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Metaplectic cusp forms and the large sieve

Alexander Dunn





# Metaplectic cusp forms and the large sieve

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*Dedicated to Chantal David on the occasion of her 60th birthday.*

We prove a power saving upper bound for the sum of Fourier coefficients  $\rho_f(\cdot)$  of a fixed cubic metaplectic cusp form  $f$  over primes. Our result is the cubic analogue of a celebrated 1990 theorem of Duke and Iwaniec, and the cuspidal analogue of a theorem due to the author and Radziwiłł for the bias in cubic Gauss sums.

The proof has two main inputs, both of independent interest. Firstly, we prove a new large sieve estimate for a bilinear form whose kernel function is  $\rho_f(\cdot)$ . The proof of the bilinear estimate uses a number field version of circle method due to Browning and Vishe, Voronoi summation, and Gauss–Ramanujan sums. Secondly, we use Voronoi summation and the cubic large sieve of Heath-Brown to prove an estimate for a linear form involving  $\rho_f(\cdot)$ . Our linear estimate overcomes a bottleneck occurring at level of distribution  $\frac{2}{3}$ .

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## 1. Introduction

**1.1. Background and statement of results.** Arithmetic functions that arise from the Fourier coefficients of automorphic forms on congruence subgroups of  $\mathrm{SL}_2(\mathbb{Z})$  encode deep arithmetic and analytic information. A famous example is the modularity theorem for elliptic curves  $E/\mathbb{Q}$  [Breuil et al. 2001], and its resolution of the Hasse–Weil conjecture for such curves.

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At a fundamental level, automorphic forms on congruence subgroups of  $SL_2(\mathbb{Z})$  are nice objects because there is an “adequate Hecke theory” available. By this, we mean the basic property that the sequence of Fourier coefficients of an integer weight cusp form restricted to values coprime to the level can be expressed as a linear combination of multiplicative functions given by the Hecke eigenvalues! It is well known that a power saving upper bound for the sum of Hecke eigenvalues  $\lambda_g(\cdot)$  over primes would yield a rectangular zero-free region in the critical strip for associated  $L$ -function  $L(s, g)$  (thanks to the Euler product). Unfortunately, the proof of such a bound is well out of reach of current technology!

The Fourier coefficients of half-integer weight modular forms also play a key role in arithmetic. An important example is the use of Dedekind’s  $\eta$ -function (holomorphic cusp form of weight  $\frac{1}{2}$  on  $SL_2(\mathbb{Z})$ ) in the proof of Rademacher’s formula [1937] for the partition function  $p(n)$ . Hecke [1983, p. 639; 1944] observed that there is not an “adequate Hecke theory” (in the naive sense above) for modular forms of half-integer weight. Wohlfahrt [1957] confirmed Hecke’s observations and essentially showed that there is an algebra of Hecke operators  $\mathbb{C}[\{T_{n^2}\}_{n=1}^\infty]$  acting on half-integer weight modular forms of weight  $k$  such that  $T_{m^2} \circ T_{n^2} = T_{m^2 n^2} = T_{n^2} \circ T_{m^2}$  for  $(m, n) = 1$ ,  $T_{p^{2a}}$  is a polynomial in  $T_{p^2}$  for each  $a \in \mathbb{Z}_{\geq 1}$  and odd prime  $p$ , and that each Hecke operator is Hermitian (on cusp forms) with respect to the standard Petersson inner product. In general, the Fourier coefficients of half-integer weight Hecke eigenforms at general integer indices are not multiplicative, unless they are squares! In foundational works, Shimura [1973] and Kohnen and Zagier [1981] studied this phenomenon in more detail. For a comprehensive summary of the theory, the reader can consult [Koblitz 1984, §4.3].

Duke and Iwaniec [1990] gave striking quantitative evidence that the Fourier coefficients of half-integer weight holomorphic cusp forms along squarefree integers are not multiplicative (unless their values are zero). In particular, suppose that  $g$  is a holomorphic cusp form on  $\Gamma_0(N)$  ( $N \equiv 0 \pmod{4}$ ) having weight  $k = \frac{1}{2} + 2\ell$ ,  $\ell \in \mathbb{Z}_{\geq 2}$ , and Fourier expansion (at  $\infty$ )

$$g(z) = \sum_{n=1}^\infty c_g(n) n^{(k-1)/2} e(nx) e^{-2\pi ny}, \quad z = x + iy \in \mathbb{H}, \tag{1-1}$$

where  $c_g(n) \in \mathbb{C}$ ,  $e(x) := e^{2\pi ix}$  for all  $x \in \mathbb{R}$ , and  $\mathbb{H} := \mathbb{R} \times \mathbb{R}_{>0}$  is the complex upper-half plane. For  $\varepsilon > 0$  and  $A, B \geq 10$ , Duke and Iwaniec [1990] proved that

$$\sum_{a \leq A} \sum_{b \leq B} \mu^2(a) \alpha_a \beta_b c_g(ab) \ll_{\varepsilon, g} (AB)^\varepsilon (B^{1/2} + AB^{1/4}) \|\alpha\|_2 \|\beta\|_2, \tag{1-2}$$

where  $\alpha, \beta$  are  $\mathbb{C}$ -valued sequences and  $\|\cdot\|$  denotes the usual  $\ell_2$ -norm. Using (1-2) together with appropriate linear estimates, Duke and Iwaniec [1990] also proved that

$$\sum_{\substack{p \leq X \\ p \text{ prime}}} c_g(p) \ll_{\varepsilon, g} X^{1-1/156+\varepsilon} \tag{1-3}$$

as  $X \rightarrow \infty$ . The result in (1-3) allows for twists by primitive characters of conductor divisible by  $N \equiv 0 \pmod{4}$ , and so one can restrict to sum to primes in an arithmetic progression (with the implied constant depending on the modulus).

The goal of this paper is to generalise the results of Duke and Iwaniec to cusp forms on the cubic metaplectic cover of  $GL_2$  (in the sense of Kubota [1969; 1971]). This is the complementary case to work in [Dunn and Radziwiłł 2024] on Patterson’s conjecture for the bias of cubic Gauss sums over primes (cubic Gauss sums are the Fourier coefficients of the cubic theta function [Patterson 1977] which is noncuspidal). The spectral theory of cubic metaplectic forms have played a key role in [Livné and Patterson 2002; Louvel 2014], on the distribution of certain cubic exponential sums. In their PhD thesis, Möhring [2004] numerically investigated the Fourier coefficients of some cuspidal cubic metaplectic forms.

Before stating our results we briefly introduce some notation. Let  $\mathbb{H}^3 := \mathbb{C} \times \mathbb{R}_{>0}$  denote hyperbolic 3-space. Let  $\omega = e^{2\pi i/3}$ , and  $\mathbb{Q}(\omega)$  denote the Eisenstein quadratic field (class number 1). This number field has ring of integers  $\mathbb{Z}[\omega]$ , discriminant  $-3$ , and the unique ramified prime is  $\lambda := \sqrt{-3} = 1 + 2\omega$ . Let  $\left(\frac{\cdot}{c}\right)_3$  denote the cubic symbol over  $\mathbb{Z}[\omega]$ , and  $\Lambda(c)$  denote the usual von Mangoldt function on  $\mathbb{Z}[\omega]$ . Consider the following congruence subgroups of  $SL_2(\mathbb{C})$ :

$$\begin{aligned} \Gamma &:= SL_2(\mathbb{Z}[\omega]), \\ \Gamma_1(3) &:= \{\gamma \in \Gamma : \gamma \equiv I \pmod{3}\}, \\ \Gamma_2 &:= \langle SL_2(\mathbb{Z}), \Gamma_1(3) \rangle. \end{aligned}$$

The cubic Kubota [1969; 1971] character  $\chi : \Gamma_1(3) \rightarrow \{1, \omega, \omega^2\}$  is defined by

$$\chi(\gamma) := \begin{cases} \left(\frac{c}{a}\right)_3 & \text{if } c \neq 0, \\ 1 & \text{if } c = 0, \end{cases} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(3), \tag{1-4}$$

and extends to a well-defined homomorphism  $\chi : \Gamma_2 \rightarrow \{1, \omega, \omega^2\}$  when one defines  $\chi|_{SL_2(\mathbb{Z})} \equiv 1$  [Patterson 1977, §2]. The group  $\Gamma_2$  is the lowest possible level for cubic metaplectic forms. Let  $f$  be a cuspidal cubic metaplectic form on  $\Gamma_2$ , i.e.,

- $f$  vanishes at all cusps of  $\Gamma_2$ ;
- $f(\gamma w) = \chi(\gamma)f(w)$  for all  $\gamma \in \Gamma_2$  and  $w \in \mathbb{H}^3$ ;
- $f$  is an eigenfunction of the hyperbolic Laplacian:  $\Delta f = -\tau_f(2 - \tau_f)$  for some  $\tau_f \in \mathbb{C}$ .

There is an algebra of Hecke operators  $\mathbb{C}[\{T_{v^3}\}_{v \in \mathbb{Z}[\omega] \setminus \{0\}}]$  acting on cubic metaplectic forms such that  $T_{\mu^3} \circ T_{v^3} = T_{\mu^3 v^3} = T_{v^3} \circ T_{\mu^3}$  for  $(\mu, v) = 1$ ,  $T_{\varpi^{3a}}$  is a polynomial in  $T_{\varpi^3}$  for each  $a \in \mathbb{Z}_{\geq 1}$  and prime  $\varpi \equiv 1 \pmod{3}$ , and that each Hecke operator is Hermitian (on cusp forms) with respect to the standard Petersson inner product [Proskurin 1998, §0.3.12]. The Fourier expansion of  $f$  (at  $\infty$ ) is given by

$$f(w) = \sum_{\substack{v \neq 0 \\ v \in \lambda^{-3}\mathbb{Z}[\omega]}} \rho_f(v) v K_{\tau_f-1}(4\pi|v|v)\check{\rho}(vz), \quad w = (z, v) \in \mathbb{H}^3, \tag{1-5}$$

where  $K_\alpha(\cdot)$  is the standard  $K$ -Bessel function of order  $\alpha \in \mathbb{C}$ ,  $\check{\rho}(z) := e^{2\pi i(z+\bar{z})}$  for  $z \in \mathbb{C}$ , and  $\rho_f(v) \in \mathbb{C}$ .

**Remark 1.1.** The cubic Shimura lift of Patterson [1998, Theorem 3.4] guarantees that one always has  $\tau_f \in 1 + i\mathbb{R}$  for cuspidal cubic metaplectic forms  $f$  on  $\Gamma_2$  (see Section 3.2).

Let  $K, M \geq 1$ , and  $W_{K,M} : (0, \infty) \rightarrow \mathbb{C}$  be a smooth function with compact support in  $[1, 2]$  such that for each  $j \in \mathbb{Z}_{\geq 0}$  we have

$$W_{K,M}^{(j)}(x) \ll_j MK^j \quad \text{for all } x > 0. \tag{1-6}$$

If  $M = 1$  then  $M$  is omitted from the notation, and we write  $W_K$ . Let  $\|\bullet\|_q$  with  $1 \leq q \leq \infty$  denote the  $\ell_q$ -norm of a  $\mathbb{C}$ -valued sequence indexed over elements of  $\mathbb{Z}[\omega]$ .

The main sums of interest in this paper are

$$\mathcal{P}_f(X, v, u; W_K) := \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv u \pmod{v}}} \rho_f(v) \Lambda(\lambda^3 v) W_K\left(\frac{N(v)}{X}\right), \tag{1-7}$$

$$\mathcal{P}_f(X, v, u) := \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv u \pmod{v} \\ N(v) \leq X}} \rho_f(v) \Lambda(\lambda^3 v), \tag{1-8}$$

$$\widetilde{\mathcal{P}}_f(X, v, u) := \sum_{\substack{\varpi \in \mathbb{Z}[\omega] \\ \varpi \text{ prime} \\ \varpi \equiv u \pmod{v} \\ N(\lambda^{-3}\varpi) \leq X}} \rho_f(\lambda^{-3}\varpi) \log N(\varpi), \tag{1-9}$$

where  $0 \neq v \in \mathbb{Z}[\omega]$  is such that  $v \equiv 0 \pmod{3}$ , and  $u \in \mathbb{Z}[\omega]/v\mathbb{Z}[\omega]$  is such that  $(u, v) = 1$  and  $u \equiv 1 \pmod{3}$ . It is technically convenient to restrict attention to  $u \equiv 1 \pmod{3}$ . The other congruence classes modulo 3 can be treated by a mild adaption of the methods of this paper.

**Theorem 1.2.** *Let  $\varepsilon > 0$  and the notation be as above. Then*

$$\mathcal{P}_f(X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^\varepsilon K^8 N(v)^4 X^{1-1/34}$$

as  $X \rightarrow \infty$ .

**Corollary 1.3.** *Let  $\varepsilon > 0$ . In the notation above we have*

$$\mathcal{P}_f(X, v, u) \ll_{\varepsilon, f, v} X^{1-1/578+\varepsilon}, \tag{1-10}$$

$$\widetilde{\mathcal{P}}_f(X, v, u) \ll_{\varepsilon, f, v} X^{1-1/578+\varepsilon} \tag{1-11}$$

as  $X \rightarrow \infty$ .

Theorem 1.2 follows from new estimates for linear and bilinear sums which we now describe. A brief sketch of the new difficulties and ideas that arise in our case (as opposed to the case in [Duke and Iwaniec 1990]) is given in Section 1.2. Let

$$\mathcal{T}_f(a, X, v, u; W_K) := \sum_{\substack{b \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \rho_f(\lambda^{-3}ab) W_K\left(\frac{N(\lambda^{-3}ab)}{X}\right) \tag{1-12}$$

denote the pointwise Type-I sum, where  $X \geq 10$  and  $a \in \mathbb{Z}[\omega]$  with  $a \equiv 1 \pmod{3}$ . Let

$$\mathcal{A}_f(\alpha, X, v, u; W_K) := \sum_{\substack{a, b \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \mu^2(a) \alpha_a \rho_f(\lambda^{-3}ab) W_K\left(\frac{N(\lambda^{-3}ab)}{X}\right) \tag{1-13}$$

denote the average (over squarefree  $a$ ) Type-I sum, where  $A, X \geq 10$  and  $\alpha := (\alpha_a)$  is a  $\mathbb{C}$ -valued sequence supported on  $a \in \mathbb{Z}[\omega]$  with  $a \equiv 1 \pmod{3}$  and  $N(a) \asymp A$ . Let

$$\mathcal{B}_f(\alpha, \beta, X, v, u; W_K) := \sum_{\substack{a, b \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \mu^2(a) \alpha_a \beta_b \rho_f(\lambda^{-3} ab) W_K \left( \frac{N(\lambda^{-3} ab)}{X} \right) \tag{1-14}$$

denote the Type-II sum, where  $A, B \geq 10$ ,  $(\alpha_a)$  is as above, and  $\beta := (\beta_b)$  is a  $\mathbb{C}$ -valued sequence supported on  $b \in \mathbb{Z}[\omega]$  with  $N(b) \asymp B$ . Note that we necessarily have  $X \asymp AB$  in (1-14), otherwise the double sum is empty.

In Section 9 we use Voronoi summation to prove the following ‘‘trivial’’ pointwise Type-I bound.

**Lemma 1.4.** *Let  $\varepsilon > 0$  and the notation be as above. Then*

$$\mathcal{T}_f(a, X; v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^\varepsilon K^4 N(v)^{1/2} N(a)^{1/2}.$$

When  $\mathcal{T}_f(a, \dots)$  is multiplied by a weight  $\alpha_a$  and the estimate in Lemma 1.4 is summed trivially over  $a \in \mathbb{Z}[\omega]$  with  $N(a) \asymp A$ , the resulting bound is acceptable when  $A \ll X^{2/3-\varepsilon}$ .

In Section 10 we use the circle method to prove the following new bilinear estimate.

**Theorem 1.5.** *Let  $\varepsilon > 0$  and the notation be as above. Then for  $A, B \geq 10$  and  $X \asymp AB$  we have*

$$\mathcal{B}_f(\alpha, \beta, X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^\varepsilon K^8 N(v)^4 ((AB)^{1/2} + A^{3/2} B^{1/4}) \|\mu^2 \alpha\|_\infty \|\beta\|_2.$$

Theorem 1.5 is acceptable when  $\|\mu^2 \alpha\|_\infty \ll A^\varepsilon$  and  $X^{2/3+\varepsilon} \ll B \ll X^{1-\varepsilon}$ .

We point out that Lemma 1.4 and Theorem 1.5 together *barely* misses primes. To overcome the bottleneck at level of distribution  $\asymp X^{2/3}$ , we use Voronoi summation and Heath-Brown’s cubic large sieve [2000] to prove the following estimate.

**Proposition 1.6.** *Let  $\varepsilon > 0$  and the notation be as above. Then for  $X, A \geq 10$  we have*

$$\mathcal{A}_f(\alpha, X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^\varepsilon K^{14/3} N(v)^{5/6} (AX)^{1/3} \|\mu^2 \alpha\|_2.$$

**1.2. Brief sketch of the method.** We close with a brief outline of the proofs of Theorem 1.5 and Proposition 1.6. For simplicity, we suppress smooth functions, and ignore both the units of  $\mathbb{Z}[\omega]$  and the congruence condition  $u \pmod{v}$ .

**1.2.1. Linear sums.** We apply Voronoi summation to the  $b$ -sum in (1-13) and perform a computation with the arithmetic exponential sums that appear on the dual side. We obtain a bilinear form

$$\frac{X}{A^2} \sum_{N(a) \asymp A} \sum_{N(v) \ll A^2/X} \mu^2(a) \overline{g(a)} \alpha_a \rho_f(v) \left( \frac{v}{a} \right)_3, \tag{1-15}$$

where  $g(a)$  denotes the unnormalised cubic Gauss sum over  $\mathbb{Z}[\omega]$  with modulus  $a$ . The use of Heath-Brown’s cubic large sieve [2000] (with the squarefree condition on one variable relaxed) leads to our average Type-I estimate.

**1.2.2. Bilinear sums.** After application of Cauchy–Schwarz in the  $b$ -variable to (1-14), the sum of interest is

$$\sum_{\substack{N(a_1), N(a_2) \asymp A \\ a_1, a_2 \equiv 1 \pmod{3}}} \mu^2(a_1) \alpha_{a_1} \mu^2(a_2) \overline{\alpha_{a_2}} \sum_{N(b) \sim B} \rho_f(a_1 b) \overline{\rho_f(a_2 b)}. \tag{1-16}$$

The natural approach would be to ignore the averaging over  $a_1$  and  $a_2$ , and estimate each convolution sum  $\sum_{N(b) \sim B} \rho_f(a_1 b) \overline{\rho_f(a_2 b)}$  directly. Duke and Iwaniec [1990] proved that each convolution sum is  $\ll_\varepsilon \delta_{a_1=a_2} B + (AB)^\varepsilon AB^{1/2}$  for the case of holomorphic half-integer weight cusp forms. We explain below why the additional averaging over  $a_1$  and  $a_2$  is crucial in the Maass case.

The initial move of [Duke and Iwaniec 1990] is to open one of the Fourier coefficients in terms of sums of half-integer weight Kloosterman sums that come from writing the holomorphic cusp form as a finite  $\mathbb{C}$ -linear combination of Poincaré series. This opening move is not available for Maass forms! Instead, we separate oscillations using the circle method of Browning and Vishe [2014] to obtain

$$\sum_{N(b) \sim B} \rho_f(a_1 b) \overline{\rho_f(a_2 b)} \approx \frac{1}{B} \sum_{\substack{N(v_1), N(v_2) \asymp AB \\ v_1 \equiv 0 \pmod{a_1} \\ v_2 \equiv 0 \pmod{a_2}}} \rho_f(v_1) \overline{\rho_f(v_2)} \sum_{\substack{N(c) \sim B^{1/2} \\ (c, \lambda a_1 a_2) = 1}} r(v_1/a_1 - v_2/a_2, c), \tag{1-17}$$

where  $r(n, c)$  denotes the unnormalised Ramanujan sum over  $\mathbb{Z}[\omega]$  with modulus  $c$  and shift  $n$ . In reality, one must also consider moduli  $c$  that are not coprime to  $\lambda a_1 a_2$ . This can be handled with a modification of the method below with an additional local computation involving cubic Gauss sums with moduli dividing  $\text{rad}(a_1 a_2)^\infty$ .

We detect the congruence conditions on the  $v_1, v_2$  using additive characters, apply Voronoi summation to each  $v_1, v_2$  sum, and perform a considerable computation with the exponential sums on the dual side. This leads to an expression of the shape

$$\sum_{\substack{s_1 | a_1 \\ s_2 | a_2}} \frac{1}{N(s_1 s_2)^{1/2}} \sum_{\substack{N(v_1) \ll N(s_1)^2/A \\ N(v_2) \ll N(s_2)^2/A \\ (v_1, s_1) = 1 \\ (v_2, s_2) = 1}} \rho_f(v_1) \overline{\rho_f(v_2)} \sum_{\substack{N(c) \sim B^{1/2} \\ (c, \lambda a_1 a_2) = 1}} r(s_2^2 a_1 v_1 - s_1^2 a_2 v_2, c). \tag{1-18}$$

We highlight that the squarefree property of  $a_1$  and  $a_2$  simplifies the computations considerably. One can apply Cauchy–Schwarz and Rankin–Selberg bounds to estimate the off-diagonal ( $s_2^2 a_1 v_1 \neq s_1^2 a_2 v_2$ ) contribution in (1-18) by  $(AB)^\varepsilon AB^{1/2}$ . The diagonal term is more subtle. The diagonal equation  $s_2^2 a_1 v_1 = s_1^2 a_2 v_2$  is equivalent to  $s_2(a_1/s_1)v_1 = s_1(a_2/s_2)v_2$ . The conditions  $(v_1, s_1) = (v_2, s_2) = 1$  together with the squarefree hypothesis on  $a_1$  and  $a_2$  imply that  $s_1 = s_2 =: s \mid (a_1, a_2)$ . Thus the diagonal contribution in (1-18) has the shape

$$B \sum_{s | (a_1, a_2)} \frac{1}{N(s)} \sum_{\substack{N(v) \ll N(s)^3 N((a_1/s, a_2/s))/A^2 \\ (v, s) = 1}} \rho_f\left(\frac{a_2/s}{(a_1/s, a_2/s)} v\right) \overline{\rho_f\left(\frac{a_1/s}{(a_1/s, a_2/s)} v\right)}. \tag{1-19}$$

At this point there is no cancellation to be realistically exploited in (1-19), and so we apply the triangle inequality and place absolute values around the Fourier coefficients. It is tempting to apply a “Deligne-type” bound for  $\rho_f(\cdot)$  to estimate the diagonal by  $(AB)^\varepsilon \cdot B \cdot (N((a_1, a_2))/A)^2$  (which is of acceptable size). However, no such bound for  $\rho_f(\cdot)$  is known unconditionally, and the author is not aware of *any* nontrivial bound for  $\rho_f(\cdot)$  stronger than the bound implied by Rankin–Selberg. There is no “Waldspurger-type” formula known for the coefficients of cubic metaplectic cusp forms (on  $GL_2$ ). Hence the strategy for bounding these Fourier coefficients via subconvexity for twisted  $L$ -values is not available (this strategy is used the half-integer weight case; see [Conrey and Iwaniec 2000]). To overcome this, we substitute (1-19) into (1-16), take absolute values and the supremum norm of the  $\alpha$  terms, and exploit the additional averaging over  $a_1$  and  $a_2$  using Cauchy–Schwarz and Rankin–Selberg bounds. This yields the acceptable estimate  $(AB)^\varepsilon AB \|\mu^2 \alpha\|_\infty^2$  for the diagonal of the averaged sum. It is interesting to note that an argument of Nelson [2020] could potentially be adapted to estimate the sparse convolution sum in (1-19). We refrain from this additional work.

**1.3. Conventions.** For  $n \in \mathbb{N}$  and  $N > 0$ , we use  $n \sim N$  to mean  $N < n \leq 2N$ , and  $n \asymp N$  to mean that there exists constants  $c_1, c_2 > 0$  such that  $c_1 N \leq n \leq c_2 N$ .

Dependence of implied constants on parameters will be indicated in statements of results, but suppressed throughout the body of the paper (i.e., proofs). Implied constants in the body of the paper are allowed to depend on  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$ ,  $\varepsilon, D > 0$  (possibly different in each instance), and the implicit constants in the statements  $N(a) \asymp A$  and  $N(b) \asymp B$ .

Whenever we write  $r \mid q$  with  $0 \neq r, q \in \mathbb{Z}[\omega]$  and  $q \equiv 1 \pmod{3}$ , it is our convention that  $r \equiv 1 \pmod{3}$ . For any integer  $b$  we let  $\mathbb{Z}_{\geq b} := \{n \in \mathbb{Z} : n \geq b\}$ .

Unless otherwise specified, it should be clear from context whether  $\bar{x}$  means modular inverse (with respect to an appropriate modulus) or complex conjugation.

Unless otherwise specified, it should be clear from context whether  $v$  refers to the modulus of an arithmetic progression or the real component of a quaternion element  $w = (z, v)$ .

## 2. Preliminaries and background

**2.1. Eisenstein quadratic field and cubic Gauss sums.** We include some brief background on  $\mathbb{Q}(\omega)$  and cubic Gauss sums. For more details see [Dunn and Radziwiłł 2024; Patterson 1977; Proskurin 1998].

Let  $\mathbb{Q}(\omega)$  be the Eisenstein quadratic number field, where  $\omega$  is identified with  $e^{2\pi i/3} \in \mathbb{C}$ . This quadratic number field has ring of integers  $\mathbb{Z}[\omega]$ , discriminant  $-3$ , and class number 1. Let  $N(x) := N_{\mathbb{Q}(\omega)/\mathbb{Q}}(x) = |x|^2$  denote the norm form on  $\mathbb{Q}(\omega)/\mathbb{Q}$ . The dual of  $\mathbb{Z}[\omega]$  is

$$\mathbb{Z}[\omega]^* := \{z \in \mathbb{C} : \check{z}(zz') = 1 \text{ for all } z' \in \mathbb{Z}[\omega]\} = \lambda^{-1} \mathbb{Z}[\omega].$$

It is well known that any nonzero element of  $\mathbb{Z}[\omega]$  can be uniquely written as  $\zeta \lambda^k c$  with  $\zeta \in \langle -\omega \rangle$  a unit (i.e.,  $\zeta^6 = 1$ ),  $\lambda := \sqrt{-3} = 1 + 2\omega$  the unique ramified prime in  $\mathbb{Z}[\omega]$ ,  $k \in \mathbb{Z}_{\geq 0}$ , and  $c \in \mathbb{Z}[\omega]$  with  $c \equiv 1 \pmod{3}$ . If  $p \equiv 1 \pmod{3}$  is a rational prime, then  $p = \varpi \bar{\varpi}$  in  $\mathbb{Z}[\omega]$  with  $N(\varpi) = p$  and  $\bar{\varpi}$

a prime in  $\mathbb{Z}[\omega]$ . If  $p \equiv 2 \pmod{3}$  is a rational prime, then  $p = \varpi$  is inert in  $\mathbb{Z}[\omega]$ , and  $N(\varpi) = p^2$ . Thus we have  $N(\varpi) \equiv 1 \pmod{3}$  for all primes  $\varpi$  with  $(\varpi) \neq (\lambda)$ .

The cubic Jacobi symbol defined for  $a \equiv 1 \pmod{3}$  and  $\varpi \equiv 1 \pmod{3}$  prime is defined by

$$\left(\frac{a}{\varpi}\right)_3 \equiv a^{(N(\varpi)-1)/3} \pmod{\varpi},$$

and the condition it take values in  $\{1, \omega, \omega^2\}$ . The cubic symbol is clearly multiplicative in  $a$  and can be extended multiplicatively in  $b$  by setting

$$\left(\frac{a}{b}\right)_3 = \prod_i \left(\frac{a}{\varpi_i}\right)$$

for any  $b = \prod_i \varpi_i$  with  $\varpi_i \equiv 1 \pmod{3}$  primes. The cubic symbol obeys cubic reciprocity: given  $a, b \equiv 1 \pmod{3}$  we have

$$\left(\frac{a}{b}\right)_3 = \left(\frac{b}{a}\right)_3. \tag{2-1}$$

There are also supplementary laws for units and the ramified prime. Given

$$d \equiv 1 + \alpha_2 \lambda^2 + \alpha_3 \lambda^3 \pmod{9} \quad \text{with} \quad \alpha_2, \alpha_3 \in \{-1, 0, 1\}, \tag{2-2}$$

we have

$$\left(\frac{\omega}{d}\right)_3 = \omega^{\alpha_2} \quad \text{and} \quad \left(\frac{\lambda}{d}\right)_3 = \omega^{-\alpha_3}. \tag{2-3}$$

We follow the standard convention for an empty product,

$$\left(\frac{a}{1}\right)_3 = 1 \quad \text{for all} \quad a \in \mathbb{Z}[\omega]. \tag{2-4}$$

Let

$$\check{e}(z) := e^{2\pi i \operatorname{Tr}_{\mathbb{C}/\mathbb{R}}(z)} = e^{2\pi i(z+\bar{z})}, \quad z \in \mathbb{C}.$$

For  $\mu \in \mathbb{Z}[\omega]^* = \lambda^{-1}\mathbb{Z}[\omega]$  and  $c \in \mathbb{Z}[\omega]$  with  $c \equiv 1 \pmod{3}$ , the cubic Gauss sum (with shift  $\mu$ ) is defined by

$$g(\mu, c) := \sum_{d \pmod{c}} \left(\frac{d}{c}\right)_3 \check{e}\left(\frac{\mu d}{c}\right). \tag{2-5}$$

We write  $g(c) := g(1, c)$  for short. Making a change of variable in the Gauss sum we see that

$$g(\mu, c) := \left(\frac{\lambda}{c}\right)_3 g(\lambda\mu, c), \tag{2-6}$$

and so for the rest of this section it suffices to consider only  $\mu \in \mathbb{Z}[\omega]$ , which we now assume. We have

$$g(\mu, c) = \overline{\left(\frac{\mu}{c}\right)_3} g(1, c) \quad \text{for} \quad (\mu, c) = 1. \tag{2-7}$$

The Chinese remainder theorem implies the twisted multiplicativity property

$$g(\mu, ab) = \overline{\left(\frac{a}{b}\right)_3} g(\mu, a) g(\mu, b) \quad \text{for} \quad a, b \equiv 1 \pmod{3} \text{ such that } (a, b) = 1. \tag{2-8}$$

By (2-7) and (2-8) it suffices to understand  $g(\varpi^k, \varpi^\ell)$  for  $\varpi \equiv 1 \pmod{3}$  prime and  $k, \ell \in \mathbb{Z}_{\geq 0}$ . A specialisation of [Proskurin 1998, property (h), p. 7] yields

$$g(\varpi^k, \varpi^\ell) = \begin{cases} 1 & \text{if } \ell = 0, \\ \varphi(\varpi^\ell) & \text{if } 1 \leq \ell \leq k, \ell \equiv 0 \pmod{3}, \\ -N(\varpi)^k & \text{if } \ell = k + 1, \ell \equiv 0 \pmod{3}, \\ N(\varpi)^k g(\varpi) & \text{if } \ell = k + 1, \ell \equiv 1 \pmod{3}, \\ N(\varpi)^k \overline{g(\varpi)} & \text{if } \ell = k + 1, \ell \equiv 2 \pmod{3}, \\ 0 & \text{otherwise.} \end{cases} \tag{2-9}$$

For  $\varpi \equiv 1 \pmod{3}$  prime we have the formula for the cube [Hasse 1950, pp. 443–445],

$$g(\varpi)^3 = -\varpi^2 \overline{\varpi}. \tag{2-10}$$

Observe that (2-7)–(2-9) and (2-10) imply that

$$|g(c)| = \mu^2(c) N(c)^{1/2} \tag{2-11}$$

for  $c \equiv 1 \pmod{3}$ . We denote the normalised cubic Gauss sum (with shift  $\mu \in \mathbb{Z}[\omega]$ ) by

$$\tilde{g}(\mu, c) := N(c)^{-1/2} g(\mu, c). \tag{2-12}$$

The following two lemmas follow directly from combining (2-7)–(2-9).

**Lemma 2.1.** *Let  $\mu \in \mathbb{Z}[\omega]$  and  $c \in \mathbb{Z}[\omega]$  such that  $c \equiv 1 \pmod{3}$  is squarefree. Then*

$$g(\mu, c) = 0 \quad \text{unless } (\mu, c) = 1.$$

**Lemma 2.2.** *Let  $c \in \mathbb{Z}[\omega]$  with  $c \equiv 1 \pmod{3}$  and  $\varpi, \mu \in \mathbb{Z}[\omega]$  be such that  $\varpi \equiv 1 \pmod{3}$  is prime and  $\varpi^2 \mid c$ . Then*

$$g(\mu, c) = 0 \quad \text{unless } \varpi \mid \mu.$$

The next lemma follows directly from combining (2-7)–(2-9) and (2-11).

**Lemma 2.3.** *Let  $\mu, c \in \mathbb{Z}[\omega]$  with  $c \equiv 1 \pmod{3}$ . Then*

$$|g(\mu, c)| \leq N(c)^{1/2} \cdot N((\mu, c))^{1/2}.$$

**Remark 2.4.** We emphasise that  $c \in \mathbb{Z}[\omega]$  is not necessarily squarefree in Lemma 2.3.

For  $b \in \mathbb{R}$ , and  $q \in \mathbb{Z}[\omega]$  with  $q \equiv 1 \pmod{3}$ , let

$$\sigma_b(q) := \sum_{d \mid q} N(d)^b. \tag{2-13}$$

For a given  $\varepsilon > 0$ , we have the standard divisor bound

$$\sigma_0(q) \ll_\varepsilon N(q)^\varepsilon. \tag{2-14}$$

The following lemma is immediate.

**Lemma 2.5.** *Let  $q \in \mathbb{Z}[\omega]$  with  $q \equiv 1 \pmod{3}$  and  $b \in \mathbb{R}$ . Then for  $Y \geq 1$  we have*

$$\sum_{\substack{\mu \in \mathbb{Z}[\omega] \\ 1 \leq N(\mu) \leq Y}} N((\mu, q))^b \leq Y \sigma_{b-1}(q),$$

where  $\sigma_b(q)$  is as given in (2-13).

**Lemma 2.6.** *Let  $q \in \mathbb{Z}[\omega]$  with  $q \equiv 1 \pmod{3}$ . Then for  $X \geq 1$  and  $\varepsilon > 0$  we have*

$$\sum_{\substack{N(r) \leq X \\ r | \text{rad}(q)^\infty}} 1 \ll_\varepsilon (N(q)X)^\varepsilon. \tag{2-15}$$

*Proof.* Without loss of generality we can assume  $X$  is an odd half-integer. By Perron’s formula (truncated) we have

$$\sum_{\substack{N(r) \leq X \\ r | \text{rad}(q)^\infty}} 1 = \int_{2-i(XN(q))^{100}}^{2+i(XN(q))^{100}} X^s \prod_{\substack{\varpi | \text{rad}(q) \\ \varpi \text{ prime} \\ \varpi \equiv 1 \pmod{3}}} (1 - N(\varpi)^{-s})^{-1} \frac{ds}{s} + O((XN(q))^{-50}).$$

The integrand is holomorphic in the half-plane  $\text{Re}(s) > 0$ . We move the contour  $\text{Re}(s) = \varepsilon$ . Taking the logarithm of the Euler product and then using the pointwise bound

$$\omega(q) \ll \frac{\log N(q)}{\log \log N(q)}, \tag{2-16}$$

we obtain (after exponentiation)

$$\prod_{\substack{\varpi | \text{rad}(q) \\ \varpi \text{ prime} \\ \varpi \equiv 1 \pmod{3}}} (1 - N(\varpi)^{-s})^{-1} \ll N(q)^\varepsilon \quad \text{for } \text{Re}(s) \geq \varepsilon.$$

The result follows from Cauchy’s residue theorem. □

**2.2. Group action on  $\mathbb{H}^3$  and Laplacian.** Let  $\mathbb{H}^3$  denote the hyperbolic 3-space  $\mathbb{C} \times \mathbb{R}_{>0}$ . Embed  $\mathbb{C}$  and  $\mathbb{H}^3$  in the Hamilton quaternions by identifying  $i = \sqrt{-1}$  with  $\hat{i}$  and  $w = (z, v) = (x + iy, v) \in \mathbb{H}^3$  with  $x + y\hat{i} + v\hat{k}$ , where  $1, \hat{i}, \hat{j}, \hat{k}$  denote the unit quaternions. The continuous action of  $\text{SL}_2(\mathbb{C})$  on  $\mathbb{H}^3$  (in quaternion arithmetic) is given by

$$\gamma w = (aw + b)(cw + d)^{-1}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \quad \text{and} \quad w \in \mathbb{H}^3.$$

The action of  $\text{SL}_2(\mathbb{C})$  on  $\mathbb{H}^3$  is transitive, and the stabiliser of a point is  $\text{SU}_2(\mathbb{C})$ . In coordinates,

$$\gamma w = \left( \frac{(az + b)\overline{(cz + d)} + a\bar{c}v^2}{|cz + d|^2 + |c|^2v^2}, \frac{v}{|cz + d|^2 + |c|^2v^2} \right), \quad w = (z, v). \tag{2-17}$$

The Laplace operator  $\Delta := v^2(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial v^2) - v\partial/\partial v$  acts on  $C^\infty(\mathbb{H}^3)$  and commutes with the action of  $\text{SL}_2(\mathbb{C})$  on  $C^\infty(\mathbb{H}^3)$ .

Consider the subgroup  $\Gamma := \text{SL}_2(\mathbb{Z}[\omega])$  of  $\text{SL}_2(\mathbb{C})$ . It has finite volume (but is not cocompact) with respect to the  $\text{SL}_2(\mathbb{C})$ -invariant Haar measure  $v^{-3} dx dy dv$  on  $\mathbb{H}^3$ . In what follows, let  $\Gamma' \subseteq \Gamma$  be a subgroup with  $[\Gamma : \Gamma'] < \infty$ . Let  $P(\Gamma') \subset \mathbb{Q}(\omega) \cup \{\infty\}$  be a complete inequivalent (finite) set of cusps for  $\Gamma'$ . Each cusp of  $\Gamma'$  can be written as  $\sigma\infty$  for some  $\sigma \in \Gamma'$ , and let

$$\Gamma'_\sigma := \{\gamma \in \Gamma' : \gamma\sigma\infty = \sigma\infty\}$$

denote the stabiliser group of the cusp  $\sigma\infty$  in  $\Gamma'$ . We have  $\Gamma'_\sigma := \sigma\Gamma'_I\sigma^{-1} \cap \Gamma'$ , and let

$$\begin{aligned} \Lambda_\sigma &:= \left\{ \mu \in \mathbb{C} : \sigma \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma^{-1} \in \Gamma' \right\}, \\ \Lambda_\sigma^* &:= \{v \in \mathbb{C} : \text{Tr}(\mu v) \in \mathbb{Z} \text{ for all } \mu \in \Lambda_\sigma\}. \end{aligned}$$

It is well known that  $\Lambda_\sigma$  and  $\Lambda_\sigma^*$  are lattices in  $\mathbb{C}$ , and that  $\Lambda_\sigma^*$  is dual to  $\Lambda_\sigma$ .

A fundamental domain for the action of  $\Gamma$  on  $\mathbb{H}^3$  is the set

$$\mathcal{F} := \{w = (z, v) \in \mathbb{H}^3 : |z|^2 + v^2 > 1 \text{ and } z \in \pm\Delta\},$$

where  $\Delta$  is the interior of the triangle with vertices  $0, (1 - \omega)^{-1}$  and  $(1 - \omega^2)^{-1}$ . The set of cusps for  $\Gamma$  is  $P(\Gamma) := \{\infty\}$ .

Other congruence subgroups of significance to this paper are given in Section 3.

**2.3. Automorphic forms (for general multipliers).** We record some facts about automorphic forms on  $\Gamma' \backslash \mathbb{H}^3$  that transform with general unitary character  $\kappa : \Gamma' \rightarrow \mathbb{C}^\times$ . For more details one may consult [Livné and Patterson 2002; Louvel 2014; Patterson 1998; Proskurin 1998]. We specialise to cubic metaplectic forms in Section 3.

Let  $\kappa : \Gamma' \rightarrow \mathbb{C}^\times$  be a unitary character that satisfies  $\kappa(-I) = 1$  if  $-I \in \Gamma'$ . The function defined by  $\mu \rightarrow \kappa(\sigma \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma^{-1}) : \Lambda_\sigma \rightarrow \mathbb{C}^\times$  is a homomorphism on the lattice  $\Lambda_\sigma$ . There exists  $h_\sigma \in \mathbb{C}$  such that

$$\kappa\left(\sigma \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma^{-1}\right) = \check{e}(h_\sigma \mu) \quad \text{for all } \mu \in \Lambda_\sigma.$$

Essential cusps with respect to  $\kappa$  are those  $\sigma$  for which we can take  $h_\sigma = 0$ .

Let

$$A(\Gamma' \backslash \mathbb{H}^3, \kappa) := \{u : \mathbb{H}^3 \rightarrow \mathbb{C} : u(\gamma w) = \kappa(\gamma)u(w) \text{ for all } \gamma \in \Gamma' \text{ and } w \in \mathbb{H}^3\}.$$

We say that  $u \in A(\Gamma' \backslash \mathbb{H}^3, \kappa)$  is an automorphic form under  $\Gamma'$  with character  $\kappa$  if it satisfies the conditions:

- $u \in C^\infty(\mathbb{H}^3)$  and is an eigenfunction of the Laplacian, i.e.,

$$\Delta u = -\tau_u(2 - \tau_u)u \quad \text{for some } \tau_u \in \mathbb{C}.$$

The quantity  $\tau_u \in \mathbb{C}$  is the spectral parameter for  $u$ , and is well-defined only up to  $\tau_u \mapsto 2 - \tau_u$ . Without loss of generality one can assume that  $\text{Re}(\tau_u) \geq 1$ .

- $u$  has moderate growth at cusps: there exists a  $D \in \mathbb{R}$  such that

$$|u(w)| < (v + (1 + |z|^2)v^{-1})^D \quad \text{for all } w = (z, v) \in \mathbb{H}^3.$$

Let  $L(\Gamma' \backslash \mathbb{H}^3, \kappa, \tau)$  denote the  $\mathbb{C}$ -vector space of automorphic forms under  $\Gamma'$  with character  $\kappa$  and spectral parameter  $\tau$ . The norm  $\|\cdot\|_2$  on  $L(\Gamma' \backslash \mathbb{H}^3, \kappa, \tau)$  is induced by the standard Petersson inner product

$$\langle u_1, u_2 \rangle := \int_{\Gamma' \backslash \mathbb{H}^3} u_1(w) \overline{u_2(w)} \frac{dx dy dv}{v^3}.$$

Let

$$L^2(\Gamma' \backslash \mathbb{H}^3, \kappa, \tau) := \{u \in L(\Gamma' \backslash \mathbb{H}^3, \kappa, \tau) : \|u\|_2 < \infty\},$$

denote the finite-dimensional Hilbert space of square integrable automorphic forms having character  $\kappa$  and spectral parameter  $\tau$ . We demand that  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  act on an automorphic form  $u$  by  $\pm 1$ , and we speak of  $u$  being even or odd respectively.

Consulting [Proskurin 1998, Theorem 0.3.1], each  $u \in L(\Gamma' \backslash \mathbb{H}^3, \kappa, \tau)$  has Fourier expansion at the cusp  $\sigma\infty$  given by

$$U_\sigma(w) := u(\sigma w) = c_{u,\sigma}(v) + \sum_{\substack{v \neq 0 \\ v \in h_\sigma + \Lambda_\sigma^*}} \rho_{u,\sigma}(v) v K_{\tau-1}(4\pi|v|v) \check{e}(vz), \quad w \in \mathbb{H}^3, \quad (2-18)$$

where  $\rho_{u,\sigma}(v) \in \mathbb{C}$ , and

$$c_{u,\sigma}(v) = \begin{cases} \rho_{u,\sigma,+}(0)v^\tau + \rho_{u,\sigma,-}(0)v^{2-\tau} & \text{if } \tau \neq 1, \\ \rho_{u,\sigma,+}^\sigma(0)v \log v + \rho_{u,\sigma,-}(0)v & \text{if } \tau = 1, \end{cases}$$

and  $\rho_{u,\sigma,+}(0), \rho_{u,\sigma,-}(0) \in \mathbb{C}$ . If  $\sigma\infty$  is essential, then one can take  $h_\sigma = 0$ . If  $\sigma\infty$  is not essential, then  $c_{u,\sigma}(v) \equiv 0$  by [Proskurin 1998, Theorem 0.3.1]. By convention, if  $\sigma = I$  then we omit it from the subscripts on the Fourier coefficients.

If  $c_{u,\sigma}(v) \equiv 0$  for all cusps  $\sigma\infty$ , then  $u$  is a cusp form (it is necessarily a Maass form since  $\mathbb{H}^3$  does not have an invariant complex structure). In particular, all cusp forms have exponential decay at the cusps, and consequently are square integrable on  $\Gamma' \backslash \mathbb{H}^3$ .

The following crude Rankin–Selberg bound follows from a standard argument that uses Plancherel’s theorem. The proof is analogous to that of [Iwaniec 1995, Theorem 3.2], and is omitted.

**Lemma 2.7.** *Let  $\tau \in \mathbb{C}$  with  $\text{Re}(\tau) \geq 1$ ,  $u \in L^2(\Gamma' \backslash \mathbb{H}^3, \kappa, \tau)$  be a cusp form,  $\sigma$  a cusp of  $\Gamma'$ , and  $\varepsilon > 0$ . Then for all  $X \geq 100$  we have*

$$\sum_{\substack{v \in h_\sigma + \Lambda_\sigma^* \\ N(v) \leq X}} |\rho_{u,\sigma}(v)|^2 \ll_{u,\sigma,\varepsilon} X^{1+\varepsilon}.$$

An application of the Cauchy–Schwarz inequality and Lemma 2.7 give the following  $L^1$ -bound.

**Lemma 2.8.** *In the notation of Lemma 2.7 we have*

$$\sum_{\substack{v \in h_\sigma + \Lambda_\sigma^* \\ N(v) \leq X}} |\rho_{u,\sigma}(v)| \ll_{u,\sigma,\varepsilon} X^{1+\varepsilon}.$$

The following Wilton-type bound follows from a standard argument using Fourier convolution with the Dirichlet kernel. The proof is analogous to that of [Epstein et al. 1985, Theorem 3.1] and is omitted.

**Lemma 2.9.** *Let the notation be as in Lemma 2.7 and suppose that  $\operatorname{Re}(\tau) = 1$ . Then*

$$\sum_{\substack{v \in h_\sigma + \Lambda_\sigma^* \\ N(v) \leq X}} \rho_{u,\sigma}(v) \check{e}(\alpha v) \ll_{u,\sigma,\varepsilon} X^{1/2+\varepsilon}$$

for any  $\alpha \in \mathbb{C}$ . The implied constant is uniform with respect to  $\alpha$ .

A direct consequence of partial summation and Lemma 2.9 is the following smoothed Wilton bound.

**Lemma 2.10.** *Let the notation be as in Lemma 2.7,  $\operatorname{Re}(\tau) = 1$ ,  $K, M \geq 1$ , and  $W_{K,M} : (0, \infty) \rightarrow \mathbb{C}$  be a smooth function with compact support in  $[1, 2]$  that satisfies (1-6). Then*

$$\sum_{\substack{v \in h_\sigma + \Lambda_\sigma^* \\ N(v) \leq X}} \rho_{u,\sigma}(v) \check{e}(\alpha v) W_{K,M}\left(\frac{N(v)}{X}\right) \ll_{u,\sigma,\varepsilon} MKX^{1/2+\varepsilon}$$

for any  $\alpha \in \mathbb{C}$ . The implied constant is uniform with respect to  $\alpha$ .

### 3. Cubic metaplectic forms

**3.1. Cubic Kubota character.** Recall that  $\Gamma := \operatorname{SL}_2(\mathbb{Z}[\omega])$ . It is well known that  $\Gamma = \langle P, T, E \rangle$ , where

$$P := \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let  $0 \neq C \in \mathbb{Z}[\omega]$  satisfy  $C \equiv 0 \pmod{3}$ , and

$$\Gamma_1(C) := \{\gamma \in \Gamma : \gamma \equiv I \pmod{C}\}.$$

Observe that  $\Gamma_1(C)$  is a normal subgroup of  $\Gamma$  since it is the kernel of the reduction modulo  $C$  map. Let

$$\Gamma_2 := \langle \operatorname{SL}_2(\mathbb{Z}), \Gamma_1(3) \rangle = \operatorname{SL}_2(\mathbb{Z})\Gamma_1(3) = \Gamma_1(3)\operatorname{SL}_2(\mathbb{Z}), \tag{3-1}$$

where the last two equalities follow because  $\Gamma_1(3)$  is normal in  $\Gamma$ . We also have  $[\Gamma : \Gamma_2] = 27$  (see [Patterson 1977, §2] for the calculation). Recall that  $\chi : \Gamma_2 \rightarrow \{1, \omega, \omega^2\}$  is the cubic Kubota character defined in Section 1.1. The cusps of  $\Gamma_2$  are  $P(\Gamma_2) = \{\infty, \omega, \omega^2\}$ , and the only essential cusp of  $\Gamma_2$  with respect to  $\chi$  is  $\infty$ .

**3.2. Cubic Shimura lift.** Suppose  $\Gamma' \subseteq \Gamma_2$  is a subgroup with  $[\Gamma_2 : \Gamma'] < \infty$ . If  $h \in L(\Gamma' \backslash \mathbb{H}^3, \chi, \tau)$ , then  $h$  is said to be a cubic metaplectic form on  $\Gamma'$  with spectral parameter  $\tau$  (abbreviated to cubic metaplectic form). In this section we specialise to the lowest possible level  $\Gamma' = \Gamma_2$ , and focus on the finite-dimensional subspace  $L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau) \subset L(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$  that contains square integrable cubic metaplectic forms.

We say that  $h \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$  is a Hecke eigenform if it is an eigenfunction for all Hecke operators  $\{T_{\nu^3}\}_{\nu \in \mathbb{Z}[\omega] \setminus \{0\}}$ , i.e.,  $T_{\nu^3}h = \tilde{\lambda}_h(\nu^3)h$  for some  $\tilde{\lambda}_h(\nu^3) \in \mathbb{C}$  and all  $\nu \in \mathbb{Z}[\omega] \setminus \{0\}$ . There is an orthonormal basis (with respect to the Petersson inner product) of  $L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$  consisting of Hecke eigenforms. Two automorphic forms are identified if they are constant multiples of one another. The

discrete spectrum of  $\Delta$  on  $L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  is completely determined via the cubic Shimura correspondence of Flicker [1980] and Patterson [1998, Theorem 3.4]. In particular, there is a bijective correspondence between even (resp. odd) Hecke eigenforms  $h \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$  and even (resp. odd) Hecke eigenforms  $g \in L^2(\Gamma \backslash \mathbb{H}^3, \mathbf{1}, 3\tau - 2)$ , where in the latter case the Hecke operators are the standard ones  $\{\mathcal{T}_\nu\}_{\nu \in \mathbb{Z}[\omega] \setminus \{0\}}$  on the trivial cover of  $\Gamma$ , i.e.,  $\mathcal{T}_\nu g = \lambda_g(\nu)g$  for some  $\lambda_g(\nu) \in \mathbb{C}$  and all  $\nu \in \mathbb{Z}[\omega] \setminus \{0\}$ . Under this correspondence one also has

$$N(\nu^3)^{-1/2} \tilde{\lambda}_h(\nu^3) = N(\nu)^{-1/2} \lambda_g(\nu).$$

The only noncuspidal Hecke eigenform in  $L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  is the cubic theta function of Patterson [1977],

$$\vartheta_3(w) := \text{Res}_{s=4/3} E_3(w, s) \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \frac{4}{3}),$$

where  $E_3(w, s)$  is the Kubota cubic Eisenstein series for  $\Gamma_1(3)$  at the cusp  $\infty$ . Its Shimura correspondent is the constant function  $1 \in L^2(\Gamma \backslash \mathbb{H}^3, \mathbf{1}, 2)$ . The countably many other Hecke eigenforms  $h_k \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  are Maass cusp forms, whose Shimura correspondents  $g_k \in L^2(\Gamma \backslash \mathbb{H}^3, \mathbf{1})$  are also Maass cusp forms. All spectral parameters are nonexceptional, i.e.,  $\text{Re}(\tau_{h_k}) = \text{Re}(\tau_{g_k}) = 1$  for  $k = 1, 2, \dots$ . We also have  $0 \leq \text{Im}(\tau_{f_1}) \leq \text{Im}(\tau_{f_2}) \leq \dots$ , where  $\text{Im}(\tau_{h_k}) \rightarrow \infty$  as  $k \rightarrow \infty$ .

**3.3. Cubic Kloosterman sums.** We will encounter cubic Kloosterman sums attached to the cubic Kubota character in our computations.

Let  $\Gamma' \subseteq \Gamma_2$  with  $[\Gamma_2 : \Gamma'] < \infty$ , and let  $\sigma, \xi \in \text{SL}_2(\mathbb{Z}[\omega])$  denote cusps of  $\Gamma'$ . Let

$$\mathcal{C}(\sigma, \xi) := \left\{ c \in \mathbb{Z}[\omega] \setminus \{0\} : \sigma \begin{pmatrix} * & * \\ c & * \end{pmatrix} \xi^{-1} \in \Gamma' \right\}$$

be the set of allowable moduli for the cusp pair  $(\sigma, \xi)$ . For  $m \in \Lambda_\sigma^*$ ,  $n \in \Lambda_\xi^*$ , and  $c \in \mathcal{C}(\sigma, \xi)$ , the cubic Kloosterman sum is

$$K_{\Gamma', \sigma, \xi}(m, n, c) := \sum_{\substack{a \pmod{c\Lambda_\sigma} \\ d \pmod{c\Lambda_\xi} \\ \sigma \begin{pmatrix} a & * \\ c & d \end{pmatrix} \xi^{-1} \in \Gamma'}} \overline{\chi \left( \sigma \begin{pmatrix} a & * \\ c & d \end{pmatrix} \xi^{-1} \right)} \check{e} \left( \frac{ma + nd}{c} \right), \tag{3-2}$$

where  $\chi : \Gamma_2 \rightarrow \{1, \omega, \omega^2\}$  is the cubic Kubota character. We have the following Weil bound [1948].

**Lemma 3.1** [Livné and Patterson 2002, Proposition 5.1; Louvel 2014, (2.6)]. *Let the notation be as above. Then for  $m, n \in \mathbb{Z}[\omega]$  and  $c \in \mathcal{C}(\sigma, \xi)$ , we have*

$$|K_{\Gamma', \sigma, \xi}(m, n, c)| \leq 2^{\omega(c)} N((m, n, c)) N(c)^{1/2},$$

where  $\omega(c)$  denotes the number of distinct prime divisors of  $c$ .

**Remark 3.2.** In [Livné and Patterson 2002, Proposition 5.1] (and propagated in [Louvel 2014, §2]), it appears the bound in Lemma 3.1 is stated suboptimally with a factor  $N((m, n, c))$  instead of  $N((m, n, c))^{1/2}$ . This makes no difference to us because  $(m, n, c) = 1$  in any instance when Lemma 3.1 is used in this paper.

**Lemma 3.3.** *Suppose that  $\Gamma' = \Gamma_1(3)$  and  $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then*

$$K_{\Gamma_1(3),\sigma,\sigma}(m, n, c) = \sum_{\substack{a,d \pmod{3c} \\ a,d \equiv 1 \pmod{3} \\ ad \equiv 1 \pmod{c}}} \left(\frac{c}{d}\right)_3 \check{e}\left(\frac{ma+nd}{c}\right), \tag{3-3}$$

for any  $c \in 3\mathbb{Z}[\omega] \setminus \{0\}$ , and  $m, n \in \lambda^{-3}\mathbb{Z}[\omega]$ .

*Proof.* Observe that  $\Lambda_\sigma = 3\mathbb{Z}[\omega]$ ,  $\Lambda_\sigma^* = 3^{-1}\mathbb{Z}[\omega]^* = \lambda^{-3}\mathbb{Z}[\omega]$ , and  $\mathcal{C}(\sigma, \sigma) = 3\mathbb{Z}[\omega] \setminus \{0\}$ . Observe that  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(3)$  if and only if  $a \equiv d \equiv 1 \pmod{3}$ ,  $b, c \equiv 0 \pmod{3}$ , and  $ad - bc = 1$ . For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(3)$  with  $c \neq 0$  we have  $\chi(\gamma) = (c/a)_3$  by (1-4). The claim now follows from (3-2), (2-2), and (2-3).  $\square$

**Lemma 3.4.** *Suppose that  $\Gamma' = \Gamma_1(3)$ ,  $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , and  $\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Then*

$$K_{\Gamma_1(3),\sigma,\xi}(m, n, c) = \sum_{\substack{a,d \pmod{3c} \\ a,d \equiv 0 \pmod{3} \\ ad \equiv 1 \pmod{c}}} \left(\frac{d}{c}\right)_3 \check{e}\left(\frac{ma+nd}{c}\right) \tag{3-4}$$

for any  $c \in \mathbb{Z}[\omega]$  such that  $c \equiv 1 \pmod{3}$ , and  $m, n \in \lambda^{-3}\mathbb{Z}[\omega]$ .

*Proof.* Observe that  $\Lambda_\sigma = \Lambda_\xi = 3\mathbb{Z}[\omega]$  and that  $\Lambda_\xi^* = \Lambda_\sigma^* = 3^{-1}\mathbb{Z}[\omega]^* = \lambda^{-3}\mathbb{Z}[\omega]$ . Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}[\omega])$ . Observe that  $\sigma\gamma\xi^{-1} \in \Gamma_1(3)$  if and only if  $a \equiv d \equiv 0 \pmod{3}$ ,  $c \equiv 1 \pmod{3}$ ,  $b \equiv -1 \pmod{3}$ , and  $ad - bc = 1$ . After recalling that  $\chi$  is homomorphism on  $\Gamma_2$  such that  $\chi|_{\text{SL}_2(\mathbb{Z})} \equiv 1$ , we see that  $\chi(\gamma\xi^{-1}) = \chi(\xi^{-1}\gamma) = (-a/c)_3 = (a/c)_3$  by (1-4) and the convention (2-4). The claim now follows from (3-2).  $\square$

#### 4. The cubic large sieve

Implicit in [Heath-Brown 2000] is a version of cubic large sieve where one of the variables is not required to be squarefree. Here we record the relevant results.

**Theorem 4.1.** *Let  $\varepsilon > 0$  be given,  $M, N \geq \frac{1}{2}$  and  $\Psi = (\Psi_c)$  be a  $\mathbb{C}$ -valued sequence supported on  $c \in \mathbb{Z}[\omega]$  with  $c \equiv 1 \pmod{3}$  and  $N(c) \sim N$ . Then*

$$\sum_{N(d) \sim M} \left| \sum_{\substack{N(c) \sim N \\ c \equiv 1 \pmod{3}}} \mu^2(c) \Psi_c \left(\frac{c}{d}\right)_3 \right|^2 \ll_\varepsilon (MN)^\varepsilon M^{1/3} (M+N) \|\mu^2 \Psi\|_2^2.$$

*Proof.* This follows from [Heath-Brown 2000, (22)] (and the display above it), (28) and the second display on p. 123],  $\square$

**Corollary 4.2.** *Let the notation be as in Theorem 4.1 and  $\Omega = (\Omega_d)$  be a  $\mathbb{C}$ -valued sequence supported on  $d \in \mathbb{Z}[\omega]$  with  $N(d) \sim M$ . Then*

$$\sum_{N(d) \sim M} \sum_{\substack{N(c) \sim N \\ c \equiv 1 \pmod{3}}} \Omega_d \mu^2(c) \Psi_c \left(\frac{d}{c}\right)_3 \ll_\varepsilon (MN)^\varepsilon M^{1/6} (M^{1/2} + N^{1/2}) \|\Omega\|_2 \|\mu^2 \Psi\|_2.$$

*Proof.* Application of the Cauchy–Schwarz inequality, unique factorisation in  $\mathbb{Z}[\omega]$ , (2-1), and Theorem 4.1 gives

$$\left| \sum_{N(d) \sim M} \sum_{\substack{N(c) \sim N \\ c \equiv 1 \pmod{3}}} \Omega_d \mu^2(c) \Psi_c \left( \frac{d}{c} \right)_3 \right|^2 \leq \|\Omega\|_2^2 \cdot \left( \sum_{\zeta} \sum_{k \geq 0} \sum_{\substack{N(\zeta \lambda^k m) \sim M \\ m \equiv 1 \pmod{3}}} \left| \sum_{\substack{N(c) \sim N \\ c \equiv 1 \pmod{3}}} \mu^2(c) \left( \frac{\zeta \lambda^k}{c} \right)_3 \Psi_c \left( \frac{c}{m} \right)_3 \right|^2 \right) \\ \ll (MN)^\varepsilon M^{1/3} (M+N) \|\Omega\|_2^2 \|\mu^2 \Psi\|_2^2,$$

as required. □

### 5. The Browning–Vishe circle method for number fields

The proof of our Type-II estimates will use a circle method over number fields due to Browning and Vishe [2014, Theorem 1.2]. Their work generalises work of Heath-Brown [1996, Theorem 1] (over  $\mathbb{Q}$ ), and ultimately relies on the  $\delta$ -function technology of Duke, Friedlander, and Iwaniec [Duke et al. 1993].

Let  $L/\mathbb{Q}$  be a number field of degree  $d \geq 2$  with ring of integers  $\mathcal{O}_L$  and unit group  $\mathcal{O}_L^\times$ . Let  $\mathfrak{a} \trianglelefteq \mathcal{O}_L$  be an integral ideal,  $N(\mathfrak{a}) := \#\mathcal{O}_L/\mathfrak{a}$  denote the ideal norm of  $\mathfrak{a}$ , and

$$\delta_L(\mathfrak{a}) := \begin{cases} 1 & \text{if } \mathfrak{a} = (0), \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 5.1.** One obtains an indicator function on  $\mathcal{O}_L$  by restricting to principal ideals, in which case one writes  $\delta_L((v)) = \delta_L(v)$  for any  $v \in \mathcal{O}_L$ . We also have  $N((v)) = N(v)$ , where the latter is the norm of an element of  $\mathcal{O}_L$ .

**Theorem 5.2** [Browning and Vishe 2014, Theorem 1.2]. *Let  $L/\mathbb{Q}$  be a number field of degree  $d \geq 2$ ,  $C \geq 1$ , and  $\mathfrak{a} \trianglelefteq \mathcal{O}_L$  be an integral ideal. Then there exists a positive constant  $k_C$  and an infinitely differentiable function  $h(x, y) : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  (depending on  $L/\mathbb{Q}$ ) such that*

$$\delta_L(\mathfrak{a}) = \frac{k_C}{C^{2d}} \sum_{(0) \neq \mathfrak{c} \subseteq \mathcal{O}_L} \sum_{\sigma \pmod{\mathfrak{c}}}^* \sigma(\mathfrak{a}) h\left(\frac{N(\mathfrak{c})}{C^d}, \frac{N(\mathfrak{a})}{C^{2d}}\right), \tag{5-1}$$

where the notation  $\sum_{\sigma \pmod{\mathfrak{c}}}^*$  means that the sum is taken over primitive additive characters (extended to ideals) modulo  $\mathfrak{c}$ . The constant  $k_C$  satisfies

$$k_C = 1 + O_{L/\mathbb{Q}, D}(C^{-D}) \quad \text{for any } D > 0. \tag{5-2}$$

Furthermore, we have

$$h(x, y) \ll_{L/\mathbb{Q}} x^{-1} \quad \text{for all } y \in \mathbb{R}, \tag{5-3}$$

$$h(x, y) \neq 0 \quad \text{only if } x \leq \max\{1, 2|y|\}. \tag{5-4}$$

**Remark 5.3.** In practice one usually chooses  $C := X^{1/(2d)}$  to detect the condition  $\mathfrak{a} = (0)$  for a sequence of ideals of  $\mathcal{O}_L$  with norm less than or equal to  $X$ . This means that for  $\mathfrak{c}$  (see (5-1)) in the generic range  $N(\mathfrak{c}) \asymp X^{1/2}$  there is no oscillation in the weight function  $h(x, y)$ .

**Lemma 5.4** [Browning and Vishe 2014, Lemma 3.1]. *Let the notation be as in Theorem 5.2. The function  $h(x, y)$  vanishes when  $x \geq 1$  and  $|y| \leq x/2$ . When  $x \leq 1$  and  $|y| \leq x/2$ , we have*

$$\frac{\partial}{\partial y} h(x, y) = 0. \tag{5-5}$$

**Lemma 5.5** [Browning and Vishe 2014, Lemma 3.2]. *Let the notation be as in Theorem 5.2. Then for  $i, j, D \in \mathbb{Z}_{\geq 0}$ , we have*

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h(x, y) \ll_{L/\mathbb{Q}, i, j, D} x^{-i-j-1} \left( x^D + \min \left\{ 1, \left( \frac{x}{|y|} \right)^D \right\} \right). \tag{5-6}$$

The term  $x^D$  on the right side of (5-6) can be omitted if  $j \neq 0$ .

**Corollary 5.6.** *Let the notation be as in Theorem 5.2. Then for any  $j \in \mathbb{Z}_{\geq 1}$  we have*

$$\frac{\partial^j}{\partial y^j} h(x, y) \ll_{L/\mathbb{Q}, j} 1. \tag{5-7}$$

*Proof.* If  $x \leq 1$  and  $|y| \leq x/2$ , then Lemma 5.4 implies that

$$\frac{\partial^j}{\partial y^j} h(x, y) = 0$$

for all  $j \in \mathbb{Z}_{\geq 1}$ . If  $x \leq 1$  and  $|y| \geq x/2$ , then Lemma 5.5 (with  $i = 0$  and  $D = j + 1$ ) gives

$$\frac{\partial^j}{\partial y^j} h(x, y) \ll_{L/\mathbb{Q}, j} 1$$

for all  $j \in \mathbb{Z}_{\geq 1}$ . If  $x \geq 1$ , then Lemma 5.4 (the vanishing condition on  $h$ ) and Lemma 5.5 (with  $i = D = 0$ ) gives

$$\frac{\partial^j}{\partial y^j} h(x, y) \ll_{L/\mathbb{Q}, j} 1,$$

for all  $j \in \mathbb{Z}_{\geq 1}$ . Putting all three cases together gives the result. □

### 6. Vaughan’s identity

Here we record a celebrated identity of Vaughan [1975] adapted to our situation.

**Proposition 6.1.** *Let  $R, S \geq 1$ . Then for any  $v \in \mathbb{Z}[\omega]$  with  $v \equiv 1 \pmod{3}$  and  $N(v) > S$ , we have*

$$\Lambda(v) = \sum_{\substack{a|v \\ N(a) \leq R}} \mu(a) \log \left( \frac{N(v)}{N(a)} \right) - \sum_{\substack{ab|v \\ N(a) \leq R \\ N(b) \leq S}} \mu(a) \Lambda(b) + \sum_{\substack{ab|v \\ N(a) > R \\ N(b) > S}} \mu(a) \Lambda(b). \tag{6-1}$$

If  $N(v) \leq S$ , the right side of (6-1) vanishes.

### 7. Proof of Theorem 1.2 and Corollary 1.3

In this section we prove Theorem 1.2 and Corollary 1.3 assuming the truth of Lemma 1.4 and the main inputs: Theorem 1.5 and Proposition 1.6.

*Proof of Theorem 1.2 assuming Lemma 1.4, Theorem 1.5, and Proposition 1.6.* Recall the definition of the quantity  $\mathcal{P}_f(X, v, u; W_K)$  given in (1-7). We apply Proposition 6.1 to (1-7). The parameters  $R, S \geq 1$  used in our application of Proposition 6.1 will be chosen at a later point in the proof and will satisfy

$$S < \frac{X}{10000} \quad \text{and} \quad 10000X < RS < 10000000X \quad \text{say,} \tag{7-1}$$

for all sufficiently large  $X$ . Since the support of  $W_K$  is contained in  $[1, 2]$  and  $S < X/10000$  by (7-1), all summands in  $\mathcal{P}_f(X, v, u; W_K)$  are automatically supported on the condition  $N(v) > S$ . Note that the right most sum in (6-1) vanishes since the support of  $W_K$  is contained in  $[1, 2]$  and  $RS > 10000X$  by (7-1). We insert a smooth partition of unity in the  $a$  and  $b$  variables in the second sum in (6-1), and then interchange these summations with the  $v$  summation after substitution of (6-1) into (1-7). We obtain

$$\mathcal{P}_f(X, v, u; W_K) = \mathcal{P}_{1f}(X, R, v, u; W_K) - \sum_{\substack{1 \ll M \ll R \\ 1 \ll N \ll S \\ M, N \text{ dyadic}}} \sum \mathcal{P}_{2f}(X, M, N, v, u; W_K), \tag{7-2}$$

where

$$\mathcal{P}_{1f}(\dots) := \sum_{\substack{a, b \equiv 1 \pmod{3} \\ ab \equiv u \pmod{v} \\ N(a) \leq R}} \sum \mu(a) \log(N(b)) \rho_f(\lambda^{-3}ab) W_K\left(\frac{N(\lambda^{-3}ab)}{X}\right), \tag{7-3}$$

$$\mathcal{P}_{2f}(\dots) := \sum_{\substack{a, b, c \equiv 1 \pmod{3} \\ abc \equiv u \pmod{v} \\ N(a) \leq R \\ N(b) \leq S}} \sum \mu(a) \Lambda(b) \rho_f(\lambda^{-3}abc) W_K\left(\frac{N(\lambda^{-3}abc)}{X}\right) U\left(\frac{N(a)}{M}\right) U\left(\frac{N(b)}{N}\right), \tag{7-4}$$

and  $U : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  is a fixed smooth function with compact support in  $[1, 2]$  such that

$$\sum_{L \text{ dyadic}} U\left(\frac{N(\ell)}{L}\right) = 1 \quad \text{for all } 0 \neq \ell \in \mathbb{Z}[\omega].$$

**7.1. Estimate for  $\mathcal{P}_{1f}(X, R, v, u; W_K)$ .** Rewriting (7-3) using additive characters we obtain

$$\begin{aligned} &\mathcal{P}_{1f}(\dots) \\ &= \sum_{\substack{a \equiv 1 \pmod{3} \\ N(a) \leq R}} \frac{\mu(a)}{N(av)} \sum_{j \pmod{av}} \check{e}\left(-\frac{j\eta}{av}\right) \sum_{v \in \lambda^{-3}\mathbb{Z}[\omega]} \rho_f(v) \check{e}\left(\frac{j\lambda^3 v}{av}\right) \log\left(\frac{N(v)}{N(\lambda^{-3}a)}\right) W_K\left(\frac{N(v)}{X}\right), \end{aligned} \tag{7-5}$$

where  $\eta \in \mathbb{Z}[\omega]$  is such that  $\eta \equiv u \pmod{v}$  and  $\eta \equiv 0 \pmod{a}$ . Applying Lemma 2.10 (while noting Remark 1.1) to the  $v$  summation and estimating the other sums trivially using the triangle inequality we get

$$\mathcal{P}_{1f}(\dots) \ll (RX)^\epsilon KRX^{1/2} \tag{7-6}$$

uniformly in the modulus  $v$ .

**7.2. Two estimates for  $\mathcal{P}_{2f}(\dots)$ .**

**7.2.1. First estimate.** For the first estimate we treat (7-4) as an average Type-I sum. That is, in (7-4) we let  $h = ab$ ,

$$\gamma'_h(M, N) := \sum_{h=ab} \mu(a) \Lambda(b) U\left(\frac{N(a)}{M}\right) U\left(\frac{N(b)}{N}\right), \tag{7-7}$$

and interpret  $c$  as the “smooth” summation variable. We then decompose

$$\mathcal{P}_{2f}(\dots) = \mathcal{P}_{2f}^*(\dots) + \mathcal{P}_{2f}^\dagger(\dots), \tag{7-8}$$

where  $\mathcal{P}_{2f}^*(\dots)$  and  $\mathcal{P}_{2f}^\dagger(\dots)$  have the factors  $\mu^2(h)$  and  $1 - \mu^2(h)$  inserted, respectively. The weight  $\mu^2(h)\gamma'_h(M, N)$  in  $\mathcal{P}_{2f}^*(\dots)$  is supported on squarefree elements of  $\mathbb{Z}[\omega]$ . We apply Proposition 1.6 (see (1-13)) to obtain

$$\mathcal{P}_{2f}^*(\dots) \ll (XKN(v))^\varepsilon K^{14/3} N(v)^{5/6} (MN)^{5/6} X^{1/3}. \tag{7-9}$$

Applying Lemma 1.4 (see (1-12)) to the  $c$ -sum in  $\mathcal{P}_{2f}^\dagger(\dots)$  we obtain

$$\begin{aligned} \mathcal{P}_{2f}^\dagger(\dots) &\ll (XKN(v))^\varepsilon K^4 N(v)^{1/2} (MN)^{1/2} \|(\mathbf{1} - \mu^2)\boldsymbol{\gamma}'(M, N)\|_1 \\ &\ll (XKN(v))^\varepsilon K^4 N(v)^{1/2} M^{3/2} N. \end{aligned} \tag{7-10}$$

Note that the support of the  $b$  variable in (7-7) imposed by the weight  $(1 - \mu^2(h))\Lambda(b) = 0$  (supported on prime powers with exponent  $\geq 2$ ) was used to obtain (7-10). Substitution of (7-9) and (7-10) into (7-8) gives

$$\mathcal{P}_{2f}(\dots) \ll (XKN(v))^\varepsilon (K^{14/3} N(v)^{5/6} (MN)^{5/6} X^{1/3} + K^4 N(v)^{1/2} M^{3/2} N). \tag{7-11}$$

**7.2.2. Second estimate.** For the second estimate we treat (7-4) as a Type-II sum. That is, we let  $h = bc$ , and

$$\gamma_h(N, X/MN) := \sum_{h=bc} \Lambda(b) U\left(\frac{N(b)}{N}\right).$$

Observe that the weight  $\mu(a)U(N(a)/M)$  is supported only on squarefree  $a$ . Thus we apply Theorem 1.5 (see (1-14)) and obtain

$$\mathcal{P}_{2f}(\dots) \ll (XKN(v))^\varepsilon K^8 N(v)^4 (XM^{-1/2} + (MX)^{3/4}). \tag{7-12}$$

**7.3. Conclusion.** We use (7-6) to estimate the first term of (7-2). Let  $1 \ll L \ll R$ . We use (7-11) (resp. (7-12)) to estimate the second term in (7-2) when  $M \leq L$  (resp.  $M \geq L$ ). The net result is

$$\mathcal{P}_f(X, v, u; W_K) \ll (XKN(v))^\varepsilon K^8 N(v)^4 (RX^{1/2} + (LS)^{5/6} X^{1/3} + L^{3/2} S + XL^{-1/2} + (RX)^{3/4}) \tag{7-13}$$

for any  $R, S \geq 1$  satisfying (7-1) and  $1 \ll L \ll R$ . The choice of parameters

$$R = 1000X^{5/17}, \quad S = 1000X^{12/17}, \quad \text{and} \quad L = X^{1/17},$$

satisfies (7-1) for all sufficiently large  $X$ , and substitution into (7-13) yields

$$\mathcal{P}_f(X, v, u; W_K) \ll (XKN(v))^\varepsilon K^8 N(v)^4 X^{1-1/34},$$

as required. □

We now remove the smoothing.

*Proof of Corollary 1.3.* Let  $\Delta := K^{-1}$  with  $K \geq 2$  and suppose that  $W_K : (0, \infty) \rightarrow \mathbb{R}$  is smooth and satisfies

$$\begin{aligned} \text{supp}(W_K) &\subset \left[\frac{5}{4} - \Delta, \frac{7}{4} + \Delta\right], \quad 0 \leq W(x) \leq 1 \quad \text{for all } x > 0, \\ W(x) &= 1 \quad \text{for } x \in \left[\frac{5}{4}, \frac{7}{4}\right], \quad \text{and} \quad W^j(x) \ll_j K^j. \end{aligned} \tag{7-14}$$

Then for any  $Z \gg 1$  we have

$$\sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv u \pmod{v} \\ 5Z/4 < N(v) \leq 7Z/4}} \rho_f(v) \Lambda(\lambda^3 v) = \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv u \pmod{v}}} \rho_f(v) \Lambda(\lambda^3 v) W_K\left(\frac{N(v)}{Z}\right) + O_v(K^{-1/2} Z^{1+\varepsilon}), \tag{7-15}$$

where the error term follows by Cauchy–Schwarz, Lemma 2.7, and the support of  $W_K$ . Applying Theorem 1.2 to the right side (7-15) gives

$$\sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv u \pmod{v} \\ 5Z/4 < N(v) \leq 7Z/4}} \rho_f(v) \Lambda(\lambda^3 v) \ll_v (ZK)^\varepsilon (K^8 Z^{1-1/34} + K^{-1/2} Z).$$

We choose  $K = Z^{1/289}$  to obtain

$$\sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv u \pmod{v} \\ 5Z/4 < N(v) \leq 7Z/4}} \rho_f(v) \Lambda(\lambda^3 v) \ll_v Z^{1-1/578+\varepsilon}. \tag{7-16}$$

Summing over intervals  $[5Z/4, 7Z/4]$  with  $7Z/4 \leq X$  yields (1-10).

To prove (1-11) we first observe that

$$\tilde{\mathcal{P}}_f(X; v, u) - \mathcal{P}_f(X; v, u) = \sum_{\substack{k \in \mathbb{Z}_{\geq 2} \\ \varpi \text{ prime} \\ \varpi^k \equiv u \pmod{v} \\ N(\lambda^{-3} \varpi^k) \leq X}} \sum_{\varpi \in \mathbb{Z}[\omega]} \rho_f(\lambda^{-3} \varpi^k) \log N(\varpi). \tag{7-17}$$

Applying Cauchy–Schwarz to the double sum in (7-17) shows that the right side of (7-17) is

$$\begin{aligned} &\ll X^{1/4+\varepsilon} \left( \sum_{\substack{k \in \mathbb{Z}_{\geq 2} \\ \varpi \text{ prime} \\ N(\lambda^{-3} \varpi^k) \leq X}} \sum_{\varpi \in \lambda^{-3}\mathbb{Z}[\omega]} |\rho_f(\lambda^{-3} \varpi^k)|^2 \right)^{1/2} \\ &\ll X^{1/4+\varepsilon} \left( \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\lambda^{-3} v) \leq X}} |\rho_f(v)|^2 \right)^{1/2} \ll X^{3/4+\varepsilon}, \end{aligned} \tag{7-18}$$

where the last inequality follows from using Lemma 2.7. The result (1-11) now follows. □

The rest of the paper will be dedicated to proving Lemma 1.4, Theorem 1.5, and Proposition 1.6.

**8. Voronoi summation formulae for twists**

In this section we develop a Voronoi summation formula for twists of a cusp form  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  with spectral parameter  $\tau_f \in 1 + i\mathbb{R}$  by appropriate non-Archimedean and Archimedean characters. Development of this formula requires some care because we are working with the group  $\Gamma_2 = \langle \text{SL}_2(\mathbb{Z}), \Gamma_1(3) \rangle$  in  $\Gamma := \text{SL}_2(\mathbb{Z}[\omega])$ .

**8.1. Twists and Dirichlet series.** We will need consider cubic metaplectic forms on groups  $\Gamma_1(C)$  with  $C \equiv 0 \pmod{9}$ , i.e., the spaces  $L^2(\Gamma_1(C) \backslash \mathbb{H}^3, \chi, \tau)$  for  $\tau \in \mathbb{C}$  with  $\text{Re}(\tau) \geq 1$ . To simplify our exposition we focus on the nonexceptional case, i.e.,  $\text{Re}(\tau) = 1$ . Suppose that  $\Psi : \lambda^{-1}\mathbb{Z}[\omega] \rightarrow \mathbb{C}$  is periodic modulo  $(\lambda^m r)(\lambda^{-1}\mathbb{Z}[\omega])$ . The  $\Psi$ -twist (at  $\infty$ ) of a cusp form  $F \in L^2(\Gamma_1(C) \backslash \mathbb{H}^3, \chi, \tau)$  is defined by

$$F(w; \Psi) := \sum_{0 \neq v \in (\lambda C)^{-1}\mathbb{Z}[\omega]} \rho_F(v) \Psi(Cv) v K_{\tau-1}(4\pi|v|v) \check{e}(vz), \quad w = (z, v) \in \mathbb{H}^3, \tag{8-1}$$

also denoted by  $(F \otimes \Psi)(w)$ . By [Proskurin 1998, Theorem 0.3.12] and its proof we have

$$F(\cdot; \Psi) \in L^2(\Gamma_1(\lambda^{2m} r^2 C) \backslash \mathbb{H}^3, \chi, \tau) \quad \text{is a cusp form.} \tag{8-2}$$

**Remark 8.1.** For the purposes of twisting we view the cusp form  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  in the larger space  $L^2(\Gamma_1(\lambda^4) \backslash \mathbb{H}^3, \chi)$ . This is immaterial in the final results and only involves extra fixed powers of the prime  $\lambda$  in the formulae.

In what follows it will be instructive to open the definition  $\check{e}(z) := e(z + \bar{z})$ ,  $z \in \mathbb{C}$ . We remind the reader that the function  $F(w; \Psi)$  in (8-1) is a function in  $z, \bar{z}$ , and  $v$  (although the notation suppresses this). For  $n \in \mathbb{Z}$ , we define

$$F(w; \Psi, n) := \frac{1}{(2\pi i)^{|n|}} \cdot \begin{cases} \left(\frac{\partial}{\partial z}\right)^n F(w; \Psi) & \text{if } n > 0, \\ F(w; \Psi) & \text{if } n = 0, \\ \left(\frac{\partial}{\partial \bar{z}}\right)^{|n|} F(w; \Psi) & \text{if } n < 0, \end{cases} \quad w = (z, v) \in \mathbb{H}^3. \tag{8-3}$$

To complement (8-1), we have the Fourier expansions (at  $\infty$ ) for  $n \in \mathbb{Z} \setminus \{0\}$ ,

$$F(w; \Psi, n) := \sum_{0 \neq v \in (\lambda C)^{-1}\mathbb{Z}[\omega]} \begin{cases} \rho_F(v) v^n \Psi(Cv) v K_{\tau-1}(4\pi|v|v) e(vz + \bar{v}\bar{z}) & \text{if } n > 0, \\ \rho_F(v) \bar{v}^{|n|} \Psi(Cv) v K_{\tau-1}(4\pi|v|v) e(vz + \bar{v}\bar{z}) & \text{if } n < 0. \end{cases} \tag{8-4}$$

Suppose that  $\psi : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  is periodic modulo  $\lambda^m r$ . The (normalised) Fourier transform  $\hat{\psi} : \lambda^{-1}\mathbb{Z}[\omega] \rightarrow \mathbb{C}$  is given by

$$\hat{\psi}(x) := \frac{1}{N(\lambda^m r)} \sum_{u \pmod{\lambda^m r}} \psi(u) \check{e}\left(\frac{ux}{\lambda^m r}\right), \quad x \in \lambda^{-1}\mathbb{Z}[\omega], \tag{8-5}$$

and is periodic modulo  $(\lambda^m r)(\lambda^{-1}\mathbb{Z}[\omega])$ . Fourier inversion asserts that

$$\psi(u) := \sum_{x \in \lambda^{-1}\mathbb{Z}[\omega]/(\lambda^m r)(\lambda^{-1}\mathbb{Z}[\omega])} \hat{\psi}(x) \check{e}\left(-\frac{xu}{\lambda^m r}\right) \quad \text{for } u \in \mathbb{Z}[\omega]. \tag{8-6}$$

For  $n \in \mathbb{Z}$  consider the Dirichlet series

$$\mathcal{D}(s, F; \Psi, n) := \sum_{\substack{v \neq 0 \\ v \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \frac{\rho_F(v)\Psi(Cv)\left(\frac{v}{|v|}\right)^n}{N(v)^s}, \quad \operatorname{Re}(s) > 1,$$

and the associated Mellin transform

$$\Lambda(s, F; \Psi, n) := \int_0^\infty F(v; \Psi, n)v^{2s+|n|-2} dv,$$

where we let  $v$  denote  $(0, v)$  for  $v > 0$ . Let

$$G_\infty(s, \tau, n) := \frac{1}{4}(2\pi)^{-2s-|n|}\Gamma\left(s + \frac{1}{2}|n| - \frac{1}{2}(\tau - 1)\right)\Gamma\left(s + \frac{1}{2}|n| + \frac{1}{2}(\tau - 1)\right), \quad s \in \mathbb{C}. \quad (8-7)$$

**Lemma 8.2.** *Let  $\tau \in \mathbb{C}$  with  $\operatorname{Re}(\tau) = 1$ ,  $C \in \mathbb{Z}[\omega]$  with  $C \equiv 0 \pmod{9}$ ,  $F \in L^2(\Gamma_1(C)\backslash\mathbb{H}^3, \chi, \tau)$  be a cusp form, and  $n \in \mathbb{Z}$ . For  $\operatorname{Re}(s) > 1$  we have*

$$\Lambda(s, F; \Psi, n) = G_\infty(s, \tau, n)\mathcal{D}(s, F; \Psi, n),$$

where  $G_\infty(s, \tau, n)$  is given by (8-7)

*Proof.* The proofs for the cases  $n > 0$ ,  $n = 0$ , and  $n < 0$  are analogous. We give details for the case  $n > 0$ . For  $\operatorname{Re}(s) > 1$  and  $n > 0$  we have

$$\begin{aligned} \Lambda(s, F; \Psi, n) &= \int_0^\infty \sum_{\substack{v \neq 0 \\ v \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \rho_F(v)v^n\Psi(Cv)K_{\tau-1}(4\pi|v|v)v^{2s+n-1} dv \\ &= \frac{1}{(4\pi)^{2s+n}} \sum_{\substack{v \neq 0 \\ v \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \frac{\rho_F(v)\Psi(Cv)\left(\frac{v}{|v|}\right)^n}{N(v)^s} \int_0^\infty K_{\tau-1}(T)T^{2s+n-1} dT \\ &= \frac{1}{4}(2\pi)^{-2s-n}\Gamma\left(s + \frac{1}{2}n - \frac{1}{2}(\tau - 1)\right)\Gamma\left(s + \frac{1}{2}n + \frac{1}{2}(\tau - 1)\right) \sum_{\substack{v \neq 0 \\ v \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \frac{\rho_F(v)\Psi(Cv)\left(\frac{v}{|v|}\right)^n}{N(v)^s}. \end{aligned} \quad (8-8)$$

The interchange of summation and integration above for  $\operatorname{Re}(s) > 1$  is justified by absolute convergence (see Lemma 2.8 and [Olver et al. 2018, (10.25.3), (10.45.7)]). Furthermore, (8-8) follows from [Olver et al. 2018, (10.43.19)].  $\square$

**8.2. A special case.** Recall that  $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$  is a cusp form with spectral parameter  $\tau_f \in 1 + i\mathbb{R}$ . For  $\ell \in \mathbb{Z}_{\geq 0}$ ,  $q \in \mathbb{Z}[\omega]$  with  $q \equiv 1 \pmod{3}$ , and  $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q\mathbb{Z}[\omega]$ , let

$$f(w; \lambda^\ell q, \eta) := \sum_{\substack{v \neq 0 \\ v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv \eta \pmod{\lambda^\ell q}}} \rho_f(v)vK_{\tau_f-1}(4\pi|v|v)\check{e}(vz). \quad (8-9)$$

Following (8-1)–(8-4) we also have the functions  $f(w; \lambda^\ell q, \eta, n)$  and their associated Fourier expansions for each  $n \in \mathbb{Z}$ . We ultimately need a Voronoi formulae for the Fourier coefficients of  $f(w; \lambda^\ell q, \eta, n)$ . Consider the Dirichlet series

$$\mathcal{D}(s, f; \lambda^\ell q, \eta, n) := \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv \eta \pmod{\lambda^\ell q}}} \frac{\rho_f(v) \left(\frac{v}{|v|}\right)^n}{N(v)^s}, \quad \text{Re}(s) > 1,$$

and the associated Mellin transform

$$\Lambda(s, f; \lambda^\ell q, \eta, n) := \int_0^\infty f(v; \lambda^\ell q, \eta, n) v^{2s+|n|-2} dv.$$

Then Lemma 8.2 asserts that

$$\Lambda(s, f; \lambda^\ell q, \eta, n) = G_\infty(s, \tau_f, n) \mathcal{D}(s, f; \lambda^\ell q, \eta, n) \quad \text{for } \text{Re}(s) > 1, \tag{8-10}$$

where  $G_\infty(s, \tau, n)$  is given by (8-7).

We detect the congruence condition in (8-9) using Fourier transforms. For  $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q \mathbb{Z}[\omega]$ ,  $0 \leq m \leq \ell$ , and  $r \mid q$ , let

$$\psi_{\lambda^m r}(u)_\eta := \mathbf{1}_{\lambda^m r}(u) \cdot \check{e}\left(-\frac{\eta u}{\lambda^m r}\right), \tag{8-11}$$

where  $\mathbf{1}_{\lambda^m r}(\cdot)$  is the principal character modulo  $\lambda^m r$ . As a shorthand we write  $\psi_{\lambda^m r}(u) := \psi_{\lambda^m r}(u)_0$ . The function  $\psi_{\lambda^m r}(\cdot)_\eta$  is periodic modulo  $\lambda^m r$ . The Fourier transform is

$$\widehat{\psi_{\lambda^m r}(\cdot)_\eta}(k) = \frac{1}{N(\lambda^m r)} \sum_{\substack{u \pmod{\lambda^m r} \\ (u, \lambda^m r) = 1}} \check{e}\left(\frac{(k - \eta)u}{\lambda^m r}\right), \quad k \in \lambda^{-1}\mathbb{Z}[\omega]. \tag{8-12}$$

As a shorthand we write  $\widehat{\psi_{\lambda^m r}(k)} := \widehat{\psi_{\lambda^m r}(\cdot)_0}(k)$ . A straightforward computation shows the following orthogonality relation.

**Lemma 8.3.** *For  $\ell \in \mathbb{Z}_{\geq 0}$  and  $k, \eta \in \mathbb{Z}[\omega]/\lambda^\ell q \mathbb{Z}[\omega]$  we have*

$$\frac{1}{N(\lambda^\ell q)} \sum_{r \mid q} \sum_{m=0}^\ell N(\lambda^m r) \widehat{\psi_{\lambda^m r}(\cdot)_\eta}(k) = \delta_{k \equiv \eta \pmod{\lambda^\ell q}}. \tag{8-13}$$

The following lemma records the standard evaluation of Ramanujan sums.

**Lemma 8.4.** *Let  $r \in \mathbb{Z}[\omega]$  satisfy  $r \equiv 1 \pmod{3}$  and  $k \in \mathbb{Z}[\omega]$ . Then we have*

$$\hat{\psi}_r(k) := \frac{1}{N(r)} \sum_{\substack{x \pmod{r} \\ (x, r) = 1}} \check{e}\left(\frac{kx}{r}\right) = \frac{1}{N(r)} \mu\left(\frac{r}{(r, k)}\right) \frac{\varphi(r)}{\varphi\left(\frac{r}{(r, k)}\right)},$$

where  $\varphi(\cdot)$  is the Euler  $\varphi$ -function on  $\mathbb{Z}[\omega]$ .

*Proof.* This follows from the multiplicativity of Ramanujan sums in the modulus  $r$ , and the first, fourth, and eighth cases in the evaluation on [Proskurin 1998, p. 11]. □

We next prove a straightforward but crucial lemma establishing the “flatness” of Ramanujan sums when averaged over the modulus.

**Lemma 8.5.** *Let  $r, k \in \mathbb{Z}[\omega]$  satisfy  $r \equiv 1 \pmod{3}$ ,  $\widehat{\psi}_r(k)$  be the normalised Ramanujan sum as in the statement of Lemma 8.4, and  $\varepsilon > 0$ . Then for  $R \geq 1$  we have*

$$\sum_{\substack{r \in \mathbb{Z}[\omega] \\ N(r) \sim R \\ r \equiv 1 \pmod{3}}} |\widehat{\psi}_r(k)| \ll_{\varepsilon} \delta_{k=0} \cdot R + \delta_{k \neq 0} \cdot (N(k)R)^{\varepsilon}. \tag{8-14}$$

*Proof.* When  $k = 0$  we have the trivial estimate  $\ll R$ . When  $k \neq 0$  use Lemma 8.4, Möbius inversion, and the triangle inequality to obtain

$$\begin{aligned} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ N(r) \sim R \\ r \equiv 1 \pmod{3}}} |\widehat{\psi}_r(k)| &= \sum_{\substack{\gamma | k \\ \gamma \equiv 1 \pmod{3}}} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ N(r) \sim R \\ r \equiv 1 \pmod{3} \\ (r,k)=\gamma}} \frac{1}{N(r)} \frac{\varphi(r)}{\varphi(r/\gamma)} \\ &\leq \sum_{\substack{\gamma | k \\ \gamma \equiv 1 \pmod{3}}} \sum_{\substack{n, u \in \mathbb{Z}[\omega] \\ N(nu) \sim R/N(\gamma) \\ nu \equiv 1 \pmod{3}}} \frac{1}{N(\gamma nu)} \frac{\varphi(\gamma nu)}{\varphi(nu)} \\ &\ll (N(k)R)^{\varepsilon}, \end{aligned} \tag{8-15}$$

where the last display follows from standard lower bounds for the Euler  $\varphi$ -function and (2-14). □

Recall the convention for twisting a cusp form  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  in Remark 8.1. We replace the congruence  $\lambda^3 \nu \equiv \eta \pmod{\lambda^\ell q}$  with the equivalent congruence  $\lambda^4 \nu \equiv \lambda \eta \pmod{\lambda^{\ell+1} q}$ . We have the immediate consequence.

**Lemma 8.6.** *Let  $\ell \in \mathbb{Z}_{\geq 0}$ ,  $n \in \mathbb{Z}$ ,  $q \in \mathbb{Z}[\omega]$  with  $q \equiv 1 \pmod{3}$ , and  $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q \mathbb{Z}[\omega]$ . For  $\text{Re}(s) > 1$  we have*

$$\begin{aligned} \mathcal{D}(s, f; \lambda^\ell q, \eta, n) &= \frac{1}{N(\lambda^{\ell+1} q)} \sum_{r|q} \sum_{m=0}^{\ell+1} N(\lambda^m r) \mathcal{D}(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_{\lambda \eta}}, n), \\ \Lambda(s, f; \lambda^\ell q, \eta, n) &= \frac{1}{N(\lambda^{\ell+1} q)} \sum_{r|q} \sum_{m=0}^{\ell+1} N(\lambda^m r) \Lambda(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_{\lambda \eta}}, n), \end{aligned}$$

where  $\widehat{\psi_{\lambda^m r}(\cdot)_{\lambda \eta}}$  is given in (8-12).

To obtain a functional equation for  $\Lambda(s, f; \lambda^\ell q, \eta, n)$  under  $s \rightarrow 1-s$  it suffices to establish a functional equation for each  $\Lambda(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_{\lambda \eta}}, n)$ . We have two different cases according to whether  $m \in \mathbb{Z}_{\geq 6}$  or  $0 \leq m \leq 5$ .

**8.3. Functional equation 1:  $m \in \mathbb{Z}_{\geq 6}$ .** Suppose that  $\psi : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  is periodic modulo  $\lambda^m r$ , where  $r \equiv 1 \pmod{3}$ .

**Remark 8.7.** The version of the functional equation proved in this section uses the automorphy of  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  directly. It requires  $m \in \mathbb{Z}_{\geq 6}$ , and is useful for large  $m$ .

For each  $\zeta$  with  $\zeta^6 = 1$ , let  $\psi_\zeta^\# : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  be given by

$$\psi_\zeta^\#(u) := \frac{1}{N(\lambda^{m_r})} \sum_{\substack{a,d \pmod{\lambda^{m_r}} \\ a,d \equiv 1 \pmod{3} \\ ad \equiv 1 \pmod{\lambda^{m_r}}}} \psi(-\zeta^{-1}d) \left( \frac{\zeta \lambda^{m-1} r}{d} \right)_3 \check{e} \left( \frac{au}{\zeta \lambda^{m_r}} \right), \quad u \in \mathbb{Z}[\omega]. \quad (8-16)$$

The function  $\psi_\zeta^\#$  is periodic modulo  $\lambda^{m_r}$ .

**Proposition 8.8.** Let  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  be a cusp form with spectral parameter  $\tau_f \in 1 + i\mathbb{R}$ ,  $m \in \mathbb{Z}_{\geq 6}$ ,  $r \in \mathbb{Z}[\omega]$  with  $r \equiv 1 \pmod{3}$ , and  $\psi : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  be a periodic function modulo  $\lambda^{m_r}$ , supported only on residue classes coprime to  $\lambda^{m_r}$ . We have

$$f(w; \hat{\psi}) = \sum_{\zeta} f \left( -\frac{\bar{z}}{(\zeta^{-1} \lambda^{m-4} r)^2 (|z|^2 + v^2)}, \frac{v}{|\lambda^{m-4} r|^2 (|z|^2 + v^2)}; \psi_{\zeta^{-1}}^\# \right), \quad w = (z, v) \in \mathbb{H}^3, \quad (8-17)$$

where  $\hat{\psi}$  and  $\psi_\zeta^\#$  are given by (8-5) and (8-16) respectively.

*Proof.* We open the definition of the Fourier transform to obtain

$$f(w; \hat{\psi}) = \frac{1}{N(\lambda^{m_r})} \sum_{\zeta} \sum_{\substack{d \pmod{\lambda^{m_r}} \\ d \equiv 1 \pmod{3} \\ (d,r)=1}} \psi(-\zeta d) f \left( z - \frac{d}{\zeta^{-1} \lambda^{m-4} r}, v \right). \quad (8-18)$$

Given  $\zeta^{-1} \lambda^{m-4} r \in \mathbb{Z}[\omega]$  (with  $m \in \mathbb{Z}_{\geq 6}$ ), and each  $d \equiv 1 \pmod{3}$  in (8-18) with  $(d, r) = 1$ , there exists a matrix

$$\gamma := \begin{pmatrix} d & \lambda^4 b \\ -\zeta^{-1} \lambda^{m-4} r & a \end{pmatrix} \in \Gamma_1(3). \quad (8-19)$$

Note that the determinant equation of this matrix implies that  $ad \equiv 1 \pmod{\lambda^{m_r}}$ . A straightforward computation using (2-17) shows that

$$\left( z - \frac{d}{\zeta^{-1} \lambda^{m-4} r}, v \right) = \gamma \left( \frac{a}{\zeta^{-1} \lambda^{m-4} r} - \frac{\bar{z}}{(\zeta^{-1} \lambda^{m-4} r)^2 (|z|^2 + v^2)}, \frac{v}{|\lambda^{m-4} r|^2 (|z|^2 + v^2)} \right). \quad (8-20)$$

We use the fact that  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  to obtain

$$f \left( z - \frac{d}{\zeta^{-1} \lambda^{m-4} r}, v \right) = \chi(\gamma) f \left( \frac{a}{\zeta^{-1} \lambda^{m-4} r} - \frac{\bar{z}}{(\zeta^{-1} \lambda^{m-4} r)^2 (|z|^2 + v^2)}, \frac{v}{|\lambda^{m-4} r|^2 (|z|^2 + v^2)} \right), \quad (8-21)$$

where

$$\chi(\gamma) = \left( \frac{-\zeta^{-1} \lambda^{m-4} r}{d} \right)_3 = \left( \frac{\zeta^{-1} \lambda^{m-1} r}{d} \right)_3. \quad (8-22)$$

We combine (8-21)–(8-22) in (8-18). We then use the Fourier expansion (1-5) to open  $f$ , and then assemble the sum over  $d$  (equivalently  $a$ ). □

**Corollary 8.9.** *Let the notation be as in Proposition 8.8 and  $n \in \mathbb{Z}$ . For  $v > 0$  we have*

$$f(v; \hat{\psi}, n) = \frac{(-1)^n}{N(\lambda^{m-4r})^{|n|} v^{2|n|}} \left( \frac{\overline{\zeta^{-1} \lambda^{m-4r}}}{\zeta^{-1} \lambda^{m-4r}} \right)^{-n} \sum_{\zeta} f\left( \frac{1}{|\lambda^{m-4r}|^2 v}; \psi_{\zeta^{-1}}^{\#}, -n \right). \tag{8-23}$$

*Proof.* Setting  $z = 0$  in Proposition 8.8 (in particular, (8-17)) gives the result for  $n = 0$ . If  $n > 0$ , we write  $|z|^2 = z\bar{z}$  and apply the operator

$$\frac{1}{(2\pi i)^n} \left( \frac{\partial}{\partial z} \right)^n \Big|_{z=0}$$

to both sides of (8-17). If  $n < 0$ , we write  $|z|^2 = z\bar{z}$  and apply the operator

$$\frac{1}{(2\pi i)^{|n|}} \left( \frac{\partial}{\partial \bar{z}} \right)^{|n|} \Big|_{z=0}$$

to both sides of (8-17). A computation with the chain rule yields the result. □

**Proposition 8.10.** *Let the notation be as in Proposition 8.8 and  $n \in \mathbb{Z}$ . The completed Dirichlet series  $\Lambda(s, f; \hat{\psi}, n)$  and  $\Lambda(s, f; \psi_{\zeta}^{\#}, n)$  both admit meromorphic continuations to entire functions, and satisfy*

$$(-1)^n N(\lambda^{m-4r})^{2s-1} \left( \frac{\overline{\zeta^{-1} \lambda^{m-4r}}}{\zeta^{-1} \lambda^{m-4r}} \right)^n \Lambda(s, f; \hat{\psi}, n) = \sum_{\zeta} \Lambda(1-s, f; \psi_{\zeta^{-1}}^{\#}, -n). \tag{8-24}$$

*Proof.* Recall that for  $\text{Re } s > 1$  we have

$$\Lambda(s, f; \hat{\psi}, n) = \int_0^{\infty} f(v; \hat{\psi}, n) v^{2s+|n|-2} dv.$$

The function  $f(v; \hat{\psi}, n)$  has exponential decay at 0 and  $\infty$  by (8-2), (8-3), and termwise differentiation of (2-18) (with constant term identically zero). Thus  $\Lambda(s, f; \hat{\psi}, n)$  has analytic continuation to an entire function. The argument for  $f(v; \psi_{\zeta}^{\#}, n)$  is analogous.

We now prove (8-24). We have

$$\Lambda(s, f; \hat{\psi}, n) = \int_0^{N(\lambda^{m-4r})^{-1}} f(v; \hat{\psi}, n) v^{2s+|n|-2} dv + \int_{N(\lambda^{m-4r})^{-1}}^{\infty} f(v; \hat{\psi}, n) v^{2s+|n|-2} dv. \tag{8-25}$$

After applying Corollary 8.9, interchanging the order of summation and integration, and a change of variables, we obtain

$$\begin{aligned} & \int_0^{N(\lambda^{m-4r})^{-1}} f(v; \hat{\psi}, n) v^{2s+|n|-2} dv \\ &= \frac{(-1)^n}{N(\lambda^{m-4r})^{|n|}} \left( \frac{\overline{\zeta^{-1} \lambda^{m-4r}}}{\zeta^{-1} \lambda^{m-4r}} \right)^{-n} \sum_{\zeta} \int_0^{N(\lambda^{m-4r})^{-1}} f\left( \frac{1}{v|\lambda^{m-4r}|^2}; \psi_{\zeta^{-1}}^{\#}, -n \right) v^{2s-|n|-2} dv \\ &= (-1)^n N(\lambda^{m-4r})^{1-2s} \left( \frac{\overline{\zeta^{-1} \lambda^{m-4r}}}{\zeta^{-1} \lambda^{m-4r}} \right)^{-n} \sum_{\zeta} \int_1^{\infty} f(v; \psi_{\zeta^{-1}}^{\#}, -n) v^{-2s+|n|} dv \end{aligned} \tag{8-26}$$

and

$$\begin{aligned} & \int_{N(\lambda^{m-4}r)^{-1}}^{\infty} f(v; \hat{\psi}, n) v^{2s+|n|-2} dv \\ &= \frac{(-1)^n}{N(\lambda^{m-4}r)^{|n|}} \left( \frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r} \right)^{-n} \sum_{\zeta} \int_{N(\lambda^{m-4}r)^{-1}}^{\infty} f\left(\frac{1}{v|\lambda^{m-4}r|^2}; \psi_{\zeta^{-1}}^{\#}, -n\right) v^{2s-|n|-2} dv \\ &= (-1)^n N(\lambda^{m-4}r)^{1-2s} \left( \frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r} \right)^{-n} \sum_{\zeta} \int_0^1 f(v; \psi_{\zeta^{-1}}^{\#}, -n) v^{-2s+|n|} dv. \end{aligned} \tag{8-27}$$

Substituting (8-26) and (8-27) into (8-25) yields the result. □

**8.4. Functional equation 2 :  $m \in \mathbb{Z}_{\geq 0}$  absolutely bounded (in particular,  $0 \leq m \leq 5$ ).** Let  $m \in \mathbb{Z}_{\geq 0}$ . The functional equation we prove in this section is valid for all  $m \in \mathbb{Z}_{\geq 0}$ , but is really only useful when  $m$  is bounded by an absolute constant.

Recall that  $\Gamma := \text{SL}_2(\mathbb{Z}[\omega])$ . Let  $\Gamma'$  be a subgroup of  $\Gamma$  with  $[\Gamma : \Gamma'] < \infty$  and  $\Gamma' \subseteq \Gamma_1(9)$ . Then by [Proskurin 1998, Theorem 0.3.1] each cusp  $\sigma\infty$  ( $\sigma \in \Gamma$ ) of  $\Gamma'$  is essential with respect to  $\chi$ , and if  $\Gamma' := \Gamma_1(C)$  with  $C \equiv 0 \pmod{9}$ , then

$$\Gamma_1(C)_\sigma = C\mathbb{Z}[\omega] \quad \text{and} \quad \Gamma_1(C)_\sigma^* = (C\lambda)^{-1}\mathbb{Z}[\omega].$$

Suppose that  $\psi' : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  is periodic modulo  $\lambda^m$ , and that  $\psi'' : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  is periodic modulo  $r$ , where  $r \equiv 1 \pmod{3}$ . Let

$$\psi''^{*,\star}(u) := \frac{1}{N(r)} \sum_{\substack{a,d \pmod{r} \\ (\lambda^{2m+4}a)(\lambda^{2m+4}d) \equiv 1 \pmod{r}}} \psi''(-d) \left( \frac{\lambda^{2m+4}d}{r} \right)_3 \check{e}\left(\frac{au}{r}\right), \quad u \in \mathbb{Z}[\omega]. \tag{8-28}$$

The function  $\psi''^{*,\star}$  is periodic modulo  $r$ .

Let  $\gamma_{m,j} \in \Gamma_2$  for  $j = 1, \dots, [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$  be a fixed complete set of representatives for  $\Gamma_1(\lambda^{2m+4}) \backslash \Gamma_2$ . We have the convention that  $\gamma_{m,1} := I$  for all  $m \in \mathbb{Z}_{\geq 0}$ . For each  $j = 1, \dots, [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$ , let

$$(f \otimes \hat{\psi}')_j(w) := (f \otimes \hat{\psi}')(\gamma_{m,j}w), \quad w \in \mathbb{H}^3, \tag{8-29}$$

each having Fourier expansion

$$(f \otimes \hat{\psi}')_j(w) := \sum_{\substack{v \neq 0 \\ v \in \lambda^{-2m-3}\mathbb{Z}[\omega]}} \rho_{f \otimes \hat{\psi}',j}(v) v K_{\tau-1}(4\pi|v|v) \check{e}(vz), \quad w \in \mathbb{H}^3, \tag{8-30}$$

where  $\rho_{f \otimes \hat{\psi}',j}(v) \in \mathbb{C}$ .

If  $g \in \Gamma_2$ , then

$$\gamma_{m,j}g = g_{m,j}(g)\gamma_{m,k_{m,j}(g)} \quad \text{for some unique } g_{m,j}(g) \in \Gamma_1(\lambda^{2m+4}) \text{ and } 1 \leq k_{m,j}(g) \leq [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]. \tag{8-31}$$

For any  $g, h \in \Gamma_2$  we have

$$g_{m,j}(gh) = g_{m,j}(g)g_{m,k_{m,j}(g)}(h) \quad \text{and} \quad k_{m,j}(gh) = k_{m,k_{m,j}(g)}(h).$$

**Remark 8.11.** Using (8-31) we see that for  $g \in \Gamma$  we have

$$(f \otimes \hat{\psi}')_j(gw) = \chi(g_{m,j}(g))(f \otimes \hat{\psi}')_{k_{m,j}(g)}(w), \quad w \in \mathbb{H}^3. \tag{8-32}$$

Since  $\Gamma_1(\lambda^{2m+4})$  is a normal subgroup of  $\Gamma_2$  for  $m \in \mathbb{Z}_{\geq 0}$  (it is also a normal subgroup of  $\Gamma$ ) we have

$$k_{m,j}(g) = j \quad \text{for } g \in \Gamma_1(\lambda^{2m+4}) \text{ and all } j. \tag{8-33}$$

Then by [Patterson 1978, Lemma 2.1] we have

$$\chi(\gamma g \gamma^{-1}) = \chi(g) \quad \text{for } g \in \Gamma_1(\lambda^{2m+4}) \text{ and } \gamma \in \Gamma_2.$$

Thus for each  $j$  we have

$$(f \otimes \hat{\psi}')_j \in L^2(\Gamma_1(\lambda^{2m+4}) \backslash \mathbb{H}^3, \chi, \tau) \quad \text{is a cusp form.} \tag{8-34}$$

Following (8-1)–(8-4) we also have the functions  $(f \otimes \hat{\psi}')_j(\cdot, \Psi, n)$  and their associated Fourier expansions for each  $n \in \mathbb{Z}$ .

**Proposition 8.12.** *Let  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  be a cusp form with spectral parameter  $\tau_f \in 1 + i\mathbb{R}$ ,  $m \in \mathbb{Z}_{\geq 0}$ ,  $1 \leq j \leq [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$  an integer,  $r \in \mathbb{Z}[\omega]$  with  $r \equiv 1 \pmod{3}$ , and  $\psi' : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$  (resp.  $\psi'' : \mathbb{Z}[\omega] \rightarrow \mathbb{C}$ ) be periodic functions modulo  $\lambda^m$  (resp.  $r$ ). Further assume that  $\psi''$  is supported only on residue classes coprime to  $r$ .*

*Then there exist an integer  $1 \leq c(m, j; r) \leq [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$  and cube root of unity  $\omega(m, j; r)$  such that*

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'') = \omega(m, j; r)(f \otimes \hat{\psi}')_{c(m,j;r)}\left(-\frac{\bar{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}; \psi''^{\star}\right), \tag{8-35}$$

$$w = (z, v) \in \mathbb{H}^3,$$

where  $\psi''^{\star}$  is given in (8-28). Both  $c(m, j; r)$  and  $\omega(m, j; r)$  depend only on  $m \in \mathbb{Z}_{\geq 0}$ ,  $j \in \mathbb{Z}_{\geq 1}$ , and the residue class  $r \pmod{\lambda^{2m+4}}$ .

**Remark 8.13.** The reason why the functional equation proved in this section is only useful for  $m$  bounded by an absolute constant is because we use the automorphy for each  $(f \otimes \hat{\psi}')_j \in L^2(\Gamma_1(\lambda^{2m+4}) \backslash \mathbb{H}^3, \chi)$ .

*Proof.* We adapt the proof of [Dunn and Radziwiłł 2024, Lemma 5.2]. We open the definition of the Fourier transform and obtain

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'') = \frac{1}{N(r)} \sum_{\substack{d \pmod{r} \\ (d,r)=1}} \psi''(-d)(f \otimes \hat{\psi}')_j\left(z - \frac{\lambda^{2m+4}d}{r}, v\right). \tag{8-36}$$

Given  $r \equiv 1 \pmod{3}$  and each  $d \in \mathbb{Z}[\omega]$  in (8-36), we have  $(r, \lambda^{2m+4}d) = 1$ . Thus there exists a matrix

$$\begin{pmatrix} r & -\lambda^{2m+4}a \\ \lambda^{2m+4}d & b \end{pmatrix} \in \Gamma_1(3),$$

and hence there exists

$$\gamma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r & -\lambda^{2m+4}a \\ \lambda^{2m+4}d & b \end{pmatrix} = \begin{pmatrix} \lambda^{2m+4}d & b \\ -r & \lambda^{2m+4}a \end{pmatrix} \in \Gamma_2. \tag{8-37}$$

Note that we implicitly we used (3-1) in the above display. Also note we have the determinant equation

$$\lambda^{4m+8}ad + br = 1. \tag{8-38}$$

A straightforward computation using (2-17) shows that

$$\left( z - \frac{\lambda^{2m+4}d}{r}, v \right) = \gamma \left( \frac{\lambda^{2m+4}a}{r} - \frac{\bar{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)} \right). \tag{8-39}$$

We now carefully factorise the  $\gamma$  in (8-37) as a word in  $P$ ,  $T$  and  $E$  so that (8-39) and automorphy of  $(f \otimes \hat{\psi}')_j$  can be used in (8-36). For each  $x + y\omega \in \mathbb{Z}[\omega]$ ,  $x, y \in \mathbb{Z}$ , let

$$A(x + y\omega) := PT^{-x}PT^{-x+y}P = \begin{pmatrix} 1 & x + y\omega \\ 0 & 1 \end{pmatrix}.$$

For each  $r, b \in \mathbb{Z}[\omega]$  occurring in (8-37), let

$$S(r, b) := E^3 A(r)EA(b)EA(r) = \begin{pmatrix} b & -1 + br \\ 1 - br & 2r - br^2 \end{pmatrix} \in \Gamma_1(3).$$

Then

$$S(r, b)E\gamma = \begin{pmatrix} -\lambda^{2m+4}d + br + \lambda^{2m+4}bdr & -b - \lambda^{2m+4}ab + b^2r \\ r + 2\lambda^{2m+4}dr - br^2 - \lambda^{2m+4}bdr^2 & -\lambda^{2m+4}a + 2br + \lambda^{2m+4}abr - b^2r^2 \end{pmatrix} =: \tilde{\gamma}.$$

Using (8-38) we see that  $\tilde{\gamma} \in \Gamma_1(\lambda^{2m+4})$  and we write

$$\gamma = E^3 S(r, b)^{-1} \tilde{\gamma}. \tag{8-40}$$

We use (8-39), (8-40), (8-32), and (8-34) to obtain

$$\begin{aligned} & (f \otimes \hat{\psi}')_j \left( z - \frac{\lambda^{2m+4}d}{r}, v \right) \\ &= \chi(g_{m,j}(E^3 S(r, b)^{-1})) \cdot \chi(\tilde{\gamma})(f \otimes \hat{\psi}')_{k_{m,j}(E^3 S(r, b)^{-1})} \left( \frac{\lambda^{2m+4}a}{r} - \frac{\bar{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)} \right). \end{aligned} \tag{8-41}$$

By (8-33) the integer  $k_{m,j}(E^3 S(r, b)^{-1})$  depends only on  $m, j \in \mathbb{N}$  and matrix residue class

$$E^3 S(r, b)^{-1} = \begin{pmatrix} -1 + br & b \\ -2r + br^2 & -1 + br \end{pmatrix} \pmod{\lambda^{2m+4}}.$$

Thus the integer  $k_{m,j}(E^3 S(r, b)^{-1})$  depends only on  $m, j \in \mathbb{N}$  and the residue class  $r \pmod{\lambda^{2m+4}}$ , since  $b \pmod{\lambda^{2m+4}}$  is determined by (8-38). By (8-31) we have

$$g_{m,j}(E^3 S(r, b)^{-1}) = \gamma_{m,j} E^3 S(r, b)^{-1} \gamma_{m,k_{m,j}(E^3 S(r, b)^{-1})}^{-1} \in \Gamma_1(\lambda^{2m+4}),$$

and each matrix in the product on the right side is an element of  $\Gamma_2$ . Thus

$$\chi(g_{m,j}(E^3 S(r, b)^{-1})) = \chi(\gamma_{m,j})\chi(E^3)\chi(S(r, b)^{-1})\chi(\gamma_{m,k_{m,j}(E^3 S(r, b)^{-1})}^{-1}) = \chi(\gamma_{m,j})\overline{\chi(\gamma_{m,k_{m,j}(E^3 S(r, b)^{-1})})}$$

is a cube root of unity depending only on  $m \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}_{\geq 1}$  and the residue class  $r \pmod{\lambda^{2m+4}}$ . For ease of notation we relabel

$$\mathfrak{c}(m, j; r) := k_{m,j}(E^3 S(r, b)^{-1}), \tag{8-42}$$

$$\omega(m, j; r) := \chi(\gamma_{m,j})\overline{\chi(\gamma_{m,k_{m,j}(E^3 S(r, b)^{-1})})}. \tag{8-43}$$

A computation following [Dunn and Radziwiłł 2024, p. 23] establishes that

$$\chi(\tilde{\gamma}) = \left(\frac{\lambda^{2m+4}d}{r}\right)_3. \tag{8-44}$$

We combine (8-41)–(8-44) in (8-36). We then use the Fourier expansion (8-30) to open  $(f \otimes \hat{\psi}')_{c(m,j;r)}$ , and assembling the sum over  $d$  (equivalently  $a$ ) shows that

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'') = \omega(m, j; r)(f \otimes \hat{\psi}')_{c(m,j;r)}\left(-\frac{\bar{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}; \psi''^{\star}\right),$$

as required. □

**Corollary 8.14.** *Let the notation be as in Proposition 8.12 and  $n \in \mathbb{Z}$ . For  $v > 0$  we have*

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'', n) = \frac{(-1)^n \omega(m, j; r)}{N(r)^{|n|} v^{2|n|}} \left(\frac{\bar{r}}{r}\right)^{-n} (f \otimes \hat{\psi}')_{c(m,j;r)}\left(\frac{1}{|r|^2 v}; \psi''^{\star}, -n\right). \tag{8-45}$$

The proof is analogous to that of Corollary 8.9 so we omit it.

**Proposition 8.15.** *Let the notation be as in Proposition 8.12 and  $n \in \mathbb{Z}$ . The completed Dirichlet series  $\Lambda(s, (f \otimes \hat{\psi}')_j; \hat{\psi}'', n)$  and  $\Lambda(s, (f \otimes \hat{\psi}')_j; \psi''^{\star}, n)$  both admit meromorphic continuation to an entire function, and satisfy*

$$(-1)^n N(r)^{2s-1} \left(\frac{\bar{r}}{r}\right)^n \Lambda(s, (f \otimes \hat{\psi}')_j; \hat{\psi}'', n) = \omega(m, j; r) \Lambda(1-s, (f \otimes \hat{\psi}')_{c(m,j;r)}; \psi''^{\star}, -n). \tag{8-46}$$

Proposition 8.15 follows from Corollary 8.14, and the proof is analogous to that of Proposition 8.10. We omit the proof.

**8.5. Level aspect Voronoi formula.** We now prove a Voronoi summation formula for the Fourier coefficients for the form  $f(w; \lambda^\ell q, \eta)$  given in (8-9).

We recall some basic facts concerning the complex Mellin transform. Let  $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ . Let  $K, M \geq 1$  and  $V_{K,M} \in C_c^\infty(\mathbb{C}^\times)$  have compact support contained in the disc of radius 100 (say), and also satisfy

$$\frac{\partial^{i+j}}{\partial x^i \partial x^j} V_{K,M}(z) \ll_{i,j} MK^{i+j} \quad \text{for all } z \in \mathbb{C}^\times. \tag{8-47}$$

The complex Mellin transform is given by

$$\widehat{V}_{K,M}(s, n) := \int_{\mathbb{C}^\times} V_{K,M}(z) |z|^{2s} (z/|z|)^{-n} d_\times z \tag{8-48}$$

for  $s \in \mathbb{C}$  and  $n \in \mathbb{Z}$ , where  $d_\times z := |z|^{-2} dx dy$ . Note that  $\widehat{V}_{K,M}(s, n)$  is entire with respect to  $s$  for each  $n \in \mathbb{Z}$ . After making a change of variables  $z = re(\theta/2)$  with  $r \in (0, \infty)$  and  $\theta \in [0, 2\pi)$ , we obtain

$$\widehat{V}_{K,M}(s, n) = \int_0^\infty \int_0^{2\pi} V_{K,M}(re(\theta/2)) r^{2s-1} e(-n\theta/2) d\theta dr. \tag{8-49}$$

After repeated integration by parts, we obtain

$$\widehat{V}_{K,M}(s, n) \ll_{j,k} M \cdot \min \left\{ 1, \frac{K^{j+k}}{|(2s)_j|(1+|n|)^k} \right\}$$

for  $j, k \in \mathbb{Z}_{\geq 0}$ ,  $s \in \mathbb{C}$  in a fixed vertical strip, and  $n \in \mathbb{Z}$ . It follows for  $D_1, D_2 \geq 0$ , we have

$$\widehat{V}_{K,M}(s, n) \ll_{D_1, D_2} \frac{MK^{D_1+D_2}}{(1+|s|)^{D_1}(1+|n|)^{D_2}} \tag{8-50}$$

for  $s \in \mathbb{C}$  in a fixed vertical strip, and  $n \in \mathbb{Z}$ . The complex Mellin inversion formula is given by

$$V_{K,M}(z) = \frac{1}{2\pi^2 i} \sum_{n \in \mathbb{Z}} \int_{(\sigma)} \widehat{V}_{K,M}(s, n) |z|^{-2s} (z/|z|)^n ds \tag{8-51}$$

for  $\sigma > 0$ ,  $z \in \mathbb{C}$ , and  $n \in \mathbb{Z}$ .

**Remark 8.16.** Suppose further that  $V_{K,M}$  is radial, i.e.,  $V_{K,M}(re(\theta)) = V_K(r)$  for all  $\theta \in \mathbb{R}$ . Then

$$\widehat{V}_{K,M}(s, n) = \delta_{n=0} 2\pi \cdot \int_0^\infty V_{K,M}(r) r^{2s-1} dr = \delta_{n=0} 2\pi \cdot \widehat{V}_{K,M}(2s) = \delta_{n=0} \pi \cdot \widehat{W}_{K,M}(s), \tag{8-52}$$

where  $\widehat{V}_{K,M}(s)$  denotes the usual Mellin transform for functions on  $(0, \infty)$ , and  $W_{K,M}$  is such that  $W_{K,M}(r) = V_{K,M}(\sqrt{r})$ . Then (8-51) becomes the standard Mellin inversion formula for functions on  $(0, \infty)$  after a change of variable in  $s$ .

**Proposition 8.17.** Let  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  be a cusp form with spectral parameter  $\tau_f \in 1 + i\mathbb{R}$ ,  $\ell \in \mathbb{Z}_{\geq 0}$ ,  $q \in \mathbb{Z}[\omega]$  with  $q \equiv 1 \pmod{3}$ ,  $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q \mathbb{Z}[\omega]$ , and  $V_{K,M} \in C_c^\infty(\mathbb{C}^\times)$  be a smooth function with compact support in the disc of radius 100 satisfying (8-47) for some  $K, M \geq 1$ . Then for  $X > 0$  we have

$$\sum_{\substack{v \in \lambda^{-3} \mathbb{Z}[\omega] \\ \lambda^3 v \equiv \eta \pmod{\lambda^\ell q}}} \rho_f(v) V_{K,M}(v/\sqrt{X}) = \frac{X}{N(\lambda^{\ell+1} q)} \sum_{r|q} \sum_{n \in \mathbb{Z}} (-1)^n \sum_{m=0}^{\ell+1} \sum_{p=1}^2 Z_{pf}(X, \lambda^m r, \eta, n; \dot{V}_{K,M}) \tag{8-53}$$

where

$$\begin{aligned} Z_{1f}(X, \lambda^m r, \eta, n; \dot{V}_{K,M}) &:= \delta_{0 \leq m \leq \min\{5, \ell+1\}} \cdot N(\lambda^m) \left(\frac{\bar{r}}{r}\right)^{-n} \omega(m, 1; r) \\ &\times \sum_{v \in \lambda^{-2m-3} \mathbb{Z}[\omega]} \rho_{f \otimes \widehat{\psi}_{\lambda^m}(\cdot)_{\lambda\eta, c(m, 1; r)}}(v) \left(\frac{v}{|v|}\right)^{-n} \psi_r^\star(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v) \dot{V}_{K,M}\left(\frac{N(v)}{N(r)^2/X}, n\right), \end{aligned} \tag{8-54}$$

$$\begin{aligned} Z_{2f}(X, \lambda^m r, \eta, n; \dot{V}_{K,M}) &:= \delta_{6 \leq m \leq \ell+1} \cdot N(\lambda^4) \sum_{\zeta} \left(\frac{\zeta^{-1} \lambda^{m-4} r}{\zeta^{-1} \lambda^{m-4} r}\right)^{-n} \\ &\times \sum_{v \in \lambda^{-3} \mathbb{Z}[\omega]} \rho_f(v) \left(\frac{v}{|v|}\right)^{-n} \psi_{\lambda^m r}^\#(\cdot)_{\lambda\eta, \zeta^{-1}}(\lambda^4 v) \dot{V}_{K,M}\left(\frac{N(v)}{N(\lambda^{m-4} r)^2/X}, n\right), \end{aligned} \tag{8-55}$$

where  $\psi_\zeta^\#$  and  $\psi^\star$  are given in (8-16) and (8-28) (with  $\psi'' \rightarrow \psi$ ) respectively,  $\dot{V}_{K,M}(\cdot, n) : (0, \infty) \rightarrow \mathbb{R}$  is given by

$$\dot{V}_{K,M}(Y, n) := \frac{1}{2\pi^2 i} \int_{(2)} Y^{-s} \frac{G_\infty(s, \tau_f, n)}{G_\infty(1-s, \tau_f, n)} \widehat{V}_{K,M}(1-s, n) ds, \tag{8-56}$$

$G_\infty(s, \tau, n)$  is given in (8-7), and  $\omega(m, j, r)$  and  $\mathfrak{c}(m, j, r)$  are both as in Proposition 8.12.

**Remark 8.18.** From Remark 8.16 we see that if  $V_K$  is radial then only  $n = 0$  is relevant on the right side of (8-53). In this case  $n$  is omitted from the notation.

*Proof.* Recall the definition of the function  $\psi_{\lambda^m r}(\cdot)_\eta$  in (8-11), and its Fourier transform  $\widehat{\psi}_{\lambda^m r}(\cdot)_\eta$  in (8-12). We apply complex Mellin inversion (8-51) to the smooth function  $V_K$ , Lemma 8.6, and then interchange of the order of integration and summation by absolute convergence. This yields

$$\begin{aligned} & \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv \eta \pmod{\lambda^\ell q}}} \rho_f(v) V_{K,M}(v/\sqrt{X}) \\ &= \frac{1}{2\pi^2 i} \frac{1}{N(\lambda^{\ell+1}q)} \sum_{r|q} \left( \sum_{m=0}^{\min\{5, \ell+1\}} + \sum_{m=6}^{\ell+1} \right) N(\lambda^m r) \sum_{n \in \mathbb{Z}} \int_{(2)} \widehat{V}_{K,M}(s, n) X^s \mathcal{D}(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_\eta}, n) ds. \end{aligned} \tag{8-57}$$

The Chinese remainder theorem implies that

$$\widehat{\psi_{\lambda^m r}(\cdot)_\eta}(u) = \widehat{\psi_{\lambda^m}(\cdot)_\eta}(u) \widehat{\psi_r(\cdot)_\eta}(u), \quad u \in \mathbb{Z}[\omega], \tag{8-58}$$

and by a change of variables we have

$$\widehat{\psi_r(\cdot)_\eta}(\lambda^{2m}u) = \widehat{\psi_r(\cdot)_\eta}(u), \quad u \in \mathbb{Z}[\omega]. \tag{8-59}$$

Recall the definition of twisting (8-1) and the convention in Remark 8.1. Using (8-2) we see that  $f \otimes \widehat{\psi_{\lambda^m}(\cdot)_\eta} \in L^2(\Gamma_1(\lambda^{2m+4}) \backslash \mathbb{H}^3, \chi, \tau)$  is a cusp form. Using (8-58) and (8-59) we obtain

$$\begin{aligned} f(w; \widehat{\psi_{\lambda^m r}(\cdot)_\eta}, n) &= f(w; \widehat{\psi_{\lambda^m}(\cdot)_\eta} \widehat{\psi_r(\cdot)_\eta} \circ \lambda^{2m}, n) \\ &= (f \otimes \widehat{\psi_{\lambda^m}(\cdot)_\eta})(w; \widehat{\psi_r(\cdot)_\eta}, n) \end{aligned}$$

for all  $0 \leq m \leq \ell + 1$ ,  $r \mid q$ , and  $\eta \in \mathbb{Z}[\omega]$ . The analogous Dirichlet series identity reads

$$\mathcal{D}(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_\eta}, n) = \mathcal{D}(s, f \otimes \widehat{\psi_{\lambda^m}(\cdot)_\eta}; \widehat{\psi_r(\cdot)_\eta}, n), \quad \text{Re}(s) > 1. \tag{8-60}$$

Substituting (8-60) into (8-57) for  $0 \leq m \leq \min\{5, \ell + 1\}$  we see that the right side of (8-57) is equal to

$$\begin{aligned} & \frac{1}{2\pi^2 i} \frac{1}{N(\lambda^{\ell+1}q)} \sum_{r|q} \sum_{n \in \mathbb{Z}} \left( \sum_{m=0}^{\min\{5, \ell+1\}} N(\lambda^m r) \int_{(2)} \widehat{V}_{K,M}(s, n) X^s \mathcal{D}(s, f \otimes \widehat{\psi_{\lambda^m}(\cdot)_\eta}; \widehat{\psi_r(\cdot)_\eta}, n) ds \right. \\ & \quad \left. + \sum_{m=6}^{\ell+1} N(\lambda^m r) \int_{(2)} \widehat{V}_{K,M}(s, n) X^s \mathcal{D}(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_\eta}, n) ds \right). \end{aligned} \tag{8-61}$$

Both of the integrands in (8-61) are entire by Propositions 8.10 and 8.15 and Lemma 8.2. We shift the contour in (8-57) to  $\text{Re}(s) = -1$  and then use the functional equations (8-24) and (8-46). We see that (8-61) is equal to

$$\begin{aligned} & \frac{1}{2\pi^2 i} \frac{1}{N(\lambda^{\ell+1}q)} \sum_{r|q} \sum_{n \in \mathbb{Z}} (-1)^n \\ & \times \left( \sum_{m=0}^{\min\{5, \ell+1\}} N(\lambda^m) N(r)^2 \left(\frac{\bar{r}}{r}\right)^{-n} \omega(m, 1; r) \int_{(-1)} \widehat{V}_{K, M}(s, n) \left(\frac{X}{N(r)^2}\right)^s \frac{G_\infty(1-s, \tau_f, -n)}{G_\infty(s, \tau_f, n)} \right. \\ & \quad \times \mathcal{D}(1-s, (f \otimes \widehat{\psi}_{\lambda^m(\cdot)_{\lambda\eta}})_{\mathfrak{c}(m, 1; r)}; \psi_r^*(\cdot)_{\lambda^{2m+1}\eta}, -n) ds \\ & \left. + \sum_{m=6}^{\ell+1} \sum_{\zeta} N(\lambda^{2m-4}) N(r)^2 \left(\frac{\bar{\zeta}^{-1}\lambda^{m-4}r}{\zeta^{-1}\lambda^{m-4}r}\right)^{-n} \int_{(-1)} \widehat{V}_{K, M}(s, n) \left(\frac{X}{N(\lambda^{m-4}r)^2}\right)^s \right. \\ & \quad \times \left. \frac{G_\infty(1-s, \tau_f, -n)}{G_\infty(s, \tau_f, n)} \mathcal{D}(1-s, f; \psi_{\lambda^m r}^\#(\cdot)_{\lambda\eta, \zeta^{-1}}, -n) ds \right). \quad (8-62) \end{aligned}$$

We make the change of variable  $s \rightarrow 1 - s$  in both integrals in (8-62), open up both of the Dirichlet series in the region of absolute convergence, and interchange the order of summation and integration to obtain (8-53) with the transforms given by (8-54)–(8-56).  $\square$

We now compute the Archimedean and non-Archimedean transforms on the dual side of the Voronoi formula in Proposition 8.17. Recall that  $K_{\Gamma', \sigma, \xi}(m, n, c)$  denotes a cubic Kloosterman attached to the cusp pair  $(\sigma, \xi)$  of  $\Gamma'$ ; see (3-2).

**Lemma 8.19.** *Let  $m \in \mathbb{Z}_{\geq 6}$ ,  $r \in \mathbb{Z}[\omega]$  with  $r \equiv 1 \pmod{3}$ ,  $\eta \in \mathbb{Z}[\omega]/\lambda^m r \mathbb{Z}[\omega]$ ,  $\psi_{\lambda^m r}(\cdot)_\eta$  be as in (8-11), and  $\zeta$  be such that  $\zeta^6 = 1$ . Then for  $v \in \lambda^{-3} \mathbb{Z}[\omega]$  we have*

$$\psi_{\lambda^m r}^\#(\cdot)_{\lambda\eta, \zeta}(\lambda^4 v) = \frac{1}{N(\lambda^{m+3}r)} K_{\Gamma_1(3), \sigma, \sigma}(\lambda^3 v, \eta, \zeta \lambda^{m-1} r),$$

where  $\psi_\zeta^\#$  is given in (8-16),  $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , and the cubic Kloosterman sum is given in (3-2).

*Proof.* We have

$$\begin{aligned} \psi_{\lambda^m r}^\#(\cdot)_{\lambda\eta, \zeta}(\lambda^4 v) &= \frac{1}{N(\lambda^m r)} \sum_{\substack{a, d \pmod{\lambda^m r} \\ a, d \equiv 1 \pmod{3} \\ ad \equiv 1 \pmod{\lambda^m r}}} \left(\frac{\zeta \lambda^{m-1} r}{d}\right)_3 \check{e}\left(\frac{a\lambda^3 v + d\eta}{\zeta \lambda^{m-1} r}\right) \\ &= \frac{1}{N(\lambda^{m+2}r)} \sum_{\substack{a, d \pmod{\lambda^{m+1}r} \\ a, d \equiv 1 \pmod{3} \\ ad \equiv 1 \pmod{\lambda^m r}}} \left(\frac{\zeta \lambda^{m-1} r}{d}\right)_3 \check{e}\left(\frac{a\lambda^3 v + d\eta}{\zeta \lambda^{m-1} r}\right) \\ &= \frac{1}{N(\lambda^{m+3}r)} \sum_{\substack{a, d \pmod{\lambda^{m+1}r} \\ a, d \equiv 1 \pmod{3} \\ ad \equiv 1 \pmod{\lambda^{m-1}r}}} \left(\frac{\zeta \lambda^{m-1} r}{d}\right)_3 \check{e}\left(\frac{a\lambda^3 v + d\eta}{\zeta \lambda^{m-1} r}\right), \quad (8-63) \end{aligned}$$

and the result follows from Lemma 3.3.  $\square$

**Lemma 8.20.** *Let the notation be as in Lemma 8.19 and  $m \in \mathbb{Z}_{\geq 0}$ . Then for  $v \in \lambda^{-2m-3}\mathbb{Z}[\omega]$  we have*

$$\psi_r^*(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v) = \frac{1}{N(r)} K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}(\lambda^{2m+3}v), \overline{\lambda^3}\eta, r),$$

where  $\psi^*$  is given in (8-28),  $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and  $\overline{\lambda^\ell} \in \mathbb{Z}[\omega]$  is such that  $\overline{\lambda^\ell}\lambda^\ell \equiv 1 \pmod{r}$  for  $\ell \in \mathbb{Z}_{\geq 0}$ .

*Proof.* By definition

$$\psi_r^*(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v) = \frac{1}{N(r)} \sum_{\substack{a,d \pmod{r} \\ (\lambda^{2m+4}a)(\lambda^{2m+4}d) \equiv 1 \pmod{r}}} \left(\frac{\lambda^{2m+4}d}{r}\right)_3 \check{e}\left(\frac{a(\lambda^{2m+4}v) + d(\lambda^{2m+1}\eta)}{r}\right). \tag{8-64}$$

The change of variables  $a \rightarrow \overline{\lambda^{2m+4}}a \pmod{r}$  and  $d \rightarrow \overline{\lambda^{2m+4}}d \pmod{r}$  shows that the right side of (8-64) is equal to

$$\frac{1}{N(r)} \sum_{\substack{a,d \pmod{r} \\ ad \equiv 1 \pmod{r}}} \left(\frac{d}{r}\right)_3 \check{e}\left(\frac{\overline{\lambda^{2m+3}}a(\lambda^{2m+3}v) + \overline{\lambda^3}\eta d}{r}\right), \tag{8-65}$$

and we can lift this to the sum

$$\frac{1}{N(r)} \sum_{\substack{a,d \pmod{3r} \\ a,d \equiv 0 \pmod{3} \\ ad \equiv 1 \pmod{r}}} \left(\frac{d}{r}\right)_3 \check{e}\left(\frac{\overline{\lambda^{2m+3}}a(\lambda^{2m+3}v) + \overline{\lambda^3}\eta d}{r}\right),$$

and the result now follows from Lemma 3.4. □

**Lemma 8.21.** *Let  $K, M \geq 1$  and  $V_{K,M} \in C_c^\infty(\mathbb{C}^\times)$  be a smooth function with compact support in  $[1, 2]$  whose derivatives satisfy (8-47). Let  $\tau \in 1 + i\mathbb{R}$ ,  $n \in \mathbb{Z}$ ,  $G_\infty(s, \tau, n)$  be as in (8-7), and let  $\dot{V}_{K,M}(\cdot, n) : (0, \infty) \rightarrow \mathbb{C}$  be as in (8-56). Then for  $D_1 > 0$  and  $D_2 \geq 0$  we have*

$$\dot{V}_{K,M}(Y, n) \ll_{\tau, D_1, D_2} MK^{4(D_1+D_2)} Y^{-D_1} (|n| + 1)^{4D_1-4D_2-2}$$

for all  $Y > 0$ .

*Proof.* In the definition (8-56) we move the contour to  $\text{Re}(s) = D_1$ . Stirling’s formula [Olver et al. 2018, (5.11.1)] implies that

$$\frac{G_\infty(s, \tau, -n)}{G_\infty(1-s, \tau, n)} \asymp \left|s + \frac{1}{2}|n| - \frac{1}{2}(\tau - 1)\right|^{2D_1-1} \cdot \left|s + \frac{1}{2}|n| + \frac{1}{2}(\tau - 1)\right|^{2D_1-1}, \tag{8-66}$$

as  $|\text{Im}(s \pm \frac{1}{2}(\tau - 1))| \rightarrow \infty$ . Using (8-50) (with  $D_1 \rightarrow 4D_1$  and  $D_2 \rightarrow 4D_2$ ) and (8-66) in (8-56) we obtain

$$\begin{aligned} &\dot{V}_{K,M}(Y, n) \\ &\ll_{D_1, D_2} MK^{4(D_1+D_2)} Y^{-D_1} (1 + |n|)^{-4D_2} \left( \int_{(D_1)} \frac{\left|s + \frac{1}{2}|n| - \frac{1}{2}(\tau - 1)\right|^{2D_1-1} \cdot \left|s + \frac{1}{2}|n| + \frac{1}{2}(\tau - 1)\right|^{2D_1-1}}{(1 + |1-s|)^{4D_1}} |ds| \right) \\ &\ll_{\tau, D_1, D_2} MK^{4(D_1+D_2)} Y^{-D_1} (1 + |n|)^{4D_1-4D_2-2}, \end{aligned}$$

as required. □

**8.6. Level aspect Voronoi summation for multiple sums.** Here we record a Voronoi formula that is an iterated version of Proposition 8.17. Let  $\mathbf{z} = (z_1, z_2) = (x_{11} + iy_{12}, x_{21} + iy_{22}) \in (\mathbb{C}^\times)^2$ ,  $x_{11}, y_{12}, x_{21}, y_{22} \in \mathbb{R}$ . Let  $K, M \geq 1$ , and  $H_{K,M} \in C_c^\infty((\mathbb{C}^\times)^2)$  be a smooth function with compact support in a ball of radius 100 such that for any  $\mathbf{i} = (i_{11}, i_{12}, i_{21}, i_{22}) \in (\mathbb{Z}_{\geq 0})^4$  we have

$$\partial^{\mathbf{i}} H_{K,M}(\mathbf{z}) \ll_{\mathbf{i}} MK^{\sum_{1 \leq j, k \leq 2} i_{jk}}, \quad \mathbf{z} \in (\mathbb{C}^\times)^2. \tag{8-67}$$

If  $M = 1$  then  $M$  is omitted from the notation and we write  $H_K$ . For each  $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ , consider the double complex Mellin transform

$$\widehat{H}_{K,M}(\mathbf{s}, \mathbf{n}) := \iint_{(\mathbb{C}^\times)^2} H_{K,M}(\mathbf{z}) \left( \prod_{i=1}^2 |z_i|^{2s_i} \left( \frac{z_i}{|z_i|} \right)^{-n_i} \right) d_{\times} \mathbf{z}, \quad \mathbf{s} = (s_1, s_2) \in \mathbb{C}^2, \tag{8-68}$$

where  $d_{\times} \mathbf{z} := dx_1 dy_1 dx_2 dy_2 / |z_1 z_2|^2$ . For  $\mathbf{D} := (D_{11}, D_{12}, D_{21}, D_{22}) \in (\mathbb{R}_{\geq 0})^4$ , repeated integration by parts using polar coordinates yields the bound

$$\widehat{H}_{K,M}(\mathbf{s}, \mathbf{n}) \ll_{\tau, \mathbf{D}} MK^{\sum_{1 \leq i, j \leq 2} D_{ij}} \cdot \prod_{i=1}^2 (1 + |s_i|)^{-D_{i1}} (1 + |n_i|)^{-D_{i2}}. \tag{8-69}$$

Consider the function  $\ddot{H}_{K,M}(\cdot, \mathbf{n}) : (0, \infty)^2 \rightarrow \mathbb{R}$  given by

$$\begin{aligned} \ddot{H}_{K,M}(\mathbf{Y}, \mathbf{n}) &:= \frac{1}{(2\pi^2 i)^2} \int_{(2)} \int_{(2)} \left( \prod_{i=1}^2 Y_i^{-s_i} \frac{G_\infty(s_i, \tau_f, -n_i)}{G_\infty(1 - s_i, \tau_f, n_i)} \right) \widehat{H}_{K,M}(\mathbf{1} - \mathbf{s}, \mathbf{n}) ds, \\ \mathbf{Y} &= (Y_1, Y_2) \in (0, \infty)^2, \end{aligned} \tag{8-70}$$

where  $G_\infty(s, \tau, n)$  is given by (8-7), and  $ds := ds_1 ds_2$ . After moving the contours in (8-70) to  $\text{Re}(s_1) = D_{11} > 0$  and  $\text{Re}(s_2) = D_{21} > 0$ , observe that (8-66) and (8-69) applied to (8-70) imply that

$$\ddot{H}_{K,M}(\mathbf{Y}, \mathbf{n}) \ll_{\tau, \mathbf{D}} MK^{4(\sum_{1 \leq i, j \leq 2} D_{ij})} \cdot \prod_{i=1}^2 Y_i^{-D_{i1}} (|n_i| + 1)^{4D_{i1} - 4D_{i2} - 2}, \quad \mathbf{Y} \in (0, \infty)^2. \tag{8-71}$$

Mellin inversion and an iterated application of the functional equation in Proposition 8.17 yields the following result. We omit the proof for the sake of brevity.

**Proposition 8.22.** *Let  $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$  be a cusp form with spectral parameter  $\tau_f \in 1 + i\mathbb{R}$ ,  $\boldsymbol{\ell} = (\ell_1, \ell_2) \in (\mathbb{Z}_{\geq 0})^2$ ,  $\mathbf{q} = (q_1, q_2) \in (\mathbb{Z}[\omega])^2$  with  $q_1, q_2 \equiv 1 \pmod{3}$ , and  $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathbb{Z}[\omega] / \lambda^{\ell_1} q_1 \mathbb{Z}[\omega] \times \mathbb{Z}[\omega] / \lambda^{\ell_2} q_2 \mathbb{Z}[\omega]$ . Let  $H_{K,M} \in C_c^\infty((\mathbb{C}^\times)^2)$  be a smooth function with compact support in the disc of radius 100 satisfying (8-67) for some  $K, M \geq 1$ . Then for  $\mathbf{X} = (X_1, X_2) \in (0, \infty)^2$  we have*

$$\begin{aligned} &\sum_{\substack{\mathbf{v} \in (\lambda^{-3} \mathbb{Z}[\omega])^2 \\ \forall i: \lambda^3 v_i \equiv \eta_i \pmod{\lambda^{\ell_i} q_i}}} \rho_f(v_1) \overline{\rho_f(v_2)} H_{K,M} \left( \frac{v_1}{\sqrt{X_1}}, \frac{v_2}{\sqrt{X_2}} \right) \\ &= \frac{X_1 X_2}{N(\lambda^{\ell_1+1} q_1) N(\lambda^{\ell_2+1} q_2)} \sum_{\substack{k \in (\mathbb{Z}[\omega] / \lambda^{14} \mathbb{Z}[\omega])^2 \\ \forall i: k_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{m}, \mathbf{r} \\ \forall i: 0 \leq m_i \leq \ell_i + 1 \\ \forall i: r_i | q_i \\ \forall i: r_i \equiv k_i \pmod{\lambda^{14}}}} \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1 + n_2} \sum_{p=1}^4 \mathcal{D}_{pf}(\mathbf{X}, \lambda^{\mathbf{m}} \mathbf{r}, \boldsymbol{\eta}, \mathbf{n}; \ddot{H}_{K,M}), \end{aligned} \tag{8-72}$$

where

$$\begin{aligned}
 & \mathcal{D}_{1f}(\mathbf{X}, \boldsymbol{\lambda}^m \mathbf{r}, \boldsymbol{\eta}, \mathbf{n}; \ddot{H}_{K,M}) \\
 & := \delta_{m_1 \in [0, \min\{5, \ell_1 + 1\}]} \cdot \delta_{m_2 \in [0, \min\{5, \ell_2 + 1\}]} \cdot N(\lambda^{m_1}) N(\lambda^{m_2}) \left(\frac{\bar{r}_1}{r_1}\right)^{-n_1} \left(\frac{\bar{r}_2}{r_2}\right)^{n_2} \overline{\omega(m_1; 1, k_1) \omega(m_2, 1, k_2)} \\
 & \times \sum_{\substack{v_1 \in \lambda^{-2m_1-3}\mathbb{Z}[\omega] \\ v_2 \in \lambda^{-2m_2-3}\mathbb{Z}[\omega]}} \rho_{f \otimes \widehat{\psi_{\lambda^{m_1}(\cdot)}_{\lambda\eta_1, c(m_1, 1; k_1)}}(v_1)} \overline{\rho_{f \otimes \widehat{\psi_{\lambda^{m_2}(\cdot)}_{\lambda\eta_2, c(m_2, 1; k_2)}}(v_2)}} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 & \times \psi_{r_1}^*(\cdot)_{\lambda^{2m_1+1}\eta_1} (\lambda^{2m_1+4} v_1) \overline{\psi_{r_2}^*(\cdot)_{\lambda^{2m_2+1}\eta_2} (\lambda^{2m_2+4} v_2)} \ddot{H}_{K,M} \left( \frac{N(v_1)}{N(r_1)^2/X_1}, \frac{N(v_2)}{N(r_2)^2/X_2}, \mathbf{n} \right), \tag{8-73}
 \end{aligned}$$

$$\begin{aligned}
 & \mathcal{D}_{2f}(\mathbf{X}, \boldsymbol{\lambda}^m \mathbf{r}, \boldsymbol{\eta}, \mathbf{n}; \ddot{H}_{K,M}) \\
 & := \delta_{m_1 \in [0, \min\{5, \ell_1 + 1\}]} \cdot \delta_{m_2 \in [6, \ell_2 + 1]} N(\lambda^{m_1}) N(\lambda^4) \left(\frac{\bar{r}_1}{r_1}\right)^{-n_1} \overline{\omega(m_1, 1; k_1)} \\
 & \times \sum_{\substack{v_1 \in \lambda^{-2m_1-3}\mathbb{Z}[\omega] \\ v_2 \in \lambda^{-3}\mathbb{Z}[\omega]}} \rho_{f \otimes \widehat{\psi_{\lambda^{m_1}(\cdot)}_{\lambda\eta_1, c(m_1, 1; k_1)}}(v_1)} \overline{\rho_f(v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 & \times \sum_{\zeta_2} \overline{\left(\frac{\zeta_2^{-1} \lambda^{m_2-4} r_2}{\zeta_2^{-1} \lambda^{m_2-4} r_2}\right)^{n_2}} \psi_{r_1}^*(\cdot)_{\lambda^{2m_1+1}\eta_1} (\lambda^{2m_1+4} v_1) \cdot \overline{\psi_{\lambda^{m_2} r_2}^\#(\cdot)_{\lambda\eta_2, \zeta_2^{-1}} (\lambda^4 v_2)}} \\
 & \times \ddot{H}_{K,M} \left( \frac{N(v_1)}{N(r_1)^2/X_1}, \frac{N(v_2)}{N(\lambda^{m_2-4} r_2)^2/X_2}, \mathbf{n} \right), \tag{8-74}
 \end{aligned}$$

$$\begin{aligned}
 & \mathcal{D}_{3f}(\mathbf{X}, \boldsymbol{\lambda}^m \mathbf{r}, \boldsymbol{\eta}, \boldsymbol{\eta}; \ddot{H}_{K,M}) \\
 & := \delta_{m_1 \in [6, \ell_1 + 1]} \cdot \delta_{m_2 \in [0, \min\{5, \ell_2 + 1\}]} \cdot N(\lambda^4) N(\lambda^{m_2}) \left(\frac{\bar{r}_2}{r_2}\right)^{n_2} \overline{\omega(m_2, 1; k_2)} \\
 & \times \sum_{\substack{v_1 \in \lambda^{-3}\mathbb{Z}[\omega] \\ v_2 \in \lambda^{-2m_2-3}\mathbb{Z}[\omega]}} \rho_f(v_1) \overline{\rho_{f \otimes \widehat{\psi_{\lambda^{m_2}(\cdot)}_{\lambda\eta_2, c(m_2, 1; k_2)}}(v_2)}} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 & \times \sum_{\zeta_1} \overline{\left(\frac{\zeta_1^{-1} \lambda^{m_1-4} r_1}{\zeta_1^{-1} \lambda^{m_1-4} r_1}\right)^{-n_1}} \psi_{\lambda^{m_1} r_1}^\#(\cdot)_{\lambda\eta_1, \zeta_1^{-1}} (\lambda^4 v_1) \cdot \overline{\psi_{r_2}^*(\cdot)_{\lambda^{2m_2+1}\eta_2} (\lambda^{2m_2+4} v_2)}} \\
 & \times \ddot{H}_{K,M} \left( \frac{N(v_1)}{N(\lambda^{m_1-4} r_1)^2/X_1}, \frac{N(v_2)}{N(r_2)^2/X_2}, \mathbf{n} \right), \tag{8-75}
 \end{aligned}$$

$$\begin{aligned}
 & \mathcal{D}_{4f}(\mathbf{X}, \boldsymbol{\lambda}^m \mathbf{r}, \boldsymbol{\eta}, \mathbf{n}; \ddot{H}_{K,M}) \\
 & := \delta_{m_1 \in [6, \ell_1 + 1]} \cdot \delta_{m_2 \in [6, \ell_2 + 1]} \cdot N(\lambda^8) \sum_{v \in (\lambda^{-3}\mathbb{Z}[\omega])^2} \rho_f(v_1) \overline{\rho_f(v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 & \times \sum_{\zeta} \overline{\left(\frac{\zeta_1^{-1} \lambda^{m_1-4} r_1}{\zeta_1^{-1} \lambda^{m_1-4} r_1}\right)^{-n_1} \left(\frac{\zeta_2^{-1} \lambda^{m_2-4} r_2}{\zeta_2^{-1} \lambda^{m_2-4} r_2}\right)^{n_2}} \psi_{\lambda^{m_1} r_1}^\#(\cdot)_{\lambda\eta_1, \zeta_1^{-1}} (\lambda^4 v_1) \cdot \overline{\psi_{\lambda^{m_2} r_2}^\#(\cdot)_{\lambda\eta_2, \zeta_2^{-1}} (\lambda^4 v_2)}} \\
 & \times \ddot{H}_{K,M} \left( \frac{N(v_1)}{N(\lambda^{m_1-4} r_1)^2/X_1}, \frac{N(v_2)}{N(\lambda^{m_2-4} r_2)^2/X_2}, \mathbf{n} \right), \tag{8-76}
 \end{aligned}$$

$\psi_\zeta^\#$  and  $\psi^*$  are given in (8-16) and (8-28) (with  $\psi'' \rightarrow \psi$ ) respectively, and  $\ddot{H}_{K,M}(\cdot, \mathbf{n}) : (0, \infty)^2 \rightarrow \mathbb{R}$  is given by (8-70), and  $\omega(m, j, r)$  and  $c(m, j, r)$  are both as in Proposition 8.12.

### 9. Type-I estimates

Recall the notation from Section 1, in particular (1-12) and (1-13).

**Remark 9.1.** We can uniquely factorise  $v = \lambda^{e_v} \zeta_v v_0$ , where  $e_v \in \mathbb{Z}_{\geq 2}$ ,  $\zeta_v^6 = 1$ , and  $v_0 \equiv 1 \pmod{3}$ . In view of the congruence condition  $ab \equiv u \pmod{v}$ , we can assume without loss of generality that  $v = \lambda^{e_v} v_0$  with  $v_0 \equiv 1 \pmod{3}$ . In particular, since  $(u, v) = 1$  and  $ab \equiv u \pmod{v}$  in (1-12) and (1-13), we have  $(a, v) = 1$ .

*Proof of Lemma 1.4.* We write (1-12) as

$$\mathcal{S}_f(a, X, v, u; W_K) = \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv 0 \pmod{a} \\ \lambda^3 v \equiv u \pmod{\lambda^{e_v} v_0}}} \rho_f(v) W_K\left(\frac{N(v)}{X}\right). \tag{9-1}$$

Since  $(a, \lambda^{e_v} v_0) = 1$ , we let  $\bar{a} \in \mathbb{Z}[\omega]$  be such that  $a\bar{a} \equiv 1 \pmod{\lambda^{e_v} v_0}$ . The congruence conditions placed on  $v$  in (9-1) are equivalent to  $\lambda^3 v \equiv ua\bar{a} \pmod{\lambda^{e_v} v_0 a}$  by the Chinese remainder theorem.

**9.1. Application of Voronoi summation.** Applying Voronoi summation (Proposition 8.17) we obtain

$$\mathcal{S}_f(a, X; v, u; W_K) = \frac{X}{N(\lambda^{e_v+1} v_0 a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k \equiv 1 \pmod{3}}} \sum_{\substack{m, r, t \\ 0 \leq m \leq e_v + 1 \\ r|a, t|v_0 \\ rt \equiv k \pmod{\lambda^{14}}}} \sum_{p=1}^2 Z_{pf}(X, \lambda^m r t, \eta, 0; \dot{W}_K), \tag{9-2}$$

where  $Z_{pf}(\dots)$  for  $p = 1, 2$  are given in (8-54) and (8-55) respectively. The weight functions involved are radial, see Remarks 8.16 and 8.18, so only  $n = 0$  occurs on the dual side of Voronoi summation.

**9.2. Evaluation and bounds for arithmetic exponential sums.** We now consider the arithmetic exponential sum  $\psi_{\lambda^m r t}^\#(\cdot)_{\lambda\eta, \zeta^{-1}}(\lambda^4 v)$  for  $v \in \lambda^{-3}\mathbb{Z}[\omega]$  that occurs in  $Z_{2f}(\dots)$ . Throughout this computation we will repeatedly use the facts  $\eta \equiv ua\bar{a} \pmod{\lambda^{e_v} v_0 a}$ ,  $a\bar{a} \equiv 1 \pmod{\lambda^{e_v} v_0}$ ,  $0 \leq m \leq e_v + 1$ ,  $r | a$ , and  $t | v_0$ , without further reference. Using Lemma 8.19 we have

$$\psi_{\lambda^m r t}^\#(\cdot)_{\lambda\eta, \zeta^{-1}}(\lambda^4 v) = \frac{1}{N(\lambda^{m+3} r t)} K_{\Gamma_1(3), \sigma, \sigma}(\zeta(\lambda^3 v), \zeta\eta, \lambda^{m-1} r t), \tag{9-3}$$

where  $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . After opening the cubic Kloosterman sum in (9-3), we then perform a computation using the Chinese remainder theorem (with coprime moduli  $\lambda^{m-1} t$  and  $r$ ), (2-1), and (2-7), to obtain

$$\psi_{\lambda^m r t}^\#(\cdot)_{\lambda\eta, \zeta^{-1}}(\lambda^4 v) = \frac{1}{N(r)^{1/2} N(\lambda^{m+3} t)} \left(\frac{\zeta^{-1} \lambda^{m-1} t}{r}\right)_3 \overline{\tilde{g}(\lambda^3 v, r)} K_{\Gamma_1(3), \sigma, \sigma}(\zeta\bar{r}(\lambda^3 v), \zeta\bar{r}u, \lambda^{m-1} t). \tag{9-4}$$

The bound

$$|\psi_{\lambda^m r t}^\#(\cdot)_{\lambda\eta, \zeta^{-1}}(\lambda^4 v)| \ll N(\lambda^m r t)^{-1/2+\varepsilon} \cdot N((\lambda^3 v, r))^{1/2} \tag{9-5}$$

for  $v \in \lambda^{-3}\mathbb{Z}[\omega]$  and  $m \in \mathbb{Z}_{\geq 6}$  follows from using Lemmas 2.3 and 3.1 in (9-4), (2-16), and the fact  $(\zeta \bar{r}(\lambda^3 v), \zeta \bar{r}u, \lambda^{m-1}t) = 1$ .

We now give a similar treatment of the arithmetic sum  $\psi_{rt}^*(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v)$  that occurs in  $Z_{1f}(\dots)$ . Using Lemma 8.20 we have

$$\psi_{rt}^*(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v) = \frac{1}{N(rt)} K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}(\lambda^{2m+3}v), \overline{\lambda^3}\eta, rt), \tag{9-6}$$

where  $\sigma$  is as above, and  $\xi := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . After opening the cubic Kloosterman sum in (9-6), we then perform a computation using the Chinese remainder theorem (with coprime moduli  $t$  and  $r$ ) and (2-7), to obtain

$$\psi_{rt}^*(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v) = \frac{1}{N(r)^{1/2}N(t)} \left(\frac{\lambda^{2m}t}{r}\right)_3 \overline{\tilde{g}(\lambda^{2m+3}v, r)} K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}\bar{r}(\lambda^{2m+3}v), \overline{\lambda^3}\bar{r}u, t). \tag{9-7}$$

The bound

$$|\psi_{rt}^*(\cdot)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}v)| \ll N(rt)^{-1/2+\varepsilon} N((\lambda^{2m+3}v, r))^{1/2} \tag{9-8}$$

for  $v \in \lambda^{-2m-3}\mathbb{Z}[\omega]$  follows from using Lemmas 2.3 and 3.1 in (9-6), (2-16), and the fact

$$(\overline{\lambda^{2m+3}}\bar{r}(\lambda^{2m+3}v), \overline{\lambda^3}\bar{r}u, t) = 1.$$

**9.3. Truncations and conclusion.** We substitute (9-7) and (9-4) into  $Z_{pf}(\dots)$  for  $p = 1, 2$  respectively. We recall Remark 8.18, use Lemma 8.21 (with  $D_1 > 0$  large and fixed and  $D_2 = 0$ ) together with Lemma 2.8 and (9-8) (resp. (9-5)) to truncate the  $v$ -sums in  $Z_{pf}(\dots)$  for  $p = 1$  (resp.  $p = 2$ ) to

$$N(v) \ll (XKN(v))^\varepsilon \cdot K^4 N(\lambda^m rt)^2 X^{-1} =: P, \tag{9-9}$$

with negligible error  $O((XKN(v))^{-2000})$ . Denote the truncated expressions by  $Z'_{pf}(\dots, P)$  for  $p = 1, 2$ . Without loss of generality, we can restrict our attention to the case  $P \gg (XKN(v))^{-\varepsilon}$  otherwise both  $Z'_{pf}(\dots, P)$  for  $p = 1, 2$  are  $O((XKN(v))^{-2000})$  by the above argument. Thus

$$\mathcal{S}_f(a, X; v, u; W_K)$$

$$= \frac{X}{N(\lambda^{e_v+1}v_0a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k \equiv 1 \pmod{3}}} \sum_{\substack{m,r,t \\ 0 \leq m \leq e_v+1 \\ r|a, t|v_0 \\ rt \equiv k \pmod{\lambda^{14}} \\ P \gg (XKN(v))^{-\varepsilon}}} \sum_{p=1}^2 Z'_{pf}(\dots, P) + O((XKN(v))^{-1000}). \tag{9-10}$$

Using the triangle inequality and (9-5), (9-8), and Lemma 8.21 (with  $D_1 = \varepsilon$  and  $D_2 = 0$ ) we obtain

$$Z'_{1f}(\dots, P) \ll (XK)^\varepsilon N(\lambda^m)^{1+\varepsilon} N(rt)^{-1/2+\varepsilon} \sum_{\substack{v \in \lambda^{-2m-3}\mathbb{Z}[\omega] \\ N(v) \ll P}} |\rho_{f \otimes \widehat{\psi_{\lambda^m(\cdot)}_{\lambda u, c(m,1;k)}}}(v)| N((\lambda^{2m+3}v, r))^{1/2} \tag{9-11}$$

for  $0 \leq m \leq \min\{5, e_v+1\}$ ,

$$Z'_{2f}(\dots, P) \ll (XK)^\varepsilon N(\lambda^m rt)^{-1/2+\varepsilon} \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v) \ll P}} |\rho_f(v)| N((\lambda^3v, r))^{1/2} \tag{9-12}$$

for  $6 \leq m \leq e_v+1$ .

We now bound (9-11) and (9-12) by applying the Cauchy–Schwarz inequality to the  $\nu$ -sums, Lemma 2.5, Lemma 2.7, and (2-14). Substitution of the result into (9-10) gives

$$\begin{aligned} \mathcal{S}_f(a, X; \nu, u; W_K) &\ll \frac{X(XKN(\nu))^\varepsilon}{N(\lambda^{e_\nu+1}v_0a)} \sum_{\substack{m,r,t \\ 0 \leq m \leq e_\nu+1 \\ r|a, t|v_0 \\ P \gg (XKN(\nu))^{-\varepsilon}}} \left( N(\lambda^m rt)^{-1/2} \cdot \frac{K^4 N(\lambda^m rt)^2}{X} \right) + (XKN(\nu))^{-1000} \\ &\ll (XKN(\nu))^\varepsilon K^4 N(\nu)^{1/2} N(a)^{1/2}, \end{aligned} \tag{9-13}$$

as required. □

*Proof of Proposition 1.6.* We multiply (9-1) by  $\mu^2(a)\alpha_a$  and sum over  $a \in \mathbb{Z}[\omega]$ . We repeat the same steps on the  $\nu$  sum as in the proof of Lemma 1.4 up to the display (9-2). We then insert a smooth dyadic partition of unity in  $r$  variable. We obtain

$$\mathcal{A}_f(\dots) = \sum_{\substack{1 \ll R \ll A \\ R \text{ dyadic}}} \mathcal{A}_f(\dots, R), \tag{9-14}$$

where

$$\begin{aligned} \mathcal{A}_f(\dots, R) &:= \frac{X}{N(\lambda^{e_\nu+1}v_0)} \sum_{a \in \mathbb{Z}[\omega]} \frac{\mu^2(a)\alpha_a}{N(a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k \equiv 1 \pmod{3}}} \sum_{\substack{m,r,t \\ 0 \leq m \leq e_\nu+1 \\ r|a, t|v_0 \\ rt \equiv k \pmod{\lambda^{14}}}} U\left(\frac{N(r)}{R}\right) \sum_{p=1}^2 Z_{pf}(X, \lambda^m rt, \eta, 0; \dot{W}_K), \end{aligned} \tag{9-15}$$

where the  $Z_{pf}(\dots)$  for  $p = 1, 2$  are given by (8-54) and (8-55) respectively. We recall Remark 8.18, use Lemma 8.21 (with  $D_1 > 0$  large and fixed and  $D_2 = 0$ ) together with Lemma 2.8 and (9-8) (resp. (9-5)) to truncate the  $\nu$ -sums in  $Z_{pf}(\dots)$  for  $p = 1$  (resp.  $p = 2$ ) with  $N(r) \sim R$  to obtain

$$N(\nu) \ll (XKN(\nu))^\varepsilon \cdot K^4 R^2 N(\lambda^m t)^2 X^{-1} =: P_0, \tag{9-16}$$

with negligible error  $O((XKN(\nu))^{-2000})$ . Denote the truncated expressions by  $Z'_{pf}(\dots, P_0)$  for  $p = 1, 2$ . Without loss of generality, we can restrict our attention to the case that  $P_0 \gg (XKN(\nu))^{-\varepsilon}$  otherwise both  $Z'_{pf}(\dots, P_0)$  are  $O((XKN(\nu))^{-2000})$  by the above argument. Thus (9-15) becomes

$$\begin{aligned} \mathcal{A}_f(\dots, R) &= \frac{X}{N(\lambda^{e_\nu+1}v_0)} \sum_{a \in \mathbb{Z}[\omega]} \frac{\mu^2(a)\alpha_a}{N(a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k \equiv 1 \pmod{3}}} \sum_{\substack{m,r,t \\ 0 \leq m \leq e_\nu+1 \\ r|a, t|v_0 \\ rt \equiv k \pmod{\lambda^{14}} \\ P_0 \gg (XKN(\nu))^{-\varepsilon}}} U\left(\frac{N(r)}{R}\right) \sum_{p=1}^2 Z'_{pf}(\dots, P_0) \\ &\quad + O((XKN(\nu))^{-1000} \|\mu^2 \alpha\|_2). \end{aligned} \tag{9-17}$$

**9.4. Further simplification using the squarefree support of  $\alpha$ .** We further open each  $Z'_{pf}(\dots, P_0)$  in (9-17) and manipulate them by further simplifying (9-4) and (9-7) under the assumption that  $r \equiv 1 \pmod{3}$  is squarefree (as is the case in (9-17)). For  $r$  squarefree and  $\mu \in \mathbb{Z}[\omega]$ , Lemma 2.1 guarantees that  $g(\mu, r) = 0$  unless  $(\mu, r) = 1$ . When  $(\mu, r) = 1$  we note that (2-7) implies that

$$\tilde{g}(\mu, r) = \left(\frac{\mu}{r}\right)_3 \tilde{g}(r).$$

Thus (9-4) becomes

$$\begin{aligned} & \psi_{\lambda^m r t}^\#(\cdot)_{\lambda \eta, \zeta^{-1}(\lambda^4 \nu)} \\ &= \begin{cases} N(r)^{-1/2} \left( \frac{\zeta^{-1} \lambda^{m-1} t}{r} \right)_3 \overline{\tilde{g}(r)} \left( \frac{\lambda^3 \nu}{r} \right)_3 \cdot N(\lambda^{m+3} t)^{-1} K_{\Gamma_1(3), \sigma, \sigma}(\zeta \bar{r}(\lambda^3 \nu), \zeta \bar{r} u, \lambda^{m-1} t) & \text{if } (\lambda^3 \nu, r) = 1, \\ 0 & \text{otherwise} \end{cases} \quad (9-18) \end{aligned}$$

for all  $m \in \mathbb{Z}_{\geq 6}$  and  $\nu \in \lambda^{-3} \mathbb{Z}[\omega]$ . Similarly, (9-7) becomes

$$\begin{aligned} & \psi_{r t}^*(\cdot)_{\lambda^{2m+1} \eta}(\lambda^{2m+4} \nu) \\ &= \begin{cases} N(r)^{-1/2} \left( \frac{\lambda^{2m} t}{r} \right)_3 \overline{\tilde{g}(r)} \left( \frac{\lambda^{2m+3} \nu}{r} \right)_3 \cdot N(t)^{-1} K_{\Gamma_1(3), \sigma, \xi}(\lambda^{2m+3} \bar{r}(\lambda^{2m+3} \nu), \lambda^3 \bar{r} u, t) & \text{if } (\lambda^{2m+3} \nu, r) = 1, \\ 0 & \text{otherwise} \end{cases} \quad (9-19) \end{aligned}$$

for all  $m \in \mathbb{Z}_{\geq 0}$  and  $\nu \in \lambda^{-2m-3} \mathbb{Z}[\omega]$ .

**9.5. Preparations for the cubic large sieve.** We substitute (9-19) and (9-18) into the expressions for  $Z'_{pf}(\dots, P_0)$  for  $p = 1$  (resp.  $p = 2$ ) in (9-17), insert a smooth dyadic partition of unity in the  $\nu$  variable, open the transforms  $\widehat{W}_K(\cdot)$  with (8-56) and move the resulting contour integral to  $\text{Re}(s) = \varepsilon$ , resolve the  $r, \lambda^3 \nu, \lambda^{2m+3} \nu$  variables into congruence classes modulo  $\lambda^{\max\{4, m-1\}} t$ , and interchange the order of summation/integration by absolute convergence (see (8-50) and (8-66)). Then (9-17) becomes

$$\mathcal{A}_f(\dots, R) = \mathcal{A}'_f(\dots, R) + \mathcal{A}''_f(\dots, R) + O((XKN(v))^{-1000} \|\mu^2 \alpha\|_2), \quad (9-20)$$

where

$$\begin{aligned} & \mathcal{A}'_f(\dots, R) \\ &:= \frac{X}{N(\lambda^{e_v+1} v_0)} \sum_{\substack{m, t \\ 0 \leq m \leq \min\{5, e_v+1\} \\ t | v_0 \\ P_0 \gg (XKN(v))^{-\varepsilon}}} \frac{N(\lambda^m)}{N(t)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k \equiv 1 \pmod{3}}} \omega(m, 1; k) \sum_{\substack{j \in (\mathbb{Z}[\omega]/9t\mathbb{Z}[\omega])^2 \\ j_1 \equiv 1 \pmod{3} \\ (j_1, t) = 1}} \left( \frac{\lambda^{2m} t}{j_1} \right)_3 \\ & \times K_{\Gamma_1(3), \sigma, \xi}(\lambda^{2m+3} \bar{j}_1 \bar{j}_2, \lambda^3 \bar{j}_1 u, t) \cdot \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{G_\infty(s, \tau_f, 0)}{G_\infty(1-s, \tau_f, 0)} \widehat{W}_K(1-s) X^{-s} \\ & \times \sum_{\substack{1 \ll S \ll P_0 \\ S \text{ dyadic}}} \left( \sum_{\substack{v \in \lambda^{-2m-3} \mathbb{Z}[\omega] \\ \lambda^{2m+3} v \equiv j_2 \pmod{9t} \\ N(v) \sim S}} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv j_1 \pmod{9t} \\ rt \equiv k \pmod{\lambda^{14}} \\ N(r) \sim R}} \Omega'_v(s, \lambda^m, k, S) \Psi_r(s, R) \left( \frac{\lambda^{2m+3} \nu}{r} \right)_3 \right) ds; \quad (9-21) \end{aligned}$$

$$\begin{aligned} & \mathcal{A}_f''(\dots, R) \\ & := \frac{X}{N(\lambda^{e_v+1}v_0)} \sum_{\substack{\zeta, m, t \\ 6 \leq m \leq e_v+1 \\ t|v_0 \\ P_0 \gg (XKN(v))^{-\varepsilon}}} \frac{1}{N(\lambda^{m-1}t)} \sum_{\substack{j \in (\mathbb{Z}[\omega]/\lambda^{m-1}t\mathbb{Z}[\omega])^2 \\ j_1 \equiv 1 \pmod{3} \\ (j_1, t) = 1}} \overline{\left( \frac{\zeta^{-1}\lambda^{m-1}t}{j_1} \right)}_3 \\ & \times K_{\Gamma_1(3), \sigma, \sigma}(\zeta \bar{j}_1 \bar{j}_2, \zeta \bar{j}_1 u, \lambda^{m-1}t) \cdot \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{G_\infty(s, \tau_f, 0)}{G_\infty(1-s, \tau_f, 0)} \widehat{W}_K(1-s) X^{-s} N(\lambda^{m-4})^{2s} \\ & \times \sum_{\substack{1 \ll S \ll P_0 \\ S \text{ dyadic}}} \left( \sum_{\substack{v \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 v \equiv j_2 \pmod{\lambda^{m-1}t} \\ N(v) \sim S}} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv j_1 \pmod{\lambda^{m-1}t} \\ N(r) \sim R}} \Omega_v''(s, S) \Psi_r(s, R) \left( \frac{\lambda^3 v}{r} \right)_3 \right) ds, \end{aligned} \tag{9-22}$$

$$\Psi_r(s, R) := N(r)^{-1/2} N(r)^{2s} \overline{g(r)} U\left(\frac{N(r)}{R}\right) \sum_{a \equiv 0 \pmod{r}} \frac{\mu^2(a) \alpha_a}{N(a)}; \tag{9-23}$$

$$\Omega_v'(s, \lambda^m, k, S) := N(v)^{-s} U\left(\frac{N(v)}{S}\right) \rho_{f \otimes \widehat{\psi_{\lambda^m}(\cdot)_{\lambda u, c(m, 1; k)}}}(v);$$

$$\Omega_v''(s, S) := N(v)^{-s} U\left(\frac{N(v)}{S}\right) \rho_f(v).$$

Observe that the weights  $\Psi_r(s, R)$  in (9-23) are supported on squarefree  $r$  (see (2-11)).

**9.6. Application of the cubic large sieve and conclusion.** Consider the bilinear form in  $v$  and  $r$  and in the last display of (9-21). Using Corollary 4.2 (the cubic large sieve) we obtain

$$\begin{aligned} & \sum_{\substack{v \in \lambda^{-2m-3}\mathbb{Z}[\omega] \\ \lambda^{2m+3}v \equiv j_2 \pmod{9t} \\ N(v) \sim S}} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv j_1 \pmod{9t} \\ rt \equiv k \pmod{\lambda^{14}} \\ N(r) \sim R}} \Omega_v'(s, \lambda^m, k, S) \Psi_r(s, R) \left( \frac{\lambda^{2m+3}v}{r} \right)_3 \\ & \ll (RS)^\varepsilon S^{1/6} (S^{1/2} + R^{1/2}) \left( \sum_{v \in \lambda^{-2m-3}\mathbb{Z}[\omega]} |\Omega_v'(s, \lambda^m, k, S)|^2 \right)^{1/2} \left( \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv 1 \pmod{3}}} \mu^2(r) |\Psi_r(s, R)|^2 \right)^{1/2}, \end{aligned} \tag{9-24}$$

where we dropped some of the congruence conditions in the  $L^2$ -norms by positivity. Lemma 2.7 gives

$$\sum_{v \in \lambda^{-2m-3}\mathbb{Z}[\omega]} |\Omega_v'(s, \lambda^m, k, S)|^2 \ll S^{1+\varepsilon} \tag{9-25}$$

for each  $0 \leq m \leq \min\{5, e_v + 1\}$  and  $S \gg 1$ . Using (2-11) and (2-12) we compute

$$\sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv 1 \pmod{3}}} \mu^2(r) |\Psi_r(s, R)|^2 = \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv 1 \pmod{3}}} \frac{\mu^2(r)}{N(r)^{1-4\text{Re}(s)}} \left| U\left(\frac{N(r)}{R}\right) \right|^2 \cdot \left| \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 0 \pmod{r}}} \frac{\mu^2(a) \alpha_a}{N(a)} \right|^2$$

$$\begin{aligned} &\ll R^{-1+\varepsilon} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv 1 \pmod{3} \\ N(r) \sim R}} \left| \sum_{\substack{a \in \mathbb{Z}[\omega] \\ a \equiv 0 \pmod{r}}} \frac{\mu^2(a)\alpha_a}{N(a)} \right|^2 \\ &\ll A^{-1} R^{-2+\varepsilon} \sum_{\substack{u, r \in \mathbb{Z}[\omega] \\ u, r \equiv 1 \pmod{3}}} \mu^2(ur) |\alpha_{ur}|^2 \ll (AR)^\varepsilon A^{-1} R^{-2} \|\mu^2 \alpha\|_2^2, \end{aligned} \tag{9-26}$$

where the penultimate display follows from the Cauchy–Schwarz inequality, a change of variables, and the last display follows from (2-14). Substituting (9-25) and (9-26) into (9-24), and summing both sides of the result inequality over dyadic values of  $S$  yields we obtain (for each  $0 \leq m \leq \min\{5, e_v + 1\}$ )

$$\begin{aligned} &\sum_{\substack{1 \ll S \ll P_0 \\ S \text{ dyadic}}} \left| \sum_{\substack{v \in \lambda^{-2m-3} \mathbb{Z}[\omega] \\ \lambda^{2m+3} v \equiv j_2 \pmod{9t} \\ N(v) \sim S}} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ r \equiv j_1 \pmod{9t} \\ rt \equiv k \pmod{\lambda^{14}} \\ N(r) \sim R}} \Omega'_v(s, \lambda^m, k, S) \Psi_r(s, R) \left( \frac{\lambda^{2m+3} v}{r} \right)_3 \right| \\ &\ll (XKN(v))^\varepsilon (K^{14/3} N(t)^{7/3} R^{4/3} A^{-1/2} X^{-7/6} + K^{8/3} N(t)^{4/3} R^{5/6} A^{-1/2} X^{-2/3}) \|\mu^2 \alpha\|_2, \end{aligned} \tag{9-27}$$

where (9-16) was used to obtain the last display. We insert the bound (9-27) into (9-21), and then use (8-50), (8-66), and Lemma 3.1 to obtain

$$\mathcal{A}'_f(\dots, R) \ll (XKN(v))^\varepsilon (K^{14/3} N(v)^{5/6} R^{4/3} A^{-1/2} X^{-1/6} + K^{8/3} N(v)^{-1/6} R^{5/6} A^{-1/2} X^{1/3}) \|\mu^2 \alpha\|_2. \tag{9-28}$$

An analogous computation shows that  $\mathcal{A}''_f(\dots, R)$  satisfies the same bound as that in (9-28). After substituting these bounds into (9-20), we then substitute the result into (9-14) to obtain

$$\begin{aligned} \mathcal{A}_f(\dots) &\ll (XKN(v))^\varepsilon (K^{14/3} N(v)^{5/6} X^{-1/6} A^{5/6} + K^{8/3} N(v)^{-1/6} (AX)^{1/3}) \|\mu^2 \alpha\|_2 \\ &\ll (XKN(v))^\varepsilon K^{14/3} N(v)^{5/6} (AX)^{1/3} \|\mu^2 \alpha\|_2, \end{aligned} \tag{9-29}$$

where the last inequality follows since  $A \ll X$ . The result follows. □

### 10. Type-II estimates via average (homogenous) convolution

Recall the notation from Section 1, in particular (1-14). The first result in this section bounds the Type-II sum in terms of a homogeneous average convolution problem.

**Lemma 10.1.** *Let the notation be as above and  $X \asymp AB$ . Then*

$$|\mathcal{B}_f(\alpha, \beta, X, v, u; W_K)| \leq \|\beta\|_2 \cdot \left( \sum_a \mu^2(a_1) \alpha_{a_1} \mu^2(a_2) \overline{\alpha_{a_2}} \mathcal{L}_f(\mathbf{a}, X, v, u; W_K) \right)^{1/2},$$

where

$$\mathcal{L}_f(\mathbf{a}, X, v, u; W_K) := \sum_{\substack{b \in \mathbb{Z}[\omega] \\ a_1 b \equiv u \pmod{v} \\ a_2 b \equiv u \pmod{v}}} \rho_f(\lambda^{-3} a_1 b) \overline{\rho_f(\lambda^{-3} a_2 b)} W_K \left( \frac{N(\lambda^{-3} a_1 b)}{X} \right) \overline{W_K \left( \frac{N(\lambda^{-3} a_2 b)}{X} \right)}. \tag{10-II}$$

*Proof.* We apply the Cauchy–Schwarz inequality to the  $b$ -sum in (1-14) to obtain

$$|\mathcal{B}_f(\dots)| \leq \|\beta\|_2 \cdot \left( \sum_{b \in \mathbb{Z}[\omega]} \left| \sum_{\substack{a \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \mu^2(a) \alpha_a \rho_f(\lambda^{-3}ab) W_K \left( \frac{N(\lambda^{-3}ab)}{X} \right) \right|^2 \right)^{1/2}.$$

The result follows from expanding the square modulus in the above expression and interchanging the order of summation.  $\square$

**Proposition 10.2.** *Let the notation be as above and  $X \asymp AB$ . Then*

$$\sum_a \mu^2(a_1) \alpha_{a_1} \mu^2(a_2) \overline{\alpha_{a_2}} \mathcal{L}_f(\mathbf{a}, X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^\varepsilon K^{16} N(v)^8 (AB + A^3 B^{1/2}) \|\mu^2 \alpha\|_\infty^2.$$

**Remark 10.3.** It will be helpful to remember the normalisation in (8-12) throughout the proof. We also use the same notation and convention as Remark 9.1.

*Proof.* We begin by separating oscillations using the circle method.

**10.1. Application of the circle method.** Rewriting (10-1) we obtain

$$\mathcal{L}_f(\mathbf{a}, \dots) = \sum_{\substack{\mathbf{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: \lambda^3 v_i \equiv 0 \pmod{a_i} \\ \forall i: \lambda^3 v_i \equiv u \pmod{\lambda^{e_v} v_0}}} \rho_f(v_1) \overline{\rho_f(v_2)} W_K \left( \frac{N(v_1)}{X} \right) \overline{W_K \left( \frac{N(v_2)}{X} \right)} \delta_{\mathbb{Q}(\omega)} \left( \frac{\lambda^3 v_2}{a_2} - \frac{\lambda^3 v_1}{a_1} \right). \quad (10-2)$$

After noting Remark 5.3 we choose  $C > 0$  such that

$$C^4 := X/A \asymp B. \quad (10-3)$$

We use Theorem 5.2 and Remark 5.1 to obtain

$$\begin{aligned} & \delta_{\mathbb{Q}(\omega)} \left( \frac{\lambda^3 v_2}{a_2} - \frac{\lambda^3 v_1}{a_1} \right) \\ &= \frac{k_C}{C^4} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c \equiv 1 \pmod{3}}} N(\lambda^\ell c) \hat{\psi}_{\lambda^\ell c} \left( \frac{\lambda^3 v_2}{a_2} - \frac{\lambda^3 v_1}{a_1} \right) h \left( \frac{N(\lambda^\ell c)}{C^2}, \frac{N(\lambda^3 v_2/a_2 - \lambda^3 v_1/a_1)}{C^4} \right) \end{aligned} \quad (10-4)$$

for any  $\mathbf{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2$  such that  $\lambda^3 v_i \equiv 0 \pmod{a_i}$  for  $i = 1, 2$ , and where  $\psi_{\lambda^\ell c}$  denotes the principal character modulo  $\lambda^\ell c$ . Let  $\ell_0 := \max\{\ell, e_v\}$ . We substitute (10-4) into (10-2), interchange the order of summation, and resolve  $\lambda^3 v_i$  into congruence classes  $\pmod{\lambda^{\ell_0} a_i v_0 c}$  for  $i = 1, 2$ . We obtain

$$\begin{aligned} \mathcal{L}_f(\mathbf{a}, \dots) &= \frac{k_C}{C^4} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c \equiv 1 \pmod{3}}} \sum_{\substack{j \in \prod_{i=1}^2 \mathbb{Z}[\omega]/\lambda^{\ell_0} a_i v_0 c \mathbb{Z}[\omega] \\ \forall i: j_i \equiv u \pmod{\lambda^{e_v} v_0}}} N(\lambda^\ell c) \hat{\psi}_{\lambda^\ell c} \left( \frac{j_2}{a_2} - \frac{j_1}{a_1} \right) \\ &\quad \times \sum_{\substack{\mathbf{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: \lambda^3 v_i \equiv j_i \pmod{\lambda^{\ell_0} a_i v_0 c}}} \rho_f(v_1) \overline{\rho_f(v_2)} H_{K, C^2/N(\lambda^\ell c)} \left( \frac{v_1}{\sqrt{X}}, \frac{v_2}{\sqrt{X}} \right), \end{aligned} \quad (10-5)$$

where  $H_{K,C^2/N(\lambda^\ell c)}(\mathbf{z}) := H_{K,C^2/N(\lambda^\ell c)}(\mathbf{z}; \mathbf{a}, \lambda^\ell c, X, C)$  is given by

$$H_{K,C^2/N(\lambda^\ell c)}(\mathbf{z}) = W_K(|z_1|^2)W_K(|z_2|^2)h\left(\frac{N(\lambda^\ell c)}{C^2}, \frac{X|\lambda^3 z_1/a_1 - \lambda^3 z_2/a_2|^2}{C^4}\right). \tag{10-6}$$

We now justify the subscripts for the function  $H_{K,C^2/N(\lambda^\ell c)}(\mathbf{z})$  (see (8-47)). Recall that  $C^4 := X/A \asymp B$ ,  $N(a_i) \asymp A$ , and  $|z_i| \asymp 1$  for  $i = 1, 2$ . Thus

$$X|\lambda^3 z_1/a_1 - \lambda^3 z_2/a_2|^2/C^4 \ll 1. \tag{10-7}$$

Observe that (10-7) and (5-4) imply that

$$h\left(\frac{N(\lambda^\ell c)}{C^2}, \frac{X|\lambda^3 z_1/a_1 - \lambda^3 z_2/a_2|^2}{C^4}\right) \neq 0 \quad \text{only if } N(\lambda^\ell c) \ll C^2. \tag{10-8}$$

The chain rule, (1-6) (with  $M = 1$ ), (5-3), Corollary 5.6, (10-8) and the fact that  $K \geq 1$  together imply that for any  $\mathbf{i} = (i_{11}, i_{12}, i_{21}, i_{22}) \in (\mathbb{Z}_{\geq 0})^4$  we have

$$\partial^{\mathbf{i}} H_{K,C^2/N(\lambda^\ell c)}(\mathbf{z}) \ll_{\mathbf{i}} \frac{C^2}{N(\lambda^\ell c)} \cdot K^{i_{11}+i_{12}+i_{21}+i_{22}}. \tag{10-9}$$

**10.2. Double application of Voronoi summation.** We use (double)-Voronoi summation (Proposition 8.22). By abuse of notation we denote  $(X, X)$  by  $X$ . We obtain

$$\begin{aligned} & \sum_{\substack{\mathbf{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: \lambda^3 v_i \equiv j_i \pmod{\lambda^{\ell_0} a_i v_0 c}}} \rho_f(v_1) \overline{\rho_f(v_2)} H_{K,C^2/N(\lambda^\ell c)}\left(\frac{v_1}{\sqrt{X}}, \frac{v_2}{\sqrt{X}}\right) \\ &= \frac{X^2}{N(\lambda^{\ell_0+1} v_0 c)^2 N(a_1 a_2)} \\ & \times \sum_{\substack{\mathbf{k} \in (\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2 \\ \forall i: k_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{m}, \mathbf{r} \\ \forall i: 0 \leq m_i \leq \ell_0 + 1 \\ \forall i: r_i | a_i v_0 c \\ \forall i: r_i \equiv k_i \pmod{\lambda^{14}}}} \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1+n_2} \sum_{p=1}^4 \mathcal{D}_{pf}(X, \lambda^{\mathbf{m}} \mathbf{r}, \mathbf{j}, \mathbf{n}; \ddot{H}_{K,C^2/N(\lambda^\ell c)}), \end{aligned} \tag{10-10}$$

where the  $\mathcal{D}_{pf}(\dots)$  are given by (8-73)–(8-76). We substitute (10-10) into (10-5) to obtain

$$\mathcal{L}_f(\mathbf{a}, \dots) = \sum_{p=1}^4 \mathcal{M}_{pf}(\mathbf{a}, \dots), \tag{10-11}$$

where

$$\begin{aligned} \mathcal{M}_{pf}(\mathbf{a}, \dots) &:= \frac{k_C}{C^4} \frac{X^2}{N(a_1 a_2) N(v_0)^2} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c \equiv 1 \pmod{3}}} \frac{N(\lambda^\ell)}{N(\lambda^{\ell_0+1})^2} \frac{1}{N(c)} \sum_{\substack{\mathbf{j} \in \prod_{i=1}^2 \mathbb{Z}[\omega]/\lambda^{\ell_0} a_i v_0 c \mathbb{Z}[\omega] \\ \forall i: j_i \equiv 0 \pmod{a_i} \\ \forall i: j_i \equiv u \pmod{\lambda^{\ell v} v_0}}} \hat{\psi}_{\lambda^\ell c}\left(\frac{j_2}{a_2} - \frac{j_1}{a_1}\right) \\ & \times \sum_{\substack{\mathbf{k} \in (\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2 \\ \forall i: k_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{m}, \mathbf{r} \\ \forall i: 0 \leq m_i \leq \ell_0 + 1 \\ \forall i: r_i | a_i v_0 c \\ \forall i: r_i \equiv k_i \pmod{\lambda^{14}}}} \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1+n_2} \mathcal{D}_{pf}(X, \lambda^{\mathbf{m}} \mathbf{r}, \mathbf{j}, \mathbf{n}; \ddot{H}_{K,C^2/N(\lambda^\ell c)}). \end{aligned} \tag{10-12}$$

We now make a sequence of manipulations to  $\mathcal{M}_{pf}(\mathbf{a}, \dots)$  in (10-12). First we make a change of variable  $j_i \rightarrow a_i j_i$  for  $i = 1, 2$  (the new  $j_i$  variables run  $(\text{mod } \lambda^{\ell_0} v_0 c)$ ). We then uniquely factorise each  $c \in \mathbb{Z}[\omega]$  with  $c \equiv 1 \pmod{3}$  as  $c = tq'q''$ , where  $t, q', q'' \in \mathbb{Z}[\omega]$  satisfy

$$t, q', q'' \equiv 1 \pmod{3}, \quad t \mid \text{rad}(v_0)^\infty, \quad q' \mid \text{rad}(a_1 a_2)^\infty, \quad \text{and} \quad (q'', a_1 a_2 v_0) = 1. \quad (10-13)$$

Note this factorisation exists and is unique since  $(v_0, a_1 a_2) = 1$ . We also uniquely factorise each  $r_i \mid a_i v_0 c$  with  $r_i \equiv k_i \pmod{\lambda^{14}}$  as  $r_i = t_i r'_i r''_i$ , where  $t_i, r'_i, r''_i$  satisfy

$$t_i, r'_i, r''_i \equiv 1 \pmod{3}, \quad t_i \mid v_0 t, \quad r'_i \mid a_i q', \quad r''_i \mid q'', \quad \text{and} \quad t_i r'_i r''_i \equiv k_i \quad \text{for } i = 1, 2. \quad (10-14)$$

We use the Chinese remainder theorem on the new  $j_i$  variables (with the pairwise coprime moduli  $\lambda^{\ell_0} v_0 t, q'$ , and  $q''$ ) and  $i = 1, 2$  to write

$$\begin{aligned} \mathbf{j} &:= q' q'' \overline{q' q''} \mathbf{J} + \lambda^{\ell_0} v_0 t q'' \overline{\lambda^{\ell_0} v_0 t q''} \mathbf{J}' + \lambda^{\ell_0} v_0 t q' \overline{\lambda^{\ell_0} v_0 t q'} \mathbf{J}'' \\ \mathbf{J}_i &\equiv \overline{a_i} \mathbf{u} \pmod{\lambda^{e_v} v_0} \quad \text{for } i = 1, 2, \end{aligned} \quad (10-15)$$

where  $\overline{a_i}, \overline{q' q''}, \overline{\lambda^{\ell_0} v_0 t q''}, \overline{\lambda^{\ell_0} v_0 t q'} \in \mathbb{Z}[\omega]$  are such that  $a_i \overline{a_i} \equiv 1 \pmod{\lambda^{e_v} v_0}, \overline{q' q''} q' q'' \equiv 1 \pmod{\lambda^{\ell_0} v_0 t}, \overline{\lambda^{\ell_0} v_0 t q'} \lambda^{\ell_0} v_0 t q' \equiv 1 \pmod{q''},$  and  $\overline{\lambda^{\ell_0} v_0 t q''} \lambda^{\ell_0} v_0 t q'' \equiv 1 \pmod{q'}$ . Without loss of generality we may assume that  $e_v \geq 14$ . We further make the change of variable

$$\mathbf{J} \rightarrow \lambda^{e_v} v_0 \mathbf{J} + (Y_1 \mathbf{u}, Y_2 \mathbf{u}) \quad (10-16)$$

in (10-15), where  $Y_i \in \mathbb{Z}[\omega]$  is such that  $Y_i \equiv \overline{a_i} \pmod{\lambda^{e_v} v_0}$ . Observe that the new  $J_1, J_2$  variables run  $(\text{mod } \lambda^{\ell_0 - e_v} t)$ . We also use the multiplicativity of Ramanujan sums  $\hat{\psi}_{\lambda^{\ell} c}(\cdot) = \hat{\psi}_{\lambda^{\ell} t}(\cdot) \hat{\psi}_{q'}(\cdot) \hat{\psi}_{q''}(\cdot)$ , and interchange the order of summation by absolute convergence. The net result is

$$\begin{aligned} &\mathcal{M}_{pf}(\mathbf{a}, \dots) \\ &:= \frac{k_C}{C^4} \frac{X^2}{N(a_1 a_2) N(v_0)^2} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{tq'q'' \in \mathbb{Z}[\omega] \\ (10-13)}} \frac{N(\lambda^\ell)}{N(\lambda^{\ell_0+1})^2} \frac{1}{N(tq'q'')} \\ &\times \sum_{\substack{\mathbf{k} \in (\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2 \\ \forall i: k_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{m}, \mathbf{t}, \mathbf{r}', \mathbf{r}'' \\ \forall i: 0 \leq m_i \leq \ell_0 + 1 \\ (10-14)}} \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1 + n_2} S_{pf}(\mathbf{a}, \lambda^\ell tq'q'', \lambda^{\mathbf{m}} \mathbf{t} \mathbf{r}' \mathbf{r}'', \mathbf{n}; \ddot{H}_{K, C^2/N(\lambda^\ell tq'q'')}), \end{aligned} \quad (10-17)$$

where

$$\begin{aligned} S_{1f}(\dots) &:= \delta_{\mathbf{m} \in [0, \min\{5, \ell_0 + 1\}]^2} N(\lambda^{m_1}) N(\lambda^{m_2}) \left(\frac{\overline{r_1}}{r_1}\right)^{-n_1} \left(\frac{\overline{r_2}}{r_2}\right)^{n_2} \omega(m_1; 1, k_1) \overline{\omega(m_2, 1, k_2)} \\ &\times \sum_{\substack{v_1 \in \lambda^{-2m_1-3}\mathbb{Z}[\omega] \\ v_2 \in \lambda^{-2m_2-3}\mathbb{Z}[\omega]}} \rho_{f \otimes \widehat{\psi_{\lambda^{m_1}(\cdot)}_{\lambda u, c(m_1, 1; k_1)}}}(v_1) \overline{\rho_{f \otimes \widehat{\psi_{\lambda^{m_2}(\cdot)}_{\lambda u, c(m_2, 1; k_2)}}}(v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\overline{v_2}}{|v_2|}\right)^{-n_2} \\ &\times \ddot{H}_{K, C^2/N(\lambda^\ell tq'q'')} \left(\frac{N(v_1)}{N(t_1 r'_1 r''_1)^2 / X}, \frac{N(v_2)}{N(t_2 r'_2 r''_2)^2 / X}, \mathbf{n}\right) \\ &\times \mathcal{E}_1(\mathbf{a}, \mathbf{v}, \lambda^\ell tq'q'', \lambda^{\mathbf{m}} \mathbf{t} \mathbf{r}' \mathbf{r}''); \end{aligned} \quad (10-18)$$

$$\begin{aligned}
 S_{2f}(\dots) &:= \delta_{\mathbf{m} \in [0, \min\{5, \ell_0 + 1\}] \times [6, \ell_0 + 1]} N(\lambda^{m_1}) N(\lambda^4) \left(\frac{\bar{r}_1}{r_1}\right)^{-n_1} \omega(\mathbf{m}_1; 1, k_1) \\
 &\times \sum_{\substack{v_1 \in \lambda^{-2m_1-3}\mathbb{Z}[\omega] \\ v_2 \in \lambda^{-3}\mathbb{Z}[\omega]}} \rho_{f \otimes \widehat{\psi_{\lambda^{m_1}(\cdot)}_{\lambda u, c(m_1, 1; k_1)}}} (v_1) \overline{\rho_f(v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 &\times \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')} \left(\frac{N(v_1)}{N(t_1 r'_1 r''_2)^2/X}, \frac{N(v_2)}{N(\lambda^{m_2-4} t_2 r'_2 r''_2)^2/X}, \mathbf{n}\right) \\
 &\times \mathcal{C}_2(\mathbf{a}, \mathbf{v}, \lambda^\ell t q' q'', \boldsymbol{\lambda}^m \mathbf{t} \mathbf{r}' \mathbf{r}''),
 \end{aligned} \tag{10-19}$$

$$\begin{aligned}
 S_{3f}(\dots) &:= \delta_{\mathbf{m} \in [6, \ell_0 + 1] \times [0, \min\{5, \ell_0 + 1\}]} N(\lambda^4) N(\lambda^{m_2}) \left(\frac{\bar{r}_2}{r_2}\right)^{n_2} \overline{\omega(\mathbf{m}_2; 1, k_2)} \\
 &\times \sum_{\substack{v_1 \in \lambda^{-3}\mathbb{Z}[\omega] \\ v_2 \in \lambda^{-2m_2-3}\mathbb{Z}[\omega]}} \rho_f(v_1) \overline{\rho_{f \otimes \widehat{\psi_{\lambda^{m_2}(\cdot)}_{\lambda u, c(m_2, 1; k_2)}}} (v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 &\times \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')} \left(\frac{N(v_1)}{N(\lambda^{m_1-4} t_1 r'_1 r''_2)^2/X}, \frac{N(v_2)}{N(t_2 r'_2 r''_2)^2/X}, \mathbf{n}\right) \\
 &\times \mathcal{C}_3(\mathbf{a}, \mathbf{v}, \lambda^\ell t q' q'', \boldsymbol{\lambda}^m \mathbf{t} \mathbf{r}' \mathbf{r}''),
 \end{aligned} \tag{10-20}$$

$$\begin{aligned}
 S_{4f}(\dots) &:= \delta_{\mathbf{m} \in [6, \ell_0 + 1]^2} \cdot N(\lambda^8) \\
 &\times \sum_{v_1, v_2 \in \lambda^{-3}\mathbb{Z}[\omega]} \rho_f(v_1) \overline{\rho_f(v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\
 &\times \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')} \left(\frac{N(v_1)}{N(\lambda^{m_1-4} t_1 r'_1 r''_2)^2/X}, \frac{N(v_2)}{N(\lambda^{m_2-4} t_2 r'_2 r''_2)^2/X}, \mathbf{n}\right) \\
 &\times \mathcal{C}_4(\mathbf{a}, \mathbf{v}, \lambda^\ell t q' q'', \boldsymbol{\lambda}^m \mathbf{t} \mathbf{r}' \mathbf{r}''),
 \end{aligned} \tag{10-21}$$

$$\begin{aligned}
 \mathcal{C}_1(\dots) &= \sum_{\substack{J \in (\mathbb{Z}[\omega]/\lambda^{\ell_0 - ev} t \mathbb{Z}[\omega])^2 \\ J' \in (\mathbb{Z}[\omega]/q' \mathbb{Z}[\omega])^2 \\ J'' \in (\mathbb{Z}[\omega]/q'' \mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^\ell t}(\lambda^{ev} v_0 (J_2 - J_1) + u(Y_2 - Y_1)) \hat{\psi}_{q'}(J'_2 - J'_1) \hat{\psi}_{q''}(J''_2 - J''_1) \\
 &\times \psi_{t_1 r'_1 r''_1}^* (\cdot)_{\lambda^{2m_1+1} a_1 j_1} (\lambda^{2m_1+4} v_1) \overline{\psi_{t_2 r'_2 r''_2}^* (\cdot)_{\lambda^{2m_2+1} a_2 j_2} (\lambda^{2m_2+4} v_2)},
 \end{aligned} \tag{10-22}$$

$$\begin{aligned}
 \mathcal{C}_2(\dots) &:= \sum_{\xi_2} \left(\frac{\bar{\xi}_2^{-1} \lambda^{m_2-4} r_2}{\xi_2^{-1} \lambda^{m_2-4} r_2}\right)^{n_2} \\
 &\times \sum_{\substack{J \in (\mathbb{Z}[\omega]/\lambda^{\ell_0 - ev} t \mathbb{Z}[\omega])^2 \\ J' \in (\mathbb{Z}[\omega]/q' \mathbb{Z}[\omega])^2 \\ J'' \in (\mathbb{Z}[\omega]/q'' \mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^\ell t}(\lambda^{ev} v_0 (J_2 - J_1) + u(Y_2 - Y_1)) \hat{\psi}_{q'}(J'_2 - J'_1) \hat{\psi}_{q''}(J''_2 - J''_1) \\
 &\times \psi_{t_1 r'_1 r''_1}^* (\cdot)_{\lambda^{2m_1+1} a_1 j_1} (\lambda^{2m_1+4} v_1) \overline{\psi_{\lambda^{m_2} t_2 r'_2 r''_2}^\# (\cdot)_{\lambda a_2 j_2, \xi_2^{-1}} (\lambda^4 v_2)},
 \end{aligned} \tag{10-23}$$

$$\begin{aligned} \mathcal{C}_3(\dots) &:= \sum_{\zeta_1} \left( \frac{\overline{\zeta_1^{-1} \lambda^{m_1-4} r_1}}{\zeta_1^{-1} \lambda^{m_1-4} r_1} \right)^{-n_1} \\ &\times \sum_{\substack{J \in (\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v} t \mathbb{Z}[\omega])^2 \\ J' \in (\mathbb{Z}[\omega]/q' \mathbb{Z}[\omega])^2 \\ J'' \in (\mathbb{Z}[\omega]/q'' \mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^{\ell_t}}(\lambda^{e_v} v_0(J_2 - J_1) + u(Y_2 - Y_1)) \hat{\psi}_{q'}(J'_2 - J'_1) \hat{\psi}_{q''}(J''_2 - J''_1) \\ &\times \psi_{\lambda^{m_1 t_1 r'_1 r''_1}}^\#(\cdot)_{\lambda a_1 j_1, \zeta_1^{-1}(\lambda^4 v_1)} \overline{\psi_{\lambda^{m_2+1} a_2 j_2}^\star(\cdot)_{\lambda^{2m_2+1} a_2 j_2}(\lambda^{2m_2+4} v_2)}}, \end{aligned} \tag{10-24}$$

$$\begin{aligned} \mathcal{C}_4(\dots) &:= \sum_{\zeta} \left( \frac{\overline{\zeta_1^{-1} \lambda^{m_1-4} r_1}}{\zeta_1^{-1} \lambda^{m_1-4} r_1} \right)^{-n_1} \left( \frac{\overline{\zeta_2^{-1} \lambda^{m_2-4} r_2}}{\zeta_2^{-1} \lambda^{m_2-4} r_2} \right)^{n_2} \\ &\times \sum_{\substack{J \in (\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v} t \mathbb{Z}[\omega])^2 \\ J' \in (\mathbb{Z}[\omega]/q' \mathbb{Z}[\omega])^2 \\ J'' \in (\mathbb{Z}[\omega]/q'' \mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^{\ell_t}}(\lambda^{e_v} v_0(J_2 - J_1) + u(Y_2 - Y_1)) \hat{\psi}_{q'}(J'_2 - J'_1) \hat{\psi}_{q''}(J''_2 - J''_1) \\ &\times \psi_{\lambda^{m_1 t_1 r'_1 r''_1}}^\#(\cdot)_{\lambda a_1 j_1, \zeta_1^{-1}(\lambda^4 v_1)} \overline{\psi_{\lambda^{m_2 t_2 r'_2 r''_2}}^\#(\cdot)_{\lambda a_2 j_2, \zeta_2^{-1}(\lambda^4 v_2)}}, \end{aligned} \tag{10-25}$$

and  $j$  is given by (10-15) with subsequent change of variable (10-16).

**Remark 10.4.** Recalling (10-11) and recalling the averaging over  $\mathbf{a}$  we have

$$\sum_{\mathbf{a}} \mu^2(a_1) \alpha_{a_1} \mu^2(a_2) \overline{\alpha_{a_2}} \mathcal{L}_f(\mathbf{a}, \dots) = \sum_{p=1}^4 \sum_{\mathbf{a}} \mu^2(a_1) \alpha_{a_1} \mu^2(a_2) \overline{\alpha_{a_2}} \mathcal{M}_{pf}(\mathbf{a}, \dots). \tag{10-26}$$

The following arguments focus on the case  $p = 4$  on the right side of (10-26). The cases  $p = 1, 2, 3$  will follow mutatis mutandis, and will be omitted for the sake of brevity.

**10.3. Evaluation and bounds for arithmetic exponential sums.** We first compute and bound  $\mathcal{C}_4(\dots)$  in (10-25).

A computation using Lemma 8.19, (8-63), (10-15), the Chinese remainder theorem (with pairwise coprime moduli  $\zeta_i \lambda^{m_i-1} t_i, r'_i$  and  $r''_i$  for  $i = 1, 2$ ), cubic reciprocity, and Lemma 3.4 yields

$$\mathcal{C}_4(\dots) = \sum_{\zeta} \left( \frac{\overline{\zeta_1 \lambda^{m_1-4} r_1}}{\zeta_1 \lambda^{m_1-4} r_1} \right)^{-n_1} \left( \frac{\overline{\zeta_2 \lambda^{m_2-4} r_2}}{\zeta_2 \lambda^{m_2-4} r_2} \right)^{n_2} \prod_{i=1}^3 G_{4i}(\mathbf{a}, \mathbf{v}, \lambda^{\ell_t} t q' q'', \boldsymbol{\zeta} \lambda^m \mathbf{t} r' r''), \tag{10-27}$$

where

$$\begin{aligned} G_{41}(\dots) &:= \frac{1}{N(\lambda^{m_1+3} t_1) N(\lambda^{m_2+3} t_2)} \sum_{J \in (\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v} t \mathbb{Z}[\omega])^2} \hat{\psi}_{\lambda^{\ell_t}}(\lambda^{e_v} v_0(J_2 - J_1) + u(Y_2 - Y_1)) \\ &\times K_{\Gamma_1(3), \sigma, \sigma}(\overline{r'_1 r''_1}(\lambda^3 v_1), \overline{r'_1 r''_1}(a_1 \lambda^{e_v+1} v_0 J_1 + \lambda a_1 Y_1 u), \zeta_1 \lambda^{m_1-1} t_1) \\ &\times \overline{K_{\Gamma_1(3), \sigma, \sigma}(\overline{r'_2 r''_2}(\lambda^3 v_2), \overline{r'_2 r''_2}(a_2 \lambda^{e_v+1} v_0 J_2 + \lambda a_2 Y_2 u), \zeta_2 \lambda^{m_2-1} t_2)}, \end{aligned} \tag{10-28}$$

$$\begin{aligned}
 G_{42}(\dots) &:= \frac{1}{N(r'_1 r'_2)} \sum_{J' \in (\mathbb{Z}[\omega]/q'\mathbb{Z}[\omega])^2} \hat{\psi}_{q'}(J'_2 - J'_1) \\
 &\quad \times \overline{K_{\Gamma_1(3), \sigma, \xi}(\zeta_1 \lambda^{m_1-1} t_1 r'_1 (\lambda^3 v_1), \zeta_1 \lambda^{m_1-1} t_1 r'_1 (\lambda a_1 J'_1), r'_1)} \\
 &\quad \times \overline{K_{\Gamma_1(3), \sigma, \xi}(\zeta_2 \lambda^{m_2-1} t_2 r'_2 (\lambda^3 v_2), \zeta_2 \lambda^{m_2-1} t_2 r'_2 (\lambda a_2 J'_2), r'_2)}, \tag{10-29}
 \end{aligned}$$

$$\begin{aligned}
 G_{43}(\dots) &:= \frac{1}{N(r''_1 r''_2)} \sum_{J'' \in (\mathbb{Z}[\omega]/q''\mathbb{Z}[\omega])^2} \hat{\psi}_{q''}(J''_2 - J''_1) \\
 &\quad \times \overline{K_{\Gamma_1(3), \sigma, \xi}(\zeta_1 \lambda^{m_1-1} t_1 r'_1 (\lambda^3 v_1), \zeta_1 \lambda^{m_1-1} t_1 r'_1 (\lambda a_1 J''_1), r''_1)} \\
 &\quad \times \overline{K_{\Gamma_1(3), \sigma, \xi}(\zeta_2 \lambda^{m_2-1} t_2 r'_2 (\lambda^3 v_2), \zeta_2 \lambda^{m_2-1} t_2 r'_2 (\lambda a_2 J''_2), r''_2)}. \tag{10-30}
 \end{aligned}$$

We now evaluate and bound each (10-28)–(10-30).

**10.3.1. Treatment of (10-29).** We open the normalised Ramanujan sums and the cubic Kloosterman sums in (10-29), use orthogonality in  $J'$ , and then reassemble the result to obtain

$$\begin{aligned}
 &G_{42}(\dots) \\
 &= N(q') \left( \prod_{i=1}^2 \delta_{(a_i q' / r'_i, q')=1} \frac{1}{N(r'_i)} \left( \frac{\overline{\zeta_i \lambda^{m_i-1} t_i r'_i}}{r'_i} \right)_3 \left( \frac{\zeta_i \lambda^{m_i-1} t_i r'_i}{r'_i} \right)_3 \right) \\
 &\quad \times \sum_{\substack{x \in (\mathbb{Z}[\omega]/r'_1 \mathbb{Z}[\omega]) \times (\mathbb{Z}[\omega]/r'_2 \mathbb{Z}[\omega]) \\ \zeta_2 \lambda^{m_2-1} t_2 r'_2 (a_1 q' / r'_1) x_2 \equiv \\ \zeta_1 \lambda^{m_1-1} t_1 r'_1 (a_2 q' / r'_2) x_1 \pmod{q'}}} \left( \frac{x_1}{r'_1} \right)_3 \left( \frac{x_2}{r'_2} \right)_3 \check{e} \left( \frac{\zeta_2 \lambda^{m_2-1} t_2 r'_2 \lambda^3 v_1 x_1 - \zeta_1 \lambda^{m_1-1} t_1 r'_1 \lambda^3 v_2 x_2}{r'_1 r'_2} \right). \tag{10-31}
 \end{aligned}$$

The delta conditions in (10-31) are nonzero only if  $q' \mid r'_i$  for  $i = 1, 2$ . We make the change of variables  $r' \rightarrow q' s'$  where  $s'_i \mid a_i$  for  $i = 1, 2$ . We detect the congruence with additive characters and reassemble to get

$$\begin{aligned}
 G_{42}(\dots) &= \frac{1}{N(q')} \left( \prod_{i=1}^2 \delta_{(a_i / s'_i, q')=1} \frac{1}{N(s'_i)^{1/2}} \left( \frac{\overline{\zeta_i \lambda^{m_i-1} t_i r'_i}}{q' s'_i} \right)_3 \left( \frac{\zeta_i \lambda^{m_i-1} t_i r'_i}{q' s'_i} \right)_3 \right) \\
 &\quad \times \sum_{k \pmod{q'}} \overline{\tilde{g}(y_1 \lambda^3 v_1 + k z_1, q' s'_1)} \tilde{g}(y_2 \lambda^3 v_2 + k z_2, q' s'_2).
 \end{aligned}$$

where

$$y_1 = \zeta_2 \lambda^{m_2-1} t_2 r''_2, \quad z_1 = \zeta_1 \lambda^{m_1-1} t_1 r''_1 (a_2 / s'_2) s'_1, \tag{10-32}$$

$$y_2 = \zeta_1 \lambda^{m_1-1} t_1 r''_1, \quad z_2 = \zeta_2 \lambda^{m_2-1} t_2 r''_2 (a_1 / s'_1) s'_2. \tag{10-33}$$

We then factorise  $q' s'_i = q' (s'_i, q') \cdot (s'_i / (s'_i, q'))$ . Since  $a_i$  is squarefree and  $s'_i \mid a_i$  for  $i = 1, 2$ , the pair of moduli  $q' (s'_i, q')$  and  $s'_i / (s'_i, q')$  are coprime. Thus (2-8), Lemma 2.1, and (2-7) imply that

$$\begin{aligned}
 G_{42}(\dots) &= \frac{1}{N(q')} \left( \prod_{i=1}^2 \delta_{(a_i/s'_i, q')=1} \cdot \delta_{(\lambda^3 v_i, s'_i/(s'_i, q'))=1} \frac{1}{N(s'_i)^{1/2}} \left( \frac{\overline{\zeta_i \lambda^{m_i-1} t_i r''_i}}{q' s'_i} \right)_3 \left( \frac{\zeta_i \lambda^{m_i-1} t_i r''_i}{q' s'_2} \right)_3 \right) \\
 &\quad \times \overline{\tilde{g}(s'_1/(s'_1, q'))} \tilde{g}(s'_2/(s'_2, q')) \left( \frac{q'(s'_1, q')}{s'_1/(s'_1, q')} \right)_3 \overline{\left( \frac{q'(s'_2, q')}{s'_2/(s'_2, q')} \right)_3} \left( \frac{\zeta_2 \lambda^{m_2-1} t_2 r''_2 \lambda^3 v_1}{s'_1/(s'_1, q')} \right)_3 \\
 &\quad \times \left( \frac{\overline{\zeta_2 \lambda^{m_2-1} t_2 r''_2 \lambda^3 v_2}}{s'_2/(s'_2, q')} \right)_3 \sum_{k \pmod{q'}} \overline{\tilde{g}(y_1 \lambda^3 v_1 + k z_1, q'(s'_1, q'))} \tilde{g}(y_2 \lambda^3 v_2 + k z_2, q'(s'_2, q')). \quad (10-34)
 \end{aligned}$$

Observe that Lemma 2.2 applied to the last two Gauss sums in the previous display imply that  $G_{42}(v, \lambda^\ell t q' q'', \xi \lambda^m t q' s' r'') \neq 0$  only if  $\lambda^3 v_i \equiv 0 \pmod{(s'_i, q')}$  for  $i = 1, 2$ . Thus

$$\begin{aligned}
 G_{42}(\dots) &= \frac{1}{N(q')} \left( \prod_{i=1}^2 \delta_{(a_i/s'_i, q')=1} \cdot \delta_{(\lambda^3 v_i, s'_i/(s'_i, q'))=1} \cdot \delta_{\lambda^3 v_i \equiv 0 \pmod{(s'_i, q')}} \right) \\
 &\quad \times \frac{1}{N(s'_i)^{1/2}} \left( \frac{\overline{\zeta_i \lambda^{m_i-1} t_i r''_i}}{q' s'_i} \right)_3 \left( \frac{\zeta_i \lambda^{m_i-1} t_i r''_i}{q' s'_2} \right)_3 \overline{\tilde{g}(s'_1/(s'_1, q'))} \tilde{g}(s'_2/(s'_2, q')) \\
 &\quad \times \left( \frac{q'(s'_1, q')}{s'_1/(s'_1, q')} \right)_3 \overline{\left( \frac{q'(s'_2, q')}{s'_2/(s'_2, q')} \right)_3} \left( \frac{\zeta_2 \lambda^{m_2-1} t_2 r''_2 \lambda^3 v_1}{s'_1/(s'_1, q')} \right)_3 \overline{\left( \frac{\zeta_2 \lambda^{m_2-1} t_2 r''_2 \lambda^3 v_2}{s'_2/(s'_2, q')} \right)_3} \\
 &\quad \times \sum_{k \pmod{q'}} \overline{\tilde{g}(y_1 \lambda^3 v_1 + k z_1, q'(s'_1, q'))} \tilde{g}(y_2 \lambda^3 v_2 + k z_2, q'(s'_2, q')). \quad (10-35)
 \end{aligned}$$

Using Lemma 2.3 (noting the normalisation in (2-12)) gives

$$\begin{aligned}
 &\sum_{k \pmod{q'}} |\tilde{g}(y_1 \lambda^3 v_1 + k z_1, q'(s'_1, q'))| \cdot |\tilde{g}(y_2 \lambda^3 v_2 + k z_2, q'(s'_2, q'))| \\
 &\leq \left( \prod_{i=1}^2 \delta_{\lambda^3 v_i \equiv 0 \pmod{(s'_i, q')}} \cdot N((s'_i, q'))^{1/2} \right) \\
 &\quad \times \sum_{k \pmod{q'}} N\left( \left( y_1 \frac{\lambda^3 v_1}{(s'_1, q')} + k \frac{z_1}{(s'_1, q')}, q' \right) \right)^{1/2} N\left( \left( y_2 \frac{\lambda^3 v_2}{(s'_2, q')} + k \frac{z_2}{(s'_2, q')}, q' \right) \right)^{1/2} \\
 &\ll N(q')^{1+\varepsilon} \left( \prod_{i=1}^2 \delta_{\lambda^3 v_i \equiv 0 \pmod{(s'_i, q')}} \cdot N((s'_i, q'))^{1/2} \right), \quad (10-36)
 \end{aligned}$$

where the last display follows from using Cauchy–Schwarz in  $k$  and then a change of variable to  $k \pmod{q'}$  in each resulting bracket (the change of variable is valid since  $(z_1/(s'_1, q'), q') = (z_2/(s'_2, q'), q') = 1$ ). We use the triangle inequality in (10-34), substitute (10-36), and then change variables back  $s' \rightarrow (1/q')r'$  to obtain

$$\begin{aligned}
 &|G_{42}(\dots)| \\
 &\ll N(q')^{1+\varepsilon} \left( \prod_{i=1}^2 \delta_{(a_i q'/r'_i, q')=1} \cdot \delta_{(\lambda^3 v_i, (r'_i/q')/(r'_i/q', q'))=1} \delta_{\lambda^3 v_i \equiv 0 \pmod{(r'_i/q', q')}} \frac{N((r'_i/q', q'))^{1/2}}{N(r'_i)^{1/2}} \right). \quad (10-37)
 \end{aligned}$$

**10.3.2. Treatment of (10-30).** We open the normalised Ramanujan sums and the cubic Kloosterman sums in (10-30), use orthogonality in the  $J_1'', J_2''$  variables, and by a similar argument to the above we reassemble the result to obtain

$$G_{43}(\dots) := \left( \prod_{i=1}^2 \delta_{r_i''=q''} \right) \cdot \left( \frac{a_1}{q''} \right)_3 \left( \frac{\zeta_1 \lambda^{m_1-1} t_1 r_1'}{q''} \right)_3 \left( \frac{a_2}{q''} \right)_3 \left( \frac{\zeta_2 \lambda^{m_2-1} t_2 r_2'}{q''} \right)_3 \hat{\psi}_{q''}(P_1 \lambda^3 v_1 - P_2 \lambda^3 v_2), \quad (10-38)$$

where

$$P_1 := (\zeta_2 \lambda^{m_2-1} t_2 r_2')^2 a_1 \quad \text{and} \quad P_2 := (\zeta_1 \lambda^{m_1-1} t_1 r_1')^2 a_2. \quad (10-39)$$

We have the bound

$$|G_{43}(\dots)| \leq \left( \prod_{i=1}^2 \delta_{r_i''=q''} \right) \cdot |\hat{\psi}_{q''}(P_1 \lambda^3 v_1 - P_2 \lambda^3 v_2)|. \quad (10-40)$$

**10.3.3. Treatment of (10-28).** Recall that  $\ell_0 := \max\{\ell, e_v\}$ . We open the normalised Ramanujan sums and the cubic Kloosterman sums in (10-28), use orthogonality in the  $J_1, J_2$  variables, and then reassemble the result to obtain

$$G_{41}(\dots) = \frac{N(\lambda^{\ell_0-e_v} t)^2}{N(\lambda^{\ell} t)} \left( \prod_{i=1}^2 \frac{1}{N(\lambda^{m_i+3} t_i)} \right) \sum_{\substack{k \pmod{\lambda^{\ell} t} \\ (k, \lambda^{\ell} t) = 1}} \check{e} \left( \frac{ku(Y_2 - Y_1)}{\lambda^{\ell} t} \right) \sum_{\mathbf{x} \in \mathcal{B}_1(k) \times \mathcal{B}_2(k)} \left( \frac{\zeta_1 \lambda^{m_1-1} t_1}{x_1} \right)_3 \left( \frac{\zeta_2 \lambda^{m_2-1} t_2}{x_2} \right)_3 \\ \times \check{e} \left( \frac{r_1' r_1'' (\lambda^3 v_1 \bar{x}_1 + \lambda a_1 Y_1 u x_1)}{\zeta_1 \lambda^{m_1-1} t_1} - \frac{r_2' r_2'' (\lambda^3 v_2 \bar{x}_2 + \lambda a_2 Y_2 u x_2)}{\zeta_2 \lambda^{m_2-1} t_2} \right), \quad (10-41)$$

where for  $i = 1, 2$  we have

$$\mathcal{B}_i(k) := \left\{ x_i \pmod{\lambda^{m_i+1} t_i} : (x_i, \lambda t_i) = 1, x_i \equiv 1 \pmod{3}, \overline{r_i' r_i'' \zeta_i a_i \lambda^{\ell_0-m_i+2} (v_0 t / t_i)} x_i \equiv k \lambda^{\ell_0-\ell} v_0 \pmod{\lambda^{\ell_0-e_v} t} \right\}. \quad (10-42)$$

For a given  $k \in \mathbb{Z}[\omega]$  with  $(k, \lambda^{\ell} t) = 1$ , any solution  $y_i \pmod{\lambda^{\ell_0-e_v} t}$  to the congruence

$$\overline{r_i' r_i'' \zeta_i a_i \lambda^{\ell_0-m_i+2} (v_0 t / t_i)} y_i \equiv k \lambda^{\ell_0-\ell} v_0 \pmod{\lambda^{\ell_0-e_v} t} \quad (10-43)$$

corresponds to  $N(\lambda^{\max\{0, m_i+1-\ell_0+e_v\}}) N(t_i / (t, t_i))$  distinct solutions  $x_i \pmod{\lambda^{m_i+1} t_i}$ . The congruence in (10-43) has a solution  $y_i \pmod{\lambda^{\ell_0-e_v} t}$  if and only if

$$\left( \overline{r_i' r_i'' \zeta_i a_i \lambda^{\ell_0-m_i+2} (v_0 t / t_i)}, \lambda^{\ell_0-e_v} t \right) \mid k \lambda^{\ell_0-\ell} v_0. \quad (10-44)$$

Since  $t \mid \text{rad}(v_0)^\infty, t_i \mid v_0 t, \overline{r_i' r_i'' \zeta_i a_i, \lambda v_0} = (\lambda, v_0) = 1$ , we have

$$\left( \overline{r_i' r_i'' \zeta_i a_i \lambda^{\ell_0-m_i+2} (v_0 t / t_i)}, \lambda^{\ell_0-e_v} t \right) = \lambda^{\min\{\ell_0-m_i+2, \ell_0-e_v\}} ((v_0 t / t_i), t) = \lambda^{\min\{\ell_0-m_i+2, \ell_0-e_v\}} t (v_0, t_i) / t_i \quad (10-45)$$

for  $i = 1, 2$ . Observe that (10-45) and the fact  $(k, \lambda^\ell t) = 1$  (recall that  $\ell \geq 1$ ) imply that (10-44) is equivalent to the two conditions

$$t \mid [v_0, t_i] \quad \text{and} \quad \min\{\ell_0 - m_i + 2, \ell_0 - e_v\} \leq \ell_0 - \ell \tag{10-46}$$

for  $i = 1, 2$ . Under the restriction  $0 \leq m_i \leq \ell_0 + 1$ , the conditions in (10-46) are equivalent to

$$t \mid [v_0, t_i] \quad \text{and} \quad 1 \leq \ell \leq e_v = \ell_0 \tag{10-47}$$

for  $i = 1, 2$ . Thus (10-41) becomes

$$\begin{aligned} G_{41}(\dots) &= \delta_{1 \leq \ell \leq e_v} \cdot \frac{N(t)^2}{N(\lambda^\ell t)} \cdot \left( \prod_{i=1}^2 \delta_{t \mid [v_0, t_i]} \cdot \frac{1}{N(\lambda^{m_i+3} t_i)} \right) \\ &\times \sum_{\substack{k \pmod{\lambda^\ell t} \\ (k, \lambda^\ell t) = 1}} \check{e} \left( \frac{ku(Y_2 - Y_1)}{\lambda^\ell t} \right) \sum_{x \in \mathcal{B}_1(k) \times \mathcal{B}_2(k)} \left( \frac{\zeta_1 \lambda^{m_1-1} t_1}{x_1} \right)_3 \left( \frac{\overline{\zeta_2 \lambda^{m_2-1} t_2}}{x_2} \right)_3 \\ &\times \check{e} \left( \frac{\overline{r'_1 r''_1 (\lambda^3 v_1 \bar{x}_1 + \lambda a_1 Y_1 u x_1)}}{\zeta_1 \lambda^{m_1-1} t_1} - \frac{\overline{r'_2 r''_2 (\lambda^3 v_2 \bar{x}_2 + \lambda a_2 Y_2 u x_2)}}{\zeta_2 \lambda^{m_2-1} t_2} \right). \end{aligned} \tag{10-48}$$

Furthermore, under the conditions in (10-47) and  $0 \leq m_i \leq e_v + 1$  for  $i = 1, 2$ , (10-45) and the sentence containing (10-43) imply that for each  $k \in \mathbb{Z}[\omega]$  with  $(k, \lambda^\ell t) = 1$  we have

$$|\mathcal{B}_1(k) \times \mathcal{B}_2(k)| \leq \left( \prod_{i=1}^2 \delta_{t \mid [v_0, t_i]} N(\lambda^{m_i+1}) N\left(\frac{t(v_0, t_i)}{(t, t_i)}\right) \right). \tag{10-49}$$

Using (10-49), we bound (10-48) trivially by

$$|G_{41}(\dots)| \leq \delta_{1 \leq \ell \leq e_v} \cdot \left( \prod_{i=1}^2 \delta_{t \mid [v_0, t_i]} \cdot N\left(\frac{t^2(v_0, t_i)}{t_i(t, t_i)}\right) \right) \leq \delta_{1 \leq \ell \leq e_v} \cdot N(v_0)^6 \left( \prod_{i=1}^2 \delta_{t \mid [v_0, t_i]} \right). \tag{10-50}$$

**10.4. Further technical manipulations and insertion of smooth dyadic partitions of unity.** We substitute (10-21) into (10-17) to obtain

$$\begin{aligned} \mathcal{M}_{4f}(\mathbf{a}, \dots) &= \frac{N(\lambda^8) k_C}{C^4} \frac{X^2}{N(a_1 a_2) N(v_0)^2} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{t q' q'' \in \mathbb{Z}[\omega] \\ (10-13)}} \frac{N(\lambda^\ell)}{N(\lambda^{\ell_0+1})^2} \frac{1}{N(t q' q'')} \\ &\times \sum_{\substack{k \in (\mathbb{Z}[\omega]/\lambda^{14} \mathbb{Z}[\omega])^2 \\ \forall i: k_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{m}, \mathbf{t}, \mathbf{r}', \mathbf{r}'' \\ \forall i: 6 \leq m_i \leq \ell_0 + 1 \\ (10-14)}} \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1 + n_2} \sum_{\mathbf{v} \in (\lambda^{-3} \mathbb{Z}[\omega])^2} \rho_f(v_1) \overline{\rho_f(v_2)} \left( \frac{v_1}{|v_1|} \right)^{-n_1} \left( \frac{\overline{v_2}}{|v_2|} \right)^{-n_2} \\ &\times \check{H}_{K, C^2/N(\lambda^\ell t q' q'')} \left( \frac{N(v_1)}{N(\lambda^{m_1-4} t_1 r'_1 r''_2)^2 / X}, \frac{N(v_2)}{N(\lambda^{m_2-4} t_2 r'_2 r''_2)^2 / X}, \mathbf{n} \right) \\ &\times \mathcal{C}_4(\mathbf{a}, \mathbf{v}, \lambda^\ell t q' q'', \lambda^{\mathbf{m}} \mathbf{t} \mathbf{r}' \mathbf{r}''), \end{aligned} \tag{10-51}$$

where  $\mathcal{C}_4(\dots)$  is given by (10-25) (and (10-27)). Note that the summands  $\mathcal{M}_{4f}(\dots)$  do not depend on the congruence classes  $k_i \pmod{\lambda^{14}}$  (unlike the other  $\mathcal{M}_p(\dots)$  for  $p = 1, 2, 3$ ). Thus the sum over  $\mathbf{k}$

in (10-51), and the last condition in (10-14) can be dropped. Equality (10-27) and the delta conditions in (10-31) (resp. (10-38)) imply that we can make the change of variable  $\mathbf{r}' \rightarrow q's'$  where  $s'_i \mid a_i$  and  $(a_i/s'_i, q') = 1$  (resp.  $\mathbf{r}'' \rightarrow \mathbf{q}''$  where  $\mathbf{q}'' = (q'', q'')$ ) in (10-51). The delta conditions in (10-48) tells us that  $1 \leq \ell \leq e_v$  and  $t \mid [v_0, t_i]$ . Thus the multiple summation  $\sum_{\mathbf{m}, t, \mathbf{r}, \mathbf{r}''}$  in (10-51) subject to  $6 \leq m_i \leq \ell_0 + 1$  for  $i = 1, 2$  and (10-14), can be written as  $\sum_{\mathbf{m}, t, s'}$  subject to  $6 \leq m_i \leq e_v + 1$  for  $i = 1, 2$ , and

$$t_i \mid v_0 t, \quad t \mid [v_0, t_i], \quad s'_i \mid a_i, \quad (a_i/s'_i, q') = 1 \quad \text{for } i = 1, 2. \tag{10-52}$$

We further note that the delta conditions in (10-35) imply that  $\mathbf{v}$  sum in (10-51) is supported on the conditions

$$\lambda^3 v_i \equiv 0 \pmod{(s'_i, q')} \quad \text{and} \quad \left( \lambda^3 v_i, \frac{s'_i}{(s'_i, q')} \right) = 1 \quad \text{for } i = 1, 2. \tag{10-53}$$

We then insert a smooth partition of unity in the variables  $t, q'$ , and  $q''$  in (10-51). Thus

$$\mathcal{M}_{4f}(\mathbf{a}, \dots) = \sum_{\substack{1 \leq \ell \leq e_v \\ 1/2 \leq T, Q', Q'' \text{ dyadic} \\ N(\lambda^\ell) T Q' Q'' \ll C^2}} \mathcal{M}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q''), \tag{10-54}$$

where

$$\begin{aligned} & \mathcal{M}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'') \\ & := \frac{N(\lambda^8) k_C}{C^4} \frac{X^2 N(\lambda^\ell)}{N(a_1 a_2) N(\lambda^{e_v+1} v_0)^2} \sum_{\substack{t q' q'' \in \mathbb{Z}[\omega] \\ (10-13)}} \frac{1}{N(t q' q'')} U\left(\frac{N(t)}{T}\right) U\left(\frac{N(q')}{Q'}\right) U\left(\frac{N(q'')}{Q''}\right) \\ & \quad \times \sum_{\substack{\mathbf{m}, t, s' \\ 6 \leq m_i \leq e_v + 1 \\ (10-52)}} \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1 + n_2} \sum_{\substack{\mathbf{v} \in (\lambda^{-3} \mathbb{Z}[\omega])^2 \\ (10-53)}} \rho_f(v_1) \overline{\rho_f(v_2)} \left(\frac{v_1}{|v_1|}\right)^{-n_1} \left(\frac{\bar{v}_2}{|v_2|}\right)^{-n_2} \\ & \quad \times \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')} \left( \frac{N(v_1)}{N(\lambda^{m_1-4} t_1 s'_1 q' q'')^2 / X}, \frac{N(v_2)}{N(\lambda^{m_2-4} t_2 s'_2 q' q'')^2 / X}, \mathbf{n} \right) \\ & \quad \times \mathcal{E}_i(\mathbf{a}, \mathbf{v}, \lambda^\ell t q' q'', \lambda^m t q' s' q''). \end{aligned} \tag{10-55}$$

The restriction

$$N(\lambda^\ell) T Q' Q'' \ll C^2 \tag{10-56}$$

in (10-54) follows from (10-8).

Using (10-9), (8-71) (with  $M \rightarrow C^2 / (N(\lambda^\ell) T Q' Q'')$ ,  $D_{i1} = D_{i2} > 0$  large and fixed, and  $D_{(i+1)1} = D_{(i+1)2} = \varepsilon$  small and fixed), Lemma 2.8, (10-37), (10-40), and (10-50), we truncate the  $v_i$ -sum in (10-55) by

$$N(v_i) \ll (XKN(v))^\varepsilon K^8 \cdot (N(\lambda^{m_i} t_i s'_i) Q' Q'')^2 X^{-1} =: \Xi_i, \tag{10-57}$$

with negligible error  $O((XKN(v))^{-2000})$ . Without loss of generality we can restrict our attention to the case  $\Xi_i \gg (XKN(v))^{-\varepsilon}$ , otherwise  $\mathcal{M}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'')$  is a negligible  $O((XKN(v))^{-2000})$ . Observe that (8-71) with  $D_{11} = D_{12} = D_{21} = D_{22} = \varepsilon > 0$  small and fixed, (10-3), (10-56), and (10-57)

imply that

$$\begin{aligned} \ddot{H}_{K,C^2/N(\lambda^\ell t q' q'')} &\left( \frac{N(v_1)}{N(\lambda^{m_1-4} t_1 s'_1 q' q'')^2 / X}, \frac{N(v_2)}{N(\lambda^{m_2-4} t_2 s'_2 q' q'')^2 / X}, \mathbf{n} \right) \\ &\ll (XKN(v))^\varepsilon \cdot \frac{C^2}{N(\lambda^\ell t q' q'')} \cdot \prod_{i=1}^2 (|n_i| + 1)^{-2+\varepsilon}. \end{aligned} \quad (10-58)$$

We apply the triangle inequality in (10-55), and then use (10-57), (10-27), (10-37), (10-40), (10-50), and (10-58) to obtain

$$\begin{aligned} \mathcal{M}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'') &\ll (XKN(v))^\varepsilon \cdot \left( \frac{XN(v_0)^2}{CATQ'Q''N(\lambda^{e_v+1})} \right)^2 \\ &\times \sum_{\substack{tq' \in \mathbb{Z}[\omega] \\ N(t) \sim T, N(q') \sim Q' \\ t|\text{rad}(v_0)^\infty \\ q'|\text{rad}(a_1 a_2)^\infty}} \sum_{\substack{\xi, m, t, s' \\ 6 \leq m_i \leq e_v + 1 \\ (10-52)}} \prod_{i=1}^2 \frac{N((s'_i, q'))^{1/2}}{N(s'_i)^{1/2}} \sum_{\substack{v \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(v_i) \ll \Xi_i \\ (10-53)}} |\rho_f(v_1)| |\rho_f(v_2)| \\ &\times \sum_{\substack{q'' \in \mathbb{Z}[\omega] \\ q'' \equiv 1 \pmod{3} \\ N(q'') \sim Q'' \\ (q'', a_1 a_2 v_0) = 1}} |\hat{\psi}_{q''}(P_1 \lambda^3 v_1 - P_2 \lambda^3 v_2)| + O((XKN(v))^{-2000}), \end{aligned} \quad (10-59)$$

where

$$P_1 := (\zeta_2 \lambda^{m_2-1} t_2 q' s'_2)^2 a_1 \quad \text{and} \quad P_2 := (\zeta_1 \lambda^{m_1-1} t_1 q' s'_1)^2 a_2. \quad (10-60)$$

We drop the condition  $(q'', a_1 a_2 v_0) = 1$  in (10-59) by positivity, and use Lemma 8.5 to obtain

$$\sum_{\substack{q'' \in \mathbb{Z}[\omega] \\ q'' \equiv 1 \pmod{3} \\ N(q'') \sim Q''}} |\hat{\psi}_{q''}(P_1 \lambda^3 v_1 - P_2 \lambda^3 v_2)| \ll \delta_{P_1 \lambda^3 v_1 = P_2 \lambda^3 v_2} \cdot Q'' + \delta_{P_1 \lambda^3 v_1 \neq P_2 \lambda^3 v_2} \cdot (XKN(v))^\varepsilon. \quad (10-61)$$

We substitute the bound (10-61) into (10-59), and obtain

$$\mathcal{M}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'') \ll \mathcal{N}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'') + \mathcal{E}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q''), \quad (10-62)$$

where the terms on the right correspond to the diagonal and off-diagonal respectively. Using (10-54) and (10-62) it suffices to estimate

$$\sum_{\substack{1 \leq \ell \leq e_v \\ 1/2 \leq T, Q', Q'' \text{ dyadic} \\ N(\lambda^\ell) T Q' Q'' \ll C^2}} \sum_{\mathbf{a}} \mu^2(a_1) |\alpha_{a_1}| \mu^2(a_2) |\alpha_{a_2}| \mathcal{N}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q''), \quad (10-63)$$

$$\sum_{\substack{1 \leq \ell \leq e_v \\ 1/2 \leq T, Q', Q'' \text{ dyadic} \\ N(\lambda^\ell) T Q' Q'' \ll C^2}} \sum_{\mathbf{a}} \mu^2(a_1) |\alpha_{a_1}| \mu^2(a_2) |\alpha_{a_2}| \mathcal{E}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q''), \quad (10-64)$$

with  $C$  given by (10-3).

**10.5. Off-diagonal: (10-64).** We drop the condition  $P_1\lambda^3v_1 \neq P_2\lambda^3v_2$  and  $(\lambda^3v_i, s'_i/(s'_i, q')) = 1$  for  $i = 1, 2$  (see (10-53)) by positivity, and then use the Cauchy–Schwarz inequality,  $\rho_f(0) = 0$ , and Lemma 2.7 to obtain

$$\sum_{\substack{v_i \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_i) \ll \Xi_i \\ \lambda^3v_i \equiv 0 \pmod{(s'_i, q')}}} |\rho_f(v_i)| \leq \left( \sum_{\substack{0 \neq v_i \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_i) \ll \Xi_i \\ \lambda^3v_i \equiv 0 \pmod{(s'_i, q')}}} 1 \right)^{1/2} \left( \sum_{\substack{v_i \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_i) \ll \Xi_i}} |\rho_f(v_i)|^2 \right)^{1/2} \ll \frac{\Xi_i^{1+\varepsilon}}{N((s'_i, q'))^{1/2}} \quad (10-65)$$

for  $i = 1, 2$ .

We use (10-65), (10-52), and Lemma 2.6 to conclude that

$$\mathcal{E}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell)TQ'Q'') \ll (XKN(v))^\varepsilon K^{16}N(v_0)^8 N(\lambda^{e_v})^2 AC^{-2}(TQ'Q'')^2. \quad (10-66)$$

Substituting (10-66) into (10-64) and using Cauchy–Schwarz we see that (10-64) is

$$\begin{aligned} &\ll (XKN(v))^\varepsilon K^{16}N(v)^8 A^2 B^{1/2} \|\mu^2 \alpha\|_2^2 \\ &\ll (XKN(v))^\varepsilon K^{16}N(v)^8 A^3 B^{1/2} \|\mu^2 \alpha\|_\infty^2. \end{aligned} \quad (10-67)$$

**10.6. Diagonal: (10-63).** Consulting (10-53) we make the change of variable

$$\lambda^3v_i = (s'_i, q')\lambda^3\mu_i \quad \text{such that} \quad 0 \neq \mu_i \in \lambda^{-3}\mathbb{Z}[\omega] \quad \text{and} \quad \left( (s'_i, q')\lambda^3\mu_i, \frac{s'_i}{(s'_i, q')} \right) = 1 \quad (10-68)$$

for  $i = 1, 2$ . Since  $a_i$  is squarefree and  $s'_i \mid a_i$ , the coprimality condition in (10-68) is equivalent to

$$\left( \lambda^3\mu_i, \frac{s'_i}{(s'_i, q')} \right) = 1 \quad (10-69)$$

for  $i = 1, 2$ . The diagonal equation  $P_1\lambda^3v_1 = P_2\lambda^3v_2$  with  $P_1$  and  $P_2$  given in (10-60) is equivalent to

$$(\zeta_2\lambda^{m_2-1}t_2)^2 \frac{s'_2}{(s'_2, q')} \frac{a_1}{s'_1} \lambda^3\mu_1 = (\zeta_1\lambda^{m_1-1}t_1)^2 \frac{s'_1}{(s'_1, q')} \frac{a_2}{s'_2} \lambda^3\mu_2, \quad (10-70)$$

where  $0 \neq \lambda^3\mu_i$  satisfies (10-69) for  $i = 1, 2$ . The hypothesis that the  $a_i$  are squarefree for  $i = 1, 2$  guarantees that

$$\left( \frac{s'_1}{(s'_1, q')}, \lambda^{m_2-1}t_2 \frac{a_1}{s'_1} \right) = \left( \frac{s'_2}{(s'_2, q')}, \lambda^{m_1-1}t_1 \frac{a_2}{s'_2} \right) = 1. \quad (10-71)$$

Using (10-69) and (10-71) we conclude from (10-70) that

$$\tilde{s} := \frac{s'_1}{(s'_1, q')} = \frac{s'_2}{(s'_2, q')} \mid (a_1, a_2), \quad (10-72)$$

and thus (10-70) is equivalent to

$$(\zeta_2\lambda^{m_2-1}t_2)^2 \frac{a_1}{s'_1} \lambda^3\mu_1 = (\zeta_1\lambda^{m_1-1}t_1)^2 \frac{a_2}{s'_2} \lambda^3\mu_2, \quad (10-73)$$

where  $0 \neq \lambda^3\mu_i$  satisfies (10-69) for  $i = 1, 2$ .

We use (10-68)–(10-73) to rewrite (10-63), set  $g_i := (s_i, q')$  and release using Möbius inversion, and then interchange the order of summation. We obtain

$$\begin{aligned} & \sum_a \mu^2(a_1) |\alpha_{a_1}| \mu^2(a_2) |\alpha_{a_2}| \mathcal{N}_{4f}(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'') \\ &= (XKN(v))^\varepsilon \cdot \left( \frac{XN(v_0)^2}{CATQ'N(\lambda^{e_v+1})} \right)^2 \frac{1}{Q''} \\ & \times \sum_{\substack{\xi, \mathbf{m}, \mathbf{t} \\ \forall i: 6 \leq m_i \leq e_v+1 \\ \forall i: t_i | \text{rad}(v_0)^\infty}} \sum_{\substack{\mathbf{h}, \mathbf{d}, \mathbf{g}, \mathbf{r} \\ \forall i: h_i, d_i, g_i, r_i \equiv 1 \pmod{3} \\ h_1 d_1 = h_2 d_2}} \frac{\mu(h_1) \mu(h_2)}{N(h_1 d_1 h_2 d_2)^{1/2}} \mu^2(h_1 d_1 g_1 r_1) |\alpha_{h_1 d_1 g_1 r_1}| \mu^2(h_2 d_2 g_2 r_2) |\alpha_{h_2 d_2 g_2 r_2}| \\ & \times \sum_{\substack{\mathbf{v} \in (\lambda^{-3} \mathbb{Z}[\omega])^2 \\ \forall i: N(v_i) \ll \Xi'_i \\ (10-76) \\ (10-77)}} |\rho_f(v_1)| \cdot |\rho_f(v_2)| \sum_{\substack{t q' \in \mathbb{Z}[\omega] \\ N(t) \sim T, N(q') \sim Q' \\ [t_1, t_2] | v_0 t \\ t | ([v_0, t_1], [v_0, t_2]) \\ [h_1 g_1, h_2 g_2] | q' | \text{rad}(h_1 d_1 g_1 h_2 d_2 g_2)^\infty}} 1, \quad (10-74) \end{aligned}$$

where (see (10-57))

$$\Xi'_i := (XKN(v))^\varepsilon K^8 \cdot (N(\lambda^{m_i} t_i h_i d_i g_i) Q' Q'')^2 X^{-1} \quad \text{for } i = 1, 2, \quad (10-75)$$

$$(\lambda^3 v_i, h_i d_i) = 1 \quad \text{and} \quad \lambda^3 v_i \equiv 0 \pmod{g_i} \quad \text{for } i = 1, 2, \quad (10-76)$$

$$(\zeta_2 \lambda^{m_2-1} t_2)^2 r_1 \frac{\lambda^3 v_1}{g_1} = (\zeta_1 \lambda^{m_1-1} t_1)^2 r_2 \frac{\lambda^3 v_2}{g_2}. \quad (10-77)$$

We dyadically partition all of the auxiliary variables, i.e.,

$$N(h_i) \sim H_i, \quad N(d_i) \sim D_i, \quad N(g_i) \sim G_i, \quad N(r_i) \sim R_i, \quad N(t_i) \sim T_i,$$

such that

$$H_i D_i G_i R_i \asymp A, \quad H_i G_i \ll Q', \quad \text{and} \quad T_i \ll N(v_0) T \quad \text{for } i = 1, 2. \quad (10-78)$$

We estimate the sum over  $t$  and  $q'$  in (10-74) by  $(XKN(v))^\varepsilon$  using (2-14) and Lemma 2.6 respectively.

We then apply the bound  $|\mu^2(a) \alpha_a| \leq \|\mu^2 \alpha\|_\infty$ . We see that the entirety of (10-74) is

$$\begin{aligned} & \ll (XKN(v))^\varepsilon \|\mu^2 \alpha\|_\infty^2 \cdot \left( \frac{XN(v_0)^2}{CATQ'N(\lambda^{e_v+1})} \right)^2 \frac{1}{Q''} \\ & \times \sum_{\substack{\xi, \mathbf{m} \\ \forall i: 6 \leq m_i \leq e_v+1}} \sum_{\substack{\forall i: H_i, D_i, G_i, R_i, T_i \\ \text{dyadic} \\ (10-78)}} \frac{1}{(H_1 D_1 H_2 D_2)^{1/2}} \\ & \times \sum_{\substack{\mathbf{t} \\ \forall i: N(t_i) \sim T_i \\ t_i | \text{rad}(v_0)^\infty}} \sum_{\substack{\mathbf{r} \\ \forall i: N(r_i) \sim R_i \\ r_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{h}, \mathbf{d} \\ h_1 d_1 = h_2 d_2 \\ \forall i: N(h_i) \sim H_i \\ \forall i: N(d_i) \sim D_i \\ \forall i: h_i, d_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{g} \\ \forall i: N(g_i) \sim G_i \\ \forall i: g_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{v} \in (\lambda^{-3} \mathbb{Z}[\omega])^2 \\ \forall i: N(v_i) \ll \Xi''_i \\ (10-76) \\ (10-77)}} |\rho_f(v_1)| \cdot |\rho_f(v_2)|, \quad (10-79) \end{aligned}$$

where (see (10-75))

$$\Xi_i'' := (XKN(v))^\varepsilon K^8 \cdot (N(\lambda^{m_i})T_i H_i D_i G_i Q' Q'')^2 X^{-1} \tag{10-80}$$

for  $i = 1, 2$ .

We apply Cauchy–Schwarz to the sum over  $\mathbf{g}$  and  $\mathbf{v}$  in (10-79), and then rearrange the order of summation to obtain

$$\begin{aligned} & \sum_{\substack{\mathbf{g} \\ \forall i: N(g_i) \sim G_i \\ \forall i: g_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(v_i) \ll \Xi_i'' \\ (10-76) \\ (10-77)}} |\rho_f(v_1)| \cdot |\rho_f(v_2)| \\ & \leq \left( \sum_{\substack{v_1 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_1) \ll \Xi_1'' \\ (\lambda^3 v_1, h_1 d_1) = 1}} |\rho_f(v_1)|^2 \sum_{\substack{N(g_1) \sim G_1 \\ g_1 | \lambda_3 v_1 \\ g_1 \equiv 1 \pmod{3}}} \sum_{\substack{N(g_2) \sim G_2 \\ g_2 \equiv 1 \pmod{3}}} \sum_{\substack{v_2 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_2) \ll \Xi_2'' \\ (\lambda^3 v_2, h_2 d_2) = 1 \\ \lambda^3 v_2 \equiv 0 \pmod{g_2} \\ (10-77)}} 1 \right)^{1/2} \\ & \quad \times \left( \sum_{\substack{v_2 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_2) \ll \Xi_2'' \\ (\lambda^3 v_2, h_2 d_2) = 1}} |\rho_f(v_2)|^2 \sum_{\substack{N(g_2) \sim G_2 \\ g_2 | \lambda_3 v_2 \\ g_2 \equiv 1 \pmod{3}}} \sum_{\substack{N(g_1) \sim G_1 \\ g_1 \equiv 1 \pmod{3}}} \sum_{\substack{v_1 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(v_1) \ll \Xi_1'' \\ (\lambda^3 v_1, h_1 d_1) = 1 \\ \lambda^3 v_1 \equiv 0 \pmod{g_1} \\ (10-77)}} 1 \right)^{1/2}. \end{aligned} \tag{10-81}$$

Consider the bracketed expression in (10-81). The conditions on the  $v_2$ -sum imply that the  $v_2$ -sum is bounded by 1. We then estimate the sum over  $g_2$  trivially, and then apply the divisor bound (2-14) to estimate the sum over  $g_1$ . Thus the sum over  $g_2, g_1$  and  $v_2$  satisfies  $\ll X^\varepsilon G_2$ . We use this bound, drop the condition  $(\lambda^3 v_1, h_1 d_1) = 1$  by positivity, and then apply Lemma 2.7 to estimate the  $v_1$ -sum. We obtain that the entire bracketed expression in (10-81) satisfies  $\ll X^\varepsilon G_2 \Xi_1$ . The analogous argument can be applied to obtain a bound of  $\ll X^\varepsilon G_1 \Xi_2$  for the bracketed expression in (10-82). We deduce that

$$\sum_{\substack{\mathbf{g} \\ \forall i: N(g_i) \sim G_i \\ \forall i: g_i \equiv 1 \pmod{3}}} \sum_{\substack{\mathbf{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(v_i) \ll \Xi_i'' \\ (10-76) \\ (10-77)}} |\rho_f(v_1)| \cdot |\rho_f(v_2)| \ll X^\varepsilon (G_2 \Xi_1)^{1/2} (G_1 \Xi_2)^{1/2}. \tag{10-83}$$

Substituting (10-83) into (10-79), bounding the remaining sums trivially (using Lemma 2.6 for the  $t_1, t_2$  sums), and recalling that  $X \asymp AB$  we obtain

$$\begin{aligned} & \sum_a \mu^2(a_1) |\alpha_{a_1}| \mu^2(a_2) |\alpha_{a_2}| \mathcal{N}_4 f(\mathbf{a}, \dots, N(\lambda^\ell) T Q' Q'') \\ & \ll (XKN(v))^\varepsilon \|\mu^2 \boldsymbol{\alpha}\|_\infty^2 K^8 N(v_0)^4 X Q'' C^{-2} A^{-2} T^{-2} \sum_{\substack{\forall i: H_i, D_i, G_i, R_i, T_i \\ \text{dyadic} \\ (10-78)}} (H_1 D_1 R_1 T_1 G_1^{3/2} H_2 D_2 R_2 T_2 G_2^{3/2}) \\ & \ll (XKN(v))^\varepsilon K^8 N(v_0)^6 ABC^{-2} Q' Q'' \|\mu^2 \boldsymbol{\alpha}\|_\infty^2. \end{aligned} \tag{10-84}$$

Substituting (10-84) into (10-63) we see that (10-63) is

$$\ll (XKN(v))^\varepsilon K^8 N(v_0)^6 AB \|\mu^2 \alpha\|_\infty^2. \quad (10-85)$$

Combining (10-67) and (10-85), and then using  $N(v_0) \leq N(v)$ , yields the result after recalling (10-26) and Remark 10.4.  $\square$

*Proof of Theorem 1.5.* This follows immediately from Lemma 10.1 and Proposition 10.2.  $\square$

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adunn61@gatech.edu

*School of Mathematics, Georgia Institute of Technology, Atlanta, GA,  
United States*

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