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Baiqing Zhu

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Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$

Baiqing Zhu

We establish the arithmetic Siegel–Weil formula on the modular curve $\mathcal{X}_0(N)$ for arbitrary level N, i.e., we relate the arithmetic degrees of special cycles on $\mathcal{X}_0(N)$ to the derivatives of Fourier coefficients of a genus-2 Eisenstein series. We prove this formula by a precise identity between the local arithmetic intersection numbers on the Rapoport–Zink space associated to $\mathcal{X}_0(N)$ and the derivatives of local representation densities of quadratic forms. When N is odd and square-free, this gives a different proof of the main results in work of Sankaran, Shi and Yang. This local identity is proved by relating it to an identity in one dimension higher, but at hyperspecial level.

1. Introduction

1.1. ***Background.*** The classical Siegel–Weil formula relates certain Siegel Eisenstein series to the arithmetic of quadratic forms, namely, it expresses special values of these series as theta functions, generating series of representation numbers of quadratic forms. Kudla initiated an influential program to establish the arithmetic Siegel–Weil formula relating certain Siegel Eisenstein series to objects in arithmetic geometry.

In this article, we study the case of modular curves. Let N be a positive integer. The classical modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ over $\mathbb{C}$ is defined as the smooth 1-dimensional complex curve

$$\mathcal{Y}_0(N)_{\mathbb{C}} \coloneqq \mathrm{GL}_2(\mathbb{Q}) \setminus \mathbb{H}_1^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f) / \Gamma_0(N)(\hat{\mathbb{Z}}) \simeq \Gamma_0(N) \setminus \mathbb{H}_1^{\pm},$$

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where $\mathbb{H}_1^{\pm} = \mathbb{C} \setminus \mathbb{R}$ and $\mathbb{H}_1^{+} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$ is the upper half plane. The group $\Gamma_0(N)(\hat{\mathbb{Z}})$ is the open compact subgroup

$$\Gamma_0(N)(\hat{\mathbb{Z}}) = \left\{ x = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}) : a, b, c, d \in \hat{\mathbb{Z}} \right\}$$

of $\mathrm{GL}_2(\mathbb{A}_f)$, and $\Gamma_0(N) = \Gamma_0(N)(\hat{\mathbb{Z}}) \bigcap \mathrm{GL}_2(\mathbb{Z})$. Notice that the determinant of an element in the group $\Gamma_0(N)$ can be either 1 or -1 rather than only 1 in the classical setting because the space $\mathbb{H}_1^{\pm}$ has two connected components.

The smooth curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ is not proper, its compactification $\mathcal{X}_0(N)_{\mathbb{C}} := \mathcal{Y}_0(N)_{\mathbb{C}} \cup \{\text{cusps}\}$ is a smooth projective curve over $\mathbb{C}$. It is a classical fact that the curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ parameterizes cyclic isogenies between elliptic curves over $\mathbb{C}$. Here an isogeny $\pi : E \to E'$ between two elliptic curves over $\mathbb{C}$ is called cyclic if $\ker(\pi)$ is a cyclic group.

Katz and Mazur [1985] extended the concept of cyclic isogeny to an arbitrary base scheme: an isogeny $\pi : E \to E'$ between two elliptic curves is called cyclic if $\ker(\pi)$ is a cyclic group scheme (see Definition 4.1.2). They also defined the $\Gamma_0(N)$-level structures on elliptic curves. In this article, we mainly work on a 2-dimensional regular flat Deligne–Mumford stack $\mathcal{X}_0(N)$, defined in [Česnavičius 2017], which is the moduli stack of generalized elliptic curves with $\Gamma_0(N)$-level structures and whose fiber over $\mathbb{C}$ is $\mathcal{X}_0(N)_{\mathbb{C}}$. We define the (arithmetic) special cycles on $\mathcal{X}_0(N)$ and study their intersection numbers. Finally, we prove that these intersection numbers are identified with the derivatives of Fourier coefficients of certain Siegel Eisenstein series of genus 2.

When N is an odd, square-free positive integer, the relation has already been obtained in the work of Sankaran, Shi and Yang [Sankaran et al. 2023, Theorem 2.14] by computing both sides explicitly based on [Yang 1998; Kudla et al. 2006]. In this article, we use a different method and work with arbitrary level N. We introduce a formal scheme $\mathcal{N}_0(N)$ which is the Rapoport–Zink space associated to $\mathcal{X}_0(N)$. Via formal uniformization of the supersingular locus of the stack $\mathcal{X}_0(N)$ and its special cycles, we reduce the identity, which relates intersection numbers on $\mathcal{X}_0(N)$ and derivatives of Fourier coefficients of Eisenstein series, to a local identity between local arithmetic intersection numbers on $\mathcal{N}_0(N)$ and derivatives of local densities of quadratic forms. Now the key observation is that both sides of the local identity, regardless of the level N, can be related to another intersection problem on Rapoport–Zink space of 1 dimension higher, but in a hyperspecial level, while the computation of the latter has been done in [Gross and Keating 1993, Proposition 5.4; Wedhorn 2007, §2.16; Rapoport 2007, Theorem 1.1] (see also [Li and Zhang 2022, Theorem 1.2.1]).

1.2. *Summary of main results.*

1.2.A. *Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$.* Let $\Delta(N)$ be the rank-3 quadratic lattice

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \tag{1}$$

over $\mathbb{Z}$, equipped with the quadratic form $x \mapsto \det(x)$.

We use v to denote a place of $\mathbb{Q}$. For every finite place v, let $\Delta_v(N) = \Delta(N) \otimes_{\mathbb{Z}} \mathbb{Z}_v$ be a rank-3 quadratic lattice over $\mathbb{Z}_v$. Let $\mathbb{A}$ be the ring of adèles over $\mathbb{Q}$. Let $\mathbb{V} = \{\mathbb{V}_v\}$ be the incoherent collection of quadratic spaces of $\mathbb{A}$ of rank 3 nearby $\Delta(N)$ at ∞, i.e.,

$$\mathbb{V}_v = \Delta_v(N) \otimes \mathbb{Q}_v \quad \text{if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.} \tag{2}$$

Consider a finite place v. Let $\mathbb{V}_f := \mathbb{V} \otimes \mathbb{A}_f$ (resp. $\mathbb{V}_f^v := \mathbb{V} \otimes \mathbb{A}_f^v$) be the quadratic space of rank 3 over $\mathbb{A}_f$ (resp. $\mathbb{A}_f^v$). Let $\mathscr{S}(\mathbb{V}^2)$ (resp. $\mathscr{S}(\mathbb{V}_f^2)$, $\mathscr{S}((\mathbb{V}_f^v)^2)$) be the space of Schwartz functions on $\mathbb{V}^2$ (resp. $\mathbb{V}_f^2$, $(\mathbb{V}_f^v)^2$). Associated to $\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} \otimes \boldsymbol{\varphi}_\infty \in \mathscr{S}(\mathbb{V}^2)$, where $\boldsymbol{\varphi}_\infty$ is the Gaussian function on $\mathbb{V}_\infty^2$ and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$, there is a classical incoherent Eisenstein series $E(\mathsf{z}, s, \boldsymbol{\varphi})$ (see Section 3.4) on the Siegel upper half plane,

$$\mathbb{H}_2 = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} : \mathsf{x} \in \mathrm{Sym}_2(\mathbb{R}),\ \mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}\}.$$

This is essentially the Siegel Eisenstein series associated to a standard Siegel–Weil section of the degenerate principal series. The Eisenstein series here has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$. The central value $E(\mathsf{z}, s, \boldsymbol{\varphi})$ is 0 by the incoherence. We thus consider its central derivative

$$\partial\,\mathrm{Eis}(\mathsf{z}, \boldsymbol{\varphi}) := \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} E(\mathsf{z}, s, \boldsymbol{\varphi}).$$

Associated to the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$, it has a decomposition into the central derivatives of the Fourier coefficients

$$\partial\,\mathrm{Eis}(\mathsf{z}, \boldsymbol{\varphi}) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} \partial\,\mathrm{Eis}_T(\mathsf{z}, \boldsymbol{\varphi}).$$

On the geometric side, there is a regular integral model of the modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ over $\mathbb{Z}$ defined by Katz and Mazur: for any scheme S, the groupoid $\mathcal{Y}_0(N)(S)$ consists of objects $(E \xrightarrow{\pi} E')$, where E and E' are elliptic curves over S and π is a cyclic isogeny such that $\pi^\vee \circ \pi = N$. They proved that $\mathcal{Y}_0(N)$ is a 2-dimensional regular flat Deligne–Mumford stack (see [Katz and Mazur 1985, Theorem 5.1.1]), but $\mathcal{Y}_0(N)$ is not proper. There is a moduli stack $\mathcal{X}_0(N)$ defined in [Česnavičius 2017] which serves as a "compactification" of $\mathcal{Y}_0(N)$. It is a proper regular flat 2-dimensional Deligne–Mumford stack which contains $\mathcal{Y}_0(N)$ as an open substack, so we can consider the arithmetic intersection theory on $\mathcal{X}_0(N)$ following the lines in [Gillet 2009].

The key concept is that of a special cycle. A typical special cycle is of the form $\mathcal{Z}(T, \boldsymbol{\varphi})$, where T is a 2×2 symmetric matrix and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ is the characteristic function of some open compact subset of $\mathbb{V}_f^2$. It is a Deligne–Mumford stack finite unramified over $\mathcal{X}_0(N)$. For an object $(E \xrightarrow{\pi} E')$ of $\mathcal{Y}_0(N)(S)$, the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ parameterizes pairs of isogenies between E and E' with inner product matrix T and orthogonal to the cyclic isogeny π, along with some level structures given by the Schwartz function $\boldsymbol{\varphi}$ which is $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant (cf. Definition 4.3.5). For every nonsingular $T \in \mathrm{Sym}_2(\mathbb{Q})$ and prime number l, we say T is represented by $\Delta(N) \otimes \mathbb{Q}_l$ if there exist two vectors $x_1, x_2 \in \Delta(N) \otimes \mathbb{Q}_l$ such that $T = \frac{1}{2}((x_i, x_j))$. Define the difference set

$$\mathrm{Diff}(T, \Delta(N)) = \{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l\}.$$

Let $\widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$ be the codimension-2 arithmetic Chow group with complex coefficients of the regular flat Deligne–Mumford stack $\mathcal{X}_0(N)$. We also construct arithmetic special cycles on the stack $\mathcal{X}_0(N)$. They are elements of the form $\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}) \in \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$, where $T \in \mathrm{Sym}_2(\mathbb{Q})$ is nonsingular, $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})$ is positive definite and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ is a Schwartz function. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function, roughly meaning $\boldsymbol{\varphi}$ is invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}})$ and for every $p \in \mathrm{Diff}(T, \Delta(N))$, $\boldsymbol{\varphi} = \boldsymbol{\varphi}^p \otimes \boldsymbol{\varphi}_p$, where $\boldsymbol{\varphi}^p \in \mathscr{S}((\mathbb{V}_f^p)^2)$ and $\boldsymbol{\varphi}_p = c \cdot \mathbf{1}_{\Delta_p(N)^2} \in \mathscr{S}(\mathbb{V}_p^2)$ for some $c \in \mathbb{C}$. Our main goal is to relate $\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))$ to derivatives of the Fourier coefficients of a genus-2 Siegel Eisenstein series, where $\widehat{\deg} : \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)) \to \mathbb{C}$ is the arithmetic degree map (see (14)).

Theorem 1.2.1. *Let N be a positive integer. Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular symmetric matrix. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Then*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))q^T = \frac{\psi(N)}{24} \cdot \partial\, \mathrm{Eis}_T(\mathsf{z}, \boldsymbol{\varphi})$$

for any $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$. Here $\psi(N) = N \cdot \prod_{l \mid N}(1 + l^{-1})$ and $q^T = e^{2\pi i \,\mathrm{tr}(T\mathsf{z})}$.

1.2.B. *The local arithmetic Siegel–Weil formula with level N.* Fix a prime number p. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let W be the integer ring of the completion of the maximal unramified extension of $\mathbb{Q}_p$.

On the geometry side, let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ of dimension 1 and height 2. Let $\mathbb{B}$ be the unique division quaternion algebra over $\mathbb{Q}_p$, so $\mathrm{End}^0(\mathbb{X}) \simeq \mathbb{B}$ as quadratic spaces. The Rapoport–Zink space associated to $\mathcal{X}_0(N)$ is the deformation space $\mathcal{N}_0(N)$ such that, for a W-scheme S where p is locally nilpotent and an element $x_0 \in \mathbb{B}$ such that $x_0^\vee \circ x_0 = N$, the set $\mathcal{N}_0(N)(S)$ consists of elements $(X \xrightarrow{\pi} X')$ where X, X' are deformations over S of $\mathbb{X}$ with certain restrictions on polarizations (see Section 5.1), the morphism π is a cyclic isogeny deforming x_0 and $\pi^\vee \circ \pi = N$.

Let $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$ be the subspace of quasi-isogenies which are orthogonal to x_0. For any $x \in \mathbb{W}$, there is a closed formal subscheme $\mathcal{Z}(x)$ of $\mathcal{N}_0(N)$ over which the quasi-isogeny x lifts to an isogeny. This is an example of a special cycle (see Definition 5.2.5) on $\mathcal{N}_0(N)$. For a rank-2 lattice $M \subset \mathbb{W}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of M, then define the local arithmetic intersection number of M on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \chi\big(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}\big).$$

This number is independent of the choice of the basis $\{x_1, x_2\}$ of M.

On the analytic side, for any two integral quadratic $\mathbb{Z}_p$-lattices L and M, let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, a $\mathbb{Z}_p$-scheme such that for any $\mathbb{Z}_p$-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{QHom}(L \otimes_{\mathbb{Z}_p} R, M \otimes_{\mathbb{Z}_p} R),$$

where QHom denotes the set of quadratic module homomorphisms. The local density of integral representations is defined to be

$$\mathrm{Den}(M, L) = \lim_{d\to\infty} \frac{\#\,\mathrm{Rep}_{M,L}(\mathbb{Z}_p/p^d)}{p^{d\cdot \dim(\mathrm{Rep}_{M,L})_{\mathbb{Q}_p}}}.$$

Let $H_2^+ = \mathbb{Z}_p^2$ be the rank-2 quadratic $\mathbb{Z}_p$-lattice equipped with the quadratic form $q_{H_2^+}(x, y) = xy$. For any $k \geq 0$, let $H_{2k}^+ := (H_2^+)^{\oplus n}$ be a rank-$2k$ quadratic $\mathbb{Z}_p$-lattice. For any $\mathbb{Z}_p$-lattice $M \subset \mathbb{W}$ of rank 2, define the local density of M with level N to be the polynomial $\mathrm{Den}_{\Delta_p(N)}(X, M)$ such that for all $k \geq 0$

$$\mathrm{Den}_{\Delta_p(N)}(X, M)\big|_{X=p^{-k}} = \begin{cases} \dfrac{\mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M)}{\mathrm{Nor}^+(p^{-k}, 1)} & \text{when } p \mid N, \\[2ex] \dfrac{\mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M)}{\mathrm{Nor}^{(N,p)_p}(p^{-k}, 2)} & \text{when } p \nmid N, \end{cases} \tag{3}$$

where $(\cdot\,,\cdot)_p$ is the Hilbert symbol at p, the polynomials $\mathrm{Nor}^{\varepsilon}(X, n)$ are normalizing factors defined in Definition 2.2.6. Then $\mathrm{Den}_{\Delta_p(N)}(1, M) = 0$ since M can't be isometrically embedded into the quadratic lattice $\Delta_p(N)$. We define the derived local density of M with level N to be

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M) = -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}_{\Delta_p(N)}(X, M).$$

The local arithmetic Siegel–Weil formula with level N is an exact identity between the two integers just defined.

Theorem 1.2.2. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\,\mathrm{Den}_{\Delta_p(N)}(M).$$

We refer to $\mathrm{Int}_{\mathcal{N}_0(N)}(M)$ as the geometric side of the identity (related to the geometry of Rapoport–Zink spaces and Shimura varieties) and $\partial\,\mathrm{Den}_{\Delta_p(N)}(M)$ as the analytic side (related to the derivatives of Eisenstein series and L-functions).

1.2.C. *Formal uniformization.* For any prime p, let $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ be the supersingular locus of the stack $\mathcal{X}_0(N)$, i.e., those $\mathbb{F}$-points of $\mathcal{X}_0(N)$ which are isogenous to a supersingular elliptic curve. Let B be the unique quaternion algebra which ramifies exactly at p and ∞. Let $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$ be the completion of the stack $\mathcal{X}_0(N)$ along the closed substack $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Let $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ be the group $\prod_{v\neq\infty,p}\Gamma_0(N)(\mathbb{Z}_v)$. We have the following formal uniformization theorem of the stack $\mathcal{X}_0(N)$.

Proposition 1.2.3. *There is an isomorphism of formal stacks over W,*

$$\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{X}_0(N)}} B^\times(\mathbb{Q})_0 \backslash \big[\mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)\big],$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

This proposition was previously known only in the case that N is odd and square-free (see [Kim 2018, Theorem 4.7] for the case $p \nmid N$ and [Oki 2020, Theorem 6.1] for $p \mid N$). As a corollary, let $\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})$

be the completion of $\mathcal{Z}(T,\boldsymbol{\varphi})$ along its supersingular locus $\mathcal{Z}^{\mathrm{ss}}(T,\boldsymbol{\varphi}) := \mathcal{Z}(T,\boldsymbol{\varphi}) \times_{\mathcal{X}_0(N)} \mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Let $\Delta(N)^{(p)}$ be the unique quadratic space over $\mathbb{Q}$ (up to isometry) such that

(1) it is positive definite at ∞;

(2) for finite primes $l \neq p$, $\Delta(N)^{(p)} \otimes \mathbb{Q}_l$ is isometric to $\Delta_l(N) \otimes \mathbb{Q}_l$;

(3) $\Delta(N)^{(p)} \otimes \mathbb{Q}_p$ is isometric to $\mathbb{W}$.

For a pair of vectors $\boldsymbol{x} = (x_1, x_2) \in (\Delta(N)^{(p)})^2$, let $T(\boldsymbol{x}) = \left(\frac{1}{2}(x_i, x_j)\right)$ be the inner product matrix. We have the following formal uniformization theorem of the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi})$.

Corollary 1.2.4. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular symmetric matrix, and $\mathrm{Diff}(T, \Delta(N)) = \{p\}$. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Let $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_{\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}}$-modules. Then we have the identity*

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T,\boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \; \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \Theta^{-1}_{\mathcal{X}_0(N)}(\mathcal{Z}(\boldsymbol{x}), g)$$

in $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$, where $B_{\boldsymbol{x}}^\times \subset B^\times$ is the stabilizer of $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$.

1.3. *Strategy of the proof of main results.*

1.3.A. *Difference formula at the geometric side.* Let $\mathcal{N}$ be the deformation functor such that, for a W-scheme S where p is locally nilpotent, the set $\mathcal{N}(S)$ consists of elements (X, X'), where both X and X' are deformations over S of $\mathbb{X}$ with certain restrictions on polarizations (see Section 5.1). For a nonzero integral element $x \in \mathbb{B}$, i.e., $0 \le \nu_p(x^\vee \circ x) < \infty$, there is a closed formal subscheme $\mathcal{Z}^\sharp(x)$ of $\mathcal{N}$ over which the quasi-isogeny x lifts to an isogeny. This is an example of a special cycle (see Definition 5.2.1) on $\mathcal{N}$.

For a rank-3 lattice $L \subset \mathbb{B}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2, x_3\}$ of L, then define the local arithmetic intersection number of L on $\mathcal{N}$ to be

$$\mathrm{Int}^\sharp(L) = \chi\left(\mathcal{N}, \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}\right).$$

This number is independent of the choice of the basis $\{x_1, x_2, x_3\}$ of the lattice L.

The special cycle $\mathcal{Z}^\sharp(x)$ is cut out by a single element $f_x \in \mathfrak{m} = (p, t_1, t_2) \subset W[\![t_1, t_2]\!]$, and when $\nu_p(x^\vee \circ x) \ge 2$, we have $f_{p^{-1}x} \mid f_x$. We define $d_x = f_x / f_{p^{-1}x} \in W[\![t_1, t_2]\!]$ when $\nu_p(x^\vee \circ x) \ge 2$, and $d_x = f_x$ when $\nu_p(x^\vee \circ x) = 0$ or 1. The divisor

$$\mathcal{D}(x) := \mathrm{Spf}\, W[\![t_1, t_2]\!]/d_x$$

is called the difference divisor associated to x (see Definition 6.2.1), which was originally introduced in [Terstiege 2011].

Fix $x_0 \in \mathbb{B}$ such that $x_0^\vee \circ x_0 = N$, recall that we have defined the deformation function $\mathcal{N}_0(N)$. In Theorem 6.2.3, we prove that $\mathcal{N}_0(N)$ is identified with the difference divisor $\mathcal{D}(x_0)$, i.e., there is an isomorphism of formal schemes

$$\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N).$$

Let $x_0^{\mathrm{univ}} : X^{\mathrm{univ}} \to X'^{\mathrm{univ}}$ be the universal isogeny deforming x_0 over the special cycle $\mathcal{Z}^\sharp(x_0)$. We will prove that the base change of x_0^{univ} to $\mathcal{D}(x_0)$ is cyclic, and therefore there is a natural morphism $\mathcal{D}(x_0) \to \mathcal{N}_0(N)$. The natural morphism is an isomorphism because both sides of the morphism are closed formal subschemes of $\mathcal{N}$ and are represented by 2-dimensional regular local rings. The identification of $\mathcal{D}(x_0)$ and $\mathcal{N}_0(N)$ implies the following difference formula of local arithmetic intersection numbers:

Theorem 1.3.1. *For any rank-2 lattice $M \subset \mathbb{W}$,*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

We refer to this formula as the difference formula at the geometric side.

1.3.B. *Difference formula at the analytic side.* For any rank-3 quadratic $\mathbb{Z}_p$-lattice $L \subset \mathbb{B}$, define the local density of L to be the polynomial $\mathrm{Den}(X, L) \in \mathbb{Z}[X]$ such that for all $k \geq 0$,

$$\mathrm{Den}(X, L)\big|_{X=p^{-k}} = \frac{\mathrm{Den}(H^+_{2k+4}, L)}{\mathrm{Nor}^+(p^{-k}, 3)}.$$

Then $\mathrm{Den}(1, L) = 0$ since L can't be isometrically embedded into the quadratic lattice H_4^+. We define the derived local density of L to be

$$\partial\, \mathrm{Den}(L) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(X, L).$$

Theorem 1.3.2. *For any rank-2 lattice $M \subset \mathbb{W}$, the identity*

$$\mathrm{Den}_{\Delta_p(N)}(X, M) = \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0)$$

holds. Since the lattice $M \oplus \mathbb{Z}_p \cdot x_0$ can't be isometrically embedded into the lattice H_4^+,

$$\partial\, \mathrm{Den}_{\Delta_p(N)}(M) = \partial\, \mathrm{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial\, \mathrm{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

The theorem is proved in a more general form in Theorem 7.2.6. We refer to this formula as the difference formula at the analytic side.

1.3.C. *Proof of Theorem 1.2.1.* The following local arithmetic Siegel–Weil formula is proved in [Wedhorn 2007, §2.16] (see also [Li and Zhang 2022, Theorem 1.2.1] when p is odd).

Theorem 1.3.3. *For any rank-3 lattice $L \subset \mathbb{B}$, we have the identity*

$$\mathrm{Int}^\sharp(L) = \partial\, \mathrm{Den}(L).$$

For a rank-2 lattice $M \subset \mathbb{W}$, let $L = M \oplus \mathbb{Z}_p \cdot x_0 \subset \mathbb{B}$. The local arithmetic Siegel–Weil formula with level N in Theorem 1.2.2 follows immediately from $\mathrm{Int}^\sharp(L) = \partial\,\mathrm{Den}(L)$ and two difference formulas we stated before (Theorems 1.3.1 and 1.3.2).

1.3.D. *Proof of Theorem 1.2.2.* Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible function. When T is not positive definite, the arithmetic special cycle $\hat{\mathcal{Z}}(T, \mathrm{y}, \boldsymbol{\varphi})$ is essentially a $(1,1)$-current on the proper smooth complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$. The number $\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathrm{y}, \boldsymbol{\varphi}))$ has been computed explicitly in [Sankaran et al. 2023, Theorem 4.10].

We focus on the case that T is positive definite. In this case, $\hat{\mathcal{Z}}(T, \mathrm{y}, \boldsymbol{\varphi}) = [(\mathcal{Z}(T, \varphi), 0)]$, where $\mathcal{Z}(T, \boldsymbol{\varphi})$ is a cycle of codimension 2 on $\mathcal{X}_0(N)$. Moreover, $\mathcal{Z}(T, \boldsymbol{\varphi}) \neq \varnothing$ only if $\mathrm{Diff}(T, \Delta(N)) = \{p\}$ for some prime number p; in this case the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ is concentrated in the supersingular locus of $\mathcal{X}_0(N)$ in characteristic p. Suppose that the 2×2 matrix T has diagonal elements t_1 and t_2, and $\boldsymbol{\varphi} = \varphi_1 \times \varphi_2 \in \mathscr{S}(\mathbb{V}_f^2)$, where $\varphi_i \in \mathscr{S}(\mathbb{V}_f)$. We will show that

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathrm{y}, \boldsymbol{\varphi})) = \chi\big(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}\big) \cdot \log(p),$$

By the formal uniformization of the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ in Corollary 1.2.4, the Euler characteristic $\chi(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)})$ is a weighted linear combination of local arithmetic intersection numbers on $\mathcal{N}_0(N)$. Theorem 1.2.1 follows from the local arithmetic Siegel–Weil formula with level N at the place p and the classical local Siegel–Weil formula at other places.

1.4. ***Supplement.*** By the windows theory developed in [Zink 2001], if $\nu_p(N) \geq 1$, we prove that the special fiber $\mathcal{Z}(x_0)_p$ of $\mathcal{Z}(x_0)$ has the following explicit description (cf. Theorem 6.2.6, Corollary 6.2.7):

$$\mathcal{Z}(x_0)_p \simeq \mathrm{Spf}\,\mathbb{F}[\![t_1, t_2]\!] \Big/ \Big(\prod_{\substack{a+b=n \\ a,b\geq 0}} (t_1^{p^a} - t_2^{p^b}) \Big).$$

Based on the isomorphism $\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N)$, the special fiber $\mathcal{N}_0(N)_p$ of $\mathcal{N}_0(N)$ can be described by

$$\mathcal{N}_0(N)_p \simeq \mathrm{Spf}\,\mathbb{F}[\![t_1, t_2]\!] \Big/ \Big((t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{\substack{a+b=n \\ a,b\geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \Big).$$

Both these two isomorphisms are proved in [Katz and Mazur 1985, Theorems 13.4.6 and 13.4.7] by a totally different method.

2. Quadratic lattices and local densities

2.1. ***Notations on quadratic lattices.*** Let p be a prime number. Let F be a nonarchimedean local field of residue characteristic p, with ring of integers $\mathcal{O}_F$, residue field $\kappa = \mathbb{F}_q$ of size q, and uniformizer π. Let $\nu_\pi : F \to \mathbb{Z} \cup \{\infty\}$ be the valuation on F and $|\cdot| : F \to \mathbb{R}_{\geq 0}$ be the normalized absolute value on F. Let $\chi_F = \left(\frac{\cdot}{\pi}\right)_F : F^\times/(F^\times)^2 \to \{\pm 1, 0\}$ be the quadratic residue symbol.

A quadratic space (U, q_U) over F is a finite-dimensional vector space U over F equipped with a quadratic form $q_U : U \to F$ inducing a symmetric bilinear form given by

$$(\cdot\,,\cdot) : U \times U \to F, \quad (x, y) \mapsto q_U(x+y) - q_U(x) - q_U(y). \tag{4}$$

An isometry between two quadratic spaces (U, q_U) and $(U', q_{U'})$ is a linear isomorphism $\phi : U \to U'$ preserving quadratic forms, i.e., $q_{U'}(\phi(x)) = q_U(x)$ for any $x \in U$. In that case, we say U and U' are isometric.

A quadratic lattice (L, q_L) is a finite free $\mathcal{O}_F$-module equipped with a quadratic form $q_L : L \to F$. The quadratic form q_L also induces a symmetric bilinear form $L \times L \xrightarrow{(\cdot,\cdot)} F$ by a formula similar to (4). Let $L^\vee = \{x \in L \otimes_{\mathcal{O}_F} F : (x, L) \subset \mathcal{O}_F\}$. We say a quadratic lattice is integral if $q_L(x) \in \mathcal{O}_F$ for all $x \in L$, and is self-dual if it is integral and $L = L^\vee$.

Let's assume that $\dim_F U = n$ and the symmetric bilinear form $(\cdot\,,\cdot)$ is nondegenerate. Let $\{x_i\}_{i=1}^n$ be a basis of U, and $t_{ij} = \frac{1}{2}(x_i, x_j)$. We define the discriminant of the quadratic space U to be

$$\operatorname{disc}(U) = (-1)^{n(n-1)/2} \det((t_{ij})) \in F^\times/(F^\times)^2.$$

If $\{x_i\}_{i=1}^n$ is an orthogonal basis of U then $t_{ij} = 0$ if $i \neq j$ and $t_{ii} \neq 0$ by the nondegeneracy of $(\cdot\,,\cdot)$. The Hasse invariant of the quadratic space U is

$$\epsilon(U) = \prod_{i<j} (t_{ii}, t_{jj})_F,$$

For a quadratic lattice L, we use $\operatorname{disc}(L)$ and $\epsilon(L)$ to denote the corresponding invariants on the quadratic space $L_F = L \otimes_{\mathcal{O}_F} F$. Recall that when p is odd, quadratic spaces U over F are classified by the three invariants

$$\dim_F U, \quad \operatorname{disc}(U), \quad \epsilon(U),$$

i.e., two quadratic spaces U and U' are isometric if and only if the above three invariants for U and U' are the same.

For a quadratic space U, define $\chi_F(U) := \chi_F(\operatorname{disc}(U))$. For a quadratic lattice L, define $\chi(L) := \chi(L \otimes_{\mathcal{O}_F} F)$. When p is odd, the quadratic space U admits a self-dual sublattice if and only if $\epsilon(U) = +1$ and $\chi_F(U) \neq 0$. We use H_k^ε to denote the unique self-dual lattice of rank k and

$$\chi_F(H_k^\varepsilon) = \varepsilon.$$

When $p = 2$, let $H_{2n}^+ = (H_2^+)^{\oplus n}$ be a self-dual lattice of rank $2n$, where the quadratic form on $H_2^+ = \mathcal{O}_F^2$ is given by $(x, y) \in \mathcal{O}_F^2 \mapsto xy$.

Example 2.1.1. Let $N \in \mathcal{O}_F$. Let $\Delta_F(N)$ be the rank-3 quadratic lattice

$$\Delta_F(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathcal{O}_F \right\}$$

over $\mathcal{O}_F$, equipped with the quadratic form induced by $x \mapsto \det(x)$. Under the basis

$$e_1 = \begin{pmatrix} -N & \\ & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad e_3 = \begin{pmatrix} & \\ 1 & \end{pmatrix}$$

of $\Delta_F(N)$, the quadratic form can be represented by the symmetric matrix

$$T = \begin{pmatrix} -N & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Therefore, $\mathrm{disc}(\Delta_F(N)) = -\frac{1}{4}N \equiv -N$, $\epsilon(\Delta_F(N)) = (-N, -1)_F$. Moreover,

$$\Delta_F(N)^\vee = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a \in \frac{1}{2N}\mathcal{O}_F, b, c \in \mathcal{O}_F \right\}.$$

Therefore, $\Delta_F(N)^\vee / \Delta_F(N) \simeq \mathcal{O}_F/2N$.

Throughout this article, we mainly focus on the case that $F = \mathbb{Q}_p$. In this case, we simply use $\Delta_p(N)$ to denote the lattice $\Delta_{\mathbb{Q}_p}(N)$ (as we did in the introduction).

2.2. *Local densities of quadratic lattices.*

Definition 2.2.1. Let L, M be two quadratic $\mathcal{O}_F$-lattices. Let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, an $\mathcal{O}_F$-scheme such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{QHom}(L \otimes_{\mathcal{O}_F} R, M \otimes_{\mathcal{O}_F} R),$$

where QHom denotes the set of injective module homomorphisms which preserve the quadratic forms. The local density of integral representations is defined to be

$$\mathrm{Den}(M, L) = \lim_{d \to \infty} \frac{\#\,\mathrm{Rep}_{M,L}(\mathcal{O}_F/\pi^d)}{q^{d \cdot \dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.2.2. If L, M have rank n, m, respectively, and the generic fiber $(\mathrm{Rep}_{M,L})_F \neq \varnothing$, then $n \leq m$ and

$$\dim(\mathrm{Rep}_{M,L})_F = \dim \mathrm{O}_m - \dim \mathrm{O}_{m-n} = \binom{m}{2} - \binom{m-n}{2} = mn - \frac{n(n+1)}{2}.$$

Definition 2.2.3. Let L, M be two quadratic $\mathcal{O}_F$-lattices. Let $\mathrm{PRep}_{M,L}$ be the $\mathcal{O}_F$-scheme of primitive integral representations such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{PRep}_{M,L}(R) = \{\phi \in \mathrm{Rep}_{M,L}(R) : \phi \text{ is an isomorphism between } L_R \text{ and a direct summand of } M_R\},$$

where L_R (resp. M_R) is $L \otimes_{\mathcal{O}_F} R$ (resp. $M \otimes_{\mathcal{O}_F} R$). The primitive local density is defined to be

$$\mathrm{Pden}(M, L) = \lim_{d \to \infty} \frac{\#\,\mathrm{PRep}_{M,L}(\mathcal{O}_F/\pi^d)}{q^{d \cdot \dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.2.4. For any positive integer d, a homomorphism $\phi \in \mathrm{Rep}_{M,L}(\mathcal{O}_F/\pi^d)$ or $\mathrm{Rep}_{M,L}(\mathcal{O}_F)$ is primitive if and only if $\overline{\phi} := \phi \bmod \pi \in \mathrm{PRep}(\mathcal{O}_F/\pi)$, which is equivalent to

$$\dim_{\mathbb{F}_q}(\phi(L) + \pi \cdot M)/\pi \cdot M = \mathrm{rank}_{\mathcal{O}_F}(L).$$

Lemma 2.2.5. *Let H be a self-dual quadratic lattice. Let L be a quadratic $\mathcal{O}_F$-lattice and k any positive integer. Then we have the stratification*

$$\mathrm{Rep}_{H,L}(\mathcal{O}_F) = \bigsqcup_{L \subset L' \subset L'^\vee} \mathrm{PRep}_{H,L'}(\mathcal{O}_F).$$

Proof. This is proved in [Cho and Yamauchi 2020, (3.1)]. □

Definition 2.2.6. Let $n \geq 0$. For $\varepsilon \in \{\pm 1\}$, we define the normalizing factors to be

$$\mathrm{Nor}^{\varepsilon}(X, n) = \left(1 - \frac{1 + (-1)^{n+1}}{2} \cdot \varepsilon q^{-(n+1)/2} X\right) \prod_{1 \leq i < (n+1)/2} (1 - q^{-2i} X^2).$$

It is well known (see [Li and Zhang 2022, §3.4]) that for a quadratic lattice L of rank n, there exists a polynomial $\mathrm{Den}(X, L) \in \mathbb{Q}[X]$ such that

$$\mathrm{Den}(X, L)|_{X = q^{-m}} = \frac{\mathrm{Den}(H^+_{n+1+2k}, L)}{\mathrm{Nor}^+(X, n)}$$

for all integers $m \geq 0$. If the lattice L can't be isometrically embedded into the lattice H^+_{n+1}, define the derived local density of L to be

$$\partial\, \mathrm{Den}(L) = -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(X, L).$$

3. Incoherent Eisenstein series and the main theorem

3.1. ***Incoherent Eisenstein series.*** Let W be the standard symplectic space over $\mathbb{Q}$ of dimension 4. Let $P = MN \subset \mathrm{Sp}(\mathrm{W})$ be the standard Siegel parabolic subgroup, which takes the following form under the standard basis of W:

$$M(\mathbb{Q}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathrm{GL}_2(\mathbb{Q}) \right\},$$

$$N(\mathbb{Q}) = \left\{ n(b) = \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} : b \in \mathrm{Sym}_2(\mathbb{Q}) \right\}.$$

Let $\mathbb{A}$ be the adèle ring over $\mathbb{Q}$. Let $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$ be the metaplectic extension

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}(\mathrm{W}_{\mathbb{A}}) \to \mathrm{Sp}(\mathrm{W})(\mathbb{A}) \to 1$$

of $\mathrm{Sp}(\mathrm{W})(\mathbb{A})$, where $\mathbb{C}^1 = \{z \in \mathbb{C}^\times : |z| = 1\}$. There is an isomorphism $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}}) \xrightarrow{\sim} \mathrm{Sp}(\mathrm{W})(\mathbb{A}) \times \mathbb{C}^1$ with the multiplication on the latter given by the global Rao cycle. Therefore, we can write an element of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$ as (g, t), where $g \in \mathrm{Sp}(\mathrm{W})(\mathbb{A})$ and $t \in \mathbb{C}^1$.

Let $P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})$ be the standard Siegel parabolic subgroup of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$, where

$$M(\mathbb{A}) = \{(m(a), t) : a \in \mathrm{GL}_2(\mathbb{A}), t \in \mathbb{C}^1\},$$
$$N(\mathbb{A}) = \{n(b) : b \in \mathrm{Sym}_2(\mathbb{A})\}.$$

Recall the incoherent collection of rank-3 quadratic spaces $\mathbb{V} = \{\mathbb{V}_v\}$ over $\mathbb{A}$ we defined in (2),

$$\mathbb{V}_v = \Delta_v(N) \otimes \mathbb{Q}_v \quad \text{if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.}$$

Then we can verify immediately that $\prod_v \epsilon(\mathbb{V}_v) = -1$.

Let $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be the quadratic character given by $\chi(x) = \prod_{v \leq \infty} \chi_v(x_v) = \prod_{v \leq \infty} (x_v, -N)_v$ for all $x = (x_v) \in \mathbb{A}^\times$. Fix the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ such that $\psi_\infty(x) = e^{2\pi i x}$. We may view χ as a character on $M(\mathbb{A})$ by

$$\chi(m(a), t) = \chi(\det(a)) \cdot \gamma(\det(a), \psi)^{-1} \cdot t,$$

and extend it to $P(\mathbb{A})$ trivially on $N(\mathbb{A})$. Here $\gamma(\det(a), \psi)$ is the Weil index (see [Kudla 1997, p. 548]). We define the degenerate principal series to be the unnormalized smooth induction

$$I(s, \chi) := \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})} (\chi \cdot |\cdot|_{\mathbb{Q}}^{s+3/2}), \quad s \in \mathbb{C}.$$

For a standard section $\Phi(-, s) \in I(s, \chi)$ (i.e., its restriction to the standard maximal compact subgroup of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$ is independent of s), we define the associated Siegel Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{Sp}(\mathbb{Q})} \Phi(\gamma g, s),$$

which converges for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$.

Recall that $\mathscr{S}(\mathbb{V}^2)$ is the space of Schwartz functions on $\mathbb{V}^2$. The fixed choice of χ and ψ gives a Weil representation $\omega = \omega_{\chi,\psi}$ of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}}) \times \mathrm{O}(\mathbb{V})$ on $\mathscr{S}(\mathbb{V}^2)$. For $\tilde{\boldsymbol{\varphi}} \in \mathscr{S}(\mathbb{V}^2)$, define a function

$$\Phi_{\tilde{\boldsymbol{\varphi}}}(g) := \omega(g)\tilde{\boldsymbol{\varphi}}(0), \quad g \in \mathrm{Mp}(\mathrm{W}_{\mathbb{A}}).$$

Then $\Phi_{\tilde{\boldsymbol{\varphi}}}(g) \in I(0, \chi)$. Let $\Phi_{\tilde{\boldsymbol{\varphi}}}(-, s) \in I(s, \chi)$ be the associated standard section, known as the standard Siegel–Weil section associated to $\tilde{\boldsymbol{\varphi}}$. For $\tilde{\boldsymbol{\varphi}} \in \mathscr{S}(\mathbb{V}^2)$, we write $E(g, s, \tilde{\boldsymbol{\varphi}}) := E(g, s, \Phi_{\tilde{\boldsymbol{\varphi}}})$.

3.2. *Fourier coefficients and derivatives.* We have a Fourier expansion of the Siegel Eisenstein series defined above:

$$E(g, s, \Phi) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} E_T(g, s, \Phi),$$

where

$$E_T(g, s, \Phi) = \int_{\mathrm{Sym}_2(\mathbb{Q}) \backslash \mathrm{Sym}_2(\mathbb{A})} E(n(b)g, s, \Phi)\psi(-\mathrm{tr}(Tb))\, \mathrm{d}n(b).$$

The Haar measure $\mathrm{d}n(b)$ is normalized to be self-dual with respect to ψ. When T is nonsingular, for factorizable $\Phi = \otimes_v \Phi_v$, we have a factorization of the Fourier coefficient into a product

$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v),$$

where the product ranges over all places v of $\mathbb{Q}$ and the local Whittaker function $W_{T,v}(g_v, s, \Phi_v)$ is defined by

$$W_{T,v}(g_v, s, \Phi_v) = \int_{\mathrm{Sym}_2(\mathbb{Q}_v)} \Phi_v(w^{-1}n(b)g_v, s) \cdot \psi_v(-\mathrm{tr}(Tb))\,\mathrm{d}n(b), \quad w = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}. \tag{5}$$

It has analytic continuation to $s \in \mathbb{C}$. Thus we have a decomposition of the derivative of a nonsingular Fourier coefficient,

$$E'_T(g, s, \Phi) = \sum_v E'_{T,v}(g, s, \Phi),$$

where

$$E'_{T,v}(g, s, \Phi) = W'_{T,v}(g_v, s, \Phi_v) \cdot \prod_{v' \neq v} W_{T,v'}(g_{v'}, s, \Phi_{v'}). \tag{6}$$

3.3. ***Whittaker functions and local densities.*** Let v be a finite place of $\mathbb{Q}$. Define the local degenerate principal series to be the unnormalized smooth induction

$$I_v(s, \chi_v) := \mathrm{Ind}_{P(\mathbb{Q}_v)}^{\mathrm{Mp}(\mathrm{W}_v)}(\chi_v \cdot |\cdot|_v^{s+3/2}), \quad s \in \mathbb{C}.$$

The fixed choice of χ_v and ψ_v gives a local Weil representation $\omega_v = \omega_{\chi_v,\psi_v}$ of $\mathrm{Mp}(\mathrm{W}_v) \times \mathrm{O}(\mathbb{V}_v)$ on the Schwartz function space $\mathscr{S}(\mathbb{V}_v^2)$. We define the local Whittaker function associated to $\boldsymbol{\varphi}_v$ and $T \in \mathrm{Sym}_2(\mathbb{Q}_v)$ to be

$$W_{T,v}(g_v, s, \boldsymbol{\varphi}_v) := W_{T,v}(g_v, s, \Phi_{\boldsymbol{\varphi}_v}),$$

where $\Phi_{\boldsymbol{\varphi}_v}(g_v) := \omega_v(g_v)\boldsymbol{\varphi}_v(0) \in I_v(0, \chi_v)$ and $\Phi_{\boldsymbol{\varphi}_v}(-, s)$ is the associated standard section.

The relationship between Whittaker functions and local densities is encoded in the following proposition.

Proposition 3.3.1. *Suppose $v \neq \infty$. Let M be an integral $\mathbb{Z}_v$-quadratic lattice of rank* 3 *contained in $\mathbb{V}_v$. Let L be an integral quadratic $\mathbb{Z}_v$-lattice of rank* 2*. Suppose that the quadratic form of L is represented by a matrix $T \in \mathrm{Sym}_2(\mathbb{Q}_v)$ after a choice of $\mathbb{Z}_v$-basis of L. We have the identity*

$$W_{T,v}(1, k, 1_{M^2}) = |M^\vee/M|_v \cdot \gamma(\mathbb{V}_v)^2 \cdot |2|_v^{1/2} \cdot \mathrm{Den}(M \oplus H_{2k}^+, L), \tag{7}$$

where the constant $\gamma(\mathbb{V}_v) = \gamma(\det(\mathbb{V}_v), \psi_v)^{-1} \cdot \epsilon(\mathbb{V}_v) \cdot \gamma(\psi_v)^{-3}$, $\gamma(\det(\mathbb{V}_v), \psi_v)$ and $\gamma(\psi_v)$ are Weil indexes [Ranga Rao 1993, Appendix].

Proof. This is proved in [Kudla et al. 2006, Lemma 5.7.1]. ◻

3.4. ***Classical incoherent Eisenstein series.*** The hermitian symmetric domain for Sp(W) is the Siegel upper half space

$$\mathbb{H}_2 = \{\mathrm{z} = \mathrm{x} + i\mathrm{y} \mid \mathrm{x} \in \mathrm{Sym}_2(\mathbb{R}),\ \mathrm{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}\}.$$

Let $\mathrm{z} = \mathrm{x} + i\mathrm{y} \in \mathbb{H}_2$ with $\mathrm{x}, \mathrm{y} \in \mathrm{Sym}_2(\mathbb{R})$ and $\mathrm{y} = {}^t a \cdot a$ positive definite. Define the classical incoherent Eisenstein series to be

$$E(\mathrm{z}, s, \tilde{\varphi}) = \chi_\infty(m(a))^{-1} |\det(m(a))|^{-3/2} \cdot E(g_{\mathrm{z}}, s, \tilde{\varphi}), \quad g_{\mathrm{z}} = n(x)m(a) \in \mathrm{Mp}(\mathrm{W}_{\mathbb{A}}).$$

Notice that $E(\mathrm{z}, s, \tilde{\varphi})$ doesn't depend on the choice of χ. We write the central derivatives as

$$\partial\,\mathrm{Eis}(\mathrm{z}, \tilde{\varphi}) := E'(\mathrm{z}, 0, \tilde{\varphi}), \quad \partial\,\mathrm{Eis}_T(\mathrm{z}, \tilde{\varphi}) := E'_T(\mathrm{z}, 0, \tilde{\varphi}). \tag{8}$$

Then we have a Fourier expansion

$$\partial\,\mathrm{Eis}(\mathrm{z}, \tilde{\varphi}) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} \partial\,\mathrm{Eis}_T(\mathrm{z}, \tilde{\varphi}).$$

For the open compact subgroup $\Gamma_0(N)(\hat{\mathbb{Z}}) \subset \mathrm{GL}_2(\mathbb{A}_f)$, we choose

$$\tilde{\varphi} = \varphi \otimes \varphi_\infty \in \mathscr{S}(\mathbb{V}^2)$$

such that $\varphi \in \mathscr{S}(\mathbb{V}_f^2)$ is $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant and φ_∞ is the Gaussian function

$$\varphi_\infty(x) = e^{-\pi\,\mathrm{tr}\,T(x)}.$$

For our fixed choice of Gaussian φ_∞, we write

$$E(\mathrm{z}, s, \varphi) = E(\mathrm{z}, s, \varphi \otimes \varphi_\infty), \quad \partial\,\mathrm{Eis}(\mathrm{z}, \varphi) = \partial\,\mathrm{Eis}(\mathrm{z}, \varphi \otimes \varphi_\infty), \quad \partial\,\mathrm{Eis}_T(\mathrm{z}, \varphi) = \partial\,\mathrm{Eis}_T(\mathrm{z}, \varphi \otimes \varphi_\infty), \tag{9}$$

and so on for short.

4. The modular curve $\mathcal{X}_0(N)$ and special cycles

4.1. ***Cyclic group schemes.*** Let S be a scheme. Let G/S be a finite locally free group scheme over S. On every connected component of S, the rank of G is a constant, if the rank is a same number N for every connected component, we say that G has order N.

Let $\mathcal{O}_S$ be the structure sheaf of the scheme S. Let G/S be a finite locally free group scheme of order N. Then the structure sheaf $\mathcal{O}_G$ of G is finite locally free of rank N as an $\mathcal{O}_S$-module. Any element $f \in \mathcal{O}_G$ acts on itself by left multiplication. This defines an $\mathcal{O}_S$-linear endomorphism of $\mathcal{O}_G$, and the characteristic polynomial of this endomorphism

$$\det(T - f) = T^N - \mathrm{tr}(f)T^{N-1} + \cdots + (-1)^N \mathrm{N}(f)$$

is a monic polynomial in $\mathcal{O}_S[T]$ of degree N.

Definition 4.1.1. We say that a set of N not necessarily distinct points $\{P_i\}_{i=1}^N$ in $G(S)$ is a full set of sections of G/S if the following condition is fulfilled: for any element $f \in \mathcal{O}_G$, the equality

$$\det(T-f)=\prod_{i=1}^{N}(T-f(P_i))$$

of polynomials with coefficients in $\mathcal{O}_S$ holds.

Definition 4.1.2. We say a finite locally free group scheme G/S of rank N is cyclic over S if there exists a section $P \in G(S)$ such that $\{aP\}_{a=1}^N$ forms a full set of sections of G/S. We call P a generator of G over S. We say G/S is *cyclic* if G_T is cyclic over T after some fppf covering by some scheme $T \to S$.

Remark 4.1.3. The cyclicity of a group scheme is preserved under base change by the definition, i.e., if G/S is cyclic, then for any morphism $S' \to S$, the base change group scheme $G \times_S S'/S'$ is also cyclic.

Proposition 4.1.4. *Let S be a scheme, E/S an elliptic curve over S, and $G \subset E[N]$ a finite locally free group scheme of order N over S. Then there exists a closed subscheme $S^{\mathrm{cyc}} \subset S$ which is universal for the condition "G is cyclic", in the sense that for any morphism $T \to S$, the base change G_T/T is cyclic if and only if the morphism $T \to S$ factors through the closed subscheme S^{cyc}.*

Proof. This is proved in [Katz and Mazur 1985, Theorem 6.4.1]. □

Lemma 4.1.5. *Let W be a discrete valuation ring with residue characteristic p and uniformizer π. Let S be a reduced, noetherian, quasiseparated and flat scheme over W. Let G be a finite locally free group scheme of order p^n over S which is also embedded into an elliptic curve E/S. If, for every generic point ξ of S, G_ξ doesn't factor through the multiplication-by-p morphism of E_ξ, then G is a cyclic group scheme.*

Proof. Since S is quasiseparated, quasicompact and flat over W, then $S[\pi^{-1}]$ is dense in S since the scheme-theoretic image commutes with flat base change, therefore every generic point ξ lies in the open dense subscheme $S[\pi^{-1}]$. Let $\kappa(\xi)$ be the residue field of ξ; it has characteristic 0.

The group scheme G_ξ is of order p^n over the characteristic 0 field $\kappa(\xi)$. Hence $G_\xi \simeq \prod_{i=1}^k \mathbb{Z}/p^{a_i}\mathbb{Z}$, where $\sum_{i=1}^k a_i = n$. The fact that G_ξ doesn't factor through the multiplication-by-p morphism of E_ξ is equivalent to saying that $E[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2 \not\subset G$. Hence the only possibility is $k=1$ and $G_\xi \simeq \mathbb{Z}/p^n\mathbb{Z}$.

Let S^{cyc} be the closed subscheme described by Proposition 4.1.4. We know that every generic point is contained in the closed subscheme S^{cyc}, and hence $S^{\mathrm{cyc}} = S$ since S is reduced. □

Corollary 4.1.6. *Let W be a discrete valuation ring with residue characteristic p and uniformizer π. Let S be an integral noetherian scheme, quasiseparated and flat over W. Let G be a finite locally free group scheme of order p^n over S which is also embedded into an elliptic curve E/S. If the isogeny $\pi_G : E \to E/G$ doesn't factor through the multiplication-by-p morphism of E, then G is a cyclic group scheme.*

Proof. The isogeny $\pi_G : E \to E/G$ factors through the multiplication-by-p morphism of E if and only if $\ker([p]_E)$ is contained (as a Cartier divisor on E) in G. This is a closed condition on the base scheme S by [Katz and Mazur 1985, Lemma 1.3.4]. We use $\mathcal{I} \neq 0$ (since the morphism π_G doesn't factor through the multiplication-by-p morphism of E) to denote the ideal sheaf of this closed subscheme of S; it is functorial with respect to the base change of S.

Let ξ be the only generic point of S, then G_ξ doesn't factor through the multiplication-by-p morphism because otherwise $\mathcal{I}_\xi = 0$, but the injection $\mathcal{I} \to \mathcal{I}_\xi$ will imply that $\mathcal{I} = 0$, which is a contradiction. Then the corollary follows from Lemma 4.1.5. □

4.2. $\Gamma_0(N)$*-structures on elliptic curves.* Let S be a scheme. We say a scheme C over S is a smooth curve over S if the structure morphism $C \to S$ is a smooth proper morphism of relative dimension 1.

Definition 4.2.1. A closed immersion $i : D \to C$ is called an effective Cartier divisor if the following conditions hold:

(i) The closed subscheme D is flat over S.

(ii) The ideal sheaf $\mathcal{I}(D)$ defining D is an invertible $\mathcal{O}_C$-module.

Lemma 4.2.2. *If C/S is a smooth curve, then any section $s \in C(S)$ defines an effective Cartier divisor on C, denoted by $[s]$.*

Proof. This is proved in [Katz and Mazur 1985, Lemma 1.2.2]. □

Given two effective Cartier divisors D and D' on C/S, we can define their sum $D+D'$. It is an effective Cartier divisor on C/S defined locally by the product of the defining equations of D and D'. Explicitly, if $S = \operatorname{Spec} R$ and if over an affine open subscheme $\operatorname{Spec} A$ of C, the Cartier divisor D (resp. D') is defined by an element $f \in A$ (resp. $g \in A$), then the Cartier divisor $D + D'$ is defined by the equation fg.

Lemma 4.2.3. *Suppose E/S and E'/S are two elliptic curves over S and $\pi : E \to E'$ is an isogeny, i.e., π is surjective and $\ker(\pi)$ is a finite flat group scheme locally of finite presentation over S. Then $\ker(\pi) \to E$ is an effective Cartier divisor.*

Proof. By the cancellation theorem of morphisms of locally finite presentation, any morphism between abelian schemes are locally of finite presentation. Hence π is locally of finite presentation, and therefore $\ker(\pi)$ is also locally of finite presentation over S. Then the lemma follows from [Katz and Mazur 1985, Lemma 1.2.3]. □

Definition 4.2.4. We say an isogeny $\pi : E \to E'$ between two elliptic curves E and E' is a cyclic N-isogeny if $\pi^\vee \circ \pi = N$, and there exists an fppf covering of S by a scheme $T \to S$ with a point $P \in \ker(\pi)(T)$ such that the equality

$$\ker(\pi)_T = \sum_{a=1}^{N} [aP]$$

of Cartier divisors on E_T holds. A $\Gamma_0(N)$-structure on an elliptic curve E/S is a cyclic N-isogeny $E \xrightarrow{\pi} E'$.

Lemma 4.2.5. *Let $\pi : E \to E'$ be an isogeny between two elliptic curves E and E', the isogeny π is an N-cyclic isogeny if and only if $\ker(\pi)$ is a cyclic group scheme of order N.*

Proof. By [Katz and Mazur 1985, Theorem 1.10.1], the set $\{aP\}_{a=1}^N$ (where $P \in \ker(\pi)(S)$) forms a full set of sections of $\ker(\pi)$ if and only if we have the equality

$$\ker(\pi) = \sum_{a=1}^{N} [aP]$$

of effective Cartier divisors in E/S, which is exactly the definition of the cyclicity of a N-isogeny. □

Example 4.2.6. (a) Suppose $\tau = x + iy \in \mathbb{H}_1^+$. We consider the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and a finite subgroup K generated by $1/N$ inside E_τ. Then $\pi : E_\tau \to E_\tau/K$ is a cyclic isogeny.

(b) Suppose E/S is an elliptic curve over an $\mathbb{F}_p$-scheme S. Then for any $n \geq 1$, the n-th iterated relative Frobenius

$$F^n : E \to E^{(p^n)}$$

is a cyclic p^n-isogeny. The origin $P = 0$ is a generator of $\ker(F^n)$ because $\ker(F^n) \simeq \mathcal{O}_S[T]/(T^{p^n})$ Zariski locally (cf. [Katz and Mazur 1985, Lemma 12.2.1]).

Let $\mathcal{E}ll$ be the stack of elliptic curves, i.e., for an arbitrary scheme S, $\mathcal{E}ll(S)$ is a groupoid whose objects are elliptic curves $p : E \to S$ and morphisms are isomorphisms of elliptic curves over S. We use $\mathcal{Y}_0(N)$ to denote the stack which consists of all the $\Gamma_0(N)$-structures on elliptic curves, i.e., for a scheme S, $\mathcal{Y}_0(N)(S)$ is a groupoid whose objects are cyclic N-isogenies $(E \xrightarrow{\pi} E')$ where E and E' are elliptic curves over S, and a morphism between two cyclic isogenies $(E_1 \xrightarrow{\pi_1} E_1')$ and $(E_2 \xrightarrow{\pi_2} E_2')$ is a pair of isomorphisms of elliptic curves $a : E_1 \xrightarrow{\sim} E_2$ and $a' : E_1' \xrightarrow{\sim} E_2'$ such that $a' \circ \pi_1 = \pi_2 \circ a$. We have functors

$$s : \mathcal{Y}_0(N) \to \mathcal{E}ll, \quad (E/S \xrightarrow{\pi} E'/S) \mapsto E/S.$$

Lemma 4.2.7. *Both $\mathcal{Y}_0(N)$ and $\mathcal{E}ll$ are 2-dimensional Deligne–Mumford stacks. The above functor $s : \mathcal{Y}_0(N) \to \mathcal{E}ll$ is finite flat of degree $\psi(N) = N \cdot \prod_{l|N}(1 + l^{-1})$, and representable by schemes. Also, s is étale over* $\operatorname{Spec} \mathbb{Z}[1/N]$.

Proof. This is proved in [Katz and Mazur 1985, Theorem 5.1.1]. The key input is that a finite order group scheme is automatically étale if the order is invertible in the base scheme. □

For a $\mathbb{Z}_{(p)}$-scheme S, a geometric point $\bar{s}$ of S and an elliptic curve E over S, let $E_{\bar{s}}$ be the base change of E to $\bar{s}$. Let $T^p(E_{\bar{s}})$ (resp. $V^p(E_{\bar{s}})$) be the integral (resp. rational) Tate module of the elliptic curve $E_{\bar{s}}$. A $\mathbb{Z}_{(p)}^\times$-isogeny $f : E \to E'$ over S is a quasi-isogeny and there exists a prime-to-p number M such that $M \circ f$ is an isogeny. Let $V^p(f)$ be the homomorphism on rational Tate modules induced by f.

Lemma 4.2.8. *Let $\mathcal{E}ll_{(p)}$ be the localization of the stack $\mathcal{E}ll$ to* $\operatorname{Spec} \mathbb{Z}_{(p)}$. *Then $\mathcal{E}ll_{(p)}$ can be described by the following stack: for every $\mathbb{Z}_{(p)}$-scheme S, $\mathcal{E}ll_{(p)}(S)$ is a groupoid whose objects are pairs $(E/S, \overline{\eta^p})$, where $\overline{\eta^p}$ is a $\pi_1(S, \bar{s})$-invariant* $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$*-equivalence class of an isomorphism*

$$\eta^p : V^p(E_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2.$$

A morphism between two objects $(E/S, \overline{\eta^p})$ *and* $(E'/S, \overline{\eta'^p})$ *is a* $\mathbb{Z}_{(p)}^{\times}$*-isogeny* $f : E \to E'$ *over* S *such that* $\overline{\eta^p} = \overline{V^p(f) \circ \eta'^p}$.

Proof. We temporarily use $\mathcal{E}ll'$ to denote the stack described in the lemma. It suffices to show that for a connected scheme S over $\operatorname{Spec} \mathbb{Z}_{(p)}$, there is a category equivalence between $\mathcal{E}ll(S)$ and $\mathcal{E}ll'(S)$. We first construct a functor F from $\mathcal{E}ll(S)$ to $\mathcal{E}ll'(S)$. Given an elliptic curve E over S and a geometric point $\bar{s}$ of S, we choose an isomorphism

$$\eta^p : T^p(E_{\bar{s}}) \simeq (\hat{\mathbb{Z}}^p)^2.$$

Then clearly the $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$-orbit of $\overline{\eta^p}$ is $\pi_1(S, \bar{s})$-invariant (because $\pi_1(S, \bar{s})$ acts linearly on $T^p(E_{\bar{s}})$). We define $F(E) = (E, \overline{\eta^p})$; this functor is independent of the choice of η^p.

Now we prove that this functor is essentially surjective and fully faithful. For essential surjectivity, we pick an arbitrary object $(E/S, \overline{\eta^p})$ of $\mathcal{E}ll'(S)$. By [Lan 2013, Corollary 1.3.5.4], there is a $\mathbb{Z}_{(p)}^{\times}$-isogeny $f : E' \to E$ such that $\eta'^p = \eta^p \circ V^p(f) : V^p(E'_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$ maps $T^p(E'_{\bar{s}})$ to $(\hat{\mathbb{Z}}^p)^2$. Therefore the object $(E/S, \overline{\eta^p})$ is isomorphic to $(E'/S, \overline{\eta'^p})$, which is the essential image of $E' \in \operatorname{Ob} \mathcal{E}ll(S)$.

Next we show that there is an isomorphism

$$\operatorname{Hom}_{\mathcal{E}ll(S)}(E, E') \simeq \operatorname{Hom}_{\mathcal{E}ll'(S)}((E, \overline{\eta^p}), (E', \overline{\eta'^p})). \tag{10}$$

This is clearly injective by the above discussion. Now we pick an arbitrary element f from the right-hand side. Then f is a $\mathbb{Z}_{(p)}^{\times}$-isogeny, and $\eta'^p = \eta^p \circ V^p(f)$. There exists an integer M prime to p such that $\tilde{f} = M \circ f$ is an isogeny from E to E'. We claim that this isogeny factors through the multiplication-by-M map, i.e., f itself is an isogeny. By the relation $\eta'^p = \eta^p \circ V^p(f)$ and the construction above, $V^p(f)$ maps $T^p(E_{\bar{s}})$ isomorphically to $T^p(E'_{\bar{s}})$, then obviously $\tilde{f}$ maps $E'_{\bar{s}}[M] \simeq E'[M]_{\bar{s}}$ to 0. This holds for every geometric point $\bar{s}$ of S, so since S is a $\mathbb{Z}_{(p)}$-scheme and by the rigidity result [Mumford and Fogarty 1982, Proposition 6.1], we know the isogeny $\tilde{f}$ vanishes on $E'[M]$. Hence f itself is an isogeny. Now $\ker(f)$ is a finite flat group scheme over S of order prime to p, but since $V^p(f)$ maps $T^p(E_{\bar{s}})$ isomorphically to $T^p(E'_{\bar{s}})$, this group scheme must be trivial, i.e., f is an isomorphism, and therefore it comes from an element of the left-hand side of (10). □

Remark 4.2.9. We consider the Deligne–Mumford stack

$$\mathcal{H} = \mathcal{E}ll \times_{\mathbb{Z}} \mathcal{E}ll.$$

For any prime p, we use $\mathcal{H}_{(p)}$ to denote the localization of $\mathcal{H}$ to $\operatorname{Spec} \mathbb{Z}_{(p)}$.

There is a similar description of the stack $\mathcal{H}_{(p)}$: for any $\mathbb{Z}_{(p)}$-scheme S, the groupoid $\mathcal{H}_{(p)}(S)$ consists of pairs $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}))$, where $\overline{\eta^p}$ (resp. $\overline{\eta'^p}$) is a $\pi_1(S, \bar{s})$-invariant $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$-equivalence class of an isomorphism $V^p(E_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$ (resp. $V^p(E'_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$).

For any $N \in \mathbb{Z}_{>0}$, let w_N be the 2×2 matrix

$$w_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the following stack $\mathcal{Y}_0(N)'_{(p)}$ over $\operatorname{Spec}\mathbb{Z}_{(p)}$: for every $\mathbb{Z}_{(p)}$-scheme S, $\mathcal{Y}_0(N)'_{(p)}(S)$ is a groupoid whose objects are pairs

$$(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}),$$

where $E \xrightarrow{\pi} E'$ is a cyclic N-isogeny and $\overline{(\eta^p, \eta'^p)}$ is a pair of $\pi_1(S, \bar{s})$-invariant $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-equivalence classes (we specify the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ in (12)) of isomorphisms

$$\eta^p : V^p(E_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2, \quad \eta'^p : V^p(E'_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2,$$

which maps $T^p(E_{\bar{s}})$ and $T^p(E'_{\bar{s}})$ to $(\hat{\mathbb{Z}}^p)^2$, and the isomorphism η'^p is determined by the commutative diagram

$$\begin{CD} V^p(E_{\bar{s}}) @>{\eta^p}>> (\mathbb{A}_f^p)^2 \\ @V{V^p(\pi)}VV @VV{w_N}V \\ V^p(E'_{\bar{s}}) @>{\eta'^p}>> (\mathbb{A}_f^p)^2 \end{CD} \tag{11}$$

A morphism from $(E_1 \xrightarrow{\pi_1} E'_1, \overline{(\eta_1^p, \eta_1'^p)})$ to $(E_2 \xrightarrow{\pi_2} E'_2, \overline{(\eta_2^p, \eta_2'^p)})$ is a pair (f, f') of isomorphisms $f : E_1 \to E_2$ and $f' : E'_1 \to E'_2$ such that

$$f' \circ \pi_1 = \pi_2 \circ f \quad \text{and} \quad \overline{(\eta_1^p, \eta_1'^p)} = \overline{(\eta_2^p \circ V^p(f), \eta_2'^p \circ V^p(f'))}$$

as $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-orbits. The action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ on the pair (η^p, η'^p) is given by

$$g \cdot ((\eta^p, \eta'^p)) = (g \circ \eta^p, w_N g w_N^{-1} \circ \eta'^p). \tag{12}$$

Lemma 4.2.10. *Let $\mathcal{Y}_0(N)_{(p)}$ be the localization of $\mathcal{Y}_0(N)$ to $\mathbb{Z}_{(p)}$. Then there is an isomorphism $G : \mathcal{Y}_0(N)_{(p)} \to \mathcal{Y}_0(N)'_{(p)}$ of stacks over* $\operatorname{Spec}\mathbb{Z}_{(p)}$.

Proof. Let S be a scheme over $\operatorname{Spec}\mathbb{Z}_{(p)}$, and $(E \xrightarrow{\pi} E')$ an object in the groupoid $\mathcal{Y}_0(N)_{(p)}(S)$. For a geometric point $\bar{s}$ of S, the cyclicity of π implies that $\pi_{\bar{s}}$ is also cyclic. Since l is invertible in $\operatorname{Spec}\kappa(\bar{s})$ if $l \neq p$, there exist isomorphisms $\eta^p : T^p(E_{\bar{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$ and $\eta'^p : T^p(E'_{\bar{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$ such that $\omega_N \circ \eta^p = \eta'^p \circ T^p(\pi)$. Now we consider a different choice of (η^p, η'^p), say $(\tilde{\eta}^p, \tilde{\eta}'^p)$, satisfying the above conditions. Then $\tilde{\eta}'^p$ differs from η^p by an element $g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$, i.e., $\tilde{\eta}^p = g \circ \eta^p$, and correspondingly $\tilde{\eta}'^p = \omega_N g \omega_N^{-1} \circ \eta'^p$. However, $\omega_N g \omega_N^{-1} \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ since both η'^p and $\tilde{\eta}'^p$ give isomorphisms from $T^p(E_{\bar{s}})$ to $(\hat{\mathbb{Z}}^p)^2$; therefore, $g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \cap \omega_N^{-1}\mathrm{GL}_2(\hat{\mathbb{Z}}^p)\omega_N = \Gamma_0(N)(\hat{\mathbb{Z}}^p)$. Thus the $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-orbit $\overline{(\eta^p, \eta'^p)}$ is well-defined. We define $G((E \xrightarrow{\pi} E')) = ((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)})$. For a pair of isomorphisms (f, f'), where $f : E_1 \to E'_1$ and $f' : E_2 \to E'_2$, define $G((f, f')) = (f, f')$.

It suffices to show that for a connected scheme S over $\operatorname{Spec}\mathbb{Z}_{(p)}$, the functor

$$G(S) : \mathcal{Y}_0(N)_{(p)}(S) \to \mathcal{Y}_0(N)_{(p)}(S)$$

is an equivalence of categories. This functor is essentially surjective by definition; now we show that it is fully faithful, i.e., the following morphism between sets is bijective:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{Y}_0(N)_{(p)}(S)}&\big((E_1 \xrightarrow{\pi_1} E_1'), (E_2 \xrightarrow{\pi_2} E_2')\big)\\ &\xrightarrow{G} \mathrm{Hom}_{\mathcal{Y}_0(N)_{(p)}(S)}\big((E_1 \xrightarrow{\pi_1} E_1', \overline{(\eta_1^p, \eta_1'^p)}), (E_2 \xrightarrow{\pi_2} E_2', \overline{(\eta_2^p, \eta_2'^p)})\big),\\ &(f, f') \mapsto (f, f'),\end{aligned}$$

but this is clearly bijective by the definition. □

There is a natural morphism from $\mathcal{Y}_0(N)_{(p)}$ to $\mathcal{H}_{(p)}$, i.e., $(E \xrightarrow{\pi} E') \to (E, E')$. By Remark 4.2.9 and Lemma 4.2.10, we can also describe it as

$$\mathcal{Y}_0(N)_{(p)} \to \mathcal{H}_{(p)}, \quad (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}) \mapsto ((E, E'), (\overline{\eta^p}, \overline{\eta'^p})). \tag{13}$$

4.2.A. *Compactification of $\mathcal{Y}_0(N)$.* Next we introduce the compactification of the moduli stack $\mathcal{Y}_0(N)$. Let S be a scheme. We first introduce the notion of Néron n-gons.

Definition 4.2.11. For any integer $n \geq 1$ and a scheme S, the Néron n-gon over S is the coequalizer of

$$\bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} S \rightrightarrows \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} \mathbb{P}^1_S,$$

where the top (resp. the bottom) closed immersion includes the i-th copy of S as the 0 (resp. ∞) section of the i-th (resp. $(i+1)$-st) copy of $\mathbb{P}^1_S$.

Definition 4.2.12. A generalized elliptic curve over a scheme S consists of the following data:

- A proper, flat, finitely presented morphism $E \to S$ each of whose geometric fibers is either a smooth connected curve of genus 1 or a Néron n-gon for some $n \geq 1$.
- An S-morphism $E^{\mathrm{sm}} \times_S E \xrightarrow{+} E$ that restricts to a commutative S-group scheme structure on E^{sm} for which $+$ becomes an S-group action such that via the pullback of line bundles the action $+$ induces the trivial action of E^{sm} on $\mathrm{Pic}^0_{E/S}$.

We use $\mathcal{X}$ to denote the moduli stack consisting of generalized elliptic curves whose degenerate fibers are Néron 1-gons, i.e., for a scheme S, $\mathcal{X}(S)$ is a groupoid whose objects are generalized elliptic curves E over S and whose geometric fibers are either elliptic curves or Néron 1-gons. The following result is proved in [Česnavičius 2017].

Lemma 4.2.13. *$\mathcal{X}$ is a proper smooth 2-dimensional Deligne–Mumford stack.*

Proof. This is proved in [Česnavičius 2017, Theorem 3.1.6]. □

We have a natural morphism of Deligne–Mumford stacks $\mathcal{E}ll \to \mathcal{X}$, which sends an elliptic curve E over S to itself. This morphism is an open immersion, i.e., the stack $\mathcal{E}ll$ is an open substack of $\mathcal{X}$. Recall that we have a finite flat representable morphism $\mathcal{Y}_0(N) \to \mathcal{E}ll$ by Lemma 4.2.7. Let $\mathcal{X}_0(N)$ be the normalization of $\mathcal{Y}_0(N)$ with respect to this morphism. A moduli description of $\mathcal{X}_0(N)$ in terms of level structures on the generalized elliptic curves can be found in [Česnavičius 2017, §5.9]. The stack $\mathcal{Y}_0(N)$ can be realized as an open substack of the stack $\mathcal{X}_0(N)$ based on this description. We also have the following theorem:

Theorem 4.2.14. *$\mathcal{X}_0(N)$ is a regular proper 2-dimensional Deligne–Mumford stack. It is finite flat over $\mathcal{X}$.*

Proof. This is proved in [Česnavičius 2017, Theorem 5.13]. □

4.3. *Special cycles on $\mathcal{H}$ and $\mathcal{X}_0(N)$.* Let p be a prime number, we first define the special cycles on the stack $\mathcal{H}_{(p)}$.

Definition 4.3.1. For every symmetric $n\times n$ matrix $T = (T_{ik})$, let $\tilde{\boldsymbol{\varphi}}^p$ be the characteristic function of an open compact subset $\tilde{\boldsymbol{\omega}}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)\times\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. We consider the stack $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)$, whose fibered category over a $\mathbb{Z}_{(p)}$-scheme S consists of the objects

$$((E, E'), (\overline{\eta^p}, \overline{\eta'^p}), \boldsymbol{j}),$$

where $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}))$ is an object in $\mathcal{H}_{(p)}(S)$, $\boldsymbol{j} = (j_1, j_2, \dots, j_n) \in (\mathrm{Hom}(E, E')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})^n$ and $\eta^*(\boldsymbol{j}) := \eta'^p\circ V^p(\boldsymbol{j})\circ(\eta^p)^{-1}\in\tilde{\boldsymbol{\omega}}^p$. Moreover,

$$T_{ik} = \tfrac{1}{2}(\deg(j_i + j_k) - \deg(j_i) - \deg(j_k)) = \tfrac{1}{2}(j_i\circ j_k^\vee + j_k\circ j_i^\vee).$$

The special cycle $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)$ may be empty.

For every symmetric $n\times n$ matrix T, we have a natural finite unramified morphism $i_n^\sharp : \mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)\to\mathcal{H}_{(p)}$ by forgetting the morphisms $\boldsymbol{j}$ of an object $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}), \boldsymbol{j})$ of $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)$. We recall the following definition of generalized Cartier divisor from [Howard and Madapusi 2022, Definition 2.4.1].

Definition 4.3.2. Suppose $D\to X$ is any finite, unramified and relatively representable morphism of Deligne–Mumford stacks. Then there is an étale cover $U\to X$ by a scheme such that the pullback $D_U\to U$ is a finite disjoint union

$$D_U = \bigsqcup_i D_U^i$$

with each map $D_U^i\to U$ a closed immersion. If each of these closed immersions is an effective Cartier divisor on U in the usual sense (the corresponding ideal sheaves are invertible), then we call $D\to X$ a generalized Cartier divisor.

Proposition 4.3.3. *Let $\tilde{\varphi}^p$ be the characteristic function of an open compact subset $\tilde{\omega}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)\times\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. For any positive number $d\in\mathbb{Q}$, the finite unramified morphism $i_1^\sharp : \mathcal{Z}^\sharp(d, \tilde{\varphi}^p)\to\mathcal{H}_{(p)}$ is a generalized Cartier divisor.*

Proof. This is proved in [Howard and Madapusi 2020, Proposition 6.5.2] (see also [Howard and Madapusi 2022, Proposition 2.4.3]). □

Now let's turn to the special cycles on the stack $\mathcal{X}_0(N)_{(p)}$ and $\mathcal{Y}_0(N)_{(p)}$. We first introduce the notion of special morphisms for the moduli stack $\mathcal{Y}_0(N)_{(p)}$.

Definition 4.3.4. Let S be a scheme over $\operatorname{Spec}\mathbb{Z}_{(p)}$. For an object $((E\xrightarrow{\pi}E'), \overline{(\eta^p, \eta'^p)})$ in $\mathcal{Y}_0(N)_{(p)}(S)$, a special morphism of this object is an element $j\in\mathrm{Hom}(E, E')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ satisfying

$$j\circ\pi^\vee + \pi\circ j^\vee = 0.$$

We denote this space by $S(E, \pi)$.

Definition 4.3.5. For every symmetric $n \times n$ matrix $T = (T_{ik})$, let $\boldsymbol{\varphi}^p$ be the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. We consider the stack $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$, whose fibered category over a $\mathbb{Z}_{(p)}$-scheme S consists of the objects

$$((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)}, \boldsymbol{j}),$$

where $((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)})$ is an object in $\mathcal{Y}_0(N)_{(p)}(S)$, $\boldsymbol{j} = (j_1, j_2, \ldots, j_n) \in S(E, \pi)^n$ and $\eta^*(\boldsymbol{j}) := \eta'^p \circ V^p(\boldsymbol{j}) \circ (\eta^p)^{-1} \in \boldsymbol{\omega}^p$. Moreover,

$$T_{ik} = \tfrac{1}{2}(\deg(j_i + j_k) - \deg(j_i) - \deg(j_k)) = \tfrac{1}{2}(j_i \circ j_k^\vee + j_k \circ j_i^\vee).$$

The special cycle $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$ may be empty.

For every symmetric $n \times n$ matrix T, we have a natural morphism $i_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ by forgetting the special morphisms.

Remark 4.3.6. Let $T \in \mathrm{Sym}_n(\mathbb{Q})$. Let $\tilde{\boldsymbol{\varphi}}^p$ be the characteristic function of an open compact subset $\tilde{\boldsymbol{\omega}}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) \times \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. Let $\boldsymbol{\varphi}^p$ be the restriction of $\tilde{\boldsymbol{\varphi}}^p$ to the subspace $(\mathbb{V}_f^p)^n$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$. Then $\boldsymbol{\varphi}^p$ is the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$, and the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$ is a union of some connected components of the fiber product $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p) \times_{\mathcal{H}_{(p)}} \mathcal{Y}_0(N)_{(p)}$. Therefore the morphism $i_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ is also finite unramified. In particular, for $n = 1$ and $T = d \in \mathbb{Q}_{>0}$, the morphism $i_1 : \mathcal{Z}(d, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ is a generalized Cartier divisor by Proposition 4.3.3.

We show next that the composite $\tilde{i}_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \xrightarrow{i_n} \mathcal{Y}_0(N)_{(p)} \to \mathcal{X}_0(N)_{(p)}$ is also finite unramified. We start with the case that $n = 1$.

Proposition 4.3.7. *Let φ^p be the characteristic function of an open compact subset ω^p of $\mathbb{V}_f^p$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. For any positive number $d \in \mathbb{Q}$, the morphism $\tilde{i}_1 : \mathcal{Z}(d, \varphi^p) \to \mathcal{X}_0(N)_{(p)}$ is finite unramified, and $\mathcal{Z}(d, \varphi^p)$ is a generalized Cartier divisor.*

Proof. The morphism $\tilde{i}_1$ is unramified since i_1 is unramified and the open immersion $\mathcal{Y}_0(N)_{(p)} \to \mathcal{X}_0(N)_{(p)}$ is also unramified. Therefore we only need to show the finiteness of $\tilde{i}_1$.

We first prove that the stack $\mathcal{Z}(d, \varphi^p)$ is flat over $\mathbb{Z}_{(p)}$. Since the morphism $\mathcal{Z}(d, \varphi^p)\mathcal{Y}_0(N)_{(p)}$ is a generalized Cartier divisor by Remark 4.3.6, the flatness of $\mathcal{Z}(d, \varphi^p)$ is equivalent to the fact that its local equation is not divisible by p since the stack $\mathcal{Y}_0(N)_{(p)}$ is flat over $\mathbb{Z}_{(p)}$. We assume the converse and suppose that there exists a point $z \in \mathcal{Z}(d, \varphi^p)(\overline{\mathbb{F}}_p)$ such that the equation of $\mathcal{Z}(d, \varphi^p)$ in the étale local ring $\mathcal{O}^{\text{ét}}_{\mathcal{Y}_0(N),z}$ is divisible by p. Then the stack $\mathcal{Z}(d, \varphi^p)$ contains an irreducible component of $\mathcal{Y}_0(N)_{\mathbb{F}_p}$ in an étale neighborhood of z. Let $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)})$ be the object corresponding to the generic point of this irreducible component. Then $\mathrm{End}(E) \simeq \mathbb{Z}$ since the j-invariant of E must be transcendental over $\mathbb{F}_p$ (by the description of the stack $\mathcal{Y}_0(N)_{\mathbb{F}_p}$ in [Katz and Mazur 1985, Proposition 13.4.5 and Theorem 13.4.7]). There also exists an isogeny $j \in \mathrm{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $j^\vee \circ \pi + \pi^\vee \circ j = 0$. Let $\alpha = j^{-1} \circ \pi \in \mathrm{End}^\circ(E) := \mathrm{End}(E) \otimes \mathbb{Q} \simeq \mathbb{Q}$. Then $\alpha^2 = -Nd^{-1} < 0$, contradicting the fact that $\mathrm{End}^\circ(E) \simeq \mathbb{Q}$.

Therefore, the stack $\mathcal{Z}(d,\varphi^p)$ is flat over $\mathbb{Z}_{(p)}$, and hence equals the flat closure of its generic fiber $\mathcal{Z}(d,\varphi^p)_{\mathbb{Q}} := \mathcal{Z}(d,\varphi^p)\times_{\mathbb{Z}_{(p)}}\mathbb{Q}$. The stack $\mathcal{Z}(d,\varphi^p)_{\mathbb{Q}}$ consists of finitely many points whose residue fields are finite extensions of $\mathbb{Q}$. Therefore the structure sheaf $\mathcal{O}_{\mathcal{Z}(d,\varphi^p)}$ of $\mathcal{Z}(d,\varphi^p)$ is a finite product of subrings of the integer rings of these residue fields. Hence the stack $\mathcal{Z}(d,\varphi^p)$ is finite over $\mathbb{Z}_{(p)}$, so $\tilde{i}_1:\mathcal{Z}(d,\varphi^p)\to\mathcal{X}_0(N)_{(p)}$ is proper since $\mathcal{X}_0(N)_{(p)}$ is proper over $\mathbb{Z}_{(p)}$. The morphism $\tilde{i}_1$ is obviously quasifinite by the finiteness of $\mathcal{Z}(d,\varphi^p)$ over $\mathbb{Z}_{(p)}$, and hence $\tilde{i}_1$ is finite.

We already know that the morphism $\tilde{i}_1$ is a generalized Cartier divisor over the open substack $\mathcal{Y}_0(N)_{(p)}$ of $\mathcal{X}_0(N)_{(p)}$. Moreover, étale locally around a cusp point of $\mathcal{X}_0(N)_{(p)}$, the stack $\mathcal{Z}(d,\varphi^p)$ is cut out by 1 since $\tilde{i}_1$ factors through the noncuspidal locus $\mathcal{Y}_0(N)_{(p)}$. Thus the finite unramified morphism $\tilde{i}_1:\mathcal{Z}(d,\varphi^p)\to\mathcal{X}_0(N)_{(p)}$ is a generalized Cartier divisor on the stack $\mathcal{X}_0(N)_{(p)}$. □

Corollary 4.3.8. *Let $\boldsymbol{\varphi}^p=\prod_{i=1}^n\varphi_i^p$ be the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. For any matrix $T\in\mathrm{Sym}_n(\mathbb{Q})_{>0}$, the morphism $\tilde{i}_n:\mathcal{Z}(T,\boldsymbol{\varphi}^p)\to\mathcal{X}_0(N)_{(p)}$ is finite unramified.*

Proof. Suppose the diagonal elements of T are $d_1,\dots,d_n$. Proposition 4.3.7 implies that the morphism $\mathcal{Z}(d_1,\varphi_1^p)\times_{\mathcal{X}_0(N)_{(p)}}\cdots\times_{\mathcal{X}_0(N)_{(p)}}\mathcal{Z}(d_n,\varphi_n^p)\to\mathcal{X}_0(N)_{(p)}$ is finite unramified. The stack $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is a connected component of $\mathcal{Z}(d_1,\varphi_1^p)\times_{\mathcal{X}_0(N)_{(p)}}\cdots\times_{\mathcal{X}_0(N)_{(p)}}\mathcal{Z}(d_n,\varphi_n^p)$, so the morphism $\tilde{i}_n:\mathcal{Z}(T,\boldsymbol{\varphi}^p)\to\mathcal{X}_0(N)_{(p)}$ is finite unramified. □

We mainly focus on the case that T is a nonsingular 2×2 symmetric matrix with coefficients in $\mathbb{Q}$. For every such matrix T, recall that we have defined the difference set to be

$$\mathrm{Diff}(T,\Delta(N))=\{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N)\otimes\mathbb{Q}_l\}.$$

Proposition 4.3.9. *Let $T\in\mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. If $\mathcal{Z}(T,\boldsymbol{\varphi}^p)(\overline{\mathbb{F}}_p)\neq\varnothing$ for some prime p, then T is positive definite, and*

$$\mathrm{Diff}(T,\Delta(N))=\{p\}.$$

Moreover, in this case, the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is supported in the supersingular locus of the special fiber $\mathcal{Y}_0(N)_{\mathbb{F}_p}$.

Proof. Since $\mathcal{Z}(T,\boldsymbol{\varphi}^p)(\overline{\mathbb{F}}_p)\neq\varnothing$, Corollary 4.3.8 implies that there are two elliptic curves $\mathbb{E}$ and $\mathbb{E}'$ over $\overline{\mathbb{F}}_p$, a cyclic isogeny $\pi\in\mathrm{Hom}(\mathbb{E},\mathbb{E}')$ and two isogenies $x_1,x_2\in\mathrm{Hom}(\mathbb{E},\mathbb{E}')_{(p)}$ such that

$$T=\left(\tfrac{1}{2}(x_i,x_j)\right)\quad\text{and}\quad(x_1,\pi)=(x_2,\pi)=0.$$

Therefore, T must be positive definite and both $\mathbb{E}$ and $\mathbb{E}'$ are supersingular elliptic curves over $\overline{\mathbb{F}}_p$ since $\dim_{\mathbb{Q}}\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}\geq3$. The quadratic space $\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}_p$ is isometric to the underlying quadratic space of the unique division quaternion algebra $\mathbb{B}$ over $\mathbb{Q}_p$.

The isogenies x_1,x_2 lie in $\{\pi\}^\perp\subset\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}_p\simeq\mathbb{B}$, where $\pi^\vee\circ\pi=N$. However, $\{\pi\}^\perp$ and $\Delta(N)\otimes\mathbb{Q}_p$ have the same discriminant $-N$ but opposite Hasse invariants. Therefore $p\in\mathrm{Diff}(T,\Delta(N))$.

At the same time, by choosing some level structures on $\mathbb{E}$ and $\mathbb{E}'$ away from p, we get that T can be realized in $\Delta(N)\otimes\mathbb{Q}_l$ for any finite prime $l\neq p$. Therefore p is the only prime in the set $\mathrm{Diff}(T,\Delta(N))$. □

Remark 4.3.10. Proposition 4.3.9 implies that the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is also finite unramified over the stack $\mathcal{X}_0(N)_{(p)}$ because the scheme-theoretic image $\widetilde{\mathcal{Z}}(T,\boldsymbol{\varphi}^p)$ of $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ in $\mathcal{X}_0(N)_{(p)}$ is supported in the supersingular locus of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$, which equals the supersingular locus of the special fiber $\mathcal{Y}_0(N)_{\mathbb{F}_p}$. Hence $\widetilde{\mathcal{Z}}(T,\boldsymbol{\varphi}^p)$ is contained in $\mathcal{Y}_0(N)_{(p)}$, and therefore equals the scheme-theoretic image of $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ in $\mathcal{Y}_0(N)_{(p)}$, over which $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is finite unramified.

For any nonsingular 2×2 symmetric matrix $T\in\mathrm{Sym}_2(\mathbb{Q})$, a Schwartz function $\boldsymbol{\varphi}=\bigotimes_{v<\infty}\boldsymbol{\varphi}_v\in\mathscr{S}(\mathbb{V}_f^2)$ is called T-admissible if $\boldsymbol{\varphi}$ is invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}})$, $\boldsymbol{\varphi}=\varphi_1\times\varphi_2$ for $\varphi_i\in\mathscr{S}(\mathbb{V}_f)$ and

- T is not positive definite, or
- T is positive definite and $|\mathrm{Diff}(T,\Delta(N))|\neq 1$, or
- T is positive definite, $\mathrm{Diff}(T,\Delta(N))=\{p\}$ for some prime number p, and $\boldsymbol{\varphi}=\boldsymbol{\varphi}^p\otimes\boldsymbol{\varphi}_p$, where $\boldsymbol{\varphi}^p\in\mathscr{S}((\mathbb{V}_f^p)^2)$ and $\boldsymbol{\varphi}_p=c\cdot\mathbf{1}_{\Delta_p(N)^2}$ for some $c\in\mathbb{C}$.

Definition 4.3.11. For a nonsingular 2×2 matrix $T\in\mathrm{Sym}_2(\mathbb{Q})$ and a T-admissible Schwartz function $\boldsymbol{\varphi}\in\mathscr{S}(\mathbb{V}_f^2)$ which is also a characteristic function of a $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant open compact subset $\boldsymbol{\omega}$ of $\mathbb{V}_f^2$, we define a stack finite unramified over $\mathcal{X}_0(N)$ as

$$\mathcal{Z}(T,\boldsymbol{\varphi}):=\mathcal{Z}(T,\boldsymbol{\varphi}^p)\to\mathcal{X}_0(N)_{(p)}\hookrightarrow\mathcal{X}_0(N),$$

where $p\in\mathrm{Diff}(T,\Delta(N))$. If $|\mathrm{Diff}(T,\Delta(N))|\neq 1$, we define $\mathcal{Z}(T,\boldsymbol{\varphi})=\varnothing$.

Remark 4.3.12. By Proposition 4.3.9, $\mathcal{Z}(T,\boldsymbol{\varphi})$ is nonempty only if $|\mathrm{Diff}(T,\Delta(N))|=1$, so the above definition makes sense.

Remark 4.3.13. If we view $\mathcal{Z}(T,\boldsymbol{\varphi})$ as an element in $\mathrm{CH}^2_{\mathbb{C}}(\mathcal{X}_0(N))$, we can drop the restrictions in Definition 4.3.11 that the Schwartz function $\boldsymbol{\varphi}$ is the characteristic function of an open compact subset of $\mathbb{V}_f^2$. Since any T-admissible Schwartz function $\boldsymbol{\varphi}$ on $\mathbb{V}_f^2$ is a finite linear combination of $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant characteristic functions of some open compact subsets, we can define $\mathcal{Z}(T,\boldsymbol{\varphi})$ as the corresponding linear combination of elements in $\mathrm{CH}^2_{\mathbb{C}}(\mathcal{X}_0(N))$.

4.3.A. *Comparison with* [Sankaran et al. 2023, §2.2]. Another kind of special cycle of $\mathcal{X}_0(N)$ is defined in [Sankaran et al. 2023, §2.2] as follows,

Definition 4.3.14. For $m\in\mathbb{Z}$, let $\mathcal{Z}(m)$ denote the moduli stack whose S points, for a base scheme S, are given by

$$\mathcal{Z}(m)(S):=\{(E\xrightarrow{\pi}E',\alpha)\},$$

where $(E\xrightarrow{\pi}E')\in\mathcal{Y}_0(N)(S)$ and $\alpha\in\mathrm{End}(E)$ satisfies the following conditions:

(a) $\alpha^\vee\circ\alpha=mN$ and $\alpha^\vee+\alpha=0$.

(b) $\alpha\circ\pi^{-1}\in\mathrm{Hom}(E',E)$.

(c) $\pi\circ\alpha\circ\pi^{-1}\in\mathrm{End}(E')$.

Lemma 4.3.15. *For every prime number p, let $\mathcal{Z}(m)_{(p)} := \mathcal{Z}(m) \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then we have an isomorphism of stacks*

$$T : \mathcal{Z}(m)_{(p)} \xrightarrow{\sim} \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p}), \quad (E \xrightarrow{\pi} E', \alpha) \mapsto (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, (\alpha \circ \pi^{-1})^\vee).$$

Proof. We first prove that T is well-defined. For any connected $\mathbb{Z}_{(p)}$-scheme S, let $\bar{s}$ be a geometric point of S. We can choose trivializations $\eta^p : V^p(E_{\bar{s}}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ and $\eta'^p : V^p(E'_{\bar{s}}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ such that $T^p(E_{\bar{s}})$ and $T^p(E'_{\bar{s}})$ are mapped isomorphically to $(\hat{\mathbb{Z}}^p)^2$, and $\eta'^p \circ V^p(\pi) \circ (\eta^p)^{-1} = w_N$ by the cyclicity of π. Moreover,

$$\begin{aligned}(\alpha \circ \pi^{-1})^\vee \circ \pi + \pi^\vee \circ (\alpha \circ \pi^{-1}) &= \frac{1}{N}\pi^\vee \circ \alpha^\vee \circ \pi + \frac{1}{N}\pi^\vee \circ \alpha \circ \pi \\ &= \frac{1}{N}\pi^\vee \circ (\alpha^\vee + \alpha) \circ \pi = 0.\end{aligned}$$

Hence $(\alpha \circ \pi^{-1})^\vee \in S(E, \pi)$, so (b) implies that $\eta'^p \circ V^p((\alpha \circ \pi^{-1})^\vee) \circ (\eta^p)^{-1} \in \Delta(N) \otimes \hat{\mathbb{Z}}^p \subset \mathbb{V}_f^p$. Therefore $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, (\alpha \circ \pi^{-1})^\vee) \in \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p})(S)$.

We define the morphism

$$R : \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p}) \to \mathcal{Z}(m)_{(p)}, \quad (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, j) \mapsto (E \xrightarrow{\pi} E', j^\vee \circ \pi).$$

We show that R is well-defined. For any connected $\mathbb{Z}_{(p)}$-scheme S, an object $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, j)$ being in $\mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p})(S)$ means that $j \in \mathrm{Hom}(E, E') \otimes \mathbb{Z}_{(p)}$ and $j^\vee \circ \pi + \pi^\vee \circ j = 0$, and the fact that $\eta'^p \circ V^p(j) \circ (\eta^p)^{-1}$ is in $\Delta(N) \otimes \hat{\mathbb{Z}}^p$ implies that $j \in \mathrm{Hom}(E, E')$. Then $j^\vee \circ \pi \in \mathrm{End}(E)$, $(j^\vee \circ \pi)^\vee \circ (j^\vee \circ \pi) = \pi^\vee \circ j \circ j^\vee \circ \pi = mN$ and $(j^\vee \circ \pi)^\vee + j^\vee \circ \pi = \pi^\vee \circ j + j^\vee \circ \pi = 0$, which is exactly (a). Moreover, (b) and (c) are easily verified. Hence $(E \xrightarrow{\pi} E', j^\vee \circ \pi) \in \mathcal{Z}(m)(S)$, so the morphism R is well-defined. It's easy to see that T and R are inverse to each other, and therefore the lemma is proved. □

4.3.B. *Arithmetic special cycles on $\mathcal{X}_0(N)$.* We apply the arithmetic intersection theory developed in [Gillet 1984; 2009] to the regular proper flat Deligne–Mumford stack $\mathcal{X}_0(N)$. We obtain the arithmetic Chow ring of $\mathcal{X}_0(N)$,

$$\widehat{\mathrm{CH}}^\bullet_{\mathbb{C}}(\mathcal{X}_0(N)) = \bigoplus_{n=0}^{2} \widehat{\mathrm{CH}}^n_{\mathbb{C}}(\mathcal{X}_0(N)).$$

Roughly speaking, a class in $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(\mathcal{X}_0(N))$ is represented by an arithmetic cycle $(\mathcal{Z}, g_{\mathcal{Z}})$, where $\mathcal{Z}$ is a closed substack of $\mathcal{X}_0(N)$ of codimension n with $\mathbb{C}$-coefficients, and $g_{\mathcal{Z}}$ is a Green current for $\mathcal{Z}(\mathbb{C})$, i.e., $g_{\mathcal{Z}}$ is a current on the proper smooth complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$ of degree $(n-1, n-1)$ for which there exists a smooth ω such that

$$dd^c(g) + \delta_\zeta = [\omega].$$

Here $[\omega]$ is the current defined by integration against the smooth form ω. The rational arithmetic cycles are those of the form $\widehat{\mathrm{div}}(f) = \big(\mathrm{div}(f), \iota_*[-\log(|\tilde{f}|^2)]\big)$, where $f \in \kappa(\mathcal{Z})^\times$ is a rational function on a codimension-$(n-1)$ integral substack $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}_0(N)$, together with classes of the form $(0, \partial\eta + \bar{\partial}\eta')$. By definition, the arithmetic Chow group $\mathrm{CH}^n_{\mathbb{C}}(\mathcal{X}_0(N))$ is the quotient of the space of arithmetic cycles by the $\mathbb{C}$-subspace spanned by those rational cycles.

Let $\mathcal{Z}$ be an irreducible codimension-2 cycle on $\mathcal{X}_0(N)$. Then $\mathcal{Z}$ is a Deligne–Mumford stack over $\mathbb{F}_p$ for some prime number p and the groupoid $\mathcal{Z}(\overline{\mathbb{F}}_p)$ is a singleton with a finite automorphism group $\mathrm{Aut}(\mathcal{Z})$. The rational function field $\kappa(\mathcal{Z})$ of $\mathcal{Z}$ is a finite extension of $\mathbb{F}_p$. Clearly $\delta_{\mathcal{Z}} = 0$ because $\mathcal{Z}(\mathbb{C}) = \varnothing$.

Let $(\mathcal{Z}, g) = \left(\sum_i n_i[\mathcal{Z}_i], g\right)$ be an arithmetic cycle of codimension 2, where each $\mathcal{Z}_i$ is an irreducible codimension-2 cycle on $\mathcal{X}_0(N)$. We define the degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)) \to \mathbb{C}, \quad [(\mathcal{Z}, g)] \mapsto \sum_i n_i \cdot \frac{\log|\kappa(\mathcal{Z}_i)|}{|\mathrm{Aut}(\mathcal{Z}_i)|} + \frac{1}{2} \int_{\mathcal{X}_0(N)(\mathbb{C})} g. \tag{14}$$

Here the integration $\int_{\mathcal{X}_0(N)_{\mathbb{C}}} g$ is the integration of the constant function 1 on $\mathcal{X}_0(N)_{\mathbb{C}}$ against the $(1, 1)$-current g. It is a finite number since the stack $\mathcal{X}_0(N)$ is proper. This number is independent of the choice of representing element $(\mathcal{Z}, g)$ as a consequence of the product formula [Kudla et al. 2006, §2.1].

Now we are going to construct Green currents for the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$. Let

$$\mathbb{D} = \{z \in \Delta(N) \otimes_{\mathbb{Z}} \mathbb{C} : (z, z) = 0,\ (z, \bar{z}) < 0\} \,/\, \mathbb{C}^* \subset \mathbb{P}(\Delta(N) \otimes \mathbb{C}).$$

We have the $\mathrm{GL}_2(\mathbb{R})$-equivariant identification

$$\mathbb{H}_1^{\pm} \xrightarrow{\sim} \mathbb{D}, \quad \tau \mapsto \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} -N\tau & -N\tau^2 \\ 1 & \tau \end{pmatrix}\right\}.$$

Next, we associate to any nonsingular $T \in \mathrm{Sym}_2(\mathbb{Q})$ and T-admissible Schwartz function $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ an element in $\widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$. Let $\mathrm{y} = {}^t a \cdot a \in \mathrm{Sym}_2(\mathbb{R})$ be a positive definite matrix, where $a \in \mathrm{GL}_2(\mathbb{R})$.

- For a positive definite T and T-admissible Schwartz function $\boldsymbol{\varphi}$, we consider the element

$$\hat{\mathcal{Z}}(T, \mathrm{y}, \boldsymbol{\varphi}) = (\mathcal{Z}(T, \boldsymbol{\varphi}), 0) \in \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)).$$

- For another nonsingular T which is not positive definite, we apply the general machine developed in [Garcia and Sankaran 2019], which is made explicit in [Sankaran et al. 2023]. For any $x \in \mathbb{V}_\infty$ and $[z] \in \mathbb{D}$, let $\mathrm{R}(x, [z]) = -|(x, z)|^2 \cdot (z, \bar{z})^{-1}$. We define an element in $\mathscr{S}(\mathbb{V}_\infty^2) \otimes \mathcal{A}^{1,1}(\mathbb{H}_1^{\pm})$ by letting, for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{V}_\infty^2$ and $[z] \in \mathbb{D}$,

$$\nu(\boldsymbol{x}, [z]) = \left(-\pi^{-1} + 2\sum_{i=1}^{2}(\mathrm{R}(x_i, [z]) + (x_i, x_i))\right) \exp\left(-2\pi \sum_{i=1}^{2}\left(\mathrm{R}(x_i, [z]) + \frac{1}{2}(x_i, x_i)\right)\right) \cdot \frac{dx \wedge dy}{y^2}.$$

Then we define a smooth $(1, 1)$-form $\mathfrak{g}(T, \mathrm{y}, \boldsymbol{\varphi})$ on $\mathbb{D}$ by letting its value at the point $[z] \in \mathbb{D}$ be

$$\mathfrak{g}(T, \mathrm{y}, \boldsymbol{\varphi})([z]) = \sum_{\substack{\boldsymbol{x} \in (\Delta(N) \otimes \mathbb{Q})^2 \\ T(\boldsymbol{x}) = T}} \boldsymbol{\varphi}(\boldsymbol{x}) \cdot \int_1^\infty \nu(t^{1/2}\boldsymbol{x} \cdot {}^t a, [z]) \cdot \frac{\mathrm{d}t}{t}.$$

The sum converges absolutely, and descends to a smooth $(1, 1)$-form on the modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$.

Lemma 4.3.16. *For nonsingular $T \in \mathrm{Sym}_2(\mathbb{R})$ which is not positive definite, the form $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ is absolutely integrable on $\mathcal{X}_0(N)_{\mathbb{C}}$. Hence $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ defines a $(1, 1)$-current on $\mathcal{X}_0(N)_{\mathbb{C}}$.*

Proof. This is proved in [Sankaran et al. 2023, Lemma 2.9]. □

To sum up, let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ a T-admissible Schwartz function. We define

$$\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}) = \begin{cases} ([\mathcal{Z}(T, \boldsymbol{\varphi})], 0) & \text{when } T \text{ is positive definite,} \\ (0, \mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})) & \text{when } T \text{ is not positive definite.} \end{cases} \tag{15}$$

It is an element in $\widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$.

4.4. *Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$.* Now we can state the main theorem of this article, which proves an identity between arithmetic intersection numbers on $\mathcal{X}_0(N)$ and derivatives of Fourier coefficients of Eisenstein series,

Theorem 4.4.1. *Let N be a positive integer, $T \in \mathrm{Sym}_2(\mathbb{Q})$ a nonsingular symmetric matrix, and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ a T-admissible Schwartz function. Then*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))q^T = \frac{\psi(N)}{24} \cdot \partial\, \mathrm{Eis}_T(\mathsf{z}, \boldsymbol{\varphi})$$

for any $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$. Here $\psi(N) = N \cdot \prod_{l \mid N}(1 + l^{-1})$, $q^T = e^{2\pi i\, \mathrm{tr}(T\mathsf{z})}$.

The article [Sankaran et al. 2023] proves this formula in the case that T is not positive definite without any restrictions on the level N, and the case that T is positive definite but with the restriction that N is odd and square-free. We give a proof of the case that T is positive definite and N is arbitrary in Section 8.3.

5. Rapoport–Zink spaces and special cycles

5.1. *$\Gamma_0(N)$-structures on p-divisible groups.* For a prime p, let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$, W the completion of the maximal unramified extension of $\mathbb{Q}_p$ and Nilp_W the category of schemes S over $\mathrm{Spec}\, W$ such that p is locally nilpotent on S. Let $\bar{S}$ be the closed subscheme of S defined by the ideal sheaf $p\mathcal{O}_S$. For a p-divisible group X, we use $X^\vee$ to denote the dual p-divisible group. We introduce two Rapoport–Zink spaces in this chapter. They are essentially isomorphic to the completed local rings of supersingular points in characteristic p of the moduli stacks $\mathcal{H}$ and $\mathcal{X}_0(N)$.

Let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ of dimension 1 and height 2. The associated filtered isocrystal $\mathbb{D}(\mathbb{X})_{\mathbb{Q}}$ has pure slope $\frac{1}{2}$, e.g., we can take $\mathbb{X}$ to be $\mathbb{E}[p^\infty]$, where $\mathbb{E}$ is a supersingular elliptic curve over $\mathbb{F}$. Let $\lambda_0 : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ be a principal polarization. We consider the following functor $\mathcal{N}$ on the category Nilp_W: for any $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}(S)$ is the isomorphism classes of tuples $((X, \rho, \lambda), (X', \rho', \lambda'))$, where

(1) X and X' are two p-divisible groups over S, and ρ, ρ' are two quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X \times_S \bar{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X' \times_S \bar{S}$;

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations such that Zariski locally on $\bar{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0$$

for some $c(\rho), c(\rho') \in \mathbb{Z}_p^\times$.

Proposition 5.1.1. *The functor $\mathcal{N}$ is represented by the formal scheme* $\operatorname{Spf} W[\![t_1, t_2]\!]$ *over* $\operatorname{Spf} W$.

Proof. When p is odd, this is explained in [Li and Zhang 2022, Example 4.5.3(ii)]. In general, the deformation space of the supersingular elliptic curve $\mathbb{E}$ is isomorphic to $\operatorname{Spf} W[\![t]\!]$. By the Serre–Tate theorem, this is also the deformation space of the p-divisible group $\mathbb{X}$ with certain restrictions on the polarization, as in the definition of the deformation functor $\mathcal{N}$. Therefore,

$$\mathcal{N} \simeq \operatorname{Spf} W[\![t_1]\!] \times_{\operatorname{Spf} W} \operatorname{Spf} W[\![t_2]\!] \simeq \operatorname{Spf} W[\![t_1, t_2]\!]. \qquad \square$$

Let $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ be the universal p-divisible group over $\mathcal{N} = \operatorname{Spf} W[\![t_1, t_2]\!]$. By Lemma 6.1.3 below, the category of p-divisible groups over $\operatorname{Spf} W[\![t_1, t_2]\!]$ is equivalent to the category of p-divisible groups over $\operatorname{Spec} W[\![t_1, t_2]\!]$. We still use $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ to denote the corresponding p-divisible group over $\operatorname{Spec} W[\![t_1, t_2]\!]$.

Next we fix an N-isogeny $x_0 : \mathbb{X} \to \mathbb{X}$, i.e., $x_0 \circ x_0^\vee = N$. $\mathcal{N}_0(N)$ is a contravariant set-valued functor defined over Nilp_W. For every $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of the isomorphism classes of elements of the form $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda'))$, where

(1) X and X' are two p-divisible groups over S, and ρ, ρ' are two quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X \times_S \bar{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X' \times_S \bar{S}$;

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations, such that Zariski locally on $\bar{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0$$

for some $c(\rho), c(\rho') \in \mathbb{Z}_p^\times$;

(3) $x : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over S) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

We will prove later that the functor $\mathcal{N}_0(N)$ is represented by a closed formal subscheme of $\operatorname{Spf} W[\![t_1, t_2]\!]$ cut out by a single equation (see Theorem 6.2.3).

5.2. *Special cycles on $\mathcal{N}$ and $\mathcal{N}_0(N)$.* Now we give the definition of special cycles on the formal schemes $\mathcal{N}$ and $\mathcal{N}_0(N)$. Recall that $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ is the universal p-divisible group over $\mathcal{N}$, and $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$ is the unique division quaternion algebra over $\mathbb{Q}_p$, whose Hasse invariant as a quadratic space is -1.

Definition 5.2.1. For any subset $L \subset \mathbb{B}$, define the special cycle $\mathcal{Z}^\sharp(L)$ to be the closed formal subscheme of $\mathcal{N}$ where the groupoid $\mathcal{Z}^\sharp(L)(S)$, for an object $S \in \mathrm{Nilp}_W$, consists of pairs $((X, \rho, \lambda), (X', \rho', \lambda')) \in \mathcal{N}(S)$ such that the quasi-isogeny $\rho' \circ x \circ \rho^{-1}$ is an isogeny from X to X'.

Remark 5.2.2. The special cycle $\mathcal{Z}^\sharp(L)$ only depends on the $\mathbb{Z}_p$-linear span of L in $\mathbb{B}$, and is nonempty only when this span is an integral quadratic $\mathbb{Z}_p$-lattice in $\mathbb{B}$.

Proposition 5.2.3. *Let $x \in \mathbb{B}$ be a nonzero and integral element, i.e., $0 \le \nu_p(x^\vee \circ x) < \infty$. Then $\mathcal{Z}^\sharp(x)$ is a Cartier divisor on $\mathcal{N}$, i.e., it is defined by a single nonzero element $f_x \in W[[t_1, t_2]]$. Moreover, $\mathcal{Z}^\sharp(x)$ is also flat over* $\operatorname{Spf} W$, *i.e., $p \nmid f_x$.*

Proof. When p is odd, the formal scheme $\mathcal{N}$ is an example of GSpin Rapoport–Zink space [Li and Zhang 2022, Example 4.5.3(ii)], and the proposition has been proved for every GSpin Rapoport–Zink space in [Li and Zhang 2022, Proposition 4.10.1]. For all p (especially $p = 2$), this is proved in [Katz and Mazur 1985, Theorem 6.8.1]. $\square$

Now let's come to the special cycles on $\mathcal{N}_0(N)$. Firstly, we give the definition of the space of special quasi-isogenies. Recall that we have fixed an N-isogeny x_0 when we define the formal scheme $\mathcal{N}_0(N)$.

Definition 5.2.4. We call a quasi-isogeny $x \in \mathbb{B} = \operatorname{End}^0(\mathbb{X})$ special to x_0 if

$$x \circ x_0^\vee + x_0 \circ x^\vee = 0.$$

By definition, the space of quasi-isogenies special to x_0 is just the quadratic space $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$. By Witt's theorem, it is a 3-dimensional quadratic space over $\mathbb{Q}_p$ whose isometric class is independent of the choice of the N-isogeny x_0.

Definition 5.2.5. Let $(\breve{X} \xrightarrow{\breve{x}} \breve{X}', (\breve{\rho}, \breve{\rho}'), (\breve{\lambda}, \breve{\lambda}'))$ be the universal object over $\mathcal{N}_0(N)$. For any subset $M \subset \mathbb{W}$, define the special cycle $\mathcal{Z}(M) \subset \mathcal{N}_0(N)$ to be the closed formal subscheme cut out by the conditions

$$\breve{\rho}' \circ x \circ \breve{\rho}^{-1} \in \operatorname{Hom}(\breve{X}, \breve{X}') \quad \text{for any } x \in M.$$

For any subset $M \subset \mathbb{W} \subset \mathbb{B}$, we have the Cartesian diagram

$$\begin{tikzcd} \mathcal{Z}(M) \arrow[r] \arrow[d] & \mathcal{N}_0(N) \arrow[d] \\ \mathcal{Z}^\sharp(M) \arrow[r] & \mathcal{N} \end{tikzcd}$$

5.3. *Formal uniformization of $\mathcal{X}_0(N)$ and the special cycle $\mathcal{Z}(T, \varphi)$.* Let B be the unique quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞. Then $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{B}$ is the unique division quaternion algebra over $\mathbb{Q}_p$. Let $\mathbb{E}$ be a supersingular elliptic curve over $\mathbb{F}$ and $\mathbb{X} = \mathbb{E}[p^\infty]$ the p-divisible group of $\mathbb{E}$. Then $B \simeq \operatorname{End}^0(\mathbb{E})$ and $\mathbb{B} \simeq \operatorname{End}^0(\mathbb{X})$. Suppose $x_0 \in \mathbb{B}$ comes from a cyclic N-isogeny of $\mathbb{E}$ under the above isomorphism $\operatorname{End}^0(\mathbb{E}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{B}$.

We first state and explain the formal uniformization theorem of the supersingular locus $\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}}$ of $\mathcal{H}_{\mathbb{F}_p}$. We use $\hat{\mathcal{H}}/_{(\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}})}$ to denote the completion of $\mathcal{H}$ along the closed substack $\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}}$.

Theorem 5.3.1. *There is an isomorphism of formal stacks over W*

$$\hat{\mathcal{H}}/_{(\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}})} \xrightarrow[\sim]{\Theta_{\mathcal{H}}} B^\times(\mathbb{Q})_0^2 \backslash [\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2 / \mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2], \tag{16}$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

Theorem 5.3.1 is proved in [Rapoport and Zink 1996, Theorem 6.24]. Here we only describe the isomorphism, especially the group action on the right-hand side of (16). Let $\eta_0^p : V^p(\mathbb{E}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ be a prime-to-p level structure of $\mathbb{E}$. Let $\widetilde{\mathbb{E}}$ be a deformation of $\mathbb{E}$ to W, and let $\widetilde{\mathbb{X}} := \widetilde{\mathbb{E}}[p^\infty]$ be the corresponding deformation of $\mathbb{X}$ to W. For some object $S \in \mathrm{Nilp}_W$, we pick an object

$$((X, \rho, \lambda), (X', \rho', \lambda'), (g, g')) \in \mathcal{N}(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)^2.$$

The quasi-isogeny ρ (resp. ρ') gives rise to a quasi-isogeny $\tilde{\rho} : \widetilde{\mathbb{X}}_S \to X$ (resp. $\tilde{\rho}' : \widetilde{\mathbb{X}}_S \to X'$). Then there exists an elliptic curve E (resp. E') up to prime-to-p isogeny over S and a quasi-isogeny $\rho_E : \widetilde{\mathbb{E}}_S \to E$ (resp. $\rho_{E'} : \widetilde{\mathbb{E}}_S \to E'$) such that $E[p^\infty] \simeq X$ (resp. $E'[p^\infty] \simeq X'$) and ρ_E (resp. $\rho_{E'}$) induces $\tilde{\rho}$ (resp. $\tilde{\rho}'$) under this isomorphism. The object $((X, \rho, \lambda), (X', \rho', \lambda'), (g, g'))$ is mapped to

$$\big((E, E'), \big(\overline{g^{-1}\eta_0^p \circ V^p(\rho_E^{-1})}, \overline{g'^{-1}\eta_0^p \circ V^p(\rho_{E'}^{-1})}\big)\big) \in \mathcal{H}(S).$$

The group action is given, for a pair of elements $(b, b') \in B^\times(\mathbb{Q})_0 \times B^\times(\mathbb{Q})_0$, by the map

$$B(\mathbb{Q}) \to B(\mathbb{Q}_p) \simeq \mathrm{End}^0(\mathbb{X}) \xrightarrow{\rho^*} \mathrm{End}^0(X) \quad (\text{resp. } \xrightarrow{\rho'^*} \mathrm{End}^0(X')),$$

and a fixed isomorphism $B(\mathbb{A}_f^p) \simeq \mathrm{GL}_2(\mathbb{A}_f^p)$. We obtain another triple

$$(b, b')_*\big(((X, \rho, \lambda), (X', \rho', \lambda'), (g, g'))\big) := \big((X, \rho \circ b^{-1}, \lambda), (X', \rho' \circ b'^{-1}, \lambda'), (bg, b'g')\big).$$

Now let $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$) be the supersingular locus of $\mathcal{X}_0(N)_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)_{\mathbb{F}_p}$). Let $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$ (resp. $\hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$) be the completion of $\mathcal{X}_0(N)$ (resp. $\mathcal{Y}_0(N)$) along the closed substack $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$). By the definition of $\mathcal{X}_0(N)$, we have $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p} = \mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ and therefore $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \simeq \hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Proposition 5.3.2. *There is an isomorphism of formal stacks over W,*

$$\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{X}_0(N)}} B^\times(\mathbb{Q})_0 \backslash [\mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)], \tag{17}$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

Proof. The following diagram is Cartesian, with all arrows closed immersions:

$$\begin{tikzcd} \mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p} \arrow[r] \arrow[d] & \mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p} \arrow[d] \\ \mathcal{Y}_0(N) \arrow[r] & \mathcal{H} \end{tikzcd}$$

this diagram gives a closed immersion $i : \hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \simeq \hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \to \hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Recall that we have the isomorphism

$$\hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{H}}} B^\times(\mathbb{Q})_0^2 \backslash [\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2/\mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2].$$

Let S be an object in Nilp_W, and let $(z, (g, g')) \in \mathcal{N}(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)^2$ be a point in the closed formal substack $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Then clearly $z \in \mathcal{N}_0(N)(S)$. Suppose z corresponds to a cyclic isogeny $E \xrightarrow{\pi} E'$ by our description of the isomorphism $\Theta_{\mathcal{H}}$. Then g' is determined by g by the diagram

$$\begin{tikzcd} V^p(E_{\bar{s}}) \arrow[r, "g^{-1}\eta_0^p\circ V^p(\rho_E^{-1})"] \arrow[d, "V^p(\pi)"] & (\mathbb{A}_f^p)^2 \arrow[d, "w_N"] \\ V^p(E'_{\bar{s}}) \arrow[r, "g'^{-1}\eta_0^p\circ V^p(\rho_{E'}^{-1})"] & (\mathbb{A}_f^p)^2 \end{tikzcd} \tag{18}$$

Thus we only focus on the pair $(z, g) \in \mathcal{N}_0(N)(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)$. Consider the morphism

$$\Theta : \mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p) \to \hat{\mathcal{H}}/_{(\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}})}, \quad (z, g) \mapsto \Theta_{\mathcal{H}}^{-1}(z, (g, g')).$$

The image of Θ lies in the closed formal substack $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})}$.

Let $(z_1, g_1), (z_2, g_2) \in \mathcal{N}_0(N)(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)$ be two points. Then $\Theta(z_1, g_1) = \Theta(z_2, g_2)$ if and only if there exist $b, b' \in B^\times(\mathbb{Q})_0$ and $k_1, k_1' \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ such that $(z_2, (g_2, g_2')) = ((b, b')_* z_1, (bg_1k_1, b'g_1'k_1'))$. We still use $E \xrightarrow{\pi} E'$ to denote the corresponding point of z_2 under $\Theta_{\mathcal{H}}$. Notice that $(z_2, (g_2, g_2')) = (z_2, (bg_1, b'g_1'))$ in the quotient stack $[\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2 / \mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2]$. Therefore

$$\Theta_{\mathcal{H}}(z_2, (g_2, g_2')) = \Theta_{\mathcal{H}}(z_2, (bg_1, b'g_1')) \in \hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})}(S),$$

and hence both $(g_2 = bg_1k_1, g_2' = b'g_1'k_1')$ and $(bg_1, b'g_1')$ satisfy the commutative diagram (18). Then

$$k_1' = w_N k_1 w_N^{-1}.$$

Since both k_1 and k_1' belongs to $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) := \prod_{v \neq \infty, p} \mathrm{GL}_2(\mathbb{Z}_v)$, there exist $a, b, c, d \in \hat{\mathbb{Z}}^p$ such that

$$k_1 = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)(\hat{\mathbb{Z}}^p).$$

Moreover, the element b' is also determined by b by the diagram (18). Therefore $\Theta(z_1, g_1) = \Theta(z_2, g_2)$ if and only if there exists $b \in B^\times(\mathbb{Q})_0$ and $k \in \Gamma_0(N)(\hat{\mathbb{Z}}^p)$ such that $(z_2, g_2) = (b_* z_1, bg_1k)$. □

Let $\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})$ be the completion of $\mathcal{Z}(T, \boldsymbol{\varphi})$ along its supersingular locus

$$\mathcal{Z}^{\mathrm{ss}}(T, \boldsymbol{\varphi}) := \mathcal{Z}(T, \boldsymbol{\varphi}) \times_{\mathcal{X}_0(N)} \mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}}.$$

Let $\Delta(N)^{(p)}$ be the unique quadratic space over $\mathbb{Q}$ (up to isometry) such that

(1) it is positive definite at ∞;

(2) for finite prime $l \neq p$, $\Delta(N)^{(p)} \otimes \mathbb{Q}_l$ is isometric to $\Delta_l(N) \otimes \mathbb{Q}_l$;

(3) $\Delta(N)^{(p)} \otimes \mathbb{Q}_p$ is isometric to $\mathbb{W}$.

As a corollary of the formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$ (see Proposition 5.3.2), we have the following formal uniformization of the special cycles on $\mathcal{X}_0(N)$.

Corollary 5.3.3. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular symmetric matrix, and $\mathrm{Diff}(T, \Delta(N)) = \{p\}$. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Let $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_{\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}}$-modules. Then in $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$ we have the identity*

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \Theta^{-1}_{\mathcal{X}_0(N)}(\mathcal{Z}(\boldsymbol{x}), g),$$

where $B_{\boldsymbol{x}}^\times \subset B^\times$ is the stabilizer of $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$.

Proof. We only need to prove the corollary when $\boldsymbol{\varphi}$ is the characteristic function of some open compact subset $\boldsymbol{\omega}$ of $\mathbb{V}_f^2$. Let S be an object in Nilp_W. Suppose $\Theta^{-1}_{\mathcal{X}_0(N)}(z, g) \in \hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})(S)$ for some $z \in \mathcal{N}_0(N)(S)$. Then z gives rise to a cyclic isogeny $E \xrightarrow{\pi} E'$, along with two isogenies $x_1, x_2 \in \mathrm{Hom}(E, E')_{(p)}$ such that

$$T = \left(\tfrac{1}{2}(x_i, x_j)\right) \quad \text{and} \quad (x_1, \pi) = (x_2, \pi) = 0.$$

Then x_1, x_2 and π induce endomorphisms of the corresponding p-divisible groups, and hence endomorphisms of $\mathbb{X}$. We still use x_1, x_2 to denote the endomorphisms of $\mathbb{X}$. Let $T(\boldsymbol{x}) := \left(\tfrac{1}{2}(x_i, x_j)\right)$ be the inner product matrix of $\boldsymbol{x} = (x_1, x_2)$). We have

$$T = T(\boldsymbol{x}) \quad \text{and} \quad (x_1, x_0) = (x_2, x_0) = 0,$$

i.e., $x_1, x_2 \in \{x_0\}^\perp = \mathbb{W} \simeq \Delta(N)^{(p)} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We can also identify x_1 and x_2 as elements in $\Delta(N)^{(p)} \otimes \mathbb{A}_f^p$ via the level structures $\eta_0^p \circ V^p(\rho_E^{-1})$ and $\eta_0^p \circ V^p(\rho_{E'}^{-1})$ of E and E'. The positivity assumption on T makes it embeddable into $\Delta(N)^{(p)} \otimes_{\mathbb{Q}} \mathbb{R}$. By carefully choosing the isometry $\mathbb{W} \simeq \Delta(N)^{(p)} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, we can find $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$ which induces x_1 and x_2 locally at every place of $\mathbb{Q}$.

Then the condition $\Theta^{-1}_{\mathcal{X}_0(N)}(z, g) \in \hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})(S)$ implies that

$$z \in \mathcal{Z}(\boldsymbol{x}) \quad \text{and} \quad g^{-1}\boldsymbol{x} \in \boldsymbol{\omega} \quad (\text{here } g \in \mathrm{GL}_2(\mathbb{A}_f) \text{ with } g_p = 1),$$

and this is exactly the meaning of the identity in the theorem. □

6. Difference formula at the geometric side

6.1. *p-divisible groups over adic noetherian rings.*

Definition 6.1.1. A topological ring R is an adic noetherian ring if it is noetherian as a ring and it has a topological basis consisting of all translations of the neighborhoods of zero of the form I^n $(n > 0)$, where $I \subset R$ is a fixed ideal of R, and R is Hausdorff and complete in that topology. A choice of such an ideal is said to be the defining ideal of the topological ring R.

Lemma 6.1.2. *Let A be an adic noetherian local ring whose defining ideal is the maximal ideal $\mathfrak{m}$. Then any ideal $I \subset A$ is complete in the topological ring A, i.e.,*

$$I = \bigcap_n (I + \mathfrak{m}^n).$$

Moreover, A/I is an adic noetherian ring with defining ideal $\mathfrak{m}/I$.

Proof. Nakayama's lemma implies that $\bigcap_n \mathfrak{m}^n I = 0$. Then we can apply [Stacks, Lemma 031B] to conclude that I is $\mathfrak{m}$-adically complete, i.e., $I \simeq \hat{I} := \varprojlim_n I/\mathfrak{m}^n I$.

We have the exact sequence

$$0 \to I \to A \to A/I \to 0.$$

Since A is noetherian, after taking completion with respect to the maximal ideal $\mathfrak{m}$, we get

$$0 \to \hat{I} \to \hat{A} \to \widehat{A/I} \to 0.$$

However, $A = \hat{A}$ and $I = \hat{I}$, and hence $\widehat{A/I} \simeq A/I$. We conclude A/I is an adic noetherian ring.

By definition, $\widehat{A/I} = \varprojlim_n A/(I + \mathfrak{m}^n)$. Hence $\widehat{A/I} \simeq A/I$ implies that $I = \bigcap_n (I + \mathfrak{m}^n)$. □

Lemma 6.1.3. *Let A be an adic noetherian ring whose defining ideal is I. Then the functor*

$$\{\text{category of } p\text{-divisible groups over } \operatorname{Spec} A\} \to \{\text{category of } p\text{-divisible groups over } \operatorname{Spf}(A)\},$$
$$G = (G_n/A) \mapsto (G_k = (G_k(n) = G(n) \times_A A/I^k))_{k \geq 1}.$$

is an equivalence.

Proof. This is proved in [de Jong 1995, Lemma 2.4.4]. □

6.2. ***Difference divisors on $\mathcal{N}$.*** Recall that for every nonzero integral $x \in \mathbb{B}$, we define the special divisor $\mathcal{Z}^\sharp(x)$ on $\mathcal{N}$ as the closed formal subscheme of $\mathcal{N}$ over where x lifts to an isogeny (cf. Definition 5.2.1 and Proposition 5.2.3). It is cut out by an element $f_x \in W[\![t_1, t_2]\!]$.

For any nonzero $x \in \mathbb{B}$ such that $\nu_p(x^\vee \circ x) \geq 2$, there is a closed immersion

$$i : \mathcal{Z}^\sharp(p^{-1}x) \to \mathcal{Z}^\sharp(x),$$

by composing every deformation of $p^{-1}x$ with the multiplication-by-p morphism. Since $W[\![t_1, t_2]\!]$ is a unique factorization domain, we get $f_{p^{-1}x} \mid f_x$. Define $d_x = f_x/f_{p^{-1}x} \in W[\![t_1, t_2]\!]$ when $\nu_p(x^\vee \circ x) \geq 2$ and $d_x = f_x$ when $\nu_p(x^\vee \circ x) = 0$ or 1.

Definition 6.2.1. Let $x \in \mathbb{B}$ be a nonzero and integral element. The difference divisor associated to x is

$$\mathcal{D}(x) = \operatorname{Spf} W[\![t_1, t_2]\!]/d_x.$$

The notion of difference divisor was first introduced in [Terstiege 2011]. Proposition 5.2.3 implies that $p \nmid f_x$, so $p \nmid d_x$. Therefore the difference divisor $\mathcal{D}(x)$ is flat over $\operatorname{Spf} W$. The following theorem asserts that $\mathcal{D}(x)$ is regular.

Theorem 6.2.2. *Let $x \in \mathbb{B}$ be a nonzero and integral element. Let $\mathfrak{m} = (p, t_1, t_2)$ be the maximal ideal of $W[\![t_1, t_2]\!]$. Then $d_x \in \mathfrak{m} \setminus \mathfrak{m}^2$, i.e., the difference divisor $\mathcal{D}(x)$ is regular. Moreover, for any $i \geq 1$, d_x and $d_{p^{-i}x}$ are coprime to each other if $p^{-i}x$ is also integral.*

Proof. Let $n \geq 0$ be the p-adic valuation of $x^\vee \circ x$. We first prove this result when $n = 0$. In this case the result follows from [Li and Zhu 2018, Lemma 3.2.2] (p odd) and [Rapoport 2007, Lemma 3.1] ($p = 2$), and $W[\![t_1, t_2]\!]/f_x \simeq W[\![t]\!]$ is even smooth over W.

Now we suppose $n \geq 1$. We can always find an element $x' \in \mathbb{B}$ such that $x'^{\vee} \circ x'$ has p-adic valuation 0 and $(x, x') = 0$. We consider the formal closed subscheme $\mathcal{Z}^\sharp(x) \times_{\mathcal{N}} \mathcal{Z}^\sharp(x')$. It is cut out by the ideal $(f_x, f_{x'}) \subset \mathfrak{m}$; it is also a formal closed subscheme of $\mathcal{Z}^\sharp(x') \simeq \operatorname{Spf} W[\![t]\!]$ cut out by the image $\tilde{f}_x$ of f_x under the surjective map $A \to W[\![t]\!]$. By [Gross and Keating 1993, (5.10)] (see also [Li and Zhang 2022, §5.1]), we have the following decomposition of $\mathcal{Z}^\sharp(x) \times_{\mathcal{N}} \mathcal{Z}^\sharp(x')$ into Cartier divisors on $\mathcal{Z}^\sharp(x')$:

$$\mathcal{Z}^\sharp(x) \times_{\mathcal{N}} \mathcal{Z}^\sharp(x') = \sum_{i=0}^{[n/2]} \mathcal{Z}_i, \tag{19}$$

with each $\mathcal{Z}_i \simeq \operatorname{Spf} \mathcal{O}_{\breve{K},i}$, where $\mathcal{O}_{\breve{K},i}$ is the ring of integers of some nonarchimedean local field. Hence it is a regular local ring, and they are different from each other. Let $d_i \in W[\![t]\!]$ be the function defining the divisor $\mathcal{Z}_i$ on $\mathcal{Z}^\sharp(x')$. Then we have the identity

$$\tilde{f}_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} d_i. \tag{20}$$

The regularity of $\mathcal{O}_{\breve{K},i}$ implies that $d_i \in (p, t) \setminus (p, t)^2$.

Let $\tilde{d}_{p^{-i}x}$ be the image of $d_{p^{-i}x}$ under the surjective map $A \to A/(f_{x'}) \simeq W[\![t]\!]$. By definition we have $f_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} d_{p^{-i}x}$. Therefore,

$$\tilde{f}_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} \tilde{d}_{p^{-i}x}. \tag{21}$$

We induct on n to conclude that $\tilde{d}_x = (\text{unit}) \times d_{[n/2]} \in (p, t) \setminus (p, t)^2$. When $n = 1$, we simply get $\tilde{d}_x = (\text{unit}) \times d_0 \in (p, t) \setminus (p, t)^2$. Let's assume the claim is true for $n < m$ for some $m \geq 2$. We prove the result for $n = m$. For this, we just need to compare (20) and (21) for $p^{-1}x$ and x.

Therefore we have proved that $A/(d_x, f_{x'})$ is a regular local ring, and hence we conclude that $d_x \in \mathfrak{m} \backslash \mathfrak{m}^2$ and $\mathcal{D}(x) \simeq \operatorname{Spf} A/(d_x)$ is regular. Moreover, since all pieces on the right-hand side of (19) are different from each other, we conclude that d_x and $d_{p^{-i}x}$ are coprime to each other. □

Fix an N-isogeny $x_0 \in \mathbb{B}$. Recall that we have defined the deformation functor $\mathcal{N}_0(N)$ in Section 5.1. Compare the moduli interpretations of $\mathcal{N}_0(N)$ and $\mathcal{Z}^\sharp(x_0)$. We have a natural functor

$$i : \mathcal{N}_0(N) \to \mathcal{Z}^\sharp(x_0), \quad (X \xrightarrow{x \text{ cyclic}} X', (\rho, \rho'), (\lambda, \lambda')) \mapsto (X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda')).$$

Theorem 6.2.3. *The natural functor i is a closed immersion, and induces an isomorphism*

$$\mathcal{N}_0(N) \xrightarrow{\sim} \mathcal{D}(x_0).$$

Proof. By Proposition 5.2.3, $\mathcal{Z}^\sharp(x_0)$ is represented by $\operatorname{Spf} W[\![t_1, t_2]\!]/f_{x_0}$. We consider the maximal ideal $\mathfrak{m} = (p, t_1, t_2)$ of $W[\![t_1, t_2]\!]$ and a projective system of rings $\varprojlim_n R_n$, where $R_n = W[\![t_1, t_2]\!]/(f_{x_0} + \mathfrak{m}^n)$. We use $(X_n \xrightarrow{x_n} X'_n, (\rho_n, \rho'_n), (\lambda_n, \lambda'_n))$ to denote the corresponding object in $\mathcal{Z}^\sharp(x_0)(R_n)$ by the natural morphism $W[\![t_1, t_2]\!]/f_{x_0} \to R_n$, which is essentially the base change from $\mathcal{Z}^\sharp(x_0)$ to $\operatorname{Spec} R_n$ of the universal object

$$(X^{\text{univ}} \xrightarrow{x_0^{\text{univ}}} X'^{\text{univ}}, (\rho^{\text{univ}}, \rho'^{\text{univ}}), (\lambda^{\text{univ}}, \lambda'^{\text{univ}})).$$

The following diagram is commutative:

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "x_n"] & X_{n+1} \arrow[d, "x_{n+1}"] \\ X'_n \arrow[r] & X'_{n+1} \end{tikzcd}$$

By [de Jong 1995, Lemma 2.4.4], x_n fits together to be an isogeny of p-divisible groups $x_0^{\mathrm{univ}}: X^{\mathrm{univ}} \to X'^{\mathrm{univ}}$ over $\operatorname{Spec} W[\![t_1, t_2]\!]/f_{x_0}$.

Now we apply the Serre–Tate theorem [Serre 2015] to the projective system $\varprojlim_n R_n$. We obtain a direct system of elliptic curves E_n, E'_n over $\operatorname{Spec} R_n$ and $\tilde{x}_n \in \operatorname{End}_{R_n}(E_n, E'_n)$ such that

(i) there exist isomorphisms $i_n : E_n[p^\infty] \simeq X_n$ and $i'_n : E'_n[p^\infty] \simeq X'_n$,

(ii) $x_n = i'_n \circ \tilde{x}_n[p^\infty] \circ i_n^{-1}$.

Since every elliptic curve is equipped with a canonical ample line bundle given by the unit section, we can apply Grothendieck's algebraization theorem [Stacks, Theorem 089A, Lemma 0A42] to obtain a triple $(E^{\mathrm{univ}} \xrightarrow{\tilde{x}_0^{\mathrm{univ}}} E'^{\mathrm{univ}}, (\rho^{\mathrm{univ}}, \rho'^{\mathrm{univ}}), (\lambda^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$, where E^{univ} and E'^{univ} are two elliptic curves over $\operatorname{Spec} W[\![t_1, t_2]\!]/f_{x_0}$ with the isomorphisms

$$i^{\mathrm{univ}} : E^{\mathrm{univ}}[p^\infty] \simeq X^{\mathrm{univ}}, \quad i'^{\mathrm{univ}} : E'^{\mathrm{univ}}[p^\infty] \simeq X'^{\mathrm{univ}},$$

and $x_0^{\mathrm{univ}} = i'^{\mathrm{univ}} \circ \tilde{x}_0^{\mathrm{univ}}[p^\infty] \circ (i^{\mathrm{univ}})^{-1}$. Then we have

$$\ker(x_0^{\mathrm{univ}}) \simeq \ker(\tilde{x}^{\mathrm{univ}}[p^\infty]) = \ker(\tilde{x}_0^{\mathrm{univ}})[p^\infty] \hookrightarrow E^{\mathrm{univ}},$$

where $\ker(\tilde{x}_0^{\mathrm{univ}})[p^\infty]$ is the p-torsion subgroup scheme of the finite locally free group scheme $\ker(\tilde{x}_0^{\mathrm{univ}})$. Therefore, the universal kernel $\ker(x_0^{\mathrm{univ}})$ is embedded into an elliptic curve. We can apply Proposition 4.1.4 and conclude that there is an ideal $\mathcal{I}^{\mathrm{cyc}}(x_0) \subset W[\![t_1, t_2]\!]$ containing f_{x_0} such that for $S \in \mathrm{Nilp}_W$ and an object $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda')) \in \mathcal{Z}^\sharp(x_0)(S)$, the isogeny x is a cyclic isogeny if and only if the morphism $S \to \operatorname{Spf} W[\![t_1, t_2]\!]/f_{x_0}$ factors through $\operatorname{Spf} W[\![t_1, t_2]\!]/\mathcal{I}^{\mathrm{cyc}}(x_0)$. We conclude from here that $\mathcal{N}_0(N)$ is represented by the formal scheme $\operatorname{Spf} W[\![t_1, t_2]\!]/\mathcal{I}^{\mathrm{cyc}}(x_0)$ and the natural functor i is a closed immersion.

Recall that we use d_{x_0} to denote the equation that cuts out the difference divisor $\mathcal{D}(x_0)$. In the following, we use $\mathcal{D}$ to denote the difference divisor $\mathcal{D}(x_0)$. Let $x_{\mathcal{D}} : X_{\mathcal{D}} \to X'_{\mathcal{D}}$ be the base change of $x_0^{\mathrm{univ}} : X^{\mathrm{univ}} \to X'^{\mathrm{univ}}$ to $\mathcal{D}$. We first show that $x_{\mathcal{D}}$ doesn't factor through the multiplication-by-p morphism of $X_{\mathcal{D}}$. Let's assume the converse, i.e., $x_{\mathcal{D}} = p \circ x'_{\mathcal{D}}$, where $x'_{\mathcal{D}} : X_{\mathcal{D}} \to X'_{\mathcal{D}}$ is an isogeny. Let $\mathcal{D}_n = \operatorname{Spec} W[\![t_1, t_2]\!]/(d_{x_0} + \mathfrak{m}^n)$. The base change of $x'_{\mathcal{D}}$ from $\mathcal{D}$ to $\mathcal{D}_n$ is a deformation of $p^{-1}x_0$, and hence the natural morphism $\mathcal{D}_n \to \mathcal{Z}^\sharp(x_0)$ factors through $\mathcal{Z}^\sharp(p^{-1}x_0) \simeq \operatorname{Spf} W[\![t_1, t_2]\!]/(f_{p^{-1}x_0})$. Since $W[\![t_1, t_2]\!]/(d_{x_0}) \simeq \varprojlim_n W[\![t_1, t_2]\!]/(d_{x_0} + \mathfrak{m}^n)$ by Lemma 6.1.2, we get a ring homomorphism $W[\![t_1, t_2]\!]/(f_{p^{-1}x_0}) \to W[\![t_1, t_2]\!]/(d_{x_0})$. However, d_{x_0} is coprime to $f_{p^{-1}x_0}$ by Theorem 6.2.2, a contradiction. Hence $x_{\mathcal{D}}$ doesn't factor through the multiplication-by-p morphism of $X_{\mathcal{D}}$.

Lemma 4.1.5 and Corollary 4.1.6 imply that $\ker(x_{\mathcal{D}})$ is a cyclic group scheme since $\mathcal{D}$ is an integral noetherian scheme which is also separated and flat over W. Hence there exists a natural morphism from

Spec $W[\![t_1,t_2]\!]/d_{x_0}$ to Spec $W[\![t_1,t_2]\!]/\mathcal{I}^{\mathrm{cyc}}(x_0)$. Therefore, we conclude that $\mathcal{I}^{\mathrm{cyc}}(x_0)\subset(d_{x_0})\subset W[\![t_1,t_2]\!]$. This shows that the closed immersion $\mathcal{D}(x_0)\to\mathcal{Z}^\sharp(x_0)$ decomposes as

$$\mathcal{D}(x_0)\to\mathcal{N}_0(N)\to\mathcal{Z}^\sharp(x_0).$$

Therefore, we have an inclusion of ideals $(f_{x_0})\subset\mathcal{I}^{\mathrm{cyc}}(x_0)\subset(d_{x_0})\in W[\![t_1,t_2]\!]$. Theorem 6.6.1 of [Katz and Mazur 1985] (see also their Case II of 5.3.2.1) asserts that $W[\![t_1,t_2]\!]/\mathcal{I}^{\mathrm{cyc}}(x_0)$ is a 2-dimensional regular local ring. Recall that we have already proved in Theorem 6.2.2 that $W[\![t_1,t_2]\!]/d_{x_0}$ is also a regular local ring. Hence we must have $\mathcal{I}^{\mathrm{cyc}}(x_0)=(d_{x_0})$, i.e., $\mathcal{D}(x_0)\simeq\mathcal{N}_0(N)$. □

6.2.A. *Special Fibers.* In this part we use the identification $\mathcal{N}_0(N)\xrightarrow{\sim}\mathcal{D}(x_0)$ to explicitly describe the special fiber of the local ring $\mathcal{N}_0(N)$. The main results of this part will not be used in the following calculations, so readers can skip on first reading.

Let $\mathfrak{a}=(t_1,t_2)\subset W[\![t_1,t_2]\!]$. Let $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\mathbb{F}[\![t_1,t_2]\!]$. Let $A_n=W[\![t_1,t_2]\!]/\mathfrak{a}^n$ and $R_n=\mathbb{F}[\![t_1,t_2]\!]/\bar{\mathfrak{a}}^n$, with $A_0=W[\![t_1,t_2]\!]$ and $R_0=\mathbb{F}[\![t_1,t_2]\!]$. Equip each A_n with a morphism σ which extends the Frobenius morphism on W and maps t_1 to t_1^p, t_2 to t_2^p. Then A_n is a frame for R_n in the sense of [Zink 2001, Definition 1]. For any $n\geq 0$, let (M,M_1,Φ) be an A_n-window in the sense of [Zink 2001, Definition 2]. Since $\Phi(M_1)\subset p\cdot M$ and p is not a zero-divisor in A_n, we define $\Phi_1:M_1\to M$ to be $p^{-1}\Phi$. The morphism Φ_1 is σ-linear and induces an isomorphism $\Phi_1^\sigma:M_1^\sigma\to M$ because both sides are free A_n-modules of the same rank and Φ_1^σ is surjective by the definition of windows [Zink 2001, Definition 2(ii)]. Let α be the injective A_n-morphism

$$\alpha:M_1\hookrightarrow M\xrightarrow{(\Phi_1^\sigma)^{-1}}M_1^\sigma.$$

Theorem 6.2.4. *For any $n\geq 0$, we have the category equivalences*

$$\{A_n\text{-window }(M,M_1,\Phi)\}\xleftrightarrow{\sim}\{\text{formal }p\text{-divisible groups over }R_n\}.$$

Moreover, both these two categories are equivalent to the category

$$\{\text{pairs }(M_1,\alpha:M_1\to M_1^\sigma)\text{ such that Coker}(\alpha)\text{ is a free }R_n\text{-module}\},$$

where the functor from A_n-windows (M,M_1,Φ) to pairs $(M_1,\alpha:M_1\to M_1^\sigma)$ is given by the constructions above.

Proof. This is proved in [Zink 2001, Theorem 4]. □

Let $((\overline{X},\bar{\rho},\bar{\lambda}),(\overline{X}',\bar{\rho}',\bar{\lambda}'))$ be the universal object in $\mathcal{N}(\mathbb{F}[\![t_1,t_2]\!])$, i.e., the base change of the universal object $((X^{\mathrm{univ}},\rho^{\mathrm{univ}},\lambda^{\mathrm{univ}}),(X'^{\mathrm{univ}},\rho'^{\mathrm{univ}},\lambda'^{\mathrm{univ}}))$ over $W[\![t_1,t_2]\!]$ to $\mathbb{F}[\![t_1,t_2]\!]$. The corresponding window can be described as follows. Let $\mathbb{D}=W\cdot e+W\cdot f$ be the Dieudonne module of $\mathbb{X}$, where $Fe=Ve=f$, $Ff=Vf=p\cdot e$ (F and V are the Frobenius and Verschiebung morphisms on $\mathbb{D}$). Then we let $M=\mathbb{D}\otimes_W W[\![t]\!]$ and $M_1=(W\cdot f+pW\cdot e)\otimes_W W[\![t]\!]$. We still use σ to denote the Frobenius action on $W[\![t]\!]$ which sends t to t^p and extends the Frobenius morphism on W. Let Φ be the σ-linear map from M to M such that $\Phi(e\otimes 1)=t\cdot(e\otimes 1)+f\otimes 1$, $\Phi(f\otimes 1)=p\cdot(e\otimes 1)$. Then (M,M_1,Φ) is the $W[\![t]\!]$-window corresponding to the universal deformation of $\mathbb{X}$ over $\mathbb{F}[\![t]\!]$ (see [Zink 2002, (86)]). Let (M', M_1', Φ') be the corresponding window for $\mathbb{X}'$. Then the $W[\![t_1, t_2]\!]$-window corresponding to the universal deformation of $\mathbb{X}\times_{\mathbb{F}}\mathbb{X}'$ over $\mathbb{F}[\![t_1, t_2]\!]$ is given by $(M\oplus M', M_1\oplus M_1', \Phi\oplus\Phi')$, or $(M_1\oplus M_1', \alpha)$, where under the basis $\{p\cdot(e\otimes 1), f\otimes 1, p(e'\otimes 1), f'\otimes 1\}$, the map α is given by the matrix

$$\alpha=\begin{pmatrix} & 1 & & \\ p & -t_1 & & \\ & & & 1\\ & & p & -t_2\end{pmatrix}.$$

Any quasi-isogeny $x\in\mathbb{B}$ induces the endomorphism

$$\mathbb{D}(x)=\begin{pmatrix} & & \sigma(a) & -\sigma(b)\\ & & -p\cdot b & a\\ a & \sigma(b) & & \\ p\cdot b & \sigma(a) & & \end{pmatrix}$$

of the window $M_1\oplus M_1'$ of $\mathbb{X}\times_{\mathbb{F}}\mathbb{X}'$ under the basis $\{p\cdot e, f, p\cdot e', f'\}$, where $a, b\in\mathbb{Q}_{p^2}$.

Let $M_1(n)=M_1\otimes_{A_0}A_n$, $M_1'(n)=M_1'\otimes_{A_0}A_n$, $\alpha(n)=\alpha\otimes_{A_0}A_n$. By Theorem 6.2.4, a quasi-isogeny x lifts to an isogeny over R_n if and only if there exists $x(n)\in\mathrm{End}\big((M_1(n)\oplus M_1'(n), \alpha(n))\big)$ such that $x(1)=\mathbb{D}(x)$ and the following diagram commutes:

$$\begin{tikzcd} M_1(n)\oplus M_1'(n) \arrow[r, "\alpha(n)"] \arrow[d, "x(n)"] & M_1(n)^\sigma\oplus M_1'(n)^\sigma \arrow[d, "\sigma(x(n))"] \\ M_1(n)\oplus M_1'(n) \arrow[r, "\alpha(n)"] & M_1(n)^\sigma\oplus M_1'(n)^\sigma \end{tikzcd}$$

Under the basis $\{p\cdot(e\otimes 1), f\otimes 1, p(e'\otimes 1), f'\otimes 1\}$, the morphism $x(n)$ has the form

$$x(n)=\begin{pmatrix} A(n) & Y(n)\\ X(n) & B(n)\end{pmatrix},$$

where $X(n), Y(n), A(n), B(n)\in\mathrm{M}_2(A_n)$ satisfy the equations,

$$X(n)=p^{-1}U'(t_2)\cdot\sigma(X(n))\cdot U(t_1),\quad Y(n)=p^{-1}U'(t_1)\cdot\sigma(Y(n))\cdot U(t_2),$$
$$A(n)=p^{-1}U'(t_1)\cdot\sigma(A(n))\cdot U(t_1),\quad B(n)=p^{-1}U'(t_2)\cdot\sigma(B(n))\cdot U(t_2),$$

where

$$U(t)=\begin{pmatrix} & 1\\ p & -t\end{pmatrix}\quad\text{and}\quad U'(t)=\begin{pmatrix} t & 1\\ p & \end{pmatrix}.$$

Since $A(1)=B(1)=0$, we conclude (by comparing degrees of t_1 and t_2) that $A(n)=B(n)=0$.

For any $A\in\mathrm{M}_2(A_n\otimes_{\mathbb{Z}}\mathbb{Q})$, the matrix $\sigma(A)$ is a well-defined element in $\mathrm{M}_2(A_{pn}\otimes_{\mathbb{Z}}\mathbb{Q})$. Therefore, starting from $X(1)$ and $Y(1)$, we can define successively

$$X(p^{l+1})=p^{-1}U'(t_2)\cdot\sigma(X(p^l))\cdot U(t_1),\quad Y(p^{l+1})=p^{-1}U'(t_1)\cdot\sigma(Y(p^l))\cdot U(t_2). \tag{22}$$

Since the local ring $\mathcal{O}_{\mathcal{Z}(x)}$ only depends (up to noncanonical isomorphisms) on the valuation of x, we take the following specific choice of x and $\mathbb{D}(x)$ in the following computations:

- When $\mathrm{ord}_p(x^\vee \circ x) = 2k$ for some $k \geq 0$, we take

$$X(1) = Y(1) = \begin{pmatrix} p^k & \\ & p^k \end{pmatrix}.$$

By computation based on the recursion formula (22), it turns out that for any $l \geq 1$,

$$X(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{(p^{l-1}-1)/(p-1)}\left(t_2^{p^{l-1}} - t_1^{p^{l-1}}\right) \\ 0 & 0 \end{pmatrix} + p \cdot C\right),$$

$$Y(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{(p^{l-1}-1)/(p-1)}\left(t_2^{p^{l-1}} - t_1^{p^{l-1}}\right) \\ 0 & 0 \end{pmatrix} + p \cdot D\right)$$

for some matrices $C, D \in \mathrm{M}_2(A_{p^l})$.

- When $\mathrm{ord}_p(x^\vee \circ x) = 2k+1$ for some $k \geq 0$, we take

$$X(1) = -Y(1) = \begin{pmatrix} & p^k \\ p^{k+1} & \end{pmatrix}.$$

By computation based on the recursion formula (22), it turns out that for any $l \geq 1$,

$$X(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l}(t_1t_2)^{(p^{l}-1)/(p-1)} \\ 0 & 0 \end{pmatrix} + p \cdot C'\right),$$

$$Y(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{(p^{l}-1)/(p-1)} \\ 0 & 0 \end{pmatrix} + p \cdot D'\right)$$

for some matrices $C', D' \in \mathrm{M}_2(A_{p^l})$.

Proposition 6.2.5. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation n and induces $X(1)$ and $Y(1)$ as described above. Let $f_x \in W[[t_1, t_2]]$ be the element cutting out $\mathcal{Z}(x)$. Then*

$$\bar{f}_x := f_x \bmod p = (\text{unit}) \times \begin{cases} (t_1t_2)^{(p^{n/2}-1)/(p-1)}\left(t_2^{p^{n/2}} - t_1^{p^{n/2}}\right) \bmod (t_1, t_2)^{p^{n/2+1}} & \text{when } n \text{ is even}, \\ (t_1t_2)^{(p^{(n+1)/2}-1)/(p-1)} \bmod (t_1, t_2)^{p^{(n+1)/2}} & \text{when } n \text{ is odd}. \end{cases} \tag{23}$$

Proof. By the above formula for $X(p^l)$ and $Y(p^l)$, we can conclude that x can be lifted to an isogeny over $R_{p^{[n/2]}}$ but not over $R_{p^{[n/2]+1}}$. Then the formula for $X(p^{[n/2]+1})$ and $Y(p^{[n/2]+1})$ imply (23). □

Theorem 6.2.6. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation n and induces $X(1)$ and $Y(1)$ as described above. Let $f_x \in W[[t_1, t_2]]$ be the element cutting out $\mathcal{Z}(x)$. Then $\bar{f}_x$ is divisible by*

$$t_1 - t_2^{p^a}, \quad t_1^{p^a} - t_2 \quad \text{for } 0 \leq a \leq n \text{ and } a \equiv n \bmod 2.$$

Moreover, $\bar{f}_x$ has no other irreducible factors and the multiplicity of $t_1 - t_2^{p^a}$, $t_1^{p^a} - t_2$ in $\bar{f}_x$ is $p^{(n-a)/2}$.

Proof. We first prove that $t_1^{p^{k_1}} - t_2^{p^{k_2}}$ divides $\bar{f}_x$, where $k_1, k_2 \geq 0$ and $k_1 + k_2 = n$. We prove this by showing that $X(p^l), Y(p^l) \bmod (t_1^{p^{k_1}} - t_2^{p^{k_2}}) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}}))$ for any $l \geq 0$.

- When $n=2k$ is even, the recursion formula (22) implies that,

$$X(p^l)=p^{k-l}U'(t_2)U'(t_2^p)\cdots U'(t_2^{p^{l-1}})U(t_1^{p^{l-1}})\cdots U(t_1^p)U(t_1),$$
$$Y(p^l)=p^{k-l}U'(t_1)U'(t_1^p)\cdots U'(t_1^{p^{l-1}})U(t_2^{p^{l-1}})\cdots U(t_2^p)U(t_2).$$

Let's assume first that $k_1\le k_2$. For any $0\le t\le l-k_1$, we have the relation $t_2^{p^{l-t}}=t_1^{p^{k_2-k_1+l-t}}$. Hence

$$U'(t_2^{p^{l-t}})U(t_1^{p^{k_2-k_1+l-t}})=p\cdot I_2.$$

Moreover, when $1\le t\le k_2-k_1$, we have $t_2^{p^{l-t}}=t_1^{p^{k_2-k_1+l-t}}=0$. Hence $U(t_2^{p^{l-t}})=U(0)$ and we get

$$X(p^l)=U'(t_2)U'(t_2^p)\cdots U'(t_2^{p^{k_2-1}})U(t_1^{p^{k_1-1}})\cdots U(t_1^p)U(t_1)\in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}}-t_2^{p^{k_2}})),$$
$$Y(p^l)=U'(t_1)U'(t_1^p)\cdots U'(t_1^{p^{k_1-1}})U(t_2^{p^{k_2-1}})\cdots U(t_2^p)U(t_2)\in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}}-t_2^{p^{k_2}})).$$

The proof of the case $k_1>k_2$ is similar and we get the same formula for $X(p^l)$ and $Y(p^l)$ as above. Therefore, we conclude that $t_1^{p^{k_1}}-t_2^{p^{k_2}}$ divides $\bar{f}_x$ when $k_1,k_2\ge 0$ and $k_1+k_2=2k$ by Theorem 6.2.4. Hence $\bar{f}_x$ is divisible by the polynomial

$$(t_1-t_2)^{p^k}\cdot\prod_{a=1}^{k}\big((t_1-t_2^{p^{2a}})(t_2-t_1^{p^{2a}})\big)^{p^{k-a}}.$$

We also know that

$$(t_1-t_2)^{p^k}\cdot\prod_{a=1}^{k}\big((t_1-t_2^{p^{2a}})(t_2-t_1^{p^{2a}})\big)^{p^{k-a}}\equiv(t_1t_2)^{(p^k-1)/(p-1)}\cdot(t_2^{p^k}-t_1^{p^k}) \bmod (t_1,t_2)^{p^{k+1}}.$$

The lemma follows by comparing this formula with (23).

- When $n=2k+1$ is odd, the recursion formula (22) implies

$$X(p^l)=p^{k-l}U'(t_2)U'(t_2^p)\cdots U'(t_2^{p^{l-1}})\begin{pmatrix}0&1\\p&0\end{pmatrix}U(t_1^{p^{l-1}})\cdots U(t_1^p)U(t_1),$$
$$Y(p^l)=p^{k-l}U'(t_1)U'(t_1^p)\cdots U'(t_1^{p^{l-1}})\begin{pmatrix}0&1\\p&0\end{pmatrix}U(t_2^{p^{l-1}})\cdots U(t_2^p)U(t_2).$$

Let's assume $k_1<k_2$. For any $0\le t\le l-k_1$, we have the relation $t_2^{p^{l-t}}=t_1^{p^{k_2-k_1+l-t}}$, and hence $U'(t_2^{p^{l-t}})U(t_1^{p^{k_2-k_1+l-t}})=p\cdot I_2$. Moreover, when $1\le t\le k_2-k_1$, we have $t_2^{p^{l-t}}=t_1^{p^{k_2-k_1+l-t}}=0$. So $U(t_2^{p^{l-t}})=U(0)$, and we get

$$X(p^l)=U'(t_2)U'(t_2^p)\cdots U'(t_2^{p^{k_2-1}})U(t_1^{p^{k_1-1}})\cdots U(t_1^p)U(t_1)\in \mathrm{M}_2\big(A_{p^l}/(t_1^{p^{k_1}}-t_2^{p^{k_2}})\big),$$
$$Y(p^l)=U'(t_1)U'(t_1^p)\cdots U'(t_1^{p^{k_1-1}})U(t_2^{p^{k_2-1}})\cdots U(t_2^p)U(t_2)\in \mathrm{M}_2\big(A_{p^l}/(t_1^{p^{k_1}}-t_2^{p^{k_2}})\big).$$

Therefore we conclude that $t_1^{p^{k_1}}-t_2^{p^{k_2}}$ divides $\bar{f}_x$ when $k_1,k_2\ge 0$ and $k_1+k_2=2k+1$ by Theorem 6.2.4.

Hence $\bar{f}_x$ is divisible by the polynomial

$$\prod_{a=0}^{k}\big((t_1-t_2^{p^{2a+1}})(t_2-t_1^{p^{2a+1}})\big)^{p^{k-a}}.$$

We also know that $\prod_{a=0}^{k}\big((t_1-t_2^{p^{2a+1}})(t_2-t_1^{p^{2a+1}})\big)^{p^{k-a}} \equiv (t_1t_2)^{(p^{k+1}-1)/(p-1)} \bmod (t_1,t_2)^{p^{k+1}}$. The lemma follows by comparing this formula with (23). □

Corollary 6.2.7. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation $n \geq 1$. Let $\mathcal{Z}(x)_p$ be special fiber of $\mathcal{Z}(x)$. Then*

$$\mathcal{Z}(x)_p \simeq \operatorname{Spf} \mathbb{F}[\![t_1,t_2]\!] \Big/ \Bigg(\prod_{\substack{a+b=n\\ a,b\geq 0}} (t_1^{p^a}-t_2^{p^b})\Bigg).$$

Let $\mathcal{D}(x)_p$ (resp. $\mathcal{N}_0(N)_p$) be the base change of $\mathcal{D}(x)$ (resp. $\mathcal{N}_0(N)$) to $\mathbb{F}[\![t_1,t_2]\!]$. Then

$$\mathcal{N}_0(N)_p \simeq \mathcal{D}(x)_p \simeq \operatorname{Spf} \mathbb{F}[\![t_1,t_2]\!] \Big/ \Bigg((t_1-t_2^{p^n})\cdot(t_2-t_1^{p^n})\cdot \prod_{\substack{a+b=n\\ a,b\geq 1}} (t_1^{p^{a-1}}-t_2^{p^{b-1}})^{p-1}\Bigg).$$

Proof. The statement for $\mathcal{Z}(x)_p$ follows from Theorem 6.2.6. The statement for $\mathcal{D}(x)_p$ follows from the definition of difference divisors. □

Remark 6.2.8. The same formula has been proved in [Katz and Mazur 1985, Theorems 13.4.6 and 13.4.7] by a totally different method.

6.3. ***Local arithmetic intersection numbers.*** Now we give the definition of the local arithmetic intersection numbers.

Definition 6.3.1. For any rank-3 lattice $L \subset \mathbb{B}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2, x_3\}$ of L. Let $\mathcal{O}_{\mathcal{Z}^\sharp(x_i)}$ be the structure sheaf of the special cycle $\mathcal{Z}^\sharp(x_i)$. Let $\mathcal{O}_{\mathcal{N}}$ be the structure sheaf of the formal scheme $\mathcal{N}$. Let $-\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}-$ be the derived tensor product functor in the derived category of coherent sheaves on $\mathcal{N}$. Define the local arithmetic intersection number of L on $\mathcal{N}$ to be

$$\operatorname{Int}^\sharp(L) = \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}).$$

This number is finite and independent of the choice of the basis $\{x_i\}_{i=1}^3$ of L because of the following result.

Lemma 6.3.2. *Let $x, y \in \mathbb{B}$ be linearly independent. Then the tor sheaves $\underline{\operatorname{Tor}}_i^{\mathcal{O}_{\mathcal{N}}}(\mathcal{O}_{\mathcal{Z}^\sharp(x)}, \mathcal{O}_{\mathcal{Z}^\sharp(y)})$ vanish for all $i \geq 1$. In particular,*

$$\mathcal{O}_{\mathcal{Z}^\sharp(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(y)} = \mathcal{O}_{\mathcal{Z}^\sharp(x)} \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(y)}.$$

Moreover, the same formula holds if $\mathcal{Z}^\sharp(x)$ or $\mathcal{Z}^\sharp(y)$ (or both) are replaced by $\mathcal{D}(x)$ or $\mathcal{D}(y)$, respectively.

Let $L \subset \mathbb{B}$ be an integral quadratic lattice of rank 3 over $\mathbb{Z}_p$ with basis $\{x_1, x_2, x_3\}$. Then the derived tensor product $\mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}$ is independent of the choice of the basis.

Proof. This is proved in [Terstiege 2011, Lemma 4.1 and Proposition 4.2]. □

Now let's come to the local arithmetic intersection numbers on $\mathcal{N}_0(N)$. For a fixed N-isogeny x_0 of $\mathbb{X}$, recall that we have defined the space of quasi-isogenies of $\mathbb{X}$ special to x_0 (see Definition 5.2.4) to be those $x \in \mathbb{B}$ such that

$$x \circ x_0^\vee + x_0 \circ x^\vee = 0.$$

Recall that we use $\mathbb{W}$ to denote this space.

Definition 6.3.3. For any rank-2 lattice $M \subset \mathbb{W}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of M. Let $\mathcal{O}_{\mathcal{Z}(x_i)}$ be the structure sheaf of the special cycle $\mathcal{Z}(x_i)$. Let $\mathcal{O}_{\mathcal{N}_0(N)}$ be the structure sheaf of the formal scheme $\mathcal{N}_0(N)$. Let $-\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} -$ be the derived tensor product functor in the derived category of coherent sheaves on $\mathcal{N}_0(N)$. Define the local arithmetic intersection number of M on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}).$$

This number is independent of the choice of the basis $\{x_1, x_2\}$ of M because of Lemma 6.3.2 and Theorem 6.2.3. We relate it to the derivative of the local density of the quadratic lattice M with level N. The following theorem is the starting point of our calculation.

Theorem 6.3.4. *For any prime number p, let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3. Then*

$$\mathrm{Int}^\sharp(L) = \partial\, \mathrm{Den}(L).$$

Proof. In [Gross and Keating 1993, §4], the Gross–Keating invariants (a_1, a_2, a_3) of the rank-3 quadratic lattice L are defined. Then the local arithmetic intersection number $\mathrm{Int}^\sharp(L)$ is computed explicitly in terms of these invariants (see also [Rapoport 2007, Theorem 1.1]). In [Wedhorn 2007, §2.11], the local density $\mathrm{Den}^+(X, L)$ is also expressed explicitly in terms of the Gross–Keating invariants (a_1, a_2, a_3), hence the derived local density $\partial\, \mathrm{Den}^+(L)$. The theorem is proved by comparing the expressions of both sides in terms of (a_1, a_2, a_3) (see [Wedhorn 2007, §2.16]). See also [Li and Zhang 2022] for a recent new proof when p is odd. □

6.4. ***Difference formula of the local arithmetic intersection numbers.*** Fix an N-isogeny $x_0 \in \mathrm{End}(\mathbb{X})$, and recall that $\mathbb{W} = \{x_0\}^\perp \to \mathbb{B}$.

Theorem 6.4.1. *For any rank-2 lattice $M \subset \mathbb{W}$, we have the identity*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

Proof. Let $\{x_1, x_2\}$ be a basis of M. By Lemma 6.3.2 and the isomorphism $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$, we have the following isomorphism as complexes of coherent sheaves on $\mathcal{N}$:

$$\begin{aligned}\mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{D}(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}.\end{aligned}$$

When $\nu_p(N)=0$ or 1, the difference divisor $\mathcal{D}(x_0)$ is just $\mathcal{Z}^\sharp(x_0)$. Hence $\mathrm{Int}_{\mathcal{N}_0(N)}(M)=\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot x_0)$ and $\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot p^{-1}x_0)=0$ since $p^{-1}x_0$ is not integral, and therefore

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M)=\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot x_0)-\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot p^{-1}x_0).$$

When $\nu_p(N)\geq 2$, we have the exact sequence

$$0\to\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\xrightarrow{\times d_{x_0}}\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\to\mathcal{O}_{\mathcal{D}(x_0)}\simeq\mathcal{O}_{\mathcal{N}_0(N)}\to 0.$$

Tensoring the above exact sequence with the complex $\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}$ in the derived category of coherent sheaves on $\mathcal{N}$, we get an exact triangle

$$\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}\to\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}$$
$$\to\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}\to.$$

Hence we have the identity

$$\begin{aligned}&\chi(\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})\\&\quad=\chi(\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})+\chi(\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}).\end{aligned}$$

We already know that $\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}\simeq\mathcal{O}_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}}\mathcal{O}_{\mathcal{Z}(x_2)}$ since $\mathcal{N}_0(N)\simeq\mathcal{D}(x_0)$. Hence

$$\begin{aligned}\mathrm{Int}_{\mathcal{N}_0(N)}(M)&=\chi(\mathcal{O}_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{D}(x_0)}}\mathcal{O}_{\mathcal{Z}(x_2)})=\chi(\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})\\&=\chi(\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})-\chi(\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})\\&=\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot x_0)-\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot p^{-1}x_0).\end{aligned}$$ □

7. Difference formula at the analytic side

Let p be a prime number. Let F be a nonarchimedean local field of residue characteristic p, with ring of integers $\mathcal{O}_F$, residue field $\kappa=\mathbb{F}_q$ of size q, and uniformizer π.

7.1. ***Primitive decomposition.*** Let $N\in F$. Recall that we use $(\langle N\rangle, q_{\langle N\rangle})$ to denote the rank-1 quadratic lattice over $\mathcal{O}_F$ with an $\mathcal{O}_F$-generator l_N such that $q_{\langle N\rangle}(l_N)=N$. Then $\langle N\rangle$ is an integral quadratic lattice if and only if $N\in\mathcal{O}_F$. Let $n=\nu_\pi(N)$. All the rank-1 integral quadratic lattices L' containing $\langle N\rangle$ have the form

$$L'=\pi^{-i}\langle N\rangle\simeq\langle\pi^{-2i}N\rangle\quad\text{for } i=0,1,\dots,\left[\frac{n}{2}\right].$$

Let H be a self-dual quadratic $\mathcal{O}_F$-lattice of finite rank. Since $q_H(x)\in\mathcal{O}_F$ for every $x\in H$, Lemma 2.2.5 gives the decomposition

$$\mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F)=\bigsqcup_{i=0}^{[n/2]}\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F).$$

Now for every $0\leq i\leq[n/2]$, we pick an arbitrary $\phi\in\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$ and consider the following sublattice of H:

$$H(\phi):=\{x\in H:(x,\phi(l_N))=0\}.$$

Lemma 7.1.1. *The isometric class of $H(\phi)$ is independent of the choice of $\phi \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$.*

Proof. Let $\phi' \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$ be another element. The homomorphisms ϕ and ϕ' are totally determined by $x := \phi(l_{\pi^{-2i}N})$ and $x' := \phi'(l_{\pi^{-2i}N})$. The fact that ϕ and ϕ' are primitive implies that $x \notin \pi\cdot H$ and $x' \notin \pi\cdot H$. Therefore,

$$(x, H) = \mathcal{O}_F, \quad (x', H) = \mathcal{O}_F,$$

where we use $(\cdot\,,\cdot)$ to denote the associated bilinear form on H. Since H is self-dual, then by [Morin-Strom 1979, Theorem 5.3], there exists $\varphi \in \mathrm{O}(H)(\mathcal{O}_F)$ such that $\varphi(x) = x'$. The homomorphism φ also induces an isometry between $H(\phi)$ and $H(\phi')$ because $H(\phi) = x^{\perp}\cap H$ and $H(\phi') = x'^{\perp}\cap H$. □

Let $N \in \mathcal{O}_F$ be an element of valuation n. For every $0 \le i \le [n/2]$ and $\phi \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$, we use $H(N, i)$ to denote the quadratic lattice $H(\phi)$.

Example 7.1.2. Let $N \in \mathcal{O}_F$ have valuation n. When $k > 4$, we have an orthogonal decomposition

$$H_k^{\varepsilon} \simeq H_4^{+} \oplus H_{k-4}^{\varepsilon}.$$

Recall that the symbol H_k^{ε} is understood in the following way: when p is odd, k can be any positive integer, and $\varepsilon \in \{\pm 1\}$ is arbitrary; when $p = 2$, k is even and $\varepsilon = +1$. The lattice $\mathrm{M}_2(\mathcal{O}_F)$ is equipped with the quadratic form induced by the determinant; it is self-dual and $\chi_F(\mathrm{M}_2(\mathcal{O}_F)) = 1$, and hence we can view $\mathrm{M}_2(\mathcal{O}_F)$ as a model lattice for H_4^{+}.

Let's consider the element $\phi \in \mathrm{PRep}_{H_k^{\varepsilon},\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$ given by

$$\phi_i : \langle\pi^{-2i}N\rangle \to \mathrm{M}_2(\mathcal{O}_F) \simeq H_4^{+} \hookrightarrow H_k^{\varepsilon}, \quad l_{\pi^{-2i}N} \mapsto \begin{pmatrix} \pi^{-2i}N & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding element in $\mathrm{Rep}_{H_k^{\varepsilon},\langle N\rangle}(\mathcal{O}_F)$ sends

$$l_N \mapsto \begin{pmatrix} \pi^{-i}N & 0 \\ 0 & \pi^{i} \end{pmatrix}.$$

Lemma 7.1.1 implies that the following quadratic lattices are isometric:

$$H_k^{\varepsilon}(N, i) = H_k^{\varepsilon}(\phi_i) \simeq \phi_i(l_{\pi^{-2i}N})^{\perp} \oplus H_{k-4}^{\varepsilon},$$

where $\phi_i(l_{\pi^{-2i}N})^{\perp}$ is the space of elements in $\mathrm{M}_2(\mathcal{O}_F)$ that are orthogonal to $\phi_i(l_{\pi^{-2i}N})$, it can be described explicitly as

$$\phi_i(l_{\pi^{-2i}N})^{\perp} = \left\{ x = \begin{pmatrix} -\pi^{-2i}Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathcal{O}_F \right\}.$$

It is exactly the lattice $\Delta_F(\pi^{-2i}N)$ defined in Example 2.1.1. Therefore

$$H_k^{\varepsilon}(N, i) \simeq \Delta_F(\pi^{-2i}N) \oplus H_{k-4}^{\varepsilon}.$$

7.2. ***Difference formula of local densities.***

Theorem 7.2.1. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice of finite rank k. Let M be an integral quadratic $\mathcal{O}_F$-lattice of finite rank r. Let $N \in \mathcal{O}_F$ be an element of valuation n. Then*

$$\mathrm{Den}(H, M \oplus \langle N\rangle) = \sum_{i=0}^{[n/2]} q^{(2-k+r)i} \cdot \mathrm{Pden}(H, \langle \pi^{-2i} N\rangle) \cdot \mathrm{Den}(H(N, i), M).$$

The proof of this theorem is based on the following lemmas.

Lemma 7.2.2. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice. Let $N \in \mathcal{O}_F$ be an element of valuation n. Then there is a bijective map*

$$D : \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d) \xrightarrow{\sim} \bigsqcup_{i=0}^{[n/2]} \bigsqcup_{\bar{x}\in\mathcal{O}_F/\pi^i} \mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$$

when the positive integer d is large enough.

Proof. Let l_N be a generator of the rank-1 $\mathcal{O}_F$-module $\langle N\rangle$ such that $q_{\langle N\rangle}(l_N) = N$. Any f in $\mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d)$ is determined by $f(\overline{l_N}) \in H/\pi^d H$. There is a natural filtration on $H/\pi^d H$ given as

$$0 \subset \pi^{d-1}H/\pi^d H \subset \pi^{d-2}H/\pi^d H \subset \cdots \subset \pi^2 H/\pi^d H \subset \pi H/\pi^d H \subset H/\pi^d H.$$

Let i be the minimal integer such that $f(\overline{l_N}) \in \pi^i H/\pi^d H$. Then $0 \le i \le [n/2]$ since $\nu_\pi(N) = n$. So there exists $l \in H$ such that $f(\overline{l_N}) = \overline{\pi^i l} \in \pi^i H/\pi^d H$; the image of l in $H/\pi^{d-i}H$ is uniquely determined by f. Let q be the quadratic form on H. Then

$$N \bmod pi^d = \overline{q_{\langle N\rangle}}(\overline{l_N}) = \bar{q}(f(\overline{l_N})) = \pi^{2i}\bar{q}(\bar{l}) = \overline{\pi^{2i}q(l)}.$$

Hence $\overline{\pi^{2i}q(l)} \equiv \pi^{-2i}N \bmod \pi^{d-2i}$. Therefore there exists $x \in \mathcal{O}_F$ such that $q(l) = \pi^{-2i}N + \pi^{d-2i}x$. Next we show that $\bar{x} \in \mathcal{O}_F/\pi^i$ is independent of the choice of $l \in H$ when d is large enough. Suppose l' is another element of H such that $f(\overline{l_N}) = \overline{\pi^i l'}$. Then there exists $\delta \in H$ such that $l' - l = \pi^{d-i}\delta$. Therefore, when d is large enough,

$$q(l') = q(l + \pi^{d-i}\delta) = q(l) + \pi^{d-i}(l, \delta) + \pi^{2d-2i}q(\delta) \equiv q(l) \bmod \pi^{d-i}.$$

Suppose $q(l') = \pi^{-2i}N + \pi^{d-2i}x'$ for some $x' \in \mathcal{O}_F$. The above congruence between $q(l)$ and $q(l')$ implies $x' \equiv x \bmod \pi^i$. The above construction gives the homomorphism $D(f)$ in $\mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$ sending the generator $\overline{l_{\pi^{-2i}N+\pi^{d-2i}x}}$ of $\langle \pi^{-2i}N + \pi^{d-2i}x\rangle/\pi^{d-i}\langle \pi^{-2i}N + \pi^{d-2i}x\rangle$ to $\bar{l} \in H/\pi^{d-i}H$.

Now for any element $\varphi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, we consider the morphism

$$\tilde{\varphi} : \langle N\rangle/\pi^d\langle N\rangle \to H/\pi^d H, \quad \overline{l_N} \mapsto \pi^i\varphi(\overline{l_{\pi^{-2i}N+\pi^{d-2i}x}}).$$

Then $\tilde{\varphi} \in \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d)$ because $\bar{q}(\tilde{\varphi}(\overline{l_N})) = \overline{\pi^{2i}(\pi^{-2i}N + \pi^{d-2i}x)} = N \bmod \pi^d$. This construction gives the inverse map of D. $\square$

Let M be an integral quadratic $\mathcal{O}_F$-lattice of finite rank. Let $N \in \mathcal{O}_F$ be an element of valuation n. Let $M^\sharp = M \oplus \langle N \rangle$ be a quadratic $\mathcal{O}_F$-lattice of one rank higher than M. For any positive integer d and any self-dual quadratic $\mathcal{O}_F$-lattice H, there is a natural restriction map

$$\mathrm{res} : \mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d) \to \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d),$$

given by composing any element in the set $\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)$ with the natural inclusion of $\langle N\rangle/\pi^d\langle N\rangle$ in $M^\sharp/\pi^d M^\sharp$. The next lemma describes the fiber of the map $D \circ \mathrm{res}$.

Lemma 7.2.3. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice and M an integral quadratic $\mathcal{O}_F$-lattice of finite rank r. For $N \in \mathcal{O}_F$ an element of valuation n, let $M^\sharp = M \oplus \langle N\rangle$ be a quadratic $\mathcal{O}_F$-lattice of rank $r+1$. Let $0 \le i \le [n/2]$ be an integer. Given $\varphi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N + \pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, for d large enough we have*

$$\#(D\circ \mathrm{res})^{-1}(\varphi) = q^{ir} \cdot \#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d).$$

Proof. Let f be an element in $\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)$ such that $D\circ \mathrm{res}(f) = \varphi$. By the proof of Lemma 7.2.2, there exists $l'_N \in H \setminus \pi H$ such that $f(\overline{l_N}) = \overline{\pi^i l'_N}$, and $q(l'_N) = \pi^{-2i}N$ when d is large enough.

Let $\{e_i\}_{i=1}^r$ be an $\mathcal{O}_F$-basis of M. Then f is determined by $\{x_i := f(\overline{e_i}) \in H/\pi^d H\}_{i=1}^r$. Therefore $(D\circ\mathrm{res})^{-1}(\varphi)$ can be described by the set

$$\begin{aligned}(D\circ\mathrm{res})^{-1}(\varphi) = \big\{(x_1,\dots,x_r) \in (H/\pi^d H)^r : (x_i, \overline{\pi^i l'_N}) = 0,\ (x_i,x_j) = (\overline{e_i},\overline{e_j}) \text{ for } i\neq j,\\ \text{and } \bar q(x_i) = \overline{q_M}(\overline{e_i}) \text{ for every } i.\big\}.\end{aligned} \tag{24}$$

Let L be the rank-1 sublattice of H generated by l'_N. We have the exact sequence

$$0 \to L \oplus H(N,i) \xrightarrow{\theta} H \to Q := H/L\oplus H(N,i) \to 0,$$

where θ is the natural inclusion map. After tensoring the above exact sequence with $\mathcal{O}_F/\pi^d$, we get the following exact sequence by the flatness of H over $\mathcal{O}_F$:

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_F}(Q,\mathcal{O}_F/\pi^d) \to L/\pi^d L \oplus H(N,i)/\pi^d H(N,i) \xrightarrow{\bar\theta} H/\pi^d H \to Q/\pi^d Q \to 0. \tag{25}$$

Claim. *Let $K = \{x \in H/\pi^d H : (x,\overline{\pi^i l'_N}) = 0\}$. When d is large enough, for every $\bar x \in K$ there exists $x' \in L$ and $x'' \in H(N,i)$ such that the image of $\overline{x'} + \overline{x''} \in L/\pi^d L \oplus H(N,i)/\pi^d H(N,i)$ under $\bar\theta$ in $H/\pi^d H$ is $\bar x$.*

Proof of the claim. We have the decomposition

$$x = x' + x''$$

in the quadratic space $V = H\otimes_{\mathcal{O}_F} F$, where $x' \in L_F := L\otimes_{\mathcal{O}_F} F$ and $x'' \in (L_F)^\perp \subset V$.

The fact that $\bar x \in K$ implies that $(x', l_N) = (x, l_N) \in (\pi^d)$. Therefore $x' \in (\pi^{d-n})\cdot l_N \in L_F$. It turns out that $x' \in L \subset H$ when d is large enough, and hence $x' = x - x' \in H \cap \{l_N\}^\perp = H(N,i)$. ☐

We get the following description of the inverse image of the set $(D\circ\mathrm{res})^{-1}(\varphi)$ under $\Theta := \bar\theta\times\cdots\times\bar\theta$ by (24):

$$\Theta^{-1}((D\circ r)^{-1}(\varphi)) = (\pi^{d+i-n}L/\pi^d L)^r \times \mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d). \tag{26}$$

The claim implies that the map $\Theta^{-1}((D\circ\mathrm{res})^{-1}(\varphi)) \xrightarrow{\Theta} (D\circ\mathrm{res})^{-1}(\varphi)$ is surjective.

Now we compute $\#\ker(\Theta)$, which equals $(\#\ker(\theta))^r$ by definition. By the exact sequence (25), $\#\ker(\bar{\theta}) = \#\mathrm{Tor}^1_{\mathcal{O}_F}(Q, \mathcal{O}_F/\pi^d) = \#Q/\pi^d Q$. Therefore, when d is large enough, $Q/\pi^d Q = Q$. Since $l'_N \notin \pi H$, there exists $y \in H$ such that $(l'_N, y) = 1$. The existence of y implies the exact sequence

$$0 \to H(N,i) \xrightarrow{\theta} H \to L^\vee \to 0, \quad x \mapsto l(x) : v \in L \mapsto (x, v).$$

Therefore, $H \simeq L^\vee \oplus H(N,i)$ as $\mathcal{O}_F$-modules, and $Q \simeq L^\vee/L \simeq \pi^{2i-n}L/L$. Then, by (26),

$$\#(D \circ \mathrm{res})^{-1}(\varphi) = \frac{q^{r(n-i)}}{q^{r(n-2i)}} \cdot \#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d) = q^{ir} \cdot \#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d). \qquad \square$$

Proof of Theorem 7.2.1. By Lemmas 7.2.2 and 7.2.3, we only need to know the size of the set $\mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$ when $x \in \mathcal{O}_F$. We first show that when d is large enough,

$$\#\mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i}) = \#\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})$$

holds for any $x \in \mathcal{O}_F$, because when d is large enough, we could find $c_x \in \mathcal{O}_F^\times$ such that $c_x^{-2} = 1+\pi^d N^{-1}x \mod \pi^{d-i}$; then for any element $l \in \mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, $c_x \cdot l \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})$. Let $M^\sharp = M \oplus \langle N\rangle$. We have

$$\begin{aligned}\mathrm{Den}(H, M \oplus \langle N\rangle) &= \lim_{d\to\infty} \frac{\#\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)}{q^{d(k(r+1)-(r+1)(r+2)/2)}} \\ &= \lim_{d\to\infty} \sum_{i=0}^{[n/2]} q^i \cdot \frac{\#\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})}{q^{(d-i)(k-1)}} \cdot \frac{q^{ir}}{q^{i(k-1)}} \cdot \frac{\#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d)}{q^{d((k-1)r-r(r+1)/2)}} \\ &= \sum_{i=0}^{[n/2]} q^{(2-k+r)i} \cdot \mathrm{Pden}(H, \langle\pi^{-2i}N\rangle) \cdot \mathrm{Den}(H(N,i), M). \qquad \square\end{aligned}$$

Remark 7.2.4. When p odd, it has been calculated explicitly (see [Li and Zhang 2022, (3.3.2.1)]) that for any $N \in \mathcal{O}_F$

$$\mathrm{Pden}(H_k^\varepsilon, \langle N\rangle) = \begin{cases} 1-q^{1-k} & \text{when } k \text{ is odd and } \pi \mid N, \\ 1+\varepsilon\chi_F(N)q^{(1-k)/2} & \text{when } k \text{ is odd and } \pi \nmid N, \\ (1-\varepsilon q^{-k/2})(1+\varepsilon q^{1-k/2}) & \text{when } k \text{ is even and } \pi \mid N, \\ 1-\varepsilon q^{-k/2} & \text{when } k \text{ is even and } \pi \nmid N. \end{cases} \tag{27}$$

When $p = 2$, the same formula makes sense and holds true only in the case that k is even and $\varepsilon = +1$.

Definition 7.2.5. Let $N \in \mathcal{O}_F$. Let M be a quadratic lattice of rank $r \geq 2$ over $\mathcal{O}_F$. Define the local density of M with level N to be a polynomial $\mathrm{Den}_{\Delta_F(N)}(X, M)$ satisfying, for $m \geq 0$,

$$\mathrm{Den}_{\Delta_F(N)}(X, M)\big|_{X=q^{-m}} = \begin{cases} \dfrac{\mathrm{Den}(\Delta_F(N) \oplus H^+_{2m+r-2}, M)}{\mathrm{Nor}^+(q^{-m}, r-1)} & \text{when } \pi \mid N, \\ \dfrac{\mathrm{Den}(\Delta_F(N) \oplus H^+_{2m+r-2}, M)}{\mathrm{Nor}^{\chi_F(N)}(q^{-m}, r)} & \text{when } \pi \nmid N. \end{cases}$$

Moreover, if the lattice $M \oplus \langle N \rangle$ can't be isometrically embedded into the self-dual lattice H_{r+2}^{+}, define the derived local density of M with level N to be

$$\partial\,\mathrm{Den}_{\Delta_F(N)}(M) = -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}_{\Delta_F(N)}(X, M).$$

Theorem 7.2.6. *Let $N \in \mathcal{O}_F$. Let M be a quadratic lattice of rank $r \geq 2$ over $\mathcal{O}_F$. Then we have*

$$\mathrm{Den}_{\Delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N \rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2} N \rangle).$$

Moreover, if the lattice $M \oplus \langle N \rangle$ can't be isometrically embedded into the self-dual lattice H_{r+2}^{+}, then

$$\partial\,\mathrm{Den}_{\Delta_F(N)}(M) = \partial\,\mathrm{Den}(M \oplus \langle N \rangle) - \partial\,\mathrm{Den}(M \oplus \langle \pi^{-2} N \rangle).$$

Proof. Recall the definition of the polynomial $\mathrm{Nor}^{\varepsilon}(X, n)$ in Definition 2.2.6. We can verify immediately by formula (27) that, for any $x \in \mathcal{O}_F$,

$$\mathrm{Nor}^{+}(q^{-m}, r+1) = \begin{cases} \mathrm{Pden}(H_{2m+r+2}^{\varepsilon}, \langle x \rangle) \cdot \mathrm{Nor}^{\chi_F(x)}(q^{-m}, r) & \text{when } \pi \nmid x, \\ \mathrm{Pden}(H_{2m+r+2}^{\varepsilon}, \langle x \rangle) \cdot \mathrm{Nor}^{+}(q^{-m}, r-1) & \text{when } \pi \mid x. \end{cases}$$

Let $n = \nu_\pi(N)$. Theorem 7.2.1 and Example 7.1.2 imply the decomposition

$$\begin{aligned} \mathrm{Den}(H_{2m+r+2}^{+}, M \oplus \langle N \rangle) &= \sum_{i=0}^{[n/2]} q^{-2mi} \cdot \mathrm{Pden}(H_{2m+r+2}^{+}, \langle \pi^{-2i} N \rangle) \cdot \mathrm{Den}(H_{2m+r+2}^{+}(N, i), M) \\ &= \sum_{i=0}^{[n/2]} q^{-2mi} \cdot \mathrm{Pden}(H_{2m+r+2}^{+}, \langle \pi^{-2i} N \rangle) \cdot \mathrm{Den}(\Delta_F(\pi^{-2i} N) \oplus H_{2m+r-2}^{+}, M). \end{aligned}$$

By Definition 7.2.5, when p is odd, we have the formula

$$\mathrm{Den}(X, M \oplus \langle N \rangle) = \sum_{i=0}^{[n/2]} X^{2i} \cdot \mathrm{Den}_{\Delta_F(\pi^{-2i} N)}(X, M). \tag{28}$$

When $n = 0$ or 1, $\mathrm{Den}(X, M \oplus \langle N \rangle) = \mathrm{Den}_{\Delta_F(N)}(X, M)$ and $\mathrm{Den}(X, M \oplus \langle \pi^{-2} N \rangle) = 0$ since $\pi^{-2} N$ is not in $\mathcal{O}_F$. Therefore $\mathrm{Den}_{\Delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N \rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2} N \rangle)$. When $n \geq 2$, $\mathrm{Den}_{\Delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N \rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2} N \rangle)$ follows from the formula (28).

The fact that the lattice $M \oplus \langle N \rangle$ can't be isometrically embedded into the quadratic space H_{r+2}^{+} implies that $\mathrm{Den}(1, M \oplus \langle N \rangle) = \mathrm{Den}(1, M \oplus \langle \pi^{-2} N \rangle) = 0$. The second formula in the theorem follows from the first one and the definitions of the symbols $\partial\,\mathrm{Den}_{\Delta_F(N)}$ and $\partial\,\mathrm{Den}$. □

Now we apply Theorem 7.2.6 to the case that we are interested in, i.e., $F = \mathbb{Q}_p$ and $r = 2$. Let $N \in \mathbb{Z}_p$. We get a difference formula of local density functions:

$$\mathrm{Den}_{\Delta_p(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N \rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle p^{-2} N \rangle). \tag{29}$$

Note that the lattice $M \oplus \langle N \rangle$ is a sublattice of $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$, which is the unique division quaternion algebra over $\mathbb{Q}_p$. Hence the lattice $M \oplus \langle N \rangle$ can't be isometrically embedded into the quadratic space

$H_4^+ \otimes \mathbb{Q}_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$. Therefore, Theorem 7.2.6 implies the difference formula

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M) = \partial\,\mathrm{Den}(M \oplus \langle N\rangle) - \partial\,\mathrm{Den}(M \oplus \langle p^{-2}N\rangle) \tag{30}$$

of the derivatives of local densities.

7.3. ***Examples.*** Assume p is odd. In the following example, we compute an explicit example of local densities and compare our formulas with known formulas given in [Wedhorn 2007; Sankaran et al. 2023].

Example 7.3.1. Let $N = N_0$ be a positive integer with $\nu_p(N_0) = 0$ or 1. Let M be a rank-2 $\mathbb{Z}_p$-lattice such that M is isometrically embedded into $\mathbb{W}$ and is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent to the matrix $\mathrm{diag}\{\varepsilon_1 p^2, \varepsilon_2 p^3\}$, where $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_p^\times$. Let $N_k = p^{2k}N_0$, where N_0 is a positive integer with $\nu_p(N_0) = 0$ or 1 and $k \geq 1$ is an integer. By the formula in [Wedhorn 2007, §2.11],

$$\begin{aligned}\mathrm{Den}(X, M \oplus \langle N_k\rangle) &\\ = 1 + pX + (p+p^2)X^2 + p^2X^3 + p^2X^4 - p^2X^{2k+1+\nu_p(N_0)} - p^2X^{2k+2+\nu_p(N_0)} &\\ - (p+p^2)X^{2k+3+\nu_p(N_0)} - pX^{2k+4+\nu_p(N_0)} - X^{2k+5+\nu_p(N_0)} &\quad \text{when } k \geq 3.\end{aligned}$$

The formula (29) implies

$$\mathrm{Den}_{\Delta_p(N_k)}(X, M) = 1 + pX + (p^2+p-1)X^2 + (p^2-p)X^3 - pX^4 - p^2X^4 - p^2X^5 - p^2X^6$$

when $k \geq 3$. Therefore, $\partial\,\mathrm{Den}_{\Delta_p(N_0)}(M) = 2 + 4p + 6p^2$ when $k \geq 3$.

We double-check our results by comparing with the formulas of local density given in [Yang 1998, Theorem 7.1]. The theorem implies that for a sufficiently large positive integer m,

$$\mathrm{Den}(\Delta_p(N_k) \oplus H_{2m}^+, M) = 1 + R_{1,k}(X) + R_{2,k}(X)\big|_{X=p^{-m}},$$

where

$$R_{1,k}(X) = \sum_{i=1}^{8} I_{1,i,k}(X) \quad \text{and} \quad R_{2,k}(X) = (1-p^{-1})\sum_{i=1}^{8} I_{2,i,k}(X) + p^{-1}I_{2,6,k}(X).$$

$I_{1,i,k}(X)$ and $I_{2,i,k}(X)$ are polynomials explicitly constructed at the beginning of Section 7 of [Yang 1998]. In our case, when $k \geq 3$,

$$\begin{gathered}I_{1,1,k}(X) = (p-p^{-1})X + (p^2-1)X^2, \quad I_{1,2,k}(X) = -X^3, \quad I_{1,3,k}(X) = 0, \quad I_{1,4,k} = -p^2X^4,\\ I_{2,1,k}(X) = (p^2-1)X^3, \quad I_{2,3,k}(X) = I_{2,5,k}(X) = I_{2,7,k}(X) = 0,\\ I_{2,2,k}(X) = -pX^4 - pX^5, \quad I_{2,4,k}(X) = -p^2X^5 - p^2X^6, \quad I_{2,6,k}(X) = pX^7, \quad I_{2,8,k}(X) = pX^2 + p^2X^4.\end{gathered}$$

Therefore, when m is sufficiently large,

$$\begin{aligned}\mathrm{Den}(\Delta_p(N_k) \oplus H_{2m}^+, M) = 1 + (p-p^{-1})X + (p^2+p-2)X^2 + (p^2-2p+p^{-1}-1)X^3 &\\ - (2p-1)X^4 + (1-p^2)X^5 + (p-p^2)X^6 + pX^7\,\big|_{X=p^{-m}}.&\end{aligned}$$

By Definition 7.2.5, when $k \geq 3$,

$$\mathrm{Den}_{\Delta_p(N_k)}(X, M)\big|_{X=p^{-m}} = \frac{\mathrm{Den}(\Delta_p(N_k) \oplus H_{2m}^+, M)}{1-p^{-m-1}}$$
$$= 1+pX+(p^2+p-1)X^2+(p^2-p)X^3-pX^4-p^2X^4-p^2X^5-p^2X^6\big|_{X=p^{-m}}.$$

Hence,

$$\mathrm{Den}_{\Delta_p(N_k)}(X, M) = 1 + pX + (p^2 + p - 1)X^2 + (p^2 - p)X^3 - pX^4 - p^2X^4 - p^2X^5 - p^2X^6$$

when $k \geq 3$. This agrees with our previous calculations.

8. Proof of the arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$

8.1. *Local arithmetic Siegel–Weil formula with level N.* Let p be a prime number. The difference formulas at the analytic side and the geometric side are combined to prove the following theorem.

Theorem 8.1.1. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\,\mathrm{Den}_{\Delta_p(N)}(M). \tag{31}$$

Proof. Theorem 6.4.1 gives the difference formula of local arithmetic intersection numbers

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).$$

We also have the difference formula of the derived local densities (see (30))

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M) = \partial\,\mathrm{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial\,\mathrm{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).$$

Theorem 6.3.4 implies that $\mathrm{Int}^\sharp(L) = \partial\,\mathrm{Den}(L)$ for any rank-3 lattice $L \subset \mathbb{B}$. Therefore (31) holds by combining the above two difference formulas. □

8.2. *Intersection numbers and Whittaker functions.* Let p be a prime number.

Proposition 8.2.1. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$W_T'(1, 0, 1_{\Delta_p(N)^2}) = c_p \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(M) \cdot \log(p), \tag{32}$$

where the constant c_p is given as

$$c_p = \begin{cases} (1-p^{-1}) \cdot (N, -1)_p \cdot |N|_p \cdot |2|_p^{3/2} & \text{when } p \mid N, \\ (1-p^{-2}) \cdot (N, -1)_p \cdot |N|_p \cdot |2|_p^{3/2} & \text{when } p \nmid N. \end{cases}$$

Proof. Recall that $\Delta_p(N)^\vee/\Delta_p(N) \simeq \mathbb{Z}_p/2N\mathbb{Z}_p$ (see Example 2.1.1). By Proposition 3.3.1 and the explicit formula given in the appendix of [Ranga Rao 1993],

$$W_T(1, k, 1_{\Delta_p(N)^2}) = |2N|_p \cdot \gamma(\Delta_p(N) \otimes \mathbb{Q}_p)^2 \cdot |2|_p^{1/2} \cdot \mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M)$$
$$= |N|_p \cdot (N, -1)_p \cdot |2|_p^{3/2} \cdot \mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M). \tag{33}$$

Taking derivatives of both sides of (33),

$$W_T'(1, 0, 1_{\Delta_p(N)^2}) = c_p \cdot \partial\,\mathrm{Den}_{\Delta_p(N)}(M) \cdot \log(p).$$

The formula (32) follows from Theorem 8.1.1. □

8.3. *Proof of the main theorem.*

Proposition 8.3.1. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a positive definite symmetric matrix. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Suppose $\boldsymbol{\varphi} = \varphi_1 \times \varphi_2$, where $\varphi_i \in \mathscr{S}(\mathbb{V}_f)$. Then for any $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, we have*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi})) = \begin{cases} \chi(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}) \cdot \log(p) & \text{when } \mathrm{Diff}(T, \Delta(N)) = \{p\}, \\ 0 & \text{when } \#\,\mathrm{Diff}(T, \Delta(N)) \neq 1. \end{cases}$$

Proof. By definition (see (15)), the arithmetic special cycle $\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi})$ is $([\mathcal{Z}(T, \boldsymbol{\varphi})], 0)$. Therefore $\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))$ is independent of y. We can assume $\mathrm{Diff}(T, \Delta(N)) = \{p\}$ for some prime number p since otherwise both sides are 0 since $\mathcal{Z}(T, \boldsymbol{\varphi})$ would be an empty stack.

Let $x \in \mathcal{Z}(T, \boldsymbol{\varphi})(\overline{\mathbb{F}}_p)$ be a geometric point. It is contained in $\mathcal{Y}_0(N)$ by Corollary 4.3.8, and hence the special divisors $\mathcal{Z}(t_1, \varphi_1)$ and $\mathcal{Z}(t_2, \varphi_2)$ intersect properly at x because T is nonsingular. Then $\chi(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}) \cdot \log(p)$ is the sum of the length of local rings $\mathcal{O}_{\mathcal{X}_0(N),x}$ cut out by these two divisors times $\log(p)$, which is exactly $\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))$ by definition of the degree homomorphism. □

Proof of Theorem 4.4.1. We first consider the case that T is positive definite. By Proposition 4.3.9, we only need to consider the case $\mathrm{Diff}(T, \Delta(N)) = \{p\}$ for some prime number p because otherwise both sides are 0. The same proposition and Corollary 4.3.8 imply that the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ lies in the supersingular locus of $\mathcal{X}_0(N)_{\mathbb{F}_p}$. Then by the definition of special cycles and the formal uniformization of the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ (see Corollary 5.3.3),

$$\begin{aligned} &\chi(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}) \cdot \log(p) \\ &\qquad = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p) / \Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(\boldsymbol{x}) \cdot \log(p). \end{aligned}$$

It is known (see (32)) that

$$W_T'(1, 0, 1_{\Delta_p(N)^2}) = c_p \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(\boldsymbol{x}) \cdot \log(p),$$

with constants c_p given by Proposition 8.2.1.

There exists a Haar measure on $\mathrm{GL}_2(\mathbb{A}_f^p)$ such that

$$\sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p) / \Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) = \frac{1}{\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}}^p))} \cdot \int_{\mathrm{SO}(\Delta(N)^{(p)})(\mathbb{A}_f^p)} \boldsymbol{\varphi}^p(g^{-1}\boldsymbol{x})\, \mathrm{d}g.$$

By definition, the last integral is a product of "local" integrals

$$\int_{\mathrm{SO}(\Delta(N)^{(p)})(\mathbb{A}_f^p)} \boldsymbol{\varphi}^p(g^{-1}\boldsymbol{x})\, \mathrm{d}g = \prod_{v \neq p, \infty} \int_{\mathrm{SO}(\Delta_v(N))(\mathbb{Q}_v)} \boldsymbol{\varphi}_v(g_v^{-1}\boldsymbol{x})\, \mathrm{d}g_v.$$

By the classical local Siegel–Weil formula, made explicit in [Kudla et al. 2006, Proposition 5.3.3], for every place v of $\mathbb{Q}$ there exists a number $d_v \in \mathbb{R}^\times$ such that

$$\int_{\mathrm{SO}(\Delta_v(N))(\mathbb{Q}_v)} \boldsymbol{\varphi}_v(g_v^{-1}\boldsymbol{x})\,\mathrm{d}g_v = d_v \cdot W_{T,v}(1,0,\boldsymbol{\varphi}_v),$$

with $\prod_{v\le\infty} d_v = 1$. Moreover, [Kudla et al. 2006, Lemma 5.3.9] implies

$$\mathrm{vol}(\Gamma_0(N)_v, dg_v) = d_v \cdot \gamma(\Delta_v(N))^2 \cdot |2|_v^{3/2} \cdot \begin{cases} (1-v^{-2}) & \text{when } v \nmid N, \\ |N|_v^{-1}(1+v^{-1}) & \text{when } v \mid N. \end{cases}$$

It can be checked immediately that

$$\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}}^p)) \cdot d_p d_\infty \cdot c_p = 2^{-1/2}\psi(N)^{-1} \cdot \frac{3}{\pi^2}.$$

Suppose $\mathsf{z} = \mathsf{x} + i\mathsf{y}$. It's a classical result that

$$W_{T,\infty}(g_{\mathsf{z}}, 0, \Phi_\infty^{3/2}) = -2^{7/2}\pi^2 \cdot \det(\mathsf{y})^{3/4} q^T.$$

Combining these with the definitions made in previous sections (see (6) and (8)) and Proposition 8.3.1, we get the formula stated in the theorem.

When T is not positive definite, the equality follows from [Sankaran et al. 2023, §4.2] and our computations of the volume of $\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}})) = \prod_{v<\infty} \mathrm{vol}(\Gamma_0(N)_v, dg_v)$ above. $\square$

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bz2393@columbia.edu *Columbia University, New York, NY, United States*

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