Algebra & Number Theory

Volume 20
2026
No. 1

The average Mordell–Weil rank of elliptic surfaces over number fields

Remke Kloosterman

msp

ALGEBRA AND NUMBER THEORY 20:1 (2026)
https://doi.org/10.2140/ant.2026.20.1

The average Mordell–Weil rank of elliptic surfaces over number fields

Remke Kloosterman

Let K be a finitely generated field over $\mathbb{Q}$. Let $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$ be a family of nontrivial elliptic surfaces over K such that the configuration of singular fibers of $\mathcal{X}_b \to \mathcal{C}_b$ is the same for each closed point $b \in |\mathcal{B}|$. Let r be the minimum of the Mordell–Weil rank in this family. Then we show that the locus inside $|\mathcal{B}|$ where the Mordell–Weil rank is at least $r+1$ is a sparse subset.

In this way we prove Cowan's conjecture on the average Mordell–Weil rank of elliptic surfaces over $\mathbb{Q}$ and prove a similar result for elliptic surfaces over arbitrary number fields.

1. introduction

In a recent paper Alex Cowan [4] formulated the following conjecture on the average rank of elliptic curves over $\mathbb{Q}(t)$. Let μ be the Mahler measure on $\mathbb{Z}[t]$ and let $P_d(M) = \{p \in \mathbb{Z}[T] \mid \deg(p) \leq d, \mu(p) < M\}$. Define $S_{m,n}(M)$ to be the set

$$\left\{ E_{A,B} : y^2 = x^3 + A(t)x + B(t) \;\middle|\; \begin{array}{c} A \in P_m(M^2),\, B \in P_n(M^3), \\ 4A(t)^3 + 27B(t)^2 \neq 0 \end{array} \right\}.$$

Conjecture 1.1 (Cowan). *For every pair of positive integers m, n we have*

$$\lim_{M\to\infty} \frac{1}{\#S_{m,n}(M)} \sum_{E \in S_{m,n}(M)} \operatorname{rank} E(\mathbb{Q}(t)) = 0.$$

Battistoni, Bettin and Delaunay proved this conjecture for $1 \leq m, n \leq 2$ and for certain unirational subfamilies of $S_{2,2}$; see [2].

In this paper we will prove Cowan's conjecture for all (m, n) with $m, n \geq 1$, and generalize this to arbitrary number fields. The main ingredient for the proof holds in a more general context: Fix a field K, finitely generated over $\mathbb{Q}$. Fix a smooth geometrically irreducible base variety $\mathcal{B}/K$. Let $\mathcal{X} \to \mathcal{B}$ be a family of elliptic surfaces with a section, with $\mathcal{C} \to \mathcal{B}$ the base curve of the elliptic fibration (cf. Section 2). Let η be the generic point of $\mathcal{B}$. Then the generic fiber of $\mathcal{X}_\eta \to \mathcal{C}_\eta$ is an elliptic curve $E_\eta/K(\eta)(\mathcal{C}_\eta)$.

The author thanks Jaap Top, Matthias Schütt and the referee for comments on a previous version of this paper. The author is a member of PRIN-2020 Project "Curves, Ricci flat Varieties and their Interactions" and of INdAM-GNSAGA.
MSC2020: primary 11G05; secondary 14G05, 14J27.
Keywords: Mordell–Weil rank of elliptic surfaces, families of elliptic surfaces.

Similarly, for a closed point $b \in |\mathcal{B}|$ the generic fiber of $\mathcal{X}_b \to \mathcal{C}_b$ is an elliptic curve $E_b/K(b)(\mathcal{C}_b)$. There is a specialization map

$$\sigma : E_\eta(K(\eta)(\mathcal{C}_\eta)) \to E_b(K(b)(\mathcal{C}_b))$$

and a second map if one passes to algebraic closures of $K(\eta)$ and $K(b)$.

Theorem 1.2. *Let K be a finitely generated field over $\mathbb{Q}$, and let $\mathcal{X} \to \mathcal{B}$ be a family of elliptic surfaces with a section over K. Then there exists a sparse set $Z \subset |\mathcal{B}|$ such that for all $b \in |\mathcal{B}| \setminus Z$ the specialization maps*

$$\bar{\sigma} : E_\eta(\overline{K(\eta)}(\mathcal{C}_\eta)) \to E_b(\overline{K(b)}(\mathcal{C}_b)) \quad \text{and} \quad \sigma : E_\eta(K(\eta)(\mathcal{C}_\eta)) \to E_b(K(b)(\mathcal{C}_b))$$

are bijective.

This result is a combination of Proposition 2.17 and Theorem 2.19. For a definition of thin subsets and of sparse subsets see Section 2.

The proof for the fact that $\bar{\sigma}$ is an isomorphism relies heavily on [6, Theorem 8.3] and the main result of [1]. The morphism σ is a restriction of $\bar{\sigma}$ and therefore injective. It remains to show that σ is surjective. For this we use an argument relying on an appropriate form of Hilbert's irreducibility theorem.

In order to prove Cowan's conjecture it remains to show that the universal elliptic surface over $S_{m,n}$ has finite (arithmetic) Mordell–Weil group $E_\eta(K(\eta)(C_\eta))$. If $m = 4k$ and $n = 6k$ for some integer $k > 1$ then it follows from Noether–Lefschetz theory [5] that the (a priori) larger group $E_\eta(\overline{K(\eta)}(C_\eta))$ is trivial. This yields Cowan's conjecture almost immediately for these values of (m, n). However, for certain other values of (m, n) we were not able to determine $E_\eta(\overline{K(\eta)}(C_\eta))$ and, moreover, in the range $1 \leq m \leq 4$; $1 \leq n \leq 6$ this larger geometric Mordell–Weil group has positive rank.

To circumvent these issues we will use an argument involving quadratic twisting to show that for all choices of m and n the universal elliptic curve over $S_{m,n}$ has arithmetic Mordell–Weil rank zero. More precisely, suppose $\mathcal{X} \to \mathcal{B}$ is a family of elliptic surfaces over a number field K. Given a rational point $b \in \mathcal{B}(K)$ consider

$$\operatorname{tw}(b) := \left\{ d \in K^*/(K^*)^2 \;\middle|\; \begin{array}{c} \text{there exists a point } b' \in \mathcal{B}(K) \\ \text{such that } \mathcal{X}_b/\mathcal{C}_b \not\cong \mathcal{X}_{b'}/\mathcal{C}_{b'} \\ \text{and } (\mathcal{X}_b/\mathcal{C}_b)_{K(\sqrt{d})} \cong (\mathcal{X}_{b'}/\mathcal{C}_{b'})_{K(\sqrt{d})} \end{array} \right\},$$

the set of d such that the quadratic twist of $\mathcal{X}_b$ by d is also contained in the family $\mathcal{X} \to \mathcal{B}$. Then we show the following:

Theorem 1.3. *Let K be a finitely generated field over $\mathbb{Q}$ and let $\mathcal{X} \to \mathcal{B}$ be a family of nontrivial elliptic surfaces with a section.*

Suppose there exists a point $b \in \mathcal{B}(K)$ such that $\operatorname{tw}(b)$ is infinite and such that the configuration of singular fibers of $\mathcal{X}_b \to \mathcal{C}_b$ is the same as $\mathcal{X}_\eta \to \mathcal{C}_\eta$. Then $E_\eta(K(\eta)(C_\eta))$ is finite.

These two results are sufficient to prove Cowan's conjecture over $\mathbb{Q}$. In order to generalize Cowan's conjecture to number fields, we have to make several small adjustments. The Mahler measure of a polynomial is usually defined for polynomials with complex coefficients. For a number field K let M_K be the set of embeddings $\tau : K \to \mathbb{C}$. Given a polynomial $p \in K[t]$ we define the Mahler measure of p to be the maximum of the Mahler measures of $\tau(p)$, where $\tau \in M_K$. This allows us to define for any $M \geq 0$ the finite set

$$P_{d,K}(M) = \{p \in \mathcal{O}_K[T] \mid \deg(p) \leq d,\ \max_{\tau \in M_K} \mu(\tau(p)) < M\}.$$

We define $S_{m,n,K}(M)$ to be the set

$$\left\{ E_{A,B} : y^2 = x^3 + A(t)x + B(t) \;\middle|\; \begin{array}{c} A \in P_{m,K}(M^2),\ B \in P_{n,K}(M^3), \\ 4A(t)^3 + 27B(t)^2 \neq 0 \end{array} \right\}$$

and $S_{m,n,K} = \bigcup_{M \in \mathbb{Z}_{>0}} S_{m,n,K}(M)$. The above two theorems combined show that on the complement of a thin subset of $S_{m,n,K}$ the Mordell–Weil rank is zero, in particular, we have:

Corollary 1.4. *Let K be a number field. Fix positive integers m, n. The set of elliptic surfaces in $S_{m,n,K}$ with Mordell–Weil rank zero has density one.*

Corollary 1.5 (Cowan's conjecture). *Fix positive integers m, n. Then the average rank of elliptic surfaces in $S_{m,n,\mathbb{Q}}$ equals zero.*

We will also prove a generalization of Cowan's conjecture to arbitrary number fields, see Corollary 3.19. However, in this case we have to adjust the definition of $S_{m,n,K}$ in order to exclude trivial elliptic fibrations, i.e., elliptic surfaces which are birational to $E \times \mathbb{P}^1$ as fibered surfaces.

The organization of this paper is as follows. In Section 2 we recall some standard results on families of elliptic surfaces. Moreover, we apply results from [6] to determine the locus in the family where the specialization map on the Mordell–Weil group is injective, and then to describe the locus where this map is bijective. In Section 3 we study the universal Weierstrass equation over $S_{m,n}$ and use the results from the previous section to show that for $K = \mathbb{Q}$ the average rank is zero.

2. Families of elliptic surfaces

Let K be a field of characteristic zero.

Definition 2.1. An *elliptic surface* $\pi : X \to C$ over K consists of a geometrically irreducible smooth projective surface X/K, a geometrically irreducible smooth projective curve C/K and a flat K-morphism $\pi : X \to C$ such that the generic fiber of π is a smooth projective curve of genus 1, and none of the fibers of π contains a (-1)-curve.

An *elliptic surface with a section* is an elliptic surface $\pi : X \to C$ together with a section $\sigma_0 : C \to X$, defined over K.

Let $E/K(C)$ be the generic fiber. Then the group $E(K(C))$ of $K(C)$-valued points of E can be naturally identified with the set of rational sections of π. This latter set can be made into an abelian group by fiberwise addition, where the zero element of the fiber over a point $p \in C$ is $\sigma_0(p)$.

We call an elliptic surface $\pi : X \to C$ *trivial* if there is an elliptic curve E/K and a birational map $\psi : X \dashrightarrow E \times C$ such that $\pi = \mathrm{pr}_2 \circ \psi$ as rational maps.

Since C is a smooth curve we can extend every rational section to a section of π, hence $E(K(C))$ is also the set of sections of π.

Definition 2.2. Let S be a smooth projective surface over an algebraically closed field K. Denote with $\mathrm{NS}(S)$ the Néron–Severi group of S, the group of divisors on S modulo algebraic equivalence.

Definition 2.3. Let $\pi : X \to C$ be an elliptic surface with a section. Let $T \subset \mathrm{NS}(X_{\overline{K}})$ be the *trivial subgroup*, generated by the irreducible components (over $\overline{K}$) of the singular fibers not intersecting $\sigma_0(C)$, the class of a smooth fiber and the image of the zero section. ([8, Section 6.1])

We have the following results, which contains both the Mordell–Weil theorem and the Shioda–Tate formula:

Proposition 2.4. *Let $\pi : X \to C$ be an elliptic surface with a section. If X is not a trivial elliptic surface then there is a natural isomorphism of groups*

$$\mathrm{NS}(X_{\overline{K}})/T \cong E(\overline{K}(C)).$$

In particular, $E(\overline{K}(C))$ is finitely generated.

Proof. See [8, Theorem 6.5]. □

Notation 2.5. Let $\mathcal{B}$ be a K-variety. Denote with $|\mathcal{B}|$ the set of closed points of $\mathcal{B}$ and with η the generic point of $\mathcal{B}$, with residue field $K(\eta) = K(\mathcal{B})$.

Definition 2.6. A *family of elliptic surfaces with a section* consists of a smooth geometrically irreducible K-variety $\mathcal{B}$, a smooth geometrically irreducible projective surface $\mathcal{X} \to \mathcal{B}$, and a smooth geometrical irreducible projective curve $\mathcal{C} \to \mathcal{B}$ together with morphisms $\pi : \mathcal{X} \to \mathcal{C}$ and $\sigma_0 : \mathcal{C} \to \mathcal{X}$ of $\mathcal{B}$-schemes, such that for each closed point $b \in |\mathcal{B}|$ we have that $\pi_b : \mathcal{X}_b \to \mathcal{C}_b$ is an elliptic surface over $K(b)$ with zero-section $(\sigma_0)_b : \mathcal{C}_b \to \mathcal{X}_b$.

Remark 2.7. Let $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$ be a family of elliptic surfaces. Then $\mathcal{X}_\eta$ is an elliptic surface over $K(\eta)$, and for every $b \in |\mathcal{B}|$ the surface $\mathcal{X}_b$ is an elliptic surface over $K(b)$. We denote with $\mathcal{X}_{\bar{\eta}}$ the base change of $\mathcal{X}_\eta$ to $\overline{K(\eta)}$ and with $\mathcal{X}_{\bar{b}}$ the base change of $\mathcal{X}_b$ to $\overline{K(b)} = \overline{K}$.

Note that the base curve of the elliptic fibration is always assumed to be projective, but that the base variety of the family of elliptic surfaces is only required to be geometrically irreducible.

Proposition 2.8. *Let $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$ be a family of elliptic surfaces. Suppose that there exists a geometric point $\bar{b}$ of $\mathcal{B}$ such that the elliptic surface $\mathcal{X}_{\bar{b}} \to \mathcal{C}_{\bar{b}}$ is not trivial, then there exists a nonempty open subscheme $\mathcal{B}'$ of $\mathcal{B}$ such that for all $b \in |\mathcal{B}'|$ the elliptic surface $\mathcal{X}_b \to \mathcal{C}_b$ is not trivial.*

Proof. Suppose first that there is a geometric point of $\bar{b}$ such that the j-invariant of $\mathcal{X}_{\bar{b}} \to \mathcal{C}_{\bar{b}}$ is not in $K(\bar{b})$, i.e., is nonconstant. Then we can take for $\mathcal{B}'$ the locus of elliptic surfaces with nonconstant j-invariant, which is open in $\mathcal{B}$.

Suppose now that for every geometric point $\bar{b}$ of $\mathcal{B}$ the j-invariant is constant, i.e., is an element of $K(\bar{b}) \subset \overline{K}$. Consider now the invariants $c_4, c_6 \in K(\mathcal{C}_{\bar{b}})$ of a Weierstrass equation for $\mathcal{X}_{\bar{b}} \to \mathcal{C}_{\bar{b}}$. Since the j-invariant is constant we have that one of $c_4 = 0$, $c_6 = 0$, $c_4^3 = \lambda c_6^2$ for some $\lambda \in \overline{K}^*$ holds. The locus with trivial elliptic fibration is the sublocus where c_4 is a perfect fourth power and c_6 is a perfect sixth power. These conditions are closed conditions in $\mathcal{B}$, hence we can take $\mathcal{B}'$ to be the complement. □

Let $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$ be a family of elliptic surfaces. Suppose that there exists a geometric point $\bar{b}$ of $\mathcal{B}$ such that the elliptic surface $\mathcal{X}_{\bar{b}} \to \mathcal{C}_{\bar{b}}$ is not trivial. From the above proposition it follows that after replacing the base of the family of elliptic surfaces by an open subset we may assume that all elliptic surfaces in the family are not trivial.

Assumption 2.9. For the rest of this section assume that $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$ is a family of elliptic surfaces such that for all $b \in |\mathcal{B}|$ the elliptic surface $\mathcal{X}_b \to \mathcal{C}_b$ is not trivial.

We have the following result by Maulik and Poonen [6, Proposition 3.6(a)]:

Proposition 2.10. *With the same notation as before, the specialization map*

$$\mathrm{sp} : \mathrm{NS}(\mathcal{X}_{\bar{\eta}}) \to \mathrm{NS}(\mathcal{X}_{\bar{b}})$$

is injective with torsion-free cokernel for all $b \in |\mathcal{B}|$.

One easily shows the following

Lemma 2.11. *With the same notation as before. Then* $\mathrm{sp}(T_\eta) \subset T_b$, *for all* $b \in |\mathcal{B}|$ *such that* $\mathcal{X}_b \to \mathcal{C}_b$ *is not trivial.*

Proof. This specialization map obviously maps the class of the zero section to the class of the section zero and the class of a fiber to a class of a fiber. Moreover, the specialization of a fiber component is contained in a single fiber of the specialized surface, hence $\mathrm{sp}(T_\eta) \subset T_b$. □

The following statements can be found in [8, Chapter 5 and Section 6.1]: Let $\pi : X \to C$ be an elliptic surface over a field K. Let Δ be the discriminant. For $p \in \Delta(\overline{K})$ let C_0 be the irreducible component of $(\pi^{-1}(p))_{\overline{K}}$ intersecting the image of the zero-section. Consider now the intersection paring on $\mathrm{NS}(X)$ restricted to the subgroup Λ_p generated by the irreducible components of $\overline{\pi^{-1}(p)_{\overline{K}} \setminus C_0}$. Then this is a lattice of type $A_n(-1)$ (for some $n \in \mathbb{Z}_{>0}$), $D_m(-1)$ (for some $m \in \mathbb{Z}_{>3}$), $E_6(-1)$, $E_7(-1)$ or $E_8(-1)$. The type can be easily read off from a Weierstrass equation for the elliptic fibration π, namely it is determined by the valuations of the standard invariants j, c_4 and c_6 at the point p. Then the trivial lattice T associated with the elliptic fibration π is isomorphic with

$$\Lambda' \oplus^{\perp}_{p \in \Delta(\overline{K})} \Lambda_p,$$

where Λ' is rank two lattice generated by classes F and Z, with $F^2 = 0$, $Z^2 = -\chi(\mathcal{O}_X)$, $F.Z = 1$.

Definition 2.12. Let $\pi : X \to C$ be an elliptic surface with section over a field K. The *configuration of singular fibers* is a finite multiset $\mathcal{M}$, whose elements are formal symbols $A_n(-1)$, with $n \in \mathbb{Z}_{>0}$,

$D_m(-1)$, with $m \in \mathbb{Z}_{>3}$, $E_6(-1)$, $E_7(-1)$, $E_8(-1)$, such that the multiplicity of Λ in $\mathcal{M}$ equals the number of $p \in C(\overline{K})$ such that Λ_p equals Λ.

Lemma 2.13. *Let $\mathcal{B}'$ be the subscheme of $\mathcal{B}$ such that for all $b \in |\mathcal{B}'|$ the configuration of singular fibers of $\pi_b : \mathcal{X}_b \to \mathcal{C}_b$ equals the configuration of singular fibers of $\pi_\eta : \mathcal{X}_\eta \to \mathcal{C}_\eta$. Then $\mathcal{B}'$ is nonempty open and for all closed points $b \in |\mathcal{B}'|$ we have*

$$\mathrm{sp}(T_\eta) = T_b.$$

In particular, for all $b \in |\mathcal{B}'|$ the specialization maps

$$\bar{\sigma} : E_\eta(\overline{K(\eta)}(C_\eta)) \to E_b(\overline{K(b)}(C_b)) \text{ and } \sigma : E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))$$

are injective.

Proof. Let $b \in |\mathcal{B}'|$ then $\mathrm{sp}(T_\eta)$ is a sublattice of T_b by the previous lemma. Since the configuration of singular fibers are the same we have that $\mathrm{rank}(T_b) = \mathrm{rank}(\mathrm{sp}(T_\eta))$ and $\det(T_b) = \det(\mathrm{sp}(T_\eta))$ hold, hence $\mathrm{sp}(T_\eta) = T_b$ for all $b \in \mathcal{B}'$.

Using Tate's algorithm [10] one easily sees that the type of singular fibers can be determined by the valuation of three standard invariants of a Weierstrass equation $(v(j), v(c_4), v(c_6))$. In particular, the locus $\mathcal{B}'$ is nonempty and open in $\mathcal{B}$. Moreover, for all $b \in |\mathcal{B}'|$ we have the following chain of morphisms:

$$E_\eta(\overline{K(\eta)}(C_\eta)) \xrightarrow{\sim} \mathrm{NS}(\mathcal{X}_{\bar{\eta}})/T_\eta \hookrightarrow \mathrm{NS}(\mathcal{X}_{\bar{b}})/T_b \xrightarrow{\sim} E_b(\overline{K(b)}(C_b)).$$

The first and third map are isomorphisms by Proposition 2.4. This shows that $\bar{\sigma}$ is injective. The second (arithmetic) specialization map σ is a restriction of $\bar{\sigma}$ and is also injective. □

Definition 2.14. Let K be a finitely generated field over $\mathbb{Q}$. Let $\mathcal{B}$ be a K-variety. Call a subset S of $|\mathcal{B}|$ *sparse* if there exists a dominant and generically finite morphism $\pi : \mathcal{B}_0 \to \mathcal{B}$ of irreducible K-varieties, such that for each $s \in S$ the fiber $\pi^{-1}(s)$ is empty or contains at least two closed points.

Definition 2.15. Let K be a field of characteristic zero. Let V/K be a K-variety. A subset S of $V(K)$ is called a *thin subset of type I* if S is contained in a Zariski closed subset of $V(K)$. A subset S of $V(K)$ is a called a *thin subset of type II* if there exists another K-variety V' such that $\dim V = \dim V'$ and a finite morphism $\varphi : V' \to V$ of degree at least 2, such that $S \subset \varphi(V'(K))$.

A subset S of $V(K)$ is *thin* if it is a subset of a finite union of thin subsets of type *I* and type *II*.

Remark 2.16. If K is finitely generated over $\mathbb{Q}$ and S is a sparse subset of $|\mathbb{A}^n|$ then $S \cap \mathbb{A}^n(K)$ is a thin set; see [6, Proposition 8.5].

Proposition 2.17. *Let K be a finitely generated field over $\mathbb{Q}$. Then there is a sparse subset $Z \subset |\mathcal{B}|$ such that for each $b \in |\mathcal{B}| \setminus Z$ the specialization map*

$$\bar{\sigma} : E_\eta(\overline{K(\eta)}(C_\eta)) \to E_b(\overline{K(b)}(C_b))$$

is an isomorphism.

Proof. Let $\mathcal{B}'$ be as in the previous lemma. Let $Z' = |\mathcal{B}| \setminus |\mathcal{B}'|$. Since $\mathcal{X} \to B$ is smooth and projective, we have by [6, Proposition 3.6 and Theorem 8.3] that there is a sparse subset $Z'' \subset |\mathcal{B}|$, such that

$$\mathrm{sp} : \mathrm{NS}(\mathcal{X}_{\bar{\eta}}) \to \mathrm{NS}(\mathcal{X}_{\bar{b}})$$

is an isomorphism for all $b \in |\mathcal{B}| \setminus Z''$. We can factor $\bar{\sigma}$ as in the previous proof, i.e., for all $b \in |\mathcal{B}| \setminus (Z' \cup Z'')$ we now have isomorphisms

$$E_\eta(\overline{K(\eta)}(C_\eta)) \xrightarrow{\sim} \mathrm{NS}(\mathcal{X}_{\bar{\eta}})/T_\eta \xrightarrow{\sim} \mathrm{NS}(\mathcal{X}_{\bar{b}})/T_b \xrightarrow{\sim} E_b(\overline{K(b)}(C_b)).$$

Since Z' is a Zariski-closed proper subset of $|\mathcal{B}|$ and Z'' is sparse it follows from [6, Proposition 8.5] that $Z' \cup Z''$ is sparse in $|\mathcal{B}|$. □

We will now prove a similar statement for the specialization on the arithmetic Mordell–Weil groups,

$$\sigma : E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b)).$$

For this we aim to compare the $\mathrm{Gal}(\overline{k(\eta)}/k(\eta))$-action on $E_\eta(\overline{K(\eta)}(C_\eta))$ and the $\mathrm{Gal}(\overline{k(b)}/k(b))$-action on $E_b(\overline{K(b)}(C_b))$.

Remark 2.18. Fix a finite Galois extension L of $K(\eta) = K(\mathcal{B})$ with Galois group G. Let L_0 be the algebraic closure of K in L. Suppose first that $L_0 \neq L$.

Then $L = K(\mathcal{B}')$, for some integral K-variety $\mathcal{B}'$, which may be not geometrically integral. From the fact that $L_0 \neq L$ it follows that there is a finite rational map $\tau : \mathcal{B}' \to \mathcal{B}$ of degree at least 2. By shrinking $\mathcal{B}$ and $\mathcal{B}'$ if necessary we may assume that τ is an unramified finite flat morphism. In particular, for every $p \in |\mathcal{B}|$ we have that $\tau^{-1}(p)$ consists of finitely many closed points $p_1, \ldots, p_t$. Each point p_i yields a unramified field extension $K(p_i)/K(p)$. The locus of points for which $t > 1$ holds, defines a sparse subset S of $|\mathcal{B}| \setminus Z$.

Hence for each point $p \in |\mathcal{B}| \setminus (Z \cup S)$ we have that $t = 1$ and that $K(p_1)/K(p)$ is Galois with group G. In this case we call $K(p_1)/K(p)$ the specialization of the Galois extension $L/K(\eta)$.

If $L_0 = L$ then for every $b \in |\mathcal{B}|$ the extension $L(b)/K(b)$ is Galois with group G.

Theorem 2.19. *Let K be a finitely generated field over $\mathbb{Q}$. Let $\mathcal{X} \to \mathcal{B}$ be a family of elliptic surfaces with base curve $\mathcal{C}$. Then there is a sparse subset $Z \subset |\mathcal{B}|$, such that for every $b \in |\mathcal{B}| \setminus Z$ the specialization map*

$$\sigma : E_\eta(K(\eta)(\mathcal{C}_\eta)) \to E_b(K(b)(\mathcal{C}_b))$$

is an isomorphism.

Proof. Let Z' be the sparse subset of $|\mathcal{B}|$ such that on $|\mathcal{B}| \setminus Z'$ the geometric specialization map

$$\bar{\sigma} : E_\eta(\overline{K(\eta)}(\mathcal{C}_\eta)) \to E_b(\overline{K(b)}(\mathcal{C}_b))$$

is an isomorphism.

Recall that the group $E_\eta(\overline{K(\eta)}(\mathcal{C}_\eta))$ is finitely generated. Moreover, for each $P \in E_\eta(\overline{K(\eta)}(\mathcal{C}_\eta))$ we have that $K(\eta)(\mathcal{C}_\eta)(P) \subset \overline{K(\eta)}(\mathcal{C}_\eta)$ is algebraic and finitely generated over $K(\eta)(\mathcal{C}_\eta)$. In particular, there exists a minimal finite extension L of $K(\eta)$ such that $L/K(\eta)$ is Galois and

$$E_\eta(L(\mathcal{C}_\eta)) = E_\eta(\overline{K(\eta)}(\mathcal{C}_\eta)).$$

Let $G = \mathrm{Gal}(L/K(\eta))$. Then there is a thin set Z' such that for all $b \in |\mathcal{B}| \setminus Z'$ the specialization L' of L to $K(b)$ exists. Then $L'/K(b)$ is Galois with group G and $E_b(L'(\mathcal{C}_b)) = E_b(\overline{K(b)}(\mathcal{C}_b))$.

Let $P_1, \ldots, P_s$ be generators for $E_\eta(K(\eta)(\mathcal{C}_\eta))$. Let H be a subgroup of the group $\mathrm{Gal}(L/K(\eta))$. If $E(\overline{K(\eta)}(\mathcal{C}_\eta))^H = E_\eta(K(\eta)(\mathcal{C}_\eta))$ then we call H irrelevant, otherwise we call H relevant.

For every relevant subgroup H of $\mathrm{Gal}(L/K(\eta))$, let

$$Q_{H,1}, \ldots Q_{H,t_H} \in E(\overline{K(\eta)}(\mathcal{C}_\eta))^H$$

be points such that $P_1, \ldots, P_s, Q_{H,1}, \ldots, Q_{H,t_H}$ generate $E(\overline{K(\eta)}(C_\eta))^H$.

For a relevant subgroup H of G let $Z_H \subset |\mathcal{B}|$ be the locus of all $b \in |\mathcal{B}|$ such that $\mathrm{sp}(Q_{H,i}) \in K(b)$ for all $i = 1 \ldots, t_H$. Then for every $b \in |\mathcal{B}| \setminus Z_H$ we have that $E_b(K(b)(\mathcal{C}_b)) \neq E_\eta(L(\mathcal{C}_\eta))^H$. Let

$$Z = Z' \cup \bigcup_{H \text{relevant}} Z_H.$$

Then for all $b \in |\mathcal{B}| \setminus Z$ we have $E_b(K(b)(\mathcal{C}_b)) = E_\eta(L(\mathcal{C}_\eta))^H$ for some subgroup H of G but since $b \notin Z$, this group H is irrelevant. In particular,

$$E_b(K(b)(\mathcal{C}_b)) = E_\eta(L(\mathcal{C}_\eta))^H = E_\eta(K(\eta)(\mathcal{C}_\eta)).$$

From the construction of Z_H it follows that Z_H is sparse and therefore Z is sparse, since is a finite union of sparse subsets. □

3. Cowan's conjecture

In this section we want to use Theorem 2.19 to prove Cowan's conjecture and generalize this result to number fields. As explained in the Introduction we will use quadratic twists for this. For the moment let K be a field of characteristic zero.

Proposition 3.1. *Let V be a $\mathbb{Q}$-vector space. Let $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(V)$ be a Galois representation, such that ρ factors through a finite group. For any $d \in K^*/(K^*)^2$, let χ_d be the quadratic character associated with the field extension $K(\sqrt{d})/K$. Let $\rho^{(d)} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(V)$ be the twisted Galois representation $\rho^{(d)}(g) = \rho(g) \circ (\chi_d(g) Id_V)$.*

Then there exists at most $\dim V$ *many $d \in K^*/(K^*)^2$ such that $\rho^{(d)}$ has a nontrivial invariant subspace.*

Proof. Suppose first that V is the trivial representation then for all $d \neq 1$ we have that $\rho^{(d)}(\cdot) = \chi_d(\cdot) Id_V$ has no nontrivial invariant subspace. Similarly, if V has a twist which is trivial then all other twists (including the trivial one) have no nontrivial invariant subspace.

We proceed now by induction on $\dim V$. The case $\dim V = 1$ is covered by the above. Suppose now that $\dim V > 1$. Suppose that for some d there is a nontrivial invariant subspace then by Maschke's theorem we can decompose $V^{(d)} = \mathbb{Q} \oplus V'$. Then there are at most $d-1$ twists of V' which have a nontrivial invariant subspace and only the trivial twist of $\mathbb{Q}$ has a nontrivial invariant subspace. □

Consider now a family of elliptic surfaces $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$. Given a rational point $b \in \mathcal{B}(K)$ consider

$$\mathrm{tw}(b) := \left\{ d \in K^*/(K^*)^2 \;\middle|\; \begin{array}{c} \text{there exists a point } b' \in \mathcal{B}(K) \\ \text{such that } \mathcal{X}_b/\mathcal{C}_b \not\cong \mathcal{X}_{b'}/\mathcal{C}_{b'} \\ \text{and } (\mathcal{X}_b/\mathcal{C}_b)_{K(\sqrt{d})} \cong (\mathcal{X}_{b'}/\mathcal{C}_{b'})_{K(\sqrt{d})} \end{array} \right\},$$

the set of d such that the quadratic twist of $\mathcal{X}_b$ by d is also contained in the family $\mathcal{X} \to \mathcal{B}$.

Proposition 3.2. *Let K be a finitely generated field over $\mathbb{Q}$ and let $\mathcal{X} \to \mathcal{C} \to \mathcal{B}$ be a family of elliptic surfaces with a section, such that for all geometric points $\bar{b}$ of $\mathcal{B}$ the elliptic surface $\mathcal{X}_{\bar{b}} \to \mathcal{C}_{\bar{b}}$ is not trivial.*

Suppose there exists a rational point $b \in \mathcal{B}(K)$ such that $\mathrm{tw}(b)$ *is infinite and the configuration of singular fibers of $\mathcal{X}_b \to \mathcal{C}_b$ is the same as $\mathcal{X}_\eta \to \mathcal{C}_\eta$. Then $E_\eta(K(\eta)(\mathcal{C}_\eta))$ is finite.*

Proof. Our assumption on the configuration of singular fibers and Lemma 2.13 yield that the specialization map

$$\sigma : E_\eta(K(\eta)(\mathcal{C}_\eta)) \to E_b(K(\mathcal{C}_b))$$

is injective. Moreover, the assumption on the configuration of singular fibers is formulated over $\overline{K}$, hence for all $d \in \mathrm{tw}(b)$ we have that

$$\sigma : E_\eta(K(\eta)(\mathcal{C}_\eta)) \to E_b^{(d)}(K(\mathcal{C}_b))$$

is injective.

Consider now the $\mathrm{Gal}(\overline{K}/K)$-representation $V = E_\eta(\overline{K}(t)) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Galois representation on $E_b^{(d)}(K(\mathcal{C}_b)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is precisely $V^{(d)}$. Hence if the rank of $E_b^{(d)}(K(\mathcal{C}_b))$ is positive then $V^{(d)}$ has a nontrivial invariant subspace. Since there are only finitely many $d \in K^*/(K^*)^2$ with this property it follows that there is a $d \in \mathrm{tw}(b)$ with $E_b^{(d)}(K(\mathcal{C}_b))$ finite and therefore $E_\eta(K(\eta)(\mathcal{C}_\eta))$ is finite. □

Assumption 3.3. For the rest of this section let K be a number field, with ring of integers $\mathcal{O}_K$.

Notation 3.4. Let R be an integral domain of characteristic zero. Let $R[t]_d$ be the R-module of polynomials in t with coefficients from R and of degree at most d.

Let m, n be integers and let

$$S_{m,n,K} = \{(A, B) \in \mathcal{O}_K[t]_m \times \mathcal{O}_K[t]_n | 4A^3 + 27B^2 \neq 0\}.$$

Let $k = \lceil \max(m/4, n/6) \rceil$. To a pair $(A, B) \in S_{m,n,K}$ we can associate an elliptic curve $E_{(A,B)}/K(t)$ with Weierstrass equation

$$y^2 = x^3 + A(t)x + B(t).$$

Moreover we can associate a hypersurface $W_{A,B}$ in the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-2k)\oplus\mathcal{O}(-3k))$ over $\mathbb{P}^1$, given by

$$-Y^2Z+X^3+s^{4k}A\left(\frac{t}{s}\right)XZ^2+s^{6k}B\left(\frac{t}{s}\right)Z^3=0.$$

Remark 3.5. Consider the projection morphism $\psi : W_{A,B}\to\mathbb{P}^1$. Then for all $p\in\mathbb{P}^1$ the fiber over p is irreducible and the general fiber is an elliptic curve. If we resolve the singularities of $W_{A,B}$ and successively collapse all -1 curves then we obtain an elliptic surface $X_{A,B}\to\mathbb{P}^1$ [7, Lecture III.3]. The collapsing of -1 curves is only necessary if the original Weierstrass equation is not minimal, or if both $\deg(A)\leq 4k-4$ and $\deg(B)\leq 6k-6$ hold.

Notation 3.6. We define now the following subset

$$U_{m,n,K}=\left\{(A,B)\in S_{m,n,K}\;\middle|\;\begin{array}{c}4A^3+27B^2 \text{ is smooth in } K[t] \text{ and}\\ \deg(4A^3+27B^2)=\max(3m,2n)\end{array}\right\}.$$

In the following we will identify $R[t]_n\times R[t]_m$ with $\mathbb{A}^{m+n+2}(R)$.

Remark 3.7. Let $(A,B)\in U_{m,n,K}$. Since $4A^3+27B^2$ is smooth and of positive degree, the associated elliptic surfaces has $\max(3m,2n)$ fibers of type I_1. In particular, the j-function is not constant, and the elliptic surface is not trivial.

Lemma 3.8. *There exists subvarieties Z' and Z'' of $\mathbb{A}^{m+n+2}$ such that $S_{m,n,K}=\mathbb{A}^{m+n+2}(\mathcal{O}_K)\setminus Z'(\mathcal{O}_K)$ and $U_{m,n,K}=\mathbb{A}^{m+n+2}(\mathcal{O}_K)\setminus Z''(\mathcal{O}_K)$. Moreover, $U_{m,n,K}\neq\varnothing$.*

Proof. The subvariety Z' is defined by the property that all coefficients in $4A^3+27B^2$ vanish. Recall that for every $(A,B)\in S_{m,n,K}$ we have that $\deg(4A^3+27B^2)\leq\max(2n,3m)$. Hence Z'' is defined by the vanishing of the coefficient of t^d in $4A^3+27B^2$, where $d=\max(2n,3m)$. It remains to show that $U_{m,n,K}$ is nonempty: if $2n\geq 3m$ then $(1,t^n)\in U_{m,n,K}$, if $2n<3m$ then $(t^m,1)\in U_{m,n,K}$. □

Notation 3.9. One easily checks that there is a scheme $\mathcal{U}_{m,n}\subset\mathbb{A}^{n+m+2}$, defined over $\mathbb{Z}$ such that $\mathcal{U}_{m,n}(\mathcal{O}_L)=U_{m,n,L}$ for all number fields L.

For fixed (m,n) the singularities of $W_{A,B}$ depend on $(A,B)\in S_{m,n,K}$. Resolving these singularities may lead to a nonflat family, and therefore is not a family of elliptic surfaces as in Definition 2.6. We avoid this issue by using $\mathcal{U}_{m,n}$ as the base for our family of elliptic surfaces. We will show that $W_{A,B}$ has only one singular point for $(A,B)\in|\mathcal{U}_{m,n}|$ and that the type of singularity is completely determined by (m,n). Moreover, we will show that the configuration of singular fibers is constant on $\mathcal{U}_{m,n}$.

Recall that $k=\lceil\max(m/4,n/6)\rceil$. Let $\alpha=4k-m$ and let $\beta=6k-n$. Note that $\alpha,\beta\geq 0$ and that $\alpha<4$ or $\beta<6$.

Lemma 3.10. *Let $(A, B) \in U_{m,n,K}$ then $p_g(X_{A,B}) = k - 1$. Moreover, all singular fibers are of type I_1, except possibly for the fiber at $t = \infty$. At $t = \infty$ we have the following fiber depending only on (α, β):*

α β	0 ≥ 0	≥ 0 0	≥ 1 1	1 ≥ 2	≥ 2 2	2 ≥ 3	≥ 2 3	≥ 3 4	3 ≥ 5	≥ 4 5
fiber-type singularity	I_0 A_0	I_0 A_0	II A_0	III A_1	IV A_2	I_0^* D_4	I_0^* D_4	IV^* E_6	III^* E_7	II^* E_8

Proof. The statement about $p_g(X)$ is [7, Lemma IV.1.1]. For the fiber over infinity we have $v(c_4) = \alpha$, $v(c_6) = \beta$ and $v(\Delta) = 12k - \max(3m, 2n) = \min(3\alpha, 2\beta)$. A straight-forward application of Tate's algorithm yields the type of fiber and the type of singularity; see [10]. □

For a given $(A, B) \in U_{m,n,K}$ we constructed a Weierstrass model $W_{A,B}$. This yields a universal family over $\mathcal{U}_{m,n}$ i.e., a morphism of schemes $\mathcal{W}_{n,m} \to \mathcal{U}_{n,m}$.

Proposition 3.11. *Consider the universal Weierstrass model $\psi : \mathcal{W}_{m,n} \to \mathcal{U}_{m,n}$. Then the resolution of the singularity of the generic fiber yields a family of elliptic surfaces with a section $\mathcal{X}_{m,n} \to \mathcal{U}_{m,n}$ such that for all $b \in |\mathcal{U}_{m,n}|$ we have $T_\eta = T_b$. Moreover, for all $b \in |\mathcal{U}_{m,n}|$ we have that the configuration of singular fibers of $\mathcal{X}_b \to \mathcal{C}_b$ is the same as the configuration of singular fibers of $\mathcal{X}_\eta \to \mathcal{C}_\eta$.*

Proof. From the previous lemma it follows that over $\mathcal{U}_{m,n}$ all Weierstrass models are smooth (if $\alpha = 0$ or if $\beta \in \{0, 1\}$) or all Weierstrass models have a single singular point, and the type of the singularity is the same for all members of the family (if $\alpha > 0$ and $\beta > 1$). In particular, if we manage to resolve the singularities of the morphism $\mathcal{W}_{m,n} \to U_{m,n}$ uniformly then the trivial lattice is constant, i.e., $\mathrm{sp}(T_\eta) = T_b$ for all $b \in |\mathcal{U}_{m,n}|$.

The singular point is located at the point $((X : Y : Z), (s : t)) = ((0 : 0 : 1), (0 : 1))$. In the sequel we will work with the Weierstrass equation in the chart with local coordinates $x = X/Z$, $y = Y/Z$ and s, where by abuse of notation s is the quotient s/t of the two coordinates on $\mathbb{P}^1$.

Suppose that the singularity is of type A_1 or type A_2 then a single blow-up suffices to resolve the singularity, hence this can be done simultaneously for the whole family. Suppose now that the singularity is of type D_4 then we have a local equation of type

$$-y^2 + x^3 + s^2 g(s)x + s^3 h(s) = 0,$$

where $4g(s)^3 + 27h(s)^2$ does not vanish at $s = 0$. To investigate the singularities of the blow-up, we blow-up the ambient space and consider the strict transform. The blow-up can be covered by three affine charts, each of which is isomorphic to $\mathbb{A}^3$. In the first chart the blow-up morphism is given by $(x_1, y_1, s_1) \to (x_1, x_1 y_1, x_1 s_1)$ and the exceptional divisor is given by $x_1 = 0$, in the other two charts we have $(x_2, y_2, s_2) \to (y_2 x_2, y_2, s_2 y_2)$ and $(x_3, y_3, s_3) \mapsto (s_3 x_3, s_3 y_3, s_3)$. In the sequel we will abuse our notation and replace (x_i, y_i, s_i) by (x, y, s) if no confusion arises.

To continue with the family with D_4-singularities, substitute $y = sy$, $x = sx$ in the local equation. We then find that the strict transform has equation

$$-y^2 + s(x^3 + g(s)x + h(s)) = 0$$

in the third chart. Since $4g(0) + 27h(0) \neq 0$ holds, this surface has A_1 singularities at the three points in $y = s = x^3 + g(0)x + h(0) = 0$. The other two charts do not contain singularities already contained in this chart. Hence this singularity can be also resolved simultaneously.

One can proceed similarly for the remaining singularities E_6, E_7, E_8. We will do only the most complicated one, E_8, the other two can be treated very similar. For the E_8-singularity we have a Weierstrass equation of the form

$$-y^2 + x^3 + s^4 g(s)x + s^5 h(s) = 0,$$

with $h(0) \neq 0$. We first blow-up the origin. This leads to three different charts. However, it is straight forward to check that the strict transform is smooth in two of the three charts. The local equation in the third chart can be obtained by the substitutions $y = sy$, $x = sx$ and factoring out s^2. In particular, we obtain that the strict transform has local equation

$$-y^2 + s(x^3 + s^2 g(s)x + s^2 h(s)) = 0.$$

The strict transform is smooth away from the origin. Blowing-up the origin yields again three charts, in two of which the strict transform is smooth. In the third chart we obtain

$$-y^2 + sx(x + s^2 g(sx)x + s^2 h(sx)) = 0.$$

as the equation for the strict transform. This affine hypersurface has a singularity at the origin, and is smooth everywhere else. Blowing up the origin yields again a strict transform smooth in two charts, and singular in one:

$$-y^2 + sx(x + s^2 g(s^2 x)x + sh(s^2 x)) = 0.$$

This is a D_4 singularity. Blowing-up this singularity yields a strict transform with three A_1 singularities. The local equation in the chart obtained by the substitution $x = xs$, $y = ys$ is

$$-y^2 + xs(x(1 + s^2 g(s^3 x)) + h(s^3 x)) = 0.$$

The strict transform is singular at points where two of the factors of $xs(x(1+s^2 g(s^3 x)) + h(s^3 x))$ vanish, i.e., $(x, y, s) = (-h(0), 0, 0)$ and $(x, y, s) = (0, 0, 0)$. Both singularities are of type A_1. (Recall that we assumed that $h(0) \neq 0$.) The third singularity lies in the other chart with local equation

$$-y^2 + xs(1 + x^2 s^2 g(s^3 x^4) + sh(s^3 x^4)).$$

Here we have A_1-singularities at the points $(x, y, s) = (0, 0, 0)$ and $(x, y, s) = (0, 0, \frac{-1}{h(0)})$. The latter point is in the intersection of the two charts, hence we have three singularities in total. Blowing-up these three A_1 singularities yields a smooth model. Therefore the condition $h(0) \neq 0$ is sufficient to have a uniform desingularization. In particular, if we resolve the singularity at $((0:0:1),(0:1))$ then we obtain a family $\mathcal{X}_{m,n} \to \mathcal{U}_{m,n}$ of elliptic surfaces over $\mathbb{P}^1$.

For the final claim, take a closed point $b \in |\mathcal{U}_{m,n}|$ then the discriminant of the associated elliptic surface is smooth of degree $\max(3m, 2n)$ and therefore has precisely $\max(3m, 2n)$ singular fibers of type I_1 (over $\overline{K(b)}$) and at most one further singular fiber, which is at $t = \infty$ and whose type is determined by Lemma 3.10. The universal family over $\mathcal{U}_{m,n}$ has also a smooth discriminant of degree $\max(3m, 2n)$ and therefore over $\overline{K(\eta)}$ the same number of fibers of type I_1 and the same fiber type at $t = \infty$. □

Theorem 3.12. *Let K be a finitely generated field over $\mathbb{Q}$. Let m, n be positive integers. Let $\mathcal{X}_{m,n}$ be the universal elliptic curve over $\mathcal{B} = \mathcal{U}_{m,n}$. Then $E_\eta(K(\eta)(t))$ is finite.*

Proof. Fix a pair $(A, B) \in U_{m,n,K} \subset \mathcal{U}_{m,n}(K)$. Take now an element $d \in K^*$. Then the Weierstrass equation of $E^{(d)}_{A,B}$ equals

$$-YZ^2 + X^3 + d^2AXZ^2 + d^3Z^3.$$

In particular, we have $E^{(d)}_{A,B} \cong E_{d^2A,d^3B}$ as elliptic curves over $K(t)$.

Recall that from $(A, B) \in \mathcal{U}_{m,n}(K)$ it follows that $\Delta(A, B) := 4A^3 + 27B^2$ has degree $\max(3m, 2n)$ and is smooth. Since $\Delta(d^2A, d^3B) = d^6\Delta(A, B)$ holds, also $\Delta(d^2A, d^43B)$ is smooth and of degree $\max(3m, 2n)$. Hence $(d^2A, d^3B) \in \mathcal{U}_{m,n}(K)$ and therefore $\mathrm{tw}((A, B)) = K^*/(K^*)^2$.

From Remark 3.7 it follows that $E_{A,B}$ is not trivial for all $(A, B) \in \mathcal{U}_{m,n}(\overline{K})$ and the previous proposition shows that the configuration of singular fibers is constant on $\mathcal{U}_{m,n}$, hence we may apply Proposition 3.2 to conclude that $E_\eta(K(\eta)(t))$ is finite. □

Remark 3.13. We would like to recall that Cowan's conjecture is formulated over $\mathbb{Z}[t]$ rather than over $\mathbb{Q}[t]$, and for this reason we work with $\mathcal{O}_K[t]$ rather than with $K[t]$ in the sequel. This is only of minor relevance: Suppose $A, B \in K[t]$ then there exists a $u \in K^*$ such that both u^4A and u^6B are elements of $\mathcal{O}_K[t]$. Now $E_{A,B}$ and E_{u^4A,u^6B} are isomorphic, hence to study all elliptic surfaces over K we may restrict to $(A, B) \in S_{m,n,K}$.

Definition 3.14. For a polynomial $p(t) = a\prod_{i=1}^{d}(t - \alpha_i) \in \mathbb{C}[t]$ we define its Mahler measure $\mu(p)$ by

$$|a| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

In particular, $\mu(0) = 0$.

Fix positive integers n, m, d. For any $M \geq 0$ we define the finite set

$$P_{d,K}(M) = \{p \in \mathcal{O}_K[T] \mid \deg(p) \leq d,\ \max_{\tau \in M_K} \mu(\tau(p)) < M\},$$

where M_K is the set of embeddings $K \to \mathbb{C}$,

We define $S_{m,n,K}(M)$ to be the set

$$\left\{ E_{A,B} : y^2 = x^3 + A(t)x + B(t) \;\middle|\; \begin{array}{c} A \in P_{m,K}(M^2),\, B \in P_{n,K}(M^3), \\ 4A(t)^3 + 27B(t)^2 \neq 0 \end{array} \right\}.$$

Note that $S_{m,n,K}$ (as defined before) equals $\bigcup_M S_{m,n,K}(M)$.

Remark 3.15. Let $V \subset S_{m,n,K}$ be a subset and $f : V \to L$ then density of V in $S_{m,n,K}$ equals

$$\lim_{M\to\infty} \frac{\#V \cap S_{m,n,k}(M)}{\#S_{m,n,k}(M)}$$

if this limit exists and the average value of f on V equals

$$\lim_{M\to\infty} \frac{\sum_{x\in V\cap S_{m,n,k}(M)} f(x)}{\#V \cap S_{m,n,k}(M)}$$

whenever this limit exists.

In the following we will use results from [9] concerning the density of thin subsets. In these statements the measure on $\mathcal{O}_K[t]_m \times \mathcal{O}_K[t]_n$ is the naive height h. I.e., for $p \in \mathcal{O}_K[t]_d$ with $p(t) = \sum_{i=0}^d a_i t^i$ we have $h(p) = \max_{i,\sigma:K\to\mathbb{C}}\{|\sigma(a_i)|\}$. A standard exercise shows that

$$\binom{d}{\lfloor d/2 \rfloor}^{-1} h(p) \le \mu(p) \le \sqrt{d+1}\, h(p).$$

In particular, a subset of $S_{m,n,K}$ has density zero with respect to the Mahler measure if and only if it has density zero with respect to the naive height. This enables us to use the results from [9].

Corollary 3.16. *The set*

$$\{(A,B) \in S_{m,n,K} \mid \operatorname{rank} E_{A,B}(K(t)) = 0\}$$

is the complement of a thin set. In particular, this set has density one in $S_{m,n,K}$.

Proof. Combining Proposition 3.11 with Theorem 2.19 yields that there is a thin subset Z of $U_{m,n,K}$ such that for all $(A,B) \in U_{m,n,K} \setminus Z$ the rank of $E_{A,B}(K(t))$ vanishes. Now the complement of $U_{m,n,K}$ in $\mathcal{O}_K[t]_m \times \mathcal{O}_K[t]_n \cong \mathbb{A}^{m+n+2}(\mathcal{O}_K)$ is a Zariski closed subset. From [9, Theorem 13.1.3] it follows that $\mathcal{U}_{m,n}(\mathcal{O}_K) \setminus Z$ has density one in $\mathcal{O}_K[t]_m \times \mathcal{O}_K[t]_n$, and therefore also density one in the smaller set $S_{m,n,K}$. □

Let $s = \lfloor \min(m/4, n/6) \rfloor$. Define

$$Z^{tr}_{m,n,K} = \left\{ (\lambda u^4, \mu u^6) \mid \lambda, \mu \in \mathcal{O}_K, 4\lambda^3 + 27\mu^2 \neq 0, u \in \mathcal{O}_K[t]_s \setminus \{0\} \right\}.$$

Proposition 3.17. *Let K be a field. Let m,n be positive integers and $k = \lceil \max(m/4, n/6) \rceil$. Let $(A,B) \in S_{m,n,K} \setminus Z^{tr}_{m,n,K}$. Then*

$$\operatorname{rank} E_{A,B}(K(t)) \le 10k - 2.$$

Proof. Let $(A,B) \in S_{m,n,K}$. There exists a $u \in K[t]$ such that $(A/u^4, B/u^6)$ defines a minimal Weierstrass equation. From [7, Lemma IV.1.1] it follows that either $p_g(X) = k - 1 - \deg(u)$ or $X_{A,B}$ is trivial.

In the former case we may apply the Shioda–Tate formula (Proposition 2.4) and obtain $\operatorname{rank} E(\overline{K}(t)) \le h^{1,1} - 2$. From [7, Lemma IV.1.1] it follows that $h^{1,1} = 10(p_g + 1) \le 10k$ and we are done.

If $X_{A,B}$ is trivial then a minimal Weierstrass equation of the generic fiber $E_{A,B}/K(t)$ is

$$y^2 = x^3 + \lambda x + \mu,$$

with $\lambda, \mu \in K$. Every other short Weierstrass equation for E, such as the equation for $W_{A,B}$, is of the shape

$$y^2 = x^3 + \lambda u^4 x + \mu u^6,$$

with $u \in K[t]$. In particular $(A, B) \in Z^{tr}_{m,n,K}$, which we excluded. □

Corollary 3.18. *Let m, n be positive integers. Then the average rank of $S_{m,n,\mathbb{Q}}$ is zero.*

Proof. From Corollary 3.16 it follows that there is a thin subset $Z \subset S_{m,n,\mathbb{Q}}$ such that on $S_{m,n,\mathbb{Q}} \setminus Z$ the rank is zero. Let $Z' = Z \cap Z^{tr}_{m,n,\mathbb{Q}}$. Let $Z'' = Z \setminus Z'$. By [9, Theorem 13.1.1] both sets Z' and Z'' have density zero in $S_{m,n,\mathbb{Q}}$.

On Z'' the Mordell–Weil rank can be bounded by $10k - 2$ by Proposition 3.17. Since this set has density zero it does not contribute to the average rank.

It is not known whether the rank on Z' is bounded or not. However, from [3] it follows that on $Z^{tr}_{m,n,\mathbb{Q}}$ the average rank exists and is finite (cf. [2, Section 3]). Since this set has density zero, it does not contribute to the average rank. □

We will now generalize Cowan's conjecture to an arbitrary number field K. However we cannot control the average rank on $Z^{tr}_{m,n,K}$ as in the previous proof, so we need to exclude this density zero set.

Corollary 3.19. *Let m, n be positive integers. Let K be a number field with ring of integers $\mathcal{O}_K$. Then the average rank of $S_{m,n,K} \setminus Z^{tr}_{m,n,K}$ is zero.*

Proof. As in the proof of the previous corollary there exists a thin subset $Z \subset S_{m,n,K} \setminus Z^{tr}_{m,n,K}$ such that on $S_{m,n,K} \setminus Z$ the rank is zero. By Proposition 3.17 we can bound the Mordell–Weil rank on Z by $10k - 2$. From [9, Theorem 13.1.1] it follows that the density of Z equals zero. Hence the average rank is zero. □

This approach to Cowan's conjecture shows that if for a given member in the family there are many twists contained in the family then the average rank is zero. We will now give an example of a family where the average rank is positive.

Example 3.20. Fix a positive integer k. Consider the following subfamily $\mathcal{X}$ of $S_{2k,3k}$ given by

$$\{y^2 - x^3 + g^3 - h^2 \mid g \in \mathbb{Q}[t]_{2k}, h \in \mathbb{Q}[t]_{3k},\ g^3 - h^2 \text{ has } 6k \text{ distinct factors}\}.$$

In this case we have $\mathrm{tw}(b) = \{1\}$ for any $b \in |\mathcal{B}|$.

One then easily checks that every member of $\mathcal{X}$ has 6 fibers of type II, and therefore a torsion-free Mordell–Weil group. Moreover, the Mordell–Weil group contains a nontrivial point $(x, y) = (g, h)$. In particular, the average rank is at least one.

References

[1] Y. André, "Pour une théorie inconditionnelle des motifs", *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 5–49. MR

[2] F. Battistoni, S. Bettin, and C. Delaunay, "On the typical rank of elliptic curves over $\mathbb{Q}(T)$", *Res. Number Theory* **8**:4 (2022), art. id. 69. MR

[3] M. Bhargava and A. Shankar, "Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves", *Ann. of Math.* (2) **181**:1 (2015), 191–242. MR

[4] A. Cowan, "Conjecture: 100% of elliptic surfaces over $\mathbb{Q}$ have rank zero", pp. 335–342 in *Arithmetic geometry*, *number theory*, *and computation*, edited by J. S. Balakrishnan et al., Springer, 2021. MR

[5] D. A. Cox, "The Noether–Lefschetz locus of regular elliptic surfaces with section and $p_g \geq 2$", *Amer. J. Math.* **112**:2 (1990), 289–329. MR

[6] D. Maulik and B. Poonen, "Néron–Severi groups under specialization", *Duke Math. J.* **161**:11 (2012), 2167–2206. MR

[7] R. Miranda, "The basic theory of elliptic surfaces", course notes, University of Pisa, 1988, https://www.math.colostate.edu/~miranda/BTES-Miranda.pdf.

[8] M. Schütt and T. Shioda, *Mordell–Weil lattices*, Ergebnisse der Math. (3) **70**, Springer, 2019. MR

[9] J.-P. Serre, *Lectures on the Mordell–Weil theorem*, Aspects of Math. **E15**, Vieweg & Sohn, Braunschweig, Germany, 1989. MR

[10] J. Tate, "Algorithm for determining the type of a singular fiber in an elliptic pencil", pp. 33–52 in *Modular functions of one variable*, *IV* (Antwerp, 1972), edited by B. J. Birch and W. Kuyk, Lecture Notes in Math. **476**, Springer, 1975. MR

Communicated by Bjorn Poonen
Received 2022-04-29 Revised 2024-07-29 Accepted 2025-01-20

klooster@math.unipd.it *Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, 35121 Padova, Italy*

Algebra & Number Theory
msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2026 is US $590/year for the electronic version, and $865/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

Algebra & Number Theory

Volume 20 No. 1 2026