Algebra & Number Theory

Volume 19
2025
No. 9

msp

Algebra & Number Theory

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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http://msp.org/

ALGEBRA AND NUMBER THEORY 19:9 (2025)
https://doi.org/10.2140/ant.2025.19.1671

Rigidity of modular morphisms via Fujita decomposition

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We prove that the Torelli, Prym and spin-Torelli morphisms, as well as covering maps between moduli stacks of smooth projective curves, cannot be deformed. The proofs use properties of the Fujita decomposition of the Hodge bundle of families of curves.

1. Introduction

Let $\mathcal{M}$ and $\mathcal{A}$ be Deligne–Mumford stacks. A nonconstant morphism $\mathcal{M} \to \mathcal{A}$ is *globally rigid* if it is the unique nonconstant morphism between $\mathcal{M}$ and $\mathcal{A}$, *locally rigid* if it does not admit nontrivial local deformations with fixed target and domain, and *infinitesimally rigid* if it does not admit nontrivial first-order deformations with fixed target and domain. In particular, global and infinitesimal rigidity both imply local rigidity. However, they do not imply each other, as certain first-order deformations may not extend to local deformations, and a discrete set of morphisms may all have no first-order deformations. From the point of view of the corresponding moduli stack of nonconstant morphisms with fixed target and domain, global rigidity forces the moduli to be just one point, while infinitesimal rigidity determines whether the point is reduced.

In Section 3, we prove the following.

Theorem 1.1. *For any $g \geq 3$ the Torelli morphism $\tau : \mathcal{M}_g \to \mathcal{A}_g$, from the moduli stack of genus g smooth projective curves $\mathcal{M}_g$ to the moduli stack of principally polarized abelian g-folds, is infinitesimally rigid.*

Farb [2024a] proved the global rigidity, i.e., uniqueness, for the Torelli morphism for $g \geq 3$ in the category of complex orbifolds (as in [Farb 2024a, Remark 2.1]). Following his proof one obtains also uniqueness in the category of stacks. Uniqueness as a map between coarse moduli spaces is still open.

Let us recall that the infinitesimal rigidity of $\mathcal{M}_g$ is still unknown. To the best of our knowledge, the most recent work in this direction is by Hacking [2008]: he proved infinitesimal rigidity for the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable curves of arithmetic genus g with n marked points (a result that has been extended to positive characteristic in [Fantechi and Massarenti 2017]), and leaves the infinitesimal rigidity of $\mathcal{M}_{g,n}$ as an open question.

Codogni would like to thank Julius Ross, who nicely raised the question about the rigidity of the Torelli map about ten years ago. We hope he will enjoy this late answer. We thank Benson Farb for private communications. We also thank the referee and the editors for their very valuable comments. Codogni is partially supported by the MIUR Excellence Department Project MatMod@TOV, CUP E83C23000330006, awarded to the Department of Mathematics, University of Rome Tor Vergata, and he is a member of the GNSAGA group of INdAM. Torelli is supported by the Alexander von Humboldt Foundation.
MSC2020: primary 14H10; secondary 32G20.
Keywords: moduli spaces, rigidity of period maps, Fujita decomposition, curves, abelian varieties.

The rigidity of $\mathcal{A}_g$ is also open. To best of our knowledge, just the following is known. Let $\mathcal{A}'_g$ be a finite cover of $\mathcal{A}_g$ constructed as the moduli space of ppav with a level structure, we choose a level structure such that there is no difference between the coarse moduli space and the stack. Let $\overline{\mathcal{A}'}_g$ be a good toroidal compactification of $\mathcal{A}'_g$. Then, building on [Calabi and Vesentini 1960], in [Peters 2017, Theorem 4.3] it is shown that the pair $(\overline{\mathcal{A}'}_g, \partial\overline{\mathcal{A}'}_g)$ is rigid. From an arithmetic point of view, similar rigidity results are proven in [Faltings 1984].

Our second result concerns infinitesimal rigidity of the Prym morphism $\mathrm{Pr}:\mathcal{R}_g \to \mathcal{A}_{g-1}$, from the moduli stack $\mathcal{R}_g$ of pairs (C, η), where C is a smooth projective curve of genus g and $\eta \in J(C)$ is a nontrivial line bundle on C with $\eta^{\otimes 2} \cong \mathcal{O}_C$. This morphism maps each pair (C, η) to its Prym variety $\mathrm{Pr}(C, \eta)$ (see Section 4 for some details). It is never an immersion but it is generically injective for $g \geq 7$, namely as soon as the dimension of the target is larger than the dimension of the domain (see, e.g., [Donagi 1992]).

As for $\mathcal{M}_g$, any rigidity for $\mathcal{R}_g$ is still unknown. However, the global rigidity of the Prym morphism was established in [Serván 2022] and answers a question posed in [Farb 2024a]. Again, our result of infinitesimal rigidity combined with the previous result on global rigidity provides a complete answer to the problem of rigidity with fixed target and domain. The following theorem is proven in Section 4.

Theorem 1.2. *For any $g \geq 3$ the Prym morphism $\mathrm{Pr}:\mathcal{R}_g \to \mathcal{A}_{g-1}$ does not admit any nontrivial first-order deformation with fixed domain and target.*

Our third result concerns the infinitesimal rigidity of the spin-Torelli morphism $\mathrm{s}\tau : \mathcal{S}_g \to \mathcal{N}_g$, from the moduli stack $\mathcal{S}_g$ of spin curves of genus g to the moduli stack $\mathcal{N}_g$ of pairs (A, Θ) of an abelian variety together with an effective symmetric divisor with $h^0(\mathcal{O}_A(\Theta)) = 1$. Closed points in $\mathcal{S}_g$ are pairs (C, ϑ) of a smooth projective curve C of genus g together with a theta-characteristic ϑ (i.e., a line bundle such that $\vartheta^{\otimes 2} \cong \omega_C$). The theta-characteristic allows to construct a unique *symmetric* divisor $\Theta \subseteq J(C)$ and therefore a unique closed point in $\mathcal{N}_g$ (see Section 5 for a more detailed description). This construction defines an injective morphism, the spin-Torelli morphism. We are not aware of any result of rigidity regarding the moduli stacks $\mathcal{S}_g$ or the spin-Torelli morphism; the state of the art about the rigidity of $\mathcal{N}_g$ is analogous to the case of $\mathcal{A}_g$.

The proof of the following theorem is contained in Section 5.

Theorem 1.3. *For any $g \geq 3$ the spin-Torelli morphism $\mathrm{s}\tau : \mathcal{S}_g \to \mathcal{N}_g$ does not admit any nontrivial first-order deformation with fixed domain and target.*

We now focus on morphisms between moduli stacks of curves. By [Royden 1971], the only automorphism $\mathcal{M}_g \to \mathcal{M}_g$ is the identity. In [Massarenti 2014], building on [Gibney et al. 2002], it is shown that the automorphism group of $\overline{\mathcal{M}}_{g,n}$ is the symmetric group acting on the marked points, except for some low genera cases explicitly described in [loc. cit.]. These problems are also reviewed in [Farkas 2009, Question 4.6].

Our next and last result regards infinitesimal rigidity of certain morphisms from $\mathcal{M}_g$ to another moduli stack of curves $\mathcal{M}_h$ of some genus $h \geq g$ constructed as follows.

Let X_g (resp. X_h) be a closed orientable real surface of genus g (resp. h). An unramified finite covering $p : X_h \to X_g$ gives a map $p^* : \mathcal{T}_g \to \mathcal{T}_h$ between the corresponding Teichmüller spaces by pulling back the complex structures. The cover p is called *characteristic* if $p_*(\pi_1(X_h))$ is a characteristic subgroup of $\pi_1(X_g)$, i.e., $p_*(\pi_1(X_h))$ is left invariant by $\mathrm{Aut}(\pi_1(X_g))$. Topologically, these are coverings such that every homeomorphism of X_g lifts to a homeomorphism of X_h, and the lifting process defines a homomorphism $L_p : \mathrm{Aut}(\pi_1(X_g)) \to \mathrm{Aut}(\pi_1(X_h))$. Because of this, the map $p^* : \mathcal{T}_g \to \mathcal{T}_h$ defined by a characteristic cover descends to a morphism $p^* : \mathcal{M}_g \to \mathcal{M}_h$ (see [Biswas and Nag 1997, III.1 and III.2] for more details).

The study of the global rigidity of these morphisms is stated as an open question in [Farb 2024b] (see Question 4.5, attributed to C. McMullen). All these problems extend to the Deligne–Mumford compactification of $\mathcal{M}_g$ given the studies on the augmented Teichmüller space (see [Biswas and Nag 1997; Hu et al. 2021] for details).

Our contribution is to prove infinitesimal rigidity of all the morphisms $p^* : \mathcal{M}_g \to \mathcal{M}_h$ induced by characteristic covers $p : X_h \to X_g$.

Theorem 1.4. *For any $g \geq 3$, the morphism $p^* : \mathcal{M}_g \to \mathcal{M}_h$ does not admit any nontrivial first-order deformation with fixed domain and target.*

It is natural to ask whether the compositions $\mathcal{M}_g \to \mathcal{A}_h$ of the morphisms $p^* : \mathcal{M}_g \to \mathcal{M}_h$ with the Torelli morphisms $\tau : \mathcal{M}_h \to \mathcal{A}_h$ are also rigid. The question on global rigidity is raised in [Farb 2024b]. With our techniques, we cannot solve the corresponding question on infinitesimal rigidity for the moment. The main obstacle is discussed after the proof of Theorem 1.4.

Let us stress that our results are for morphisms between stacks. The situation for the induced morphism between coarse spaces is discussed in Section 7.

Let us briefly explain the structure of the proofs. We first study the space of sections of the pullback of the tangent bundle of the target via the investigated morphism. For both smooth Deligne–Mumford stacks and normal varieties, the vanishing of all these sections suffices to conclude infinitesimal rigidity (see Lemma 2.1). To prove this vanishing, we study those sections restricted to a complete curve B in $\mathcal{M}_g$ through a general point of $\mathcal{M}_g$ and with a general tangent direction (see Lemma 2.3 for details). We then conclude by relating these sections to the Hodge bundle associated to the family of genus g curves over B, and using the positivity properties provided by the associated Fujita decomposition [Fujita 1978; Catanese and Dettweiler 2017]. We explain this in detail in Section 2.2. Notice that such a complete curve B exists for $g \geq 3$ because $\mathcal{M}_g$ admits a compactification with boundary of codimension 2, the Satake compactification (see for instance [Oort 1974]).

In genus $g = 2$, the moduli space of curves is affine; hence nontrivial sheaves on $\mathcal{M}_2$ have plenty of sections. This indicates that the above results should not hold in this case.

We conclude the introduction with a couple of words about infinitesimal rigidity for moduli spaces of surfaces. On the one hand, moduli spaces of surfaces might deform [Hacking 2008, Section 6]. On the other hand, for higher-dimensional varieties, the Torelli map generalizes to period maps. If we consider a rigid surface S with strictly positive geometric genus or irregularity, then the domain of the period map is a point (the local deformation space of S) but the codomain has positive dimension (see, for instance [Carlson et al. 2017, Sections 4.4 and 4.5, Example 4.4.5]). Hence, in this case the period map admits nontrivial deformations. Examples of these surfaces are the BCD surfaces constructed by Bauer and Catanese [2008; 2018] or the surfaces with $p_g = q = 2$ constructed by Polizzi, Rito, and Roulleau [Polizzi et al. 2020]. We do not know under which hypotheses the period maps of higher-dimensional manifolds with positive-dimensional moduli are rigid.

2. Preliminaries

We work over the field of complex numbers.

2.1. ***First-order deformations of morphisms.*** In this paper, we are concerned about first order infinitesimal deformations of certain morphisms of stacks or normal varieties $f : X \to Y$ with fixed source and target. If Y is a smooth variety, it is known that these deformations are classified by the global sections of $f^*\mathcal{T}_Y$ [Sernesi 2006, Proposition 3.4.2, page 158]. This fact also holds in quite more general settings. Since we have not been able to find in the literature the statement in the generality we need, we include here a short proof (the same proof works in greater generality, but we give the statement only for our set-up).

Lemma 2.1. *Let $f : X \to Y$ be a morphism of either smooth Deligne–Mumford stacks or of normal varieties, and let $\mathcal{T}_Y$ denote the tangent sheaf of Y. If $H^0(X, f^*\mathcal{T}_Y) = 0$, then all first-order deformations of f with fixed source and target are trivial.*

Proof. Let D be the spectrum of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, and denote by $\{o\}$ its closed point. A first order deformation of $f : X \to Y$ is a D-morphism $\hat{f} : X \times D \to Y \times D$ which, when restricted to the central fiber $X_o := X \times \{o\} \subset X \times D$ is equal to f.

Note that the ideal sheaf of the central fiber X_o squares to zero, so that any D-morphism $X \times D \to Y \times D$ is uniquely determined by its restriction to X_o and the restriction of its differential to X_o. Moreover, in the case of a first-order deformation $\hat{f}$, its restriction to the central fiber is given by f; hence one only needs to study $d\hat{f}_{|X_o}$.

Since $\mathcal{T}_{X\times D} = \mathcal{T}_X \boxplus \mathcal{T}_D$ and analogously $\mathcal{T}_{Y\times D} = \mathcal{T}_Y \boxplus \mathcal{T}_D$, we have

$$\mathcal{T}_X \boxplus \mathcal{T}_D \cong \mathcal{T}_{X\times D} \xrightarrow{d\hat{f}} \hat{f}^*\mathcal{T}_{Y\times D} \cong f^*\mathcal{T}_Y \boxplus \mathcal{T}_D.$$

Hence we can describe $d\hat{f}_{|X_o}$ as

$$\left(\begin{array}{c|c} df & v \\ \hline 0 & 1 \end{array}\right),$$

where $v := (d\hat{f})\big(\frac{d}{d\varepsilon}\big)_{|X_o}$ and the 1 in the low-right corner follows from $\hat{f}$ being a D-morphism.

From the hypothesis $H^0(X, f^*\mathcal{T}_Y) = 0$ follows that $v = 0$. Thus $\hat{f}$ has the same differential as $f \times \mathrm{id}_D : X \times D \to Y \times D$, and by the above remark it follows $\hat{f} = f \times \mathrm{id}_D$ is the trivial deformation of f. □

2.2. ***Fujita decompositions on the Hodge bundle.*** Let $f : S \to B$ be a fibration from a smooth projective surface S to a smooth projective curve B, namely a family of projective curves of arithmetic genus g over a smooth projective curve B whose general fiber is smooth. Denote by q_f the relative irregularity of f, defined as the difference $q_f = q(S) - q(B)$ of the irregularities of S and B. To any such f one can associate a Hodge bundle $f_*\omega_{S/B} = f_*\omega_S \otimes \omega_B^\vee$ whose general fiber is of rank g isomorphic to $H^0(C_b, \omega_{C_b})$, where $C_b = f^{-1}(b)$.

Theorem 2.2 [Fujita 1978; Catanese and Dettweiler 2017]. *The Hodge bundle $f_*\omega_{S/B}$ has decompositions of vector bundles*

$$f_*\omega_{S/B} = \mathcal{O}_B^{q_f} \oplus \mathcal{V} = \mathcal{U} \oplus \mathcal{A}, \tag{1}$$

where $\mathcal{A}$ is ample and $\mathcal{U}$ is unitary flat, which are compatible in the sense that $\mathcal{O}_B^{q_f} \subset \mathcal{U}$ as vector bundle provides a splitting $\mathcal{U} = \mathcal{O}_B^{q_f} \oplus \mathcal{U}'$.

Fujita decompositions are strongly related to the infinitesimal variation of the Hodge structure, namely with the coboundary morphism $\theta_b : H^0(C_b, \omega_b) \to H^1(C_b, \mathcal{O}_{C_b})$ of the short exact sequence attached to the first order deformation $\xi_b \in \mathrm{Ext}^1(\omega_{C_b}, \mathcal{O}_{C_b}) \cong H^1(T_{C_b})$ induced by f on the fiber C_b. Suppose that f is semistable, namely that the relative canonical bundle is f-ample and the singular fibers are reduced with at most nodal singularities; then if $\Gamma \subset B$ denotes the set of critical values and $\Upsilon = f^*\Gamma$, there is a canonical isomorphism $f_*\omega_{S/B} \simeq f_*\Omega^1_{C/B}(\log \Upsilon)$, where the latter bundle is defined by the short exact sequence

$$0 \to f^*\omega_B(\log \Gamma) \to \Omega^1_C(\log \Upsilon) \to \Omega^1_{C/B}(\log \Upsilon) \to 0.$$

The connecting homomorphism

$$\theta : f_*\omega_{S/B} \simeq f_*\Omega^1_{C/B}(\log \Upsilon) \to R^1 f_*\mathcal{O}_C \otimes \omega_B(\log \Gamma) \tag{2}$$

is a morphism of locally free sheaves which on the fibers over $b \notin \Gamma$ coincides with θ_b. The kernel $\mathcal{K} = \ker \theta$ is a vector subbundle of $f_*\omega_{S/B}$ whose fiber over a general $b \in B$ is $\ker \theta_b$. There are natural inclusions $\mathcal{U} \subseteq \mathcal{K} \subseteq f_*\omega_{S/B}$.

We refer to [González-Alonso and Torelli 2021] for more details on the last paragraph, and a treatment of the non-semistable case, which requires more care and it is not used in this note.

Lemma 2.3. *Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus g. A general complete curve $\pi : B \to \overline{\mathcal{M}}_g$ corresponds to a semistable fibration with $\mathcal{U} = 0$ (more precisely, there exists an open dense subset U of the tangent bundle $\mathcal{T}\overline{\mathcal{M}}_g$ such that if the image of $d\pi$ intersects U, then $\mathcal{U} = 0$).*

Proof. Curves in $\overline{\mathcal{M}}_g$ correspond by construction to semistable fibrations. For a general smooth curve $[C_b]$ in $\mathcal{M}_g$ and a general direction $\xi_b \in T_{[C_b]}\overline{\mathcal{M}}_g \simeq H^1(C_b, T_{C_b}) \simeq \mathrm{Ext}^1(\omega_{C_b}, \mathcal{O}_{C_b})$, the induced linear map $\theta_b : H^0(C_b, \omega_{C_b}) \to H^1(C_b, \mathcal{O}_{C_b})$ has maximal rank (see for example [González-Alonso and Torelli 2021; Lee and Pirola 2016, Lemma 2.4]), so the fiber $\mathcal{K}_b$ is zero. As $\mathcal{U}$ is locally free and contained in $\mathcal{K}$, we obtain the statement. □

2.3. ***Ample vector bundles on curves.***

Lemma 2.4. *If $\mathcal{A}$ is an ample vector bundle over a smooth projective curve B, then $H^0(B, \mathrm{Sym}^n \mathcal{A}^\vee) = 0$ for every $n > 0$.*

Proof. Note first that if $\mathcal{A}$ is ample, then so is $\mathrm{Sym}^n \mathcal{A}$ [Lazarsfeld 2004, Theorem 6.1.15]. In particular any quotient line bundle $\mathrm{Sym}^n \mathcal{A} \twoheadrightarrow Q$ is ample on B [Lazarsfeld 2004, Proposition 6.1.2], i.e., $\deg Q > 0$.

Suppose $H^0(\mathrm{Sym}^n \mathcal{A}^\vee) \neq 0$ and let σ be a nonzero section, which induces a morphism of sheaves $\sigma : \mathcal{O}_B \to \mathrm{Sym}^n \mathcal{A}^\vee$. Dualizing it we obtain a nonzero map $\mathrm{Sym}^n \mathcal{A} \to \mathcal{O}_B$, whose image is a quotient of $\mathrm{Sym}^n \mathcal{A}$ and a nonzero subsheaf $Q \subseteq \mathcal{O}_B$. In particular Q is torsion-free, and hence a locally free sheaf because B is a smooth curve. Moreover $\deg Q \leq \deg \mathcal{O}_B = 0$, contradicting the amplitude of $\mathrm{Sym}^n \mathcal{A}$. □

3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1, asserting that the Torelli morphism $\tau : \mathcal{M}_g \to \mathcal{A}_g$ does not admit nontrivial first-order deformations.

Recall that $\mathcal{M}_g$ denotes the moduli stack of smooth projective curves of genus g, $\mathcal{A}_g$ the moduli stack of principally polarized abelian varieties of dimension g, and $\tau : \mathcal{M}_g \to \mathcal{A}_g$ the Torelli morphism, which at the level of points maps (the isomorphism class of) a smooth projective curve C to its Jacobian variety $J(C) \cong H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z})$ with its natural principal polarization Θ_C.

The tangent space to $\mathcal{M}_g$ at $[C]$ is $H^1(C, T_C)$, and the tangent space to $\mathcal{A}_g$ at $[J(C), \Theta_C]$ is

$$\mathrm{Sym}^2 H^1(C, \mathcal{O}_C) \cong \mathrm{Sym}^2 H^0(C, \omega_C)^\vee \cong \mathrm{Hom}^s(H^0(C, \omega_C), H^1(C, \mathcal{O}_C)),$$

where Hom^s denotes the set of symmetric (i.e., self-dual) linear maps.

Moreover the image of $\xi \in H^1(C, T_C) \cong T_{[C]}\mathcal{M}_g$ under the differential of τ can be identified (up to nonzero scalar) with the multiplication map (cup-product followed by contraction)

$$H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C), \quad \alpha \mapsto \xi \cdot \alpha.$$

By Lemma 2.1, Theorem 1.1 follows from the following vanishing:

Theorem 3.1. *If $g \geq 3$, then $H^0(\mathcal{M}_g, \tau^*\mathcal{T}_{\mathcal{A}_g}) = 0$.*

Proof. Since $\tau^*\mathcal{T}_{\mathcal{A}_g}$ is locally free and $\mathcal{M}_g$ is reduced, if we show that for every point in a dense subset of $\mathcal{M}_g$ there exists a curve $\pi : B \to \mathcal{M}_g$ such that $h^0(B, \pi^*\tau^*\mathcal{T}_{\mathcal{A}_g}) = 0$, then $h^0(\mathcal{M}_g, \tau^*\mathcal{T}_{\mathcal{A}_g}) = 0$.

When $g \geq 3$, the coarse moduli space of $\mathcal{M}_g$ admits a normal projective compactification whose boundary has codimension two, namely the Satake compactification. In $\mathcal{M}_g$, we can look at the open subset M_g^0 of curves with trivial automorphism group, which is represented by a smooth scheme and whose complement in the Satake compactification has also codimension two. Because of this, for every point $[C]$ of M_g^0 and every tangent direction v in $\mathcal{T}_{[C]}\mathcal{M}_g$, we can find a smooth projective curve $\pi : B \to \mathcal{M}_g$ passing through $[C]$ and tangent to v.

Consider such a curve B, the corresponding family of curves $f : \mathcal{C} \to B$ and the Hodge bundle $E = f_*\Omega^1_{\mathcal{C}/B}$, whose fiber over a point $[C] \in B$ is $H^0(C, \omega_C)$. By the above discussion, the restriction of $\pi^*\tau^*\mathcal{T}_{\mathcal{A}_g}$ to B is naturally isomorphic to $\mathrm{Sym}^2 E^\vee$. Now by Theorem 2.2, E carries a decomposition $E = \mathcal{A} \oplus \mathcal{U}$ with $\mathcal{A}$ ample and $\mathcal{U}$ unitary flat, and by Lemma 2.3, $\mathcal{U}$ is zero for general B and v. By Lemma 2.4, $h^0(B, \pi^*\tau^*\mathcal{T}_{\mathcal{A}_g}) = 0$. □

4. Proof of Theorem 1.2

As already recalled in the introduction, we denote by $\mathcal{R}_g$ the moduli stack of pairs (C, η), where C is a smooth projective curve of genus g, and $\eta \in J(C)$ is a nontrivial line bundle of order two (i.e., $\eta^{\otimes 2} \cong \mathcal{O}_C$). By standard theory, such a pair is equivalent to an étale double cover $\pi : C' \to C$, where C' is a connected smooth projective curve and

$$\pi_*\mathcal{O}_{C'} = \mathcal{O}_C \oplus \eta. \tag{3}$$

More precisely, since π is finite, there is a trace morphism $\mathrm{Tr} : \pi_*\mathcal{O}_{C'} \to \mathcal{O}_C$ that splits the structure morphism $\mathcal{O}_C \to \pi_*\mathcal{O}_{C'}$, and $\eta = \ker \mathrm{Tr}$.

One way to define the Prym variety $\mathrm{Pr}(C, \eta)$ of the pair (C, η) (or the cover $\pi : C' \to C$) is as the cokernel of the pull-back map

$$\pi^* : J(C) \to J(C'),$$

which has dimension $\dim \mathrm{Pr}(C, \eta) = g(C') - g(C) = g - 1$ (note that $g(C') = 2g(C) - 1$ by the Hurwitz formula).

Alternatively $\mathrm{Pr}(C, \eta)$ can be defined as the connected component through $[\mathcal{O}_{C'}]$ of the norm map $\mathrm{Nm} : J(C') \to J(C)$.

The natural principal polarization of $J(C')$ induces twice a principal polarization Ξ on $\mathrm{Pr}(C, \eta)$. The Prym morphism $\mathrm{Pr} : \mathcal{R}_g \to \mathcal{A}_{g-1}$ is then defined (at the level of $\mathbb{C}$-points) as

$$[C, \eta] \to [\mathrm{Pr}(C, \eta), \Xi].$$

Since $\mathcal{R}_g$ is an étale cover of $\mathcal{M}_g$ (of degree $2^{2g} - 1$), the tangent space of $\mathcal{R}_g$ at a point $[C, \eta]$ is naturally isomorphic to $T_{[C]}\mathcal{M}_g \cong H^1(C, T_C)$.

On the other hand, the tangent spaces of the Jacobians $J(C)$ and $J(C')$ (at the corresponding neutral elements) are $H^1(C, \mathcal{O}_C)$ and $H^1(C', \mathcal{O}_{C'})$. Thus the tangent space of $\mathrm{Pr}(C, \eta)$ is naturally isomorphic to

$$H^1(C', \mathcal{O}_{C'})/\pi^* H^1(C, \mathcal{O}_C) \cong H^1(C, \eta).$$

In the last isomorphism we have combined (3) and the fact that π is a finite morphism to obtain

$$H^1(C', \mathcal{O}_{C'}) \cong H^1(C, \pi_*\mathcal{O}_{C'}) \cong H^1(C, \mathcal{O}_C) \oplus H^1(C, \eta).$$

Therefore the tangent space of $\mathcal{A}_{g-1}$ at $\mathrm{Pr}(C, \eta)$ can be naturally identified with

$$\begin{aligned}\mathrm{Sym}^2 T_0 \mathrm{Pr}(C, \eta) &\cong \mathrm{Sym}^2 H^1(C, \eta) \cong \mathrm{Hom}^s(H^1(C, \eta)^\vee, H^1(C, \eta)) \\ &\cong \mathrm{Hom}^s(H^0(C, \omega_C \otimes \eta), H^1(C, \eta)),\end{aligned}$$

where the last isomorphism follows from Serre duality and $\eta^{\otimes 2} \cong \mathcal{O}_C$.

Finally the differential of the Prym morphism at $[C, \eta]$,

$$d\,\mathrm{Pr}_{[C,\eta]} : H^1(C, T_C) \to \mathrm{Sym}^2 H^1(C, \eta) \cong \mathrm{Hom}^s(H^0(C, \omega_C \otimes \eta), H^1(C, \eta))$$

is induced by cup-product (up to a nonzero scalar).

As in the above section, Theorem 1.2 follows from Lemma 2.1 and the following vanishing:

Theorem 4.1. *When $g \geq 3$, it holds that $H^0(\mathcal{R}_g, \mathrm{Pr}^* \mathcal{T}_{\mathcal{A}_{g-1}}) = 0$.*

Proof. By construction, for a given curve C there are $2^{2g} - 1$ choices of η, and indeed this gives a natural étale morphism $\varphi : \mathcal{R}_g \to \mathcal{M}_g$ of degree $2^{2g} - 1$. Set $R_g^0 = \varphi^{-1}(M_g^0)$ to be the local chart of $\mathcal{R}_g$ corresponding to coverings $C' \to C$, where C has trivial automorphism group. Moreover, since M_g^0 can be covered by smooth projective curves, so can R_g^0 by taking the connected components of the preimages under φ.

Let now $B \subseteq R_g^0$ be a general smooth curve, which corresponds to a family of coverings $f' : \mathcal{C}' \xrightarrow{\pi} \mathcal{C} \xrightarrow{f} B$. The induced morphism π is also a étale double cover of surfaces. The trace of π gives a splitting $\pi_* \mathcal{O}_{\mathcal{C}'} \cong \mathcal{O}_{\mathcal{C}} \oplus \mathcal{L}$, where $\mathcal{L} = \ker \mathrm{Tr}$ restricts to η on a fiber $C' \to C$. In particular we also have

$$R^1 f'_* \mathcal{O}_{\mathcal{C}'} \cong R^1 f_* \mathcal{O}_{\mathcal{C}} \oplus R^1 f_* \mathcal{L}, \tag{4}$$

and by the above discussion on tangent spaces, there is a natural identification

$$\mathrm{Pr}^* \mathcal{T}_{\mathcal{A}_{g-1}} \cong \mathrm{Sym}^2(R^1 f'_* \mathcal{O}_{\mathcal{C}'} / R^1 f_* \mathcal{O}_{\mathcal{C}}) \cong \mathrm{Sym}^2 R^1 f_* \mathcal{L}.$$

By relative duality, equation (4) gives

$$f'_* \Omega^1_{\mathcal{C}'/B} \cong f_* \Omega^1_{\mathcal{C}/B} \oplus (R^1 f_* \mathcal{L})^\vee$$

so that $(R^1 f_* \mathcal{L})^\vee$ is isomorphic to a quotient of the Hodge bundle $E' = f'_* \Omega^1_{\mathcal{C}'/B}$ of f'.

We can now adapt the proof of Lemma 2.3 to show that E' is ample for a general B, and then Lemma 2.4 concludes the proof of the theorem.

Let $C' \to C$ be one of the coverings of the family π, with corresponding $\eta \in \mathrm{Pic}^0(C)$, and let $\xi \in H^1(C, T_C) \cong T_{[C]} M_g^0 \cong T_{[C' \to C]} R_g^0$ be a tangent vector to B. Using the decompositions

$$H^0(C', \omega'_C) = H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta) \quad \text{and} \quad H^1(C', \mathcal{O}_{C'}) = H^1(C, \mathcal{O}_C) \oplus H^1(C, \eta)$$

the infinitesimal variation of Hodge structure $\mathcal{C}' \to B$ at this point is "diagonal", given by multiplication with η on each component. By [Lee and Pirola 2016, Lemma 2.4], a general $\xi \in H^1(C, T_C)$ gives isomorphisms in both components. Taking a smooth projective curve in M_g^0 through $[C]$ with tangent vector ξ, and then B the appropriate connected component of its preimage in R_0^g induces a family $f' : \mathcal{C}' \to B$ with ample Hodge bundle, as wanted. □

5. Proof of Theorem 1.3

Recall from the introduction that $\mathcal{S}_g$ is the moduli stack of pairs (C, ϑ) of projective curves of genus g with a theta characteristic ϑ, i.e., $\vartheta \in \mathrm{Pic}^{g-1}(C)$ such that $\vartheta^{\otimes 2} \cong \omega_C$. Since two theta characteristics differ by a 2-torsion element of the g-dimensional abelian variety $\mathrm{Pic}^0(C)$, the natural forgetful morphism $\pi : \mathcal{S}_g \to \mathcal{M}_g$ defined on points by $(C, \vartheta) \to C$ is étale of degree 2^{2g}.

On the other side, $\mathcal{N}_g$ denotes the moduli stack of pairs (A, Θ), where A is an abelian variety of dimension g and $\Theta \subseteq A$ is a symmetric divisor inducing a principal polarization on A, i.e., $-\Theta = \Theta$ and $h^0(\mathcal{O}_A(\Theta)) = 1$.

In order to describe the morphism $\mathrm{s}\tau : \mathcal{S}_g \to \mathcal{N}_g$, let's first quickly recall one construction of the principal polarization on the jacobian variety $J(C) = \mathrm{Pic}^0(C)$ of a projective curve C of genus g. There is a natural morphism

$$\varphi : C^{g-1} \to \mathrm{Pic}^{g-1}(C), \quad (p_1, \dots, p_{g-1}) \mapsto [\mathcal{O}_C(p_1 + \cdots + p_{g-1})].$$

By the Riemann parametrization theorem, φ is birational onto its image, which is thus a divisor. Moreover, its image is precisely the set $W^0_{g-1} = \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(L) > 0\}$. Any fixed $\vartheta \in \mathrm{Pic}^{g-1}(C)$ induces an isomorphism (of algebraic varieties)

$$\mathrm{Pic}^{g-1}(C) \to \mathrm{Pic}^0(C) = J(C), \quad [L] \mapsto [L] - [\vartheta] := [L \otimes \vartheta^\vee].$$

The image $\Theta_\vartheta := W^0_{g-1} - [\vartheta]$ of W^0_{g-1} is thus a divisor in $J(C)$ and induces the principal polarization used in the Torelli morphism τ. An easy application of Riemann–Roch shows that Θ_ϑ is symmetric if and only if $\vartheta^{\otimes 2} \cong \omega_C$.

The morphism $\mathrm{s}\tau$ is defined by mapping $[S, \vartheta]$ to the pair $(J(C), \Theta_\vartheta)$.

The above discussion also shows that the natural principal polarization on $J(C)$ can be represented by exactly 2^{2g} symmetric divisors; hence the forgetful morphism $\pi' : \mathcal{N}_g \to \mathcal{A}_g$ is also étale of degree 2^{2g}.

As in the two preceding cases, Theorem 1.3 follows from Lemma 2.1 and the following vanishing:

Theorem 5.1. *When $g \geq 3$, it holds that $H^0(\mathcal{S}_g, \mathrm{s}\tau^* \mathcal{T}_{\mathcal{N}_g}) = 0$.*

Proof. As in the previous two proofs, it is enough to show that a general point of $\mathcal{S}_g$ is contained in a projective curve $B \subseteq \mathcal{S}_g$ such that $H^0(B, \mathrm{s}\tau^* \mathcal{T}_{\mathcal{N}_g}) = 0$.

To this aim consider the natural commutative diagram with étale vertical arrows

$$\begin{array}{ccc} \mathcal{S}_g & \xrightarrow{\mathrm{s}\tau} & \mathcal{N}_g \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathcal{M}_g & \xrightarrow{\tau} & \mathcal{A}_g \end{array}$$

Note that $\mathcal{T}_{\mathcal{N}_g} = (\pi')^* \mathcal{T}_{\mathcal{A}_g}$ because π' is étale, and thus $\mathrm{s}\tau^* \mathcal{T}_{\mathcal{N}_g} = \pi^* \tau^* \mathcal{T}_{\mathcal{A}_g}$.

By the proof of Theorem 1.1, the general point of M^0_g is contained in a smooth projective curve $B' \subseteq M^0_g$ such that the Hodge bundle $E' = (f')_* \Omega^1_{\mathcal{C}'/B'}$ of the corresponding family of curves $f' : \mathcal{C}' \to B'$ is ample. Any connected component $B \subseteq \pi^{-1}(B')$ corresponds to a family of smooth projective curves $f : \mathcal{C} \to B$ (with a family of theta characteristics), which is nothing but the pull-back of f' by the étale morphism $\varphi = \pi_{|B} : B' \to B$. In particular, the Hodge bundle of f,

$$E := f_* \Omega^1_{\mathcal{C}/B} \cong \varphi^* E',$$

is also ample.

Thus a general point of $S_g^0 = \pi^{-1}(M_g^0)$ is contained in a smooth projective curve $B \subseteq S_g^0$ such that $(\pi^*\tau^*\mathcal{A}_g)_{|B} \cong \mathrm{Sym}^2 E^\vee$ with ample E. Lemma 2.4 implies that $H^0(B, \pi^*\tau^*\mathcal{A}_g) = 0$, as wanted. □

Remark 5.2 (super Torelli morphism). It is possible also to define a period map for the moduli space of Supersymmetric Riemann surfaces. Its target is again $\mathcal{N}_g$. As explained in [Codogni and Viviani 2019], this map is rational and factors through a nonreduced classical stack M ($= \mathfrak{M}_g^+/\Gamma$, in the notation of [loc. cit.]). The reduced stack underlying M is the irreducible component $\mathcal{S}_g^+$ of $\mathcal{S}_g$ where the spin structure has an even number of sections. The restriction of the period map to $\mathcal{S}_g^+$ is the spin-Torelli map studied also in this note. We do not know if this generalization of the spin-Torelli map is rigid.

6. Proof of Theorem 1.4

Let X be a closed orientable real surface of genus g. An unramified finite covering $p : X' \to X$ is called *characteristic* if it corresponds to a characteristic subgroup of the fundamental group $\pi_1(X)$, namely $\pi_1(X')$ as a subgroup of $\pi_1(X)$ must be left invariant by every element of $\mathrm{Aut}(\pi_1(X))$. Topologically, these are coverings such that every homeomorphism of X lifts to a homeomorphism of X' and the lifting process defines a homomorphism $L_p : \mathrm{Aut}(\pi_1(X)) \to \mathrm{Aut}(\pi_1(X'))$.

The moduli $\mathcal{M}_i$ of curves of genus i is realized as the quotient of the Teichmüller space $\mathcal{T}_i$ by the mapping class modular group DM_i. Any characteristic cover $p : X' \to X$ defines a map $\mathcal{T}_g \to \mathcal{T}_h$ (where $g = g(X)$ and $h = g(X')$). By using L_p, such a morphism descends to a morphism $p^* : \mathcal{M}_g \to \mathcal{M}_h$ (see [Biswas and Nag 1997, III.1 and III.2] for more details).

The statement of Theorem 1.4 follows now from Lemma 2.1 and the following theorem.

Theorem 6.1. *When $g \geq 3$, it holds that $H^0(\mathcal{M}_g, p^*\mathcal{T}_{\mathcal{M}_h}) = 0$.*

Proof. We use the same strategy as in the previous proofs. A smooth projective curve $B \subseteq \mathcal{M}_g$ corresponds to a nonisotrivial family $\pi : \mathcal{C} \to B$ of smooth projective curves of genus g, and the morphism $\tau_p : \mathcal{M}_g \to \mathcal{M}_h$ produces a nonisotrivial family $\pi' : \mathcal{C}' \to B$ of curves of genus h such that over $b \in B$ there is covering $C'_b \to C_b$ induced by p. The fiber of $p^*\mathcal{T}_{\mathcal{M}_h}$ over b is $H^1(C'_b, T_{C'_b})$ (the tangent space to $\mathcal{M}_h$ at $C'_b = \tau_p(C_b)$), and hence

$$p^*\mathcal{T}_{\mathcal{M}_h|B} = R^1\pi'_* T_{\mathcal{C}'/B} \cong (\pi'_*\omega_{\mathcal{C}'/B}^{\otimes 2})^\vee.$$

By [Esnault and Viehweg 1990, Theorem 3.1], the bundle $\pi'_*\omega_{\mathcal{C}'/B}^{\otimes 2}$ is ample on B; hence by Lemma 2.4 we have $H^0(B, p^*\mathcal{T}_{\mathcal{M}_h|B}) = 0$. The proof finishes as in the previous cases, by noticing that $\mathcal{M}_g$ can be covered by such smooth projective curves B and $p^*\mathcal{T}_{\mathcal{M}_h}$ is torsion-free, so that $H^0(B, p^*\mathcal{T}_{\mathcal{M}_h}) = 0$ for a general B implies $H^0(\mathcal{M}_g, p^*\mathcal{T}_{\mathcal{M}_h}) = 0$ as wanted. □

Composing the above studied map $p^* : \mathcal{M}_g \to \mathcal{M}_h$, with the Torelli map $\mathcal{M}_h \to \mathcal{A}_h$ we obtain a morphism $\mathcal{M}_g \to \mathcal{A}_h$. Our methods do not apply immediately to the study of its rigidity, one needs a more sophisticated understanding of the inclusion $p^*(\mathcal{M}_g) \subseteq \mathcal{M}_h$ and its tangent spaces. More precisely, to prove the relevant generalization of Lemma 2.3, given a étale covering $\pi : C_h \to C_g$, it would be necessary to understand if a general first order deformation of C_h *compatible with* π also satisfies the properties of [Lee and Pirola 2016, Lemma 2.4].

Given a possibly covering $\pi : C_h \to C_g$ as above, one could also consider the generalized Prym variety $\mathrm{Pic}^0(C_h)/\pi^* \mathrm{Pic}^0(C_g)$ (which inherits a polarization of a certain type δ depending on the topological type of π) and thus construct a generalized Prym morphism $\mathcal{M}_g \to \mathcal{A}^{\delta}_{h-g}$ from the moduli stack of curves to that of $(h-g)$-dimensional abelian varieties with polarization of type δ. The study of its rigidity presents the same difficulties of that of $\mathcal{M}_g \to \mathcal{A}_h$ introduced above.

7. Remarks about rigidity of coarse morphisms

Given an (infinitesimally) rigid morphism of stacks $F : \mathcal{X} \to \mathcal{Y}$, one can ask if the corresponding morphism of coarse spaces $f : X \to Y$ is also (infinitesimally) rigid. The answer in this generality is negative, as the following example shows.

Example 7.1. Let G be the group $\mathbb{Z}_2$, $\mathcal{X} = BG$ be the quotient stack of a point by G, and $\mathcal{Y}$ the quotient stack of the affine line by the action of G which maps x to $-x$. There is a unique morphism $F : \mathcal{X} \to \mathcal{Y}$, which maps BG to the fixed point of the action and is infinitesimally rigid. However, the corresponding map of coarse spaces does deform. (To make contact with the forthcoming Proposition 7.2, note that in this case $F^{-1}(U)$ is the empty set.)

It is natural to wonder if the coarse version of the modular morphisms considered in this paper are rigid. Concerning infinitesimal rigidity, taking into account the criterion given in Lemma 2.1, we can ask under which conditions $H^0(\mathcal{X}, F^*T_{\mathcal{Y}}) = 0$ implies $H^0(X, f^*T_Y) = 0$.

The following definition will be useful. Let V be a sheaf on a variety X, and $T(V)$ the torsion subsheaf of V. The inclusion $T(V) \hookrightarrow V$ induces an inclusion $i : H^0(X, T(V)) \hookrightarrow H^0(X, V)$. We say that a global section of V is a torsion section if it is in the image of i. We have the following partial result.

Proposition 7.2. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a morphism of smooth Deligne–Mumford stacks such that $H^0(\mathcal{X}, F^*T_{\mathcal{Y}}) = 0$. Let $p : \mathcal{X} \to X$ and $q : \mathcal{Y} \to Y$ be the maps to coarse spaces and $f : X \to Y$ the morphism induced by F. Let $U \subseteq \mathcal{Y}$ be the open subset where q is étale, and assume that $F^{-1}(U) \subseteq \mathcal{X}$ is a big open subset* (*i.e., its complement has codimension at least* 2). *Then all sections $H^0(X, f^*T_Y)$ are torsion sections.*

Proof. Over a field of characteristic zero, coarse moduli spaces of DM stacks are good; hence the adjunction morphism $f^*\mathcal{T}_Y \to p_* p^* f^* \mathcal{T}_Y$ is an isomorphism, and thus

$$H^0(X, f^*\mathcal{T}_Y) = H^0(X, p_* p^* f^* \mathcal{T}_Y) = H^0(\mathcal{X}, p^* f^* \mathcal{T}_Y).$$

We have $p^* f^* \mathcal{T}_Y = F^* q^* \mathcal{T}_Y$. On U, we have $q^* \mathcal{T}_Y = \mathcal{T}_{\mathcal{Y}}$. Assume by contradiction that there exists an element s of $H^0(\mathcal{X}, p^* f^* \mathcal{T}_Y)$ which is not torsion. Since it is not torsion, its restriction to $F^{-1}(U)$ is not zero; hence $H^0(F^{-1}(U), p^* f^* \mathcal{T}_Y|_{F^{-1}(U)}) \neq 0$. Then $H^0(F^{-1}(U), F^* \mathcal{T}_{\mathcal{Y}}|_{F^{-1}(U)}) \neq 0$.

As $F^{-1}(U)$ is big in $\mathcal{X}$ and $F^* \mathcal{T}_{\mathcal{Y}}$ is locally free, we obtain that $H^0(\mathcal{X}, F^* \mathcal{T}_{\mathcal{Y}}) \neq 0$, contradicting the hypothesis. □

Let us check whether the hypotheses of Proposition 7.2 are satisfied in our cases. On the one hand, the map from the moduli stack of ppav (and its variants discussed in this paper) to its coarse moduli space is étale over the closed points with automorphism group exactly $\{\pm 1\}$. The preimage of this open set via the various Torelli, spin and Prym maps is the open set of curves without automorphisms, which is big in the moduli space of smooth curves and its variants when $g \geq 4$; see, e.g., [Hacking 2008, Lemma 5.3].

On the other hand, characteristic covers are Galois, so the image of p^* lies in the locus of genus h curves with nontrivial automorphisms. This means that Proposition 7.2 cannot be applied to the morphisms induced by characteristic covers.

Note that torsion sections of $f^*\mathcal{T}_Y$ might exist even in simple cases, as the following example shows, so we cannot exclude that the Torelli morphism has infinitesimal deformations.

Example 7.3. Let $Y \subseteq \mathbb{C}^3$ be a quadratic cone, which has a normal singularity at the vertex. Take as X a line through the vertex, and f the inclusion. Then $f^*\mathcal{T}_Y$ is a rank two sheaf on $X \cong \mathbb{C}$ with torsion at the origin (a direct computation reveals that the torsion subsheaf is a skyscraper sheaf with two-dimensional fiber).

Unfortunately, we do not know of any systematic study of infinitesimal deformations coming from torsion sections. Let us pose the following general question in deformation theory, whose study goes beyond the scope of this paper.

Question 7.4. Let $f : X \to Y$ be a morphism of normal varieties. If all global sections of $f^*\mathcal{T}_Y$ are torsion, is f locally rigid?

A variant of the phenomenon encountered here is studied in [Arbarello and Cornalba 1981, Section 6]; in the spirit of [loc. cit.], we might speculate that the answer to Question 7.4 is positive.

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Communicated by Gavril Farkas
Received 2023-05-16 Revised 2024-07-11 Accepted 2024-09-13

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ALGEBRA AND NUMBER THEORY 19:9 (2025)
https://doi.org/10.2140/ant.2025.19.1685

Prismatic G-displays and descent theory

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For a smooth affine group scheme G over the ring of p-adic integers $\mathbb{Z}_p$ and a cocharacter μ of G, we study G-μ-displays over the prismatic site of Bhatt and Scholze. In particular, we obtain several descent results for them. If $G = \mathrm{GL}_n$, then our G-μ-displays can be thought of as Breuil–Kisin modules with some additional conditions. The relation between our G-μ-displays and prismatic F-gauges introduced by Drinfeld and Bhatt–Lurie is also discussed.

In fact, our main results are formulated and proved for smooth affine group schemes over the ring of integers $\mathcal{O}_E$ of any finite extension E of $\mathbb{Q}_p$ by using $\mathcal{O}_E$-prisms, which are $\mathcal{O}_E$-analogues of prisms.

1. Introduction

Bhatt and Scholze [2022] introduced the theory of prisms. The category of (bounded) prisms with the flat topology is called the absolute prismatic site. It has been observed that *prismatic F-crystals* on the absolute prismatic site introduced in [Bhatt and Scholze 2023] play significant roles in various aspects of arithmetic geometry. For a smooth affine group scheme G over the ring of p-adic integers $\mathbb{Z}_p$, we provide a systematic study of prismatic F-crystals with certain G-actions, which we call *prismatic G-μ-displays*. The results obtained here will be used to study the deformation theory of prismatic G-μ-displays in [Ito 2025]. We also discuss the relation between prismatic G-μ-displays and *prismatic F-gauges* introduced in [Drinfeld 2024; Bhatt and Lurie 2022a; 2022b; Bhatt 2022].

MSC2020: primary 14F30; secondary 14G45, 14L05.
Keywords: prisms, displays, p-divisible groups, prismatic F-gauges.

1.1. ***Prismatic Dieudonné crystals.*** Anschütz and Le Bras [2023] introduced *prismatic Dieudonné crystals*, which are prismatic F-crystals with additional conditions, and showed that prismatic Dieudonné crystals can be used to classify p-divisible groups in mixed characteristic. The notion of prismatic G-μ-displays can be seen as a generalization of that of prismatic Dieudonné crystals. Before discussing prismatic G-μ-displays, let us state our main result for prismatic Dieudonné crystals.

Let k be a perfect field of characteristic $p > 0$ and let $W(k)$ be the ring of p-typical Witt vectors of k. Let R be a complete regular local ring over $W(k)$ with residue field k. There exists a pair

$$(A, I) = (W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E})),$$

with an isomorphism $R \simeq A/I$ over $W(k)$, where $\mathcal{E} \in W(k)[\![t_1, \dots, t_n]\!]$ is a formal power series whose constant term is p. Here A admits a Frobenius endomorphism $\phi : A \to A$ such that it acts on $W(k)$ as the usual Frobenius and sends t_i to t_i^p for each i. The pair (A, I) is a typical example of a prism. Let $(R)_{\Delta}$ be the absolute prismatic site of R (where R is equipped with the p-adic topology). We regard (A, I) as an object of $(R)_{\Delta}$. We will prove (and generalize) the following result.

Theorem 1.1.1 (Proposition 7.1.1). *The category of prismatic Dieudonné crystals on $(R)_{\Delta}$ is equivalent to the category of minuscule Breuil–Kisin modules over (A, I).*

Anschütz and Le Bras [2023, Theorem 5.12] proved Theorem 1.1.1 when the dimension of R is ≤ 1 (or equivalently $n \leq 1$) and stated that their result should be generalized to R of arbitrary dimension.

Remark 1.1.2. A *minuscule Breuil–Kisin module* over (A, I) is a free A-module M of finite rank equipped with an A-linear homomorphism

$$F_M : \phi^* M := A \otimes_{\phi, A} M \to M$$

whose cokernel is killed by I. For a prismatic Dieudonné crystal $\mathcal{M}$ on $(R)_{\Delta}$, the value $\mathcal{M}(A, I)$ at $(A, I) \in (R)_{\Delta}$ is by definition a minuscule Breuil–Kisin module over (A, I), and the construction $\mathcal{M} \mapsto \mathcal{M}(A, I)$ induces an equivalence between the two categories in Theorem 1.1.1.

Anschütz and Le Bras [2023, Theorem 4.74] showed that the category of prismatic Dieudonné crystals on $(R)_{\Delta}$ is equivalent to the category of p-divisible groups over R. In fact, such an equivalence is obtained not only for R but also for any quasisyntomic ring (in the sense of [Bhatt et al. 2019, Definition 4.10]), where we need to replace prismatic Dieudonné crystals by *admissible* prismatic Dieudonné crystals [Anschütz and Le Bras 2023, Definition 4.5].

Although (admissible) prismatic Dieudonné crystals are theoretically important, it is difficult to describe them explicitly in general. Theorem 1.1.1 provides a practical way to study prismatic Dieudonné crystals on $(R)_{\Delta}$. For example, this shows that giving a prismatic Dieudonné crystal on $(R)_{\Delta}$ is equivalent to giving a minuscule Breuil–Kisin module over (A, I). The latter can be carried out in a much simpler way than the former.

1.2. ***Prismatic G-μ-displays.*** Let G be a smooth affine group scheme over $\mathbb{Z}_p$ and $\mu : \mathbb{G}_m \to G_{W(k)}$ a cocharacter defined over $W(k)$, where $G_{W(k)} := G \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} W(k)$. We will generalize Theorem 1.1.1 to prismatic G-μ-displays, or equivalently G-Breuil–Kisin modules of type μ, as explained below.

Let (A, I) be a bounded prism in the sense of [Bhatt and Scholze 2022] such that A is a $W(k)$-algebra. A G-Breuil–Kisin module over (A, I) is a G-torsor $\mathcal{P}$ over Spec A with an isomorphism

$$F_{\mathcal{P}} : (\phi^*\mathcal{P})[1/I] \xrightarrow{\sim} \mathcal{P}[1/I]$$

of G-torsors over Spec $A[1/I]$, where $\phi^*\mathcal{P}$ is the base change of $\mathcal{P}$ along the Frobenius $\phi : A \to A$. We say that $\mathcal{P}$ is *of type* μ if, (p, I)-completely étale locally on A, there exists some trivialization $\mathcal{P} \simeq G_A$ under which the isomorphism $F_{\mathcal{P}}$ is given by $g \mapsto Xg$ for an element X in the double coset

$$G(A)\mu(d)G(A) \subset G(A[1/I]),$$

where $d \in I$ is a generator. The notion of G-Breuil–Kisin modules of type μ is important in the study of integral models of (local) Shimura varieties; see Section 1.3.

We will study G-Breuil–Kisin modules of type μ via the theory of higher frames and G-μ-displays developed in [Lau 2021]. More precisely, we introduce and study the groupoid

$$G\text{-}\operatorname{Disp}_\mu(A, I)$$

of *G-μ-displays over* (A, I). It turns out that $G\text{-}\operatorname{Disp}_\mu(A, I)$ is equivalent to the groupoid of G-Breuil–Kisin modules of type μ over (A, I) (Proposition 5.3.8). For a p-adically complete ring R, the groupoid of *prismatic G-μ-displays over R* is defined to be

$$G\text{-}\operatorname{Disp}_\mu((R)_{\Delta}) := 2 - \varprojlim_{(A,I)\in(R)_{\Delta}} G\text{-}\operatorname{Disp}_\mu(A, I).$$

The main result of this paper is as follows. Let R be a complete regular local ring over $W(k)$ with residue field k. As in Section 1.1, there exists a prism $(W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E}))$ with an isomorphism $R \simeq W(k)[\![t_1, \dots, t_n]\!]/\mathcal{E}$ over $W(k)$.

Theorem 1.2.1 (Theorem 6.1.3). *Assume that the cocharacter μ is 1-bounded. Then the following natural functor is an equivalence*:

$$G\text{-}\operatorname{Disp}_\mu((R)_{\Delta}) \xrightarrow{\sim} G\text{-}\operatorname{Disp}_\mu(W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E})).$$

See Definition 4.2.3 for the definition of 1-bounded cocharacters. If G is reductive, then μ is 1-bounded if and only if μ is minuscule. A minuscule Breuil–Kisin module of rank N over $(W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E}))$ can be regarded as a GL_N-Breuil–Kisin module of type μ over $(W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E}))$ for a minuscule cocharacter μ. Theorem 1.1.1 is a special case of Theorem 1.2.1; see Section 7 for details.

We make a few comments on the proof of Theorem 1.2.1. To simplify the notation, we set $(A, I) := (W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E}))$. As in the proof of [Anschütz and Le Bras 2023, Theorem 5.12], the key part of the proof is to show that every G-μ-display $\mathcal{Q}$ over (A, I) admits a unique descent datum. More precisely, let $(A^{(2)}, I^{(2)})$ be the coproduct of two copies of (A, I) in $(R)_{\Delta}$ and let $p_1, p_2 : (A, I) \to (A^{(2)}, I^{(2)})$ be the associated morphisms. Then we will prove that there exists a unique isomorphism

$$\epsilon : p_1^* \mathcal{Q} \xrightarrow{\sim} p_2^* \mathcal{Q}$$

of G-μ-displays over $(A^{(2)}, I^{(2)})$ satisfying the usual cocycle condition over the coproduct $(A^{(3)}, I^{(3)})$ of three copies of (A, I). In the case where $G = \mathrm{GL}_N$, the proof of this claim goes along the same lines as that of [Anschütz and Le Bras 2023], but it requires some additional arguments when $n \geq 2$. For general G, we will use some techniques from the proof of [Lau 2021, Proposition 7.1.5].

We also give some basic definitions and results on prismatic G-μ-displays. In particular, we establish several descent results for prismatic G-μ-displays, such as flat descent (Proposition 5.2.8) and p-complete arc-descent (Corollary 5.6.10). We also introduce the *Hodge filtration* and the *underlying G-ϕ-module* of a prismatic G-μ-display. These notions will be needed in the Grothendieck–Messing deformation theory for prismatic G-μ-displays studied in [Ito 2025].

Remark 1.2.2. In fact, Theorem 1.2.1 will be formulated and proved for a smooth affine group scheme G over the ring of integers $\mathcal{O}_E$ of any finite extension E of $\mathbb{Q}_p$. For this, we will use $\mathcal{O}_E$-analogues of prisms, called *$\mathcal{O}_E$-prisms*. This notion was already introduced in [Marks 2023] (in which these objects are called E-typical prisms). Section 2 is devoted to discussing results analogous to those of [Bhatt and Scholze 2022, Sections 2 and 3] for $\mathcal{O}_E$-prisms. We will define G-μ-displays for bounded $\mathcal{O}_E$-prisms in the same way, and prove the above results for them. As explained in Remark 1.3.3 below, it will be convenient to establish our results in this generality, but the reader (who is familiar with the theory of prisms) may assume that $\mathcal{O}_E = \mathbb{Z}_p$ and skip Section 2 on a first reading. The arguments for general $\mathcal{O}_E$ are the same as for the case where $\mathcal{O}_E = \mathbb{Z}_p$.

Remark 1.2.3. G-Breuil–Kisin modules of type μ may be more familiar to readers than prismatic G-μ-displays. However, in order to prove Theorems 1.1.1 and 1.2.1, and other descent results (e.g., Corollaries 5.3.9 and 5.6.10), it is essential to work with prismatic G-μ-displays.

Remark 1.2.4. We briefly discuss how our results are related to the theory of G-objects in crystalline $\mathbb{Z}_p$-local systems. Here we follow the terminology of [Imai et al. 2024]. Let $R = \mathcal{O}_K$ be the ring of integers of a finite totally ramified extension K of $W(k)[1/p]$. Bhatt and Scholze [2023] proved that the category of prismatic F-crystals on $(\mathcal{O}_K)_{\Delta}$ is equivalent to the category $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(K)$ of free $\mathbb{Z}_p$-modules T of finite rank with a continuous $\mathrm{Gal}(\overline{K}/K)$-action such that $T[1/p]$ is crystalline. (Here $\mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K.) Using this result, together with [Imai et al. 2024], one can prove that there is an equivalence of groupoids

$$G\text{-}\mathrm{Disp}_{\mu}((\mathcal{O}_K)_{\Delta}) \xrightarrow{\sim} G\text{-}\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,\mu}(K) \tag{1-1}$$

if G is reductive, where $G\text{-}\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,\mu}(K)$ is the groupoid of G-objects in $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(K)$ having cocharacter μ in the sense of [Imai et al. 2024]. More specifically, this follows from [loc. cit., Theorem 2, Proposition 3.13] and [Ito 2025, Proposition 5.1.16]. Let $\mathcal{E} \in W(k)[[t]]$ be the Eisenstein polynomial of a uniformizer

$\varpi \in \mathcal{O}_K$. Then (1-1) and Theorem 1.2.1 give an equivalence

$$G\text{-}\mathrm{Disp}_\mu(W(k)[\![t]\!], (\mathcal{E})) \xrightarrow{\sim} G\text{-}\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,\mu}(K).$$

A similar result was previously obtained in [Levin 2015, Corollary 4.3.8] by a completely different method. (The result of [Bhatt and Scholze 2023] was generalized to higher-dimensional smooth p-adic formal schemes over $\mathcal{O}_K$; see [Du et al. 2024; Guo and Reinecke 2024]. However, since a higher-dimensional complete regular local ring R as in Theorem 1.2.1 is in general not topologically of finite type over $W(k)$ with respect to the *p-adic* topology, we cannot directly apply those results to obtain an analogue of (1-1) for R. We do not pursue this issue here.)

We will also discuss (in the case where $\mathcal{O}_E = \mathbb{Z}_p$) the relation between prismatic G-μ-displays and prismatic F-gauges in vector bundles introduced in [Drinfeld 2024; Bhatt and Lurie 2022a; 2022b; Bhatt 2022]. In particular, for a quasisyntomic ring S over $W(k)$, we introduce[1] the groupoid

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(S)$$

of *prismatic G-F-gauges of type μ over S* and construct a fully faithful functor

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(S) \to G\text{-}\mathrm{Disp}_\mu((S)_{\Delta}).$$

See Section 8 for details. This functor can be thought of as a generalization of the fully faithful functor from the category of admissible prismatic Dieudonné crystals on $(S)_{\Delta}$ to the category of prismatic Dieudonné crystals on $(S)_{\Delta}$ (see Example 8.1.15). If S is a complete regular local ring over $W(k)$ with residue field k, then the above functor is an equivalence (Corollary 8.2.12). Thus, we can rephrase Theorem 1.2.1 as follows:

Corollary 1.2.5 (Theorem 6.1.3, Corollary 8.2.12). *Let the notation be as in Theorem 1.2.1. Assume that μ is 1-bounded. Then we have a natural equivalence*

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(W(k)[\![t_1, \dots, t_n]\!], (\mathcal{E})).$$

1.3. *Motivation.* The primary motivation behind this work is to understand some classification results on p-divisible groups and the local structure of integral local Shimura varieties defined in [Scholze and Weinstein 2020], by using the theory of prisms. In the following, we explain this in more detail with a brief review of previous studies.

We first explain the motivation for $G = \mathrm{GL}_N$. Let $\mathcal{O}_K$ be the ring of integers of a finite totally ramified extension K of $W(k)[1/p]$. Let $\mathcal{E} \in W(k)[\![t]\!]$ be the Eisenstein polynomial of a uniformizer $\varpi \in \mathcal{O}_K$.

Remark 1.3.1. Anschütz and Le Bras [2023, Theorem 5.12] obtained the equivalence of categories

$$\left\{\begin{matrix} p\text{-divisible groups} \\ \text{over } \mathcal{O}_K \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \text{minuscule Breuil–Kisin modules} \\ \text{over } (W(k)[\![t]\!], (\mathcal{E})) \end{matrix}\right\} \tag{1-2}$$

[1] After this work was completed, Gardner and Madapusi [2024] announced that they defined (certain objects which are essentially equivalent to) prismatic G-F-gauges of type μ for more general p-adically complete rings, using the stacky approach of Drinfeld and Bhatt–Lurie. See also [Imai et al. 2023] for the relation between our prismatic G-F-gauges of type μ and those defined in [Gardner and Madapusi 2024].

by combining the classification theorem [Anschütz and Le Bras 2023, Theorem 4.74] with Theorem 1.1.1 for $R = \mathcal{O}_K$. This result was conjectured in [Breuil 1998], proved in [Kisin 2006; 2009] when $p \geq 3$, and proved in [Kim 2012; Liu 2013; Lau 2014] for all $p > 0$.

We consider the pair $(W(k)[[t]]/t^n, (\mathcal{E}))$, which is naturally a bounded prism for every $n \geq 1$. Lau [2014] obtained the equivalence of categories

$$\left\{ \begin{matrix} p\text{-divisible groups} \\ \text{over } \mathcal{O}_K/\varpi^n \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{minuscule Breuil–Kisin modules} \\ \text{over } (W(k)[[t]]/t^n, (\mathcal{E})) \end{matrix} \right\} \tag{1-3}$$

by a deformation-theoretic argument and then proved (1-2) by taking the limit over n; see [Lau 2014, Corollary 5.4, Theorem 6.6]. His proof uses the theory of *displays*, which was initiated by Zink and developed by many authors, including Zink [2001; 2002] and Lau [2008; 2014]. This classification result over $\mathcal{O}_K/\varpi^n$ is important in its own right. For example, this is a key ingredient in the construction of integral canonical models of Shimura varieties of abelian type with hyperspecial level structure in characteristic $p = 2$; see [Kim and Madapusi Pera 2016] for details.

Contrary to Lau's approach, it is not clear whether the results in [Anschütz and Le Bras 2023] imply (1-3) since the Grothendieck–Messing deformation theory does not apply directly to prismatic Dieudonné crystals. This point is discussed in [Ito 2025], where we develop the deformation theory for prismatic Dieudonné crystals, or more generally for prismatic G-μ-displays when μ is 1-bounded. In [loc. cit.], we construct universal deformations of prismatic G-μ-displays over k as certain prismatic G-μ-displays over complete regular local rings of higher dimension, where Theorem 1.2.1 plays a crucial role. As a result, we can give an alternative proof of the equivalences (1-2) and (1-3).

Remark 1.3.2. In the proof of [Anschütz and Le Bras 2023, Theorem 4.74] (and hence in the proof of the equivalence (1-2) of that work), they use [Scholze and Weinstein 2013, Theorem B], which says that for an algebraically closed complete extension C of $\mathbb{Q}_p$, the category of p-divisible groups over $\mathcal{O}_C$ is equivalent to the category of free $\mathbb{Z}_p$-modules T of finite rank together with a C-subspace of $T \otimes_{\mathbb{Z}_p} C$. In [Ito 2025], we also give an alternative proof of this result.

We now explain our motivation for general G. The notion of G-μ-displays ("displays with G-μ-structures") was first introduced in [Bültel 2008; Bültel and Pappas 2020] to study Rapoport–Zink spaces and integral models of Shimura varieties (where G is reductive and μ is minuscule). The theory of G-μ-displays has been developed in various settings; see for example [Langer and Zink 2019; Pappas 2023; Lau 2021; Daniels 2021; Bartling 2022]. In fact, the notion of G-μ-displays over *perfect* prisms was already used in [Bartling 2022]. Bartling used G-μ-displays over perfect prisms to prove the local representability and the formal smoothness of integral local Shimura varieties with hyperspecial level structure, under a certain nilpotence assumption (introduced in [Bültel and Pappas 2020, Definition 3.4.2]). In [Ito 2025], we prove the same assertion without any nilpotence assumptions, by using the universal deformations of prismatic G-μ-displays over k.

Remark 1.3.3. In [Ito 2025], we establish the above results not only when G is defined over $\mathbb{Z}_p$ but also when G is defined over $\mathcal{O}_E$, where E is any finite extension of $\mathbb{Q}_p$. For this, it will be convenient to work with $\mathcal{O}_E$-prisms.

The theory of G-μ-displays also has applications to K3 surfaces and related varieties; see [Langer and Zink 2019; Lau 2021; Inoue 2024]. In a future work, we plan to employ prismatic G-μ-displays to investigate the deformation theory for these varieties.

1.4. ***Outline of this paper.*** This paper is organized as follows. In Section 2, we collect some basic definitions and facts about $\mathcal{O}_E$-prisms. In Section 3, we discuss the notion of displayed Breuil–Kisin modules (of type μ), which will serve as examples of prismatic G-μ-displays. In Section 4, we introduce and study the *display group* $G_\mu(A, I)$, which is used in the definition of prismatic G-μ-displays. The structural results about $G_\mu(A, I)$ obtained there play crucial roles in the study of prismatic G-μ-displays.

Sections 5 and 6 are the main parts of this paper. In Section 5, we introduce prismatic G-μ-displays, give some basic definitions (e.g., Hodge filtrations and underlying G-ϕ-modules), and establish several descent results. In Section 6, we prove our main result (Theorem 1.2.1).

In Section 7, we make a few remarks on prismatic Dieudonné crystals, and prove Theorem 1.1.1. Finally, in Section 8, we provide a comparison between prismatic G-μ-displays and prismatic F-gauges. In particular, we introduce the notion of prismatic G-F-gauges of type μ for quasisyntomic rings over $W(k)$.

Notation. In this paper, all rings are commutative and unital. For a module M over a ring R and a ring homomorphism $f : R \to R'$, the tensor product $M \otimes_R R'$ is denoted by $M_{R'}$ or f^*M. For a scheme X over R, the base change $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is denoted by $X_{R'}$ or f^*X. We use similar notation for the base change of group schemes, p-divisible groups, etc. Moreover, all actions of groups will be right actions, unless otherwise stated.

2. Preliminaries on $\mathcal{O}_E$-prisms

Throughout this paper, we fix a prime number p. Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}_q$. Here $\mathbb{F}_q$ is a finite field with q elements. We fix a uniformizer $\pi \in \mathcal{O}_E$.

In this section, we study an "$\mathcal{O}_E$-analogue" of the notion of prisms. Such objects are called $\mathcal{O}_E$-prisms in this paper. This notion was also introduced in [Marks 2023] (in which $\mathcal{O}_E$-prisms are called *E-typical prisms*). We discuss some properties of $\mathcal{O}_E$-prisms which we need in the sequel. We hope that this section will also help readers unfamiliar with [Bhatt and Scholze 2022] to understand some basic facts about prisms.

2.1. ***Prisms.*** We first recall the definition of bounded prisms.

Let A be a $\mathbb{Z}_{(p)}$-algebra. A *δ-structure* on A is a map $\delta : A \to A$ of sets with the following properties:

(1) $\delta(1) = 0$.

(2) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.

(3) $\delta(x + y) = \delta(x) + \delta(y) + (x^p + y^p - (x + y)^p)/p$.

A *δ-ring* is a pair (A, δ) of a $\mathbb{Z}_{(p)}$-algebra A and a δ-structure $\delta : A \to A$. The above equalities imply that

$$\phi : A \to A, \quad x \mapsto x^p + p\delta(x),$$

is a ring homomorphism which is a lift of the Frobenius $A/p \to A/p$, $x \mapsto x^p$.

In the following, for a ring A and an ideal $I \subset A$, we say that an A-module M is I-adically complete (or x-adically complete if I is generated by an element $x \in I$) if the natural homomorphism

$$M \to \widehat{M} := \varprojlim_n M/I^n M$$

is bijective.

Definition 2.1.1 [Bhatt and Scholze 2022]. A *bounded prism* is a pair (A, I) of a δ-ring A and a Cartier divisor $I \subset A$ with the following properties:

(1) A is (p, I)-adically complete.

(2) A/I has bounded p-torsion, i.e., $(A/I)[p^\infty] = (A/I)[p^n]$ for some integer $n > 0$.

(3) We have $p \in (I, \phi(I))$.

We say that a bounded prism (A, I) is *orientable* if I is principal.

Remark 2.1.2. Under the condition that A/I has bounded p^∞-torsion, the requirement that A is (p, I)-adically complete is equivalent to saying that A is *derived* (p, I)-adically complete; see [Bhatt and Scholze 2022, Lemma 3.7]. We refer to [loc. cit., Section 1.2] and [Stacks 2005–, Tag 091N] for the notion of derived complete modules (or complexes). For a ring A and a finitely generated ideal $I \subset A$, if an A-module M is I-adically complete, then M is derived I-adically complete; see [Stacks 2005–, Tag 091T] or [Positselski 2023, Lemma 2.3].

2.2. *δ_E-rings.* In this subsection, we recall the notion of δ_E-rings, which is an "$\mathcal{O}_E$-analogue" of the notion of a δ-ring. We define

$$\delta_{\mathcal{O}_E,\pi} : \mathcal{O}_E \to \mathcal{O}_E, \quad x \mapsto (x - x^q)/\pi.$$

Definition 2.2.1 [Marks 2023, Definition 2.2]. (1) Let A be an $\mathcal{O}_E$-algebra. A *δ_E-structure* on A is a map $\delta_E : A \to A$ of sets with the following properties:

(a) $\delta_E(xy) = x^q\delta_E(y) + y^q\delta_E(x) + \pi\delta_E(x)\delta_E(y)$.

(b) $\delta_E(x + y) = \delta_E(x) + \delta_E(y) + (x^q + y^q - (x + y)^q)/\pi$.

(c) $\delta_E : A \to A$ is compatible with $\delta_{\mathcal{O}_E,\pi}$, i.e., we have $\delta_E(x) = \delta_{\mathcal{O}_E,\pi}(x)$ for any $x \in \mathcal{O}_E$.

A *δ_E-ring* is a pair (A, δ_E) of an $\mathcal{O}_E$-algebra A and a δ_E-structure $\delta_E : A \to A$.

(2) A homomorphism $f : (A, \delta_E) \to (A', \delta'_E)$ of δ_E-rings is a homomorphism $f : A \to A'$ of $\mathcal{O}_E$-algebras such that $f \circ \delta_E = \delta'_E \circ f$.

The term $(x^q + y^q - (x+y)^q)/\pi$ in (b) makes sense since we can write it as

$$(x^q + y^q - (x+y)^q)/\pi = -\sum_{0<i<q} \left(\binom{q}{i}/\pi\right) x^i y^{q-i}.$$

We usually denote a δ_E-ring (A, δ_E) simply by A.

Remark 2.2.2. The notion of δ_E-rings also appeared in [Borger 2011, Remark 1.19; Li 2022] for example. In the end of the latter work, Li suggests using δ_E-structures for the study of prismatic sites of higher level over ramified bases.

Remark 2.2.3. The notion of δ_E-rings is essentially independent of the choice of π. More precisely, let $\pi' \in \mathcal{O}_E$ be another uniformizer. We write $\pi = u\pi'$ for a unique unit $u \in \mathcal{O}_E^\times$. If an $\mathcal{O}_E$-algebra A is equipped with a δ_E-structure $\delta_E : A \to A$ with respect to π, then it also admits a δ_E-structure with respect to π', defined by $x \mapsto u\delta_E(x)$.

For a δ_E-ring A, we define

$$\phi_A : A \to A, \quad x \mapsto x^q + \pi\delta_E(x).$$

We see that ϕ_A is a homomorphism of $\mathcal{O}_E$-algebras and is a lift of the q-th power Frobenius $A/\pi \to A/\pi$, $x \mapsto x^q$. The homomorphism ϕ_A is called the *Frobenius* of the δ_E-ring A. When there is no ambiguity, we omit the subscript and simply write $\phi = \phi_A$.

Remark 2.2.4. If A is a π-torsion-free $\mathcal{O}_E$-algebra, then the construction $\delta_E \mapsto \phi$ gives a bijection between the set of δ_E-structures on A and the set of homomorphisms $\phi : A \to A$ over $\mathcal{O}_E$ that are lifts of $A/\pi \to A/\pi$, $x \mapsto x^q$.

Example 2.2.5 (free δ_E-rings). We define an endomorphism ϕ of the polynomial ring $\mathcal{O}_E[X_0, X_1, X_2, \dots]$ by $X_i \mapsto X_i^q + \pi X_{i+1}$ ($i \geq 0$). By Remark 2.2.4, we get the corresponding δ_E-structure on the ring $\mathcal{O}_E[X_0, X_1, X_2, \dots]$, which sends X_i to X_{i+1}. We write

$$\mathcal{O}_E\{X\}$$

for the resulting δ_E-ring. As in the proof of [Bhatt and Scholze 2022, Lemma 2.11], one can check that $\mathcal{O}_E\{X\}$ has the following property: For a δ_E-ring A and an element $x \in A$, there exists a unique homomorphism $f : \mathcal{O}_E\{X\} \to A$ of δ_E-rings with $f(X_0) = x$. In other words, the δ_E-ring $\mathcal{O}_E\{X\}$ is a free object with basis $X := X_0$ in the category of δ_E-rings.

The same argument as in the proof of [loc. cit., Lemma 2.11] also shows that the Frobenius $\phi : \mathcal{O}_E\{X\} \to \mathcal{O}_E\{X\}$ is faithfully flat; this fact will be used in Section 2.6.

Lemma 2.2.6. *For a δ_E-ring A, the Frobenius $\phi : A \to A$ is a homomorphism of δ_E-rings.*

Proof. Let $x \in A$ be an element. We have to show that $\phi(\delta_E(x)) = \delta_E(\phi(x))$. Since there exists a (unique) homomorphism $f : \mathcal{O}_E\{X\} \to A$ of δ_E-rings with $f(X) = x$, it suffices to prove the assertion for $A = \mathcal{O}_E\{X\}$, which is clear since A is π-torsion-free and $\phi : A \to A$ is ϕ-equivariant. □

Following [loc. cit., Remark 2.4], we shall give a characterization of δ_E-rings in terms of *ramified* Witt vectors. For an $\mathcal{O}_E$-algebra A, let

$$W_{\mathcal{O}_E,\pi,2}(A)$$

denote the *ring of π-typical Witt vectors of length* 2: the underlying set of $W_{\mathcal{O}_E,\pi,2}(A)$ is $A \times A$, and for $(x_0, x_1), (y_0, y_1) \in W_{\mathcal{O}_E,\pi,2}(A)$, we have

$$(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1 + (x_0^q + y_0^q - (x_0 + y_0)^q)/\pi),$$
$$(x_0, x_1) \cdot (y_0, y_1) = (x_0 y_0, x_0^q y_1 + y_0^q x_1 + \pi x_1 y_1).$$

If $\mathcal{O}_E = \mathbb{Z}_p$ and $\pi = p$, then $W_{\mathcal{O}_E,\pi,2}(A)$ is the ring $W_2(A)$ of p-typical Witt vectors of length 2. For a detailed treatment of the rings of π-typical Witt vectors (of any length), we refer to [Schneider 2017, Section 1.1; Borger 2011].

Remark 2.2.7 (cf. [Bhatt and Scholze 2022, Remark 2.4]). The map

$$\mathcal{O}_E \to W_{\mathcal{O}_E,\pi,2}(A), \quad x \mapsto (x, \delta_{\mathcal{O}_E,\pi}(x)),$$

is a ring homomorphism for any $\mathcal{O}_E$-algebra A. We regard $W_{\mathcal{O}_E,\pi,2}(A)$ as an $\mathcal{O}_E$-algebra by this homomorphism. Let

$$\epsilon : W_{\mathcal{O}_E,\pi,2}(A) \to A, \quad (x_0, x_1) \mapsto x_0,$$

denote the projection map, which is a homomorphism of $\mathcal{O}_E$-algebras. For a δ_E-structure $\delta_E : A \to A$, the map $s : A \mapsto W_{\mathcal{O}_E,\pi,2}(A)$ defined by $x \mapsto (x, \delta_E(x))$ is a homomorphism of $\mathcal{O}_E$-algebras such that $\epsilon \circ s = \mathrm{id}_A$. By this procedure, we obtain a bijection between the set of δ_E-structures on A and the set of homomorphisms $s : A \to W_{\mathcal{O}_E,\pi,2}(A)$ of $\mathcal{O}_E$-algebras satisfying $\epsilon \circ s = \mathrm{id}_A$.

Remark 2.2.8 (cf. [Bhatt and Scholze 2022, Remark 2.7]). It follows from Remark 2.2.7 that the category of δ_E-rings admits all limits and colimits, and they are preserved by the forgetful functor from the category of δ_E-rings to the category of $\mathcal{O}_E$-algebras.

The following two lemmas will be used frequently in the sequel.

Lemma 2.2.9. *Let $A = (A, \delta_E)$ be a δ_E-ring and $I \subset A$ an ideal. Then I is stable under δ_E if and only if A/I admits a δ_E-structure that is compatible with the one on A. If such a δ_E-structure on A/I exists, then it is unique.*

Proof. This follows immediately from the definition of δ_E-structures (see the proof of [Bhatt and Scholze 2022, Lemma 2.9]). □

Lemma 2.2.10. *Let A be a δ_E-ring and let $I \subset A$ be a finitely generated ideal containing π. Then, for any integer $n \geq 1$, there exists an integer $m \geq 1$ such that, for any $x \in A$, we have $\delta_E(x + I^m) \subset \delta_E(x) + I^n$. In particular, the I-adic completion of A admits a unique δ_E-structure that is compatible with the one on A.*

Proof. The proof is the same as that of [Bhatt and Scholze 2022, Lemma 2.17]. □

We shall discuss some properties of perfect δ_E-rings, which are defined as follows.

Definition 2.2.11. We say that a δ_E-ring A is *perfect* if the Frobenius $\phi : A \to A$ is bijective.

Lemma 2.2.12 [Marks 2023, Lemma 2.11]. *A perfect δ_E-ring A is π-torsion-free.*

Proof. This is proved in [Marks 2023, Lemma 2.11], and follows from the same argument as in the proof of [Bhatt and Scholze 2022, Lemma 2.28]. □

Example 2.2.13. Let R be an $\mathbb{F}_q$-algebra. Assume that R is perfect (i.e., $R \to R$, $x \mapsto x^p$, is bijective). Let $W(R)$ be the ring of p-typical Witt vectors and we define

$$W_{\mathcal{O}_E}(R) := W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E.$$

Let $\phi : W_{\mathcal{O}_E}(R) \to W_{\mathcal{O}_E}(R)$ denote the base change of the q-th power Frobenius of $W(R)$. This is a lift of the q-th power Frobenius of $W_{\mathcal{O}_E}(R)/\pi = R$. Since $W_{\mathcal{O}_E}(R)$ is π-torsion-free, we obtain the corresponding δ_E-structure on $W_{\mathcal{O}_E}(R)$. It is clear that $W_{\mathcal{O}_E}(R)$ is a perfect δ_E-ring.

Lemma 2.2.14. *The functor $R \mapsto W_{\mathcal{O}_E}(R)$ from the category of perfect $\mathbb{F}_q$-algebras to the category of π-adically complete $\mathcal{O}_E$-algebras admits a right adjoint given by $A \mapsto \varprojlim_{x \mapsto x^p} A/\pi A$.*

Proof. This is well known in the case where $\mathcal{O}_E = \mathbb{Z}_p$ (see [Szamuely and Zábrádi 2018, Proposition 3.12] for example). The general case follows from this special case. □

Corollary 2.2.15 [Marks 2023, Proposition 2.13]. *The following categories are equivalent*:

- *The category $\mathcal{C}_1$ of π-adically complete perfect δ_E-rings (A, δ_E).*
- *The category $\mathcal{C}_2$ of π-adically complete and π-torsion-free $\mathcal{O}_E$-algebras A such that $A/\pi A$ is perfect.*
- *The category $\mathcal{C}_3$ of perfect $\mathbb{F}_q$-algebras R.*

More precisely, the functors $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3 \to \mathcal{C}_1$, defined by $(A, \delta_E) \mapsto A$, $A \mapsto A/\pi$, $R \mapsto W_{\mathcal{O}_E}(R)$, respectively, are equivalences.

Proof. Using Lemma 2.2.14, one can prove the assertion in exactly the same way as [Bhatt and Scholze 2022, Corollary 2.31]. □

Corollary 2.2.16. *Let A be a perfect δ_E-ring and B a π-adically complete δ_E-ring. Then any homomorphism $A \to B$ of $\mathcal{O}_E$-algebras is a homomorphism of δ_E-rings.*

Proof. We may assume that A is π-adically complete. It then follows from Lemma 2.2.14 and Corollary 2.2.15 that $A \to B$ is ϕ-equivariant. We recall that $\phi : B \to B$ is a homomorphism of δ_E-rings by Lemma 2.2.6. Let B^{perf} be a limit of the diagram

$$B \xleftarrow{\phi} B \xleftarrow{\phi} B \longleftarrow \cdots$$

in the category of δ_E-rings, which is a perfect δ_E-ring. Since A is perfect, $A \to B$ factors through a ϕ-equivariant homomorphism $A \to B^{\mathrm{perf}}$ of $\mathcal{O}_E$-algebras. It follows from the π-torsion-freeness of B^{perf} (see Lemma 2.2.12) that $A \to B^{\mathrm{perf}}$ is a homomorphism of δ_E-rings. Since $A \to B$ is the composition $A \to B^{\mathrm{perf}} \to B$, the assertion follows. □

2.3. ***$\mathcal{O}_E$-prisms.*** We now introduce $\mathcal{O}_E$-prisms.

Definition 2.3.1 [Marks 2023, Definition 3.1]. (1) An *$\mathcal{O}_E$-prism* is a pair (A, I) of a δ_E-ring A and a Cartier divisor $I \subset A$ such that A is derived (π, I)-adically complete and $\pi \in I + \phi(I)A$.

(2) We say that an $\mathcal{O}_E$-prism (A, I) is *bounded* if A/I has bounded p^∞-torsion.

(3) We say that an $\mathcal{O}_E$-prism (A, I) is *orientable* if I is principal.

(4) An *oriented and bounded* $\mathcal{O}_E$-prism (A, d) is an orientable and bounded $\mathcal{O}_E$-prism (A, I) with a choice of a generator $d \in I$.

(5) A map $f : (A, I) \to (A', I')$ of $\mathcal{O}_E$-prisms is a homomorphism $f : A \to A'$ of δ_E-rings such that $f(I) \subset I'$.

If $\mathcal{O}_E = \mathbb{Z}_p$, then bounded $\mathcal{O}_E$-prisms are nothing but bounded prisms.

Remark 2.3.2. Let (A, I) be a bounded $\mathcal{O}_E$-prism. By [Tian 2023, Proposition 2.5(1)] (see also Lemma 2.5.1 below), we see that A is (π, I)-adically complete. Moreover, since A/I is derived π-adically complete and has bounded p^∞-torsion, it follows that A/I is π-adically complete (see [Bhatt et al. 2019, Lemma 4.7] for example).

Let A be a δ_E-ring. Following [Bhatt and Scholze 2022, Definition 2.19], we say that an element $d \in A$ is *distinguished* if $\delta_E(d) \in A^\times$, i.e., $\delta_E(d)$ is a unit. Since $\delta_{\mathcal{O}_E,\pi}(\pi) = 1 - \pi^{q-1} \in \mathcal{O}_E^\times$, we see that $\pi \in A$ is distinguished.

Lemma 2.3.3. *Let A be a δ_E-ring and $d \in A$ an element. Assume that π is contained in the Jacobson radical* $\mathrm{rad}(A)$ *of A.*

(1) *Assume that $d = fh$ for some elements $f, h \in A$ with $f \in \mathrm{rad}(A)$. If d is distinguished, then f is distinguished and $h \in A^\times$.*

(2) *Assume that $d \in \mathrm{rad}(A)$. Then d is distinguished if and only if $\pi \in (d, \phi(d))$.*

Proof. This can be proved exactly in the same way as [Bhatt and Scholze 2022, Lemmas 2.24 and 2.25]. See also [Marks 2023, Lemma 2.9]. □

The following rigidity property plays a fundamental role in the theory of $\mathcal{O}_E$-prisms.

Lemma 2.3.4 (cf. [Bhatt and Scholze 2022, Lemma 3.5]). *Let $(A, I) \to (A', I')$ be a map of $\mathcal{O}_E$-prisms. Then the natural homomorphism $I \otimes_A A' \to IA'$ is an isomorphism and $IA' = I'$.*

Proof. By using [Marks 2023, Lemma 3.4], this follows from the same argument as in the proof of [Bhatt and Scholze 2022, Lemma 3.5]. We recall the argument in the case where both (A, I) and (A', I') are orientable. It follows from Lemma 2.3.3(2) that any generator $d \in I$ is distinguished. Let $d' \in I'$ be a generator. Then Lemma 2.3.3(1) implies that d is mapped to ud' for some $u \in A'^\times$. In particular, the image of d in A' is a nonzerodivisor, and we obtain $I \otimes_A A' \xrightarrow{\sim} IA'$ and $IA' = I'$. □

The following lemma will be used several times in this paper.

Lemma 2.3.5 (cf. the proof of [Bhatt and Scholze 2022, Lemma 4.8]). *Let A be a perfect δ_E-ring and (B, I) a bounded $\mathcal{O}_E$-prism. Then any homomorphism $A \to B/I$ of $\mathcal{O}_E$-algebras lifts uniquely to a homomorphism $A \to B$ of δ_E-rings.*

Proof. By Corollary 2.2.16, it is enough to check that the homomorphism $A \to B/I$ lifts uniquely to a homomorphism $A \to B$ of $\mathcal{O}_E$-algebras. We may assume that A is π-adically complete, and then $A \simeq W_{\mathcal{O}_E}(R)$ for some perfect $\mathbb{F}_q$-algebra R by Corollary 2.2.15. Since B is (π, I)-adically complete and B/I is π-adically complete, it suffices to prove that, for every integer $n \geq 1$, any homomorphism $W_n(R) \to B/(\pi^n, I)$ of $W_n(\mathbb{F}_q)$-algebras lifts uniquely to a homomorphism $W_n(R) \to B/(\pi^n, I^n)$ of $W_n(\mathbb{F}_q)$-algebras (here $W_n(R) = W(R)/p^n$). This follows from the fact that the cotangent complex $L_{W_n(R)/W_n(\mathbb{F}_q)}$ is acyclic [Szamuely and Zábrádi 2018, Lemma 3.27(1)]. □

We give some examples of $\mathcal{O}_E$-prisms.

Example 2.3.6 (cf. [Bhatt and Scholze 2022, Example 1.3(1)]). Let A be a π-adically complete and π-torsion-free $\mathcal{O}_E$-algebra. Let $\phi : A \to A$ be a homomorphism over $\mathcal{O}_E$ which is a lift of the q-th power Frobenius of A/π. This homomorphism induces a δ_E-structure on A, and the pair $(A, (\pi))$ is a bounded $\mathcal{O}_E$-prism.

Definition 2.3.7 ($\mathcal{O}_E$-prism over $\mathcal{O}$). Let k be a perfect field containing $\mathbb{F}_q$. We will write

$$\mathcal{O} := W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$$

instead of $W_{\mathcal{O}_E}(k)$. An $\mathcal{O}_E$-prism *over* $\mathcal{O}$ is an $\mathcal{O}_E$-prism (A, I) with a homomorphism $\mathcal{O} \to A$ of δ_E-rings.

Let $\mathcal{O} = W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ be as in Definition 2.3.7. Let

$$\mathfrak{S}_{\mathcal{O}} := \mathcal{O}[[t_1, \dots, t_n]]$$

(where $n \geq 0$) and let $\phi : \mathfrak{S}_{\mathcal{O}} \to \mathfrak{S}_{\mathcal{O}}$ be the homomorphism such that $\phi(t_i) = t_i^q$ $(1 \leq i \leq n)$ and its restriction to $\mathcal{O}$ agrees with the Frobenius of $\mathcal{O}$. Since $\mathfrak{S}_{\mathcal{O}}$ is π-torsion-free, ϕ gives rise to a δ_E-structure on $\mathfrak{S}_{\mathcal{O}}$.

Proposition 2.3.8 (cf. [Bhatt and Scholze 2022, Example 1.3(3)]). *Let $\mathcal{E} \in \mathfrak{S}_{\mathcal{O}}$ be a formal power series whose constant term is a uniformizer of $\mathcal{O}$. Then the pair $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ is a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.*

Proof. We shall show that $\pi \in (\mathcal{E}, \phi(\mathcal{E}))$; the other required conditions are clearly satisfied. It is enough to check that $\delta_E(\mathcal{E}) \in \mathfrak{S}_{\mathcal{O}}^\times$. For this, it suffices to show that the image of $\delta_E(\mathcal{E})$ in $\mathfrak{S}_{\mathcal{O}}/(t_1, \dots, t_n) = \mathcal{O}$ is a unit, which is clear since the constant term of $\mathcal{E}$ is a uniformizer of $\mathcal{O}$. □

We call $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ an $\mathcal{O}_E$-prism *of Breuil–Kisin type* in this paper. Here n could be any nonnegative integer. Such a pair is also considered in [Cheng 2018].

2.4. *Perfectoid rings and $\mathcal{O}_E$-prisms.* The notion of (integral) perfectoid rings in the sense of [Bhatt et al. 2018, Definition 3.5] plays a central role in the theory of prismatic G-μ-displays. We refer to [loc. cit., Section 3] and [Česnavičius and Scholze 2024, Section 2] for basic properties of perfectoid rings. We recall the definition of perfectoid rings and some notation from [Bhatt et al. 2018, Section 3].

A ring R is a *perfectoid ring* if there exists an element $\varpi \in R$ such that $p \in (\varpi)^p$ and R is ϖ-adically complete, the Frobenius $R/p \to R/p$, $x \mapsto x^p$ is surjective, and the kernel of $\theta : W(R^\flat) \to R$ is principal. Here

$$R^\flat := \varprojlim_{x \mapsto x^p} R/p$$

and $\theta : W(R^\flat) \to R$ is the unique homomorphism whose reduction modulo p is the projection $R^\flat \to R/p$, $(x_0, x_1, \dots) \mapsto x_0$. The homomorphism θ is the counit of the adjunction given in Lemma 2.2.14 (in the case where $\mathcal{O}_E = \mathbb{Z}_p$). By [Bhatt et al. 2018, Lemma 3.9], there is an element $\varpi^\flat \in R^\flat$ such that $\theta([\varpi^\flat])$ is a unit multiple of ϖ, where $[-]$ denotes the Teichmüller lift.

Example 2.4.1. (1) An $\mathbb{F}_p$-algebra R is a perfectoid ring if and only if it is perfect; see [Bhatt et al. 2018, Example 3.15].

(2) Let V be a p-adically complete valuation ring with algebraically closed fraction field. Then V is a perfectoid ring. This follows from [loc. cit., Lemma 3.10].

Let $\mathcal{O}$ be as in Definition 2.3.7. If R is a perfectoid ring over $\mathcal{O}$ (i.e., R is a perfectoid ring with a ring homomorphism $\mathcal{O} \to R$), then $R^\flat$ is naturally a k-algebra, and $W_{\mathcal{O}_E}(R^\flat) = W(R^\flat) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ is an $\mathcal{O}$-algebra. Let

$$\theta_{\mathcal{O}_E} : W_{\mathcal{O}_E}(R^\flat) \to R$$

be the homomorphism induced from θ.

Lemma 2.4.2 (cf. [Fargues and Scholze 2021, Proposition II.1.4]). *The kernel* $\operatorname{Ker}\theta_{\mathcal{O}_E}$ *of* $\theta_{\mathcal{O}_E}$ *is a principal ideal. Moreover, any generator of* $\operatorname{Ker}\theta_{\mathcal{O}_E}$ *is a nonzerodivisor in* $W_{\mathcal{O}_E}(R^\flat)$.

Proof. Let $\varpi \in R$ be an element such that R is ϖ-adically complete and $p \in (\varpi)^p$. By [Bhatt et al. 2018, Lemma 3.10(i)], after replacing ϖ by $\theta([(\varpi^\flat)^{1/p^n}])$ for some integer $n > 0$, we have $\pi \in (\varpi)$. Then we can write $\pi = \theta([\varpi^\flat]x)$ for some element $x \in W(R^\flat)$ since θ is surjective. We shall show that $\pi - [\varpi^\flat]x$ generates $\operatorname{Ker}\theta_{\mathcal{O}_E}$.

Let $\mathcal{E}(T) \in W(k)[T]$ be the (monic) Eisenstein polynomial of $\pi \in \mathcal{O}$ so that we have $W(k)[T]/\mathcal{E}(T) \xrightarrow{\sim} \mathcal{O}$, $T \mapsto \pi$. We see that

$$\begin{aligned} W_{\mathcal{O}_E}(R^\flat)/(\pi - [\varpi^\flat]x) &\simeq W(R^\flat)[T]/(\mathcal{E}(T), T - [\varpi^\flat]x) \\ &\simeq W(R^\flat)/\mathcal{E}([\varpi^\flat]x). \end{aligned}$$

It thus suffices to show that $\mathcal{E}([\varpi^\flat]x)$ is a generator of the kernel $\operatorname{Ker}\theta$ of θ. It is clear that $\mathcal{E}([\varpi^\flat]x) \in \operatorname{Ker}\theta$. Since $\mathcal{E}([\varpi^\flat]x)$ is a unit multiple of an element of the form $p + [\varpi^\flat]y$, the proof of [Bhatt et al. 2018, Lemma 3.10] shows that $\mathcal{E}([\varpi^\flat]x) \in \operatorname{Ker}\theta$ is a generator.

It remains to prove that any generator $\xi \in \operatorname{Ker}\theta_{\mathcal{O}_E}$ is a nonzerodivisor. We recall that $\operatorname{Ker}\theta$ is generated by a nonzerodivisor $\xi' \in W(R^\flat)$. Since $W(\mathbb{F}_q) \to \mathcal{O}_E$ is flat, the element ξ' is a nonzerodivisor in $W_{\mathcal{O}_E}(R^\flat)$. This implies that ξ is a nonzerodivisor since we have $\xi' \in (\xi) = \operatorname{Ker}\theta_{\mathcal{O}_E}$. □

Proposition 2.4.3 (cf. [Bhatt and Scholze 2022, Example 1.3(2)]). *Let R be a perfectoid ring over $\mathcal{O}$ and we write $I_R := \operatorname{Ker}\theta_{\mathcal{O}_E}$. Then the pair*

$$(W_{\mathcal{O}_E}(R^\flat), I_R)$$

is an orientable and bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Proof. By the proof of Lemma 2.4.2, we know that I_R is generated by a nonzerodivisor of the form $\xi = \pi - [(\varpi')^\flat]b$, where $\varpi' \in R$ is such that R is ϖ'-adically complete and $p \in (\varpi')^p$. In order to show that $W_{\mathcal{O}_E}(R^\flat)$ is (π, ξ)-adically complete, it suffices to show that $W(R^\flat)$ is $(p, [(\varpi')^\flat])$-adically complete, which is easy to check (see also the proof of [Česnavičius and Scholze 2024, Proposition 2.1.11(b)]). Moreover $W_{\mathcal{O}_E}(R^\flat)/\xi = R$ has bounded p^∞-torsion by [Bhatt and Scholze 2022, Lemma 3.8].

It remains to show that $\pi \in (\xi, \phi(\xi))$. It suffices to prove that $\delta_E(\xi) \in W_{\mathcal{O}_E}(R^\flat)^\times$. The image of $\delta_E(\xi)$ in $W_{\mathcal{O}_E}(R^\flat)/[(\varpi')^\flat]$ is equal to $1 - \pi^{q-1}$ (we note that $W_{\mathcal{O}_E}(R^\flat)/[(\varpi')^\flat]$ is π-torsion-free) and hence is a unit, which in turn implies that $\delta_E(\xi) \in W_{\mathcal{O}_E}(R^\flat)^\times$. □

A homomorphism $R \to S$ of perfectoid rings over $\mathcal{O}$ induces a map $(W_{\mathcal{O}_E}(R^\flat), I_R) \to (W_{\mathcal{O}_E}(S^\flat), I_S)$ of $\mathcal{O}_E$-prisms over $\mathcal{O}$.

Remark 2.4.4. We say that an $\mathcal{O}_E$-prism (A, I) is *perfect* if the δ_E-ring A is perfect. By [Marks 2023, Lemma 3.10], a perfect $\mathcal{O}_E$-prism (A, I) is bounded and orientable. Moreover, in [loc. cit., Theorem 3.18], it is proved that A/I is a perfectoid ring. These facts, together with Lemma 2.3.5 and Proposition 2.4.3, imply that the functor $(A, I) \mapsto A/I$ from the category of perfect $\mathcal{O}_E$-prisms to that of perfectoid rings over $\mathcal{O}_E$ is an equivalence, whose inverse is given by $R \mapsto (W_{\mathcal{O}_E}(R^\flat), I_R)$. This is an analogue of [Bhatt and Scholze 2022, Theorem 3.10].

2.5. *Prismatic sites.* For a ring A, let $D(A)$ denote the derived category of A-modules. Let $I \subset A$ be a finitely generated ideal. We say that a complex $M \in D(A)$ is *I-completely flat* (resp. *I-completely faithfully flat*) if $M \otimes^{\mathbb{L}}_A A/I$ is concentrated in degree 0 and it is a flat (resp. faithfully flat) A/I-module. One can easily check that this definition is equivalent to the one introduced in [Bhatt and Scholze 2022, Section 1.2].

Lemma 2.5.1. *Let (A, I) be a bounded $\mathcal{O}_E$-prism.*

(1) *For a complex $M \in D(A)$, the derived (π, I)-adic completion of M is isomorphic to*

$$R\varprojlim_n (M \otimes^{\mathbb{L}}_A A/(\pi, I)^n).$$

In particular, if M is (π, I)-completely flat, then the derived (π, I)-adic completion of M is concentrated in degree 0.

(2) *Let M be an A-module. Assume that M is (π, I)-completely flat and derived (π, I)-adically complete. Then M is (π, I)-adically complete. Moreover the natural homomorphism $M \otimes_A I \to M$ is injective and $M/I^n M$ has bounded p^∞-torsion for any n.*

Proof. (1) The assertion follows from [Tian 2023, Proposition 2.5(1)] or the proof of [Bhatt and Scholze 2022, Lemma 3.7(1)]. This can also be deduced from the results discussed in [Yekutieli 2021]; by

[Yekutieli 2021, Corollary 3.5, Theorem 3.11], it suffices to prove that the ideal $(\pi, I) \subset A$ is weakly proregular in the sense of [loc. cit., Definition 3.2], which follows from the same argument as in the proof of [loc. cit., Theorem 7.3].

(2) It follows from (1) that M is (π, I)-adically complete. The second statement can be proved in the same way as [Bhatt and Scholze 2022, Lemma 3.7(2)]. (In [loc. cit.], we should assume that M is derived (p, I)-adically complete.) □

We say that a map $f : (A, I) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms is a *(faithfully) flat map* if $A \to A'$ is (π, I)-completely (faithfully) flat. If f is a faithfully flat map, then we say that (A', I') is a flat covering of (A, I).

Remark 2.5.2. For a map $f : (A, I) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms, we have $A' \otimes^{\mathbb{L}}_A A/I \simeq A'/I'$ by Lemma 2.3.4, which in turn implies that

$$A' \otimes^{\mathbb{L}}_A A/(\pi, I) \simeq A'/I' \otimes^{\mathbb{L}}_{A/I} A/(\pi, I). \tag{2-1}$$

In particular f is a (faithfully) flat map if and only if $A/I \to A'/I'$ is π-completely (faithfully) flat.

Definition 2.5.3. Let R be a π-adically complete $\mathcal{O}_E$-algebra. Let

$$(R)_{\Delta, \mathcal{O}_E}$$

denote the category of bounded $\mathcal{O}_E$-prisms (A, I) together with a homomorphism $R \to A/I$ of $\mathcal{O}_E$-algebras. The morphisms $f : (A, I) \to (A', I')$ in $(R)_{\Delta, \mathcal{O}_E}$ are the maps of $\mathcal{O}_E$-prisms such that $A/I \to A'/I'$ is a homomorphism of R-algebras. We endow the opposite category $(R)^{\mathrm{op}}_{\Delta, \mathcal{O}_E}$ with the topology generated by the faithfully flat maps. This topology is called the *flat topology*.

We note that $(\mathcal{O}_E)_{\Delta, \mathcal{O}_E}$ is just the category of bounded $\mathcal{O}_E$-prisms. If $\mathcal{O}_E = \mathbb{Z}_p$, then $(R)_{\Delta, \mathcal{O}_E}$ is the category $(R)_{\Delta}$ introduced in [Bhatt and Scholze 2022, Remark 4.7]. The category $(R)_{\Delta}$ (or its opposite) is called the *absolute prismatic site of* R.

Remark 2.5.4. A diagram

$$(A_2, I_2) \xleftarrow{g} (A_1, I_1) \xrightarrow{f} (A_3, I_3)$$

in $(R)_{\Delta, \mathcal{O}_E}$ such that g is a flat map, admits a colimit (i.e., a pushout). Indeed, by Lemma 2.5.1(1), the (π, I_3)-adic completion $A := (A_2 \otimes_{A_1} A_3)^\wedge$ is isomorphic to the derived (π, I_3)-adic completion of $A_2 \otimes^{\mathbb{L}}_{A_1} A_3$. In particular A is (π, I_3)-completely flat over A_3. (Here we use that J-complete flatness is preserved under base change and taking derived J-adic completions.) It follows from Remark 2.2.8 and Lemma 2.2.10 that A admits a unique δ_E-structure that is compatible with the δ_E-structures on A_2 and A_3. By Lemma 2.5.1(2), we see that $(A, I_3 A)$ is a bounded $\mathcal{O}_E$-prism. By construction, $(A, I_3 A)$ is a colimit of the above diagram. As a result, it follows that $(R)^{\mathrm{op}}_{\Delta, \mathcal{O}_E}$ is indeed a site.

Remark 2.5.5 (cf. [Bhatt and Scholze 2022, Corollary 3.12]). A faithfully flat map $(A, I) \to (A', I')$ induces faithfully flat homomorphisms $A/(\pi, I)^n \to A'/(\pi, I')^n$ and $A/(\pi^n, I) \to A'/(\pi^n, I')$ for any n.

It follows that the functors

$$\mathcal{O}_{\Delta} : (R)_{\Delta,\mathcal{O}_E} \to \text{Set}, \quad (A, I) \mapsto A,$$
$$\mathcal{O}_{\bar{\Delta}} : (R)_{\Delta,\mathcal{O}_E} \to \text{Set}, \quad (A, I) \mapsto A/I,$$

form sheaves with respect to the flat topology. Here Set is the category of sets.

More generally, we have the following descent result.

Proposition 2.5.6. *The fibered category over $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to each $(A, I) \in (\mathcal{O}_E)_{\Delta,\mathcal{O}_E}$ the category of finite projective A-modules satisfies descent with respect to the flat topology. The same holds for finite projective A/I-modules.*

Proof. For a ring B and an ideal $J \subset B$ such that B is J-adically complete, the natural functor

$$\{\text{finite projective } B\text{-modules}\} \longrightarrow 2-\varprojlim_n \{\text{finite projective } B/J^n\text{-modules}\}$$

is an equivalence; see for example [Bhatt 2016, Lemma 4.11]. The assertions of the proposition follow from this fact and faithfully flat descent for finite projective modules over $A/(\pi, I)^n$ and $A/(\pi^n, I)$, respectively. See also [Anschütz and Le Bras 2023, Lemma A.1, Proposition A.3]. □

Definition 2.5.7. For a bounded $\mathcal{O}_E$-prism (A, I), let

$$(A, I)_{\Delta}$$

be the category of bounded $\mathcal{O}_E$-prisms (B, J) with a map $(A, I) \to (B, J)$. We endow $(A, I)^{\mathrm{op}}_{\Delta}$ with the flat topology induced from $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$.

Example 2.5.8. (1) Let $\mathcal{O}$ be as in Definition 2.3.7. The category $(\mathcal{O})_{\Delta,\mathcal{O}_E}$ is the same as the category of bounded $\mathcal{O}_E$-prisms over $\mathcal{O}$ by Lemma 2.3.5.

(2) Let R be a perfectoid ring over $\mathcal{O}_E$. It follows from Lemma 2.3.5 that $(R)_{\Delta,\mathcal{O}_E}$ is the same as the category $(W_{\mathcal{O}_E}(R^\flat), I_R)_{\Delta}$.

Let $A \to B$ be a ring homomorphism and $I \subset A$ a finitely generated ideal. We say that $A \to B$ is *I-completely étale* if B is derived I-adically complete, $B \otimes^{\mathbb{L}}_A A/I$ is concentrated in degree 0, and B/IB is étale over A/I. We write $A_{I\text{-ét}}$ for the category of I-completely étale A-algebras. If $I = 0$, then $A_{I\text{-ét}}$ is just the category $A_{\text{ét}}$ of étale A-algebras.

Lemma 2.5.9. *Let (A, I) be a bounded $\mathcal{O}_E$-prism.*

(1) *A ring homomorphism $A/I \to C$ is π-completely étale if and only if C is π-adically complete and C/π^n is étale over $A/(\pi^n, I)$ for every integer $n \geq 1$. If this is the case, then C has bounded p^∞-torsion.*

(2) *A ring homomorphism $A \to B$ is (π, I)-completely étale if and only if B is (π, I)-adically complete and $B/(\pi, I)^n$ is étale over $A/(\pi, I)^n$ for every $n \geq 1$. If this is the case, then $B \otimes^{\mathbb{L}}_A A/I \xrightarrow{\sim} B/IB$ and $A/I \to B/IB$ is π-completely étale.*

(3) *The functors*

$$A_{(\pi,I)\text{-ét}} \to (A/I)_{\pi\text{-ét}} \to (A/(\pi, I))_{\text{ét}},$$

where the first one is defined by $B \mapsto B/IB$ *and the second one is defined by* $C \mapsto C/\pi$, *are equivalences.*

Proof. This result is well known to specialists, but we include a proof for the convenience of the reader.

(1) Assume that $A/I \to C$ is π-completely étale. Then, since A/I has bounded p^∞-torsion, [Bhatt et al. 2019, Lemma 4.7] implies that C is π-adically complete and has bounded p^∞-torsion. Since C/π^n is flat over $A/(\pi^n, I)$ and C/π is étale over $A/(\pi, I)$, it follows that C/π^n is étale over $A/(\pi^n, I)$ for any n.

We next prove the "if" direction, so we assume that C is π-adically complete and C/π^n is étale over $A/(\pi^n, I)$ for any n. We want to show that $C \otimes^{\mathbb{L}}_{A/I} A/(\pi, I)$ is concentrated in degree 0. There exists an étale A/I-algebra C_0 such that $C_0/\pi \simeq C/\pi$ over $A/(\pi, I)$; see for example [Arabia 2001, Section 1.1] or [Stacks 2005–, Tag 04D1] (this is known as a special case of Elkik's result [1973]). One easily sees that the derived π-adic completion of C_0, the π-adic completion of C_0, and C are isomorphic to each other. Then we obtain

$$C \otimes^{\mathbb{L}}_{A/I} A/(\pi, I) \simeq C_0 \otimes^{\mathbb{L}}_{A/I} A/(\pi, I) \simeq C_0/\pi.$$

This proves the assertion.

(2) Assume that $A \to B$ is (π, I)-completely étale. We easily see that $B/(\pi, I)^n$ is étale over $A/(\pi, I)^n$. It follows from Lemma 2.5.1 that B is (π, I)-adically complete and we have $B \otimes^{\mathbb{L}}_A A/I \xrightarrow{\sim} B/IB$. It is then clear that $A/I \to B/IB$ is π-completely étale.

The "if" direction can be proved by the same argument as in (1). Suppose that B is (π, I)-adically complete and $B/(\pi, I)^n$ is étale over $A/(\pi, I)^n$ for any n. As above, there exists an étale A-algebra B_0 such that the (π, I)-adic completion of B_0 is isomorphic to B. It follows from Lemma 2.5.1(1) that B is isomorphic to the derived (π, I)-adic completion of B_0, which in turn implies that B is (π, I)-completely étale over A.

(3) This follows from the proofs of (1) and (2). □

Lemma 2.5.10 (cf. [Bhatt and Scholze 2022, Lemma 2.18]). *Let* (A, I) *be a bounded* $\mathcal{O}_E$*-prism and* $A \to B$ *a* (π, I)*-completely étale homomorphism. Then* B *admits a unique* δ_E*-structure compatible with that on* A*. Moreover, the pair* (B, IB) *is a bounded* $\mathcal{O}_E$*-prism.*

Proof. It suffices to prove the first statement by Lemma 2.5.1. For this, we proceed as in the proof of [Bhatt and Scholze 2022, Lemma 2.18].

We regard $W_{\mathcal{O}_E,\pi,2}(B)$ as an A-algebra via the composition $A \to W_{\mathcal{O}_E,\pi,2}(A) \to W_{\mathcal{O}_E,\pi,2}(B)$, where $A \to W_{\mathcal{O}_E,\pi,2}(A)$ is the homomorphism corresponding to the δ_E-structure on A (Remark 2.2.7). Then $W_{\mathcal{O}_E,\pi,2}(B)$ is (π, I)-adically complete. Indeed, we have an exact sequence of A-modules

$$0 \longrightarrow \phi_* B \xrightarrow{V} W_{\mathcal{O}_E,\pi,2}(B) \xrightarrow{\epsilon} B \longrightarrow 0,$$

where we write $\phi_* B$ for B regarded as an A-algebra via the composition $A \xrightarrow{\phi} A \to B$, and $V : \phi_* B \to W_{\mathcal{O}_E,\pi,2}(B)$ is defined by $x \mapsto (0, x)$. Since $B \otimes^{\mathbb{L}}_A A/(\pi, I)^n$ is concentrated in degree 0 and both $\phi_* B$ and B are (π, I)-adically complete, we can conclude that $W_{\mathcal{O}_E,\pi,2}(B)$ is (π, I)-adically complete.

As in the proof of Lemma 2.5.9, there exists an étale A-algebra B_0 such that the (π, I)-adic completion of B_0 is isomorphic to B. Since $W_{\mathcal{O}_E,\pi,2}(B)$ is $(\operatorname{Ker} \epsilon)$-adically complete and $A \to B_0$ is étale, there exists a unique homomorphism $s_0 : B_0 \to W_{\mathcal{O}_E,\pi,2}(B)$ of A-algebras such that $\epsilon \circ s_0$ coincides with $B_0 \to B$. Then, since $W_{\mathcal{O}_E,\pi,2}(B)$ is (π, I)-adically complete, we see that $s_0 : B_0 \to W_{\mathcal{O}_E,\pi,2}(B)$ extends to a unique homomorphism $s : B \to W_{\mathcal{O}_E,\pi,2}(B)$ of A-algebras such that $\epsilon \circ s = \mathrm{id}_B$, which corresponds to a unique δ_E-structure on B compatible with that on A by virtue of Remark 2.2.7. □

Example 2.5.11. Let R be a perfectoid ring over $\mathcal{O}_E$ and let $R \to S$ be a π-completely étale (or equivalently, p-completely étale) homomorphism. By [Anschütz and Le Bras 2023, Corollary 2.10] or [Lau 2018, Lemma 8.11], we see that S is a perfectoid ring. Moreover, the isomorphism (2-1) implies that $W_{\mathcal{O}_E}(R^\flat) \to W_{\mathcal{O}_E}(S^\flat)$ is (π, I_R)-completely étale.

Let (A, I) be a bounded $\mathcal{O}_E$-prism. We say that a homomorphism $B \to B'$ of (π, I)-completely étale A-algebras is a (π, I)-*completely étale covering* if

$$\operatorname{Spec} B'/(\pi, I) \to \operatorname{Spec} B/(\pi, I)$$

is surjective, or equivalently, the homomorphism $B \to B'$ is (π, I)-completely faithfully flat. We note that $B \to B'$ is automatically (π, I)-completely étale.

Definition 2.5.12. We write

$$(A, I)_{\text{ét}}$$

for the category of (π, I)-completely étale A-algebras instead of $A_{(\pi,I)\text{-ét}}$. We endow the opposite category $(A, I)^{\mathrm{op}}_{\text{ét}}$ with the topology generated by the (π, I)-completely étale coverings, which is called the (π, I)-*completely étale topology*.

The category $(A, I)^{\mathrm{op}}_{\text{ét}}$ admits fiber products. Indeed, a colimit of the diagram $C \leftarrow B \to D$ in $(A, I)_{\text{ét}}$ is given by the (π, I)-adic completion of $C \otimes_B D$; see Remark 2.5.4. It follows that $(A, I)^{\mathrm{op}}_{\text{ét}}$ is a site.

Remark 2.5.13. Recall that, for a (π, I)-completely étale A-algebra $B \in (A, I)_{\text{ét}}$, the pair (B, IB) is naturally a bounded $\mathcal{O}_E$-prism by Lemma 2.5.10. We can regard $(A, I)_{\text{ét}}$ as a full subcategory of the category $(A, I)_{\Delta}$. The (π, I)-completely étale topology on $(A, I)^{\mathrm{op}}_{\text{ét}}$ coincides with the one induced from the flat topology.

Remark 2.5.14. Any bounded $\mathcal{O}_E$-prism (A, I) admits a (π, I)-completely étale covering $A \to B$ such that (B, IB) is orientable. Indeed, there exists an étale and faithfully flat homomorphism $A \to A'$ such that IA' is principal. The (π, I)-adic completion B of A' is a (π, I)-completely étale covering of A (by Lemma 2.5.1). Since IA' is principal, the bounded $\mathcal{O}_E$-prism (B, IB) is orientable.

2.6. ***Prismatic envelopes for regular sequences.*** The existence and the flatness of prismatic envelopes for regular sequences are proved in [Bhatt and Scholze 2022, Proposition 3.13]. In this subsection, we give an analogous result for $\mathcal{O}_E$-prisms. We will freely use the formalism of *animated rings* here. For the definition and properties of animated rings, see for example [Česnavičius and Scholze 2024, Section 5] and [Bhatt and Lurie 2022a, Appendix A]. (See also [Lurie 2016, Chapter 25], where animated rings are called simplicial rings.)

We recall some terminology from [Česnavičius and Scholze 2024; Bhatt and Scholze 2022]. To an animated ring A, we can attach a graded ring of homotopy groups $\bigoplus_{n\geq 0}\pi_n(A)$. For an animated ring A, the *derived quotient* of A with respect to a sequence $x_1,\dots,x_n\in\pi_0(A)$ is defined by

$$A/^{\mathbb{L}}(x_1,\dots,x_n):=A\otimes^{\mathbb{L}}_{\mathbb{Z}[X_1,\dots,X_n]}\mathbb{Z}[X_1,\dots,X_n]/(X_1,\dots,X_n).$$

Here $\mathbb{Z}[X_1,\dots,X_n]\to A$ is a morphism such that the induced ring homomorphism $\mathbb{Z}[X_1,\dots,X_n]\to\pi_0(A)$ is given by $X_i\mapsto x_i$. In [Bhatt and Scholze 2022], the derived quotient is denoted by $\mathrm{Kos}(A;x_1,\dots,x_n)$. We say that a morphism $A\to B$ of animated rings is *flat* (resp. *faithfully flat*) if $\pi_0(B)$ is flat (resp. faithfully flat) over $\pi_0(A)$ and we have $\pi_n(A)\otimes_{\pi_0(A)}\pi_0(B)\xrightarrow{\sim}\pi_n(B)$ for any $n\geq 0$.

Before stating the result, let us quickly recall the definition of an $\mathcal{O}_E$-PD structure, and its relation to δ_E-structures.

Definition 2.6.1 [Hopkins and Gross 1994, Section 10; Faltings 2002, Definition 14]. Let A be an $\mathcal{O}_E$-algebra and $I\subset A$ an ideal. An *$\mathcal{O}_E$-PD structure* on I is a map $\gamma_\pi: I\to I$ of sets with the following properties:

(1) $\pi\gamma_\pi(x)=x^q$.

(2) $\gamma_\pi(ax)=a^q\gamma_\pi(x)$, where $a\in A$.

(3) $\gamma_\pi(x+y)=\gamma_\pi(x)+\gamma_\pi(y)+(x^q+y^q-(x+y)^q)/\pi$.

Example 2.6.2 [Faltings 2002, Section 7]. Let $n\geq 0$ be an integer and let $\mathcal{O}_E[(X_{i,j})]$ be the polynomial ring with variables $X_{i,j}$ indexed by integers i, j with $1\leq i\leq n$ and $j\geq 0$. We write

$$\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$$

for the quotient of $\mathcal{O}_E[(X_{i,j})]$ by the ideal generated by the elements $X_{i,j}^q-\pi X_{i,j+1}$ for all i, j. The image of $X_{i,0}$ in $\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$ is denoted by X_i. We see that $\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$ is canonically isomorphic to the $\mathcal{O}_E$-subalgebra of $E[X_1,\dots,X_n]$ generated by $X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}}$ ($1\leq i\leq n$ and $j\geq 0$). The ideal $I^{\mathrm{PD}}\subset\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}$ generated by the elements $X_{i,j}$ admits an $\mathcal{O}_E$-PD structure γ_π such that $\gamma_\pi(X_{i,j})=X_{i,j+1}$. In fact, the pair $(\mathcal{O}_E[X_1,\dots,X_n]^{\mathrm{PD}}, I^{\mathrm{PD}})$ is the $\mathcal{O}_E$-PD envelope (in the usual sense) of the polynomial ring $\mathcal{O}_E[X_1,\dots,X_n]$ with respect to the ideal $(X_1,\dots,X_n)$.

Lemma 2.6.3 (cf. [Bhatt and Scholze 2022, Lemma 2.38]). *Let B be a π-torsion-free $\mathcal{O}_E$-algebra. Let $x_1,\dots,x_n\in B$ be a sequence such that $(B/\pi)/^{\mathbb{L}}(\bar{x}_1,\dots,\bar{x}_n)$ is concentrated in degree* 0, *where*

$\bar{x}_1, \dots, \bar{x}_n \in B/\pi$ are the images of $x_1, \dots, x_n \in B$. We set

$$C := B \otimes^{\mathbb{L}}_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}},$$

where $\mathcal{O}_E[X_1, \dots, X_n] \to B$ is defined by $X_i \mapsto x_i$. Then C is concentrated in degree 0 and $\pi_0(C)$ is π-torsion-free. Moreover the pair $(\pi_0(C), I^{\mathrm{PD}}\pi_0(C))$ is the $\mathcal{O}_E$-PD envelope of B with respect to the ideal $(x_1, \dots, x_n)$.

In the following, we will write

$$D_{(x_1,\dots,x_n)}(B) := \pi_0(C) = B \otimes_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}}.$$

Proof. The proof is identical to that of [Bhatt and Scholze 2022, Lemma 2.38]. □

We also need the following construction. Let B be a δ_E-ring. Let $d \in B$ be an element and be $x_1, \dots, x_n \in B$ a sequence. We set $B\{X\} := B \otimes_{\mathcal{O}_E} \mathcal{O}_E\{X\}$ and let

$$B\{X_1, \dots, X_n\}$$

be the n-th tensor power of $B\{X\}$ over B. We consider the diagram of δ_E-rings

$$B \xleftarrow{f} B\{X_1, \dots, X_n\} \xrightarrow{g} B\{Y_1, \dots, Y_n\},$$

where f is defined by $X_i \mapsto x_i$ and g is defined by $X_i \mapsto dY_i$. Let

$$B\{x_1/d, \dots, x_n/d\}$$

denote the pushout of this diagram, which is a δ_E-ring over B with the following property: For a homomorphism $B \to C$ of δ_E-rings such that the image of d is a nonzerodivisor in C and $x_i \in dC$ for all i, there exists a unique homomorphism $B\{x_1/d, \dots, x_n/d\} \to C$ of δ_E-rings over B. We let $x_i/d \in B\{x_1/d, \dots, x_n/d\}$ denote the image of $Y_i \in B\{Y_1, \dots, Y_n\}$.

Using this construction, we can relate δ_E-structures to $\mathcal{O}_E$-PD structures.

Lemma 2.6.4 (cf. [Bhatt and Scholze 2022, Lemma 2.36]). *We have a natural isomorphism*

$$(\mathcal{O}_E\{X_1, \dots, X_n\})\{\phi(X_1)/\pi, \dots, \phi(X_n)/\pi\} \simeq D_{(X_1,\dots,X_n)}(\mathcal{O}_E\{X_1, \dots, X_n\})$$

of $\mathcal{O}_E\{X_1, \dots, X_n\}$-algebras.

Proof. This can be proved in the same way as [Bhatt and Scholze 2022, Lemma 2.36]. We include a sketch of the proof.

We set $B := \mathcal{O}_E\{X_1, \dots, X_n\}$. Since the Frobenius $\phi : B \to B$ is faithfully flat by Example 2.2.5, it follows that $C := B\{\phi(X_1)/\pi, \dots, \phi(X_n)/\pi\}$ is π-torsion-free. Since $\phi(X_i)/\pi = X_i^q/\pi + \delta_E(X_i)$, C can be regarded as the smallest δ_E-subring of $B[1/\pi]$ which contains B and X_i^q/π $(1 \le i \le n)$. On the other hand, since $D := D_{(X_1,\dots,X_n)}(B)$ is π-torsion-free by Lemma 2.6.3, we see that D is the $\mathcal{O}_E$-subalgebra of $B[1/\pi]$ generated by B and $X_i^{q^j}/\pi^{1+q+\dots+q^{j-1}}$ $(1 \le i \le n$ and $j \ge 1)$.

We shall prove that $C = D$ in $B[1/\pi]$. Let us first show that $D \subset C$. For this, it suffices to prove that, for every $1 \le i \le n$, we have $X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}} \in C$ for all $j \ge 1$. We proceed by induction on j. The assertion is clear for $j = 1$. Assume that the assertion is true for some $j \ge 1$. Then we have

$$\begin{aligned}\delta_E(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}}) &= \phi(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}})/\pi - (X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}})^q/\pi \\ &= \phi(X_i)^{q^j}/\pi^{2+q+\cdots+q^{j-1}} - X_i^{q^{j+1}}/\pi^{1+q+\cdots+q^j} \in C.\end{aligned}$$

To prove that $X_i^{q^{j+1}}/\pi^{1+q+\cdots+q^j} \in C$, it is enough to show that $\phi(X_i)^{q^j}/\pi^{2+q+\cdots+q^{j-1}} \in C$. Since $\phi(X_i)/\pi = X_i^q/\pi + \delta_E(X_i)$ is contained in C, the assertion now follows from the inequality $q^j \ge 2 + q + \cdots + q^{j-1}$.

It remains to prove that $C \subset D$. Since $\phi(X_i)/\pi = X_i^q/\pi + \delta_E(X_i)$ is contained in D, the inequality $q^j \ge 2 + q + \cdots + q^{j-1}$ for $j \ge 1$ implies that

$$\phi(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}}) = \phi(X_i)^{q^j}/\pi^{1+q+\cdots+q^{j-1}} \in \pi D$$

for $j \ge 1$. Since

$$(X_i^{q^j}/\pi^{1+q+\cdots+q^{j-1}})^q = \pi(X_i^{q^{j+1}}/\pi^{1+q+\cdots+q^j}) \in \pi D,$$

we see that ϕ preserves D and that the reduction modulo π of $\phi : D \to D$ is the q-th power Frobenius. This implies that $C \subset D$. □

Corollary 2.6.5 (cf. [Bhatt and Scholze 2022, Corollary 2.39]). *Let B be a π-torsion-free δ_E-ring. Let $x_1, \dots, x_n \in B$ be a sequence such that $(B/\pi)/^{\mathbb{L}}(\bar{x}_1, \dots, \bar{x}_n)$ is concentrated in degree 0. We set $D := B \otimes^{\mathbb{L}}_{\mathcal{O}_E\{X_1,\dots,X_n\}} \mathcal{O}_E\{Y_1, \dots, Y_n\}$ where $\mathcal{O}_E\{X_1, \dots, X_n\} \to B$ is defined by $X_i \mapsto \phi(x_i)$ and $\mathcal{O}_E\{X_1, \dots, X_n\} \to \mathcal{O}_E\{Y_1, \dots, Y_n\}$ is defined by $X_i \mapsto \pi Y_i$. Then D is concentrated in degree 0. Moreover*

$$\pi_0(D) = B\{\phi(x_1)/\pi, \dots, \phi(x_n)/\pi\}$$

is π-torsion-free, and is isomorphic to $D_{(x_1,\dots,x_n)}(B)$ as a B-algebra.

Proof. Since $\phi : \mathcal{O}_E\{X_1, \dots, X_n\} \to \mathcal{O}_E\{X_1, \dots, X_n\}$ is faithfully flat by Example 2.2.5, we have an identification

$$D = B \otimes^{\mathbb{L}}_{\mathcal{O}_E\{X_1,\dots,X_n\}} (\mathcal{O}_E\{X_1, \dots, X_n\})\{\phi(X_1)/\pi, \dots, \phi(X_n)/\pi\},$$

where $\mathcal{O}_E\{X_1, \dots, X_n\} \to B$ is defined by $X_i \mapsto x_i$. Then Lemma 2.6.4 implies that

$$D \simeq B \otimes^{\mathbb{L}}_{\mathcal{O}_E\{X_1,\dots,X_n\}} D_{(X_1,\dots,X_n)}(\mathcal{O}_E\{X_1, \dots, X_n\}).$$

By Lemma 2.6.3, we have

$$D_{(X_1,\dots,X_n)}(\mathcal{O}_E\{X_1, \dots, X_n\}) \simeq \mathcal{O}_E\{X_1, \dots, X_n\} \otimes^{\mathbb{L}}_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}}$$

and thus $D \simeq B \otimes^{\mathbb{L}}_{\mathcal{O}_E[X_1,\dots,X_n]} \mathcal{O}_E[X_1, \dots, X_n]^{\mathrm{PD}}$. The assertion then follows by applying Lemma 2.6.3 again. □

Now we can state the desired result:

Proposition 2.6.6 (cf. [Bhatt and Scholze 2022, Proposition 3.13]). *Assume that (A, I) is an orientable and bounded $\mathcal{O}_E$-prism. Let $d \in I$ be a generator. Let B be a δ_E-ring over A. Let $x_1, \ldots, x_n \in B$ be a sequence such that the induced morphism*

$$A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \ldots, x_n) \tag{2-2}$$

of animated rings is flat. (In other words, the sequence $x_1, \ldots, x_n \in B$ is (π, I)-completely regular relative to A in the sense of [Bhatt and Scholze 2022, Definition 2.42].*) We set $J := (d, x_1, \ldots, x_n) \subset B$. Then the following assertions hold*:

(1) *The (π, I)-adic completion $B\{J/I\}^\wedge$ of $B\{x_1/d, \ldots, x_n/d\}$ is (π, I)-completely flat over A. In particular, the pair*

$$(B\{J/I\}^\wedge, IB\{J/I\}^\wedge)$$

is an orientable and bounded $\mathcal{O}_E$-prism. Moreover $B\{J/I\}^\wedge$ is (π, I)-completely faithfully flat over A if the morphism (2-2) is faithfully flat.

(2) *For a bounded $\mathcal{O}_E$-prism (D, ID) over (A, I) and a homomorphism $B \to D$ of δ_E-rings over A such that $JD \subset ID$, there exists a unique map of $\mathcal{O}_E$-prisms*

$$(B\{J/I\}^\wedge, IB\{J/I\}^\wedge) \to (D, ID)$$

over B. Moreover, the formation of $B\{J/I\}^\wedge$ commutes with base change along any map $(A, I) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms, and also commutes with base change along any (π, I)-completely flat homomorphism $B \to B'$ of δ_E-rings.

See [Bhatt and Scholze 2022, Proposition 3.13(3)] for the precise meaning of the last statement.

Proof. Let $C := B \otimes^{\mathbb{L}}_{B\{X_1,\ldots,X_n\}} B\{Y_1, \ldots, Y_n\}$ be the pushout of the diagram $B \leftarrow B\{X_1, \ldots, X_n\} \to B\{Y_1, \ldots, Y_n\}$ in the ∞-category of animated rings, where the first map is defined by $X_i \mapsto x_i$ and the second one is defined by $X_i \mapsto dY_i$. It suffices to prove that if the morphism (2-2) is flat (resp. faithfully flat), then C is (π, I)-completely flat (resp. (π, I)-completely faithfully flat) over A. Indeed, if this is true, then the derived (π, I)-adic completion of C is isomorphic to $B\{J/I\}^\wedge$, and in particular $B\{J/I\}^\wedge$ is (π, I)-completely flat (resp. (π, I)-completely faithfully flat) over A. It is then easy to see that $B\{J/I\}^\wedge$ has the desired properties.

In order to prove that C is (π, I)-completely flat (resp. (π, I)-completely faithfully flat) over A, one can argue as in the proof of [loc. cit., Proposition 3.13]. (The faithful flatness is not discussed in [loc. cit.], but the same argument works.) The only difference is that we have to use $\mathcal{O}_E$-PD structures, instead of usual PD structures. Here we need the results established above (e.g., Corollary 2.6.5). The details are left to the reader. □

The bounded $\mathcal{O}_E$-prism $(B\{J/I\}^\wedge, IB\{J/I\}^\wedge)$ is called the *prismatic envelope* of B over (A, I) with respect to the ideal J.

Remark 2.6.7. As in [Bhatt and Scholze 2022, Proposition 3.13], we need to use animated δ_E-rings in the proof of Proposition 2.6.6. (For example, in the proof of [loc. cit., Proposition 3.13], the notion of animated δ_E-rings is used to obtain the description of the bottom right vertex C'' of the diagram appearing there.) One can define the notion of animated δ_E-rings in the same way as in [Mao 2024, Section 5] (i.e., by animating δ_E-rings). Alternatively, we can follow the approach employed in [Bhatt and Lurie 2022b, Appendix A].

3. Displayed Breuil–Kisin modules

In this section, we study Breuil–Kisin modules for bounded $\mathcal{O}_E$-prisms. We introduce the notions of a displayed Breuil–Kisin module and of a minuscule Breuil–Kisin module. These objects serve as examples of prismatic G-μ-displays introduced in Section 5.

3.1. *Displayed Breuil–Kisin modules.* We use the following notation. Let A be a ring. For A-modules M and N, the set of A-linear homomorphisms $M \to N$ is denoted by $\mathrm{Hom}_A(M, N)$. Let $I \subset A$ be a Cartier divisor. For an integer $n \geq 1$, we define $I^{-n} := \mathrm{Hom}_A(I^n, A)$. We have a natural injection $I^{-n} \hookrightarrow I^{-n-1}$ for any integer n. We then define

$$A[1/I] := \varinjlim_n I^{-n},$$

which is an A-algebra. For an A-module M, we set $M[1/I] := M \otimes_A A[1/I]$. If I is generated by a nonzerodivisor d, then we have $A[1/I] = A[1/d]$ and $I^{-n} = d^{-n}A$.

Lemma 3.1.1. *Let M, N be finite projective A-modules and let $F : N[1/I] \xrightarrow{\sim} M[1/I]$ be an $A[1/I]$-linear isomorphism. For an integer i, we set*

$$\mathrm{Fil}^i(N) := \{x \in N \mid F(x) \in M \otimes_A I^i\},$$

where we view $M \otimes_A I^i$ as a subset of $M[1/I]$. Let m be an integer. Then the following are equivalent:

(1) $\mathrm{Fil}^{m+1}(N) \subset IN$.

(2) $M \otimes_A I^m \subset F(N)$.

If these equivalent conditions are satisfied, then F restricts to an isomorphism $\mathrm{Fil}^m(N) \xrightarrow{\sim} M \otimes_A I^m$, and in particular $\mathrm{Fil}^m(N)$ is a finite projective A-module.

Proof. The final statement clearly follows from (2). We shall prove that (1) and (2) are equivalent. For this, we can reduce to the case where $I = (d)$ is principal.

Assume that (1) holds. Let $x \in M$. We want to show that $d^m x \in F(N)$. For a large enough integer n, we have $d^n x \in F(N)$. Let $y \in N$ be an element such that $F(y) = d^n x$. If $n \geq m+1$, then we have $y \in \mathrm{Fil}^{m+1}(N) \subset IN$, which in turn implies that $d^{n-1}x \in F(N)$. From this observation, we can conclude that $d^m x \in F(N)$.

Assume that (2) holds. Let $y \in \mathrm{Fil}^{m+1}(N)$. There exists an element $x \in M$ such that $F(y) = d^{m+1}x$. The condition (2) implies that $d^m x = F(z)$ for some $z \in N$. It then follows that $y = dz \in IN$. □

Let (A, I) be a bounded $\mathcal{O}_E$-prism.

Definition 3.1.2. A *Breuil–Kisin module* over (A, I) is a pair (M, F_M) consisting of a finite projective A-module M and an $A[1/I]$-linear isomorphism

$$F_M : (\phi^* M)[1/I] \xrightarrow{\sim} M[1/I],$$

where $\phi^* M := A \otimes_{\phi, A} M$. When there is no possibility of confusion, we simply write M instead of (M, F_M). For an integer i, we set

$$\mathrm{Fil}^i(\phi^* M) := \{x \in \phi^* M \mid F_M(x) \in M \otimes_A I^i\}.$$

Let $P^i \subset (\phi^* M)/I(\phi^* M)$ be the image of $\mathrm{Fil}^i(\phi^* M)$. We often write

$$M_{\mathrm{dR}} := (\phi^* M)/I(\phi^* M).$$

Remark 3.1.3. If $F_M(\phi^* M) \subset M$, then we say that M is *effective*. In this case, the induced homomorphism $\phi^* M \to M$ is again denoted by F_M. The cokernel of $F_M : \phi^* M \to M$ is killed by some power of I.

Conversely, for a finite projective A-module M and a homomorphism $F_M : \phi^* M \to M$ of A-modules whose cokernel is killed by some power of I, the induced homomorphism $(\phi^* M)[1/I] \to M[1/I]$ is an isomorphism. Indeed, it is clear that $(\phi^* M)[1/I] \to M[1/I]$ is surjective, which in turn implies that it is an isomorphism since (the vector bundles on Spec A associated with) $\phi^* M$ and M have the same rank. In particular, it follows that $F_M : \phi^* M \to M$ is injective.

Remark 3.1.4. For any integer i, we have $I\,\mathrm{Fil}^{i-1}(\phi^* M) = \mathrm{Fil}^i(\phi^* M) \cap I(\phi^* M)$. In other words, the natural homomorphism $\mathrm{Fil}^i(\phi^* M)/I\,\mathrm{Fil}^{i-1}(\phi^* M) \to P^i$ is bijective. We have $P^i = M_{\mathrm{dR}}$ for small enough i and $P^i = 0$ for large enough i.

Definition 3.1.5. Let M be a Breuil–Kisin module over (A, I). We say that M is *displayed* if the A/I-submodule $P^i \subset M_{\mathrm{dR}}$ is a direct summand for every i. In this case, the filtration $\{P^i\}_{i \in \mathbb{Z}}$ is called the *Hodge filtration*. We say that M is *minuscule* if it is displayed, and if we have $P^i = M_{\mathrm{dR}}$ for any $i \leq 0$ and $P^i = 0$ for any $i \geq 2$.

The following proposition, which is basically a consequence of [Anschütz and Le Bras 2023, Remark 4.25], shows that the definition of a minuscule Breuil–Kisin module given in Definition 3.1.5 agrees with the usual one employed in the literature (for example in [Kisin 2006, Section 2.2] and [Anschütz and Le Bras 2023, Definition 4.24]).

Proposition 3.1.6. *Let M be a Breuil–Kisin module over (A, I). The following statements are equivalent*:

(1) *M is minuscule.*

(2) *M is effective, and the cokernel* Coker F_M *of $F_M : \phi^* M \to M$ is killed by I.*

Proof. Assume that (1) holds. It follows from $P^0 = M_{\mathrm{dR}}$ and Nakayama's lemma that $\mathrm{Fil}^0(\phi^* M) = \phi^* M$. Moreover, we have $IM \subset F_M(\phi^* M)$ by Lemma 3.1.1. This proves that (1) implies (2).

Assume that (2) holds. It follows from Lemma 3.1.1 that $\mathrm{Fil}^2(\phi^*M) \subset I(\phi^*M)$, and hence $P^i = 0$ for any $i \geq 2$. Since M is effective, we have $P^i = M_{\mathrm{dR}}$ for any $i \leq 0$. It remains to prove that P^1 is a direct summand of M_{dR}. For this, it suffices to show that $(\phi^*M)/\mathrm{Fil}^1(\phi^*M)$ is projective as an A/I-module. Since we have the exact sequence of A/I-modules

$$0 \to (\phi^*M)/\mathrm{Fil}^1(\phi^*M) \to M/IM \to \mathrm{Coker}\, F_M \to 0,$$

it suffices to prove that $\mathrm{Coker}\, F_M$ is a projective A/I-module. With Lemma 3.1.7 below, this follows from the same argument as in [Anschütz and Le Bras 2023, Remark 4.25]. □

Lemma 3.1.7. *Let (A, I) be an $\mathcal{O}_E$-prism. For a perfect field k containing $\mathbb{F}_q$ and a homomorphism $g : A/I \to k$ of $\mathcal{O}_E$-algebras, there exists a map $(A, I) \to (\mathcal{O}, (\pi))$ of $\mathcal{O}_E$-prisms which induces g, where $\mathcal{O} := W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.*

Proof. Let $A_{\mathrm{perf}} := \varinjlim_{\phi} A$ be a colimit of the diagram $A \xrightarrow{\phi} A \xrightarrow{\phi} A \to \cdots$, which is a perfect δ_E-ring. Since k is perfect, the homomorphism $A/\pi \to k$ induced by the composition $A \to A/I \to k$ factors through a homomorphism $A_{\mathrm{perf}}/\pi \to k$. This homomorphism lifts uniquely to a homomorphism $A_{\mathrm{perf}} \to \mathcal{O}$ of δ_E-rings by Lemma 2.3.5. The composition $A \to A_{\mathrm{perf}} \to \mathcal{O}$ gives a map $(A, I) \to (\mathcal{O}, (\pi))$ which induces g, as desired. □

We shortly discuss the relation between the notion of minuscule Breuil–Kisin modules and that of windows introduced by Zink and Lau. We recall the notion of windows, adapted to our context. Let (A, d) be an oriented and bounded $\mathcal{O}_E$-prism.

Definition 3.1.8. A *window* over (A, d) is a quadruple

$$\underline{N} = (N, \mathrm{Fil}^1(N), \Phi, \Phi_1),$$

where N is a finite projective A-module, $\mathrm{Fil}^1(N) \subset N$ is an A-submodule, $\Phi : N \to N$ and $\Phi_1 : \mathrm{Fil}^1(N) \to N$ are ϕ-linear homomorphisms, such that the following conditions hold:

(1) We have $dN \subset \mathrm{Fil}^1(N)$, and $\Phi(x) = \Phi_1(dx)$ for every $x \in N$.

(2) The image $P^1 \subset N/dN$ of $\mathrm{Fil}^1(N)$ is a direct summand of N/dN.

(3) The linearization $1 \otimes \Phi_1 : \phi^* \mathrm{Fil}^1(N) \to N$ of Φ_1 is an isomorphism.

Proposition 3.1.9 (cf. [Cais and Lau 2017, Lemma 2.1.16; Anschütz and Le Bras 2023, Proposition 4.26]). *For a window $\underline{N}$ over (A, d), the pair $(\mathrm{Fil}^1(N), F)$, where $F : \phi^* \mathrm{Fil}^1(N) \to \mathrm{Fil}^1(N)$ is defined by $F = d(1 \otimes \Phi_1)$, is a minuscule Breuil–Kisin module over $(A, (d))$. This construction gives an equivalence between the category of windows over (A, d) and the category of minuscule Breuil–Kisin modules over $(A, (d))$.*

Proof. By virtue of Proposition 3.1.6, we can use the same argument as in the proof of [Cais and Lau 2017, Lemma 2.1.16]. □

We study the structure of displayed Breuil–Kisin modules. For this, we introduce the following definition. Let (A, I) be a bounded $\mathcal{O}_E$-prism.

Definition 3.1.10. Let M be a Breuil–Kisin module over (A, I). A decomposition $\phi^*M = \bigoplus_{j\in\mathbb{Z}} L_j$ is called a *normal decomposition* if the isomorphism $F_M : (\phi^*M)[1/I] \xrightarrow{\sim} M[1/I]$ restricts to an isomorphism

$$\bigoplus_{j\in\mathbb{Z}} (L_j \otimes_A I^{-j}) \xrightarrow{\sim} M$$

of A-modules.

Remark 3.1.11. If $\phi^*M = \bigoplus_{j\in\mathbb{Z}} L_j$ is a normal decomposition, then we have

$$\mathrm{Fil}^i(\phi^*M) = \Big(\bigoplus_{j\geq i} L_j\Big) \oplus \Big(\bigoplus_{j<i} I^{i-j} L_j\Big)$$

for every $i \in \mathbb{Z}$. In particular, a Breuil–Kisin module over (A, I) which admits a normal decomposition is displayed. In the next lemma, we shall prove that the converse is also true.

Lemma 3.1.12. *Let M be a displayed Breuil–Kisin module over (A, I). Then there exists a normal decomposition $\phi^*M = \bigoplus_{j\in\mathbb{Z}} L_j$.*

Proof. We choose a decomposition $(\phi^*M)/I(\phi^*M) = \bigoplus_{j\in\mathbb{Z}} K_j$ such that $P^i = \bigoplus_{j\geq i} K_j$ for every i. Since K_j is a finite projective A/I-module and A is I-adically complete, there exists a finite projective A-module L_j such that $L_j/IL_j \simeq K_j$ for every j; see [Stacks 2005–, Tag 0D4A] or [Greco 1968, Theorem 5.1] for example. Moreover we have $L_j = 0$ for all but finitely many j. Since $\mathrm{Fil}^i(\phi^*M) \to P^i$ is surjective, there exists a homomorphism $L_i \to \mathrm{Fil}^i(\phi^*M)$ which fits into the following commutative diagram:

$$\begin{array}{ccc} L_i & \longrightarrow & K_i \\ \downarrow & & \downarrow \\ \mathrm{Fil}^i(\phi^*M) & \longrightarrow & P^i \end{array}$$

The induced homomorphism $\bigoplus_{j\in\mathbb{Z}} L_j \to \phi^*M$ is an isomorphism since it is a lift of the isomorphism $\bigoplus_{j\in\mathbb{Z}} K_j \xrightarrow{\sim} (\phi^*M)/I(\phi^*M)$. We shall prove that, under this isomorphism, $\mathrm{Fil}^i(\phi^*M)$ coincides with $\big(\bigoplus_{j\geq i} L_j\big) \oplus \big(\bigoplus_{j<i} I^{i-j} L_j\big)$ for any $i \in \mathbb{Z}$. This implies that $\bigoplus_{j\in\mathbb{Z}} L_j$ is a normal decomposition.

We proceed by induction on i. The assertion clearly holds for small enough i. Let us assume that the assertion holds for an integer i. Since

$$\Big(\bigoplus_{j\geq i} IL_j\Big) \oplus \Big(\bigoplus_{j<i} I^{i+1-j} L_j\Big) = I\,\mathrm{Fil}^i(\phi^*M) \subset \mathrm{Fil}^{i+1}(\phi^*M)$$

and $\bigoplus_{j\geq i+1} L_j \subset \mathrm{Fil}^{i+1}(\phi^*M)$ by construction, we obtain

$$\Big(\bigoplus_{j\geq i+1} L_j\Big) \oplus \Big(\bigoplus_{j<i+1} I^{i+1-j} L_j\Big) \subset \mathrm{Fil}^{i+1}(\phi^*M).$$

The left-hand side contains $I\,\mathrm{Fil}^i(\phi^*M)$ and the quotient by $I\,\mathrm{Fil}^i(\phi^*M)$ is equal to P^{i+1}. The same holds for the right-hand side by Remark 3.1.4. Therefore, this inclusion is actually an equality. □

Let $f:(A,I)\to(A',I')$ be a map of bounded $\mathcal{O}_E$-prisms. For a Breuil–Kisin module (M,F_M) over (A,I), let $F_{M_{A'}}:(\phi^*(M_{A'}))[1/I']\xrightarrow{\sim}M_{A'}[1/I']$ be the base change of F_M, where $M_{A'}:=M\otimes_A A'$. We also write f^*M for $(M_{A'},F_{M_{A'}})$.

Proposition 3.1.13. *Let (M,F_M) be a Breuil–Kisin module over (A,I).*

(1) *Assume that (M,F_M) is displayed. Then $(M_{A'},F_{M_{A'}})$ is a displayed Breuil–Kisin module over (A',I'), and we have $\mathrm{Fil}^i(\phi^*M)\otimes_A A'\xrightarrow{\sim}\mathrm{Fil}^i(\phi^*(M_{A'}))$ for any integer i.*

(2) *Assume that $(M_{A'},F_{M_{A'}})$ is displayed and $f:(A,I)\to(A',I')$ is a faithfully flat map of $\mathcal{O}_E$-prisms. Then (M,F_M) is displayed.*

Proof. (1) This follows from Remark 3.1.11, Lemma 3.1.12, and the fact that normal decompositions are preserved under base change.

(2) We note that $\mathrm{Fil}^i(\phi^*(M_{A'}))$ is a finite projective A'-module for any i by Lemma 3.1.12 and Remark 3.1.11. Since $\mathrm{Fil}^i(\phi^*(M_{A'}))$ is stable under the natural descent datum of $\phi^*(M_{A'})$ (with respect to the flat covering $(A,I)\to(A',I')$) by (1), it follows from Proposition 2.5.6 that there is a descending filtration $\{\mathrm{Fil}^i\}_{i\in\mathbb{Z}}$ of ϕ^*M by finite projective A-submodules such that $\mathrm{Fil}^i\otimes_A A'\to\phi^*(M_{A'})$ induces an isomorphism $\mathrm{Fil}^i\otimes_A A'\xrightarrow{\sim}\mathrm{Fil}^i(\phi^*(M_{A'}))$ for any i.

Let m be an integer such that $M\otimes_A I^m\subset F_M(\phi^*M)$. Then $\mathrm{Fil}^m=\mathrm{Fil}^m(\phi^*M)$ (see Lemma 3.1.1). Moreover, we have $I\,\mathrm{Fil}^{i-1}\subset\mathrm{Fil}^i$ for any i, and $I\,\mathrm{Fil}^{i-1}=\mathrm{Fil}^i$ for $i\geq m+1$. In particular, we obtain $\mathrm{Fil}^i=\mathrm{Fil}^i(\phi^*M)$ for $i\geq m$.

Let i be any integer. We claim that the natural homomorphism of A/I-modules

$$\iota:\mathrm{Fil}^i/I\,\mathrm{Fil}^{i-1}\to(\phi^*M)/I(\phi^*M)$$

is injective and its cokernel is a finite projective A/I-module. Indeed, it suffices to show that for every closed point $x\in\operatorname{Spec}A/I$, the base change of ι to the residue field $k(x)$ is injective. Since x is contained in $\operatorname{Spec}A/(\pi,I)$ and $\operatorname{Spec}A'/(\pi,I')\to\operatorname{Spec}A/(\pi,I)$ is surjective, it is enough to prove that the base change of ι along $A/I\to A'/I'$ is injective and its cokernel is a finite projective A'/I'-module. This follows from the assumption that $(M_{A'},F_{M_{A'}})$ is displayed.

It follows from the claim that $I\,\mathrm{Fil}^{i-1}=I(\phi^*M)\cap\mathrm{Fil}^i$, or equivalently, $\mathrm{Fil}^{i-1}=\phi^*M\cap(\mathrm{Fil}^i\otimes_A I^{-1})$. Since $\mathrm{Fil}^i=\mathrm{Fil}^i(\phi^*M)$ for $i\geq m$, we can conclude that $\mathrm{Fil}^i=\mathrm{Fil}^i(\phi^*M)$ for any i. This, together with the claim, shows that (M,F_M) is displayed. □

Remark 3.1.14. The functor

$$(\mathcal{O}_E)_{\Delta,\mathcal{O}_E}\to\mathrm{Set},\quad (B,J)\mapsto B[1/J],$$

forms a sheaf (with respect to the flat topology) by Lemma 2.3.4 and Remark 2.5.5. Thus for finite projective A-modules M,M', the functor $(A,I)_\Delta\to\mathrm{Set}$ which associates to each $(B,J)\in(A,I)_\Delta$ the set of isomorphisms $M_B[1/J]\xrightarrow{\sim}M'_B[1/J]$ forms a sheaf. This fact, together with Proposition 2.5.6, implies that the fibered category over $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to a bounded $\mathcal{O}_E$-prism (A,I) the category of Breuil–Kisin modules over (A,I) satisfies descent with respect to the flat topology.

Corollary 3.1.15. *The fibered category over $(\mathcal{O}_E)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to a bounded $\mathcal{O}_E$-prism (A, I) the category of displayed Breuil–Kisin modules over (A, I) satisfies descent with respect to the flat topology.*

Proof. This follows from Proposition 3.1.13 and Remark 3.1.14. □

We finish this subsection by giving an example of a Breuil–Kisin module which is not displayed.

Example 3.1.16. Let (A, I) be an orientable and bounded $\mathcal{O}_E$-prism. We assume that A/I is π-torsion-free and $A/I \neq 0$. We set $M := A^2$ and let $F_M : \phi^* M \to M$ be the homomorphism defined by the matrix $\begin{pmatrix} \pi & d \\ d & d^2 \end{pmatrix}$ for a generator $d \in I$. The pair (M, F_M) is a Breuil–Kisin module over (A, I). We claim that P^1/P^2 is not π-torsion-free, and thus (M, F_M) is not displayed. Indeed, since $(d, 1) \in \mathrm{Fil}^1(\phi^* M)$, we have $(0, 1) \in P^1 \subset M_{\mathrm{dR}}$. One can check that the image of $(0, 1)$ in P^1/P^2 is not zero and is killed by π.

3.2. *Breuil–Kisin modules of type μ.* Here we introduce the notion of Breuil–Kisin modules *of type* μ. Let

$$\mu : \mathbb{G}_m \to \mathrm{GL}_{n,\mathcal{O}}$$

be a cocharacter defined over $\mathcal{O}$, where $\mathcal{O} = W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ is as in Definition 2.3.7. There is a unique tuple $(m_1, \dots, m_n)$ of integers $m_1 \geq \cdots \geq m_n$ such that μ is conjugate to the cocharacter defined by $t \mapsto \mathrm{diag}\,(t^{m_1}, \dots, t^{m_n})$. By abuse of notation, the tuple $(m_1, \dots, m_n)$ is also denoted by μ. Let $r_i \in \mathbb{Z}_{\geq 0}$ be the number of occurrences of i in $(m_1, \dots, m_n)$. We set $L := \mathcal{O}_E^n$ and $L_{\mathcal{O}} := L \otimes_{\mathcal{O}_E} \mathcal{O}$. The cocharacter μ induces an action of $\mathbb{G}_m$ on $L_{\mathcal{O}}$. We have the weight decomposition

$$L_{\mathcal{O}} = \bigoplus_{j \in \mathbb{Z}} L_{\mu,j},$$

where an element $t \in \mathbb{G}_m(\mathcal{O}) = \mathcal{O}^\times$ acts on $L_{\mu,j}$ by $x \mapsto t^j x$ for every $j \in \mathbb{Z}$. (See for example [Conrad et al. 2015, Lemma A.8.8] for the existence of the weight decomposition over a ring.) The rank of $L_{\mu,j}$ is equal to r_j.

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Definition 3.2.1. Let M be a displayed Breuil–Kisin module over (A, I). We say that M is *of type* μ if, for the Hodge filtration $\{P^i\}_{i \in \mathbb{Z}}$, the successive quotient P^i/P^{i+1} is of rank r_i (i.e., the corresponding vector bundle on $\mathrm{Spec}\, A/I$ has constant rank r_i) for every i. We say that M is *banal* if all successive quotients P^i/P^{i+1} are free A/I-modules.

We write $\mathrm{BK}_\mu(A, I)$ (resp. $\mathrm{BK}_\mu(A, I)_{\mathrm{banal}}$) for the category of Breuil–Kisin modules over (A, I) of type μ (resp. banal Breuil–Kisin modules over (A, I) of type μ).

Remark 3.2.2. We set

$$\mathrm{Fil}^i_\mu := \Big(\bigoplus_{j \geq i} (L_{\mu,j})_A\Big) \oplus \Big(\bigoplus_{j < i} I^{i-j} (L_{\mu,j})_A\Big) \subset A^n,$$

where $(L_{\mu,j})_A := L_{\mu,j} \otimes_{\mathcal{O}} A$. Let $M \in \mathrm{BK}_\mu(A, I)_{\mathrm{banal}}$. Let $\phi^* M = \bigoplus_{j \in \mathbb{Z}} L_j$ be a normal decomposition. Then, each L_j is a free A-module of rank r_j. Thus there is an isomorphism $A^n \simeq \phi^* M$ such that the filtration $\{\mathrm{Fil}^i_\mu\}_{i \in \mathbb{Z}}$ coincides with $\{\mathrm{Fil}^i(\phi^* M)\}_{i \in \mathbb{Z}}$.

Remark 3.2.3. Let M be a displayed Breuil–Kisin module over (A, I). Then there exists a (π, I)-completely étale covering $A \to A_1 \times \cdots \times A_m$ such that for each $1 \le j \le m$, the base change of M to the bounded $\mathcal{O}_E$-prism (A_j, IA_j) (see Lemma 2.5.10) is of type μ for some μ and banal. Indeed, by Lemma 2.5.9, it suffices to prove that there exists an étale and faithfully flat homomorphism $A/(\pi, I) \to B_1 \times \cdots \times B_m$ such that, for all $1 \le j \le m$ and i, the base change $P^i/P^{i+1} \otimes_{A/I} B_j$ is free over B_j, which is clear.

4. Display group

Let G be a smooth affine group scheme over $\mathcal{O}_E$. Let k be a perfect field containing $\mathbb{F}_q$ and we set $\mathcal{O} := W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. Let

$$\mu : \mathbb{G}_m \to G_{\mathcal{O}} := G \times_{\operatorname{Spec} \mathcal{O}_E} \operatorname{Spec} \mathcal{O}$$

be a cocharacter defined over $\mathcal{O}$. In this section, we introduce the display group $G_\mu(A, I)$ for an orientable and bounded $\mathcal{O}_E$-prism (A, I) over $\mathcal{O}$. The display group will be used in the definition of prismatic G-μ-displays.

4.1. *Definition of the display group.* Let A be an $\mathcal{O}$-algebra with an ideal $I \subset A$ which is generated by a nonzerodivisor $d \in A$.

Definition 4.1.1. We define

$$G_\mu(A, I) := \{g \in G(A) \mid \mu(d) g \mu(d)^{-1} \text{ lies in } G(A) \subset G(A[1/I])\}.$$

The group $G_\mu(A, I)$ is called the *display group*. We note that $G_\mu(A, I)$ does not depend on the choice of d.

Remark 4.1.2. The definition of the display group given here is a translation of the one given in [Lau 2021] to our setting; see Remark 5.2.3 for details. If G is reductive and μ is minuscule, such a group was also considered in [Bültel and Pappas 2020].

For the cocharacter $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$, we endow $G_{\mathcal{O}}$ with the action of $\mathbb{G}_m$ defined by

$$G_{\mathcal{O}}(R) \times \mathbb{G}_m(R) \to G_{\mathcal{O}}(R), \quad (g, t) \mapsto \mu(t)^{-1} g \mu(t), \tag{4-1}$$

for every $\mathcal{O}$-algebra R. We note that this action is the inverse of the one used in Definition 4.1.1. We write $G = \operatorname{Spec} A'_G$ and $A_G := A'_G \otimes_{\mathcal{O}_E} \mathcal{O}$, so that $G_{\mathcal{O}} = \operatorname{Spec} A_G$. Let

$$A_G = \bigoplus_{i \in \mathbb{Z}} A_{G,i}$$

be the weight decomposition with respect to the action of $\mathbb{G}_m$. An element $t \in \mathbb{G}_m(R) = R^\times$ acts on $A_{G,i} \otimes_{\mathcal{O}} R$ by $x \mapsto t^i x$.

Remark 4.1.3. Let R be an $\mathcal{O}$-algebra. For any $t \in \mathbb{G}_m(R)$ and any $g \in G_{\mathcal{O}}(R)$ with corresponding homomorphism $g^* : A_G \to R$, the homomorphism

$$(\mu(t)^{-1} g \mu(t))^* : A_G \to R$$

corresponding to $\mu(t)^{-1} g \mu(t) \in G_{\mathcal{O}}(R)$ sends an element $x \in A_{G,i}$ to $t^i g^*(x) \in R$.

Lemma 4.1.4. *Let $g \in G(A)$ be an element. Then $g \in G_\mu(A, I)$ if and only if $g^*(x) \in I^i A$ for every $i > 0$ and every $x \in A_{G,i}$.*

Proof. This follows from Remark 4.1.3. □

Example 4.1.5. Assume that $G = \mathrm{GL}_n$. Let $\mu : \mathbb{G}_m \to \mathrm{GL}_{n,\mathcal{O}}$ be a cocharacter and let $(m_1, \dots, m_n)$ be the corresponding tuple of integers $m_1 \geq \cdots \geq m_n$ as in Section 3.2. Let $\{\mathrm{Fil}^i_\mu\}_{i \in \mathbb{Z}}$ be the filtration of $M := A^n$ defined as in Remark 3.2.2. Then we have

$$(\mathrm{GL}_n)_\mu(A, I) = \{g \in \mathrm{GL}_n(A) \mid g(\mathrm{Fil}^i_\mu) = \mathrm{Fil}^i_\mu \text{ for every } i \in \mathbb{Z}\}.$$

Let $d \in I$ be a generator. For any $g \in (\mathrm{GL}_n)_\mu(A, I)$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Fil}^{m_1}_\mu & \xrightarrow{\quad g \quad} & \mathrm{Fil}^{m_1}_\mu \\ \simeq\downarrow & & \downarrow\simeq \\ M & \xrightarrow{\mu(d) g \mu(d)^{-1}} & M \end{array}$$

where $\mathrm{Fil}^{m_1}_\mu \xrightarrow{\sim} M$ is defined by $d^{-m_1}\mu(d)$.

4.2. *Properties of the display group.* For the cocharacter $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$, we consider the closed subgroup schemes $P_\mu, U^-_\mu \subset G_{\mathcal{O}}$ over $\mathcal{O}$ defined by, for every $\mathcal{O}$-algebra R,

$$P_\mu(R) = \bigl\{g \in G(R) \mid \lim_{t \to 0} \mu(t) g \mu(t)^{-1} \text{ exists}\bigr\},$$
$$U^-_\mu(R) = \bigl\{g \in G(R) \mid \lim_{t \to 0} \mu(t)^{-1} g \mu(t) = 1\bigr\}.$$

(We refer to [Conrad et al. 2015, Lemma 2.1.4] for the definition of $\lim_{t \to 0} \mu(t) g \mu(t)^{-1}$.) We see that P_μ and U^-_μ are stable under the action of $\mathbb{G}_m$ on $G_{\mathcal{O}}$ given by (4-1). The group schemes P_μ and U^-_μ are smooth over $\mathcal{O}$. Moreover, the multiplication map

$$U^-_\mu \times_{\operatorname{Spec} \mathcal{O}} P_\mu \to G_{\mathcal{O}}$$

is an open immersion. See [Conrad et al. 2015, Section 2.1], especially Proposition 2.1.8 of that work, for details.

Remark 4.2.1. We have employed slightly different notation than in [Lau 2021]. For example, in that work, the subgroup P_μ (resp. U^-_μ) is denoted by P^- (resp. U^+).

Lemma 4.2.2. (1) *Let R be an $\mathcal{O}$-algebra and $g \in G_{\mathcal{O}}(R)$ an element. Then $g \in P_\mu(R)$ if and only if $g^*(x) = 0$ for every $i > 0$ and every $x \in A_{G,i}$.*

(2) *We have $P_\mu(A) \subset G_\mu(A, I)$, and the image of $G_\mu(A, I)$ in $G(A/I)$ under the projection $G(A) \to G(A/I)$ is contained in $P_\mu(A/I)$. Moreover $\mu(d) P_\mu(A) \mu(d)^{-1}$ is contained in $P_\mu(A)$.*

Proof. Remark 4.1.3 immediately implies (1). Assertion (2) follows from (1) and Lemma 4.1.4. □

Definition 4.2.3 [Lau 2021, Definition 6.3.1]. The action of $\mathbb{G}_m$ on $G_{\mathcal{O}}$ given in (4-1) induces an action of $\mathbb{G}_m$ on the Lie algebra $\mathrm{Lie}(G_{\mathcal{O}})$. Let

$$\mathrm{Lie}(G_{\mathcal{O}}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$$

be the weight decomposition with respect to the action of $\mathbb{G}_m$. We say that the cocharacter $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ is 1-*bounded* if $\mathfrak{g}_i = 0$ for $i \geq 2$.

In general, the Lie algebra $\mathrm{Lie}(U_\mu^-)$ of U_μ^- coincides with $\bigoplus_{i\geq 1} \mathfrak{g}_i$. (We also note that $\mathrm{Lie}(P_\mu) = \bigoplus_{i\leq 0} \mathfrak{g}_i$.) Thus μ is 1-bounded if and only if $\mathrm{Lie}(U_\mu^-) = \mathfrak{g}_1$.

Remark 4.2.4. If G is a reductive group scheme over $\mathcal{O}_E$, then μ is 1-bounded if and only if μ is minuscule, that is, the equality $\mathrm{Lie}(G_{\mathcal{O}}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ holds.

Example 4.2.5. Assume that $G = \mathrm{GL}_n$. The cocharacter $\mathbb{G}_m \to \mathrm{GL}_{n,\mathcal{O}}$ defined by

$$t \mapsto \mathrm{diag}\,(\underbrace{t^m, \dots, t^m}_{s}, \underbrace{t^{m-1}, \dots, t^{m-1}}_{n-s})$$

for some integers m and s $(0 \leq s \leq n)$ is 1-bounded. In fact, any 1-bounded cocharacter of $\mathrm{GL}_{n,\mathcal{O}}$ is conjugate to a cocharacter of this form.

For a free $\mathcal{O}$-module M of finite rank, we let $V(M)$ denote the group scheme over $\mathcal{O}$ defined by $R \mapsto M \otimes_{\mathcal{O}} R$ for every $\mathcal{O}$-algebra R.

Lemma 4.2.6. *There exists a* $\mathbb{G}_m$*-equivariant isomorphism*

$$\log : U_\mu^- \xrightarrow{\sim} V(\mathrm{Lie}(U_\mu^-))$$

of schemes over $\mathcal{O}$ *which induces the identity on the Lie algebras. If* μ *is* 1*-bounded, then the isomorphism* log *is unique, and it is an isomorphism of group schemes over* $\mathcal{O}$.

Proof. The same arguments as in the proofs of [Lau 2021, Lemmas 6.1.1 and 6.3.2] work here. □

Remark 4.2.7. (1) An isomorphism $\log : U_\mu^- \xrightarrow{\sim} V(\mathrm{Lie}(U_\mu^-))$ as in Lemma 4.2.6 induces a bijection

$$U_\mu^-(A) \cap G_\mu(A, I) \xrightarrow{\sim} \bigoplus_{i\geq 1} I^i(\mathfrak{g}_i \otimes_{\mathcal{O}} A).$$

(2) If μ is 1-bounded, then we identify U_μ^- with $V(\mathrm{Lie}(U_\mu^-))$ by the unique isomorphism log. In particular, we view $\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A$ as a subgroup of $G(A)$. We then obtain

$$I(\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A) = (\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A) \cap G_\mu(A, I).$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} I(\mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A) & \hookrightarrow & G_\mu(A, I) \\ \downarrow{\scriptstyle dv\mapsto v} & & \downarrow{\scriptstyle g\mapsto \mu(d)g\mu(d)^{-1}} \\ \mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} A & \hookrightarrow & G(A) \end{array}$$

Proposition 4.2.8 (cf. [Lau 2021, Lemma 6.2.2]). *Assume that A is I-adically complete. Then the multiplication map*

$$(U_\mu^-(A) \cap G_\mu(A, I)) \times P_\mu(A) \to G_\mu(A, I) \tag{4-2}$$

is bijective.

Proof. Since $P_\mu(A) \subset G_\mu(A, I)$ by Lemma 4.2.2, the map (4-2) is well-defined. Since the map $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu \to G_\mathcal{O}$ is an open immersion, the map (4-2) is injective.

We shall show that the map (4-2) is surjective. Let $g \in G_\mu(A, I)$ be an element. By Lemma 4.2.2, the image of g in $G(A/I)$ is contained in $P_\mu(A/I)$. Since P_μ is smooth and A is I-adically complete, there exists an element $t \in P_\mu(A)$ whose image in $P_\mu(A/I)$ coincides with the image of g. The restriction of the morphism $gt^{-1} : \operatorname{Spec} A \to G_\mathcal{O}$ to $\operatorname{Spec} A/I$ factors through the open subscheme $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu$. Since $I \subset \operatorname{rad}(A)$, it follows that $gt^{-1} : \operatorname{Spec} A \to G_\mathcal{O}$ itself factors through $U_\mu^- \times_{\operatorname{Spec} \mathcal{O}} P_\mu$. In other words, there are elements $u \in U_\mu^-(A)$ and $t' \in P_\mu(A)$ such that $g = ut't$. We note that $u \in G_\mu(A, I)$. In conclusion, we have shown that g is contained in the image of the map (4-2). □

Proposition 4.2.9. *Assume that $\mu : \mathbb{G}_m \to G_\mathcal{O}$ is 1-bounded. Assume further that A is I-adically complete. Then the multiplication map*

$$I(\operatorname{Lie}(U_\mu^-) \otimes_\mathcal{O} A) \times P_\mu(A) \to G_\mu(A, I) \tag{4-3}$$

is bijective. Moreover $G_\mu(A, I)$ coincides with the inverse image of $P_\mu(A/I)$ in $G(A)$ under the projection $G(A) \to G(A/I)$, and we have the bijection

$$G(A)/G_\mu(A, I) \xrightarrow{\sim} G(A/I)/P_\mu(A/I). \tag{4-4}$$

Proof. It follows from Remark 4.2.7 and Proposition 4.2.8 that the map (4-3) is bijective. Let $G'_\mu \subset G(A)$ be the inverse image of $P_\mu(A/I)$. We have $G_\mu(A, I) \subset G'_\mu$. By the same argument as in the proof of Proposition 4.2.8, one can show that $G'_\mu \subset I(\operatorname{Lie}(U_\mu^-) \otimes_\mathcal{O} A) \times P_\mu(A)$. Thus, we obtain $G_\mu(A, I) = G'_\mu$.

It remains to prove that the map (4-4) is bijective. Since G is smooth and A is I-adically complete, the projection $G(A) \to G(A/I)$ is surjective, which in turn implies the surjectivity of (4-4). The injectivity follows from the equality $G_\mu(A, I) = G'_\mu$. □

For an integer $m \geq 0$, let $G^{\geq m}(A)$ be the kernel of $G(A) \to G(A/I^m)$. We set

$$G_\mu^{\geq m}(A, I) := G_\mu(A, I) \cap G^{\geq m}(A).$$

We record a structural result about the quotient $G_\mu^{\geq m}(A, I)/G_\mu^{\geq m+1}(A, I)$.

Lemma 4.2.10. *Assume that A is I-adically complete. Then we have the isomorphisms of groups*

$$G^{\geq m}(A)/G^{\geq m+1}(A) \simeq \begin{cases} G(A/I) & (m = 0), \\ \operatorname{Lie}(G_\mathcal{O}) \otimes_\mathcal{O} I^m/I^{m+1} & (m \geq 1), \end{cases}$$

$$G_\mu^{\geq m}(A, I)/G_\mu^{\geq m+1}(A, I) \simeq \begin{cases} P_\mu(A/I) & (m = 0), \\ \left(\bigoplus_{i \leq m} \mathfrak{g}_i\right) \otimes_\mathcal{O} I^m/I^{m+1} & (m \geq 1). \end{cases}$$

Proof. Since A is I-adically complete and G is smooth, the map $G(A) \to G(A/I^m)$ is surjective. It follows that $G^{\geq m}(A)/G^{\geq m+1}(A)$ is isomorphic to the kernel $\mathrm{Ker}(G(A/I^{m+1}) \to G(A/I^m))$ of $G(A/I^{m+1}) \to G(A/I^m)$. This is equal to $G(A/I)$ when $m = 0$. If $m \geq 1$, then we have a canonical identification

$$\mathrm{Ker}(G(A/I^{m+1}) \to G(A/I^m)) = \mathrm{Lie}(G_{\mathcal{O}}) \otimes_{\mathcal{O}} I^m/I^{m+1}$$

since $I^m/I^{m+1} \subset A/I^{m+1}$ is a square zero ideal. This proves the first assertion.

Since $P_\mu(A) \to P_\mu(A/I)$ is surjective (as A is I-adically complete and P_μ is smooth), it follows from Lemma 4.2.2 that $G_\mu(A, I)/G_\mu^{\geq 1}(A, I) \simeq P_\mu(A/I)$. To prove the second assertion, it then suffices to show that the image of the natural homomorphism

$$G_\mu^{\geq m}(A, I) \to \mathrm{Ker}(G(A/I^{m+1}) \to G(A/I^m)) = \mathrm{Lie}(G_{\mathcal{O}}) \otimes_{\mathcal{O}} I^m/I^{m+1}$$

is $\left(\bigoplus_{i \leq m} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m/I^{m+1}$ for any $m \geq 1$. By Proposition 4.2.8, we may identify $G_\mu^{\geq m}(A, I)$ with

$$(U_\mu^-(A) \cap G_\mu^{\geq m}(A, I)) \times P_\mu^{\geq m}(A),$$

where $P_\mu^{\geq m}(A) := P_\mu(A) \cap G^{\geq m}(A)$. By the same argument as above, we have

$$P_\mu^{\geq m}(A)/P_\mu^{\geq m+1}(A) \simeq \mathrm{Lie}(P_\mu) \otimes_{\mathcal{O}} I^m/I^{m+1} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m/I^{m+1}.$$

It now suffices to prove that the image of the natural homomorphism

$$U_\mu^-(A) \cap G_\mu^{\geq m}(A, I) \to \mathrm{Ker}(U_\mu^-(A/I^{m+1}) \to U_\mu^-(A/I^m)) = \mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} I^m/I^{m+1} \tag{4-5}$$

is $\left(\bigoplus_{1 \leq i \leq m} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m/I^{m+1}$. For this, we fix an isomorphism $\log : U_\mu^- \xrightarrow{\sim} V(\mathrm{Lie}(U_\mu^-))$ as in Lemma 4.2.6. Since log induces the identity on the Lie algebras, the isomorphism

$$\mathrm{Ker}(U_\mu^-(A/I^{m+1}) \to U_\mu^-(A/I^m)) \xrightarrow{\sim} \mathrm{Lie}(U_\mu^-) \otimes_{\mathcal{O}} I^m/I^{m+1}$$

induced by log is the same as the one in (4-5). Since log induces

$$U_\mu^-(A) \cap G_\mu^{\geq m}(A, I) \xrightarrow{\sim} \left(\bigoplus_{1 \leq i \leq m} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^m \oplus \left(\bigoplus_{i \geq m+1} \mathfrak{g}_i\right) \otimes_{\mathcal{O}} I^i,$$

by Remark 4.2.7(1), the result follows. □

4.3. ***Display groups on prismatic sites.*** In this subsection, for a bounded $\mathcal{O}_E$-prism (A, I) over $\mathcal{O}$, we define the display group sheaf $G_{\mu,A,I}$ on the site $(A, I)_{\text{ét}}^{\text{op}}$ and discuss some basic results on $G_{\mu,A,I}$-torsors.

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. We begin with a comparison result between torsors over $\mathrm{Spec}\, A$ (or $\mathrm{Spec}\, A/I$) with respect to the usual étale topology, and torsors on the sites $(A, I)_{\text{ét}}^{\text{op}}$ and $(A, I)_{\Delta}^{\text{op}}$ from Section 2.5. To an affine scheme X over $\mathcal{O}$ (or A), we attach a functor

$$X_{\Delta,A} : (A, I)_{\Delta} \to \mathrm{Set}, \quad (B, J) \mapsto X(B).$$

This forms a sheaf (with respect to the flat topology) by Remark 2.5.5 since $X(B)$ can be regarded as the set of homomorphisms $R \to B$ of $\mathcal{O}$-algebras (or A-algebras) where $X = \mathrm{Spec}\, R$. Similarly, to an affine scheme X over $\mathcal{O}$ (or A/I), we attach a sheaf

$$X_{\bar{\Delta},A} : (A, I)_{\Delta} \to \mathrm{Set}, \quad (B, J) \mapsto X(B/J).$$

The restrictions of these sheaves to $(A, I)_{\text{ét}}$ are denoted by the same notation (see also Remark 2.5.13).

Proposition 4.3.1. *Let H be a smooth affine group scheme over $\mathcal{O}$.*

(1) *For an $H_{A/I}$-torsor $\mathcal{P}$ over* Spec A/I *with respect to the étale topology, which is an affine scheme over A/I, the sheaf $\mathcal{P}_{\bar{\Delta},A}$ on $(A, I)_{\Delta}^{\mathrm{op}}$ is an $H_{\bar{\Delta},A}$-torsor with respect to the flat topology. The functor*

$$\mathcal{P} \mapsto \mathcal{P}_{\bar{\Delta},A}$$

is an equivalence from the groupoid of $H_{A/I}$-torsors over Spec A/I *to the groupoid of $H_{\bar{\Delta},A}$-torsors on $(A, I)_{\Delta}^{\mathrm{op}}$. The same holds if we replace $(A, I)_{\Delta}^{\mathrm{op}}$ by $(A, I)_{\text{ét}}^{\mathrm{op}}$.*

(2) *The construction*

$$\mathcal{P} \mapsto \mathcal{P}_{\Delta,A}$$

gives an equivalence from the groupoid of H_A-torsors over Spec A *to the groupoid of $H_{\Delta,A}$-torsors on $(A, I)_{\Delta}^{\mathrm{op}}$. The same holds if we replace $(A, I)_{\Delta}^{\mathrm{op}}$ by $(A, I)_{\text{ét}}^{\mathrm{op}}$.*

Proof. (1) It follows from Lemma 2.5.9 that $\mathcal{P}_{\bar{\Delta},A}$ is trivialized by a (π, I)-completely étale covering of A. Thus $\mathcal{P}_{\bar{\Delta},A}$ is an $H_{\bar{\Delta},A}$-torsor on both $(A, I)_{\Delta}^{\mathrm{op}}$ and $(A, I)_{\text{ét}}^{\mathrm{op}}$. It then suffices to prove that the fibered category over $(A, I)_{\Delta}^{\mathrm{op}}$ which associates to each $(B, J) \in (A, I)_{\Delta}$ the groupoid of $H_{B/J}$-torsors over Spec B/J is a stack with respect to the flat topology.

It is known that, for any $\mathcal{O}$-algebra R, the groupoid of H_R-torsors over Spec R is equivalent to the groupoid of exact tensor functors $\mathrm{Rep}_{\mathcal{O}}(H) \to \mathrm{Vect}(R)$, where $\mathrm{Rep}_{\mathcal{O}}(H)$ is the category of algebraic representations of H on free $\mathcal{O}$-modules of finite rank, and $\mathrm{Vect}(R)$ is the category of finite projective R-modules; see [Scholze and Weinstein 2020, Theorem 19.5.1] and [Broshi 2013, Theorem 1.2]. (Although this result is stated only for the case where $\mathcal{O} = \mathbb{Z}_p$ in [Scholze and Weinstein 2020, Theorem 19.5.1], the proof also works for general $\mathcal{O}$.) Using this Tannakian perspective, the desired claim follows from Proposition 2.5.6 and the following fact: For a π-completely faithfully flat homomorphism $C \to C'$ of π-adically complete $\mathcal{O}$-algebras, a complex

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of finite projective C-modules is exact if the base change

$$0 \to M_1 \otimes_C C' \to M_2 \otimes_C C' \to M_3 \otimes_C C' \to 0$$

is exact. (This fact follows from the following criterion: A complex $0 \to M_1 \to M_2 \to M_3 \to 0$ of finite projective modules over a ring C is exact if for every closed point $x \in \operatorname{Spec} C$, its base change to the residue field $k(x)$ is exact.)

(2) This can be proved in the same way as (1). □

Definition 4.3.2. Let $(A, I)_{\Delta,\mathrm{ori}}$ be the category of *orientable* and bounded $\mathcal{O}_E$-prisms (B, J) with a map $(A, I) \to (B, J)$. We endow $(A, I)^{\mathrm{op}}_{\Delta,\mathrm{ori}}$ with the flat topology. If (A, I) is orientable, then we have $(A, I)_{\Delta,\mathrm{ori}} = (A, I)_{\Delta}$.

Remark 4.3.3. By Remark 2.5.14, the objects in $(A, I)^{\mathrm{op}}_{\Delta,\mathrm{ori}}$ form a basis for $(A, I)^{\mathrm{op}}_{\Delta}$. We may identify sheaves on $(A, I)^{\mathrm{op}}_{\Delta,\mathrm{ori}}$ with sheaves on $(A, I)^{\mathrm{op}}_{\Delta}$.

Definition 4.3.4. We define the functor

$$G_{\Delta,A} : (A, I)_{\Delta} \to \mathrm{Set}, \quad (B, J) \mapsto G(B).$$

As explained above, the functor $G_{\Delta,A}$ forms a group sheaf. We also define the functor

$$G_{\mu,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G_\mu(B, J).$$

Since the functor $(A, I)_{\Delta} \to \mathrm{Set}$, $(B, J) \mapsto G(B[1/J])$ forms a group sheaf (Remark 3.1.14), it follows that $G_{\mu,A,I}$ forms a group sheaf. We regard $G_{\mu,A,I}$ as a group sheaf on $(A, I)^{\mathrm{op}}_{\Delta}$. The restrictions of $G_{\Delta,A}$ and $G_{\mu,A,I}$ to $(A, I)_{\text{ét}}$ will be denoted by the same notation.

We remark that Proposition 4.3.1 cannot be applied directly to $G_{\mu,A,I}$-torsors. However, it is still useful for analyzing $G_{\mu,A,I}$-torsors in several places below, since we have the following lemma. For the notation used below, see Lemma 4.2.10.

Lemma 4.3.5. (1) *For an integer $m \geq 0$, the functor*

$$G^{=m}_{\mu,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G^{\geq m}_\mu(B, J)/G^{\geq m+1}_\mu(B, J),$$

forms a group sheaf, and it is isomorphic to $(P_\mu)_{\bar{\Delta},A}$ (resp. $V\big(\bigoplus_{i\leq m} \mathfrak{g}_i\big)_{\bar{\Delta},A}$) if $m = 0$ (resp. $m \geq 1$). Moreover, the functor

$$G^{<m}_{\mu,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G_\mu(B, J)/G^{\geq m}_\mu(B, J),$$

forms a group sheaf.

(2) *For a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\Delta}$, we write $\mathcal{Q}^{<m}$ for the pushout of $\mathcal{Q}$ along $G_{\mu,A,I} \to G^{<m}_{\mu,A,I}$* (see Remark 4.3.6 below). *Then we have $\mathcal{Q} \xrightarrow{\sim} \varprojlim_m \mathcal{Q}^{<m}$. The same holds for $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\text{ét}}$.*

Proof. (1) The statement about $G^{=m}_{\mu,A,I}$ follows from Lemma 4.2.10. Using the exact sequence

$$1 \mapsto G^{=m}_{\mu,A,I}(B) \to G^{<m+1}_{\mu,A,I}(B) \to G^{<m}_{\mu,A,I}(B) \to 1,$$

the statement about $G^{<m}_{\mu,A,I}$ then follows by induction on m.

(2) We may assume that $\mathcal{Q}$ is a trivial $G_{\mu,A,I}$-torsor and (A, I) is orientable. Then it is enough to prove that $G_{\mu,A,I} \xrightarrow{\sim} \varprojlim_m G^{<m}_{\mu,A,I}$ on $(A, I)^{\mathrm{op}}_{\Delta}$. By Proposition 4.2.8, the multiplication map

$$(U^-_\mu(A) \cap G_\mu(A, I)) \times P_\mu(A) \to G_\mu(A, I)$$

is bijective. Note that $G_\mu(A, I)/G^{\geq m}_\mu(A, I)$ can be identified with the image of $G_\mu(A, I)$ in $G(A/I^m)$. Let $U^{<m}$ be the image of $U^-_\mu(A) \cap G_\mu(A, I)$ in $U^-_\mu(A/I^m)$. Then the multiplication map induces a

bijection

$$U^{<m} \times P_\mu(A/I^m) \xrightarrow{\sim} G_\mu(A, I)/G_\mu^{\geq m}(A, I).$$

We have $P_\mu(A) \xrightarrow{\sim} \varprojlim P_\mu(A/I^m)$. Moreover, using Lemma 4.2.6, one can check that

$$U_\mu^-(A) \cap G_\mu(A, I) \xrightarrow{\sim} \varprojlim U^{<m}.$$

Thus, we obtain $G_\mu(A, I) \xrightarrow{\sim} \varprojlim G_\mu(A, I)/G_\mu^{\geq m}(A, I)$. The same assertion holds for any $(B, J) \in (A, I)_{\Delta}$, and hence $G_{\mu,A,I} \xrightarrow{\sim} \varprojlim G_{\mu,A,I}^{<m}$. □

Remark 4.3.6. Let $f : H \to H'$ be a homomorphism of groups and Q a set with an H-action. We can attach to Q a set Q^f with an H'-action and an H-equivariant map $Q \to Q^f$ with the following universal property: for any set Q' with an H'-action and any H-equivariant map $Q \to Q'$, the map $Q \to Q'$ factors through a unique H'-equivariant map $Q^f \to Q'$. Explicitly, we can define Q^f as the contracted product

$$Q^f = (Q \times H')/H,$$

where the action of an $h \in H$ on $Q \times H'$ is defined by $(x, h') \mapsto (xh, f(h)^{-1}h')$. We call Q^f the pushout of Q along $f : H \to H'$.

Similarly, for a homomorphism $f : H \to H'$ of group sheaves on a site and a sheaf Q with an action of H, we can form the pushout Q^f with the same properties as above. If Q is an H-torsor, then Q^f is an H'-torsor.

We will use the following notation. Let us denote the inclusion $G_{\mu,A,I} \hookrightarrow G_{\Delta,A}$ by τ. The composition of τ with the projection map $G_{\Delta,A} \to G_{\bar{\Delta},A}$ is denoted by $\bar{\tau}$. (Here $G_{\bar{\Delta},A} := (G_{\mathcal{O}})_{\bar{\Delta},A}$.) By Lemma 4.2.2, the homomorphism $\bar{\tau}$ factors through a homomorphism $\bar{\tau}_P : G_{\mu,A,I} \to (P_\mu)_{\bar{\Delta},A}$. In summary, we have the following commutative diagram of group sheaves on $(A, I)_{\Delta}^{\mathrm{op}}$ (or on $(A, I)_{\text{ét}}^{\mathrm{op}}$):

$$\begin{array}{ccc} G_{\mu,A,I} & \xrightarrow{\tau} & G_{\Delta,A} \\ {\scriptstyle \bar{\tau}_P}\downarrow & \searrow^{\bar{\tau}} & \downarrow \\ (P_\mu)_{\bar{\Delta},A} & \longrightarrow & G_{\bar{\Delta},A} \end{array} \tag{4-6}$$

Corollary 4.3.7. *A $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)_{\Delta}^{\mathrm{op}}$ is trivial if the pushout of $\mathcal{Q}$ along $\bar{\tau}_P : G_{\mu,A,I} \to (P_\mu)_{\bar{\Delta},A}$ is trivial as a $(P_\mu)_{\bar{\Delta},A}$-torsor on $(A, I)_{\Delta}^{\mathrm{op}}$. The same holds if we replace $(A, I)_{\Delta}^{\mathrm{op}}$ by $(A, I)_{\text{ét}}^{\mathrm{op}}$.*

Proof. We prove the assertion for $(A, I)_{\Delta}^{\mathrm{op}}$; the argument for $(A, I)_{\text{ét}}^{\mathrm{op}}$ is similar. By Lemma 4.3.5(2), it suffices to show that $\mathcal{Q}^{<m}$ is trivial as a $G_{\mu,A,I}^{<m}$-torsor for any m. We proceed by induction on m. The assertion is true for $m = 1$ by our assumption. We assume that $\mathcal{Q}^{<m}$ is trivial for an integer $m \geq 1$, so that there exists an element $x \in \mathcal{Q}^{<m}(A)$. The fiber of the morphism $\mathcal{Q}^{<m+1} \to \mathcal{Q}^{<m}$ at x is a $G_{\mu,A,I}^{=m}$-torsor. Lemma 4.3.5(1) shows that $G_{\mu,A,I}^{=m} \simeq V\big(\bigoplus_{i \leq m} \mathfrak{g}_i\big)_{\bar{\Delta},A}$. By Proposition 4.3.1, the fiber arises from a $V\big(\bigoplus_{i \leq m} \mathfrak{g}_i\big)_{A/I}$-torsor over $\operatorname{Spec} A/I$, which is trivial since $\operatorname{Spec} A/I$ is affine. This implies that the $G_{\mu,A,I}^{<m+1}$-torsor $\mathcal{Q}^{<m+1}$ is trivial, as desired. □

Corollary 4.3.8. *A $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\Delta}$ is trivialized by a (π, I)-completely étale covering $A \to B$, i.e., the restriction of $\mathcal{Q}$ to $(B, IB)^{\mathrm{op}}_{\Delta}$ is trivial. Moreover, the restriction functor induces an equivalence from the groupoid of $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\Delta}$ to the groupoid of $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\text{ét}}$.*

Proof. The first assertion follows from Corollary 4.3.7 since any $(P_\mu)_{\bar{\Delta},A}$-torsor $\mathcal{P}$ on $(A, I)^{\mathrm{op}}_{\Delta}$ arises from a $(P_\mu)_{A/I}$-torsor over Spec A/I with respect to the étale topology, which in turn implies that $\mathcal{P}$ is trivialized by a (π, I)-completely étale covering $A \to B$ (see also Lemma 2.5.9). The second assertion is a formal consequence of the first one. □

Remark 4.3.9. To a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\Delta}$, we can associate the $G_{\Delta,A}$-torsor $\mathcal{Q}^\tau$ and the $(P_\mu)_{\bar{\Delta},A}$-torsor $\mathcal{Q}^{\bar{\tau}_P}$ on $(A, I)^{\mathrm{op}}_{\Delta}$, and there is a canonical isomorphism between the $G_{\bar{\Delta},A}$-torsors associated with $\mathcal{Q}^\tau$ and $\mathcal{Q}^{\bar{\tau}_P}$. We assume that μ is 1-bounded. Then, by Proposition 4.2.9, this construction induces an equivalence from the groupoid of $G_{\mu,A,I}$-torsors on $(A, I)^{\mathrm{op}}_{\Delta}$ to the groupoid of triples consisting of a $G_{\Delta,A}$-torsor, a $(P_\mu)_{\bar{\Delta},A}$-torsor, and an isomorphism between the $G_{\bar{\Delta},A}$-torsors associated with them. The same holds if we replace $(A, I)^{\mathrm{op}}_{\Delta}$ by $(A, I)^{\mathrm{op}}_{\text{ét}}$. Corollaries 4.3.7 and 4.3.8 also follow from this fact and Proposition 4.3.1 when μ is 1-bounded.

5. Prismatic G-μ-displays

In this section, we come to the heart of this paper, namely prismatic G-μ-displays. We first discuss the notion of G-Breuil–Kisin modules of type μ in Section 5.1. Then we introduce and study prismatic G-μ-displays in Sections 5.2–5.7. Our prismatic G-μ-displays are essentially equivalent to G-Breuil–Kisin modules of type μ, and the latter may be more familiar to readers. Nevertheless, in many cases, such as the proof of the main result (Theorem 6.1.3) of this paper, it will be crucial to work with prismatic G-μ-displays.

We retain the notation of Section 4. Recall that G is a smooth affine group scheme over $\mathcal{O}_E$ and $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ is a cocharacter defined over $\mathcal{O} = W(k) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.

5.1. *G-Breuil–Kisin modules of type μ.*

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Definition 5.1.1. A *G-Breuil–Kisin module* over (A, I) is a pair $(\mathcal{P}, F_{\mathcal{P}})$ consisting of a G_A-torsor $\mathcal{P}$ over Spec A (with respect to the étale topology) and an isomorphism

$$F_{\mathcal{P}} : (\phi^*\mathcal{P})[1/I] \xrightarrow{\sim} \mathcal{P}[1/I]$$

of $G_{A[1/I]}$-torsors over Spec $A[1/I]$.

Here, for a G_A-torsor $\mathcal{P}$ over Spec A, we let $\phi^*\mathcal{P}$ denote the base change of $\mathcal{P}$ along the Frobenius $\phi : A \to A$. Since ϕ is $\mathcal{O}_E$-linear and G is defined over $\mathcal{O}_E$, we have $\phi^*G_A = G_A$, and hence $\phi^*\mathcal{P}$ is a G_A-torsor over Spec A. Moreover, we write $\mathcal{P}[1/I] := \mathcal{P} \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A[1/I]$. When there is no ambiguity, we simply write $\mathcal{P} = (\mathcal{P}, F_{\mathcal{P}})$.

Example 5.1.2. Assume that $G = \mathrm{GL}_n$. Let (M, F_M) be a Breuil–Kisin module of rank n over (A, I). Let

$$\mathcal{P}(M) := \underline{\mathrm{Isom}}(A^n, M)$$

be the $\mathrm{GL}_{n,A}$-torsor over Spec A defined by sending an A-algebra B to the set of isomorphisms $B^n \simeq M_B$. Together with the isomorphism $(\phi^*\mathcal{P}(M))[1/I] \xrightarrow{\sim} \mathcal{P}(M)[1/I]$ induced by F_M, we regard $\mathcal{P}(M)$ as a GL_n-Breuil–Kisin module over (A, I). This construction $M \mapsto \mathcal{P}(M)$ induces an equivalence between the groupoid of Breuil–Kisin modules of rank n over (A, I) and the groupoid of GL_n-Breuil–Kisin modules over (A, I).

Remark 5.1.3. Let $\mathcal{P}$ and $\mathcal{P}'$ be G_A-torsors over Spec A. Using that the functor $(\mathcal{O}_E)_{\mathbb{\Delta},\mathcal{O}_E} \to \mathrm{Set}$, $(B, J) \mapsto B[1/J]$, forms a sheaf (see Remark 3.1.14) and that $\mathcal{P}, \mathcal{P}'$ are affine and flat over Spec A, one can show that the functor $(A, I)_{\mathbb{\Delta}} \to \mathrm{Set}$ which associates to each $(B, J) \in (A, I)_{\mathbb{\Delta}}$ the set of isomorphisms $\mathcal{P}_B[1/J] \xrightarrow{\sim} \mathcal{P}'_B[1/J]$ of $G_{B[1/J]}$-torsors forms a sheaf. This, together with Proposition 4.3.1, implies that the fibered category over $(A, I)_{\mathbb{\Delta}}^{\mathrm{op}}$ which associates to each $(B, J) \in (A, I)_{\mathbb{\Delta}}$ the groupoid of G-Breuil–Kisin modules over (B, J) is a stack with respect to the flat topology.

We introduce G-Breuil–Kisin modules of type μ. Recall that for a (π, I)-completely étale A-algebra $B \in (A, I)_{\text{ét}}$, the pair (B, IB) is naturally a bounded $\mathcal{O}_E$-prism; see Lemma 2.5.10.

Definition 5.1.4 (G-Breuil–Kisin module of type μ). We say that a G-Breuil–Kisin module $(\mathcal{P}, F_{\mathcal{P}})$ over (A, I) is *of type* μ if there exists a (π, I)-completely étale covering $A \to B$ such that (B, IB) is orientable, the base change $\mathcal{P}_B$ is a trivial G_B-torsor, and via some (and hence any) trivialization $\mathcal{P}_B \simeq G_B$, the isomorphism $F_{\mathcal{P}}$ is given by $g \mapsto Yg$ for an element Y in the double coset

$$G(B)\mu(d)G(B) \subset G(B[1/IB]),$$

where $d \in IB$ is a generator. If these conditions are satisfied for $B = A$, then we say that $(\mathcal{P}, F_{\mathcal{P}})$ is *banal*.

We write

$$G\text{-}\mathrm{BK}_\mu(A, I) \quad \text{and} \quad G\text{-}\mathrm{BK}_\mu(A, I)_{\mathrm{banal}}$$

for the groupoid of G-Breuil–Kisin modules of type μ over (A, I) and the groupoid of banal G-Breuil–Kisin modules of type μ over (A, I) (when (A, I) is orientable), respectively.

Remark 5.1.5. By Remark 5.1.3, the fibered category over $(A, I)_{\text{ét}}^{\mathrm{op}}$ which associates to each $B \in (A, I)_{\text{ét}}$ the groupoid of G-Breuil–Kisin modules of type μ over (B, IB) is a stack with respect to the (π, I)-completely étale topology. We will prove that the same result holds for the flat topology in Corollary 5.3.9 below, using G-μ-displays introduced in the next subsection.

Example 5.1.6. Let M be a Breuil–Kisin module of rank n over (A, I) and let $\mathcal{P}(M)$ be the associated GL_n-Breuil–Kisin module over (A, I) (see Example 5.1.2). If $\mathcal{P}(M)$ is of type μ, then M is of type μ in the sense of Definition 3.2.1 by Proposition 3.1.13. We will prove that the converse is also true in Example 5.3.10.

5.2. *G-μ-displays.* We now introduce prismatic G-μ-displays. To an orientable and bounded $\mathcal{O}_E$-prism (A, I) over $\mathcal{O}$, we attach the display group $G_\mu(A, I)$ as in Definition 4.1.1. Since G is defined over $\mathcal{O}_E$, the Frobenius ϕ of A induces a homomorphism $\phi : G(A) \to G(A)$. For each generator $d \in I$, we define the homomorphism

$$\sigma_{\mu,d} : G_\mu(A, I) \to G(A), \quad g \mapsto \phi(\mu(d)g\mu(d)^{-1}). \tag{5-1}$$

We endow $G(A)$ with the following action of $G_\mu(A, I)$:

$$G(A) \times G_\mu(A, I) \to G(A), \quad (X, g) \mapsto X \cdot g := g^{-1} X \sigma_{\mu,d}(g). \tag{5-2}$$

We write $G(A) = G(A)_d$ when we regard $G(A)$ as a set with this action of $G_\mu(A, I)$. For another generator $d' \in I$, we have $d = ud'$ for a unique element $u \in A^\times$. The map $G(A)_d \to G(A)_{d'}$ defined by $X \mapsto X\phi(\mu(u))$ is $G_\mu(A, I)$-equivariant. Then we define the set

$$G(A)_I := \varprojlim_d G(A)_d$$

equipped with a natural action of $G_\mu(A, I)$, where d runs over the set of generators $d \in I$. The projection map $G(A)_I \to G(A)_d$ is an isomorphism. For an element $X \in G(A)_I$, let

$$X_d \in G(A)_d$$

denote the image of X. Although $G(A)_I$ depends on the cocharacter μ, we omit it from the notation. We hope that this will not cause any confusion.

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. We recall the category $(A, I)_{\Delta,\mathrm{ori}}$ from Definition 4.3.2. We define the functor

$$G_{\Delta,A,I} : (A, I)_{\Delta,\mathrm{ori}} \to \mathrm{Set}, \quad (B, J) \mapsto G(B)_J.$$

This forms a sheaf. We regard $G_{\Delta,A,I}$ as a sheaf on $(A, I)_{\Delta}^{\mathrm{op}}$ (see Remark 4.3.3). The sheaf $G_{\Delta,A,I}$ is equipped with a natural action of the group sheaf $G_{\mu,A,I}$ on $(A, I)_{\Delta}^{\mathrm{op}}$ defined in Definition 4.3.4.

The restriction of $G_{\Delta,A,I}$ to $(A, I)_{\text{ét}}$ is denoted by the same notation. We define prismatic G-μ-displays, using the (π, I)-completely étale topology, as follows.

Definition 5.2.1 (G-μ-display). Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

(1) A *G-μ-display* over (A, I) is a pair

$$(\mathcal{Q}, \alpha_{\mathcal{Q}}),$$

where $\mathcal{Q}$ is a $G_{\mu,A,I}$-torsor on $(A, I)_{\text{ét}}^{\mathrm{op}}$ and $\alpha_{\mathcal{Q}} : \mathcal{Q} \to G_{\Delta,A,I}$ is a $G_{\mu,A,I}$-equivariant map of sheaves. The $G_{\mu,A,I}$-torsor $\mathcal{Q}$ is called the *underlying $G_{\mu,A,I}$-torsor* of $(\mathcal{Q}, \alpha_{\mathcal{Q}})$. We say that $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ is *banal* if $\mathcal{Q}$ is trivial as a $G_{\mu,A,I}$-torsor. When there is no possibility of confusion, we write $\mathcal{Q}$ instead of $(\mathcal{Q}, \alpha_{\mathcal{Q}})$.

(2) An isomorphism $g : (\mathcal{Q}, \alpha_{\mathcal{Q}}) \to (\mathcal{R}, \alpha_{\mathcal{R}})$ of G-μ-displays over (A, I) is an isomorphism $g : \mathcal{Q} \xrightarrow{\sim} \mathcal{R}$ of $G_{\mu,A,I}$-torsors such that $\alpha_{\mathcal{R}} \circ g = \alpha_{\mathcal{Q}}$.

We write

$$G\text{-}\mathrm{Disp}_\mu(A, I) \quad \text{and} \quad G\text{-}\mathrm{Disp}_\mu(A, I)_{\mathrm{banal}}$$

for the groupoid of G-μ-displays over (A, I) and the groupoid of banal G-μ-displays over (A, I), respectively.

Remark 5.2.2. The notion of G-μ-displays was originally introduced in [Bültel 2008; Bültel and Pappas 2020; Lau 2021] in different settings. The definition given here is an adaptation of Lau's approach to the context of ($\mathcal{O}_E$-)prisms; see also Remark 5.2.3 below. If $\mathcal{O}_E = \mathbb{Z}_p$ and μ is 1-bounded, the notion of G-μ-displays for an oriented perfect prism has already appeared in [Bartling 2022]. He also claimed that the same construction should work for more general oriented prisms in [loc. cit., Remark 14].

Remark 5.2.3. Assume that (A, I) is orientable. We consider the graded ring

$$\mathrm{Rees}(I^\bullet) := \Bigl(\bigoplus_{i\geq 0} I^i t^{-i}\Bigr) \oplus \Bigl(\bigoplus_{i<0} At^{-i}\Bigr) \subset A[t, t^{-1}],$$

where the degree of t is -1. Let $\tau : \mathrm{Rees}(I^\bullet) \to A$ be the homomorphism of A-algebras defined by $t \mapsto 1$. For a generator $d \in I$, let $\sigma_d : \mathrm{Rees}(I^\bullet) \to A$ be the homomorphism defined by $a_i t^{-i} \mapsto \phi(a_i d^{-i})$ for any $i \in \mathbb{Z}$. The triple

$$(\mathrm{Rees}(I^\bullet), \sigma_d, \tau)$$

can be viewed as an analogue of a higher frame introduced in [Lau 2021, Definition 2.0.1]. We note that by Lemma 4.1.4, the homomorphism τ induces an isomorphism between the display group $G_\mu(A, I)$ and the subgroup

$$G(\mathrm{Rees}(I^\bullet))^0 \subset G(\mathrm{Rees}(I^\bullet))$$

consisting of homomorphisms $g^* : A_G \to \mathrm{Rees}(I^\bullet)$ of graded $\mathcal{O}$-algebras. Under this isomorphism, the homomorphism $\sigma_{\mu,d}$ agrees with the one $G(\mathrm{Rees}(I^\bullet))^0 \to G(A)$ induced by σ_d. Therefore, the action (5-2) is consistent with the one considered in [loc. cit., (5-2)].

Remark 5.2.4. Let $\tilde{k}$ be a perfect field containing k. We set $\widetilde{\mathcal{O}} := W(\tilde{k}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. Let $\tilde{\mu} : \mathbb{G}_m \to G_{\widetilde{\mathcal{O}}}$ be the base change of μ. Then, for a bounded $\mathcal{O}_E$-prism (A, I) over $\widetilde{\mathcal{O}}$, a G-$\tilde{\mu}$-display over (A, I) is the same as a G-μ-display over (A, I).

We have the following alternative description of banal G-μ-displays, which we will use frequently in the sequel.

Remark 5.2.5. Assume that (A, I) is orientable. Let

$$[G(A)_I / G_\mu(A, I)]$$

denote the groupoid whose objects are the elements $X \in G(A)_I$ and whose morphisms are defined by $\mathrm{Hom}(X, X') = \{g \in G_\mu(A, I) \mid X' \cdot g = X\}$. Here $(-) \cdot g$ denotes the action of $g \in G_\mu(A, I)$ on $G(A)_I$. To each $X \in G(A)_I$, we attach a banal G-μ-display

$$\mathcal{Q}_X := (G_{\mu,A,I}, \alpha_X)$$

over (A, I), where $\alpha_X : G_{\mu,A,I} \to G_{\Delta,A,I}$ is given by $1 \mapsto X$. We obtain an equivalence

$$[G(A)_I / G_\mu(A, I)] \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(A, I)_{\mathrm{banal}}, \quad X \mapsto \mathcal{Q}_X,$$

of groupoids.

We discuss the notion of base change for G-μ-displays. Let $f : (A, I) \to (A', I')$ be a map of orientable and bounded $\mathcal{O}_E$-prisms over $\mathcal{O}$. We have natural homomorphisms $f : G(A) \to G(A')$ and

$f : G_\mu(A, I) \to G_\mu(A', I')$. Let $d \in I$ and $d' \in I'$ be generators and let $u \in A'^\times$ be the unique element satisfying $f(d) = ud'$. Then the composition of $G_\mu(A, I)$-equivariant maps

$$G(A)_I \simeq G(A)_d \to G(A')_{d'} \simeq G(A')_{I'},$$

where the second map is defined by $X \mapsto f(X)\phi(\mu(u))$, is independent of the choices of d and d', and is also denoted by f.

We now consider a map $f : (A, I) \to (A', I')$ of (not necessarily orientable) bounded $\mathcal{O}_E$-prisms over $\mathcal{O}$. The functor $(A, I)^{\mathrm{op}}_{\text{ét}} \to (A', I')^{\mathrm{op}}_{\text{ét}}$ sending $B \in (A, I)^{\mathrm{op}}_{\text{ét}}$ to the (π, I)-adic completion B' of $B \otimes_A A'$ induces a morphism of the associated topoi

$$f : ((A', I')^{\mathrm{op}}_{\text{ét}})^\sim \to ((A, I)^{\mathrm{op}}_{\text{ét}})^\sim$$

(since it sends (π, I)-completely étale coverings to (π, I')-completely étale coverings, sends final objects to final objects, and commutes with fiber products). We have a natural homomorphism $f : f^{-1}G_{\mu,A,I} \to G_{\mu,A',I'}$ of group sheaves. Moreover, the maps $G(A)_I \to G(A')_{I'}$ defined in the orientable case glue together to a morphism $f : f^{-1}G_{\Delta,A,I} \to G_{\Delta,A',I'}$ of sheaves.

Definition 5.2.6. Let $(\mathcal{Q}, \alpha_\mathcal{Q})$ be a G-μ-display over (A, I). Let $f^*\mathcal{Q}$ be the pushout of the $f^{-1}G_{\mu,A,I}$-torsor $f^{-1}\mathcal{Q}$ along $f : f^{-1}G_{\mu,A,I} \to G_{\mu,A',I'}$. By the universal property of $f^*\mathcal{Q}$, the composition

$$f^{-1}\mathcal{Q} \xrightarrow{f^{-1}(\alpha_\mathcal{Q})} f^{-1}G_{\Delta,A,I} \to G_{\Delta,A',I'}$$

factors through a unique $G_{\mu,A',I'}$-equivariant map $f^*(\alpha_\mathcal{Q}) : f^*\mathcal{Q} \to G_{\Delta,A',I'}$. The base change of $(\mathcal{Q}, \alpha_\mathcal{Q})$ along $f : (A, I) \to (A', I')$ is defined to be $(f^*\mathcal{Q}, f^*(\alpha_\mathcal{Q}))$.

Example 5.2.7. Assume that (A, I) is orientable. For the banal G-μ-display $\mathcal{Q}_X$ associated with an element $X \in G(A)_I$ (see Remark 5.2.5), we have $f^*(\mathcal{Q}_X) = \mathcal{Q}_{f(X)}$.

By definition, it is clear that G-μ-displays form a stack with respect to the (π, I)-completely étale topology. In fact, we can prove the following flat descent result, which is an analogue of [Lau 2021, Lemma 5.4.2].

Proposition 5.2.8 (flat descent). *The fibered category over $(\mathcal{O})^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ which associates to each $(A, I) \in (\mathcal{O})_{\Delta,\mathcal{O}_E}$ the groupoid $G\text{-}\mathrm{Disp}_\mu(A, I)$ is a stack with respect to the flat topology.*

Proof. It suffices to prove that $G\text{-}\mathrm{Disp}_\mu(A, I)$ is equivalent to the groupoid of pairs $(\mathcal{Q}, \alpha_\mathcal{Q})$, where $\mathcal{Q}$ is a $G_{\mu,A,I}$-torsor on $(A, I)^{\mathrm{op}}_\Delta$ (with respect to the flat topology) and $\alpha_\mathcal{Q} : \mathcal{Q} \to G_{\Delta,A,I}$ is a $G_{\mu,A,I}$-equivariant map of sheaves on $(A, I)^{\mathrm{op}}_\Delta$. This follows from Corollary 4.3.8. □

5.3. *G-μ-displays and G-Breuil–Kisin modules of type μ.* Here we shall show that G-μ-displays are essentially equivalent to G-Breuil–Kisin modules of type μ. Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$.

Definition 5.3.1. To a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$, we attach a G_A-torsor $\mathcal{Q}_{\mathrm{BK}}$ over Spec A as follows. We first assume that (A, I) is orientable. Let $d \in I$ be a generator. Let $\mathcal{Q}_{\mathrm{BK},d}$ be the pushout of $\mathcal{Q}$ along the homomorphism

$$G_{\mu,A,I} \hookrightarrow G_{\Delta,A}, \quad g \mapsto \mu(d)g\mu(d)^{-1}.$$

Let $d' \in I$ be another generator and let $u \in A^\times$ be the unique element such that $d = ud'$. We define $\mathrm{ad}(\mu(u))$: $G_{\Delta,A} \xrightarrow{\sim} G_{\Delta,A}$ by $g \mapsto \mu(u)g\mu(u)^{-1}$. The pushout $(\mathcal{Q}_{\mathrm{BK},d'})^{\mathrm{ad}(\mu(u))}$ can be identified with $\mathcal{Q}_{\mathrm{BK},d}$. The composition

$$\mathcal{Q}_{\mathrm{BK},d'} \to (\mathcal{Q}_{\mathrm{BK},d'})^{\mathrm{ad}(\mu(u))} = \mathcal{Q}_{\mathrm{BK},d} \xrightarrow{x \mapsto x\cdot\mu(u)} \mathcal{Q}_{\mathrm{BK},d} \tag{5-3}$$

is an isomorphism of $G_{\Delta,A}$-torsors. (See Remark 4.3.6 for the first map.) Then we define

$$\mathcal{Q}_{\mathrm{BK}} := \varprojlim_{d} \mathcal{Q}_{\mathrm{BK},d},$$

where d runs over the set of generators $d \in I$.

In general, the sheaves constructed in the banal case glue together to a $G_{\Delta,A}$-torsor $\mathcal{Q}_{\mathrm{BK}}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$. By Proposition 4.3.1, we regard $\mathcal{Q}_{\mathrm{BK}}$ as a G_A-torsor over Spec A.

Remark 5.3.2. Recall that $\tau : G_{\mu,A,I} \hookrightarrow G_{\Delta,A}$ is the natural inclusion. For a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$, let

$$\mathcal{Q}_A := \mathcal{Q}^\tau$$

be the pushout of $\mathcal{Q}$ along τ, regarded as a G_A-torsor over Spec A (by Proposition 4.3.1). There exists a canonical isomorphism

$$\mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I]$$

of $G_{A[1/I]}$-torsors over Spec $A[1/I]$ obtained as follows. We first assume that (A, I) is orientable. Let $d \in I$ be a generator. Similarly to (5-3), the composition

$$\mathcal{Q}_A[1/I] \to (\mathcal{Q}_A[1/I])^{\mathrm{ad}(\mu(d))} = \mathcal{Q}_{\mathrm{BK},d}[1/I] \xrightarrow{x \mapsto x\cdot\mu(d)} \mathcal{Q}_{\mathrm{BK},d}[1/I]$$

is an isomorphism of $G_{A[1/I]}$-torsors, where $\mathrm{ad}(\mu(d)) : G_{A[1/I]} \xrightarrow{\sim} G_{A[1/I]}$ is defined by $g \mapsto \mu(d)g\mu(d)^{-1}$. We then obtain the desired isomorphism as

$$\mathcal{Q}_A[1/I] \simeq \mathcal{Q}_{\mathrm{BK},d}[1/I] \simeq \mathcal{Q}_{\mathrm{BK}}[1/I],$$

which does not depend on the choice of $d \in I$. By Remark 5.1.3, the isomorphisms in the banal case glue together to an isomorphism $\mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I]$.

Example 5.3.3. Assume that $G = \mathrm{GL}_n$. Let the notation be as in Example 4.1.5. Let M be a Breuil–Kisin module of type μ over (A, I). Recall the filtration $\{\mathrm{Fil}^i(\phi^*M)\}_{i\in\mathbb{Z}}$ of ϕ^*M from Definition 3.1.2. Let $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ be the filtration of A^n defined in Remark 3.2.2. The functor

$$\mathcal{Q}(M) := \underline{\mathrm{Isom}}_{\mathrm{Fil}}(A^n, \phi^*M) : (A, I)_{\text{ét}} \to \mathrm{Set}$$

sending $B \in (A, I)_{\text{ét}}$ to the set of isomorphisms $h : B^n \xrightarrow{\sim} (\phi^*M)_B$ preserving the filtrations is a $(\mathrm{GL}_n)_{\mu,A,I}$-torsor by Remark 3.2.2, Example 4.1.5, and the fact that M is (π, I)-completely étale locally on A banal. We note that

$$\mathcal{Q}(M)_A = \underline{\mathrm{Isom}}(A^n, \phi^* M).$$

We set $\widetilde{M} := \mathrm{Fil}^{m_1}(\phi^* M) \otimes_A I^{-m_1}$. Then we have a canonical identification

$$\mathcal{Q}(M)_{\mathrm{BK}} = \underline{\mathrm{Isom}}(A^n, \widetilde{M}).$$

If A is orientable and $d \in I$ is a generator, then $\mathcal{Q}(M)_{\mathrm{BK},d} = \underline{\mathrm{Isom}}(A^n, \widetilde{M})$ and the natural map $\mathcal{Q}(M) \to \mathcal{Q}(M)_{\mathrm{BK},d}$ sends $h \in \mathcal{Q}(M)(A)$ to the composition of isomorphisms

$$A^n \xrightarrow{\mu(d)^{-1}} \mathrm{Fil}^{m_1}_{\mu} \otimes_A I^{-m_1} \xrightarrow{h} \widetilde{M}.$$

If $d' \in I$ is another generator, then the isomorphism $\mathcal{Q}(M)_{\mathrm{BK},d'} \xrightarrow{\sim} \mathcal{Q}(M)_{\mathrm{BK},d}$ from (5-3) is the identity $\underline{\mathrm{Isom}}(A^n, \widetilde{M}) \to \underline{\mathrm{Isom}}(A^n, \widetilde{M})$.

The isomorphism $\mathcal{Q}(M)_A[1/I] \xrightarrow{\sim} \mathcal{Q}(M)_{\mathrm{BK}}[1/I]$ defined in Remark 5.3.2 agrees with the one induced from the equality $(\phi^* M)[1/I] = \widetilde{M}[1/I]$.

To construct G-Breuil–Kisin modules of type μ from G-μ-displays, we use the following proposition, which also gives an alternative description of G-μ-displays.

Proposition 5.3.4. *Let $\mathcal{Q}$ be a $G_{\mu,A,I}$-torsor on $(A, I)^{\mathrm{op}}_{\text{ét}}$. Then there is a natural bijection $\alpha \mapsto \alpha'$ from the set of $G_{\mu,A,I}$-equivariant maps $\alpha : \mathcal{Q} \to G_{\Delta,A,I}$ to the set of isomorphisms $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ of G_A-torsors over* Spec A.

Proof. We shall construct the bijection when (A, I) is orientable and $\mathcal{Q}$ is a trivial $G_{\mu,A,I}$-torsor; the general case follows by gluing. Let $\alpha : \mathcal{Q} \to G_{\Delta,A,I}$ be a $G_{\mu,A,I}$-equivariant map. We choose a trivialization $\mathcal{Q} \simeq G_{\mu,A,I}$. Then α can be regarded as a $G_{\mu,A,I}$-equivariant map $G_{\mu,A,I} \to G_{\Delta,A,I}$, which is determined by the image $X \in G(A)_I$ of $1 \in G_{\mu}(A, I)$. We may also identify $\mathcal{Q}_A$ with G_A. Let $d \in I$ be a generator. Then we may identify $\mathcal{Q}_{\mathrm{BK}}$ with G_A by

$$\mathcal{Q}_{\mathrm{BK}} \simeq \mathcal{Q}_{\mathrm{BK},d} \simeq (G_{\mu,A,I})_{\mathrm{BK},d} = G_A.$$

(See Definition 5.3.1 for $\mathcal{Q}_{\mathrm{BK},d}$.) Via these identifications, we define $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ by

$$\phi^*(\mathcal{Q}_{\mathrm{BK}}) = \phi^* G_A = G_A \xrightarrow{\sim} G_A = \mathcal{Q}_A, \quad g \mapsto X_d \cdot g,$$

where $X_d \in G(A) = G(A)_d$ is the image of $X \in G(A)_I$. One can check that the resulting isomorphism α' does not depend on the choices of $\mathcal{Q} \simeq G_{\mu,A,I}$ and $d \in I$. It is clear that the map $\alpha \mapsto \alpha'$ is a bijection. □

Remark 5.3.5. By Proposition 5.3.4, a G-μ-display over (A, I) can be thought of as a pair $(\mathcal{Q}, \alpha')$ of a $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$ and an isomorphism $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ of G_A-torsors over Spec A.

Definition 5.3.6. Let $(\mathcal{Q}, \alpha_{\mathcal{Q}})$ be a G-μ-display over (A, I) and let $(\alpha_{\mathcal{Q}})' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ be the corresponding isomorphism. We denote by F the composition

$$(\phi^*(\mathcal{Q}_{\mathrm{BK}}))[1/I] \xrightarrow{(\alpha_{\mathcal{Q}})'} \mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I],$$

where the second isomorphism is constructed in Remark 5.3.2. By construction, we see that $\mathcal{Q}_{\mathrm{BK}}$, together with the isomorphism F, is a G-Breuil–Kisin module of type μ. (See also Example 5.3.7 below.) We have a functor

$$G\text{-}\mathrm{Disp}_{\mu}(A, I) \to G\text{-}\mathrm{BK}_{\mu}(A, I), \quad \mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{BK}}. \tag{5-4}$$

Example 5.3.7. Assume that (A, I) is orientable and let $d \in I$ be a generator. Let $\mathcal{Q}_X$ be the banal G-μ-display associated with an element $X \in G(A)_I$ (Remark 5.2.5). The trivial G_A-torsor G_A with the isomorphism

$$(\phi^* G_A)[1/I] = G_A[1/I] \xrightarrow{\sim} G_A[1/I], \quad g \mapsto (\mu(d)X_d)g,$$

is a banal G-Breuil–Kisin module of type μ over (A, I), which is denoted by $\mathcal{P}_{X_d}$. By construction, we have $(\mathcal{Q}_X)_{\mathrm{BK}} \xrightarrow{\sim} \mathcal{P}_{X_d}$.

Proposition 5.3.8. *Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. The functor (5-4)*

$$G\text{-}\mathrm{Disp}_{\mu}(A, I) \to G\text{-}\mathrm{BK}_{\mu}(A, I), \quad \mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{BK}},$$

is an equivalence.

Proof. By Remark 2.5.14, Remark 5.1.5, and (π, I)-completely étale descent for G-μ-displays, it suffices to prove that the functor

$$G\text{-}\mathrm{Disp}_{\mu}(A, I)_{\mathrm{banal}} \to G\text{-}\mathrm{BK}_{\mu}(A, I)_{\mathrm{banal}}, \quad \mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{BK}},$$

is an equivalence when (A, I) is orientable.

We shall prove that the functor is fully faithful. It suffices to prove that, for all $X, X' \in G(A)_I$ and the associated banal G-μ-displays $\mathcal{Q}_X, \mathcal{Q}_{X'}$ over (A, I), we have

$$\mathrm{Hom}(\mathcal{Q}_X, \mathcal{Q}_{X'}) \xrightarrow{\sim} \mathrm{Hom}((\mathcal{Q}_X)_{\mathrm{BK}}, (\mathcal{Q}_{X'})_{\mathrm{BK}}). \tag{5-5}$$

We fix a generator $d \in I$. The left-hand side can be identified with

$$\{g \in G_{\mu}(A, I) \mid g^{-1} X'_d \phi(\mu(d) g \mu(d)^{-1}) = X_d\}.$$

(See Remark 5.2.5.) By Example 5.3.7, we have $(\mathcal{Q}_X)_{\mathrm{BK}} \xrightarrow{\sim} \mathcal{P}_{X_d}$ and $(\mathcal{Q}_{X'})_{\mathrm{BK}} \xrightarrow{\sim} \mathcal{P}_{X'_d}$. Thus the right-hand side of (5-5) can be identified with

$$\{h \in G(A) \mid h^{-1} \mu(d) X'_d \phi(h) = \mu(d) X_d\}.$$

The map (5-5) is given by $g \mapsto \mu(d) g \mu(d)^{-1}$ under these identifications. In particular, the map is injective. For surjectivity, let $h \in G(A)$ be an element such that $h^{-1}\mu(d) X'_d \phi(h) = \mu(d) X_d$. The element $g := \mu(d)^{-1} h \mu(d) = X'_d \phi(h) X_d^{-1}$ belongs to $G(A)$, and hence $g \in G_{\mu}(A, I)$. It follows that $g \in \mathrm{Hom}(\mathcal{Q}_X, \mathcal{Q}_{X'})$, and g is mapped to h.

It remains to prove that the functor is essentially surjective. It is enough to show that a banal G-Breuil–Kisin module $\mathcal{P}$ of type μ over (A, I), such that $\mathcal{P} = G_A$ and $F_{\mathcal{P}}$ corresponds to an element $Y \in G(A)\mu(d)G(A)$, is isomorphic to $\mathcal{P}_{X_d}$ for some $X \in G(A)_I$. After changing the trivialization $\mathcal{P} = G_A$, we may assume that $Y \in \mu(d)G(A)$. Then the result is clear. □

Corollary 5.3.9. *The fibered category over* $(\mathcal{O})^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ *which associates to each* $(A, I) \in (\mathcal{O})_{\Delta,\mathcal{O}_E}$ *the groupoid* $G\text{-}\mathrm{BK}_\mu(A, I)$ *of* G*-Breuil–Kisin modules of type* μ *over* (A, I) *is a stack with respect to the flat topology.*

Proof. This follows from Propositions 5.2.8 and 5.3.8. □

Example 5.3.10. Assume that $G = \mathrm{GL}_n$. We retain the notation of Example 5.3.3. Let M be a Breuil–Kisin module of type μ over (A, I). Since M is of type μ, it follows from Lemma 3.1.1 that F_M restricts to an isomorphism $\widetilde{M} \xrightarrow{\sim} M$. The base change $\phi^*(F_M) : \phi^*\widetilde{M} \xrightarrow{\sim} \phi^*M$ induces an isomorphism

$$\alpha' : \phi^*(\mathcal{Q}(M)_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}(M)_A$$

of $\mathrm{GL}_{n,A}$-torsors over $\operatorname{Spec} A$. The $(\mathrm{GL}_n)_{\mu,A,I}$-torsor $\mathcal{Q}(M)$ with α' is a GL_n-μ-display over (A, I).

By construction, the GL_n-Breuil–Kisin module $\mathcal{Q}(M)_{\mathrm{BK}}$ agrees with the one $\mathcal{P}(\widetilde{M})$ associated with the Breuil–Kisin module $(\widetilde{M}, F_{\widetilde{M}})$, where the isomorphism $F_{\widetilde{M}}$ is

$$(\phi^*\widetilde{M})[1/I] \xrightarrow{\phi^*(F_M)} (\phi^*M)[1/I] = \widetilde{M}[1/I].$$

(See $\mathcal{P}(\widetilde{M})$ for Example 5.1.2.) We note that $F_M : \widetilde{M} \xrightarrow{\sim} M$ is an isomorphism of Breuil–Kisin modules. Since $\mathcal{Q}(M)_{\mathrm{BK}}$ is of type μ, it follows that $\mathcal{P}(M)$ is of type μ.

Corollary 5.3.11. *Let* (A, I) *be a bounded* $\mathcal{O}_E$*-prism over* $\mathcal{O}$*. We have equivalences of groupoids*

$$\mathrm{BK}_\mu(A, I)^{\simeq} \xrightarrow{\sim} \mathrm{GL}_n\text{-}\mathrm{BK}_\mu(A, I), \qquad M \mapsto \mathcal{P}(M),$$
$$\mathrm{BK}_\mu(A, I)^{\simeq} \xrightarrow{\sim} \mathrm{GL}_n\text{-}\mathrm{Disp}_\mu(A, I), \qquad M \mapsto \mathcal{Q}(M).$$

Here $\mathrm{BK}_\mu(A, I)^{\simeq}$ *is the groupoid of Breuil–Kisin modules of type* μ *over* (A, I).

Proof. The first equivalence follows from Examples 5.1.2, 5.1.6, and 5.3.10. We shall prove that the functor $M \mapsto \mathcal{Q}(M)$ is an equivalence. It follows from Example 5.3.10 that the composition of this functor with the functor (5-4) is isomorphic to the functor $M \mapsto \mathcal{P}(M)$. Since (5-4) is an equivalence by Proposition 5.3.8, the result follows. □

5.4. ***Hodge filtrations.*** We define the Hodge filtrations for G-μ-displays, following [Lau 2021, Section 7.4]. Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. We recall the commutative diagram (4-6) from Section 4.3.

Definition 5.4.1 (Hodge filtration). Let $\mathcal{Q}$ be a G-μ-display over (A, I). We write

$$\mathcal{Q}_{A/I} := \mathcal{Q}^{\bar{\tau}} \quad (\text{resp. } P(\mathcal{Q})_{A/I} := \mathcal{Q}^{\bar{\tau}_P})$$

for the pushout of the underlying $G_{\mu,A,I}$-torsor $\mathcal{Q}$ on $(A, I)^{\mathrm{op}}_{\text{ét}}$ along $\bar{\tau}$ (resp. $\bar{\tau}_P$), which is a $G_{\bar{\Delta},A}$-torsor (resp. a $(P_\mu)_{\bar{\Delta},A}$-torsor) on $(A, I)^{\mathrm{op}}_{\text{ét}}$. There is a natural $(P_\mu)_{\bar{\Delta},A}$-equivariant injection

$$P(\mathcal{Q})_{A/I} \hookrightarrow \mathcal{Q}_{A/I}.$$

We call $P(\mathcal{Q})_{A/I}$ (or the injection $P(\mathcal{Q})_{A/I} \hookrightarrow \mathcal{Q}_{A/I}$) the *Hodge filtration* of $\mathcal{Q}_{A/I}$. If there is no risk of confusion, we also say that $P(\mathcal{Q})_{A/I}$ is the Hodge filtration of $\mathcal{Q}$.

Example 5.4.2. Assume that $G = \mathrm{GL}_n$ and let the notation be as in Example 5.3.3. Let M be a Breuil–Kisin module over (A, I) of type μ and let $\mathcal{Q} = \mathcal{Q}(M)$ be the associated GL_n-μ-display over (A, I) given in Example 5.3.10. Recall that the filtration $\{\mathrm{Fil}^i(\phi^*M)\}_{i\in\mathbb{Z}}$ defines the Hodge filtration $\{P^i\}_{i\in\mathbb{Z}}$ of $M_{\mathrm{dR}} = (\phi^*M)/I(\phi^*M)$. Similarly, the filtration $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ of A^n induces a filtration of $(A/I)^n$. Let $\underline{\mathrm{Isom}}((A/I)^n, M_{\mathrm{dR}})$ (resp. $\underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}})$) be the functor sending $B \in (A, I)_{\text{ét}}$ to the set of isomorphisms $(B/IB)^n \xrightarrow{\sim} (M_{\mathrm{dR}})_{B/IB}$ (resp. the set of isomorphisms $(B/IB)^n \xrightarrow{\sim} (M_{\mathrm{dR}})_{B/IB}$ preserving the filtrations). Since M is of type μ, we see that $\underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}})$ is naturally a $(P_\mu)_{\bar{\Delta},A}$-torsor. It follows that the natural morphism

$$\mathcal{Q} = \underline{\mathrm{Isom}}_{\mathrm{Fil}}(A^n, \phi^*M) \to \underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}})$$

induces an isomorphism

$$P(\mathcal{Q})_{A/I} \xrightarrow{\sim} \underline{\mathrm{Isom}}_{\mathrm{Fil}}((A/I)^n, M_{\mathrm{dR}}).$$

Similarly, we obtain $\mathcal{Q}_{A/I} \xrightarrow{\sim} \underline{\mathrm{Isom}}((A/I)^n, M_{\mathrm{dR}})$.

Remark 5.4.3. Let $\mathcal{Q}$ be a G-μ-display over (A, I). By Proposition 4.3.1, the $G_{\bar{\Delta},A}$-torsor $\mathcal{Q}_{A/I}$ (resp. the $(P_\mu)_{\bar{\Delta},A}$-torsor $P(\mathcal{Q})_{A/I}$) corresponds to a $G_{A/I}$-torsor (resp. a $(P_\mu)_{A/I}$-torsor) over $\operatorname{Spec} A/I$, which will be denoted by the same symbol.

Example 5.4.4. Assume that (A, I) is orientable. Let $X \in G(A)_I$ be an element. Then the Hodge filtration associated with $\mathcal{Q}_X$ can be identified with the natural inclusion $(P_\mu)_{A/I} \hookrightarrow G_{A/I}$.

Proposition 5.4.5. *A G-μ-display $\mathcal{Q}$ over (A, I) is banal if and only if the Hodge filtration $P(\mathcal{Q})_{A/I}$ is a trivial $(P_\mu)_{A/I}$-torsor over* $\operatorname{Spec} A/I$.

Proof. This is a restatement of Corollary 4.3.7 in the current context. □

5.5. *Underlying G-ϕ-modules.* Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$ and let (M, F_M) be a Breuil–Kisin module over (A, I). Since $\{\mathrm{Fil}^i(\phi^*M)\}_{i\in\mathbb{Z}}$ is the filtration of ϕ^*M, it is sometimes reasonable to consider ϕ^*M (rather than M) as "the underlying A-module" of the Breuil–Kisin module (M, F_M). The same applies to G-Breuil–Kisin modules $\mathcal{P}$ over (A, I). In fact, the Frobenius of $\phi^*\mathcal{P}$ will also be important. For example, this can be observed in the Grothendieck–Messing deformation theory studied in [Ito 2025].

It will be convenient to make the following definition. We assume that (A, I) is orientable for simplicity. We set $A[1/\phi(I)] := A[1/\phi(d)]$ for a generator $d \in I$, which does not depend on the choice of d.

Definition 5.5.1. A *G-ϕ-module* over (A, I) is a pair $(\mathcal{P}, \phi_{\mathcal{P}})$ consisting of a G_A-torsor $\mathcal{P}$ over $\operatorname{Spec} A$ and an isomorphism

$$\phi_{\mathcal{P}} : (\phi^*\mathcal{P})[1/\phi(I)] \xrightarrow{\sim} \mathcal{P}[1/\phi(I)]$$

of $G_{A[1/\phi(I)]}$-torsors over $\operatorname{Spec} A[1/\phi(I)]$. (Here $\mathcal{P}[1/\phi(I)] := \mathcal{P} \times_{\operatorname{Spec} A} \operatorname{Spec} A[1/\phi(I)]$.) If there is no possibility of confusion, we write $\mathcal{P} = (\mathcal{P}, \phi_{\mathcal{P}})$.

Here we explain how to attach a G-ϕ-module over (A, I) to a G-μ-display $\mathcal{Q}$ over (A, I). Recall $\mathcal{Q}_A := \mathcal{Q}^\tau$ from Remark 5.3.2, which we regard as a G_A-torsor over Spec A. We define

$$\phi_{\mathcal{Q}_A} : (\phi^*(\mathcal{Q}_A))[1/\phi(I)] \xrightarrow{\sim} \mathcal{Q}_A[1/\phi(I)]$$

as the composition

$$(\phi^*(\mathcal{Q}_A))[1/\phi(I)] \xrightarrow{\sim} (\phi^*(\mathcal{Q}_{\mathrm{BK}}))[1/\phi(I)] \xrightarrow{\sim} \mathcal{Q}_A[1/\phi(I)],$$

where the first isomorphism is the base change of $\mathcal{Q}_A[1/I] \xrightarrow{\sim} \mathcal{Q}_{\mathrm{BK}}[1/I]$ given in Remark 5.3.2 along $\phi : A[1/I] \to A[1/\phi(I)]$, and the second one is the base change of $(\alpha_{\mathcal{Q}})' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ given in Proposition 5.3.4 along the natural homomorphism $A \to A[1/\phi(I)]$.

Definition 5.5.2 (underlying G-ϕ-module). Let $\mathcal{Q}$ be a G-μ-display over (A, I). The G-ϕ-module

$$\mathcal{Q}_\phi := (\mathcal{Q}_A, \phi_{\mathcal{Q}_A})$$

over (A, I) is called the *underlying G-ϕ-module* of $\mathcal{Q}$.

Example 5.5.3. Let $\mathcal{Q}_X$ be the banal G-μ-display associated with an element $X \in G(A)_I$. The underlying G-ϕ-module $(\mathcal{Q}_X)_\phi$ of $\mathcal{Q}_X$ is the trivial G_A-torsor G_A with the isomorphism

$$(\phi^* G_A)[1/\phi(I)] = G_A[1/\phi(I)] \xrightarrow{\sim} G_A[1/\phi(I)], \quad g \mapsto X_d \phi(\mu(d)) g,$$

for a generator $d \in I$. We note that the element $X_d \phi(\mu(d)) \in G(A[1/\phi(I)])$ is independent of the choice of $d \in I$.

Remark 5.5.4. Let $\mathcal{Q}$ be a G-μ-display over (A, I). The base change $\phi^*(\mathcal{Q}_{\mathrm{BK}})$ of the associated G-Breuil–Kisin module $\mathcal{Q}_{\mathrm{BK}}$ is naturally a G-ϕ-module over (A, I). We note that $(\alpha_{\mathcal{Q}})'$ gives an isomorphism $\phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_\phi$ of G-ϕ-modules. Therefore, one can also define the underlying G-ϕ-module of $\mathcal{Q}$ as $\phi^*(\mathcal{Q}_{\mathrm{BK}})$. However, the construction of $\mathcal{Q}_\phi$ is more natural and will be useful in [Ito 2025].

5.6. *G-μ-displays for perfectoid rings.* Let R be a perfectoid ring over $\mathcal{O}$. We discuss p-complete arc-descent results for G-μ-displays over the $\mathcal{O}_E$-prism $(W_{\mathcal{O}_E}(R^\flat), I_R)$.

Remark 5.6.1. Assume that $\mathcal{O}_E = \mathbb{Z}_p$. In [Bartling 2022], the notion of G-Breuil–Kisin modules over $(W(R^\flat), I_R)$ *of type* μ was introduced in a different way; namely, a G-Breuil–Kisin module $\mathcal{P}$ over $(W(R^\flat), I_R)$ is said to be of type μ if for any homomorphism $R \to V$ with V a p-adically complete valuation ring of rank ≤ 1 whose fraction field is algebraically closed, the base change $\mathcal{P}_{W(V^\flat)}$ is of type μ in the sense of Definition 5.1.4. In Proposition 5.6.11 below, we will prove that this notion agrees with the one introduced in Definition 5.1.4.

Let Perfd_R be the category of perfectoid rings over R. We endow $\mathrm{Perfd}_R^{\mathrm{op}}$ with the topology generated by the *π-complete* arc-*coverings* (or equivalently, the p-complete arc-coverings) in the sense of [Česnavičius and Scholze 2024, Section 2.2.1]. This topology is called the *π-complete* arc-*topology*.

Remark 5.6.2. We quickly review the notion of a π-complete arc-covering.

(1) We say that a homomorphism $R \to S$ of perfectoid rings over $\mathcal{O}$ is a π-complete arc-covering if for any homomorphism $R \to V$ with V a π-adically complete valuation ring of rank ≤ 1, there exist an extension $V \hookrightarrow W$ of π-adically complete valuation rings of rank ≤ 1 and a homomorphism $S \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

(2) The category $\mathrm{Perfd}_R^{\mathrm{op}}$ admits fiber products; a colimit of the diagram $S_2 \leftarrow S_1 \to S_3$ in Perfd_R is given by the π-adic completion of $S_2 \otimes_{S_1} S_3$ (see [Česnavičius and Scholze 2024, Proposition 2.1.11]). We see that $\mathrm{Perfd}_R^{\mathrm{op}}$ is indeed a site.

(3) Let $R \to S$ be a π-completely étale covering. Then S is perfectoid as explained in Example 2.5.11, and $R \to S$ is a π-complete arc-covering; see [loc. cit., Section 2.2.1].

(4) There exists a π-complete arc-covering of the form $R \to \prod_{i \in I} V_i$, where V_i are π-adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields; see [Česnavičius and Scholze 2024, Lemma 2.2.3].

Proposition 5.6.3 [Ito 2023, Corollary 4.2]. *The fibered category over* $\mathrm{Perfd}_R^{\mathrm{op}}$ *which associates to a perfectoid ring S over R the category of finite projective S-modules satisfies descent with respect to the π-complete* arc*-topology. In particular, the functor* $\mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto S$, *forms a sheaf.*

Proof. See [Ito 2023, Corollary 4.2]. The second assertion was previously proved in [Bhatt and Scholze 2022, Proposition 8.10]. □

Remark 5.6.4. In fact, it is proved in [Ito 2023, Theorem 1.2] that the functor on Perfd_R associating to each $S \in \mathrm{Perfd}_R$ the ∞-category $\mathrm{Perf}(S)$ of perfect complexes over S satisfies π-complete arc-hyperdescent. Using this, we can prove that for any integer $n \geq 1$, the functor $S \mapsto \mathrm{Perf}(W_{\mathcal{O}_E}(S^\flat)/I_S^n)$ on Perfd_R satisfies π-complete arc-hyperdescent, by induction on n. This implies that the functor $S \mapsto \mathrm{Perf}(W_{\mathcal{O}_E}(S^\flat))$ satisfies π-complete arc-hyperdescent as well. See the discussion in [loc. cit., Section 4.1].

Corollary 5.6.5. *The fibered category over* $\mathrm{Perfd}_R^{\mathrm{op}}$ *which associates to a perfectoid ring S over R the category of finite projective $W_{\mathcal{O}_E}(S^\flat)$-modules satisfies descent with respect to the π-complete* arc*-topology. The same holds for finite projective $W_{\mathcal{O}_E}(S^\flat)/I_S^n$-modules.*

Proof. By the same argument as in the proof of [loc. cit., Corollary 4.2], we can deduce the assertion from Remark 5.6.4. □

In particular, the functor $\mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto W_{\mathcal{O}_E}(S^\flat)$ forms a sheaf. This fact also follows from [Česnavičius and Scholze 2024, Lemma 4.2.6] or the proof of [Bhatt and Scholze 2022, Proposition 8.10] (using that $W(\mathbb{F}_q) \to \mathcal{O}_E$ is flat).

Remark 5.6.6. In the case where $\mathcal{O}_E = \mathbb{Z}_p$, the first assertion of Corollary 5.6.5 is proved in [Ito 2023, Corollary 4.2]. The general case can also be deduced from this special case, using that a module over $W_{\mathcal{O}_E}(S^\flat)$ is finite projective if and only if it is finite projective over $W(S^\flat)$.

For an affine scheme X over $\mathcal{O}$ (or R), we define a functor $X_{\bar{\Delta}} : \mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto X(S)$. By Proposition 5.6.3, this forms a sheaf. Similarly, for an affine scheme X over $\mathcal{O}$ (or $W_{\mathcal{O}_E}(R^\flat)$), we define a functor $X_{\Delta} : \mathrm{Perfd}_R \to \mathrm{Set}$, $S \mapsto X(W_{\mathcal{O}_E}(S^\flat))$, which forms a sheaf by Corollary 5.6.5. We have the following analogue of Proposition 4.3.1.

Proposition 5.6.7. *Let H be a smooth affine group scheme over $\mathcal{O}$.*

(1) *The functor $\mathcal{P} \mapsto \mathcal{P}_{\bar{\Delta}}$ from the groupoid of H_R-torsors over* $\mathrm{Spec}\, R$ *to the groupoid of $H_{\bar{\Delta}}$-torsors on* $\mathrm{Perfd}_R^{\mathrm{op}}$ *is an equivalence.*

(2) *The functor $\mathcal{P} \mapsto \mathcal{P}_{\Delta}$ from the groupoid of $H_{W_{\mathcal{O}_E}(R^\flat)}$-torsors over* $\mathrm{Spec}\, W_{\mathcal{O}_E}(R^\flat)$ *to the groupoid of H_{Δ}-torsors on* $\mathrm{Perfd}_R^{\mathrm{op}}$ *is an equivalence.*

Proof. This can be proved by the same argument as in the proof of Proposition 4.3.1, using Proposition 5.6.3 and Corollary 5.6.5. □

Remark 5.6.8. Arguing as in Remark 5.1.3, we see that the fibered category over $\mathrm{Perfd}_R^{\mathrm{op}}$ which associates to each $S \in \mathrm{Perfd}_R$ the groupoid of G-Breuil–Kisin modules over $(W_{\mathcal{O}_E}(S^\flat), I_S)$ is a stack with respect to the π-complete arc-topology.

As in Section 5.2, the functors

$$G_{\mu,I} : \mathrm{Perfd}_R \to \mathrm{Set}, \quad S \mapsto G_\mu(W_{\mathcal{O}_E}(S^\flat), I_S),$$
$$G_{\Delta,I} : \mathrm{Perfd}_R \to \mathrm{Set}, \quad S \mapsto G(W_{\mathcal{O}_E}(S^\flat))_{I_S},$$

form sheaves, and the group sheaf $G_{\mu,I}$ acts on $G_{\Delta,I}$.

Lemma 5.6.9. *Let $\mathcal{Q}$ be a $G_{\mu,I}$-torsor with respect to the π-complete* arc*-topology. Then $\mathcal{Q}$ is trivialized by a π-completely étale covering $R \to S$.*

Proof. We claim that if the pushout of $\mathcal{Q}$ along the homomorphism $G_{\mu,I} \to (P_\mu)_{\bar{\Delta}}$ is trivial as a $(P_\mu)_{\bar{\Delta}}$-torsor, then $\mathcal{Q}$ is itself trivial. Indeed, one can prove the analogue of Lemma 4.3.5 for $G_{\mu,I}$, and then the argument as in the proof of Corollary 4.3.7 works.

By the claim, it suffices to prove that any $(P_\mu)_{\bar{\Delta}}$-torsor with respect to the π-complete arc-topology can be trivialized by a π-completely étale covering $R \to S$. This is a consequence of Proposition 5.6.7. □

Corollary 5.6.10. *The fibered category over* $\mathrm{Perfd}_R^{\mathrm{op}}$ *which associates to a perfectoid ring S over R the groupoid of G-μ-displays over $(W_{\mathcal{O}_E}(S^\flat), I_S)$ is a stack with respect to the π-complete* arc*-topology. The same holds for G-Breuil–Kisin modules of type μ over $(W_{\mathcal{O}_E}(S^\flat), I_S)$.*

Proof. The first assertion can be deduced from Lemma 5.6.9 by the same argument as in the proof of Proposition 5.2.8. The second assertion follows from the first one, together with Proposition 5.3.8. □

Now we are ready to prove the following result:

Proposition 5.6.11. *For a G-Breuil–Kisin module $\mathcal{P}$ over $(W_{\mathcal{O}_E}(R^\flat), I_R)$, the following conditions are equivalent*:

(1) *$\mathcal{P}$ is of type μ* (in the sense of Definition 5.1.4).

(2) *There exists a π-complete* arc-*covering $R \to S$ such that the base change of $\mathcal{P}$ along $(W_{\mathcal{O}_E}(R^\flat), I_R) \to (W_{\mathcal{O}_E}(S^\flat), I_S)$ is of type μ.*

(3) *For any homomorphism $R \to V$ with V a π-adically complete valuation ring of rank ≤ 1 whose fraction field is algebraically closed, the base change of $\mathcal{P}$ along $(W_{\mathcal{O}_E}(R^\flat), I_R) \to (W_{\mathcal{O}_E}(V^\flat), I_V)$ is of type μ.*

Proof. It is clear (1) implies (2) and (3). By Remark 5.6.8 and Corollary 5.6.10, we see (2) implies (1).

Assume that the condition (3) is satisfied. We want to show that this implies (2), which will conclude the proof of the proposition. By Remark 5.6.2(4), there exists a π-complete arc-covering $R \to S = \prod_i V_i$, where V_i are π-adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields. Since $W_{\mathcal{O}_E}(V_i^\flat)$ is strictly henselian, the base change $\mathcal{P}_{W_{\mathcal{O}_E}(V_i^\flat)}$ is a trivial $G_{W_{\mathcal{O}_E}(V_i^\flat)}$-torsor. Since $\mathcal{P}_{W_{\mathcal{O}_E}(S^\flat)}$ is affine and $W_{\mathcal{O}_E}(S^\flat) = \prod_i W_{\mathcal{O}_E}(V_i^\flat)$, it follows that $\mathcal{P}_{W_{\mathcal{O}_E}(S^\flat)}$ has a $W_{\mathcal{O}_E}(S^\flat)$-valued point, and hence is a trivial $G_{W_{\mathcal{O}_E}(S^\flat)}$-torsor. We fix a trivialization $\mathcal{P}_{W_{\mathcal{O}_E}(S^\flat)} \simeq G_{W_{\mathcal{O}_E}(S^\flat)}$. Let $\xi \in I_S$ be a generator. The condition (3) implies that, for each i, the base change of $F_{\mathcal{P}}$ along $W_{\mathcal{O}_E}(R^\flat) \to W_{\mathcal{O}_E}(V_i^\flat)$ corresponds to an element of $G(W_{\mathcal{O}_E}(V_i^\flat)[1/\xi])$ which is of the form $Z_i\mu(\xi)Z'_i$ for some $Z_i, Z'_i \in G(W_{\mathcal{O}_E}(V_i^\flat))$, via the induced trivialization $\mathcal{P}_{W_{\mathcal{O}_E}(V_i^\flat)} \simeq G_{W_{\mathcal{O}_E}(V_i^\flat)}$. We set

$$Z := (Z_i)_i \in G(W_{\mathcal{O}_E}(S^\flat)) = \prod_i G(W_{\mathcal{O}_E}(V_i^\flat)),$$

and similarly let $Z' := (Z'_i)_i \in G(W_{\mathcal{O}_E}(S^\flat))$. Then the base change of $F_{\mathcal{P}}$ along $W_{\mathcal{O}_E}(R^\flat) \to W_{\mathcal{O}_E}(S^\flat)$ corresponds to the element $Z\mu(\xi)Z'$. This means that the condition (2) is satisfied. □

5.7. *Examples.* We discuss some examples of G-μ-displays and G-Breuil–Kisin modules of type μ for certain pairs (G, μ).

We first discuss a pair (G, μ) of Hodge type. Let G be a connected reductive group scheme over $\mathcal{O}_E$ and $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ a cocharacter. We assume that there exists a closed immersion $G \hookrightarrow \mathrm{GL}_n$ over $\mathcal{O}_E$ such that the composition $\mathbb{G}_m \to G_{\mathcal{O}} \to \mathrm{GL}_{n,\mathcal{O}}$ is conjugate to the cocharacter defined by

$$t \mapsto \mathrm{diag}\,(\underbrace{t, \dots, t}_{s}, \underbrace{1, \dots, 1}_{n-s})$$

for some s. In particular μ is 1-bounded. We set $L := \mathcal{O}_E^n$. By [Kisin 2010, Proposition 1.3.2], there exists a finite set of tensors $\{s_\alpha\}_{\alpha\in\Lambda} \subset L^\otimes$ such that $G \hookrightarrow \mathrm{GL}_n$ is the pointwise stabilizer of $\{s_\alpha\}_{\alpha\in\Lambda}$, where $L^\otimes$ is the direct sum of all $\mathcal{O}_E$-modules obtained from L by taking tensor products, duals, symmetric powers, and exterior powers. Let

$$L_{\mathcal{O}} = L_{\mu,1} \oplus L_{\mu,0}$$

be the weight decomposition with respect to μ. (Here the composition $\mathbb{G}_m \to G_{\mathcal{O}} \to \mathrm{GL}_{n,\mathcal{O}}$ is also denoted by μ.)

Let (A, I) be a bounded $\mathcal{O}_E$-prism over $\mathcal{O}$. Let M be a Breuil–Kisin module of type μ over (A, I). We note that M is minuscule in the sense of Definition 3.1.5, and that the rank of the Hodge filtration $P^1 \subset (\phi^* M)/I(\phi^* M)$ is s. For a finite set of tensors $\{s_{\alpha,M}\}_{\alpha\in\Lambda} \subset M^{\otimes}$ which are F_M-invariant, we say that the pair $(M, \{s_{\alpha,M}\}_{\alpha\in\Lambda})$ is *G-μ-adapted* if there exist a (π, I)-completely étale covering $A \to B$ and an isomorphism $\psi : L_B \xrightarrow{\sim} M_B$ such that ψ carries s_α to $s_{\alpha,M}$ for each $\alpha \in \Lambda$ and the reduction modulo I of $\phi^*\psi$ identifies $(L_{\mu,1})_{B/IB} \subset L_{B/IB}$ with the Hodge filtration $(P^1)_{B/IB}$.

Proposition 5.7.1. *With the notation above, the groupoid $G\text{-}\mathrm{Disp}_\mu(A, I)$ is equivalent to the groupoid of G-μ-adapted pairs $(M, \{s_{\alpha,M}\}_{\alpha\in\Lambda})$ over (A, I).*

Proof. We shall construct a functor from the groupoid of G-μ-adapted pairs over (A, I) to $G\text{-}\mathrm{Disp}_\mu(A, I)$. Let $(M, \{s_{\alpha,M}\}_{\alpha\in\Lambda})$ be a G-μ-adapted pair over (A, I). Let

$$\mathcal{Q} := \underline{\mathrm{Isom}}_{\mathrm{Fil},\{s_\alpha\}}(L_A, \phi^* M) : (A, I)_{\text{ét}} \to \mathrm{Set}$$

be the functor sending $B \in (A, I)_{\text{ét}}$ to the set of isomorphisms $h : L_B \xrightarrow{\sim} (\phi^* M)_B$ preserving the filtrations and carrying s_α to $1 \otimes s_{\alpha,M}$ for each $\alpha \in \Lambda$. Here L_A is equipped with the filtration $\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}}$ given in Remark 3.2.2. We claim that $\mathcal{Q}$ is a $G_{\mu,A,I}$-torsor. For this, we may assume that there exists an isomorphism $\psi : L_A \xrightarrow{\sim} M$ such that ψ carries s_α to $s_{\alpha,M}$ for each $\alpha \in \Lambda$ and the reduction modulo I of $h := \phi^*\psi$ identifies $(L_{\mu,1})_{A/I}$ with P^1. Under the isomorphism $h : L_A \xrightarrow{\sim} \phi^* M$, we have

$$\{\mathrm{Fil}^i_\mu\}_{i\in\mathbb{Z}} = \{\mathrm{Fil}^i(\phi^* M)\}_{i\in\mathbb{Z}},$$

which in turn implies that $h \in \mathcal{Q}(A)$. To see this, it suffices to prove that $\mathrm{Fil}^1_\mu = \mathrm{Fil}^1(\phi^* M)$ since M is minuscule. We observe that Fil^1_μ and $\mathrm{Fil}^1(\phi^* M)$ are the inverse images of $(L_{\mu,1})_{A/I} \subset L_{A/I}$ and $P^1 \subset (\phi^* M)/I(\phi^* M)$, respectively. It then follows that $\mathrm{Fil}^1_\mu = \mathrm{Fil}^1(\phi^* M)$.

We define $\widetilde{M} := \mathrm{Fil}^1(\phi^* M) \otimes_A I^{-1}$. Since M is of type μ, we see that F_M restricts to an isomorphism $\widetilde{M} \xrightarrow{\sim} M$, and we may regard $\{1 \otimes s_{\alpha,M}\}_{\alpha\in\Lambda}$ as tensors of $\widetilde{M}$. Similarly to Example 5.3.3, we have

$$\mathcal{Q}_{\mathrm{BK}} = \underline{\mathrm{Isom}}_{\{s_\alpha\}}(L_A, \widetilde{M}),$$

where $\underline{\mathrm{Isom}}_{\{s_\alpha\}}(L_A, \widetilde{M})$ is the G_A-torsor over $\operatorname{Spec} A$ defined by sending an A-algebra B to the set of isomorphisms $L_B \xrightarrow{\sim} \widetilde{M}_B$ carrying s_α to $1 \otimes s_{\alpha,M}$ for each $\alpha \in \Lambda$. Moreover, we have

$$\mathcal{Q}_A = \underline{\mathrm{Isom}}_{\{s_\alpha\}}(L_A, \phi^* M).$$

The base change $\phi^*(F_M) : \phi^*\widetilde{M} \xrightarrow{\sim} \phi^* M$ induces an isomorphism $\alpha' : \phi^*(\mathcal{Q}_{\mathrm{BK}}) \xrightarrow{\sim} \mathcal{Q}_A$ of G_A-torsors. The $G_{\mu,A,I}$-torsor $\mathcal{Q}$ with α' is a G-μ-display over (A, I); see Remark 5.3.5. In this way, we obtain a functor from the groupoid of G-μ-adapted pairs over (A, I) to $G\text{-}\mathrm{Disp}_\mu(A, I)$.

One can prove that this functor is an equivalence in the same way as in the case of $G = \mathrm{GL}_n$; see Section 5.3. □

Remark 5.7.2. In [Kisin 2010, Proposition 1.3.4], [Kim and Madapusi Pera 2016, Theorem 2.5], and [Imai et al. 2023], it is observed that G-μ-adapted pairs naturally arise from crystalline Galois representations associated with integral canonical models of Shimura varieties of Hodge type with hyperspecial level structure. The notion of G-μ-adapted pairs plays a central role in the construction of integral canonical models in [Kisin 2010, Proposition 1.5.8] and [Kim and Madapusi Pera 2016, Section 3].

We include the following two important examples. The details will be presented elsewhere.

Example 5.7.3 (G-shtuka). Let G be a connected reductive group scheme over $\mathcal{O}_E$. We assume that k is an algebraic closure of $\mathbb{F}_q$ and let $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ be a 1-bounded (or equivalently, minuscule) cocharacter. Let C be an algebraically closed nonarchimedean field over $\mathcal{O}[1/\pi]$ with ring of integers $\mathcal{O}_C$. We consider the perfectoid space $S = \mathrm{Spa}(C, \mathcal{O}_C)$ and its tilt $S^\flat = \mathrm{Spa}(C^\flat, \mathcal{O}_C^\flat)$. We can show that the groupoid of G-μ-displays over $(W_{\mathcal{O}_E}(\mathcal{O}_C^\flat), I_{\mathcal{O}_C})$ is equivalent to the groupoid of G-*shtukas* over $S^\flat$ with one leg at S which are bounded by μ (or bounded by μ^{-1}, depending on the sign convention) introduced in [Scholze and Weinstein 2020]. See [Ito 2025, Section 5.1] for details.

Example 5.7.4 (orthogonal Breuil–Kisin module). Let $n = 2m$ be an even positive integer and we set $L := \mathcal{O}_E^n$. We define the quadratic form

$$Q : L \to \mathcal{O}_E$$

by $(a_1, \ldots, a_{2m}) \mapsto \sum_{i=1}^m a_i a_{2m-i+1}$. The quadratic form Q is perfect in the sense that the bilinear form on L defined by $(x, y) \mapsto Q(x+y) - Q(x) - Q(y)$ is perfect. Let $G := \mathrm{O}(Q) \subset \mathrm{GL}_n$ be the orthogonal group of Q, which is a smooth affine group scheme over $\mathcal{O}_E$. Let $\mu : \mathbb{G}_m \to G \subset \mathrm{GL}_n$ be the cocharacter defined by

$$t \mapsto \mathrm{diag}\,(t, 1, \ldots, 1, t^{-1}).$$

Let (A, I) be a bounded $\mathcal{O}_E$-prism. An *orthogonal Breuil–Kisin module of type* μ over (A, I) is a Breuil–Kisin module M of type μ over (A, I) together with a perfect quadratic form $Q_M : M \to A$ which is compatible with F_M in the sense that for every $x \in M$, we have $\phi(Q_M(x)) = Q_M(F_M(1 \otimes x))$ in $A[1/I]$. Let

$$\mathcal{P} := \underline{\mathrm{Isom}}_Q(L_A, M)$$

be the G_A-torsor over $\mathrm{Spec}\, A$ defined by sending an A-algebra B to the set of isomorphisms $L_B \simeq M_B$ of quadratic spaces. One can show that $\mathcal{P}$, together with the isomorphism $F_{\mathcal{P}} : (\phi^*\mathcal{P})[1/I] \xrightarrow{\sim} \mathcal{P}[1/I]$ induced by F_M, forms a G-Breuil–Kisin module of type μ over (A, I). This construction gives an equivalence between the groupoid of orthogonal Breuil–Kisin modules of type μ over (A, I) and the groupoid G-$\mathrm{BK}_\mu(A, I)$. Thus, by Proposition 5.3.8, the groupoid of orthogonal Breuil–Kisin modules of type μ over (A, I) is equivalent to the groupoid $G\text{-}\mathrm{Disp}_\mu(A, I)$. The details will be presented in a forthcoming paper.

Remark 5.7.5. Let the notation be as in Example 5.7.4. Our main result (Theorem 6.1.3) cannot be applied to Breuil–Kisin modules of type μ since μ is not 1-bounded as a cocharacter of GL_n. However, since μ is 1-bounded as a cocharacter of $G = \mathrm{O}(Q)$, the result can be applied to *orthogonal* Breuil–Kisin modules of type μ. Such an observation was made in [Lau 2021] in the context of the deformation theory of K3 surfaces.

6. Prismatic G-μ-displays over complete regular local rings

In this section, we prove the main result (Theorem 6.1.3) of this paper, which we state in Section 6.1. The proof will be given in Section 6.5. In Sections 6.2–6.4, we discuss a few technical results that will be used in the proof.

6.1. ***G-μ-displays on absolute prismatic sites.*** In this paper, we use the following definition.

Definition 6.1.1. Let R be a π-adically complete $\mathcal{O}$-algebra. A *prismatic G-μ-display over R* is defined to be an object of the groupoid

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta,\mathcal{O}_E}) := 2-\varprojlim_{(A,I)\in(R)_{\Delta,\mathcal{O}_E}} G\text{-}\mathrm{Disp}_\mu(A,I).$$

Remark 6.1.2. Giving a prismatic G-μ-display $\mathfrak{Q}$ over R is equivalent to giving a G-μ-display $\mathfrak{Q}_{(A,I)}$ over (A,I) for each $(A,I)\in(R)_{\Delta,\mathcal{O}_E}$ and an isomorphism

$$\gamma_f : f^*(\mathfrak{Q}_{(A,I)}) \xrightarrow{\sim} \mathfrak{Q}_{(A',I')}$$

for each morphism $f:(A,I)\to(A',I')$ in $(R)_{\Delta,\mathcal{O}_E}$, such that $\gamma_{f'}\circ(f'^*\gamma_f)=\gamma_{f'\circ f}$ (via the natural identification $f'^*\circ f^*\simeq(f'\circ f)^*$) for two morphisms $f:(A,I)\to(A',I')$ and $f':(A',I')\to(A'',I'')$. We call $\mathfrak{Q}_{(A,I)}$ the *value* of $\mathfrak{Q}$ at $(A,I)\in(R)_{\Delta,\mathcal{O}_E}$.

Assume that R is a complete regular local ring over $\mathcal{O}$ with residue field k. Let $(\mathcal{O}[[t_1,\dots,t_n]],(\mathcal{E}))$ be an $\mathcal{O}_E$-prism of Breuil–Kisin type with an isomorphism $R\simeq\mathcal{O}[[t_1,\dots,t_n]]/\mathcal{E}$ over $\mathcal{O}$ (where $n\geq 0$ is the dimension of R). Such an $\mathcal{O}_E$-prism exists; see for example [Cheng 2018, Section 3.3]. We set $\mathfrak{S}_\mathcal{O}:=\mathcal{O}[[t_1,\dots,t_n]]$. Our goal is to prove the following result.

Theorem 6.1.3. *Assume that the cocharacter μ is 1-bounded. Then the functor*

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta,\mathcal{O}_E})\to G\text{-}\mathrm{Disp}_\mu(\mathfrak{S}_\mathcal{O},(\mathcal{E})),\quad \mathfrak{Q}\mapsto\mathfrak{Q}_{(\mathfrak{S}_\mathcal{O},(\mathcal{E}))},$$

given by evaluation at $(\mathfrak{S}_\mathcal{O},(\mathcal{E}))$ is an equivalence.

The rest of this section is devoted to the proof of Theorem 6.1.3.

6.2. ***Coproducts of Breuil–Kisin prisms.*** In this subsection, we establish some properties of the object

$$(\mathfrak{S}_\mathcal{O},(\mathcal{E}))\in(R)_{\Delta,\mathcal{O}_E}.$$

We begin with the following result.

Proposition 6.2.1. *For any $(A,I)\in(R)_{\Delta,\mathcal{O}_E}$, there exists a flat covering $(A,I)\to(A',I')$ in $(R)_{\Delta,\mathcal{O}_E}$ such that (A',I') admits a morphism $(\mathfrak{S}_\mathcal{O},(\mathcal{E}))\to(A',I')$ in $(R)_{\Delta,\mathcal{O}_E}$.*

Proof. We may assume that (A,I) is orientable by Remark 2.5.14. Let $d\in I$ be a generator. Let $v_1,\dots,v_n\in A$ be elements such that each v_i is a lift of the image of $t_i\in\mathfrak{S}_\mathcal{O}$ under the composition $\mathfrak{S}_\mathcal{O}\to R\to A/I$. Let $B:=A\otimes_\mathcal{O}\mathfrak{S}_\mathcal{O}$. We set

$$x_i := 1\otimes t_i - v_i\otimes 1\in B.$$

Then the morphism $A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ of animated rings is faithfully flat. Indeed, using that the natural homomorphism $\mathcal{O}[t_1, \dots, t_n] \to \mathfrak{S}_{\mathcal{O}}$ is flat, we see that $A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ is flat. Since the composition $\mathfrak{S}_{\mathcal{O}} \to R \to A/(\pi, d)$ induces a homomorphism $B/(\pi, d, x_1, \dots, x_n) \to A/(\pi, d)$ over $A/(\pi, d)$, it follows that $A/^{\mathbb{L}}(\pi, d) \to B/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ is faithfully flat.

Let $\mathfrak{S}_{\mathcal{O},\infty}$ be the $(\pi, \mathcal{E})$-adic completion of a colimit $\varinjlim_{\phi} \mathfrak{S}_{\mathcal{O}}$ of the diagram

$$\mathfrak{S}_{\mathcal{O}} \xrightarrow{\phi} \mathfrak{S}_{\mathcal{O}} \xrightarrow{\phi} \mathfrak{S}_{\mathcal{O}} \to \cdots.$$

Since $\phi : \mathfrak{S}_{\mathcal{O}} \to \mathfrak{S}_{\mathcal{O}}$ is faithfully flat, we see that $\mathfrak{S}_{\mathcal{O}} \to \mathfrak{S}_{\mathcal{O},\infty}$ is $(\pi, \mathcal{E})$-completely faithfully flat. In fact, it is faithfully flat by [Yekutieli 2018, Theorem 1.5]. We set $B' := B \otimes_{\mathfrak{S}_{\mathcal{O}}} \mathfrak{S}_{\mathcal{O},\infty}$. Then $A/^{\mathbb{L}}(\pi, d) \to B'/^{\mathbb{L}}(\pi, d, x_1, \dots, x_n)$ is faithfully flat as well. Thus, by Proposition 2.6.6, we can consider the prismatic envelope

$$(A', I') := (B'\{J/I\}^{\wedge}, IB'\{J/I\}^{\wedge})$$

of B' over (A, I) with respect to the ideal $J := (d, x_1, \dots, x_n) \subset B'$. The map $(A, I) \to (A', I')$ is a flat covering.

We shall construct a morphism $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A', I')$ in $(R)_{\Delta, \mathcal{O}_E}$. We remark that since A'/I' is not necessarily $\mathfrak{m}$-adically complete for the maximal ideal $\mathfrak{m} \subset R$, it is not clear that the natural homomorphism $\mathfrak{S}_{\mathcal{O}} \to A'$ induces a morphism $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A', I')$ in $(R)_{\Delta, \mathcal{O}_E}$. Instead, we construct a morphism $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A', I')$ as follows. Since $\mathfrak{S}_{\mathcal{O},\infty}$ can be identified with the $(\pi, \mathcal{E})$-adic completion of

$$\bigcup_{m \geq 0} \mathfrak{S}_{\mathcal{O}}[t_1^{1/q^m}, \dots, t_n^{1/q^m}],$$

the quotient $R_\infty := \mathfrak{S}_{\mathcal{O},\infty}/\mathcal{E}$ is the π-adic completion of $\bigcup_{m \geq 0} R[\bar{t}_1^{1/q^m}, \dots, \bar{t}_n^{1/q^m}]$, where $\bar{t}_i \in R$ is the image of t_i. Here

$$\mathfrak{S}_{\mathcal{O}}[t_1^{1/q^m}, \dots, t_n^{1/q^m}] = \mathfrak{S}_{\mathcal{O}}[X_1, \dots, X_n]/(X_1^{q^m} - t_1, \dots, X_n^{q^m} - t_n)$$

and similarly for $R[\bar{t}_1^{1/q^m}, \dots, \bar{t}_n^{1/q^m}]$. The composition $R \to A/I \to A'/I'$ factors through the homomorphism

$$g : R_\infty \to A'/I'$$

defined by sending $\bar{t}_i^{1/q^m}$ to the image of $1 \otimes t_i^{1/q^m} \in A'$, which is well-defined since $1 \otimes t_i = v_i \otimes 1$ in A'/I'. By Lemma 2.3.5, there exists a unique map $(\mathfrak{S}_{\mathcal{O},\infty}, (\mathcal{E})) \to (A', I')$ of bounded $\mathcal{O}_E$-prisms which induces g. By construction, the composition

$$(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (\mathfrak{S}_{\mathcal{O},\infty}, (\mathcal{E})) \to (A', I')$$

is a morphism in $(R)_{\Delta, \mathcal{O}_E}$. □

Remark 6.2.2. Assume that $\mathcal{O}_E = \mathbb{Z}_p$. In this case, Proposition 6.2.1 is proved in [Bhatt and Scholze 2022, Example 7.13] and [Anschütz and Le Bras 2023, Lemma 5.14], using [Bhatt and Scholze 2022, Proposition 7.11]. Moreover, if $\mathfrak{S}_{\mathcal{O}} = W(k)[[t]]$ and $\mathcal{E} \in W(k)[[t]]$ is an Eisenstein polynomial, then an alternative argument using prismatic envelopes is given in [Bhatt and Scholze 2023, Example 2.6]. Our argument is similar to the one given there, but we have to modify it slightly in order to treat the case where A/I is not $\mathfrak{m}$-adically complete.

In a similar way, we obtain the following result:

Lemma 6.2.3. *Let $(A, I) \in (R)_{\Delta,\mathcal{O}_E}$ and let $f_1, f_2 : (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A, I)$ be two morphisms in $(R)_{\Delta,\mathcal{O}_E}$. If $f_1(t_i) = f_2(t_i)$ in A for any i, then we have $f = g$.*

Proof. As in the proof of Proposition 6.2.1, let $\mathfrak{S}_{\mathcal{O},\infty}$ be the $(\pi, \mathcal{E})$-adic completion of $\varinjlim_{\phi} \mathfrak{S}_{\mathcal{O}}$, which is faithfully flat over $\mathfrak{S}_{\mathcal{O}}$. After replacing (A, I) by a flat covering, we may assume that $f_1 : (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E})) \to (A, I)$ factors through a map $\tilde{f}_1 : (\mathfrak{S}_{\mathcal{O},\infty}, (\mathcal{E})) \to (A, I)$ of $\mathcal{O}_E$-prisms. Since $\mathfrak{S}_{\mathcal{O},\infty}$ is the $(\pi, \mathcal{E})$-adic completion of $\bigcup_{m\geq 0} \mathfrak{S}_{\mathcal{O}}[t_1^{1/q^m}, \dots, t_n^{1/q^m}]$, there exists a map $\tilde{f}_2 : \mathfrak{S}_{\mathcal{O},\infty} \to A$ extending f_2 such that

$$\tilde{f}_2(t_i^{1/q^m}) = \tilde{f}_1(t_i^{1/q^m})$$

for all m and i. The map $\tilde{f}_2$ preserves the δ_E-structures by Corollary 2.2.16. It suffices to prove that $\tilde{f}_1 = \tilde{f}_2$. We note that both f_1 and f_2 induce the same homomorphism $R \to A/I$. Since $R_\infty = \mathfrak{S}_{\mathcal{O},\infty}/\mathcal{E}$ is the π-adic completion of $\bigcup_{m\geq 0} R[\bar{t}_1^{1/q^m}, \dots, \bar{t}_n^{1/q^m}]$, it follows that the homomorphism $R_\infty \to A/I$ induced by $\tilde{f}_1$ agrees with the one induced by $\tilde{f}_2$. Then, by Lemma 2.3.5, we conclude that $\tilde{f}_1 = \tilde{f}_2$. □

We next study a coproduct of two copies of $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ in the category $(R)_{\Delta,\mathcal{O}_E}$. To simplify the notation, we write

$$(A, I) := (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$$

in the rest of this section. We set

$$B := A[\![x_1, \dots, x_n]\!]$$

and let $p'_1 : A \to B$ be the natural homomorphism. There exists a unique δ_E-structure on B such that p'_1 is a homomorphism of δ_E-rings and the associated Frobenius $\phi : B \to B$ sends x_i to $(x_i + t_i)^q - t_i^q$ for every i. We consider the prismatic envelope

$$(A^{(2)}, I^{(2)})$$

of B over (A, I) with respect to the ideal $(\mathcal{E}, x_1, \dots, x_n) \subset B$ as in Proposition 2.6.6. Let $p_1 : (A, I) \to (A^{(2)}, I^{(2)})$ denote the natural map. We view $(A^{(2)}, I^{(2)})$ as an object of $(R)_{\Delta,\mathcal{O}_E}$ via the homomorphism $\bar{p}_1 : R \to A^{(2)}/I^{(2)}$ induced by p_1.

The homomorphism $p'_2 : A \to B$ over $\mathcal{O}$ defined by $t_i \mapsto x_i + t_i$ is a homomorphism of δ_E-rings. Let $p_2 : A \to A^{(2)}$ be the composition of p'_2 with the natural homomorphism $B \to A^{(2)}$.

Lemma 6.2.4. *Let the notation be as above.*

(1) *We have $p_2(I) \subset I^{(2)}$, and the induced map $p_2 : (A, I) \to (A^{(2)}, I^{(2)})$ is a morphism in $(R)_{\Delta,\mathcal{O}_E}$.*

(2) *The object $(A^{(2)}, I^{(2)}) \in (R)_{\Delta,\mathcal{O}_E}$ with the morphisms $p_1, p_2 : (A, I) \to (A^{(2)}, I^{(2)})$ is a coproduct of two copies of (A, I) in the category $(R)_{\Delta,\mathcal{O}_E}$.*

Proof. (1) It suffices to show that the composition of $p_2 : A \to A^{(2)}$ with $A^{(2)} \to A^{(2)}/I^{(2)}$ coincides with the composition of $A \to R$ with $\bar{p}_1 : R \to A^{(2)}/I^{(2)}$. For any $h \in A$, the element $p'_2(h) - p'_1(h) \in B$ is contained in the ideal $(x_1, \dots, x_n) \subset B$. Since the image of x_i in $A^{(2)}$ is contained in $I^{(2)}$, we have $p_2(h) - p_1(h) \in I^{(2)}$, which implies the assertion.

(2) We have to show that for any $(A', I') \in (R)_{\Delta,\mathcal{O}_E}$ and two morphisms $f_1, f_2 : (A, I) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$, there exists a unique morphism

$$f : (A^{(2)}, I^{(2)}) \to (A', I')$$

in $(R)_{\Delta,\mathcal{O}_E}$ such that $f \circ p_1 = f_1$ and $f \circ p_2 = f_2$.

We first prove the uniqueness of f. Let $f' : B \to A'$ be the composition of f with $B \to A^{(2)}$. Then we have $f' \circ p'_j = f_j$ ($j = 1, 2$), and f' sends $x_i = p'_2(t_i) - p'_1(t_i)$ to

$$f_2(t_i) - f_1(t_i) \in I' \subset A'$$

for any i. Since A' is I'-adically complete, such a homomorphism f' of δ_E-rings is uniquely determined (if it exists). The uniqueness of f now follows from the universal property of the prismatic envelope $(A^{(2)}, I^{(2)})$.

We next prove the existence of f. Since $f_2(t_i) - f_1(t_i) \in I' \subset A'$ and A' is I'-adically complete, there exists a unique homomorphism $f' : B \to A'$ over $\mathcal{O}$ such that $f' \circ p'_1 = f_1$ and $f'(x_i) = f_2(t_i) - f_1(t_i)$ for every i.

We claim that f' is a homomorphism of δ_E-rings. Indeed, as in the proof of Proposition 6.2.1, let A_∞ be the $(\pi, \mathcal{E})$-adic completion of $\varinjlim_\phi A$. Then A_∞ is faithfully flat over A. After replacing (A', I') by a flat covering, we may assume that f_j factors through a morphism $\tilde{f}_j : (A_\infty, IA_\infty) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$ for each $j = 1, 2$. For an integer $m \geq 0$ and i, we set

$$x_{i,m} := \tilde{f}_2(t_i^{1/q^m}) - \tilde{f}_1(t_i^{1/q^m}) \in A'.$$

Since we have $x_{i,m}^{q^m} \in (\pi, I')$, it follows that A' is $(x_{1,m}, \dots, x_{n,m})$-adically complete for every m (see [Stacks 2005–, Tag 090T] for example). Thus, for each $m \geq 0$, there exists a unique homomorphism $f'(m) : B \to A'$ such that $f'(m) \circ p'_1$ is the composition

$$A \to A_\infty \xrightarrow{\phi^{-m}} A_\infty \xrightarrow{\tilde{f}_1} A'$$

and $f'(m)(x_i) = x_{i,m}$ for any i. Since $f'(m) = f'(m+1) \circ \phi$, they give rise to a homomorphism $\tilde{f}' : \varinjlim_\phi B \to A'$. By Corollary 2.2.16, $\tilde{f}'$ is a homomorphism of δ_E-rings. Since f' is the composition $B \to \varinjlim_\phi B \to A'$, we conclude that f' is a homomorphism of δ_E-rings.

By the universal property of the prismatic envelope $(A^{(2)}, I^{(2)})$, the homomorphism f' extends to a unique morphism $f : (A^{(2)}, I^{(2)}) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$. By construction, we have $f \circ p_1 = f_1$. It follows from Lemma 6.2.3 that $f \circ p_2 = f_2$.

The proof of Lemma 6.2.4 is complete. □

Remark 6.2.5. It might be more natural to expect that the prismatic envelope (C, IC) of $A \otimes_{\mathcal{O}} A$ over (A, I) with respect to the ideal $(\mathcal{E} \otimes 1, t_1 \otimes 1 - 1 \otimes t_1, \dots, t_n \otimes 1 - 1 \otimes t_n)$ is a coproduct of two copies of (A, I) in the category $(R)_{\Delta, \mathcal{O}_E}$, where we regard $A \otimes_{\mathcal{O}} A$ as an A-algebra via the homomorphism $a \mapsto a \otimes 1$. However, this does not seem to be the case in general. For example, it is not clear that the homomorphism $A \to C$, $a \mapsto 1 \otimes a$, induces a morphism $(A, I) \to (C, IC)$ in $(R)_{\Delta, \mathcal{O}_E}$ (see the proof of Proposition 6.2.1).

Let

$$m : (A^{(2)}, I^{(2)}) \to (A, I)$$

be the unique morphism in $(R)_{\Delta, \mathcal{O}_E}$ such that $m \circ p_1 = m \circ p_2 = \mathrm{id}_{(A,I)}$. Let K be the kernel of $m : A^{(2)} \to A$. Let $d \in I^{(2)}$ be a generator.

Lemma 6.2.6 (cf. [Anschütz and Le Bras 2023, Lemma 5.15]). *We have $\phi(K) \subset dK$.*

Proof. It suffices to show that $\phi(K) \subset dA^{(2)}$. Indeed, let $x \in K$, and we assume that $\phi(x) = dy$ for some $y \in A^{(2)}$. Then, since $m(\phi(x)) = 0$ and $m(d) \in A$ is a nonzerodivisor, we have $y \in K$.

We shall prove that $\phi(K) \subset dA^{(2)}$. We may assume that $d = p_1(\mathcal{E})$. The image $p_1'(\mathcal{E}) \in B$ is also denoted by d. It follows from Proposition 2.6.6 that $A^{(2)}$ can be identified with the (π, d)-adic completion of

$$C := B\{x_1/d, \dots, x_n/d\}.$$

We write $y_i := x_i/d$. The composition $C \to A^{(2)} \to A$ sends $\delta_E^j(y_i)$ to 0 for any $j \geq 0$ and any i. Here δ_E^j is the j-th iterate of the map $\delta_E : C \to C$. Since the kernel of the homomorphism $B \to A$ defined by $x_i \mapsto 0$ $(1 \leq i \leq n)$ coincides with $(x_1, \dots, x_n)$, it follows that the kernel K_0 of $C \to A$ is generated by

$$\{\delta_E^j(y_i)\}_{1 \leq i \leq n, j \geq 0}.$$

We note that K can be identified with the (π, d)-adic completion of K_0. We also note that $dA^{(2)} = \bigcap_{l \geq 0}(dA^{(2)} + (\pi, d)^l A^{(2)})$ since $A^{(2)}/d$ is π-adically complete (see Remark 2.3.2). It then suffices to show that $\phi(\delta_E^j(y_i)) \in dA^{(2)}$ for any $j \geq 0$ and any i. This can be proved by the same argument as in the first paragraph of the proof of [Anschütz and Le Bras 2023, Lemma 5.15] when $\mathcal{E} \in A$ is not contained in πA. A similar argument holds when $\mathcal{E} \in \pi A$. We include the argument in this case for the convenience of the reader.

We may assume that $\mathcal{E} = \pi$. In fact, we prove a more general statement: for any $j \geq 0$, we have $\phi^l(\delta_E^j(y_i)) \in \pi^l A^{(2)}$ for any $l \geq 1$ and any i. We proceed by induction on j. Let $u_i := x_i + t_i = \pi y_i + t_i \in A^{(2)}$. Then we have

$$\phi^l(x_i) = u_i^{q^l} - t_i^{q^l} = \sum_{0 \leq h \leq q^l - 1} \binom{q^l}{h} (\pi y_i)^{q^l - h} t_i^h \in \pi^{l+1} A^{(2)}.$$

Thus, we obtain $\phi^l(y_i) \in \pi^l A^{(2)}$, which proves the assertion in the case where $j = 0$. Suppose that the assertion holds for some $j \geq 0$. Since

$$\pi \phi^l(\delta_E^{j+1}(y_i)) = \phi^l(\pi \delta_E^{j+1}(y_i)) = \phi^l(\phi(\delta_E^j(y_i)) - \delta_E^j(y_i)^q) = \phi^{l+1}(\delta_E^j(y_i)) - \phi^l(\delta_E^j(y_i))^q,$$

the induction hypothesis implies that $\pi \phi^l(\delta_E^{j+1}(y_i)) \in \pi^{l+1} A^{(2)}$, whence $\phi^l(\delta_E^{j+1}(y_i)) \in \pi^l A^{(2)}$. $\square$

The following lemma plays a crucial role in the proof of Theorem 6.1.3 (especially in the proof of Proposition 6.4.1 below). As in the proof of Lemma 6.2.6, we set $y_i := x_i/d \in A^{(2)}$.

Lemma 6.2.7. *Let $M \subset A^{(2)}$ be the ideal generated by $\phi(y_1)/d, \dots, \phi(y_n)/d \in K$. Then we have inclusions*

$$\phi(K) \subset dM + d(\pi, d)K \quad \textit{and} \quad \phi(M) \subset d(t_1, \dots, t_n)M + d(\pi, d)K.$$

The proof of Lemma 6.2.7 will be given in Section 6.3.

Remark 6.2.8. There exists a coproduct $(A^{(3)}, I^{(3)})$ of three copies of (A, I) in the category $(R)_{\Delta,\mathcal{O}_E}$. Indeed, one can define $(A^{(3)}, I^{(3)})$ as a pushout of the diagram

$$(A^{(2)}, I^{(2)}) \xleftarrow{p_2} (A, I) \xrightarrow{p_1} (A^{(2)}, I^{(2)}),$$

which exists since p_1 is a flat map (see Remark 2.5.4). Let $q_1, q_2, q_3 : (A, I) \to (A^{(3)}, I^{(3)})$ denote the associated three morphisms. For $1 \le i < j \le 3$, let $p_{ij} : (A^{(2)}, I^{(2)}) \to (A^{(3)}, I^{(3)})$ be the unique morphism such that $p_{ij} \circ p_1 = q_i$ and $p_{ij} \circ p_2 = q_j$.

Corollary 6.2.9. *Let $m : (A^{(3)}, I^{(3)}) \to (A, I)$ be the unique morphism in $(R)_{\Delta,\mathcal{O}_E}$ such that $m \circ q_i = \mathrm{id}_{(A,I)}$ for $i = 1, 2, 3$. Let L be the kernel of $m : A^{(3)} \to A$. Let $d \in I^{(3)}$ be a generator. Then the following assertions hold*:

(1) *We have $\phi(L) \subset dL$.*

(2) *Let $N \subset A^{(3)}$ be the ideal generated by $\{\phi(p_{12}(y_l))/d, \phi(p_{23}(y_l))/d\}_{1 \le l \le n} \subset L$. Then we have inclusions*

$$\phi(L) \subset dN + d(\pi, d)L \quad \textit{and} \quad \phi(N) \subset d(q_1(t_1), \dots, q_1(t_n))N + d(\pi, d)L.$$

Proof. We may assume that d is the image of a generator of $I^{(2)}$, again denoted by d, under the homomorphism p_{12}. As in Remark 6.2.8, we identify $A^{(3)}$ with the (π, d)-adic completion of $A_0^{(3)} := A^{(2)} \otimes_{p_2,A,p_1} A^{(2)}$. Under this identification, the homomorphism p_{12} (resp. p_{23}) is induced by the homomorphism $A^{(2)} \to A_0^{(3)}$ defined by $a \mapsto a \otimes 1$ (resp. $a \mapsto 1 \otimes a$). The kernel L_0 of the natural homomorphism $A_0^{(3)} \to A$ coincides with $K \otimes_A A^{(2)} + A^{(2)} \otimes_A K$, and L is the (π, d)-adic completion of L_0.

In order to prove the assertion (1), it suffices to show that for any element $x \in L$ which lies in the image of $L_0 \to L$, we have $\phi(x) \in dA^{(3)}$. (Note that $A^{(3)}/d$ is π-adically complete by Remark 2.3.2.) This follows from Lemma 6.2.6. Similarly, the assertion (2) follows from Lemma 6.2.7. We note here that, since $q_j(t_l) - q_i(t_l) = p_{ij}(x_l) \in dL$ for $1 \le i < j \le 3$, the ideal $d(q_1(t_1), \dots, q_1(t_n))N + d(\pi, d)L$ is unchanged if we replace q_1 by q_i $(1 \le i \le 3)$. □

Remark 6.2.10. Assume that $\mathcal{O}_E = \mathbb{Z}_p$. Under the assumption that $n = 1$ and R is p-torsion-free, Anschütz and Le Bras [2023, Section 5.2] gave a proof of the analogue of Theorem 6.1.3 for minuscule Breuil–Kisin modules. (We will come back to this result in Section 7.1.) In the proof, they use that the map $K \to K$, $x \mapsto \phi(x)/d$ is topologically nilpotent with respect to the (p, d)-adic topology [loc. cit., Lemma 5.15]. This topological nilpotence may not be true if $n \ge 2$ or $p = 0$ in R. We will use Lemma 6.2.7, Corollary 6.2.9, and the fact that the local ring A is complete and noetherian to overcome this issue; see Section 6.4.

6.3. *Proof of Lemma 6.2.7.* The proof of Lemma 6.2.7 will require some preliminary results. We first introduce some notation.

If $\mathcal{E}$ is not contained in πA, then the image of $\mathcal{E}$ in A/π is a nonzerodivisor (since A/π is an integral domain). In this case, the δ_E-ring $A\{\phi(\mathcal{E})/\pi\}$ is π-torsion-free, and is isomorphic to the $\mathcal{O}_E$-PD envelope $D_{(\mathcal{E})}(A)$ of A with respect to the ideal $(\mathcal{E})$; see Corollary 2.6.5. Let A'' be the π-adic completion of $A\{\phi(\mathcal{E})/\pi\}$, and let $g : A \to A''$ be the natural homomorphism. We note that A'' is also π-torsion-free. We consider the following pushout squares of δ_E-rings:

$$\begin{array}{ccccccc} A & \xrightarrow{p_1'} & B & \longrightarrow & A^{(2)} & \xrightarrow{m} & A \\ \phi\downarrow & & \downarrow & & \downarrow & & \downarrow\phi \\ A & \longrightarrow & B' & \longrightarrow & A^{(2)'} & \longrightarrow & A \\ g\downarrow & & \downarrow & & \downarrow & & \downarrow g \\ A'' & \longrightarrow & B_0'' & \longrightarrow & A_0^{(2)''} & \longrightarrow & A'' \end{array}$$

Let $A^{(2)''}$ be the π-adic completion of $A_0^{(2)''}$ and K'' the kernel of the induced homomorphism $A^{(2)''} \to A''$. Since $A \to A^{(2)}$ is flat (by Proposition 2.6.6 and [Yekutieli 2018, Theorem 1.5]), so is $A'' \to A_0^{(2)''}$. It follows that $A^{(2)''}$ is π-torsion-free. In the case where $\mathcal{E} \in \pi A$, we set $A^{(2)''} := A^{(2)}$ and $K'' := K$.

Lemma 6.3.1. *Let the notation be as above. Then the following assertions hold*:

(1) *We have $\phi(K'') \subset \pi K''$.*

(2) *We have $x_i \in \pi K''$ for any $1 \le i \le n$. (Here we denote the image of $x_i \in B$ in $A^{(2)''}$ again by x_i.) We set $w_i := x_i/\pi \in K''$. Then $K''/\pi K''$ is generated by the images of $\{\delta_E^j(w_i)\}_{1\le i\le n, j\ge 0}$ as an $A^{(2)''}$-module.*

Proof. If $\mathcal{E} \in \pi A$, then the assertions follow from Lemma 6.2.6 and its proof. Thus, we may assume that $\mathcal{E}$ is not contained in πA. Let $h : A^{(2)} \to A^{(2)''}$ denote the natural homomorphism. Since $g(\phi(\mathcal{E}))/\pi \in A''$ is a unit by Lemma 2.3.3(1), it follows that $h(d) \in A^{(2)''}$ is a unit multiple of π. The kernel K'' of $A^{(2)''} \to A''$ can be identified with the π-adic completion of h^*K. Therefore, the assertion (1) follows from Lemma 6.2.6.

Using that the image of $g(\phi(\mathcal{E}))$ in B_0'' is a unit multiple of π, we see that $A^{(2)''}$ agrees with the π-adic completion of $B_0''\{x_1/\pi, \dots, x_n/\pi\}$. Since the kernel of $B_0'' \to A''$ is generated by $x_1, \dots, x_n$, it follows that the kernel of $B_0''\{x_1/\pi, \dots, x_n/\pi\} \to A''$ is generated by $\{\delta_E^j(x_i/\pi)\}_{1\le i\le n, j\ge 0}$, which implies (2). □

Lemma 6.3.2. *We define*

$$\phi_1 : K'' \to K'', \quad x \mapsto \phi(x)/\pi.$$

The induced ϕ-linear homomorphism $K''/\pi K'' \to K''/\pi K''$ is denoted by the same symbol ϕ_1. Let $\overline{M}'' \subset K''/\pi K''$ be the $A^{(2)''}$-submodule generated by the images of $\phi_1(w_1), \dots, \phi_1(w_n) \in K''$. Then we have inclusions

$$\phi_1(K''/\pi K'') \subset \overline{M}'' \quad \text{and} \quad \phi_1(\overline{M}'') \subset (t_1, \dots, t_n)\overline{M}'',$$

where we denote the image of $t_i \in A^{(2)}$ in $A^{(2)''}$ again by t_i.

Proof. We have $x^q = \phi(x) - \pi\delta_E(x) \in \pi K''$ for every $x \in K''$ by Lemma 6.3.1. Let $J \subset K''$ be the ideal generated by $\{x^q/\pi\}_{x\in K''}$. For any $x \in K''$, we have

$$\phi_1(x^q/\pi) = \phi(x^q/\pi)/\pi = \phi(x)^q/\pi^2 = \pi^{q-1}(\phi_1(x)^q/\pi) \in \pi J,$$

and thus we obtain $\phi_1(J) \subset \pi J$.

We shall prove that $K''/(J + \pi K'')$ is generated by the images of $w_1, \dots, w_n$ as an $A^{(2)''}$-module. By Lemma 6.3.1, it suffices to show that for any $j \geq 0$ and any i, the image of $\delta_E^j(w_i)$ in $K''/(J + \pi K'')$ is contained in the $A^{(2)''}$-submodule of $K''/(J + \pi K'')$ generated by the images of $w_1, \dots, w_n$. We proceed by induction on j. If $j = 0$, then the assertion holds trivially. Assume that the assertion is true for some $j \geq 0$. Since

$$\phi_1(w_i) = \phi(x_i)/\pi^2 = ((x_i + t_i)^q - t_i^q)/\pi^2 = ((\pi w_i + t_i)^q - t_i^q)/\pi^2,$$

we can write $\phi_1(w_i)$ as

$$\phi_1(w_i) = \pi^{q-2} w_i^q + (q/\pi) t_i^{q-1} w_i + \pi b_i \tag{6-1}$$

for some element $b_i \in K''$. For any $x \in K''$, we have $\delta_E(x) = \phi_1(x)$ in K''/J. Thus the image of $\delta_E^{j+1}(w_i)$ in $K''/(J + \pi K'')$ agrees with the one of $\phi_1(\delta_E^j(w_i))$, which is contained in the $A^{(2)''}$-submodule of $K''/(J + \pi K'')$ generated by the images of $\phi_1(w_1), \dots, \phi_1(w_n)$ by the induction hypothesis. Then (6-1) implies the assertion for $j + 1$.

We have shown that every $x \in K''$ can be written as

$$x = \left(\sum_{1\leq i\leq n} a_i w_i\right) + b + \pi c$$

for some $a_i \in A^{(2)''}$ $(1 \leq i \leq n)$, $b \in J$, and $c \in K''$. Since $\phi_1(b) \in \pi J$, the image of $\phi_1(x)$ in $K''/\pi K''$ coincides with that of $\sum_{1\leq i\leq n} \phi(a_i)\phi_1(w_i)$. This proves that $\phi_1(K''/\pi K'') \subset \overline{M}''$. Moreover, since $\phi_1(w_i^q) = \phi(w_i)\phi_1(w_i^{q-1})$ is contained in $\pi K''$, it follows from (6-1) that the image of $\phi_1(\phi_1(w_i))$ in $K''/\pi K''$ is equal to that of $\phi_1((q/\pi)t_i^{q-1} w_i) = (q/\pi)t_i^{q(q-1)}\phi_1(w_i)$. This proves that $\phi_1(\overline{M}'') \subset (t_1, \dots, t_n)\overline{M}''$. □

We now prove Lemma 6.2.7.

Proof of Lemma 6.2.7. We first treat the case where $\mathcal{E} \in \pi A$. In this case d is a unit multiple of π. Thus, the assertion follows from Lemma 6.3.2.

We now assume that $\mathcal{E}$ is not contained in πA. We define

$$\phi_1 : K \to K, \quad x \mapsto \phi(x)/d.$$

The induced ϕ-linear homomorphism $K/(\pi, d)K \to K/(\pi, d)K$ is also denoted by ϕ_1. Let $\overline{M} \subset K/(\pi, d)K$ be the $A^{(2)}$-submodule generated by the images of $\phi_1(y_1), \dots, \phi_1(y_n) \in K$. It suffices to prove that $\phi_1(K/(\pi, d)K) \subset \overline{M}$ and $\phi_1(\overline{M}) \subset (t_1, \dots, t_n)\overline{M}$.

Let $f : A^{(2)} \to A^{(2)'}$ denote the natural homomorphism. Let K' be the kernel of the homomorphism $A^{(2)'} \to A$, which can be identified with f^*K. We define $\phi'_1 : K' \to K'$ by $x \mapsto \phi(x)/f(d)$, and let

$\overline{M}' \subset K'/(\pi, f(d))K'$ be the $A^{(2)'}$-submodule generated by the images of $\phi'_1(f(y_1)), \dots, \phi'_1(f(y_n)) \in K'$. Since $\phi : A \to A$ is faithfully flat, so is f. Therefore, in order to prove the assertion, it is enough to prove that

$$\phi'_1(K'/(\pi, f(d))K') \subset \overline{M}' \quad \text{and} \quad \phi'_1(\overline{M}') \subset (f(t_1), \dots, f(t_n))\overline{M}'. \tag{6-2}$$

The homomorphism $A^{(2)'} \to A^{(2)''}$ induced by $g : A \to A''$ is again denoted by g. The element $g(f(d))$ is a unit multiple of π in $A^{(2)''}$. Thus, for $\phi_1 : K'' \to K''$ defined in Lemma 6.3.2, the element $g(\phi'_1(x))$ is a unit multiple of $\phi_1(g(x))$ for any $x \in K'$. Also, the induced homomorphism $A^{(2)'}/(\pi, f(d)) \to A^{(2)''}/\pi$, again denoted by g, sends $\overline{M}'$ into $\overline{M}''$. It follows from Lemma 6.3.2 that, for any $x \in K'/(\pi, f(d))K'$ (resp. $x \in \overline{M}'$), we have

$$g(\phi'_1(x)) \in \overline{M}'' \quad (\text{resp. } g(\phi'_1(x)) \in (g(f(t_1)), \dots, g(f(t_n)))\overline{M}''). \tag{6-3}$$

Since $A''/\pi \simeq D_{(\mathcal{E})}(A)/\pi$, we can find a homomorphism

$$s : A''/\pi \to A/(\pi, \phi(\mathcal{E}))$$

of $\mathcal{O}_E$-algebras such that the composition $A/(\pi, \phi(\mathcal{E})) \xrightarrow{g} A''/\pi \xrightarrow{s} A/(\pi, \phi(\mathcal{E}))$ is the identity; see Example 2.6.2 and Lemma 2.6.3. We consider the following pushout squares of $\mathcal{O}_E$-algebras:

$$\begin{array}{ccccc} A''/\pi & \longrightarrow & A^{(2)''}/\pi & \longrightarrow & A''/\pi \\ \downarrow s & & \downarrow \tilde{s} & & \downarrow s \\ A/(\pi, \phi(\mathcal{E})) & \longrightarrow & A^{(2)'}/(\pi, f(d)) & \longrightarrow & A/(\pi, \phi(\mathcal{E})) \end{array}$$

The homomorphism $g : A^{(2)'}/(\pi, f(d)) \to A^{(2)''}/\pi$ is a section of $\tilde{s}$. We observe that $\tilde{s}(K''/\pi K'') \subset K'/(\pi, f(d))K'$ and $\tilde{s}(\overline{M}'') \subset \overline{M}'$. It follows from (6-3) that, for any $x \in K'/(\pi, f(d))K'$ (resp. $x \in \overline{M}'$), its image $\phi'_1(x) = \tilde{s}(g(\phi'_1(x)))$ belongs to $\overline{M}'$ (resp. $(f(t_1), \dots, f(t_n))\overline{M}'$). This proves (6-2), and the proof of Lemma 6.2.7 is now complete. □

6.4. *Deformations of isomorphisms.* As in Section 6.2, we write $(A, I) = (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$. In this subsection, as a preparation for the proof of Theorem 6.1.3, we study deformations of isomorphisms of G-μ-displays over (A, I) along the morphisms $m : (A^{(2)}, I^{(2)}) \to (A, I)$ and $m : (A^{(3)}, I^{(3)}) \to (A, I)$ defined in Section 6.2. Throughout this subsection, we assume that μ is 1-bounded.

Our setup is as follows. Let $(A', I') := (A^{(2)}, I^{(2)})$ (resp. $(A', I') := (A^{(3)}, I^{(3)})$). Let $m : (A', I') \to (A, I)$ denote $m : (A^{(2)}, I^{(2)}) \to (A, I)$ (resp. $m : (A^{(3)}, I^{(3)}) \to (A, I)$). Let $f_1, f_2 \in \{p_1, p_2\}$ (resp. $f_1, f_2 \in \{q_1, q_2, q_3\}$). We do not exclude the case where $f_1 = f_2$.

The purpose of this subsection is to prove the following result:

Proposition 6.4.1. *Assume μ is 1-bounded. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be G-μ-displays over (A, I). Then the map*

$$m^* : \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_1), f_2^*(\mathcal{Q}_2)) \to \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A,I)}(\mathcal{Q}_1, \mathcal{Q}_2) \tag{6-4}$$

induced by the base change functor $m^* : G\text{-}\mathrm{Disp}_\mu(A', I') \to G\text{-}\mathrm{Disp}_\mu(A, I)$ is bijective.

We need some preliminary results for the proof of Proposition 6.4.1. We will use the following notation. Let H be a group scheme over $\mathcal{O}$. For an ideal $J \subset A'$, we write

$$H(J) := \mathrm{Ker}(H(A') \to H(A'/J))$$

for the kernel of the homomorphism $H(A') \to H(A'/J)$. If $H = G_{\mathcal{O}}$, then we simply write $G(J) := G_{\mathcal{O}}(J)$.

Let K denote the kernel of $m : A' \to A$. (We note that if $A' = A^{(3)}$, then this kernel was denoted by L in Corollary 6.2.9.) Let $d \in I'$ be a generator.

Lemma 6.4.2. *Let $J \subset A'$ be an ideal such that $J \subset dK$ and, for any $x \in J$, we have $\phi(x/d) \in J$. Then the homomorphism $\sigma_{\mu,d} : G_{\mu}(A', I') \to G(A')$ (see (5-1)) sends $G(J) \subset G_{\mu}(A', I')$ into itself.*

Proof. We note that, by Proposition 4.2.9, we have $G(J) \subset G(dK) \subset G_{\mu}(A', I')$, and the multiplication map $U^{-}_{\mu} \times_{\mathrm{Spec}\, \mathcal{O}} P_{\mu} \to G_{\mathcal{O}}$ induces a bijection

$$(\mathrm{Lie}(U^{-}_{\mu}) \otimes_{\mathcal{O}} J) \times P_{\mu}(J) \xrightarrow{\sim} G(J).$$

Thus, it suffices to prove that $\sigma_{\mu,d}(P_{\mu}(J)) \subset G(J)$ and $\sigma_{\mu,d}(\mathrm{Lie}(U^{-}_{\mu}) \otimes_{\mathcal{O}} J) \subset G(J)$.

By Remark 4.1.3 and Lemma 4.2.2, we have $\mu(d)P_{\mu}(J)\mu(d)^{-1} \subset P_{\mu}(J)$. (In fact, this holds for any ideal $J \subset A'$.) Since $\phi(J) \subset J$, we have $\phi(G(J)) \subset G(J)$. It follows that $\sigma_{\mu,d}(P_{\mu}(J)) \subset G(J)$.

Since μ is 1-bounded, the homomorphism $G_{\mu}(A', I') \to G(A')$, $g \mapsto \mu(d)g\mu(d)^{-1}$, restricts to a homomorphism

$$\mathrm{Lie}(U^{-}_{\mu}) \otimes_{\mathcal{O}} J \to \mathrm{Lie}(U^{-}_{\mu}) \otimes_{\mathcal{O}} \frac{1}{d}J, \quad v \mapsto v/d.$$

(See Remark 4.2.7.) Since $\phi((1/d)J) \subset J$, we obtain $\sigma_{\mu,d}(\mathrm{Lie}(U^{-}_{\mu}) \otimes_{\mathcal{O}} J) \subset G(J)$. □

Definition 6.4.3. Let $J \subset A'$ be an ideal as in Lemma 6.4.2. For an element $X \in G(A')$, we define a homomorphism

$$\mathcal{U}_{d,X} : G(J) \to G(J), \quad g \mapsto X\sigma_{\mu,d}(g)X^{-1}.$$

We also define a map of sets

$$\mathcal{V}_{d,X} : G(J) \to G(J), \quad g \mapsto \mathcal{U}_{d,X}(g)g^{-1}.$$

Let $J_2 \subset J_1 \subset A'$ be two ideals which satisfy the assumption of Lemma 6.4.2. Then $\mathcal{U}_{d,X} : G(J_1) \to G(J_1)$ induces a homomorphism

$$G(J_1)/G(J_2) \to G(J_1)/G(J_2),$$

which we denote by the same symbol $\mathcal{U}_{d,X}$. Let $\mathcal{V}_{d,X} : G(J_1)/G(J_2) \to G(J_1)/G(J_2)$ be the map of sets defined by $g \mapsto \mathcal{U}_{d,X}(g)g^{-1}$.

By Lemma 6.2.6 and Corollary 6.2.9, we have $\phi(K) \subset dK$. Thus, the ideal $dK \subset A'$ satisfies the assumption of Lemma 6.4.2. We shall prove (in Proposition 6.4.7 below) that $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective for any $X \in G(A')$, from which we will deduce Proposition 6.4.1. For this purpose, we need the following lemmas.

Lemma 6.4.4. *Let $J_2 \subset J_1 \subset A'$ be two ideals which satisfy the assumption of Lemma 6.4.2. Assume that for any $x \in J_1$, we have $\phi(x/d) \in J_2$. Then we have*

$$\sigma_{\mu,d}(G(J_1)) \subset G(J_2).$$

In particular, the map $\mathcal{V}_{d,X} : G(J_1)/G(J_2) \to G(J_1)/G(J_2)$ is equal to the map $g \mapsto g^{-1}$ for any $X \in G(A')$.

Proof. The same argument as in the proof of Lemma 6.4.2 shows that $\sigma_{\mu,d}(G(J_1)) \subset G(J_2)$. The second assertion immediately follows from the first one. □

Lemma 6.4.5. *Let $J_3 \subset J_2 \subset J_1 \subset A'$ be three ideals which satisfy the assumption of Lemma 6.4.2. Let $X \in G(A')$. If the maps*

$$\mathcal{V}_{d,X} : G(J_1)/G(J_2) \to G(J_1)/G(J_2) \quad \text{and} \quad \mathcal{V}_{d,X} : G(J_2)/G(J_3) \to G(J_2)/G(J_3)$$

are bijective, then $\mathcal{V}_{d,X} : G(J_1)/G(J_3) \to G(J_1)/G(J_3)$ is also bijective.

Proof. Let us prove the surjectivity. Let $h \in G(J_1)/G(J_3)$ be an element. The image $h' \in G(J_1)/G(J_2)$ of h can be written as $h' = \mathcal{V}_{d,X}(g')$ for some element $g' \in G(J_1)/G(J_2)$. We choose some $g \in G(J_1)/G(J_3)$ which is a lift of g'. Then we see that $\mathcal{U}_{d,X}(g)^{-1}hg$ is contained in $G(J_2)/G(J_3)$, so that there exists an element $g'' \in G(J_2)/G(J_3)$ such that

$$\mathcal{V}_{d,X}(g'') = \mathcal{U}_{d,X}(g'')g''^{-1} = \mathcal{U}_{d,X}(g)^{-1}hg.$$

It follows that $h = \mathcal{V}_{d,X}(gg'')$. This proves that $\mathcal{V}_{d,X} : G(J_1)/G(J_3) \to G(J_1)/G(J_3)$ is surjective. The proof of the injectivity is similar. □

Lemma 6.4.6. *Let $l \geq 0$ be an integer. For any $X \in G(A')$, the map*

$$\mathcal{V}_{d,X} : G((\pi, d)^l dK)/G((\pi, d)^{l+1} dK) \to G((\pi, d)^l dK)/G((\pi, d)^{l+1} dK)$$

is bijective.

Proof. Step 1. We set $K_l := (\pi, d)^l K$. We consider the ideal $K^- := K^2 + (\pi, d)K$ and let $K_l^- := (\pi, d)^l K^-$. All of dK_l, dK_{l+1}, dK_l^- satisfy the assumption of Lemma 6.4.2. Since

$$\phi(K^2) \subset d^2K^2 \subset d(\pi, d)K,$$

we have $\phi(K_l^-) \subset dK_{l+1}$. Thus, it follows from Lemma 6.4.4 that $\mathcal{V}_{d,X}$ is bijective for $G(dK_l^-)/G(dK_{l+1})$. By Lemma 6.4.5, it now suffices to show that $\mathcal{V}_{d,X}$ is bijective for $G(dK_l)/G(dK_l^-)$.

Step 2. By Lemma 6.2.7 and Corollary 6.2.9, there exists a finitely generated ideal $M \subset A'$ which is contained in K such that $\phi(K) \subset dM + dK^-$ and $\phi(M) \subset (t_1, \dots, t_n)dM + dK^-$, where we abuse notation and denote the image of $t_i \in A$ under the morphism $p_1 : A \to A'$ (resp. $q_1 : A \to A'$) if $A' = A^{(2)}$ (resp. if $A' = A^{(3)}$) by the same symbol. We set $M_l := (\pi, d)^l M \subset K_l$. Then we have inclusions

$$\phi(K_l) \subset dM_l + dK_l^- \quad \text{and} \quad \phi(M_l) \subset (t_1, \dots, t_n)dM_l + dK_l^-. \tag{6-5}$$

In particular, the ideals $dM_l + dK_l^- \subset dK_l$ satisfy the assumption of Lemma 6.4.4, so that $\mathcal{V}_{d,X}$ is bijective for $G(dK_l)/G(dM_l + dK_l^-)$. By Lemma 6.4.5, it is enough to prove that $\mathcal{V}_{d,X}$ is bijective for $G(dM_l + dK_l^-)/G(dK_l^-)$.

Step 3. We shall prove that

$$G(dM_l + dK_l^-)/G(dK_l^-) \xrightarrow{\sim} \varprojlim_{r\geq 0} G(dM_l + dK_l^-)/G((t_1, \dots, t_n)^r dM_l + dK_l^-). \tag{6-6}$$

To simplify the notation, we set $C_1 := A'/(dM_l + dK_l^-)$ and $C_2 := A'/dK_l^-$. Let $N \subset C_2$ be the image of $dM_l + dK_l^-$. We first claim that

$$A' \xrightarrow{\sim} \varprojlim_{r\geq 0} A'/dK_r, \tag{6-7}$$

$$C_2 \xrightarrow{\sim} \varprojlim_{r\geq 0} C_2/(t_1, \dots, t_n)^r N. \tag{6-8}$$

Since d is a nonzerodivisor and K is (π, d)-adically complete, it follows that $dK \xrightarrow{\sim} \varprojlim_{r\geq 0} dK/dK_r$. This implies (6-7). Since N is killed by K, we see that N is a finitely generated module over $A'/K \xrightarrow{\sim} A$. Since A is noetherian and is $(t_1, \dots, t_n)$-adically complete, it follows that N is also $(t_1, \dots, t_n)$-adically complete, which means that $N \xrightarrow{\sim} \varprojlim_{r\geq 0} N/(t_1, \dots, t_n)^r N$. This implies (6-8).

We next show that $G(A') \to G(C_2)$ is surjective. Indeed, by (6-7) and the fact that $G(A'/dK_{r+1}) \to G(A'/dK_r)$ is surjective (as G is smooth), it follows that $G(A') \to G(A'/dK_r)$ is surjective for any r. Since we have $(dK_l^-)^2 \subset dK_{l+1} \subset dK_l^-$, we see that $G(A'/dK_{l+1}) \to G(C_2)$ is surjective (again by the smoothness of G). Therefore $G(A') \to G(C_2)$ is surjective, as desired. Similarly, it follows from (6-8) that $G(C_2) \to G(C_2/(t_1, \dots, t_n)^r N)$ is surjective.

Using the results obtained in the previous paragraph, we see that

$$G(dM_l + dK_l^-)/G(dK_l^-) \xrightarrow{\sim} \operatorname{Ker}(G(C_2) \to G(C_1)),$$

$$G(dM_l + dK_l^-)/G((t_1, \dots, t_n)^r dM_l + dK_l^-) \xrightarrow{\sim} \operatorname{Ker}(G(C_2/(t_1, \dots, t_n)^r N) \to G(C_1)).$$

Now (6-6) follows from (6-8).

Step 4. We claim that $\mathcal{V}_{d,X}$ is bijective for

$$G((t_1, \dots, t_n)^r dM_l + dK_l^-)/G((t_1, \dots, t_n)^{r+1} dM_l + dK_l^-)$$

for any $r \geq 0$. Indeed, the second inclusion of (6-5) shows that the assumption of Lemma 6.4.4 is satisfied in this case, and hence the assertion follows.

Using Lemma 6.4.5 repeatedly, we see that $\mathcal{V}_{d,X}$ is bijective for

$$G(dM_l + dK_l^-)/G((t_1, \dots, t_n)^r dM_l + dK_l^-)$$

for any $r \geq 0$. It then follows from (6-6) that $\mathcal{V}_{d,X}$ is bijective for $G(dM_l + dK_l^-)/G(dK_l^-)$ as well. □

Let us now prove the desired result.

Proposition 6.4.7. *For any $X \in G(A')$, the map $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective.*

Proof. By (6-7) in the proof of Lemma 6.4.6, we have

$$G(dK) \xrightarrow{\sim} \varprojlim_{l\geq 0} G(dK)/G((\pi,d)^l dK).$$

In order to show that $\mathcal{V}_{d,X}: G(dK) \to G(dK)$ is bijective, it suffices to check that

$$\mathcal{V}_{d,X}: G(dK)/G((\pi,d)^l dK) \to G(dK)/G((\pi,d)^l dK)$$

is bijective for any $l \geq 0$. This follows from Lemma 6.4.6 by using Lemma 6.4.5 repeatedly. □

We also need the following lemma:

Lemma 6.4.8. *Let $\mathcal{Q}$ be a G-μ-display over (A, I). Then there exists a finite extension $\tilde{k}$ of k such that the base change of $\mathcal{Q}$ to $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ is banal, where $\widetilde{\mathcal{O}} := W(\tilde{k}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ and $A_{\widetilde{\mathcal{O}}} := A \otimes_{\mathcal{O}} \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}[[t_1, \dots, t_n]]$.*

Proof. The Hodge filtration $P(\mathcal{Q})_{A/I} = P(\mathcal{Q})_R$ of $\mathcal{Q}$ is a $(P_\mu)_R$-torsor over $\operatorname{Spec} R$. There exists a finite extension $\tilde{k}$ of k such that $P(\mathcal{Q})_R \times_{\operatorname{Spec} R} \operatorname{Spec} \tilde{k}$ is trivial. Since P_μ is smooth and $R \otimes_{\mathcal{O}} \widetilde{\mathcal{O}}$ is a complete local ring, it follows that $P(\mathcal{Q})_R$ is trivial over $R \otimes_{\mathcal{O}} \widetilde{\mathcal{O}}$. By Proposition 5.4.5, the base change of $\mathcal{Q}$ to $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ is banal. □

Proof of Proposition 6.4.1. By Lemma 6.4.8, there exists a finite Galois extension $\tilde{k}$ of k such that the base changes of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ are banal. Here $\widetilde{\mathcal{O}} := W(\tilde{k}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ and $A_{\widetilde{\mathcal{O}}} := A \otimes_{\mathcal{O}} \widetilde{\mathcal{O}}$; we use the same notation for $\mathcal{O}$-algebras. We can identify $(A'_{\widetilde{\mathcal{O}}}, I'A'_{\widetilde{\mathcal{O}}})$ with a coproduct of two (resp. three) copies of $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$ in $(R_{\widetilde{\mathcal{O}}})_{\Delta,\mathcal{O}_E}$ if $A' = A^{(2)}$ (resp. if $A' = A^{(3)}$). By Galois descent for G-μ-displays, it suffices to prove the same statement for banal G-μ-displays over $(A_{\widetilde{\mathcal{O}}}, IA_{\widetilde{\mathcal{O}}})$. We may thus assume without loss of generality that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are banal G-μ-displays over (A, I).

If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are not isomorphic to each other, then the assertion holds trivially. Thus, we may assume that $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_Y$ for some $Y \in G(A)_I$. Let $d := f_2(\mathcal{E})$. We have $f_1(\mathcal{E}) = ud$ for some $u \in A'^{\times}$. With the choice of $d \in I'$, the G-μ-displays $f_1^*(\mathcal{Q}_Y)$ and $f_2^*(\mathcal{Q}_Y)$ correspond to the elements $f_1(Y_{\mathcal{E}})\phi(\mu(u))$, $f_2(Y_{\mathcal{E}}) \in G(A')_d$, respectively. Thus we can identify $\operatorname{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_Y), f_2^*(\mathcal{Q}_Y))$ with the set

$$\{g \in G_\mu(A', I') \mid g^{-1} f_2(Y_{\mathcal{E}}) \sigma_{\mu,d}(g) = f_1(Y_{\mathcal{E}})\phi(\mu(u))\}.$$

We set $X := f_2(Y_{\mathcal{E}})$. We shall prove that the map (6-4) is injective. Let $g, h \in G_\mu(A', I')$ be two elements in $\operatorname{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_1), f_2^*(\mathcal{Q}_2))$ such that $m(g) = m(h)$ in $G_\mu(A, I)$. We set $\beta := gh^{-1}$. Since $m(\beta) = 1$, we have $\mu(d)\beta\mu(d)^{-1} \in G(K)$. It then follows from $\phi(K) \subset dK$ that $\sigma_{\mu,d}(\beta) \in G(dK)$. The equalities

$$g^{-1} X \sigma_{\mu,d}(g) = f_1(Y_{\mathcal{E}})\phi(\mu(u)) = h^{-1} X \sigma_{\mu,d}(h)$$

imply that $\beta = X\sigma_{\mu,d}(\beta)X^{-1}$. It follows that $\beta \in G(dK)$, and we have $\mathcal{V}_{d,X}(\beta) = 1$ for the map $\mathcal{V}_{d,X}: G(dK) \to G(dK)$. Since $\mathcal{V}_{d,X}$ is bijective by Proposition 6.4.7, we obtain $\beta = 1$.

It still remains to prove that the map (6-4) is surjective. For this, it is sufficient to prove that $\operatorname{Hom}_{G\text{-}\mathrm{Disp}_\mu(A',I')}(f_1^*(\mathcal{Q}_Y), f_2^*(\mathcal{Q}_Y))$ is not empty. (Once we have obtained an isomorphism $g: f_1^*(\mathcal{Q}_Y) \xrightarrow{\sim} f_2^*(\mathcal{Q}_Y)$, we can write any isomorphism $h: \mathcal{Q}_Y \xrightarrow{\sim} \mathcal{Q}_Y$ as $m^*(f_2^*(h \circ m^*(g^{-1})) \circ g)$.) We claim that

$\phi(\mu(u)) \in G(dK)$ and $\gamma := f_2(Y_{\mathcal{E}})^{-1} f_1(Y_{\mathcal{E}}) \in G(dK)$. Indeed, since $m(u) = 1$, we have $\mu(u) \in G(K)$, which in turn implies that $\phi(\mu(u)) \in G(dK)$. Since the morphisms f_1 and f_2 induce the same homomorphism $R \to A'/I'$, we see that $\gamma \in G(I')$. Using that $I' \cap K = dK$, we then obtain $\gamma \in G(dK)$. Since $\mathcal{V}_{d,X} : G(dK) \to G(dK)$ is bijective, there exists an element $g \in G(dK)$ such that $\mathcal{V}_{d,X}(g^{-1}) = X\phi(\mu(u))^{-1}\gamma^{-1}X^{-1}$, or equivalently

$$g^{-1} f_2(Y_{\mathcal{E}})\sigma_{\mu,d}(g) = f_1(Y_{\mathcal{E}})\phi(\mu(u)).$$

In other words, the element g gives an isomorphism $f_1^*(\mathcal{Q}_Y) \xrightarrow{\sim} f_2^*(\mathcal{Q}_Y)$. □

6.5. ***Proof of Theorem 6.1.3.*** In this section, we prove Theorem 6.1.3 using our previous results.

As in Section 6.2, we write $(A, I) = (\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$. Let

$$G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I)$$

be the groupoid of pairs $(\mathcal{Q}, \epsilon)$ consisting of a G-μ-display $\mathcal{Q}$ over (A, I) and an isomorphism $\epsilon : p_1^*\mathcal{Q} \xrightarrow{\sim} p_2^*\mathcal{Q}$ of G-μ-displays over $(A^{(2)}, I^{(2)})$ satisfying the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$. An isomorphism $(\mathcal{Q}, \epsilon) \xrightarrow{\sim} (\mathcal{Q}', \epsilon')$ is an isomorphism $f : \mathcal{Q} \xrightarrow{\sim} \mathcal{Q}'$ of G-μ-displays over (A, I) such that $\epsilon' \circ (p_1^* f) = (p_2^* f) \circ \epsilon$.

For a prismatic G-μ-display $\mathfrak{Q}$ over R, we have the associated isomorphism

$$\gamma_{p_i} : p_i^*(\mathfrak{Q}_{(A,I)}) \xrightarrow{\sim} \mathfrak{Q}_{(A^{(2)}, I^{(2)})}$$

for $i = 1, 2$. Let $\epsilon := \gamma_{p_2}^{-1} \circ \gamma_{p_1}$. Then ϵ satisfies the cocycle condition, so that the pair $(\mathfrak{Q}_{(A,I)}, \epsilon)$ is an object of $\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I)$. This construction induces a functor

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta,\mathcal{O}_E}) \to G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I), \quad \mathfrak{Q} \mapsto (\mathfrak{Q}_{(A,I)}, \epsilon).$$

Proposition 6.5.1. *The functor* $G\text{-}\mathrm{Disp}_\mu((R)_{\Delta,\mathcal{O}_E}) \to G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I)$ *is an equivalence.*

Proof. This is a formal consequence of Propositions 5.2.8 and 6.2.1. □

Proof of Theorem 6.1.3. We assume that μ is 1-bounded. By virtue of Proposition 6.5.1, it suffices to show that the forgetful functor

$$G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I) \to G\text{-}\mathrm{Disp}_\mu(A, I)$$

is an equivalence. Let $m : (A^{(2)}, I^{(2)}) \to (A, I)$ be the unique morphism in $(R)_{\Delta,\mathcal{O}_E}$ such that $m \circ p_i = \mathrm{id}_{(A,I)}$ for $i = 1, 2$, and let $m' : (A^{(3)}, I^{(3)}) \to (A, I)$ be the unique morphism in $(R)_{\Delta,\mathcal{O}_E}$ such that $m \circ q_i = \mathrm{id}_{(A,I)}$ for $i = 1, 2, 3$. Let $\mathcal{Q}$ be a G-μ-display over (A, I). We claim that an isomorphism $\epsilon : p_1^*\mathcal{Q} \xrightarrow{\sim} p_2^*\mathcal{Q}$ satisfies the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$ if and only if $m^*\epsilon = \mathrm{id}_{\mathcal{Q}}$. Indeed, since the map

$$m'^* : \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A^{(3)}, I^{(3)})}(q_1^*\mathcal{Q}, q_3^*\mathcal{Q}) \to \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A,I)}(\mathcal{Q}, \mathcal{Q})$$

is bijective by Proposition 6.4.1, we see that ϵ satisfies the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$ if and only if $m^*\epsilon = m^*\epsilon \circ m^*\epsilon$, which is equivalent to saying that $m^*\epsilon = \mathrm{id}_{\mathcal{Q}}$.

By Proposition 6.4.1, the map

$$m^* : \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A^{(2)},I^{(2)})}(p_1^*\mathcal{Q}, p_2^*\mathcal{Q}) \to \mathrm{Hom}_{G\text{-}\mathrm{Disp}_\mu(A,I)}(\mathcal{Q}, \mathcal{Q})$$

is bijective. Therefore, for any G-μ-display $\mathcal{Q}$ over (A, I), there exists a unique isomorphism $\epsilon : p_1^*\mathcal{Q} \xrightarrow{\sim} p_2^*\mathcal{Q}$ satisfying the cocycle condition $p_{13}^*\epsilon = p_{23}^*\epsilon \circ p_{12}^*\epsilon$, and ϵ is characterized by the condition that $m^*\epsilon = \mathrm{id}_{\mathcal{Q}}$. It follows that the forgetful functor $G\text{-}\mathrm{Disp}_\mu^{\mathrm{DD}}(A, I) \to G\text{-}\mathrm{Disp}_\mu(A, I)$ is an equivalence. □

7. *p*-divisible groups and prismatic Dieudonné crystals

In this section, we make a few remarks on prismatic Dieudonné crystals, which are introduced in [Anschütz and Le Bras 2023].

7.1. ***A remark on prismatic Dieudonné crystals.*** Let R be a π-adically complete $\mathcal{O}_E$-algebra. Recall the sheaf $\mathcal{O}_{\Delta}$ on the site $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ from Remark 2.5.5.

We say that an $\mathcal{O}_{\Delta}$-module $\mathcal{M}$ on $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ is a *prismatic crystal in vector bundles* if $\mathcal{M}(A, I)$ is a finite projective A-module for any $(A, I) \in (R)_{\Delta,\mathcal{O}_E}$, and for any morphism $(A, I) \to (A', I')$ in $(R)_{\Delta,\mathcal{O}_E}$, the natural homomorphism

$$\mathcal{M}(A, I) \otimes_A A' \to \mathcal{M}(A', I')$$

is bijective. A *prismatic Dieudonné crystal* on $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ (or on $(R)_{\Delta,\mathcal{O}_E}$) is a prismatic crystal $\mathcal{M}$ in vector bundles on $(R)^{\mathrm{op}}_{\Delta,\mathcal{O}_E}$ equipped with a ϕ-linear homomorphism

$$\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$$

such that for any $(A, I) \in (R)_{\Delta,\mathcal{O}_E}$, the finite projective A-module $\mathcal{M}(A, I)$ with the linearization $1 \otimes \varphi_{\mathcal{M}} : \phi^*(\mathcal{M}(A, I)) \to \mathcal{M}(A, I)$ is a minuscule Breuil–Kisin module over (A, I) in the sense of Definition 3.1.5 (see also Proposition 3.1.6). For a bounded $\mathcal{O}_E$-prism (A, I), let

$$\mathrm{BK}_{\mathrm{min}}(A, I)$$

be the category of minuscule Breuil–Kisin modules over (A, I). Then the category of prismatic Dieudonné crystals on $(R)_{\Delta,\mathcal{O}_E}$ is equivalent to the category

$$2-\varprojlim_{(A,I)\in(R)_{\Delta,\mathcal{O}_E}} \mathrm{BK}_{\mathrm{min}}(A, I).$$

As in Section 6, let R be a complete regular local ring over $\mathcal{O}$ with residue field k. Let $(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ be an $\mathcal{O}_E$-prism of Breuil–Kisin type, where $\mathfrak{S}_{\mathcal{O}} = \mathcal{O}[[t_1, \dots, t_n]]$, with an isomorphism $R \simeq \mathfrak{S}_{\mathcal{O}}/\mathcal{E}$ over $\mathcal{O}$. By using the results of Section 6, we can prove the following proposition, which is obtained in the proof of [Anschütz and Le Bras 2023, Theorem 5.12] if $n \leq 1$ (and $\mathcal{O}_E = \mathbb{Z}_p$).

Proposition 7.1.1. *The functor $\mathcal{M} \mapsto \mathcal{M}(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ from the category of prismatic Dieudonné crystals on $(R)_{\Delta,\mathcal{O}_E}$ to the category $\mathrm{BK}_{\mathrm{min}}(\mathfrak{S}_{\mathcal{O}}, (\mathcal{E}))$ is an equivalence.*

Proof. This follows from Corollary 5.3.11, Theorem 6.1.3, and the following fact: a functor of additive categories is an equivalence if and only if it induces an equivalence of the associated groupoids. This fact follows since homomorphisms $f : X \to Y$ in an additive category can be completely described in terms of automorphisms of $X \oplus Y$ by considering $\left(\begin{smallmatrix} \mathrm{id}_X & 0 \\ f & \mathrm{id}_Y \end{smallmatrix}\right)$. □

7.2. *Quasisyntomic rings.* In the following (and in Section 8 below), we will need the notions of *quasisyntomic rings* in the sense of [Bhatt et al. 2019, Definition 4.10] and *quasiregular semiperfectoid rings* in the sense of [loc. cit., Definition 4.20]. Let

$$\mathrm{QSyn}$$

be the category of quasisyntomic rings and let

$$\mathrm{QRSPerfd} \subset \mathrm{QSyn}$$

be the full subcategory spanned by quasiregular semiperfectoid rings. We endow both $\mathrm{QSyn}^{\mathrm{op}}$ and $\mathrm{QRSPerfd}^{\mathrm{op}}$ with the quasisyntomic topology, i.e., the topology generated by the quasisyntomic coverings; see [loc. cit., Definition 4.10]. We will assume that the reader is familiar with basic properties of QSyn and QRSPerfd discussed in [loc. cit., Section 4]. Here we just recall that quasiregular semiperfectoid rings form a basis for QSyn; see [loc. cit., Lemma 4.28].

Example 7.2.1. A p-adically complete regular local ring is a quasisyntomic ring (see [Anschütz and Le Bras 2023, Example 3.17]). A perfectoid ring is a quasiregular semiperfectoid ring (see [Bhatt et al. 2019, Example 4.24]).

Remark 7.2.2. Let $R \in \mathrm{QSyn}$ be a quasisyntomic ring. In [Anschütz and Le Bras 2023, Definition 4.5], Anschütz–Le Bras defined prismatic Dieudonné crystals over R as sheaves on the quasisyntomic site of R. By virtue of [loc. cit., Proposition 4.4], the category of prismatic Dieudonné crystals on $(R)_{\Delta}$ in our sense is equivalent to the category of prismatic Dieudonné crystals over R in the sense of [loc. cit., Definition 4.5].

7.3. *p-divisible groups and minuscule Breuil-Kisin modules.* In this subsection, we consider the case where $\mathcal{O}_E = \mathbb{Z}_p$. Let R be a p-adically complete ring, and let $\mathcal{G}$ be a p-divisible group over Spec R. We define the functors

$$\underline{\mathcal{G}} : (R)_{\Delta} \to \mathrm{Set}, \quad (A, I) \mapsto \mathcal{G}(A/I),$$
$$\underline{\mathcal{G}[p^n]} : (R)_{\Delta} \to \mathrm{Set}, \quad (A, I) \mapsto \mathcal{G}[p^n](A/I).$$

These functors form sheaves on the site $(R)_{\Delta}^{\mathrm{op}}$. In [loc. cit., Proposition 4.69], it is proved that the $\mathcal{O}_{\Delta}$-module

$$\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$$

on $(R)_{\Delta}^{\mathrm{op}}$ is a prismatic crystal in vector bundles. (Here we simply write $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ rather than $\mathcal{E}xt^1_{(R)_{\Delta}^{\mathrm{op}}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$.)

Remark 7.3.1. (1) For an integer $n \geq 1$, the map $[p^n] : \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ induced by multiplication by p^n is surjective. This follows from [loc. cit., Corollary 3.25].

(2) We have $\mathcal{H}om_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) = 0$. Indeed, since $[p] : \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ is surjective and the topos associated with $(R)_{\Delta}^{\mathrm{op}}$ is replete in the sense of [Bhatt and Scholze 2015, Definition 3.1.1], the projection $\varprojlim_{[p]} \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ is surjective. Since $\mathcal{O}_{\Delta}(A, I) = A$ is p-adically complete for any $(A, I) \in (R)_{\Delta}$, we can conclude that $\mathcal{H}om_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) = 0$. As a consequence, the local-to-global spectral sequence implies that

$$\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})(A, I)$$

for any $(A, I) \in (R)_{\Delta}$. Here we regard the site $(A, I)_{\Delta}^{\mathrm{op}}$ as the localization of $(R)_{\Delta}^{\mathrm{op}}$ at (A, I), and the restriction of $\underline{\mathcal{G}}$ to $(A, I)_{\Delta}^{\mathrm{op}}$ is denoted by the same symbol. In particular $\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ is a finite projective A-module and its formation commutes with base change along any morphism $(A, I) \to (A', I')$ in $(R)_{\Delta}$.

We assume that R is quasisyntomic. In [Anschütz and Le Bras 2023, Theorem 4.71], it is proved that $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ with the ϕ-linear homomorphism $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ induced by the Frobenius $\phi : \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}$ is a prismatic Dieudonné crystal. More precisely, they showed that $\mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ is *admissible* in the sense of [loc. cit., Definition 4.5]. (See also Remark 7.2.2.) We shall recall the argument.

Proposition 7.3.2 [Anschütz and Le Bras 2023, Theorem 4.71]. *Let $R \in \mathrm{QSyn}$ and $(A, I) \in (R)_{\Delta}$. We write $M := \mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$. Then M with the induced homomorphism $F_M : \phi^* M \to M$ is a minuscule Breuil–Kisin module over (A, I).*

Proof. By the fact that the formation of $\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ commutes with base change along any morphism $(A, I) \to (A', I')$ (see Remark 7.3.1) and Corollary 3.1.15, the assertion can be checked (p, I)-completely flat locally. Let $R \to R'$ be a quasisyntomic covering with R' a quasiregular semiperfectoid ring. Applying [Bhatt and Scholze 2022, Proposition 7.11] to the p-adic completion of $A/I \otimes_R R'$, which is a quasisyntomic covering of A/I, we can find a flat covering $(A, I) \to (A', I')$ in $(R)_{\Delta}$ such that there exists a homomorphism $R' \to A'/I'$ over R. After replacing R by R' and replacing (A, I) by (A', I'), we may assume that R is a quasiregular semiperfectoid ring. Then, by choosing a surjective homomorphism from a perfectoid ring to R and using [Anschütz and Le Bras 2023, Corollary 2.10, Lemma 4.70], we may assume that R is a perfectoid ring and $(A, I) = (W(R^\flat), I_R)$. (Here we regard $(W(R^\flat), I_R)$ as an object of $(R)_{\Delta}$ via the homomorphism $\theta : W(R^\flat) \to R$. In [loc. cit.], the composition $\theta \circ \phi^{-1}$ is used instead.)

Let $\xi \in I_R$ be a generator. By Proposition 3.1.6, it suffices to prove that the cokernel of F_M is killed by ξ. By Remark 5.6.2(4) and p-complete arc-descent (Proposition 5.6.3), we may further assume that R is a p-adically complete valuation ring of rank ≤ 1 with algebraically closed fraction field.

If $p = 0$ in R, then R is perfect by Example 2.4.1. In this case, the Frobenius F_M can be identified with the homomorphism

$$\mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{(\phi^*\mathcal{G})}, \mathcal{O}_{\Delta}) \to \mathrm{Ext}^1_{(A,I)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$$

induced by the relative Frobenius $\mathcal{G} \to \phi^*\mathcal{G}$. Thus, the Verschiebung homomorphism $\phi^*\mathcal{G} \to \mathcal{G}$ induces a $W(R)$-linear homomorphism $V_M : M \to \phi^* M$ such that $F_M \circ V_M = p$, which in turn implies the assertion.

It remains to treat the case where p is a nonzerodivisor in R, so that R is the ring of integers $\mathcal{O}_C$ of an algebraically closed nonarchimedean extension C of $\mathbb{Q}_p$. We set

$$M_n := \mathrm{Ext}^1_{(W(\mathcal{O}_C^\flat), I_{\mathcal{O}_C})_{\Delta}}(\underline{\mathcal{G}[p^n]}, \mathcal{O}_{\Delta}).$$

By the proof of [Anschütz and Le Bras 2023, Proposition 4.69], the natural homomorphism $M \to M_n$ induces an isomorphism $M/p^n \xrightarrow{\sim} M_n$ for any $n \geq 1$. In particular, we obtain

$$M \xrightarrow{\sim} \varprojlim_n M_n$$

and M_n is a free $W_n(\mathcal{O}_C^\flat)$-module of finite rank. We claim that the cokernel of the Frobenius F_{M_n} : $\phi^* M_n \to M_n$ is killed by ξ. Indeed, there is an embedding $\mathcal{G}[p^n] \hookrightarrow X$ into an abelian scheme X over $\operatorname{Spec} \mathcal{O}_C$; see [Berthelot et al. 1982, Théorème 3.1.1]. Let Y be the p-adic completion of X, which is a smooth p-adic formal scheme over $\operatorname{Spf} \mathcal{O}_C$. It follows from the proofs of [Anschütz and Le Bras 2023, Theorem 4.62, Proposition 4.66] that there exists a surjection

$$H^1_{\Delta}(Y/W(\mathcal{O}_C^\flat)) \to M_n$$

which is compatible with Frobenius homomorphisms. Here $H^1_{\Delta}(Y/W(\mathcal{O}_C^\flat))$ is the first prismatic cohomology of Y (with respect to $(W(\mathcal{O}_C^\flat), I_{\mathcal{O}_C})$) defined in [Bhatt and Scholze 2022]. By [loc. cit., Theorem 1.8(6)], the cokernel of the Frobenius

$$\phi^* H^1_{\Delta}(Y/W(\mathcal{O}_C^\flat)) \to H^1_{\Delta}(Y/W(\mathcal{O}_C^\flat))$$

is killed by ξ, which in turn implies the claim. Since the image of ξ in $W_n(\mathcal{O}_C^\flat)$ is a nonzerodivisor, it follows that F_{M_n} is injective. Since $F_M = \varprojlim_n F_{M_n}$, we can conclude that the cokernel of F_M is killed by ξ. □

Remark 7.3.3. Our proof of Proposition 7.3.2 in the case where $A = \mathcal{O}_C$ is slightly different from that given in [Anschütz and Le Bras 2023]. For example, we do not use [Scholze and Weinstein 2020, Proposition 14.9.4] (see the proof of [Anschütz and Le Bras 2023, Proposition 4.48]).

Finally, we recall the following classification theorem for p-divisible groups given in [Anschütz and Le Bras 2023]. Let R be a complete regular local ring with perfect residue field k of characteristic p. Let $(\mathfrak{S}, (\mathcal{E}))$ be a prism of Breuil–Kisin type, where $\mathfrak{S} := W(k)[[t_1, \dots, t_n]]$, with an isomorphism $R \simeq \mathfrak{S}/\mathcal{E}$ which lifts $\mathrm{id}_k : k \to k$.

Theorem 7.3.4 [Anschütz and Le Bras 2023, Theorem 4.74, Theorem 5.12].

(1) *The contravariant functor*

$$\{p\text{-divisible groups over } \operatorname{Spec} R\} \to \{\text{prismatic Dieudonné crystals on } (R)_{\Delta}\}$$

defined by $\mathcal{G} \mapsto \mathcal{E}xt^1_{(R)_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ *is an antiequivalence of categories.*

(2) *The contravariant functor*

$$\{p\text{-divisible groups over } \operatorname{Spec} R\} \to \{\text{minuscule Breuil–Kisin modules over } (\mathfrak{S}, (\mathcal{E}))\}$$

defined by $\mathcal{G} \mapsto \mathrm{Ext}^1_{(\mathfrak{S}, (\mathcal{E}))_{\Delta}}(\underline{\mathcal{G}}, \mathcal{O}_{\Delta})$ *is an antiequivalence of categories.*

Proof. (1) This is a consequence of [Anschütz and Le Bras 2023, Theorem 4.74, Proposition 5.10].

(2) The assertion follows from (1) and Proposition 7.1.1. This result was already stated in [loc. cit., Theorem 5.12], and the proof was given in the case where $n \leq 1$. □

8. Comparison with prismatic F-gauges

For the sake of completeness, we discuss the relation between our prismatic G-μ-displays and *prismatic F-gauges* introduced in [Drinfeld 2024, 1.8.1; Bhatt and Lurie 2022a; 2022b; Bhatt 2022, Definition 6.1.1]. For simplicity, we assume that $\mathcal{O}_E = \mathbb{Z}_p$ throughout this section, and we restrict ourselves to the case where base rings R are quasisyntomic. In this case, Guo and Li [2023] studied prismatic F-gauges over R in a slightly different way, without using the original stacky approach. Here we follow the approach employed in [Guo and Li 2023]. In Section 8.1, we compare prismatic F-gauges in vector bundles with displayed Breuil–Kisin modules. In Section 8.2, we introduce *prismatic G-F-gauges of type μ* and explain their relation to prismatic G-μ-displays.

We work with the category QSyn of quasisyntomic rings and the full subcategory QRSPerfd $\subset$ QSyn spanned by quasiregular semiperfectoid rings (see Section 7.2).

8.1. *Prismatic F-gauges in vector bundles.* We recall the definition of prismatic F-gauges in vector bundles over quasisyntomic rings, following [Guo and Li 2023].

Let $S \in$ QRSPerfd be a quasiregular semiperfectoid ring. By [Bhatt and Scholze 2022, Proposition 7.10], the category $(S)_{\Delta}$ admits an initial object

$$(\Delta_S, I_S) \in (S)_{\Delta}.$$

Moreover the bounded prism (Δ_S, I_S) is orientable. We often omit the subscript and simply write $I = I_S$. Following [loc. cit., Definition 12.1], we define

$$\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S) := \{x \in \Delta_S \mid \phi(x) \in I^i \Delta_S\}$$

for a nonnegative integer $i \geq 0$. For a negative integer $i < 0$, we set $\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S) = \Delta_S$. The filtration $\{\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)\}_{i \in \mathbb{Z}}$ is called the *Nygaard filtration*. We recall the following terminology from [Bhatt 2022, Section 5.5].

Definition 8.1.1. The *extended Rees algebra* $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$ of the Nygaard filtration $\{\mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)\}_{i \in \mathbb{Z}}$ is defined by

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^i_{\mathcal{N}}(\Delta_S) t^{-i} \subset \Delta_S[t, t^{-1}].$$

We view $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$ as a graded ring, where the degree of t is -1. Let

$$\tau : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \Delta_S$$

be the homomorphism of Δ_S-algebras defined by $t \mapsto 1$. We consider the graded ring $\bigoplus_{i \in \mathbb{Z}} I^i t^{-i} \subset \Delta_S[1/I][t, t^{-1}]$. Let

$$\sigma : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \bigoplus_{i \in \mathbb{Z}} I^i t^{-i}$$

be the graded homomorphism defined by $a_i t^{-i} \mapsto \phi(a_i) t^{-i}$ for any $i \in \mathbb{Z}$.

Remark 8.1.2. For the grading of $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$, our sign convention is opposite to that of [Bhatt 2022], where the degree of t is defined to be 1. Our grading is chosen to be consistent with the convention of [Lau 2021].

Definition 8.1.3 (Drinfeld, Bhatt–Lurie). Let $S \in \mathrm{QRSPerfd}$. A *prismatic F-gauge in vector bundles* over S is a pair (N, F_N) consisting of a graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module N which is finite projective as a $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module, and an isomorphism

$$F_N : (\sigma^* N)_0 \xrightarrow{\sim} \tau^* N$$

of Δ_S-modules. Here $(\sigma^* N)_0$ is the degree-0 part of the graded $\bigoplus_{i \in \mathbb{Z}} I^i t^{-i}$-module $\sigma^* N$.

Let $F\text{-Gauge}^{\mathrm{vect}}(S)$ be the category of prismatic F-gauges in vector bundles over S.

Remark 8.1.4. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded $\bigoplus_{i \in \mathbb{Z}} I^i t^{-i}$-module. For any $i \in \mathbb{Z}$, we have a natural isomorphism $M_0 \otimes_{\Delta_S} I^i t^{-i} \xrightarrow{\sim} M_i$ of Δ_S-modules. It follows that the functor $M \mapsto M_0$ from the category of graded $\bigoplus_{i \in \mathbb{Z}} I^i t^{-i}$-modules to the category of Δ_S-modules is an equivalence, whose inverse is given by $L \mapsto L \otimes_{\Delta_S} \left(\bigoplus_{i \in \mathbb{Z}} I^i t^{-i}\right)$.

Remark 8.1.5. The notion of prismatic F-gauges in vector bundles is closely related to the notion of (higher) displays in the sense of [Lau 2021, Definition 3.2.1]. See Remark 8.2.7 for more details.

We collect some useful facts about graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-modules.

Remark 8.1.6. Let $N = \bigoplus_{i \in \mathbb{Z}} N_i$ be a graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module which is finite projective as a $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module. Then each degree-i part N_i is a direct summand of a Δ_S-module of the form $\bigoplus_{j=1}^{m} \mathrm{Fil}^{i_j}_{\mathcal{N}}(\Delta_S) t^{-i_j}$, and in particular N_i is (p, I)-adically complete. This follows from the following fact: for a graded ring A, a graded A-module N is projective as an A-module if and only if N is projective in the category of graded A-modules; see [Lau 2021, Lemma 3.0.1].

Let

$$\rho : \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to \Delta_S / \mathrm{Fil}^{1}_{\mathcal{N}}(\Delta_S) \tag{8-1}$$

be the composition of the projection $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ with the natural homomorphism $\Delta_S \to \Delta_S / \mathrm{Fil}^{1}_{\mathcal{N}}(\Delta_S)$. The map ρ is a ring homomorphism. For an integer $n \geq 1$, we write

$$\Delta^{\mathcal{N}}_{S,n} := \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \otimes_{\Delta_S} \Delta_S/(p, I)^n.$$

Let $\bar{\rho} : \Delta^{\mathcal{N}}_{S,n} \to \Delta_S / (\mathrm{Fil}^{1}_{\mathcal{N}}(\Delta_S) + (p, I))$ be the homomorphism induced by ρ.

Lemma 8.1.7 (cf. [Lau 2021, Lemma 3.1.1, Corollary 3.1.2]).

(1) *Let M be a finite graded $\Delta^{\mathcal{N}}_{S,n}$-module. If $\bar{\rho}^* M = 0$, then we have $M = 0$.*

(2) *Let M and N be finite graded $\Delta^{\mathcal{N}}_{S,n}$-modules. Assume that N is projective as a $\Delta^{\mathcal{N}}_{S,n}$-module. Then a homomorphism $f : M \to N$ of graded $\Delta^{\mathcal{N}}_{S,n}$-modules is an isomorphism if $\bar{\rho}^* f : \bar{\rho}^* M \to \bar{\rho}^* N$ is an isomorphism.*

Proof. (1) By [Anschütz and Le Bras 2023, Lemma 4.28], the pair $(\Delta_S, \mathrm{Fil}^1_{\mathcal{N}}(\Delta_S))$ is henselian. In particular we have $\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) \subset \mathrm{rad}(\Delta_S)$. Using this fact, we can prove the assertion by the same argument as in the proof of [Lau 2021, Lemma 3.1.1].

(2) By (1), we see that f is surjective. Since N is projective as a graded $\Delta^{\mathcal{N}}_{S,n}$-module (Remark 8.1.6), we have $M \simeq N \oplus \mathrm{Ker}\, f$ as graded $\Delta^{\mathcal{N}}_{S,n}$-modules. Thus $\mathrm{Ker}\, f$ is a finite graded $\Delta^{\mathcal{N}}_{S,n}$-module such that $\bar{\rho}^* \,\mathrm{Ker}\, f = 0$. By (1) again, we have $\mathrm{Ker}\, f = 0$. □

Corollary 8.1.8. *Let $N = \bigoplus_{i \in \mathbb{Z}} N_i$ be a graded $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$-module with the following properties*:

(1) *The degree-i part N_i is (p, I)-adically complete for every $i \in \mathbb{Z}$.*

(2) *$N^n := N/(p, I)^n N$ is a finite projective $\Delta^{\mathcal{N}}_{S,n}$-module for every $n \geq 1$.*

Then there exists a graded finite projective Δ_S-module L with an isomorphism

$$L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \simeq N$$

of graded $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$-modules. In particular N is a finite projective $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$-module.

Proof. Since Δ_S is henselian with respect to both ideals $\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S)$ and (p, I), there exists a graded finite projective Δ_S-module L with an isomorphism

$$L/(\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) + (p, I))L \xrightarrow{\sim} \bar{\rho}^* N^1$$

of graded modules (by [Stacks 2005–, Tag 0D4A] or [Greco 1968, Theorem 5.1]). This isomorphism lifts to a homomorphism

$$f : L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to N$$

of graded $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$-modules (see Remark 8.1.6). By Lemma 8.1.7, the reduction modulo $(p, I)^n$ of f is bijective for every n. Since degree-i parts of $L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$ and N are (p, I)-adically complete for every $i \in \mathbb{Z}$, it follows that f is an isomorphism. □

For a bounded prism (A, I), let $\mathrm{BK}_{\mathrm{disp}}(A, I)$ be the category of displayed Breuil–Kisin modules over (A, I) (see Definition 3.1.5). Prismatic F-gauges in vector bundles over S can be related to displayed Breuil–Kisin modules over (Δ_S, I_S) as follows.

Proposition 8.1.9. *Let $S \in$ QRSPerfd. There exists a fully faithful functor*

$$F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S) \to \mathrm{BK}_{\mathrm{disp}}(\Delta_S, I_S)$$

which is compatible with base change along any homomorphism $S \to S'$ in QRSPerfd.

Proof. To each $(N, F_N) \in F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S)$, we attach a displayed Breuil–Kisin module (M, F_M) over (Δ_S, I) as follows. Let $M := \tau^* N$. The kernel of $\tau : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ is generated by $t - 1$, so that $M = N/(t-1)N$. It follows that the natural homomorphism $N_i \to M$ of Δ_S-modules is injective, whose image is denoted by $\mathrm{Fil}^i(M) \subset M$. We have $\mathrm{Fil}^{i+1}(M) \subset \mathrm{Fil}^i(M)$, and the corresponding map $N_{i+1} \to N_i$ is given by $x \mapsto tx$. Moreover we have $M = \bigcup_i \mathrm{Fil}^i(M)$. Let i be a small enough integer such that $\mathrm{Fil}^i(M) = M$. We define $\phi^* M \to M[1/I]$ to be the composition

$$\phi^* M = \phi^* \mathrm{Fil}^i(M) \xrightarrow{\sim} \phi^* N_i \to (\sigma^* N)_i \xrightarrow{\sim} (\sigma^* N)_0 \otimes_{\Delta_S} I^i t^{-i} \xrightarrow{F_N} M \otimes_{\Delta_S} I^i t^{-i} \xrightarrow{t \mapsto 1} M[1/I].$$

(For the isomorphism $(\sigma^* N)_i \xrightarrow{\sim} (\sigma^* N)_0 \otimes_{\Delta_S} I^i t^{-i}$, see Remark 8.1.4.) This homomorphism $\phi^* M \to M[1/I]$ is independent of the choice of i. Let $F_M : (\phi^* M)[1/I] \to M[1/I]$ be the induced homomorphism.

We shall prove that (M, F_M) is a displayed Breuil–Kisin module over (Δ_S, I). By Corollary 8.1.8, we may assume that

$$N = L \otimes_{\Delta_S} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S))$$

for some graded finite projective Δ_S-module $L = \bigoplus_{j \in \mathbb{Z}} L_j^{(-1)}$. Then we have $M = L$ and

$$\mathrm{Fil}^i(M) = \Big(\bigoplus_{j \geq i} L_j^{(-1)}\Big) \oplus \Big(\bigoplus_{j < i} \mathrm{Fil}^{i-j}_{\mathcal{N}}(\Delta_S) L_j^{(-1)}\Big) \tag{8-2}$$

for every $i \in \mathbb{Z}$. We set $L_j := \phi^* L_j^{(-1)}$. By sending t to 1, we obtain $(\sigma^* N)_0 \xrightarrow{\sim} \bigoplus_{j \in \mathbb{Z}} (L_j \otimes_{\Delta_S} I^{-j})$, and the isomorphism F_N can be written as

$$F_N : \bigoplus_{j \in \mathbb{Z}} (L_j \otimes_{\Delta_S} I^{-j}) \xrightarrow{\sim} M.$$

Now $F_M : \bigoplus_{j \in \mathbb{Z}} L_j[1/I] \to M[1/I]$ is the base change of F_N. (In particular F_M is an isomorphism.) Recall the filtration $\{\mathrm{Fil}^i(\phi^* M)\}_{i \in \mathbb{Z}}$ of $\phi^* M$ from Definition 3.1.2. We see that $\mathrm{Fil}^i(\phi^* M) \subset \phi^* M$ is the intersection of $\phi^* M = \bigoplus_{j \in \mathbb{Z}} L_j$ with $\bigoplus_{j \in \mathbb{Z}} (L_j \otimes_{\Delta_S} I^{i-j})$, and thus

$$\mathrm{Fil}^i(\phi^* M) = \Big(\bigoplus_{j \geq i} L_j\Big) \oplus \Big(\bigoplus_{j < i} I^{i-j} L_j\Big). \tag{8-3}$$

From this description, we see that (M, F_M) is a displayed Breuil–Kisin module. (Moreover $\phi^* M = \bigoplus_{j \in \mathbb{Z}} L_j$ is a normal decomposition in the sense of Definition 3.1.10.)

We have constructed a functor $F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S) \to \mathrm{BK}_{\mathrm{disp}}(\Delta_S, I_S)$. The full faithfulness of this functor follows from [Guo and Li 2023, Corollary 2.53]. The important point is that the filtration $\{\mathrm{Fil}^i(M)\}_{i \in \mathbb{Z}}$ of M can be recovered from (M, F_M). Namely, $\mathrm{Fil}^i(M)$ agrees with the inverse image of $M \otimes_{\Delta_S} I^i$ under the composition

$$M \xrightarrow{x \mapsto 1 \otimes x} (\phi^* M)[1/I] \xrightarrow{F_M} M[1/I].$$

(Compare (8-2) with (8-3).) Since the full faithfulness of the functor also follows from Example 5.3.10, Example 8.2.9, and Proposition 8.2.11 below (compare with the argument in the proof of Proposition 7.1.1), we omit the details here. □

The following result is essential for the definition of prismatic F-gauges in vector bundles over a quasisyntomic ring.

Proposition 8.1.10. *The fibered category over* $\mathrm{QRSPerfd}^{\mathrm{op}}$ *which associates to each* $S \in \mathrm{QRSPerfd}$ *the category* $F\text{-}\mathrm{Gauge}^{\mathrm{vect}}(S)$ *satisfies descent with respect to the quasisyntomic topology.*

Proof. This was originally proved by Bhatt and Lurie (see [Bhatt 2022, Remark 5.5.18]). In [Guo and Li 2023, Proposition 2.29], this result was obtained by a slightly different method. We briefly recall the argument given in that proposition.

Let $S \to S'$ be a quasisyntomic covering in QRSPerfd. By the proof of [loc. cit., Proposition 2.29], the induced homomorphism $\Delta^{\mathcal{N}}_{S,n} \to \Delta^{\mathcal{N}}_{S',n}$ is faithfully flat for every $n \geq 1$, and for a homomorphism $S \to S_1$ in QRSPerfd, the following natural homomorphism of graded rings is an isomorphism:

$$\Delta^{\mathcal{N}}_{S',n} \otimes_{\Delta^{\mathcal{N}}_{S,n}} \Delta^{\mathcal{N}}_{S_1,n} \xrightarrow{\sim} \Delta^{\mathcal{N}}_{S' \widehat{\otimes}_S S_1, n},$$

where $S' \widehat{\otimes}_S S_1 \in$ QRSPerfd is the p-adic completion of $S' \otimes_S S_1$. Using these results, we can prove that the natural functor from the category of prismatic F-gauges in vector bundles over S to the category of prismatic F-gauges in vector bundles over S' with a descent datum (with respect to $S \to S'$) is an equivalence. We only prove the essential surjectivity of the functor.

Let $(N', F_{N'}) \in F\text{-Gauge}^{\mathrm{vect}}(S')$ with a descent datum. By the results recalled in the previous paragraph and by faithfully flat descent, we see that, for every $n \geq 1$, the graded $\Delta^{\mathcal{N}}_{S',n}$-module $N'/(p, I)^n N'$ with the descent datum arises from a graded $\Delta^{\mathcal{N}}_{S,n}$-module $N^n = \bigoplus_{i \in \mathbb{Z}} N^n_i$ such that N^n is finite projective as a $\Delta^{\mathcal{N}}_{S,n}$-module. Let $N_i := \varprojlim_n N^n_i$ and we define $N := \bigoplus_{i \in \mathbb{Z}} N_i$, which is a graded $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module. We have $N/(p, I)^n N = N^n$ for every $n \geq 1$. By Corollary 8.1.8, we see that N is a finite projective $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$-module. Moreover the natural homomorphism

$$N \otimes_{\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))} \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_{S'})) \to N'$$

is an isomorphism (as its reduction modulo $(p, I)^n$ is an isomorphism for every n). Since $(\Delta_S, I_S) \to (\Delta_{S'}, I_{S'})$ is a faithfully flat map of $\mathcal{O}_E$-prisms and the (p, I)-adic completion of $\Delta_{S'} \otimes_{\Delta_S} \Delta_{S'}$ is isomorphic to $\Delta_{S' \widehat{\otimes}_S S'}$ (see the proof of [Guo and Li 2023, Proposition 2.29]), the isomorphism $F_{N'}$ descends to an isomorphism $F_N : (\sigma^* N)_0 \xrightarrow{\sim} \tau^* N$ by Proposition 2.5.6. This shows that $(N', F_{N'})$ with the descent datum arises from $(N, F_N) \in F\text{-Gauge}^{\mathrm{vect}}(S)$. □

Proposition 8.1.10, together with [Bhatt et al. 2019, Proposition 4.31], shows that the fibered category $S \mapsto F\text{-Gauge}^{\mathrm{vect}}(S)$ over $\mathrm{QRSPerfd}^{\mathrm{op}}$ extends uniquely to a fibered category

$$R \mapsto F\text{-Gauge}^{\mathrm{vect}}(R)$$

over the category $\mathrm{QSyn}^{\mathrm{op}}$ that satisfies descent with respect to the quasisyntomic topology.

Definition 8.1.11 (Drinfeld, Bhatt–Lurie, Guo–Li). Let $R \in$ QSyn. An object $N \in F\text{-Gauge}^{\mathrm{vect}}(R)$ is called a *prismatic F-gauge in vector bundles* over R. For a homomorphism $R \to S$ with $S \in$ QRSPerfd, the image of N in $F\text{-Gauge}^{\mathrm{vect}}(S)$ is denoted by (N_S, F_{N_S}).

A fully faithful functor from $F\text{-Gauge}^{\mathrm{vect}}(R)$ to the category of prismatic F-crystals on $(R)_{\Delta}$ (in the sense of [Bhatt and Scholze 2023]) is obtained in [Guo and Li 2023, Corollary 2.53]. More precisely, we have the following result.

Proposition 8.1.12. *Let $R \in$ QSyn. There exists a fully faithful functor*

$$F\text{-Gauge}^{\mathrm{vect}}(R) \to 2 - \varprojlim_{(A,I)\in(R)_{\Delta}} \mathrm{BK}_{\mathrm{disp}}(A, I). \tag{8-4}$$

This functor is compatible with base change along any homomorphism $R \to R'$ in QSyn.

Proof. By Proposition 8.1.10, the left-hand side of (8-4) satisfies quasisyntomic descent. By [Bhatt and Scholze 2022, Proposition 7.11] and Corollary 3.1.15, the right-hand side of (8-4) also satisfies quasisyntomic descent; see also the proof of [Bhatt and Scholze 2023, Proposition 2.14]. Thus, the assertion follows from Proposition 8.1.9. □

As in Section 3.2, let $\mu = (m_1, \dots, m_n)$ be a tuple of integers $m_1 \geq \cdots \geq m_n$. Let $r_i \in \mathbb{Z}_{\geq 0}$ be the number of occurrences of i in $(m_1, \dots, m_n)$. We want to compare prismatic F-gauges in vector bundles with (displayed) Breuil–Kisin modules of type μ in the sense of Definition 3.2.1.

Remark 8.1.13. Let $S \in$ QRSPerfd. By [Bhatt and Scholze 2022, Theorem 12.2], the Frobenius $\phi : \Delta_S \to \Delta_S$ induces an isomorphism

$$\Delta_S / \mathrm{Fil}^1_{\mathcal{N}}(\Delta_S) \xrightarrow{\sim} S.$$

Using this, we regard the homomorphism ρ (see (8-1)) as $\rho : \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to S$.

Definition 8.1.14. Let $R \in$ QSyn. Let $N \in F\text{-Gauge}^{\mathrm{vect}}(R)$. We say that N is *of type* μ if for any $R \to S$ with $S \in$ QRSPerfd, the degree-i part $(\rho^* N_S)_i$ of the graded S-module $\rho^* N_S$ is of rank r_i for any $i \in \mathbb{Z}$.

Let

$$F\text{-Gauge}_{\mu}(R) \subset F\text{-Gauge}^{\mathrm{vect}}(R)$$

be the full subcategory spanned by those objects of type μ. The property of being of type μ can be checked locally in the quasisyntomic topology. Thus the fibered category $R \mapsto F\text{-Gauge}_{\mu}(R)$ over $\mathrm{QSyn}^{\mathrm{op}}$ satisfies descent with respect to the quasisyntomic topology.

By construction, the functor (8-4) induces a fully faithful functor

$$F\text{-Gauge}_{\mu}(R) \to 2 - \varprojlim_{(A,I)\in(R)_{\Delta}} \mathrm{BK}_{\mu}(A, I) \tag{8-5}$$

for any $R \in$ QSyn. (Recall that $\mathrm{BK}_{\mu}(A, I)$ is the category of Breuil–Kisin modules over (A, I) of type μ.) We will prove later that the functors (8-4) and (8-5) are equivalences if R is a perfectoid ring or a complete regular local ring with perfect residue field k of characteristic p; see Corollary 8.2.13 below.

Example 8.1.15. Let $R \in$ QSyn. Let $F\text{-Gauge}^{\mathrm{vect}}_{[0,1]}(R) \subset F\text{-Gauge}^{\mathrm{vect}}(R)$ be the full subcategory of those $N \in F\text{-Gauge}^{\mathrm{vect}}(R)$ such that for any homomorphism $R \to S$ with $S \in$ QRSPerfd, we have $(\rho^* N_S)_i = 0$ for all $i \neq 0, 1$. The functor (8-4) induces a fully faithful functor

$$F\text{-Gauge}^{\mathrm{vect}}_{[0,1]}(R) \to 2 - \varprojlim_{(A,I)\in(R)_{\Delta}} \mathrm{BK}_{\min}(A, I).$$

The right-hand side can be identified with the category of prismatic Dieudonné crystals on $(R)_{\Delta}$; see Section 7.1. By [Guo and Li 2023, Theorem 2.54], the essential image of this functor is the full subcategory of admissible prismatic Dieudonné crystals on $(R)_{\Delta}$. If R is a perfectoid ring or a complete regular local ring with perfect residue field k of characteristic p, then any prismatic Dieudonné crystal on $(R)_{\Delta}$ is admissible by [Anschütz and Le Bras 2023, Propositions 4.12 and 5.10], and hence the above functor is an equivalence in this case. This fact also follows from Corollary 8.2.13.

8.2. ***Prismatic G-F-gauges of type μ.*** Let G be a smooth affine group scheme over $\mathbb{Z}_p$. Let $\mu : \mathbb{G}_m \to G_{W(k)}$ be a cocharacter where k is a perfect field of characteristic p. We introduce prismatic G-F-gauges of type μ in the same way as for prismatic G-μ-displays.

We retain the notation of Section 4. For the cocharacter μ, we have the action (4-1) of $\mathbb{G}_m$ on $G_{W(k)} = \operatorname{Spec} A_G$. Let $A_G = \bigoplus_{i\in\mathbb{Z}} A_{G,i}$ be the weight decomposition. We define $A^{(-1)}_{G,i} := (\phi^{-1})^* A_{G,i}$, where $\phi^{-1} : W(k) \to W(k)$ is the inverse of the Frobenius. Since $(\phi^{-1})^* A_G = A_G$, we have

$$A_G = \bigoplus_{i\in\mathbb{Z}} A^{(-1)}_{G,i}.$$

Let $\mu^{(-1)} : \mathbb{G}_m \to G_{W(k)}$ be the base change of μ along ϕ^{-1}. Then $A_G = \bigoplus_{i\in\mathbb{Z}} A^{(-1)}_{G,i}$ is the weight decomposition with respect to the action of $\mathbb{G}_m$ induced by $\mu^{(-1)}$.

Let S be a quasiregular semiperfectoid ring over $W(k)$.

Definition 8.2.1. Let

$$G_{\mu,\mathcal{N}}(S) \subset G(\Delta_S)$$

be the subgroup consisting of homomorphisms $g^* : A_G \to \Delta_S$ of $W(k)$-algebras such that $g^*(A^{(-1)}_{G,i}) \subset \mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)$ for any $i \in \mathbb{Z}$. The group $G_{\mu,\mathcal{N}}(S)$ is called the *gauge group*.

Remark 8.2.2. It follows from Lemma 4.1.4 that $G_{\mu,\mathcal{N}}(S) \subset G(\Delta_S)$ is the inverse image of the display group $G_\mu(\Delta_S, I_S) \subset G(\Delta_S)$ under the homomorphism $\phi : G(\Delta_S) \to G(\Delta_S)$.

For a generator $d \in I_S$, we have the homomorphism

$$\sigma_{\mu,\mathcal{N},d} : G_{\mu,\mathcal{N}}(S) \to G(\Delta_S), \quad g \mapsto \mu(d)\phi(g)\mu(d)^{-1},$$

by Remark 8.2.2. Let $G(\Delta_S)_{\mathcal{N},d}$ be the set $G(\Delta_S)$ together with the following action of $G_{\mu,\mathcal{N}}(S)$:

$$G(\Delta_S) \times G_{\mu,\mathcal{N}}(S) \to G(\Delta_S), \quad (X, g) \mapsto X \cdot g := g^{-1} X \sigma_{\mu,\mathcal{N},d}(g). \tag{8-6}$$

For another generator $d' \in I_S$ and the unique element $u \in \Delta_S^\times$ such that $d = ud'$, the bijection $G(\Delta_S)_{\mathcal{N},d} \to G(\Delta_S)_{\mathcal{N},d'}$, $X \mapsto X\mu(u)$, is $G_{\mu,\mathcal{N}}(S)$-equivariant. Then we set

$$G(\Delta_S)_{\mathcal{N}} := \varprojlim_{d} G(\Delta_S)_{\mathcal{N},d},$$

which is equipped with a natural action of $G_{\mu,\mathcal{N}}(S)$. Here d runs over the set of generators $d \in I_S$. Although $G(\Delta_S)_{\mathcal{N}}$ depends on μ, we omit it from the notation.

Remark 8.2.3. We recall some notation from [Bhatt and Scholze 2023, Definition 2.9]. Let R be a quasisyntomic ring. Let $(R)_{\mathrm{QSyn}}$ (resp. $(R)_{\mathrm{qrsp}}$) be the category of quasisyntomic rings R' (resp. quasiregular semiperfectoid rings R') with a quasisyntomic map $R \to R'$. We endow both $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$ and $(R)^{\mathrm{op}}_{\mathrm{qrsp}}$ with the quasisyntomic topology. Since quasiregular semiperfectoid rings form a basis for QSyn, we may identify sheaves on $(R)_{\mathrm{QSyn}}^{\mathrm{op}}$ with sheaves on $(R)_{\mathrm{qrsp}}^{\mathrm{op}}$. On the site $(R)_{\mathrm{QSyn}}^{\mathrm{op}}$, we have the sheaves $\Delta_\bullet$ and $I_\bullet$ such that

$$\Delta_\bullet(S) = \Delta_S \quad \text{and} \quad I_\bullet(S) = I_S$$

for each $S \in (R)_{\mathrm{qrsp}}$.

Lemma 8.2.4. *Let R be a quasisyntomic ring over $W(k)$. The functors*

$$G_{\mu,\mathcal{N}} : (R)_{\mathrm{qrsp}} \to \mathrm{Set}, \quad S \mapsto G_{\mu,\mathcal{N}}(S),$$
$$G_{\Delta,\mathcal{N}} : (R)_{\mathrm{qrsp}} \to \mathrm{Set}, \quad S \mapsto G(\Delta_S)_{\mathcal{N}},$$

form sheaves with respect to the quasisyntomic topology.

Proof. As $\Delta_\bullet$ is a sheaf, so is $G_{\Delta,\mathcal{N}}$. Since $I_\bullet$ is a sheaf, it follows that the functor $S \mapsto \mathrm{Fil}^i_{\mathcal{N}}(\Delta_S)$ forms a sheaf for any $i \in \mathbb{Z}$. This implies that $G_{\mu,\mathcal{N}}$ is a sheaf. □

We regard $G_{\mu,\mathcal{N}}$ and $G_{\Delta,\mathcal{N}}$ as sheaves on $(R)_{\mathrm{QSyn}}^{\mathrm{op}}$. The sheaf $G_{\Delta,\mathcal{N}}$ is equipped with an action of $G_{\mu,\mathcal{N}}$.

Definition 8.2.5 (prismatic G-F-gauge of type μ). Let R be a quasisyntomic ring over $W(k)$. A *prismatic G-F-gauge of type μ* over R is a pair

$$(\mathscr{Q}, \alpha_{\mathscr{Q}}),$$

where $\mathscr{Q}$ is a $G_{\mu,\mathcal{N}}$-torsor on $(R)_{\mathrm{QSyn}}^{\mathrm{op}}$ and $\alpha_{\mathscr{Q}} : \mathscr{Q} \to G_{\Delta,\mathcal{N}}$ is a $G_{\mu,\mathcal{N}}$-equivariant map. We say that $(\mathscr{Q}, \alpha_{\mathscr{Q}})$ is *banal* if $\mathscr{Q}$ is trivial as a $G_{\mu,\mathcal{N}}$-torsor. When there is no possibility of confusion, we write $\mathscr{Q}$ instead of $(\mathscr{Q}, \alpha_{\mathscr{Q}})$. An isomorphism of prismatic G-F-gauges of type μ over R is defined in the same way as in Definition 5.2.1.

Let

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R)$$

be the groupoid of prismatic G-F-gauges of type μ over R. For a homomorphism $f : R \to R'$ of quasisyntomic rings over $W(k)$, we have a base change functor

$$f^* : G\text{-}F\text{-}\mathrm{Gauge}_\mu(R) \to G\text{-}F\text{-}\mathrm{Gauge}_\mu(R')$$

defined in the same way as in Definition 5.2.6.

Remark 8.2.6. A "truncated analogue" of the notion of prismatic G-F-gauges of type μ was introduced by Drinfeld [2023, Appendix C] for a p-adic formal scheme $\mathcal{X}$ which is formally of finite type over $\mathrm{Spf}\,\mathbb{Z}_p$, in terms of certain torsors on the *syntomification* of $\mathcal{X}$ in the sense of Drinfeld and Bhatt–Lurie. It should be possible to define prismatic G-F-gauges of type μ over any p-adic formal scheme by using certain torsors on syntomifications, but we will not discuss this here.[2]

Remark 8.2.7. Let S be a quasiregular semiperfectoid ring over $W(k)$. For a generator $d \in I_S$, let $\sigma_d : \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ be the homomorphism defined by $a_i t^{-i} \mapsto \phi(a_i) d^{-i}$ for any $i \in \mathbb{Z}$. Recall the

[2] After this work was completed, and during the refereeing process, this has been carried out by Gardner and Madapusi [2024].

homomorphism $\tau : \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \to \Delta_S$ from Definition 8.1.1. Similarly to the triple $(\mathrm{Rees}(I^{\bullet}), \sigma_d, \tau)$ in Remark 5.2.3, the triple

$$(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)), \sigma_d, \tau)$$

is an analogue of a higher frame in the sense of Lau. The homomorphism τ induces an isomorphism

$$G(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))^0 \xrightarrow{\sim} G_{\mu,\mathcal{N}}(S),$$

where $G(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))^0$ is the group of elements $g \in G(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))$ such that

$$g^* : A_G = \bigoplus_{i \in \mathbb{Z}} A^{(-1)}_{G,i} \to \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$$

is a homomorphism of graded $W(k)$-algebras. Via this isomorphism, the homomorphism $\sigma_{\mu,\mathcal{N},d}$ agrees with the one induced by σ_d. Thus, the action (8-6) is consistent with the one considered in [Lau 2021, (5-2)].

Roughly speaking, prismatic F-gauges in vector bundles (resp. prismatic G-F-gauges of type μ) over S can be considered as displays (resp. G-μ-displays) over the "higher frame" $(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)), \sigma_d, \tau)$. On the other hand, displayed Breuil–Kisin modules (resp. prismatic G-μ-displays) over (Δ_S, I_S) can be thought of as displays (resp. G-μ-displays) over the "higher frame" $(\mathrm{Rees}(I^{\bullet}_S), \sigma_d, \tau)$. See also [Lau 2021, Section 3.7] where the relation between displays over higher frames and Frobenius gauges in the sense of [Fontaine and Jannsen 2021, Section 2.2] is discussed.

Let us discuss the relation between prismatic F-gauges in vector bundles of type μ and prismatic GL_n-F-gauges of type μ.

Example 8.2.8. Let $\mu : \mathbb{G}_m \to \mathrm{GL}_{n,W(k)}$ be a cocharacter and let $(m_1, \dots, m_n)$ be the corresponding tuple of integers $m_1 \geq \cdots \geq m_n$ as in Section 3.2. We retain the notation of Section 3.2. Let $L_{W(k)} = \bigoplus_{j \in \mathbb{Z}} L_{\mu,j}$ be the weight decomposition with respect to the action of $\mathbb{G}_m$ on $L_{W(k)} = W(k)^n$ induced by μ. We set $L^{(-1)}_{\mu,j} := (\phi^{-1})^* L_{\mu,j}$. By the decomposition $L_{W(k)} = \bigoplus_{j \in \mathbb{Z}} L^{(-1)}_{\mu,j}$, we regard $L_{W(k)}$ as a graded module. Let S be a quasiregular semiperfectoid ring over $W(k)$. Then, via the isomorphism

$$\mathrm{GL}_n(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)))^0 \xrightarrow{\sim} (\mathrm{GL}_n)_{\mu,\mathcal{N}}(S)$$

given in Remark 8.2.7, we may identify $(\mathrm{GL}_n)_{\mu,\mathcal{N}}(S)$ with the group of graded automorphisms of $L_{W(k)} \otimes_{W(k)} \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S))$.

Example 8.2.9. Let the notation be as in Example 8.2.8. Let R be a quasisyntomic ring over $W(k)$. We shall construct an equivalence

$$F\text{-}\mathrm{Gauge}_{\mu}(R)^{\simeq} \xrightarrow{\sim} \mathrm{GL}_n\text{-}F\text{-}\mathrm{Gauge}_{\mu}(R), \quad N \mapsto \mathcal{Q}(N), \tag{8-7}$$

where $F\text{-}\mathrm{Gauge}_{\mu}(R)^{\simeq}$ is the groupoid of prismatic F-gauges in vector bundles of type μ over R. Let $N \in F\text{-}\mathrm{Gauge}_{\mu}(R)$. We consider the sheaf

$$\mathcal{Q}(N) : (R)_{\mathrm{QSyn}} \to \mathrm{Set}$$

sending $S \in (R)_{\mathrm{qrsp}}$ to the set of graded isomorphisms

$$h : L_{W(k)} \otimes_{W(k)} \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}(\Delta_S)) \xrightarrow{\sim} N_S.$$

Such an isomorphism h exists locally in the quasisyntomic topology by (the proof of) Corollary 8.1.8. By Example 8.2.8, the sheaf $\mathscr{Q}(N)$ then admits the structure of a $(\mathrm{GL}_n)_{\mu,\mathcal{N}}$-torsor. Let $d \in I_S$ be a generator. We fix an isomorphism h as above. Then we have the isomorphisms

$$(\sigma^* N)_0 \simeq \bigoplus_{j\in\mathbb{Z}} (L_{\mu,j} \otimes_{W(k)} I_S^{-j}) \simeq L_{\Delta_S},$$

where the second isomorphism is given by $\mu(d)$. We also have $\tau^* N \simeq L_{\Delta_S}$. Thus, the isomorphism F_{N_S} gives an element $\alpha(h)_d \in \mathrm{GL}_n(\Delta_S) = \mathrm{GL}_n(\Delta_S)_{\mathcal{N},d}$. The element $\alpha(h) \in \mathrm{GL}_n(\Delta_S)_{\mathcal{N}}$ corresponding to $\alpha(h)_d$ does not depend on the choice of d. In this way, we get a $(\mathrm{GL}_n)_{\mu,\mathcal{N}}$-equivariant map $\alpha : \mathscr{Q}(N) \to (\mathrm{GL}_n)_{\Delta,\mathcal{N}}$, so that the pair $(\mathscr{Q}(N), \alpha)$ belongs to GL_n -F-$\mathrm{Gauge}_\mu(R)$. This construction gives the functor (8-7). Using quasisyntomic descent, one can check that this functor is an equivalence.

We now compare prismatic G-μ-displays with prismatic G-F-gauges of type μ. We first note the following result.

Proposition 8.2.10. *Let R be a quasisyntomic ring over $W(k)$. The fibered category over $(R)^{\mathrm{op}}_{\mathrm{QSyn}}$ which associates to each $R' \in (R)_{\mathrm{QSyn}}$ the groupoid G-$\mathrm{Disp}_\mu((R')_\Delta)$ is a stack with respect to the quasisyntomic topology.*

Proof. This follows from [Bhatt and Scholze 2022, Proposition 7.11] and Proposition 5.2.8 by a standard argument. (See also the proof of [Bhatt and Scholze 2023, Proposition 2.14].) □

Proposition 8.2.11. *Let R be a quasisyntomic ring over $W(k)$. There exists a fully faithful functor*

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(R) \to G\text{-}\mathrm{Disp}_\mu((R)_\Delta). \tag{8-8}$$

This functor is compatible with base change along any homomorphism $R \to R'$ in QSyn.

Proof. It is clear that the left-hand side of (8-8) satisfies quasisyntomic descent. By Proposition 8.2.10, the right-hand side of (8-8) also satisfies quasisyntomic descent. It thus suffices to construct, for each quasiregular semiperfectoid ring S over $W(k)$, a fully faithful functor

$$G\text{-}F\text{-}\mathrm{Gauge}_\mu(S)_{\mathrm{banal}} \to G\text{-}\mathrm{Disp}_\mu(\Delta_S, I_S)_{\mathrm{banal}} \tag{8-9}$$

that is compatible with base change along any homomorphism $S \to S'$ in the subcategory QRSPerfd. Here G-F-$\mathrm{Gauge}_\mu(S)_{\mathrm{banal}}$ is the groupoid of banal prismatic G-F-gauges of type μ over S. By Remark 5.2.5, we may identify G-$\mathrm{Disp}_\mu(\Delta_S, I_S)_{\mathrm{banal}}$ with the groupoid $[G(\Delta_S)_{I_S}/G_\mu(\Delta_S, I_S)]$. Similarly, we may identify G-F-$\mathrm{Gauge}_\mu(S)_{\mathrm{banal}}$ with the groupoid

$$[G(\Delta_S)_{\mathcal{N}}/G_{\mu,\mathcal{N}}(S)]$$

whose objects are the elements $X \in G(\Delta_S)_{\mathcal{N}}$ and whose morphisms are defined by $\mathrm{Hom}(X, X') = \{g \in G_{\mu,\mathcal{N}}(S) \mid X' \cdot g = X\}$. The map $\phi : G(\Delta_S) \to G(\Delta_S)$ induces a map $\phi : G(\Delta_S)_{\mathcal{N}} \to G(\Delta_S)_{I_S}$ such that, for every $X \in G(\Delta_S)_{\mathcal{N}}$ and every $g \in G_{\mu,\mathcal{N}}(S)$, we have $\phi(X \cdot g) = \phi(X) \cdot \phi(g)$, where

$\phi(g) \in G_\mu(\Delta_S, I_S)$ is the image of g under the natural homomorphism $\phi : G_{\mu,\mathcal{N}}(S) \to G_\mu(\Delta_S, I_S)$. Then we define the functor (8-9) as

$$[G(\Delta_S)_{\mathcal{N}}/G_{\mu,\mathcal{N}}(S)] \to [G(\Delta_S)_{I_S}/G_\mu(\Delta_S, I_S)], \quad X \mapsto \phi(X). \tag{8-10}$$

This functor is fully faithful. Indeed, let $d \in I_S$ be a generator. It suffices to prove that for all $X, X' \in G(\Delta_S)$, the map

$$\{g \in G_{\mu,\mathcal{N}}(S) \mid g^{-1}X'\sigma_{\mu,\mathcal{N},d}(g) = X\} \to \{h \in G_\mu(\Delta_S, I_S) \mid h^{-1}\phi(X')\sigma_{\mu,d}(h) = \phi(X)\}$$

defined by $g \mapsto \phi(g)$ is bijective. (Recall $\sigma_{\mu,\mathcal{N},d}(g) = \mu(d)\phi(g)\mu(d)^{-1}$ and $\sigma_{\mu,d}(h) = \phi(\mu(d)h\mu(d)^{-1})$.) One can check that the map $h \mapsto X'\mu(d)h\mu(d)^{-1}X^{-1}$ gives the inverse of the above map. Indeed, for an element $g \in G_{\mu,\mathcal{N}}(S)$ in the left-hand side, we have

$$X'\mu(d)\phi(g)\mu(d)^{-1}X^{-1} = X'\sigma_{\mu,\mathcal{N},d}(g)X^{-1} = g.$$

Similarly, for an element $h \in G_\mu(\Delta_S, I_S)$ in the right-hand side, we have

$$\phi(X'\mu(d)h\mu(d)^{-1}X^{-1}) = \phi(X')\sigma_{\mu,d}(h)\phi(X)^{-1} = h.$$

The proof of Proposition 8.2.11 is complete. □

Corollary 8.2.12. *Let R be a perfectoid ring over $W(k)$ or a complete regular local ring over $W(k)$ with residue field k. Then the functor (8-8) is an equivalence*:

$$G\text{-}F\text{-Gauge}_\mu(R) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu((R)_{\Delta}).$$

Proof. Since we already know that this functor is fully faithful (Proposition 8.2.11), it suffices to prove the essential surjectivity. The assertion can be checked locally in the quasisyntomic topology.

We first assume that R is a perfectoid ring over $W(k)$. In this case, we have

$$G\text{-}\mathrm{Disp}_\mu((R)_{\Delta}) \xrightarrow{\sim} G\text{-}\mathrm{Disp}_\mu(W(R^\flat), I_R).$$

Since every G-μ-display over $(W(R^\flat), I_R)$ is banal over a p-completely étale covering $R \to R'$ with R' a perfectoid ring (Example 2.5.11), it suffices to prove that the functor (8-10) given in the proof of Proposition 8.2.11 is essentially surjective when $S = R$. This follows since $(\Delta_R, I_R) \simeq (W(R^\flat), I_R)$ and the Frobenius $\phi : W(R^\flat) \to W(R^\flat)$ is bijective.

The case where R is a complete regular local ring over $W(k)$ with residue field k follows from the previous case since there exists a quasisyntomic covering $R \to R'$ with R' a perfectoid ring by [Anschütz and Le Bras 2023, Proposition 5.8]. □

Corollary 8.2.13. *The functors (8-4) and (8-5) are equivalences if R is a perfectoid ring or a complete regular local ring with perfect residue field k of characteristic p.*

Proof. We need to prove that (8-4) and (8-5) are essentially surjective. For (8-5), this follows from Corollary 8.2.12 together with Examples 5.3.10 and 8.2.9. For (8-4), we argue as follows. As in the proof of Corollary 8.2.12, it suffices to treat the case where R is a perfectoid ring. Then we have

$$2-\varprojlim_{(A,I)\in(R)_{\Delta}}\mathrm{BK}_{\mathrm{disp}}(A,I)\xrightarrow{\sim}\mathrm{BK}_{\mathrm{disp}}(W(R^{\flat}),I_R).$$

For each $M\in\mathrm{BK}_{\mathrm{disp}}(W(R^{\flat}),I_R)$, there exists a p-completely étale covering $R\to R_1\times\cdots\times R_m$ with $R_1,\ldots,R_m$ perfectoid rings such that, for any $1\le i\le m$, the base change $M_{(W(R_i^{\flat}),I_{R_i})}$ is of type μ for some μ; see Example 2.5.11 and Remark 3.2.3. Since (8-5) is essentially surjective, we can conclude that (8-4) is also essentially surjective by using p-completely étale descent. □

Remark 8.2.14. Let R be a quasisyntomic ring over $W(k)$. For a bounded prism $(A,I)\in(R)_{\Delta}$, we defined the groupoid G-$\mathrm{BK}_{\mu}(A,I)$ of G-Breuil–Kisin modules of type μ over (A,I) in Section 5.1 and showed that it is equivalent to G-$\mathrm{Disp}_{\mu}(A,I)$ in Proposition 5.3.8. Thus the fully faithful functor (8-8) can be written as

$$G\text{-}F\text{-}\mathrm{Gauge}_{\mu}(R)\to G\text{-}\mathrm{BK}_{\mu}((R)_{\Delta}):=2-\varprojlim_{(A,I)\in(R)_{\Delta}}G\text{-}\mathrm{BK}_{\mu}(A,I).$$

The essential image of this functor consists of those $\mathcal{P}\in G\text{-}\mathrm{BK}_{\mu}((R)_{\Delta})$ such that for some quasisyntomic covering $R\to S$ with S a quasiregular semiperfectoid ring, the image $\mathcal{P}_{(\Delta_S,I_S)}\in G\text{-}\mathrm{BK}_{\mu}(\Delta_S,I_S)$ of $\mathcal{P}$ is a trivial G_{Δ_S}-torsor, and via some trivialization $\mathcal{P}_{(\Delta_S,I_S)}\simeq G_{\Delta_S}$, the isomorphism $F_{\mathcal{P}_{(\Delta_S,I_S)}}$ is given by $g\mapsto Yg$ for an element Y in

$$\mu(d)\phi(G(\Delta_S))\subset G(\Delta_S[1/I_S]),$$

where $d\in I_S$ is a generator. Therefore, we can simply define a prismatic G-F-gauge of type μ over R as an object $\mathcal{P}\in G\text{-}\mathrm{BK}_{\mu}((R)_{\Delta})$ that satisfies the above condition. However, similarly to prismatic G-μ-displays, it should be more technically convenient to work with the one introduced in Definition 8.2.5.

We shall give an example which shows that the functors (8-4) and (8-8) are not essentially surjective in general. This also shows that there exists a nonadmissible prismatic Dieudonné crystal (see Example 8.1.15).

Let $\mathcal{O}_C$ be the ring of integers of an algebraically closed nonarchimedean extension C of $\mathbb{Q}_p$. Then the quotient $S=\mathcal{O}_C/p$ is a quasiregular semiperfectoid ring. The natural homomorphism $S\to\Delta_S/I_S$ is injective, and the Frobenius $\phi:\Delta_S\to\Delta_S$ induces an isomorphism $\Delta_S/\mathrm{Fil}^1_{\mathcal{N}}(\Delta_S)\xrightarrow{\sim}S$ (see [Bhatt and Scholze 2022, Theorem 12.2]). The Hodge–Tate comparison theorem for the conjugate filtration with respect to the natural homomorphism $\mathcal{O}_C\to S$ shows that $S\to\Delta_S/I_S$ is not surjective; see [loc. cit., Section 12.1]. We fix a generator $d\in I_S$.

Example 8.2.15. Let the notation be as above. We assume that $G=\mathrm{GL}_2$ and $\mu:\mathbb{G}_m\to\mathrm{GL}_2$ is the 1-bounded cocharacter defined by $t\mapsto\mathrm{diag}\,(t,1)$. We choose an element $x\in\Delta_S$ whose image $\bar{x}\in\Delta_S/I_S$ is not contained in S. Let $X\in G(\Delta_S)_{I_S}$ be the element such that $X_d=\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right)\in G(\Delta_S)$. We shall show that X is not contained in the essential image of the functor (8-10). If X is contained in the essential image, then there are $Y\in G(\Delta_S)_{\mathcal{N}}$ and $g\in G_{\mu}(\Delta_S,I_S)$ such that the equality $X\cdot g=\phi(Y)$ holds in $G(\Delta_S)_{I_S}$. In particular, we see that $g^{-1}X_d$ belongs to the image of $\phi:G(\Delta_S)\to G(\Delta_S)$. We write

$$g=\begin{pmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{pmatrix}\in G_{\mu}(\Delta_S,I_S).$$

By Proposition 4.2.9, we have $g_{21} \in I_S$ and $g_{11}, g_{22} \in \Delta_S^\times$. By computing the image of $g^{-1}X_d$ in $G(\Delta_S/I_S)$ and using that $g^{-1}X_d \in \phi(G(\Delta_S))$, it follows that $\bar{x}/\bar{g}_{22}, 1/\bar{g}_{22} \in \Delta_S/I_S$ are contained in S. We thus have $\bar{x} \in S$, which leads to a contradiction.

Acknowledgements

The author would like to thank Kentaro Inoue, Hiroki Kato, Arthur-César Le Bras, Kimihiko Li, Samuel Marks, and Alex Youcis for helpful discussions and comments. The author is also grateful to Teruhisa Koshikawa for answering some questions on prisms. Finally, the author would like to thank the referees for helpful comments. The author especially would like to thank one of the referees for the suggestion that a discussion of the relation between prismatic G-μ-displays and prismatic F-gauges should be included, and one of the referees for explaining an alternative description of G-μ-displays, which simplifies some constructions given in the previous version of this paper. The work of the author was supported by JSPS KAKENHI grant numbers 22K20332 and 24K16887.

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Communicated by Bhargav Bhatt
Received 2023-06-11 Revised 2024-08-31 Accepted 2024-10-18

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ALGEBRA AND NUMBER THEORY 19:9 (2025)
https://doi.org/10.2140/ant.2025.19.1771

Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$

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We establish the arithmetic Siegel–Weil formula on the modular curve $\mathcal{X}_0(N)$ for arbitrary level N, i.e., we relate the arithmetic degrees of special cycles on $\mathcal{X}_0(N)$ to the derivatives of Fourier coefficients of a genus-2 Eisenstein series. We prove this formula by a precise identity between the local arithmetic intersection numbers on the Rapoport–Zink space associated to $\mathcal{X}_0(N)$ and the derivatives of local representation densities of quadratic forms. When N is odd and square-free, this gives a different proof of the main results in work of Sankaran, Shi and Yang. This local identity is proved by relating it to an identity in one dimension higher, but at hyperspecial level.

1. Introduction

1.1. ***Background.*** The classical Siegel–Weil formula relates certain Siegel Eisenstein series to the arithmetic of quadratic forms, namely, it expresses special values of these series as theta functions, generating series of representation numbers of quadratic forms. Kudla initiated an influential program to establish the arithmetic Siegel–Weil formula relating certain Siegel Eisenstein series to objects in arithmetic geometry.

In this article, we study the case of modular curves. Let N be a positive integer. The classical modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ over $\mathbb{C}$ is defined as the smooth 1-dimensional complex curve

$$\mathcal{Y}_0(N)_{\mathbb{C}} := \mathrm{GL}_2(\mathbb{Q}) \backslash \mathbb{H}_1^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/\Gamma_0(N)(\hat{\mathbb{Z}}) \simeq \Gamma_0(N) \backslash \mathbb{H}_1^{\pm},$$

MSC2020: 11F27, 11F30, 11F32, 11F41, 14G40.
Keywords: modular curves, arithmetic intersection, local densities, Eisenstein series.

where $\mathbb{H}_1^{\pm} = \mathbb{C} \setminus \mathbb{R}$ and $\mathbb{H}_1^{+} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$ is the upper half plane. The group $\Gamma_0(N)(\hat{\mathbb{Z}})$ is the open compact subgroup

$$\Gamma_0(N)(\hat{\mathbb{Z}}) = \left\{ x = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}) : a, b, c, d \in \hat{\mathbb{Z}} \right\}$$

of $\mathrm{GL}_2(\mathbb{A}_f)$, and $\Gamma_0(N) = \Gamma_0(N)(\hat{\mathbb{Z}}) \bigcap \mathrm{GL}_2(\mathbb{Z})$. Notice that the determinant of an element in the group $\Gamma_0(N)$ can be either 1 or -1 rather than only 1 in the classical setting because the space $\mathbb{H}_1^{\pm}$ has two connected components.

The smooth curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ is not proper, its compactification $\mathcal{X}_0(N)_{\mathbb{C}} := \mathcal{Y}_0(N)_{\mathbb{C}} \cup \{\text{cusps}\}$ is a smooth projective curve over $\mathbb{C}$. It is a classical fact that the curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ parameterizes cyclic isogenies between elliptic curves over $\mathbb{C}$. Here an isogeny $\pi : E \to E'$ between two elliptic curves over $\mathbb{C}$ is called cyclic if $\ker(\pi)$ is a cyclic group.

Katz and Mazur [1985] extended the concept of cyclic isogeny to an arbitrary base scheme: an isogeny $\pi : E \to E'$ between two elliptic curves is called cyclic if $\ker(\pi)$ is a cyclic group scheme (see Definition 4.1.2). They also defined the $\Gamma_0(N)$-level structures on elliptic curves. In this article, we mainly work on a 2-dimensional regular flat Deligne–Mumford stack $\mathcal{X}_0(N)$, defined in [Česnavičius 2017], which is the moduli stack of generalized elliptic curves with $\Gamma_0(N)$-level structures and whose fiber over $\mathbb{C}$ is $\mathcal{X}_0(N)_{\mathbb{C}}$. We define the (arithmetic) special cycles on $\mathcal{X}_0(N)$ and study their intersection numbers. Finally, we prove that these intersection numbers are identified with the derivatives of Fourier coefficients of certain Siegel Eisenstein series of genus 2.

When N is an odd, square-free positive integer, the relation has already been obtained in the work of Sankaran, Shi and Yang [Sankaran et al. 2023, Theorem 2.14] by computing both sides explicitly based on [Yang 1998; Kudla et al. 2006]. In this article, we use a different method and work with arbitrary level N. We introduce a formal scheme $\mathcal{N}_0(N)$ which is the Rapoport–Zink space associated to $\mathcal{X}_0(N)$. Via formal uniformization of the supersingular locus of the stack $\mathcal{X}_0(N)$ and its special cycles, we reduce the identity, which relates intersection numbers on $\mathcal{X}_0(N)$ and derivatives of Fourier coefficients of Eisenstein series, to a local identity between local arithmetic intersection numbers on $\mathcal{N}_0(N)$ and derivatives of local densities of quadratic forms. Now the key observation is that both sides of the local identity, regardless of the level N, can be related to another intersection problem on Rapoport–Zink space of 1 dimension higher, but in a hyperspecial level, while the computation of the latter has been done in [Gross and Keating 1993, Proposition 5.4; Wedhorn 2007, §2.16; Rapoport 2007, Theorem 1.1] (see also [Li and Zhang 2022, Theorem 1.2.1]).

1.2. *Summary of main results.*

1.2.A. *Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$.* Let $\Delta(N)$ be the rank-3 quadratic lattice

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \tag{1}$$

over $\mathbb{Z}$, equipped with the quadratic form $x \mapsto \det(x)$.

We use v to denote a place of $\mathbb{Q}$. For every finite place v, let $\Delta_v(N) = \Delta(N) \otimes_{\mathbb{Z}} \mathbb{Z}_v$ be a rank-3 quadratic lattice over $\mathbb{Z}_v$. Let $\mathbb{A}$ be the ring of adèles over $\mathbb{Q}$. Let $\mathbb{V} = \{\mathbb{V}_v\}$ be the incoherent collection of quadratic spaces of $\mathbb{A}$ of rank 3 nearby $\Delta(N)$ at ∞, i.e.,

$$\mathbb{V}_v = \Delta_v(N) \otimes \mathbb{Q}_v \quad \text{if } v < \infty\text{, and } \mathbb{V}_\infty \text{ is positive definite.} \tag{2}$$

Consider a finite place v. Let $\mathbb{V}_f := \mathbb{V} \otimes \mathbb{A}_f$ (resp. $\mathbb{V}_f^v := \mathbb{V} \otimes \mathbb{A}_f^v$) be the quadratic space of rank 3 over $\mathbb{A}_f$ (resp. $\mathbb{A}_f^v$). Let $\mathscr{S}(\mathbb{V}^2)$ (resp. $\mathscr{S}(\mathbb{V}_f^2)$, $\mathscr{S}((\mathbb{V}_f^v)^2)$) be the space of Schwartz functions on $\mathbb{V}^2$ (resp. $\mathbb{V}_f^2$, $(\mathbb{V}_f^v)^2$). Associated to $\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} \otimes \boldsymbol{\varphi}_\infty \in \mathscr{S}(\mathbb{V}^2)$, where $\boldsymbol{\varphi}_\infty$ is the Gaussian function on $\mathbb{V}_\infty^2$ and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$, there is a classical incoherent Eisenstein series $E(\mathsf{z}, s, \boldsymbol{\varphi})$ (see Section 3.4) on the Siegel upper half plane,

$$\mathbb{H}_2 = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} : \mathsf{x} \in \mathrm{Sym}_2(\mathbb{R}),\ \mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}\}.$$

This is essentially the Siegel Eisenstein series associated to a standard Siegel–Weil section of the degenerate principal series. The Eisenstein series here has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$. The central value $E(\mathsf{z}, s, \boldsymbol{\varphi})$ is 0 by the incoherence. We thus consider its central derivative

$$\partial\,\mathrm{Eis}(\mathsf{z}, \boldsymbol{\varphi}) := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\mathsf{z}, s, \boldsymbol{\varphi}).$$

Associated to the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$, it has a decomposition into the central derivatives of the Fourier coefficients

$$\partial\,\mathrm{Eis}(\mathsf{z}, \boldsymbol{\varphi}) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} \partial\,\mathrm{Eis}_T(\mathsf{z}, \boldsymbol{\varphi}).$$

On the geometric side, there is a regular integral model of the modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ over $\mathbb{Z}$ defined by Katz and Mazur: for any scheme S, the groupoid $\mathcal{Y}_0(N)(S)$ consists of objects $(E \xrightarrow{\pi} E')$, where E and E' are elliptic curves over S and π is a cyclic isogeny such that $\pi^\vee \circ \pi = N$. They proved that $\mathcal{Y}_0(N)$ is a 2-dimensional regular flat Deligne–Mumford stack (see [Katz and Mazur 1985, Theorem 5.1.1]), but $\mathcal{Y}_0(N)$ is not proper. There is a moduli stack $\mathcal{X}_0(N)$ defined in [Česnavičius 2017] which serves as a “compactification” of $\mathcal{Y}_0(N)$. It is a proper regular flat 2-dimensional Deligne–Mumford stack which contains $\mathcal{Y}_0(N)$ as an open substack, so we can consider the arithmetic intersection theory on $\mathcal{X}_0(N)$ following the lines in [Gillet 2009].

The key concept is that of a special cycle. A typical special cycle is of the form $\mathcal{Z}(T, \boldsymbol{\varphi})$, where T is a 2×2 symmetric matrix and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ is the characteristic function of some open compact subset of $\mathbb{V}_f^2$. It is a Deligne–Mumford stack finite unramified over $\mathcal{X}_0(N)$. For an object $(E \xrightarrow{\pi} E')$ of $\mathcal{Y}_0(N)(S)$, the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ parameterizes pairs of isogenies between E and E' with inner product matrix T and orthogonal to the cyclic isogeny π, along with some level structures given by the Schwartz function $\boldsymbol{\varphi}$ which is $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant (cf. Definition 4.3.5). For every nonsingular $T \in \mathrm{Sym}_2(\mathbb{Q})$ and prime number l, we say T is represented by $\Delta(N) \otimes \mathbb{Q}_l$ if there exist two vectors $x_1, x_2 \in \Delta(N) \otimes \mathbb{Q}_l$ such that $T = \frac{1}{2}((x_i, x_j))$. Define the difference set

$$\mathrm{Diff}(T, \Delta(N)) = \{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l\}.$$

Let $\widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$ be the codimension-2 arithmetic Chow group with complex coefficients of the regular flat Deligne–Mumford stack $\mathcal{X}_0(N)$. We also construct arithmetic special cycles on the stack $\mathcal{X}_0(N)$. They are elements of the form $\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}) \in \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$, where $T \in \mathrm{Sym}_2(\mathbb{Q})$ is nonsingular, $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})$ is positive definite and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ is a Schwartz function. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function, roughly meaning $\boldsymbol{\varphi}$ is invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}})$ and for every $p \in \mathrm{Diff}(T, \Delta(N))$, $\boldsymbol{\varphi} = \boldsymbol{\varphi}^p \otimes \boldsymbol{\varphi}_p$, where $\boldsymbol{\varphi}^p \in \mathscr{S}((\mathbb{V}_f^p)^2)$ and $\boldsymbol{\varphi}_p = c \cdot \mathbf{1}_{\Delta_p(N)^2} \in \mathscr{S}(\mathbb{V}_p^2)$ for some $c \in \mathbb{C}$. Our main goal is to relate $\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))$ to derivatives of the Fourier coefficients of a genus-2 Siegel Eisenstein series, where $\widehat{\deg} : \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)) \to \mathbb{C}$ is the arithmetic degree map (see (14)).

Theorem 1.2.1. *Let N be a positive integer. Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular symmetric matrix. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Then*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))q^T = \frac{\psi(N)}{24} \cdot \partial\, \mathrm{Eis}_T(\mathsf{z}, \boldsymbol{\varphi})$$

for any $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$. Here $\psi(N) = N \cdot \prod_{l|N}(1 + l^{-1})$ and $q^T = e^{2\pi i\, \mathrm{tr}(T\mathsf{z})}$.

1.2.B. *The local arithmetic Siegel–Weil formula with level N.* Fix a prime number p. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let W be the integer ring of the completion of the maximal unramified extension of $\mathbb{Q}_p$.

On the geometry side, let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ of dimension 1 and height 2. Let $\mathbb{B}$ be the unique division quaternion algebra over $\mathbb{Q}_p$, so $\mathrm{End}^0(\mathbb{X}) \simeq \mathbb{B}$ as quadratic spaces. The Rapoport–Zink space associated to $\mathcal{X}_0(N)$ is the deformation space $\mathcal{N}_0(N)$ such that, for a W-scheme S where p is locally nilpotent and an element $x_0 \in \mathbb{B}$ such that $x_0^\vee \circ x_0 = N$, the set $\mathcal{N}_0(N)(S)$ consists of elements $(X \xrightarrow{\pi} X')$ where X, X' are deformations over S of $\mathbb{X}$ with certain restrictions on polarizations (see Section 5.1), the morphism π is a cyclic isogeny deforming x_0 and $\pi^\vee \circ \pi = N$.

Let $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$ be the subspace of quasi-isogenies which are orthogonal to x_0. For any $x \in \mathbb{W}$, there is a closed formal subscheme $\mathcal{Z}(x)$ of $\mathcal{N}_0(N)$ over which the quasi-isogeny x lifts to an isogeny. This is an example of a special cycle (see Definition 5.2.5) on $\mathcal{N}_0(N)$. For a rank-2 lattice $M \subset \mathbb{W}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of M, then define the local arithmetic intersection number of M on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \chi\big(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}\big).$$

This number is independent of the choice of the basis $\{x_1, x_2\}$ of M.

On the analytic side, for any two integral quadratic $\mathbb{Z}_p$-lattices L and M, let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, a $\mathbb{Z}_p$-scheme such that for any $\mathbb{Z}_p$-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{QHom}(L \otimes_{\mathbb{Z}_p} R, M \otimes_{\mathbb{Z}_p} R),$$

where QHom denotes the set of quadratic module homomorphisms. The local density of integral representations is defined to be

$$\mathrm{Den}(M, L) = \lim_{d\to\infty} \frac{\#\,\mathrm{Rep}_{M,L}(\mathbb{Z}_p/p^d)}{p^{d\cdot\dim(\mathrm{Rep}_{M,L})_{\mathbb{Q}_p}}}.$$

Let $H_2^+ = \mathbb{Z}_p^2$ be the rank-2 quadratic $\mathbb{Z}_p$-lattice equipped with the quadratic form $q_{H_2^+}(x, y) = xy$. For any $k \geq 0$, let $H_{2k}^+ := (H_2^+)^{\oplus n}$ be a rank-$2k$ quadratic $\mathbb{Z}_p$-lattice. For any $\mathbb{Z}_p$-lattice $M \subset \mathbb{W}$ of rank 2, define the local density of M with level N to be the polynomial $\mathrm{Den}_{\Delta_p(N)}(X, M)$ such that for all $k \geq 0$

$$\mathrm{Den}_{\Delta_p(N)}(X, M)\big|_{X=p^{-k}} = \begin{cases} \dfrac{\mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M)}{\mathrm{Nor}^+(p^{-k}, 1)} & \text{when } p \mid N, \\[2ex] \dfrac{\mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M)}{\mathrm{Nor}^{(N,p)_p}(p^{-k}, 2)} & \text{when } p \nmid N, \end{cases} \tag{3}$$

where $(\cdot\,,\cdot)_p$ is the Hilbert symbol at p, the polynomials $\mathrm{Nor}^\varepsilon(X, n)$ are normalizing factors defined in Definition 2.2.6. Then $\mathrm{Den}_{\Delta_p(N)}(1, M) = 0$ since M can't be isometrically embedded into the quadratic lattice $\Delta_p(N)$. We define the derived local density of M with level N to be

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M) = -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}_{\Delta_p(N)}(X, M).$$

The local arithmetic Siegel–Weil formula with level N is an exact identity between the two integers just defined.

Theorem 1.2.2. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\,\mathrm{Den}_{\Delta_p(N)}(M).$$

We refer to $\mathrm{Int}_{\mathcal{N}_0(N)}(M)$ as the geometric side of the identity (related to the geometry of Rapoport–Zink spaces and Shimura varieties) and $\partial\,\mathrm{Den}_{\Delta_p(N)}(M)$ as the analytic side (related to the derivatives of Eisenstein series and L-functions).

1.2.C. *Formal uniformization.* For any prime p, let $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ be the supersingular locus of the stack $\mathcal{X}_0(N)$, i.e., those $\mathbb{F}$-points of $\mathcal{X}_0(N)$ which are isogenous to a supersingular elliptic curve. Let B be the unique quaternion algebra which ramifies exactly at p and ∞. Let $\hat{\mathcal{X}}_0(N)_{/(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$ be the completion of the stack $\mathcal{X}_0(N)$ along the closed substack $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Let $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ be the group $\prod_{v\neq\infty,p}\Gamma_0(N)(\mathbb{Z}_v)$. We have the following formal uniformization theorem of the stack $\mathcal{X}_0(N)$.

Proposition 1.2.3. *There is an isomorphism of formal stacks over W,*

$$\hat{\mathcal{X}}_0(N)_{/(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{X}_0(N)}} B^\times(\mathbb{Q})_0 \backslash \big[\mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p)/\,\Gamma_0(N)(\hat{\mathbb{Z}}^p)\big],$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

This proposition was previously known only in the case that N is odd and square-free (see [Kim 2018, Theorem 4.7] for the case $p \nmid N$ and [Oki 2020, Theorem 6.1] for $p \mid N$). As a corollary, let $\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})$

be the completion of $\mathcal{Z}(T,\boldsymbol{\varphi})$ along its supersingular locus $\mathcal{Z}^{\mathrm{ss}}(T,\boldsymbol{\varphi}) := \mathcal{Z}(T,\boldsymbol{\varphi}) \times_{\mathcal{X}_0(N)} \mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Let $\Delta(N)^{(p)}$ be the unique quadratic space over $\mathbb{Q}$ (up to isometry) such that

(1) it is positive definite at ∞;

(2) for finite primes $l \neq p$, $\Delta(N)^{(p)} \otimes \mathbb{Q}_l$ is isometric to $\Delta_l(N) \otimes \mathbb{Q}_l$;

(3) $\Delta(N)^{(p)} \otimes \mathbb{Q}_p$ is isometric to $\mathbb{W}$.

For a pair of vectors $\boldsymbol{x} = (x_1, x_2) \in (\Delta(N)^{(p)})^2$, let $T(\boldsymbol{x}) = \big(\frac{1}{2}(x_i, x_j)\big)$ be the inner product matrix. We have the following formal uniformization theorem of the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi})$.

Corollary 1.2.4. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular symmetric matrix, and $\mathrm{Diff}(T, \Delta(N)) = \{p\}$. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Let $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_{\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}}$-modules. Then we have the identity*

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T,\boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{x}\in B^\times(\mathbb{Q})_0\backslash(\Delta(N)^{(p)})^2\\ T(\boldsymbol{x})=T}} \sum_{g\in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0\backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \Theta^{-1}_{\mathcal{X}_0(N)}(\mathcal{Z}(\boldsymbol{x}), g)$$

in $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$, where $B_{\boldsymbol{x}}^\times \subset B^\times$ is the stabilizer of $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$.

1.3. *Strategy of the proof of main results.*

1.3.A. *Difference formula at the geometric side.* Let $\mathcal{N}$ be the deformation functor such that, for a W-scheme S where p is locally nilpotent, the set $\mathcal{N}(S)$ consists of elements (X, X'), where both X and X' are deformations over S of $\mathbb{X}$ with certain restrictions on polarizations (see Section 5.1). For a nonzero integral element $x \in \mathbb{B}$, i.e., $0 \leq \nu_p(x^\vee \circ x) < \infty$, there is a closed formal subscheme $\mathcal{Z}^\sharp(x)$ of $\mathcal{N}$ over which the quasi-isogeny x lifts to an isogeny. This is an example of a special cycle (see Definition 5.2.1) on $\mathcal{N}$.

For a rank-3 lattice $L \subset \mathbb{B}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2, x_3\}$ of L, then define the local arithmetic intersection number of L on $\mathcal{N}$ to be

$$\mathrm{Int}^\sharp(L) = \chi\big(\mathcal{N}, \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}\big).$$

This number is independent of the choice of the basis $\{x_1, x_2, x_3\}$ of the lattice L.

The special cycle $\mathcal{Z}^\sharp(x)$ is cut out by a single element $f_x \in \mathfrak{m} = (p, t_1, t_2) \subset W[\![t_1, t_2]\!]$, and when $\nu_p(x^\vee \circ x) \geq 2$, we have $f_{p^{-1}x} \mid f_x$. We define $d_x = f_x / f_{p^{-1}x} \in W[\![t_1, t_2]\!]$ when $\nu_p(x^\vee \circ x) \geq 2$, and $d_x = f_x$ when $\nu_p(x^\vee \circ x) = 0$ or 1. The divisor

$$\mathcal{D}(x) := \mathrm{Spf}\, W[\![t_1, t_2]\!]/d_x$$

is called the difference divisor associated to x (see Definition 6.2.1), which was originally introduced in [Terstiege 2011].

Fix $x_0 \in \mathbb{B}$ such that $x_0^\vee \circ x_0 = N$, recall that we have defined the deformation function $\mathcal{N}_0(N)$. In Theorem 6.2.3, we prove that $\mathcal{N}_0(N)$ is identified with the difference divisor $\mathcal{D}(x_0)$, i.e., there is an isomorphism of formal schemes

$$\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N).$$

Let $x_0^{\mathrm{univ}} : X^{\mathrm{univ}} \to X'^{\mathrm{univ}}$ be the universal isogeny deforming x_0 over the special cycle $\mathcal{Z}^\sharp(x_0)$. We will prove that the base change of x_0^{univ} to $\mathcal{D}(x_0)$ is cyclic, and therefore there is a natural morphism $\mathcal{D}(x_0) \to \mathcal{N}_0(N)$. The natural morphism is an isomorphism because both sides of the morphism are closed formal subschemes of $\mathcal{N}$ and are represented by 2-dimensional regular local rings. The identification of $\mathcal{D}(x_0)$ and $\mathcal{N}_0(N)$ implies the following difference formula of local arithmetic intersection numbers:

Theorem 1.3.1. *For any rank-2 lattice $M \subset \mathbb{W}$,*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

We refer to this formula as the difference formula at the geometric side.

1.3.B. *Difference formula at the analytic side.* For any rank-3 quadratic $\mathbb{Z}_p$-lattice $L \subset \mathbb{B}$, define the local density of L to be the polynomial $\mathrm{Den}(X, L) \in \mathbb{Z}[X]$ such that for all $k \geq 0$,

$$\mathrm{Den}(X, L)\big|_{X=p^{-k}} = \frac{\mathrm{Den}(H_{2k+4}^+, L)}{\mathrm{Nor}^+(p^{-k}, 3)}.$$

Then $\mathrm{Den}(1, L) = 0$ since L can't be isometrically embedded into the quadratic lattice H_4^+. We define the derived local density of L to be

$$\partial\,\mathrm{Den}(L) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(X, L).$$

Theorem 1.3.2. *For any rank-2 lattice $M \subset \mathbb{W}$, the identity*

$$\mathrm{Den}_{\Delta_p(N)}(X, M) = \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0)$$

holds. Since the lattice $M \oplus \mathbb{Z}_p \cdot x_0$ can't be isometrically embedded into the lattice H_4^+,

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M) = \partial\,\mathrm{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial\,\mathrm{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

The theorem is proved in a more general form in Theorem 7.2.6. We refer to this formula as the difference formula at the analytic side.

1.3.C. *Proof of Theorem 1.2.1.* The following local arithmetic Siegel–Weil formula is proved in [Wedhorn 2007, §2.16] (see also [Li and Zhang 2022, Theorem 1.2.1] when p is odd).

Theorem 1.3.3. *For any rank-3 lattice $L \subset \mathbb{B}$, we have the identity*

$$\mathrm{Int}^\sharp(L) = \partial\,\mathrm{Den}(L).$$

For a rank-2 lattice $M \subset \mathbb{W}$, let $L = M \oplus \mathbb{Z}_p \cdot x_0 \subset \mathbb{B}$. The local arithmetic Siegel–Weil formula with level N in Theorem 1.2.2 follows immediately from $\mathrm{Int}^\sharp(L) = \partial\,\mathrm{Den}(L)$ and two difference formulas we stated before (Theorems 1.3.1 and 1.3.2).

1.3.D. *Proof of Theorem 1.2.2.* Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible function. When T is not positive definite, the arithmetic special cycle $\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi})$ is essentially a $(1,1)$-current on the proper smooth complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$. The number $\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))$ has been computed explicitly in [Sankaran et al. 2023, Theorem 4.10].

We focus on the case that T is positive definite. In this case, $\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}) = [(\mathcal{Z}(T, \varphi), 0)]$, where $\mathcal{Z}(T, \boldsymbol{\varphi})$ is a cycle of codimension 2 on $\mathcal{X}_0(N)$. Moreover, $\mathcal{Z}(T, \boldsymbol{\varphi}) \neq \varnothing$ only if $\mathrm{Diff}(T, \Delta(N)) = \{p\}$ for some prime number p; in this case the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ is concentrated in the supersingular locus of $\mathcal{X}_0(N)$ in characteristic p. Suppose that the 2×2 matrix T has diagonal elements t_1 and t_2, and $\boldsymbol{\varphi} = \varphi_1 \times \varphi_2 \in \mathscr{S}(\mathbb{V}_f^2)$, where $\varphi_i \in \mathscr{S}(\mathbb{V}_f)$. We will show that

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi})) = \chi\big(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}\big) \cdot \log(p),$$

By the formal uniformization of the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ in Corollary 1.2.4, the Euler characteristic $\chi(\mathcal{Z}(T, \boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)})$ is a weighted linear combination of local arithmetic intersection numbers on $\mathcal{N}_0(N)$. Theorem 1.2.1 follows from the local arithmetic Siegel–Weil formula with level N at the place p and the classical local Siegel–Weil formula at other places.

1.4. ***Supplement.*** By the windows theory developed in [Zink 2001], if $\nu_p(N) \geq 1$, we prove that the special fiber $\mathcal{Z}(x_0)_p$ of $\mathcal{Z}(x_0)$ has the following explicit description (cf. Theorem 6.2.6, Corollary 6.2.7):

$$\mathcal{Z}(x_0)_p \simeq \mathrm{Spf}\,\mathbb{F}[\![t_1, t_2]\!] \Big/ \Bigg(\prod_{\substack{a+b=n \\ a,b\geq 0}} (t_1^{p^a} - t_2^{p^b}) \Bigg).$$

Based on the isomorphism $\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N)$, the special fiber $\mathcal{N}_0(N)_p$ of $\mathcal{N}_0(N)$ can be described by

$$\mathcal{N}_0(N)_p \simeq \mathrm{Spf}\,\mathbb{F}[\![t_1, t_2]\!] \Big/ \Bigg((t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{\substack{a+b=n \\ a,b\geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \Bigg).$$

Both these two isomorphisms are proved in [Katz and Mazur 1985, Theorems 13.4.6 and 13.4.7] by a totally different method.

2. Quadratic lattices and local densities

2.1. ***Notations on quadratic lattices.*** Let p be a prime number. Let F be a nonarchimedean local field of residue characteristic p, with ring of integers $\mathcal{O}_F$, residue field $\kappa = \mathbb{F}_q$ of size q, and uniformizer π. Let $\nu_\pi : F \to \mathbb{Z} \cup \{\infty\}$ be the valuation on F and $|\cdot| : F \to \mathbb{R}_{\geq 0}$ be the normalized absolute value on F. Let $\chi_F = \big(\frac{\cdot}{\pi}\big)_F : F^\times/(F^\times)^2 \to \{\pm 1, 0\}$ be the quadratic residue symbol.

A quadratic space (U, q_U) over F is a finite-dimensional vector space U over F equipped with a quadratic form $q_U : U \to F$ inducing a symmetric bilinear form given by

$$(\cdot\,,\cdot) : U \times U \to F, \quad (x, y) \mapsto q_U(x+y) - q_U(x) - q_U(y). \tag{4}$$

An isometry between two quadratic spaces (U, q_U) and $(U', q_{U'})$ is a linear isomorphism $\phi : U \to U'$ preserving quadratic forms, i.e., $q_{U'}(\phi(x)) = q_U(x)$ for any $x \in U$. In that case, we say U and U' are isometric.

A quadratic lattice (L, q_L) is a finite free $\mathcal{O}_F$-module equipped with a quadratic form $q_L : L \to F$. The quadratic form q_L also induces a symmetric bilinear form $L \times L \xrightarrow{(\cdot,\cdot)} F$ by a formula similar to (4). Let $L^\vee = \{x \in L \otimes_{\mathcal{O}_F} F : (x, L) \subset \mathcal{O}_F\}$. We say a quadratic lattice is integral if $q_L(x) \in \mathcal{O}_F$ for all $x \in L$, and is self-dual if it is integral and $L = L^\vee$.

Let's assume that $\dim_F U = n$ and the symmetric bilinear form $(\cdot\,,\cdot)$ is nondegenerate. Let $\{x_i\}_{i=1}^n$ be a basis of U, and $t_{ij} = \frac{1}{2}(x_i, x_j)$. We define the discriminant of the quadratic space U to be

$$\operatorname{disc}(U) = (-1)^{n(n-1)/2} \det((t_{ij})) \in F^\times/(F^\times)^2.$$

If $\{x_i\}_{i=1}^n$ is an orthogonal basis of U then $t_{ij} = 0$ if $i \neq j$ and $t_{ii} \neq 0$ by the nondegeneracy of $(\cdot\,,\cdot)$. The Hasse invariant of the quadratic space U is

$$\epsilon(U) = \prod_{i<j} (t_{ii}, t_{jj})_F,$$

For a quadratic lattice L, we use $\operatorname{disc}(L)$ and $\epsilon(L)$ to denote the corresponding invariants on the quadratic space $L_F = L \otimes_{\mathcal{O}_F} F$. Recall that when p is odd, quadratic spaces U over F are classified by the three invariants

$$\dim_F U, \quad \operatorname{disc}(U), \quad \epsilon(U),$$

i.e., two quadratic spaces U and U' are isometric if and only if the above three invariants for U and U' are the same.

For a quadratic space U, define $\chi_F(U) := \chi_F(\operatorname{disc}(U))$. For a quadratic lattice L, define $\chi(L) := \chi(L \otimes_{\mathcal{O}_F} F)$. When p is odd, the quadratic space U admits a self-dual sublattice if and only if $\epsilon(U) = +1$ and $\chi_F(U) \neq 0$. We use H_k^ε to denote the unique self-dual lattice of rank k and

$$\chi_F(H_k^\varepsilon) = \varepsilon.$$

When $p = 2$, let $H_{2n}^+ = (H_2^+)^{\oplus n}$ be a self-dual lattice of rank $2n$, where the quadratic form on $H_2^+ = \mathcal{O}_F^2$ is given by $(x, y) \in \mathcal{O}_F^2 \mapsto xy$.

Example 2.1.1. Let $N \in \mathcal{O}_F$. Let $\Delta_F(N)$ be the rank-3 quadratic lattice

$$\Delta_F(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathcal{O}_F \right\}$$

over $\mathcal{O}_F$, equipped with the quadratic form induced by $x \mapsto \det(x)$. Under the basis

$$e_1 = \begin{pmatrix} -N & \\ & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad e_3 = \begin{pmatrix} & \\ 1 & \end{pmatrix}$$

of $\Delta_F(N)$, the quadratic form can be represented by the symmetric matrix

$$T = \begin{pmatrix} -N & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Therefore, $\mathrm{disc}(\Delta_F(N)) = -\frac{1}{4}N \equiv -N$, $\epsilon(\Delta_F(N)) = (-N, -1)_F$. Moreover,

$$\Delta_F(N)^\vee = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a \in \frac{1}{2N}\mathcal{O}_F, b, c \in \mathcal{O}_F \right\}.$$

Therefore, $\Delta_F(N)^\vee / \Delta_F(N) \simeq \mathcal{O}_F/2N$.

Throughout this article, we mainly focus on the case that $F = \mathbb{Q}_p$. In this case, we simply use $\Delta_p(N)$ to denote the lattice $\Delta_{\mathbb{Q}_p}(N)$ (as we did in the introduction).

2.2. *Local densities of quadratic lattices.*

Definition 2.2.1. Let L, M be two quadratic $\mathcal{O}_F$-lattices. Let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, an $\mathcal{O}_F$-scheme such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{QHom}(L \otimes_{\mathcal{O}_F} R, M \otimes_{\mathcal{O}_F} R),$$

where QHom denotes the set of injective module homomorphisms which preserve the quadratic forms. The local density of integral representations is defined to be

$$\mathrm{Den}(M, L) = \lim_{d \to \infty} \frac{\# \mathrm{Rep}_{M,L}(\mathcal{O}_F/\pi^d)}{q^{d \cdot \dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.2.2. If L, M have rank n, m, respectively, and the generic fiber $(\mathrm{Rep}_{M,L})_F \neq \varnothing$, then $n \leq m$ and

$$\dim(\mathrm{Rep}_{M,L})_F = \dim \mathrm{O}_m - \dim \mathrm{O}_{m-n} = \binom{m}{2} - \binom{m-n}{2} = mn - \frac{n(n+1)}{2}.$$

Definition 2.2.3. Let L, M be two quadratic $\mathcal{O}_F$-lattices. Let $\mathrm{PRep}_{M,L}$ be the $\mathcal{O}_F$-scheme of primitive integral representations such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{PRep}_{M,L}(R) = \{\phi \in \mathrm{Rep}_{M,L}(R) : \phi \text{ is an isomorphism between } L_R \text{ and a direct summand of } M_R\},$$

where L_R (resp. M_R) is $L \otimes_{\mathcal{O}_F} R$ (resp. $M \otimes_{\mathcal{O}_F} R$). The primitive local density is defined to be

$$\mathrm{Pden}(M, L) = \lim_{d \to \infty} \frac{\# \mathrm{PRep}_{M,L}(\mathcal{O}_F/\pi^d)}{q^{d \cdot \dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.2.4. For any positive integer d, a homomorphism $\phi \in \mathrm{Rep}_{M,L}(\mathcal{O}_F/\pi^d)$ or $\mathrm{Rep}_{M,L}(\mathcal{O}_F)$ is primitive if and only if $\overline{\phi} := \phi \bmod \pi \in \mathrm{PRep}(\mathcal{O}_F/\pi)$, which is equivalent to

$$\dim_{\mathbb{F}_q}(\phi(L)+\pi\cdot M)/\pi\cdot M = \mathrm{rank}_{\mathcal{O}_F}(L).$$

Lemma 2.2.5. *Let H be a self-dual quadratic lattice. Let L be a quadratic $\mathcal{O}_F$-lattice and k any positive integer. Then we have the stratification*

$$\mathrm{Rep}_{H,L}(\mathcal{O}_F) = \bigsqcup_{L\subset L'\subset L'^\vee} \mathrm{PRep}_{H,L'}(\mathcal{O}_F).$$

Proof. This is proved in [Cho and Yamauchi 2020, (3.1)]. □

Definition 2.2.6. Let $n \geq 0$. For $\varepsilon \in \{\pm 1\}$, we define the normalizing factors to be

$$\mathrm{Nor}^\varepsilon(X,n) = \left(1 - \frac{1+(-1)^{n+1}}{2}\cdot \varepsilon q^{-(n+1)/2}X\right) \prod_{1\leq i<(n+1)/2} (1-q^{-2i}X^2).$$

It is well known (see [Li and Zhang 2022, §3.4]) that for a quadratic lattice L of rank n, there exists a polynomial $\mathrm{Den}(X, L) \in \mathbb{Q}[X]$ such that

$$\mathrm{Den}(X,L)|_{X=q^{-m}} = \frac{\mathrm{Den}(H^+_{n+1+2k}, L)}{\mathrm{Nor}^+(X,n)}$$

for all integers $m \geq 0$. If the lattice L can't be isometrically embedded into the lattice H^+_{n+1}, define the derived local density of L to be

$$\partial\,\mathrm{Den}(L) = -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(X,L).$$

3. Incoherent Eisenstein series and the main theorem

3.1. *Incoherent Eisenstein series.* Let W be the standard symplectic space over $\mathbb{Q}$ of dimension 4. Let $P = MN \subset \mathrm{Sp}(\mathrm{W})$ be the standard Siegel parabolic subgroup, which takes the following form under the standard basis of W:

$$M(\mathbb{Q}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathrm{GL}_2(\mathbb{Q}) \right\},$$
$$N(\mathbb{Q}) = \left\{ n(b) = \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} : b \in \mathrm{Sym}_2(\mathbb{Q}) \right\}.$$

Let $\mathbb{A}$ be the adèle ring over $\mathbb{Q}$. Let $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$ be the metaplectic extension

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}(\mathrm{W}_{\mathbb{A}}) \to \mathrm{Sp}(\mathrm{W})(\mathbb{A}) \to 1$$

of $\mathrm{Sp}(\mathrm{W})(\mathbb{A})$, where $\mathbb{C}^1 = \{z \in \mathbb{C}^\times : |z| = 1\}$. There is an isomorphism $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}}) \xrightarrow{\sim} \mathrm{Sp}(\mathrm{W})(\mathbb{A}) \times \mathbb{C}^1$ with the multiplication on the latter given by the global Rao cycle. Therefore, we can write an element of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$ as (g, t), where $g \in \mathrm{Sp}(\mathrm{W})(\mathbb{A})$ and $t \in \mathbb{C}^1$.

Let $P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})$ be the standard Siegel parabolic subgroup of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$, where

$$M(\mathbb{A}) = \{(m(a), t) : a \in \mathrm{GL}_2(\mathbb{A}), t \in \mathbb{C}^1\},$$
$$N(\mathbb{A}) = \{n(b) : b \in \mathrm{Sym}_2(\mathbb{A})\}.$$

Recall the incoherent collection of rank-3 quadratic spaces $\mathbb{V} = \{\mathbb{V}_v\}$ over $\mathbb{A}$ we defined in (2),

$$\mathbb{V}_v = \Delta_v(N) \otimes \mathbb{Q}_v \quad \text{if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.}$$

Then we can verify immediately that $\prod_v \epsilon(\mathbb{V}_v) = -1$.

Let $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be the quadratic character given by $\chi(x) = \prod_{v \le \infty} \chi_v(x_v) = \prod_{v \le \infty}(x_v, -N)_v$ for all $x = (x_v) \in \mathbb{A}^\times$. Fix the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ such that $\psi_\infty(x) = e^{2\pi i x}$. We may view χ as a character on $M(\mathbb{A})$ by

$$\chi(m(a), t) = \chi(\det(a)) \cdot \gamma(\det(a), \psi)^{-1} \cdot t,$$

and extend it to $P(\mathbb{A})$ trivially on $N(\mathbb{A})$. Here $\gamma(\det(a), \psi)$ is the Weil index (see [Kudla 1997, p. 548]). We define the degenerate principal series to be the unnormalized smooth induction

$$I(s, \chi) := \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})}(\chi \cdot |\cdot|_{\mathbb{Q}}^{s+3/2}), \quad s \in \mathbb{C}.$$

For a standard section $\Phi(-, s) \in I(s, \chi)$ (i.e., its restriction to the standard maximal compact subgroup of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}})$ is independent of s), we define the associated Siegel Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{Sp}(\mathbb{Q})} \Phi(\gamma g, s),$$

which converges for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$.

Recall that $\mathscr{S}(\mathbb{V}^2)$ is the space of Schwartz functions on $\mathbb{V}^2$. The fixed choice of χ and ψ gives a Weil representation $\omega = \omega_{\chi,\psi}$ of $\mathrm{Mp}(\mathrm{W}_{\mathbb{A}}) \times \mathrm{O}(\mathbb{V})$ on $\mathscr{S}(\mathbb{V}^2)$. For $\tilde{\boldsymbol{\varphi}} \in \mathscr{S}(\mathbb{V}^2)$, define a function

$$\Phi_{\tilde{\boldsymbol{\varphi}}}(g) := \omega(g)\tilde{\boldsymbol{\varphi}}(0), \quad g \in \mathrm{Mp}(\mathrm{W}_{\mathbb{A}}).$$

Then $\Phi_{\tilde{\boldsymbol{\varphi}}}(g) \in I(0, \chi)$. Let $\Phi_{\tilde{\boldsymbol{\varphi}}}(-, s) \in I(s, \chi)$ be the associated standard section, known as the standard Siegel–Weil section associated to $\tilde{\boldsymbol{\varphi}}$. For $\tilde{\boldsymbol{\varphi}} \in \mathscr{S}(\mathbb{V}^2)$, we write $E(g, s, \tilde{\boldsymbol{\varphi}}) := E(g, s, \Phi_{\tilde{\boldsymbol{\varphi}}})$.

3.2. ***Fourier coefficients and derivatives.*** We have a Fourier expansion of the Siegel Eisenstein series defined above:

$$E(g, s, \Phi) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} E_T(g, s, \Phi),$$

where

$$E_T(g, s, \Phi) = \int_{\mathrm{Sym}_2(\mathbb{Q})\backslash \mathrm{Sym}_2(\mathbb{A})} E(n(b)g, s, \Phi)\psi(-\mathrm{tr}(Tb))\,\mathrm{d}n(b).$$

The Haar measure $\mathrm{d}n(b)$ is normalized to be self-dual with respect to ψ. When T is nonsingular, for factorizable $\Phi = \otimes_v \Phi_v$, we have a factorization of the Fourier coefficient into a product

$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v),$$

where the product ranges over all places v of $\mathbb{Q}$ and the local Whittaker function $W_{T,v}(g_v, s, \Phi_v)$ is defined by

$$W_{T,v}(g_v, s, \Phi_v) = \int_{\mathrm{Sym}_2(\mathbb{Q}_v)} \Phi_v(w^{-1}n(b)g_v, s) \cdot \psi_v(-\mathrm{tr}(Tb))\,\mathrm{d}n(b), \quad w = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}. \tag{5}$$

It has analytic continuation to $s \in \mathbb{C}$. Thus we have a decomposition of the derivative of a nonsingular Fourier coefficient,

$$E'_T(g, s, \Phi) = \sum_v E'_{T,v}(g, s, \Phi),$$

where

$$E'_{T,v}(g, s, \Phi) = W'_{T,v}(g_v, s, \Phi_v) \cdot \prod_{v' \neq v} W_{T,v'}(g_{v'}, s, \Phi_{v'}). \tag{6}$$

3.3. ***Whittaker functions and local densities.*** Let v be a finite place of $\mathbb{Q}$. Define the local degenerate principal series to be the unnormalized smooth induction

$$I_v(s, \chi_v) := \mathrm{Ind}_{P(\mathbb{Q}_v)}^{\mathrm{Mp}(\mathrm{W}_v)}(\chi_v \cdot |\cdot|_v^{s+3/2}), \quad s \in \mathbb{C}.$$

The fixed choice of χ_v and ψ_v gives a local Weil representation $\omega_v = \omega_{\chi_v,\psi_v}$ of $\mathrm{Mp}(\mathrm{W}_v) \times \mathrm{O}(\mathbb{V}_v)$ on the Schwartz function space $\mathscr{S}(\mathbb{V}_v^2)$. We define the local Whittaker function associated to $\boldsymbol{\varphi}_v$ and $T \in \mathrm{Sym}_2(\mathbb{Q}_v)$ to be

$$W_{T,v}(g_v, s, \boldsymbol{\varphi}_v) := W_{T,v}(g_v, s, \Phi_{\boldsymbol{\varphi}_v}),$$

where $\Phi_{\boldsymbol{\varphi}_v}(g_v) := \omega_v(g_v)\boldsymbol{\varphi}_v(0) \in I_v(0, \chi_v)$ and $\Phi_{\boldsymbol{\varphi}_v}(-, s)$ is the associated standard section.

The relationship between Whittaker functions and local densities is encoded in the following proposition.

Proposition 3.3.1. *Suppose $v \neq \infty$. Let M be an integral $\mathbb{Z}_v$-quadratic lattice of rank 3 contained in $\mathbb{V}_v$. Let L be an integral quadratic $\mathbb{Z}_v$-lattice of rank 2. Suppose that the quadratic form of L is represented by a matrix $T \in \mathrm{Sym}_2(\mathbb{Q}_v)$ after a choice of $\mathbb{Z}_v$-basis of L. We have the identity*

$$W_{T,v}(1, k, 1_{M^2}) = |M^\vee/M|_v \cdot \gamma(\mathbb{V}_v)^2 \cdot |2|_v^{1/2} \cdot \mathrm{Den}(M \oplus H_{2k}^+, L), \tag{7}$$

where the constant $\gamma(\mathbb{V}_v) = \gamma(\det(\mathbb{V}_v), \psi_v)^{-1} \cdot \epsilon(\mathbb{V}_v) \cdot \gamma(\psi_v)^{-3}$, $\gamma(\det(\mathbb{V}_v), \psi_v)$ and $\gamma(\psi_v)$ are Weil indexes [Ranga Rao 1993, Appendix].

Proof. This is proved in [Kudla et al. 2006, Lemma 5.7.1]. ☐

3.4. ***Classical incoherent Eisenstein series.*** The hermitian symmetric domain for Sp(W) is the Siegel upper half space

$$\mathbb{H}_2 = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} \mid \mathsf{x} \in \mathrm{Sym}_2(\mathbb{R}), \mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}\}.$$

Let $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$ with $\mathsf{x}, \mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})$ and $\mathsf{y} = {}^t a \cdot a$ positive definite. Define the classical incoherent Eisenstein series to be

$$E(\mathsf{z}, s, \tilde{\varphi}) = \chi_\infty(m(a))^{-1}|\det(m(a))|^{-3/2} \cdot E(g_{\mathsf{z}}, s, \tilde{\varphi}), \quad g_{\mathsf{z}} = n(x)m(a) \in \mathrm{Mp}(\mathrm{W}_{\mathbb{A}}).$$

Notice that $E(\mathsf{z}, s, \tilde{\varphi})$ doesn't depend on the choice of χ. We write the central derivatives as

$$\partial\,\mathrm{Eis}(\mathsf{z}, \tilde{\varphi}) := E'(\mathsf{z}, 0, \tilde{\varphi}), \quad \partial\,\mathrm{Eis}_T(\mathsf{z}, \tilde{\varphi}) := E'_T(\mathsf{z}, 0, \tilde{\varphi}). \tag{8}$$

Then we have a Fourier expansion

$$\partial\,\mathrm{Eis}(\mathsf{z}, \tilde{\varphi}) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} \partial\,\mathrm{Eis}_T(\mathsf{z}, \tilde{\varphi}).$$

For the open compact subgroup $\Gamma_0(N)(\hat{\mathbb{Z}}) \subset \mathrm{GL}_2(\mathbb{A}_f)$, we choose

$$\tilde{\varphi} = \varphi \otimes \varphi_\infty \in \mathscr{S}(\mathbb{V}^2)$$

such that $\varphi \in \mathscr{S}(\mathbb{V}_f^2)$ is $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant and φ_∞ is the Gaussian function

$$\varphi_\infty(x) = e^{-\pi\,\mathrm{tr}\,T(x)}.$$

For our fixed choice of Gaussian φ_∞, we write

$$E(\mathsf{z}, s, \varphi) = E(\mathsf{z}, s, \varphi \otimes \varphi_\infty), \quad \partial\,\mathrm{Eis}(\mathsf{z}, \varphi) = \partial\,\mathrm{Eis}(\mathsf{z}, \varphi \otimes \varphi_\infty), \quad \partial\,\mathrm{Eis}_T(\mathsf{z}, \varphi) = \partial\,\mathrm{Eis}_T(\mathsf{z}, \varphi \otimes \varphi_\infty), \tag{9}$$

and so on for short.

4. The modular curve $\mathcal{X}_0(N)$ and special cycles

4.1. ***Cyclic group schemes.*** Let S be a scheme. Let G/S be a finite locally free group scheme over S. On every connected component of S, the rank of G is a constant, if the rank is a same number N for every connected component, we say that G has order N.

Let $\mathcal{O}_S$ be the structure sheaf of the scheme S. Let G/S be a finite locally free group scheme of order N. Then the structure sheaf $\mathcal{O}_G$ of G is finite locally free of rank N as an $\mathcal{O}_S$-module. Any element $f \in \mathcal{O}_G$ acts on itself by left multiplication. This defines an $\mathcal{O}_S$-linear endomorphism of $\mathcal{O}_G$, and the characteristic polynomial of this endomorphism

$$\det(T - f) = T^N - \mathrm{tr}(f)T^{N-1} + \cdots + (-1)^N \mathrm{N}(f)$$

is a monic polynomial in $\mathcal{O}_S[T]$ of degree N.

Definition 4.1.1. We say that a set of N not necessarily distinct points $\{P_i\}_{i=1}^N$ in $G(S)$ is a full set of sections of G/S if the following condition is fulfilled: for any element $f \in \mathcal{O}_G$, the equality

$$\det(T-f) = \prod_{i=1}^{N}(T - f(P_i))$$

of polynomials with coefficients in $\mathcal{O}_S$ holds.

Definition 4.1.2. We say a finite locally free group scheme G/S of rank N is cyclic over S if there exists a section $P \in G(S)$ such that $\{aP\}_{a=1}^N$ forms a full set of sections of G/S. We call P a generator of G over S. We say G/S is *cyclic* if G_T is cyclic over T after some fppf covering by some scheme $T \to S$.

Remark 4.1.3. The cyclicity of a group scheme is preserved under base change by the definition, i.e., if G/S is cyclic, then for any morphism $S' \to S$, the base change group scheme $G \times_S S'/S'$ is also cyclic.

Proposition 4.1.4. *Let S be a scheme, E/S an elliptic curve over S, and $G \subset E[N]$ a finite locally free group scheme of order N over S. Then there exists a closed subscheme $S^{\mathrm{cyc}} \subset S$ which is universal for the condition "G is cyclic", in the sense that for any morphism $T \to S$, the base change G_T/T is cyclic if and only if the morphism $T \to S$ factors through the closed subscheme S^{cyc}.*

Proof. This is proved in [Katz and Mazur 1985, Theorem 6.4.1]. □

Lemma 4.1.5. *Let W be a discrete valuation ring with residue characteristic p and uniformizer π. Let S be a reduced, noetherian, quasiseparated and flat scheme over W. Let G be a finite locally free group scheme of order p^n over S which is also embedded into an elliptic curve E/S. If, for every generic point ξ of S, G_ξ doesn't factor through the multiplication-by-p morphism of E_ξ, then G is a cyclic group scheme.*

Proof. Since S is quasiseparated, quasicompact and flat over W, then $S[\pi^{-1}]$ is dense in S since the scheme-theoretic image commutes with flat base change, therefore every generic point ξ lies in the open dense subscheme $S[\pi^{-1}]$. Let $\kappa(\xi)$ be the residue field of ξ; it has characteristic 0.

The group scheme G_ξ is of order p^n over the characteristic 0 field $\kappa(\xi)$. Hence $G_\xi \simeq \prod_{i=1}^k \mathbb{Z}/p^{a_i}\mathbb{Z}$, where $\sum_{i=1}^k a_i = n$. The fact that G_ξ doesn't factor through the multiplication-by-p morphism of E_ξ is equivalent to saying that $E[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2 \not\subset G$. Hence the only possibility is $k=1$ and $G_\xi \simeq \mathbb{Z}/p^n\mathbb{Z}$.

Let S^{cyc} be the closed subscheme described by Proposition 4.1.4. We know that every generic point is contained in the closed subscheme S^{cyc}, and hence $S^{\mathrm{cyc}} = S$ since S is reduced. □

Corollary 4.1.6. *Let W be a discrete valuation ring with residue characteristic p and uniformizer π. Let S be an integral noetherian scheme, quasiseparated and flat over W. Let G be a finite locally free group scheme of order p^n over S which is also embedded into an elliptic curve E/S. If the isogeny $\pi_G : E \to E/G$ doesn't factor through the multiplication-by-p morphism of E, then G is a cyclic group scheme.*

Proof. The isogeny $\pi_G : E \to E/G$ factors through the multiplication-by-p morphism of E if and only if $\ker([p]_E)$ is contained (as a Cartier divisor on E) in G. This is a closed condition on the base scheme S by [Katz and Mazur 1985, Lemma 1.3.4]. We use $\mathcal{I} \neq 0$ (since the morphism π_G doesn't factor through the multiplication-by-p morphism of E) to denote the ideal sheaf of this closed subscheme of S; it is functorial with respect to the base change of S.

Let ξ be the only generic point of S, then G_ξ doesn't factor through the multiplication-by-p morphism because otherwise $\mathcal{I}_\xi = 0$, but the injection $\mathcal{I} \to \mathcal{I}_\xi$ will imply that $\mathcal{I} = 0$, which is a contradiction. Then the corollary follows from Lemma 4.1.5. ☐

4.2. $\Gamma_0(N)$*-structures on elliptic curves.* Let S be a scheme. We say a scheme C over S is a smooth curve over S if the structure morphism $C \to S$ is a smooth proper morphism of relative dimension 1.

Definition 4.2.1. A closed immersion $i : D \to C$ is called an effective Cartier divisor if the following conditions hold:

(i) The closed subscheme D is flat over S.

(ii) The ideal sheaf $\mathcal{I}(D)$ defining D is an invertible $\mathcal{O}_C$-module.

Lemma 4.2.2. *If C/S is a smooth curve, then any section $s \in C(S)$ defines an effective Cartier divisor on C, denoted by $[s]$.*

Proof. This is proved in [Katz and Mazur 1985, Lemma 1.2.2]. ☐

Given two effective Cartier divisors D and D' on C/S, we can define their sum $D + D'$. It is an effective Cartier divisor on C/S defined locally by the product of the defining equations of D and D'. Explicitly, if $S = \operatorname{Spec} R$ and if over an affine open subscheme $\operatorname{Spec} A$ of C, the Cartier divisor D (resp. D') is defined by an element $f \in A$ (resp. $g \in A$), then the Cartier divisor $D + D'$ is defined by the equation fg.

Lemma 4.2.3. *Suppose E/S and E'/S are two elliptic curves over S and $\pi : E \to E'$ is an isogeny, i.e., π is surjective and $\ker(\pi)$ is a finite flat group scheme locally of finite presentation over S. Then $\ker(\pi) \to E$ is an effective Cartier divisor.*

Proof. By the cancellation theorem of morphisms of locally finite presentation, any morphism between abelian schemes are locally of finite presentation. Hence π is locally of finite presentation, and therefore $\ker(\pi)$ is also locally of finite presentation over S. Then the lemma follows from [Katz and Mazur 1985, Lemma 1.2.3]. ☐

Definition 4.2.4. We say an isogeny $\pi : E \to E'$ between two elliptic curves E and E' is a cyclic N-isogeny if $\pi^\vee \circ \pi = N$, and there exists an fppf covering of S by a scheme $T \to S$ with a point $P \in \ker(\pi)(T)$ such that the equality

$$\ker(\pi)_T = \sum_{a=1}^{N} [aP]$$

of Cartier divisors on E_T holds. A $\Gamma_0(N)$-structure on an elliptic curve E/S is a cyclic N-isogeny $E \xrightarrow{\pi} E'$.

Lemma 4.2.5. *Let $\pi : E \to E'$ be an isogeny between two elliptic curves E and E', the isogeny π is an N-cyclic isogeny if and only if $\ker(\pi)$ is a cyclic group scheme of order N.*

Proof. By [Katz and Mazur 1985, Theorem 1.10.1], the set $\{aP\}_{a=1}^{N}$ (where $P \in \ker(\pi)(S)$) forms a full set of sections of $\ker(\pi)$ if and only if we have the equality

$$\ker(\pi) = \sum_{a=1}^{N} [aP]$$

of effective Cartier divisors in E/S, which is exactly the definition of the cyclicity of a N-isogeny. □

Example 4.2.6. (a) Suppose $\tau = x + iy \in \mathbb{H}_1^+$. We consider the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and a finite subgroup K generated by $1/N$ inside E_τ. Then $\pi : E_\tau \to E_\tau/K$ is a cyclic isogeny.

(b) Suppose E/S is an elliptic curve over an $\mathbb{F}_p$-scheme S. Then for any $n \geq 1$, the n-th iterated relative Frobenius

$$F^n : E \to E^{(p^n)}$$

is a cyclic p^n-isogeny. The origin $P = 0$ is a generator of $\ker(F^n)$ because $\ker(F^n) \simeq \mathcal{O}_S[T]/(T^{p^n})$ Zariski locally (cf. [Katz and Mazur 1985, Lemma 12.2.1]).

Let $\mathcal{E}ll$ be the stack of elliptic curves, i.e., for an arbitrary scheme S, $\mathcal{E}ll(S)$ is a groupoid whose objects are elliptic curves $p : E \to S$ and morphisms are isomorphisms of elliptic curves over S. We use $\mathcal{Y}_0(N)$ to denote the stack which consists of all the $\Gamma_0(N)$-structures on elliptic curves, i.e., for a scheme S, $\mathcal{Y}_0(N)(S)$ is a groupoid whose objects are cyclic N-isogenies $(E \xrightarrow{\pi} E')$ where E and E' are elliptic curves over S, and a morphism between two cyclic isogenies $(E_1 \xrightarrow{\pi_1} E_1')$ and $(E_2 \xrightarrow{\pi_2} E_2')$ is a pair of isomorphisms of elliptic curves $a : E_1 \xrightarrow{\sim} E_2$ and $a' : E_1' \xrightarrow{\sim} E_2'$ such that $a' \circ \pi_1 = \pi_2 \circ a$. We have functors

$$s : \mathcal{Y}_0(N) \to \mathcal{E}ll, \quad (E/S \xrightarrow{\pi} E'/S) \mapsto E/S.$$

Lemma 4.2.7. *Both $\mathcal{Y}_0(N)$ and $\mathcal{E}ll$ are 2-dimensional Deligne–Mumford stacks. The above functor $s : \mathcal{Y}_0(N) \to \mathcal{E}ll$ is finite flat of degree $\psi(N) = N \cdot \prod_{l \mid N}(1 + l^{-1})$, and representable by schemes. Also, s is étale over* Spec $\mathbb{Z}[1/N]$.

Proof. This is proved in [Katz and Mazur 1985, Theorem 5.1.1]. The key input is that a finite order group scheme is automatically étale if the order is invertible in the base scheme. □

For a $\mathbb{Z}_{(p)}$-scheme S, a geometric point $\bar{s}$ of S and an elliptic curve E over S, let $E_{\bar{s}}$ be the base change of E to $\bar{s}$. Let $T^p(E_{\bar{s}})$ (resp. $V^p(E_{\bar{s}})$) be the integral (resp. rational) Tate module of the elliptic curve $E_{\bar{s}}$. A $\mathbb{Z}_{(p)}^\times$-isogeny $f : E \to E'$ over S is a quasi-isogeny and there exists a prime-to-p number M such that $M \circ f$ is an isogeny. Let $V^p(f)$ be the homomorphism on rational Tate modules induced by f.

Lemma 4.2.8. *Let $\mathcal{E}ll_{(p)}$ be the localization of the stack $\mathcal{E}ll$ to* Spec $\mathbb{Z}_{(p)}$. *Then $\mathcal{E}ll_{(p)}$ can be described by the following stack: for every $\mathbb{Z}_{(p)}$-scheme S, $\mathcal{E}ll_{(p)}(S)$ is a groupoid whose objects are pairs $(E/S, \overline{\eta^p})$, where $\overline{\eta^p}$ is a $\pi_1(S, \bar{s})$-invariant* $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$*-equivalence class of an isomorphism*

$$\eta^p : V^p(E_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2.$$

A morphism between two objects $(E/S, \overline{\eta^p})$ *and* $(E'/S, \overline{\eta'^p})$ *is a* $\mathbb{Z}_{(p)}^{\times}$*-isogeny* $f : E \to E'$ *over* S *such that* $\overline{\eta^p} = \overline{V^p(f) \circ \eta'^p}$*.*

Proof. We temporarily use $\mathcal{E}ll'$ to denote the stack described in the lemma. It suffices to show that for a connected scheme S over $\operatorname{Spec}\mathbb{Z}_{(p)}$, there is a category equivalence between $\mathcal{E}ll(S)$ and $\mathcal{E}ll'(S)$. We first construct a functor F from $\mathcal{E}ll(S)$ to $\mathcal{E}ll'(S)$. Given an elliptic curve E over S and a geometric point $\bar{s}$ of S, we choose an isomorphism

$$\eta^p : T^p(E_{\bar{s}}) \simeq (\hat{\mathbb{Z}}^p)^2.$$

Then clearly the $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$-orbit of $\overline{\eta^p}$ is $\pi_1(S, \bar{s})$-invariant (because $\pi_1(S, \bar{s})$ acts linearly on $T^p(E_{\bar{s}})$). We define $F(E) = (E, \overline{\eta^p})$; this functor is independent of the choice of η^p.

Now we prove that this functor is essentially surjective and fully faithful. For essential surjectivity, we pick an arbitrary object $(E/S, \overline{\eta^p})$ of $\mathcal{E}ll'(S)$. By [Lan 2013, Corollary 1.3.5.4], there is a $\mathbb{Z}_{(p)}^{\times}$-isogeny $f : E' \to E$ such that $\eta'^p = \eta^p \circ V^p(f) : V^p(E'_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$ maps $T^p(E'_{\bar{s}})$ to $(\hat{\mathbb{Z}}^p)^2$. Therefore the object $(E/S, \overline{\eta^p})$ is isomorphic to $(E'/S, \overline{\eta'^p})$, which is the essential image of $E' \in \operatorname{Ob}\mathcal{E}ll(S)$.

Next we show that there is an isomorphism

$$\operatorname{Hom}_{\mathcal{E}ll(S)}(E, E') \simeq \operatorname{Hom}_{\mathcal{E}ll'(S)}((E, \overline{\eta^p}), (E', \overline{\eta'^p})). \tag{10}$$

This is clearly injective by the above discussion. Now we pick an arbitrary element f from the right-hand side. Then f is a $\mathbb{Z}_{(p)}^{\times}$-isogeny, and $\eta'^p = \eta^p \circ V^p(f)$. There exists an integer M prime to p such that $\tilde{f} = M \circ f$ is an isogeny from E to E'. We claim that this isogeny factors through the multiplication-by-M map, i.e., f itself is an isogeny. By the relation $\eta'^p = \eta^p \circ V^p(f)$ and the construction above, $V^p(f)$ maps $T^p(E_{\bar{s}})$ isomorphically to $T^p(E'_{\bar{s}})$, then obviously $\tilde{f}$ maps $E'_{\bar{s}}[M] \simeq E'[M]_{\bar{s}}$ to 0. This holds for every geometric point $\bar{s}$ of S, so since S is a $\mathbb{Z}_{(p)}$-scheme and by the rigidity result [Mumford and Fogarty 1982, Proposition 6.1], we know the isogeny $\tilde{f}$ vanishes on $E'[M]$. Hence f itself is an isogeny. Now $\ker(f)$ is a finite flat group scheme over S of order prime to p, but since $V^p(f)$ maps $T^p(E_{\bar{s}})$ isomorphically to $T^p(E'_{\bar{s}})$, this group scheme must be trivial, i.e., f is an isomorphism, and therefore it comes from an element of the left-hand side of (10). □

Remark 4.2.9. We consider the Deligne–Mumford stack

$$\mathcal{H} = \mathcal{E}ll \times_{\mathbb{Z}} \mathcal{E}ll.$$

For any prime p, we use $\mathcal{H}_{(p)}$ to denote the localization of $\mathcal{H}$ to $\operatorname{Spec}\mathbb{Z}_{(p)}$.

There is a similar description of the stack $\mathcal{H}_{(p)}$: for any $\mathbb{Z}_{(p)}$-scheme S, the groupoid $\mathcal{H}_{(p)}(S)$ consists of pairs $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}))$, where $\overline{\eta^p}$ (resp. $\overline{\eta'^p}$) is a $\pi_1(S, \bar{s})$-invariant $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$-equivalence class of an isomorphism $V^p(E_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$ (resp. $V^p(E'_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$).

For any $N \in \mathbb{Z}_{>0}$, let w_N be the 2×2 matrix

$$w_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the following stack $\mathcal{Y}_0(N)'_{(p)}$ over $\operatorname{Spec}\mathbb{Z}_{(p)}$: for every $\mathbb{Z}_{(p)}$-scheme S, $\mathcal{Y}_0(N)'_{(p)}(S)$ is a groupoid whose objects are pairs

$$(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}),$$

where $E \xrightarrow{\pi} E'$ is a cyclic N-isogeny and $\overline{(\eta^p, \eta'^p)}$ is a pair of $\pi_1(S, \bar{s})$-invariant $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-equivalence classes (we specify the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ in (12)) of isomorphisms

$$\eta^p : V^p(E_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2, \quad \eta'^p : V^p(E'_{\bar{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2,$$

which maps $T^p(E_{\bar{s}})$ and $T^p(E'_{\bar{s}})$ to $(\hat{\mathbb{Z}}^p)^2$, and the isomorphism η'^p is determined by the commutative diagram

$$\begin{array}{ccc} V^p(E_{\bar{s}}) & \xrightarrow{\eta^p} & (\mathbb{A}_f^p)^2 \\ \Big\downarrow V^p(\pi) & & \Big\downarrow w_N \\ V^p(E'_{\bar{s}}) & \xrightarrow{\eta'^p} & (\mathbb{A}_f^p)^2 \end{array} \tag{11}$$

A morphism from $(E_1 \xrightarrow{\pi_1} E'_1, \overline{(\eta_1^p, \eta_1'^p)})$ to $(E_2 \xrightarrow{\pi_2} E'_2, \overline{(\eta_2^p, \eta_2'^p)})$ is a pair (f, f') of isomorphisms $f : E_1 \to E_2$ and $f' : E'_1 \to E'_2$ such that

$$f' \circ \pi_1 = \pi_2 \circ f \quad \text{and} \quad \overline{(\eta_1^p, \eta_1'^p)} = \overline{(\eta_2^p \circ V^p(f), \eta_2'^p \circ V^p(f'))}$$

as $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-orbits. The action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ on the pair (η^p, η'^p) is given by

$$g \cdot ((\eta^p, \eta'^p)) = (g \circ \eta^p, w_N g w_N^{-1} \circ \eta'^p). \tag{12}$$

Lemma 4.2.10. *Let $\mathcal{Y}_0(N)_{(p)}$ be the localization of $\mathcal{Y}_0(N)$ to $\mathbb{Z}_{(p)}$. Then there is an isomorphism $G : \mathcal{Y}_0(N)_{(p)} \to \mathcal{Y}_0(N)'_{(p)}$ of stacks over* $\operatorname{Spec}\mathbb{Z}_{(p)}$.

Proof. Let S be a scheme over $\operatorname{Spec}\mathbb{Z}_{(p)}$, and $(E \xrightarrow{\pi} E')$ an object in the groupoid $\mathcal{Y}_0(N)_{(p)}(S)$. For a geometric point $\bar{s}$ of S, the cyclicity of π implies that $\pi_{\bar{s}}$ is also cyclic. Since l is invertible in $\operatorname{Spec}\kappa(\bar{s})$ if $l \neq p$, there exist isomorphisms $\eta^p : T^p(E_{\bar{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$ and $\eta'^p : T^p(E'_{\bar{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$ such that $\omega_N \circ \eta^p = \eta'^p \circ T^p(\pi)$. Now we consider a different choice of (η^p, η'^p), say $(\tilde{\eta}^p, \tilde{\eta}'^p)$, satisfying the above conditions. Then $\tilde{\eta}'^p$ differs from η^p by an element $g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$, i.e., $\tilde{\eta}^p = g \circ \eta^p$, and correspondingly $\tilde{\eta}'^p = \omega_N g \omega_N^{-1} \circ \eta'^p$. However, $\omega_N g \omega_N^{-1} \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ since both η'^p and $\tilde{\eta}'^p$ give isomorphisms from $T^p(E_{\bar{s}})$ to $(\hat{\mathbb{Z}}^p)^2$; therefore, $g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \cap \omega_N^{-1}\mathrm{GL}_2(\hat{\mathbb{Z}}^p)\omega_N = \Gamma_0(N)(\hat{\mathbb{Z}}^p)$. Thus the $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-orbit $\overline{(\eta^p, \eta'^p)}$ is well-defined. We define $G((E \xrightarrow{\pi} E')) = ((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)})$. For a pair of isomorphisms (f, f'), where $f : E_1 \to E'_1$ and $f' : E_2 \to E'_2$, define $G((f, f')) = (f, f')$.

It suffices to show that for a connected scheme S over $\operatorname{Spec}\mathbb{Z}_{(p)}$, the functor

$$G(S) : \mathcal{Y}_0(N)_{(p)}(S) \to \mathcal{Y}_0(N)_{(p)}(S)$$

is an equivalence of categories. This functor is essentially surjective by definition; now we show that it is fully faithful, i.e., the following morphism between sets is bijective:

$$\mathrm{Hom}_{\mathcal{Y}_0(N)_{(p)}(S)}\big((E_1 \xrightarrow{\pi_1} E_1'), (E_2 \xrightarrow{\pi_2} E_2')\big)$$
$$\xrightarrow{G} \mathrm{Hom}_{\mathcal{Y}_0(N)_{(p)}(S)}\big((E_1 \xrightarrow{\pi_1} E_1', \overline{(\eta_1^p, \eta_1'^p)}), (E_2 \xrightarrow{\pi_2} E_2', \overline{(\eta_2^p, \eta_2'^p)})\big),$$
$$(f, f') \mapsto (f, f'),$$

but this is clearly bijective by the definition. □

There is a natural morphism from $\mathcal{Y}_0(N)_{(p)}$ to $\mathcal{H}_{(p)}$, i.e., $(E \xrightarrow{\pi} E') \to (E, E')$. By Remark 4.2.9 and Lemma 4.2.10, we can also describe it as

$$\mathcal{Y}_0(N)_{(p)} \to \mathcal{H}_{(p)}, \quad (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}) \mapsto ((E, E'), (\overline{\eta^p}, \overline{\eta'^p})). \tag{13}$$

4.2.A. *Compactification of* $\mathcal{Y}_0(N)$. Next we introduce the compactification of the moduli stack $\mathcal{Y}_0(N)$. Let S be a scheme. We first introduce the notion of Néron n-gons.

Definition 4.2.11. For any integer $n \geq 1$ and a scheme S, the Néron n-gon over S is the coequalizer of

$$\bigsqcup_{i\in\mathbb{Z}/n\mathbb{Z}} S \rightrightarrows \bigsqcup_{i\in\mathbb{Z}/n\mathbb{Z}} \mathbb{P}^1_S,$$

where the top (resp. the bottom) closed immersion includes the i-th copy of S as the 0 (resp. ∞) section of the i-th (resp. $(i+1)$-st) copy of $\mathbb{P}^1_S$.

Definition 4.2.12. A generalized elliptic curve over a scheme S consists of the following data:

- A proper, flat, finitely presented morphism $E \to S$ each of whose geometric fibers is either a smooth connected curve of genus 1 or a Néron n-gon for some $n \geq 1$.
- An S-morphism $E^{\mathrm{sm}} \times_S E \xrightarrow{+} E$ that restricts to a commutative S-group scheme structure on E^{sm} for which $+$ becomes an S-group action such that via the pullback of line bundles the action $+$ induces the trivial action of E^{sm} on $\mathrm{Pic}^0_{E/S}$.

We use $\mathcal{X}$ to denote the moduli stack consisting of generalized elliptic curves whose degenerate fibers are Néron 1-gons, i.e., for a scheme S, $\mathcal{X}(S)$ is a groupoid whose objects are generalized elliptic curves E over S and whose geometric fibers are either elliptic curves or Néron 1-gons. The following result is proved in [Česnavičius 2017].

Lemma 4.2.13. *$\mathcal{X}$ is a proper smooth 2-dimensional Deligne–Mumford stack.*

Proof. This is proved in [Česnavičius 2017, Theorem 3.1.6]. □

We have a natural morphism of Deligne–Mumford stacks $\mathcal{E}ll \to \mathcal{X}$, which sends an elliptic curve E over S to itself. This morphism is an open immersion, i.e., the stack $\mathcal{E}ll$ is an open substack of $\mathcal{X}$. Recall that we have a finite flat representable morphism $\mathcal{Y}_0(N) \to \mathcal{E}ll$ by Lemma 4.2.7. Let $\mathcal{X}_0(N)$ be the normalization of $\mathcal{Y}_0(N)$ with respect to this morphism. A moduli description of $\mathcal{X}_0(N)$ in terms of level structures on the generalized elliptic curves can be found in [Česnavičius 2017, §5.9]. The stack $\mathcal{Y}_0(N)$ can be realized as an open substack of the stack $\mathcal{X}_0(N)$ based on this description. We also have the following theorem:

Theorem 4.2.14. *$\mathcal{X}_0(N)$ is a regular proper 2-dimensional Deligne–Mumford stack. It is finite flat over $\mathcal{X}$.*

Proof. This is proved in [Česnavičius 2017, Theorem 5.13]. □

4.3. *Special cycles on $\mathcal{H}$ and $\mathcal{X}_0(N)$.* Let p be a prime number, we first define the special cycles on the stack $\mathcal{H}_{(p)}$.

Definition 4.3.1. For every symmetric $n\times n$ matrix $T=(T_{ik})$, let $\tilde{\boldsymbol{\varphi}}^p$ be the characteristic function of an open compact subset $\tilde{\boldsymbol{\omega}}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)\times\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. We consider the stack $\mathcal{Z}^\sharp(T,\tilde{\boldsymbol{\varphi}}^p)$, whose fibered category over a $\mathbb{Z}_{(p)}$-scheme S consists of the objects

$$((E,E'),(\overline{\eta^p},\overline{\eta'^p}),\boldsymbol{j}),$$

where $((E,E'),(\overline{\eta^p},\overline{\eta'^p}))$ is an object in $\mathcal{H}_{(p)}(S)$, $\boldsymbol{j}=(j_1,j_2,\dots,j_n)\in(\mathrm{Hom}(E,E')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})^n$ and $\eta^*(\boldsymbol{j}):=\eta'^p\circ V^p(\boldsymbol{j})\circ(\eta^p)^{-1}\in\tilde{\boldsymbol{\omega}}^p$. Moreover,

$$T_{ik}=\tfrac{1}{2}(\deg(j_i+j_k)-\deg(j_i)-\deg(j_k))=\tfrac{1}{2}(j_i\circ j_k^\vee+j_k\circ j_i^\vee).$$

The special cycle $\mathcal{Z}^\sharp(T,\tilde{\boldsymbol{\varphi}}^p)$ may be empty.

For every symmetric $n\times n$ matrix T, we have a natural finite unramified morphism $i_n^\sharp:\mathcal{Z}^\sharp(T,\tilde{\boldsymbol{\varphi}}^p)\to\mathcal{H}_{(p)}$ by forgetting the morphisms $\boldsymbol{j}$ of an object $((E,E'),(\overline{\eta^p},\overline{\eta'^p}),\boldsymbol{j})$ of $\mathcal{Z}^\sharp(T,\tilde{\boldsymbol{\varphi}}^p)$. We recall the following definition of generalized Cartier divisor from [Howard and Madapusi 2022, Definition 2.4.1].

Definition 4.3.2. Suppose $D\to X$ is any finite, unramified and relatively representable morphism of Deligne–Mumford stacks. Then there is an étale cover $U\to X$ by a scheme such that the pullback $D_U\to U$ is a finite disjoint union

$$D_U=\bigsqcup_i D_U^i$$

with each map $D_U^i\to U$ a closed immersion. If each of these closed immersions is an effective Cartier divisor on U in the usual sense (the corresponding ideal sheaves are invertible), then we call $D\to X$ a generalized Cartier divisor.

Proposition 4.3.3. *Let $\tilde{\varphi}^p$ be the characteristic function of an open compact subset $\tilde{\omega}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)\times\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. For any positive number $d\in\mathbb{Q}$, the finite unramified morphism $i_1^\sharp:\mathcal{Z}^\sharp(d,\tilde{\varphi}^p)\to\mathcal{H}_{(p)}$ is a generalized Cartier divisor.*

Proof. This is proved in [Howard and Madapusi 2020, Proposition 6.5.2] (see also [Howard and Madapusi 2022, Proposition 2.4.3]). □

Now let's turn to the special cycles on the stack $\mathcal{X}_0(N)_{(p)}$ and $\mathcal{Y}_0(N)_{(p)}$. We first introduce the notion of special morphisms for the moduli stack $\mathcal{Y}_0(N)_{(p)}$.

Definition 4.3.4. Let S be a scheme over $\mathrm{Spec}\,\mathbb{Z}_{(p)}$. For an object $((E\xrightarrow{\pi}E'),\overline{(\eta^p,\eta'^p)})$ in $\mathcal{Y}_0(N)_{(p)}(S)$, a special morphism of this object is an element $j\in\mathrm{Hom}(E,E')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ satisfying

$$j\circ\pi^\vee+\pi\circ j^\vee=0.$$

We denote this space by $S(E,\pi)$.

Definition 4.3.5. For every symmetric $n \times n$ matrix $T = (T_{ik})$, let $\boldsymbol{\varphi}^p$ be the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. We consider the stack $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$, whose fibered category over a $\mathbb{Z}_{(p)}$-scheme S consists of the objects

$$((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)}, \boldsymbol{j}),$$

where $((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)})$ is an object in $\mathcal{Y}_0(N)_{(p)}(S)$, $\boldsymbol{j} = (j_1, j_2, \dots, j_n) \in S(E, \pi)^n$ and $\eta^*(\boldsymbol{j}) := \eta'^p \circ V^p(\boldsymbol{j}) \circ (\eta^p)^{-1} \in \boldsymbol{\omega}^p$. Moreover,

$$T_{ik} = \tfrac{1}{2}(\deg(j_i + j_k) - \deg(j_i) - \deg(j_k)) = \tfrac{1}{2}(j_i \circ j_k^\vee + j_k \circ j_i^\vee).$$

The special cycle $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$ may be empty.

For every symmetric $n \times n$ matrix T, we have a natural morphism $i_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ by forgetting the special morphisms.

Remark 4.3.6. Let $T \in \mathrm{Sym}_n(\mathbb{Q})$. Let $\tilde{\boldsymbol{\varphi}}^p$ be the characteristic function of an open compact subset $\tilde{\boldsymbol{\omega}}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) \times \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. Let $\boldsymbol{\varphi}^p$ be the restriction of $\tilde{\boldsymbol{\varphi}}^p$ to the subspace $(\mathbb{V}_f^p)^n$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$. Then $\boldsymbol{\varphi}^p$ is the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$, and the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$ is a union of some connected components of the fiber product $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p) \times_{\mathcal{H}_{(p)}} \mathcal{Y}_0(N)_{(p)}$. Therefore the morphism $i_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ is also finite unramified. In particular, for $n = 1$ and $T = d \in \mathbb{Q}_{>0}$, the morphism $i_1 : \mathcal{Z}(d, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ is a generalized Cartier divisor by Proposition 4.3.3.

We show next that the composite $\tilde{i}_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \xrightarrow{i_n} \mathcal{Y}_0(N)_{(p)} \to \mathcal{X}_0(N)_{(p)}$ is also finite unramified. We start with the case that $n = 1$.

Proposition 4.3.7. *Let φ^p be the characteristic function of an open compact subset ω^p of $\mathbb{V}_f^p$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. For any positive number $d \in \mathbb{Q}$, the morphism $\tilde{i}_1 : \mathcal{Z}(d, \varphi^p) \to \mathcal{X}_0(N)_{(p)}$ is finite unramified, and $\mathcal{Z}(d, \varphi^p)$ is a generalized Cartier divisor.*

Proof. The morphism $\tilde{i}_1$ is unramified since i_1 is unramified and the open immersion $\mathcal{Y}_0(N)_{(p)} \to \mathcal{X}_0(N)_{(p)}$ is also unramified. Therefore we only need to show the finiteness of $\tilde{i}_1$.

We first prove that the stack $\mathcal{Z}(d, \varphi^p)$ is flat over $\mathbb{Z}_{(p)}$. Since the morphism $\mathcal{Z}(d, \varphi^p)\mathcal{Y}_0(N)_{(p)}$ is a generalized Cartier divisor by Remark 4.3.6, the flatness of $\mathcal{Z}(d, \varphi^p)$ is equivalent to the fact that its local equation is not divisible by p since the stack $\mathcal{Y}_0(N)_{(p)}$ is flat over $\mathbb{Z}_{(p)}$. We assume the converse and suppose that there exists a point $z \in \mathcal{Z}(d, \varphi^p)(\overline{\mathbb{F}}_p)$ such that the equation of $\mathcal{Z}(d, \varphi^p)$ in the étale local ring $\mathcal{O}^{\text{ét}}_{\mathcal{Y}_0(N),z}$ is divisible by p. Then the stack $\mathcal{Z}(d, \varphi^p)$ contains an irreducible component of $\mathcal{Y}_0(N)_{\mathbb{F}_p}$ in an étale neighborhood of z. Let $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)})$ be the object corresponding to the generic point of this irreducible component. Then $\mathrm{End}(E) \simeq \mathbb{Z}$ since the j-invariant of E must be transcendental over $\mathbb{F}_p$ (by the description of the stack $\mathcal{Y}_0(N)_{\mathbb{F}_p}$ in [Katz and Mazur 1985, Proposition 13.4.5 and Theorem 13.4.7]). There also exists an isogeny $j \in \mathrm{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $j^\vee \circ \pi + \pi^\vee \circ j = 0$. Let $\alpha = j^{-1} \circ \pi \in \mathrm{End}^\circ(E) := \mathrm{End}(E) \otimes \mathbb{Q} \simeq \mathbb{Q}$. Then $\alpha^2 = -Nd^{-1} < 0$, contradicting the fact that $\mathrm{End}^\circ(E) \simeq \mathbb{Q}$.

Therefore, the stack $\mathcal{Z}(d,\varphi^p)$ is flat over $\mathbb{Z}_{(p)}$, and hence equals the flat closure of its generic fiber $\mathcal{Z}(d,\varphi^p)_{\mathbb{Q}} := \mathcal{Z}(d,\varphi^p)\times_{\mathbb{Z}_{(p)}}\mathbb{Q}$. The stack $\mathcal{Z}(d,\varphi^p)_{\mathbb{Q}}$ consists of finitely many points whose residue fields are finite extensions of $\mathbb{Q}$. Therefore the structure sheaf $\mathcal{O}_{\mathcal{Z}(d,\varphi^p)}$ of $\mathcal{Z}(d,\varphi^p)$ is a finite product of subrings of the integer rings of these residue fields. Hence the stack $\mathcal{Z}(d,\varphi^p)$ is finite over $\mathbb{Z}_{(p)}$, so $\tilde{i}_1:\mathcal{Z}(d,\varphi^p)\to\mathcal{X}_0(N)_{(p)}$ is proper since $\mathcal{X}_0(N)_{(p)}$ is proper over $\mathbb{Z}_{(p)}$. The morphism $\tilde{i}_1$ is obviously quasifinite by the finiteness of $\mathcal{Z}(d,\varphi^p)$ over $\mathbb{Z}_{(p)}$, and hence $\tilde{i}_1$ is finite.

We already know that the morphism $\tilde{i}_1$ is a generalized Cartier divisor over the open substack $\mathcal{Y}_0(N)_{(p)}$ of $\mathcal{X}_0(N)_{(p)}$. Moreover, étale locally around a cusp point of $\mathcal{X}_0(N)_{(p)}$, the stack $\mathcal{Z}(d,\varphi^p)$ is cut out by 1 since $\tilde{i}_1$ factors through the noncuspidal locus $\mathcal{Y}_0(N)_{(p)}$. Thus the finite unramified morphism $\tilde{i}_1:\mathcal{Z}(d,\varphi^p)\to\mathcal{X}_0(N)_{(p)}$ is a generalized Cartier divisor on the stack $\mathcal{X}_0(N)_{(p)}$. □

Corollary 4.3.8. *Let $\boldsymbol{\varphi}^p=\prod_{i=1}^n\varphi_i^p$ be the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. For any matrix $T\in\mathrm{Sym}_n(\mathbb{Q})_{>0}$, the morphism $\tilde{i}_n:\mathcal{Z}(T,\boldsymbol{\varphi}^p)\to\mathcal{X}_0(N)_{(p)}$ is finite unramified.*

Proof. Suppose the diagonal elements of T are $d_1,\dots,d_n$. Proposition 4.3.7 implies that the morphism $\mathcal{Z}(d_1,\varphi_1^p)\times_{\mathcal{X}_0(N)_{(p)}}\cdots\times_{\mathcal{X}_0(N)_{(p)}}\mathcal{Z}(d_n,\varphi_n^p)\to\mathcal{X}_0(N)_{(p)}$ is finite unramified. The stack $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is a connected component of $\mathcal{Z}(d_1,\varphi_1^p)\times_{\mathcal{X}_0(N)_{(p)}}\cdots\times_{\mathcal{X}_0(N)_{(p)}}\mathcal{Z}(d_n,\varphi_n^p)$, so the morphism $\tilde{i}_n:\mathcal{Z}(T,\boldsymbol{\varphi}^p)\to\mathcal{X}_0(N)_{(p)}$ is finite unramified. □

We mainly focus on the case that T is a nonsingular 2×2 symmetric matrix with coefficients in $\mathbb{Q}$. For every such matrix T, recall that we have defined the difference set to be

$$\mathrm{Diff}(T,\Delta(N))=\{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N)\otimes\mathbb{Q}_l\}.$$

Proposition 4.3.9. *Let $T\in\mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. If $\mathcal{Z}(T,\boldsymbol{\varphi}^p)(\overline{\mathbb{F}}_p)\neq\varnothing$ for some prime p, then T is positive definite, and*

$$\mathrm{Diff}(T,\Delta(N))=\{p\}.$$

Moreover, in this case, the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is supported in the supersingular locus of the special fiber $\mathcal{Y}_0(N)_{\mathbb{F}_p}$.

Proof. Since $\mathcal{Z}(T,\boldsymbol{\varphi}^p)(\overline{\mathbb{F}}_p)\neq\varnothing$, Corollary 4.3.8 implies that there are two elliptic curves $\mathbb{E}$ and $\mathbb{E}'$ over $\overline{\mathbb{F}}_p$, a cyclic isogeny $\pi\in\mathrm{Hom}(\mathbb{E},\mathbb{E}')$ and two isogenies $x_1,x_2\in\mathrm{Hom}(\mathbb{E},\mathbb{E}')_{(p)}$ such that

$$T=\left(\tfrac{1}{2}(x_i,x_j)\right)\quad\text{and}\quad(x_1,\pi)=(x_2,\pi)=0.$$

Therefore, T must be positive definite and both $\mathbb{E}$ and $\mathbb{E}'$ are supersingular elliptic curves over $\overline{\mathbb{F}}_p$ since $\dim_{\mathbb{Q}}\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}\geq 3$. The quadratic space $\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}_p$ is isometric to the underlying quadratic space of the unique division quaternion algebra $\mathbb{B}$ over $\mathbb{Q}_p$.

The isogenies x_1,x_2 lie in $\{\pi\}^\perp\subset\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}_p\simeq\mathbb{B}$, where $\pi^\vee\circ\pi=N$. However, $\{\pi\}^\perp$ and $\Delta(N)\otimes\mathbb{Q}_p$ have the same discriminant $-N$ but opposite Hasse invariants. Therefore $p\in\mathrm{Diff}(T,\Delta(N))$.

At the same time, by choosing some level structures on $\mathbb{E}$ and $\mathbb{E}'$ away from p, we get that T can be realized in $\Delta(N)\otimes\mathbb{Q}_l$ for any finite prime $l\neq p$. Therefore p is the only prime in the set $\mathrm{Diff}(T,\Delta(N))$. □

Remark 4.3.10. Proposition 4.3.9 implies that the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is also finite unramified over the stack $\mathcal{X}_0(N)_{(p)}$ because the scheme-theoretic image $\widetilde{\mathcal{Z}}(T,\boldsymbol{\varphi}^p)$ of $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ in $\mathcal{X}_0(N)_{(p)}$ is supported in the supersingular locus of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$, which equals the supersingular locus of the special fiber $\mathcal{Y}_0(N)_{\mathbb{F}_p}$. Hence $\widetilde{\mathcal{Z}}(T,\boldsymbol{\varphi}^p)$ is contained in $\mathcal{Y}_0(N)_{(p)}$, and therefore equals the scheme-theoretic image of $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ in $\mathcal{Y}_0(N)_{(p)}$, over which $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is finite unramified.

For any nonsingular 2×2 symmetric matrix $T\in\mathrm{Sym}_2(\mathbb{Q})$, a Schwartz function $\boldsymbol{\varphi}=\bigotimes_{v<\infty}\boldsymbol{\varphi}_v\in\mathscr{S}(\mathbb{V}_f^2)$ is called T-admissible if $\boldsymbol{\varphi}$ is invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}})$, $\boldsymbol{\varphi}=\varphi_1\times\varphi_2$ for $\varphi_i\in\mathscr{S}(\mathbb{V}_f)$ and

- T is not positive definite, or
- T is positive definite and $|\mathrm{Diff}(T,\Delta(N))|\neq 1$, or
- T is positive definite, $\mathrm{Diff}(T,\Delta(N))=\{p\}$ for some prime number p, and $\boldsymbol{\varphi}=\boldsymbol{\varphi}^p\otimes\boldsymbol{\varphi}_p$, where $\boldsymbol{\varphi}^p\in\mathscr{S}((\mathbb{V}_f^p)^2)$ and $\boldsymbol{\varphi}_p=c\cdot\mathbf{1}_{\Delta_p(N)^2}$ for some $c\in\mathbb{C}$.

Definition 4.3.11. For a nonsingular 2×2 matrix $T\in\mathrm{Sym}_2(\mathbb{Q})$ and a T-admissible Schwartz function $\boldsymbol{\varphi}\in\mathscr{S}(\mathbb{V}_f^2)$ which is also a characteristic function of a $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant open compact subset $\boldsymbol{\omega}$ of $\mathbb{V}_f^2$, we define a stack finite unramified over $\mathcal{X}_0(N)$ as

$$\mathcal{Z}(T,\boldsymbol{\varphi}):=\mathcal{Z}(T,\boldsymbol{\varphi}^p)\to\mathcal{X}_0(N)_{(p)}\hookrightarrow\mathcal{X}_0(N),$$

where $p\in\mathrm{Diff}(T,\Delta(N))$. If $|\mathrm{Diff}(T,\Delta(N))|\neq 1$, we define $\mathcal{Z}(T,\boldsymbol{\varphi})=\varnothing$.

Remark 4.3.12. By Proposition 4.3.9, $\mathcal{Z}(T,\boldsymbol{\varphi})$ is nonempty only if $|\mathrm{Diff}(T,\Delta(N))|=1$, so the above definition makes sense.

Remark 4.3.13. If we view $\mathcal{Z}(T,\boldsymbol{\varphi})$ as an element in $\mathrm{CH}^2_{\mathbb{C}}(\mathcal{X}_0(N))$, we can drop the restrictions in Definition 4.3.11 that the Schwartz function $\boldsymbol{\varphi}$ is the characteristic function of an open compact subset of $\mathbb{V}_f^2$. Since any T-admissible Schwartz function $\boldsymbol{\varphi}$ on $\mathbb{V}_f^2$ is a finite linear combination of $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant characteristic functions of some open compact subsets, we can define $\mathcal{Z}(T,\boldsymbol{\varphi})$ as the corresponding linear combination of elements in $\mathrm{CH}^2_{\mathbb{C}}(\mathcal{X}_0(N))$.

4.3.A. *Comparison with* [Sankaran et al. 2023, §2.2]. Another kind of special cycle of $\mathcal{X}_0(N)$ is defined in [Sankaran et al. 2023, §2.2] as follows,

Definition 4.3.14. For $m\in\mathbb{Z}$, let $\mathcal{Z}(m)$ denote the moduli stack whose S points, for a base scheme S, are given by

$$\mathcal{Z}(m)(S):=\{(E\xrightarrow{\pi}E',\alpha)\},$$

where $(E\xrightarrow{\pi}E')\in\mathcal{Y}_0(N)(S)$ and $\alpha\in\mathrm{End}(E)$ satisfies the following conditions:

(a) $\alpha^\vee\circ\alpha=mN$ and $\alpha^\vee+\alpha=0$.

(b) $\alpha\circ\pi^{-1}\in\mathrm{Hom}(E',E)$.

(c) $\pi\circ\alpha\circ\pi^{-1}\in\mathrm{End}(E')$.

Lemma 4.3.15. *For every prime number p, let $\mathcal{Z}(m)_{(p)} := \mathcal{Z}(m) \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then we have an isomorphism of stacks*

$$T : \mathcal{Z}(m)_{(p)} \xrightarrow{\sim} \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p}), \quad (E \xrightarrow{\pi} E', \alpha) \mapsto (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, (\alpha \circ \pi^{-1})^\vee).$$

Proof. We first prove that T is well-defined. For any connected $\mathbb{Z}_{(p)}$-scheme S, let $\bar{s}$ be a geometric point of S. We can choose trivializations $\eta^p : V^p(E_{\bar{s}}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ and $\eta'^p : V^p(E'_{\bar{s}}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ such that $T^p(E_{\bar{s}})$ and $T^p(E'_{\bar{s}})$ are mapped isomorphically to $(\hat{\mathbb{Z}}^p)^2$, and $\eta'^p \circ V^p(\pi) \circ (\eta^p)^{-1} = w_N$ by the cyclicity of π. Moreover,

$$\begin{aligned}(\alpha \circ \pi^{-1})^\vee \circ \pi + \pi^\vee \circ (\alpha \circ \pi^{-1}) &= \frac{1}{N}\pi^\vee \circ \alpha^\vee \circ \pi + \frac{1}{N}\pi^\vee \circ \alpha \circ \pi \\ &= \frac{1}{N}\pi^\vee \circ (\alpha^\vee + \alpha) \circ \pi = 0.\end{aligned}$$

Hence $(\alpha \circ \pi^{-1})^\vee \in S(E, \pi)$, so (b) implies that $\eta'^p \circ V^p((\alpha \circ \pi^{-1})^\vee) \circ (\eta^p)^{-1} \in \Delta(N) \otimes \hat{\mathbb{Z}}^p \subset \mathbb{V}_f^p$. Therefore $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, (\alpha \circ \pi^{-1})^\vee) \in \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p})(S)$.

We define the morphism

$$R : \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p}) \to \mathcal{Z}(m)_{(p)}, \quad (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, j) \mapsto (E \xrightarrow{\pi} E', j^\vee \circ \pi).$$

We show that R is well-defined. For any connected $\mathbb{Z}_{(p)}$-scheme S, an object $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, j)$ being in $\mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p})(S)$ means that $j \in \mathrm{Hom}(E, E') \otimes \mathbb{Z}_{(p)}$ and $j^\vee \circ \pi + \pi^\vee \circ j = 0$, and the fact that $\eta'^p \circ V^p(j) \circ (\eta^p)^{-1}$ is in $\Delta(N) \otimes \hat{\mathbb{Z}}^p$ implies that $j \in \mathrm{Hom}(E, E')$. Then $j^\vee \circ \pi \in \mathrm{End}(E)$, $(j^\vee \circ \pi)^\vee \circ (j^\vee \circ \pi) = \pi^\vee \circ j \circ j^\vee \circ \pi = mN$ and $(j^\vee \circ \pi)^\vee + j^\vee \circ \pi = \pi^\vee \circ j + j^\vee \circ \pi = 0$, which is exactly (a). Moreover, (b) and (c) are easily verified. Hence $(E \xrightarrow{\pi} E', j^\vee \circ \pi) \in \mathcal{Z}(m)(S)$, so the morphism R is well-defined. It's easy to see that T and R are inverse to each other, and therefore the lemma is proved. □

4.3.B. *Arithmetic special cycles on $\mathcal{X}_0(N)$.* We apply the arithmetic intersection theory developed in [Gillet 1984; 2009] to the regular proper flat Deligne–Mumford stack $\mathcal{X}_0(N)$. We obtain the arithmetic Chow ring of $\mathcal{X}_0(N)$,

$$\widehat{\mathrm{CH}}^\bullet_{\mathbb{C}}(\mathcal{X}_0(N)) = \bigoplus_{n=0}^{2} \widehat{\mathrm{CH}}^n_{\mathbb{C}}(\mathcal{X}_0(N)).$$

Roughly speaking, a class in $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(\mathcal{X}_0(N))$ is represented by an arithmetic cycle $(\mathcal{Z}, g_{\mathcal{Z}})$, where $\mathcal{Z}$ is a closed substack of $\mathcal{X}_0(N)$ of codimension n with $\mathbb{C}$-coefficients, and $g_{\mathcal{Z}}$ is a Green current for $\mathcal{Z}(\mathbb{C})$, i.e., $g_{\mathcal{Z}}$ is a current on the proper smooth complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$ of degree $(n-1, n-1)$ for which there exists a smooth ω such that

$$dd^c(g) + \delta_\zeta = [\omega].$$

Here $[\omega]$ is the current defined by integration against the smooth form ω. The rational arithmetic cycles are those of the form $\widehat{\mathrm{div}}(f) = \big(\mathrm{div}(f), \iota_*[-\log(|\tilde{f}|^2)]\big)$, where $f \in \kappa(\mathcal{Z})^\times$ is a rational function on a codimension-$(n-1)$ integral substack $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}_0(N)$, together with classes of the form $(0, \partial\eta + \bar{\partial}\eta')$. By definition, the arithmetic Chow group $\mathrm{CH}^n_{\mathbb{C}}(\mathcal{X}_0(N))$ is the quotient of the space of arithmetic cycles by the $\mathbb{C}$-subspace spanned by those rational cycles.

Let $\mathcal{Z}$ be an irreducible codimension-2 cycle on $\mathcal{X}_0(N)$. Then $\mathcal{Z}$ is a Deligne–Mumford stack over $\mathbb{F}_p$ for some prime number p and the groupoid $\mathcal{Z}(\overline{\mathbb{F}}_p)$ is a singleton with a finite automorphism group $\mathrm{Aut}(\mathcal{Z})$. The rational function field $\kappa(\mathcal{Z})$ of $\mathcal{Z}$ is a finite extension of $\mathbb{F}_p$. Clearly $\delta_{\mathcal{Z}} = 0$ because $\mathcal{Z}(\mathbb{C}) = \varnothing$.

Let $(\mathcal{Z}, g) = \left(\sum_i n_i[\mathcal{Z}_i], g\right)$ be an arithmetic cycle of codimension 2, where each $\mathcal{Z}_i$ is an irreducible codimension-2 cycle on $\mathcal{X}_0(N)$. We define the degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)) \to \mathbb{C}, \quad [(\mathcal{Z}, g)] \mapsto \sum_i n_i \cdot \frac{\log|\kappa(\mathcal{Z}_i)|}{|\mathrm{Aut}(\mathcal{Z}_i)|} + \frac{1}{2} \int_{\mathcal{X}_0(N)(\mathbb{C})} g. \tag{14}$$

Here the integration $\int_{\mathcal{X}_0(N)_{\mathbb{C}}} g$ is the integration of the constant function 1 on $\mathcal{X}_0(N)_{\mathbb{C}}$ against the $(1, 1)$-current g. It is a finite number since the stack $\mathcal{X}_0(N)$ is proper. This number is independent of the choice of representing element $(\mathcal{Z}, g)$ as a consequence of the product formula [Kudla et al. 2006, §2.1].

Now we are going to construct Green currents for the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$. Let

$$\mathbb{D} = \{z \in \Delta(N) \otimes_{\mathbb{Z}} \mathbb{C} : (z, z) = 0,\ (z, \bar{z}) < 0\} / \mathbb{C}^* \subset \mathbb{P}(\Delta(N) \otimes \mathbb{C}).$$

We have the $\mathrm{GL}_2(\mathbb{R})$-equivariant identification

$$\mathbb{H}_1^{\pm} \xrightarrow{\sim} \mathbb{D}, \quad \tau \mapsto \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} -N\tau & -N\tau^2 \\ 1 & \tau \end{pmatrix}\right\}.$$

Next, we associate to any nonsingular $T \in \mathrm{Sym}_2(\mathbb{Q})$ and T-admissible Schwartz function $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ an element in $\widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$. Let $\mathrm{y} = {}^t a \cdot a \in \mathrm{Sym}_2(\mathbb{R})$ be a positive definite matrix, where $a \in \mathrm{GL}_2(\mathbb{R})$.

- For a positive definite T and T-admissible Schwartz function $\boldsymbol{\varphi}$, we consider the element

$$\hat{\mathcal{Z}}(T, \mathrm{y}, \boldsymbol{\varphi}) = (\mathcal{Z}(T, \boldsymbol{\varphi}), 0) \in \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)).$$

- For another nonsingular T which is not positive definite, we apply the general machine developed in [Garcia and Sankaran 2019], which is made explicit in [Sankaran et al. 2023]. For any $x \in \mathbb{V}_\infty$ and $[z] \in \mathbb{D}$, let $\mathrm{R}(x, [z]) = -|(x, z)|^2 \cdot (z, \bar{z})^{-1}$. We define an element in $\mathscr{S}(\mathbb{V}_\infty^2) \otimes \mathcal{A}^{1,1}(\mathbb{H}_1^{\pm})$ by letting, for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{V}_\infty^2$ and $[z] \in \mathbb{D}$,

$$\nu(\boldsymbol{x}, [z]) = \left(-\pi^{-1} + 2\sum_{i=1}^{2}(\mathrm{R}(x_i, [z]) + (x_i, x_i))\right) \exp\left(-2\pi \sum_{i=1}^{2}\left(\mathrm{R}(x_i, [z]) + \frac{1}{2}(x_i, x_i)\right)\right) \cdot \frac{dx \wedge dy}{y^2}.$$

Then we define a smooth $(1, 1)$-form $\mathfrak{g}(T, \mathrm{y}, \boldsymbol{\varphi})$ on $\mathbb{D}$ by letting its value at the point $[z] \in \mathbb{D}$ be

$$\mathfrak{g}(T, \mathrm{y}, \boldsymbol{\varphi})([z]) = \sum_{\substack{\boldsymbol{x} \in (\Delta(N) \otimes \mathbb{Q})^2 \\ T(\boldsymbol{x}) = T}} \boldsymbol{\varphi}(\boldsymbol{x}) \cdot \int_1^\infty \nu(t^{1/2}\boldsymbol{x} \cdot {}^t a, [z]) \cdot \frac{\mathrm{d}t}{t}.$$

The sum converges absolutely, and descends to a smooth $(1, 1)$-form on the modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$.

Lemma 4.3.16. *For nonsingular $T \in \mathrm{Sym}_2(\mathbb{R})$ which is not positive definite, the form $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ is absolutely integrable on $\mathcal{X}_0(N)_{\mathbb{C}}$. Hence $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ defines a $(1,1)$-current on $\mathcal{X}_0(N)_{\mathbb{C}}$.*

Proof. This is proved in [Sankaran et al. 2023, Lemma 2.9]. □

To sum up, let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ a T-admissible Schwartz function. We define

$$\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}) = \begin{cases} ([\mathcal{Z}(T, \boldsymbol{\varphi})], 0) & \text{when } T \text{ is positive definite,} \\ (0, \mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})) & \text{when } T \text{ is not positive definite.} \end{cases} \tag{15}$$

It is an element in $\widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N))$.

4.4. *Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$.* Now we can state the main theorem of this article, which proves an identity between arithmetic intersection numbers on $\mathcal{X}_0(N)$ and derivatives of Fourier coefficients of Eisenstein series,

Theorem 4.4.1. *Let N be a positive integer, $T \in \mathrm{Sym}_2(\mathbb{Q})$ a nonsingular symmetric matrix, and $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ a T-admissible Schwartz function. Then*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}))q^T = \frac{\psi(N)}{24} \cdot \partial\, \mathrm{Eis}_T(\mathsf{z}, \boldsymbol{\varphi})$$

for any $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$. Here $\psi(N) = N \cdot \prod_{l \mid N}(1 + l^{-1})$, $q^T = e^{2\pi i\, \mathrm{tr}(T\mathsf{z})}$.

The article [Sankaran et al. 2023] proves this formula in the case that T is not positive definite without any restrictions on the level N, and the case that T is positive definite but with the restriction that N is odd and square-free. We give a proof of the case that T is positive definite and N is arbitrary in Section 8.3.

5. Rapoport–Zink spaces and special cycles

5.1. $\boldsymbol{\Gamma_0(N)}$*-structures on p-divisible groups.* For a prime p, let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$, W the completion of the maximal unramified extension of $\mathbb{Q}_p$ and Nilp_W the category of schemes S over $\mathrm{Spec}\, W$ such that p is locally nilpotent on S. Let $\bar{S}$ be the closed subscheme of S defined by the ideal sheaf $p\mathcal{O}_S$. For a p-divisible group X, we use $X^\vee$ to denote the dual p-divisible group. We introduce two Rapoport–Zink spaces in this chapter. They are essentially isomorphic to the completed local rings of supersingular points in characteristic p of the moduli stacks $\mathcal{H}$ and $\mathcal{X}_0(N)$.

Let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ of dimension 1 and height 2. The associated filtered isocrystal $\mathbb{D}(\mathbb{X})_{\mathbb{Q}}$ has pure slope $\frac{1}{2}$, e.g., we can take $\mathbb{X}$ to be $\mathbb{E}[p^\infty]$, where $\mathbb{E}$ is a supersingular elliptic curve over $\mathbb{F}$. Let $\lambda_0 : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ be a principal polarization. We consider the following functor $\mathcal{N}$ on the category Nilp_W: for any $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}(S)$ is the isomorphism classes of tuples $((X, \rho, \lambda), (X', \rho', \lambda'))$, where

(1) X and X' are two p-divisible groups over S, and ρ, ρ' are two quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X \times_S \bar{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X' \times_S \bar{S}$;

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations such that Zariski locally on $\bar{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0$$

for some $c(\rho), c(\rho') \in \mathbb{Z}_p^\times$.

Proposition 5.1.1. *The functor* $\mathcal{N}$ *is represented by the formal scheme* $\operatorname{Spf} W[\![t_1, t_2]\!]$ *over* $\operatorname{Spf} W$.

Proof. When p is odd, this is explained in [Li and Zhang 2022, Example 4.5.3(ii)]. In general, the deformation space of the supersingular elliptic curve $\mathbb{E}$ is isomorphic to $\operatorname{Spf} W[\![t]\!]$. By the Serre–Tate theorem, this is also the deformation space of the p-divisible group $\mathbb{X}$ with certain restrictions on the polarization, as in the definition of the deformation functor $\mathcal{N}$. Therefore,

$$\mathcal{N} \simeq \operatorname{Spf} W[\![t_1]\!] \times_{\operatorname{Spf} W} \operatorname{Spf} W[\![t_2]\!] \simeq \operatorname{Spf} W[\![t_1, t_2]\!]. \qquad \square$$

Let $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ be the universal p-divisible group over $\mathcal{N} = \operatorname{Spf} W[\![t_1, t_2]\!]$. By Lemma 6.1.3 below, the category of p-divisible groups over $\operatorname{Spf} W[\![t_1, t_2]\!]$ is equivalent to the category of p-divisible groups over $\operatorname{Spec} W[\![t_1, t_2]\!]$. We still use $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ to denote the corresponding p-divisible group over $\operatorname{Spec} W[\![t_1, t_2]\!]$.

Next we fix an N-isogeny $x_0 : \mathbb{X} \to \mathbb{X}$, i.e., $x_0 \circ x_0^\vee = N$. $\mathcal{N}_0(N)$ is a contravariant set-valued functor defined over Nilp_W. For every $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of the isomorphism classes of elements of the form $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda'))$, where

(1) X and X' are two p-divisible groups over S, and ρ, ρ' are two quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X \times_S \bar{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \bar{S} \to X' \times_S \bar{S}$;

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations, such that Zariski locally on $\bar{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0$$

for some $c(\rho), c(\rho') \in \mathbb{Z}_p^\times$;

(3) $x : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over S) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

We will prove later that the functor $\mathcal{N}_0(N)$ is represented by a closed formal subscheme of $\operatorname{Spf} W[\![t_1, t_2]\!]$ cut out by a single equation (see Theorem 6.2.3).

5.2. ***Special cycles on $\mathcal{N}$ and $\mathcal{N}_0(N)$.*** Now we give the definition of special cycles on the formal schemes $\mathcal{N}$ and $\mathcal{N}_0(N)$. Recall that $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ is the universal p-divisible group over $\mathcal{N}$, and $\mathbb{B} \simeq \operatorname{End}^0(\mathbb{X})$ is the unique division quaternion algebra over $\mathbb{Q}_p$, whose Hasse invariant as a quadratic space is -1.

Definition 5.2.1. For any subset $L \subset \mathbb{B}$, define the special cycle $\mathcal{Z}^\sharp(L)$ to be the closed formal subscheme of $\mathcal{N}$ where the groupoid $\mathcal{Z}^\sharp(L)(S)$, for an object $S \in \mathrm{Nilp}_W$, consists of pairs $((X, \rho, \lambda), (X', \rho', \lambda')) \in \mathcal{N}(S)$ such that the quasi-isogeny $\rho' \circ x \circ \rho^{-1}$ is an isogeny from X to X'.

Remark 5.2.2. The special cycle $\mathcal{Z}^\sharp(L)$ only depends on the $\mathbb{Z}_p$-linear span of L in $\mathbb{B}$, and is nonempty only when this span is an integral quadratic $\mathbb{Z}_p$-lattice in $\mathbb{B}$.

Proposition 5.2.3. *Let $x \in \mathbb{B}$ be a nonzero and integral element, i.e., $0 \le \nu_p(x^\vee \circ x) < \infty$. Then $\mathcal{Z}^\sharp(x)$ is a Cartier divisor on $\mathcal{N}$, i.e., it is defined by a single nonzero element $f_x \in W[[t_1, t_2]]$. Moreover, $\mathcal{Z}^\sharp(x)$ is also flat over* Spf W, *i.e., $p \nmid f_x$.*

Proof. When p is odd, the formal scheme $\mathcal{N}$ is an example of GSpin Rapoport–Zink space [Li and Zhang 2022, Example 4.5.3(ii)], and the proposition has been proved for every GSpin Rapoport–Zink space in [Li and Zhang 2022, Proposition 4.10.1]. For all p (especially $p = 2$), this is proved in [Katz and Mazur 1985, Theorem 6.8.1]. □

Now let's come to the special cycles on $\mathcal{N}_0(N)$. Firstly, we give the definition of the space of special quasi-isogenies. Recall that we have fixed an N-isogeny x_0 when we define the formal scheme $\mathcal{N}_0(N)$.

Definition 5.2.4. We call a quasi-isogeny $x \in \mathbb{B} = \mathrm{End}^0(\mathbb{X})$ special to x_0 if

$$x \circ x_0^\vee + x_0 \circ x^\vee = 0.$$

By definition, the space of quasi-isogenies special to x_0 is just the quadratic space $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$. By Witt's theorem, it is a 3-dimensional quadratic space over $\mathbb{Q}_p$ whose isometric class is independent of the choice of the N-isogeny x_0.

Definition 5.2.5. Let $(\breve{X} \xrightarrow{\breve{x}} \breve{X}', (\breve{\rho}, \breve{\rho}'), (\breve{\lambda}, \breve{\lambda}'))$ be the universal object over $\mathcal{N}_0(N)$. For any subset $M \subset \mathbb{W}$, define the special cycle $\mathcal{Z}(M) \subset \mathcal{N}_0(N)$ to be the closed formal subscheme cut out by the conditions

$$\breve{\rho}' \circ x \circ \breve{\rho}^{-1} \in \mathrm{Hom}(\breve{X}, \breve{X}') \quad \text{for any } x \in M.$$

For any subset $M \subset \mathbb{W} \subset \mathbb{B}$, we have the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}(M) & \longrightarrow & \mathcal{N}_0(N) \\ \downarrow & & \downarrow \\ \mathcal{Z}^\sharp(M) & \longrightarrow & \mathcal{N} \end{array}$$

5.3. *Formal uniformization of $\mathcal{X}_0(N)$ and the special cycle $\mathcal{Z}(T, \varphi)$.* Let B be the unique quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞. Then $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{B}$ is the unique division quaternion algebra over $\mathbb{Q}_p$. Let $\mathbb{E}$ be a supersingular elliptic curve over $\mathbb{F}$ and $\mathbb{X} = \mathbb{E}[p^\infty]$ the p-divisible group of $\mathbb{E}$. Then $B \simeq \mathrm{End}^0(\mathbb{E})$ and $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$. Suppose $x_0 \in \mathbb{B}$ comes from a cyclic N-isogeny of $\mathbb{E}$ under the above isomorphism $\mathrm{End}^0(\mathbb{E}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{B}$.

We first state and explain the formal uniformization theorem of the supersingular locus $\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p}$ of $\mathcal{H}_{\mathbb{F}_p}$. We use $\hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})}$ to denote the completion of $\mathcal{H}$ along the closed substack $\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p}$.

Theorem 5.3.1. *There is an isomorphism of formal stacks over W*

$$\hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{H}}} B^\times(\mathbb{Q})_0^2 \backslash [\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2 / \mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2], \tag{16}$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

Theorem 5.3.1 is proved in [Rapoport and Zink 1996, Theorem 6.24]. Here we only describe the isomorphism, especially the group action on the right-hand side of (16). Let $\eta_0^p : V^p(\mathbb{E}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ be a prime-to-p level structure of $\mathbb{E}$. Let $\widetilde{\mathbb{E}}$ be a deformation of $\mathbb{E}$ to W, and let $\widetilde{\mathbb{X}} := \widetilde{\mathbb{E}}[p^\infty]$ be the corresponding deformation of $\mathbb{X}$ to W. For some object $S \in \mathrm{Nilp}_W$, we pick an object

$$((X, \rho, \lambda), (X', \rho', \lambda'), (g, g')) \in \mathcal{N}(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)^2.$$

The quasi-isogeny ρ (resp. ρ') gives rise to a quasi-isogeny $\tilde{\rho} : \widetilde{\mathbb{X}}_S \to X$ (resp. $\tilde{\rho}' : \widetilde{\mathbb{X}}_S \to X'$). Then there exists an elliptic curve E (resp. E') up to prime-to-p isogeny over S and a quasi-isogeny $\rho_E : \widetilde{\mathbb{E}}_S \to E$ (resp. $\rho_{E'} : \widetilde{\mathbb{E}}_S \to E'$) such that $E[p^\infty] \simeq X$ (resp. $E'[p^\infty] \simeq X'$) and ρ_E (resp. $\rho_{E'}$) induces $\tilde{\rho}$ (resp. $\tilde{\rho}'$) under this isomorphism. The object $((X, \rho, \lambda), (X', \rho', \lambda'), (g, g'))$ is mapped to

$$\big((E, E'), \big(\overline{g^{-1}\eta_0^p \circ V^p(\rho_E^{-1})}, \overline{g'^{-1}\eta_0^p \circ V^p(\rho_{E'}^{-1})}\big)\big) \in \mathcal{H}(S).$$

The group action is given, for a pair of elements $(b, b') \in B^\times(\mathbb{Q})_0 \times B^\times(\mathbb{Q})_0$, by the map

$$B(\mathbb{Q}) \to B(\mathbb{Q}_p) \simeq \mathrm{End}^0(\mathbb{X}) \xrightarrow{\rho^*} \mathrm{End}^0(X) \quad (\text{resp. } \xrightarrow{\rho'^*} \mathrm{End}^0(X')),$$

and a fixed isomorphism $B(\mathbb{A}_f^p) \simeq \mathrm{GL}_2(\mathbb{A}_f^p)$. We obtain another triple

$$(b, b')_*\big(((X, \rho, \lambda), (X', \rho', \lambda'), (g, g'))\big) := \big((X, \rho \circ b^{-1}, \lambda), (X', \rho' \circ b'^{-1}, \lambda'), (bg, b'g')\big).$$

Now let $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$) be the supersingular locus of $\mathcal{X}_0(N)_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)_{\mathbb{F}_p}$). Let $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$ (resp. $\hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$) be the completion of $\mathcal{X}_0(N)$ (resp. $\mathcal{Y}_0(N)$) along the closed substack $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$). By the definition of $\mathcal{X}_0(N)$, we have $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p} = \mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ and therefore $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \simeq \hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Proposition 5.3.2. *There is an isomorphism of formal stacks over W,*

$$\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{X}_0(N)}} B^\times(\mathbb{Q})_0 \backslash [\mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)], \tag{17}$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

Proof. The following diagram is Cartesian, with all arrows closed immersions:

$$\begin{array}{ccc}
\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p} & \longrightarrow & \mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p} \\
\downarrow & & \downarrow \\
\mathcal{Y}_0(N) & \longrightarrow & \mathcal{H}
\end{array}$$

this diagram gives a closed immersion $i : \hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \simeq \hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \to \hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Recall that we have the isomorphism

$$\hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{H}}} B^\times(\mathbb{Q})_0^2 \backslash [\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2/\mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2].$$

Let S be an object in Nilp_W, and let $(z, (g, g')) \in \mathcal{N}(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)^2$ be a point in the closed formal substack $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Then clearly $z \in \mathcal{N}_0(N)(S)$. Suppose z corresponds to a cyclic isogeny $E \xrightarrow{\pi} E'$ by our description of the isomorphism $\Theta_{\mathcal{H}}$. Then g' is determined by g by the diagram

$$\begin{array}{ccc} V^p(E_{\bar{s}}) & \xrightarrow{g^{-1}\eta_0^p \circ V^p(\rho_E^{-1})} & (\mathbb{A}_f^p)^2 \\ \Big\downarrow V^p(\pi) & & \Big\downarrow w_N \\ V^p(E'_{\bar{s}}) & \xrightarrow{g'^{-1}\eta_0^p \circ V^p(\rho_{E'}^{-1})} & (\mathbb{A}_f^p)^2 \end{array} \tag{18}$$

Thus we only focus on the pair $(z, g) \in \mathcal{N}_0(N)(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)$. Consider the morphism

$$\Theta : \mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p) \to \hat{\mathcal{H}}/_{(\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}})}, \quad (z, g) \mapsto \Theta_{\mathcal{H}}^{-1}(z, (g, g')).$$

The image of Θ lies in the closed formal substack $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})}$.

Let $(z_1, g_1), (z_2, g_2) \in \mathcal{N}_0(N)(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)$ be two points. Then $\Theta(z_1, g_1) = \Theta(z_2, g_2)$ if and only if there exist $b, b' \in B^\times(\mathbb{Q})_0$ and $k_1, k_1' \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ such that $(z_2, (g_2, g_2')) = ((b, b')_* z_1, (bg_1k_1, b'g_1'k_1'))$. We still use $E \xrightarrow{\pi} E'$ to denote the corresponding point of z_2 under $\Theta_{\mathcal{H}}$. Notice that $(z_2, (g_2, g_2')) = (z_2, (bg_1, b'g_1'))$ in the quotient stack $[\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2 / \mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2]$. Therefore

$$\Theta_{\mathcal{H}}(z_2, (g_2, g_2')) = \Theta_{\mathcal{H}}(z_2, (bg_1, b'g_1')) \in \hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})}(S),$$

and hence both $(g_2 = bg_1k_1, g_2' = b'g_1'k_1')$ and $(bg_1, b'g_1')$ satisfy the commutative diagram (18). Then

$$k_1' = w_N k_1 w_N^{-1}.$$

Since both k_1 and k_1' belongs to $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) := \prod_{v \neq \infty, p} \mathrm{GL}_2(\mathbb{Z}_v)$, there exist $a, b, c, d \in \hat{\mathbb{Z}}^p$ such that

$$k_1 = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)(\hat{\mathbb{Z}}^p).$$

Moreover, the element b' is also determined by b by the diagram (18). Therefore $\Theta(z_1, g_1) = \Theta(z_2, g_2)$ if and only if there exists $b \in B^\times(\mathbb{Q})_0$ and $k \in \Gamma_0(N)(\hat{\mathbb{Z}}^p)$ such that $(z_2, g_2) = (b_* z_1, bg_1k)$. □

Let $\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})$ be the completion of $\mathcal{Z}(T, \boldsymbol{\varphi})$ along its supersingular locus

$$\mathcal{Z}^{\mathrm{ss}}(T, \boldsymbol{\varphi}) := \mathcal{Z}(T, \boldsymbol{\varphi}) \times_{\mathcal{X}_0(N)} \mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}}.$$

Let $\Delta(N)^{(p)}$ be the unique quadratic space over $\mathbb{Q}$ (up to isometry) such that

(1) it is positive definite at ∞;

(2) for finite prime $l \neq p$, $\Delta(N)^{(p)} \otimes \mathbb{Q}_l$ is isometric to $\Delta_l(N) \otimes \mathbb{Q}_l$;

(3) $\Delta(N)^{(p)} \otimes \mathbb{Q}_p$ is isometric to $\mathbb{W}$.

As a corollary of the formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$ (see Proposition 5.3.2), we have the following formal uniformization of the special cycles on $\mathcal{X}_0(N)$.

Corollary 5.3.3. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular symmetric matrix, and $\mathrm{Diff}(T, \Delta(N)) = \{p\}$. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Let $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_{\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}}$-modules. Then in $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})})$ we have the identity*

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \Theta^{-1}_{\mathcal{X}_0(N)}(\mathcal{Z}(\boldsymbol{x}), g),$$

where $B_{\boldsymbol{x}}^\times \subset B^\times$ is the stabilizer of $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$.

Proof. We only need to prove the corollary when $\boldsymbol{\varphi}$ is the characteristic function of some open compact subset $\boldsymbol{\omega}$ of $\mathbb{V}_f^2$. Let S be an object in Nilp_W. Suppose $\Theta^{-1}_{\mathcal{X}_0(N)}(z, g) \in \hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})(S)$ for some $z \in \mathcal{N}_0(N)(S)$. Then z gives rise to a cyclic isogeny $E \xrightarrow{\pi} E'$, along with two isogenies $x_1, x_2 \in \mathrm{Hom}(E, E')_{(p)}$ such that

$$T = \left(\tfrac{1}{2}(x_i, x_j)\right) \quad \text{and} \quad (x_1, \pi) = (x_2, \pi) = 0.$$

Then x_1, x_2 and π induce endomorphisms of the corresponding p-divisible groups, and hence endomorphisms of $\mathbb{X}$. We still use x_1, x_2 to denote the endomorphisms of $\mathbb{X}$. Let $T(\boldsymbol{x}) := \left(\tfrac{1}{2}(x_i, x_j)\right)$ be the inner product matrix of $\boldsymbol{x} = (x_1, x_2)$). We have

$$T = T(\boldsymbol{x}) \quad \text{and} \quad (x_1, x_0) = (x_2, x_0) = 0,$$

i.e., $x_1, x_2 \in \{x_0\}^\perp = \mathbb{W} \simeq \Delta(N)^{(p)} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We can also identify x_1 and x_2 as elements in $\Delta(N)^{(p)} \otimes \mathbb{A}_f^p$ via the level structures $\eta_0^p \circ V^p(\rho_E^{-1})$ and $\eta_0^p \circ V^p(\rho_{E'}^{-1})$ of E and E'. The positivity assumption on T makes it embeddable into $\Delta(N)^{(p)} \otimes_{\mathbb{Q}} \mathbb{R}$. By carefully choosing the isometry $\mathbb{W} \simeq \Delta(N)^{(p)} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, we can find $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$ which induces x_1 and x_2 locally at every place of $\mathbb{Q}$.

Then the condition $\Theta^{-1}_{\mathcal{X}_0(N)}(z, g) \in \hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})(S)$ implies that

$$z \in \mathcal{Z}(\boldsymbol{x}) \quad \text{and} \quad g^{-1}\boldsymbol{x} \in \boldsymbol{\omega} \quad (\text{here } g \in \mathrm{GL}_2(\mathbb{A}_f) \text{ with } g_p = 1),$$

and this is exactly the meaning of the identity in the theorem. □

6. Difference formula at the geometric side

6.1. *p-divisible groups over adic noetherian rings.*

Definition 6.1.1. A topological ring R is an adic noetherian ring if it is noetherian as a ring and it has a topological basis consisting of all translations of the neighborhoods of zero of the form I^n $(n > 0)$, where $I \subset R$ is a fixed ideal of R, and R is Hausdorff and complete in that topology. A choice of such an ideal is said to be the defining ideal of the topological ring R.

Lemma 6.1.2. *Let A be an adic noetherian local ring whose defining ideal is the maximal ideal $\mathfrak{m}$. Then any ideal $I \subset A$ is complete in the topological ring A, i.e.,*

$$I = \bigcap_n (I + \mathfrak{m}^n).$$

Moreover, A/I is an adic noetherian ring with defining ideal $\mathfrak{m}/I$.

Proof. Nakayama's lemma implies that $\bigcap_n \mathfrak{m}^n I = 0$. Then we can apply [Stacks, Lemma 031B] to conclude that I is $\mathfrak{m}$-adically complete, i.e., $I \simeq \hat{I} := \varprojlim_n I/\mathfrak{m}^n I$.

We have the exact sequence

$$0 \to I \to A \to A/I \to 0.$$

Since A is noetherian, after taking completion with respect to the maximal ideal $\mathfrak{m}$, we get

$$0 \to \hat{I} \to \hat{A} \to \widehat{A/I} \to 0.$$

However, $A = \hat{A}$ and $I = \hat{I}$, and hence $\widehat{A/I} \simeq A/I$. We conclude A/I is an adic noetherian ring.

By definition, $\widehat{A/I} = \varprojlim_n A/(I + \mathfrak{m}^n)$. Hence $\widehat{A/I} \simeq A/I$ implies that $I = \bigcap_n (I + \mathfrak{m}^n)$. □

Lemma 6.1.3. *Let A be an adic noetherian ring whose defining ideal is I. Then the functor*

$$\{\text{category of } p\text{-divisible groups over } \operatorname{Spec} A\} \to \{\text{category of } p\text{-divisible groups over } \operatorname{Spf}(A)\},$$
$$G = (G_n/A) \mapsto (G_k = (G_k(n) = G(n) \times_A A/I^k))_{k \geq 1}.$$

is an equivalence.

Proof. This is proved in [de Jong 1995, Lemma 2.4.4]. □

6.2. ***Difference divisors on $\mathcal{N}$.*** Recall that for every nonzero integral $x \in \mathbb{B}$, we define the special divisor $\mathcal{Z}^\sharp(x)$ on $\mathcal{N}$ as the closed formal subscheme of $\mathcal{N}$ over where x lifts to an isogeny (cf. Definition 5.2.1 and Proposition 5.2.3). It is cut out by an element $f_x \in W[[t_1, t_2]]$.

For any nonzero $x \in \mathbb{B}$ such that $\nu_p(x^\vee \circ x) \geq 2$, there is a closed immersion

$$i : \mathcal{Z}^\sharp(p^{-1}x) \to \mathcal{Z}^\sharp(x),$$

by composing every deformation of $p^{-1}x$ with the multiplication-by-p morphism. Since $W[[t_1, t_2]]$ is a unique factorization domain, we get $f_{p^{-1}x} \mid f_x$. Define $d_x = f_x/f_{p^{-1}x} \in W[[t_1, t_2]]$ when $\nu_p(x^\vee \circ x) \geq 2$ and $d_x = f_x$ when $\nu_p(x^\vee \circ x) = 0$ or 1.

Definition 6.2.1. Let $x \in \mathbb{B}$ be a nonzero and integral element. The difference divisor associated to x is

$$\mathcal{D}(x) = \operatorname{Spf} W[[t_1, t_2]]/d_x.$$

The notion of difference divisor was first introduced in [Terstiege 2011]. Proposition 5.2.3 implies that $p \nmid f_x$, so $p \nmid d_x$. Therefore the difference divisor $\mathcal{D}(x)$ is flat over $\operatorname{Spf} W$. The following theorem asserts that $\mathcal{D}(x)$ is regular.

Theorem 6.2.2. *Let $x \in \mathbb{B}$ be a nonzero and integral element. Let $\mathfrak{m} = (p, t_1, t_2)$ be the maximal ideal of $W[[t_1, t_2]]$. Then $d_x \in \mathfrak{m} \setminus \mathfrak{m}^2$, i.e., the difference divisor $\mathcal{D}(x)$ is regular. Moreover, for any $i \geq 1$, d_x and $d_{p^{-i}x}$ are coprime to each other if $p^{-i}x$ is also integral.*

Proof. Let $n \geq 0$ be the p-adic valuation of $x^\vee \circ x$. We first prove this result when $n = 0$. In this case the result follows from [Li and Zhu 2018, Lemma 3.2.2] (p odd) and [Rapoport 2007, Lemma 3.1] ($p = 2$), and $W[[t_1, t_2]]/f_x \simeq W[[t]]$ is even smooth over W.

Now we suppose $n \geq 1$. We can always find an element $x' \in \mathbb{B}$ such that $x'^{\vee} \circ x'$ has p-adic valuation 0 and $(x, x') = 0$. We consider the formal closed subscheme $\mathcal{Z}^{\sharp}(x) \times_{\mathcal{N}} \mathcal{Z}^{\sharp}(x')$. It is cut out by the ideal $(f_x, f_{x'}) \subset \mathfrak{m}$; it is also a formal closed subscheme of $\mathcal{Z}^{\sharp}(x') \simeq \operatorname{Spf} W[\![t]\!]$ cut out by the image $\tilde{f}_x$ of f_x under the surjective map $A \to W[\![t]\!]$. By [Gross and Keating 1993, (5.10)] (see also [Li and Zhang 2022, §5.1]), we have the following decomposition of $\mathcal{Z}^{\sharp}(x) \times_{\mathcal{N}} \mathcal{Z}^{\sharp}(x')$ into Cartier divisors on $\mathcal{Z}^{\sharp}(x')$:

$$\mathcal{Z}^{\sharp}(x) \times_{\mathcal{N}} \mathcal{Z}^{\sharp}(x') = \sum_{i=0}^{[n/2]} \mathcal{Z}_i, \tag{19}$$

with each $\mathcal{Z}_i \simeq \operatorname{Spf} \mathcal{O}_{\breve{K},i}$, where $\mathcal{O}_{\breve{K},i}$ is the ring of integers of some nonarchimedean local field. Hence it is a regular local ring, and they are different from each other. Let $d_i \in W[\![t]\!]$ be the function defining the divisor $\mathcal{Z}_i$ on $\mathcal{Z}^{\sharp}(x')$. Then we have the identity

$$\tilde{f}_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} d_i. \tag{20}$$

The regularity of $\mathcal{O}_{\breve{K},i}$ implies that $d_i \in (p, t) \setminus (p, t)^2$.

Let $\tilde{d}_{p^{-i}x}$ be the image of $d_{p^{-i}x}$ under the surjective map $A \to A/(f_{x'}) \simeq W[\![t]\!]$. By definition we have $f_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} d_{p^{-i}x}$. Therefore,

$$\tilde{f}_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} \tilde{d}_{p^{-i}x}. \tag{21}$$

We induct on n to conclude that $\tilde{d}_x = (\text{unit}) \times d_{[n/2]} \in (p, t) \setminus (p, t)^2$. When $n = 1$, we simply get $\tilde{d}_x = (\text{unit}) \times d_0 \in (p, t) \setminus (p, t)^2$. Let's assume the claim is true for $n < m$ for some $m \geq 2$. We prove the result for $n = m$. For this, we just need to compare (20) and (21) for $p^{-1}x$ and x.

Therefore we have proved that $A/(d_x, f_{x'})$ is a regular local ring, and hence we conclude that $d_x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\mathcal{D}(x) \simeq \operatorname{Spf} A/(d_x)$ is regular. Moreover, since all pieces on the right-hand side of (19) are different from each other, we conclude that d_x and $d_{p^{-i}x}$ are coprime to each other. □

Fix an N-isogeny $x_0 \in \mathbb{B}$. Recall that we have defined the deformation functor $\mathcal{N}_0(N)$ in Section 5.1. Compare the moduli interpretations of $\mathcal{N}_0(N)$ and $\mathcal{Z}^{\sharp}(x_0)$. We have a natural functor

$$i : \mathcal{N}_0(N) \to \mathcal{Z}^{\sharp}(x_0), \quad (X \xrightarrow{x \text{ cyclic}} X', (\rho, \rho'), (\lambda, \lambda')) \mapsto (X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda')).$$

Theorem 6.2.3. *The natural functor i is a closed immersion, and induces an isomorphism*

$$\mathcal{N}_0(N) \xrightarrow{\sim} \mathcal{D}(x_0).$$

Proof. By Proposition 5.2.3, $\mathcal{Z}^{\sharp}(x_0)$ is represented by $\operatorname{Spf} W[\![t_1, t_2]\!]/f_{x_0}$. We consider the maximal ideal $\mathfrak{m} = (p, t_1, t_2)$ of $W[\![t_1, t_2]\!]$ and a projective system of rings $\varprojlim_n R_n$, where $R_n = W[\![t_1, t_2]\!]/(f_{x_0} + \mathfrak{m}^n)$. We use $(X_n \xrightarrow{x_n} X'_n, (\rho_n, \rho'_n), (\lambda_n, \lambda'_n))$ to denote the corresponding object in $\mathcal{Z}^{\sharp}(x_0)(R_n)$ by the natural morphism $W[\![t_1, t_2]\!]/f_{x_0} \to R_n$, which is essentially the base change from $\mathcal{Z}^{\sharp}(x_0)$ to $\operatorname{Spec} R_n$ of the universal object

$$(X^{\text{univ}} \xrightarrow{x_0^{\text{univ}}} X'^{\text{univ}}, (\rho^{\text{univ}}, \rho'^{\text{univ}}), (\lambda^{\text{univ}}, \lambda'^{\text{univ}})).$$

The following diagram is commutative:

$$\begin{array}{ccc} X_n & \longrightarrow & X_{n+1} \\ \downarrow{\scriptstyle x_n} & & \downarrow{\scriptstyle x_{n+1}} \\ X'_n & \longrightarrow & X'_{n+1} \end{array}$$

By [de Jong 1995, Lemma 2.4.4], x_n fits together to be an isogeny of p-divisible groups $x_0^{\text{univ}}: X^{\text{univ}} \to X'^{\text{univ}}$ over $\operatorname{Spec} W[[t_1, t_2]]/f_{x_0}$.

Now we apply the Serre–Tate theorem [Serre 2015] to the projective system $\varprojlim_n R_n$. We obtain a direct system of elliptic curves E_n, E'_n over $\operatorname{Spec} R_n$ and $\tilde{x}_n \in \operatorname{End}_{R_n}(E_n, E'_n)$ such that

(i) there exist isomorphisms $i_n : E_n[p^\infty] \simeq X_n$ and $i'_n : E'_n[p^\infty] \simeq X'_n$,

(ii) $x_n = i'_n \circ \tilde{x}_n[p^\infty] \circ i_n^{-1}$.

Since every elliptic curve is equipped with a canonical ample line bundle given by the unit section, we can apply Grothendieck's algebraization theorem [Stacks, Theorem 089A, Lemma 0A42] to obtain a triple $(E^{\text{univ}} \xrightarrow{\tilde{x}_0^{\text{univ}}} E'^{\text{univ}}, (\rho^{\text{univ}}, \rho'^{\text{univ}}), (\lambda^{\text{univ}}, \lambda'^{\text{univ}}))$, where E^{univ} and E'^{univ} are two elliptic curves over $\operatorname{Spec} W[[t_1, t_2]]/f_{x_0}$ with the isomorphisms

$$i^{\text{univ}} : E^{\text{univ}}[p^\infty] \simeq X^{\text{univ}}, \quad i'^{\text{univ}} : E'^{\text{univ}}[p^\infty] \simeq X'^{\text{univ}},$$

and $x_0^{\text{univ}} = i'^{\text{univ}} \circ \tilde{x}_0^{\text{univ}}[p^\infty] \circ (i^{\text{univ}})^{-1}$. Then we have

$$\ker(x_0^{\text{univ}}) \simeq \ker(\tilde{x}^{\text{univ}}[p^\infty]) = \ker(\tilde{x}_0^{\text{univ}})[p^\infty] \hookrightarrow E^{\text{univ}},$$

where $\ker(\tilde{x}_0^{\text{univ}})[p^\infty]$ is the p-torsion subgroup scheme of the finite locally free group scheme $\ker(\tilde{x}_0^{\text{univ}})$. Therefore, the universal kernel $\ker(x_0^{\text{univ}})$ is embedded into an elliptic curve. We can apply Proposition 4.1.4 and conclude that there is an ideal $\mathcal{I}^{\text{cyc}}(x_0) \subset W[[t_1, t_2]]$ containing f_{x_0} such that for $S \in \operatorname{Nilp}_W$ and an object $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda')) \in \mathcal{Z}^\sharp(x_0)(S)$, the isogeny x is a cyclic isogeny if and only if the morphism $S \to \operatorname{Spf} W[[t_1, t_2]]/f_{x_0}$ factors through $\operatorname{Spf} W[[t_1, t_2]]/\mathcal{I}^{\text{cyc}}(x_0)$. We conclude from here that $\mathcal{N}_0(N)$ is represented by the formal scheme $\operatorname{Spf} W[[t_1, t_2]]/\mathcal{I}^{\text{cyc}}(x_0)$ and the natural functor i is a closed immersion.

Recall that we use d_{x_0} to denote the equation that cuts out the difference divisor $\mathcal{D}(x_0)$. In the following, we use $\mathcal{D}$ to denote the difference divisor $\mathcal{D}(x_0)$. Let $x_{\mathcal{D}} : X_{\mathcal{D}} \to X'_{\mathcal{D}}$ be the base change of $x_0^{\text{univ}} : X^{\text{univ}} \to X'^{\text{univ}}$ to $\mathcal{D}$. We first show that $x_{\mathcal{D}}$ doesn't factor through the multiplication-by-p morphism of $X_{\mathcal{D}}$. Let's assume the converse, i.e., $x_{\mathcal{D}} = p \circ x'_{\mathcal{D}}$, where $x'_{\mathcal{D}} : X_{\mathcal{D}} \to X'_{\mathcal{D}}$ is an isogeny. Let $\mathcal{D}_n = \operatorname{Spec} W[[t_1, t_2]]/(d_{x_0} + \mathfrak{m}^n)$. The base change of $x'_{\mathcal{D}}$ from $\mathcal{D}$ to $\mathcal{D}_n$ is a deformation of $p^{-1}x_0$, and hence the natural morphism $\mathcal{D}_n \to \mathcal{Z}^\sharp(x_0)$ factors through $\mathcal{Z}^\sharp(p^{-1}x_0) \simeq \operatorname{Spf} W[[t_1, t_2]]/(f_{p^{-1}x_0})$. Since $W[[t_1, t_2]]/(d_{x_0}) \simeq \varprojlim_n W[[t_1, t_2]]/(d_{x_0} + \mathfrak{m}^n)$ by Lemma 6.1.2, we get a ring homomorphism $W[[t_1, t_2]]/(f_{p^{-1}x_0}) \to W[[t_1, t_2]]/(d_{x_0})$. However, d_{x_0} is coprime to $f_{p^{-1}x_0}$ by Theorem 6.2.2, a contradiction. Hence $x_{\mathcal{D}}$ doesn't factor through the multiplication-by-p morphism of $X_{\mathcal{D}}$.

Lemma 4.1.5 and Corollary 4.1.6 imply that $\ker(x_{\mathcal{D}})$ is a cyclic group scheme since $\mathcal{D}$ is an integral noetherian scheme which is also separated and flat over W. Hence there exists a natural morphism from

Spec $W[\![t_1,t_2]\!]/d_{x_0}$ to Spec $W[\![t_1,t_2]\!]/\mathcal{I}^{\mathrm{cyc}}(x_0)$. Therefore, we conclude that $\mathcal{I}^{\mathrm{cyc}}(x_0)\subset(d_{x_0})\subset W[\![t_1,t_2]\!]$. This shows that the closed immersion $\mathcal{D}(x_0)\to\mathcal{Z}^{\sharp}(x_0)$ decomposes as

$$\mathcal{D}(x_0)\to\mathcal{N}_0(N)\to\mathcal{Z}^{\sharp}(x_0).$$

Therefore, we have an inclusion of ideals $(f_{x_0})\subset\mathcal{I}^{\mathrm{cyc}}(x_0)\subset(d_{x_0})\in W[\![t_1,t_2]\!]$. Theorem 6.6.1 of [Katz and Mazur 1985] (see also their Case II of 5.3.2.1) asserts that $W[\![t_1,t_2]\!]/\mathcal{I}^{\mathrm{cyc}}(x_0)$ is a 2-dimensional regular local ring. Recall that we have already proved in Theorem 6.2.2 that $W[\![t_1,t_2]\!]/d_{x_0}$ is also a regular local ring. Hence we must have $\mathcal{I}^{\mathrm{cyc}}(x_0)=(d_{x_0})$, i.e., $\mathcal{D}(x_0)\simeq\mathcal{N}_0(N)$. □

6.2.A. *Special Fibers.* In this part we use the identification $\mathcal{N}_0(N)\xrightarrow{\sim}\mathcal{D}(x_0)$ to explicitly describe the special fiber of the local ring $\mathcal{N}_0(N)$. The main results of this part will not be used in the following calculations, so readers can skip on first reading.

Let $\mathfrak{a}=(t_1,t_2)\subset W[\![t_1,t_2]\!]$. Let $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\mathbb{F}[\![t_1,t_2]\!]$. Let $A_n=W[\![t_1,t_2]\!]/\mathfrak{a}^n$ and $R_n=\mathbb{F}[\![t_1,t_2]\!]/\bar{\mathfrak{a}}^n$, with $A_0=W[\![t_1,t_2]\!]$ and $R_0=\mathbb{F}[\![t_1,t_2]\!]$. Equip each A_n with a morphism σ which extends the Frobenius morphism on W and maps t_1 to t_1^p, t_2 to t_2^p. Then A_n is a frame for R_n in the sense of [Zink 2001, Definition 1]. For any $n\geq 0$, let (M,M_1,Φ) be an A_n-window in the sense of [Zink 2001, Definition 2]. Since $\Phi(M_1)\subset p\cdot M$ and p is not a zero-divisor in A_n, we define $\Phi_1:M_1\to M$ to be $p^{-1}\Phi$. The morphism Φ_1 is σ-linear and induces an isomorphism $\Phi_1^{\sigma}:M_1^{\sigma}\to M$ because both sides are free A_n-modules of the same rank and Φ_1^{σ} is surjective by the definition of windows [Zink 2001, Definition 2(ii)]. Let α be the injective A_n-morphism

$$\alpha:M_1\hookrightarrow M\xrightarrow{(\Phi_1^{\sigma})^{-1}}M_1^{\sigma}.$$

Theorem 6.2.4. *For any $n\geq 0$, we have the category equivalences*

$$\{A_n\text{-window }(M,M_1,\Phi)\}\xleftrightarrow{\sim}\{\text{formal }p\text{-divisible groups over }R_n\}.$$

Moreover, both these two categories are equivalent to the category

$$\{\text{pairs }(M_1,\alpha:M_1\to M_1^{\sigma})\text{ such that Coker}(\alpha)\text{ is a free }R_n\text{-module}\},$$

where the functor from A_n-windows (M,M_1,Φ) to pairs $(M_1,\alpha:M_1\to M_1^{\sigma})$ is given by the constructions above.

Proof. This is proved in [Zink 2001, Theorem 4]. □

Let $((\overline{X},\bar{\rho},\bar{\lambda}),(\overline{X}',\bar{\rho}',\bar{\lambda}'))$ be the universal object in $\mathcal{N}(\mathbb{F}[\![t_1,t_2]\!])$, i.e., the base change of the universal object $((X^{\mathrm{univ}},\rho^{\mathrm{univ}},\lambda^{\mathrm{univ}}),(X'^{\mathrm{univ}},\rho'^{\mathrm{univ}},\lambda'^{\mathrm{univ}}))$ over $W[\![t_1,t_2]\!]$ to $\mathbb{F}[\![t_1,t_2]\!]$. The corresponding window can be described as follows. Let $\mathbb{D}=W\cdot e+W\cdot f$ be the Dieudonne module of $\mathbb{X}$, where $Fe=Ve=f$, $Ff=Vf=p\cdot e$ (F and V are the Frobenius and Verschiebung morphisms on $\mathbb{D}$). Then we let $M=\mathbb{D}\otimes_W W[\![t]\!]$ and $M_1=(W\cdot f+pW\cdot e)\otimes_W W[\![t]\!]$. We still use σ to denote the Frobenius action on $W[\![t]\!]$ which sends t to t^p and extends the Frobenius morphism on W. Let Φ be the σ-linear map from M to M such that $\Phi(e\otimes 1)=t\cdot(e\otimes 1)+f\otimes 1$, $\Phi(f\otimes 1)=p\cdot(e\otimes 1)$. Then (M,M_1,Φ) is the $W[\![t]\!]$-window corresponding to the universal deformation of $\mathbb{X}$ over $\mathbb{F}[\![t]\!]$ (see [Zink 2002, (86)]). Let (M', M_1', Φ') be the corresponding window for $\mathbb{X}'$. Then the $W[\![t_1, t_2]\!]$-window corresponding to the universal deformation of $\mathbb{X}\times_{\mathbb{F}}\mathbb{X}'$ over $\mathbb{F}[\![t_1, t_2]\!]$ is given by $(M\oplus M', M_1\oplus M_1', \Phi\oplus\Phi')$, or $(M_1\oplus M_1', \alpha)$, where under the basis $\{p\cdot(e\otimes 1), f\otimes 1, p(e'\otimes 1), f'\otimes 1\}$, the map α is given by the matrix

$$\alpha=\begin{pmatrix} & 1 & & \\ p & -t_1 & & \\ & & & 1\\ & & p & -t_2\end{pmatrix}.$$

Any quasi-isogeny $x\in\mathbb{B}$ induces the endomorphism

$$\mathbb{D}(x)=\begin{pmatrix} & & \sigma(a) & -\sigma(b)\\ & & -p\cdot b & a\\ a & \sigma(b) & & \\ p\cdot b & \sigma(a) & & \end{pmatrix}$$

of the window $M_1\oplus M_1'$ of $\mathbb{X}\times_{\mathbb{F}}\mathbb{X}'$ under the basis $\{p\cdot e, f, p\cdot e', f'\}$, where $a, b\in\mathbb{Q}_{p^2}$.

Let $M_1(n)=M_1\otimes_{A_0}A_n$, $M_1'(n)=M_1'\otimes_{A_0}A_n$, $\alpha(n)=\alpha\otimes_{A_0}A_n$. By Theorem 6.2.4, a quasi-isogeny x lifts to an isogeny over R_n if and only if there exists $x(n)\in\mathrm{End}\big((M_1(n)\oplus M_1'(n), \alpha(n))\big)$ such that $x(1)=\mathbb{D}(x)$ and the following diagram commutes:

$$\begin{array}{ccc} M_1(n)\oplus M_1'(n) & \xrightarrow{\alpha(n)} & M_1(n)^\sigma\oplus M_1'(n)^\sigma\\ \downarrow{\scriptstyle x(n)} & & \downarrow{\scriptstyle \sigma(x(n))}\\ M_1(n)\oplus M_1'(n) & \xrightarrow{\alpha(n)} & M_1(n)^\sigma\oplus M_1'(n)^\sigma \end{array}$$

Under the basis $\{p\cdot(e\otimes 1), f\otimes 1, p(e'\otimes 1), f'\otimes 1\}$, the morphism $x(n)$ has the form

$$x(n)=\begin{pmatrix} A(n) & Y(n)\\ X(n) & B(n)\end{pmatrix},$$

where $X(n), Y(n), A(n), B(n)\in\mathrm{M}_2(A_n)$ satisfy the equations,

$$X(n)=p^{-1}U'(t_2)\cdot\sigma(X(n))\cdot U(t_1),\quad Y(n)=p^{-1}U'(t_1)\cdot\sigma(Y(n))\cdot U(t_2),$$
$$A(n)=p^{-1}U'(t_1)\cdot\sigma(A(n))\cdot U(t_1),\quad B(n)=p^{-1}U'(t_2)\cdot\sigma(B(n))\cdot U(t_2),$$

where

$$U(t)=\begin{pmatrix} & 1\\ p & -t\end{pmatrix}\quad\text{and}\quad U'(t)=\begin{pmatrix} t & 1\\ p & \end{pmatrix}.$$

Since $A(1)=B(1)=0$, we conclude (by comparing degrees of t_1 and t_2) that $A(n)=B(n)=0$.

For any $A\in\mathrm{M}_2(A_n\otimes_{\mathbb{Z}}\mathbb{Q})$, the matrix $\sigma(A)$ is a well-defined element in $\mathrm{M}_2(A_{pn}\otimes_{\mathbb{Z}}\mathbb{Q})$. Therefore, starting from $X(1)$ and $Y(1)$, we can define successively

$$X(p^{l+1})=p^{-1}U'(t_2)\cdot\sigma(X(p^l))\cdot U(t_1),\quad Y(p^{l+1})=p^{-1}U'(t_1)\cdot\sigma(Y(p^l))\cdot U(t_2).\tag{22}$$

Since the local ring $\mathcal{O}_{\mathcal{Z}(x)}$ only depends (up to noncanonical isomorphisms) on the valuation of x, we take the following specific choice of x and $\mathbb{D}(x)$ in the following computations:

- When $\mathrm{ord}_p(x^\vee \circ x) = 2k$ for some $k \geq 0$, we take

$$X(1) = Y(1) = \begin{pmatrix} p^k & \\ & p^k \end{pmatrix}.$$

By computation based on the recursion formula (22), it turns out that for any $l \geq 1$,

$$X(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{(p^{l-1}-1)/(p-1)}\left(t_2^{p^{l-1}} - t_1^{p^{l-1}}\right) \\ 0 & 0 \end{pmatrix} + p \cdot C\right),$$
$$Y(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{(p^{l-1}-1)/(p-1)}\left(t_2^{p^{l-1}} - t_1^{p^{l-1}}\right) \\ 0 & 0 \end{pmatrix} + p \cdot D\right)$$

for some matrices $C, D \in \mathrm{M}_2(A_{p^l})$.

- When $\mathrm{ord}_p(x^\vee \circ x) = 2k+1$ for some $k \geq 0$, we take

$$X(1) = -Y(1) = \begin{pmatrix} & p^k \\ p^{k+1} & \end{pmatrix}.$$

By computation based on the recursion formula (22), it turns out that for any $l \geq 1$,

$$X(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l}(t_1t_2)^{(p^{l}-1)/(p-1)} \\ 0 & 0 \end{pmatrix} + p \cdot C'\right),$$
$$Y(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{(p^{l}-1)/(p-1)} \\ 0 & 0 \end{pmatrix} + p \cdot D'\right)$$

for some matrices $C', D' \in \mathrm{M}_2(A_{p^l})$.

Proposition 6.2.5. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation n and induces $X(1)$ and $Y(1)$ as described above. Let $f_x \in W[\![t_1, t_2]\!]$ be the element cutting out $\mathcal{Z}(x)$. Then*

$$\bar{f}_x := f_x \bmod p = (\text{unit}) \times \begin{cases} (t_1t_2)^{(p^{n/2}-1)/(p-1)}\left(t_2^{p^{n/2}} - t_1^{p^{n/2}}\right) \bmod (t_1, t_2)^{p^{n/2+1}} & \text{when } n \text{ is even,} \\ (t_1t_2)^{(p^{(n+1)/2}-1)/(p-1)} \bmod (t_1, t_2)^{p^{(n+1)/2}} & \text{when } n \text{ is odd.} \end{cases} \tag{23}$$

Proof. By the above formula for $X(p^l)$ and $Y(p^l)$, we can conclude that x can be lifted to an isogeny over $R_{p^{[n/2]}}$ but not over $R_{p^{[n/2]+1}}$. Then the formula for $X(p^{[n/2]+1})$ and $Y(p^{[n/2]+1})$ imply (23). □

Theorem 6.2.6. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation n and induces $X(1)$ and $Y(1)$ as described above. Let $f_x \in W[\![t_1, t_2]\!]$ be the element cutting out $\mathcal{Z}(x)$. Then $\bar{f}_x$ is divisible by*

$$t_1 - t_2^{p^a}, \quad t_1^{p^a} - t_2 \quad \text{for } 0 \leq a \leq n \text{ and } a \equiv n \bmod 2.$$

Moreover, $\bar{f}_x$ has no other irreducible factors and the multiplicity of $t_1 - t_2^{p^a}$, $t_1^{p^a} - t_2$ in $\bar{f}_x$ is $p^{(n-a)/2}$.

Proof. We first prove that $t_1^{p^{k_1}} - t_2^{p^{k_2}}$ divides $\bar{f}_x$, where $k_1, k_2 \geq 0$ and $k_1 + k_2 = n$. We prove this by showing that $X(p^l), Y(p^l) \bmod (t_1^{p^{k_1}} - t_2^{p^{k_2}}) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}}))$ for any $l \geq 0$.

- When $n = 2k$ is even, the recursion formula (22) implies that,

$$X(p^l) = p^{k-l} U'(t_2) U'(t_2^p) \cdots U'(t_2^{p^{l-1}}) U(t_1^{p^{l-1}}) \cdots U(t_1^p) U(t_1),$$
$$Y(p^l) = p^{k-l} U'(t_1) U'(t_1^p) \cdots U'(t_1^{p^{l-1}}) U(t_2^{p^{l-1}}) \cdots U(t_2^p) U(t_2).$$

Let's assume first that $k_1 \le k_2$. For any $0 \le t \le l - k_1$, we have the relation $t_2^{p^{l-t}} = t_1^{p^{k_2-k_1+l-t}}$. Hence

$$U'(t_2^{p^{l-t}}) U(t_1^{p^{k_2-k_1+l-t}}) = p \cdot I_2.$$

Moreover, when $1 \le t \le k_2 - k_1$, we have $t_2^{p^{l-t}} = t_1^{p^{k_2-k_1+l-t}} = 0$. Hence $U(t_2^{p^{l-t}}) = U(0)$ and we get

$$X(p^l) = U'(t_2) U'(t_2^p) \cdots U'(t_2^{p^{k_2-1}}) U(t_1^{p^{k_1-1}}) \cdots U(t_1^p) U(t_1) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}})),$$
$$Y(p^l) = U'(t_1) U'(t_1^p) \cdots U'(t_1^{p^{k_1-1}}) U(t_2^{p^{k_2-1}}) \cdots U(t_2^p) U(t_2) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}})).$$

The proof of the case $k_1 > k_2$ is similar and we get the same formula for $X(p^l)$ and $Y(p^l)$ as above. Therefore, we conclude that $t_1^{p^{k_1}} - t_2^{p^{k_2}}$ divides $\bar{f}_x$ when $k_1, k_2 \ge 0$ and $k_1 + k_2 = 2k$ by Theorem 6.2.4. Hence $\bar{f}_x$ is divisible by the polynomial

$$(t_1 - t_2)^{p^k} \cdot \prod_{a=1}^{k} \big((t_1 - t_2^{p^{2a}})(t_2 - t_1^{p^{2a}})\big)^{p^{k-a}}.$$

We also know that

$$(t_1 - t_2)^{p^k} \cdot \prod_{a=1}^{k} \big((t_1 - t_2^{p^{2a}})(t_2 - t_1^{p^{2a}})\big)^{p^{k-a}} \equiv (t_1 t_2)^{(p^k-1)/(p-1)} \cdot (t_2^{p^k} - t_1^{p^k}) \bmod (t_1, t_2)^{p^{k+1}}.$$

The lemma follows by comparing this formula with (23).

- When $n = 2k+1$ is odd, the recursion formula (22) implies

$$X(p^l) = p^{k-l} U'(t_2) U'(t_2^p) \cdots U'(t_2^{p^{l-1}}) \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} U(t_1^{p^{l-1}}) \cdots U(t_1^p) U(t_1),$$
$$Y(p^l) = p^{k-l} U'(t_1) U'(t_1^p) \cdots U'(t_1^{p^{l-1}}) \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} U(t_2^{p^{l-1}}) \cdots U(t_2^p) U(t_2).$$

Let's assume $k_1 < k_2$. For any $0 \le t \le l - k_1$, we have the relation $t_2^{p^{l-t}} = t_1^{p^{k_2-k_1+l-t}}$, and hence $U'(t_2^{p^{l-t}}) U(t_1^{p^{k_2-k_1+l-t}}) = p \cdot I_2$. Moreover, when $1 \le t \le k_2 - k_1$, we have $t_2^{p^{l-t}} = t_1^{p^{k_2-k_1+l-t}} = 0$. So $U(t_2^{p^{l-t}}) = U(0)$, and we get

$$X(p^l) = U'(t_2) U'(t_2^p) \cdots U'(t_2^{p^{k_2-1}}) U(t_1^{p^{k_1-1}}) \cdots U(t_1^p) U(t_1) \in \mathrm{M}_2\big(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}})\big),$$
$$Y(p^l) = U'(t_1) U'(t_1^p) \cdots U'(t_1^{p^{k_1-1}}) U(t_2^{p^{k_2-1}}) \cdots U(t_2^p) U(t_2) \in \mathrm{M}_2\big(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}})\big).$$

Therefore we conclude that $t_1^{p^{k_1}} - t_2^{p^{k_2}}$ divides $\bar{f}_x$ when $k_1, k_2 \ge 0$ and $k_1 + k_2 = 2k+1$ by Theorem 6.2.4.

Hence $\bar{f}_x$ is divisible by the polynomial

$$\prod_{a=0}^{k}\big((t_1 - t_2^{p^{2a+1}})(t_2 - t_1^{p^{2a+1}})\big)^{p^{k-a}}.$$

We also know that $\prod_{a=0}^{k}\big((t_1 - t_2^{p^{2a+1}})(t_2 - t_1^{p^{2a+1}})\big)^{p^{k-a}} \equiv (t_1t_2)^{(p^{k+1}-1)/(p-1)} \bmod (t_1, t_2)^{p^{k+1}}$. The lemma follows by comparing this formula with (23). □

Corollary 6.2.7. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation $n \geq 1$. Let $\mathcal{Z}(x)_p$ be special fiber of $\mathcal{Z}(x)$. Then*

$$\mathcal{Z}(x)_p \simeq \operatorname{Spf} \mathbb{F}[\![t_1, t_2]\!] \Big/ \Big(\prod_{\substack{a+b=n \\ a,b \geq 0}} (t_1^{p^a} - t_2^{p^b}) \Big).$$

Let $\mathcal{D}(x)_p$ (resp. $\mathcal{N}_0(N)_p$) be the base change of $\mathcal{D}(x)$ (resp. $\mathcal{N}_0(N)$) to $\mathbb{F}[\![t_1, t_2]\!]$. Then

$$\mathcal{N}_0(N)_p \simeq \mathcal{D}(x)_p \simeq \operatorname{Spf} \mathbb{F}[\![t_1, t_2]\!] \Big/ \Big((t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{\substack{a+b=n \\ a,b \geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \Big).$$

Proof. The statement for $\mathcal{Z}(x)_p$ follows from Theorem 6.2.6. The statement for $\mathcal{D}(x)_p$ follows from the definition of difference divisors. □

Remark 6.2.8. The same formula has been proved in [Katz and Mazur 1985, Theorems 13.4.6 and 13.4.7] by a totally different method.

6.3. ***Local arithmetic intersection numbers.*** Now we give the definition of the local arithmetic intersection numbers.

Definition 6.3.1. For any rank-3 lattice $L \subset \mathbb{B}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2, x_3\}$ of L. Let $\mathcal{O}_{\mathcal{Z}^\sharp(x_i)}$ be the structure sheaf of the special cycle $\mathcal{Z}^\sharp(x_i)$. Let $\mathcal{O}_{\mathcal{N}}$ be the structure sheaf of the formal scheme $\mathcal{N}$. Let $- \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} -$ be the derived tensor product functor in the derived category of coherent sheaves on $\mathcal{N}$. Define the local arithmetic intersection number of L on $\mathcal{N}$ to be

$$\operatorname{Int}^\sharp(L) = \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}).$$

This number is finite and independent of the choice of the basis $\{x_i\}_{i=1}^3$ of L because of the following result.

Lemma 6.3.2. *Let $x, y \in \mathbb{B}$ be linearly independent. Then the tor sheaves $\underline{\operatorname{Tor}}_i^{\mathcal{O}_{\mathcal{N}}}(\mathcal{O}_{\mathcal{Z}^\sharp(x)}, \mathcal{O}_{\mathcal{Z}^\sharp(y)})$ vanish for all $i \geq 1$. In particular,*

$$\mathcal{O}_{\mathcal{Z}^\sharp(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(y)} = \mathcal{O}_{\mathcal{Z}^\sharp(x)} \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(y)}.$$

Moreover, the same formula holds if $\mathcal{Z}^\sharp(x)$ or $\mathcal{Z}^\sharp(y)$ (or both) are replaced by $\mathcal{D}(x)$ or $\mathcal{D}(y)$, respectively.

Let $L \subset \mathbb{B}$ be an integral quadratic lattice of rank 3 over $\mathbb{Z}_p$ with basis $\{x_1, x_2, x_3\}$. Then the derived tensor product $\mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}$ is independent of the choice of the basis.

Proof. This is proved in [Terstiege 2011, Lemma 4.1 and Proposition 4.2]. □

Now let's come to the local arithmetic intersection numbers on $\mathcal{N}_0(N)$. For a fixed N-isogeny x_0 of $\mathbb{X}$, recall that we have defined the space of quasi-isogenies of $\mathbb{X}$ special to x_0 (see Definition 5.2.4) to be those $x \in \mathbb{B}$ such that

$$x \circ x_0^\vee + x_0 \circ x^\vee = 0.$$

Recall that we use $\mathbb{W}$ to denote this space.

Definition 6.3.3. For any rank-2 lattice $M \subset \mathbb{W}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of M. Let $\mathcal{O}_{\mathcal{Z}(x_i)}$ be the structure sheaf of the special cycle $\mathcal{Z}(x_i)$. Let $\mathcal{O}_{\mathcal{N}_0(N)}$ be the structure sheaf of the formal scheme $\mathcal{N}_0(N)$. Let $-\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}}-$ be the derived tensor product functor in the derived category of coherent sheaves on $\mathcal{N}_0(N)$. Define the local arithmetic intersection number of M on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}).$$

This number is independent of the choice of the basis $\{x_1, x_2\}$ of M because of Lemma 6.3.2 and Theorem 6.2.3. We relate it to the derivative of the local density of the quadratic lattice M with level N. The following theorem is the starting point of our calculation.

Theorem 6.3.4. *For any prime number p, let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3. Then*

$$\mathrm{Int}^\sharp(L) = \partial\,\mathrm{Den}(L).$$

Proof. In [Gross and Keating 1993, §4], the Gross–Keating invariants (a_1, a_2, a_3) of the rank-3 quadratic lattice L are defined. Then the local arithmetic intersection number $\mathrm{Int}^\sharp(L)$ is computed explicitly in terms of these invariants (see also [Rapoport 2007, Theorem 1.1]). In [Wedhorn 2007, §2.11], the local density $\mathrm{Den}^+(X, L)$ is also expressed explicitly in terms of the Gross–Keating invariants (a_1, a_2, a_3), hence the derived local density $\partial\,\mathrm{Den}^+(L)$. The theorem is proved by comparing the expressions of both sides in terms of (a_1, a_2, a_3) (see [Wedhorn 2007, §2.16]). See also [Li and Zhang 2022] for a recent new proof when p is odd. □

6.4. ***Difference formula of the local arithmetic intersection numbers.*** Fix an N-isogeny $x_0 \in \mathrm{End}(\mathbb{X})$, and recall that $\mathbb{W} = \{x_0\}^\perp \to \mathbb{B}$.

Theorem 6.4.1. *For any rank-2 lattice $M \subset \mathbb{W}$, we have the identity*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).$$

Proof. Let $\{x_1, x_2\}$ be a basis of M. By Lemma 6.3.2 and the isomorphism $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$, we have the following isomorphism as complexes of coherent sheaves on $\mathcal{N}$:

$$\begin{aligned}\mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{D}(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}.\end{aligned}$$

When $\nu_p(N)=0$ or 1, the difference divisor $\mathcal{D}(x_0)$ is just $\mathcal{Z}^\sharp(x_0)$. Hence $\mathrm{Int}_{\mathcal{N}_0(N)}(M)=\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot x_0)$ and $\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot p^{-1}x_0)=0$ since $p^{-1}x_0$ is not integral, and therefore

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M)=\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot x_0)-\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot p^{-1}x_0).$$

When $\nu_p(N)\geq 2$, we have the exact sequence

$$0\to\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\xrightarrow{\times d_{x_0}}\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\to\mathcal{O}_{\mathcal{D}(x_0)}\simeq\mathcal{O}_{\mathcal{N}_0(N)}\to 0.$$

Tensoring the above exact sequence with the complex $\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}$ in the derived category of coherent sheaves on $\mathcal{N}$, we get an exact triangle

$$\begin{aligned}\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}&\to\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}\\&\to\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}\to .\end{aligned}$$

Hence we have the identity

$$\begin{aligned}&\chi(\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})\\&\quad=\chi(\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})+\chi(\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}).\end{aligned}$$

We already know that $\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)}\simeq\mathcal{O}_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}}\mathcal{O}_{\mathcal{Z}(x_2)}$ since $\mathcal{N}_0(N)\simeq\mathcal{D}(x_0)$. Hence

$$\begin{aligned}\mathrm{Int}_{\mathcal{N}_0(N)}(M)&=\chi(\mathcal{O}_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{D}(x_0)}}\mathcal{O}_{\mathcal{Z}(x_2)})=\chi(\mathcal{O}_{\mathcal{N}_0(N)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})\\&=\chi(\mathcal{O}_{\mathcal{Z}^\sharp(x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})-\chi(\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}^\sharp(x_2)})\\&=\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot x_0)-\mathrm{Int}^\sharp(M\oplus\mathbb{Z}_p\cdot p^{-1}x_0).\end{aligned}$$

□

7. Difference formula at the analytic side

Let p be a prime number. Let F be a nonarchimedean local field of residue characteristic p, with ring of integers $\mathcal{O}_F$, residue field $\kappa=\mathbb{F}_q$ of size q, and uniformizer π.

7.1. ***Primitive decomposition.*** Let $N\in F$. Recall that we use $(\langle N\rangle, q_{\langle N\rangle})$ to denote the rank-1 quadratic lattice over $\mathcal{O}_F$ with an $\mathcal{O}_F$-generator l_N such that $q_{\langle N\rangle}(l_N)=N$. Then $\langle N\rangle$ is an integral quadratic lattice if and only if $N\in\mathcal{O}_F$. Let $n=\nu_\pi(N)$. All the rank-1 integral quadratic lattices L' containing $\langle N\rangle$ have the form

$$L'=\pi^{-i}\langle N\rangle\simeq\langle\pi^{-2i}N\rangle\quad\text{for } i=0,1,\dots,\left[\frac{n}{2}\right].$$

Let H be a self-dual quadratic $\mathcal{O}_F$-lattice of finite rank. Since $q_H(x)\in\mathcal{O}_F$ for every $x\in H$, Lemma 2.2.5 gives the decomposition

$$\mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F)=\bigsqcup_{i=0}^{[n/2]}\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F).$$

Now for every $0\leq i\leq[n/2]$, we pick an arbitrary $\phi\in\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$ and consider the following sublattice of H:

$$H(\phi)\coloneqq\{x\in H:(x,\phi(l_N))=0\}.$$

Lemma 7.1.1. *The isometric class of $H(\phi)$ is independent of the choice of $\phi \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$.*

Proof. Let $\phi' \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$ be another element. The homomorphisms ϕ and ϕ' are totally determined by $x := \phi(l_{\pi^{-2i}N})$ and $x' := \phi'(l_{\pi^{-2i}N})$. The fact that ϕ and ϕ' are primitive implies that $x \notin \pi \cdot H$ and $x' \notin \pi \cdot H$. Therefore,

$$(x, H) = \mathcal{O}_F, \quad (x', H) = \mathcal{O}_F,$$

where we use $(\cdot\,,\cdot)$ to denote the associated bilinear form on H. Since H is self-dual, then by [Morin-Strom 1979, Theorem 5.3], there exists $\varphi \in \mathrm{O}(H)(\mathcal{O}_F)$ such that $\varphi(x) = x'$. The homomorphism φ also induces an isometry between $H(\phi)$ and $H(\phi')$ because $H(\phi) = x^{\perp} \cap H$ and $H(\phi') = x'^{\perp} \cap H$. □

Let $N \in \mathcal{O}_F$ be an element of valuation n. For every $0 \le i \le [n/2]$ and $\phi \in \mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$, we use $H(N, i)$ to denote the quadratic lattice $H(\phi)$.

Example 7.1.2. Let $N \in \mathcal{O}_F$ have valuation n. When $k > 4$, we have an orthogonal decomposition

$$H_k^{\varepsilon} \simeq H_4^{+} \oplus H_{k-4}^{\varepsilon}.$$

Recall that the symbol H_k^{ε} is understood in the following way: when p is odd, k can be any positive integer, and $\varepsilon \in \{\pm 1\}$ is arbitrary; when $p = 2$, k is even and $\varepsilon = +1$. The lattice $\mathrm{M}_2(\mathcal{O}_F)$ is equipped with the quadratic form induced by the determinant; it is self-dual and $\chi_F(\mathrm{M}_2(\mathcal{O}_F)) = 1$, and hence we can view $\mathrm{M}_2(\mathcal{O}_F)$ as a model lattice for H_4^{+}.

Let's consider the element $\phi \in \mathrm{PRep}_{H_k^{\varepsilon},\langle\pi^{-2i}N\rangle}(\mathcal{O}_F)$ given by

$$\phi_i : \langle\pi^{-2i}N\rangle \to \mathrm{M}_2(\mathcal{O}_F) \simeq H_4^{+} \hookrightarrow H_k^{\varepsilon}, \quad l_{\pi^{-2i}N} \mapsto \begin{pmatrix} \pi^{-2i}N & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding element in $\mathrm{Rep}_{H_k^{\varepsilon},\langle N\rangle}(\mathcal{O}_F)$ sends

$$l_N \mapsto \begin{pmatrix} \pi^{-i}N & 0 \\ 0 & \pi^{i} \end{pmatrix}.$$

Lemma 7.1.1 implies that the following quadratic lattices are isometric:

$$H_k^{\varepsilon}(N, i) = H_k^{\varepsilon}(\phi_i) \simeq \phi_i(l_{\pi^{-2i}N})^{\perp} \oplus H_{k-4}^{\varepsilon},$$

where $\phi_i(l_{\pi^{-2i}N})^{\perp}$ is the space of elements in $\mathrm{M}_2(\mathcal{O}_F)$ that are orthogonal to $\phi_i(l_{\pi^{-2i}N})$, it can be described explicitly as

$$\phi_i(l_{\pi^{-2i}N})^{\perp} = \left\{ x = \begin{pmatrix} -\pi^{-2i}Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathcal{O}_F \right\}.$$

It is exactly the lattice $\Delta_F(\pi^{-2i}N)$ defined in Example 2.1.1. Therefore

$$H_k^{\varepsilon}(N, i) \simeq \Delta_F(\pi^{-2i}N) \oplus H_{k-4}^{\varepsilon}.$$

7.2. ***Difference formula of local densities.***

Theorem 7.2.1. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice of finite rank k. Let M be an integral quadratic $\mathcal{O}_F$-lattice of finite rank r. Let $N \in \mathcal{O}_F$ be an element of valuation n. Then*

$$\mathrm{Den}(H, M \oplus \langle N\rangle) = \sum_{i=0}^{[n/2]} q^{(2-k+r)i} \cdot \mathrm{Pden}(H, \langle \pi^{-2i} N\rangle) \cdot \mathrm{Den}(H(N, i), M).$$

The proof of this theorem is based on the following lemmas.

Lemma 7.2.2. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice. Let $N \in \mathcal{O}_F$ be an element of valuation n. Then there is a bijective map*

$$D : \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d) \xrightarrow{\sim} \bigsqcup_{i=0}^{[n/2]} \bigsqcup_{\bar{x}\in\mathcal{O}_F/\pi^i} \mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$$

when the positive integer d is large enough.

Proof. Let l_N be a generator of the rank-1 $\mathcal{O}_F$-module $\langle N\rangle$ such that $q_{\langle N\rangle}(l_N) = N$. Any f in $\mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d)$ is determined by $f(\overline{l_N}) \in H/\pi^d H$. There is a natural filtration on $H/\pi^d H$ given as

$$0 \subset \pi^{d-1}H/\pi^d H \subset \pi^{d-2}H/\pi^d H \subset \cdots \subset \pi^2 H/\pi^d H \subset \pi H/\pi^d H \subset H/\pi^d H.$$

Let i be the minimal integer such that $f(\overline{l_N}) \in \pi^i H/\pi^d H$. Then $0 \le i \le [n/2]$ since $\nu_\pi(N) = n$. So there exists $l \in H$ such that $f(\overline{l_N}) = \overline{\pi^i l} \in \pi^i H/\pi^d H$; the image of l in $H/\pi^{d-i}H$ is uniquely determined by f. Let q be the quadratic form on H. Then

$$N \bmod pi^d = \overline{q_{\langle N\rangle}}(\overline{l_N}) = \bar{q}(f(\overline{l_N})) = \pi^{2i}\bar{q}(\bar{l}) = \overline{\pi^{2i}q(l)}.$$

Hence $\overline{\pi^{2i}q(l)} \equiv \pi^{-2i}N \bmod \pi^{d-2i}$. Therefore there exists $x \in \mathcal{O}_F$ such that $q(l) = \pi^{-2i}N + \pi^{d-2i}x$. Next we show that $\bar{x} \in \mathcal{O}_F/\pi^i$ is independent of the choice of $l \in H$ when d is large enough. Suppose l' is another element of H such that $f(\overline{l_N}) = \overline{\pi^i l'}$. Then there exists $\delta \in H$ such that $l' - l = \pi^{d-i}\delta$. Therefore, when d is large enough,

$$q(l') = q(l + \pi^{d-i}\delta) = q(l) + \pi^{d-i}(l, \delta) + \pi^{2d-2i}q(\delta) \equiv q(l) \bmod \pi^{d-i}.$$

Suppose $q(l') = \pi^{-2i}N + \pi^{d-2i}x'$ for some $x' \in \mathcal{O}_F$. The above congruence between $q(l)$ and $q(l')$ implies $x' \equiv x \bmod \pi^i$. The above construction gives the homomorphism $D(f)$ in $\mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$ sending the generator $\overline{l_{\pi^{-2i}N+\pi^{d-2i}x}}$ of $\langle \pi^{-2i}N + \pi^{d-2i}x\rangle/\pi^{d-i}\langle \pi^{-2i}N + \pi^{d-2i}x\rangle$ to $\bar{l} \in H/\pi^{d-i}H$.

Now for any element $\varphi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, we consider the morphism

$$\tilde{\varphi} : \langle N\rangle/\pi^d\langle N\rangle \to H/\pi^d H, \quad \overline{l_N} \mapsto \pi^i\varphi(\overline{l_{\pi^{-2i}N+\pi^{d-2i}x}}).$$

Then $\tilde{\varphi} \in \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d)$ because $\bar{q}(\tilde{\varphi}(\overline{l_N})) = \overline{\pi^{2i}(\pi^{-2i}N + \pi^{d-2i}x)} = N \bmod \pi^d$. This construction gives the inverse map of D. ☐

Let M be an integral quadratic $\mathcal{O}_F$-lattice of finite rank. Let $N \in \mathcal{O}_F$ be an element of valuation n. Let $M^\sharp = M \oplus \langle N \rangle$ be a quadratic $\mathcal{O}_F$-lattice of one rank higher than M. For any positive integer d and any self-dual quadratic $\mathcal{O}_F$-lattice H, there is a natural restriction map

$$\mathrm{res} : \mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d) \to \mathrm{Rep}_{H,\langle N \rangle}(\mathcal{O}_F/\pi^d),$$

given by composing any element in the set $\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)$ with the natural inclusion of $\langle N \rangle/\pi^d\langle N \rangle$ in $M^\sharp/\pi^d M^\sharp$. The next lemma describes the fiber of the map $D \circ \mathrm{res}$.

Lemma 7.2.3. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice and M an integral quadratic $\mathcal{O}_F$-lattice of finite rank r. For $N \in \mathcal{O}_F$ an element of valuation n, let $M^\sharp = M \oplus \langle N \rangle$ be a quadratic $\mathcal{O}_F$-lattice of rank $r+1$. Let $0 \le i \le [n/2]$ be an integer. Given $\varphi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N + \pi^{d-2i}x \rangle}(\mathcal{O}_F/\pi^{d-i})$, for d large enough we have*

$$\#(D \circ \mathrm{res})^{-1}(\varphi) = q^{ir} \cdot \#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d).$$

Proof. Let f be an element in $\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)$ such that $D \circ \mathrm{res}(f) = \varphi$. By the proof of Lemma 7.2.2, there exists $l'_N \in H \setminus \pi H$ such that $f(\overline{l_N}) = \overline{\pi^i l'_N}$, and $q(l'_N) = \pi^{-2i}N$ when d is large enough.

Let $\{e_i\}_{i=1}^r$ be an $\mathcal{O}_F$-basis of M. Then f is determined by $\{x_i := f(\overline{e_i}) \in H/\pi^d H\}_{i=1}^r$. Therefore $(D \circ \mathrm{res})^{-1}(\varphi)$ can be described by the set

$$(D \circ \mathrm{res})^{-1}(\varphi) = \big\{(x_1, \dots, x_r) \in (H/\pi^d H)^r : (x_i, \overline{\pi^i l'_N}) = 0,\ (x_i, x_j) = (\overline{e_i}, \overline{e_j}) \text{ for } i \neq j, \\ \text{and } \bar{q}(x_i) = \overline{q_M}(\overline{e_i}) \text{ for every } i.\big\}. \quad (24)$$

Let L be the rank-1 sublattice of H generated by l'_N. We have the exact sequence

$$0 \to L \oplus H(N,i) \xrightarrow{\theta} H \to Q := H/L \oplus H(N,i) \to 0,$$

where θ is the natural inclusion map. After tensoring the above exact sequence with $\mathcal{O}_F/\pi^d$, we get the following exact sequence by the flatness of H over $\mathcal{O}_F$:

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_F}(Q, \mathcal{O}_F/\pi^d) \to L/\pi^d L \oplus H(N,i)/\pi^d H(N,i) \xrightarrow{\bar{\theta}} H/\pi^d H \to Q/\pi^d Q \to 0. \quad (25)$$

Claim. *Let $K = \big\{x \in H/\pi^d H : (x, \overline{\pi^i l'_N}) = 0\big\}$. When d is large enough, for every $\bar{x} \in K$ there exists $x' \in L$ and $x'' \in H(N,i)$ such that the image of $\overline{x'} + \overline{x''} \in L/\pi^d L \oplus H(N,i)/\pi^d H(N,i)$ under $\bar{\theta}$ in $H/\pi^d H$ is $\bar{x}$.*

Proof of the claim. We have the decomposition

$$x = x' + x''$$

in the quadratic space $V = H \otimes_{\mathcal{O}_F} F$, where $x' \in L_F := L \otimes_{\mathcal{O}_F} F$ and $x'' \in (L_F)^\perp \subset V$.

The fact that $\bar{x} \in K$ implies that $(x', l_N) = (x, l_N) \in (\pi^d)$. Therefore $x' \in (\pi^{d-n}) \cdot l_N \in L_F$. It turns out that $x' \in L \subset H$ when d is large enough, and hence $x' = x - x' \in H \cap \{l_N\}^\perp = H(N,i)$. □

We get the following description of the inverse image of the set $(D \circ \mathrm{res})^{-1}(\varphi)$ under $\Theta := \bar{\theta} \times \cdots \times \bar{\theta}$ by (24):

$$\Theta^{-1}((D \circ r)^{-1}(\varphi)) = (\pi^{d+i-n} L/\pi^d L)^r \times \mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d). \quad (26)$$

The claim implies that the map $\Theta^{-1}((D \circ \mathrm{res})^{-1}(\varphi)) \xrightarrow{\Theta} (D \circ \mathrm{res})^{-1}(\varphi)$ is surjective.

Now we compute $\#\ker(\Theta)$, which equals $(\#\ker(\theta))^r$ by definition. By the exact sequence (25), $\#\ker(\bar{\theta}) = \#\mathrm{Tor}^1_{\mathcal{O}_F}(Q, \mathcal{O}_F/\pi^d) = \#Q/\pi^d Q$. Therefore, when d is large enough, $Q/\pi^d Q = Q$. Since $l'_N \notin \pi H$, there exists $y \in H$ such that $(l'_N, y) = 1$. The existence of y implies the exact sequence

$$0 \to H(N,i) \xrightarrow{\theta} H \to L^\vee \to 0, \quad x \mapsto l(x) : v \in L \mapsto (x, v).$$

Therefore, $H \simeq L^\vee \oplus H(N,i)$ as $\mathcal{O}_F$-modules, and $Q \simeq L^\vee/L \simeq \pi^{2i-n}L/L$. Then, by (26),

$$\#(D \circ \mathrm{res})^{-1}(\varphi) = \frac{q^{r(n-i)}}{q^{r(n-2i)}} \cdot \#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d) = q^{ir} \cdot \#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d). \qquad \square$$

Proof of Theorem 7.2.1. By Lemmas 7.2.2 and 7.2.3, we only need to know the size of the set $\mathrm{PRep}_{H,\langle \pi^{-2i}N + \pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$ when $x \in \mathcal{O}_F$. We first show that when d is large enough,

$$\#\mathrm{PRep}_{H,\langle \pi^{-2i}N + \pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i}) = \#\mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})$$

holds for any $x \in \mathcal{O}_F$, because when d is large enough, we could find $c_x \in \mathcal{O}_F^\times$ such that $c_x^{-2} = 1 + \pi^d N^{-1} x \bmod \pi^{d-i}$; then for any element $l \in \mathrm{PRep}_{H,\langle \pi^{-2i}N + \pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, $c_x \cdot l \in \mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})$. Let $M^\sharp = M \obot \langle N\rangle$. We have

$$\begin{aligned}
\mathrm{Den}(H, M \obot \langle N\rangle) &= \lim_{d\to\infty} \frac{\#\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)}{q^{d(k(r+1)-(r+1)(r+2)/2)}} \\
&= \lim_{d\to\infty} \sum_{i=0}^{[n/2]} q^i \cdot \frac{\#\mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})}{q^{(d-i)(k-1)}} \cdot \frac{q^{ir}}{q^{i(k-1)}} \cdot \frac{\#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d)}{q^{d((k-1)r - r(r+1)/2)}} \\
&= \sum_{i=0}^{[n/2]} q^{(2-k+r)i} \cdot \mathrm{Pden}(H, \langle \pi^{-2i}N\rangle) \cdot \mathrm{Den}(H(N,i), M). \qquad \square
\end{aligned}$$

Remark 7.2.4. When p odd, it has been calculated explicitly (see [Li and Zhang 2022, (3.3.2.1)]) that for any $N \in \mathcal{O}_F$

$$\mathrm{Pden}(H_k^\varepsilon, \langle N\rangle) = \begin{cases} 1 - q^{1-k} & \text{when } k \text{ is odd and } \pi \mid N, \\ 1 + \varepsilon \chi_F(N) q^{(1-k)/2} & \text{when } k \text{ is odd and } \pi \nmid N, \\ (1 - \varepsilon q^{-k/2})(1 + \varepsilon q^{1-k/2}) & \text{when } k \text{ is even and } \pi \mid N, \\ 1 - \varepsilon q^{-k/2} & \text{when } k \text{ is even and } \pi \nmid N. \end{cases} \tag{27}$$

When $p = 2$, the same formula makes sense and holds true only in the case that k is even and $\varepsilon = +1$.

Definition 7.2.5. Let $N \in \mathcal{O}_F$. Let M be a quadratic lattice of rank $r \geq 2$ over $\mathcal{O}_F$. Define the local density of M with level N to be a polynomial $\mathrm{Den}_{\Delta_F(N)}(X, M)$ satisfying, for $m \geq 0$,

$$\mathrm{Den}_{\Delta_F(N)}(X, M)\big|_{X=q^{-m}} = \begin{cases} \dfrac{\mathrm{Den}(\Delta_F(N) \obot H^+_{2m+r-2}, M)}{\mathrm{Nor}^+(q^{-m}, r-1)} & \text{when } \pi \mid N, \\ \dfrac{\mathrm{Den}(\Delta_F(N) \obot H^+_{2m+r-2}, M)}{\mathrm{Nor}^{\chi_F(N)}(q^{-m}, r)} & \text{when } \pi \nmid N. \end{cases}$$

Moreover, if the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the self-dual lattice H_{r+2}^+, define the derived local density of M with level N to be

$$\partial\,\mathrm{Den}_{\Delta_F(N)}(M) = -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}_{\Delta_F(N)}(X, M).$$

Theorem 7.2.6. *Let $N \in \mathcal{O}_F$. Let M be a quadratic lattice of rank $r \geq 2$ over $\mathcal{O}_F$. Then we have*

$$\mathrm{Den}_{\Delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle).$$

Moreover, if the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the self-dual lattice H_{r+2}^+, then

$$\partial\,\mathrm{Den}_{\Delta_F(N)}(M) = \partial\,\mathrm{Den}(M \oplus \langle N\rangle) - \partial\,\mathrm{Den}(M \oplus \langle \pi^{-2}N\rangle).$$

Proof. Recall the definition of the polynomial $\mathrm{Nor}^{\varepsilon}(X, n)$ in Definition 2.2.6. We can verify immediately by formula (27) that, for any $x \in \mathcal{O}_F$,

$$\mathrm{Nor}^{+}(q^{-m}, r+1) = \begin{cases} \mathrm{Pden}(H_{2m+r+2}^{\varepsilon}, \langle x\rangle) \cdot \mathrm{Nor}^{\chi_F(x)}(q^{-m}, r) & \text{when } \pi \nmid x, \\ \mathrm{Pden}(H_{2m+r+2}^{\varepsilon}, \langle x\rangle) \cdot \mathrm{Nor}^{+}(q^{-m}, r-1) & \text{when } \pi \mid x. \end{cases}$$

Let $n = \nu_\pi(N)$. Theorem 7.2.1 and Example 7.1.2 imply the decomposition

$$\begin{aligned} \mathrm{Den}(H_{2m+r+2}^+, M \oplus \langle N\rangle) &= \sum_{i=0}^{[n/2]} q^{-2mi} \cdot \mathrm{Pden}(H_{2m+r+2}^+, \langle \pi^{-2i}N\rangle) \cdot \mathrm{Den}(H_{2m+r+2}^+(N, i), M) \\ &= \sum_{i=0}^{[n/2]} q^{-2mi} \cdot \mathrm{Pden}(H_{2m+r+2}^+, \langle \pi^{-2i}N\rangle) \cdot \mathrm{Den}(\Delta_F(\pi^{-2i}N) \oplus H_{2m+r-2}^+, M). \end{aligned}$$

By Definition 7.2.5, when p is odd, we have the formula

$$\mathrm{Den}(X, M \oplus \langle N\rangle) = \sum_{i=0}^{[n/2]} X^{2i} \cdot \mathrm{Den}_{\Delta_F(\pi^{-2i}N)}(X, M). \tag{28}$$

When $n = 0$ or 1, $\mathrm{Den}(X, M \oplus \langle N\rangle) = \mathrm{Den}_{\Delta_F(N)}(X, M)$ and $\mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle) = 0$ since $\pi^{-2}N$ is not in $\mathcal{O}_F$. Therefore $\mathrm{Den}_{\Delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle)$. When $n \geq 2$, $\mathrm{Den}_{\Delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle)$ follows from the formula (28).

The fact that the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the quadratic space H_{r+2}^+ implies that $\mathrm{Den}(1, M \oplus \langle N\rangle) = \mathrm{Den}(1, M \oplus \langle \pi^{-2}N\rangle) = 0$. The second formula in the theorem follows from the first one and the definitions of the symbols $\partial\,\mathrm{Den}_{\Delta_F(N)}$ and $\partial\,\mathrm{Den}$. □

Now we apply Theorem 7.2.6 to the case that we are interested in, i.e., $F = \mathbb{Q}_p$ and $r = 2$. Let $N \in \mathbb{Z}_p$. We get a difference formula of local density functions:

$$\mathrm{Den}_{\Delta_p(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle p^{-2}N\rangle). \tag{29}$$

Note that the lattice $M \oplus \langle N\rangle$ is a sublattice of $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$, which is the unique division quaternion algebra over $\mathbb{Q}_p$. Hence the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the quadratic space

$H_4^+\otimes\mathbb{Q}_p\simeq \mathrm{M}_2(\mathbb{Q}_p)$. Therefore, Theorem 7.2.6 implies the difference formula

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M)=\partial\,\mathrm{Den}(M\oplus\langle N\rangle)-\partial\,\mathrm{Den}(M\oplus\langle p^{-2}N\rangle) \tag{30}$$

of the derivatives of local densities.

7.3. *Examples.* Assume p is odd. In the following example, we compute an explicit example of local densities and compare our formulas with known formulas given in [Wedhorn 2007; Sankaran et al. 2023].

Example 7.3.1. Let $N=N_0$ be a positive integer with $\nu_p(N_0)=0$ or 1. Let M be a rank-2 $\mathbb{Z}_p$-lattice such that M is isometrically embedded into $\mathbb{W}$ and is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent to the matrix $\mathrm{diag}\{\varepsilon_1 p^2,\,\varepsilon_2 p^3\}$, where $\varepsilon_1,\varepsilon_2\in\mathbb{Z}_p^\times$. Let $N_k=p^{2k}N_0$, where N_0 is a positive integer with $\nu_p(N_0)=0$ or 1 and $k\geq 1$ is an integer. By the formula in [Wedhorn 2007, §2.11],

$$\begin{aligned}\mathrm{Den}(X,\,&M\oplus\langle N_k\rangle)\\ &=1+pX+(p+p^2)X^2+p^2X^3+p^2X^4-p^2X^{2k+1+\nu_p(N_0)}-p^2X^{2k+2+\nu_p(N_0)}\\ &\qquad-(p+p^2)X^{2k+3+\nu_p(N_0)}-pX^{2k+4+\nu_p(N_0)}-X^{2k+5+\nu_p(N_0)}\quad\text{when } k\geq 3.\end{aligned}$$

The formula (29) implies

$$\mathrm{Den}_{\Delta_p(N_k)}(X,M)=1+pX+(p^2+p-1)X^2+(p^2-p)X^3-pX^4-p^2X^4-p^2X^5-p^2X^6$$

when $k\geq 3$. Therefore, $\partial\,\mathrm{Den}_{\Delta_p(N_0)}(M)=2+4p+6p^2$ when $k\geq 3$.

We double-check our results by comparing with the formulas of local density given in [Yang 1998, Theorem 7.1]. The theorem implies that for a sufficiently large positive integer m,

$$\mathrm{Den}(\Delta_p(N_k)\oplus H_{2m}^+,\,M)=1+R_{1,k}(X)+R_{2,k}(X)\big|_{X=p^{-m}},$$

where

$$R_{1,k}(X)=\sum_{i=1}^{8}I_{1,i,k}(X)\quad\text{and}\quad R_{2,k}(X)=(1-p^{-1})\sum_{i=1}^{8}I_{2,i,k}(X)+p^{-1}I_{2,6,k}(X).$$

$I_{1,i,k}(X)$ and $I_{2,i,k}(X)$ are polynomials explicitly constructed at the beginning of Section 7 of [Yang 1998]. In our case, when $k\geq 3$,

$$\begin{gathered}I_{1,1,k}(X)=(p-p^{-1})X+(p^2-1)X^2,\quad I_{1,2,k}(X)=-X^3,\quad I_{1,3,k}(X)=0,\quad I_{1,4,k}=-p^2X^4,\\ I_{2,1,k}(X)=(p^2-1)X^3,\quad I_{2,3,k}(X)=I_{2,5,k}(X)=I_{2,7,k}(X)=0,\\ I_{2,2,k}(X)=-pX^4-pX^5,\quad I_{2,4,k}(X)=-p^2X^5-p^2X^6,\quad I_{2,6,k}(X)=pX^7,\quad I_{2,8,k}(X)=pX^2+p^2X^4.\end{gathered}$$

Therefore, when m is sufficiently large,

$$\begin{aligned}\mathrm{Den}(\Delta_p(N_k)\oplus H_{2m}^+,\,M)&=1+(p-p^{-1})X+(p^2+p-2)X^2+(p^2-2p+p^{-1}-1)X^3\\ &\qquad-(2p-1)X^4+(1-p^2)X^5+(p-p^2)X^6+pX^7\,\big|_{X=p^{-m}}.\end{aligned}$$

By Definition 7.2.5, when $k \geq 3$,

$$\begin{aligned}\mathrm{Den}_{\Delta_p(N_k)}(X, M)\big|_{X=p^{-m}} &= \frac{\mathrm{Den}(\Delta_p(N_k) \oplus H_{2m}^+, M)}{1-p^{-m-1}} \\ &= 1+pX+(p^2+p-1)X^2+(p^2-p)X^3-pX^4-p^2X^4-p^2X^5-p^2X^6\big|_{X=p^{-m}}.\end{aligned}$$

Hence,

$$\mathrm{Den}_{\Delta_p(N_k)}(X, M) = 1 + pX + (p^2 + p - 1)X^2 + (p^2 - p)X^3 - pX^4 - p^2X^4 - p^2X^5 - p^2X^6$$

when $k \geq 3$. This agrees with our previous calculations.

8. Proof of the arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$

8.1. *Local arithmetic Siegel–Weil formula with level N.* Let p be a prime number. The difference formulas at the analytic side and the geometric side are combined to prove the following theorem.

Theorem 8.1.1. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\,\mathrm{Den}_{\Delta_p(N)}(M). \tag{31}$$

Proof. Theorem 6.4.1 gives the difference formula of local arithmetic intersection numbers

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).$$

We also have the difference formula of the derived local densities (see (30))

$$\partial\,\mathrm{Den}_{\Delta_p(N)}(M) = \partial\,\mathrm{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial\,\mathrm{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).$$

Theorem 6.3.4 implies that $\mathrm{Int}^\sharp(L) = \partial\,\mathrm{Den}(L)$ for any rank-3 lattice $L \subset \mathbb{B}$. Therefore (31) holds by combining the above two difference formulas. □

8.2. *Intersection numbers and Whittaker functions.* Let p be a prime number.

Proposition 8.2.1. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$W_T'(1, 0, 1_{\Delta_p(N)^2}) = c_p \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(M) \cdot \log(p), \tag{32}$$

where the constant c_p is given as

$$c_p = \begin{cases} (1-p^{-1}) \cdot (N, -1)_p \cdot |N|_p \cdot |2|_p^{3/2} & \text{when } p \mid N, \\ (1-p^{-2}) \cdot (N, -1)_p \cdot |N|_p \cdot |2|_p^{3/2} & \text{when } p \nmid N. \end{cases}$$

Proof. Recall that $\Delta_p(N)^\vee/\Delta_p(N) \simeq \mathbb{Z}_p/2N\mathbb{Z}_p$ (see Example 2.1.1). By Proposition 3.3.1 and the explicit formula given in the appendix of [Ranga Rao 1993],

$$\begin{aligned}W_T(1, k, 1_{\Delta_p(N)^2}) &= |2N|_p \cdot \gamma(\Delta_p(N) \otimes \mathbb{Q}_p)^2 \cdot |2|_p^{1/2} \cdot \mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M) \\ &= |N|_p \cdot (N, -1)_p \cdot |2|_p^{3/2} \cdot \mathrm{Den}(\Delta_p(N) \oplus H_{2k}^+, M).\end{aligned} \tag{33}$$

Taking derivatives of both sides of (33),

$$W_T'(1,0,1_{\Delta_p(N)^2}) = c_p \cdot \partial\,\mathrm{Den}_{\Delta_p(N)}(M)\cdot \log(p).$$

The formula (32) follows from Theorem 8.1.1. □

8.3. *Proof of the main theorem.*

Proposition 8.3.1. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a positive definite symmetric matrix. Let $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Suppose $\boldsymbol{\varphi} = \varphi_1 \times \varphi_2$, where $\varphi_i \in \mathscr{S}(\mathbb{V}_f)$. Then for any $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, we have*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi})) = \begin{cases} \chi(\mathcal{Z}(T,\boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)})\cdot\log(p) & \text{when } \mathrm{Diff}(T,\Delta(N)) = \{p\},\\ 0 & \text{when } \#\,\mathrm{Diff}(T,\Delta(N)) \neq 1.\end{cases}$$

Proof. By definition (see (15)), the arithmetic special cycle $\hat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi})$ is $([\mathcal{Z}(T,\boldsymbol{\varphi})],0)$. Therefore $\widehat{\deg}(\hat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi}))$ is independent of y. We can assume $\mathrm{Diff}(T,\Delta(N)) = \{p\}$ for some prime number p since otherwise both sides are 0 since $\mathcal{Z}(T,\boldsymbol{\varphi})$ would be an empty stack.

Let $x \in \mathcal{Z}(T,\boldsymbol{\varphi})(\overline{\mathbb{F}}_p)$ be a geometric point. It is contained in $\mathcal{Y}_0(N)$ by Corollary 4.3.8, and hence the special divisors $\mathcal{Z}(t_1,\varphi_1)$ and $\mathcal{Z}(t_2,\varphi_2)$ intersect properly at x because T is nonsingular. Then $\chi(\mathcal{Z}(T,\boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)})\cdot\log(p)$ is the sum of the length of local rings $\mathcal{O}_{\mathcal{X}_0(N),x}$ cut out by these two divisors times $\log(p)$, which is exactly $\widehat{\deg}(\hat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi}))$ by definition of the degree homomorphism. □

Proof of Theorem 4.4.1. We first consider the case that T is positive definite. By Proposition 4.3.9, we only need to consider the case $\mathrm{Diff}(T,\Delta(N)) = \{p\}$ for some prime number p because otherwise both sides are 0. The same proposition and Corollary 4.3.8 imply that the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi})$ lies in the supersingular locus of $\mathcal{X}_0(N)_{\mathbb{F}_p}$. Then by the definition of special cycles and the formal uniformization of the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi})$ (see Corollary 5.3.3),

$$\chi(\mathcal{Z}(T,\boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)})\cdot\log(p) = \sum_{\substack{\boldsymbol{x}\in B^\times(\mathbb{Q})_0\backslash(\Delta(N)^{(p)})^2\\ T(\boldsymbol{x})=T}} \sum_{g\in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0\backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x})\cdot \mathrm{Int}_{\mathcal{N}_0(N)}(\boldsymbol{x})\cdot\log(p).$$

It is known (see (32)) that

$$W_T'(1,0,1_{\Delta_p(N)^2}) = c_p\cdot \mathrm{Int}_{\mathcal{N}_0(N)}(\boldsymbol{x})\cdot\log(p),$$

with constants c_p given by Proposition 8.2.1.

There exists a Haar measure on $\mathrm{GL}_2(\mathbb{A}_f^p)$ such that

$$\sum_{\substack{\boldsymbol{x}\in B^\times(\mathbb{Q})_0\backslash(\Delta(N)^{(p)})^2\\ T(\boldsymbol{x})=T}} \sum_{g\in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0\backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) = \frac{1}{\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}}^p))}\cdot \int_{\mathrm{SO}(\Delta(N)^{(p)})(\mathbb{A}_f^p)} \boldsymbol{\varphi}^p(g^{-1}\boldsymbol{x})\,\mathrm{d}g.$$

By definition, the last integral is a product of "local" integrals

$$\int_{\mathrm{SO}(\Delta(N)^{(p)})(\mathbb{A}_f^p)} \boldsymbol{\varphi}^p(g^{-1}\boldsymbol{x})\,\mathrm{d}g = \prod_{v\neq p,\infty} \int_{\mathrm{SO}(\Delta_v(N))(\mathbb{Q}_v)} \boldsymbol{\varphi}_v(g_v^{-1}\boldsymbol{x})\,\mathrm{d}g_v.$$

By the classical local Siegel–Weil formula, made explicit in [Kudla et al. 2006, Proposition 5.3.3], for every place v of $\mathbb{Q}$ there exists a number $d_v \in \mathbb{R}^\times$ such that

$$\int_{\mathrm{SO}(\Delta_v(N))(\mathbb{Q}_v)} \boldsymbol{\varphi}_v(g_v^{-1}\boldsymbol{x})\, \mathrm{d}g_v = d_v \cdot W_{T,v}(1, 0, \boldsymbol{\varphi}_v),$$

with $\prod_{v\le\infty} d_v = 1$. Moreover, [Kudla et al. 2006, Lemma 5.3.9] implies

$$\mathrm{vol}(\Gamma_0(N)_v, dg_v) = d_v \cdot \gamma(\Delta_v(N))^2 \cdot |2|_v^{3/2} \cdot \begin{cases} (1-v^{-2}) & \text{when } v \nmid N, \\ |N|_v^{-1}(1+v^{-1}) & \text{when } v \mid N. \end{cases}$$

It can be checked immediately that

$$\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}}^p)) \cdot d_p d_\infty \cdot c_p = 2^{-1/2}\psi(N)^{-1} \cdot \frac{3}{\pi^2}.$$

Suppose $\mathsf{z} = \mathsf{x} + i\mathsf{y}$. It's a classical result that

$$W_{T,\infty}(g_{\mathsf{z}}, 0, \Phi_\infty^{3/2}) = -2^{7/2}\pi^2 \cdot \det(\mathsf{y})^{3/4} q^T.$$

Combining these with the definitions made in previous sections (see (6) and (8)) and Proposition 8.3.1, we get the formula stated in the theorem.

When T is not positive definite, the equality follows from [Sankaran et al. 2023, §4.2] and our computations of the volume of $\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}})) = \prod_{v<\infty} \mathrm{vol}(\Gamma_0(N)_v, dg_v)$ above. □

Acknowledgements

First, I would like to express my deep gratitude to Professor Chao Li, who suggested I consider the question, and had countless instructive conversations with me during this project, without which this article would never exist. Next, I would like to thank Michael Rapoport, Tonghai Yang and Qiao He for their careful reading of the manuscript and many helpful comments. Moreover, I thank Cathy Chen for her unwavering support. The author is supported by the Department of Mathematics at Columbia University in the city of New York.

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Communicated by Shou-Wu Zhang
Received 2023-12-15 Revised 2024-06-22 Accepted 2024-09-03

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ALGEBRA AND NUMBER THEORY 19:9 (2025)
https://doi.org/10.2140/ant.2025.19.1823

Metaplectic cusp forms and the large sieve

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Dedicated to Chantal David on the occasion of her 60th birthday.

We prove a power saving upper bound for the sum of Fourier coefficients $\rho_f(\cdot)$ of a fixed cubic metaplectic cusp form f over primes. Our result is the cubic analogue of a celebrated 1990 theorem of Duke and Iwaniec, and the cuspidal analogue of a theorem due to the author and Radziwiłł for the bias in cubic Gauss sums.

The proof has two main inputs, both of independent interest. Firstly, we prove a new large sieve estimate for a bilinear form whose kernel function is $\rho_f(\cdot)$. The proof of the bilinear estimate uses a number field version of circle method due to Browning and Vishe, Voronoi summation, and Gauss–Ramanujan sums. Secondly, we use Voronoi summation and the cubic large sieve of Heath-Brown to prove an estimate for a linear form involving $\rho_f(\cdot)$. Our linear estimate overcomes a bottleneck occurring at level of distribution $\frac{2}{3}$.

1. Introduction

1.1. ***Background and statement of results.*** Arithmetic functions that arise from the Fourier coefficients of automorphic forms on congruence subgroups of $SL_2(\mathbb{Z})$ encode deep arithmetic and analytic information. A famous example is the modularity theorem for elliptic curves $E/\mathbb{Q}$ [Breuil et al. 2001], and its resolution of the Hasse–Weil conjecture for such curves.

MSC2020: 11F27, 11F30, 11L05, 11L20, 11N36.
Keywords: cubic metaplectic forms, primes, large sieve, circle method.

At a fundamental level, automorphic forms on congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are nice objects because there is an "adequate Hecke theory" available. By this, we mean the basic property that the sequence of Fourier coefficients of an integer weight cusp form restricted to values coprime to the level can be expressed as a linear combination of multiplicative functions given by the Hecke eigenvalues! It is well known that a power saving upper bound for the sum of Hecke eigenvalues $\lambda_g(\cdot)$ over primes would yield a rectangular zero-free region in the critical strip for associated L-function $L(s, g)$ (thanks to the Euler product). Unfortunately, the proof of such a bound is well out of reach of current technology!

The Fourier coefficients of half-integer weight modular forms also play a key role in arithmetic. An important example is the use of Dedekind's η-function $\big($holomorphic cusp form of weight $\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})\big)$ in the proof of Rademacher's formula [1937] for the partition function $p(n)$. Hecke [1983, p. 639; 1944] observed that there is not an "adequate Hecke theory" (in the naive sense above) for modular forms of half-integer weight. Wohlfahrt [1957] confirmed Hecke's observations and essentially showed that there is an algebra of Hecke operators $\mathbb{C}[\{T_{n^2}\}_{n=1}^{\infty}]$ acting on half-integer weight modular forms of weight k such that $T_{m^2} \circ T_{n^2} = T_{m^2n^2} = T_{n^2} \circ T_{m^2}$ for $(m, n) = 1$, $T_{p^{2a}}$ is a polynomial in T_{p^2} for each $a \in \mathbb{Z}_{\geq 1}$ and odd prime p, and that each Hecke operator is Hermitian (on cusp forms) with respect to the standard Petersson inner product. In general, the Fourier coefficients of half-integer weight Hecke eigenforms at general integer indices are not multiplicative, unless they are squares! In foundational works, Shimura [1973] and Kohnen and Zagier [1981] studied this phenomenon in more detail. For a comprehensive summary of the theory, the reader can consult [Koblitz 1984, §4.3].

Duke and Iwaniec [1990] gave striking quantitative evidence that the Fourier coefficients of half-integer weight holomorphic cusp forms along squarefree integers are not multiplicative (unless their values are zero). In particular, suppose that g is a holomorphic cusp form on $\Gamma_0(N)$ ($N \equiv 0 \pmod 4$) having weight $k = \frac{1}{2} + 2\ell$, $\ell \in \mathbb{Z}_{\geq 2}$, and Fourier expansion (at ∞)

$$g(z) = \sum_{n=1}^{\infty} c_g(n) n^{(k-1)/2} e(nx) e^{-2\pi n y}, \quad z = x + iy \in \mathbb{H}, \tag{1-1}$$

where $c_g(n) \in \mathbb{C}$, $e(x) := e^{2\pi i x}$ for all $x \in \mathbb{R}$, and $\mathbb{H} := \mathbb{R} \times \mathbb{R}_{>0}$ is the complex upper-half plane. For $\varepsilon > 0$ and $A, B \geq 10$, Duke and Iwaniec [1990] proved that

$$\sum_{a \leq A} \sum_{b \leq B} \mu^2(a) \alpha_a \beta_b c_g(ab) \ll_{\varepsilon, g} (AB)^{\varepsilon} (B^{1/2} + AB^{1/4}) \|\boldsymbol{\alpha}\|_2 \|\boldsymbol{\beta}\|_2, \tag{1-2}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are $\mathbb{C}$-valued sequences and $\|\bullet\|$ denotes the usual ℓ_2-norm. Using (1-2) together with appropriate linear estimates, Duke and Iwaniec [1990] also proved that

$$\sum_{\substack{p \leq X \\ p \text{ prime}}} c_g(p) \ll_{\varepsilon, g} X^{1-1/156+\varepsilon} \tag{1-3}$$

as $X \to \infty$. The result in (1-3) allows for twists by primitive characters of conductor divisible by $N \equiv 0 \pmod 4$, and so one can restrict to sum to primes in an arithmetic progression (with the implied constant depending on the modulus).

The goal of this paper is to generalise the results of Duke and Iwaniec to cusp forms on the cubic metaplectic cover of GL_2 (in the sense of Kubota [1969; 1971]). This is the complementary case to work in [Dunn and Radziwiłł 2024] on Patterson's conjecture for the bias of cubic Gauss sums over primes (cubic Gauss sums are the Fourier coefficients of the cubic theta function [Patterson 1977] which is noncuspidal). The spectral theory of cubic metaplectic forms have played a key role in [Livné and Patterson 2002; Louvel 2014], on the distribution of certain cubic exponential sums. In their PhD thesis, Möhring [2004] numerically investigated the Fourier coefficients of some cuspidal cubic metaplectic forms.

Before stating our results we briefly introduce some notation. Let $\mathbb{H}^3 := \mathbb{C} \times \mathbb{R}_{>0}$ denote hyperbolic 3-space. Let $\omega = e^{2\pi i/3}$, and $\mathbb{Q}(\omega)$ denote the Eisenstein quadratic field (class number 1). This number field has ring of integers $\mathbb{Z}[\omega]$, discriminant -3, and the unique ramified prime is $\lambda := \sqrt{-3} = 1 + 2\omega$. Let $\left(\frac{\bullet}{c}\right)_3$ denote the cubic symbol over $\mathbb{Z}[\omega]$, and $\Lambda(c)$ denote the usual von Mangoldt function on $\mathbb{Z}[\omega]$. Consider the following congruence subgroups of $\mathrm{SL}_2(\mathbb{C})$:

$$\begin{aligned}\Gamma &:= \mathrm{SL}_2(\mathbb{Z}[\omega]),\\ \Gamma_1(3) &:= \{\gamma \in \Gamma : \gamma \equiv I \pmod 3\},\\ \Gamma_2 &:= \langle \mathrm{SL}_2(\mathbb{Z}), \Gamma_1(3)\rangle.\end{aligned}$$

The cubic Kubota [1969; 1971] character $\chi : \Gamma_1(3) \to \{1, \omega, \omega^2\}$ is defined by

$$\chi(\gamma) := \begin{cases} \left(\frac{c}{a}\right)_3 & \text{if } c \neq 0,\\ 1 & \text{if } c = 0,\end{cases} \qquad \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_1(3), \tag{1-4}$$

and extends to a well-defined homomorphism $\chi : \Gamma_2 \to \{1, \omega, \omega^2\}$ when one defines $\chi|_{\mathrm{SL}_2(\mathbb{Z})} \equiv 1$ [Patterson 1977, §2]. The group Γ_2 is the lowest possible level for cubic metaplectic forms. Let f be a cuspidal cubic metaplectic form on Γ_2, i.e.,

- f vanishes at all cusps of Γ_2;
- $f(\gamma w) = \chi(\gamma) f(w)$ for all $\gamma \in \Gamma_2$ and $w \in \mathbb{H}^3$;
- f is an eigenfunction of the hyperbolic Laplacian: $\Delta f = -\tau_f(2 - \tau_f)$ for some $\tau_f \in \mathbb{C}$.

There is an algebra of Hecke operators $\mathbb{C}[\{T_{\nu^3}\}_{\nu \in \mathbb{Z}[\omega]\setminus\{0\}}]$ acting on cubic metaplectic forms such that $T_{\mu^3} \circ T_{\nu^3} = T_{\mu^3\nu^3} = T_{\nu^3} \circ T_{\mu^3}$ for $(\mu, \nu) = 1$, $T_{\varpi^{3a}}$ is a polynomial in T_{ϖ^3} for each $a \in \mathbb{Z}_{\geq 1}$ and prime $\varpi \equiv 1 \pmod 3$, and that each Hecke operator is Hermitian (on cusp forms) with respect to the standard Petersson inner product [Proskurin 1998, §0.3.12]. The Fourier expansion of f (at ∞) is given by

$$f(w) = \sum_{\substack{\nu \neq 0\\ \nu \in \lambda^{-3}\mathbb{Z}[\omega]}} \rho_f(\nu) v K_{\tau_f - 1}(4\pi|\nu|v)\check{e}(\nu z), \qquad w = (z, v) \in \mathbb{H}^3, \tag{1-5}$$

where $K_\alpha(\cdot)$ is the standard K-Bessel function of order $\alpha \in \mathbb{C}$, $\check{e}(z) := e^{2\pi i(z+\bar{z})}$ for $z \in \mathbb{C}$, and $\rho_f(\nu) \in \mathbb{C}$.

Remark 1.1. The cubic Shimura lift of Patterson [1998, Theorem 3.4] guarantees that one always has $\tau_f \in 1 + i\mathbb{R}$ for cuspidal cubic metaplectic forms f on Γ_2 (see Section 3.2).

Let $K, M \geq 1$, and $W_{K,M} : (0,\infty) \to \mathbb{C}$ be a smooth function with compact support in $[1, 2]$ such that for each $j \in \mathbb{Z}_{\geq 0}$ we have

$$W_{K,M}^{(j)}(x) \ll_j MK^j \quad \text{for all } x > 0. \tag{1-6}$$

If $M = 1$ then M is omitted from the notation, and we write W_K. Let $\|\bullet\|_q$ with $1 \leq q \leq \infty$ denote the ℓ_q-norm of a $\mathbb{C}$-valued sequence indexed over elements of $\mathbb{Z}[\omega]$.

The main sums of interest in this paper are

$$\mathscr{P}_f(X, v, u; W_K) := \sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3\nu \equiv u \pmod{v}}} \rho_f(\nu)\Lambda(\lambda^3\nu) W_K\left(\frac{N(\nu)}{X}\right), \tag{1-7}$$

$$\mathscr{P}_f(X, v, u) := \sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3\nu \equiv u \pmod{v} \\ N(\nu) \leq X}} \rho_f(\nu)\Lambda(\lambda^3\nu), \tag{1-8}$$

$$\widetilde{\mathscr{P}}_f(X, v, u) := \sum_{\substack{\varpi \in \mathbb{Z}[\omega] \\ \varpi \text{ prime} \\ \varpi \equiv u \pmod{v} \\ N(\lambda^{-3}\varpi) \leq X}} \rho_f(\lambda^{-3}\varpi) \log N(\varpi), \tag{1-9}$$

where $0 \neq v \in \mathbb{Z}[\omega]$ is such that $v \equiv 0 \pmod 3$, and $u \in \mathbb{Z}[\omega]/v\mathbb{Z}[\omega]$ is such that $(u, v) = 1$ and $u \equiv 1 \pmod 3$. It is technically convenient to restrict attention to $u \equiv 1 \pmod 3$. The other congruence classes modulo 3 can be treated by a mild adaption of the methods of this paper.

Theorem 1.2. *Let $\varepsilon > 0$ and the notation be as above. Then*

$$\mathscr{P}_f(X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^{\varepsilon} K^8 N(v)^4 X^{1-1/34}$$

as $X \to \infty$.

Corollary 1.3. *Let $\varepsilon > 0$. In the notation above we have*

$$\mathscr{P}_f(X, v, u) \ll_{\varepsilon, f, v} X^{1-1/578+\varepsilon}, \tag{1-10}$$

$$\widetilde{\mathscr{P}}_f(X, v, u) \ll_{\varepsilon, f, v} X^{1-1/578+\varepsilon} \tag{1-11}$$

as $X \to \infty$.

Theorem 1.2 follows from new estimates for linear and bilinear sums which we now describe. A brief sketch of the new difficulties and ideas that arise in our case (as opposed to the case in [Duke and Iwaniec 1990]) is given in Section 1.2. Let

$$\mathscr{T}_f(a, X, v, u; W_K) := \sum_{\substack{b \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \rho_f(\lambda^{-3}ab) W_K\left(\frac{N(\lambda^{-3}ab)}{X}\right) \tag{1-12}$$

denote the pointwise Type-I sum, where $X \geq 10$ and $a \in \mathbb{Z}[\omega]$ with $a \equiv 1 \pmod 3$. Let

$$\mathscr{A}_f(\boldsymbol{\alpha}, X, v, u; W_K) := \sum\sum_{\substack{a,b \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \mu^2(a)\alpha_a \rho_f(\lambda^{-3}ab) W_K\left(\frac{N(\lambda^{-3}ab)}{X}\right) \tag{1-13}$$

denote the average (over squarefree a) Type-I sum, where $A, X \geq 10$ and $\boldsymbol{\alpha} := (\alpha_a)$ is a $\mathbb{C}$-valued sequence supported on $a \in \mathbb{Z}[\omega]$ with $a \equiv 1 \pmod 3$ and $N(a) \asymp A$. Let

$$\mathscr{B}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, X, v, u; W_K) := \sum_{\substack{a,b \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod v}} \sum \mu^2(a) \alpha_a \beta_b \rho_f(\lambda^{-3} ab) W_K\left(\frac{N(\lambda^{-3}ab)}{X}\right) \tag{1-14}$$

denote the Type-II sum, where $A, B \geq 10$, (α_a) is as above, and $\boldsymbol{\beta} := (\beta_b)$ is a $\mathbb{C}$-valued sequence supported on $b \in \mathbb{Z}[\omega]$ with $N(b) \asymp B$. Note that we necessarily have $X \asymp AB$ in (1-14), otherwise the double sum is empty.

In Section 9 we use Voronoi summation to prove the following "trivial" pointwise Type-I bound.

Lemma 1.4. *Let $\varepsilon > 0$ and the notation be as above. Then*

$$\mathscr{T}_f(a, X; v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^{\varepsilon} K^4 N(v)^{1/2} N(a)^{1/2}.$$

When $\mathscr{T}_f(a, \dots)$ is multiplied by a weight α_a and the estimate in Lemma 1.4 is summed trivially over $a \in \mathbb{Z}[\omega]$ with $N(a) \asymp A$, the resulting bound is acceptable when $A \ll X^{2/3-\varepsilon}$.

In Section 10 we use the circle method to prove the following new bilinear estimate.

Theorem 1.5. *Let $\varepsilon > 0$ and the notation be as above. Then for $A, B \geq 10$ and $X \asymp AB$ we have*

$$\mathscr{B}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^{\varepsilon} K^8 N(v)^4 ((AB)^{1/2} + A^{3/2} B^{1/4}) \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_{\infty} \|\boldsymbol{\beta}\|_2.$$

Theorem 1.5 is acceptable when $\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_\infty \ll A^{\varepsilon}$ and $X^{2/3+\varepsilon} \ll B \ll X^{1-\varepsilon}$.

We point out that Lemma 1.4 and Theorem 1.5 together *barely* misses primes. To overcome the bottleneck at level of distribution $\asymp X^{2/3}$, we use Voronoi summation and Heath-Brown's cubic large sieve [2000] to prove the following estimate.

Proposition 1.6. *Let $\varepsilon > 0$ and the notation be as above. Then for $X, A \geq 10$ we have*

$$\mathscr{A}_f(\boldsymbol{\alpha}, X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^{\varepsilon} K^{14/3} N(v)^{5/6} (AX)^{1/3} \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_2.$$

1.2. *Brief sketch of the method.* We close with a brief outline of the proofs of Theorem 1.5 and Proposition 1.6. For simplicity, we suppress smooth functions, and ignore both the units of $\mathbb{Z}[\omega]$ and the congruence condition $u \pmod v$.

1.2.1. *Linear sums.* We apply Voronoi summation to the b-sum in (1-13) and perform a computation with the arithmetic exponential sums that appear on the dual side. We obtain a bilinear form

$$\frac{X}{A^2} \sum_{N(a) \asymp A} \sum_{N(\nu) \ll A^2/X} \mu^2(a) \overline{g(a)} \alpha_a \rho_f(\nu) \left(\frac{\nu}{a}\right)_3, \tag{1-15}$$

where $g(a)$ denotes the unnormalised cubic Gauss sum over $\mathbb{Z}[\omega]$ with modulus a. The use of Heath-Brown's cubic large sieve [2000] (with the squarefree condition on one variable relaxed) leads to our average Type-I estimate.

1.2.2. *Bilinear sums.* After application of Cauchy–Schwarz in the b-variable to (1-14), the sum of interest is

$$\sum_{\substack{N(a_1),N(a_2)\asymp A\\ a_1,a_2\equiv 1 \pmod 3}} \mu^2(a_1)\alpha_{a_1}\mu^2(a_2)\overline{\alpha_{a_2}} \sum_{N(b)\sim B} \rho_f(a_1 b)\overline{\rho_f(a_2 b)}. \tag{1-16}$$

The natural approach would be to ignore the averaging over a_1 and a_2, and estimate each convolution sum $\sum_{N(b)\sim B} \rho_f(a_1 b)\overline{\rho_f(a_2 b)}$ directly. Duke and Iwaniec [1990] proved that each convolution sum is $\ll_\varepsilon \delta_{a_1=a_2} B + (AB)^\varepsilon AB^{1/2}$ for the case of holomorphic half-integer weight cusp forms. We explain below why the additional averaging over a_1 and a_2 is crucial in the Maass case.

The initial move of [Duke and Iwaniec 1990] is to open one of the Fourier coefficients in terms of sums of half-integer weight Kloosterman sums that come from writing the holomorphic cusp form as a finite $\mathbb{C}$-linear combination of Poincaré series. This opening move is not available for Maass forms! Instead, we separate oscillations using the circle method of Browning and Vishe [2014] to obtain

$$\sum_{N(b)\sim B} \rho_f(a_1 b)\overline{\rho_f(a_2 b)} \approx \frac{1}{B} \sum_{\substack{N(\nu_1),N(\nu_2)\asymp AB\\ \nu_1\equiv 0 \pmod{a_1}\\ \nu_2\equiv 0 \pmod{a_2}}} \rho_f(\nu_1)\overline{\rho_f(\nu_2)} \sum_{\substack{N(c)\sim B^{1/2}\\ (c,\lambda a_1 a_2)=1}} r(\nu_1/a_1-\nu_2/a_2, c), \tag{1-17}$$

where $r(n,c)$ denotes the unnormalised Ramanujan sum over $\mathbb{Z}[\omega]$ with modulus c and shift n. In reality, one must also consider moduli c that are not coprime to $\lambda a_1 a_2$. This can be handled with an modification of the method below with an additional local computation involving cubic Gauss sums with moduli dividing $\operatorname{rad}(a_1 a_2)^\infty$.

We detect the congruence conditions on the ν_1, ν_2 using additive characters, apply Voronoi summation to each ν_1, ν_2 sum, and perform a considerable computation with the exponential sums on the dual side. This leads to an expression of the shape

$$\sum_{\substack{s_1\mid a_1\\ s_2\mid a_2}} \frac{1}{N(s_1 s_2)^{1/2}} \sum_{\substack{N(\nu_1)\ll N(s_1)^2/A\\ N(\nu_2)\ll N(s_2)^2/A\\ (\nu_1,s_1)=1\\ (\nu_2,s_2)=1}} \rho_f(\nu_1)\overline{\rho_f(\nu_2)} \sum_{\substack{N(c)\sim B^{1/2}\\ (c,\lambda a_1 a_2)=1}} r(s_2^2 a_1\nu_1 - s_1^2 a_2 \nu_2, c). \tag{1-18}$$

We highlight that the squarefree property of a_1 and a_2 simplifies the computations considerably. One can apply Cauchy–Schwarz and Rankin–Selberg bounds to estimate the off-diagonal ($s_2^2 a_1\nu_1 \neq s_1^2 a_2\nu_2$) contribution in (1-18) by $(AB)^\varepsilon AB^{1/2}$. The diagonal term is more subtle. The diagonal equation $s_2^2 a_1 \nu_1 = s_1^2 a_2 \nu_2$ is equivalent to $s_2(a_1/s_1)\nu_1 = s_1(a_2/s_2)\nu_2$. The conditions $(\nu_1,s_1)=(\nu_2,s_2)=1$ together with the squarefree hypothesis on a_1 and a_2 imply that $s_1=s_2=: s \mid (a_1,a_2)$. Thus the diagonal contribution in (1-18) has the shape

$$B \sum_{s\mid(a_1,a_2)} \frac{1}{N(s)} \sum_{\substack{N(\nu)\ll N(s)^3 N((a_1/s,a_2/s))/A^2\\ (\nu,s)=1}} \rho_f\Big(\frac{a_2/s}{(a_1/s,a_2/s)}\nu\Big)\overline{\rho_f\Big(\frac{a_1/s}{(a_1/s,a_2/s)}\nu\Big)}. \tag{1-19}$$

At this point there is no cancellation to be realistically exploited in (1-19), and so we apply the triangle inequality and place absolute values around the Fourier coefficients. It is tempting to apply a "Deligne-type" bound for $\rho_f(\cdot)$ to estimate the diagonal by $(AB)^{\varepsilon} \cdot B \cdot (N((a_1, a_2))/A)^2$ (which is of acceptable size). However, no such bound for $\rho_f(\cdot)$ is known unconditionally, and the author is not aware of *any* nontrivial bound for $\rho_f(\cdot)$ stronger than the bound implied by Rankin–Selberg. There is no "Waldspurger-type" formula known for the coefficients of cubic metaplectic cusp forms (on GL_2). Hence the strategy for bounding these Fourier coefficients via subconvexity for twisted L-values is not available (this strategy is used the half-integer weight case; see [Conrey and Iwaniec 2000]). To overcome this, we substitute (1-19) into (1-16), take absolute values and the supremum norm of the $\boldsymbol{\alpha}$ terms, and exploit the additional averaging over a_1 and a_2 using Cauchy–Schwarz and Rankin–Selberg bounds. This yields the acceptable estimate $(AB)^{\varepsilon} AB \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_{\infty}^2$ for the diagonal of the averaged sum. It is interesting to note that an argument of Nelson [2020] could potentially be adapted to estimate the sparse convolution sum in (1-19). We refrain from this additional work.

1.3. *Conventions.* For $n \in \mathbb{N}$ and $N > 0$, we use $n \sim N$ to mean $N < n \leq 2N$, and $n \asymp N$ to mean that there exists constants $c_1, c_2 > 0$ such that $c_1 N \leq n \leq c_2 N$.

Dependence of implied constants on parameters will be indicated in statements of results, but suppressed throughout the body of the paper (i.e., proofs). Implied constants in the body of the paper are allowed to depend on $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$, ε, $D > 0$ (possibly different in each instance), and the implicit constants in the statements $N(a) \asymp A$ and $N(b) \asymp B$.

Whenever we write $r \mid q$ with $0 \neq r, q \in \mathbb{Z}[\omega]$ and $q \equiv 1 \pmod 3$, it is our convention that $r \equiv 1 \pmod 3$. For any integer b we let $\mathbb{Z}_{\geq b} := \{n \in \mathbb{Z} : n \geq b\}$.

Unless otherwise specified, it should be clear from context whether $\bar{x}$ means modular inverse (with respect to an appropriate modulus) or complex conjugation.

Unless otherwise specified, it should be clear from context whether v refers to the modulus of an arithmetic progression or the real component of a quaternion element $w = (z, v)$.

2. Preliminaries and background

2.1. *Eisenstein quadratic field and cubic Gauss sums.* We include some brief background on $\mathbb{Q}(\omega)$ and cubic Gauss sums. For more details see [Dunn and Radziwiłł 2024; Patterson 1977; Proskurin 1998].

Let $\mathbb{Q}(\omega)$ be the Eisenstein quadratic number field, where ω is identified with $e^{2\pi i/3} \in \mathbb{C}$. This quadratic number field has ring of integers $\mathbb{Z}[\omega]$, discriminant -3, and class number 1. Let $N(x) := N_{\mathbb{Q}(\omega)/\mathbb{Q}}(x) = |x|^2$ denote the norm form on $\mathbb{Q}(\omega)/\mathbb{Q}$. The dual of $\mathbb{Z}[\omega]$ is

$$\mathbb{Z}[\omega]^* := \{z \in \mathbb{C} : \check{e}(zz') = 1 \text{ for all } z' \in \mathbb{Z}[\omega]\} = \lambda^{-1}\mathbb{Z}[\omega].$$

It is well known that any nonzero element of $\mathbb{Z}[\omega]$ can be uniquely written as $\zeta \lambda^k c$ with $\zeta \in \langle -\omega \rangle$ a unit (i.e., $\zeta^6 = 1$), $\lambda := \sqrt{-3} = 1 + 2\omega$ the unique ramified prime in $\mathbb{Z}[\omega]$, $k \in \mathbb{Z}_{\geq 0}$, and $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$. If $p \equiv 1 \pmod 3$ is a rational prime, then $p = \varpi \overline{\varpi}$ in $\mathbb{Z}[\omega]$ with $N(\varpi) = p$ and ϖ

a prime in $\mathbb{Z}[\omega]$. If $p \equiv 2 \pmod 3$ is a rational prime, then $p = \varpi$ is inert in $\mathbb{Z}[\omega]$, and $N(\varpi) = p^2$. Thus we have $N(\varpi) \equiv 1 \pmod 3$ for all primes ϖ with $(\varpi) \neq (\lambda)$.

The cubic Jacobi symbol defined for $a \equiv 1 \pmod 3$ and $\varpi \equiv 1 \pmod 3$ prime is defined by

$$\left(\frac{a}{\varpi}\right)_3 \equiv a^{(N(\varpi)-1)/3} \pmod{\varpi},$$

and the condition it take values in $\{1, \omega, \omega^2\}$. The cubic symbol is clearly multiplicative in a and can be extended multiplicatively in b by setting

$$\left(\frac{a}{b}\right)_3 = \prod_i \left(\frac{a}{\varpi_i}\right)$$

for any $b = \prod_i \varpi_i$ with $\varpi_i \equiv 1 \pmod 3$ primes. The cubic symbol obeys cubic reciprocity: given $a, b \equiv 1 \pmod 3$ we have

$$\left(\frac{a}{b}\right)_3 = \left(\frac{b}{a}\right)_3. \tag{2-1}$$

There are also supplementary laws for units and the ramified prime. Given

$$d \equiv 1 + \alpha_2\lambda^2 + \alpha_3\lambda^3 \pmod 9 \quad \text{with} \quad \alpha_2, \alpha_3 \in \{-1, 0, 1\}, \tag{2-2}$$

we have

$$\left(\frac{\omega}{d}\right)_3 = \omega^{\alpha_2} \quad \text{and} \quad \left(\frac{\lambda}{d}\right)_3 = \omega^{-\alpha_3}. \tag{2-3}$$

We follow the standard convention for an empty product,

$$\left(\frac{a}{1}\right)_3 = 1 \quad \text{for all } a \in \mathbb{Z}[\omega]. \tag{2-4}$$

Let

$$\check{e}(z) := e^{2\pi i \,\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(z)} = e^{2\pi i (z+\bar{z})}, \quad z \in \mathbb{C}.$$

For $\mu \in \mathbb{Z}[\omega]^* = \lambda^{-1}\mathbb{Z}[\omega]$ and $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$, the cubic Gauss sum (with shift μ) is defined by

$$g(\mu, c) := \sum_{d \pmod c} \left(\frac{d}{c}\right)_3 \check{e}\left(\frac{\mu d}{c}\right). \tag{2-5}$$

We write $g(c) := g(1, c)$ for short. Making a change of variable in the Gauss sum we see that

$$g(\mu, c) := \left(\frac{\lambda}{c}\right)_3 g(\lambda\mu, c), \tag{2-6}$$

and so for the rest of this section it suffices to consider only $\mu \in \mathbb{Z}[\omega]$, which we now assume. We have

$$g(\mu, c) = \overline{\left(\frac{\mu}{c}\right)_3} g(1, c) \quad \text{for } (\mu, c) = 1. \tag{2-7}$$

The Chinese remainder theorem implies the twisted multiplicativity property

$$g(\mu, ab) = \overline{\left(\frac{a}{b}\right)_3} g(\mu, a) g(\mu, b) \quad \text{for } a, b \equiv 1 \pmod 3 \text{ such that } (a, b) = 1. \tag{2-8}$$

By (2-7) and (2-8) is suffices to understand $g(\varpi^k,\varpi^\ell)$ for $\varpi \equiv 1 \pmod 3$ prime and $k,\ell \in \mathbb{Z}_{\geq 0}$. A specialisation of [Proskurin 1998, property (h), p. 7] yields

$$g(\varpi^k,\varpi^\ell) = \begin{cases} 1 & \text{if } \ell = 0, \\ \varphi(\varpi^\ell) & \text{if } 1 \leq \ell \leq k,\ \ \ell \equiv 0 \pmod 3, \\ -N(\varpi)^k & \text{if } \ell = k+1,\ \ \ell \equiv 0 \pmod 3, \\ N(\varpi)^k g(\varpi) & \text{if } \ell = k+1,\ \ \ell \equiv 1 \pmod 3, \\ N(\varpi)^k \overline{g(\varpi)} & \text{if } \ell = k+1,\ \ \ell \equiv 2 \pmod 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2-9}$$

For $\varpi \equiv 1 \pmod 3$ prime we have the formula for the cube [Hasse 1950, pp. 443–445],

$$g(\varpi)^3 = -\varpi^2\overline{\varpi}. \tag{2-10}$$

Observe that (2-7)–(2-9) and (2-10) imply that

$$|g(c)| = \mu^2(c) N(c)^{1/2} \tag{2-11}$$

for $c \equiv 1 \pmod 3$. We denote the normalised cubic Gauss sum (with shift $\mu \in \mathbb{Z}[\omega]$) by

$$\tilde{g}(\mu,c) := N(c)^{-1/2} g(\mu,c). \tag{2-12}$$

The following two lemmas follow directly from combining (2-7)–(2-9).

Lemma 2.1. *Let $\mu \in \mathbb{Z}[\omega]$ and $c \in \mathbb{Z}[\omega]$ such that $c \equiv 1 \pmod 3$ is squarefree. Then*

$$g(\mu,c) = 0 \quad \textit{unless} \quad (\mu,c) = 1.$$

Lemma 2.2. *Let $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$ and $\varpi,\mu \in \mathbb{Z}[\omega]$ be such that $\varpi \equiv 1 \pmod 3$ is prime and $\varpi^2 \mid c$. Then*

$$g(\mu,c) = 0 \quad \textit{unless} \quad \varpi \mid \mu.$$

The next lemma follows directly from combining (2-7)–(2-9) and (2-11).

Lemma 2.3. *Let $\mu, c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$. Then*

$$|g(\mu,c)| \leq N(c)^{1/2} \cdot N((\mu,c))^{1/2}.$$

Remark 2.4. We emphasise that $c \in \mathbb{Z}[\omega]$ is not necessarily squarefree in Lemma 2.3.

For $b \in \mathbb{R}$, and $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$, let

$$\sigma_b(q) := \sum_{d \mid q} N(d)^b. \tag{2-13}$$

For a given $\varepsilon > 0$, we have the standard divisor bound

$$\sigma_0(q) \ll_\varepsilon N(q)^\varepsilon. \tag{2-14}$$

The following lemma is immediate.

Lemma 2.5. *Let $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$ and $b \in \mathbb{R}$. Then for $Y \geq 1$ we have*

$$\sum_{\substack{\mu \in \mathbb{Z}[\omega] \\ 1 \leq N(\mu) \leq Y}} N((\mu, q))^b \leq Y \sigma_{b-1}(q),$$

where $\sigma_b(q)$ is as given in (2-13).

Lemma 2.6. *Let $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$. Then for $X \geq 1$ and $\varepsilon > 0$ we have*

$$\sum_{\substack{N(r) \leq X \\ r \mid \mathrm{rad}(q)^\infty}} 1 \ll_\varepsilon (N(q)X)^\varepsilon. \tag{2-15}$$

Proof. Without loss of generality we can assume X is an odd half-integer. By Perron's formula (truncated) we have

$$\sum_{\substack{N(r) \leq X \\ r \mid \mathrm{rad}(q)^\infty}} 1 = \int_{2-i(XN(q))^{100}}^{2+i(XN(q))^{100}} X^s \prod_{\substack{\varpi \mid \mathrm{rad}(q) \\ \varpi \text{ prime} \\ \varpi \equiv 1 \pmod 3}} (1 - N(\varpi)^{-s})^{-1} \frac{ds}{s} + O((XN(q))^{-50}).$$

The integrand is holomorphic in the half-plane $\mathrm{Re}(s) > 0$. We move the contour $\mathrm{Re}(s) = \varepsilon$. Taking the logarithm of the Euler product and then using the pointwise bound

$$\omega(q) \ll \frac{\log N(q)}{\log \log N(q)}, \tag{2-16}$$

we obtain (after exponentiation)

$$\prod_{\substack{\varpi \mid \mathrm{rad}(q) \\ \varpi \text{ prime} \\ \varpi \equiv 1 \pmod 3}} (1 - N(\varpi)^{-s})^{-1} \ll N(q)^\varepsilon \quad \text{for } \mathrm{Re}(s) \geq \varepsilon.$$

The result follows from Cauchy's residue theorem. □

2.2. ***Group action on $\mathbb{H}^3$ and Laplacian.*** Let $\mathbb{H}^3$ denote the hyperbolic 3-space $\mathbb{C} \times \mathbb{R}_{>0}$. Embed $\mathbb{C}$ and $\mathbb{H}^3$ in the Hamilton quaternions by identifying $i = \sqrt{-1}$ with $\hat{i}$ and $w = (z, v) = (x + iy, v) \in \mathbb{H}^3$ with $x + y\hat{i} + v\hat{k}$, where $1, \hat{i}, \hat{j}, \hat{k}$ denote the unit quaternions. The continuous action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}^3$ (in quaternion arithmetic) is given by

$$\gamma w = (aw + b)(cw + d)^{-1}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \quad \text{and} \quad w \in \mathbb{H}^3.$$

The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}^3$ is transitive, and the stabiliser of a point is $\mathrm{SU}_2(\mathbb{C})$. In coordinates,

$$\gamma w = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}v^2}{|cz + d|^2 + |c|^2 v^2}, \frac{v}{|cz + d|^2 + |c|^2 v^2} \right), \quad w = (z, v). \tag{2-17}$$

The Laplace operator $\Delta := v^2(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial v^2) - v\partial/\partial v$ acts on $C^\infty(\mathbb{H}^3)$ and commutes with the action of $\mathrm{SL}_2(\mathbb{C})$ on $C^\infty(\mathbb{H}^3)$.

Consider the subgroup $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\omega])$ of $\mathrm{SL}_2(\mathbb{C})$. It has finite volume (but is not cocompact) with respect to the $\mathrm{SL}_2(\mathbb{C})$-invariant Haar measure $v^{-3}\,dx\,dy\,dv$ on $\mathbb{H}^3$. In what follows, let $\Gamma' \subseteq \Gamma$ be a subgroup with $[\Gamma : \Gamma'] < \infty$. Let $P(\Gamma') \subset \mathbb{Q}(\omega) \cup \{\infty\}$ be a complete inequivalent (finite) set of cusps for Γ'. Each cusp of Γ' can be written as $\sigma\infty$ for some $\sigma \in \Gamma$, and let

$$\Gamma'_\sigma := \{\gamma \in \Gamma' : \gamma\sigma\infty = \sigma\infty\}$$

denote the stabiliser group of the cusp $\sigma\infty$ in Γ'. We have $\Gamma'_\sigma := \sigma\Gamma'_I\sigma^{-1} \cap \Gamma'$, and let

$$\Lambda_\sigma := \left\{\mu \in \mathbb{C} : \sigma\left(\begin{smallmatrix}1 & \mu\\ 0 & 1\end{smallmatrix}\right)\sigma^{-1} \in \Gamma'\right\},$$
$$\Lambda^*_\sigma := \{\nu \in \mathbb{C} : \mathrm{Tr}(\mu\nu) \in \mathbb{Z} \text{ for all } \mu \in \Lambda_\sigma\}.$$

It is well known that Λ_σ and Λ^*_σ are lattices in $\mathbb{C}$, and that Λ^*_σ is dual to Λ_σ.

A fundamental domain for the action of Γ on $\mathbb{H}^3$ is the set

$$\mathcal{F} := \{w = (z, v) \in \mathbb{H}^3 : |z|^2 + v^2 > 1 \text{ and } z \in \pm\triangle\},$$

where $\triangle$ is the interior of the triangle with vertices 0, $(1-\omega)^{-1}$ and $(1-\omega^2)^{-1}$. The set of cusps for Γ is $P(\Gamma) := \{\infty\}$.

Other congruence subgroups of significance to this paper are given in Section 3.

2.3. *Automorphic forms (for general multipliers).* We record some facts about automorphic forms on $\Gamma'\backslash\mathbb{H}^3$ that transform with general unitary character $\kappa : \Gamma' \to \mathbb{C}^\times$. For more details one may consult [Livné and Patterson 2002; Louvel 2014; Patterson 1998; Proskurin 1998]. We specialise to cubic metaplectic forms in Section 3.

Let $\kappa : \Gamma' \to \mathbb{C}^\times$ be a unitary character that satisfies $\kappa(-I) = 1$ if $-I \in \Gamma'$. The function defined by $\mu \to \kappa\left(\sigma\left(\begin{smallmatrix}1 & \mu\\ 0 & 1\end{smallmatrix}\right)\sigma^{-1}\right) : \Lambda_\sigma \to \mathbb{C}^\times$ is a homomorphism on the lattice Λ_σ. There exists $h_\sigma \in \mathbb{C}$ such that

$$\kappa\left(\sigma\left(\begin{smallmatrix}1 & \mu\\ 0 & 1\end{smallmatrix}\right)\sigma^{-1}\right) = \check{e}(h_\sigma\mu) \quad \text{for all } \mu \in \Lambda_\sigma.$$

Essential cusps with respect to κ are those σ for which we can take $h_\sigma = 0$.

Let

$$A(\Gamma'\backslash\mathbb{H}^3, \kappa) := \{u : \mathbb{H}^3 \to \mathbb{C} : u(\gamma w) = \kappa(\gamma)u(w) \text{ for all } \gamma \in \Gamma' \text{ and } w \in \mathbb{H}^3\}.$$

We say that $u \in A(\Gamma'\backslash\mathbb{H}^3, \kappa)$ is an automorphic form under Γ' with character κ if it satisfies the conditions:

- $u \in C^\infty(\mathbb{H}^3)$ and is an eigenfunction of the Laplacian, i.e.,

$$\Delta u = -\tau_u(2-\tau_u)u \quad \text{for some } \tau_u \in \mathbb{C}.$$

  The quantity $\tau_u \in \mathbb{C}$ is the spectral parameter for u, and is well-defined only up to $\tau_u \mapsto 2-\tau_u$. Without loss of generality one can assume that $\mathrm{Re}(\tau_u) \ge 1$.

- u has moderate growth at cusps: there exists a $D \in \mathbb{R}$ such that

$$|u(w)| < (v + (1+|z|^2)v^{-1})^D \quad \text{for all } w = (z, v) \in \mathbb{H}^3.$$

Let $L(\Gamma'\backslash\mathbb{H}^3,\kappa,\tau)$ denote the $\mathbb{C}$-vector space of automorphic forms under Γ' with character κ and spectral parameter τ. The norm $\|\cdot\|_2$ on $L(\Gamma'\backslash\mathbb{H}^3,\kappa,\tau)$ is induced by the standard Petersson inner product

$$\langle u_1,u_2\rangle := \int_{\Gamma'\backslash\mathbb{H}^3} u_1(w)\overline{u_2(w)}\frac{dx\,dy\,dv}{v^3}.$$

Let

$$L^2(\Gamma'\backslash\mathbb{H}^3,\kappa,\tau) := \{u\in L(\Gamma'\backslash\mathbb{H}^3,\kappa,\tau) : \|u\|_2<\infty\},$$

denote the finite-dimensional Hilbert space of square integrable automorphic forms having character κ and spectral parameter τ. We demand that $\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ act on an automorphic form u by ± 1, and we speak of u being even or odd respectively.

Consulting [Proskurin 1998, Theorem 0.3.1], each $u\in L(\Gamma'\backslash\mathbb{H}^3,\kappa,\tau)$ has Fourier expansion at the cusp $\sigma\infty$ given by

$$U_\sigma(w) := u(\sigma w) = c_{u,\sigma}(v) + \sum_{\substack{\nu\neq 0\\ \nu\in h_\sigma+\Lambda_\sigma^*}} \rho_{u,\sigma}(\nu)vK_{\tau-1}(4\pi|\nu|v)\check{e}(\nu z),\qquad w\in\mathbb{H}^3, \tag{2-18}$$

where $\rho_{u,\sigma}(\nu)\in\mathbb{C}$, and

$$c_{u,\sigma}(v) = \begin{cases}\rho_{u,\sigma,+}(0)v^\tau + \rho_{u,\sigma,-}(0)v^{2-\tau} & \text{if } \tau\neq 1,\\ \rho^\sigma_{u,\sigma,+}(0)v\log v + \rho_{u,\sigma,-}(0)v & \text{if } \tau = 1,\end{cases}$$

and $\rho_{u,\sigma,+}(0),\rho_{u,\sigma,-}(0)\in\mathbb{C}$. If $\sigma\infty$ is essential, then one can take $h_\sigma=0$. If $\sigma\infty$ is not essential, then $c_{u,\sigma}(v)\equiv 0$ by [Proskurin 1998, Theorem 0.3.1]. By convention, if $\sigma = I$ then we omit it from the subscripts on the Fourier coefficients.

If $c_{u,\sigma}(v)\equiv 0$ for all cusps $\sigma\infty$, then u is a cusp form (it is necessarily a Maass form since $\mathbb{H}^3$ does not have an invariant complex structure). In particular, all cusp forms have exponential decay at the cusps, and consequently are square integrable on $\Gamma'\backslash\mathbb{H}^3$.

The following crude Rankin–Selberg bound follows from a standard argument that uses Plancherel's theorem. The proof is analogous to that of [Iwaniec 1995, Theorem 3.2], and is omitted.

Lemma 2.7. *Let $\tau\in\mathbb{C}$ with $\mathrm{Re}(\tau)\geq 1$, $u\in L^2(\Gamma'\backslash\mathbb{H}^3,\kappa,\tau)$ be a cusp form, σ a cusp of Γ', and $\varepsilon>0$. Then for all $X\geq 100$ we have*

$$\sum_{\substack{\nu\in h_\sigma+\Lambda_\sigma^*\\ N(\nu)\leq X}} |\rho_{u,\sigma}(\nu)|^2 \ll_{u,\sigma,\varepsilon} X^{1+\varepsilon}.$$

An application of the Cauchy–Schwarz inequality and Lemma 2.7 give the following L^1-bound.

Lemma 2.8. *In the notation of Lemma 2.7 we have*

$$\sum_{\substack{\nu\in h_\sigma+\Lambda_\sigma^*\\ N(\nu)\leq X}} |\rho_{u,\sigma}(\nu)| \ll_{u,\sigma,\varepsilon} X^{1+\varepsilon}.$$

The following Wilton-type bound follows from a standard argument using Fourier convolution with the Dirichlet kernel. The proof is analogous to that of [Epstein et al. 1985, Theorem 3.1] and is omitted.

Lemma 2.9. *Let the notation be as in Lemma 2.7 and suppose that* $\mathrm{Re}(\tau) = 1$. *Then*

$$\sum_{\substack{\nu \in h_\sigma + \Lambda_\sigma^* \\ N(\nu) \le X}} \rho_{u,\sigma}(\nu) \check{e}(\alpha \nu) \ll_{u,\sigma,\varepsilon} X^{1/2+\varepsilon}$$

for any $\alpha \in \mathbb{C}$. *The implied constant is uniform with respect to* α.

A direct consequence of partial summation and Lemma 2.9 is the following smoothed Wilton bound.

Lemma 2.10. *Let the notation be as in Lemma 2.7,* $\mathrm{Re}(\tau) = 1$, $K, M \ge 1$, *and* $W_{K,M} : (0, \infty) \to \mathbb{C}$ *be a smooth function with compact support in* $[1, 2]$ *that satisfies* (1-6). *Then*

$$\sum_{\substack{\nu \in h_\sigma + \Lambda_\sigma^* \\ N(\nu) \le X}} \rho_{u,\sigma}(\nu) \check{e}(\alpha \nu) W_{K,M}\left(\frac{N(\nu)}{X}\right) \ll_{u,\sigma,\varepsilon} MKX^{1/2+\varepsilon}$$

for any $\alpha \in \mathbb{C}$. *The implied constant is uniform with respect to* α.

3. Cubic metaplectic forms

3.1. *Cubic Kubota character.* Recall that $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\omega])$. It is well known that $\Gamma = \langle P, T, E \rangle$, where

$$P := \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $0 \ne C \in \mathbb{Z}[\omega]$ satisfy $C \equiv 0 \pmod 3$, and

$$\Gamma_1(C) := \{\gamma \in \Gamma : \gamma \equiv I \pmod C\}.$$

Observe that $\Gamma_1(C)$ is a normal subgroup of Γ since it is the kernel of the reduction modulo C map. Let

$$\Gamma_2 := \langle \mathrm{SL}_2(\mathbb{Z}), \Gamma_1(3) \rangle = \mathrm{SL}_2(\mathbb{Z})\Gamma_1(3) = \Gamma_1(3)\,\mathrm{SL}_2(\mathbb{Z}), \tag{3-1}$$

where the last two equalities follow because $\Gamma_1(3)$ is normal in Γ. We also have $[\Gamma : \Gamma_2] = 27$ (see [Patterson 1977, §2] for the calculation). Recall that $\chi : \Gamma_2 \to \{1, \omega, \omega^2\}$ is the cubic Kubota character defined in Section 1.1. The cusps of Γ_2 are $P(\Gamma_2) = \{\infty, \omega, \omega^2\}$, and the only essential cusp of Γ_2 with respect to χ is ∞.

3.2. *Cubic Shimura lift.* Suppose $\Gamma' \subseteq \Gamma_2$ is a subgroup with $[\Gamma_2 : \Gamma'] < \infty$. If $h \in L(\Gamma' \backslash \mathbb{H}^3, \chi, \tau)$, then h is said to be a cubic metaplectic form on Γ' with spectral parameter τ (abbreviated to cubic metaplectic form). In this section we specialise to the lowest possible level $\Gamma' = \Gamma_2$, and focus on the finite-dimensional subspace $L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau) \subset L(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$ that contains square integrable cubic metaplectic forms.

We say that $h \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$ is a Hecke eigenform if it is an eigenfunction for all Hecke operators $\{T_{\nu^3}\}_{\nu \in \mathbb{Z}[\omega] \setminus \{0\}}$, i.e., $T_{\nu^3} h = \tilde{\lambda}_h(\nu^3) h$ for some $\tilde{\lambda}_h(\nu^3) \in \mathbb{C}$ and all $\nu \in \mathbb{Z}[\omega] \setminus \{0\}$. There is an orthonormal basis (with respect to the Petersson inner product) of $L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi, \tau)$ consisting of Hecke eigenforms. Two automorphic forms are identified if they are constant multiples of one another. The discrete spectrum of Δ on $L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ is completely determined via the cubic Shimura correspondence of Flicker [1980] and Patterson [1998, Theorem 3.4]. In particular, there is a bijective correspondence between even (resp. odd) Hecke eigenforms $h \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi, \tau)$ and even (resp. odd) Hecke eigenforms $g \in L^2(\Gamma\backslash\mathbb{H}^3, \mathbf{1}, 3\tau-2)$, where in the latter case the Hecke operators are the standard ones $\{\mathcal{T}_\nu\}_{\nu\in\mathbb{Z}[\omega]\setminus\{0\}}$ on the trivial cover of Γ, i.e., $\mathcal{T}_\nu g = \lambda_g(\nu)g$ for some $\lambda_g(\nu) \in \mathbb{C}$ and all $\nu \in \mathbb{Z}[\omega]\setminus\{0\}$. Under this correspondence one also has

$$N(\nu^3)^{-1/2}\tilde{\lambda}_h(\nu^3) = N(\nu)^{-1/2}\lambda_g(\nu).$$

The only noncuspidal Hecke eigenform in $L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ is the cubic theta function of Patterson [1977],

$$\vartheta_3(w) := \mathrm{Res}_{s=4/3} E_3(w, s) \in L^2\big(\Gamma_2\backslash\mathbb{H}^3, \chi, \tfrac{4}{3}\big),$$

where $E_3(w, s)$ is the Kubota cubic Eisenstein series for $\Gamma_1(3)$ at the cusp ∞. Its Shimura correspondent is the constant function $1 \in L^2(\Gamma\backslash\mathbb{H}^3, \mathbf{1}, 2)$. The countably many other Hecke eigenforms $h_k \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ are Maass cusp forms, whose Shimura correspondents $g_k \in L^2(\Gamma\backslash\mathbb{H}^3, \mathbf{1})$ are also Maass cusp forms. All spectral parameters are nonexceptional, i.e., $\mathrm{Re}(\tau_{h_k}) = \mathrm{Re}(\tau_{g_k}) = 1$ for $k = 1, 2, \ldots$. We also have $0 \le \mathrm{Im}(\tau_{f_1}) \le \mathrm{Im}(\tau_{f_2}) \le \ldots$, where $\mathrm{Im}(\tau_{h_k}) \to \infty$ as $k \to \infty$.

3.3. *Cubic Kloosterman sums.* We will encounter cubic Kloosterman sums attached to the cubic Kubota character in our computations.

Let $\Gamma' \subseteq \Gamma_2$ with $[\Gamma_2 : \Gamma'] < \infty$, and let $\sigma, \xi \in \mathrm{SL}_2(\mathbb{Z}[\omega])$ denote cusps of Γ'. Let

$$\mathcal{C}(\sigma, \xi) := \big\{c \in \mathbb{Z}[\omega]\setminus\{0\} : \sigma\big(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\big)\xi^{-1} \in \Gamma'\big\}$$

be the set of allowable moduli for the cusp pair (σ, ξ). For $m \in \Lambda_\sigma^*$, $n \in \Lambda_\xi^*$, and $c \in \mathcal{C}(\sigma, \xi)$, the cubic Kloosterman sum is

$$K_{\Gamma',\sigma,\xi}(m, n, c) := \sum_{\substack{a \,(\mathrm{mod}\, c\Lambda_\sigma) \\ d \,(\mathrm{mod}\, c\Lambda_\xi) \\ \sigma\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)\xi^{-1} \in \Gamma'}} \overline{\chi\left(\sigma\begin{pmatrix} a & * \\ c & d \end{pmatrix}\xi^{-1}\right)}\, \check{e}\left(\frac{ma+nd}{c}\right), \tag{3-2}$$

where $\chi : \Gamma_2 \to \{1, \omega, \omega^2\}$ is the cubic Kubota character. We have the following Weil bound [1948].

Lemma 3.1 [Livné and Patterson 2002, Proposition 5.1; Louvel 2014, (2.6)]. *Let the notation be as above. Then for $m, n \in \mathbb{Z}[\omega]$ and $c \in \mathcal{C}(\sigma, \xi)$, we have*

$$|K_{\Gamma',\sigma,\xi}(m, n, c)| \le 2^{\omega(c)} N((m, n, c)) N(c)^{1/2},$$

where $\omega(c)$ denotes the number of distinct prime divisors of c.

Remark 3.2. In [Livné and Patterson 2002, Proposition 5.1] (and propagated in [Louvel 2014, §2]), it appears the bound in Lemma 3.1 is stated suboptimally with a factor $N((m, n, c))$ instead of $N((m, n, c))^{1/2}$. This makes no difference to us because $(m, n, c) = 1$ in any instance when Lemma 3.1 is used in this paper.

Lemma 3.3. *Suppose that* $\Gamma' = \Gamma_1(3)$ *and* $\sigma = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. *Then*

$$K_{\Gamma_1(3),\sigma,\sigma}(m,n,c) = \sum_{\substack{a,d \ (\mathrm{mod}\ 3c)\\ a,d\equiv 1 \ (\mathrm{mod}\ 3)\\ ad\equiv 1 \ (\mathrm{mod}\ c)}} \left(\frac{c}{d}\right)_3 \check{e}\left(\frac{ma+nd}{c}\right), \tag{3-3}$$

for any $c \in 3\mathbb{Z}[\omega] \setminus \{0\}$, *and* $m, n \in \lambda^{-3}\mathbb{Z}[\omega]$.

Proof. Observe that $\Lambda_\sigma = 3\mathbb{Z}[\omega]$, $\Lambda_\sigma^* = 3^{-1}\mathbb{Z}[\omega]^* = \lambda^{-3}\mathbb{Z}[\omega]$, and $\mathcal{C}(\sigma,\sigma) = 3\mathbb{Z}[\omega] \setminus \{0\}$. Observe that $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma_1(3)$ if and only if $a \equiv d \equiv 1 \pmod 3$, $b, c \equiv 0 \pmod 3$, and $ad - bc = 1$. For $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma_1(3)$ with $c \neq 0$ we have $\chi(\gamma) = (c/a)_3$ by (1-4). The claim now follows from (3-2), (2-2), and (2-3). □

Lemma 3.4. *Suppose that* $\Gamma' = \Gamma_1(3)$, $\sigma = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, *and* $\xi = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. *Then*

$$K_{\Gamma_1(3),\sigma,\xi}(m,n,c) = \sum_{\substack{a,d \ (\mathrm{mod}\ 3c)\\ a,d\equiv 0 \ (\mathrm{mod}\ 3)\\ ad\equiv 1 \ (\mathrm{mod}\ c)}} \left(\frac{d}{c}\right)_3 \check{e}\left(\frac{ma+nd}{c}\right) \tag{3-4}$$

for any $c \in \mathbb{Z}[\omega]$ *such that* $c \equiv 1 \pmod 3$, *and* $m, n \in \lambda^{-3}\mathbb{Z}[\omega]$.

Proof. Observe that $\Lambda_\sigma = \Lambda_\xi = 3\mathbb{Z}[\omega]$ and that $\Lambda_\xi^* = \Lambda_\sigma^* = 3^{-1}\mathbb{Z}[\omega]^* = \lambda^{-3}\mathbb{Z}[\omega]$. Let $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}[\omega])$. Observe that $\sigma\gamma\xi^{-1} \in \Gamma_1(3)$ if and only if $a \equiv d \equiv 0 \pmod 3$, $c \equiv 1 \pmod 3$, $b \equiv -1 \pmod 3$, and $ad - bc = 1$. After recalling that χ is homomorphism on Γ_2 such that $\chi|_{\mathrm{SL}_2(\mathbb{Z})} \equiv 1$, we see that $\chi(\gamma\xi^{-1}) = \chi(\xi^{-1}\gamma) = (-a/c)_3 = (a/c)_3$ by (1-4) and the convention (2-4). The claim now follows from (3-2). □

4. The cubic large sieve

Implicit in [Heath-Brown 2000] is a version of cubic large sieve where one of the variables is not required to be squarefree. Here we record the relevant results.

Theorem 4.1. *Let* $\varepsilon > 0$ *be given*, $M, N \geq \frac{1}{2}$ *and* $\boldsymbol{\Psi} = (\Psi_c)$ *be a* $\mathbb{C}$*-valued sequence supported on* $c \in \mathbb{Z}[\omega]$ *with* $c \equiv 1 \pmod 3$ *and* $N(c) \sim N$. *Then*

$$\sum_{N(d)\sim M} \left| \sum_{\substack{N(c)\sim N\\ c\equiv 1 \ (\mathrm{mod}\ 3)}} \mu^2(c)\Psi_c \left(\frac{c}{d}\right)_3 \right|^2 \ll_\varepsilon (MN)^\varepsilon M^{1/3}(M+N)\|\boldsymbol{\mu}^2\boldsymbol{\Psi}\|_2^2.$$

Proof. This follows from [Heath-Brown 2000, (22) (and the display above it), (28) and the second display on p. 123], □

Corollary 4.2. *Let the notation be as in Theorem 4.1 and* $\boldsymbol{\Omega} = (\Omega_d)$ *be a* $\mathbb{C}$*-valued sequence supported on* $d \in \mathbb{Z}[\omega]$ *with* $N(d) \sim M$. *Then*

$$\sum_{N(d)\sim M} \sum_{\substack{N(c)\sim N\\ c\equiv 1 \ (\mathrm{mod}\ 3)}} \Omega_d \mu^2(c)\Psi_c \left(\frac{d}{c}\right)_3 \ll_\varepsilon (MN)^\varepsilon M^{1/6}(M^{1/2}+N^{1/2})\|\boldsymbol{\Omega}\|_2\|\boldsymbol{\mu}^2\boldsymbol{\Psi}\|_2.$$

Proof. Application of the Cauchy–Schwarz inequality, unique factorisation in $\mathbb{Z}[\omega]$, (2-1), and Theorem 4.1 gives

$$\Bigg|\sum_{N(d)\sim M}\sum_{\substack{N(c)\sim N\\ c\equiv 1 \pmod 3}}\Omega_d\mu^2(c)\Psi_c\left(\frac{d}{c}\right)_3\Bigg|^2\le \|\mathbf{\Omega}\|_2^2\cdot\Bigg(\sum_{\zeta}\sum_{k\ge 0}\sum_{\substack{N(\zeta\lambda^k m)\sim M\\ m\equiv 1 \pmod 3}}\Bigg|\sum_{\substack{N(c)\sim N\\ c\equiv 1 \pmod 3}}\mu^2(c)\left(\frac{\zeta\lambda^k}{c}\right)_3\Psi_c\left(\frac{c}{m}\right)_3\Bigg|^2\Bigg)$$
$$\ll (MN)^{\varepsilon}M^{1/3}(M+N)\|\mathbf{\Omega}\|_2^2\|\boldsymbol{\mu}^2\mathbf{\Psi}\|_2^2,$$

as required. □

5. The Browning–Vishe circle method for number fields

The proof of our Type-II estimates will use a circle method over number fields due to Browning and Vishe [2014, Theorem 1.2]. Their work generalises work of Heath-Brown [1996, Theorem 1] (over $\mathbb{Q}$), and ultimately relies on the δ-function technology of Duke, Friedlander, and Iwaniec [Duke et al. 1993].

Let $L/\mathbb{Q}$ be a number field of degree $d\ge 2$ with ring of integers $\mathcal{O}_L$ and unit group $\mathcal{O}_L^{\times}$. Let $\mathfrak{a}\trianglelefteq\mathcal{O}_L$ be an integral ideal, $N(\mathfrak{a}) := \#\mathcal{O}_L/\mathfrak{a}$ denote the ideal norm of $\mathfrak{a}$, and

$$\delta_L(\mathfrak{a}) := \begin{cases} 1 & \text{if } \mathfrak{a}=(0),\\ 0 & \text{otherwise.}\end{cases}$$

Remark 5.1. One obtains an indicator function on $\mathcal{O}_L$ by restricting to principal ideals, in which case one writes $\delta_L((\nu))=\delta_L(\nu)$ for any $\nu\in\mathcal{O}_L$. We also have $N((\nu))=N(\nu)$, where the latter is the norm of an element of $\mathcal{O}_L$.

Theorem 5.2 [Browning and Vishe 2014, Theorem 1.2]. *Let $L/\mathbb{Q}$ be a number field of degree $d\ge 2$, $C\ge 1$, and $\mathfrak{a}\trianglelefteq\mathcal{O}_L$ be an integral ideal. Then there exists a positive constant k_C and an infinitely differentiable function $h(x,y):(0,\infty)\times\mathbb{R}\to\mathbb{R}$ (depending on $L/\mathbb{Q}$) such that*

$$\delta_L(\mathfrak{a}) = \frac{k_C}{C^{2d}}\sum_{(0)\ne\mathfrak{c}\subseteq\mathcal{O}_L}\ \sideset{}{^*}\sum_{\sigma \pmod{\mathfrak{c}}}\sigma(\mathfrak{a})h\left(\frac{N(\mathfrak{c})}{C^d},\frac{N(\mathfrak{a})}{C^{2d}}\right), \tag{5-1}$$

where the notation $\Sigma^*_{\sigma \pmod{\mathfrak{c}}}$ means that the sum is taken over primitive additive characters (extended to ideals) modulo $\mathfrak{c}$. The constant k_C satisfies

$$k_C = 1+O_{L/\mathbb{Q},D}(C^{-D}) \quad \textit{for any } D>0. \tag{5-2}$$

Furthermore, we have

$$h(x,y)\ll_{L/\mathbb{Q}} x^{-1} \quad \textit{for all } y\in\mathbb{R}, \tag{5-3}$$
$$h(x,y)\ne 0 \quad \textit{only if} \quad x\le\max\{1,2|y|\}. \tag{5-4}$$

Remark 5.3. In practice one usually chooses $C := X^{1/(2d)}$ to detect the condition $\mathfrak{a}=(0)$ for a sequence of ideals of $\mathcal{O}_L$ with norm less than or equal to X. This means that for $\mathfrak{c}$ (see (5-1)) in the generic range $N(\mathfrak{c})\asymp X^{1/2}$ there is no oscillation in the weight function $h(x,y)$.

Lemma 5.4 [Browning and Vishe 2014, Lemma 3.1]. *Let the notation be as in Theorem 5.2. The function $h(x, y)$ vanishes when $x \geq 1$ and $|y| \leq x/2$. When $x \leq 1$ and $|y| \leq x/2$, we have*

$$\frac{\partial}{\partial y} h(x, y) = 0. \tag{5-5}$$

Lemma 5.5 [Browning and Vishe 2014, Lemma 3.2]. *Let the notation be as in Theorem 5.2. Then for $i, j, D \in \mathbb{Z}_{\geq 0}$, we have*

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h(x, y) \ll_{L/\mathbb{Q},i,j,D} x^{-i-j-1}\left(x^D + \min\left\{1, \left(\frac{x}{|y|}\right)^D\right\}\right). \tag{5-6}$$

The term x^D on the right side of (5-6) can be omitted if $j \neq 0$.

Corollary 5.6. *Let the notation be as in Theorem 5.2. Then for any $j \in \mathbb{Z}_{\geq 1}$ we have*

$$\frac{\partial^j}{\partial y^j} h(x, y) \ll_{L/\mathbb{Q},j} 1. \tag{5-7}$$

Proof. If $x \leq 1$ and $|y| \leq x/2$, then Lemma 5.4 implies that

$$\frac{\partial^j}{\partial y^j} h(x, y) = 0$$

for all $j \in \mathbb{Z}_{\geq 1}$. If $x \leq 1$ and $|y| \geq x/2$, then Lemma 5.5 (with $i = 0$ and $D = j + 1$) gives

$$\frac{\partial^j}{\partial y^j} h(x, y) \ll_{L/\mathbb{Q},j} 1$$

for all $j \in \mathbb{Z}_{\geq 1}$. If $x \geq 1$, then Lemma 5.4 (the vanishing condition on h) and Lemma 5.5 (with $i = D = 0$) gives

$$\frac{\partial^j}{\partial y^j} h(x, y) \ll_{L/\mathbb{Q},j} 1,$$

for all $j \in \mathbb{Z}_{\geq 1}$. Putting all three cases together gives the result. □

6. Vaughan's identity

Here we record a celebrated identity of Vaughan [1975] adapted to our situation.

Proposition 6.1. *Let $R, S \geq 1$. Then for any $\nu \in \mathbb{Z}[\omega]$ with $\nu \equiv 1 \pmod 3$ and $N(\nu) > S$, we have*

$$\Lambda(\nu) = \sum_{\substack{a|\nu \\ N(a)\leq R}} \mu(a) \log\left(\frac{N(\nu)}{N(a)}\right) - \sum\sum_{\substack{ab|\nu \\ N(a)\leq R \\ N(b)\leq S}} \mu(a)\Lambda(b) + \sum\sum_{\substack{ab|\nu \\ N(a)>R \\ N(b)>S}} \mu(a)\Lambda(b). \tag{6-1}$$

If $N(\nu) \leq S$, the right side of (6-1) vanishes.

7. Proof of Theorem 1.2 and Corollary 1.3

In this section we prove Theorem 1.2 and Corollary 1.3 assuming the truth of Lemma 1.4 and the main inputs: Theorem 1.5 and Proposition 1.6.

Proof of Theorem 1.2 assuming Lemma 1.4, Theorem 1.5, and Proposition 1.6. Recall the definition of the quantity $\mathscr{P}_f(X, v, u; W_K)$ given in (1-7). We apply Proposition 6.1 to (1-7). The parameters $R, S \geq 1$ used in our application of Proposition 6.1 will be chosen at a later point in the proof and will satisfy

$$S < \frac{X}{10000} \quad \text{and} \quad 10000X < RS < 10000000X \quad \text{say}, \tag{7-1}$$

for all sufficiently large X. Since the support of W_K is contained in $[1, 2]$ and $S < X/10000$ by (7-1), all summands in $\mathscr{P}_f(X, v, u; W_K)$ are automatically supported on the condition $N(\nu) > S$. Note that the right most sum in (6-1) vanishes since the support of W_K is contained in $[1, 2]$ and $RS > 10000X$ by (7-1). We insert a smooth partition of unity in the a and b variables in the second sum in (6-1), and then interchange these summations with the ν summation after substitution of (6-1) into (1-7). We obtain

$$\mathscr{P}_f(X, v, u; W_K) = \mathscr{P}_{1f}(X, R, v, u; W_K) - \sum_{\substack{1 \ll M \ll R \\ 1 \ll N \ll S \\ M, N \text{ dyadic}}} \sum \mathscr{P}_{2f}(X, M, N, v, u; W_K), \tag{7-2}$$

where

$$\mathscr{P}_{1f}(\cdots) := \sum_{\substack{a, b \equiv 1 \pmod 3 \\ ab \equiv u \pmod v \\ N(a) \leq R}} \sum \mu(a) \log(N(b)) \rho_f(\lambda^{-3} ab) W_K\left(\frac{N(\lambda^{-3} ab)}{X}\right), \tag{7-3}$$

$$\mathscr{P}_{2f}(\cdots) := \sum_{\substack{a, b, c \equiv 1 \pmod 3 \\ abc \equiv u \pmod v \\ N(a) \leq R \\ N(b) \leq S}} \sum \sum \mu(a) \Lambda(b) \rho_f(\lambda^{-3} abc) W_K\left(\frac{N(\lambda^{-3} abc)}{X}\right) U\left(\frac{N(a)}{M}\right) U\left(\frac{N(b)}{N}\right), \tag{7-4}$$

and $U : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a fixed smooth function with compact support in $[1, 2]$ such that

$$\sum_{L \text{ dyadic}} U\left(\frac{N(\ell)}{L}\right) = 1 \quad \text{for all } 0 \neq \ell \in \mathbb{Z}[\omega].$$

7.1. ***Estimate for $\mathscr{P}_{1f}(X, R, v, u; W_K)$.*** Rewriting (7-3) using additive characters we obtain

$$\mathscr{P}_{1f}(\cdots) = \sum_{\substack{a \equiv 1 \pmod 3 \\ N(a) \leq R}} \frac{\mu(a)}{N(av)} \sum_{j \pmod{av}} \check{e}\left(-\frac{j\eta}{av}\right) \sum_{\nu \in \lambda^{-3}\mathbb{Z}[\omega]} \rho_f(\nu) \check{e}\left(\frac{j\lambda^3 \nu}{av}\right) \log\left(\frac{N(\nu)}{N(\lambda^{-3} a)}\right) W_K\left(\frac{N(\nu)}{X}\right), \tag{7-5}$$

where $\eta \in \mathbb{Z}[\omega]$ is such that $\eta \equiv u \pmod v$ and $\eta \equiv 0 \pmod a$. Applying Lemma 2.10 (while noting Remark 1.1) to the ν summation and estimating the other sums trivially using the triangle inequality we get

$$\mathscr{P}_{1f}(\cdots) \ll (RX)^{\varepsilon} KRX^{1/2} \tag{7-6}$$

uniformly in the modulus v.

7.2. ***Two estimates for $\mathscr{P}_{2f}(\cdots)$.***

7.2.1. *First estimate.* For the first estimate we treat (7-4) as an average Type-I sum. That is, in (7-4) we let $h = ab$,

$$\gamma'_h(M, N) := \sum_{h = ab} \mu(a) \Lambda(b) U\left(\frac{N(a)}{M}\right) U\left(\frac{N(b)}{N}\right), \tag{7-7}$$

and interpret c as the "smooth" summation variable. We then decompose

$$\mathscr{P}_{2f}(\cdots) = \mathscr{P}^{\star}_{2f}(\cdots) + \mathscr{P}^{\dagger}_{2f}(\cdots), \tag{7-8}$$

where $\mathscr{P}^{\star}_{2f}(\cdots)$ and $\mathscr{P}^{\dagger}_{2f}(\cdots)$ have the factors $\mu^2(h)$ and $1-\mu^2(h)$ inserted, respectively. The weight $\mu^2(h)\gamma'_h(M,N)$ in $\mathscr{P}^{\star}_{2}(\cdots)$ is supported on squarefree elements of $\mathbb{Z}[\omega]$. We apply Proposition 1.6 (see (1-13)) to obtain

$$\mathscr{P}^{\star}_{2f}(\cdots) \ll (XKN(v))^{\varepsilon} K^{14/3} N(v)^{5/6}(MN)^{5/6} X^{1/3}. \tag{7-9}$$

Applying Lemma 1.4 (see (1-12)) to the c-sum in $\mathscr{P}^{\dagger}(\cdots)$ we obtain

$$\begin{aligned}\mathscr{P}^{\dagger}_{2f}(\cdots) &\ll (XKN(v))^{\varepsilon} K^{4} N(v)^{1/2}(MN)^{1/2} \|(\mathbf{1}-\boldsymbol{\mu}^2)\boldsymbol{\gamma}'(M,N)\|_1 \\ &\ll (XKN(v))^{\varepsilon} K^{4} N(v)^{1/2} M^{3/2} N.\end{aligned} \tag{7-10}$$

Note that the support of the b variable in (7-7) imposed by the weight $(1-\mu^2(h))\Lambda(b)=0$ (supported on prime powers with exponent ≥ 2) was used to obtain (7-10). Substitution of (7-9) and (7-10) into (7-8) gives

$$\mathscr{P}_{2f}(\cdots) \ll (XKN(v))^{\varepsilon}(K^{14/3} N(v)^{5/6}(MN)^{5/6} X^{1/3} + K^{4} N(v)^{1/2} M^{3/2} N). \tag{7-11}$$

7.2.2. *Second estimate.* For the second estimate we treat (7-4) as a Type-II sum. That is, we let $h=bc$, and

$$\gamma_h(N, X/MN) := \sum_{h=bc} \Lambda(b) U\left(\frac{N(b)}{N}\right).$$

Observe that the weight $\mu(a)U(N(a)/M)$ is supported only on squarefree a. Thus we apply Theorem 1.5 (see (1-14)) and obtain

$$\mathscr{P}_{2f}(\cdots) \ll (XKN(v))^{\varepsilon} K^{8} N(v)^{4}(XM^{-1/2} + (MX)^{3/4}). \tag{7-12}$$

7.3. *Conclusion.* We use (7-6) to estimate the first term of (7-2). Let $1 \ll L \ll R$. We use (7-11) (resp. (7-12)) to estimate the second term in (7-2) when $M \leq L$ (resp. $M \geq L$). The net result is

$$\mathscr{P}_f(X, v, u; W_K) \ll (XKN(v))^{\varepsilon} K^{8} N(v)^{4}(RX^{1/2} + (LS)^{5/6} X^{1/3} + L^{3/2} S + XL^{-1/2} + (RX)^{3/4}) \tag{7-13}$$

for any $R, S \geq 1$ satisfying (7-1) and $1 \ll L \ll R$. The choice of parameters

$$R = 1000X^{5/17}, \quad S = 1000X^{12/17}, \quad \text{and} \quad L = X^{1/17},$$

satisfies (7-1) for all sufficiently large X, and substitution into (7-13) yields

$$\mathscr{P}_f(X, v, u; W_K) \ll (XKN(v))^{\varepsilon} K^{8} N(v)^{4} X^{1-1/34},$$

as required. □

We now remove the smoothing.

Proof of Corollary 1.3. Let $\Delta := K^{-1}$ with $K \geq 2$ and suppose that $W_K : (0, \infty) \to \mathbb{R}$ is smooth and satisfies

$$\begin{gathered}\operatorname{supp}(W_K) \subset \left[\tfrac{5}{4} - \Delta, \tfrac{7}{4} + \Delta\right], \quad 0 \leq W(x) \leq 1 \quad \text{for all } x > 0, \\ W(x) = 1 \quad \text{for } x \in \left[\tfrac{5}{4}, \tfrac{7}{4}\right], \quad \text{and} \quad W^{j}(x) \ll_j K^{j}.\end{gathered} \tag{7-14}$$

Then for any $Z \gg 1$ we have

$$\sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 \nu \equiv u \ (\mathrm{mod}\ v) \\ 5Z/4 < N(\nu) \leq 7Z/4}} \rho_f(\nu) \Lambda(\lambda^3 \nu) = \sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 \nu \equiv u \ (\mathrm{mod}\ v)}} \rho_f(\nu) \Lambda(\lambda^3 \nu) W_K\left(\frac{N(\nu)}{Z}\right) + O_v(K^{-1/2} Z^{1+\varepsilon}), \tag{7-15}$$

where the error term follows by Cauchy–Schwarz, Lemma 2.7, and the support of W_K. Applying Theorem 1.2 to the right side (7-15) gives

$$\sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 \nu \equiv u \ (\mathrm{mod}\ v) \\ 5Z/4 < N(\nu) \leq 7Z/4}} \rho_f(\nu) \Lambda(\lambda^3 \nu) \ll_v (ZK)^{\varepsilon} (K^8 Z^{1-1/34} + K^{-1/2} Z).$$

We choose $K = Z^{1/289}$ to obtain

$$\sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 \nu \equiv u \ (\mathrm{mod}\ v) \\ 5Z/4 < N(\nu) \leq 7Z/4}} \rho_f(\nu) \Lambda(\lambda^3 \nu) \ll_v Z^{1-1/578+\varepsilon}. \tag{7-16}$$

Summing over intervals $[5Z/4, 7Z/4]$ with $7Z/4 \leq X$ yields (1-10).

To prove (1-11) we first observe that

$$\widetilde{\mathscr{P}}_f(X; v, u) - \mathscr{P}_f(X; v, u) = \sum\sum_{\substack{k \in \mathbb{Z}_{\geq 2}, \varpi \in \mathbb{Z}[\omega] \\ \varpi \text{ prime} \\ \varpi^k \equiv u \ (\mathrm{mod}\ v) \\ N(\lambda^{-3} \varpi^k) \leq X}} \rho_f(\lambda^{-3} \varpi^k) \log N(\varpi). \tag{7-17}$$

Applying Cauchy–Schwarz to the double sum in (7-17) shows that the right side of (7-17) is

$$\begin{aligned} &\ll X^{1/4+\varepsilon} \Biggl(\sum\sum_{\substack{k \in \mathbb{Z}_{\geq 2}, \varpi \in \lambda^{-3}\mathbb{Z}[\omega] \\ \varpi \text{ prime} \\ N(\lambda^{-3} \varpi^k) \leq X}} |\rho_f(\lambda^{-3} \varpi^k)|^2 \Biggr)^{1/2} \\ &\ll X^{1/4+\varepsilon} \Biggl(\sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\lambda^{-3} \nu) \leq X}} |\rho_f(\nu)|^2 \Biggr)^{1/2} \ll X^{3/4+\varepsilon}, \end{aligned} \tag{7-18}$$

where the last inequality follows from using Lemma 2.7. The result (1-11) now follows. □

The rest of the paper will be dedicated to proving Lemma 1.4, Theorem 1.5, and Proposition 1.6.

8. Voronoi summation formulae for twists

In this section we develop a Voronoi summation formula for twists of a cusp form $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ with spectral parameter $\tau_f \in 1+i\mathbb{R}$ by appropriate non-Archimedean and Archimedean characters. Development of this formula requires some care because we are working with the group $\Gamma_2 = \langle \mathrm{SL}_2(\mathbb{Z}), \Gamma_1(3)\rangle$ in $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\omega])$.

8.1. ***Twists and Dirichlet series.*** We will need consider cubic metaplectic forms on groups $\Gamma_1(C)$ with $C \equiv 0 \pmod 9$, i.e., the spaces $L^2(\Gamma_1(C)\backslash\mathbb{H}^3, \chi, \tau)$ for $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) \geq 1$. To simplify our exposition we focus on the nonexceptional case, i.e., $\mathrm{Re}(\tau) = 1$. Suppose that $\Psi : \lambda^{-1}\mathbb{Z}[\omega] \to \mathbb{C}$ is periodic modulo $(\lambda^m r)(\lambda^{-1}\mathbb{Z}[\omega])$. The Ψ-twist (at ∞) of a cusp form $F \in L^2(\Gamma_1(C)\backslash\mathbb{H}^3, \chi, \tau)$ is defined by

$$F(w;\Psi) := \sum_{0\neq\nu\in(\lambda C)^{-1}\mathbb{Z}[\omega]} \rho_F(\nu)\Psi(C\nu)vK_{\tau-1}(4\pi|\nu|v)\check{e}(\nu z), \qquad w=(z,v)\in\mathbb{H}^3, \tag{8-1}$$

also denoted by $(F\otimes\Psi)(w)$. By [Proskurin 1998, Theorem 0.3.12] and its proof we have

$$F(\,\cdot\,;\Psi) \in L^2(\Gamma_1(\lambda^{2m}r^2C)\backslash\mathbb{H}^3, \chi, \tau) \quad \text{is a cusp form.} \tag{8-2}$$

Remark 8.1. For the purposes of twisting we view the cusp form $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ in the larger space $L^2(\Gamma_1(\lambda^4)\backslash\mathbb{H}^3, \chi)$. This is immaterial in the final results and only involves extra fixed powers of the prime λ in the formulae.

In what follows it will be instructive to open the definition $\check{e}(z) := e(z+\bar{z})$, $z \in \mathbb{C}$. We remind the reader that the function $F(w;\Psi)$ in (8-1) is a function in z, $\bar{z}$, and v (although the notation suppresses this). For $n \in \mathbb{Z}$, we define

$$F(w;\Psi,n) := \frac{1}{(2\pi i)^{|n|}}\cdot\begin{cases} \left(\dfrac{\partial}{\partial z}\right)^n F(w;\Psi) & \text{if } n>0,\\ F(w;\Psi) & \text{if } n=0,\\ \left(\dfrac{\partial}{\partial \bar{z}}\right)^{|n|} F(w;\Psi) & \text{if } n<0,\end{cases} \qquad w=(z,v)\in\mathbb{H}^3. \tag{8-3}$$

To complement (8-1), we have the Fourier expansions (at ∞) for $n \in \mathbb{Z}\setminus\{0\}$,

$$F(w;\Psi,n) := \sum_{0\neq\nu\in(\lambda C)^{-1}\mathbb{Z}[\omega]} \begin{cases} \rho_F(\nu)\nu^n\Psi(C\nu)vK_{\tau-1}(4\pi|\nu|v)e(\nu z+\overline{\nu z}) & \text{if } n>0,\\ \rho_F(\nu)\bar{\nu}^{|n|}\Psi(C\nu)vK_{\tau-1}(4\pi|\nu|v)e(\nu z+\overline{\nu z}) & \text{if } n<0.\end{cases} \tag{8-4}$$

Suppose that $\psi : \mathbb{Z}[\omega] \to \mathbb{C}$ is periodic modulo $\lambda^m r$. The (normalised) Fourier transform $\hat{\psi} : \lambda^{-1}\mathbb{Z}[\omega] \to \mathbb{C}$ is given by

$$\hat{\psi}(x) := \frac{1}{N(\lambda^m r)} \sum_{u \pmod{\lambda^m r}} \psi(u)\check{e}\left(\frac{ux}{\lambda^m r}\right), \qquad x \in \lambda^{-1}\mathbb{Z}[\omega], \tag{8-5}$$

and is periodic modulo $(\lambda^m r)(\lambda^{-1}\mathbb{Z}[\omega])$. Fourier inversion asserts that

$$\psi(u) := \sum_{x\in\lambda^{-1}\mathbb{Z}[\omega]/(\lambda^m r)(\lambda^{-1}\mathbb{Z}[\omega])} \hat{\psi}(x)\check{e}\left(-\frac{xu}{\lambda^m r}\right) \quad \text{for } u \in \mathbb{Z}[\omega]. \tag{8-6}$$

For $n \in \mathbb{Z}$ consider the Dirichlet series

$$\mathcal{D}(s, F; \Psi, n) := \sum_{\substack{\nu \neq 0 \\ \nu \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \frac{\rho_F(\nu)\Psi(C\nu)\left(\frac{\nu}{|\nu|}\right)^n}{N(\nu)^s}, \quad \mathrm{Re}(s) > 1,$$

and the associated Mellin transform

$$\Lambda(s, F; \Psi, n) := \int_0^\infty F(v; \Psi, n) v^{2s+|n|-2}\, dv,$$

where we let v denote $(0, v)$ for $v > 0$. Let

$$G_\infty(s, \tau, n) := \tfrac{1}{4}(2\pi)^{-2s-|n|}\Gamma\big(s + \tfrac{1}{2}|n| - \tfrac{1}{2}(\tau - 1)\big)\Gamma\big(s + \tfrac{1}{2}|n| + \tfrac{1}{2}(\tau - 1)\big), \quad s \in \mathbb{C}. \tag{8-7}$$

Lemma 8.2. *Let $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) = 1$, $C \in \mathbb{Z}[\omega]$ with $C \equiv 0 \pmod 9$, $F \in L^2(\Gamma_1(C)\backslash\mathbb{H}^3, \chi, \tau)$ be a cusp form, and $n \in \mathbb{Z}$. For $\mathrm{Re}(s) > 1$ we have*

$$\Lambda(s, F; \Psi, n) = G_\infty(s, \tau, n)\mathcal{D}(s, F; \Psi, n),$$

where $G_\infty(s, \tau, n)$ is given by (8-7)

Proof. The proofs for the cases $n > 0$, $n = 0$, and $n < 0$ are analogous. We give details for the case $n > 0$. For $\mathrm{Re}(s) > 1$ and $n > 0$ we have

$$\begin{aligned}
\Lambda(s, F; \Psi, n) &= \int_0^\infty \sum_{\substack{\nu \neq 0 \\ \nu \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \rho_F(\nu)\nu^n \Psi(C\nu) K_{\tau-1}(4\pi|\nu|v) v^{2s+n-1}\, dv \\
&= \frac{1}{(4\pi)^{2s+n}} \sum_{\substack{\nu \neq 0 \\ \nu \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \frac{\rho_F(\nu)\Psi(C\nu)\left(\frac{\nu}{|\nu|}\right)^n}{N(\nu)^s} \int_0^\infty K_{\tau-1}(T) T^{2s+n-1}\, dT \\
&= \tfrac{1}{4}(2\pi)^{-2s-n}\Gamma\big(s + \tfrac{1}{2}n - \tfrac{1}{2}(\tau - 1)\big)\Gamma\big(s + \tfrac{1}{2}n + \tfrac{1}{2}(\tau - 1)\big) \sum_{\substack{\nu \neq 0 \\ \nu \in (\lambda C)^{-1}\mathbb{Z}[\omega]}} \frac{\rho_F(\nu)\Psi(C\nu)\left(\frac{\nu}{|\nu|}\right)^n}{N(\nu)^s}.
\end{aligned} \tag{8-8}$$

The interchange of summation and integration above for $\mathrm{Re}(s) > 1$ is justified by absolute convergence (see Lemma 2.8 and [Olver et al. 2018, (10.25.3), (10.45.7)]). Furthermore, (8-8) follows from [Olver et al. 2018, (10.43.19)]. □

8.2. ***A special case.*** Recall that $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ is a cusp form with spectral parameter $\tau_f \in 1 + i\mathbb{R}$. For $\ell \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$, and $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q\mathbb{Z}[\omega]$, let

$$f(w; \lambda^\ell q, \eta) := \sum_{\substack{\nu \neq 0 \\ \nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3\nu \equiv \eta \pmod{\lambda^\ell q}}} \rho_f(\nu) v K_{\tau_f - 1}(4\pi|\nu|v)\check{e}(\nu z). \tag{8-9}$$

Following (8-1)–(8-4) we also have the functions $f(w; \lambda^\ell q, \eta, n)$ and their associated Fourier expansions for each $n \in \mathbb{Z}$. We ultimately need a Voronoi formulae for the Fourier coefficients of $f(w; \lambda^\ell q, \eta, n)$. Consider the Dirichlet series

$$\mathcal{D}(s, f; \lambda^\ell q, \eta, n) := \sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3 \nu \equiv \eta \pmod{\lambda^\ell q}}} \frac{\rho_f(\nu)\left(\frac{\nu}{|\nu|}\right)^n}{N(\nu)^s}, \quad \mathrm{Re}(s) > 1,$$

and the associated Mellin transform

$$\Lambda(s, f; \lambda^\ell q, \eta, n) := \int_0^\infty f(v; \lambda^\ell q, \eta, n) v^{2s+|n|-2}\, dv.$$

Then Lemma 8.2 asserts that

$$\Lambda(s, f; \lambda^\ell q, \eta, n) = G_\infty(s, \tau_f, n)\mathcal{D}(s, f; \lambda^\ell q, \eta, n) \quad \text{for } \mathrm{Re}(s) > 1, \tag{8-10}$$

where $G_\infty(s, \tau, n)$ is given by (8-7).

We detect the congruence condition in (8-9) using Fourier transforms. For $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q\mathbb{Z}[\omega]$, $0 \le m \le \ell$, and $r \mid q$, let

$$\psi_{\lambda^m r}(u)_\eta := \mathbf{1}_{\lambda^m r}(u) \cdot \check{e}\left(-\frac{\eta u}{\lambda^m r}\right), \tag{8-11}$$

where $\mathbf{1}_{\lambda^m r}(\cdot)$ is the principal character modulo $\lambda^m r$. As a shorthand we write $\psi_{\lambda^m r}(u) := \psi_{\lambda^m r}(u)_0$. The function $\psi_{\lambda^m r}(\cdot)_\eta$ is periodic modulo $\lambda^m r$. The Fourier transform is

$$\widehat{\psi_{\lambda^m r}(\cdot)_\eta}(k) = \frac{1}{N(\lambda^m r)} \sum_{\substack{u \pmod{\lambda^m r} \\ (u, \lambda^m r) = 1}} \check{e}\left(\frac{(k-\eta)u}{\lambda^m r}\right), \quad k \in \lambda^{-1}\mathbb{Z}[\omega]. \tag{8-12}$$

As a shorthand we write $\widehat{\psi_{\lambda^m r}}(k) := \widehat{\psi_{\lambda^m r}(\cdot)_0}(k)$. A straightforward computation shows the following orthogonality relation.

Lemma 8.3. *For $\ell \in \mathbb{Z}_{\ge 0}$ and $k, \eta \in \mathbb{Z}[\omega]/\lambda^\ell q\mathbb{Z}[\omega]$ we have*

$$\frac{1}{N(\lambda^\ell q)} \sum_{r \mid q} \sum_{m=0}^{\ell} N(\lambda^m r) \widehat{\psi_{\lambda^m r}(\cdot)_\eta}(k) = \delta_{k \equiv \eta \pmod{\lambda^\ell q}}. \tag{8-13}$$

The following lemma records the standard evaluation of Ramanujan sums.

Lemma 8.4. *Let $r \in \mathbb{Z}[\omega]$ satisfy $r \equiv 1 \pmod 3$ and $k \in \mathbb{Z}[\omega]$. Then we have*

$$\hat{\psi}_r(k) := \frac{1}{N(r)} \sum_{\substack{x \pmod r \\ (x,r)=1}} \check{e}\left(\frac{kx}{r}\right) = \frac{1}{N(r)} \mu\left(\frac{r}{(r,k)}\right) \frac{\varphi(r)}{\varphi\left(\frac{r}{(r,k)}\right)},$$

where $\varphi(\cdot)$ is the Euler φ-function on $\mathbb{Z}[\omega]$.

Proof. This follows from the multiplicativity of Ramanujan sums in the modulus r, and the first, fourth, and eighth cases in the evaluation on [Proskurin 1998, p. 11]. □

We next prove a straightforward but crucial lemma establishing the "flatness" of Ramanujan sums when averaged over the modulus.

Lemma 8.5. *Let $r, k \in \mathbb{Z}[\omega]$ satisfy $r \equiv 1 \pmod 3$, $\widehat{\psi_r}(k)$ be the normalised Ramanujan sum as in the statement of Lemma 8.4, and $\varepsilon > 0$. Then for $R \geq 1$ we have*

$$\sum_{\substack{r \in \mathbb{Z}[\omega] \\ N(r) \sim R \\ r \equiv 1 \ (\mathrm{mod}\ 3)}} |\widehat{\psi_r}(k)| \ll_\varepsilon \delta_{k=0} \cdot R + \delta_{k \neq 0} \cdot (N(k)R)^{\varepsilon}. \tag{8-14}$$

Proof. When $k = 0$ we have the trivial estimate $\ll R$. When $k \neq 0$ use Lemma 8.4, Möbius inversion, and the triangle inequality to obtain

$$\begin{aligned}\sum_{\substack{r \in \mathbb{Z}[\omega] \\ N(r) \sim R \\ r \equiv 1 \ (\mathrm{mod}\ 3)}} |\widehat{\psi_r}(k)| &= \sum_{\substack{\gamma \mid k \\ \gamma \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{r \in \mathbb{Z}[\omega] \\ N(r) \sim R \\ r \equiv 1 \ (\mathrm{mod}\ 3) \\ (r,k) = \gamma}} \frac{1}{N(r)} \frac{\varphi(r)}{\varphi(r/\gamma)} \\ &\leq \sum_{\substack{\gamma \mid k \\ \gamma \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{n, u \in \mathbb{Z}[\omega] \\ N(nu) \sim R/N(\gamma) \\ nu \equiv 1 \ (\mathrm{mod}\ 3)}} \frac{1}{N(\gamma n u)} \frac{\varphi(\gamma n u)}{\varphi(nu)} \\ &\ll (N(k)R)^{\varepsilon}, \end{aligned} \tag{8-15}$$

where the last display follows from standard lower bounds for the Euler φ-function and (2-14). □

Recall the convention for twisting a cusp form $f \in L^2(\Gamma_2 \backslash \mathbb{H}^3, \chi)$ in Remark 8.1. We replace the congruence $\lambda^3 \nu \equiv \eta \pmod{\lambda^\ell q}$ with the equivalent congruence $\lambda^4 \nu \equiv \lambda \eta \pmod{\lambda^{\ell+1} q}$. We have the immediate consequence.

Lemma 8.6. *Let $\ell \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$, $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$, and $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q \mathbb{Z}[\omega]$. For $\mathrm{Re}(s) > 1$ we have*

$$\mathcal{D}(s, f; \lambda^\ell q, \eta, n) = \frac{1}{N(\lambda^{\ell+1} q)} \sum_{r \mid q} \sum_{m=0}^{\ell+1} N(\lambda^m r) \mathcal{D}(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_{\lambda\eta}}, n),$$

$$\Lambda(s, f; \lambda^\ell q, \eta, n) = \frac{1}{N(\lambda^{\ell+1} q)} \sum_{r \mid q} \sum_{m=0}^{\ell+1} N(\lambda^m r) \Lambda(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_{\lambda\eta}}, n),$$

where $\widehat{\psi_{\lambda^m r}(\cdot)_\eta}$ is given in (8-12).

To obtain a functional equation for $\Lambda(s, f; \lambda^\ell q, \eta, n)$ under $s \to 1-s$ it suffices to establish a functional equation for each $\Lambda(s, f; \widehat{\psi_{\lambda^m r}(\cdot)_\eta}, n)$. We have two different cases according to whether $m \in \mathbb{Z}_{\geq 6}$ or $0 \leq m \leq 5$.

8.3. *Functional equation 1:* $m \in \mathbb{Z}_{\geq 6}$. Suppose that $\psi : \mathbb{Z}[\omega] \to \mathbb{C}$ is periodic modulo $\lambda^m r$, where $r \equiv 1 \pmod 3$.

Remark 8.7. The version of the functional equation proved in this section uses the automorphy of $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ directly. It requires $m \in \mathbb{Z}_{\geq 6}$, and is useful for large m.

For each ζ with $\zeta^6 = 1$, let $\psi_\zeta^\# : \mathbb{Z}[\omega] \to \mathbb{C}$ be given by

$$\psi_\zeta^\#(u) := \frac{1}{N(\lambda^m r)} \sum_{\substack{a,d \ (\mathrm{mod}\ \lambda^m r) \\ a,d \equiv 1 \ (\mathrm{mod}\ 3) \\ ad \equiv 1 \ (\mathrm{mod}\ \lambda^m r)}} \psi(-\zeta^{-1} d) \left(\frac{\zeta \lambda^{m-1} r}{d}\right)_3 \check{e}\left(\frac{au}{\zeta \lambda^m r}\right), \quad u \in \mathbb{Z}[\omega]. \tag{8-16}$$

The function $\psi_\zeta^\#$ is periodic modulo $\lambda^m r$.

Proposition 8.8. *Let $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ be a cusp form with spectral parameter $\tau_f \in 1 + i\mathbb{R}$, $m \in \mathbb{Z}_{\geq 6}$, $r \in \mathbb{Z}[\omega]$ with $r \equiv 1 \pmod 3$, and $\psi : \mathbb{Z}[\omega] \to \mathbb{C}$ be a periodic function modulo $\lambda^m r$, supported only on residue classes coprime to $\lambda^m r$. We have*

$$f(w; \hat{\psi}) = \sum_\zeta f\left(-\frac{\bar{z}}{(\zeta^{-1}\lambda^{m-4} r)^2(|z|^2+v^2)}, \frac{v}{|\lambda^{m-4} r|^2(|z|^2+v^2)}; \psi_{\zeta^{-1}}^\#\right), \quad w = (z, v) \in \mathbb{H}^3, \tag{8-17}$$

where $\hat{\psi}$ and $\psi_\zeta^\#$ are given by (8-5) and (8-16) respectively.

Proof. We open the definition of the Fourier transform to obtain

$$f(w; \hat{\psi}) = \frac{1}{N(\lambda^m r)} \sum_\zeta \sum_{\substack{d \ (\mathrm{mod}\ \lambda^m r) \\ d \equiv 1 \ (\mathrm{mod}\ 3) \\ (d,r)=1}} \psi(-\zeta d) f\left(z - \frac{d}{\zeta^{-1}\lambda^{m-4} r}, v\right). \tag{8-18}$$

Given $\zeta^{-1}\lambda^{m-4} r \in \mathbb{Z}[\omega]$ (with $m \in \mathbb{Z}_{\geq 6}$), and each $d \equiv 1 \pmod 3$ in (8-18) with $(d, r) = 1$, there exists a matrix

$$\gamma := \begin{pmatrix} d & \lambda^4 b \\ -\zeta^{-1}\lambda^{m-4} r & a \end{pmatrix} \in \Gamma_1(3). \tag{8-19}$$

Note that the determinant equation of this matrix implies that $ad \equiv 1 \pmod{\lambda^m r}$. A straightforward computation using (2-17) shows that

$$\left(z - \frac{d}{\zeta^{-1}\lambda^{m-4} r}, v\right) = \gamma\left(\frac{a}{\zeta^{-1}\lambda^{m-4} r} - \frac{\bar{z}}{(\zeta^{-1}\lambda^{m-4} r)^2(|z|^2+v^2)}, \frac{v}{|\lambda^{m-4} r|^2(|z|^2+v^2)}\right). \tag{8-20}$$

We use the fact that $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ to obtain

$$f\left(z - \frac{d}{\zeta^{-1}\lambda^{m-4} r}, v\right) = \chi(\gamma) f\left(\frac{a}{\zeta^{-1}\lambda^{m-4} r} - \frac{\bar{z}}{(\zeta^{-1}\lambda^{m-4} r)^2(|z|^2+v^2)}, \frac{v}{|\lambda^{m-4} r|^2(|z|^2+v^2)}\right), \tag{8-21}$$

where

$$\chi(\gamma) = \left(\frac{-\zeta^{-1}\lambda^{m-4} r}{d}\right)_3 = \left(\frac{\zeta^{-1}\lambda^{m-1} r}{d}\right)_3. \tag{8-22}$$

We combine (8-21)–(8-22) in (8-18). We then use the Fourier expansion (1-5) to open f, and then assemble the sum over d (equivalently a). □

Corollary 8.9. *Let the notation be as in Proposition 8.8 and $n \in \mathbb{Z}$. For $v > 0$ we have*

$$f(v; \hat{\psi}, n) = \frac{(-1)^n}{N(\lambda^{m-4}r)^{|n|} v^{2|n|}} \left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r} \right)^{-n} \sum_{\zeta} f\left(\frac{1}{|\lambda^{m-4}r|^2 v}; \psi^{\#}_{\zeta^{-1}}, -n \right). \tag{8-23}$$

Proof. Setting $z = 0$ in Proposition 8.8 (in particular, (8-17)) gives the result for $n = 0$. If $n > 0$, we write $|z|^2 = z\bar{z}$ and apply the operator

$$\frac{1}{(2\pi i)^n} \left(\frac{\partial}{\partial z} \right)^n \bigg|_{z=0}$$

to both sides of (8-17). If $n < 0$, we write $|z|^2 = z\bar{z}$ and apply the operator

$$\frac{1}{(2\pi i)^{|n|}} \left(\frac{\partial}{\partial \bar{z}} \right)^{|n|} \bigg|_{z=0}$$

to both sides of (8-17) A computation with the chain rule yields the result. □

Proposition 8.10. *Let the notation be as in Proposition 8.8 and $n \in \mathbb{Z}$. The completed Dirichlet series $\Lambda(s, f; \hat{\psi}, n)$ and $\Lambda(s, f; \psi^{\#}_{\zeta}, n)$ both admit meromorphic continuations to entire functions, and satisfy*

$$(-1)^n N(\lambda^{m-4}r)^{2s-1} \left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r} \right)^n \Lambda(s, f; \hat{\psi}, n) = \sum_{\zeta} \Lambda(1-s, f; \psi^{\#}_{\zeta^{-1}}, -n). \tag{8-24}$$

Proof. Recall that for $\operatorname{Re} s > 1$ we have

$$\Lambda(s, f; \hat{\psi}, n) = \int_0^{\infty} f(v; \hat{\psi}, n) v^{2s+|n|-2} \, dv.$$

The function $f(v; \hat{\psi}, n)$ has exponential decay at 0 and ∞ by (8-2), (8-3), and termwise differentiation of (2-18) (with constant term identically zero). Thus $\Lambda(s, f; \hat{\psi}, n)$ has analytic continuation to an entire function. The argument for $f(v; \psi^{\#}_{\zeta}, n)$ is analogous.

We now prove (8-24). We have

$$\Lambda(s, f; \hat{\psi}, n) = \int_0^{N(\lambda^{m-4}r)^{-1}} f(v; \hat{\psi}, n) v^{2s+|n|-2} \, dv + \int_{N(\lambda^{m-4}r)^{-1}}^{\infty} f(v; \hat{\psi}, n) v^{2s+|n|-2} \, dv. \tag{8-25}$$

After applying Corollary 8.9, interchanging the order of summation and integration, and a change of variables, we obtain

$$\begin{aligned} &\int_0^{N(\lambda^{m-4}r)^{-1}} f(v; \hat{\psi}, n) v^{2s+|n|-2} \, dv \\ &\quad = \frac{(-1)^n}{N(\lambda^{m-4}r)^{|n|}} \left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r} \right)^{-n} \sum_{\zeta} \int_0^{N(\lambda^{m-4}r)^{-1}} f\left(\frac{1}{v|\lambda^{m-4}r|^2}; \psi^{\#}_{\zeta^{-1}}, -n \right) v^{2s-|n|-2} \, dv \\ &\quad = (-1)^n N(\lambda^{m-4}r)^{1-2s} \left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r} \right)^{-n} \sum_{\zeta} \int_1^{\infty} f(v; \psi^{\#}_{\zeta^{-1}}, -n) v^{-2s+|n|} \, dv \end{aligned} \tag{8-26}$$

and

$$\int_{N(\lambda^{m-4}r)^{-1}}^{\infty} f(v;\hat{\psi},n)v^{2s+|n|-2}\,dv$$
$$= \frac{(-1)^n}{N(\lambda^{m-4}r)^{|n|}}\left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r}\right)^{-n}\sum_{\zeta}\int_{N(\lambda^{m-4}r)^{-1}}^{\infty} f\left(\frac{1}{v|\lambda^{m-4}r|^2};\psi^{\#}_{\zeta^{-1}},-n\right)v^{2s-|n|-2}\,dv$$
$$= (-1)^n N(\lambda^{m-4}r)^{1-2s}\left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r}\right)^{-n}\sum_{\zeta}\int_0^1 f(v;\psi^{\#}_{\zeta^{-1}},-n)v^{-2s+|n|}\,dv. \tag{8-27}$$

Substituting (8-26) and (8-27) into (8-25) yields the result. □

8.4. ***Functional equation 2 : $m \in \mathbb{Z}_{\geq 0}$ absolutely bounded (in particular, $0 \leq m \leq 5$).*** Let $m \in \mathbb{Z}_{\geq 0}$. The functional equation we prove in this section is valid for all $m \in \mathbb{Z}_{\geq 0}$, but is really only useful when m is bounded by an absolute constant.

Recall that $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\omega])$. Let Γ' be a subgroup of Γ with $[\Gamma : \Gamma'] < \infty$ and $\Gamma' \subseteq \Gamma_1(9)$. Then by [Proskurin 1998, Theorem 0.3.1] each cusp $\sigma\infty$ ($\sigma \in \Gamma$) of Γ' is essential with respect to χ, and if $\Gamma' := \Gamma_1(C)$ with $C \equiv 0 \pmod 9$, then

$$\Gamma_1(C)_\sigma = C\mathbb{Z}[\omega] \quad \text{and} \quad \Gamma_1(C)^*_\sigma = (C\lambda)^{-1}\mathbb{Z}[\omega].$$

Suppose that $\psi' : \mathbb{Z}[\omega] \to \mathbb{C}$ is periodic modulo λ^m, and that $\psi'' : \mathbb{Z}[\omega] \to \mathbb{C}$ is periodic modulo r, where $r \equiv 1 \pmod 3$. Let

$$\psi''^{,\star}(u) := \frac{1}{N(r)} \sum_{\substack{a,d \pmod r \\ (\lambda^{2m+4}a)(\lambda^{2m+4}d)\equiv 1 \pmod r}} \psi''(-d)\left(\frac{\lambda^{2m+4}d}{r}\right)_3 \check{e}\left(\frac{au}{r}\right), \quad u \in \mathbb{Z}[\omega]. \tag{8-28}$$

The function $\psi''^{,\star}$ is periodic modulo r.

Let $\gamma_{m,j} \in \Gamma_2$ for $j = 1, \ldots, [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$ be a fixed complete set of representatives for $\Gamma_1(\lambda^{2m+4})\backslash\Gamma_2$. We have the convention that $\gamma_{m,1} := I$ for all $m \in \mathbb{Z}_{\geq 0}$. For each $j = 1, \ldots, [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$, let

$$(f \otimes \widehat{\psi'})_j(w) := (f \otimes \widehat{\psi'})(\gamma_{m,j} w), \quad w \in \mathbb{H}^3, \tag{8-29}$$

each having Fourier expansion

$$(f \otimes \hat{\psi}')_j(w) := \sum_{\substack{\nu \neq 0 \\ \nu \in \lambda^{-2m-3}\mathbb{Z}[\omega]}} \rho_{f\otimes\hat{\psi}',j}(\nu)\, v K_{\tau-1}(4\pi|\nu|v)\check{e}(\nu z), \quad w \in \mathbb{H}^3, \tag{8-30}$$

where $\rho_{f\otimes\hat{\psi}',j}(\nu) \in \mathbb{C}$.

If $g \in \Gamma_2$, then

$$\gamma_{m,j} g = g_{m,j}(g)\gamma_{m,k_{m,j}(g)} \quad \text{for some unique } g_{m,j}(g) \in \Gamma_1(\lambda^{2m+4}) \text{ and } 1 \leq k_{m,j}(g) \leq [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]. \tag{8-31}$$

For any $g, h \in \Gamma_2$ we have

$$g_{m,j}(gh) = g_{m,j}(g) g_{m,k_{m,j}(g)}(h) \quad \text{and} \quad k_{m,j}(gh) = k_{m,k_{m,j}(g)}(h).$$

Remark 8.11. Using (8-31) we see that for $g \in \Gamma$ we have

$$(f \otimes \hat{\psi}')_j(gw) = \chi(g_{m,j}(g))(f \otimes \hat{\psi}')_{k_{m,j}(g)}(w), \quad w \in \mathbb{H}^3. \tag{8-32}$$

Since $\Gamma_1(\lambda^{2m+4})$ is a normal subgroup of Γ_2 for $m \in \mathbb{Z}_{\geq 0}$ (it is also a normal subgroup of Γ) we have

$$k_{m,j}(g) = j \quad \text{for } g \in \Gamma_1(\lambda^{2m+4}) \text{ and all } j. \tag{8-33}$$

Then by [Patterson 1978, Lemma 2.1] we have

$$\chi(\gamma g \gamma^{-1}) = \chi(g) \quad \text{for } g \in \Gamma_1(\lambda^{2m+4}) \text{ and } \gamma \in \Gamma_2.$$

Thus for each j we have

$$(f \otimes \hat{\psi}')_j \in L^2(\Gamma_1(\lambda^{2m+4})\backslash\mathbb{H}^3, \chi, \tau) \quad \text{is a cusp form.} \tag{8-34}$$

Following (8-1)–(8-4) we also have the functions $(f \otimes \hat{\psi}')_j(\,\cdot\,, \Psi, n)$ and their associated Fourier expansions for each $n \in \mathbb{Z}$.

Proposition 8.12. *Let $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ be a cusp form with spectral parameter $\tau_f \in 1 + i\mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $1 \leq j \leq [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$ an integer, $r \in \mathbb{Z}[\omega]$ with $r \equiv 1 \pmod 3$, and $\psi' : \mathbb{Z}[\omega] \to \mathbb{C}$ (resp. $\psi'' : \mathbb{Z}[\omega] \to \mathbb{C}$) be periodic functions modulo λ^m (resp. r). Further assume that ψ'' is supported only on residue classes coprime to r.*

Then there exist an integer $1 \leq \mathfrak{c}(m, j; r) \leq [\Gamma_2 : \Gamma_1(\lambda^{2m+4})]$ and cube root of unity $\omega(m, j; r)$ such that

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'') = \omega(m, j; r)(f \otimes \hat{\psi}')_{\mathfrak{c}(m,j;r)}\left(-\frac{\bar{z}}{r^2(|z|^2 + v^2)}, \frac{v}{|r|^2(|z|^2 + v^2)}; \psi^{\prime\prime,\star}\right), \quad w = (z, v) \in \mathbb{H}^3, \tag{8-35}$$

where $\psi^{\prime\prime,\star}$ is given in (8-28). Both $\mathfrak{c}(m, j; r)$ and $\omega(m, j; r)$ depend only on $m \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}_{\geq 1}$, and the residue class $r \pmod{\lambda^{2m+4}}$.

Remark 8.13. The reason why the functional equation proved in this section is only useful for m bounded by an absolute constant is because we use the automorphy for each $(f \otimes \hat{\psi}')_j \in L^2(\Gamma_1(\lambda^{2m+4})\backslash\mathbb{H}^3, \chi)$.

Proof. We adapt the proof of [Dunn and Radziwiłł 2024, Lemma 5.2]. We open the definition of the Fourier transform and obtain

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'') = \frac{1}{N(r)} \sum_{\substack{d \pmod r \\ (d,r)=1}} \psi''(-d)(f \otimes \hat{\psi}')_j\left(z - \frac{\lambda^{2m+4}d}{r}, v\right). \tag{8-36}$$

Given $r \equiv 1 \pmod 3$ and each $d \in \mathbb{Z}[\omega]$ in (8-36), we have $(r, \lambda^{2m+4}d) = 1$. Thus there exists a matrix

$$\begin{pmatrix} r & -\lambda^{2m+4}a \\ \lambda^{2m+4}d & b \end{pmatrix} \in \Gamma_1(3),$$

and hence there exists

$$\gamma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} r & -\lambda^{2m+4}a \\ \lambda^{2m+4}d & b \end{pmatrix} = \begin{pmatrix} \lambda^{2m+4}d & b \\ -r & \lambda^{2m+4}a \end{pmatrix} \in \Gamma_2. \tag{8-37}$$

Note that we implicitly we used (3-1) in the above display. Also note we have the determinant equation

$$\lambda^{4m+8}ad+br=1. \tag{8-38}$$

A straightforward computation using (2-17) shows that

$$\left(z-\frac{\lambda^{2m+4}d}{r},v\right)=\gamma\left(\frac{\lambda^{2m+4}a}{r}-\frac{\bar{z}}{r^2(|z|^2+v^2)},\frac{v}{|r|^2(|z|^2+v^2)}\right). \tag{8-39}$$

We now carefully factorise the γ in (8-37) as a word in P, T and E so that (8-39) and automorphy of $(f\otimes\hat{\psi}')_j$ can be used in (8-36). For each $x+y\omega\in\mathbb{Z}[\omega]$, $x,y\in\mathbb{Z}$, let

$$A(x+y\omega):=PT^{-x}PT^{-x+y}P=\begin{pmatrix}1 & x+y\omega\\ 0 & 1\end{pmatrix}.$$

For each $r,b\in\mathbb{Z}[\omega]$ occurring in (8-37), let

$$S(r,b):=E^3A(r)EA(b)EA(r)=\begin{pmatrix}b & -1+br\\ 1-br & 2r-br^2\end{pmatrix}\in\Gamma_1(3).$$

Then

$$S(r,b)E\gamma=\begin{pmatrix}-\lambda^{2m+4}d+br+\lambda^{2m+4}bdr & -b-\lambda^{2m+4}ab+b^2r\\ r+2\lambda^{2m+4}dr-br^2-\lambda^{2m+4}bdr^2 & -\lambda^{2m+4}a+2br+\lambda^{2m+4}abr-b^2r^2\end{pmatrix}=:\tilde{\gamma}.$$

Using (8-38) we see that $\tilde{\gamma}\in\Gamma_1(\lambda^{2m+4})$ and we write

$$\gamma=E^3S(r,b)^{-1}\tilde{\gamma}. \tag{8-40}$$

We use (8-39), (8-40), (8-32), and (8-34) to obtain

$$\begin{aligned}&(f\otimes\hat{\psi}')_j\left(z-\frac{\lambda^{2m+4}d}{r},v\right)\\ &=\chi(g_{m,j}(E^3S(r,b)^{-1}))\cdot\chi(\tilde{\gamma})(f\otimes\hat{\psi}')_{k_{m,j}(E^3S(r,b)^{-1})}\left(\frac{\lambda^{2m+4}a}{r}-\frac{\bar{z}}{r^2(|z|^2+v^2)},\frac{v}{|r|^2(|z|^2+v^2)}\right).\end{aligned} \tag{8-41}$$

By (8-33) the integer $k_{m,j}(E^3S(r,b)^{-1})$ depends only on $m,j\in\mathbb{N}$ and matrix residue class

$$E^3S(r,b)^{-1}=\begin{pmatrix}-1+br & b\\ -2r+br^2 & -1+br\end{pmatrix}\pmod{\lambda^{2m+4}}.$$

Thus the integer $k_{m,j}(E^3S(r,b)^{-1})$ depends only on $m,j\in\mathbb{N}$ and the residue class $r\pmod{\lambda^{2m+4}}$, since $b\pmod{\lambda^{2m+4}}$ is determined by (8-38). By (8-31) we have

$$g_{m,j}(E^3S(r,b)^{-1})=\gamma_{m,j}E^3S(r,b)^{-1}\gamma^{-1}_{m,k_{m,j}(E^3S(r,b)^{-1})}\in\Gamma_1(\lambda^{2m+4}),$$

and each matrix in the product on the right side is an element of Γ_2. Thus

$$\chi(g_{m,j}(E^3S(r,b)^{-1}))=\chi(\gamma_{m,j})\chi(E^3)\chi(S(r,b)^{-1})\chi(\gamma^{-1}_{m,k_{m,j}(E^3S(r,b)^{-1})})=\chi(\gamma_{m,j})\overline{\chi(\gamma_{m,k_{m,j}(E^3S(r,b)^{-1})})}$$

is a cube root of unity depending only on $m\in\mathbb{Z}_{\geq 0}$, $j\in\mathbb{Z}_{\geq 1}$ and the residue class $r\pmod{\lambda^{2m+4}}$. For ease of notation we relabel

$$\mathfrak{c}(m,j;r):=k_{m,j}(E^3S(r,b)^{-1}), \tag{8-42}$$

$$\omega(m,j;r):=\chi(\gamma_{m,j})\overline{\chi(\gamma_{m,k_{m,j}(E^3S(r,b)^{-1})})}. \tag{8-43}$$

A computation following [Dunn and Radziwiłł 2024, p. 23] establishes that

$$\chi(\tilde{\gamma}) = \left(\frac{\lambda^{2m+4}d}{r}\right)_3. \tag{8-44}$$

We combine (8-41)–(8-44) in (8-36). We then use the Fourier expansion (8-30) to open $(f \otimes \hat{\psi}')_{\mathfrak{c}(m,j;r)}$, and assembling the sum over d (equivalently a) shows that

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'') = \omega(m, j; r)(f \otimes \hat{\psi}')_{\mathfrak{c}(m,j;r)}\left(-\frac{\bar{z}}{r^2(|z|^2+v^2)}, \frac{v}{|r|^2(|z|^2+v^2)}; \psi''^{,\star}\right),$$

as required. □

Corollary 8.14. *Let the notation be as in Proposition 8.12 and $n \in \mathbb{Z}$. For $v > 0$ we have*

$$(f \otimes \hat{\psi}')_j(w; \hat{\psi}'', n) = \frac{(-1)^n \omega(m, j; r)}{N(r)^{|n|} v^{2|n|}}\left(\frac{\bar{r}}{r}\right)^{-n}(f \otimes \hat{\psi}')_{\mathfrak{c}(m,j;r)}\left(\frac{1}{|r|^2 v}; \psi''^{,\star}, -n\right). \tag{8-45}$$

The proof is analogous to that of Corollary 8.9 so we omit it.

Proposition 8.15. *Let the notation be as in Proposition 8.12 and $n \in \mathbb{Z}$. The completed Dirichlet series $\Lambda(s, (f \otimes \hat{\psi}')_j; \hat{\psi}'', n)$ and $\Lambda(s, (f \otimes \hat{\psi}')_j; \psi''^{,\star}, n)$ both admit meromorphic continuation to an entire function, and satisfy*

$$(-1)^n N(r)^{2s-1}\left(\frac{\bar{r}}{r}\right)^n \Lambda(s, (f \otimes \hat{\psi}')_j; \hat{\psi}'', n) = \omega(m, j; r)\Lambda\big(1-s, (f \otimes \hat{\psi}')_{\mathfrak{c}(m,j;r)}; \psi''^{,\star}, -n\big). \tag{8-46}$$

Proposition 8.15 follows from Corollary 8.14, and the proof is analogous to that of Proposition 8.10. We omit the proof.

8.5. ***Level aspect Voronoi formula.*** We now prove a Voronoi summation formula for the Fourier coefficients for the form $f(w; \lambda^\ell q, \eta)$ given in (8-9).

We recall some basic facts concerning the complex Mellin transform. Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Let $K, M \geq 1$ and $V_{K,M} \in C_c^\infty(\mathbb{C}^\times)$ have compact support contained in the disc of radius 100 (say), and also satisfy

$$\frac{\partial^{i+j}}{\partial x^i \partial x^j} V_{K,M}(z) \ll_{i,j} MK^{i+j} \quad \text{for all } z \in \mathbb{C}^\times. \tag{8-47}$$

The complex Mellin transform is given by

$$\widehat{V}_{K,M}(s, n) := \int_{\mathbb{C}^\times} V_{K,M}(z)|z|^{2s}(z/|z|)^{-n}\, d_\times z \tag{8-48}$$

for $s \in \mathbb{C}$ and $n \in \mathbb{Z}$, where $d_\times z := |z|^{-2}\, dx\, dy$. Note that $\widehat{V}_{K,M}(s, n)$ is entire with respect to s for each $n \in \mathbb{Z}$. After making a change of variables $z = re(\theta/2)$ with $r \in (0, \infty)$ and $\theta \in [0, 2\pi)$, we obtain

$$\widehat{V}_{K,M}(s, n) = \int_0^\infty \int_0^{2\pi} V_{K,M}(re(\theta/2))r^{2s-1}e(-n\theta/2)\, d\theta\, dr. \tag{8-49}$$

After repeated integration by parts, we obtain

$$\widehat{V}_{K,M}(s,n) \ll_{j,k} M \cdot \min\left\{1, \frac{K^{j+k}}{|(2s)_j|(1+|n|)^k}\right\}$$

for $j, k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{C}$ in a fixed vertical strip, and $n \in \mathbb{Z}$. It follows for $D_1, D_2 \geq 0$, we have

$$\widehat{V}_{K,M}(s,n) \ll_{D_1,D_2} \frac{MK^{D_1+D_2}}{(1+|s|)^{D_1}(1+|n|)^{D_2}} \tag{8-50}$$

for $s \in \mathbb{C}$ in a fixed vertical strip, and $n \in \mathbb{Z}$. The complex Mellin inversion formula is given by

$$V_{K,M}(z) = \frac{1}{2\pi^2 i} \sum_{n\in\mathbb{Z}} \int_{(\sigma)} \widehat{V}_{K,M}(s,n)|z|^{-2s}(z/|z|)^n \, ds \tag{8-51}$$

for $\sigma > 0$, $z \in \mathbb{C}$, and $n \in \mathbb{Z}$.

Remark 8.16. Suppose further that $V_{K,M}$ is radial, i.e., $V_{K,M}(re(\theta)) = V_K(r)$ for all $\theta \in \mathbb{R}$. Then

$$\widehat{V}_{K,M}(s,n) = \delta_{n=0} 2\pi \cdot \int_0^\infty V_{K,M}(r) r^{2s-1} \, dr = \delta_{n=0} 2\pi \cdot \widehat{V}_{K,M}(2s) = \delta_{n=0} \pi \cdot \widehat{W}_{K,M}(s), \tag{8-52}$$

where $\widehat{V}_{K,M}(s)$ denotes the usual Mellin transform for functions on $(0, \infty)$, and $W_{K,M}$ is such that $W_{K,M}(r) = V_{K,M}(\sqrt{r})$. Then (8-51) becomes the standard Mellin inversion formula for functions on $(0, \infty)$ after a change of variable in s.

Proposition 8.17. *Let $f \in L^2(\Gamma_2\backslash\mathbb{H}^3, \chi)$ be a cusp form with spectral parameter $\tau_f \in 1 + i\mathbb{R}$, $\ell \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{Z}[\omega]$ with $q \equiv 1 \pmod 3$, $\eta \in \mathbb{Z}[\omega]/\lambda^\ell q\mathbb{Z}[\omega]$, and $V_{K,M} \in C_c^\infty(\mathbb{C}^\times)$ be a smooth function with compact support in the disc of radius 100 satisfying (8-47) for some $K, M \geq 1$. Then for $X > 0$ we have*

$$\sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3\nu \equiv \eta \ (\mathrm{mod}\ \lambda^\ell q)}} \rho_f(\nu) V_{K,M}(\nu/\sqrt{X}) = \frac{X}{N(\lambda^{\ell+1}q)} \sum_{r|q} \sum_{n\in\mathbb{Z}} (-1)^n \sum_{m=0}^{\ell+1} \sum_{p=1}^{2} Z_{pf}(X, \lambda^m r, \eta, n; \dot{V}_{K,M}) \tag{8-53}$$

where

$$\begin{aligned} &Z_{1f}(X, \lambda^m r, \eta, n; \dot{V}_{K,M}) \\ &\quad := \delta_{0\leq m\leq \min\{5,\ell+1\}} \cdot N(\lambda^m) \left(\frac{\bar{r}}{r}\right)^{-n} \omega(m, 1; r) \\ &\qquad \times \sum_{\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega]} \rho_{f\otimes\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}}, \mathfrak{c}(m,1;r)}(\nu) \left(\frac{\nu}{|\nu|}\right)^{-n} \psi_r^\star(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu) \dot{V}_{K,M}\left(\frac{N(\nu)}{N(r)^2/X}, n\right), \end{aligned} \tag{8-54}$$

$$\begin{aligned} &Z_{2f}(X, \lambda^m r, \eta, n; \dot{V}_{K,M}) \\ &\quad := \delta_{6\leq m\leq \ell+1} \cdot N(\lambda^4) \sum_{\zeta} \left(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r}\right)^{-n} \\ &\qquad\qquad \times \sum_{\nu\in\lambda^{-3}\mathbb{Z}[\omega]} \rho_f(\nu) \left(\frac{\nu}{|\nu|}\right)^{-n} \psi^{\#}_{\lambda^m r}(\,\cdot\,)_{\lambda\eta,\zeta^{-1}}(\lambda^4\nu) \dot{V}_{K,M}\left(\frac{N(\nu)}{N(\lambda^{m-4}r)^2/X}, n\right), \end{aligned} \tag{8-55}$$

where $\psi^{\#}_{\zeta}$ and $\psi^{\star}$ are given in (8-16) and (8-28) (with $\psi'' \to \psi$) respectively, $\dot{V}_{K,M}(\,\cdot\,,n):(0,\infty)\to\mathbb{R}$ is given by

$$\dot{V}_{K,M}(Y,n) := \frac{1}{2\pi^2 i}\int_{(2)} Y^{-s}\frac{G_\infty(s,\tau_f,n)}{G_\infty(1-s,\tau_f,n)}\widehat{V}_{K,M}(1-s,n)\,ds, \tag{8-56}$$

$G_\infty(s,\tau,n)$ is given in (8-7), and $\omega(m,j,r)$ and $\mathfrak{c}(m,j,r)$ are both as in Proposition 8.12.

Remark 8.18. From Remark 8.16 we see that if V_K is radial then only $n=0$ is relevant on the right side of (8-53). In this case n is omitted from the notation.

Proof. Recall the definition of the function $\psi_{\lambda^m r}(\,\cdot\,)_\eta$ in (8-11), and its Fourier transform $\hat{\psi}_{\lambda^m r}(\,\cdot\,)_\eta$ in (8-12). We apply complex Mellin inversion (8-51) to the smooth function V_K, Lemma 8.6, and then interchange of the order of integration and summation by absolute convergence. This yields

$$\sum_{\substack{\nu\in\lambda^{-3}\mathbb{Z}[\omega]\\ \lambda^3\nu\equiv\eta \ (\mathrm{mod}\ \lambda^\ell q)}} \rho_f(\nu)V_{K,M}(\nu/\sqrt{X})$$

$$= \frac{1}{2\pi^2 i}\frac{1}{N(\lambda^{\ell+1}q)}\sum_{r\mid q}\Bigg(\sum_{m=0}^{\min\{5,\ell+1\}}+\sum_{m=6}^{\ell+1}\Bigg)N(\lambda^m r)\sum_{n\in\mathbb{Z}}\int_{(2)}\widehat{V}_{K,M}(s,n)X^s\mathcal{D}(s,f;\widehat{\psi_{\lambda^m r}(\,\cdot\,)_{\lambda\eta}},n)\,ds. \tag{8-57}$$

The Chinese remainder theorem implies that

$$\widehat{\psi_{\lambda^m r}(\,\cdot\,)_{\lambda\eta}}(u) = \widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}}(u)\widehat{\psi_r(\,\cdot\,)_{\lambda\eta}}(u), \quad u\in\mathbb{Z}[\omega], \tag{8-58}$$

and by a change of variables we have

$$\widehat{\psi_r(\,\cdot\,)_{\lambda^{2m+1}\eta}}(\lambda^{2m}u) = \widehat{\psi_r(\,\cdot\,)_{\lambda\eta}}(u), \quad u\in\mathbb{Z}[\omega]. \tag{8-59}$$

Recall the definition of twisting (8-1) and the convention in Remark 8.1. Using (8-2) we see that $f\otimes\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}}\in L^2(\Gamma_1(\lambda^{2m+4})\backslash\mathbb{H}^3,\chi,\tau)$ is a cusp form. Using (8-58) and (8-59) we obtain

$$\begin{aligned} f(w;\widehat{\psi_{\lambda^m r}(\,\cdot\,)_{\lambda\eta}},n) &= f(w;\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}}\widehat{\psi_r(\,\cdot\,)_{\lambda^{2m+1}\eta}}\circ\lambda^{2m},n)\\ &= (f\otimes\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}})(w;\widehat{\psi_r(\,\cdot\,)_{\lambda^{2m+1}\eta}},n)\end{aligned}$$

for all $0\le m\le \ell+1$, $r\mid q$, and $\eta\in\mathbb{Z}[\omega]$. The analogous Dirichlet series identity reads

$$\mathcal{D}(s,f;\widehat{\psi_{\lambda^m r}(\,\cdot\,)_{\lambda\eta}},n) = \mathcal{D}(s,f\otimes\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}};\widehat{\psi_r(\,\cdot\,)_{\lambda^{2m+1}\eta}},n), \quad \mathrm{Re}(s)>1. \tag{8-60}$$

Substituting (8-60) into (8-57) for $0\le m\le\min\{5,\ell+1\}$ we see that the right side of (8-57) is equal to

$$\begin{aligned}\frac{1}{2\pi^2 i}\frac{1}{N(\lambda^{\ell+1}q)}\sum_{r\mid q}\sum_{n\in\mathbb{Z}}\Bigg(&\sum_{m=0}^{\min\{5,\ell+1\}}N(\lambda^m r)\int_{(2)}\widehat{V}_{K,M}(s,n)X^s\mathcal{D}(s,f\otimes\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}};\widehat{\psi_r(\,\cdot\,)_{\lambda^{2m+1}\eta}},n)\,ds\\ &+\sum_{m=6}^{\ell+1}N(\lambda^m r)\int_{(2)}\widehat{V}_{K,M}(s,n)X^s\mathcal{D}(s,f;\widehat{\psi_{\lambda^m r}(\,\cdot\,)_{\lambda\eta}},n)\,ds\Bigg).\end{aligned} \tag{8-61}$$

Both of the integrands in (8-61) are entire by Propositions 8.10 and 8.15 and Lemma 8.2. We shift the contour in (8-57) to $\mathrm{Re}(s) = -1$ and then use the functional equations (8-24) and (8-46). We see that (8-61) is equal to

$$\frac{1}{2\pi^2 i}\frac{1}{N(\lambda^{\ell+1}q)}\sum_{r|q}\sum_{n\in\mathbb{Z}}(-1)^n$$
$$\times\Bigg(\sum_{m=0}^{\min\{5,\ell+1\}} N(\lambda^m)N(r)^2\Big(\frac{\bar r}{r}\Big)^{-n}\omega(m,1;r)\int_{(-1)}\widehat{V}_{K,M}(s,n)\Big(\frac{X}{N(r)^2}\Big)^s\frac{G_\infty(1-s,\tau_f,-n)}{G_\infty(s,\tau_f,n)}$$
$$\times\mathcal{D}\big(1-s,(f\otimes\widehat{\psi_{\lambda^m}(\,\cdot\,)_{\lambda\eta}})_{\mathfrak{c}(m,1;r)};\psi_r^{\star}(\,\cdot\,)_{\lambda^{2m+1}\eta},-n\big)\,ds$$
$$+\sum_{m=6}^{\ell+1}\sum_{\zeta}N(\lambda^{2m-4})N(r)^2\Big(\frac{\overline{\zeta^{-1}\lambda^{m-4}r}}{\zeta^{-1}\lambda^{m-4}r}\Big)^{-n}\int_{(-1)}\widehat{V}_{K,M}(s,n)\Big(\frac{X}{N(\lambda^{m-4}r)^2}\Big)^s$$
$$\times\frac{G_\infty(1-s,\tau_f,-n)}{G_\infty(s,\tau_f,n)}\mathcal{D}(1-s,f;\psi^{\#}_{\lambda^m r}(\,\cdot\,)_{\lambda\eta,\zeta^{-1}},-n)\,ds\Bigg). \tag{8-62}$$

We make the change of variable $s\to 1-s$ in both integrals in (8-62), open up both of the Dirichlet series in the region of absolute convergence, and interchange the order of summation and integration to obtain (8-53) with the transforms given by (8-54)–(8-56). □

We now compute the Archimedean and non-Archimedean transforms on the dual side of the Voronoi formula in Proposition 8.17. Recall that $K_{\Gamma',\sigma,\xi}(m,n,c)$ denotes a cubic Kloosterman attached to the cusp pair (σ,ξ) of Γ'; see (3-2).

Lemma 8.19. *Let $m\in\mathbb{Z}_{\geq 6}$, $r\in\mathbb{Z}[\omega]$ with $r\equiv 1 \pmod 3$, $\eta\in\mathbb{Z}[\omega]/\lambda^m r\mathbb{Z}[\omega]$, $\psi_{\lambda^m r}(\,\cdot\,)_\eta$ be as in (8-11), and ζ be such that $\zeta^6=1$. Then for $\nu\in\lambda^{-3}\mathbb{Z}[\omega]$ we have*

$$\psi^{\#}_{\lambda^m r}(\,\cdot\,)_{\lambda\eta,\zeta}(\lambda^4\nu)=\frac{1}{N(\lambda^{m+3}r)}K_{\Gamma_1(3),\sigma,\sigma}(\lambda^3\nu,\eta,\zeta\lambda^{m-1}r),$$

where $\psi^{\#}_\zeta$ is given in (8-16), $\sigma=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and the cubic Kloosterman sum is given in (3-2).

Proof. We have

$$\psi^{\#}_{\lambda^m r}(\,\cdot\,)_{\lambda\eta,\zeta}(\lambda^4\nu)=\frac{1}{N(\lambda^m r)}\sum_{\substack{a,d \pmod{\lambda^m r}\\ a,d\equiv 1 \pmod 3\\ ad\equiv 1 \pmod{\lambda^m r}}}\Big(\frac{\zeta\lambda^{m-1}r}{d}\Big)_3\check{e}\Big(\frac{a\lambda^3\nu+d\eta}{\zeta\lambda^{m-1}r}\Big)$$
$$=\frac{1}{N(\lambda^{m+2}r)}\sum_{\substack{a,d \pmod{\lambda^{m+1}r}\\ a,d\equiv 1 \pmod 3\\ ad\equiv 1 \pmod{\lambda^m r}}}\Big(\frac{\zeta\lambda^{m-1}r}{d}\Big)_3\check{e}\Big(\frac{a\lambda^3\nu+d\eta}{\zeta\lambda^{m-1}r}\Big)$$
$$=\frac{1}{N(\lambda^{m+3}r)}\sum_{\substack{a,d \pmod{\lambda^{m+1}r}\\ a,d\equiv 1 \pmod 3\\ ad\equiv 1 \pmod{\lambda^{m-1}r}}}\Big(\frac{\zeta\lambda^{m-1}r}{d}\Big)_3\check{e}\Big(\frac{a\lambda^3\nu+d\eta}{\zeta\lambda^{m-1}r}\Big), \tag{8-63}$$

and the result follows from Lemma 3.3. □

Lemma 8.20. *Let the notation be as in Lemma 8.19 and $m \in \mathbb{Z}_{\geq 0}$. Then for $\nu \in \lambda^{-2m-3}\mathbb{Z}[\omega]$ we have*

$$\psi_r^\star(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu) = \frac{1}{N(r)} K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}(\lambda^{2m+3}\nu), \overline{\lambda^3}\eta, r),$$

where $\psi^\star$ is given in (8-28), $\sigma = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\xi = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and $\overline{\lambda^\ell} \in \mathbb{Z}[\omega]$ is such that $\overline{\lambda^\ell}\lambda^\ell \equiv 1 \pmod r$ for $\ell \in \mathbb{Z}_{\geq 0}$.

Proof. By definition

$$\psi_r^\star(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu) = \frac{1}{N(r)} \sum_{\substack{a,d \pmod r \\ (\lambda^{2m+4}a)(\lambda^{2m+4}d)\equiv 1 \pmod r}} \left(\frac{\lambda^{2m+4}d}{r}\right)_3 \check{e}\left(\frac{a(\lambda^{2m+4}\nu)+d(\lambda^{2m+1}\eta)}{r}\right). \tag{8-64}$$

The change of variables $a \to \overline{\lambda^{2m+4}}a \pmod r$ and $d \to \overline{\lambda^{2m+4}}d \pmod r$ shows that the right side of (8-64) is equal to

$$\frac{1}{N(r)} \sum_{\substack{a,d \pmod r \\ ad\equiv 1 \pmod r}} \left(\frac{d}{r}\right)_3 \check{e}\left(\frac{\overline{\lambda^{2m+3}}a(\lambda^{2m+3}\nu)+\overline{\lambda^3}\eta d}{r}\right), \tag{8-65}$$

and we can lift this to the sum

$$\frac{1}{N(r)} \sum_{\substack{a,d \pmod{3r} \\ a,d\equiv 0 \pmod 3 \\ ad\equiv 1 \pmod r}} \left(\frac{d}{r}\right)_3 \check{e}\left(\frac{\overline{\lambda^{2m+3}}a(\lambda^{2m+3}\nu)+\overline{\lambda^3}\eta d}{r}\right),$$

and the result now follows from Lemma 3.4. □

Lemma 8.21. *Let $K, M \geq 1$ and $V_{K,M} \in C_c^\infty(\mathbb{C}^\times)$ be a smooth function with compact support in $[1,2]$ whose derivatives satisfy (8-47). Let $\tau \in 1+i\mathbb{R}$, $n \in \mathbb{Z}$, $G_\infty(s,\tau,n)$ be as in (8-7), and let $\dot{V}_{K,M}(\cdot,n) : (0,\infty) \to \mathbb{C}$ be as in (8-56). Then for $D_1 > 0$ and $D_2 \geq 0$ we have*

$$\dot{V}_{K,M}(Y,n) \ll_{\tau,D_1,D_2} MK^{4(D_1+D_2)}Y^{-D_1}(|n|+1)^{4D_1-4D_2-2}$$

for all $Y > 0$.

Proof. In the definition (8-56) we move the contour to $\mathrm{Re}(s) = D_1$. Stirling's formula [Olver et al. 2018, (5.11.1)] implies that

$$\frac{G_\infty(s,\tau,-n)}{G_\infty(1-s,\tau,n)} \asymp \left|s+\tfrac{1}{2}|n|-\tfrac{1}{2}(\tau-1)\right|^{2D_1-1} \cdot \left|s+\tfrac{1}{2}|n|+\tfrac{1}{2}(\tau-1)\right|^{2D_1-1}, \tag{8-66}$$

as $\left|\mathrm{Im}(s\pm\tfrac{1}{2}(\tau-1))\right| \to \infty$. Using (8-50) (with $D_1 \to 4D_1$ and $D_2 \to 4D_2$) and (8-66) in (8-56) we obtain

$$\begin{aligned}
&\dot{V}_{K,M}(Y,n) \\
&\ll_{D_1,D_2} MK^{4(D_1+D_2)}Y^{-D_1}(1+|n|)^{-4D_2}\left(\int_{(D_1)} \frac{\left|s+\frac{1}{2}|n|-\frac{1}{2}(\tau-1)\right|^{2D_1-1}\cdot\left|s+\frac{1}{2}|n|+\frac{1}{2}(\tau-1)\right|^{2D_1-1}}{(1+|1-s|)^{4D_1}}|ds|\right) \\
&\ll_{\tau,D_1,D_2} MK^{4(D_1+D_2)}Y^{-D_1}(1+|n|)^{4D_1-4D_2-2},
\end{aligned}$$

as required. □

8.6. ***Level aspect Voronoi summation for multiple sums.*** Here we record a Voronoi formula that is an iterated version of Proposition 8.17. Let $\boldsymbol{z}=(z_1,z_2)=(x_{11}+iy_{12},x_{21}+iy_{22})\in(\mathbb{C}^\times)^2$, $x_{11},y_{12},x_{21},y_{22}\in\mathbb{R}$. Let $K,M\geq 1$, and $H_{K,M}\in C_c^\infty((\mathbb{C}^\times)^2)$ be a smooth function with compact support in a ball of radius 100 such that for any $\boldsymbol{i}=(i_{11},i_{12},i_{21},i_{22})\in(\mathbb{Z}_{\geq 0})^4$ we have

$$\partial^{\boldsymbol{i}} H_{K,M}(\boldsymbol{z}) \ll_{\boldsymbol{i}} MK^{\sum_{1\leq j,k\leq 2} i_{jk}}, \quad \boldsymbol{z}\in(\mathbb{C}^\times)^2. \tag{8-67}$$

If $M=1$ then M is omitted from the notation and we write H_K. For each $\boldsymbol{n}=(n_1,n_2)\in\mathbb{Z}^2$, consider the double complex Mellin transform

$$\widehat{\widehat{H}}_{K,M}(\boldsymbol{s},\boldsymbol{n}) := \iint_{(\mathbb{C}^\times)^2} H_{K,M}(\boldsymbol{z})\left(\prod_{i=1}^{2}|z_i|^{2s_i}\left(\frac{z_i}{|z_i|}\right)^{-n_i}\right) d_\times \boldsymbol{z}, \quad \boldsymbol{s}=(s_1,s_2)\in\mathbb{C}^2, \tag{8-68}$$

where $d_\times \boldsymbol{z} := dx_1\,dy_1\,dx_2\,dy_2/|z_1z_2|^2$. For $\boldsymbol{D}:=(D_{11},D_{12},D_{21},D_{22})\in(\mathbb{R}_{\geq 0})^4$, repeated integration by parts using polar coordinates yields the bound

$$\widehat{\widehat{H}}_{K,M}(\boldsymbol{s},\boldsymbol{n}) \ll_{\tau,\boldsymbol{D}} MK^{\sum_{1\leq i,j\leq 2} D_{ij}}\cdot\prod_{i=1}^{2}(1+|s_i|)^{-D_{i1}}(1+|n_i|)^{-D_{i2}}. \tag{8-69}$$

Consider the function $\ddot{H}_{K,M}(\cdot,\boldsymbol{n}):(0,\infty)^2\to\mathbb{R}$ given by

$$\ddot{H}_{K,M}(\boldsymbol{Y},\boldsymbol{n}) := \frac{1}{(2\pi^2 i)^2}\int_{(2)}\int_{(2)}\left(\prod_{i=1}^{2} Y_i^{-s_i}\frac{G_\infty(s_i,\tau_f,-n_i)}{G_\infty(1-s_i,\tau_f,n_i)}\right)\widehat{\widehat{H}}_{K,M}(\boldsymbol{1}-\boldsymbol{s},\boldsymbol{n})\,d\boldsymbol{s}, \quad \boldsymbol{Y}=(Y_1,Y_2)\in(0,\infty)^2, \tag{8-70}$$

where $G_\infty(s,\tau,n)$ is given by (8-7), and $d\boldsymbol{s}:=ds_1\,ds_2$. After moving the contours in (8-70) to $\mathrm{Re}(s_1)=D_{11}>0$ and $\mathrm{Re}(s_2)=D_{21}>0$, observe that (8-66) and (8-69) applied to (8-70) imply that

$$\ddot{H}_{K,M}(\boldsymbol{Y},\boldsymbol{n}) \ll_{\tau,\boldsymbol{D}} MK^{4(\sum_{1\leq i,j\leq 2} D_{ij})}\cdot\prod_{i=1}^{2} Y_i^{-D_{i1}}(|n_i|+1)^{4D_{i1}-4D_{i2}-2}, \quad \boldsymbol{Y}\in(0,\infty)^2. \tag{8-71}$$

Mellin inversion and an iterated application of the functional equation in Proposition 8.17 yields the following result. We omit the proof for the sake of brevity.

Proposition 8.22. *Let $f\in L^2(\Gamma_2\backslash\mathbb{H}^3,\chi)$ be a cusp form with spectral parameter $\tau_f\in 1+i\mathbb{R}$, $\boldsymbol{\ell}=(\ell_1,\ell_2)\in(\mathbb{Z}_{\geq 0})^2$, $\boldsymbol{q}=(q_1,q_2)\in(\mathbb{Z}[\omega])^2$ with $q_1,q_2\equiv 1 \pmod 3$, and $\boldsymbol{\eta}=(\eta_1,\eta_2)\in\mathbb{Z}[\omega]/\lambda^{\ell_1}q_1\mathbb{Z}[\omega]\times\mathbb{Z}[\omega]/\lambda^{\ell_2}q_2\mathbb{Z}[\omega]$. Let $H_{K,M}\in C_c^\infty((\mathbb{C}^\times)^2)$ be a smooth function with compact support in the disc of radius 100 satisfying (8-67) for some $K,M\geq 1$. Then for $\boldsymbol{X}=(X_1,X_2)\in(0,\infty)^2$ we have*

$$\sum_{\substack{\boldsymbol{\nu}\in(\lambda^{-3}\mathbb{Z}[\omega])^2\\ \forall i:\lambda^3\nu_i\equiv\eta_i \pmod{\lambda^{\ell_i}q_i}}} \rho_f(\nu_1)\overline{\rho_f(\nu_2)}H_{K,M}\left(\frac{\nu_1}{\sqrt{X_1}},\frac{\nu_2}{\sqrt{X_2}}\right)$$
$$= \frac{X_1X_2}{N(\lambda^{\ell_1+1}q_1)N(\lambda^{\ell_2+1}q_2)}\sum_{\substack{\boldsymbol{k}\in(\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2\\ \forall i:k_i\equiv 1 \pmod 3}}\;\sum_{\substack{\boldsymbol{m},\boldsymbol{r}\\ \forall i:0\leq m_i\leq \ell_i+1\\ \forall i:r_i\mid q_i\\ \forall i:r_i\equiv k_i \pmod{\lambda^{14}}}}\;\sum_{\boldsymbol{n}\in\mathbb{Z}^2}(-1)^{n_1+n_2}\sum_{p=1}^{4}\mathscr{D}_{pf}(\boldsymbol{X},\boldsymbol{\lambda}^{\boldsymbol{m}}\boldsymbol{r},\boldsymbol{\eta},\boldsymbol{n};\ddot{H}_{K,M}), \tag{8-72}$$

where

$$\begin{aligned}
&\mathscr{D}_{1f}(\boldsymbol{X},\boldsymbol{\lambda^m r},\boldsymbol{\eta},\boldsymbol{n};\ddot{H}_{K,M})\\
&:=\delta_{m_1\in[0,\min\{5,\ell_1+1\}]}\cdot\delta_{m_2\in[0,\min\{5,\ell_2+1\}]}\cdot N(\lambda^{m_1})N(\lambda^{m_2})\Big(\frac{\overline{r_1}}{r_1}\Big)^{-n_1}\Big(\frac{\overline{r_2}}{r_2}\Big)^{n_2}\omega(m_1;1,k_1)\overline{\omega(m_2,1,k_2)}\\
&\quad\times\sum_{\substack{\nu_1\in\lambda^{-2m_1-3}\mathbb{Z}[\omega]\\ \nu_2\in\lambda^{-2m_2-3}\mathbb{Z}[\omega]}}\rho_{f\otimes\widehat{\psi_{\lambda^{m_1}}(\,\cdot\,)_{\lambda\eta_1}},\mathfrak{c}(m_1,1;k_1)}(\nu_1)\overline{\rho_{f\otimes\widehat{\psi_{\lambda^{m_2}}(\,\cdot\,)_{\lambda\eta_2}},\mathfrak{c}(m_2,1;k_2)}(\nu_2)}\Big(\frac{\nu_1}{|\nu_1|}\Big)^{-n_1}\Big(\frac{\overline{\nu_2}}{|\nu_2|}\Big)^{-n_2}\\
&\quad\times\psi^\star_{r_1}(\,\cdot\,)_{\lambda^{2m_1+1}\eta_1}(\lambda^{2m_1+4}\nu_1)\overline{\psi^\star_{r_2}(\,\cdot\,)_{\lambda^{2m_2+1}\eta_2}(\lambda^{2m_2+4}\nu_2)}\ddot{H}_{K,M}\Big(\frac{N(\nu_1)}{N(r_1)^2/X_1},\frac{N(\nu_2)}{N(r_2)^2/X_2},\boldsymbol{n}\Big),
\end{aligned}\tag{8-73}$$

$$\begin{aligned}
&\mathscr{D}_{2f}(\boldsymbol{X},\boldsymbol{\lambda^m r},\boldsymbol{\eta},\boldsymbol{n};\ddot{H}_{K,M})\\
&:=\delta_{m_1\in[0,\min\{5,\ell_1+1\}]}\cdot\delta_{m_2\in[6,\ell_2+1]}N(\lambda^{m_1})N(\lambda^{4})\Big(\frac{\overline{r_1}}{r_1}\Big)^{-n_1}\omega(m_1,1;k_1)\\
&\quad\times\sum_{\substack{\nu_1\in\lambda^{-2m_1-3}\mathbb{Z}[\omega]\\ \nu_2\in\lambda^{-3}\mathbb{Z}[\omega]}}\rho_{f\otimes\widehat{\psi_{\lambda^{m_1}}(\,\cdot\,)_{\lambda\eta_1}},\mathfrak{c}(m_1,1;k_1)}(\nu_1)\overline{\rho_f(\nu_2)}\Big(\frac{\nu_1}{|\nu_1|}\Big)^{-n_1}\Big(\frac{\overline{\nu_2}}{|\nu_2|}\Big)^{-n_2}\\
&\quad\times\sum_{\zeta_2}\Big(\frac{\overline{\zeta_2^{-1}\lambda^{m_2-4}r_2}}{\zeta_2^{-1}\lambda^{m_2-4}r_2}\Big)^{n_2}\psi^\star_{r_1}(\,\cdot\,)_{\lambda^{2m_1+1}\eta_1}(\lambda^{2m_1+4}\nu_1)\cdot\overline{\psi^\#_{\lambda^{m_2}r_2}(\,\cdot\,)_{\lambda\eta_2,\zeta_2^{-1}}(\lambda^4\nu_2)}\\
&\quad\times\ddot{H}_{K,M}\Big(\frac{N(\nu_1)}{N(r_1)^2/X_1},\frac{N(\nu_2)}{N(\lambda^{m_2-4}r_2)^2/X_2},\boldsymbol{n}\Big),
\end{aligned}\tag{8-74}$$

$$\begin{aligned}
&\mathscr{D}_{3f}(\boldsymbol{X},\boldsymbol{\lambda^m r},\boldsymbol{\eta},\boldsymbol{\eta};\ddot{H}_{K,M})\\
&:=\delta_{m_1\in[6,\ell_1+1]}\cdot\delta_{m_2\in[0,\min\{5,\ell_2+1\}]}\cdot N(\lambda^{4})N(\lambda^{m_2})\Big(\frac{\overline{r_2}}{r_2}\Big)^{n_2}\overline{\omega(m_2,1;k_2)}\\
&\quad\times\sum_{\substack{\nu_1\in\lambda^{-3}\mathbb{Z}[\omega]\\ \nu_2\in\lambda^{-2m_2-3}\mathbb{Z}[\omega]}}\rho_f(\nu_1)\overline{\rho_{f\otimes\widehat{\psi_{\lambda^{m_2}}(\,\cdot\,)_{\lambda\eta_2}},\mathfrak{c}(m_2,1;k_2)}(\nu_2)}\Big(\frac{\nu_1}{|\nu_1|}\Big)^{-n_1}\Big(\frac{\overline{\nu_2}}{|\nu_2|}\Big)^{-n_2}\\
&\quad\times\sum_{\zeta_1}\Big(\frac{\overline{\zeta_1^{-1}\lambda^{m_1-4}r_1}}{\zeta_1^{-1}\lambda^{m_1-4}r_1}\Big)^{-n_1}\psi^\#_{\lambda^{m_1}r_1}(\,\cdot\,)_{\lambda\eta_1,\zeta_1^{-1}}(\lambda^4\nu_1)\cdot\overline{\psi^\star_{r_2}(\,\cdot\,)_{\lambda^{2m_2+1}\eta_2}(\lambda^{2m_2+4}\nu_2)}\\
&\quad\times\ddot{H}_{K,M}\Big(\frac{N(\nu_1)}{N(\lambda^{m_1-4}r_1)^2/X_1},\frac{N(\nu_2)}{N(r_2)^2/X_2},\boldsymbol{n}\Big),
\end{aligned}\tag{8-75}$$

$$\begin{aligned}
&\mathscr{D}_{4f}(\boldsymbol{X},\boldsymbol{\lambda^m r},\boldsymbol{\eta},\boldsymbol{n};\ddot{H}_{K,M})\\
&:=\delta_{m_1\in[6,\ell_1+1]}\cdot\delta_{m_2\in[6,\ell_2+1]}\cdot N(\lambda^{8})\sum_{\boldsymbol{\nu}\in(\lambda^{-3}\mathbb{Z}[\omega])^2}\rho_f(\nu_1)\overline{\rho_f(\nu_2)}\Big(\frac{\nu_1}{|\nu_1|}\Big)^{-n_1}\Big(\frac{\overline{\nu_2}}{|\nu_2|}\Big)^{-n_2}\\
&\quad\times\sum_{\boldsymbol{\zeta}}\Big(\frac{\overline{\zeta_1^{-1}\lambda^{m_1-4}r_1}}{\zeta_1^{-1}\lambda^{m_1-4}r_1}\Big)^{-n_1}\Big(\frac{\overline{\zeta_2^{-1}\lambda^{m_2-4}r_2}}{\zeta_2^{-1}\lambda^{m_2-4}r_2}\Big)^{n_2}\psi^\#_{\lambda^{m_1}r_1}(\,\cdot\,)_{\lambda\eta_1,\zeta_1^{-1}}(\lambda^4\nu_1)\cdot\overline{\psi^\#_{\lambda^{m_2}r_2}(\,\cdot\,)_{\lambda\eta_2,\zeta_2^{-1}}(\lambda^4\nu_2)}\\
&\quad\times\ddot{H}_{K,M}\Big(\frac{N(\nu_1)}{N(\lambda^{m_1-4}r_1)^2/X_1},\frac{N(\nu_2)}{N(\lambda^{m_2-4}r_2)^2/X_2},\boldsymbol{n}\Big),
\end{aligned}\tag{8-76}$$

$\psi_\zeta^{\#}$ and $\psi^\star$ are given in (8-16) and (8-28) (with $\psi'' \to \psi$) respectively, and $\ddot{H}_{K,M}(\,\cdot\,,\boldsymbol{n}) : (0,\infty)^2 \to \mathbb{R}$ is given by (8-70), and $\omega(m,j,r)$ and $\mathfrak{c}(m,j,r)$ are both as in Proposition 8.12.

9. Type-I estimates

Recall the notation from Section 1, in particular (1-12) and (1-13).

Remark 9.1. We can uniquely factorise $v = \lambda^{e_v}\zeta_v v_0$, where $e_v \in \mathbb{Z}_{\geq 2}$, $\zeta_v^6 = 1$, and $v_0 \equiv 1 \pmod 3$. In view of the congruence condition $ab \equiv u \pmod v$, we can assume without loss of generality that $v = \lambda^{e_v} v_0$ with $v_0 \equiv 1 \pmod 3$. In particular, since $(u,v) = 1$ and $ab \equiv u \pmod v$ in (1-12) and (1-13), we have $(a,v) = 1$.

Proof of Lemma 1.4. We write (1-12) as

$$\mathscr{S}_f(a,X,v,u;W_K) = \sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3\nu \equiv 0 \pmod a \\ \lambda^3\nu \equiv u \pmod{\lambda^{e_v} v_0}}} \rho_f(\nu) W_K\left(\frac{N(\nu)}{X}\right). \tag{9-1}$$

Since $(a, \lambda^{e_v} v_0) = 1$, we let $\bar{a} \in \mathbb{Z}[\omega]$ be such that $a\bar{a} \equiv 1 \pmod{\lambda^{e_v} v_0}$. The congruence conditions placed on ν in (9-1) are equivalent to $\lambda^3\nu \equiv ua\bar{a} \pmod{\lambda^{e_v} v_0 a}$ by the Chinese remainder theorem.

9.1. *Application of Voronoi summation.*

Applying Voronoi summation (Proposition 8.17) we obtain

$$\mathscr{S}_f(a,X;v,u;W_K) = \frac{X}{N(\lambda^{e_v+1} v_0 a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k \equiv 1 \pmod 3}} \sum_{\substack{m,r,t \\ 0 \leq m \leq e_v+1 \\ r \mid a,\ t \mid v_0 \\ rt \equiv k \pmod{\lambda^{14}}}} \sum_{p=1}^{2} Z_{pf}(X, \lambda^m rt, \eta, 0; \dot{W}_K), \tag{9-2}$$

where $Z_{pf}(\cdots)$ for $p = 1,2$ are given in (8-54) and (8-55) respectively. The weight functions involved are radial, see Remarks 8.16 and 8.18, so only $n = 0$ occurs on the dual side of Voronoi summation.

9.2. *Evaluation and bounds for arithmetic exponential sums.*

We now consider the arithmetic exponential sum $\psi^{\#}_{\lambda^m rt}(\cdot)_{\lambda\eta,\zeta^{-1}}(\lambda^4\nu)$ for $\nu \in \lambda^{-3}\mathbb{Z}[\omega]$ that occurs in $Z_{2f}(\cdots)$. Throughout this computation we will repeatedly use the facts $\eta \equiv ua\bar{a} \pmod{\lambda^{e_v} v_0 a}$, $a\bar{a} \equiv 1 \pmod{\lambda^{e_v} v_0}$, $0 \leq m \leq e_v + 1$, $r \mid a$, and $t \mid v_0$, without further reference. Using Lemma 8.19 we have

$$\psi^{\#}_{\lambda^m rt}(\cdot)_{\lambda\eta,\zeta^{-1}}(\lambda^4\nu) = \frac{1}{N(\lambda^{m+3} rt)} K_{\Gamma_1(3),\sigma,\sigma}(\zeta(\lambda^3\nu), \zeta\eta, \lambda^{m-1} rt), \tag{9-3}$$

where $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. After opening the cubic Kloosterman sum in (9-3), we then perform a computation using the Chinese remainder theorem (with coprime moduli $\lambda^{m-1}t$ and r), (2-1), and (2-7), to obtain

$$\psi^{\#}_{\lambda^m rt}(\cdot)_{\lambda\eta,\zeta^{-1}}(\lambda^4\nu) = \frac{1}{N(r)^{1/2} N(\lambda^{m+3} t)} \overline{\left(\frac{\zeta^{-1}\lambda^{m-1}t}{r}\right)_3}\, \overline{\tilde{g}(\lambda^3\nu, r)}\, K_{\Gamma_1(3),\sigma,\sigma}(\zeta\bar{r}(\lambda^3\nu), \zeta\bar{r}u, \lambda^{m-1}t). \tag{9-4}$$

The bound

$$|\psi^{\#}_{\lambda^m rt}(\cdot)_{\lambda\eta,\zeta^{-1}}(\lambda^4\nu)| \ll N(\lambda^m rt)^{-1/2+\varepsilon} \cdot N((\lambda^3\nu, r))^{1/2} \tag{9-5}$$

for $\nu \in \lambda^{-3}\mathbb{Z}[\omega]$ and $m \in \mathbb{Z}_{\geq 6}$ follows from using Lemmas 2.3 and 3.1 in (9-4), (2-16), and the fact $(\zeta\bar{r}(\lambda^3\nu), \zeta\bar{r}u, \lambda^{m-1}t) = 1$.

We now give a similar treatment of the arithmetic sum $\psi^{\star}_{rt}(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu)$ that occurs in $Z_{1f}(\cdots)$. Using Lemma 8.20 we have

$$\psi^{\star}_{rt}(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu) = \frac{1}{N(rt)} K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}(\lambda^{2m+3}\nu), \overline{\lambda^3}\eta, rt), \tag{9-6}$$

where σ is as above, and $\xi := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. After opening the cubic Kloosterman sum in (9-6), we then perform a computation using the Chinese remainder theorem (with coprime moduli t and r) and (2-7), to obtain

$$\psi^{\star}_{rt}(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu) = \frac{1}{N(r)^{1/2}N(t)} \overline{\left(\frac{\lambda^{2m}t}{r}\right)_3}\, \overline{\tilde{g}(\lambda^{2m+3}\nu, r)} K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}\bar{r}(\lambda^{2m+3}\nu), \overline{\lambda^3}\bar{r}u, t). \tag{9-7}$$

The bound

$$|\psi^{\star}_{rt}(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu)| \ll N(rt)^{-1/2+\varepsilon} N((\lambda^{2m+3}\nu, r))^{1/2} \tag{9-8}$$

for $\nu \in \lambda^{-2m-3}\mathbb{Z}[\omega]$ follows from using Lemmas 2.3 and 3.1 in (9-6), (2-16), and the fact

$$(\overline{\lambda^{2m+3}}\bar{r}(\lambda^{2m+3}\nu), \overline{\lambda^3}\bar{r}u, t) = 1.$$

9.3. *Truncations and conclusion.* We substitute (9-7) and (9-4) into $Z_{pf}(\cdots)$ for $p = 1, 2$ respectively. We recall Remark 8.18, use Lemma 8.21 (with $D_1 > 0$ large and fixed and $D_2 = 0$) together with Lemma 2.8 and (9-8) (resp. (9-5)) to truncate the ν-sums in $Z_{pf}(\cdots)$ for $p = 1$ (resp. $p = 2$) to

$$N(\nu) \ll (XKN(v))^{\varepsilon} \cdot K^4 N(\lambda^m rt)^2 X^{-1} =: P, \tag{9-9}$$

with negligible error $O((XKN(v))^{-2000})$. Denote the truncated expressions by $Z'_{pf}(\cdots, P)$ for $p = 1, 2$. Without loss of generality, we can restrict our attention to the case $P \gg (XKN(v))^{-\varepsilon}$ otherwise both $Z'_{pf}(\cdots, P)$ for $p = 1, 2$. are $O((XKN(v))^{-2000})$ by the above argument. Thus

$$\mathscr{S}_f(a, X; v, u; W_K) = \frac{X}{N(\lambda^{e_v+1}v_0 a)} \sum_{\substack{k \ (\mathrm{mod}\ \lambda^{14}) \\ k \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{m,r,t \\ 0 \leq m \leq e_v+1 \\ r \mid a,\ t \mid v_0 \\ rt \equiv k \ (\mathrm{mod}\ \lambda^{14}) \\ P \gg (XKN(v))^{-\varepsilon}}} \sum_{p=1}^{2} Z'_{pf}(\cdots, P) + O((XKN(v))^{-1000}). \tag{9-10}$$

Using the triangle inequality and (9-5), (9-8), and Lemma 8.21 (with $D_1 = \varepsilon$ and $D_2 = 0$) we obtain

$$Z'_{1f}(\cdots, P) \ll (XK)^{\varepsilon} N(\lambda^m)^{1+\varepsilon} N(rt)^{-1/2+\varepsilon} \sum_{\substack{\nu \in \lambda^{-2m-3}\mathbb{Z}[\omega] \\ N(\nu) \ll P}} |\rho_{f \otimes \widehat{\psi_{\lambda^m}(\cdot)_{\lambda u}}, \mathfrak{c}(m,1;k)}(\nu)| N((\lambda^{2m+3}\nu, r))^{1/2} \quad \text{for } 0 \leq m \leq \min\{5, e_v+1\}, \tag{9-11}$$

$$Z'_{2f}(\cdots, P) \ll (XK)^{\varepsilon} N(\lambda^m rt)^{-1/2+\varepsilon} \sum_{\substack{\nu \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu) \ll P}} |\rho_f(\nu)| N((\lambda^3\nu, r))^{1/2} \quad \text{for } 6 \leq m \leq e_v+1. \tag{9-12}$$

We now bound (9-11) and (9-12) by applying the Cauchy–Schwarz inequality to the ν-sums, Lemma 2.5, Lemma 2.7, and (2-14). Substitution of the result into (9-10) gives

$$\mathscr{S}_f(a, X; v, u; W_K) \ll \frac{X(XKN(v))^{\varepsilon}}{N(\lambda^{e_v+1}v_0 a)} \sum_{\substack{m,r,t \\ 0\le m\le e_v+1 \\ r\mid a,\ t\mid v_0 \\ P\gg (XKN(v))^{-\varepsilon}}} \left(N(\lambda^m rt)^{-1/2} \cdot \frac{K^4 N(\lambda^m rt)^2}{X} \right) + (XKN(v))^{-1000}$$
$$\ll (XKN(v))^{\varepsilon} K^4 N(v)^{1/2} N(a)^{1/2}, \tag{9-13}$$

as required. □

Proof of Proposition 1.6. We multiply (9-1) by $\mu^2(a)\alpha_a$ and sum over $a \in \mathbb{Z}[\omega]$. We repeat the same steps on the ν sum as in the proof of Lemma 1.4 up to the display (9-2). We then insert a smooth dyadic partition of unity in r variable. We obtain

$$\mathscr{A}_f(\cdots) = \sum_{\substack{1\ll R\ll A \\ R \text{ dyadic}}} \mathscr{A}_f(\cdots, R), \tag{9-14}$$

where

$$\mathscr{A}_f(\cdots, R) := \frac{X}{N(\lambda^{e_v+1}v_0)} \sum_{a\in\mathbb{Z}[\omega]} \frac{\mu^2(a)\alpha_a}{N(a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k\equiv 1 \pmod 3}} \sum_{\substack{m,r,t \\ 0\le m\le e_v+1 \\ r\mid a,\ t\mid v_0 \\ rt\equiv k \pmod{\lambda^{14}}}} U\left(\frac{N(r)}{R}\right) \sum_{p=1}^{2} Z_{pf}(X, \lambda^m rt, \eta, 0; \dot{W}_K), \tag{9-15}$$

where the $Z_{pf}(\cdots)$ for $p = 1, 2$ are given by (8-54) and (8-55) respectively. We recall Remark 8.18, use Lemma 8.21 (with $D_1 > 0$ large and fixed and $D_2 = 0$) together with Lemma 2.8 and (9-8) (resp. (9-5)) to truncate the ν-sums in $Z_{pf}(\cdots)$ for $p = 1$ (resp. $p = 2$) with $N(r) \sim R$ to obtain

$$N(\nu) \ll (XKN(v))^{\varepsilon} \cdot K^4 R^2 N(\lambda^m t)^2 X^{-1} =: P_0, \tag{9-16}$$

with negligible error $O((XKN(v))^{-2000})$. Denote the truncated expressions by $Z'_{pf}(\cdots, P_0)$ for $p = 1, 2$. Without loss of generality, we can restrict our attention to the case that $P_0 \gg (XKN(v))^{-\varepsilon}$ otherwise both $Z'_{pf}(\cdots, P_0)$ are $O((XKN(v))^{-2000})$ by the above argument. Thus (9-15) becomes

$$\mathscr{A}_f(\cdots, R) = \frac{X}{N(\lambda^{e_v+1}v_0)} \sum_{a\in\mathbb{Z}[\omega]} \frac{\mu^2(a)\alpha_a}{N(a)} \sum_{\substack{k \pmod{\lambda^{14}} \\ k\equiv 1 \pmod 3}} \sum_{\substack{m,r,t \\ 0\le m\le e_v+1 \\ r\mid a,\ t\mid v_0 \\ rt\equiv k \pmod{\lambda^{14}} \\ P_0\gg (XKN(v))^{-\varepsilon}}} U\left(\frac{N(r)}{R}\right) \sum_{p=1}^{2} Z'_{pf}(\cdots, P_0) + O((XKN(v))^{-1000}\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2). \tag{9-17}$$

9.4. ***Further simplification using the squarefree support of $\boldsymbol{\alpha}$.*** We further open each $Z'_{pf}(\cdots, P_0)$ in (9-17) and manipulate them by further simplifying (9-4) and (9-7) under the assumption that $r \equiv 1 \pmod 3$ is squarefree (as is the case in (9-17)). For r squarefree and $\mu \in \mathbb{Z}[\omega]$, Lemma 2.1 guarantees that $g(\mu, r) = 0$ unless $(\mu, r) = 1$. When $(\mu, r) = 1$ we note that (2-7) implies that

$$\tilde{g}(\mu, r) = \overline{\left(\frac{\mu}{r}\right)_3}\, \tilde{g}(r).$$

Thus (9-4) becomes

$$\psi^{\#}_{\lambda^m rt}(\,\cdot\,)_{\lambda\eta,\zeta^{-1}}(\lambda^4\nu) = \begin{cases} N(r)^{-1/2}\overline{\left(\frac{\zeta^{-1}\lambda^{m-1}t}{r}\right)_3}\,\overline{\tilde{g}(r)}\left(\frac{\lambda^3\nu}{r}\right)_3 \cdot N(\lambda^{m+3}t)^{-1}K_{\Gamma_1(3),\sigma,\sigma}(\zeta\bar{r}(\lambda^3\nu),\zeta\bar{r}u,\lambda^{m-1}t) & \text{if } (\lambda^3\nu,r)=1, \\ 0 & \text{otherwise} \end{cases} \tag{9-18}$$

for all $m\in\mathbb{Z}_{\geq 6}$ and $\nu\in\lambda^{-3}\mathbb{Z}[\omega]$. Similarly, (9-7) becomes

$$\psi^{\star}_{rt}(\,\cdot\,)_{\lambda^{2m+1}\eta}(\lambda^{2m+4}\nu) = \begin{cases} N(r)^{-1/2}\overline{\left(\frac{\lambda^{2m}t}{r}\right)_3}\,\overline{\tilde{g}(r)}\left(\frac{\lambda^{2m+3}\nu}{r}\right)_3 \cdot N(t)^{-1}K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}\bar{r}(\lambda^{2m+3}\nu),\overline{\lambda^3}\bar{r}u,t) & \text{if } (\lambda^{2m+3}\nu,r)=1, \\ 0 & \text{otherwise} \end{cases} \tag{9-19}$$

for all $m\in\mathbb{Z}_{\geq 0}$ and $\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega]$.

9.5. ***Preparations for the cubic large sieve.*** We substitute (9-19) and (9-18) into the expressions for $Z'_{pf}(\cdots,P_0)$ for $p=1$ (resp. $p=2$) in (9-17), insert a smooth dyadic partition of unity in the ν variable, open the transforms $\dot{W}_K(\,\cdot\,)$ with (8-56) and move the resulting contour integral to $\mathrm{Re}(s)=\varepsilon$, resolve the r, $\lambda^3\nu$, $\lambda^{2m+3}\nu$ variables into congruence classes modulo $\lambda^{\max\{4,m-1\}}t$, and interchange the order of summation/integration by absolute convergence (see (8-50) and (8-66)). Then (9-17) becomes

$$\mathscr{A}_f(\cdots,R)=\mathscr{A}'_f(\cdots,R)+\mathscr{A}''_f(\cdots,R)+O((XKN(v))^{-1000}\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2), \tag{9-20}$$

where

$$\begin{aligned}\mathscr{A}'_f(\cdots,R) &:= \frac{X}{N(\lambda^{e_v+1}v_0)}\sum_{\substack{m,t\\ 0\leq m\leq\min\{5,e_v+1\}\\ t\mid v_0\\ P_0\gg(XKN(v))^{-\varepsilon}}}\frac{N(\lambda^m)}{N(t)}\sum_{\substack{k \pmod{\lambda^{14}}\\ k\equiv 1 \pmod 3}}\omega(m,1;k)\sum_{\substack{\boldsymbol{j}\in(\mathbb{Z}[\omega]/9t\mathbb{Z}[\omega])^2\\ j_1\equiv 1 \pmod 3\\ (j_1,t)=1}}\overline{\left(\frac{\lambda^{2m}t}{j_1}\right)_3}\\ &\times K_{\Gamma_1(3),\sigma,\xi}(\overline{\lambda^{2m+3}}\,\overline{j_1}j_2,\overline{\lambda^3}\,\overline{j_1}u,t)\cdot\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G_\infty(s,\tau_f,0)}{G_\infty(1-s,\tau_f,0)}\widehat{W}_K(1-s)X^{-s}\\ &\times\sum_{\substack{1\ll S\ll P_0\\ S \text{ dyadic}}}\Bigg(\sum_{\substack{\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega]\\ \lambda^{2m+3}\nu\equiv j_2 \pmod{9t}\\ N(\nu)\sim S}}\sum_{\substack{r\in\mathbb{Z}[\omega]\\ r\equiv j_1 \pmod{9t}\\ rt\equiv k \pmod{\lambda^{14}}\\ N(r)\sim R}}\Omega'_\nu(s,\lambda^m,k,S)\Psi_r(s,R)\left(\frac{\lambda^{2m+3}\nu}{r}\right)_3\Bigg)\,ds;\end{aligned} \tag{9-21}$$

$$\mathscr{A}''_f(\cdots, R) := \frac{X}{N(\lambda^{e_v+1}v_0)} \sum_{\substack{\zeta,m,t \\ 6\le m\le e_v+1 \\ t\mid v_0 \\ P_0\gg(XKN(v))^{-\varepsilon}}} \frac{1}{N(\lambda^{m-1}t)} \sum_{\substack{j\in(\mathbb{Z}[\omega]/\lambda^{m-1}t\mathbb{Z}[\omega])^2 \\ j_1\equiv 1 \pmod 3 \\ (j_1,t)=1}} \overline{\left(\frac{\zeta^{-1}\lambda^{m-1}t}{j_1}\right)_3}$$

$$\times K_{\Gamma_1(3),\sigma,\sigma}(\zeta\,\overline{j_1}j_2, \zeta\,\overline{j_1}u, \lambda^{m-1}t)\cdot\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G_\infty(s,\tau_f,0)}{G_\infty(1-s,\tau_f,0)}\widehat{W}_K(1-s)X^{-s}N(\lambda^{m-4})^{2s}$$

$$\times \sum_{\substack{1\ll S\ll P_0 \\ S \text{ dyadic}}}\Bigg(\sum_{\substack{\nu\in\lambda^{-3}\mathbb{Z}[\omega] \\ \lambda^3\nu\equiv j_2 \pmod{\lambda^{m-1}t} \\ N(\nu)\sim S}} \sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv j_1 \pmod{\lambda^{m-1}t} \\ N(r)\sim R}} \Omega''_\nu(s,S)\Psi_r(s,R)\left(\frac{\lambda^3\nu}{r}\right)_3\Bigg)\,ds, \tag{9-22}$$

$$\Psi_r(s,R) := N(r)^{-1/2}N(r)^{2s}\overline{\tilde{g}(r)}U\left(\frac{N(r)}{R}\right)\sum_{a\equiv 0 \pmod r}\frac{\mu^2(a)\alpha_a}{N(a)}; \tag{9-23}$$

$$\Omega'_\nu(s,\lambda^m,k,S) := N(\nu)^{-s}U\left(\frac{N(\nu)}{S}\right)\rho_{f\otimes\widehat{\psi_{\lambda^m}(\cdot)_{\lambda u}},\mathfrak{c}(m,1;k)}(\nu);$$

$$\Omega''_\nu(s,S) := N(\nu)^{-s}U\left(\frac{N(\nu)}{S}\right)\rho_f(\nu).$$

Observe that the weights $\Psi_r(s,R)$ in (9-23) are supported on squarefree r (see (2-11)).

9.6. ***Application of the cubic large sieve and conclusion.*** Consider the bilinear form in ν and r and in the last display of (9-21). Using Corollary 4.2 (the cubic large sieve) we obtain

$$\sum_{\substack{\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega] \\ \lambda^{2m+3}\nu\equiv j_2 \pmod{9t} \\ N(\nu)\sim S}} \sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv j_1 \pmod{9t} \\ rt\equiv k \pmod{\lambda^{14}} \\ N(r)\sim R}} \Omega'_\nu(s,\lambda^m,k,S)\Psi_r(s,R)\left(\frac{\lambda^{2m+3}\nu}{r}\right)_3$$

$$\ll (RS)^\varepsilon S^{1/6}(S^{1/2}+R^{1/2})\Bigg(\sum_{\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega]}|\Omega'_\nu(s,\lambda^m,k,S)|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv 1 \pmod 3}}\mu^2(r)|\Psi_r(s,R)|^2\Bigg)^{1/2}, \tag{9-24}$$

where we dropped some of the congruence conditions in the L^2-norms by positivity. Lemma 2.7 gives

$$\sum_{\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega]}|\Omega'_\nu(s,\lambda^m,k,S)|^2 \ll S^{1+\varepsilon} \tag{9-25}$$

for each $0\le m\le \min\{5, e_v+1\}$ and $S\gg 1$. Using (2-11) and (2-12) we compute

$$\sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv 1 \pmod 3}}\mu^2(r)|\Psi_r(s,R)|^2 = \sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv 1 \pmod 3}}\frac{\mu^2(r)}{N(r)^{1-4\operatorname{Re}(s)}}\left|U\left(\frac{N(r)}{R}\right)\right|^2\cdot\Bigg|\sum_{\substack{a\in\mathbb{Z}[\omega] \\ a\equiv 0 \pmod r}}\frac{\mu^2(a)\alpha_a}{N(a)}\Bigg|^2$$

$$\ll R^{-1+\varepsilon} \sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv 1 \pmod 3 \\ N(r)\sim R}} \Bigg| \sum_{\substack{a\in\mathbb{Z}[\omega] \\ a\equiv 0 \pmod r}} \frac{\mu^2(a)\alpha_a}{N(a)} \Bigg|^2$$

$$\ll A^{-1}R^{-2+\varepsilon} \sum_{\substack{u,r\in\mathbb{Z}[\omega] \\ u,r\equiv 1 \pmod 3}} \mu^2(ur)|\alpha_{ur}|^2 \ll (AR)^{\varepsilon}A^{-1}R^{-2}\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2^2, \quad (9\text{-}26)$$

where the penultimate display follows from the Cauchy–Schwarz inequality, a change of variables, and the last display follows from (2-14). Substituting (9-25) and (9-26) into (9-24), and summing both sides of the result inequality over dyadic values of S yields we obtain (for each $0 \le m \le \min\{5, e_v + 1\}$)

$$\sum_{\substack{1\ll S\ll P_0 \\ S \text{ dyadic}}} \Bigg| \sum_{\substack{\nu\in\lambda^{-2m-3}\mathbb{Z}[\omega] \\ \lambda^{2m+3}\nu\equiv j_2 \pmod{9t} \\ N(\nu)\sim S}} \sum_{\substack{r\in\mathbb{Z}[\omega] \\ r\equiv j_1 \pmod{9t} \\ rt\equiv k \pmod{\lambda^{14}} \\ N(r)\sim R}} \Omega_v'(s,\lambda^m,k,S)\Psi_r(s,R)\left(\frac{\lambda^{2m+3}\nu}{r}\right)_3 \Bigg|$$

$$\ll (XKN(v))^{\varepsilon}(K^{14/3}N(t)^{7/3}R^{4/3}A^{-1/2}X^{-7/6} + K^{8/3}N(t)^{4/3}R^{5/6}A^{-1/2}X^{-2/3})\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2, \quad (9\text{-}27)$$

where (9-16) was used to obtain the last display. We insert the bound (9-27) into (9-21), and then use (8-50), (8-66), and Lemma 3.1 to obtain

$$\mathscr{A}_f'(\cdots,R) \ll (XKN(v))^{\varepsilon}(K^{14/3}N(v)^{5/6}R^{4/3}A^{-1/2}X^{-1/6}+K^{8/3}N(v)^{-1/6}R^{5/6}A^{-1/2}X^{1/3})\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2. \quad (9\text{-}28)$$

An analogous computation shows that $\mathscr{A}_f''(\cdots,R)$ satisfies the same bound as that in (9-28). After substituting these bounds into (9-20), we then substitute the result into (9-14) to obtain

$$\mathscr{A}_f(\cdots) \ll (XKN(v))^{\varepsilon}(K^{14/3}N(v)^{5/6}X^{-1/6}A^{5/6} + K^{8/3}N(v)^{-1/6}(AX)^{1/3})\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2$$

$$\ll (XKN(v))^{\varepsilon}K^{14/3}N(v)^{5/6}(AX)^{1/3}\|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_2, \quad (9\text{-}29)$$

where the last inequality follows since $A \ll X$. The result follows. □

10. Type-II estimates via average (homogenous) convolution

Recall the notation from Section 1, in particular (1-14). The first result in this section bounds the Type-II sum in terms of a homogeneous average convolution problem.

Lemma 10.1. *Let the notation be as above and $X \asymp AB$. Then*

$$|\mathscr{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},X,v,u;W_K)| \le \|\boldsymbol{\beta}\|_2 \cdot \left(\sum_{\boldsymbol{a}} \mu^2(a_1)\alpha_{a_1}\mu^2(a_2)\overline{\alpha_{a_2}}\mathscr{L}_f(\boldsymbol{a},X,v,u;W_K)\right)^{1/2},$$

where

$$\mathscr{L}_f(\boldsymbol{a},X,v,u;W_K) := \sum_{\substack{b\in\mathbb{Z}[\omega] \\ a_1b\equiv u \pmod v \\ a_2b\equiv u \pmod v}} \rho_f(\lambda^{-3}a_1b)\overline{\rho_f(\lambda^{-3}a_2b)}W_K\left(\frac{N(\lambda^{-3}a_1b)}{X}\right)\overline{W_K\left(\frac{N(\lambda^{-3}a_2b)}{X}\right)}. \quad (10\text{-}1)$$

Proof. We apply the Cauchy–Schwarz inequality to the b-sum in (1-14) to obtain

$$|\mathscr{B}_f(\cdots)| \leq \|\boldsymbol{\beta}\|_2 \cdot \Bigg(\sum_{b \in \mathbb{Z}[\omega]} \Bigg| \sum_{\substack{a \in \mathbb{Z}[\omega] \\ ab \equiv u \pmod{v}}} \mu^2(a) \alpha_a \rho_f(\lambda^{-3} ab) W_K\Big(\frac{N(\lambda^{-3}ab)}{X}\Big) \Bigg|^2 \Bigg)^{1/2}.$$

The result follows from expanding the square modulus in the above expression and interchanging the order of summation. □

Proposition 10.2. *Let the notation be as above and $X \asymp AB$. Then*

$$\sum_{\boldsymbol{a}} \mu^2(a_1)\alpha_{a_1}\mu^2(a_2)\overline{\alpha_{a_2}} \mathscr{L}_f(\boldsymbol{a}, X, v, u; W_K) \ll_{\varepsilon, f} (XKN(v))^{\varepsilon} K^{16} N(v)^8 (AB + A^3 B^{1/2}) \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_\infty^2.$$

Remark 10.3. It will be helpful to remember the normalisation in (8-12) throughout the proof. We also use the same notation and convention as Remark 9.1.

Proof. We begin by separating oscillations using the circle method.

10.1. ***Application of the circle method.*** Rewriting (10-1) we obtain

$$\mathscr{L}_f(\boldsymbol{a}, \dots) = \sum_{\substack{\boldsymbol{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i : \lambda^3 v_i \equiv 0 \pmod{a_i} \\ \forall i : \lambda^3 v_i \equiv u \pmod{\lambda^{e_v} v_0}}} \rho_f(v_1)\overline{\rho_f(v_2)} W_K\Big(\frac{N(v_1)}{X}\Big) \overline{W_K\Big(\frac{N(v_2)}{X}\Big)} \delta_{\mathbb{Q}(\omega)}\Big(\frac{\lambda^3 v_2}{a_2} - \frac{\lambda^3 v_1}{a_1}\Big). \quad (10\text{-}2)$$

After noting Remark 5.3 we choose $C > 0$ such that

$$C^4 := X/A \asymp B. \quad (10\text{-}3)$$

We use Theorem 5.2 and Remark 5.1 to obtain

$$\delta_{\mathbb{Q}(\omega)}\Big(\frac{\lambda^3 v_2}{a_2} - \frac{\lambda^3 v_1}{a_1}\Big) = \frac{k_C}{C^4} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c \equiv 1 \pmod{3}}} N(\lambda^\ell c) \hat{\psi}_{\lambda^\ell c}\Big(\frac{\lambda^3 v_2}{a_2} - \frac{\lambda^3 v_1}{a_1}\Big) h\Big(\frac{N(\lambda^\ell c)}{C^2}, \frac{N(\lambda^3 v_2/a_2 - \lambda^3 v_1/a_1)}{C^4}\Big) \quad (10\text{-}4)$$

for any $\boldsymbol{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2$ such that $\lambda^3 v_i \equiv 0 \pmod{a_i}$ for $i = 1, 2$, and where $\psi_{\lambda^\ell c}$ denotes the principal character modulo $\lambda^\ell c$. Let $\ell_0 := \max\{\ell, e_v\}$. We substitute (10-4) into (10-2), interchange the order of summation, and resolve $\lambda^3 v_i$ into congruence classes $\pmod{\lambda^{\ell_0} a_i v_0 c}$ for $i = 1, 2$. We obtain

$$\mathscr{L}_f(\boldsymbol{a}, \dots) = \frac{k_C}{C^4} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c \equiv 1 \pmod{3}}} \sum_{\substack{\boldsymbol{j} \in \prod_{i=1}^2 \mathbb{Z}[\omega]/\lambda^{\ell_0} a_i v_0 c \mathbb{Z}[\omega] \\ \forall i : j_i \equiv u \pmod{\lambda^{e_v} v_0}}} N(\lambda^\ell c) \hat{\psi}_{\lambda^\ell c}\Big(\frac{j_2}{a_2} - \frac{j_1}{a_1}\Big)$$
$$\times \sum_{\substack{\boldsymbol{v} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i : \lambda^3 v_i \equiv j_i \pmod{\lambda^{\ell_0} a_i v_0 c}}} \rho_f(v_1)\overline{\rho_f(v_2)} H_{K, C^2/N(\lambda^\ell c)}\Big(\frac{v_1}{\sqrt{X}}, \frac{v_2}{\sqrt{X}}\Big), \quad (10\text{-}5)$$

where $H_{K,C^2/N(\lambda^\ell c)}(z) := H_{K,C^2/N(\lambda^\ell c)}(z;\boldsymbol{a},\lambda^\ell c, X, C)$ is given by

$$H_{K,C^2/N(\lambda^\ell c)}(z) = W_K(|z_1|^2)W_K(|z_2|^2)h\biggl(\frac{N(\lambda^\ell c)}{C^2}, \frac{X|\lambda^3 z_1/a_1 - \lambda^3 z_2/a_2|^2}{C^4}\biggr). \tag{10-6}$$

We now justify the subscripts for the function $H_{K,C^2/N(\lambda^\ell c)}(z)$ (see (8-47)). Recall that $C^4 := X/A \asymp B$, $N(a_i) \asymp A$, and $|z_i| \asymp 1$ for $i = 1, 2$. Thus

$$X|\lambda^3 z_1/a_1 - \lambda^3 z_2/a_2|^2/C^4 \ll 1. \tag{10-7}$$

Observe that (10-7) and (5-4) imply that

$$h\biggl(\frac{N(\lambda^\ell c)}{C^2}, \frac{X|\lambda^3 z_1/a_1 - \lambda^3 z_2/a_2|^2}{C^4}\biggr) \neq 0 \quad \text{only if } N(\lambda^\ell c) \ll C^2. \tag{10-8}$$

The chain rule, (1-6) (with $M = 1$), (5-3), Corollary 5.6, (10-8) and the fact that $K \geq 1$ together imply that for any $\boldsymbol{i} = (i_{11}, i_{12}, i_{21}, i_{22}) \in (\mathbb{Z}_{\geq 0})^4$ we have

$$\partial^{\boldsymbol{i}} H_{K,C^2/N(\lambda^\ell c)}(z) \ll_{\boldsymbol{i}} \frac{C^2}{N(\lambda^\ell c)} \cdot K^{i_{11}+i_{12}+i_{21}+i_{22}}. \tag{10-9}$$

10.2. ***Double application of Voronoi summation.*** We use (double)-Voronoi summation (Proposition 8.22). By abuse of notation we denote (X, X) by X. We obtain

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{\nu}\in(\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i:\lambda^3\nu_i\equiv j_i \pmod{\lambda^{\ell_0}a_i v_0 c}}} \rho_f(\nu_1)\overline{\rho_f(\nu_2)} H_{K,C^2/N(\lambda^\ell c)}\biggl(\frac{\nu_1}{\sqrt{X}}, \frac{\nu_2}{\sqrt{X}}\biggr) \\
&= \frac{X^2}{N(\lambda^{\ell_0+1}v_0 c)^2 N(a_1 a_2)} \\
&\quad \times \sum_{\substack{\boldsymbol{k}\in(\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2 \\ \forall i:k_i\equiv 1 \pmod 3}} \sum_{\substack{\boldsymbol{m},\boldsymbol{r} \\ \forall i:0\leq m_i\leq \ell_0+1 \\ \forall i:r_i|a_i v_0 c \\ \forall i:r_i\equiv k_i \pmod{\lambda^{14}}}} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1+n_2} \sum_{p=1}^{4} \mathscr{D}_{pf}(X, \boldsymbol{\lambda}^{\boldsymbol{m}}\boldsymbol{r}, \boldsymbol{j}, \boldsymbol{n}; \ddot{H}_{K,C^2/N(\lambda^\ell c)}),
\end{aligned} \tag{10-10}$$

where the $\mathscr{D}_{pf}(\cdots)$ are given by (8-73)–(8-76). We substitute (10-10) into (10-5) to obtain

$$\mathscr{L}_f(\boldsymbol{a},\dots) = \sum_{p=1}^{4} \mathscr{M}_{pf}(\boldsymbol{a},\dots), \tag{10-11}$$

where

$$\begin{aligned}
\mathscr{M}_{pf}(\boldsymbol{a},\dots) := \frac{k_C}{C^4}\frac{X^2}{N(a_1a_2)N(v_0)^2} \sum_{1\leq \ell \ll \log C} \sum_{\substack{c\in\mathbb{Z}[\omega] \\ c\equiv 1 \pmod 3}} \frac{N(\lambda^\ell)}{N(\lambda^{\ell_0+1})^2}\frac{1}{N(c)} \sum_{\substack{\boldsymbol{j}\in\prod_{i=1}^2 \mathbb{Z}[\omega]/\lambda^{\ell_0}a_i v_0 c\mathbb{Z}[\omega] \\ \forall i: j_i\equiv 0 \pmod{a_i} \\ \forall i: j_i \equiv u \pmod{\lambda^{e_v} v_0}}} \hat{\psi}_{\lambda^\ell c}\biggl(\frac{j_2}{a_2} - \frac{j_1}{a_1}\biggr) \\
\times \sum_{\substack{\boldsymbol{k}\in(\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2 \\ \forall i:k_i\equiv 1 \pmod 3}} \sum_{\substack{\boldsymbol{m},\boldsymbol{r} \\ \forall i:0\leq m_i\leq \ell_0+1 \\ \forall i:r_i|a_i v_0 c \\ \forall i:r_i\equiv k_i \pmod{\lambda^{14}}}} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1+n_2} \mathscr{D}_{pf}(X, \boldsymbol{\lambda}^{\boldsymbol{m}}\boldsymbol{r}, \boldsymbol{j}, \boldsymbol{n}; \ddot{H}_{K,C^2/N(\lambda^\ell c)}).
\end{aligned} \tag{10-12}$$

We now make a sequence of manipulations to $\mathscr{M}_{pf}(\boldsymbol{a},\dots)$ in (10-12). First we make a change of variable $j_i \to a_i j_i$ for $i=1,2$ (the new j_i variables run $(\operatorname{mod} \lambda^{\ell_0} v_0 c)$). We then uniquely factorise each $c \in \mathbb{Z}[\omega]$ with $c \equiv 1 \pmod 3$ as $c = tq'q''$, where $t, q', q'' \in \mathbb{Z}[\omega]$ satisfy

$$t, q', q'' \equiv 1 \pmod 3, \quad t \mid \operatorname{rad}(v_0)^\infty, \quad q' \mid \operatorname{rad}(a_1 a_2)^\infty, \quad \text{and} \quad (q'', a_1 a_2 v_0) = 1. \tag{10-13}$$

Note this factorisation exists and is unique since $(v_0, a_1 a_2) = 1$. We also uniquely factorise each $r_i \mid a_i v_0 c$ with $r_i \equiv k_i \pmod{\lambda^{14}}$ as $r_i = t_i r_i' r_i''$, where t_i, r_i', r_i'' satisfy

$$t_i, r_i', r_i'' \equiv 1 \pmod 3, \quad t_i \mid v_0 t, \quad r_i' \mid a_i q', \quad r_i'' \mid q'', \quad \text{and} \quad t_i r_i' r_i'' \equiv k_i \quad \text{for } i=1,2. \tag{10-14}$$

We use the Chinese remainder theorem on the new j_i variables (with the pairwise coprime moduli $\lambda^{\ell_0} v_0 t$, q', and q'') and $i=1,2$ to write

$$\begin{aligned} \boldsymbol{j} &:= q'q''\overline{q'q''}\boldsymbol{J} + \lambda^{\ell_0} v_0 t q'' \overline{\lambda^{\ell_0} v_0 t q''}\boldsymbol{J}' + \lambda^{\ell_0} v_0 t q' \overline{\lambda^{\ell_0} v_0 t q'}\boldsymbol{J}'', \\ J_i &\equiv \overline{a_i} u \pmod{\lambda^{e_v} v_0} \quad \text{for } i=1,2, \end{aligned} \tag{10-15}$$

where $\overline{a_i}$, $\overline{q'q''}$, $\overline{\lambda^{\ell_0} v_0 t q''}$, $\overline{\lambda^{\ell_0} v_0 t q'} \in \mathbb{Z}[\omega]$ are such that $a_i \overline{a_i} \equiv 1 \pmod{\lambda^{e_v} v_0}$, $\overline{q'q''}q'q'' \equiv 1 \pmod{\lambda^{\ell_0} v_0 t}$, $\overline{\lambda^{\ell_0} v_0 t q'}\lambda^{\ell_0} v_0 t q' \equiv 1 \pmod{q''}$, and $\overline{\lambda^{\ell_0} v_0 t q''}\lambda^{\ell_0} v_0 t q'' \equiv 1 \pmod{q'}$. Without loss of generality we may assume that $e_v \geq 14$. We further make the change of variable

$$\boldsymbol{J} \to \lambda^{e_v} v_0 \boldsymbol{J} + (Y_1 u, Y_2 u) \tag{10-16}$$

in (10-15), where $Y_i \in \mathbb{Z}[\omega]$ is such that $Y_i \equiv \overline{a_i} \pmod{\lambda^{e_v} v_0}$. Observe that the new J_1, J_2 variables run $(\operatorname{mod} \lambda^{\ell_0 - e_v} t)$. We also use the multiplicativity of Ramanujan sums $\hat{\psi}_{\lambda^\ell c}(\cdot) = \hat{\psi}_{\lambda^{\ell} t}(\cdot)\hat{\psi}_{q'}(\cdot)\hat{\psi}_{q''}(\cdot)$, and interchange the order of summation by absolute convergence. The net result is

$$\begin{aligned} &\mathscr{M}_{pf}(\boldsymbol{a},\dots) \\ &\quad := \frac{k_C}{C^4} \frac{X^2}{N(a_1 a_2) N(v_0)^2} \sum_{1 \leq \ell \ll \log C} \sum_{\substack{tq'q'' \in \mathbb{Z}[\omega] \\ (10\text{-}13)}} \frac{N(\lambda^\ell)}{N(\lambda^{\ell_0+1})^2} \frac{1}{N(tq'q'')} \\ &\qquad \times \sum_{\substack{\boldsymbol{k} \in (\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2 \\ \forall i: k_i \equiv 1 \ (\operatorname{mod} 3)}} \sum_{\substack{\boldsymbol{m}, \boldsymbol{t}, \boldsymbol{r}', \boldsymbol{r}'' \\ \forall i: 0 \leq m_i \leq \ell_0+1 \\ (10\text{-}14)}} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} (-1)^{n_1+n_2} S_{pf}(\boldsymbol{a}, \lambda^\ell t q' q'', \boldsymbol{\lambda^m t r' r''}, \boldsymbol{n}; \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')}), \end{aligned} \tag{10-17}$$

where

$$\begin{aligned} S_{1f}(\cdots) &:= \delta_{\boldsymbol{m} \in [0, \min\{5, \ell_0+1\}]^2} N(\lambda^{m_1}) N(\lambda^{m_2}) \left(\frac{\overline{r_1}}{r_1}\right)^{-n_1} \left(\frac{\overline{r_2}}{r_2}\right)^{n_2} \omega(m_1; 1, k_1) \overline{\omega(m_2, 1, k_2)} \\ &\quad \times \sum_{\substack{\nu_1 \in \lambda^{-2m_1-3}\mathbb{Z}[\omega] \\ \nu_2 \in \lambda^{-2m_2-3}\mathbb{Z}[\omega]}} \rho_{f \otimes \widehat{\psi_{\lambda^{m_1}}(\cdot)_{\lambda u}}, \mathfrak{c}(m_1, 1; k_1)}(\nu_1) \overline{\rho_{f \otimes \widehat{\psi_{\lambda^{m_2}}(\cdot)_{\lambda u}}, \mathfrak{c}(m_2, 1; k_2)}(\nu_2)} \left(\frac{\nu_1}{|\nu_1|}\right)^{-n_1} \left(\frac{\overline{\nu_2}}{|\nu_2|}\right)^{-n_2} \\ &\quad \times \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')} \left(\frac{N(\nu_1)}{N(t_1 r_1' r_2'')^2 / X}, \frac{N(\nu_2)}{N(t_2 r_2' r_2'')^2 / X}, \boldsymbol{n} \right) \\ &\quad \times \mathscr{C}_1(\boldsymbol{a}, \boldsymbol{\nu}, \lambda^\ell t q' q'', \boldsymbol{\lambda^m t r' r''}); \end{aligned} \tag{10-18}$$

$$S_{2f}(\cdots) := \delta_{\boldsymbol{m}\in[0,\min\{5,\ell_0+1\}]\times[6,\ell_0+1]} N(\lambda^{m_1}) N(\lambda^4) \left(\frac{\overline{r_1}}{r_1}\right)^{-n_1} \omega(m_1;1,k_1)$$
$$\times \sum_{\substack{\nu_1\in\lambda^{-2m_1-3}\mathbb{Z}[\omega]\\ \nu_2\in\lambda^{-3}\mathbb{Z}[\omega]}} \rho_{f\otimes\widehat{\psi_{\lambda^{m_1}}(\,\cdot\,)_{\lambda u}},\mathfrak{c}(m_1,1;k_1)}(\nu_1)\overline{\rho_f(\nu_2)} \left(\frac{\nu_1}{|\nu_1|}\right)^{-n_1}\left(\frac{\overline{\nu_2}}{|\nu_2|}\right)^{-n_2}$$
$$\times \ddot{H}_{K,C^2/N(\lambda^\ell t q' q'')}\left(\frac{N(\nu_1)}{N(t_1 r_1' r_2'')^2/X}, \frac{N(\nu_2)}{N(\lambda^{m_2-4} t_2 r_2' r_2'')^2/X}, \boldsymbol{n}\right)$$
$$\times \mathscr{C}_2(\boldsymbol{a},\boldsymbol{\nu},\lambda^\ell t q' q'', \boldsymbol{\lambda^m t r' r''}), \tag{10-19}$$

$$S_{3f}(\cdots) := \delta_{\boldsymbol{m}\in[6,\ell_0+1]\times[0,\min\{5,\ell_0+1\}]} N(\lambda^4) N(\lambda^{m_2}) \left(\frac{\overline{r_2}}{r_2}\right)^{n_2} \overline{\omega(m_2;1,k_2)}$$
$$\times \sum_{\substack{\nu_1\in\lambda^{-3}\mathbb{Z}[\omega]\\ \nu_2\in\lambda^{-2m_2-3}\mathbb{Z}[\omega]}} \rho_f(\nu_1)\overline{\rho_{f\otimes\widehat{\psi_{\lambda^{m_2}}(\,\cdot\,)_{\lambda u}},\mathfrak{c}(m_2,1;k_2)}(\nu_2)} \left(\frac{\nu_1}{|\nu_1|}\right)^{-n_1}\left(\frac{\overline{\nu_2}}{|\nu_2|}\right)^{-n_2}$$
$$\times \ddot{H}_{K,C^2/N(\lambda^\ell t q' q'')}\left(\frac{N(\nu_1)}{N(\lambda^{m_1-4} t_1 r_1' r_2'')^2/X}, \frac{N(\nu_2)}{N(t_2 r_2' r_2'')^2/X}, \boldsymbol{n}\right)$$
$$\times \mathscr{C}_3(\boldsymbol{a},\boldsymbol{\nu},\lambda^\ell t q' q'', \boldsymbol{\lambda^m t r' r''}), \tag{10-20}$$

$$S_{4f}(\cdots) := \delta_{\boldsymbol{m}\in[6,\ell_0+1]^2}\cdot N(\lambda^8)$$
$$\times \sum_{\nu_1,\nu_2\in\lambda^{-3}\mathbb{Z}[\omega]} \rho_f(\nu_1)\overline{\rho_f(\nu_2)} \left(\frac{\nu_1}{|\nu_1|}\right)^{-n_1}\left(\frac{\overline{\nu_2}}{|\nu_2|}\right)^{-n_2}$$
$$\times \ddot{H}_{K,C^2/N(\lambda^\ell t q' q'')}\left(\frac{N(\nu_1)}{N(\lambda^{m_1-4} t_1 r_1' r_2'')^2/X}, \frac{N(\nu_2)}{N(\lambda^{m_2-4} t_2 r_2' r_2'')^2/X}, \boldsymbol{n}\right)$$
$$\times \mathscr{C}_4(\boldsymbol{a},\boldsymbol{\nu},\lambda^\ell t q' q'', \boldsymbol{\lambda^m t r' r''}), \tag{10-21}$$

$$\mathscr{C}_1(\cdots) = \sum_{\substack{\boldsymbol{J}\in(\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v}t\mathbb{Z}[\omega])^2\\ \boldsymbol{J}'\in(\mathbb{Z}[\omega]/q'\mathbb{Z}[\omega])^2\\ \boldsymbol{J}''\in(\mathbb{Z}[\omega]/q''\mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^\ell t}(\lambda^{e_v} v_0(J_2-J_1)+u(Y_2-Y_1))\hat{\psi}_{q'}(J_2'-J_1')\hat{\psi}_{q''}(J_2''-J_1'')$$
$$\times \psi^\star_{t_1 r_1' r_1''}(\,\cdot\,)_{\lambda^{2m_1+1}a_1 j_1}(\lambda^{2m_1+4}\nu_1)\overline{\psi^\star_{t_2 r_2' r_2''}(\,\cdot\,)_{\lambda^{2m_2+1}a_2 j_2}(\lambda^{2m_2+4}\nu_2)}, \tag{10-22}$$

$$\mathscr{C}_2(\cdots) := \sum_{\zeta_2}\left(\frac{\overline{\zeta_2^{-1}\lambda^{m_2-4}r_2}}{\zeta_2^{-1}\lambda^{m_2-4}r_2}\right)^{n_2}$$
$$\times \sum_{\substack{\boldsymbol{J}\in(\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v}t\mathbb{Z}[\omega])^2\\ \boldsymbol{J}'\in(\mathbb{Z}[\omega]/q'\mathbb{Z}[\omega])^2\\ \boldsymbol{J}''\in(\mathbb{Z}[\omega]/q''\mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^\ell t}(\lambda^{e_v} v_0(J_2-J_1)+u(Y_2-Y_1))\hat{\psi}_{q'}(J_2'-J_1')\hat{\psi}_{q''}(J_2''-J_1'')$$
$$\times \psi^\star_{t_1 r_1' r_1''}(\,\cdot\,)_{\lambda^{2m_1+1}a_1 j_1}(\lambda^{2m_1+4}\nu_1)\overline{\psi^\#_{\lambda^{m_2}t_2 r_2' r_2''}(\,\cdot\,)_{\lambda a_2 j_2,\zeta_2^{-1}}(\lambda^4\nu_2)}, \tag{10-23}$$

$$\mathscr{C}_3(\cdots) := \sum_{\zeta_1} \left(\frac{\overline{\zeta_1^{-1}\lambda^{m_1-4}r_1}}{\zeta_1^{-1}\lambda^{m_1-4}r_1} \right)^{-n_1} \times \sum_{\substack{\boldsymbol{J}\in(\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v}t\mathbb{Z}[\omega])^2 \\ \boldsymbol{J}'\in(\mathbb{Z}[\omega]/q'\mathbb{Z}[\omega])^2 \\ \boldsymbol{J}''\in(\mathbb{Z}[\omega]/q''\mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^{\ell}t}(\lambda^{e_v}v_0(J_2-J_1)+u(Y_2-Y_1))\hat{\psi}_{q'}(J_2'-J_1')\hat{\psi}_{q''}(J_2''-J_1'') \times \psi^{\#}_{\lambda^{m_1}t_1r_1'r_1''}(\,\cdot\,)_{\lambda a_1 j_1,\zeta_1^{-1}}(\lambda^4 \nu_1)\overline{\psi^{\star}_{t_2r_2'r_2''}(\,\cdot\,)_{\lambda^{2m_2+1}a_2 j_2}(\lambda^{2m_2+4}\nu_2)}, \tag{10-24}$$

$$\mathscr{C}_4(\cdots) := \sum_{\boldsymbol{\zeta}} \left(\frac{\overline{\zeta_1^{-1}\lambda^{m_1-4}r_1}}{\zeta_1^{-1}\lambda^{m_1-4}r_1} \right)^{-n_1} \left(\frac{\overline{\zeta_2^{-1}\lambda^{m_2-4}r_2}}{\zeta_2^{-1}\lambda^{m_2-4}r_2} \right)^{n_2} \times \sum_{\substack{\boldsymbol{J}\in(\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v}t\mathbb{Z}[\omega])^2 \\ \boldsymbol{J}'\in(\mathbb{Z}[\omega]/q'\mathbb{Z}[\omega])^2 \\ \boldsymbol{J}''\in(\mathbb{Z}[\omega]/q''\mathbb{Z}[\omega])^2}} \hat{\psi}_{\lambda^{\ell}t}(\lambda^{e_v}v_0(J_2-J_1)+u(Y_2-Y_1))\hat{\psi}_{q'}(J_2'-J_1')\hat{\psi}_{q''}(J_2''-J_1'') \times \psi^{\#}_{\lambda^{m_1}t_1r_1'r_1''}(\,\cdot\,)_{\lambda a_1 j_1,\zeta_1^{-1}}(\lambda^4 \nu_1)\overline{\psi^{\#}_{\lambda^{m_2}t_2r_2'r_2''}(\,\cdot\,)_{\lambda a_2 j_2,\zeta_2^{-1}}(\lambda^4\nu_2)}, \tag{10-25}$$

and $\boldsymbol{j}$ is given by (10-15) with subsequent change of variable (10-16).

Remark 10.4. Recalling (10-11) and recalling the averaging over $\boldsymbol{a}$ we have

$$\sum_{\boldsymbol{a}} \mu^2(a_1)\alpha_{a_1}\mu^2(a_2)\overline{\alpha_{a_2}}\mathscr{L}_f(\boldsymbol{a},\dots) = \sum_{p=1}^{4}\sum_{\boldsymbol{a}} \mu^2(a_1)\alpha_{a_1}\mu^2(a_2)\overline{\alpha_{a_2}}\mathscr{M}_{pf}(\boldsymbol{a},\dots). \tag{10-26}$$

The following arguments focus on the case $p=4$ on the right side of (10-26). The cases $p=1,2,3$ will follow mutatis mutandis, and will be omitted for the sake of brevity.

10.3. *Evaluation and bounds for arithmetic exponential sums.*

We first compute and bound $\mathscr{C}_4(\cdots)$ in (10-25).

A computation using Lemma 8.19, (8-63), (10-15), the Chinese remainder theorem (with pairwise coprime moduli $\zeta_i\lambda^{m_i-1}t_i$, r_i' and r_i'' for $i=1,2$), cubic reciprocity, and Lemma 3.4 yields

$$\mathscr{C}_4(\cdots) = \sum_{\boldsymbol{\zeta}} \left(\frac{\overline{\zeta_1\lambda^{m_1-4}r_1}}{\zeta_1\lambda^{m_1-4}r_1} \right)^{-n_1} \left(\frac{\overline{\zeta_2\lambda^{m_2-4}r_2}}{\zeta_2\lambda^{m_2-4}r_2} \right)^{n_2} \prod_{i=1}^{3} G_{4i}(\boldsymbol{a},\boldsymbol{\nu},\lambda^{\ell}tq'q'',\boldsymbol{\zeta\lambda^{m}tr'r''}), \tag{10-27}$$

where

$$G_{41}(\cdots) := \frac{1}{N(\lambda^{m_1+3}t_1)N(\lambda^{m_2+3}t_2)} \sum_{\boldsymbol{J}\in(\mathbb{Z}[\omega]/\lambda^{\ell_0-e_v}t\mathbb{Z}[\omega])^2} \hat{\psi}_{\lambda^{\ell}t}(\lambda^{e_v}v_0(J_2-J_1)+u(Y_2-Y_1)) \times K_{\Gamma_1(3),\sigma,\sigma}(\overline{r_1'r_1''}(\lambda^3\nu_1),\overline{r_1'r_1''}(a_1\lambda^{e_v+1}v_0J_1+\lambda a_1Y_1u),\zeta_1\lambda^{m_1-1}t_1) \times \overline{K_{\Gamma_1(3),\sigma,\sigma}(\overline{r_2'r_2''}(\lambda^3\nu_2),\overline{r_2'r_2''}(a_2\lambda^{e_v+1}v_0J_2+\lambda a_2Y_2u),\zeta_2\lambda^{m_2-1}t_2)}, \tag{10-28}$$

$$G_{42}(\cdots) := \frac{1}{N(r_1'r_2')} \sum_{\boldsymbol{J}' \in (\mathbb{Z}[\omega]/q'\mathbb{Z}[\omega])^2} \hat{\psi}_{q'}(J_2' - J_1') \\ \times K_{\Gamma_1(3),\sigma,\xi}(\overline{\zeta_1 \lambda^{m_1-1} t_1 r_1''}(\lambda^3 \nu_1), \overline{\zeta_1 \lambda^{m_1-1} t_1 r_1''}(\lambda a_1 J_1'), r_1') \\ \times \overline{K_{\Gamma_1(3),\sigma,\xi}(\overline{\zeta_2 \lambda^{m_2-1} t_2 r_2''}(\lambda^3 \nu_2), \overline{\zeta_2 \lambda^{m_2-1} t_2 r_2''}(\lambda a_2 J_2'), r_2')}, \tag{10-29}$$

$$G_{43}(\cdots) := \frac{1}{N(r_1''r_2'')} \sum_{\boldsymbol{J}'' \in (\mathbb{Z}[\omega]/q''\mathbb{Z}[\omega])^2} \hat{\psi}_{q''}(J_2'' - J_1'') \\ \times K_{\Gamma_1(3),\sigma,\xi}(\overline{\zeta_1 \lambda^{m_1-1} t_1 r_1'}(\lambda^3 \nu_1), \overline{\zeta_1 \lambda^{m_1-1} t_1 r_1'}(\lambda a_1 J_1''), r_1'') \\ \times \overline{K_{\Gamma_1(3),\sigma,\xi}(\overline{\zeta_2 \lambda^{m_2-1} t_2 r_2'}(\lambda^3 \nu_2), \overline{\zeta_2 \lambda^{m_2-1} t_2 r_2'}(\lambda a_2 J_2''), r_2'')}. \tag{10-30}$$

We now evaluate and bound each (10-28)–(10-30).

10.3.1. *Treatment of* (10-29). We open the normalised Ramanujan sums and the cubic Kloosterman sums in (10-29), use orthogonality in $\boldsymbol{J}'$, and then reassemble the result to obtain

$$G_{42}(\cdots) \\ = N(q') \biggl(\prod_{i=1}^{2} \delta_{(a_i q'/r_i', q')=1} \frac{1}{N(r_i')} \overline{\left(\frac{\zeta_i \lambda^{m_i-1} t_i r_i''}{r_1'} \right)_3} \left(\frac{\zeta_i \lambda^{m_i-1} t_i r_i''}{r_2'} \right)_3 \biggr) \\ \times \sum_{\substack{\boldsymbol{x} \in (\mathbb{Z}[\omega]/r_1'\mathbb{Z}[\omega]) \times (\mathbb{Z}[\omega]/r_2'\mathbb{Z}[\omega]) \\ \zeta_2 \lambda^{m_2-1} t_2 r_2''(a_1 q'/r_1') x_2 \equiv \\ \zeta_1 \lambda^{m_1-1} t_1 r_1''(a_2 q'/r_2') x_1 \pmod{q'}}} \overline{\left(\frac{x_1}{r_1'} \right)_3} \left(\frac{x_2}{r_2'} \right)_3 \check{e}\left(\frac{\zeta_2 \lambda^{m_2-1} t_2 r_2'' \lambda^3 \nu_1 x_1}{r_1'} - \frac{\zeta_1 \lambda^{m_1-1} t_1 r_1'' \lambda^3 \nu_2 x_2}{r_2'} \right). \tag{10-31}$$

The delta conditions in (10-31) are nonzero only if $q' \mid r_i'$ for $i = 1, 2$. We make the change of variables $\boldsymbol{r}' \to q'\boldsymbol{s}'$ where $s_i' \mid a_i$ for $i = 1, 2$. We detect the congruence with additive characters and reassemble to get

$$G_{42}(\cdots) = \frac{1}{N(q')} \biggl(\prod_{i=1}^{2} \delta_{(a_i/s_i', q')=1} \frac{1}{N(s_i')^{1/2}} \overline{\left(\frac{\zeta_i \lambda^{m_i-1} t_i r_i''}{q's_1'} \right)_3} \left(\frac{\zeta_i \lambda^{m_i-1} t_i r_i''}{q's_2'} \right)_3 \biggr) \\ \times \sum_{k \pmod{q'}} \overline{\tilde{g}(y_1 \lambda^3 \nu_1 + k z_1, q's_1')} \tilde{g}(y_2 \lambda^3 \nu_2 + k z_2, q's_2').$$

where

$$y_1 = \zeta_2 \lambda^{m_2-1} t_2 r_2'', \quad z_1 = \zeta_1 \lambda^{m_1-1} t_1 r_1''(a_2/s_2') s_1', \tag{10-32}$$

$$y_2 = \zeta_1 \lambda^{m_1-1} t_1 r_1'', \quad z_2 = \zeta_2 \lambda^{m_2-1} t_2 r_2''(a_1/s_1') s_2'. \tag{10-33}$$

We then factorise $q's_i' = q'(s_i', q') \cdot (s_i'/(s_i', q'))$. Since a_i is squarefree and $s_i' \mid a_i$ for $i = 1, 2$, the pair of moduli $q'(s_i', q')$ and $s_i'/(s_i', q')$ are coprime. Thus (2-8), Lemma 2.1, and (2-7) imply that

$$G_{42}(\cdots)=\frac{1}{N(q')}\Biggl(\prod_{i=1}^{2}\delta_{(a_i/s_i',q')=1}\cdot\delta_{(\lambda^3\nu_i,s_i'/(s_1',q'))=1}\frac{1}{N(s_i')^{1/2}}\overline{\Bigl(\frac{\zeta_i\lambda^{m_i-1}t_ir_i''}{q's_1'}\Bigr)_3}\Bigl(\frac{\zeta_i\lambda^{m_i-1}t_ir_i''}{q's_2'}\Bigr)_3\Biggr)$$
$$\times\overline{\tilde{g}(s_1'/(s_1',q'))}\tilde{g}(s_2'/(s_2',q'))\Bigl(\frac{q'(s_1',q')}{s_1'/(s_1',q')}\Bigr)_3\overline{\Bigl(\frac{q'(s_2',q')}{s_2'/(s_2',q')}\Bigr)_3}\Bigl(\frac{\zeta_2\lambda^{m_2-1}t_2r_2''\lambda^3\nu_1}{s_1'/(s_1',q')}\Bigr)_3$$
$$\times\overline{\Bigl(\frac{\zeta_2\lambda^{m_2-1}t_2r_2''\lambda^3\nu_2}{s_2'/(s_2',q')}\Bigr)_3}\sum_{k\ (\mathrm{mod}\ q')}\overline{\tilde{g}(y_1\lambda^3\nu_1+kz_1,q'(s_1',q'))}\tilde{g}(y_2\lambda^3\nu_2+kz_2,q'(s_2',q')).\quad(10\text{-}34)$$

Observe that Lemma 2.2 applied to the last two Gauss sums in the previous display imply that $G_{42}(\boldsymbol{\nu},\lambda^\ell tq'q'',\boldsymbol{\zeta\lambda^m t}q'\boldsymbol{s'r''})\neq 0$ only if $\lambda^3\nu_i\equiv 0\ (\mathrm{mod}\ (s_i',q'))$ for $i=1,2$. Thus

$$G_{42}(\cdots)=\frac{1}{N(q')}\Biggl(\prod_{i=1}^{2}\delta_{(a_i/s_i',q')=1}\cdot\delta_{(\lambda^3\nu_i,s_i'/(s_i',q'))=1}\cdot\delta_{\lambda^3\nu_i\equiv 0\ (\mathrm{mod}\ (s_i',q'))}$$
$$\times\frac{1}{N(s_i')^{1/2}}\overline{\Bigl(\frac{\zeta_i\lambda^{m_i-1}t_ir_i''}{q's_1'}\Bigr)_3}\Bigl(\frac{\zeta_i\lambda^{m_i-1}t_ir_i''}{q's_2'}\Bigr)_3\Biggr)\overline{\tilde{g}(s_1'/(s_1',q'))}\tilde{g}(s_2'/(s_2',q'))$$
$$\times\Bigl(\frac{q'(s_1',q')}{s_1'/(s_1',q')}\Bigr)_3\overline{\Bigl(\frac{q'(s_2',q')}{s_2'/(s_2',q')}\Bigr)_3}\Bigl(\frac{\zeta_2\lambda^{m_2-1}t_2r_2''\lambda^3\nu_1}{s_1'/(s_1',q')}\Bigr)_3\overline{\Bigl(\frac{\zeta_2\lambda^{m_2-1}t_2r_2''\lambda^3\nu_2}{s_2'/(s_2',q')}\Bigr)_3}$$
$$\times\sum_{k\ (\mathrm{mod}\ q')}\overline{\tilde{g}(y_1\lambda^3\nu_1+kz_1,q'(s_1',q'))}\tilde{g}(y_2\lambda^3\nu_2+kz_2,q'(s_2',q')).\quad(10\text{-}35)$$

Using Lemma 2.3 (noting the normalisation in (2-12)) gives

$$\sum_{k\ (\mathrm{mod}\ q')}|\tilde{g}(y_1\lambda^3\nu_1+kz_1,q'(s_1',q'))|\cdot|\tilde{g}(y_2\lambda^3\nu_2+kz_2,q'(s_2',q'))|$$
$$\leq\Bigl(\prod_{i=1}^{2}\delta_{\lambda^3\nu_i\equiv 0\ (\mathrm{mod}\ (s_i',q'))}\cdot N((s_i',q'))^{1/2}\Bigr)$$
$$\times\sum_{k\ (\mathrm{mod}\ q')}N\Bigl(\Bigl(y_1\frac{\lambda^3\nu_1}{(s_1',q')}+k\frac{z_1}{(s_1',q')},q'\Bigr)\Bigr)^{1/2}N\Bigl(\Bigl(y_2\frac{\lambda^3\nu_2}{(s_2',q')}+k\frac{z_2}{(s_2',q')},q'\Bigr)\Bigr)^{1/2}$$
$$\ll N(q')^{1+\varepsilon}\Bigl(\prod_{i=1}^{2}\delta_{\lambda^3\nu_i\equiv 0\ (\mathrm{mod}\ (s_i',q'))}\cdot N((s_i',q'))^{1/2}\Bigr),\quad(10\text{-}36)$$

where the last display follows from using Cauchy–Schwarz in k and then a change of variable to $k\ (\mathrm{mod}\ q')$ in each resulting braket (the change of variable is valid since $(z_1/(s_1',q'),q')=(z_2/(s_2',q'),q')=1$). We use the triangle inequality in (10-34), substitute (10-36), and then change variables back $\boldsymbol{s}'\to(1/q')\boldsymbol{r}'$ to obtain

$$|G_{42}(\cdots)|$$
$$\ll N(q')^{1+\varepsilon}\Biggl(\prod_{i=1}^{2}\delta_{(a_iq'/r_i',q')=1}\cdot\delta_{(\lambda^3\nu_i,(r_i'/q')/(r_i'/q',q'))=1}\delta_{\lambda^3\nu_i\equiv 0\ (\mathrm{mod}\ (r_i'/q',q'))}\frac{N((r_i'/q',q'))^{1/2}}{N(r_i')^{1/2}}\Biggr).\quad(10\text{-}37)$$

10.3.2. *Treatment of* (10-30). We open the normalised Ramanujan sums and the cubic Kloosterman sums in (10-30), use orthogonality in the J_1'', J_2'' variables, and by a similar argument to the above we reassemble the result to obtain

$$G_{43}(\cdots) := \left(\prod_{i=1}^{2}\delta_{r_i''=q''}\right)\cdot\overline{\left(\frac{a_1}{q''}\right)_3}\left(\frac{\zeta_1\lambda^{m_1-1}t_1r_1'}{q''}\right)_3\left(\frac{a_2}{q''}\right)_3\overline{\left(\frac{\zeta_2\lambda^{m_2-1}t_2r_2'}{q''}\right)_3}\hat{\psi}_{q''}(P_1\lambda^3\nu_1-P_2\lambda^3\nu_2), \tag{10-38}$$

where

$$P_1 := (\zeta_2\lambda^{m_2-1}t_2r_2')^2a_1 \quad\text{and}\quad P_2 := (\zeta_1\lambda^{m_1-1}t_1r_1')^2a_2. \tag{10-39}$$

We have the bound

$$|G_{43}(\cdots)| \le \left(\prod_{i=1}^{2}\delta_{r_i''=q''}\right)\cdot|\hat{\psi}_{q''}(P_1\lambda^3\nu_1-P_2\lambda^3\nu_2)|. \tag{10-40}$$

10.3.3. *Treatment of* (10-28). Recall that $\ell_0 := \max\{\ell, e_v\}$. We open the normalised Ramanujan sums and the cubic Kloosterman sums in (10-28), use orthogonality in the J_1, J_2 variables, and then reassemble the result to obtain

$$G_{41}(\cdots) = \frac{N(\lambda^{\ell_0-e_v}t)^2}{N(\lambda^\ell t)}\left(\prod_{i=1}^{2}\frac{1}{N(\lambda^{m_i+3}t_i)}\right)\sum_{\substack{k \pmod{\lambda^\ell t}\\(k,\lambda^\ell t)=1}}\check{e}\left(\frac{ku(Y_2-Y_1)}{\lambda^\ell t}\right)\sum_{\boldsymbol{x}\in\mathscr{B}_1(k)\times\mathscr{B}_2(k)}\left(\frac{\zeta_1\lambda^{m_1-1}t_1}{x_1}\right)_3\overline{\left(\frac{\zeta_2\lambda^{m_2-1}t_2}{x_2}\right)_3}$$
$$\times\,\check{e}\left(\frac{\overline{r_1'r_1''}(\lambda^3\nu_1\overline{x_1}+\lambda a_1Y_1ux_1)}{\zeta_1\lambda^{m_1-1}t_1}-\frac{\overline{r_2'r_2''}(\lambda^3\nu_2\overline{x_2}+\lambda a_2Y_2ux_2)}{\zeta_2\lambda^{m_2-1}t_2}\right), \tag{10-41}$$

where for $i=1,2$ we have

$$\mathscr{B}_i(k) := \Big\{x_i \pmod{\lambda^{m_i+1}t_i} : (x_i,\lambda t_i)=1,\ x_i\equiv 1 \pmod 3,$$
$$\overline{r_i'r_i''\zeta_i}a_i\lambda^{\ell_0-m_i+2}(v_0t/t_i)x_i \equiv k\lambda^{\ell_0-\ell}v_0 \pmod{\lambda^{\ell_0-e_v}t}\Big\}. \tag{10-42}$$

For a given $k\in\mathbb{Z}[\omega]$ with $(k,\lambda^\ell t)=1$, any solution $y_i \pmod{\lambda^{\ell_0-e_v}t}$ to the congruence

$$\overline{r_i'r_i''\zeta_i}a_i\lambda^{\ell_0-m_i+2}(v_0t/t_i)y_i \equiv k\lambda^{\ell_0-\ell}v_0 \pmod{\lambda^{\ell_0-e_v}t} \tag{10-43}$$

corresponds to $N(\lambda^{\max\{0,m_1+1-\ell_0+e_v\}})N(t_i/(t,t_i))$ distinct solutions $x_i \pmod{\lambda^{m_i+1}t_i}$. The congruence in (10-43) has a solution $y_i \pmod{\lambda^{\ell_0-e_v}t}$ if and only if

$$(\overline{r_i'r_i''\zeta_i}a_i\lambda^{\ell_0-m_i+2}(v_0t/t_i),\lambda^{\ell_0-e_v}t) \mid k\lambda^{\ell_0-\ell}v_0. \tag{10-44}$$

Since $t\mid \mathrm{rad}(v_0)^\infty$, $t_i\mid v_0t$, $(\overline{r_i'r_i''\zeta_i}a_i,\lambda v_0)=(\lambda,v_0)=1$, we have

$$(\overline{r_i'r_i''\zeta_i}a_i\lambda^{\ell_0-m_i+2}(v_0t/t_i),\lambda^{\ell_0-e_v}t)=\lambda^{\min\{\ell_0-m_i+2,\ell_0-e_v\}}((v_0t/t_i),t)=\lambda^{\min\{\ell_0-m_i+2,\ell_0-e_v\}}t(v_0,t_i)/t_i \tag{10-45}$$

for $i=1,2$. Observe that (10-45) and the fact $(k,\lambda^\ell t)=1$ (recall that $\ell\ge 1$) imply that (10-44) is equivalent to the two conditions

$$t\mid[v_0,t_i]\quad\text{and}\quad \min\{\ell_0-m_i+2,\ell_0-e_v\}\le \ell_0-\ell \tag{10-46}$$

for $i=1,2$. Under the restriction $0\le m_i\le \ell_0+1$, the conditions in (10-46) are equivalent to

$$t\mid[v_0,t_i]\quad\text{and}\quad 1\le\ell\le e_v=\ell_0 \tag{10-47}$$

for $i=1,2$. Thus (10-41) becomes

$$\begin{aligned}G_{41}(\cdots)=\delta_{1\le\ell\le e_v}\cdot\frac{N(t)^2}{N(\lambda^\ell t)}\cdot\Bigl(\prod_{i=1}^{2}\delta_{t\mid[v_0,t_i]}\cdot\frac{1}{N(\lambda^{m_i+3}t_i)}\Bigr)\\ \times\sum_{\substack{k\ (\mathrm{mod}\ \lambda^\ell t)\\(k,\lambda^\ell t)=1}}\check{e}\Bigl(\frac{ku(Y_2-Y_1)}{\lambda^\ell t}\Bigr)\sum_{\boldsymbol{x}\in\mathscr{B}_1(k)\times\mathscr{B}_2(k)}\Bigl(\frac{\zeta_1\lambda^{m_1-1}t_1}{x_1}\Bigr)_3\overline{\Bigl(\frac{\zeta_2\lambda^{m_2-1}t_2}{x_2}\Bigr)_3}\\ \times\check{e}\Bigl(\frac{\overline{r_1'r_1''}(\lambda^3v_1\overline{x_1}+\lambda a_1Y_1ux_1)}{\zeta_1\lambda^{m_1-1}t_1}-\frac{\overline{r_2'r_2''}(\lambda^3v_2\overline{x_2}+\lambda a_2Y_2ux_2)}{\zeta_2\lambda^{m_2-1}t_2}\Bigr).\end{aligned}\tag{10-48}$$

Furthermore, under the conditions in (10-47) and $0\le m_i\le e_v+1$ for $i=1,2$, (10-45) and the sentence containing (10-43) imply that for each $k\in\mathbb{Z}[\omega]$ with $(k,\lambda^\ell t)=1$ we have

$$|\mathscr{B}_1(k)\times\mathscr{B}_2(k)|\le\Bigl(\prod_{i=1}^{2}\delta_{t\mid[v_0,t_i]}N(\lambda^{m_i+1})N\Bigl(\frac{t(v_0,t_i)}{(t,t_i)}\Bigr)\Bigr). \tag{10-49}$$

Using (10-49), we bound (10-48) trivially by

$$|G_{41}(\cdots)|\le\delta_{1\le\ell\le e_v}\cdot\Bigl(\prod_{i=1}^{2}\delta_{t\mid[v_0,t_i]}\cdot N\Bigl(\frac{t^2(v_0,t_i)}{t_i(t,t_i)}\Bigr)\Bigr)\le\delta_{1\le\ell\le e_v}\cdot N(v_0)^6\Bigl(\prod_{i=1}^{2}\delta_{t\mid[v_0,t_i]}\Bigr). \tag{10-50}$$

10.4. ***Further technical manipulations and insertion of smooth dyadic partitions of unity.*** We substitute (10-21) into (10-17) to obtain

$$\begin{aligned}\mathscr{M}_{4f}(\boldsymbol{a},\dots)=\frac{N(\lambda^8)k_C}{C^4}\frac{X^2}{N(a_1a_2)N(v_0)^2}\sum_{1\le\ell\ll\log C}\ \sum_{\substack{tq'q''\in\mathbb{Z}[\omega]\\(10\text{-}13)}}\frac{N(\lambda^\ell)}{N(\lambda^{\ell_0+1})^2}\frac{1}{N(tq'q'')}\\ \times\sum_{\substack{\boldsymbol{k}\in(\mathbb{Z}[\omega]/\lambda^{14}\mathbb{Z}[\omega])^2\\ \forall i:k_i\equiv 1\ (\mathrm{mod}\ 3)}}\ \sum_{\substack{\boldsymbol{m},\boldsymbol{t},\boldsymbol{r}',\boldsymbol{r}''\\ \forall i:6\le m_i\le\ell_0+1\\(10\text{-}14)}}\ \sum_{\boldsymbol{n}\in\mathbb{Z}^2}(-1)^{n_1+n_2}\sum_{\boldsymbol{v}\in(\lambda^{-3}\mathbb{Z}[\omega])^2}\rho_f(v_1)\overline{\rho_f(v_2)}\Bigl(\frac{v_1}{|v_1|}\Bigr)^{-n_1}\Bigl(\frac{\overline{v_2}}{|v_2|}\Bigr)^{-n_2}\\ \times\ddot{H}_{K,C^2/N(\lambda^\ell tq'q'')}\Bigl(\frac{N(v_1)}{N(\lambda^{m_1-4}t_1r_1'r_2'')^2/X},\frac{N(v_2)}{N(\lambda^{m_2-4}t_2r_2'r_2'')^2/X},\boldsymbol{n}\Bigr)\\ \times\mathscr{C}_4(\boldsymbol{a},\boldsymbol{v},\lambda^\ell tq'q'',\boldsymbol{\lambda}^{\boldsymbol{m}}\boldsymbol{t}\boldsymbol{r}'\boldsymbol{r}''),\end{aligned}\tag{10-51}$$

where $\mathscr{C}_4(\cdots)$ is given by (10-25) (and (10-27)). Note that the summands $\mathscr{M}_{4f}(\cdots)$ do not depend on the congruence classes $k_i\ (\mathrm{mod}\ \lambda^{14})$ (unlike the other $\mathscr{M}_p(\cdots)$ for $p=1,2,3$). Thus the sum over $\boldsymbol{k}$

in (10-51), and the last condition in (10-14) can be dropped. Equality (10-27) and the delta conditions in (10-31) (resp. (10-38)) imply that we can make the change of variable $\boldsymbol{r}' \to q'\boldsymbol{s}'$ where $s_i' \mid a_i$ and $(a_i/s_i', q') = 1$ (resp. $\boldsymbol{r}'' \to \boldsymbol{q}''$ where $\boldsymbol{q}'' = (q'', q'')$) in (10-51). The delta conditions in (10-48) tells us that $1 \leq \ell \leq e_v$ and $t \mid [v_0, t_i]$. Thus the multiple summation $\sum_{\boldsymbol{m},\boldsymbol{t},\boldsymbol{r},\boldsymbol{r}''}$ in (10-51) subject to $6 \leq m_i \leq \ell_0 + 1$ for $i = 1, 2$ and (10-14), can be written as $\sum_{\boldsymbol{m},\boldsymbol{t},\boldsymbol{s}'}$ subject to $6 \leq m_i \leq e_v + 1$ for $i = 1, 2$, and

$$t_i \mid v_0 t, \quad t \mid [v_0, t_i], \quad s_i' \mid a_i, \quad (a_i/s_i', q') = 1 \quad \text{for } i = 1, 2. \tag{10-52}$$

We further note that the delta conditions in (10-35) imply that $\boldsymbol{\nu}$ sum in (10-51) is supported on the conditions

$$\lambda^3 \nu_i \equiv 0 \pmod{(s_i', q')} \quad \text{and} \quad \left(\lambda^3 \nu_i, \frac{s_i'}{(s_i', q')}\right) = 1 \quad \text{for } i = 1, 2. \tag{10-53}$$

We then insert a smooth partition of unity in the variables t, q', and q'' in (10-51). Thus

$$\mathscr{M}_{4f}(\boldsymbol{a}, \dots) = \sum_{\substack{1 \leq \ell \leq e_v \\ 1/2 \leq T, Q', Q'' \text{dyadic} \\ N(\lambda^\ell) T Q' Q'' \ll C^2}} \mathscr{M}_{4f}(\boldsymbol{a}, \dots, N(\lambda^\ell) T Q' Q''), \tag{10-54}$$

where

$$\begin{aligned} &\mathscr{M}_{4f}(\boldsymbol{a}, \dots, N(\lambda^\ell) T Q' Q'') \\ &\quad := \frac{N(\lambda^8) k_C}{C^4} \frac{X^2 N(\lambda^\ell)}{N(a_1 a_2) N(\lambda^{e_v+1} v_0)^2} \sum_{\substack{t q' q'' \in \mathbb{Z}[\omega] \\ \text{(10-13)}}} \frac{1}{N(t q' q'')} U\!\left(\frac{N(t)}{T}\right) U\!\left(\frac{N(q')}{Q'}\right) U\!\left(\frac{N(q'')}{Q''}\right) \\ &\qquad \times \sum_{\substack{\boldsymbol{m}, \boldsymbol{t}, \boldsymbol{s}' \\ 6 \leq m_i \leq e_v + 1 \\ \text{(10-52)}}} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} (-1)^{n_1 + n_2} \sum_{\substack{\boldsymbol{\nu} \in (\lambda^{-3} \mathbb{Z}[\omega])^2 \\ \text{(10-53)}}} \rho_f(\nu_1) \overline{\rho_f(\nu_2)} \left(\frac{\nu_1}{|\nu_1|}\right)^{-n_1} \left(\frac{\overline{\nu_2}}{|\nu_2|}\right)^{-n_2} \\ &\qquad \times \ddot{H}_{K, C^2/N(\lambda^\ell t q' q'')}\!\left(\frac{N(\nu_1)}{N(\lambda^{m_1 - 4} t_1 s_1' q' q'')^2 / X}, \frac{N(\nu_2)}{N(\lambda^{m_2 - 4} t_2 s_2' q' q'')^2 / X}, \boldsymbol{n}\right) \\ &\qquad \times \mathscr{C}_i(\boldsymbol{a}, \boldsymbol{\nu}, \lambda^\ell t q' q'', \boldsymbol{\lambda}^{\boldsymbol{m}} t q' \boldsymbol{s}' \boldsymbol{q}''). \end{aligned} \tag{10-55}$$

The restriction

$$N(\lambda^\ell) T Q' Q'' \ll C^2 \tag{10-56}$$

in (10-54) follows from (10-8).

Using (10-9), (8-71) (with $M \to C^2/(N(\lambda^\ell) T Q' Q'')$, $D_{i1} = D_{i2} > 0$ large and fixed, and $D_{(i+1)1} = D_{(i+1)2} = \varepsilon$ small and fixed), Lemma 2.8, (10-37), (10-40), and (10-50), we truncate the ν_i-sum in (10-55) by

$$N(\nu_i) \ll (X K N(v))^\varepsilon K^8 \cdot (N(\lambda^{m_i} t_i s_i') Q' Q'')^2 X^{-1} =: \Xi_i, \tag{10-57}$$

with negligible error $O((XKN(v))^{-2000})$. Without loss of generality we can restrict our attention to the case $\Xi_i \gg (XKN(v))^{-\varepsilon}$, otherwise $\mathscr{M}_{4f}(\boldsymbol{a}, \dots, N(\lambda^\ell) T Q' Q'')$ is a negligible $O((XKN(v))^{-2000})$. Observe that (8-71) with $D_{11} = D_{12} = D_{21} = D_{22} = \varepsilon > 0$ small and fixed, (10-3), (10-56), and (10-57) imply that

$$\ddot{H}_{K,C^2/N(\lambda^\ell tq'q'')}\left(\frac{N(\nu_1)}{N(\lambda^{m_1-4}t_1s_1'q'q'')^2/X},\frac{N(\nu_2)}{N(\lambda^{m_2-4}t_2s_2'q'q'')^2/X},\boldsymbol{n}\right) \ll (XKN(v))^{\varepsilon}\cdot\frac{C^2}{N(\lambda^\ell tq'q'')}\cdot\prod_{i=1}^{2}(|n_i|+1)^{-2+\varepsilon}. \quad (10\text{-}58)$$

We apply the triangle inequality in (10-55), and then use (10-57), (10-27), (10-37), (10-40), (10-50), and (10-58) to obtain

$$\begin{aligned}\mathscr{M}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q'') &\ll (XKN(v))^{\varepsilon}\cdot\left(\frac{XN(v_0)^2}{CATQ'Q''N(\lambda^{e_v+1})}\right)^2 \\ &\times \sum_{\substack{tq'\in\mathbb{Z}[\omega]\\ N(t)\sim T,N(q')\sim Q'\\ t\mid\mathrm{rad}(v_0)^\infty\\ q'\mid\mathrm{rad}(a_1a_2)^\infty}}\ \sum_{\substack{\boldsymbol{\zeta},\boldsymbol{m},\boldsymbol{t},\boldsymbol{s}'\\ 6\le m_i\le e_v+1\\ (10\text{-}52)}}\ \prod_{i=1}^{2}\frac{N((s_i',q'))^{1/2}}{N(s_i')^{1/2}}\sum_{\substack{\boldsymbol{\nu}\in(\lambda^{-3}\mathbb{Z}[\omega])^2\\ \forall i:N(\nu_i)\ll\Xi_i\\ (10\text{-}53)}}|\rho_f(\nu_1)||\rho_f(\nu_2)| \\ &\times \sum_{\substack{q''\in\mathbb{Z}[\omega]\\ q''\equiv1\ (\mathrm{mod}\ 3)\\ N(q'')\sim Q''\\ (q'',a_1a_2v_0)=1}}|\hat{\psi}_{q''}(P_1\lambda^3\nu_1-P_2\lambda^3\nu_2)|+O((XKN(v))^{-2000}), \end{aligned}\quad (10\text{-}59)$$

where

$$P_1 := (\zeta_2\lambda^{m_2-1}t_2q's_2')^2a_1 \quad\text{and}\quad P_2 := (\zeta_1\lambda^{m_1-1}t_1q's_1')^2a_2. \quad (10\text{-}60)$$

We drop the condition $(q'',a_1a_2v_0)=1$ in (10-59) by positivity, and use Lemma 8.5 to obtain

$$\sum_{\substack{q''\in\mathbb{Z}[\omega]\\ q''\equiv1\ (\mathrm{mod}\ 3)\\ N(q'')\sim Q''}}|\hat{\psi}_{q''}(P_1\lambda^3\nu_1-P_2\lambda^3\nu_2)| \ll \delta_{P_1\lambda^3\nu_1=P_2\lambda^3\nu_2}\cdot Q''+\delta_{P_1\lambda^3\nu_1\neq P_2\lambda^3\nu_2}\cdot(XKN(v))^{\varepsilon}. \quad (10\text{-}61)$$

We substitute the bound (10-61) into (10-59), and obtain

$$\mathscr{M}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q'')\ll\mathscr{N}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q'')+\mathscr{E}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q''), \quad (10\text{-}62)$$

where the terms on the right correspond to the diagonal and off-diagonal respectively. Using (10-54) and (10-62) it suffices to estimate

$$\sum_{\substack{1\le\ell\le e_v\\ 1/2\le T,Q',Q''\,\text{dyadic}\\ N(\lambda^\ell)TQ'Q''\ll C^2}}\ \sum_{\boldsymbol{a}}\mu^2(a_1)|\alpha_{a_1}|\mu^2(a_2)|\alpha_{a_2}|\mathscr{N}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q''), \quad (10\text{-}63)$$

$$\sum_{\substack{1\le\ell\le e_v\\ 1/2\le T,Q',Q''\,\text{dyadic}\\ N(\lambda^\ell)TQ'Q''\ll C^2}}\ \sum_{\boldsymbol{a}}\mu^2(a_1)|\alpha_{a_1}|\mu^2(a_2)|\alpha_{a_2}|\mathscr{E}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q''), \quad (10\text{-}64)$$

with C given by (10-3).

10.5. ***Off-diagonal*: (10-64).** We drop the condition $P_1\lambda^3\nu_1 \neq P_2\lambda^3\nu_2$ and $(\lambda^3\nu_i, s_i'/(s_i', q')) = 1$ for $i = 1, 2$ (see (10-53)) by positivity, and then use the Cauchy–Schwarz inequality, $\rho_f(0) = 0$, and Lemma 2.7 to obtain

$$\sum_{\substack{\nu_i \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_i) \ll \Xi_i \\ \lambda^3\nu_i \equiv 0 \ (\mathrm{mod}\ (s_i', q'))}} |\rho_f(\nu_i)| \leq \Bigg(\sum_{\substack{0 \neq \nu_i \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_i) \ll \Xi_i \\ \lambda^3\nu_i \equiv 0 \ (\mathrm{mod}\ (s_i', q'))}} 1 \Bigg)^{1/2} \Bigg(\sum_{\substack{\nu_i \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_i) \ll \Xi_i}} |\rho_f(\nu_i)|^2 \Bigg)^{1/2} \ll \frac{\Xi_i^{1+\varepsilon}}{N((s_i', q'))^{1/2}} \tag{10-65}$$

for $i = 1, 2$.

We use (10-65), (10-52), and Lemma 2.6 to conclude that

$$\mathscr{E}_{4f}(\boldsymbol{a}, \dots, N(\lambda^{\ell})TQ'Q'') \ll (XKN(v))^{\varepsilon} K^{16} N(v_0)^8 N(\lambda^{e_v})^2 AC^{-2}(TQ'Q'')^2. \tag{10-66}$$

Substituting (10-66) into (10-64) and using Cauchy–Schwarz we see that (10-64) is

$$\begin{aligned} &\ll (XKN(v))^{\varepsilon} K^{16} N(v)^8 A^2 B^{1/2} \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_2^2 \\ &\ll (XKN(v))^{\varepsilon} K^{16} N(v)^8 A^3 B^{1/2} \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_\infty^2. \end{aligned} \tag{10-67}$$

10.6. ***Diagonal*: (10-63).** Consulting (10-53) we make the change of variable

$$\lambda^3\nu_i = (s_i', q')\lambda^3\mu_i \quad \text{such that} \quad 0 \neq \mu_i \in \lambda^{-3}\mathbb{Z}[\omega] \quad \text{and} \quad \left((s_i', q')\lambda^3\mu_i, \frac{s_i'}{(s_i', q')} \right) = 1 \tag{10-68}$$

for $i = 1, 2$. Since a_i is squarefree and $s_i' \mid a_i$, the coprimality condition in (10-68) is equivalent to

$$\left(\lambda^3\mu_i, \frac{s_i'}{(s_i', q')} \right) = 1 \tag{10-69}$$

for $i = 1, 2$. The diagonal equation $P_1\lambda^3\nu_1 = P_2\lambda^3\nu_2$ with P_1 and P_2 given in (10-60) is equivalent to

$$(\zeta_2\lambda^{m_2-1}t_2)^2 \frac{s_2'}{(s_2', q')} \frac{a_1}{s_1'} \lambda^3\mu_1 = (\zeta_1\lambda^{m_1-1}t_1)^2 \frac{s_1'}{(s', q')} \frac{a_2}{s_2'} \lambda^3\mu_2, \tag{10-70}$$

where $0 \neq \lambda^3\mu_i$ satisfies (10-69) for $i = 1, 2$. The hypothesis that the a_i are squarefree for $i = 1, 2$ guarantees that

$$\left(\frac{s_1'}{(s_1', q')}, \lambda^{m_2-1}t_2 \frac{a_1}{s_1'} \right) = \left(\frac{s_2'}{(s_2', q')}, \lambda^{m_1-1}t_1 \frac{a_2}{s_2'} \right) = 1. \tag{10-71}$$

Using (10-69) and (10-71) we conclude from (10-70) that

$$\tilde{s} := \frac{s_1'}{(s_1', q')} = \frac{s_2'}{(s_2', q')} \mid (a_1, a_2), \tag{10-72}$$

and thus (10-70) is equivalent to

$$(\zeta_2\lambda^{m_2-1}t_2)^2 \frac{a_1}{s_1'} \lambda^3\mu_1 = (\zeta_1\lambda^{m_1-1}t_1)^2 \frac{a_2}{s_2'} \lambda^3\mu_2, \tag{10-73}$$

where $0 \neq \lambda^3\mu_i$ satisfies (10-69) for $i = 1, 2$.

We use (10-68)–(10-73) to rewrite (10-63), set $g_i := (s_i, q')$ and release using Möbius inversion, and then interchange the order of summation. We obtain

$$\sum_{\boldsymbol{a}} \mu^2(a_1)|\alpha_{a_1}|\mu^2(a_2)|\alpha_{a_2}|\mathcal{N}_{4f}(\boldsymbol{a},\dots,N(\lambda^\ell)TQ'Q'')$$
$$= (XKN(v))^\varepsilon \cdot \left(\frac{XN(v_0)^2}{CATQ'N(\lambda^{e_v+1})}\right)^2 \frac{1}{Q''}$$
$$\times \sum_{\substack{\boldsymbol{\zeta},\boldsymbol{m},\boldsymbol{t} \\ \forall i: 6\le m_i \le e_v+1 \\ \forall i: t_i | \mathrm{rad}(v_0)^\infty}} \sum_{\substack{\boldsymbol{h},\boldsymbol{d},\boldsymbol{g},\boldsymbol{r} \\ \forall i: h_i,d_i,g_i,r_i \equiv 1 \pmod 3 \\ h_1 d_1 = h_2 d_2}} \frac{\mu(h_1)\mu(h_2)}{N(h_1d_1h_2d_2)^{1/2}} \mu^2(h_1d_1g_1r_1)|\alpha_{h_1d_1g_1r_1}|\mu^2(h_2d_2g_2r_2)|\alpha_{h_2d_2g_2r_2}|$$
$$\times \sum_{\substack{\boldsymbol{\nu}\in(\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(\nu_i)\ll \Xi_i' \\ (10\text{-}76) \\ (10\text{-}77)}} |\rho_f(\nu_1)|\cdot|\rho_f(\nu_2)| \sum_{\substack{tq'\in\mathbb{Z}[\omega] \\ N(t)\sim T, N(q')\sim Q' \\ [t_1,t_2] | v_0 t \\ t | ([v_0,t_1],[v_0,t_2]) \\ [h_1g_1,h_2g_2] | q' | \mathrm{rad}(h_1d_1g_1h_2d_2g_2)^\infty}} 1, \tag{10-74}$$

where (see (10-57))

$$\Xi_i' := (XKN(v))^\varepsilon K^8 \cdot (N(\lambda^{m_i} t_i h_i d_i g_i) Q'Q'')^2 X^{-1} \quad \text{for } i = 1, 2, \tag{10-75}$$
$$(\lambda^3 \nu_i, h_i d_i) = 1 \quad \text{and} \quad \lambda^3\nu_i \equiv 0 \pmod{g_i} \qquad \text{for } i = 1, 2, \tag{10-76}$$
$$(\zeta_2 \lambda^{m_2-1} t_2)^2 r_1 \frac{\lambda^3 \nu_1}{g_1} = (\zeta_1 \lambda^{m_1-1} t_1)^2 r_2 \frac{\lambda^3\nu_2}{g_2}. \tag{10-77}$$

We dyadically partition all of the auxiliary variables, i.e.,

$$N(h_i) \sim H_i, \quad N(d_i) \sim D_i, \quad N(g_i) \sim G_i, \quad N(r_i) \sim R_i, \quad N(t_i) \sim T_i,$$

such that

$$H_i D_i G_i R_i \asymp A, \quad H_i G_i \ll Q', \quad \text{and} \quad T_i \ll N(v_0) T \quad \text{for } i = 1, 2. \tag{10-78}$$

We estimate the sum over t and q' in (10-74) by $(XKN(v))^\varepsilon$ using (2-14) and Lemma 2.6 respectively. We then apply the bound $|\mu^2(a)\alpha_a| \le \|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_\infty$. We see that the entirety of (10-74) is

$$\ll (XKN(v))^\varepsilon \|\boldsymbol{\mu}^2\boldsymbol{\alpha}\|_\infty^2 \cdot \left(\frac{XN(v_0)^2}{CATQ'N(\lambda^{e_v+1})}\right)^2 \frac{1}{Q''}$$
$$\times \sum_{\substack{\boldsymbol{\zeta},\boldsymbol{m} \\ \forall i: 6\le m_i\le e_v+1}} \sum_{\substack{\forall i: H_i,D_i,G_i,R_i,T_i \\ \text{dyadic} \\ (10\text{-}78)}} \frac{1}{(H_1D_1H_2D_2)^{1/2}}$$
$$\times \sum_{\substack{\boldsymbol{t} \\ \forall i: N(t_i)\sim T_i \\ t_i | \mathrm{rad}(v_0)^\infty}} \sum_{\substack{\boldsymbol{r} \\ \forall i: N(r_i)\sim R_i \\ r_i \equiv 1 \pmod 3}} \sum_{\substack{\boldsymbol{h},\boldsymbol{d} \\ h_1d_1=h_2d_2 \\ \forall i: N(h_i)\sim H_i \\ \forall i: N(d_i)\sim D_i \\ \forall i: h_i,d_i\equiv 1 \pmod 3}} \sum_{\substack{\boldsymbol{g} \\ \forall i: N(g_i)\sim G_i \\ \forall i: g_i\equiv 1 \pmod 3}} \sum_{\substack{\boldsymbol{\nu}\in(\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(\nu_i)\ll \Xi_i'' \\ (10\text{-}76) \\ (10\text{-}77)}} |\rho_f(\nu_1)|\cdot|\rho_f(\nu_2)|, \tag{10-79}$$

where (see (10-75))

$$\Xi_i'' := (XKN(v))^{\varepsilon} K^8 \cdot (N(\lambda^{m_i}) T_i H_i D_i G_i Q' Q'')^2 X^{-1} \tag{10-80}$$

for $i = 1, 2$.

We apply Cauchy–Schwarz to the sum over $\boldsymbol{g}$ and $\boldsymbol{\nu}$ in (10-79), and then rearrange the order of summation to obtain

$$\sum_{\substack{\boldsymbol{g} \\ \forall i: N(g_i) \sim G_i \\ \forall i: g_i \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{\boldsymbol{\nu} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(\nu_i) \ll \Xi_i'' \\ (10\text{-}76) \\ (10\text{-}77)}} |\rho_f(\nu_1)| \cdot |\rho_f(\nu_2)|$$

$$\leq \Bigg(\sum_{\substack{\nu_1 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_1) \ll \Xi_1'' \\ (\lambda^3 \nu_1, h_1 d_1) = 1}} |\rho_f(\nu_1)|^2 \sum_{\substack{N(g_1) \sim G_1 \\ g_1 \mid \lambda_3 \nu_1 \\ g_1 \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{N(g_2) \sim G_2 \\ g_2 \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{\nu_2 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_2) \ll \Xi_2'' \\ (\lambda^3 \nu_2, h_2 d_2) = 1 \\ \lambda^3 \nu_2 \equiv 0 \ (\mathrm{mod}\ g_2) \\ (10\text{-}77)}} 1 \Bigg)^{1/2} \tag{10-81}$$

$$\times \Bigg(\sum_{\substack{\nu_2 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_2) \ll \Xi_2'' \\ (\lambda^3 \nu_2, h_2 d_2) = 1}} |\rho_f(\nu_2)|^2 \sum_{\substack{N(g_2) \sim G_2 \\ g_2 \mid \lambda_3 \nu_2 \\ g_2 \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{N(g_1) \sim G_1 \\ g_1 \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{\nu_1 \in \lambda^{-3}\mathbb{Z}[\omega] \\ N(\nu_1) \ll \Xi_1'' \\ (\lambda^3 \nu_1, h_1 d_1) = 1 \\ \lambda^3 \nu_1 \equiv 0 \ (\mathrm{mod}\ g_1) \\ (10\text{-}77)}} 1 \Bigg)^{1/2}. \tag{10-82}$$

Consider the bracketed expression in (10-81). The conditions on the ν_2-sum imply that the ν_2-sum is bounded by 1. We then estimate the sum over g_2 trivially, and then apply the divisor bound (2-14) to estimate the sum over g_1. Thus the sum over g_2, g_1 and ν_2 satisfies $\ll X^{\varepsilon} G_2$. We use this bound, drop the condition $(\lambda^3 \nu_1, h_1 d_1) = 1$ by positivity, and then apply Lemma 2.7 to estimate the ν_1-sum. We obtain that the entire bracketed expression in (10-81) satisfies $\ll X^{\varepsilon} G_2 \Xi_1$. The analogous argument can be applied to obtain a bound of $\ll X^{\varepsilon} G_1 \Xi_2$ for the bracketed expression in (10-82). We deduce that

$$\sum_{\substack{\boldsymbol{g} \\ \forall i: N(g_i) \sim G_i \\ \forall i: g_i \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{\boldsymbol{\nu} \in (\lambda^{-3}\mathbb{Z}[\omega])^2 \\ \forall i: N(\nu_i) \ll \Xi_i'' \\ (10\text{-}76) \\ (10\text{-}77)}} |\rho_f(\nu_1)| \cdot |\rho_f(\nu_2)| \ll X^{\varepsilon} (G_2 \Xi_1)^{1/2} (G_1 \Xi_2)^{1/2}. \tag{10-83}$$

Substituting (10-83) into (10-79), bounding the remaining sums trivially (using Lemma 2.6 for the t_1, t_2 sums), and recalling that $X \asymp AB$ we obtain

$$\sum_{\boldsymbol{a}} \mu^2(a_1) |\alpha_{a_1}| \mu^2(a_2) |\alpha_{a_2}| \mathcal{N}_{4f}(\boldsymbol{a}, \dots, N(\lambda^{\ell}) T Q' Q'')$$

$$\ll (XKN(v))^{\varepsilon} \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_{\infty}^2 K^8 N(v_0)^4 X Q'' C^{-2} A^{-2} T^{-2} \sum_{\substack{\forall i: H_i, D_i, G_i, R_i, T_i \\ \text{dyadic} \\ (10\text{-}78)}} (H_1 D_1 R_1 T_1 G_1^{3/2} H_2 D_2 R_2 T_2 G_2^{3/2})$$

$$\ll (XKN(v))^{\varepsilon} K^8 N(v_0)^6 ABC^{-2} Q' Q'' \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_{\infty}^2. \tag{10-84}$$

Substituting (10-84) into (10-63) we see that (10-63) is

$$\ll (XKN(v))^{\varepsilon} K^8 N(v_0)^6 AB \|\boldsymbol{\mu}^2 \boldsymbol{\alpha}\|_{\infty}^2. \tag{10-85}$$

Combining (10-67) and (10-85), and then using $N(v_0) \leq N(v)$, yields the result after recalling (10-26) and Remark 10.4. □

Proof of Theorem 1.5. This follows immediately from Lemma 10.1 and Proposition 10.2. □

Acknowledgements

The author thanks both Ikuya Kaneko and Maksym Radziwiłł for making them aware of the reference [Browning and Vishe 2014] and for useful discussions. The author thanks the referee for helpful feedback on the manuscript.

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Communicated by Roger Heath-Brown
Received 2024-03-21 Revised 2024-07-08 Accepted 2024-09-03

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Algebra & Number Theory

Volume 19 No. 9 2025