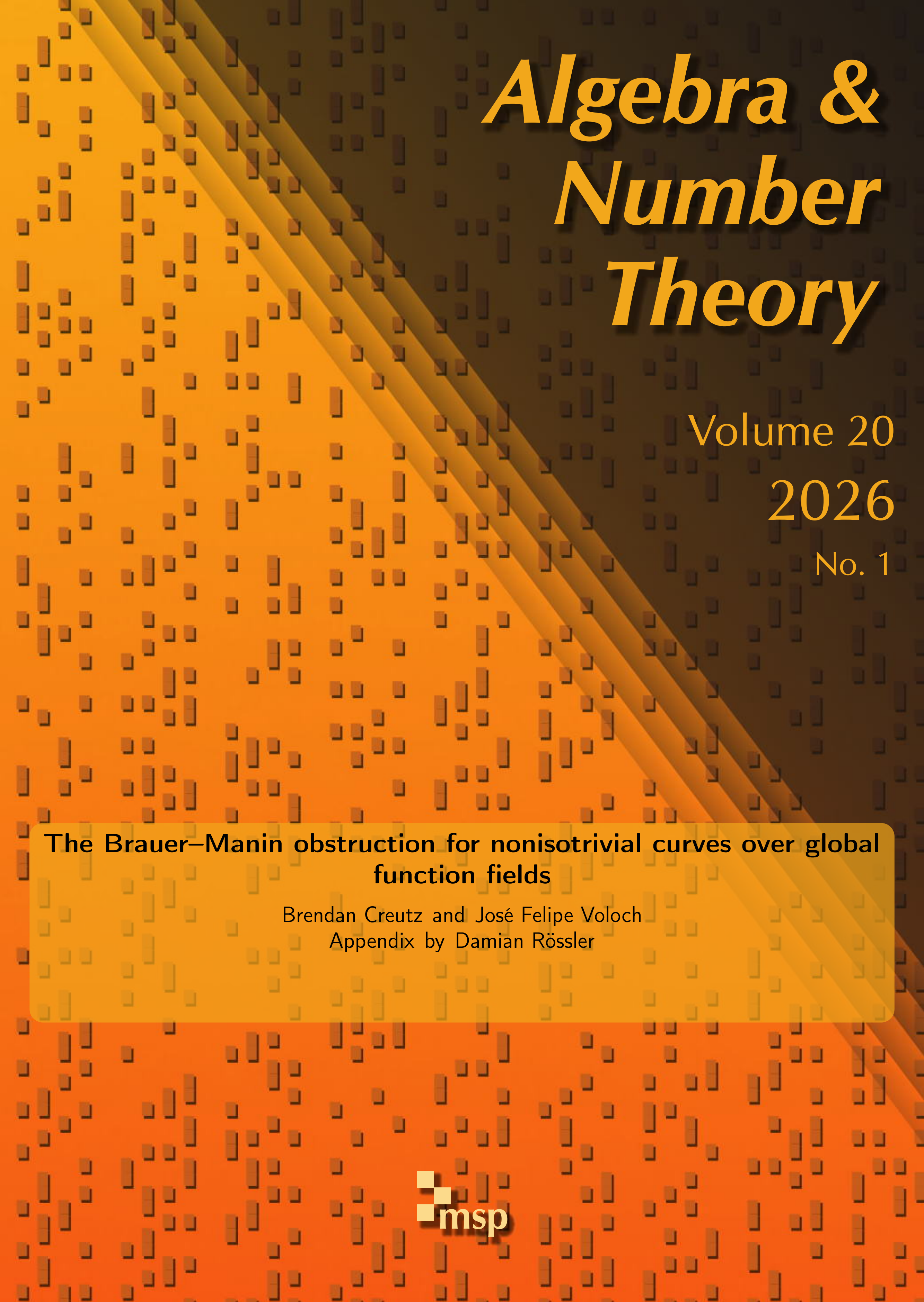

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Appendix by Damian Rössler

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The Brauer–Manin obstruction for nonisotrivial curves over global function fields

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We prove that the set of rational points on a nonisotrivial curve of genus at least 2 over a global function field is equal to the set of adelic points cut out by the Brauer–Manin obstruction.

1. Introduction

Let X/K be a smooth projective and geometrically irreducible curve of genus at least 2 over a global field K of characteristic $p > 0$. We prove that if X is not isotrivial, then the Brauer–Manin obstruction cuts out exactly the set of rational points on X.

Theorem 1.1. *Let X/K be a smooth projective curve of genus at least 2 over a global function field K. If X is not isotrivial, then $X(\mathbb{A}_K)^{\mathrm{Br}} = X(K)$.*

We refer the reader to [11] for the definition of the Brauer–Manin obstruction and the relevant background in this context. Theorem 1.1 is proved in that paper for X contained in an abelian variety A such that $A(K^{\mathrm{sep}})[p^\infty]$ is finite and no geometric isogeny factor of A is isotrivial. That result holds more generally for any "coset-free" subvariety of such an abelian variety over K. We remove the hypotheses on an abelian variety containing X, but our proof does not immediately extend to higher dimensional subvarieties of abelian varieties.

As in [11] our results are a consequence of related results concerning adelic intersections whose connection to the Brauer–Manin obstruction was first observed in [13] for curves over number fields.

Theorem 1.2. *Suppose X is a smooth, proper and nonisotrivial curve of genus at least 2 contained in an abelian variety A over a global field K of characteristic $p > 0$. Then $X(\mathbb{A}_K) \cap \overline{A(K)} = X(K)$, where $\overline{A(K)}$ denotes the topological closure of $A(K)$ in $A(\mathbb{A}_K)$.*

We follow the strategy of the proof in [11], but there are two new ingredients allowing us to remove all hypotheses on an abelian variety containing X. The first, appearing as Proposition 2.3, is based on ideas in the proof of the Mordell–Lang conjecture appearing in [1; 17]. This replaces the input from [6], which relies heavily on model theory and requires assumptions on the Jacobian of X. The second new ingredient is an isogeny constructed by Rössler in the Appendix to this paper. We use this instead of

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multiplication by p in some of the arguments appearing in [11] to prove Proposition 3.1. This removes the need for the hypothesis on $A(K^{\mathrm{sep}})[p^\infty]$ in [11, Proposition 5.3] and elsewhere.

The theorems above are expected to hold (in a slightly modified form) for any closed subvariety of an abelian variety over a global field. This was originally posed as a question in the case of curves over number fields by Scharaschkin [13] and, independently, by Skorobogatov [14]. It was later stated as a conjecture for curves over number fields in [10] and [15]. The number field case has seen little progress and remains wide open. Building on [11], this paper settles the function field analogue of these conjectures for nonisotrivial curves of genus ≥ 2. Some partial results toward the conjecture in the isotrivial case are given in [3; 4; 5], but this case too remains open.

2. Zariski dense adelic points surviving p^∞-descent

In this section we assume $X \subset A$ is a proper smooth curve of genus ≥ 2 contained in an abelian variety A over K.

Definition 2.1. Let $N \geq 1$ be an integer. An N-covering of a subvariety $X \subset A$ of an abelian variety A over K is an fppf-torsor $Y \to X$ under the N-torsion subgroup scheme $A[N]$ such that the base change of $Y \to X$ to K^{sep} is isomorphic to the pull back of multiplication by N on A. An adelic point on X is said to survive N-descent if it lifts to an adelic point on some N-covering of X.

Definition 2.2. An adelic point $(P_v)_v \in X(\mathbb{A}_K)$ is called Zariski dense if for any proper closed subvariety $Y \subsetneq X$, there exists v such that $P_v \notin Y$.

Proposition 2.3. *Suppose $X \subset A$ is a proper smooth curve of genus at least 2 contained in an abelian variety A over a global field K of characteristic $p > 0$. If there is a Zariski dense adelic point on X which survives p^n-descent for all $n \geq 1$, then X is isotrivial.*

The proof of this proposition will be given at the end of this section.

Definition 2.4. Let $L \subset K$ be a subfield. We say that X is defined over L if there exists X_0/L such that $X \simeq X_0 \times_L K$. We say that X is definable over L if there exists X_0/L such that $X \times_K \overline{K} \simeq X_0 \times_L \overline{K}$, where $\overline{K}$ denotes an algebraic closure of K containing L.

For an abelian variety A/K, multiplication by p^n factors as

$$A \xrightarrow{F^n} A^{(p^n)} \xrightarrow{V^n} A\,,$$

where F^n and V^n are the n-fold compositions of the absolute Frobenius and Verschiebung isogenies. Recall that, for any $n \geq 1$, K is a purely inseparable extension of degree p^n of its subfield $K^{p^n} := \{a^{p^n} : a \in K\} \subset K$. The abelian variety $A^{(p^n)}$ is defined over $K^{(p^n)}$.

Lemma 2.5. *Suppose X contains a Zariski dense adelic point which lifts to a p^n-covering $Y' \to X$ and let $Y \to X$ be the torsor under* $\ker(V^n : A^{(p^n)} \to A)$ *through which it factors. Then Y_{red} is geometrically reduced and definable over K^{p^n}.*

Proof. Let $(P_v)_v \in X(\mathbb{A}_K)$ be the given adelic point and let $Y' \to X$ be the p^n-covering to which $(P_v)_v$ lifts. By passing to a separable extension of K (which is harmless thanks to the equality $(K^{\text{sep}})^p \cap K = K^p$ and [7, Lemma 1.5.11]) we can assume $Y' \to X$ is the pullback of multiplication by p^n on A. In particular, it factors through the n-fold Frobenius morphism $F^n : A \to A^{(p^n)}$ and we have a commutative diagram with Y the torsor in the statement:

$$\begin{tikzcd}
Y'_{\text{red}} \arrow[r] \arrow[d] & Y_{\text{red}} \arrow[r] \arrow[d] & X \arrow[d, equal] \\
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \\
A \arrow[r, "F^n"] & A^{(p^n)} \arrow[r, "V^n"] & A
\end{tikzcd}$$

Let $(Q_v)_v \in Y'(\mathbb{A}_K)$ denote a lift of $(P_v)_v$. For any v, the point $Q_v : \text{Spec}(K_v) \to Y'$ factors through the reduced subscheme $Y'_{\text{red}} \subset Y'$, because $\text{Spec}(K_v)$ is reduced. So $(Q_v)_v$ is also a Zariski dense adelic point on Y'_{red}. Its image $(R_v)_v$ in $Y_{\text{red}}(\mathbb{A}_K)$ is a Zariski dense adelic point and by commutativity of the diagram the image of $(R_v)_v$ in $A^{(p^n)}$ lies in $F^n(A(\mathbb{A}_K))$. In particular, for each v, the point R_v lies in $A^{(p^n)}(K_v^{p^n})$. It then follows from the proof of [1, Lemma 1] that Y_{red} is defined over K^{p^n} and is geometrically reduced. Below is an alternative argument using [17], in particular, the last paragraph.

We show that Y_{red} is defined over K^{p^n} and is geometrically reduced. Assume $n = 1$, which is enough, as the argument can be repeated n times. Let U be an affine open subset of Y_{red} and f a function defined on an affine open set of $A^{(p^n)}$ which vanishes on U. We have that $f(R_v) = 0$ and differentiating this equation with respect to a derivation δ on K with kernel K^p, gives $f^\delta(R_v) = 0$. Since $(R_v)_v$ is Zariski dense on Y_{red}, we conclude that f^δ also vanishes on U. This means that δ extends to a vector field on a spreading out of Y_{red} and we conclude via [17, Lemma 1]. ☐

Remark 2.6. From the above proof, if Y_{red} is not defined over K^p, some f^δ does not vanish on Y_{red} and the equation $f^\delta = 0$ defines a proper Zariski closed subset containing $(R_v)_v$.

Lemma 2.7. *If $X' \to X$ is a torsor under an étale group scheme and X' is definable over K^{p^n}, then X is definable over K^{p^n}.*

Proof. Lemma 2 of [17] proves this for Galois covers. This gives the result, since taking a separable extension to trivialise the Galois action on the étale group scheme is harmless. ☐

Lemma 2.8. *Suppose $Y_i \subset A_i$ are geometrically integral curves contained in abelian varieties A_i over K, for $i = 1, 2$. Suppose there is an isogeny $A_1 \to A_2$ restricting to a generically purely inseparable map $Y_1 \to Y_2$. If Y_1 and A_1 are definable over K^{p^n}, then Y_2 is definable over K^{p^n}.*

Proof. Passing to a finite separable extension we can assume Y_1 is defined over K^{p^n}. In particular, Y_1 is defined over K^p, so the argument in [1, Theorem A(2)] shows that Y_2 is defined over K^p. Replacing K with K^p and repeating n times we find that Y_2 is defined over K^{p^n}. ☐

Proof of Proposition 2.3. Let $P := (P_v)_v \in X(\mathbb{A}_K)$ be a Zariski dense adelic point that survives p^n-descent for all $n \geq 1$.

Let $n \geq 1$ and let $Y' \to X$ be a p^n-covering to which P lifts. By Lemma 2.5, $Y' \to X$ factors through a torsor $Y \to X$ under the kernel of $V^n : A^{(p^n)} \to A$, with Y_{red} geometrically reduced and definable over K^{p^n}. We can factor V^n as $V_e \circ V_c$, with V_c an isogeny whose kernel is a connected abelian p-group scheme and V_e étale. Let $Y \to X_e \to X$ be the corresponding factorisation of $Y \to X$. Since $X_e \to X$ is étale and X is smooth, X_e is geometrically integral. The isogeny V_c restricts to a morphism $Y_{\text{red}} \to X_e$ which is generically purely inseparable, so X_e is definable over K^{p^n} by Lemma 2.8. Then X is definable over K^{p^n} by Lemma 2.7.

Since P survives p^n-descent for all n, we conclude that X is definable over K^{p^n} for all $n \geq 1$. This implies that X is isotrivial (see the discussion in [16, Section 0]). ☐

3. Rational points on finite subschemes of abelian varieties

Proposition 3.1. *Let Z be a finite subscheme of an abelian variety A defined over a global function field K. Then*

$$Z(K) = Z(\mathbb{A}_K) \cap \overline{A(K)} = Z(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\text{Br}}.$$

Proof. By [11, Theorem E] we have $Z(K) \subset \overline{A(K)} \subset A(\mathbb{A}_K)^{\text{Br}} \subset \widehat{\text{Sel}}(A)$, where $\widehat{\text{Sel}}(A) = \varprojlim \text{Sel}^n(A/k)$ is the projective limit of the n-Selmer groups of A. So it suffices to show that $Z(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A) \subset Z(K)$. As in the second paragraph of the proof of [11, Proposition 3.9], it suffices to show that this holds after a finite separable extension, so we can assume that Z consists of a finite set of K-points.

Replacing K by a further finite separable extension if needed, we can also assume that $A[n]$ is a constant group scheme for some n prime to p and that the Néron model of A has semiabelian connected component. In the Appendix by D. Rössler it is shown that, under these hypotheses, there exists an étale isogeny $f : A \to B$ and an isogeny $g : B \to B$ of degree > 1 such that $\ker(g)(K^{\text{sep}}) = 0$. Let $W = f(Z) \subset B$. If $B(K^{\text{sep}})[p] = 0$, then [11, Proposition 5.3] gives that $W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B) = W(K)$. Working with the given endomorphism g instead of multiplication by p, the argument there can be adapted to give the same conclusion (Details are given in Lemma 3.2 below).

Now suppose $P \in Z(\mathbb{A}_K) \cap \widehat{\text{Sel}}(A)$. It follows from the definition of the Selmer groups that $f\left(\widehat{\text{Sel}}(A)\right) \subset \widehat{\text{Sel}}(B)$. So $f(P) \in W(\mathbb{A}_K) \cap \widehat{\text{Sel}}(B) = W(K)$. For any $v \in \Omega_K$, the v-adic component of P is the image of some $Q_v \in Z(K)$. The adelic point $P - Q_v \in A(\mathbb{A}_K)$ lies in the kernel of f and in $\widehat{\text{Sel}}(A)$. So $P - Q_v \in \widehat{\text{Sel}}(A)_{\text{tors}}$. By [11, Lemma 5.1] this implies that $P - Q_v \in A(K)$. So $P \in A(K)$. ☐

Here are details of the claimed analogue of [11, Proposition 5.3] used in the proof above. We denote by $B(K^{\text{sep}})$ the group of points on an abelian variety B/K defined over the separable closure K^{sep} of K and, for an endomorphism g of B, we denote by $B(K^{\text{sep}})[g]$ the kernel of g acting on $B(K^{\text{sep}})$.

All cohomology below is fppf cohomology, i.e., faithfully flat and of finite presentation (also called “flat cohomology”; see [8], [9]).

Lemma 3.2. *Let W be a finite subscheme of an abelian variety B defined over K. Suppose there exists an endomorphism $g : B \to B$ of degree > 1 such that $B(K^{\mathrm{sep}})[g] = 0$. Then $W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B) = W(K)$.*

Proof. By [11, Theorem E] we have

$$W(K) \subset B(K) \subset B(\mathbb{A}_K)^{\mathrm{Br}} \subset \widehat{\mathrm{Sel}}(B)\,.$$

So it suffices to show that $W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B) \subset W(K)$. Moreover we can assume $W = W(K)$ as in [11, Proposition 3.9].

Suppose $P \in W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B)$. For any $v \in \Omega_K$, the v-adic component of P is the image of some point $Q_v \in W(K)$, and $P - Q_v \in \widehat{\mathrm{Sel}}(B)$ maps to 0 in $B(K_v)^{(g)} := \varprojlim_n B(K_v)/g^n(B(K_v))$. In particular, $P - Q_v$ is in the kernel of $\widehat{\mathrm{Sel}}^{(g)}(B) \to B(K_v)^{(g)}$ where $\widehat{\mathrm{Sel}}^{(g)}(B)$ denotes the inverse limit of the Selmer groups corresponding to the isogenies g^n for $n \geq 1$. Below we show that this map is injective, so the image of $P - Q_v$ in $\widehat{\mathrm{Sel}}^{(g)}(B)$ is 0.

Since this holds for any v, if v' is any other prime we have

$$Q_{v'} - Q_v \in \ker\bigl(B(K) \to \varprojlim_n B(K)/g^n B(K) \hookrightarrow \widehat{\mathrm{Sel}}^{(g)}(B)\bigr)\,.$$

In other words, $(Q_v - Q_{v'}) \in \bigcap_{n\geq 1} g^n B(K)$. Since $B(K)$ is finitely generated, this implies that $(Q_v - Q_{v'}) \in B(K)_{\mathrm{tors}}$. Again, since this holds for all v we see that $R := P - Q_v \in \widehat{\mathrm{Sel}}(B)_{\mathrm{tors}}$. By [11, Lemma 5.1] this implies that $P - Q_v \in B(K)$. So $P \in W(K)$.

It remains to prove that $\widehat{\mathrm{Sel}}^{(g)}(B) \to B(K_v)^{(g)}$ is injective. For this it suffices (as in the proof of [11, Proposition 5.2]) to prove injectivity of $\widehat{\mathrm{Sel}}'^{(g)}(B) \to B(K'_v)^{(g)}$, where $K'_v \subset K^{\mathrm{sep}}$ denotes the Henselisation with respect to v and $\widehat{\mathrm{Sel}}'^{(g)}(B)$ is defined in the same way as $\widehat{\mathrm{Sel}}^{(g)}(B)$ but using K'_v instead of K_v. Let $b \in \ker\bigl(\widehat{\mathrm{Sel}}'^{(g)}(B) \to B(K'_v)^{(g)}\bigr)$ and, for each integer $M \geq 1$, let b_M denote the image of b in $\mathrm{Sel}'^{g^M}(B) \subset \mathrm{H}^1(K'_v, B[g^M])$ (where $B[g^M]$ is the kernel of g^M acting on B). Then the image of b_M under

$$\mathrm{Sel}'^{g^M}(B) \to \frac{B(K'_v)}{g^M B(K'_v)} \subset \mathrm{H}^1(K'_v, B[g^M]) \to \mathrm{H}^1(K^{\mathrm{sep}}, B[g^M])$$

is 0. The inflation-restriction sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), B(K^{\mathrm{sep}})[g^M]) \to \mathrm{H}^1(K, B[g^M]) \to \mathrm{H}^1(K^{\mathrm{sep}}, B[g^M])$$

shows that b_M comes from an element of $\mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), B(K^{\mathrm{sep}})[g^M])$. But this group is trivial since $B(K^{\mathrm{sep}})[g^M] = 0$. Since this holds for all $M \geq 1$, we conclude that $b = 0$. □

4. Proofs of the theorems

Proof of Theorem 1.2. By [11, Theorem E] we have

$$\overline{A(K)} \subset A(\mathbb{A}_K)^{\mathrm{Br}} \subset \widehat{\mathrm{Sel}}(A) \subset A(\mathbb{A}_K).$$

Intersecting with $X(\mathbb{A}_K)$ we have

$$X(\mathbb{A}_K) \cap \overline{A(K)} \subset X(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A)\,,$$

where the rightmost set consists of the adelic points on X which survive N-descent for all $N \geq 1$ (relative to the embedding $X \subset A$ as in Definition 2.1). In particular, any $P \in X(\mathbb{A}_K) \cap \overline{A(K)}$ survives p^n-descent for all $n \geq 1$. Since X is not isotrivial, Proposition 2.3 implies that there is a finite subscheme $Z \subset X \subset A$ which contains P. Then $P \in Z(\mathbb{A}_K) \cap \overline{A(K)} = Z(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}} = X(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A) = Z(K)$, where the final equality is Proposition 3.1. ☐

Remark 4.1. The preceding proof shows that for $X \subset A$ as in Theorem 1.2 we have

$$X(K) = X(\mathbb{A}_K) \cap \overline{A(K)} = X(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}} = X(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A)\,.$$

Proof of Theorem 1.1. Let X/K be as in the statement and let $J = \mathrm{Jac}(X)$ be its Jacobian. It suffices to show that $X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(K)$. Passing to some finite separable extension L/K we can embed X_L in J_L via the Abel–Jacobi map corresponding to an L-rational point. If $P \in X(\mathbb{A}_K)^{\mathrm{Br}}$, then its image under the inclusion $X(\mathbb{A}_K) \subset X(\mathbb{A}_L) = X_L(\mathbb{A}_L)$ is orthogonal to $\mathrm{Br}(X_L)$ by [2, Lemma 3.1]. By functoriality of the Brauer pairing and Remark 4.1 we have $X_L(\mathbb{A}_L)^{\mathrm{Br}} \subset X_L(\mathbb{A}_L) \cap J_L(\mathbb{A}_L)^{\mathrm{Br}} = X_L(L)$. Then P is in $X(\mathbb{A}_K) \cap X(L)$ which is equal to $X(K)$ by [11, Lemma 3.2]. ☐

Remark 4.2. In the proof of Theorem 1.1 just given [2, Lemma 3.1] and [11, Lemma 3.2] allow us to pass to an extension over which X can be embedded in its Jacobian. Alternatively one can use the following construction suggested to one of us by Poonen. Restriction of scalars gives a map $\mathrm{Res}_{L/K}(X_L) \to \mathrm{Res}_{L/K}(J_L)$. Composing this with the canonical map $X \to \mathrm{Res}_{L/K}(X_L)$ gives a closed immersion $X \to A$ into the abelian variety $A := \mathrm{Res}_{L/K}(J_L)$ over K. To prove Theorem 1.1 one can then apply Remark 4.1 to $X \subset A$.

Appendix: On abelian varieties with an infinite group of separable p^∞-torsion points

by Damian Rössler

If $n \in \mathbb{N}$, we write $[n]$ for the multiplication by n endomorphism on an abelian variety. If h is an endomorphism with finite kernel of an abelian variety A over a field L, we write

$$A(L)[h^\ell] := \{x \in A(L) \mid h^{\circ\ell}(x) = 0\}$$

and

$$A(L)[h^\infty] := \{x \in A(L) \mid \exists n \in \mathbb{N} : h^{\circ n}(x) = 0\}.$$

Here $h^{\circ n}(x) := h(h(h(\cdots(x)\cdots)))$, where there are n pairs of brackets. The notation $A(L)[n^\ell]$ (resp. $A(L)[n^\infty]$) will be a shorthand for $A(L)[[n]^\ell]$ (resp. $A(L)[[n]^\infty]$).

Now let K_0 be the function field of a smooth and proper curve U over a finite field $\mathbb{F}$ of characteristic $p > 0$. Let B be an abelian variety over K_0. Suppose that for some $n > 3$ prime to p, the group scheme $B[n]$ is constant and that the Néron model of B over U has a semiabelian connected component.

Proposition A.1. *There exists an abelian variety C over K_0, an étale K_0-isogeny $\phi : B \to C$, an étale K_0-isogeny $f : C \to C$, a K_0-isogeny $g : C \to C$, and a natural number $r \geq 0$ such that*

(a) $g \circ f = [p^r]$ *and* $g \circ f = f \circ g$;

(b) $C(K_0^{\mathrm{sep}})[p^\infty] = C(K_0^{\mathrm{sep}})[f^\infty] = C(\bar{K}_0)[f^\infty]$;

(c) $C(K_0^{\mathrm{sep}})[g^\infty] = 0$.

Proof. For $\ell \geq 0$, define inductively

$$B_0 := B$$

and

$$B_{\ell+1} := B_\ell / (B_\ell(K_0^{\mathrm{sep}})[p]).$$

For $\ell_2 \geq \ell_1$, let $\phi_{\ell_1,\ell_2} : B_{\ell_1} \to B_{\ell_2}$ be the (étale!) morphism obtained by composing the natural morphisms $B_{\ell_1} \to B_{\ell_1+1} \to \cdots \to B_{\ell_2}$. We first claim that

$$(\ker \phi_{\ell_1,\ell_2})(K_0^{\mathrm{sep}}) = B_{\ell_1}(K_0^{\mathrm{sep}})[p^{\ell_2-\ell_1}] \tag{A-1}$$

We prove the claim by induction on $\ell_2 - \ell_1$. For $\ell_2 - \ell_1 \leq 1$, the claim is true by definition. Suppose that $\ell_2 - \ell_1 \geq 1$. Let $x \in B(K_0^{\mathrm{sep}})[p^{\ell_2-\ell_1}]$. Then $[p^{\ell_2-\ell_1-1}](x) \in B(K_0^{\mathrm{sep}})[p]$ and thus

$$\phi_{\ell_1,\ell_1+1}([p^{\ell_2-\ell_1-1}](x)) = [p^{\ell_2-\ell_1-1}](\phi_{\ell_1,\ell_1+1}(x)) = 0.$$

Applying the inductive assumption to $\phi_{\ell_1,\ell_1+1}(x)$, we see that $\phi_{\ell_1+1,\ell_2}(\phi_{\ell_1,\ell_1+1}(x)) = \phi_{\ell_1,\ell_2}(x) = 0$. This proves that $(\ker \phi_{\ell_1,\ell_2})(K_0^{\mathrm{sep}}) \supseteq B_{\ell_1}(K_0^{\mathrm{sep}})[p^{\ell_2-\ell_1}]$. To prove the opposite inclusion, let $x \in (\ker \phi_{\ell_1,\ell_2})(K_0^{\mathrm{sep}})$. We compute

$$\phi_{\ell_1,\ell_2}(x) = \phi_{\ell_1+1,\ell_2}(\phi_{\ell_1,\ell_1+1}(x)) = 0,$$

which implies (by the inductive hypothesis) that

$$[p^{\ell_2-\ell_1-1}](\phi_{\ell_1,\ell_1+1}(x)) = \phi_{\ell_1,\ell_1+1}([p^{\ell_2-\ell_1-1}](x)) = 0,$$

which in turn implies that $[p]([p^{\ell_2-\ell_1-1}](x)) = [p^{\ell_2-\ell_1}](x) = 0$. This proves that $(\ker \phi_{\ell_1,\ell_2})(K_0^{\mathrm{sep}}) \subseteq B_{\ell_1}(K_0^{\mathrm{sep}})[p^{\ell_2-\ell_1}]$ and completes the proof of the claim.

Now we know that by the reasoning made in the last page of [12], that there are only finitely many isomorphism classes of abelian varieties over K_0 in the sequence $\{B_\ell\}_{\ell \in \mathbb{N}}$. Let C be an abelian variety over K_0, which appears at least twice in the sequence $\{B_\ell\}_{\ell \in \mathbb{N}}$. Let $n_2 > n_1$ be such that $C \simeq B_{n_1} \simeq B_{n_2}$. Then by construction (under the identification $C = B_{n_1}$)

$$\phi_{n_1,n_2}^{\circ \ell} = \phi_{n_1, n_1+\ell\cdot(n_2-n_1)} \tag{A-2}$$

for any $\ell \geq 1$ and thus

$$C(K_0^{\mathrm{sep}})[p^\infty] = C(K_0^{\mathrm{sep}})[\phi_{n_1,n_2}^\infty] \tag{A-3}$$

Now define $f := \phi_{n_1,n_2}$ and $r := n_2 - n_1$. Define g as the only K_0-isogeny such that $g \circ f = [p^r]$.

Notice then that the identity $g \circ f = [p^r]$ implies the identity $f \circ g = [p^r]$. To see this last fact directly, recall first that there are natural injection of rings

$$\mathrm{End}_{K_0}(C) \hookrightarrow \mathrm{End}_{\bar{K}_0}(C_{\bar{K}_0}) \hookrightarrow \mathrm{End}_{\mathbb{Z}_t}(T_t(C(\bar{K}_0))) \hookrightarrow \mathrm{End}_{\mathbb{Q}_t}(T_t(C(\bar{K}_0)) \otimes \mathbb{Q}_t)$$

where $T_t(C(\bar{K}_0))$ is the classical Tate module of $C_{\bar{K}_0}$ and $t > 0$ is some prime number $\neq p$. Now if M and N are two square matrices of the same size with coefficients in a field of characteristic 0, such that $M \cdot N = p^r$, then $p^{-r} N$ is the inverse matrix of M and thus $N \cdot M = p^r$. This fact combined with the existence of the above injections implies that $f \circ g = [p^r]$ if $g \circ f = [p^r]$.

We have already proven (a). Point (b) is contained in (A-3).

We now prove (c). Suppose that for some $\ell \geq 0$ and some $x \in C(K_0^{\mathrm{sep}})$, we have $g^{\circ l}(x) = 0$. Let $y \in (f^{\circ \ell})^{-1}(x) \subseteq C(K_0^{\mathrm{sep}})$. Then $g^{\circ l}(f^{\circ l}(y)) = [p^{r\ell}](y) = 0$. Hence $f^{\circ l}(y) = 0 = x$ by (A-1) and (A-2). □

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