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function fields**

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# The Brauer–Manin obstruction for nonisotrivial curves over global function fields

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We prove that the set of rational points on a nonisotrivial curve of genus at least 2 over a global function field is equal to the set of adelic points cut out by the Brauer–Manin obstruction.

## 1. Introduction

Let  $X/K$  be a smooth projective and geometrically irreducible curve of genus at least 2 over a global field  $K$  of characteristic  $p > 0$ . We prove that if  $X$  is not isotrivial, then the Brauer–Manin obstruction cuts out exactly the set of rational points on  $X$ .

**Theorem 1.1.** *Let  $X/K$  be a smooth projective curve of genus at least 2 over a global function field  $K$ . If  $X$  is not isotrivial, then  $X(\mathbb{A}_K)^{\text{Br}} = X(K)$ .*

We refer the reader to [11] for the definition of the Brauer–Manin obstruction and the relevant background in this context. Theorem 1.1 is proved in that paper for  $X$  contained in an abelian variety  $A$  such that  $A(K^{\text{sep}})[p^\infty]$  is finite and no geometric isogeny factor of  $A$  is isotrivial. That result holds more generally for any “coset-free” subvariety of such an abelian variety over  $K$ . We remove the hypotheses on an abelian variety containing  $X$ , but our proof does not immediately extend to higher dimensional subvarieties of abelian varieties.

As in [11] our results are a consequence of related results concerning adelic intersections whose connection to the Brauer–Manin obstruction was first observed in [13] for curves over number fields.

**Theorem 1.2.** *Suppose  $X$  is a smooth, proper and nonisotrivial curve of genus at least 2 contained in an abelian variety  $A$  over a global field  $K$  of characteristic  $p > 0$ . Then  $X(\mathbb{A}_K) \cap \overline{A(K)} = X(K)$ , where  $\overline{A(K)}$  denotes the topological closure of  $A(K)$  in  $A(\mathbb{A}_K)$ .*

We follow the strategy of the proof in [11], but there are two new ingredients allowing us to remove all hypotheses on an abelian variety containing  $X$ . The first, appearing as Proposition 2.3, is based on ideas in the proof of the Mordell–Lang conjecture appearing in [1; 17]. This replaces the input from [6], which relies heavily on model theory and requires assumptions on the Jacobian of  $X$ . The second new ingredient is an isogeny constructed by Rössler in the Appendix to this paper. We use this instead of

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multiplication by  $p$  in some of the arguments appearing in [11] to prove Proposition 3.1. This removes the need for the hypothesis on  $A(K^{\text{sep}})[p^\infty]$  in [11, Proposition 5.3] and elsewhere.

The theorems above are expected to hold (in a slightly modified form) for any closed subvariety of an abelian variety over a global field. This was originally posed as a question in the case of curves over number fields by Scharaschkin [13] and, independently, by Skorobogatov [14]. It was later stated as a conjecture for curves over number fields in [10] and [15]. The number field case has seen little progress and remains wide open. Building on [11], this paper settles the function field analogue of these conjectures for nonisotrivial curves of genus  $\geq 2$ . Some partial results toward the conjecture in the isotrivial case are given in [3; 4; 5], but this case too remains open.

## 2. Zariski dense adelic points surviving $p^\infty$ -descent

In this section we assume  $X \subset A$  is a proper smooth curve of genus  $\geq 2$  contained in an abelian variety  $A$  over  $K$ .

**Definition 2.1.** Let  $N \geq 1$  be an integer. An  $N$ -covering of a subvariety  $X \subset A$  of an abelian variety  $A$  over  $K$  is an fppf-torsor  $Y \rightarrow X$  under the  $N$ -torsion subgroup scheme  $A[N]$  such that the base change of  $Y \rightarrow X$  to  $K^{\text{sep}}$  is isomorphic to the pull back of multiplication by  $N$  on  $A$ . An adelic point on  $X$  is said to survive  $N$ -descent if it lifts to an adelic point on some  $N$ -covering of  $X$ .

**Definition 2.2.** An adelic point  $(P_v)_v \in X(\mathbb{A}_K)$  is called Zariski dense if for any proper closed subvariety  $Y \subsetneq X$ , there exists  $v$  such that  $P_v \notin Y$ .

**Proposition 2.3.** *Suppose  $X \subset A$  is a proper smooth curve of genus at least 2 contained in an abelian variety  $A$  over a global field  $K$  of characteristic  $p > 0$ . If there is a Zariski dense adelic point on  $X$  which survives  $p^n$ -descent for all  $n \geq 1$ , then  $X$  is isotrivial.*

The proof of this proposition will be given at the end of this section.

**Definition 2.4.** Let  $L \subset K$  be a subfield. We say that  $X$  is defined over  $L$  if there exists  $X_0/L$  such that  $X \simeq X_0 \times_L K$ . We say that  $X$  is definable over  $L$  if there exists  $X_0/L$  such that  $X \times_K \bar{K} \simeq X_0 \times_L \bar{K}$ , where  $\bar{K}$  denotes an algebraic closure of  $K$  containing  $L$ .

For an abelian variety  $A/K$ , multiplication by  $p^n$  factors as

$$A \xrightarrow{F^n} A^{(p^n)} \xrightarrow{V^n} A,$$

where  $F^n$  and  $V^n$  are the  $n$ -fold compositions of the absolute Frobenius and Verschiebung isogenies. Recall that, for any  $n \geq 1$ ,  $K$  is a purely inseparable extension of degree  $p^n$  of its subfield  $K^{p^n} := \{a^{p^n} : a \in K\} \subset K$ . The abelian variety  $A^{(p^n)}$  is defined over  $K^{(p^n)}$ .

**Lemma 2.5.** *Suppose  $X$  contains a Zariski dense adelic point which lifts to a  $p^n$ -covering  $Y' \rightarrow X$  and let  $Y \rightarrow X$  be the torsor under  $\ker(V^n : A^{(p^n)} \rightarrow A)$  through which it factors. Then  $Y_{\text{red}}$  is geometrically reduced and definable over  $K^{p^n}$ .*

*Proof.* Let  $(P_v)_v \in X(\mathbb{A}_K)$  be the given adelic point and let  $Y' \rightarrow X$  be the  $p^n$ -covering to which  $(P_v)_v$  lifts. By passing to a separable extension of  $K$  (which is harmless thanks to the equality  $(K^{\text{sep}})^p \cap K = K^p$  and [7, Lemma 1.5.11]) we can assume  $Y' \rightarrow X$  is the pullback of multiplication by  $p^n$  on  $A$ . In particular, it factors through the  $n$ -fold Frobenius morphism  $F^n : A \rightarrow A^{(p^n)}$  and we have a commutative diagram with  $Y$  the torsor in the statement:

$$\begin{array}{ccccc}
 Y'_{\text{red}} & \longrightarrow & Y_{\text{red}} & \longrightarrow & X \\
 \downarrow & & \downarrow & & \parallel \\
 Y' & \longrightarrow & Y & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{F^n} & A^{(p^n)} & \xrightarrow{V^n} & A
 \end{array}$$

Let  $(Q_v)_v \in Y'(\mathbb{A}_K)$  denote a lift of  $(P_v)_v$ . For any  $v$ , the point  $Q_v : \text{Spec}(K_v) \rightarrow Y'$  factors through the reduced subscheme  $Y'_{\text{red}} \subset Y'$ , because  $\text{Spec}(K_v)$  is reduced. So  $(Q_v)_v$  is also a Zariski dense adelic point on  $Y'_{\text{red}}$ . Its image  $(R_v)_v$  in  $Y_{\text{red}}(\mathbb{A}_K)$  is a Zariski dense adelic point and by commutativity of the diagram the image of  $(R_v)_v$  in  $A^{(p^n)}$  lies in  $F^n(A(\mathbb{A}_K))$ . In particular, for each  $v$ , the point  $R_v$  lies in  $A^{(p^n)}(K_v^{p^n})$ . It then follows from the proof of [1, Lemma 1] that  $Y_{\text{red}}$  is defined over  $K^{p^n}$  and is geometrically reduced. Below is an alternative argument using [17], in particular, the last paragraph.

We show that  $Y_{\text{red}}$  is defined over  $K^{p^n}$  and is geometrically reduced. Assume  $n = 1$ , which is enough, as the argument can be repeated  $n$  times. Let  $U$  be an affine open subset of  $Y_{\text{red}}$  and  $f$  a function defined on an affine open set of  $A^{(p^n)}$  which vanishes on  $U$ . We have that  $f(R_v) = 0$  and differentiating this equation with respect to a derivation  $\delta$  on  $K$  with kernel  $K^p$ , gives  $f^\delta(R_v) = 0$ . Since  $(R_v)_v$  is Zariski dense on  $Y_{\text{red}}$ , we conclude that  $f^\delta$  also vanishes on  $U$ . This means that  $\delta$  extends to a vector field on a spreading out of  $Y_{\text{red}}$  and we conclude via [17, Lemma 1].  $\square$

**Remark 2.6.** From the above proof, if  $Y_{\text{red}}$  is not defined over  $K^p$ , some  $f^\delta$  does not vanish on  $Y_{\text{red}}$  and the equation  $f^\delta = 0$  defines a proper Zariski closed subset containing  $(R_v)_v$ .

**Lemma 2.7.** *If  $X' \rightarrow X$  is a torsor under an étale group scheme and  $X'$  is definable over  $K^{p^n}$ , then  $X$  is definable over  $K^{p^n}$ .*

*Proof.* Lemma 2 of [17] proves this for Galois covers. This gives the result, since taking a separable extension to trivialise the Galois action on the étale group scheme is harmless.  $\square$

**Lemma 2.8.** *Suppose  $Y_i \subset A_i$  are geometrically integral curves contained in abelian varieties  $A_i$  over  $K$ , for  $i = 1, 2$ . Suppose there is an isogeny  $A_1 \rightarrow A_2$  restricting to a generically purely inseparable map  $Y_1 \rightarrow Y_2$ . If  $Y_1$  and  $A_1$  are definable over  $K^{p^n}$ , then  $Y_2$  is definable over  $K^{p^n}$ .*

*Proof.* Passing to a finite separable extension we can assume  $Y_1$  is defined over  $K^{p^n}$ . In particular,  $Y_1$  is defined over  $K^p$ , so the argument in [1, Theorem A(2)] shows that  $Y_2$  is defined over  $K^p$ . Replacing  $K$  with  $K^p$  and repeating  $n$  times we find that  $Y_2$  is defined over  $K^{p^n}$ .  $\square$

*Proof of Proposition 2.3.* Let  $P := (P_v)_v \in X(\mathbb{A}_K)$  be a Zariski dense adelic point that survives  $p^n$ -descent for all  $n \geq 1$ .

Let  $n \geq 1$  and let  $Y' \rightarrow X$  be a  $p^n$ -covering to which  $P$  lifts. By Lemma 2.5,  $Y' \rightarrow X$  factors through a torsor  $Y \rightarrow X$  under the kernel of  $V^n : A^{(p^n)} \rightarrow A$ , with  $Y_{\text{red}}$  geometrically reduced and definable over  $K^{p^n}$ . We can factor  $V^n$  as  $V_e \circ V_c$ , with  $V_c$  an isogeny whose kernel is a connected abelian  $p$ -group scheme and  $V_e$  étale. Let  $Y \rightarrow X_e \rightarrow X$  be the corresponding factorisation of  $Y \rightarrow X$ . Since  $X_e \rightarrow X$  is étale and  $X$  is smooth,  $X_e$  is geometrically integral. The isogeny  $V_c$  restricts to a morphism  $Y_{\text{red}} \rightarrow X_e$  which is generically purely inseparable, so  $X_e$  is definable over  $K^{p^n}$  by Lemma 2.8. Then  $X$  is definable over  $K^{p^n}$  by Lemma 2.7.

Since  $P$  survives  $p^n$ -descent for all  $n$ , we conclude that  $X$  is definable over  $K^{p^n}$  for all  $n \geq 1$ . This implies that  $X$  is isotrivial (see the discussion in [16, Section 0]).  $\square$

### 3. Rational points on finite subschemes of abelian varieties

**Proposition 3.1.** *Let  $Z$  be a finite subscheme of an abelian variety  $A$  defined over a global function field  $K$ . Then*

$$Z(K) = Z(\mathbb{A}_K) \cap \overline{A(K)} = Z(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\text{Br}}.$$

*Proof.* By [11, Theorem E] we have  $Z(K) \subset \overline{A(K)} \subset A(\mathbb{A}_K)^{\text{Br}} \subset \widehat{\text{Sel}}(A)$ , where  $\widehat{\text{Sel}}(A) = \varprojlim \text{Sel}^n(A/k)$  is the projective limit of the  $n$ -Selmer groups of  $A$ . So it suffices to show that  $Z(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A) \subset Z(K)$ . As in the second paragraph of the proof of [11, Proposition 3.9], it suffices to show that this holds after a finite separable extension, so we can assume that  $Z$  consists of a finite set of  $K$ -points.

Replacing  $K$  by a further finite separable extension if needed, we can also assume that  $A[n]$  is a constant group scheme for some  $n$  prime to  $p$  and that the Néron model of  $A$  has semiabelian connected component. In the Appendix by D. Rössler it is shown that, under these hypotheses, there exists an étale isogeny  $f : A \rightarrow B$  and an isogeny  $g : B \rightarrow B$  of degree  $> 1$  such that  $\ker(g)(K^{\text{sep}}) = 0$ . Let  $W = f(Z) \subset B$ . If  $B(K^{\text{sep}})[p] = 0$ , then [11, Proposition 5.3] gives that  $W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B) = W(K)$ . Working with the given endomorphism  $g$  instead of multiplication by  $p$ , the argument there can be adapted to give the same conclusion (Details are given in Lemma 3.2 below).

Now suppose  $P \in Z(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A)$ . It follows from the definition of the Selmer groups that  $f(\widehat{\text{Sel}}(A)) \subset \widehat{\text{Sel}}(B)$ . So  $f(P) \in W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B) = W(K)$ . For any  $v \in \Omega_K$ , the  $v$ -adic component of  $P$  is the image of some  $Q_v \in Z(K)$ . The adelic point  $P - Q_v \in A(\mathbb{A}_K)$  lies in the kernel of  $f$  and in  $\widehat{\text{Sel}}(A)$ . So  $P - Q_v \in \widehat{\text{Sel}}(A)_{\text{tors}}$ . By [11, Lemma 5.1] this implies that  $P - Q_v \in A(K)$ . So  $P \in A(K)$ .  $\square$

Here are details of the claimed analogue of [11, Proposition 5.3] used in the proof above. We denote by  $B(K^{\text{sep}})$  the group of points on an abelian variety  $B/K$  defined over the separable closure  $K^{\text{sep}}$  of  $K$  and, for an endomorphism  $g$  of  $B$ , we denote by  $B(K^{\text{sep}})[g]$  the kernel of  $g$  acting on  $B(K^{\text{sep}})$ .

All cohomology below is fppf cohomology, i.e., faithfully flat and of finite presentation (also called “flat cohomology”; see [8], [9]).

**Lemma 3.2.** *Let  $W$  be a finite subscheme of an abelian variety  $B$  defined over  $K$ . Suppose there exists an endomorphism  $g : B \rightarrow B$  of degree  $> 1$  such that  $B(K^{\text{sep}})[g] = 0$ . Then  $W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B) = W(K)$ .*

*Proof.* By [11, Theorem E] we have

$$W(K) \subset B(K) \subset B(\mathbb{A}_K)^{\text{Br}} \subset \widehat{\text{Sel}}(B).$$

So it suffices to show that  $W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B) \subset W(K)$ . Moreover we can assume  $W = W(K)$  as in [11, Proposition 3.9].

Suppose  $P \in W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B)$ . For any  $v \in \Omega_K$ , the  $v$ -adic component of  $P$  is the image of some point  $Q_v \in W(K)$ , and  $P - Q_v \in \widehat{\text{Sel}}(B)$  maps to 0 in  $B(K_v)^{(g)} := \varprojlim_n B(K_v)/g^n(B(K_v))$ . In particular,  $P - Q_v$  is in the kernel of  $\widehat{\text{Sel}}^{(g)}(B) \rightarrow B(K_v)^{(g)}$  where  $\widehat{\text{Sel}}^{(g)}(B)$  denotes the inverse limit of the Selmer groups corresponding to the isogenies  $g^n$  for  $n \geq 1$ . Below we show that this map is injective, so the image of  $P - Q_v$  in  $\widehat{\text{Sel}}^{(g)}(B)$  is 0.

Since this holds for any  $v$ , if  $v'$  is any other prime we have

$$Q_{v'} - Q_v \in \ker(B(K) \rightarrow \varprojlim_n B(K)/g^n B(K) \hookrightarrow \widehat{\text{Sel}}^{(g)}(B)).$$

In other words,  $(Q_{v'} - Q_v) \in \bigcap_{n \geq 1} g^n B(K)$ . Since  $B(K)$  is finitely generated, this implies that  $(Q_{v'} - Q_v) \in B(K)_{\text{tors}}$ . Again, since this holds for all  $v$  we see that  $R := P - Q_v \in \widehat{\text{Sel}}(B)_{\text{tors}}$ . By [11, Lemma 5.1] this implies that  $P - Q_v \in B(K)$ . So  $P \in W(K)$ .

It remains to prove that  $\widehat{\text{Sel}}^{(g)}(B) \rightarrow B(K_v)^{(g)}$  is injective. For this it suffices (as in the proof of [11, Proposition 5.2]) to prove injectivity of  $\widehat{\text{Sel}}'^{(g)}(B) \rightarrow B(K'_v)^{(g)}$ , where  $K'_v \subset K^{\text{sep}}$  denotes the Henselisation with respect to  $v$  and  $\widehat{\text{Sel}}'^{(g)}(B)$  is defined in the same way as  $\widehat{\text{Sel}}^{(g)}(B)$  but using  $K'_v$  instead of  $K_v$ . Let  $b \in \ker(\widehat{\text{Sel}}'^{(g)}(B) \rightarrow B(K'_v)^{(g)})$  and, for each integer  $M \geq 1$ , let  $b_M$  denote the image of  $b$  in  $\text{Sel}'^{g^M}(B) \subset H^1(K'_v, B[g^M])$  (where  $B[g^M]$  is the kernel of  $g^M$  acting on  $B$ ). Then the image of  $b_M$  under

$$\text{Sel}'^{g^M}(B) \rightarrow \frac{B(K'_v)}{g^M B(K'_v)} \subset H^1(K'_v, B[g^M]) \rightarrow H^1(K^{\text{sep}}, B[g^M])$$

is 0. The inflation-restriction sequence

$$0 \rightarrow H^1(\text{Gal}(K^{\text{sep}}/K), B(K^{\text{sep}})[g^M]) \rightarrow H^1(K, B[g^M]) \rightarrow H^1(K^{\text{sep}}, B[g^M])$$

shows that  $b_M$  comes from an element of  $H^1(\text{Gal}(K^{\text{sep}}/K), B(K^{\text{sep}})[g^M])$ . But this group is trivial since  $B(K^{\text{sep}})[g^M] = 0$ . Since this holds for all  $M \geq 1$ , we conclude that  $b = 0$ .  $\square$

#### 4. Proofs of the theorems

*Proof of Theorem 1.2.* By [11, Theorem E] we have

$$\overline{A(K)} \subset A(\mathbb{A}_K)^{\text{Br}} \subset \widehat{\text{Sel}}(A) \subset A(\mathbb{A}_K).$$

Intersecting with  $X(\mathbb{A}_K)$  we have

$$X(\mathbb{A}_K) \cap \overline{A(K)} \subset X(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\text{Br}} \subset X(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A),$$

where the rightmost set consists of the adelic points on  $X$  which survive  $N$ -descent for all  $N \geq 1$  (relative to the embedding  $X \subset A$  as in Definition 2.1). In particular, any  $P \in X(\mathbb{A}_K) \cap \overline{A(K)}$  survives  $p^n$ -descent for all  $n \geq 1$ . Since  $X$  is not isotrivial, Proposition 2.3 implies that there is a finite subscheme  $Z \subset X \subset A$  which contains  $P$ . Then  $P \in Z(\mathbb{A}_K) \cap \overline{A(K)} = Z(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\text{Br}} = X(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A) = Z(K)$ , where the final equality is Proposition 3.1.  $\square$

**Remark 4.1.** The preceding proof shows that for  $X \subset A$  as in Theorem 1.2 we have

$$X(K) = X(\mathbb{A}_K) \cap \overline{A(K)} = X(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\text{Br}} = X(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A).$$

*Proof of Theorem 1.1.* Let  $X/K$  be as in the statement and let  $J = \text{Jac}(X)$  be its Jacobian. It suffices to show that  $X(\mathbb{A}_K)^{\text{Br}} \subset X(K)$ . Passing to some finite separable extension  $L/K$  we can embed  $X_L$  in  $J_L$  via the Abel–Jacobi map corresponding to an  $L$ -rational point. If  $P \in X(\mathbb{A}_K)^{\text{Br}}$ , then its image under the inclusion  $X(\mathbb{A}_K) \subset X(\mathbb{A}_L) = X_L(\mathbb{A}_L)$  is orthogonal to  $\text{Br}(X_L)$  by [2, Lemma 3.1]. By functoriality of the Brauer pairing and Remark 4.1 we have  $X_L(\mathbb{A}_L)^{\text{Br}} \subset X_L(\mathbb{A}_L) \cap J_L(\mathbb{A}_L)^{\text{Br}} = X_L(L)$ . Then  $P$  is in  $X(\mathbb{A}_K) \cap X(L)$  which is equal to  $X(K)$  by [11, Lemma 3.2].  $\square$

**Remark 4.2.** In the proof of Theorem 1.1 just given [2, Lemma 3.1] and [11, Lemma 3.2] allow us to pass to an extension over which  $X$  can be embedded in its Jacobian. Alternatively one can use the following construction suggested to one of us by Poonen. Restriction of scalars gives a map  $\text{Res}_{L/K}(X_L) \rightarrow \text{Res}_{L/K}(J_L)$ . Composing this with the canonical map  $X \rightarrow \text{Res}_{L/K}(X_L)$  gives a closed immersion  $X \rightarrow A$  into the abelian variety  $A := \text{Res}_{L/K}(J_L)$  over  $K$ . To prove Theorem 1.1 one can then apply Remark 4.1 to  $X \subset A$ .

## Appendix: On abelian varieties with an infinite group of separable $p^\infty$ -torsion points

by Damian Rössler

If  $n \in \mathbb{N}$ , we write  $[n]$  for the multiplication by  $n$  endomorphism on an abelian variety. If  $h$  is an endomorphism with finite kernel of an abelian variety  $A$  over a field  $L$ , we write

$$A(L)[h^\ell] := \{x \in A(L) \mid h^{\circ \ell}(x) = 0\}$$

and

$$A(L)[h^\infty] := \{x \in A(L) \mid \exists n \in \mathbb{N} : h^{\circ n}(x) = 0\}.$$

Here  $h^{\circ n}(x) := h(h(\dots(x)\dots))$ , where there are  $n$  pairs of brackets. The notation  $A(L)[n^\ell]$  (resp.  $A(L)[n^\infty]$ ) will be a shorthand for  $A(L)[[n]^\ell]$  (resp.  $A(L)[[n]^\infty]$ ).

Now let  $K_0$  be the function field of a smooth and proper curve  $U$  over a finite field  $\mathbb{F}$  of characteristic  $p > 0$ . Let  $B$  be an abelian variety over  $K_0$ . Suppose that for some  $n > 3$  prime to  $p$ , the group scheme  $B[n]$  is constant and that the Néron model of  $B$  over  $U$  has a semiabelian connected component.

**Proposition A.1.** *There exists an abelian variety  $C$  over  $K_0$ , an étale  $K_0$ -isogeny  $\phi : B \rightarrow C$ , an étale  $K_0$ -isogeny  $f : C \rightarrow C$ , a  $K_0$ -isogeny  $g : C \rightarrow C$ , and a natural number  $r \geq 0$  such that*

- (a)  $g \circ f = [p^r]$  and  $g \circ f = f \circ g$ ;
- (b)  $C(K_0^{\text{sep}})[p^\infty] = C(K_0^{\text{sep}})[f^\infty] = C(\bar{K}_0)[f^\infty]$ ;
- (c)  $C(K_0^{\text{sep}})[g^\infty] = 0$ .

*Proof.* For  $\ell \geq 0$ , define inductively

$$B_0 := B$$

and

$$B_{\ell+1} := B_\ell / (B_\ell(K_0^{\text{sep}})[p]).$$

For  $\ell_2 \geq \ell_1$ , let  $\phi_{\ell_1, \ell_2} : B_{\ell_1} \rightarrow B_{\ell_2}$  be the (étale!) morphism obtained by composing the natural morphisms  $B_{\ell_1} \rightarrow B_{\ell_1+1} \rightarrow \cdots \rightarrow B_{\ell_2}$ . We first claim that

$$(\ker \phi_{\ell_1, \ell_2})(K_0^{\text{sep}}) = B_{\ell_1}(K_0^{\text{sep}})[p^{\ell_2 - \ell_1}] \quad (\text{A-1})$$

We prove the claim by induction on  $\ell_2 - \ell_1$ . For  $\ell_2 - \ell_1 \leq 1$ , the claim is true by definition. Suppose that  $\ell_2 - \ell_1 \geq 1$ . Let  $x \in B(K_0^{\text{sep}})[p^{\ell_2 - \ell_1}]$ . Then  $[p^{\ell_2 - \ell_1 - 1}](x) \in B(K_0^{\text{sep}})[p]$  and thus

$$\phi_{\ell_1, \ell_1+1}([p^{\ell_2 - \ell_1 - 1}](x)) = [p^{\ell_2 - \ell_1 - 1}](\phi_{\ell_1, \ell_1+1}(x)) = 0.$$

Applying the inductive assumption to  $\phi_{\ell_1, \ell_1+1}(x)$ , we see that  $\phi_{\ell_1+1, \ell_2}(\phi_{\ell_1, \ell_1+1}(x)) = \phi_{\ell_1, \ell_2}(x) = 0$ . This proves that  $(\ker \phi_{\ell_1, \ell_2})(K_0^{\text{sep}}) \supseteq B_{\ell_1}(K_0^{\text{sep}})[p^{\ell_2 - \ell_1}]$ . To prove the opposite inclusion, let  $x \in (\ker \phi_{\ell_1, \ell_2})(K_0^{\text{sep}})$ . We compute

$$\phi_{\ell_1, \ell_2}(x) = \phi_{\ell_1+1, \ell_2}(\phi_{\ell_1, \ell_1+1}(x)) = 0,$$

which implies (by the inductive hypothesis) that

$$[p^{\ell_2 - \ell_1 - 1}](\phi_{\ell_1, \ell_1+1}(x)) = \phi_{\ell_1, \ell_1+1}([p^{\ell_2 - \ell_1 - 1}](x)) = 0,$$

which in turn implies that  $[p]([p^{\ell_2 - \ell_1 - 1}](x)) = [p^{\ell_2 - \ell_1}](x) = 0$ . This proves that  $(\ker \phi_{\ell_1, \ell_2})(K_0^{\text{sep}}) \subseteq B_{\ell_1}(K_0^{\text{sep}})[p^{\ell_2 - \ell_1}]$  and completes the proof of the claim.

Now we know that by the reasoning made in the last page of [12], that there are only finitely many isomorphism classes of abelian varieties over  $K_0$  in the sequence  $\{B_\ell\}_{\ell \in \mathbb{N}}$ . Let  $C$  be an abelian variety over  $K_0$ , which appears at least twice in the sequence  $\{B_\ell\}_{\ell \in \mathbb{N}}$ . Let  $n_2 > n_1$  be such that  $C \simeq B_{n_1} \simeq B_{n_2}$ . Then by construction (under the identification  $C = B_{n_1}$ )

$$\phi_{n_1, n_2}^{\text{ol}} = \phi_{n_1, n_1 + \ell \cdot (n_2 - n_1)} \quad (\text{A-2})$$

for any  $\ell \geq 1$  and thus

$$C(K_0^{\text{sep}})[p^\infty] = C(K_0^{\text{sep}})[\phi_{n_1, n_2}^\infty] \quad (\text{A-3})$$

Now define  $f := \phi_{n_1, n_2}$  and  $r := n_2 - n_1$ . Define  $g$  as the only  $K_0$ -isogeny such that  $g \circ f = [p^r]$ .

Notice then that the identity  $g \circ f = [p^r]$  implies the identity  $f \circ g = [p^r]$ . To see this last fact directly, recall first that there are natural injection of rings

$$\text{End}_{K_0}(C) \hookrightarrow \text{End}_{\bar{K}_0}(C_{\bar{K}_0}) \hookrightarrow \text{End}_{\mathbb{Z}_t}(T_t(C(\bar{K}_0))) \hookrightarrow \text{End}_{\mathbb{Q}_t}(T_t(C(\bar{K}_0)) \otimes \mathbb{Q}_t)$$

where  $T_t(C(\bar{K}_0))$  is the classical Tate module of  $C_{\bar{K}_0}$  and  $t > 0$  is some prime number  $\neq p$ . Now if  $M$  and  $N$  are two square matrices of the same size with coefficients in a field of characteristic 0, such that  $M \cdot N = p^r$ , then  $p^{-r}N$  is the inverse matrix of  $M$  and thus  $N \cdot M = p^r$ . This fact combined with the existence of the above injections implies that  $f \circ g = [p^r]$  if  $g \circ f = [p^r]$ .

We have already proven (a). Point (b) is contained in (A-3).

We now prove (c). Suppose that for some  $\ell \geq 0$  and some  $x \in C(K_0^{\text{sep}})$ , we have  $g^{\text{ol}}(x) = 0$ . Let  $y \in (f^{\text{ol}})^{-1}(x) \subseteq C(K_0^{\text{sep}})$ . Then  $g^{\text{ol}}(f^{\text{ol}}(y)) = [p^{\ell}](y) = 0$ . Hence  $f^{\text{ol}}(y) = 0 = x$  by (A-1) and (A-2).  $\square$

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### References

- [1] D. Abramovich and J. F. Voloch, “Toward a proof of the Mordell–Lang conjecture in characteristic  $p$ ”, *Int. Math. Res. Not.* **1992**:5 (1992), 103–115. MR
- [2] B. Creutz and B. Viray, “Quadratic points on intersections of two quadrics”, *Algebra Number Theory* **17**:8 (2023), 1411–1452. MR
- [3] B. Creutz and J. F. Voloch, “The Brauer–Manin obstruction for constant curves over global function fields”, *Ann. Inst. Fourier (Grenoble)* **72**:1 (2022), 43–58. MR
- [4] B. Creutz and J. F. Voloch, “Étale descent obstruction and anabelian geometry of curves over finite fields”, *Épjournal Géom. Algébrique* **8** (2024), art. id. 10. MR
- [5] B. Creutz, J. Pajwani, and J. F. Voloch, “Galois invariants of finite abelian descent and Brauer sets”, *Bull. Lond. Math. Soc.* **56**:10 (2024), 3240–3256. MR
- [6] E. Hrushovski, “The Mordell–Lang conjecture for function fields”, *J. Amer. Math. Soc.* **9**:3 (1996), 667–690. MR
- [7] B. S. Huggins, *Fields of moduli and fields of definition of curves*, Ph.D. thesis, University of California, Berkeley, 2005, available at <https://www.proquest.com/docview/305031041>. MR
- [8] J. S. Milne, *Étale cohomology*, Princeton Math. Ser. **33**, Princeton Univ. Press, 1980. MR
- [9] J. S. Milne, *Arithmetic duality theorems*, Perspect. Math. **1**, Academic Press, Boston, MA, 1986. MR
- [10] B. Poonen, “Heuristics for the Brauer–Manin obstruction for curves”, *Exp. Math.* **15**:4 (2006), 415–420. MR
- [11] B. Poonen and J. F. Voloch, “The Brauer–Manin obstruction for subvarieties of abelian varieties over function fields”, *Ann. of Math. (2)* **171**:1 (2010), 511–532. MR
- [12] D. Rössler, “Infinitely  $p$ -divisible points on abelian varieties defined over function fields of characteristic  $p > 0$ ”, *Notre Dame J. Form. Log.* **54**:3-4 (2013), 579–589. MR

- [13] V. Scharaschkin, *Local-global problems and the Brauer–Manin obstruction*, Ph.D. thesis, University of Michigan, 1999, available at <https://www.proquest.com/docview/304517948>. MR
- [14] A. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Math. **144**, Cambridge Univ. Press, 2001. MR
- [15] M. Stoll, “Finite descent obstructions and rational points on curves”, *Algebra Number Theory* **1**:4 (2007), 349–391. MR
- [16] L. Szpiro, “Propriétés numériques du faisceau dualisant relatif”, pp. 44–78 in *Seminar on pencils of curves of genus at least two*, Astérisque **86**, Soc. Math. France, Paris, 1981. MR
- [17] J. F. Voloch, “On the conjectures of Mordell and Lang in positive characteristics”, *Invent. Math.* **104**:3 (1991), 643–646. MR

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