

In general, the sheaves constructed in the banal case glue together to a $G_{\Delta,A}$-torsor $\mathcal{Q}_{\mathrm{BK}}$ on $(A, I)^{\mathrm{op}}_{\mathrm{\acute{e}t}}$. By Proposition 4.3.1, we regard $\mathcal{Q}_{\mathrm{BK}}$ as a G_A-torsor over $\operatorname{Spec} A$.

Remark 5.3.2. Recall that $\tau : G_{\mu,A,I} \hookrightarrow G_{\Delta,A}$ is the natural inclusion. For a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\mathrm{\acute{e}t}}$, let

$$\mathcal{Q}_A := \mathcal{Q}^\tau$$

be the pushout of $\mathcal{Q}$ along τ, regarded as a G_A-torsor over $\operatorname{Spec} A$ (by Proposition 4.3.1). There exists a canonical isomorphism

$$\mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I]$$

of $G_{A[1/I]}$-torsors over $\operatorname{Spec} A[1/I]$ obtained as follows. We first assume that (A, I) is orientable. Let $d \in I$ be a generator. Similarly to (5-3), the composition

$$\mathcal{Q}_A[1/I] \to (\mathcal{Q}_A[1/I])^{\mathrm{ad}(\mu(d))} = \mathcal{Q}_{\mathrm{BK},d}[1/I] \xrightarrow{x \mapsto x \cdot \mu(d)} \mathcal{Q}_{\mathrm{BK},d}[1/I]$$

is an isomorphism of $G_{A[1/I]}$-torsors, where $\mathrm{ad}(\mu(d)) : G_{A[1/I]} \xrightarrow{\sim} G_{A[1/I]}$ is defined by $g \mapsto \mu(d)g\mu(d)^{-1}$. We then obtain the desired isomorphism as

$$\mathcal{Q}_A[1/I] \simeq \mathcal{Q}_{\mathrm{BK},d}[1/I] \simeq \mathcal{Q}_{\mathrm{BK}}[1/I],$$

which does not depend on the choice of $d \in I$. By Remark 5.1.3, the isomorphisms in the banal case glue together to an isomorphism $\mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I]$.

Example 5.3.3. Assume that $G = \mathrm{GL}_n$. Let the notation be as in Example 4.1.5. Let M be a Breuil–Kisin module of type μ over (A, I). Recall the filtration $\{\mathrm{Fil}^i(\phi^* M)\}_{i \in \mathbb{Z}}$ of $\phi^* M$ from Definition 3.1.2. Let $\{\mathrm{Fil}^i_\mu\}_{i \in \mathbb{Z}}$ be the filtration of A^n defined in Remark 3.2.2. The functor

$$\mathcal{Q}(M) := \underline{\mathrm{Isom}}_{\mathrm{Fil}}(A^n, \phi^* M) : (A, I)_{\mathrm{\acute{e}t}} \to \mathrm{Set}$$

sending $B \in (A, I)_{\mathrm{\acute{e}t}}$ to the set of isomorphisms $h : B^n \xrightarrow{\sim} (\phi^* M)_B$ preserving the filtrations is a $(\mathrm{GL}_n)_{\mu,A,I}$-torsor by Remark 3.2.2, Example 4.1.5, and the fact that M is (π, I)-completely étale locally on A banal. We note that

$$\mathcal{Q}(M)_A = \underline{\mathrm{Isom}}(A^n, \phi^* M).$$

We set $\widetilde{M} := \mathrm{Fil}^{m_1}(\phi^* M) \otimes_A I^{-m_1}$. Then we have a canonical identification

$$\mathcal{Q}(M)_{\mathrm{BK}} = \underline{\mathrm{Isom}}(A^n, \widetilde{M}).$$

If A is orientable and $d \in I$ is a generator, then $\mathcal{Q}(M)_{\mathrm{BK},d} = \underline{\mathrm{Isom}}(A^n, \widetilde{M})$ and the natural map $\mathcal{Q}(M) \to \mathcal{Q}(M)_{\mathrm{BK},d}$ sends $h \in \mathcal{Q}(M)(A)$ to the composition of isomorphisms

$$A^n \xrightarrow{\mu(d)^{-1}} \mathrm{Fil}^{m_1}_{\mu} \otimes_A I^{-m_1} \xrightarrow{h} \widetilde{M}.$$

If $d' \in I$ is another generator, then the isomorphism $\mathcal{Q}(M)_{\mathrm{BK},d'} \xrightarrow{\sim} \mathcal{Q}(M)_{\mathrm{BK},d}$ from (5-3) is the identity $\underline{\mathrm{Isom}}(A^n, \widetilde{M}) \to \underline{\mathrm{Isom}}(A^n, \widetilde{M})$.

The isomorphism $\mathcal{Q}(M)_A[1/I] \xrightarrow{\sim} \mathcal{Q}(M)_{\mathrm{BK}}[1/I]$ defined in Remark 5.3.2 agrees with the one induced from the equality $(\phi^* M)[1/I] = \widetilde{M}[1/I]$.

To construct G-Breuil–Kisin modules of type μ from G-μ-displays, we use the following proposition, which also gives an alternative description of G-μ-displays.

Proposition 5.3.4. *Let $\mathcal{Q}$ be a $G_{\mu,A,I}$-torsor on $(A, I)^{\mathrm{op}}_{\text{ét}}$. Then there is a natural bijection $\alpha \mapsto \alpha'$ from the set of $G_{\mu,A,I}$-equivariant maps $\alpha : \mathcal{Q} \to G_{\Delta,A,I}$ to the set of isomorphisms $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ of G_A-torsors over* Spec A.

Proof. We shall construct the bijection when (A, I) is orientable and $\mathcal{Q}$ is a trivial $G_{\mu,A,I}$-torsor; the general case follows by gluing. Let $\alpha : \mathcal{Q} \to G_{\Delta,A,I}$ be a $G_{\mu,A,I}$-equivariant map. We choose a trivialization $\mathcal{Q} \simeq G_{\mu,A,I}$. Then α can be regarded as a $G_{\mu,A,I}$-equivariant map $G_{\mu,A,I} \to G_{\Delta,A,I}$, which is determined by the image $X \in G(A)_I$ of $1 \in G_\mu(A, I)$. We may also identify $\mathcal{Q}_A$ with G_A. Let $d \in I$ be a generator. Then we may identify $\mathcal{Q}_{\mathrm{BK}}$ with G_A by

$$\mathcal{Q}_{\mathrm{BK}} \simeq \mathcal{Q}_{\mathrm{BK},d} \simeq (G_{\mu,A,I})_{\mathrm{BK},d} = G_A.$$

(See Definition 5.3.1 for $\mathcal{Q}_{\mathrm{BK},d}$.) Via these identifications, we define $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ by

$$\phi^*(\mathcal{Q}_{\mathrm{BK}}) = \phi^* G_A = G_A \xrightarrow{\sim} G_A = \mathcal{Q}_A, \quad g \mapsto X_d \cdot g,$$

where $X_d \in G(A) = G(A)_d$ is the image of $X \in G(A)_I$. One can check that the resulting isomorphism α' does not depend on the choices of $\mathcal{Q} \simeq G_{\mu,A,I}$ and $d \in I$. It is clear that the map $\alpha \mapsto \alpha'$ is a bijection. □

Remark 5.3.5. By Proposition 5.3.4, a G-μ-display over (A, I) can be thought of as a pair $(\mathcal{Q}, \alpha')$ of a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$ and an isomorphism $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ of G_A-torsors over Spec A.

Definition 5.3.6. Let $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ be a G-μ-display over (A, I) and let $(\alpha_{\mathcal{Q}})' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ be the corresponding isomorphism. We denote by F the composition

$$(\phi^*(\mathcal{Q}_{\mathrm{BK}}))[1/I] \xrightarrow{(\alpha_{\mathcal{Q}})'} \mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I],$$

where the second isomorphism is constructed in Remark 5.3.2. By construction, we see that $\mathcal{Q}_{\mathrm{BK}}$, together with the isomorphism F, is a G-Breuil–Kisin module of type μ. (See also Example 5.3.7 below.) We have a functor

$$G\text{-}\mathrm{Disp}_{\mu}(A, I) \to G\text{-}\mathrm{BK}_{\mu}(A, I), \quad \mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{BK}}. \tag{5-4}$$

Example 5.3.7. Assume that (A, I) is orientable and let $d \in I$ be a generator. Let $\mathcal{Q}_X$ be the banal G-μ-display associated with an element $X \in G(A)_I$ (Remark 5.2.5). The trivial G_A-torsor G_A with the isomorphism

$$(\phi^* G_A)[1/I] = G_A[1/I] \xrightarrow{\sim} G_A[1/I], \quad g \mapsto (\mu(d)X_d)g,$$

is a banal G-Breuil–Kisin module of type μ over (A, I), which is denoted by $\mathcal{P}_{X_d}$. By construction, we have $(\mathcal{Q}_X)_{\mathrm{BK}} \xrightarrow{\sim} \mathcal{P}_{X_d}$.

Proposition 5.3.8. *Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. The functor (5-4)*

$$G\text{-}\mathrm{Disp}_{\mu}(A, I) \to G\text{-}\mathrm{BK}_{\mu}(A, I), \quad \mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{BK}},$$

is an equivalence.

Proof. By Remark 2.5.14, Remark 5.1.5, and (π, I)-completely étale descent for G-μ-displays, it suffices to prove that the functor

$$G\text{-}\mathrm{Disp}_{\mu}(A, I)_{\mathrm{banal}} \to G\text{-}\mathrm{BK}_{\mu}(A, I)_{\mathrm{banal}}, \quad \mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{BK}},$$

is an equivalence when (A, I) is orientable.

We shall prove that the functor is fully faithful. It suffices to prove that, for all $X, X' \in G(A)_I$ and the associated banal G-μ-displays $\mathcal{Q}_X, \mathcal{Q}_{X'}$ over (A, I), we have

$$\mathrm{Hom}(\mathcal{Q}_X, \mathcal{Q}_{X'}) \xrightarrow{\sim} \mathrm{Hom}((\mathcal{Q}_X)_{\mathrm{BK}}, (\mathcal{Q}_{X'})_{\mathrm{BK}}). \tag{5-5}$$

We fix a generator $d \in I$. The left-hand side can be identified with

$$\{g \in G_{\mu}(A, I) \mid g^{-1} X'_d \phi(\mu(d) g \mu(d)^{-1}) = X_d\}.$$

(See Remark 5.2.5.) By Example 5.3.7, we have $(\mathcal{Q}_X)_{\mathrm{BK}} \xrightarrow{\sim} \mathcal{P}_{X_d}$ and $(\mathcal{Q}_{X'})_{\mathrm{BK}} \xrightarrow{\sim} \mathcal{P}_{X'_d}$. Thus the right-hand side of (5-5) can be identified with

$$\{h \in G(A) \mid h^{-1} \mu(d) X'_d \phi(h) = \mu(d) X_d\}.$$

The map (5-5) is given by $g \mapsto \mu(d) g \mu(d)^{-1}$ under these identifications. In particular, the map is injective. For surjectivity, let $h \in G(A)$ be an element such that $h^{-1}\mu(d)X'_d\phi(h) = \mu(d)X_d$. The element $g := \mu(d)^{-1} h \mu(d) = X'_d \phi(h) X_d^{-1}$ belongs to $G(A)$, and hence $g \in G_{\mu}(A, I)$. It follows that $g \in \mathrm{Hom}(\mathcal{Q}_X, \mathcal{Q}_{X'})$, and g is mapped to h.

It remains to prove that the functor is essentially surjective. It is enough to show that a banal G-Breuil–Kisin module $\mathcal{P}$ of type μ over (A, I), such that $\mathcal{P} = G_A$ and $F_{\mathcal{P}}$ corresponds to an element $Y \in G(A)\mu(d)G(A)$, is isomorphic to $\mathcal{P}_{X_d}$ for some $X \in G(A)_I$. After changing the trivialization $\mathcal{P} = G_A$, we may assume that $Y \in \mu(d)G(A)$. Then the result is clear. □

Corollary 5.3.9. *The fibered category over* $(\mathcal{O})^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ *which associates to each* $(A, I) \in (\mathcal{O})_{\Delta,\mathcal{O}_E}$ *the groupoid* $G\text{-}\mathrm{BK}_\mu(A, I)$ *of* G*-Breuil–Kisin modules of type* μ *over* (A, I) *is a stack with respect to the flat topology.*

Proof. This follows from Propositions 5.2.8 and 5.3.8. □

Example 5.3.10. Assume that $G = \mathrm{GL}_n$. We retain the notation of Example 5.3.3. Let M be a Breuil–Kisin module of type μ over (A, I). Since M is of type μ, it follows from Lemma 3.1.1 that F_M restricts to an isomorphism $\widetilde{M} \xrightarrow{\sim} M$. The base change $\phi^*(F_M) : \phi^*\widetilde{M} \xrightarrow{\sim} \phi^*M$ induces an isomorphism

$$\alpha' : \phi^*(\mathcal{Q}(M)_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}(M)_A$$

of $\mathrm{GL}_{n,A}$-torsors over $\mathrm{Spec}\, A$. The $(\mathrm{GL}_n)_{\mu,A,I}$-torsor $\mathcal{Q}(M)$ with α' is a GL_n-μ-display over (A, I).

By construction, the GL_n-Breuil–Kisin module $\mathcal{Q}(M)_{\mathrm{BK}}$ agrees with the one $\mathcal{P}(\widetilde{M})$ associated with the Breuil–Kisin module $(\widetilde{M}, F_{\widetilde{M}})$, where the isomorphism $F_{\widetilde{M}}$ is

$$(\phi^*\widetilde{M})[1/I] \xrightarrow{\phi^*(F_M)} (\phi^*M)[1/I] = \widetilde{M}[1/I].$$

(See $\mathcal{P}(\widetilde{M})$ for Example 5.1.2.) We note that $F_M : \widetilde{M} \xrightarrow{\sim} M$ is an isomorphism of Breuil–Kisin modules. Since $\mathcal{Q}(M)_{\mathrm{BK}}$ is of type μ, it follows that $\mathcal{P}(M)$ is of type μ.

Corollary 5.3.11. *Let* (A, I) *be a bounded* $\mathcal{O}_E$*-prism over* $\mathcal{O}$*. We have equivalences of groupoids*

$$\mathrm{BK}_\mu(A, I)^{\simeq} \xrightarrow{\sim} \mathrm{GL}_n\text{-}\mathrm{BK}_\mu(A, I), \qquad M \mapsto \mathcal{P}(M),$$
$$\mathrm{BK}_\mu(A, I)^{\simeq} \xrightarrow{\sim} \mathrm{GL}_n\text{-}\mathrm{Disp}_\mu(A, I), \qquad M \mapsto \mathcal{Q}(M).$$

Here $\mathrm{BK}_\mu(A, I)^{\simeq}$ *is the groupoid of Breuil–Kisin modules of type* μ *over* (A, I)*.*

Proof. The first equivalence follows from Examples 5.1.2, 5.1.6, and 5.3.10. We shall prove that the functor $M \mapsto \mathcal{Q}(M)$ is an equivalence. It follows from Example 5.3.10 that the composition of this functor with the functor (5-4) is isomorphic to the functor $M \mapsto \mathcal{P}(M)$. Since (5-4) is an equivalence by Proposition 5.3.8, the result follows. □

5.4. *Hodge filtrations.*

We define the Hodge filtrations for G-μ-displays, following [Lau 2021, Section 7.4]. Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. We recall the commutative diagram (4-6) from Section 4.3.

Definition 5.4.1 (Hodge filtration). Let $\mathcal{Q}$ be a G-μ-display over (A, I). We write

$$\mathcal{Q}_{A/I} := \mathcal{Q}^{\bar{\tau}} \quad (\text{resp. } P(\mathcal{Q})_{A/I} := \mathcal{Q}^{\bar{\tau}_P})$$

for the pushout of the underlying $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$ along $\bar{\tau}$ (resp. $\bar{\tau}_P$), which is a $G_{\bar{\Delta},A}$-torsor (resp. a $(P_\mu)_{\bar{\Delta},A}$-torsor) on $(A, I)^{\mathrm{op}}_{\text{ét}}$. There is a natural $(P_\mu)_{\bar{\Delta},A}$-equivariant injection

$$P(\mathcal{Q})_{A/I} \hookrightarrow \mathcal{Q}_{A/I}.$$

We call $P(\mathcal{Q})_{A/I}$ (or the injection $P(\mathcal{Q})_{A/I} \hookrightarrow \mathcal{Q}_{A/I}$) the *Hodge filtration* of $\mathcal{Q}_{A/I}$. If there is no risk of confusion, we also say that $P(\mathcal{Q})_{A/I}$ is the Hodge filtration of $\mathcal{Q}$.

Example 5.4.2. Assume that $G = \mathrm{GL}_n$ and let the notation be as in Example 5.3.3. Let M be a Breuil–Kisin module over (A, I) of type μ and let $\mathcal{Q} = \mathcal{Q}(M)$ be the associated GL_n-μ-display over (A, I) given in Example 5.3.10. Recall that the filtration $\{\mathrm{Fil}^i(\phi^*M)\}_{i\in\mathbb{Z}}$ defines the Hodge filtration $\{P^i\}_{i\in\mathbb{Z}}$ of $M_{\mathrm{dR}} = (\phi^*M)/I(\phi^*M)$. Similarly, the filtration $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ of A^n induces a filtration of $(A/I)^n$. Let $\underline{\mathrm{Isom}}((A/I)^n, M_{\mathrm{dR}})$ (resp. $\underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}})$) be the functor sending $B \in (A, I)_{\text{ét}}$ to the set of isomorphisms $(B/IB)^n \xrightarrow{\sim} (M_{\mathrm{dR}})_{B/IB}$ (resp. the set of isomorphisms $(B/IB)^n \xrightarrow{\sim} (M_{\mathrm{dR}})_{B/IB}$ preserving the filtrations). Since M is of type μ, we see that $\underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}})$ is naturally a $(P_\mu)_{\bar{\Delta},A}$-torsor. It follows that the natural morphism

$$\mathcal{Q} = \underline{\mathrm{Isom}}_{\mathrm{Fil}}(A^n, \phi^*M) \to \underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}})$$

induces an isomorphism

$$P(\mathcal{Q})_{A/I} \xrightarrow{\sim} \underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}}).$$

Similarly, we obtain $\mathcal{Q}_{A/I} \xrightarrow{\sim} \underline{\mathrm{Isom}}((A/I)^n, M_{\mathrm{dR}})$.

Remark 5.4.3. Let $\mathcal{Q}$ be a G-μ-display over (A, I). By Proposition 4.3.1, the $G_{\bar{\Delta},A}$-torsor $\mathcal{Q}_{A/I}$ (resp. the $(P_\mu)_{\bar{\Delta},A}$-torsor $P(\mathcal{Q})_{A/I}$) corresponds to a $G_{A/I}$-torsor (resp. a $(P_\mu)_{A/I}$-torsor) over $\mathrm{Spec}\, A/I$, which will be denoted by the same symbol.

Example 5.4.4. Assume that (A, I) is orientable. Let $X \in G(A)_I$ be an element. Then the Hodge filtration associated with $\mathcal{Q}_X$ can be identified with the natural inclusion $(P_\mu)_{A/I} \hookrightarrow G_{A/I}$.

Proposition 5.4.5. *A G-μ-display $\mathcal{Q}$ over (A, I) is banal if and only if the Hodge filtration $P(\mathcal{Q})_{A/I}$ is a trivial $(P_\mu)_{A/I}$-torsor over* $\mathrm{Spec}\, A/I$.

Proof. This is a restatement of Corollary 4.3.7 in the current context. □

5.5. *Underlying G-ϕ-modules.* Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$ and let (M, F_M) be a Breuil–Kisin module over (A, I). Since $\{\mathrm{Fil}^i(\phi^*M)\}_{i\in\mathbb{Z}}$ is the filtration of ϕ^*M, it is sometimes reasonable to consider ϕ^*M (rather than M) as "the underlying A-module" of the Breuil–Kisin module (M, F_M). The same applies to G-Breuil–Kisin modules $\mathcal{P}$ over (A, I). In fact, the Frobenius of $\phi^*\mathcal{P}$ will also be important. For example, this can be observed in the Grothendieck–Messing deformation theory studied in [Ito 2025].

It will be convenient to make the following definition. We assume that (A, I) is orientable for simplicity. We set $A[1/\phi(I)] := A[1/\phi(d)]$ for a generator $d \in I$, which does not depend on the choice of d.

Definition 5.5.1. A *G-ϕ-module* over (A, I) is a pair $(\mathcal{P}, \phi_{\mathcal{P}})$ consisting of a G_A-torsor $\mathcal{P}$ over $\mathrm{Spec}\, A$ and an isomorphism

$$\phi_{\mathcal{P}} : (\phi^*\mathcal{P})[1/\phi(I)] \xrightarrow{\sim} \mathcal{P}[1/\phi(I)]$$

of $G_{A[1/\phi(I)]}$-torsors over $\mathrm{Spec}\, A[1/\phi(I)]$. (Here $\mathcal{P}[1/\phi(I)] := \mathcal{P} \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A[1/\phi(I)]$.) If there is no possibility of confusion, we write $\mathcal{P} = (\mathcal{P}, \phi_{\mathcal{P}})$.

Here we explain how to attach a G-ϕ-module over (A, I) to a G-μ-display $\mathcal{Q}$ over (A, I). Recall $\mathcal{Q}_A := \mathcal{Q}^\tau$ from Remark 5.3.2, which we regard as a G_A-torsor over Spec A. We define

$$\phi_{\mathcal{Q}_A} : (\phi^*(\mathcal{Q}_A))[1/\phi(I)] \xrightarrow{\sim} \mathcal{Q}_A[1/\phi(I)]$$

as the composition

$$(\phi^*(\mathcal{Q}_A))[1/\phi(I)] \xrightarrow{\sim} (\phi^*(\mathcal{Q}_{\mathrm{BK}}))[1/\phi(I)] \xrightarrow{\sim} \mathcal{Q}_A[1/\phi(I)],$$

where the first isomorphism is the base change of $\mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I]$ given in Remark 5.3.2 along $\phi : A[1/I] \to A[1/\phi(I)]$, and the second one is the base change of $(\alpha_{\mathcal{Q}})' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ given in Proposition 5.3.4 along the natural homomorphism $A \to A[1/\phi(I)]$.

Definition 5.5.2 (underlying G-ϕ-module). Let $\mathcal{Q}$ be a G-μ-display over (A, I). The G-ϕ-module

$$\mathcal{Q}_\phi := (\mathcal{Q}_A, \phi_{\mathcal{Q}_A})$$

over (A, I) is called the *underlying G-ϕ-module* of $\mathcal{Q}$.

Example 5.5.3. Let $\mathcal{Q}_X$ be the banal G-μ-display associated with an element $X \in G(A)_I$. The underlying G-ϕ-module $(\mathcal{Q}_X)_\phi$ of $\mathcal{Q}_X$ is the trivial G_A-torsor G_A with the isomorphism

$$(\phi^* G_A)[1/\phi(I)] = G_A[1/\phi(I)] \xrightarrow{\sim} G_A[1/\phi(I)], \quad g \mapsto X_d \phi(\mu(d)) g,$$

for a generator $d \in I$. We note that the element $X_d \phi(\mu(d)) \in G(A[1/\phi(I)])$ is independent of the choice of $d \in I$.

Remark 5.5.4. Let $\mathcal{Q}$ be a G-μ-display over (A, I). The base change $\phi^*(\mathcal{Q}_{\mathrm{BK}})$ of the associated G-Breuil–Kisin module $\mathcal{Q}_{\mathrm{BK}}$ is naturally a G-ϕ-module over (A, I). We note that $(\alpha_{\mathcal{Q}})'$ gives an isomorphism $\phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_\phi$ of G-ϕ-modules. Therefore, one can also define the underlying G-ϕ-module of $\mathcal{Q}$ as $\phi^*(\mathcal{Q}_{\mathrm{BK}})$. However, the construction of $\mathcal{Q}_\phi$ is more natural and will be useful in [Ito 2025].

5.6. *G-μ-displays for perfectoid rings.* Let R be a perfectoid ring over $\mathcal{O}$. We discuss p-complete arc-descent results for G-μ-displays over the $\mathcal{O}_E$-prism $(W_{\mathcal{O}_E}(R^\flat), I_R)$.

Remark 5.6.1. Assume that $\mathcal{O}_E = \mathbb{Z}_p$. In [Bartling 2022], the notion of G-Breuil–Kisin modules over $(W(R^\flat), I_R)$ *of type* μ was introduced in a different way; namely, a G-Breuil–Kisin module $\mathcal{P}$ over $(W(R^\flat), I_R)$ is said to be of type μ if for any homomorphism $R \to V$ with V a p-adically complete valuation ring of rank ≤ 1 whose fraction field is algebraically closed, the base change $\mathcal{P}_{W(V^\flat)}$ is of type μ in the sense of Definition 5.1.4. In Proposition 5.6.11 below, we will prove that this notion agrees with the one introduced in Definition 5.1.4.

Let Perfd_R be the category of perfectoid rings over R. We endow $\mathrm{Perfd}_R^{\mathrm{op}}$ with the topology generated by the *π-complete* arc-*coverings* (or equivalently, the p-complete arc-coverings) in the sense of [Česnavičius and Scholze 2024, Section 2.2.1]. This topology is called the *π-complete* arc-*topology*.

Remark 5.6.2. We quickly review the notion of a π-complete arc-covering.

(1) We say that a homomorphism $R \to S$ of perfectoid rings over $\mathcal{O}$ is a π-complete arc-covering if for any homomorphism $R \to V$ with V a π-adically complete valuation ring of rank ≤ 1, there exist an extension $V \hookrightarrow W$ of π-adically complete valuation rings of rank ≤ 1 and a homomorphism $S \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

(2) The category $\mathrm{Perfd}_R^{\mathrm{op}}$ admits fiber products; a colimit of the diagram $S_2 \leftarrow S_1 \to S_3$ in Perfd_R is given by the π-adic completion of $S_2 \otimes_{S_1} S_3$ (see [Česnavičius and Scholze 2024, Proposition 2.1.11]). We see that $\mathrm{Perfd}_R^{\mathrm{op}}$ is indeed a site.

(3) Let $R \to S$ be a π-completely étale covering. Then S is perfectoid as explained in Example 2.5.11, and $R \to S$ is a π-complete arc-covering; see [loc. cit., Section 2.2.1].

(4) There exists a π-complete arc-covering of the form $R \to \prod_{i \in I} V_i$, where V_i are π-adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields; see [Česnavičius and Scholze 2024, Lemma 2.2.3].

Proposition 5.6.3 [Ito 2023, Corollary 4.2]. *The fibered category over* $\mathrm{Perfd}_R^{\mathrm{op}}$ *which associates to a perfectoid ring S over R the category of finite projective S-modules satisfies descent with respect to the π-complete* arc*-topology. In particular, the functor* $\mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto S$, *forms a sheaf.*

Proof. See [Ito 2023, Corollary 4.2]. The second assertion was previously proved in [Bhatt and Scholze 2022, Proposition 8.10]. □

Remark 5.6.4. In fact, it is proved in [Ito 2023, Theorem 1.2] that the functor on Perfd_R associating to each $S \in \mathrm{Perfd}_R$ the ∞-category $\mathrm{Perf}(S)$ of perfect complexes over S satisfies π-complete arc-hyperdescent. Using this, we can prove that for any integer $n \geq 1$, the functor $S \mapsto \mathrm{Perf}(W_{\mathcal{O}_E}(S^\flat)/I_S^n)$ on Perfd_R satisfies π-complete arc-hyperdescent, by induction on n. This implies that the functor $S \mapsto \mathrm{Perf}(W_{\mathcal{O}_E}(S^\flat))$ satisfies π-complete arc-hyperdescent as well. See the discussion in [loc. cit., Section 4.1].

Corollary 5.6.5. *The fibered category over* $\mathrm{Perfd}_R^{\mathrm{op}}$ *which associates to a perfectoid ring S over R the category of finite projective $W_{\mathcal{O}_E}(S^\flat)$-modules satisfies descent with respect to the π-complete* arc*-topology. The same holds for finite projective $W_{\mathcal{O}_E}(S^\flat)/I_S^n$-modules.*

Proof. By the same argument as in the proof of [loc. cit., Corollary 4.2], we can deduce the assertion from Remark 5.6.4. □

In particular, the functor $\mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto W_{\mathcal{O}_E}(S^\flat)$ forms a sheaf. This fact also follows from [Česnavičius and Scholze 2024, Lemma 4.2.6] or the proof of [Bhatt and Scholze 2022, Proposition 8.10] (using that $W(\mathbb{F}_q) \to \mathcal{O}_E$ is flat).

Remark 5.6.6. In the case where $\mathcal{O}_E = \mathbb{Z}_p$, the first assertion of Corollary 5.6.5 is proved in [Ito 2023, Corollary 4.2]. The general case can also be deduced from this special case, using that a module over $W_{\mathcal{O}_E}(S^\flat)$ is finite projective if and only if it is finite projective over $W(S^\flat)$.

For an affine scheme X over $\mathcal{O}$ (or R), we define a functor $X_{\bar{\Delta}} : \mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto X(S)$. By Proposition 5.6.3, this forms a sheaf. Similarly, for an affine scheme X over $\mathcal{O}$ (or $W_{\mathcal{O}_E}(R^\flat)$), we define a functor $X_{\Delta} : \mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto X(W_{\mathcal{O}_E}(S^\flat))$, which forms a sheaf by Corollary 5.6.5. We have the following analogue of Proposition 4.3.1.

Proposition 5.6.7. *Let H be a smooth affine group scheme over $\mathcal{O}$.*

(1) *The functor $\mathcal{P} \mapsto \mathcal{P}_{\bar{\Delta}}$ from the groupoid of H_R-torsors over* Spec R *to the groupoid of $H_{\bar{\Delta}}$-torsors on* $\mathrm{Perfd}_R^{\mathrm{op}}$ *is an equivalence.*

(2) *The functor $\mathcal{P} \mapsto \mathcal{P}_{\Delta}$ from the groupoid of $H_{W_{\mathcal{O}_E}(R^\flat)}$-torsors over* Spec $W_{\mathcal{O}_E}(R^\flat)$ *to the groupoid of H_{Δ}-torsors on* $\mathrm{Perfd}_R^{\mathrm{op}}$ *is an equivalence.*

Proof. This can be proved by the same argument as in the proof of Proposition 4.3.1, using Proposition 5.6.3 and Corollary 5.6.5. □

Remark 5.6.8. Arguing as in Remark 5.1.3, we see that the fibered category over $\mathrm{Perfd}_R^{\mathrm{op}}$ which associates to each $S \in \mathrm{Perfd}_R$ the groupoid of G-Breuil–Kisin modules over $(W_{\mathcal{O}_E}(S^\flat), I_S)$ is a stack with respect to the π-complete arc-topology.

As in Section 5.2, the functors

$$G_{\mu,I} : \mathrm{Perfd}_R \to \mathrm{Set}, \quad S \mapsto G_\mu(W_{\mathcal{O}_E}(S^\flat), I_S),$$
$$G_{\Delta,I} : \mathrm{Perfd}_R \to \mathrm{Set}, \quad S \mapsto G(W_{\mathcal{O}_E}(S^\flat))_{I_S},$$

form sheaves, and the group sheaf $G_{\mu,I}$ acts on $G_{\Delta,I}$.

Lemma 5.6.9. *Let $\mathcal{Q}$ be a $G_{\mu,I}$-torsor with respect to the π-complete* arc*-topology. Then $\mathcal{Q}$ is trivialized by a π-completely étale covering $R \to S$.*

Proof. We claim that if the pushout of $\mathcal{Q}$ along the homomorphism $G_{\mu,I} \to (P_\mu)_{\bar{\Delta}}$ is trivial as a $(P_\mu)_{\bar{\Delta}}$-torsor, then $\mathcal{Q}$ is itself trivial. Indeed, one can prove the analogue of Lemma 4.3.5 for $G_{\mu,I}$, and then the argument as in the proof of Corollary 4.3.7 works.

By the claim, it suffices to prove that any $(P_\mu)_{\bar{\Delta}}$-torsor with respect to the π-complete arc-topology can be trivialized by a π-completely étale covering $R \to S$. This is a consequence of Proposition 5.6.7. □

Corollary 5.6.10. *The fibered category over* $\mathrm{Perfd}_R^{\mathrm{op}}$ *which associates to a perfectoid ring S over R the groupoid of G-μ-displays over $(W_{\mathcal{O}_E}(S^\flat), I_S)$ is a stack with respect to the π-complete* arc*-topology. The same holds for G-Breuil–Kisin modules of type μ over $(W_{\mathcal{O}_E}(S^\flat), I_S)$.*

Proof. The first assertion can be deduced from Lemma 5.6.9 by the same argument as in the proof of Proposition 5.2.8. The second assertion follows from the first one, together with Proposition 5.3.8. □

Now we are ready to prove the following result:

Proposition 5.6.11. *For a G-Breuil–Kisin module $\mathcal{P}$ over $(W_{\mathcal{O}_E}(R^\flat), I_R)$, the following conditions are equivalent*:

(1) *$\mathcal{P}$ is of type μ* (in the sense of Definition 5.1.4).

(2) *There exists a π-complete* arc-*covering $R \to S$ such that the base change of $\mathcal{P}$ along $(W_{\mathcal{O}_E}(R^\flat), I_R) \to (W_{\mathcal{O}_E}(S^\flat), I_S)$ is of type μ.*

(3) *For any homomorphism $R \to V$ with V a π-adically complete valuation ring of rank ≤ 1 whose fraction field is algebraically closed, the base change of $\mathcal{P}$ along $(W_{\mathcal{O}_E}(R^\flat), I_R) \to (W_{\mathcal{O}_E}(V^\flat), I_V)$ is of type μ.*

Proof. It is clear (1) implies (2) and (3). By Remark 5.6.8 and Corollary 5.6.10, we see (2) implies (1).

Assume that the condition (3) is satisfied. We want to show that this implies (2), which will conclude the proof of the proposition. By Remark 5.6.2(4), there exists a π-complete arc-covering $R \to S = \prod_i V_i$, where V_i are π-adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields. Since $W_{\mathcal{O}_E}(V_i^\flat)$ is strictly henselian, the base change $\mathcal{P}_{W_{\mathcal{O}_E}(V_i^\flat)}$ is a trivial $G_{W_{\mathcal{O}_E}(V_i^\flat)}$-torsor. Since $\mathcal{P}_{W_{\mathcal{O}_E}(S^\flat)}$ is affine and $W_{\mathcal{O}_E}(S^\flat) = \prod_i W_{\mathcal{O}_E}(V_i^\flat)$, it follows that $\mathcal{P}_{W_{\mathcal{O}_E}(S^\flat)}$ has a $W_{\mathcal{O}_E}(S^\flat)$-valued point, and hence is a trivial $G_{W_{\mathcal{O}_E}(S^\flat)}$-torsor. We fix a trivialization $\mathcal{P}_{W_{\mathcal{O}_E}(S^\flat)} \simeq G_{W_{\mathcal{O}_E}(S^\flat)}$. Let $\xi \in I_S$ be a generator. The condition (3) implies that, for each i, the base change of $F_{\mathcal{P}}$ along $W_{\mathcal{O}_E}(R^\flat) \to W_{\mathcal{O}_E}(V_i^\flat)$ corresponds to an element of $G(W_{\mathcal{O}_E}(V_i^\flat)[1/\xi])$ which is of the form $Z_i \mu(\xi) Z_i'$ for some $Z_i, Z_i' \in G(W_{\mathcal{O}_E}(V_i^\flat))$, via the induced trivialization $\mathcal{P}_{W_{\mathcal{O}_E}(V_i^\flat)} \simeq G_{W_{\mathcal{O}_E}(V_i^\flat)}$. We set

$$Z := (Z_i)_i \in G(W_{\mathcal{O}_E}(S^\flat)) = \prod_i G(W_{\mathcal{O}_E}(V_i^\flat)),$$

and similarly let $Z' := (Z_i')_i \in G(W_{\mathcal{O}_E}(S^\flat))$. Then the base change of $F_{\mathcal{P}}$ along $W_{\mathcal{O}_E}(R^\flat) \to W_{\mathcal{O}_E}(S^\flat)$ corresponds to the element $Z\mu(\xi)Z'$. This means that the condition (2) is satisfied. □

5.7. *Examples.* We discuss some examples of G-μ-displays and G-Breuil–Kisin modules of type μ for certain pairs (G, μ).

We first discuss a pair (G, μ) of Hodge type. Let G be a connected reductive group scheme over $\mathcal{O}_E$ and $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ a cocharacter. We assume that there exists a closed immersion $G \hookrightarrow \mathrm{GL}_n$ over $\mathcal{O}_E$ such that the composition $\mathbb{G}_m \to G_{\mathcal{O}} \to \mathrm{GL}_{n,\mathcal{O}}$ is conjugate to the cocharacter defined by

$$t \mapsto \mathrm{diag}\,(\underbrace{t, \dots, t}_{s}, \underbrace{1, \dots, 1}_{n-s})$$

for some s. In particular μ is 1-bounded. We set $L := \mathcal{O}_E^n$. By [Kisin 2010, Proposition 1.3.2], there exists a finite set of tensors $\{s_\alpha\}_{\alpha \in \Lambda} \subset L^\otimes$ such that $G \hookrightarrow \mathrm{GL}_n$ is the pointwise stabilizer of $\{s_\alpha\}_{\alpha \in \Lambda}$, where $L^\otimes$ is the direct sum of all $\mathcal{O}_E$-modules obtained from L by taking tensor products, duals, symmetric powers, and exterior powers. Let

$$L_{\mathcal{O}} = L_{\mu,1} \oplus L_{\mu,0}$$

be the weight decomposition with respect to μ. (Here the composition $\mathbb{G}_m \to G_{\mathcal{O}} \to \mathrm{GL}_{n,\mathcal{O}}$ is also denoted by μ.)

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. Let M be a Breuil–Kisin module of type μ over (A, I). We note that M is minuscule in the sense of Definition 3.1.5, and that the rank of the Hodge filtration $P^1 \subset (\phi^* M)/I(\phi^* M)$ is s. For a finite set of tensors $\{s_{\alpha,M}\}_{\alpha\in\Lambda} \subset M^{\otimes}$ which are F_M-invariant, we say that the pair $(M, \{s_{\alpha,M}\}_{\alpha\in\Lambda})$ is *G-μ-adapted* if there exist a (π, I)-completely étale covering $A \to B$ and an isomorphism $\psi : L_B \xrightarrow{\sim} M_B$ such that ψ carries s_α to $s_{\alpha,M}$ for each $\alpha \in \Lambda$ and the reduction modulo I of $\phi^*\psi$ identifies $(L_{\mu,1})_{B/IB} \subset L_{B/IB}$ with the Hodge filtration $(P^1)_{B/IB}$.

Proposition 5.7.1. *With the notation above, the groupoid $G\text{-}\mathrm{Disp}_\mu(A, I)$ is equivalent to the groupoid of G-μ-adapted pairs $(M, \{s_{\alpha,M}\}_{\alpha\in\Lambda})$ over (A, I).*

Proof. We shall construct a functor from the groupoid of G-μ-adapted pairs over (A, I) to $G\text{-}\mathrm{Disp}_\mu(A, I)$. Let $(M, \{s_{\alpha,M}\}_{\alpha\in\Lambda})$ be a G-μ-adapted pair over (A, I). Let

$$\mathcal{Q} := \underline{\mathrm{Isom}}_{\mathrm{Fil},\{s_\alpha\}}(L_A, \phi^* M) : (A, I)_{\text{ét}} \to \mathrm{Set}$$

be the functor sending $B \in (A, I)_{\text{ét}}$ to the set of isomorphisms $h : L_B \xrightarrow{\sim} (\phi^* M)_B$ preserving the filtrations and carrying s_α to $1 \otimes s_{\alpha,M}$ for each $\alpha \in \Lambda$. Here L_A is equipped with the filtration $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ given in Remark 3.2.2. We claim that $\mathcal{Q}$ is a $G_{\mu,A,I}$-torsor. For this, we may assume that there exists an isomorphism $\psi : L_A \xrightarrow{\sim} M$ such that ψ carries s_α to $s_{\alpha,M}$ for each $\alpha \in \Lambda$ and the reduction modulo I of $h := \phi^*\psi$ identifies $(L_{\mu,1})_{A/I}$ with P^1. Under the isomorphism $h : L_A \xrightarrow{\sim} \phi^* M$, we have

$$\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}} = \{\mathrm{Fil}^i(\phi^* M)\}_{i\in\mathbb{Z}},$$

which in turn implies that $h \in \mathcal{Q}(A)$. To see this, it suffices to prove that $\mathrm{Fil}^1_\mu = \mathrm{Fil}^1(\phi^* M)$ since M is minuscule. We observe that Fil^1_μ and $\mathrm{Fil}^1(\phi^* M)$ are the inverse images of $(L_{\mu,1})_{A/I} \subset L_{A/I}$ and $P^1 \subset (\phi^* M)/I(\phi^* M)$, respectively. It then follows that $\mathrm{Fil}^1_\mu = \mathrm{Fil}^1(\phi^* M)$.

We define $\widetilde{M} := \mathrm{Fil}^1(\phi^* M) \otimes_A I^{-1}$. Since M is of type μ, we see that F_M restricts to an isomorphism $\widetilde{M} \xrightarrow{\sim} M$, and we may regard $\{1 \otimes s_{\alpha,M}\}_{\alpha\in\Lambda}$ as tensors of $\widetilde{M}$. Similarly to Example 5.3.3, we have

$$\mathcal{Q}_{\mathrm{BK}} = \underline{\mathrm{Isom}}_{\{s_\alpha\}}(L_A, \widetilde{M}),$$

where $\underline{\mathrm{Isom}}_{\{s_\alpha\}}(L_A, \widetilde{M})$ is the G_A-torsor over $\operatorname{Spec} A$ defined by sending an A-algebra B to the set of isomorphisms $L_B \xrightarrow{\sim} \widetilde{M}_B$ carrying s_α to $1 \otimes s_{\alpha,M}$ for each $\alpha \in \Lambda$. Moreover, we have

$$\mathcal{Q}_A = \underline{\mathrm{Isom}}_{\{s_\alpha\}}(L_A, \phi^* M).$$

The base change $\phi^*(F_M) : \phi^* \widetilde{M} \xrightarrow{\sim} \phi^* M$ induces an isomorphism $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ of G_A-torsors. The $G_{\mu,A,I}$-torsor $\mathcal{Q}$ with α' is a G-μ-display over (A, I); see Remark 5.3.5. In this way, we obtain a functor from the groupoid of G-μ-adapted pairs over (A, I) to $G\text{-}\mathrm{Disp}_\mu(A, I)$.

One can prove that this functor is an equivalence in the same way as in the case of $G = \mathrm{GL}_n$; see Section 5.3. □

Remark 5.7.2. In [Kisin 2010, Proposition 1.3.4], [Kim and Madapusi Pera 2016, Theorem 2.5], and [Imai et al. 2023], it is observed that G-μ-adapted pairs naturally arise from crystalline Galois representations associated with integral canonical models of Shimura varieties of Hodge type with hyperspecial level structure. The notion of G-μ-adapted pairs plays a central role in the construction of integral canonical models in [Kisin 2010, Proposition 1.5.8] and [Kim and Madapusi Pera 2016, Section 3].

We include the following two important examples. The details will be presented elsewhere.

Example 5.7.3 (G-shtuka). Let G be a connected reductive group scheme over $\mathcal{O}_E$. We assume that k is an algebraic closure of $\mathbb{F}_q$ and let $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ be a 1-bounded (or equivalently, minuscule) cocharacter. Let C be an algebraically closed nonarchimedean field over $\mathcal{O}[1/\pi]$ with ring of integers $\mathcal{O}_C$. We consider the perfectoid space $S = \mathrm{Spa}(C, \mathcal{O}_C)$ and its tilt $S^\flat = \mathrm{Spa}(C^\flat, \mathcal{O}_C^\flat)$. We can show that the groupoid of G-μ-displays over $(W_{\mathcal{O}_E}(\mathcal{O}_C^\flat), I_{\mathcal{O}_C})$ is equivalent to the groupoid of G-*shtukas* over $S^\flat$ with one leg at S which are bounded by μ (or bounded by μ^{-1}, depending on the sign convention) introduced in [Scholze and Weinstein 2020]. See [Ito 2025, Section 5.1] for details.

Example 5.7.4 (orthogonal Breuil–Kisin module). Let $n = 2m$ be an even positive integer and we set $L := \mathcal{O}_E^n$. We define the quadratic form

$$Q : L \to \mathcal{O}_E$$

by $(a_1, \dots, a_{2m}) \mapsto \sum_{i=1}^m a_i a_{2m-i+1}$. The quadratic form Q is perfect in the sense that the bilinear form on L defined by $(x, y) \mapsto Q(x+y) - Q(x) - Q(y)$ is perfect. Let $G := \mathrm{O}(Q) \subset \mathrm{GL}_n$ be the orthogonal group of Q, which is a smooth affine group scheme over $\mathcal{O}_E$. Let $\mu : \mathbb{G}_m \to G \subset \mathrm{GL}_n$ be the cocharacter defined by

$$t \mapsto \mathrm{diag}\,(t, 1, \dots, 1, t^{-1}).$$

Let (A, I) be a bounded $\mathcal{O}_E$-prism. An *orthogonal Breuil–Kisin module of type* μ over (A, I) is a Breuil–Kisin module M of type μ over (A, I) together with a perfect quadratic form $Q_M : M \to A$ which is compatible with F_M in the sense that for every $x \in M$, we have $\phi(Q_M(x)) = Q_M(F_M(1 \otimes x))$ in $A[1/I]$. Let

$$\mathcal{P} := \underline{\mathrm{Isom}}_Q(L_A, M)$$

be the G_A-torsor over $\mathrm{Spec}\, A$ defined by sending an A-algebra B to the set of isomorphisms $L_B \simeq M_B$ of quadratic spaces. One can show that $\mathcal{P}$, together with the isomorphism $F_{\mathcal{P}} : (\phi^*\mathcal{P})[1/I] \xrightarrow{\sim} \mathcal{P}[1/I]$ induced by F_M, forms a G-Breuil–Kisin module of type μ over (A, I). This construction gives an equivalence between the groupoid of orthogonal Breuil–Kisin modules of type μ over (A, I) and the groupoid $G\text{-}\mathrm{BK}_\mu(A, I)$. Thus, by Proposition 5.3.8, the groupoid of orthogonal Breuil–Kisin modules of type μ over (A, I) is equivalent to the groupoid $G\text{-}\mathrm{Disp}_\mu(A, I)$. The details will be presented in a forthcoming paper.

Remark 5.7.5. Let the notation be as in Example 5.7.4. Our main result (Theorem 6.1.3) cannot be applied to Breuil–Kisin modules of type μ since μ is not 1-bounded as a cocharacter of GL_n. However, since μ is 1-bounded as a cocharacter of $G = \mathrm{O}(Q)$, the result can be applied to *orthogonal* Breuil–Kisin modules of type μ. Such an observation was made in [Lau 2021] in the context of the deformation theory of K3 surfaces.

6. Prismatic G-μ-displays over complete regular local rings

In this section, we prove the main result (Theorem 6.1.3) of this paper, which we state in Section 6.1. The proof will be given in Section 6.5. In Sections 6.2–6.4, we discuss a few technical results that will be used in the proof.

6.1. *G-μ-displays on absolute prismatic sites.* In this paper, we use the following definition.

Definition 6.1.1. Let R be a π-adically complete $\mathcal{O}$-algebra. A *prismatic G-μ-display over R* is defined to be an object of the groupoid

$$G\text{-}\mathrm{Disp}_{\mu}((R)_{\Delta,\mathcal{O}_E}) := 2-\varprojlim_{(A,I)\in(R)_{\Delta,\mathcal{O}_E}} G\text{-}\mathrm{Disp}_{\mu}(A,I).$$

Remark 6.1.2. Giving a prismatic G-μ-display $\mathfrak{Q}$ over R is equivalent to giving a G-μ-display $\mathfrak{Q}_{(A,I)}$ over (A,I) for each $(A,I)\in(R)_{\Delta,\mathcal{O}_E}$ and an isomorphism

$$\gamma_f : f^*(\mathfrak{Q}_{(A,I)}) \xrightarrow{\sim} \mathfrak{Q}_{(A',I')}$$

for each morphism $f:(A,I)\to(A',I')$ in $(R)_{\Delta,\mathcal{O}_E}$, such that $\gamma_{f'}\circ(f'^*\gamma_f)=\gamma_{f'\circ f}$ (via the natural identification $f'^*\circ f^*\simeq(f'\circ f)^*$) for two morphisms $f:(A,I)\to(A',I')$ and $f':(A',I')\to(A'',I'')$. We call $\mathfrak{Q}_{(A,I)}$ the *value* of $\mathfrak{Q}$ at $(A,I)\in(R)_{\Delta,\mathcal{O}_E}$.

Assume that R is a complete regular local ring over $\mathcal{O}$ with residue field k. Let $(\mathcal{O}[[t_1,\dots,t_n]],(\mathcal{E}))$ be an $\mathcal{O}_E$-prism of Breuil–Kisin type with an isomorphism $R\simeq\mathcal{O}[[t_1,\dots,t_n]]/\mathcal{E}$ over $\mathcal{O}$ (where $n\geq 0$ is the dimension of R). Such an $\mathcal{O}_E$-prism exists; see for example [Cheng 2018, Section 3.3]. We set $\mathfrak{S}_{\mathcal{O}}:=\mathcal{O}[[t_1,\dots,t_n]]$. Our goal is to prove the following result.

Theorem 6.1.3. *Assume that the cocharacter μ is 1-bounded. Then the functor*

$$G\text{-}\mathrm{Disp}_{\mu}((R)_{\Delta,\mathcal{O}_E})\to G\text{-}\mathrm{Disp}_{\mu}(\mathfrak{S}_{\mathcal{O}},(\mathcal{E})),\quad \mathfrak{Q}\mapsto\mathfrak{Q}_{(\mathfrak{S}_{\mathcal{O}},(\mathcal{E}))},$$

given by evaluation at $(\mathfrak{S}_{\mathcal{O}},(\mathcal{E}))$ is an equivalence.

The rest of this section is devoted to the proof of Theorem 6.1.3.

6.2. *Coproducts of Breuil–Kisin prisms.* In this subsection, we establish some properties of the object

$$(\mathfrak{S}_{\mathcal{O}},(\mathcal{E}))\in(R)_{\Delta,\mathcal{O}_E}.$$

We begin with the following result.

Proposition 6.2.1. *For any $(A,I)\in(R)_{\Delta,\mathcal{O}_E}$, there exists a flat covering $(A,I)\to(A',I')$ in $(R)_{\Delta,\mathcal{O}_E}$ such that (A',I') admits a morphism $(\mathfrak{S}_{\mathcal{O}},(\mathcal{E}))\to(A',I')$ in $(R)_{\Delta,\mathcal{O}_E}$.*

Proof. We may assume that (A,I) is orientable by Remark 2.5.14. Let $d\in I$ be a generator. Let $v_1,\dots,v_n\in A$ be elements such that each v_i is a lift of the image of $t_i\in\mathfrak{S}_{\mathcal{O}}$ under the composition $\mathfrak{S}_{\mathcal{O}}\to R\to A/I$. Let $B:=A\otimes_{\mathcal{O}}\mathfrak{S}_{\mathcal{O}}$. We set

$$x_i := 1\otimes t_i - v_i\otimes 1\in B.$$

Then the morphism $A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ of animated rings is faithfully flat. Indeed, using that the natural homomorphism $\mathcal{O}[t_1, \dots, t_n] \to \mathfrak{S}_{\mathcal{O}}$ is flat, we see that $A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ is flat. Since the composition $\mathfrak{S}_{\mathcal{O}} \to R \to A/(\pi, d)$ induces a homomorphism $B/(\pi, d, x_1, \dots, x_n) \to A/(\pi, d)$ over $A/(\pi, d)$, it follows that $A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ is faithfully flat.

Let $\mathfrak{S}_{\mathcal{O},\infty}$ be the $(\pi, \mathcal{E})$-adic completion of a colimit $\varinjlim_{\phi} \mathfrak{S}_{\mathcal{O}}$ of the diagram

$$\mathfrak{S}_{\mathcal{O}} \xrightarrow{\phi} \mathfrak{S}_{\mathcal{O}} \xrightarrow{\phi} \mathfrak{S}_{\mathcal{O}} \to \cdots .$$

Since $\phi : \mathfrak{S}_{\mathcal{O}} \to \mathfrak{S}_{\mathcal{O}}$ is faithfully flat, we see that $\mathfrak{S}_{\mathcal{O}} \to \mathfrak{S}_{\mathcal{O},\infty}$ is $(\pi, \mathcal{E})$-completely faithfully flat. In fact, it is faithfully flat by [Yekutieli 2018, Theorem 1.5]. We set $B' := B \otimes_{\mathfrak{S}_{\mathcal{O}}} \mathfrak{S}_{\mathcal{O},\infty}$. Then $A/^{\mathbb{L}}(\pi, d) \to B'/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ is faithfully flat as well. Thus, by Proposition 2.6.6, we can consider the prismatic envelope

$$(A', I') := (B'\{J/I\}^{\wedge}, IB'\{J/I\}^{\wedge})$$

of B' over (A, I) with respect to the ideal $J := (d, x_1, \dots, x_n) \subset B'$. The map $(A, I) \to (A', I')$ is a flat covering.

We shall construct a morphism $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$. We remark that since A'/I' is not necessarily $\mathfrak{m}$-adically complete for the maximal ideal $\mathfrak{m} \subset R$, it is not clear that the natural homomorphism $\mathfrak{S}_{\mathcal{O}} \to A'$ induces a morphism $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$. Instead, we construct a morphism $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A', I')$ as follows. Since $\mathfrak{S}_{\mathcal{O},\infty}$ can be identified with the $(\pi, \mathcal{E})$-adic completion of

$$\bigcup_{m \geq 0} \mathfrak{S}_{\mathcal{O}}[t_1^{1/q^m}, \dots, t_n^{1/q^m}],$$

the quotient $R_\infty := \mathfrak{S}_{\mathcal{O},\infty}/\mathcal{E}$ is the π-adic completion of $\bigcup_{m \geq 0} R[\bar{t}_1^{1/q^m}, \dots, \bar{t}_n^{1/q^m}]$, where $\bar{t}_i \in R$ is the image of t_i. Here

$$\mathfrak{S}_{\mathcal{O}}[t_1^{1/q^m}, \dots, t_n^{1/q^m}] = \mathfrak{S}_{\mathcal{O}}[X_1, \dots, X_n]/(X_1^{q^m} - t_1, \dots, X_n^{q^m} - t_n)$$

and similarly for $R[\bar{t}_1^{1/q^m}, \dots, \bar{t}_n^{1/q^m}]$. The composition $R \to A/I \to A'/I'$ factors through the homomorphism

$$g : R_\infty \to A'/I'$$

defined by sending $\bar{t}_i^{1/q^m}$ to the image of $1 \otimes t_i^{1/q^m} \in A'$, which is well-defined since $1 \otimes t_i = v_i \otimes 1$ in A'/I'. By Lemma 2.3.5, there exists a unique map $(\mathfrak{S}_{\mathcal{O},\infty}, (\mathcal{E})) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms which induces g. By construction, the composition

$$(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (\mathfrak{S}_{\mathcal{O},\infty}, (\mathcal{E})) \to (A', I')$$

is a morphism in $(R)_{\Delta,\mathcal{O}_E}$. □

Remark 6.2.2. Assume that $\mathcal{O}_E = \mathbb{Z}_p$. In this case, Proposition 6.2.1 is proved in [Bhatt and Scholze 2022, Example 7.13] and [Anschütz and Le Bras 2023, Lemma 5.14], using [Bhatt and Scholze 2022, Proposition 7.11]. Moreover, if $\mathfrak{S}_{\mathcal{O}} = W(k)[[t]]$ and $\mathcal{E} \in W(k)[[t]]$ is an Eisenstein polynomial, then an alternative argument using prismatic envelopes is given in [Bhatt and Scholze 2023, Example 2.6]. Our argument is similar to the one given there, but we have to modify it slightly in order to treat the case where A/I is not $\mathfrak{m}$-adically complete.

In a similar way, we obtain the following result:

Lemma 6.2.3. *Let $(A, I) \in (R)_{\Delta,\mathcal{O}_E}$ and let $f_1, f_2 : (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A, I)$ be two morphisms in $(R)_{\Delta,\mathcal{O}_E}$. If $f_1(t_i) = f_2(t_i)$ in A for any i, then we have $f = g$.*

Proof. As in the proof of Proposition 6.2.1, let $\mathfrak{S}_{\mathcal{O},\infty}$ be the $(\pi, \mathcal{E})$-adic completion of $\varinjlim_{\phi} \mathfrak{S}_{\mathcal{O}}$, which is faithfully flat over $\mathfrak{S}_{\mathcal{O}}$. After replacing (A, I) by a flat covering, we may assume that $f_1 : (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A, I)$ factors through a map $\tilde{f}_1 : (\mathfrak{S}_{\mathcal{O},\infty}, (\mathcal{E})) \to (A, I)$ of $\mathcal{O}_E$-prisms. Since $\mathfrak{S}_{\mathcal{O},\infty}$ is the $(\pi, \mathcal{E})$-adic completion of $\bigcup_{m\geq 0} \mathfrak{S}_{\mathcal{O}}[t_1^{1/q^m}, \dots, t_n^{1/q^m}]$, there exists a map $\tilde{f}_2 : \mathfrak{S}_{\mathcal{O},\infty} \to A$ extending f_2 such that

$$\tilde{f}_2(t_i^{1/q^m}) = \tilde{f}_1(t_i^{1/q^m})$$

for all m and i. The map $\tilde{f}_2$ preserves the δ_E-structures by Corollary 2.2.16. It suffices to prove that $\tilde{f}_1 = \tilde{f}_2$. We note that both f_1 and f_2 induce the same homomorphism $R \to A/I$. Since $R_\infty = \mathfrak{S}_{\mathcal{O},\infty}/\mathcal{E}$ is the π-adic completion of $\bigcup_{m\geq 0} R[\bar{t}_1^{1/q^m}, \dots, \bar{t}_n^{1/q^m}]$, it follows that the homomorphism $R_\infty \to A/I$ induced by $\tilde{f}_1$ agrees with the one induced by $\tilde{f}_2$. Then, by Lemma 2.3.5, we conclude that $\tilde{f}_1 = \tilde{f}_2$. □

We next study a coproduct of two copies of $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ in the category $(R)_{\Delta,\mathcal{O}_E}$. To simplify the notation, we write

$$(A, I) := (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$$

in the rest of this section. We set

$$B := A[\![x_1, \dots, x_n]\!]$$

and let $p'_1 : A \to B$ be the natural homomorphism. There exists a unique δ_E-structure on B such that p'_1 is a homomorphism of δ_E-rings and the associated Frobenius $\phi : B \to B$ sends x_i to $(x_i + t_i)^q - t_i^q$ for every i. We consider the prismatic envelope

$$(A^{(2)}, I^{(2)})$$

of B over (A, I) with respect to the ideal $(\mathcal{E}, x_1, \dots, x_n) \subset B$ as in Proposition 2.6.6. Let $p_1 : (A, I) \to (A^{(2)}, I^{(2)})$ denote the natural map. We view $(A^{(2)}, I^{(2)})$ as an object of $(R)_{\Delta,\mathcal{O}_E}$ via the homomorphism $\bar{p}_1 : R \to A^{(2)}/I^{(2)}$ induced by p_1.

The homomorphism $p'_2 : A \to B$ over $\mathcal{O}$ defined by $t_i \mapsto x_i + t_i$ is a homomorphism of δ_E-rings. Let $p_2 : A \to A^{(2)}$ be the composition of p'_2 with the natural homomorphism $B \to A^{(2)}$.

Lemma 6.2.4. *Let the notation be as above.*

(1) *We have $p_2(I) \subset I^{(2)}$, and the induced map $p_2 : (A, I) \to (A^{(2)}, I^{(2)})$ is a morphism in $(R)_{\Delta,\mathcal{O}_E}$.*

(2) *The object $(A^{(2)}, I^{(2)}) \in (R)_{\Delta,\mathcal{O}_E}$ with the morphisms $p_1, p_2 : (A, I) \to (A^{(2)}, I^{(2)})$ is a coproduct of two copies of (A, I) in the category $(R)_{\Delta,\mathcal{O}_E}$.*

Proof. (1) It suffices to show that the composition of $p_2 : A \to A^{(2)}$ with $A^{(2)} \to A^{(2)}/I^{(2)}$ coincides with the composition of $A \to R$ with $\bar{p}_1 : R \to A^{(2)}/I^{(2)}$. For any $h \in A$, the element $p'_2(h) - p'_1(h) \in B$ is contained in the ideal $(x_1, \dots, x_n) \subset B$. Since the image of x_i in $A^{(2)}$ is contained in $I^{(2)}$, we have $p_2(h) - p_1(h) \in I^{(2)}$, which implies the assertion.

(2) We have to show that for any $(A', I') \in (R)_{\Delta,\mathcal{O}_E}$ and two morphisms $f_1, f_2 : (A, I) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$, there exists a unique morphism

$$f : (A^{(2)}, I^{(2)}) \to (A', I')$$

in $(R)_{\Delta,\mathcal{O}_E}$ such that $f \circ p_1 = f_1$ and $f \circ p_2 = f_2$.

We first prove the uniqueness of f. Let $f' : B \to A'$ be the composition of f with $B \to A^{(2)}$. Then we have $f' \circ p'_j = f_j$ $(j = 1, 2)$, and f' sends $x_i = p'_2(t_i) - p'_1(t_i)$ to

$$f_2(t_i) - f_1(t_i) \in I' \subset A'$$

for any i. Since A' is I'-adically complete, such a homomorphism f' of δ_E-rings is uniquely determined (if it exists). The uniqueness of f now follows from the universal property of the prismatic envelope $(A^{(2)}, I^{(2)})$.

We next prove the existence of f. Since $f_2(t_i) - f_1(t_i) \in I' \subset A'$ and A' is I'-adically complete, there exists a unique homomorphism $f' : B \to A'$ over $\mathcal{O}$ such that $f' \circ p'_1 = f_1$ and $f'(x_i) = f_2(t_i) - f_1(t_i)$ for every i.

We claim that f' is a homomorphism of δ_E-rings. Indeed, as in the proof of Proposition 6.2.1, let A_∞ be the $(\pi, \mathcal{E})$-adic completion of $\varinjlim_\phi A$. Then A_∞ is faithfully flat over A. After replacing (A', I') by a flat covering, we may assume that f_j factors through a morphism $\tilde{f}_j : (A_\infty, IA_\infty) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$ for each $j = 1, 2$. For an integer $m \geq 0$ and i, we set

$$x_{i,m} := \tilde{f}_2(t_i^{1/q^m}) - \tilde{f}_1(t_i^{1/q^m}) \in A'.$$

Since we have $x_{i,m}^{q^m} \in (\pi, I')$, it follows that A' is $(x_{1,m}, \dots, x_{n,m})$-adically complete for every m (see [Stacks 2005–, Tag 090T] for example). Thus, for each $m \geq 0$, there exists a unique homomorphism $f'(m) : B \to A'$ such that $f'(m) \circ p'_1$ is the composition

$$A \to A_\infty \xrightarrow{\phi^{-m}} A_\infty \xrightarrow{\tilde{f}_1} A'$$

and $f'(m)(x_i) = x_{i,m}$ for any i. Since $f'(m) = f'(m+1) \circ \phi$, they give rise to a homomorphism $\tilde{f}' : \varinjlim_\phi B \to A'$. By Corollary 2.2.16, $\tilde{f}'$ is a homomorphism of δ_E-rings. Since f' is the composition $B \to \varinjlim_\phi B \to A'$, we conclude that f' is a homomorphism of δ_E-rings.

By the universal property of the prismatic envelope $(A^{(2)}, I^{(2)})$, the homomorphism f' extends to a unique morphism $f : (A^{(2)}, I^{(2)}) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$. By construction, we have $f \circ p_1 = f_1$. It follows from Lemma 6.2.3 that $f \circ p_2 = f_2$.

The proof of Lemma 6.2.4 is complete. □

Remark 6.2.5. It might be more natural to expect that the prismatic envelope (C, IC) of $A \otimes_{\mathcal{O}} A$ over (A, I) with respect to the ideal $(\mathcal{E} \otimes 1, t_1 \otimes 1 - 1 \otimes t_1, \ldots, t_n \otimes 1 - 1 \otimes t_n)$ is a coproduct of two copies of (A, I) in the category $(R)_{\Delta, \mathcal{O}_E}$, where we regard $A \otimes_{\mathcal{O}} A$ as an A-algebra via the homomorphism $a \mapsto a \otimes 1$. However, this does not seem to be the case in general. For example, it is not clear that the homomorphism $A \to C,\ a \mapsto 1 \otimes a$, induces a morphism $(A, I) \to (C, IC)$ in $(R)_{\Delta, \mathcal{O}_E}$ (see the proof of Proposition 6.2.1).

Let

$$m : (A^{(2)}, I^{(2)}) \to (A, I)$$

be the unique morphism in $(R)_{\Delta, \mathcal{O}_E}$ such that $m \circ p_1 = m \circ p_2 = \mathrm{id}_{(A,I)}$. Let K be the kernel of $m : A^{(2)} \to A$. Let $d \in I^{(2)}$ be a generator.

Lemma 6.2.6 (cf. [Anschütz and Le Bras 2023, Lemma 5.15]). *We have $\phi(K) \subset dK$.*

Proof. It suffices to show that $\phi(K) \subset dA^{(2)}$. Indeed, let $x \in K$, and we assume that $\phi(x) = dy$ for some $y \in A^{(2)}$. Then, since $m(\phi(x)) = 0$ and $m(d) \in A$ is a nonzerodivisor, we have $y \in K$.

We shall prove that $\phi(K) \subset dA^{(2)}$. We may assume that $d = p_1(\mathcal{E})$. The image $p_1'(\mathcal{E}) \in B$ is also denoted by d. It follows from Proposition 2.6.6 that $A^{(2)}$ can be identified with the (π, d)-adic completion of

$$C := B\{x_1/d, \ldots, x_n/d\}.$$

We write $y_i := x_i/d$. The composition $C \to A^{(2)} \to A$ sends $\delta_E^j(y_i)$ to 0 for any $j \geq 0$ and any i. Here δ_E^j is the j-th iterate of the map $\delta_E : C \to C$. Since the kernel of the homomorphism $B \to A$ defined by $x_i \mapsto 0$ $(1 \leq i \leq n)$ coincides with $(x_1, \ldots, x_n)$, it follows that the kernel K_0 of $C \to A$ is generated by

$$\{\delta_E^j(y_i)\}_{1 \leq i \leq n,\, j \geq 0}.$$

We note that K can be identified with the (π, d)-adic completion of K_0. We also note that $dA^{(2)} = \bigcap_{l \geq 0}(dA^{(2)} + (\pi, d)^l A^{(2)})$ since $A^{(2)}/d$ is π-adically complete (see Remark 2.3.2). It then suffices to show that $\phi(\delta_E^j(y_i)) \in dA^{(2)}$ for any $j \geq 0$ and any i. This can be proved by the same argument as in the first paragraph of the proof of [Anschütz and Le Bras 2023, Lemma 5.15] when $\mathcal{E} \in A$ is not contained in πA. A similar argument holds when $\mathcal{E} \in \pi A$. We include the argument in this case for the convenience of the reader.

We may assume that $\mathcal{E} = \pi$. In fact, we prove a more general statement: for any $j \geq 0$, we have $\phi^l(\delta_E^j(y_i)) \in \pi^l A^{(2)}$ for any $l \geq 1$ and any i. We proceed by induction on j. Let $u_i := x_i + t_i = \pi y_i + t_i \in A^{(2)}$. Then we have

$$\phi^l(x_i) = u_i^{q^l} - t_i^{q^l} = \sum_{0 \leq h \leq q^l - 1} \binom{q^l}{h} (\pi y_i)^{q^l - h} t_i^h \in \pi^{l+1} A^{(2)}.$$

Thus, we obtain $\phi^l(y_i) \in \pi^l A^{(2)}$, which proves the assertion in the case where $j = 0$. Suppose that the assertion holds for some $j \geq 0$. Since

$$\pi \phi^l(\delta_E^{j+1}(y_i)) = \phi^l(\pi \delta_E^{j+1}(y_i)) = \phi^l(\phi(\delta_E^j(y_i)) - \delta_E^j(y_i)^q) = \phi^{l+1}(\delta_E^j(y_i)) - \phi^l(\delta_E^j(y_i))^q,$$

the induction hypothesis implies that $\pi \phi^l(\delta_E^{j+1}(y_i)) \in \pi^{l+1} A^{(2)}$, whence $\phi^l(\delta_E^{j+1}(y_i)) \in \pi^l A^{(2)}$. □

The following lemma plays a crucial role in the proof of Theorem 6.1.3 (especially in the proof of Proposition 6.4.1 below). As in the proof of Lemma 6.2.6, we set $y_i := x_i/d \in A^{(2)}$.

Lemma 6.2.7. *Let $M \subset A^{(2)}$ be the ideal generated by $\phi(y_1)/d, \dots, \phi(y_n)/d \in K$. Then we have inclusions*

$$\phi(K) \subset dM + d(\pi, d)K \quad \textit{and} \quad \phi(M) \subset d(t_1, \dots, t_n)M + d(\pi, d)K.$$

The proof of Lemma 6.2.7 will be given in Section 6.3.

Remark 6.2.8. There exists a coproduct $(A^{(3)}, I^{(3)})$ of three copies of (A, I) in the category $(R)_{\Delta,\mathcal{O}_E}$. Indeed, one can define $(A^{(3)}, I^{(3)})$ as a pushout of the diagram

$$(A^{(2)}, I^{(2)}) \xleftarrow{p_2} (A, I) \xrightarrow{p_1} (A^{(2)}, I^{(2)}),$$

which exists since p_1 is a flat map (see Remark 2.5.4). Let $q_1, q_2, q_3 : (A, I) \to (A^{(3)}, I^{(3)})$ denote the associated three morphisms. For $1 \le i < j \le 3$, let $p_{ij} : (A^{(2)}, I^{(2)}) \to (A^{(3)}, I^{(3)})$ be the unique morphism such that $p_{ij} \circ p_1 = q_i$ and $p_{ij} \circ p_2 = q_j$.

Corollary 6.2.9. *Let $m : (A^{(3)}, I^{(3)}) \to (A, I)$ be the unique morphism in $(R)_{\Delta,\mathcal{O}_E}$ such that $m \circ q_i = \mathrm{id}_{(A,I)}$ for $i = 1, 2, 3$. Let L be the kernel of $m : A^{(3)} \to A$. Let $d \in I^{(3)}$ be a generator. Then the following assertions hold*:

(1) *We have $\phi(L) \subset dL$.*

(2) *Let $N \subset A^{(3)}$ be the ideal generated by $\{\phi(p_{12}(y_l))/d, \phi(p_{23}(y_l))/d\}_{1\le l\le n} \subset L$. Then we have inclusions*

$$\phi(L) \subset dN + d(\pi, d)L \quad \textit{and} \quad \phi(N) \subset d(q_1(t_1), \dots, q_1(t_n))N + d(\pi, d)L.$$

Proof. We may assume that d is the image of a generator of $I^{(2)}$, again denoted by d, under the homomorphism p_{12}. As in Remark 6.2.8, we identify $A^{(3)}$ with the (π, d)-adic completion of $A_0^{(3)} := A^{(2)} \otimes_{p_2,A,p_1} A^{(2)}$. Under this identification, the homomorphism p_{12} (resp. p_{23}) is induced by the homomorphism $A^{(2)} \to A_0^{(3)}$ defined by $a \mapsto a \otimes 1$ (resp. $a \mapsto 1 \otimes a$). The kernel L_0 of the natural homomorphism $A_0^{(3)} \to A$ coincides with $K \otimes_A A^{(2)} + A^{(2)} \otimes_A K$, and L is the (π, d)-adic completion of L_0.

In order to prove the assertion (1), it suffices to show that for any element $x \in L$ which lies in the image of $L_0 \to L$, we have $\phi(x) \in dA^{(3)}$. (Note that $A^{(3)}/d$ is π-adically complete by Remark 2.3.2.) This follows from Lemma 6.2.6. Similarly, the assertion (2) follows from Lemma 6.2.7. We note here that, since $q_j(t_l) - q_i(t_l) = p_{ij}(x_l) \in dL$ for $1 \le i < j \le 3$, the ideal $d(q_1(t_1), \dots, q_1(t_n))N + d(\pi, d)L$ is unchanged if we replace q_1 by q_i $(1 \le i \le 3)$. □

Remark 6.2.10. Assume that $\mathcal{O}_E = \mathbb{Z}_p$. Under the assumption that $n = 1$ and R is p-torsion-free, Anschütz and Le Bras [2023, Section 5.2] gave a proof of the analogue of Theorem 6.1.3 for minuscule Breuil–Kisin modules. (We will come back to this result in Section 7.1.) In the proof, they use that the map $K \to K$, $x \mapsto \phi(x)/d$ is topologically nilpotent with respect to the (p, d)-adic topology [loc. cit., Lemma 5.15]. This topological nilpotence may not be true if $n \ge 2$ or $p = 0$ in R. We will use Lemma 6.2.7, Corollary 6.2.9, and the fact that the local ring A is complete and noetherian to overcome this issue; see Section 6.4.

6.3. ***Proof of Lemma 6.2.7.*** The proof of Lemma 6.2.7 will require some preliminary results. We first introduce some notation.

If $\mathcal{E}$ is not contained in πA, then the image of $\mathcal{E}$ in A/π is a nonzerodivisor (since A/π is an integral domain). In this case, the δ_E-ring $A\{\phi(\mathcal{E})/\pi\}$ is π-torsion-free, and is isomorphic to the $\mathcal{O}_E$-PD envelope $D_{(\mathcal{E})}(A)$ of A with respect to the ideal $(\mathcal{E})$; see Corollary 2.6.5. Let A'' be the π-adic completion of $A\{\phi(\mathcal{E})/\pi\}$, and let $g : A \to A''$ be the natural homomorphism. We note that A'' is also π-torsion-free. We consider the following pushout squares of δ_E-rings:

$$\begin{array}{ccccccc}
A & \xrightarrow{p_1'} & B & \longrightarrow & A^{(2)} & \xrightarrow{m} & A \\
\phi\downarrow & & \downarrow & & \downarrow & & \downarrow\phi \\
A & \longrightarrow & B' & \longrightarrow & A^{(2)'} & \longrightarrow & A \\
g\downarrow & & \downarrow & & \downarrow & & \downarrow g \\
A'' & \longrightarrow & B_0'' & \longrightarrow & A_0^{(2)''} & \longrightarrow & A''
\end{array}$$

Let $A^{(2)''}$ be the π-adic completion of $A_0^{(2)''}$ and K'' the kernel of the induced homomorphism $A^{(2)''} \to A''$. Since $A \to A^{(2)}$ is flat (by Proposition 2.6.6 and [Yekutieli 2018, Theorem 1.5]), so is $A'' \to A_0^{(2)''}$. It follows that $A^{(2)''}$ is π-torsion-free. In the case where $\mathcal{E} \in \pi A$, we set $A^{(2)''} := A^{(2)}$ and $K'' := K$.

Lemma 6.3.1. *Let the notation be as above. Then the following assertions hold*:

(1) *We have $\phi(K'') \subset \pi K''$.*

(2) *We have $x_i \in \pi K''$ for any $1 \leq i \leq n$. (Here we denote the image of $x_i \in B$ in $A^{(2)''}$ again by x_i.) We set $w_i := x_i/\pi \in K''$. Then $K''/\pi K''$ is generated by the images of $\{\delta_E^j(w_i)\}_{1\leq i\leq n, j\geq 0}$ as an $A^{(2)''}$-module.*

Proof. If $\mathcal{E} \in \pi A$, then the assertions follow from Lemma 6.2.6 and its proof. Thus, we may assume that $\mathcal{E}$ is not contained in πA. Let $h : A^{(2)} \to A^{(2)''}$ denote the natural homomorphism. Since $g(\phi(\mathcal{E}))/\pi \in A''$ is a unit by Lemma 2.3.3(1), it follows that $h(d) \in A^{(2)''}$ is a unit multiple of π. The kernel K'' of $A^{(2)''} \to A''$ can be identified with the π-adic completion of h^*K. Therefore, the assertion (1) follows from Lemma 6.2.6.

Using that the image of $g(\phi(\mathcal{E}))$ in B_0'' is a unit multiple of π, we see that $A^{(2)''}$ agrees with the π-adic completion of $B_0''\{x_1/\pi, \ldots, x_n/\pi\}$. Since the kernel of $B_0'' \to A''$ is generated by $x_1, \ldots, x_n$, it follows that the kernel of $B_0''\{x_1/\pi, \ldots, x_n/\pi\} \to A''$ is generated by $\{\delta_E^j(x_i/\pi)\}_{1\leq i\leq n, j\geq 0}$, which implies (2). $\square$

Lemma 6.3.2. *We define*

$$\phi_1 : K'' \to K'', \quad x \mapsto \phi(x)/\pi.$$

The induced ϕ-linear homomorphism $K''/\pi K'' \to K''/\pi K''$ is denoted by the same symbol ϕ_1. Let $\overline{M}'' \subset K''/\pi K''$ be the $A^{(2)''}$-submodule generated by the images of $\phi_1(w_1), \ldots, \phi_1(w_n) \in K''$. Then we have inclusions

$$\phi_1(K''/\pi K'') \subset \overline{M}'' \quad \text{and} \quad \phi_1(\overline{M}'') \subset (t_1, \ldots, t_n)\overline{M}'',$$

where we denote the image of $t_i \in A^{(2)}$ in $A^{(2)''}$ again by t_i.

Proof. We have $x^q = \phi(x) - \pi\delta_E(x) \in \pi K''$ for every $x \in K''$ by Lemma 6.3.1. Let $J \subset K''$ be the ideal generated by $\{x^q/\pi\}_{x\in K''}$. For any $x \in K''$, we have

$$\phi_1(x^q/\pi) = \phi(x^q/\pi)/\pi = \phi(x)^q/\pi^2 = \pi^{q-1}(\phi_1(x)^q/\pi) \in \pi J,$$

and thus we obtain $\phi_1(J) \subset \pi J$.

We shall prove that $K''/(J+\pi K'')$ is generated by the images of $w_1, \dots, w_n$ as an $A^{(2)''}$-module. By Lemma 6.3.1, it suffices to show that for any $j \geq 0$ and any i, the image of $\delta_E^j(w_i)$ in $K''/(J+\pi K'')$ is contained in the $A^{(2)''}$-submodule of $K''/(J+\pi K'')$ generated by the images of $w_1, \dots, w_n$. We proceed by induction on j. If $j = 0$, then the assertion holds trivially. Assume that the assertion is true for some $j \geq 0$. Since

$$\phi_1(w_i) = \phi(x_i)/\pi^2 = ((x_i + t_i)^q - t_i^q)/\pi^2 = ((\pi w_i + t_i)^q - t_i^q)/\pi^2,$$

we can write $\phi_1(w_i)$ as

$$\phi_1(w_i) = \pi^{q-2} w_i^q + (q/\pi) t_i^{q-1} w_i + \pi b_i \tag{6-1}$$

for some element $b_i \in K''$. For any $x \in K''$, we have $\delta_E(x) = \phi_1(x)$ in K''/J. Thus the image of $\delta_E^{j+1}(w_i)$ in $K''/(J+\pi K'')$ agrees with the one of $\phi_1(\delta_E^j(w_i))$, which is contained in the $A^{(2)''}$-submodule of $K''/(J+\pi K'')$ generated by the images of $\phi_1(w_1), \dots, \phi_1(w_n)$ by the induction hypothesis. Then (6-1) implies the assertion for $j+1$.

We have shown that every $x \in K''$ can be written as

$$x = \Big(\sum_{1\leq i\leq n} a_i w_i\Big) + b + \pi c$$

for some $a_i \in A^{(2)''}$ $(1 \leq i \leq n)$, $b \in J$, and $c \in K''$. Since $\phi_1(b) \in \pi J$, the image of $\phi_1(x)$ in $K''/\pi K''$ coincides with that of $\sum_{1\leq i\leq n} \phi(a_i)\phi_1(w_i)$. This proves that $\phi_1(K''/\pi K'') \subset \overline{M}''$. Moreover, since $\phi_1(w_i^q) = \phi(w_i)\phi_1(w_i^{q-1})$ is contained in $\pi K''$, it follows from (6-1) that the image of $\phi_1(\phi_1(w_i))$ in $K''/\pi K''$ is equal to that of $\phi_1((q/\pi)t_i^{q-1} w_i) = (q/\pi)t_i^{q(q-1)}\phi_1(w_i)$. This proves that $\phi_1(\overline{M}'') \subset (t_1, \dots, t_n)\overline{M}''$. □

We now prove Lemma 6.2.7.

Proof of Lemma 6.2.7. We first treat the case where $\mathcal{E} \in \pi A$. In this case d is a unit multiple of π. Thus, the assertion follows from Lemma 6.3.2.

We now assume that $\mathcal{E}$ is not contained in πA. We define

$$\phi_1 : K \to K, \quad x \mapsto \phi(x)/d.$$

The induced ϕ-linear homomorphism $K/(\pi, d)K \to K/(\pi, d)K$ is also denoted by ϕ_1. Let $\overline{M} \subset K/(\pi, d)K$ be the $A^{(2)}$-submodule generated by the images of $\phi_1(y_1), \dots, \phi_1(y_n) \in K$. It suffices to prove that $\phi_1(K/(\pi, d)K) \subset \overline{M}$ and $\phi_1(\overline{M}) \subset (t_1, \dots, t_n)\overline{M}$.

Let $f : A^{(2)} \to A^{(2)'}$ denote the natural homomorphism. Let K' be the kernel of the homomorphism $A^{(2)'} \to A$, which can be identified with f^*K. We define $\phi'_1 : K' \to K'$ by $x \mapsto \phi(x)/f(d)$, and let

$\overline{M}' \subset K'/(\pi, f(d))K'$ be the $A^{(2)'}$-submodule generated by the images of $\phi_1'(f(y_1)), \dots, \phi_1'(f(y_n)) \in K'$. Since $\phi : A \to A$ is faithfully flat, so is f. Therefore, in order to prove the assertion, it is enough to prove that

$$\phi_1'(K'/(\pi, f(d))K') \subset \overline{M}' \quad \text{and} \quad \phi_1'(\overline{M}') \subset (f(t_1), \dots, f(t_n))\overline{M}'. \tag{6-2}$$

The homomorphism $A^{(2)'} \to A^{(2)''}$ induced by $g : A \to A''$ is again denoted by g. The element $g(f(d))$ is a unit multiple of π in $A^{(2)''}$. Thus, for $\phi_1 : K'' \to K''$ defined in Lemma 6.3.2, the element $g(\phi_1'(x))$ is a unit multiple of $\phi_1(g(x))$ for any $x \in K'$. Also, the induced homomorphism $A^{(2)'}/(\pi, f(d)) \to A^{(2)''}/\pi$, again denoted by g, sends $\overline{M}'$ into $\overline{M}''$. It follows from Lemma 6.3.2 that, for any $x \in K'/(\pi, f(d))K'$ (resp. $x \in \overline{M}'$), we have

$$g(\phi_1'(x)) \in \overline{M}'' \quad (\text{resp. } g(\phi_1'(x)) \in (g(f(t_1)), \dots, g(f(t_n)))\overline{M}''). \tag{6-3}$$

Since $A''/\pi \simeq D_{(\mathcal{E})}(A)/\pi$, we can find a homomorphism

$$s : A''/\pi \to A/(\pi, \phi(\mathcal{E}))$$

of $\mathcal{O}_E$-algebras such that the composition $A/(\pi, \phi(\mathcal{E})) \xrightarrow{g} A''/\pi \xrightarrow{s} A/(\pi, \phi(\mathcal{E}))$ is the identity; see Example 2.6.2 and Lemma 2.6.3. We consider the following pushout squares of $\mathcal{O}_E$-algebras:

$$\begin{array}{ccccc}
A''/\pi & \longrightarrow & A^{(2)''}/\pi & \longrightarrow & A''/\pi \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \tilde{s}} & & \downarrow{\scriptstyle s} \\
A/(\pi, \phi(\mathcal{E})) & \longrightarrow & A^{(2)'}/(\pi, f(d)) & \longrightarrow & A/(\pi, \phi(\mathcal{E}))
\end{array}$$

The homomorphism $g : A^{(2)'}/(\pi, f(d)) \to A^{(2)''}/\pi$ is a section of $\tilde{s}$. We observe that $\tilde{s}(K''/\pi K'') \subset K'/(\pi, f(d))K'$ and $\tilde{s}(\overline{M}'') \subset \overline{M}'$. It follows from (6-3) that, for any $x \in K'/(\pi, f(d))K'$ (resp. $x \in \overline{M}'$), its image $\phi_1'(x) = \tilde{s}(g(\phi_1'(x)))$ belongs to $\overline{M}'$ (resp. $(f(t_1), \dots, f(t_n))\overline{M}'$). This proves (6-2), and the proof of Lemma 6.2.7 is now complete. □

6.4. *Deformations of isomorphisms.* As in Section 6.2, we write $(A, I) = (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$. In this subsection, as a preparation for the proof of Theorem 6.1.3, we study deformations of isomorphisms of G-μ-displays over (A, I) along the morphisms $m : (A^{(2)}, I^{(2)}) \to (A, I)$ and $m : (A^{(3)}, I^{(3)}) \to (A, I)$ defined in Section 6.2. Throughout this subsection, we assume that μ is 1-bounded.

Our setup is as follows. Let $(A', I') := (A^{(2)}, I^{(2)})$ (resp. $(A', I') := (A^{(3)}, I^{(3)})$). Let $m : (A', I') \to (A, I)$ denote $m : (A^{(2)}, I^{(2)}) \to (A, I)$ (resp. $m : (A^{(3)}, I^{(3)}) \to (A, I)$). Let $f_1, f_2 \in \{p_1, p_2\}$ (resp. $f_1, f_2 \in \{q_1, q_2, q_3\}$). We do not exclude the case where $f_1 = f_2$.

The purpose of this subsection is to prove the following result:

Proposition 6.4.1. *Assume μ is 1-bounded. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be G-μ-displays over (A, I). Then the map*

$$m^* : \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A', I')}(f_1^*(\mathcal{Q}_1), f_2^*(\mathcal{Q}_2)) \to \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A, I)}(\mathcal{Q}_1, \mathcal{Q}_2) \tag{6-4}$$

induced by the base change functor $m^* : G\text{-}\mathrm{Disp}_\mu(A', I') \to G\text{-}\mathrm{Disp}_\mu(A, I)$ is bijective.

We need some preliminary results for the proof of Proposition 6.4.1. We will use the following notation. Let H be a group scheme over $\mathcal{O}$. For an ideal $J \subset A'$, we write

$$H(J) := \operatorname{Ker}(H(A') \to H(A'/J))$$

for the kernel of the homomorphism $H(A') \to H(A'/J)$. If $H = G_{\mathcal{O}}$, then we simply write $G(J) := G_{\mathcal{O}}(J)$.

Let K denote the kernel of $m : A' \to A$. (We note that if $A' = A^{(3)}$, then this kernel was denoted by L in Corollary 6.2.9.) Let $d \in I'$ be a generator.

Lemma 6.4.2. *Let $J \subset A'$ be an ideal such that $J \subset dK$ and, for any $x \in J$, we have $\phi(x/d) \in J$. Then the homomorphism $\sigma_{\mu,d} : G_\mu(A', I') \to G(A')$ (see (5-1)) sends $G(J) \subset G_\mu(A', I')$ into itself.*

Proof. We note that, by Proposition 4.2.9, we have $G(J) \subset G(dK) \subset G_\mu(A', I')$, and the multiplication map $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu \to G_{\mathcal{O}}$ induces a bijection

$$(\operatorname{Lie}(U_\mu^-) \otimes_{\mathcal{O}} J) \times P_\mu(J) \xrightarrow{\sim} G(J).$$

Thus, it suffices to prove that $\sigma_{\mu,d}(P_\mu(J)) \subset G(J)$ and $\sigma_{\mu,d}(\operatorname{Lie}(U_\mu^-) \otimes_{\mathcal{O}} J) \subset G(J)$.

By Remark 4.1.3 and Lemma 4.2.2, we have $\mu(d)P_\mu(J)\mu(d)^{-1} \subset P_\mu(J)$. (In fact, this holds for any ideal $J \subset A'$.) Since $\phi(J) \subset J$, we have $\phi(G(J)) \subset G(J)$. It follows that $\sigma_{\mu,d}(P_\mu(J)) \subset G(J)$.

Since μ is 1-bounded, the homomorphism $G_\mu(A', I') \to G(A')$, $g \mapsto \mu(d)g\mu(d)^{-1}$, restricts to a homomorphism

$$\operatorname{Lie}(U_\mu^-) \otimes_{\mathcal{O}} J \to \operatorname{Lie}(U_\mu^-) \otimes_{\mathcal{O}} \frac{1}{d} J, \quad v \mapsto v/d.$$

(See Remark 4.2.7.) Since $\phi((1/d)J) \subset J$, we obtain $\sigma_{\mu,d}(\operatorname{Lie}(U_\mu^-) \otimes_{\mathcal{O}} J) \subset G(J)$. □

Definition 6.4.3. Let $J \subset A'$ be an ideal as in Lemma 6.4.2. For an element $X \in G(A')$, we define a homomorphism

$$\mathcal{U}_{d,X} : G(J) \to G(J), \quad g \mapsto X\sigma_{\mu,d}(g)X^{-1}.$$

We also define a map of sets

$$\mathcal{V}_{d,X} : G(J) \to G(J), \quad g \mapsto \mathcal{U}_{d,X}(g)g^{-1}.$$

Let $J_2 \subset J_1 \subset A'$ be two ideals which satisfy the assumption of Lemma 6.4.2. Then $\mathcal{U}_{d,X} : G(J_1) \to G(J_1)$ induces a homomorphism

$$G(J_1)/G(J_2) \to G(J_1)/G(J_2),$$

which we denote by the same symbol $\mathcal{U}_{d,X}$. Let $\mathcal{V}_{d,X} : G(J_1)/G(J_2) \to G(J_1)/G(J_2)$ be the map of sets defined by $g \mapsto \mathcal{U}_{d,X}(g)g^{-1}$.

By Lemma 6.2.6 and Corollary 6.2.9, we have $\phi(K) \subset dK$. Thus, the ideal $dK \subset A'$ satisfies the assumption of Lemma 6.4.2. We shall prove (in Proposition 6.4.7 below) that $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective for any $X \in G(A')$, from which we will deduce Proposition 6.4.1. For this purpose, we need the following lemmas.

Lemma 6.4.4. *Let $J_2 \subset J_1 \subset A'$ be two ideals which satisfy the assumption of Lemma 6.4.2. Assume that for any $x \in J_1$, we have $\phi(x/d) \in J_2$. Then we have*

$$\sigma_{\mu,d}(G(J_1)) \subset G(J_2).$$

In particular, the map $\mathcal{V}_{d,X} : G(J_1)/G(J_2) \to G(J_1)/G(J_2)$ is equal to the map $g \mapsto g^{-1}$ for any $X \in G(A')$.

Proof. The same argument as in the proof of Lemma 6.4.2 shows that $\sigma_{\mu,d}(G(J_1)) \subset G(J_2)$. The second assertion immediately follows from the first one. □

Lemma 6.4.5. *Let $J_3 \subset J_2 \subset J_1 \subset A'$ be three ideals which satisfy the assumption of Lemma 6.4.2. Let $X \in G(A')$. If the maps*

$$\mathcal{V}_{d,X} : G(J_1)/G(J_2) \to G(J_1)/G(J_2) \quad \text{and} \quad \mathcal{V}_{d,X} : G(J_2)/G(J_3) \to G(J_2)/G(J_3)$$

are bijective, then $\mathcal{V}_{d,X} : G(J_1)/G(J_3) \to G(J_1)/G(J_3)$ is also bijective.

Proof. Let us prove the surjectivity. Let $h \in G(J_1)/G(J_3)$ be an element. The image $h' \in G(J_1)/G(J_2)$ of h can be written as $h' = \mathcal{V}_{d,X}(g')$ for some element $g' \in G(J_1)/G(J_2)$. We choose some $g \in G(J_1)/G(J_3)$ which is a lift of g'. Then we see that $\mathcal{U}_{d,X}(g)^{-1}hg$ is contained in $G(J_2)/G(J_3)$, so that there exists an element $g'' \in G(J_2)/G(J_3)$ such that

$$\mathcal{V}_{d,X}(g'') = \mathcal{U}_{d,X}(g'')g''^{-1} = \mathcal{U}_{d,X}(g)^{-1}hg.$$

It follows that $h = \mathcal{V}_{d,X}(gg'')$. This proves that $\mathcal{V}_{d,X} : G(J_1)/G(J_3) \to G(J_1)/G(J_3)$ is surjective. The proof of the injectivity is similar. □

Lemma 6.4.6. *Let $l \geq 0$ be an integer. For any $X \in G(A')$, the map*

$$\mathcal{V}_{d,X} : G((\pi, d)^l dK)/G((\pi, d)^{l+1} dK) \to G((\pi, d)^l dK)/G((\pi, d)^{l+1} dK)$$

is bijective.

Proof. Step 1. We set $K_l := (\pi, d)^l K$. We consider the ideal $K^- := K^2 + (\pi, d)K$ and let $K_l^- := (\pi, d)^l K^-$. All of dK_l, dK_{l+1}, dK_l^- satisfy the assumption of Lemma 6.4.2. Since

$$\phi(K^2) \subset d^2K^2 \subset d(\pi, d)K,$$

we have $\phi(K_l^-) \subset dK_{l+1}$. Thus, it follows from Lemma 6.4.4 that $\mathcal{V}_{d,X}$ is bijective for $G(dK_l^-)/G(dK_{l+1})$. By Lemma 6.4.5, it now suffices to show that $\mathcal{V}_{d,X}$ is bijective for $G(dK_l)/G(dK_l^-)$.

Step 2. By Lemma 6.2.7 and Corollary 6.2.9, there exists a finitely generated ideal $M \subset A'$ which is contained in K such that $\phi(K) \subset dM + dK^-$ and $\phi(M) \subset (t_1, \dots, t_n)dM + dK^-$, where we abuse notation and denote the image of $t_i \in A$ under the morphism $p_1 : A \to A'$ (resp. $q_1 : A \to A'$) if $A' = A^{(2)}$ (resp. if $A' = A^{(3)}$) by the same symbol. We set $M_l := (\pi, d)^l M \subset K_l$. Then we have inclusions

$$\phi(K_l) \subset dM_l + dK_l^- \quad \text{and} \quad \phi(M_l) \subset (t_1, \dots, t_n)dM_l + dK_l^-. \tag{6-5}$$

In particular, the ideals $dM_l + dK_l^- \subset dK_l$ satisfy the assumption of Lemma 6.4.4, so that $\mathcal{V}_{d,X}$ is bijective for $G(dK_l)/G(dM_l + dK_l^-)$. By Lemma 6.4.5, it is enough to prove that $\mathcal{V}_{d,X}$ is bijective for $G(dM_l + dK_l^-)/G(dK_l^-)$.

Step 3. We shall prove that

$$G(dM_l + dK_l^-)/G(dK_l^-) \xrightarrow{\sim} \varprojlim_{r\geq 0} G(dM_l + dK_l^-)/G((t_1, \dots, t_n)^r dM_l + dK_l^-). \tag{6-6}$$

To simplify the notation, we set $C_1 := A'/(dM_l + dK_l^-)$ and $C_2 := A'/dK_l^-$. Let $N \subset C_2$ be the image of $dM_l + dK_l^-$. We first claim that

$$A' \xrightarrow{\sim} \varprojlim_{r\geq 0} A'/dK_r, \tag{6-7}$$

$$C_2 \xrightarrow{\sim} \varprojlim_{r\geq 0} C_2/(t_1, \dots, t_n)^r N. \tag{6-8}$$

Since d is a nonzerodivisor and K is (π, d)-adically complete, it follows that $dK \xrightarrow{\sim} \varprojlim_{r\geq 0} dK/dK_r$. This implies (6-7). Since N is killed by K, we see that N is a finitely generated module over $A'/K \xrightarrow{\sim} A$. Since A is noetherian and is $(t_1, \dots, t_n)$-adically complete, it follows that N is also $(t_1, \dots, t_n)$-adically complete, which means that $N \xrightarrow{\sim} \varprojlim_{r\geq 0} N/(t_1, \dots, t_n)^r N$. This implies (6-8).

We next show that $G(A') \to G(C_2)$ is surjective. Indeed, by (6-7) and the fact that $G(A'/dK_{r+1}) \to G(A'/dK_r)$ is surjective (as G is smooth), it follows that $G(A') \to G(A'/dK_r)$ is surjective for any r. Since we have $(dK_l^-)^2 \subset dK_{l+1} \subset dK_l^-$, we see that $G(A'/dK_{l+1}) \to G(C_2)$ is surjective (again by the smoothness of G). Therefore $G(A') \to G(C_2)$ is surjective, as desired. Similarly, it follows from (6-8) that $G(C_2) \to G(C_2/(t_1, \dots, t_n)^r N)$ is surjective.

Using the results obtained in the previous paragraph, we see that

$$G(dM_l + dK_l^-)/G(dK_l^-) \xrightarrow{\sim} \mathrm{Ker}(G(C_2) \to G(C_1)),$$

$$G(dM_l + dK_l^-)/G((t_1, \dots, t_n)^r dM_l + dK_l^-) \xrightarrow{\sim} \mathrm{Ker}(G(C_2/(t_1, \dots, t_n)^r N) \to G(C_1)).$$

Now (6-6) follows from (6-8).

Step 4. We claim that $\mathcal{V}_{d,X}$ is bijective for

$$G((t_1, \dots, t_n)^r dM_l + dK_l^-)/G((t_1, \dots, t_n)^{r+1} dM_l + dK_l^-)$$

for any $r \geq 0$. Indeed, the second inclusion of (6-5) shows that the assumption of Lemma 6.4.4 is satisfied in this case, and hence the assertion follows.

Using Lemma 6.4.5 repeatedly, we see that $\mathcal{V}_{d,X}$ is bijective for

$$G(dM_l + dK_l^-)/G((t_1, \dots, t_n)^r dM_l + dK_l^-)$$

for any $r \geq 0$. It then follows from (6-6) that $\mathcal{V}_{d,X}$ is bijective for $G(dM_l + dK_l^-)/G(dK_l^-)$ as well. □

Let us now prove the desired result.

Proposition 6.4.7. *For any $X \in G(A')$, the map $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective.*

Proof. By (6-7) in the proof of Lemma 6.4.6, we have

$$G(dK) \xrightarrow{\sim} \varprojlim_{l\geq 0} G(dK)/G((\pi, d)^l dK).$$

In order to show that $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective, it suffices to check that

$$\mathcal{V}_{d,X} : G(dK)/G((\pi, d)^l dK) \to G(dK)/G((\pi, d)^l dK)$$

is bijective for any $l \geq 0$. This follows from Lemma 6.4.6 by using Lemma 6.4.5 repeatedly. □

We also need the following lemma:

Lemma 6.4.8. *Let $\mathcal{Q}$ be a G-μ-display over (A, I). Then there exists a finite extension $\tilde{k}$ of k such that the base change of $\mathcal{Q}$ to $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ is banal, where $\widetilde{\mathcal{O}} := W(\tilde{k})\otimes_{W(\mathbb{F}_q)}\mathcal{O}_E$ and $A_{\widetilde{\mathcal{O}}} := A\otimes_{\mathcal{O}}\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}[[t_1, \dots, t_n]]$.*

Proof. The Hodge filtration $P(\mathcal{Q})_{A/I} = P(\mathcal{Q})_R$ of $\mathcal{Q}$ is a $(P_\mu)_R$-torsor over $\operatorname{Spec} R$. There exists a finite extension $\tilde{k}$ of k such that $P(\mathcal{Q})_R \times_{\operatorname{Spec} R} \operatorname{Spec} \tilde{k}$ is trivial. Since P_μ is smooth and $R\otimes_{\mathcal{O}}\widetilde{\mathcal{O}}$ is a complete local ring, it follows that $P(\mathcal{Q})_R$ is trivial over $R\otimes_{\mathcal{O}}\widetilde{\mathcal{O}}$. By Proposition 5.4.5, the base change of $\mathcal{Q}$ to $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ is banal. □

Proof of Proposition 6.4.1. By Lemma 6.4.8, there exists a finite Galois extension $\tilde{k}$ of k such that the base changes of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ are banal. Here $\widetilde{\mathcal{O}} := W(\tilde{k})\otimes_{W(\mathbb{F}_q)}\mathcal{O}_E$ and $A_{\widetilde{\mathcal{O}}} := A\otimes_{\mathcal{O}}\widetilde{\mathcal{O}}$; we use the same notation for $\mathcal{O}$-algebras. We can identify $(A'_{\widetilde{\mathcal{O}}}, I'A'_{\widetilde{\mathcal{O}}})$ with a coproduct of two (resp. three) copies of $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ in $(R_{\widetilde{\mathcal{O}}})_{\Delta,\mathcal{O}_E}$ if $A' = A^{(2)}$ (resp. if $A' = A^{(3)}$). By Galois descent for G-μ-displays, it suffices to prove the same statement for banal G-μ-displays over $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$. We may thus assume without loss of generality that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are banal G-μ-displays over (A, I).

If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are not isomorphic to each other, then the assertion holds trivially. Thus, we may assume that $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_Y$ for some $Y \in G(A)_I$. Let $d := f_2(\mathcal{E})$. We have $f_1(\mathcal{E}) = ud$ for some $u \in A'^{\times}$. With the choice of $d \in I'$, the G-μ-displays $f_1^*(\mathcal{Q}_Y)$ and $f_2^*(\mathcal{Q}_Y)$ correspond to the elements $f_1(Y_\mathcal{E})\phi(\mu(u))$, $f_2(Y_\mathcal{E}) \in G(A')_d$, respectively. Thus we can identify $\operatorname{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_Y), f_2^*(\mathcal{Q}_Y))$ with the set

$$\{g \in G_\mu(A', I') \mid g^{-1} f_2(Y_\mathcal{E})\sigma_{\mu,d}(g) = f_1(Y_\mathcal{E})\phi(\mu(u))\}.$$

We set $X := f_2(Y_\mathcal{E})$. We shall prove that the map (6-4) is injective. Let $g, h \in G_\mu(A', I')$ be two elements in $\operatorname{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_1), f_2^*(\mathcal{Q}_2))$ such that $m(g) = m(h)$ in $G_\mu(A, I)$. We set $\beta := gh^{-1}$. Since $m(\beta) = 1$, we have $\mu(d)\beta\mu(d)^{-1} \in G(K)$. It then follows from $\phi(K) \subset dK$ that $\sigma_{\mu,d}(\beta) \in G(dK)$. The equalities

$$g^{-1}X\sigma_{\mu,d}(g) = f_1(Y_\mathcal{E})\phi(\mu(u)) = h^{-1}X\sigma_{\mu,d}(h)$$

imply that $\beta = X\sigma_{\mu,d}(\beta)X^{-1}$. It follows that $\beta \in G(dK)$, and we have $\mathcal{V}_{d,X}(\beta) = 1$ for the map $\mathcal{V}_{d,X} : G(dK) \to G(dK)$. Since $\mathcal{V}_{d,X}$ is bijective by Proposition 6.4.7, we obtain $\beta = 1$.

It still remains to prove that the map (6-4) is surjective. For this, it is sufficient to prove that $\operatorname{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_Y), f_2^*(\mathcal{Q}_Y))$ is not empty. (Once we have obtained an isomorphism $g : f_1^*(\mathcal{Q}_Y) \xrightarrow{\sim} f_2^*(\mathcal{Q}_Y)$, we can write any isomorphism $h : \mathcal{Q}_Y \xrightarrow{\sim} \mathcal{Q}_Y$ as $m^*(f_2^*(h \circ m^*(g^{-1})) \circ g)$.) We claim that

$\phi(\mu(u)) \in G(dK)$ and $\gamma := f_2(Y_{\mathcal{E}})^{-1} f_1(Y_{\mathcal{E}}) \in G(dK)$. Indeed, since $m(u) = 1$, we have $\mu(u) \in G(K)$, which in turn implies that $\phi(\mu(u)) \in G(dK)$. Since the morphisms f_1 and f_2 induce the same homomorphism $R \to A'/I'$, we see that $\gamma \in G(I')$. Using that $I' \cap K = dK$, we then obtain $\gamma \in G(dK)$. Since $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective, there exists an element $g \in G(dK)$ such that $\mathcal{V}_{d,X}(g^{-1}) = X\phi(\mu(u))^{-1}\gamma^{-1}X^{-1}$, or equivalently

$$g^{-1} f_2(Y_{\mathcal{E}})\sigma_{\mu,d}(g) = f_1(Y_{\mathcal{E}})\phi(\mu(u)).$$

In other words, the element g gives an isomorphism $f_1^*(\mathcal{Q}_Y) \xrightarrow{\sim} f_2^*(\mathcal{Q}_Y)$. □

6.5. *Proof of Theorem 6.1.3.* In this section, we prove Theorem 6.1.3 using our previous results.

As in Section 6.2, we write $(A, I) = (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$. Let

$$G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I)$$

be the groupoid of pairs $(\mathcal{Q}, \epsilon)$ consisting of a G-μ-display $\mathcal{Q}$ over (A, I) and an isomorphism $\epsilon : p_1^*\mathcal{Q} \xrightarrow{\sim} p_2^*\mathcal{Q}$ of G-μ-displays over $(A^{(2)}, I^{(2)})$ satisfying the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$. An isomorphism $(\mathcal{Q}, \epsilon) \xrightarrow{\sim} (\mathcal{Q}', \epsilon')$ is an isomorphism $f : \mathcal{Q} \xrightarrow{\sim} \mathcal{Q}'$ of G-μ-displays over (A, I) such that $\epsilon' \circ (p_1^* f) = (p_2^* f) \circ \epsilon$.

For a prismatic G-μ-display $\mathfrak{Q}$ over R, we have the associated isomorphism

$$\gamma_{p_i} : p_i^*(\mathfrak{Q}_{(A,I)}) \xrightarrow{\sim} \mathfrak{Q}_{(A^{(2)}, I^{(2)})}$$

for $i = 1, 2$. Let $\epsilon := \gamma_{p_2}^{-1} \circ \gamma_{p_1}$. Then ϵ satisfies the cocycle condition, so that the pair $(\mathfrak{Q}_{(A,I)}, \epsilon)$ is an object of $\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I)$. This construction induces a functor

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta,\mathcal{O}_E}) \to G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I), \quad \mathfrak{Q} \mapsto (\mathfrak{Q}_{(A,I)}, \epsilon).$$

Proposition 6.5.1. *The functor $G\text{-}\mathrm{Disp}_\mu((R)_{\Delta,\mathcal{O}_E}) \to G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I)$ is an equivalence.*

Proof. This is a formal consequence of Propositions 5.2.8 and 6.2.1. □

Proof of Theorem 6.1.3. We assume that μ is 1-bounded. By virtue of Proposition 6.5.1, it suffices to show that the forgetful functor

$$G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I) \to G\text{-}\mathrm{Disp}_\mu(A, I)$$

is an equivalence. Let $m : (A^{(2)}, I^{(2)}) \to (A, I)$ be the unique morphism in $(R)_{\Delta,\mathcal{O}_E}$ such that $m \circ p_i = \mathrm{id}_{(A,I)}$ for $i = 1, 2$, and let $m' : (A^{(3)}, I^{(3)}) \to (A, I)$ be the unique morphism in $(R)_{\Delta,\mathcal{O}_E}$ such that $m \circ q_i = \mathrm{id}_{(A,I)}$ for $i = 1, 2, 3$. Let $\mathcal{Q}$ be a G-μ-display over (A, I). We claim that an isomorphism $\epsilon : p_1^*\mathcal{Q} \xrightarrow{\sim} p_2^*\mathcal{Q}$ satisfies the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$ if and only if $m^*\epsilon = \mathrm{id}_{\mathcal{Q}}$. Indeed, since the map

$$m'^* : \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A^{(3)}, I^{(3)})}(q_1^*\mathcal{Q}, q_3^*\mathcal{Q}) \to \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A,I)}(\mathcal{Q}, \mathcal{Q})$$

is bijective by Proposition 6.4.1, we see that ϵ satisfies the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$ if and only if $m^*\epsilon = m^*\epsilon \circ m^*\epsilon$, which is equivalent to saying that $m^*\epsilon = \mathrm{id}_{\mathcal{Q}}$.

By Proposition 6.4.1, the map

$$m^* : \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A^{(2)},I^{(2)})}(p_1^*\mathcal{Q}, p_2^*\mathcal{Q}) \to \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A,I)}(\mathcal{Q}, \mathcal{Q})$$

is bijective. Therefore, for any G-μ-display $\mathcal{Q}$ over (A, I), there exists a unique isomorphism $\epsilon : p_1^*\mathcal{Q} \xrightarrow{\sim} p_2^*\mathcal{Q}$ satisfying the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$, and ϵ is characterized by the condition that $m^*\epsilon = \mathrm{id}_{\mathcal{Q}}$. It follows that the forgetful functor $G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I) \to G\text{-}\mathrm{Disp}_\mu(A, I)$ is an equivalence. □

7. *p*-divisible groups and prismatic Dieudonné crystals

In this section, we make a few remarks on prismatic Dieudonné crystals, which are introduced in [Anschütz and Le Bras 2023].

7.1. *A remark on prismatic Dieudonné crystals.* Let R be a π-adically complete $\mathcal{O}_E$-algebra. Recall the sheaf $\mathcal{O}_{\Delta}$ on the site $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ from Remark 2.5.5.

We say that an $\mathcal{O}_{\Delta}$-module $\mathcal{M}$ on $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ is a *prismatic crystal in vector bundles* if $\mathcal{M}(A, I)$ is a finite projective A-module for any $(A, I) \in (R)_{\Delta,\mathcal{O}_E}$, and for any morphism $(A, I) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$, the natural homomorphism

$$\mathcal{M}(A, I) \otimes_A A' \to \mathcal{M}(A', I')$$

is bijective. A *prismatic Dieudonné crystal* on $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ (or on $(R)_{\Delta,\mathcal{O}_E}$) is a prismatic crystal $\mathcal{M}$ in vector bundles on $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ equipped with a ϕ-linear homomorphism

$$\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$$

such that for any $(A, I) \in (R)_{\Delta,\mathcal{O}_E}$, the finite projective A-module $\mathcal{M}(A, I)$ with the linearization $1 \otimes \varphi_{\mathcal{M}} : \phi^*(\mathcal{M}(A, I)) \to \mathcal{M}(A, I)$ is a minuscule Breuil–Kisin module over (A, I) in the sense of Definition 3.1.5 (see also Proposition 3.1.6). For a bounded $\mathcal{O}_E$-prism (A, I), let

$$\mathrm{BK}_{\min}(A, I)$$

be the category of minuscule Breuil–Kisin modules over (A, I). Then the category of prismatic Dieudonné crystals on $(R)_{\Delta,\mathcal{O}_E}$ is equivalent to the category

$$2 - \varprojlim_{(A,I)\in(R)_{\Delta,\mathcal{O}_E}} \mathrm{BK}_{\min}(A, I).$$

As in Section 6, let R be a complete regular local ring over $\mathcal{O}$ with residue field k. Let $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ be an $\mathcal{O}_E$-prism of Breuil–Kisin type, where $\mathfrak{S}_{\mathcal{O}} = \mathcal{O}[[t_1, \dots, t_n]]$, with an isomorphism $R \simeq \mathfrak{S}_{\mathcal{O}}/\mathcal{E}$ over $\mathcal{O}$. By using the results of Section 6, we can prove the following proposition, which is obtained in the proof of [Anschütz and Le Bras 2023, Theorem 5.12] if $n \leq 1$ (and $\mathcal{O}_E = \mathbb{Z}_p$).

Proposition 7.1.1. *The functor $\mathcal{M} \mapsto \mathcal{M}(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ from the category of prismatic Dieudonné crystals on $(R)_{\Delta,\mathcal{O}_E}$ to the category $\mathrm{BK}_{\min}(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ is an equivalence.*

Proof. This follows from Corollary 5.3.11, Theorem 6.1.3, and the following fact: a functor of additive categories is an equivalence if and only if it induces an equivalence of the associated groupoids. This fact follows since homomorphisms $f : X \to Y$ in an additive category can be completely described in terms of automorphisms of $X \oplus Y$ by considering $\begin{pmatrix} \mathrm{id}_X & 0 \\ f & \mathrm{id}_Y \end{pmatrix}$. □

7.2. ***Quasisyntomic rings.*** In the following (and in Section 8 below), we will need the notions of *quasisyntomic rings* in the sense of [Bhatt et al. 2019, Definition 4.10] and *quasiregular semiperfectoid rings* in the sense of [loc. cit., Definition 4.20]. Let

$$\mathrm{QSyn}$$

be the category of quasisyntomic rings and let

$$\mathrm{QRSPerfd} \subset \mathrm{QSyn}$$

be the full subcategory spanned by quasiregular semiperfectoid rings. We endow both $\mathrm{QSyn}^{\mathrm{op}}$ and $\mathrm{QRSPerfd}^{\mathrm{op}}$ with the quasisyntomic topology, i.e., the topology generated by the quasisyntomic coverings; see [loc. cit., Definition 4.10]. We will assume that the reader is familiar with basic properties of QSyn and QRSPerfd discussed in [loc. cit., Section 4]. Here we just recall that quasiregular semiperfectoid rings form a basis for QSyn; see [loc. cit., Lemma 4.28].

Example 7.2.1. A p-adically complete regular local ring is a quasisyntomic ring (see [Anschütz and Le Bras 2023, Example 3.17]). A perfectoid ring is a quasiregular semiperfectoid ring (see [Bhatt et al. 2019, Example 4.24]).

Remark 7.2.2. Let $R \in \mathrm{QSyn}$ be a quasisyntomic ring. In [Anschütz and Le Bras 2023, Definition 4.5], Anschütz–Le Bras defined prismatic Dieudonné crystals over R as sheaves on the quasisyntomic site of R. By virtue of [loc. cit., Proposition 4.4], the category of prismatic Dieudonné crystals on $(R)_{\Delta}$ in our sense is equivalent to the category of prismatic Dieudonné crystals over R in the sense of [loc. cit., Definition 4.5].

7.3. ***p-divisible groups and minuscule Breuil-Kisin modules.*** In this subsection, we consider the case where $\mathcal{O}_E = \mathbb{Z}_p$. Let R be a p-adically complete ring, and let $\mathcal{G}$ be a p-divisible group over Spec R. We define the functors

$$\underline{\mathcal{G}} : (R)_{\Delta} \to \mathrm{Set}, \quad (A, I) \mapsto \mathcal{G}(A/I),$$
$$\underline{\mathcal{G}[p^n]} : (R)_{\Delta} \to \mathrm{Set}, \quad (A, I) \mapsto \mathcal{G}[p^n](A/I).$$

These functors form sheaves on the site $(R)_{\Delta}^{\mathrm{op}}$. In [loc. cit., Proposition 4.69], it is proved that the $\mathcal{O}_{\Delta}$-module

$$\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$$

on $(R)_{\Delta}^{\mathrm{op}}$ is a prismatic crystal in vector bundles. (Here we simply write $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ rather than $\mathcal{E}xt^1_{(R)_{\Delta}^{\mathrm{op}}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$.)

Remark 7.3.1. (1) For an integer $n \geq 1$, the map $[p^n] : \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ induced by multiplication by p^n is surjective. This follows from [loc. cit., Corollary 3.25].

(2) We have $\mathcal{H}om_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) = 0$. Indeed, since $[p] : \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ is surjective and the topos associated with $(R)_{\Delta}^{\mathrm{op}}$ is replete in the sense of [Bhatt and Scholze 2015, Definition 3.1.1], the projection $\varprojlim_{[p]} \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ is surjective. Since $\mathcal{O}_{\Delta}(A, I) = A$ is p-adically complete for any $(A, I) \in (R)_{\Delta}$, we can conclude that $\mathcal{H}om_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) = 0$. As a consequence, the local-to-global spectral sequence implies that

$$\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})(A, I)$$

for any $(A, I) \in (R)_{\Delta}$. Here we regard the site $(A, I)_{\Delta}^{\mathrm{op}}$ as the localization of $(R)_{\Delta}^{\mathrm{op}}$ at (A, I), and the restriction of $\underline{\mathcal{G}}$ to $(A, I)_{\Delta}^{\mathrm{op}}$ is denoted by the same symbol. In particular $\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ is a finite projective A-module and its formation commutes with base change along any morphism $(A, I) \to (A', I')$ in $(R)_{\Delta}$.

We assume that R is quasisyntomic. In [Anschütz and Le Bras 2023, Theorem 4.71], it is proved that $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ with the ϕ-linear homomorphism $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ induced by the Frobenius $\phi : \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}$ is a prismatic Dieudonné crystal. More precisely, they showed that $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ is *admissible* in the sense of [loc. cit., Definition 4.5]. (See also Remark 7.2.2.) We shall recall the argument.

Proposition 7.3.2 [Anschütz and Le Bras 2023, Theorem 4.71]. *Let $R \in \mathrm{QSyn}$ and $(A, I) \in (R)_{\Delta}$. We write $M := \mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$. Then M with the induced homomorphism $F_M : \phi^* M \to M$ is a minuscule Breuil–Kisin module over (A, I).*

Proof. By the fact that the formation of $\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ commutes with base change along any morphism $(A, I) \to (A', I')$ (see Remark 7.3.1) and Corollary 3.1.15, the assertion can be checked (p, I)-completely flat locally. Let $R \to R'$ be a quasisyntomic covering with R' a quasiregular semiperfectoid ring. Applying [Bhatt and Scholze 2022, Proposition 7.11] to the p-adic completion of $A/I \otimes_R R'$, which is a quasisyntomic covering of A/I, we can find a flat covering $(A, I) \to (A', I')$ in $(R)_{\Delta}$ such that there exists a homomorphism $R' \to A'/I'$ over R. After replacing R by R' and replacing (A, I) by (A', I'), we may assume that R is a quasiregular semiperfectoid ring. Then, by choosing a surjective homomorphism from a perfectoid ring to R and using [Anschütz and Le Bras 2023, Corollary 2.10, Lemma 4.70], we may assume that R is a perfectoid ring and $(A, I) = (W(R^{\flat}), I_R)$. (Here we regard $(W(R^{\flat}), I_R)$ as an object of $(R)_{\Delta}$ via the homomorphism $\theta : W(R^{\flat}) \to R$. In [loc. cit.], the composition $\theta \circ \phi^{-1}$ is used instead.)

Let $\xi \in I_R$ be a generator. By Proposition 3.1.6, it suffices to prove that the cokernel of F_M is killed by ξ. By Remark 5.6.2(4) and p-complete arc-descent (Proposition 5.6.3), we may further assume that R is a p-adically complete valuation ring of rank ≤ 1 with algebraically closed fraction field.

If $p = 0$ in R, then R is perfect by Example 2.4.1. In this case, the Frobenius F_M can be identified with the homomorphism

$$\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{(\phi^*\mathcal{G})}, \mathcal{O}_{\Delta}) \to \mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$$

induced by the relative Frobenius $\mathcal{G} \to \phi^*\mathcal{G}$. Thus, the Verschiebung homomorphism $\phi^*\mathcal{G} \to \mathcal{G}$ induces a $W(R)$-linear homomorphism $V_M : M \to \phi^* M$ such that $F_M \circ V_M = p$, which in turn implies the assertion.

It remains to treat the case where p is a nonzerodivisor in R, so that R is the ring of integers $\mathcal{O}_C$ of an algebraically closed nonarchimedean extension C of $\mathbb{Q}_p$. We set

$$M_n := \mathrm{Ext}^1_{(W(\mathcal{O}_C^\flat), I_{\mathcal{O}_C})_{\mathbb{\Delta}}}(\underline{\mathcal{G}[p^n]}, \mathcal{O}_{\mathbb{\Delta}}).$$

By the proof of [Anschütz and Le Bras 2023, Proposition 4.69], the natural homomorphism $M \to M_n$ induces an isomorphism $M/p^n \xrightarrow{\sim} M_n$ for any $n \geq 1$. In particular, we obtain

$$M \xrightarrow{\sim} \varprojlim_n M_n$$

and M_n is a free $W_n(\mathcal{O}_C^\flat)$-module of finite rank. We claim that the cokernel of the Frobenius $F_{M_n} : \phi^* M_n \to M_n$ is killed by ξ. Indeed, there is an embedding $\mathcal{G}[p^n] \hookrightarrow X$ into an abelian scheme X over $\mathrm{Spec}\, \mathcal{O}_C$; see [Berthelot et al. 1982, Théorème 3.1.1]. Let Y be the p-adic completion of X, which is a smooth p-adic formal scheme over $\mathrm{Spf}\, \mathcal{O}_C$. It follows from the proofs of [Anschütz and Le Bras 2023, Theorem 4.62, Proposition 4.66] that there exists a surjection

$$H^1_{\mathbb{\Delta}}(Y/W(\mathcal{O}_C^\flat)) \to M_n$$

which is compatible with Frobenius homomorphisms. Here $H^1_{\mathbb{\Delta}}(Y/W(\mathcal{O}_C^\flat))$ is the first prismatic cohomology of Y (with respect to $(W(\mathcal{O}_C^\flat), I_{\mathcal{O}_C})$) defined in [Bhatt and Scholze 2022]. By [loc. cit., Theorem 1.8(6)], the cokernel of the Frobenius

$$\phi^* H^1_{\mathbb{\Delta}}(Y/W(\mathcal{O}_C^\flat)) \to H^1_{\mathbb{\Delta}}(Y/W(\mathcal{O}_C^\flat))$$

is killed by ξ, which in turn implies the claim. Since the image of ξ in $W_n(\mathcal{O}_C^\flat)$ is a nonzerodivisor, it follows that F_{M_n} is injective. Since $F_M = \varprojlim_n F_{M_n}$, we can conclude that the cokernel of F_M is killed by ξ. □

Remark 7.3.3. Our proof of Proposition 7.3.2 in the case where $A = \mathcal{O}_C$ is slightly different from that given in [Anschütz and Le Bras 2023]. For example, we do not use [Scholze and Weinstein 2020, Proposition 14.9.4] (see the proof of [Anschütz and Le Bras 2023, Proposition 4.48]).

Finally, we recall the following classification theorem for p-divisible groups given in [Anschütz and Le Bras 2023]. Let R be a complete regular local ring with perfect residue field k of characteristic p. Let $(\mathfrak{S}, (\mathcal{E}))$ be a prism of Breuil–Kisin type, where $\mathfrak{S} := W(k)[[t_1, \dots, t_n]]$, with an isomorphism $R \simeq \mathfrak{S}/\mathcal{E}$ which lifts $\mathrm{id}_k : k \to k$.

Theorem 7.3.4 [Anschütz and Le Bras 2023, Theorem 4.74, Theorem 5.12].

(1) *The contravariant functor*

$$\{p\text{-divisible groups over } \mathrm{Spec}\, R\} \to \{\text{prismatic Dieudonné crystals on } (R)_{\mathbb{\Delta}}\}$$

defined by $\mathcal{G} \mapsto \mathcal{E}xt^1_{(R)_{\mathbb{\Delta}}}(\underline{\mathcal{G}}, \mathcal{O}_{\mathbb{\Delta}})$ *is an antiequivalence of categories.*

(2) *The contravariant functor*

$$\{p\text{-divisible groups over } \mathrm{Spec}\, R\} \to \{\text{minuscule Breuil–Kisin modules over } (\mathfrak{S}, (\mathcal{E}))\}$$

defined by $\mathcal{G} \mapsto \mathrm{Ext}^1_{(\mathfrak{S}, (\mathcal{E}))_{\mathbb{\Delta}}}(\underline{\mathcal{G}}, \mathcal{O}_{\mathbb{\Delta}})$ *is an antiequivalence of categories.*

Proof. (1) This is a consequence of [Anschütz and Le Bras 2023, Theorem 4.74, Proposition 5.10].

(2) The assertion follows from (1) and Proposition 7.1.1. This result was already stated in [loc. cit., Theorem 5.12], and the proof was given in the case where $n \leq 1$. □

8. Comparison with prismatic F-gauges

For the sake of completeness, we discuss the relation between our prismatic G-μ-displays and *prismatic F-gauges* introduced in [Drinfeld 2024, 1.8.1; Bhatt and Lurie 2022a; 2022b; Bhatt 2022, Definition 6.1.1]. For simplicity, we assume that $\mathcal{O}_E = \mathbb{Z}_p$ throughout this section, and we restrict ourselves to the case where base rings R are quasisyntomic. In this case, Guo and Li [2023] studied prismatic F-gauges over R in a slightly different way, without using the original stacky approach. Here we follow the approach employed in [Guo and Li 2023]. In Section 8.1, we compare prismatic F-gauges in vector bundles with displayed Breuil–Kisin modules. In Section 8.2, we introduce *prismatic G-F-gauges of type μ* and explain their relation to prismatic G-μ-displays.

We work with the category QSyn of quasisyntomic rings and the full subcategory QRSPerfd $\subset$ QSyn spanned by quasiregular semiperfectoid rings (see Section 7.2).

8.1. ***Prismatic F-gauges in vector bundles.*** We recall the definition of prismatic F-gauges in vector bundles over quasisyntomic rings, following [Guo and Li 2023].

Let $S \in$ QRSPerfd be a quasiregular semiperfectoid ring. By [Bhatt and Scholze 2022, Proposition 7.10], the category $(S)_{\Delta}$ admits an initial object

$$(\Delta_S, I_S) \in (S)_{\Delta}.$$

Moreover the bounded prism (Δ_S, I_S) is orientable. We often omit the subscript and simply write $I = I_S$. Following [loc. cit., Definition 12.1], we define

$$\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S) := \{x \in \Delta_S \mid \phi(x) \in I^i \Delta_S\}$$

for a nonnegative integer $i \geq 0$. For a negative integer $i < 0$, we set $\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S) = \Delta_S$. The filtration $\{\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)\}_{i \in \mathbb{Z}}$ is called the *Nygaard filtration*. We recall the following terminology from [Bhatt 2022, Section 5.5].

Definition 8.1.1. The *extended Rees algebra* $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$ of the Nygaard filtration $\{\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)\}_{i \in \mathbb{Z}}$ is defined by

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^i_{\mathcal{N}}(\Delta_S) t^{-i} \subset \Delta_S[t, t^{-1}].$$

We view $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$ as a graded ring, where the degree of t is -1. Let

$$\tau : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \Delta_S$$

be the homomorphism of Δ_S-algebras defined by $t \mapsto 1$. We consider the graded ring $\bigoplus_{i \in \mathbb{Z}} I^i t^{-i} \subset \Delta_S[1/I][t, t^{-1}]$. Let

$$\sigma : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \bigoplus_{i \in \mathbb{Z}} I^i t^{-i}$$

be the graded homomorphism defined by $a_i t^{-i} \mapsto \phi(a_i) t^{-i}$ for any $i \in \mathbb{Z}$.

Remark 8.1.2. For the grading of $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$, our sign convention is opposite to that of [Bhatt 2022], where the degree of t is defined to be 1. Our grading is chosen to be consistent with the convention of [Lau 2021].

Definition 8.1.3 (Drinfeld, Bhatt–Lurie). Let $S \in \mathrm{QRSPerfd}$. A *prismatic F-gauge in vector bundles* over S is a pair (N, F_N) consisting of a graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module N which is finite projective as a $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module, and an isomorphism

$$F_N : (\sigma^* N)_0 \xrightarrow{\sim} \tau^* N$$

of Δ_S-modules. Here $(\sigma^* N)_0$ is the degree-0 part of the graded $\bigoplus_{i\in\mathbb{Z}} I^i t^{-i}$-module $\sigma^* N$.

Let $F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S)$ be the category of prismatic F-gauges in vector bundles over S.

Remark 8.1.4. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded $\bigoplus_{i\in\mathbb{Z}} I^i t^{-i}$-module. For any $i \in \mathbb{Z}$, we have a natural isomorphism $M_0 \otimes_{\Delta_S} I^i t^{-i} \xrightarrow{\sim} M_i$ of Δ_S-modules. It follows that the functor $M \mapsto M_0$ from the category of graded $\bigoplus_{i\in\mathbb{Z}} I^i t^{-i}$-modules to the category of Δ_S-modules is an equivalence, whose inverse is given by $L \mapsto L \otimes_{\Delta_S} \left(\bigoplus_{i\in\mathbb{Z}} I^i t^{-i}\right)$.

Remark 8.1.5. The notion of prismatic F-gauges in vector bundles is closely related to the notion of (higher) displays in the sense of [Lau 2021, Definition 3.2.1]. See Remark 8.2.7 for more details.

We collect some useful facts about graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-modules.

Remark 8.1.6. Let $N = \bigoplus_{i\in\mathbb{Z}} N_i$ be a graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module which is finite projective as a $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module. Then each degree-i part N_i is a direct summand of a Δ_S-module of the form $\bigoplus_{j=1}^{m} \mathrm{Fil}^{i_j}_{\mathcal{N}}(\Delta_S) t^{-i_j}$, and in particular N_i is (p, I)-adically complete. This follows from the following fact: for a graded ring A, a graded A-module N is projective as an A-module if and only if N is projective in the category of graded A-modules; see [Lau 2021, Lemma 3.0.1].

Let

$$\rho : \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to \Delta_S / \mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) \tag{8-1}$$

be the composition of the projection $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ with the natural homomorphism $\Delta_S \to \Delta_S / \mathrm{Fil}^1_{\mathcal{N}}(\Delta_S)$. The map ρ is a ring homomorphism. For an integer $n \geq 1$, we write

$$\Delta^{\mathcal{N}}_{S,n} := \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \otimes_{\Delta_S} \Delta_S/(p, I)^n.$$

Let $\bar{\rho} : \Delta^{\mathcal{N}}_{S,n} \to \Delta_S/(\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) + (p, I))$ be the homomorphism induced by ρ.

Lemma 8.1.7 (cf. [Lau 2021, Lemma 3.1.1, Corollary 3.1.2]).

(1) *Let M be a finite graded $\Delta^{\mathcal{N}}_{S,n}$-module. If $\bar{\rho}^* M = 0$, then we have $M = 0$.*

(2) *Let M and N be finite graded $\Delta^{\mathcal{N}}_{S,n}$-modules. Assume that N is projective as a $\Delta^{\mathcal{N}}_{S,n}$-module. Then a homomorphism $f : M \to N$ of graded $\Delta^{\mathcal{N}}_{S,n}$-modules is an isomorphism if $\bar{\rho}^* f : \bar{\rho}^* M \to \bar{\rho}^* N$ is an isomorphism.*

Proof. (1) By [Anschütz and Le Bras 2023, Lemma 4.28], the pair $(\Delta_S, \mathrm{Fil}^1_{\mathcal{N}}(\Delta_S))$ is henselian. In particular we have $\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) \subset \mathrm{rad}(\Delta_S)$. Using this fact, we can prove the assertion by the same argument as in the proof of [Lau 2021, Lemma 3.1.1].

(2) By (1), we see that f is surjective. Since N is projective as a graded $\Delta^{\mathcal{N}}_{S,n}$-module (Remark 8.1.6), we have $M \simeq N \oplus \mathrm{Ker}\, f$ as graded $\Delta^{\mathcal{N}}_{S,n}$-modules. Thus $\mathrm{Ker}\, f$ is a finite graded $\Delta^{\mathcal{N}}_{S,n}$-module such that $\bar{\rho}^* \mathrm{Ker}\, f = 0$. By (1) again, we have $\mathrm{Ker}\, f = 0$. □

Corollary 8.1.8. *Let* $N = \bigoplus_{i\in\mathbb{Z}} N_i$ *be a graded* $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$*-module with the following properties*:

(1) *The degree-i part N_i is (p, I)-adically complete for every $i \in \mathbb{Z}$.*

(2) $N^n := N/(p, I)^n N$ *is a finite projective* $\Delta^{\mathcal{N}}_{S,n}$*-module for every* $n \geq 1$.

Then there exists a graded finite projective Δ_S-module L with an isomorphism

$$L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \simeq N$$

of graded $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$*-modules. In particular N is a finite projective* $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$*-module.*

Proof. Since Δ_S is henselian with respect to both ideals $\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S)$ and (p, I), there exists a graded finite projective Δ_S-module L with an isomorphism

$$L/(\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) + (p, I))L \xrightarrow{\sim} \bar{\rho}^* N^1$$

of graded modules (by [Stacks 2005–, Tag 0D4A] or [Greco 1968, Theorem 5.1]). This isomorphism lifts to a homomorphism

$$f : L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to N$$

of graded $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$-modules (see Remark 8.1.6). By Lemma 8.1.7, the reduction modulo $(p, I)^n$ of f is bijective for every n. Since degree-i parts of $L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$ and N are (p, I)-adically complete for every $i \in \mathbb{Z}$, it follows that f is an isomorphism. □

For a bounded prism (A, I), let $\mathrm{BK}_{\mathrm{disp}}(A, I)$ be the category of displayed Breuil–Kisin modules over (A, I) (see Definition 3.1.5). Prismatic F-gauges in vector bundles over S can be related to displayed Breuil–Kisin modules over (Δ_S, I_S) as follows.

Proposition 8.1.9. *Let* $S \in \mathrm{QRSPerfd}$. *There exists a fully faithful functor*

$$F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S) \to \mathrm{BK}_{\mathrm{disp}}(\Delta_S, I_S)$$

which is compatible with base change along any homomorphism $S \to S'$ in QRSPerfd.

Proof. To each $(N, F_N) \in F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S)$, we attach a displayed Breuil–Kisin module (M, F_M) over (Δ_S, I) as follows. Let $M := \tau^* N$. The kernel of $\tau : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ is generated by $t - 1$, so that $M = N/(t-1)N$. It follows that the natural homomorphism $N_i \to M$ of Δ_S-modules is injective, whose image is denoted by $\mathrm{Fil}^i(M) \subset M$. We have $\mathrm{Fil}^{i+1}(M) \subset \mathrm{Fil}^i(M)$, and the corresponding map $N_{i+1} \to N_i$ is given by $x \mapsto tx$. Moreover we have $M = \bigcup_i \mathrm{Fil}^i(M)$. Let i be a small enough integer such that $\mathrm{Fil}^i(M) = M$. We define $\phi^*M \to M[1/I]$ to be the composition

$$\phi^*M = \phi^*\,\mathrm{Fil}^i(M) \xrightarrow{\sim} \phi^*N_i \to (\sigma^*N)_i \xrightarrow{\sim} (\sigma^*N)_0 \otimes_{\Delta_S} I^i t^{-i} \xrightarrow{F_N} M \otimes_{\Delta_S} I^i t^{-i} \xrightarrow{t \mapsto 1} M[1/I].$$

(For the isomorphism $(\sigma^*N)_i \xrightarrow{\sim} (\sigma^*N)_0 \otimes_{\Delta_S} I^i t^{-i}$, see Remark 8.1.4.) This homomorphism $\phi^*M \to M[1/I]$ is independent of the choice of i. Let $F_M : (\phi^*M)[1/I] \to M[1/I]$ be the induced homomorphism.

We shall prove that (M, F_M) is a displayed Breuil–Kisin module over (Δ_S, I). By Corollary 8.1.8, we may assume that

$$N = L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$$

for some graded finite projective Δ_S-module $L = \bigoplus_{j \in \mathbb{Z}} L_j^{(-1)}$. Then we have $M = L$ and

$$\mathrm{Fil}^i(M) = \Big(\bigoplus_{j \geq i} L_j^{(-1)}\Big) \oplus \Big(\bigoplus_{j < i} \mathrm{Fil}^{i-j}_{\mathcal{N}}(\Delta_S) L_j^{(-1)}\Big) \tag{8-2}$$

for every $i \in \mathbb{Z}$. We set $L_j := \phi^* L_j^{(-1)}$. By sending t to 1, we obtain $(\sigma^*N)_0 \xrightarrow{\sim} \bigoplus_{j \in \mathbb{Z}} (L_j \otimes_{\Delta_S} I^{-j})$, and the isomorphism F_N can be written as

$$F_N : \bigoplus_{j \in \mathbb{Z}} (L_j \otimes_{\Delta_S} I^{-j}) \xrightarrow{\sim} M.$$

Now $F_M : \bigoplus_{j \in \mathbb{Z}} L_j[1/I] \to M[1/I]$ is the base change of F_N. (In particular F_M is an isomorphism.) Recall the filtration $\{\mathrm{Fil}^i(\phi^*M)\}_{i \in \mathbb{Z}}$ of ϕ^*M from Definition 3.1.2. We see that $\mathrm{Fil}^i(\phi^*M) \subset \phi^*M$ is the intersection of $\phi^*M = \bigoplus_{j \in \mathbb{Z}} L_j$ with $\bigoplus_{j \in \mathbb{Z}} (L_j \otimes_{\Delta_S} I^{i-j})$, and thus

$$\mathrm{Fil}^i(\phi^*M) = \Big(\bigoplus_{j \geq i} L_j\Big) \oplus \Big(\bigoplus_{j < i} I^{i-j} L_j\Big). \tag{8-3}$$

From this description, we see that (M, F_M) is a displayed Breuil–Kisin module. (Moreover $\phi^*M = \bigoplus_{j \in \mathbb{Z}} L_j$ is a normal decomposition in the sense of Definition 3.1.10.)

We have constructed a functor $F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S) \to \mathrm{BK}_{\mathrm{disp}}(\Delta_S, I_S)$. The full faithfulness of this functor follows from [Guo and Li 2023, Corollary 2.53]. The important point is that the filtration $\{\mathrm{Fil}^i(M)\}_{i \in \mathbb{Z}}$ of M can be recovered from (M, F_M). Namely, $\mathrm{Fil}^i(M)$ agrees with the inverse image of $M \otimes_{\Delta_S} I^i$ under the composition

$$M \xrightarrow{x \mapsto 1 \otimes x} (\phi^*M)[1/I] \xrightarrow{F_M} M[1/I].$$

(Compare (8-2) with (8-3).) Since the full faithfulness of the functor also follows from Example 5.3.10, Example 8.2.9, and Proposition 8.2.11 below (compare with the argument in the proof of Proposition 7.1.1), we omit the details here. □

The following result is essential for the definition of prismatic F-gauges in vector bundles over a quasisyntomic ring.

Proposition 8.1.10. *The fibered category over* $\mathrm{QRSPerfd}^{\mathrm{op}}$ *which associates to each* $S \in \mathrm{QRSPerfd}$ *the category* $F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S)$ *satisfies descent with respect to the quasisyntomic topology.*

Proof. This was originally proved by Bhatt and Lurie (see [Bhatt 2022, Remark 5.5.18]). In [Guo and Li 2023, Proposition 2.29], this result was obtained by a slightly different method. We briefly recall the argument given in that proposition.

Let $S \to S'$ be a quasisyntomic covering in QRSPerfd. By the proof of [loc. cit., Proposition 2.29], the induced homomorphism $\Delta^{\mathcal{N}}_{S,n} \to \Delta^{\mathcal{N}}_{S',n}$ is faithfully flat for every $n \geq 1$, and for a homomorphism $S \to S_1$ in QRSPerfd, the following natural homomorphism of graded rings is an isomorphism:

$$\Delta^{\mathcal{N}}_{S',n} \otimes_{\Delta^{\mathcal{N}}_{S,n}} \Delta^{\mathcal{N}}_{S_1,n} \xrightarrow{\sim} \Delta^{\mathcal{N}}_{S' \widehat{\otimes}_S S_1, n},$$

where $S' \widehat{\otimes}_S S_1 \in$ QRSPerfd is the p-adic completion of $S' \otimes_S S_1$. Using these results, we can prove that the natural functor from the category of prismatic F-gauges in vector bundles over S to the category of prismatic F-gauges in vector bundles over S' with a descent datum (with respect to $S \to S'$) is an equivalence. We only prove the essential surjectivity of the functor.

Let $(N', F_{N'}) \in F\text{-Gauge}^{\mathrm{vect}}(S')$ with a descent datum. By the results recalled in the previous paragraph and by faithfully flat descent, we see that, for every $n \geq 1$, the graded $\Delta^{\mathcal{N}}_{S',n}$-module $N'/(p, I)^n N'$ with the descent datum arises from a graded $\Delta^{\mathcal{N}}_{S,n}$-module $N^n = \bigoplus_{i \in \mathbb{Z}} N^n_i$ such that N^n is finite projective as a $\Delta^{\mathcal{N}}_{S,n}$-module. Let $N_i := \varprojlim_n N^n_i$ and we define $N := \bigoplus_{i \in \mathbb{Z}} N_i$, which is a graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module. We have $N/(p, I)^n N = N^n$ for every $n \geq 1$. By Corollary 8.1.8, we see that N is a finite projective $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module. Moreover the natural homomorphism

$$N \otimes_{\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))} \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_{S'})) \to N'$$

is an isomorphism (as its reduction modulo $(p, I)^n$ is an isomorphism for every n). Since $(\Delta_S, I_S) \to (\Delta_{S'}, I_{S'})$ is a faithfully flat map of $\mathcal{O}_E$-prisms and the (p, I)-adic completion of $\Delta_{S'} \otimes_{\Delta_S} \Delta_{S'}$ is isomorphic to $\Delta_{S' \widehat{\otimes}_S S'}$ (see the proof of [Guo and Li 2023, Proposition 2.29]), the isomorphism $F_{N'}$ descends to an isomorphism $F_N : (\sigma^* N)_0 \xrightarrow{\sim} \tau^* N$ by Proposition 2.5.6. This shows that $(N', F_{N'})$ with the descent datum arises from $(N, F_N) \in F\text{-Gauge}^{\mathrm{vect}}(S)$. □

Proposition 8.1.10, together with [Bhatt et al. 2019, Proposition 4.31], shows that the fibered category $S \mapsto F\text{-Gauge}^{\mathrm{vect}}(S)$ over $\mathrm{QRSPerfd}^{\mathrm{op}}$ extends uniquely to a fibered category

$$R \mapsto F\text{-Gauge}^{\mathrm{vect}}(R)$$

over the category $\mathrm{QSyn}^{\mathrm{op}}$ that satisfies descent with respect to the quasisyntomic topology.

Definition 8.1.11 (Drinfeld, Bhatt–Lurie, Guo–Li). Let $R \in$ QSyn. An object $N \in F\text{-Gauge}^{\mathrm{vect}}(R)$ is called a *prismatic F-gauge in vector bundles* over R. For a homomorphism $R \to S$ with $S \in$ QRSPerfd, the image of N in $F\text{-Gauge}^{\mathrm{vect}}(S)$ is denoted by (N_S, F_{N_S}).

A fully faithful functor from $F\text{-Gauge}^{\mathrm{vect}}(R)$ to the category of prismatic F-crystals on $(R)_{\Delta}$ (in the sense of [Bhatt and Scholze 2023]) is obtained in [Guo and Li 2023, Corollary 2.53]. More precisely, we have the following result.

Proposition 8.1.12. *Let $R \in \mathrm{QSyn}$. There exists a fully faithful functor*

$$F\text{-Gauge}^{\mathrm{vect}}(R) \to 2-\varprojlim_{(A,I)\in(R)_{\mathbb{\Delta}}} \mathrm{BK}_{\mathrm{disp}}(A,I). \tag{8-4}$$

This functor is compatible with base change along any homomorphism $R \to R'$ in QSyn.

Proof. By Proposition 8.1.10, the left-hand side of (8-4) satisfies quasisyntomic descent. By [Bhatt and Scholze 2022, Proposition 7.11] and Corollary 3.1.15, the right-hand side of (8-4) also satisfies quasisyntomic descent; see also the proof of [Bhatt and Scholze 2023, Proposition 2.14]. Thus, the assertion follows from Proposition 8.1.9. □

As in Section 3.2, let $\mu = (m_1, \ldots, m_n)$ be a tuple of integers $m_1 \geq \cdots \geq m_n$. Let $r_i \in \mathbb{Z}_{\geq 0}$ be the number of occurrences of i in $(m_1, \ldots, m_n)$. We want to compare prismatic F-gauges in vector bundles with (displayed) Breuil–Kisin modules of type μ in the sense of Definition 3.2.1.

Remark 8.1.13. Let $S \in \mathrm{QRSPerfd}$. By [Bhatt and Scholze 2022, Theorem 12.2], the Frobenius $\phi : \mathbb{\Delta}_S \to \mathbb{\Delta}_S$ induces an isomorphism

$$\mathbb{\Delta}_S / \mathrm{Fil}^1_{\mathcal{N}}(\mathbb{\Delta}_S) \xrightarrow{\sim} S.$$

Using this, we regard the homomorphism ρ (see (8-1)) as $\rho : \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\mathbb{\Delta}_S)) \to S$.

Definition 8.1.14. Let $R \in \mathrm{QSyn}$. Let $N \in F\text{-Gauge}^{\mathrm{vect}}(R)$. We say that N is *of type* μ if for any $R \to S$ with $S \in \mathrm{QRSPerfd}$, the degree-i part $(\rho^* N_S)_i$ of the graded S-module $\rho^* N_S$ is of rank r_i for any $i \in \mathbb{Z}$.

Let

$$F\text{-Gauge}_{\mu}(R) \subset F\text{-Gauge}^{\mathrm{vect}}(R)$$

be the full subcategory spanned by those objects of type μ. The property of being of type μ can be checked locally in the quasisyntomic topology. Thus the fibered category $R \mapsto F\text{-Gauge}_{\mu}(R)$ over $\mathrm{QSyn}^{\mathrm{op}}$ satisfies descent with respect to the quasisyntomic topology.

By construction, the functor (8-4) induces a fully faithful functor

$$F\text{-Gauge}_{\mu}(R) \to 2-\varprojlim_{(A,I)\in(R)_{\mathbb{\Delta}}} \mathrm{BK}_{\mu}(A,I) \tag{8-5}$$

for any $R \in \mathrm{QSyn}$. (Recall that $\mathrm{BK}_{\mu}(A,I)$ is the category of Breuil–Kisin modules over (A,I) of type μ.) We will prove later that the functors (8-4) and (8-5) are equivalences if R is a perfectoid ring or a complete regular local ring with perfect residue field k of characteristic p; see Corollary 8.2.13 below.

Example 8.1.15. Let $R \in \mathrm{QSyn}$. Let $F\text{-Gauge}^{\mathrm{vect}}_{[0,1]}(R) \subset F\text{-Gauge}^{\mathrm{vect}}(R)$ be the full subcategory of those $N \in F\text{-Gauge}^{\mathrm{vect}}(R)$ such that for any homomorphism $R \to S$ with $S \in \mathrm{QRSPerfd}$, we have $(\rho^* N_S)_i = 0$ for all $i \neq 0, 1$. The functor (8-4) induces a fully faithful functor

$$F\text{-Gauge}^{\mathrm{vect}}_{[0,1]}(R) \to 2-\varprojlim_{(A,I)\in(R)_{\mathbb{\Delta}}} \mathrm{BK}_{\mathrm{min}}(A,I).$$

The right-hand side can be identified with the category of prismatic Dieudonné crystals on $(R)_{\mathbb{\Delta}}$; see Section 7.1. By [Guo and Li 2023, Theorem 2.54], the essential image of this functor is the full subcategory of admissible prismatic Dieudonné crystals on $(R)_{\mathbb{\Delta}}$. If R is a perfectoid ring or a complete regular local ring with perfect residue field k of characteristic p, then any prismatic Dieudonné crystal on $(R)_{\Delta}$ is admissible by [Anschütz and Le Bras 2023, Propositions 4.12 and 5.10], and hence the above functor is an equivalence in this case. This fact also follows from Corollary 8.2.13.

8.2. ***Prismatic G-F-gauges of type μ.*** Let G be a smooth affine group scheme over $\mathbb{Z}_p$. Let $\mu : \mathbb{G}_m \to G_{W(k)}$ be a cocharacter where k is a perfect field of characteristic p. We introduce prismatic G-F-gauges of type μ in the same way as for prismatic G-μ-displays.

We retain the notation of Section 4. For the cocharacter μ, we have the action (4-1) of $\mathbb{G}_m$ on $G_{W(k)} = \operatorname{Spec} A_G$. Let $A_G = \bigoplus_{i\in\mathbb{Z}} A_{G,i}$ be the weight decomposition. We define $A^{(-1)}_{G,i} := (\phi^{-1})^* A_{G,i}$, where $\phi^{-1} : W(k) \to W(k)$ is the inverse of the Frobenius. Since $(\phi^{-1})^* A_G = A_G$, we have

$$A_G = \bigoplus_{i\in\mathbb{Z}} A^{(-1)}_{G,i}.$$

Let $\mu^{(-1)} : \mathbb{G}_m \to G_{W(k)}$ be the base change of μ along ϕ^{-1}. Then $A_G = \bigoplus_{i\in\mathbb{Z}} A^{(-1)}_{G,i}$ is the weight decomposition with respect to the action of $\mathbb{G}_m$ induced by $\mu^{(-1)}$.

Let S be a quasiregular semiperfectoid ring over $W(k)$.

Definition 8.2.1. Let

$$G_{\mu,\mathcal{N}}(S) \subset G(\Delta_S)$$

be the subgroup consisting of homomorphisms $g^* : A_G \to \Delta_S$ of $W(k)$-algebras such that $g^*(A^{(-1)}_{G,i}) \subset \mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)$ for any $i \in \mathbb{Z}$. The group $G_{\mu,\mathcal{N}}(S)$ is called the *gauge group*.

Remark 8.2.2. It follows from Lemma 4.1.4 that $G_{\mu,\mathcal{N}}(S) \subset G(\Delta_S)$ is the inverse image of the display group $G_\mu(\Delta_S, I_S) \subset G(\Delta_S)$ under the homomorphism $\phi : G(\Delta_S) \to G(\Delta_S)$.

For a generator $d \in I_S$, we have the homomorphism

$$\sigma_{\mu,\mathcal{N},d} : G_{\mu,\mathcal{N}}(S) \to G(\Delta_S), \quad g \mapsto \mu(d)\phi(g)\mu(d)^{-1},$$

by Remark 8.2.2. Let $G(\Delta_S)_{\mathcal{N},d}$ be the set $G(\Delta_S)$ together with the following action of $G_{\mu,\mathcal{N}}(S)$:

$$G(\Delta_S) \times G_{\mu,\mathcal{N}}(S) \to G(\Delta_S), \quad (X, g) \mapsto X \cdot g := g^{-1} X \sigma_{\mu,\mathcal{N},d}(g). \tag{8-6}$$

For another generator $d' \in I_S$ and the unique element $u \in \Delta_S^\times$ such that $d = ud'$, the bijection $G(\Delta_S)_{\mathcal{N},d} \to G(\Delta_S)_{\mathcal{N},d'}$, $X \mapsto X\mu(u)$, is $G_{\mu,\mathcal{N}}(S)$-equivariant. Then we set

$$G(\Delta_S)_{\mathcal{N}} := \varprojlim_d G(\Delta_S)_{\mathcal{N},d},$$

which is equipped with a natural action of $G_{\mu,\mathcal{N}}(S)$. Here d runs over the set of generators $d \in I_S$. Although $G(\Delta_S)_{\mathcal{N}}$ depends on μ, we omit it from the notation.

Remark 8.2.3. We recall some notation from [Bhatt and Scholze 2023, Definition 2.9]. Let R be a quasisyntomic ring. Let $(R)_{\mathrm{QSyn}}$ (resp. $(R)_{\mathrm{qrsp}}$) be the category of quasisyntomic rings R' (resp. quasiregular semiperfectoid rings R') with a quasisyntomic map $R \to R'$. We endow both $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$ and $(R)^{\mathrm{op}}_{\mathrm{qrsp}}$ with the quasisyntomic topology. Since quasiregular semiperfectoid rings form a basis for QSyn, we may identify sheaves on $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$ with sheaves on $(R)^{\mathrm{op}}_{\mathrm{qrsp}}$. On the site $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$, we have the sheaves $\Delta_\bullet$ and $I_\bullet$ such that

$$\Delta_\bullet(S) = \Delta_S \quad \text{and} \quad I_\bullet(S) = I_S$$

for each $S \in (R)_{\mathrm{qrsp}}$.

Lemma 8.2.4. *Let R be a quasisyntomic ring over $W(k)$. The functors*

$$G_{\mu,\mathcal{N}} : (R)_{\mathrm{qrsp}} \to \mathrm{Set}, \quad S \mapsto G_{\mu,\mathcal{N}}(S),$$
$$G_{\Delta,\mathcal{N}} : (R)_{\mathrm{qrsp}} \to \mathrm{Set}, \quad S \mapsto G(\Delta_S)_{\mathcal{N}},$$

form sheaves with respect to the quasisyntomic topology.

Proof. As $\Delta_\bullet$ is a sheaf, so is $G_{\Delta,\mathcal{N}}$. Since $I_\bullet$ is a sheaf, it follows that the functor $S \mapsto \mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)$ forms a sheaf for any $i \in \mathbb{Z}$. This implies that $G_{\mu,\mathcal{N}}$ is a sheaf. □

We regard $G_{\mu,\mathcal{N}}$ and $G_{\Delta,\mathcal{N}}$ as sheaves on $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$. The sheaf $G_{\Delta,\mathcal{N}}$ is equipped with an action of $G_{\mu,\mathcal{N}}$.

Definition 8.2.5 (prismatic G-F-gauge of type μ). Let R be a quasisyntomic ring over $W(k)$. A *prismatic G-F-gauge of type μ* over R is a pair

$$(\mathcal{Q}, \alpha_{\mathcal{Q}}),$$

where $\mathcal{Q}$ is a $G_{\mu,\mathcal{N}}$-torsor on $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$ and $\alpha_{\mathcal{Q}} : \mathcal{Q} \to G_{\Delta,\mathcal{N}}$ is a $G_{\mu,\mathcal{N}}$-equivariant map. We say that $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ is *banal* if $\mathcal{Q}$ is trivial as a $G_{\mu,\mathcal{N}}$-torsor. When there is no possibility of confusion, we write $\mathcal{Q}$ instead of $(\mathcal{Q}, \alpha_{\mathcal{Q}})$. An isomorphism of prismatic G-F-gauges of type μ over R is defined in the same way as in Definition 5.2.1.

Let

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R)$$

be the groupoid of prismatic G-F-gauges of type μ over R. For a homomorphism $f : R \to R'$ of quasisyntomic rings over $W(k)$, we have a base change functor

$$f^* : G\text{-}F\text{-}\mathrm{Gauge}_\mu(R) \to G\text{-}F\text{-}\mathrm{Gauge}_\mu(R')$$

defined in the same way as in Definition 5.2.6.

Remark 8.2.6. A "truncated analogue" of the notion of prismatic G-F-gauges of type μ was introduced by Drinfeld [2023, Appendix C] for a p-adic formal scheme $\mathcal{X}$ which is formally of finite type over $\mathrm{Spf}\,\mathbb{Z}_p$, in terms of certain torsors on the *syntomification* of $\mathcal{X}$ in the sense of Drinfeld and Bhatt–Lurie. It should be possible to define prismatic G-F-gauges of type μ over any p-adic formal scheme by using certain torsors on syntomifications, but we will not discuss this here.[2]

Remark 8.2.7. Let S be a quasiregular semiperfectoid ring over $W(k)$. For a generator $d \in I_S$, let $\sigma_d : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ be the homomorphism defined by $a_i t^{-i} \mapsto \phi(a_i) d^{-i}$ for any $i \in \mathbb{Z}$. Recall the

[2] After this work was completed, and during the refereeing process, this has been carried out by Gardner and Madapusi [2024].

homomorphism $\tau : \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ from Definition 8.1.1. Similarly to the triple $(\mathrm{Rees}(I^{\bullet}), \sigma_d, \tau)$ in Remark 5.2.3, the triple

$$(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)), \sigma_d, \tau)$$

is an analogue of a higher frame in the sense of Lau. The homomorphism τ induces an isomorphism

$$G(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))^0 \xrightarrow{\sim} G_{\mu,\mathcal{N}}(S),$$

where $G(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))^0$ is the group of elements $g \in G(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))$ such that

$$g^* : A_G = \bigoplus_{i \in \mathbb{Z}} A^{(-1)}_{G,i} \to \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$$

is a homomorphism of graded $W(k)$-algebras. Via this isomorphism, the homomorphism $\sigma_{\mu,\mathcal{N},d}$ agrees with the one induced by σ_d. Thus, the action (8-6) is consistent with the one considered in [Lau 2021, (5-2)].

Roughly speaking, prismatic F-gauges in vector bundles (resp. prismatic G-F-gauges of type μ) over S can be considered as displays (resp. G-μ-displays) over the "higher frame" $(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)), \sigma_d, \tau)$. On the other hand, displayed Breuil–Kisin modules (resp. prismatic G-μ-displays) over (Δ_S, I_S) can be thought of as displays (resp. G-μ-displays) over the "higher frame" $(\mathrm{Rees}(I^{\bullet}_S), \sigma_d, \tau)$. See also [Lau 2021, Section 3.7] where the relation between displays over higher frames and Frobenius gauges in the sense of [Fontaine and Jannsen 2021, Section 2.2] is discussed.

Let us discuss the relation between prismatic F-gauges in vector bundles of type μ and prismatic GL_n-F-gauges of type μ.

Example 8.2.8. Let $\mu : \mathbb{G}_m \to \mathrm{GL}_{n,W(k)}$ be a cocharacter and let $(m_1, \dots, m_n)$ be the corresponding tuple of integers $m_1 \geq \cdots \geq m_n$ as in Section 3.2. We retain the notation of Section 3.2. Let $L_{W(k)} = \bigoplus_{j \in \mathbb{Z}} L_{\mu,j}$ be the weight decomposition with respect to the action of $\mathbb{G}_m$ on $L_{W(k)} = W(k)^n$ induced by μ. We set $L^{(-1)}_{\mu,j} := (\phi^{-1})^* L_{\mu,j}$. By the decomposition $L_{W(k)} = \bigoplus_{j \in \mathbb{Z}} L^{(-1)}_{\mu,j}$, we regard $L_{W(k)}$ as a graded module. Let S be a quasiregular semiperfectoid ring over $W(k)$. Then, via the isomorphism

$$\mathrm{GL}_n(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))^0 \xrightarrow{\sim} (\mathrm{GL}_n)_{\mu,\mathcal{N}}(S)$$

given in Remark 8.2.7, we may identify $(\mathrm{GL}_n)_{\mu,\mathcal{N}}(S)$ with the group of graded automorphisms of $L_{W(k)} \otimes_{W(k)} \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$.

Example 8.2.9. Let the notation be as in Example 8.2.8. Let R be a quasisyntomic ring over $W(k)$. We shall construct an equivalence

$$F\text{-}\mathrm{Gauge}_{\mu}(R)^{\simeq} \xrightarrow{\sim} \mathrm{GL}_n\text{-}F\text{-}\mathrm{Gauge}_{\mu}(R), \quad N \mapsto \mathcal{Q}(N), \tag{8-7}$$

where $F\text{-}\mathrm{Gauge}_{\mu}(R)^{\simeq}$ is the groupoid of prismatic F-gauges in vector bundles of type μ over R. Let $N \in F\text{-}\mathrm{Gauge}_{\mu}(R)$. We consider the sheaf

$$\mathcal{Q}(N) : (R)_{\mathrm{QSyn}} \to \mathrm{Set}$$

sending $S \in (R)_{\mathrm{qrsp}}$ to the set of graded isomorphisms

$$h : L_{W(k)} \otimes_{W(k)} \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \xrightarrow{\sim} N_S.$$

Such an isomorphism h exists locally in the quasisyntomic topology by (the proof of) Corollary 8.1.8. By Example 8.2.8, the sheaf $\mathscr{Q}(N)$ then admits the structure of a $(\mathrm{GL}_n)_{\mu,\mathcal{N}}$-torsor. Let $d \in I_S$ be a generator. We fix an isomorphism h as above. Then we have the isomorphisms

$$(\sigma^* N)_0 \simeq \bigoplus_{j\in\mathbb{Z}} (L_{\mu,j} \otimes_{W(k)} I_S^{-j}) \simeq L_{\Delta_S},$$

where the second isomorphism is given by $\mu(d)$. We also have $\tau^* N \simeq L_{\Delta_S}$. Thus, the isomorphism F_{N_S} gives an element $\alpha(h)_d \in \mathrm{GL}_n(\Delta_S) = \mathrm{GL}_n(\Delta_S)_{\mathcal{N},d}$. The element $\alpha(h) \in \mathrm{GL}_n(\Delta_S)_{\mathcal{N}}$ corresponding to $\alpha(h)_d$ does not depend on the choice of d. In this way, we get a $(\mathrm{GL}_n)_{\mu,\mathcal{N}}$-equivariant map $\alpha : \mathscr{Q}(N) \to (\mathrm{GL}_n)_{\Delta,\mathcal{N}}$, so that the pair $(\mathscr{Q}(N), \alpha)$ belongs to GL_n -F-$\mathrm{Gauge}_\mu(R)$. This construction gives the functor (8-7). Using quasisyntomic descent, one can check that this functor is an equivalence.

We now compare prismatic G-μ-displays with prismatic G-F-gauges of type μ. We first note the following result.

Proposition 8.2.10. *Let R be a quasisyntomic ring over $W(k)$. The fibered category over $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$ which associates to each $R' \in (R)_{\mathrm{QSyn}}$ the groupoid G-$\mathrm{Disp}_\mu((R')_\Delta)$ is a stack with respect to the quasisyntomic topology.*

Proof. This follows from [Bhatt and Scholze 2022, Proposition 7.11] and Proposition 5.2.8 by a standard argument. (See also the proof of [Bhatt and Scholze 2023, Proposition 2.14].) □

Proposition 8.2.11. *Let R be a quasisyntomic ring over $W(k)$. There exists a fully faithful functor*

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R) \to G\text{-}\mathrm{Disp}_\mu((R)_\Delta). \tag{8-8}$$

This functor is compatible with base change along any homomorphism $R \to R'$ in QSyn.

Proof. It is clear that the left-hand side of (8-8) satisfies quasisyntomic descent. By Proposition 8.2.10, the right-hand side of (8-8) also satisfies quasisyntomic descent. It thus suffices to construct, for each quasiregular semiperfectoid ring S over $W(k)$, a fully faithful functor

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(S)_{\mathrm{banal}} \to G\text{-}\mathrm{Disp}_\mu(\Delta_S, I_S)_{\mathrm{banal}} \tag{8-9}$$

that is compatible with base change along any homomorphism $S \to S'$ in the subcategory QRSPerfd. Here G-F-$\mathrm{Gauge}_\mu(S)_{\mathrm{banal}}$ is the groupoid of banal prismatic G-F-gauges of type μ over S. By Remark 5.2.5, we may identify G-$\mathrm{Disp}_\mu(\Delta_S, I_S)_{\mathrm{banal}}$ with the groupoid $[G(\Delta_S)_{I_S}/G_\mu(\Delta_S, I_S)]$. Similarly, we may identify G-F-$\mathrm{Gauge}_\mu(S)_{\mathrm{banal}}$ with the groupoid

$$[G(\Delta_S)_{\mathcal{N}}/G_{\mu,\mathcal{N}}(S)]$$

whose objects are the elements $X \in G(\Delta_S)_{\mathcal{N}}$ and whose morphisms are defined by $\mathrm{Hom}(X, X') = \{g \in G_{\mu,\mathcal{N}}(S) \mid X' \cdot g = X\}$. The map $\phi : G(\Delta_S) \to G(\Delta_S)$ induces a map $\phi : G(\Delta_S)_{\mathcal{N}} \to G(\Delta_S)_{I_S}$ such that, for every $X \in G(\Delta_S)_{\mathcal{N}}$ and every $g \in G_{\mu,\mathcal{N}}(S)$, we have $\phi(X \cdot g) = \phi(X) \cdot \phi(g)$, where

$\phi(g) \in G_\mu(\Delta_S, I_S)$ is the image of g under the natural homomorphism $\phi : G_{\mu,\mathcal{N}}(S) \to G_\mu(\Delta_S, I_S)$. Then we define the functor (8-9) as

$$[G(\Delta_S)_{\mathcal{N}}/G_{\mu,\mathcal{N}}(S)] \to [G(\Delta_S)_{I_S}/G_\mu(\Delta_S, I_S)], \quad X \mapsto \phi(X). \tag{8-10}$$

This functor is fully faithful. Indeed, let $d \in I_S$ be a generator. It suffices to prove that for all $X, X' \in G(\Delta_S)$, the map

$$\{g \in G_{\mu,\mathcal{N}}(S) \mid g^{-1}X'\sigma_{\mu,\mathcal{N},d}(g) = X\} \to \{h \in G_\mu(\Delta_S, I_S) \mid h^{-1}\phi(X')\sigma_{\mu,d}(h) = \phi(X)\}$$

defined by $g \mapsto \phi(g)$ is bijective. (Recall $\sigma_{\mu,\mathcal{N},d}(g) = \mu(d)\phi(g)\mu(d)^{-1}$ and $\sigma_{\mu,d}(h) = \phi(\mu(d)h\mu(d)^{-1})$.) One can check that the map $h \mapsto X'\mu(d)h\mu(d)^{-1}X^{-1}$ gives the inverse of the above map. Indeed, for an element $g \in G_{\mu,\mathcal{N}}(S)$ in the left-hand side, we have

$$X'\mu(d)\phi(g)\mu(d)^{-1}X^{-1} = X'\sigma_{\mu,\mathcal{N},d}(g)X^{-1} = g.$$

Similarly, for an element $h \in G_\mu(\Delta_S, I_S)$ in the right-hand side, we have

$$\phi(X'\mu(d)h\mu(d)^{-1}X^{-1}) = \phi(X')\sigma_{\mu,d}(h)\phi(X)^{-1} = h.$$

The proof of Proposition 8.2.11 is complete. □

Corollary 8.2.12. *Let R be a perfectoid ring over $W(k)$ or a complete regular local ring over $W(k)$ with residue field k. Then the functor (8-8) is an equivalence*:

$$G\text{-}F\text{-Gauge}_\mu(R) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu((R)_{\Delta}).$$

Proof. Since we already know that this functor is fully faithful (Proposition 8.2.11), it suffices to prove the essential surjectivity. The assertion can be checked locally in the quasisyntomic topology.

We first assume that R is a perfectoid ring over $W(k)$. In this case, we have

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta}) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(W(R^\flat), I_R).$$

Since every G-μ-display over $(W(R^\flat), I_R)$ is banal over a p-completely étale covering $R \to R'$ with R' a perfectoid ring (Example 2.5.11), it suffices to prove that the functor (8-10) given in the proof of Proposition 8.2.11 is essentially surjective when $S = R$. This follows since $(\Delta_R, I_R) \simeq (W(R^\flat), I_R)$ and the Frobenius $\phi : W(R^\flat) \to W(R^\flat)$ is bijective.

The case where R is a complete regular local ring over $W(k)$ with residue field k follows from the previous case since there exists a quasisyntomic covering $R \to R'$ with R' a perfectoid ring by [Anschütz and Le Bras 2023, Proposition 5.8]. □

Corollary 8.2.13. *The functors (8-4) and (8-5) are equivalences if R is a perfectoid ring or a complete regular local ring with perfect residue field k of characteristic p.*

Proof. We need to prove that (8-4) and (8-5) are essentially surjective. For (8-5), this follows from Corollary 8.2.12 together with Examples 5.3.10 and 8.2.9. For (8-4), we argue as follows. As in the proof of Corollary 8.2.12, it suffices to treat the case where R is a perfectoid ring. Then we have

$$2-\varprojlim_{(A,I)\in(R)_{\mathbb{\Delta}}}\mathrm{BK}_{\mathrm{disp}}(A,I)\xrightarrow{\sim}\mathrm{BK}_{\mathrm{disp}}(W(R^{\flat}),I_R).$$

For each $M\in\mathrm{BK}_{\mathrm{disp}}(W(R^{\flat}),I_R)$, there exists a p-completely étale covering $R\to R_1\times\cdots\times R_m$ with $R_1,\ldots,R_m$ perfectoid rings such that, for any $1\le i\le m$, the base change $M_{(W(R_i^{\flat}),I_{R_i})}$ is of type μ for some μ; see Example 2.5.11 and Remark 3.2.3. Since (8-5) is essentially surjective, we can conclude that (8-4) is also essentially surjective by using p-completely étale descent. □

Remark 8.2.14. Let R be a quasisyntomic ring over $W(k)$. For a bounded prism $(A,I)\in(R)_{\mathbb{\Delta}}$, we defined the groupoid G-$\mathrm{BK}_\mu(A,I)$ of G-Breuil–Kisin modules of type μ over (A,I) in Section 5.1 and showed that it is equivalent to G-$\mathrm{Disp}_\mu(A,I)$ in Proposition 5.3.8. Thus the fully faithful functor (8-8) can be written as

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R)\to G\text{-}\mathrm{BK}_\mu((R)_{\mathbb{\Delta}}):=2-\varprojlim_{(A,I)\in(R)_{\mathbb{\Delta}}}G\text{-}\mathrm{BK}_\mu(A,I).$$

The essential image of this functor consists of those $\mathcal{P}\in G$-$\mathrm{BK}_\mu((R)_{\mathbb{\Delta}})$ such that for some quasisyntomic covering $R\to S$ with S a quasiregular semiperfectoid ring, the image $\mathcal{P}_{(\mathbb{\Delta}_S,I_S)}\in G$-$\mathrm{BK}_\mu(\mathbb{\Delta}_S,I_S)$ of $\mathcal{P}$ is a trivial $G_{\mathbb{\Delta}_S}$-torsor, and via some trivialization $\mathcal{P}_{(\mathbb{\Delta}_S,I_S)}\simeq G_{\mathbb{\Delta}_S}$, the isomorphism $F_{\mathcal{P}_{(\mathbb{\Delta}_S,I_S)}}$ is given by $g\mapsto Yg$ for an element Y in

$$\mu(d)\phi(G(\mathbb{\Delta}_S))\subset G(\mathbb{\Delta}_S[1/I_S]),$$

where $d\in I_S$ is a generator. Therefore, we can simply define a prismatic G-F-gauge of type μ over R as an object $\mathcal{P}\in G$-$\mathrm{BK}_\mu((R)_{\mathbb{\Delta}})$ that satisfies the above condition. However, similarly to prismatic G-μ-displays, it should be more technically convenient to work with the one introduced in Definition 8.2.5.

We shall give an example which shows that the functors (8-4) and (8-8) are not essentially surjective in general. This also shows that there exists a nonadmissible prismatic Dieudonné crystal (see Example 8.1.15).

Let $\mathcal{O}_C$ be the ring of integers of an algebraically closed nonarchimedean extension C of $\mathbb{Q}_p$. Then the quotient $S=\mathcal{O}_C/p$ is a quasiregular semiperfectoid ring. The natural homomorphism $S\to\mathbb{\Delta}_S/I_S$ is injective, and the Frobenius $\phi:\mathbb{\Delta}_S\to\mathbb{\Delta}_S$ induces an isomorphism $\mathbb{\Delta}_S/\mathrm{Fil}^1_{\mathcal{N}}(\mathbb{\Delta}_S)\xrightarrow{\sim}S$ (see [Bhatt and Scholze 2022, Theorem 12.2]). The Hodge–Tate comparison theorem for the conjugate filtration with respect to the natural homomorphism $\mathcal{O}_C\to S$ shows that $S\to\mathbb{\Delta}_S/I_S$ is not surjective; see [loc. cit., Section 12.1]. We fix a generator $d\in I_S$.

Example 8.2.15. Let the notation be as above. We assume that $G=\mathrm{GL}_2$ and $\mu:\mathbb{G}_m\to\mathrm{GL}_2$ is the 1-bounded cocharacter defined by $t\mapsto\mathrm{diag}\,(t,1)$. We choose an element $x\in\mathbb{\Delta}_S$ whose image $\bar{x}\in\mathbb{\Delta}_S/I_S$ is not contained in S. Let $X\in G(\mathbb{\Delta}_S)_{I_S}$ be the element such that $X_d=\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right)\in G(\mathbb{\Delta}_S)$. We shall show that X is not contained in the essential image of the functor (8-10). If X is contained in the essential image, then there are $Y\in G(\mathbb{\Delta}_S)_{\mathcal{N}}$ and $g\in G_\mu(\mathbb{\Delta}_S,I_S)$ such that the equality $X\cdot g=\phi(Y)$ holds in $G(\mathbb{\Delta}_S)_{I_S}$. In particular, we see that $g^{-1}X_d$ belongs to the image of $\phi:G(\mathbb{\Delta}_S)\to G(\mathbb{\Delta}_S)$. We write

$$g=\begin{pmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{pmatrix}\in G_\mu(\mathbb{\Delta}_S,I_S).$$

By Proposition 4.2.9, we have $g_{21} \in I_S$ and $g_{11}, g_{22} \in \Delta_S^\times$. By computing the image of $g^{-1}X_d$ in $G(\Delta_S/I_S)$ and using that $g^{-1}X_d \in \phi(G(\Delta_S))$, it follows that $\bar{x}/\bar{g}_{22}, 1/\bar{g}_{22} \in \Delta_S/I_S$ are contained in S. We thus have $\bar{x} \in S$, which leads to a contradiction.

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