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Malcev completions, Hodge theory, and motives

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Malcev completions, Hodge theory, and motives

Emil Jacobsen

We prove that, on a smooth, connected variety in characteristic zero admitting a rational point, local systems of geometric origin are stable under extension in the category of all local systems. As a consequence of this, we obtain a (Nori) motivic strengthening of Hain's theorem on Malcev completions of monodromy representations.

Our methods are Tannakian, and rely on an abstract criterion for "Malcev completeness", which is proved in the first part of the paper. A couple of secondary applications of this criterion are given: an alternative proof of D'Addezio–Esnault's theorem, which says that local systems of Hodge origin are stable under extension in the category of all local systems; a generalization of the theorem of Hain, mentioned above, which also affirms a conjecture of Arapura; and an alternative proof of a theorem of Lazda, which under suitable assumptions gives an isomorphism between the relative unipotent de Rham fundamental group and the unipotent de Rham fundamental group of the special fiber.

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1. Introduction

The main result of this paper can be stated as: local systems of geometric origin are stable under extension. Let us have a closer look at what is meant by this.

Fix a subfield k of $\mathbb{C}$. For a k-variety X, Ivorra and Morel defined in [24] the category of *perverse motives* on X, here denoted by $\mathrm{PM}(X)$. It is a $\mathbb{Q}$-linear abelian category and comes with a faithful exact *Betti realization functor* $\mathrm{PM}(X) \to \mathcal{P}(X)$ landing in the category $\mathcal{P}(X)$ of perverse sheaves (with rational coefficients) on X. Over the base field k, perverse motives are the same as Nori motives.

Now let X be a smooth, connected k-variety. Then, the category of local systems (with rational coefficients) $\mathrm{LS}(X)$ on X forms an abelian subcategory of $\mathcal{P}(X)$ (after shifting). This lets us define the category of *motivic local systems*, $\mathrm{MLS}(X)$, as the full subcategory of perverse motives on X generated by those whose Betti realizations are (shifted) local systems. We get an induced Betti realization functor $b_X^* : \mathrm{MLS}(X) \to \mathrm{LS}(X)$.

Remark 1.1. Apart from the results of Ivorra and Morel, the main reason that motivic local systems are well-behaved enough for our purposes is the work of Terenzi in [32], to which we return later in the introduction.

Definition 1.2. A local system on X is *of geometric origin*, or *comes from geometry*, if it is a subquotient of one in the image of the Betti realization $b_X^* : \mathrm{MLS}(X) \to \mathrm{LS}(X)$.

Our main result can then be stated as follows:

Theorem 1.3 (Theorem 7.7.1). *Let X be a smooth and connected k-variety with a rational point. Then local systems of geometric origin are stable under extension in* $\mathrm{LS}(X)$. *In other words, if*

$$0 \to L' \to L \to L'' \to 0$$

is a short exact sequence in $\mathrm{LS}(X)$ *such that L' and L'' are of geometric origin, then so is L.*

Remark 1.4. There are multiple other definitions of local systems of geometric origin in the literature. For some recent examples, see [6, Definition 1.6.1] and [26, 685]. Further back, Simson gives a vague definition in [31, 9] (using the terminology of a local system being "motivic", rather than "of geometric origin"), and we believe our definition is in the spirit of that one. As a consequence of our main theorem (Theorem 1.3), it is likely our definition of geometric origin agrees with that of Ayoub.

The introduction will continue as follows. First, in Section 1.1, we introduce the notion of Malcev completions. This lets us state our next result, which serves as a motivic version of a theorem of Hain. After that, we quickly introduce the motivic fundamental group in Section 1.2 and give an alternative formulation of the Theorem 1.3. The key technical input in our proof of the main theorem, is a criterion for "Malcev completeness", which will be discussed towards the end of the introduction (Section 1.5). Before we get there, we will go over two other applications of this criterion (Section 1.3), as well as some background which places our main results in a larger context (Section 1.4). A brief sketch of the proofs of our main results closes out the introduction in Section 1.6.

1.1. ***Motivic Malcev completions.*** Keep k as above and let X be a smooth, connected k-variety with a rational point x. Recall that $x^* : \mathrm{LS}(X) \to \mathrm{Vec}$ induces an equivalence $\mathrm{LS}(X) \simeq \mathrm{Rep}\,\pi_1(X, x)$, where the latter denotes the category of finite-dimensional $\mathbb{Q}$-linear representations of the fundamental group of X^{an} pointed at x.

Before we go further, we need to introduce the notion of Malcev completion. Suppose $P \to K$ is a homomorphism of affine group schemes over a field. Then, the *Malcev completion* of this homomorphism satisfies the following universal property: it is the initial group $\mathcal{G}$ equipped with a morphism from P and a morphism to K, such that the latter has pro-unipotent kernel and such that everything commutes (see Section 4 for a much more thorough treatment):

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{G} & \longrightarrow & K \\ & & & & \uparrow & \nearrow & \\ & & & & P & & \end{array}$$

If P is an abstract group with a homomorphism to the rational points of K, we can still talk about the Malcev completion, by first replacing P with its pro-algebraic completion.

Now let $M \in \mathrm{MLS}(X)$ be a motivic local system on X. The underlying local system $b_X^* M$ corresponds to a (monodromy) representation $\varrho : \pi_1(X, x) \to \mathrm{GL}(V)$. Let $\hat{\pi}_1^M(X, x)$ be the Malcev completion of ϱ.

We're able to obtain the next theorem as a corollary of the previous one, using basic properties of Malcev completions worked out in Section 4. It's a motivic version of Hain's theorem [21, Theorem 13.1].

Theorem 1.5 (Theorem 7.7.2). *The group $\hat{\pi}_1^M(X, x)$ is naturally a Nori motive.*

Remark 1.6. An affine $\mathbb{Q}$-group scheme G *is a Nori motive* if $\mathcal{O}(G)$ is a Hopf algebra object in the ind-category of Nori motives.

Remark 1.7. The special case of the unipotent completion of $\pi_1(X, x)$ was treated by Cushman; see [11, Theorem 3.1].

1.2. ***The motivic fundamental group.*** We keep k, X, and x as before. Let $\mathrm{LS}_{\mathrm{geo}}(X)$ denote the full subcategory of local systems of geometric origin in $\mathrm{LS}(X)$. Note that $\mathrm{LS}_{\mathrm{geo}}(X)$ is neutral Tannakian, neutralized, for instance, by (the restriction of) x^*. The Tannaka dual group of $\mathrm{LS}_{\mathrm{geo}}(X)$ is denoted by $\pi_1^{\mathrm{mot}}(X, x)$ and is called the *motivic fundamental group* of X (pointed at x). By construction, there is a homomorphism $\pi_1(X, x) \to \pi_1^{\mathrm{mot}}(X, x)$. Theorem 1.3 is equivalent to the following:

Theorem 1.8 (Theorem 7.7.1). *The homomorphism $\pi_1(X, x) \to \pi_1^{\mathrm{mot}}(X, x)$ is* ***Malcev complete****, i.e., its Malcev completion is isomorphic to $\pi_1^{\mathrm{mot}}(X, x)$ (along the canonical map).*

This can be viewed as a universal version of Theorem 1.5.

1.3. ***Other results.*** The proof of the main theorem relies on a criterion for Malcev completeness (which is discussed in Section 1.5). In this subsection, I will briefly discuss two other applications of this criterion.

1.3.1. *Hodge theory.* Let X be a smooth, geometrically connected variety over a subfield k of $\mathbb{C}$. Fix a $\mathbb{C}$-point x of X, and let $\pi_1^{\mathrm{Hdg}}(X, x)$ denote the *Hodge fundamental group* of X, i.e., the Tannaka dual of the category of admissible variations of mixed Hodge structures on X, with respect to the fiber functor induced by x (see Section 7.1).

Theorem 1.9 (Theorem 7.2).

(1) [12, Theorem 4.4] *The map $\pi_1(X, x) \to \pi_1^{\mathrm{Hdg}}(X, x)$ is Malcev complete. Equivalently, the full subcategory of local systems on X of Hodge origin is stable under extension in the category of local systems on X. In particular, every unipotent local system on X is of Hodge origin.*

(2) *Given an admissible variation of mixed Hodge structures V on X, the Malcev completion of the monodromy representation $\pi_1(X, x) \to \mathrm{GL}(V_x)$ carries a canonical (rational) mixed Hodge structure.*

Remark 1.10. An affine $\mathbb{Q}$-group scheme G *carries a mixed Hodge structure* if $\mathcal{O}(G)$ is a Hopf algebra object in the ind-category of rational, graded-polarizable, mixed Hodge structures.

The Malcev completeness of $\pi_1(X, x) \to \pi_1^{\mathrm{Hdg}}(X, x)$ has already been proved by D'Addezio and Esnault by a different method in [12, Theorem 4.4]. The second part of the theorem did not appear in [12], and is a generalization of Hain's theorem ([21, Theorem 13.1], see Section 1.4.2).

1.3.2. *De Rham fundamental groups.* Let $X \to S$ be a nice enough morphism of nice enough k-schemes (see Section 8 for the precise statements). In [27], Lazda introduces the relative (unipotent) de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S)$, and shows that it's isomorphic, via a natural map to the (unipotent) de Rham fundamental group $\pi_1^{\mathrm{dR}}(X_s)$ of the special fiber. In Section 8, Lazda's theorem is deduced using the unipotent isomorphism criterion (see Theorem 1.13 below).

1.4. ***Background.*** Let us spend some time putting the main theorem in a historical context.

1.4.1. *The fundamental short exact sequence.* Fix a field k and an algebraic closure $\bar{k}$. The étale fundamental group $\pi_1^{\text{ét}}(X, x)$ of a connected k-pointed k-variety (X, x) was introduced by Grothendieck in [18]. By functoriality, the structure map $X \to \mathrm{Spec}(k)$ and the point x induce a split surjection

$$\pi_1^{\text{ét}}(X, x) \overset{\longleftarrow}{\twoheadrightarrow} \mathrm{Gal}(\bar{k}/k). \tag{1}$$

Under suitable assumptions, Grothendieck gives a precise description of the kernel of this surjection. Namely, it is given by the geometric étale fundamental group $\pi_1^{\text{ét}}(\bar{X}, \bar{x})$ of X, where $\bar{X} := X \otimes_k \bar{k}$. Thus, there is a split exact sequence of profinite groups:

$$1 \longrightarrow \pi_1^{\text{ét}}(\bar{X}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X, x) \overset{\longleftarrow}{\longrightarrow} \mathrm{Gal}(\bar{k}/k) \longrightarrow 1. \tag{2}$$

From now on, we suppose that k is a subfield of the complex numbers. We then have access to the topological fundamental group $\pi_1(X, x)$ of the complex analytification X^{an} of X. There is a natural map

$$\pi_1(X, x) \to \pi_1^{\text{ét}}(\bar{X}, \bar{x}), \tag{3}$$

and Grothendieck also tells us exactly what it is, namely a profinite completion. I am interested in *motivic analogues* of the above results. More precisely,

(a) a motivic analogue of the split surjection (1),

(b) a description of its kernel, and

(c) some understanding of the relationship between this kernel and the topological fundamental group.

The motivic analogue of the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ of k is given by the *motivic Galois group*, here denoted by $\mathscr{G}^{\mathrm{mot}}(k)$. This affine $\mathbb{Q}$-group scheme can be defined as the Tannaka dual of the category of Nori motives, or equivalently as the spectrum of a certain *motivic Hopf algebra* constructed by Ayoub in [4; 5] (see [10] for the comparison). By equipping the derived category of perverse motives with a well-behaved tensor product, Terenzi shows in [32, Theorem 6.2] that $\mathrm{MLS}(X)$ is neutral Tannakian. We denote its Tannaka dual by $G(X, x)$: this is the motivic analogue of $\pi_1^{\text{ét}}(X, x)$.

The structure map of X and the rational point x induce a split surjection $G(X, x) \to \mathscr{G}^{\mathrm{mot}}(k)$, by way of the inverse image functors on perverse motives (see [24]). This is our motivic analogue of (1), in line with goal (a). Denote the kernel by $K(X, x)$. By definition, we have a homomorphism $\pi_1(X, x) \to K(X, x)$ and (the Zariski closure of) its image is the motivic fundamental group $\pi_1^{\mathrm{mot}}(X, x)$. Our main theorem

says that $\pi_1(X,x) \to K(X,x)$ is Malcev complete. This implies firstly that $K(X,x) = \pi_1^{\mathrm{mot}}(X,x)$, which we may view as a result in the direction of goal (b). Then, the fact that $\pi_1(X,x) \to \pi_1^{\mathrm{mot}}(X,x)$ is Malcev complete, is a result in the direction of goal (c).

In conclusion, we have a split exact sequence of affine group schemes over $\mathbb{Q}$

$$1 \longrightarrow \pi_1^{\mathrm{mot}}(X,x) \longrightarrow G(X,x) \overset{\longleftarrow}{\longrightarrow} \mathscr{G}^{\mathrm{mot}}(k) \longrightarrow 1, \tag{4}$$

and a homomorphism $\pi_1(X,x) \to \pi_1^{\mathrm{mot}}(X,x)$, which is Malcev complete.

1.4.2. *Motivic structures on fundamental groups.* The singular homology functors $H_i(-,\mathbb{Q})$ on algebraic varieties over $\mathbb{C}$ factor through the category of mixed Hodge structures, by the seminal work of Deligne. Conjecturally, they factor further, through a (Tannakian) category of motives, enjoying many useful properties. By Hurewicz, the functor $\pi_1(-,-)^{\mathrm{ab}} \otimes \mathbb{Q}$ on pointed algebraic $\mathbb{C}$-varieties is isomorphic to $H_1(-,\mathbb{Q})$, so it too should factor through motives. Put differently, the abelianization of the fundamental group of an algebraic variety is canonically a motive. It's natural to investigate whether a larger part of the fundamental group is a motive, as passing to the abelianization is rather destructive. In particular, one can ask to what extent Hodge theory can be extended to the fundamental group, i.e., what part of the fundamental group carries a natural mixed Hodge structure.

In this direction, and improving upon [28], Hain showed in [20] that the unipotent completion of the fundamental group carries a natural mixed Hodge structure (see Remark 1.10). Hain then went further in [21] and proved the following:

Theorem 1.11 [21, Theorem 13.1]. *Let X be a smooth connected variety over $\mathbb{C}$, and let x be a complex point of X. Let V be a polarizable variation of Hodge structure on X, such that the image of the monodromy representation $\pi_1(X,x) \to \mathrm{GL}(V_x)$ is Zariski dense in $\mathrm{Aut}(V_x, \langle , \rangle)$. Then the Malcev completion of this monodromy representation carries a canonical (real) mixed Hodge structure.*

Motivic strengthenings of Hain's unipotent theorem have been established in some cases. For instance, Deligne in [14] equipped the unipotent completion of the fundamental group of a rational variety with the structure of a compatible system of realizations, which is not too far from that of a motive. This was strengthened in [16], where the unipotent completion of the fundamental group of a unirational variety defined over a number field was equipped with the structure of a mixed Tate motive, in Voevodsky's sense.

A motivic strengthening of Hain's theorem on Malcev completions is achieved in Theorem 1.5. The second part of the Hodge theoretic result Theorem 1.9 generalizes Hain's theorem, by allowing arbitrary admissible variations.

1.5. ***The Malcev completeness criterion.*** The proofs of these three theorems are based on a general criterion for a homomorphism to be Malcev complete. All the groups we consider are affine group schemes over a fixed field. Let $P \to K$ be a group homomorphism, and denote the restriction functor

of (finite-dimensional) K-representations along this morphism by $\mathrm{Res}^K_P : \mathrm{Rep}\, K \to \mathrm{Rep}\, P$. It's an easy observation that $P \to K$ is Malcev complete if and only if

(i) $P \to K$ is surjective, and

(ii) the image of Res^K_P is stable under extension in $\mathrm{Rep}\, P$.

Let $K \twoheadrightarrow S$ be a quotient by a pro-unipotent normal subgroup. We write $\Gamma(K, -)$ and $\Gamma(P, -)$ for K- and P-invariants respectively. It is then not difficult to see that Malcev completeness of $P \to K$ is equivalent to

(iii) the composition $P \to K \to S$ is surjective, and

(iv) the natural maps

$$R^i\Gamma(K, -) \to R^i\Gamma(P, -) \circ \mathrm{Res}^K_P$$

are isomorphisms for $i = 0, 1$.

In applications, the following theorem offers a condition which is easier to check than (iv):

Theorem 1.12 (Malcev completeness criterion, Theorem 5.1). *Let $G = K \rtimes H$ be a semidirect product and consider a homomorphism $P \to K$. Let $K \twoheadrightarrow S$ be quotient by a pro-unipotent normal subgroup. Then $P \to K$ is Malcev complete if $P \to S$ is surjective and the natural maps*

$$R^i\Gamma(K, -) \circ \mathrm{Res}^G_K \to R^i\Gamma(P, -) \circ \mathrm{Res}^G_P$$

between functors $\mathrm{Rep}\, G \to \mathrm{Vec}$ *are isomorphisms for $i = 0, 1$.*

In other words, when checking condition (iv) above, it's enough to work with objects in the image of Res^G_K. The group H and its action on K can be chosen completely arbitrary, and therein lies the strength of the theorem. The proof of Theorem 1.12 works by reducing to the following special case:

Theorem 1.13 (Unipotent isomorphism criterion, Theorem 3.3). *Let $G = K \rtimes H$ be a semidirect product and consider a homomorphism $P \to K$. Then $P \to K$ is an isomorphism if P is pro-unipotent and the natural maps*

$$R^i\Gamma(K, -) \circ \mathrm{Res}^G_K \to R^i\Gamma(P, -) \circ \mathrm{Res}^G_P$$

between functors $\mathrm{Rep}\, G \to \mathrm{Vec}$ *are isomorphisms for $i = 0, 1$.*

Remark 1.14. Using Tannaka duality, it's easy to see that, when P is pro-unipotent, $P \to K$ is an isomorphism if and only if condition (iv) above holds. Moreover, the assumptions in Theorem 1.13 easily imply that K too is pro-unipotent, so that S is trivial and the surjectivity of $K \to S$ is automatic, really making it a special case of Theorem 1.12.

The following direct corollary to Theorem 1.12 has been suggested by Marco D'Addezio:

Corollary 1.15. *Suppose* $\operatorname{char} F = 0$. *Let* $P \to G$ *be a homomorphism whose image is contained in the pro-unipotent radical* $R_{\mathrm{u}}G$, *and suppose that* P *is pro-unipotent. Then* $P \to R_{\mathrm{u}}G$ *is an isomorphism if and only if the natural maps*

$$\operatorname{Res}_1^{G_{\mathrm{red}}} \circ R^i \operatorname{Ind}_G^{G_{\mathrm{red}}} \to R^i \Gamma(P, -) \circ \operatorname{Res}_P^G$$

are isomorphisms for $i = 0, 1$.

Remark 1.16. Keep the setup as in Theorem 1.12. The theorem takes three assumptions: the cohomological assumptions for $i = 0, 1$, and the surjectivity of $P \to S$. The result is the Malcev completeness of $P \to K$, which essentially has two parts: the surjectivity of $P \to K$ (listed as (i) above), and condition (ii) on stability under extension. If one only cares about the first consequence, it is known (private communication from Marco D'Addezio, 17 January 2023) that the condition for $i = 1$ isn't needed. More precisely, One can use results from [12, Appendix A] to lift surjectivity of $P \to S$ to surjectivity of $P \to K$ when $\Gamma(K, -) \circ \operatorname{Res}_K^G \to \Gamma(P, -) \circ \operatorname{Res}_P^G$ is an isomorphism.

1.6. ***A sketch of the proofs.*** Keep the notation as in the unipotent isomorphism criterion (Theorem 1.13). The choice of a section $H \to G$ makes $\mathcal{O}(K)$ into an algebra object in $\operatorname{Ind}\operatorname{Rep} G$. To prove Theorem 1.13, we need to show that $P \to K$ is an isomorphism, or equivalently that $\mathcal{O}(K) \to \mathcal{O}(P)$ is an isomorphism. We do this by constructing $\mathcal{O}(P)$ and $\mathcal{O}(K)$ as explicit colimits in $\operatorname{Ind}\operatorname{Rep} P$ and $\operatorname{Ind}\operatorname{Rep} G$, respectively, and then showing that the natural map between them in $\operatorname{Ind}\operatorname{Rep} P$ is an isomorphism. They are constructed as certain universal iterated extensions starting with the trivial object $\mathbf{1}$, and this is carried out in Sections 3.1 and 3.2. The rest of Section 3 is then dedicated to showing that these constructions work as intended, and finally deducing the result.

This construction of the regular representation of a pro-unipotent group as a universal iterated extension has appeared in the literature in several special cases, for example in [19; 1; 27; 9]. As far as we know, the general form of Theorem 1.13 we prove has not appeared before, though it might have been known to experts.

To prove the Malcev completeness criterion (Theorem 1.12), we reduce to the unipotent case. The details of this reduction are a bit technical, but the idea is simple: K and the Malcev completion of $P \to K$ are by assumption both extensions of S by pro-unipotent groups. With this in mind we reduce to showing that these pro-unipotent groups are isomorphic, and finish the job using Theorem 1.13. The heart of this reduction argument is Proposition 5.4, which might also be of independent interest.

In order for Theorem 1.12 to be applicable in the Hodge-theoretic context of Theorem 1.9, the assumptions need to be checked, but let us first provide a dictionary between the groups appearing in the two theorems. The role of P is played by (the pro-algebraic completion of) $\pi_1(X, x)$, while that of G is played by the Tannaka dual of the category of admissible variations on X. The group H is then taken to be the *Hodge group*, i.e., the Tannaka dual of the category of rational mixed Hodge structures. Finally, K is in this context the kernel of the natural homomorphism[1] $G \to H$, and the map from $P \to G$ is easily

[1] It's induced under Tannaka duality by the functor sending a mixed Hodge structure to the associated constant variation.

seen to factor through K. The group-cohomological assumption is then established using the theory of mixed Hodge modules, in Proposition 6.13 and Lemma 7.1, and the surjectivity assumption is established using Proposition 6.15 and Deligne's semisimplicity theorem. This gives the Malcev completeness of the relevant map, and the second part follows as a corollary, see Proposition 6.16.

In the motivic context of Theorem 1.3, we use results from [24] on perverse motives to check the cohomological assumption of Theorem 1.12 (see 7.5). The surjectivity assumption again boils down to Deligne's semisimplicity theorem, as well as a well-behaved theory of weights on perverse motives.

Part I. Affine group schemes over a field

2. Preliminaries

We fix a field F and assume all additive categories to be F-linear. The category of finite-dimensional F-vector spaces will be denoted by Vec. We write $\mathbf{1}$ for the unit object in a monoidal category. When there are several monoidal categories around, it will be understood from context which unit object we mean.

By *pro-algebraic group* we mean an affine group scheme over F.

We will use the term *tensor category* to mean an F-linear symmetric monoidal category. That is, an F-linear category $\mathcal{C}$ equipped with an F-bilinear functor $(-\otimes -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbf{1}$, as well as compatible natural isomorphisms encoding associativity, commutativity, and the unit relations, that satisfy the usual axioms. We will say a functor f between tensor categories is *monoidal* when it's equipped with an isomorphism $f(\mathbf{1}) \simeq \mathbf{1}$ and a natural isomorphism $f(X \otimes Y) \simeq f(X) \otimes f(Y)$, both compatible with the associator and commutator isomorphisms of the two categories. Finally, a natural transformation $\phi : f \to g$ between monoidal functors f and g is *monoidal* when it's compatible with the monoidal structures of f and g.

An object X in a tensor category $\mathcal{C}$ is (*strongly*) *dualizable* if there exists an object $X^\vee$ (called a *dual* of X) and maps $\mathrm{ev} : X \otimes X^\vee \to \mathbf{1}$, $\mathrm{coev} : \mathbf{1} \to X^\vee \otimes X$ that turn $X^\vee \otimes (-)$ into a right adjoint of $X \otimes (-)$. Moreover, $\mathcal{C}$ is *rigid* if every object in $\mathcal{C}$ is dualizable. Duality is reflexive, in the sense that X is a dual of $X^\vee$. Monoidal functors preserve duals.

2.1. *Unipotency.*

Definition 2.1. Let $\mathcal{A}$ be an abelian category and $f : \mathcal{C} \to \mathcal{A}$ a functor. Say an object U in $\mathcal{A}$ is *f-unipotent* if it can be built in a finite number of steps from the essential image of f, using extensions. More precisely, if there exists a filtration

$$0 = U_{-1} \subseteq U_0 \subseteq \ldots \subseteq U_n = U,$$

such that the quotients U_i / U_{i-1} are in the essential image of f. We call such a filtration an *f-unipotent filtration* of U, of *length* n. If every object of $\mathcal{A}$ is f-unipotent, then $\mathcal{A}$ is said to be *f-unipotent* (or sometimes *relatively unipotent over* $\mathcal{C}$).

Definition 2.2. Let U be an f-unipotent object, as above. Let n be the smallest integer such that U admits an f-unipotent filtration of length n. We say the *f-unipotency index* of U is n.

Remark 2.3. The usefulness of unipotency is that it allows for *unipotent induction*, that is, induction on the unipotency index. This technique lets us prove many statements by checking that they hold for a smaller class of objects, and that they behave well with respect to extension.

Lemma 2.4. *Unipotency is stable under extension.*

Proof. Let $f : \mathcal{C} \to \mathcal{A}$ be as above. Consider an extension

$$0 \to U \to A \to W \to 0,$$

where U and W are both f-unipotent. In the case when $W = f(C)$, A is unipotent by definition. We now proceed by induction on the unipotency index of W. Let

$$0 \to W' \to W \to f(C) \to 0,$$

such that W' is of a lower unipotency index than W. By induction, any extension of W' by something f-unipotent is unipotent. The pullback A' of $A \to W$ and $W' \to W$ is therefore unipotent, and A is an extension

$$0 \to A' \to A \to f(C) \to 0,$$

by standard arguments. □

Definition 2.5. With $f : \mathcal{C} \to \mathcal{A}$ as above, we say that $\mathcal{A}$ is *weakly f-unipotent* if every nonzero object of $\mathcal{A}$ admits a nonzero subobject in the essential image of f.

Unipotency immediately implies weak unipotency, but the converse is only true under a finiteness condition.

Proposition 2.6. *Let $f : \mathcal{C} \to \mathcal{A}$ be as above.*

(i) *If $\mathcal{A}$ is f-unipotent, then it is weakly f-unipotent.*

(ii) *If $\mathcal{A}$ is weakly f-unipotent, then any object in $\mathcal{A}$ of finite length is f-unipotent.*

Proof. (i) Let $A \in \mathcal{A}$ be nonzero. Fix an f-unipotent filtration A_i of A as above. Then the minimal i such that A_i is nonzero gives the desired subobject.

(ii) Let A be nonzero of finite length. We get a short exact sequence

$$0 \to f(C) \to A \to A_1 \to 0.$$

Then A_1 has a nonzero subobject $f(C_1)$, and we iterate this until, by finiteness of the length of A, we get $A_n = f(C_n)$. Using that unipotency is closed under extension (Lemma 2.4), we conclude that the A_i and finally A are all f-unipotent. □

Remark 2.7. When $\mathcal{A}$ is monoidal, $\mathcal{C} = \mathrm{Vec}$, and $f : \mathrm{Vec} \to \mathcal{A}$ is monoidal, then f is uniquely determined, up to monoidal natural isomorphism (cf. Remark 2.9). In this case, we usually just say that $\mathcal{A}$ is *unipotent*, without reference to f.

2.2. ***Neutral Tannakian categories.*** The notion of a (neutral) Tannakian category is central in the sequel. A *neutral Tannakian category* (over F) is a rigid abelian tensor category $(\mathcal{A}, \otimes, \mathbf{1})$ such that $\mathrm{End}(\mathbf{1}) = F$, and which admits a faithful, exact, monoidal functor $\mathcal{A} \to \mathrm{Vec}$: such a functor is called a (*neutral*) *fiber functor*. A *neutralized Tannakian category* is a Tannakian category together with a neutral fiber functor.

Remark 2.8. The tensor product is required to be F-bilinear, and it is automatically exact in both arguments ([17, Proposition 1.17]). The rigidity implies existence of an internal Hom functor, which is (right and left) adjoint to the tensor product, and which is hence also exact in both arguments. We denote it by $\underline{\mathrm{Hom}}(-,-)$.

By *morphism* of neutral Tannakian categories $\mathcal{A}$ and $\mathcal{B}$ we shall mean an exact, faithful, monoidal functor. We will always write such a functor decorated with a raised asterisk, e.g., $p^* : \mathcal{A} \to \mathcal{B}$. These functors always have ind-right adjoints, i.e., functors $p_* : \mathrm{Ind}\,\mathcal{B} \to \mathrm{Ind}\,\mathcal{A}$, for which we use the same letters but lowered asterisks. These ind-right adjoints are left exact and can always be right derived, as the ind-category of a neutral Tannakian category has enough injectives (see, e.g., [25, Proposition I.3.9]). By *morphism* of neutralized Tannakian categories $(\mathcal{A}, f^*)$ and $(\mathcal{B}, \varphi^*)$ we shall mean a morphism $p^* : \mathcal{A} \to \mathcal{B}$ together with a monoidal isomorphism $\varphi^* \circ p^* \simeq f^*$.

This gives 2-categories nTan and nTan_* of neutral and neutralized Tannakian categories, respectively, after we define what the 2-morphisms are. In nTan, they are the monoidal, invertible natural transformations. In nTan_*, they are the same, with the additional requirement that they be compatible with the extra data in the morphisms (the monoidal isomorphisms above).

Remark 2.9 (On sections of fiber functors). A fiber functor f^* of a Tannakian category $\mathcal{B}$ always admits a section in the 2-category above. Fix an equivalence of Vec with its skeleton, consisting of objects $\mathbf{1}^{\oplus n}$, $n \geq 0$. On these objects, we simply define the section s^* in the only way we can: $\mathbf{1}$ has to go to $\mathbf{1}$, and direct sums must be respected. This is clearly a section, since f^* is monoidal and respects direct sums. We can explicitly describe the ind-right adjoint s_*, which turns out to restrict to an honest right adjoint of s^*. Indeed, there's a natural isomorphism $s_* \simeq \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, -)$. This follows immediately from the fact that $\underline{\mathrm{Hom}}_{\mathrm{Vec}}(\mathbf{1}, -)$ is the identity functor on Vec.

Given a neutralized Tannakian category $(\mathcal{B}, f^*)$, Tannaka duality produces a pro-algebraic group, which we denote $\mathscr{G}(f^*)$, or $\mathscr{G}(\mathcal{B}, f^*)$, or $\mathscr{G}(\mathcal{B})$, depending on what we want to emphasize. We call this group the *Tannaka dual* of $(\mathcal{B}, f^*)$. This group is such that f^* naturally factors through the forgetful functor $\mathrm{Rep}\,\mathscr{G}(f^*) \to \mathrm{Vec}$, and such that $\mathcal{B} \to \mathrm{Rep}\,\mathscr{G}(f^*)$ is a (monoidal) F-linear equivalence. We also have a canonical isomorphism $\mathscr{G}(\mathrm{Rep}\,G) \simeq G$ for all G.

Remarks 2.10 (On Tannakian categories and unipotency).

- A pro-algebraic group G is pro-unipotent if and only if $\mathrm{Rep}\,G$ is unipotent.

- All objects in a neutral Tannakian category are of finite length, so for these categories, weak unipotency and unipotency coincide.
- If V is a unipotent G-representation, then any subquotient of V in $\operatorname{Rep} G$ is unipotent. (Proof: Enough to check for subobjects of V, by rigidity. Given a unipotent filtration on V, the induced filtration on a subobject is also a unipotent filtration, by the fact that any subobject of $\mathbf{1}^n$ is of the form $\mathbf{1}^m$.) □

Tannaka duality is functorial: given a morphism p^* of neutralized Tannakian categories as above, we get an induced group-scheme homomorphism $p : \mathscr{G}(\varphi^*) \to \mathscr{G}(f^*)$, which we always write using the same symbol, but without the asterisk. Conversely, given a group homomorphism $p : G \to H$, we get a morphism of Tannakian categories $p^* : \operatorname{Rep} H \to \operatorname{Rep} G$, namely restriction along p. The ind-right adjoint p_* is in this context given by induction along p (cf. Section 2.3).

The following proposition is an example of how properties of group homomorphisms are reflected on the Tannakian side:

Proposition 2.11 [17, Proposition 2.21]. *Let $p : G \to H$ be a homomorphism of pro-algebraic groups. Then*:

(1) *p is a closed immersion* (*injective*) *if and only if every object in* $\operatorname{Rep} G$ *is isomorphic to a subquotient of an object in the image of p^*.*

(2) *p is faithfully flat* (*surjective*) *if and only if for every subobject $X \hookrightarrow p^*Y$ in* $\operatorname{Rep} G$, *there exists a subobject Y' of Y such that $p^*Y' \simeq X$.*

Proof. This is essentially shown in [17, Proposition 2.21], the only difference being that the second part also includes the condition that p^* is full. However, fullness follows from having a subobject stable image. Indeed, let Y, Y' be representations of H, and let $\varphi : p^*Y \to p^*Y'$ be a morphism. It corresponds to a subobject $\Gamma_\varphi \subseteq p^*Y \oplus p^*Y' \simeq p^*(Y \oplus Y')$, namely its graph. Lifting this to a subobject of $Y \oplus Y'$ amounts exactly to lifting φ to a map $Y \to Y'$. □

With the second part of this proposition in mind, we make the following definition:

Definition 2.12. A *Tannakian subcategory* of a Tannakian category is a full abelian subcategory closed under direct sums, tensor products, taking duals, and taking subquotients.

Given a homomorphism $G \to H$ of pro-algebraic groups, we sometimes denote by $\operatorname{Res}^H_G : \operatorname{Rep} H \to \operatorname{Rep} G$ the restriction functor from H-representations to G-representations.

Lemma 2.13. *Given a homomorphism $p : G \to H$, the smallest Tannakian subcategory of* $\operatorname{Rep} G$ *containing* $\operatorname{im}(p^*)$ *is the essential image of* $\operatorname{Res}_G^{\operatorname{im}(p)}$.

Proof. Let $\mathcal{T} \subseteq \operatorname{Rep}(G)$ denote the smallest Tannakian subcategory containing $\operatorname{im}(p^*)$. We have a factorization $G \to \mathscr{G}(\mathcal{T}) \to H$ of p, and that the first map is surjective by Proposition 2.11.(2). Next, we check that $\mathscr{G}(\mathcal{T}) \to H$ is injective by Proposition 2.11.(1). Thus, we need to show that every object in $\mathcal{T}$ is a subquotient of an object in the essential image of p^*. But this follows from the following elementary

fact: if you take a full, monoidal subcategory of a Tannakian category, closed under taking duals, then the closure of this subcategory under taking subquotients is a Tannakian category. Using this fact for the monoidal subcategory $\mathrm{im}(p^*)$ of $\mathrm{Rep}\,G$, we get that $\mathcal{T}$ is its closure under forming subquotients, by the minimality of $\mathcal{T}$. □

Lemma 2.14. *A homomorphism $p : G \to H$ has pro-unipotent kernel if and only if the category* $\mathrm{Rep}\,G$ *is* $\mathrm{Res}_G^{\mathrm{im}(p)}$*-unipotent.* (*Equivalently*, $\mathrm{Rep}\,G$ *is generated under extensions and subquotients by its full subcategory* $\mathrm{im}(p^*)$.)

Proof. Assume that $\mathrm{Rep}\,G$ is $\mathrm{Res}_G^{\mathrm{im}(p)}$-unipotent, and let $K := \ker(p)$. Then it is clear that whatever is in the image of Res_K^G, is (absolutely) unipotent. By Proposition 2.11.(1), every K-representation is then a subquotient of a unipotent K-representation. But such a representation must also be unipotent (Remarks 2.10).

Now assume that $\ker(p) =: K$ is pro-unipotent. Then, given a nonzero G-representation V, the K-invariants V^K are nonzero and form a sub-G-representation (since K is normal in G). Moreover, V^K is actually the restriction of an $\mathrm{im}(p)$-representation, since $\mathrm{im}(p) = \mathrm{coker}(K \to G)$. This shows $\mathrm{Res}_G^{\mathrm{im}(p)}$-unipotency of $\mathrm{Rep}\,G$, by Remarks 2.10. □

One perspective on Tannaka duality is as follows. Let f^* be a fiber functor on a neutral Tannakian category $\mathcal{B}$ as above. Then, using the 2-section s^* of f^* described in Remark 2.9, one can prove that $f^* f_* \mathbf{1}$ is naturally a Hopf algebra, and we denote it by $\mathscr{H}(f^*) := f^* f_* \mathbf{1}$. The Tannaka dual group is then $\mathscr{G}(f^*) := \mathrm{Spec}\,\mathscr{H}(f^*)$. Note that $\mathscr{G}(f^*)$-representations are the same as $\mathscr{H}(f^*)$-comodules.

Remark 2.15. It's worth noting at this point, that if f^* denotes the canonical fiber functor on $\mathrm{Rep}\,G$, then $f^* f_* \mathbf{1}$ is $\mathcal{O}(G)$, and $f_* \mathbf{1}$ is the regular representation of G. Moreover, $f_* \mathbf{1}$ is naturally an algebra object in $\mathrm{Ind}\,\mathrm{Rep}\,G$.

2.3. ***Restriction, induction, and invariants.*** Let $p : H \to G$ be a morphism of pro-algebraic groups. We denote the restriction functor $\mathrm{Rep}\,G \to \mathrm{Rep}\,H$ by p^*, or sometimes Res_H^G (usually when the morphism is unlabelled). In the literature, induction is often only defined for p injective. We depart from this convention and call the ind-right adjoint $p_* : \mathrm{Ind}\,\mathrm{Rep}\,H \to \mathrm{Ind}\,\mathrm{Rep}\,G$ of p^* *induction* for arbitrary p. For this reason, we sometimes denote p_* by Ind_H^G (again, usually when the morphism is unlabelled). We can decompose any p as a surjection $p_0 : H \to \mathrm{im}(p)$ followed by an injection $p_1 : \mathrm{im}(p) \to G$. Our induction functor p_* is then given by the composition of the functor $\Gamma(\ker(p_0), -) = (-)^{\ker(p_0)}$ taking invariants under the action of the kernel of p_0, and the more traditional induction functor $\mathrm{Ind}_{\mathrm{im}(p)}^G$ along the injective homomorphism p_1.

Remark 2.16. The invariants functor $\Gamma(G, -) = (-)^G$ on $\mathrm{Rep}\,G$ is given by "induction" along $G \to 1$, and the trivial representation functor $\mathrm{Vec} \to \mathrm{Rep}\,G$ is given by restriction along $G \to 1$.

An important fact is that the projection morphism for the restriction-induction adjunction is an isomorphism.

Proposition 2.17 (Restriction-induction projection formula). *Let p be a homomorphism of pro-algebraic groups. Then there is an isomorphism $p_*(A \otimes p^*B) \simeq p_*A \otimes B$, natural in A and B. In particular, setting $A = \mathbf{1}$ gives a natural isomorphism $p_*p^* \simeq p_*\mathbf{1} \otimes (-)$.*

Proof. For p injective, this is [25, I.3.6]. For p surjective, it is obvious. Combining these two cases, we can conclude for arbitrary p. □

We give two trivial consequences of this result, since they will be useful in the sequel. They both concern morphisms p that admit a retraction r, and that are therefore, in particular, injective. The first result gives a description of p^* in terms of p_* and r_*. The second says that, when tensoring with $p_*\mathbf{1}$, one may first change the $\ker(r)$-action to be trivial, and then tensor with $p_*\mathbf{1}$, and get the same result.

Lemma 2.18. *Let $p : H \to G$ be a homomorphism of pro-algebraic groups, and let $r : G \to H$ be a retraction of p. Then we have a natural isomorphism $p^* \simeq r_*(p_*\mathbf{1} \otimes -)$.*

Proof. By Proposition 2.17, we have an isomorphism $p_*p^* \simeq p_*\mathbf{1} \otimes (-)$. To this isomorphism, we apply r_*, then use that $r_*p_* = \mathrm{id}$, and get $p^* \simeq r_*(p_*\mathbf{1} \otimes -)$. □

Lemma 2.19. *Let $p : H \to G$ be a homomorphism of pro-algebraic groups, and let $r : G \to H$ be a retraction of p. Then we have a natural isomorphism $p_*\mathbf{1} \otimes (-) \simeq p_*\mathbf{1} \otimes r^*p^*(-)$.*

Proof. Using the projection formula in Proposition 2.17, we compute

$$p_*\mathbf{1} \otimes (-) \simeq p_*p^*(-) \simeq p_*(p^*r^*)p^*(-) \simeq p_*\mathbf{1} \otimes r^*p^*(-).$$ □

2.4. ***Group isomorphisms.*** Consider a homomorphism of pro-algebraic groups $p : G \to H$. We have a corresponding morphism of Tannakian categories $p^* : \mathrm{Rep}\, H \to \mathrm{Rep}\, G$ (restriction along p). Let f^* and φ^* be the canonical fiber functors of $\mathrm{Rep}\, H$ and $\mathrm{Rep}\, G$, respectively. Then p^* induces a morphism of algebra objects $p_{\mathrm{alg}} : p^*f_*\mathbf{1} \to \varphi_*\mathbf{1}$, from the restriction along p of the regular representation of H to the regular representation of G (cf. Remark 2.15). This map is simply the algebra homomorphism $\mathcal{O}(H) \to \mathcal{O}(G)$ induced by p, which respects the $\mathcal{O}(G)$-coactions, making it a morphism of algebra objects in $\mathrm{Ind}\,\mathrm{Rep}\, G$.

Another way of viewing this morphism, is via the universal property of $\varphi_*\mathbf{1}$. We have a natural isomorphism $\mathrm{Hom}(-, \varphi_*\mathbf{1}) \simeq \mathrm{Hom}(\varphi^*(-), \mathbf{1})$. So, to give a map $p^*f_*\mathbf{1} \to \varphi_*\mathbf{1}$, is the same as to give a map from $\varphi^*p^*f_*\mathbf{1}$ to $\mathbf{1}$. But $\varphi^*p^*f_*\mathbf{1} \simeq f^*f_*\mathbf{1}$ has a natural map to $\mathbf{1}$: the counit it's equipped with as a Hopf algebra. In this way, we recover the algebra homomorphism $p_{\mathrm{alg}} : p^*f_*\mathbf{1} \to \varphi_*\mathbf{1}$ above. We can see this by applying φ^* to p_{alg}, and noting that the result is a morphism of Hopf algebras, and is hence compatible with the two counits. We get the following proposition:

Proposition 2.20. *A homomorphism $p : G \to H$ of pro-algebraic groups is an isomorphism if and only if the algebra morphism $p^*\mathcal{O}(H) \to \mathcal{O}(G)$ in $\mathrm{Ind}\,\mathrm{Rep}\, G$, induced by the counit of $\mathcal{O}(H)$ under the universal property of $\mathcal{O}(G)$, is an isomorphism.*

2.5. ***Categories of modules.*** Consider an abelian tensor category $\mathcal{A}$ with all filtered colimits, in which the tensor product commutes with finite colimits, and let $\mathcal{A}^{\mathrm{cpt}}$ denote the full subcategory of compact objects: objects $C \in \mathcal{A}$ such that $\mathrm{Hom}(C, -)$ commutes with filtered colimits. Take an algebra object A in $\mathcal{A}$, and consider the category $\mathrm{Mod}(A)$ of (left) A-modules in $\mathcal{A}$, i.e., objects $M \in \mathcal{A}$ together with an A-action $A \otimes M \to M$ satisfying the usual axioms. The category $\mathrm{Mod}(A)$ is an abelian tensor category (under $\otimes_A$).

Definition 2.21. Let M be an A-module, as above. Then:

- M is *strictly free of finite rank* if $M \simeq A^{\oplus n}$ for some n.
- M is *free* if $M \simeq A \otimes X$ for some X in $\mathcal{A}$.
- M is *finitely generated free* if $M \simeq A \otimes X$ for some X in $\mathcal{A}^{\mathrm{cpt}}$.
- M is *finitely generated* if M is a quotient $A \otimes X \twoheadrightarrow M$ of some finitely generated free module.

Remark 2.22. If $\mathcal{B}$ is an abelian category, then $(\mathrm{Ind}\,\mathcal{B})^{\mathrm{cpt}}$ is exactly the essential image of $\mathcal{B}$ in $\mathrm{Ind}\,\mathcal{B}$. This is the situation in which we will use these notions, i.e., $\mathcal{B}$ will be an abelian tensor category, $\mathcal{A}$ will be $\mathrm{Ind}\,\mathcal{B}$, and A will be an algebra object in $\mathrm{Ind}\,\mathcal{B}$.

2.6. ***The Standard Situation.*** We now describe the Tannaka dual of the group theoretic setup from Theorems 1.12 and 1.13 from the introduction. Consider the following diagram in nTan:[2]

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{B^*} & \mathcal{T} \\ {\scriptstyle f^*}\downarrow\uparrow{\scriptstyle s^*} & & {\scriptstyle \varphi^*}\downarrow\uparrow{\scriptstyle \sigma^*} \\ \mathcal{E} & \xrightarrow[b^*]{} & \mathrm{Vec} \end{array} \tag{5}$$

We sometimes refer to B^* and b^* as *realization functors*. We assume we're given monoidal isomorphisms $b^* f^* \simeq \varphi^* B^*$, $\varphi^* \sigma^* \simeq \mathrm{id}$, and $f^* s^* \simeq \mathrm{id}$. These combine to give various other isomorphisms, such as $b^* \simeq \varphi^* B^* s^*$. Moreover, σ^* and s^* are automatically full, since they are 2-sections of faithful functors. We denote the Tannaka duals as follows:

$$H := \mathscr{G}(\mathcal{E}), \quad G := \mathscr{G}(\mathcal{M}), \quad P := \mathscr{G}(\mathcal{T}).$$

Let us extract from diagram (5) the sequence

$$\mathcal{E} \xrightarrow{s^*} \mathcal{M} \xrightarrow{B^*} \mathcal{T},$$

of morphisms of neutral Tannakian categories. We have a dual sequence of groups

$$P \xrightarrow{B} G \xrightarrow{s} H.$$

[2] If the choice of symbols looks strange, it is because it has been made with certain applications in mind. The letters $\mathcal{M}$ and $\mathcal{T}$ were chosen with the words *motivic* and *topological* in mind, while B and b were taken from *Betti*. The letters f and φ are meant to evoke *fiber*, while s and σ simply stand for *section*.

Note that s is surjective, since the 2-retraction f^* of s^* in nTan produces a section f of s.[3] Writing $\iota : K \to G$ for the kernel of s, we get a split exact sequence of pro-algebraic groups

$$1 \longrightarrow K \xrightarrow{\iota} G \underset{s}{\overset{f}{\rightleftarrows}} H \longrightarrow 1,$$

and an isomorphism $G \simeq K \rtimes H$.

The splitting just described induces an action by H on K, telling us that $\mathcal{O}(K)$ should naturally come from $\operatorname{Ind}\mathcal{E}$. Indeed, the fact that f^* has a nice section s^*, gives rise to a natural Hopf algebra structure on $f^*f_*\mathbf{1}$ in $\operatorname{Ind}\mathcal{E}$ (see [4, Theorem 1.45]), and this Hopf algebra then describes the kernel group K. More precisely, $\operatorname{Spec}(b^*(f^*f_*\mathbf{1})) \simeq K$ (see [5, Theorem 2.7]). In analogy with the notation introduced in Section 2.2, we write $f^*f_*\mathbf{1} =: \mathscr{H}(\mathcal{M}/\mathcal{E},\, f^*,\, s^*)$, omitting some of the arguments from the notation depending on what we want to emphasize. Likewise, we write $\mathscr{G}(\mathcal{M}/\mathcal{E},\, f^*,\, s^*)$ for its pro-dual in $(\operatorname{Ind}\mathcal{E})^{\mathrm{op}} \simeq \operatorname{Pro}(\mathcal{E}^{\mathrm{op}})$. Then one can make sense of the formula $b^*\mathscr{G}(\mathcal{M}/\mathcal{E}) \simeq K$.

The composition $B^* \circ s^*$ factors through σ^*, up to a natural isomorphism. Thus, the composition $s \circ B$ is trivial, and B factors as $B = \iota \circ \varrho$, through the kernel K of s. These diagrams summarize the situation:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\iota} & G & \underset{s}{\overset{f}{\rightleftarrows}} & H & \longrightarrow & 1 \\ & & & \overset{\exists\varrho}{\nwarrow}\!\cdots & \uparrow{\scriptstyle B} & & & & \\ & & & & P & & & & \end{array} \qquad \begin{array}{ccccc} \mathcal{M} & \xrightarrow{\iota^*} & \operatorname{Rep} K & \xrightarrow{\varrho^*} & \mathcal{T} \\ {\scriptstyle f^*}\downarrow\uparrow{\scriptstyle s^*} & & {\scriptstyle g^*}\downarrow\uparrow{\scriptstyle t^*} & & {\scriptstyle \varphi^*}\downarrow\uparrow{\scriptstyle \sigma^*} \\ \mathcal{E} & \xrightarrow[b^*]{} & \operatorname{Vec} & = & \operatorname{Vec} \end{array}$$

Finally, there is a canonical map

$$b^*Rs_* \to R\sigma_*B^*, \tag{6}$$

which will be important to us. It corresponds under adjunction to the map $\sigma^*b^*Rs_* \to B^*$ which, via the isomorphism $\sigma^*b^* \simeq B^*s^*$, is induced by the counit of the $(s^*,\, Rs_*)$-adjunction. In our two main theorems, the key assumption is that the degree 0 and 1 parts of this canonical map (6) are isomorphisms.

3. The unipotent isomorphism criterion

The goal of this section is to prove Theorem 1.13. The methods are Tannakian, and the first step is to reformulate the result under Tannakian duality, via the Standard Situation (Section 2.6):

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{B^*} & \mathcal{T} \\ {\scriptstyle f^*}\downarrow\uparrow{\scriptstyle s^*} & & {\scriptstyle \varphi^*}\downarrow\uparrow{\scriptstyle \sigma^*} \\ \mathcal{E} & \xrightarrow[b^*]{} & \operatorname{Vec} \end{array}$$

[3] It is somewhat unfortunate that f denotes the section of s, instead of the other way around. But since the bulk of our arguments takes place on the Tannakian side of things, it seemed reasonable to give priority in the notation to the fact that s^* is a section of f^*.

We may now state the first key result.

Theorem 3.1. *Suppose $\mathcal{T}$ is unipotent. If the canonical maps $b^*R^is_* \to R^i\sigma_*B^*$, for $i = 0, 1$, are isomorphisms, then the natural algebra homomorphism $B^*f_*\mathbf{1} \to \varphi_*\mathbf{1}$ (see Section 2.4) is an isomorphism.*

Remark 3.2. The assumptions of Theorem 3.1 imply that $\mathcal{M}$ is s^*-unipotent, as follows. By Remarks 2.10, it's enough to show weak unipotency. But given a nonzero object X of $\mathcal{M}$, the subobject s^*s_*X must be nonzero. (The fact that it's a subobject follows from the surjectivity of $\mathscr{G}(\mathcal{M}) \to \mathscr{G}(\mathcal{E})$.) To check this, we apply B^*, yielding $\sigma^*\sigma_*B^*X$, Now the first assumption give that this is nonzero, since B^*X is nonzero and $\sigma^*\sigma_*B^*X$ is the largest subobject of B^*X in the essential image of σ^*.

Recall that in the Standard Situation (Section 2.6), there is an induced group homomorphism

$$\varrho : P = \mathscr{G}(\varphi^*) \to b^*\mathscr{G}(f^*) = K.$$

Using Theorem 3.1 and Proposition 2.20, we get Theorem 1.13 from the introduction, which in the present context reads as follows.

Theorem 3.3 (Unipotent isomorphism criterion). *Suppose P is pro-unipotent. If the canonical maps $b^*R^is_* \to R^i\sigma_*B^*$, for $i = 0, 1$, are isomorphisms, then ϱ is an isomorphism.* □

We prove Theorem 3.1 by constructing $U \simeq \varphi_*\mathbf{1}$ in $\operatorname{Ind}\mathcal{T}$ (Section 3.1 and Proposition 3.9) and $W \simeq f_*\mathbf{1}$ in $\operatorname{Ind}\mathcal{M}$ (Section 3.2 and Proposition 3.14) as well as an isomorphism $B^*W \simeq U$ (Proposition 3.6). After all of that, we only have to check some compatibilities, which we do in Section 3.4.

3.1. ***Construction of U.*** The relevant context in this section is that of a Tannakian category $\mathcal{T}$ with fiber functor φ^*, which has a section σ^* as in Remark 2.9. We need no further assumptions for the construction.

We begin with some motivation. When $\mathcal{T}$ is assumed to be unipotent, every object is (by definition) a successive extension of trivial objects, starting from $\mathbf{1}$. The ind-object $\varphi_*\mathbf{1}$ is also (ind-)unipotent in this situation. It moreover (always) satisfies a universal property that makes it as "large", in the sense that every object in $\mathcal{T}$ appears as a subquotient of some $(\varphi_*\mathbf{1})^{\oplus n}$. The idea, when constructing U, is therefore to start from $\mathbf{1}$ and iterate a kind of universal extension-of-a-trivial-object operation, to get a large, universal, ind-unipotent object.

Let $U_0 := \mathbf{1}_{\mathcal{T}}$ and construct U_{n+1} inductively:

$$0 \to U_n \to U_{n+1} \to \sigma^*V \to 0, \tag{7}$$

i.e., define U_{n+1} as an extension of a trivial object by U_n. We have to say what V is and what the extension class is.

Note that

$$\operatorname{Ext}^1(\sigma^*V, U_n) = \operatorname{Hom}(\sigma^*V, U_n[1]) = \operatorname{Hom}(V, R\sigma_*U_n[1]) = \operatorname{Hom}(V, R^1\sigma_*U_n),$$

where the last equality is due to Vec being semisimple. We therefore take V to be $R^1\sigma_*U_n$ and the extension class to be given by the identity in $\operatorname{End}(R^1\sigma_*U_n)$. Set $U := \operatorname{colim}_{n\geq 0} U_n$ in $\operatorname{Ind}\mathcal{T}$.

Remark 3.4. These are *universal extensions* in the sense of [12, Definition 3.1], but we won't use this.

We now record two key properties of this construction. Apply $\sigma_* \simeq \mathrm{Hom}(\mathbf{1}, -)$ to the defining sequence (7) of U_{n+1} to get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathbf{1}, U_n) &\xrightarrow{a} \mathrm{Hom}(\mathbf{1}, U_{n+1}) \longrightarrow R^1\sigma_* U_n \\ &\xrightarrow{\delta} \mathrm{Ext}^1(\mathbf{1}, U_n) \xrightarrow{b} \mathrm{Ext}^1(\mathbf{1}, U_{n+1}) \longrightarrow \cdots . \end{aligned}$$

By definition, δ is an isomorphism. Therefore, a is an isomorphism and b is the zero map. This establishes the following lemma:

Lemma 3.5. *We have a canonical isomorphism $\mathbf{1} \simeq \sigma_* U_0$, and the canonical maps $\sigma_* U_n \to \sigma_* U_{n+1}$, $n \geq 0$, are isomorphisms. The pushout extension along $U_n \to U_{n+1}$ of any extension of a trivial object by U_n is split.* □

3.2. *Construction of W.* For this construction, we work with Tannakian categories $\mathcal{M}$ and $\mathcal{E}$ and a morphism $f^* : \mathcal{M} \to \mathcal{E}$ with a section s^*, as introduced above. We need no further assumptions for the construction.

The idea for this construction is simply to implement the construction of U in this more general context. The difficulty arises from the failure of $\mathcal{E}$ to be semisimple (in contrast to Vec). While not every distinguished triangle in the derived category $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}$ splits, the ones we need to consider, it turns out, do split. The functor f^* plays a key role in finding these splittings.

Let $W_0 := \mathbf{1}_{\mathcal{M}}$. We define W_{n+1} inductively from W_n as an extension

$$0 \to W_n \to W_{n+1} \to s^* R^1 s_* W_n \to 0. \tag{8}$$

Such an extension is given by a map

$$s^* R^1 s_* W_n \to W_n[1],$$

in the derived category. The counit of the (s^*, s_*)-adjunction gives us a map $s^* R s_* W_n[1] \to W_n[1]$, and what's missing is a map $s^* R^1 s_* W_n \to s^* R s_* W_n[1]$. This map will be the image under s^* of a map $R^1 s_* W_n \to R s_* W_n[1]$ which we now spend some time constructing.

We shall make two auxiliary induction hypotheses.

(IH-1) We have a canonical isomorphism $\mathbf{1} \simeq s_* W_0$, and the canonical maps

$$s_* W_0 \to s_* W_1 \to \cdots \to s_* W_n,$$

are isomorphisms.

(IH-2) We have a map $\alpha_n : f^* W_n \to \mathbf{1}$, such that the composition

$$\mathbf{1} \simeq s_* W_n \simeq f^* s^* s_* W_n \to f^* W_n \xrightarrow{\alpha_n} \mathbf{1},$$

is the identity map.

These hypotheses are satisfied for the base case $n = 0$: $s_*\mathbf{1} \simeq s_*s^*\mathbf{1} \simeq \mathbf{1}$, since s^* is monoidal and fully faithful; and we let $\alpha_0 : f^*\mathbf{1} \simeq \mathbf{1}$ be the map coming from the monoidal structure of f^*.

We are now ready to construct the missing map $R^1s_*W_n \to Rs_*W_n[1]$, using the auxiliary assumptions on W_n. They allow us to split off s_*W_n from Rs_*W_n.

Note that (IH-1) gives us a map $\mathbf{1} \to Rs_*W_n$, which is an isomorphism on the zeroth cohomology. Now consider a cone C of this map:

$$\mathbf{1} \to Rs_*W_n \to C \to . \tag{9}$$

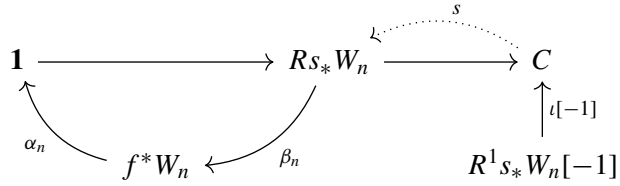
Consider also The map $\beta_n : Rs_*W_n \to f^*W_n$ defined as the composition

$$Rs_*W_n \simeq f^*s^*Rs_*W_n \to f^*W_n.$$

By (IH-2), the triangle (9) is split by retracting the left map, $\mathbf{1} \to Rs_*W_n$, with

$$Rs_*W_n \xrightarrow{\beta_n} f^*W_n \xrightarrow{\alpha_n} \mathbf{1},$$

and we hence get a section $s : C \to Rs_*W_n$. By construction, the first nonzero cohomology of C is $H^1C = R^1s_*W_n$ and we get a map $\iota : R^1s_*W_n \to C[1]$. Composing with $s[1] : C[1] \to Rs_*W_n[1]$ gives the map $R^1s_*W_n \to Rs_*W_n[1]$ we are looking for. The following diagram may clarify the situation:



What is left is to establish the induction hypotheses (IH-1) and (IH-2) for W_{n+1}. We start with lifting α_n to W_{n+1}. Applying f^* and then the contravariant $\mathrm{Hom}(-, \mathbf{1})$ to the defining sequence (8) of W_{n+1} yields an exact sequence

$$\mathrm{Hom}(f^*W_{n+1}, \mathbf{1}) \longrightarrow \mathrm{Hom}(f^*W_n, \mathbf{1}) \xrightarrow{a} \mathrm{Ext}^1(R^1s_*W_n, \mathbf{1})$$

with $\alpha_{n+1} \mapsto \alpha_n \mapsto a(\alpha_n)$; hence the existence of α_{n+1} is equivalent to $a(\alpha_n)$ being zero. But $a(\alpha_n) = (\alpha_n[1]) \circ \omega_n$, where ω_n is the extension class of f^*W_{n+1}, i.e.,

$$\omega_n : R^1s_*W_n \xrightarrow{\iota} C[1] \xrightarrow{s[1]} Rs_*W_n[1] \xrightarrow{\beta_n[1]} f^*W_n[1].$$

Composing this with $\alpha_n[1]$ gives zero, because $\alpha_n \circ \beta_n \circ s = 0$, as this was how we split the triangle (9) defining C. Finally, consider the following diagram:

$$\begin{array}{ccccccccc}
\mathbf{1} & \xrightarrow{\sim} & s_*W_n & \xrightarrow{\sim} & f^*s^*s_*W_n & \longrightarrow & f^*W_n & \xrightarrow{\alpha_n} & \mathbf{1} \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
\mathbf{1} & \longrightarrow & s_*W_{n+1} & \longrightarrow & f^*s^*s_*W_{n+1} & \longrightarrow & f^*W_{n+1} & \xrightarrow{\alpha_{n+1}} & \mathbf{1}
\end{array}$$

It is commutative, and hence the bottom row is the identity as required, establishing (IH-2) for W_{n+1}.

Lastly, as the full faithfulness of s^* implies $s_*s^* \simeq \mathrm{id}$, the long exact sequence after applying s_* to (8) is

$$\begin{aligned} 0 \longrightarrow s_*W_n \xrightarrow{a} s_*W_{n+1} \longrightarrow R^1s_*W_n \\ \xrightarrow{\delta} R^1s_*W_n \xrightarrow{b} R^1s_*W_{n+1} \longrightarrow \cdots . \end{aligned}$$

Because δ is the identity map by construction, we get that $a : s_*W_n \simeq s_*W_{n+1}$ is an isomorphism, establishing (IH-1) for W_{n+1}. This finishes the construction of $W := \mathrm{colim}_{n\geq 0} W_n$.

3.3. ***Properties.*** In this section, we establish some key properties of U and W. Most importantly, their isomorphisms with $\varphi_*\mathbf{1}$ and $f_*\mathbf{1}$, respectively, are proved in Proposition 3.9 and Proposition 3.14.

3.3.1. *W realizes to U.* We begin with establishing that B^*W is isomorphic to U.

Proposition 3.6. *If the canonical map $b^*R^1s_* \to R^1\sigma_*B^*$ is an isomorphism, then $B^*W \simeq U$.*

Proof. We give an isomorphism of inductive systems $(B^*W_n)_n \simeq (U_n)_n$. Clearly $B^*W_0 \simeq U_0$, since B^* is monoidal. We now proceed by induction, and assume that $B^*W_n \simeq U_n$. Applying B^* to the defining sequence (8) for W_{n+1} yields an isomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^*W_n & \longrightarrow & B^*W_{n+1} & \longrightarrow & B^*(s^*R^1s_*W_n) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & U_n & \longrightarrow & E & \longrightarrow & \sigma^*R^1\sigma_*U_n & \longrightarrow & 0 \end{array}$$

where we've used that realization commutes with pushforwards on the rightmost term. We have to check that the extension class is correct. Applying B^* to the composition

$$s^*R^1s_*W_n \to s^*C[1] \to s^*Rs_*W_n[1] \to W_n[1],$$

gives

$$\sigma^*R^1\sigma_*U_n \to \sigma^*b^*C[1] \to \sigma^*b^*Rs_*W_n[1] \to U_n[1].$$

Under adjunction this corresponds to

$$R^1\sigma_*U_n \to b^*C[1] \to b^*Rs_*W_n[1] \to R\sigma_*U_n[1].$$

This map is in the semisimple category $\mathcal{D}(\mathrm{Vec})$, and the source is concentrated in degree zero, so it is equivalent to its zeroth cohomology

$$R^1\sigma_*U_n \to H^1(b^*C) \to b^*R^1s_*W_n \to R^1\sigma_*U_n,$$

and, looking at the construction of U, we have to see that this composition is the identity. This is easy to see, using the definition of C together with the definition of the first two maps, as well as the assumption of the proposition (to deal with the last map). □

3.3.2. *Characterization of U.* The universal property of $\varphi_*\mathbf{1}$ is that the natural transformation

$$\mathrm{Hom}(-, \varphi_*\mathbf{1}) \to \mathrm{Hom}(\varphi^*(-), \mathbf{1}),$$

is an isomorphism. This transformation takes a map on the left hand side, applies φ^* to it, and then composes with $\varphi^*\varphi_*\mathbf{1} \to \mathbf{1}$. In order to show the isomorphism $U \simeq \varphi_*\mathbf{1}$, we simply have to establish the same universal property for U.

A key property of $\varphi_*\mathbf{1}$ (which we won't use) is the following:

Proposition 3.7. *Every short exact sequence*

$$0 \to \varphi_*\mathbf{1} \to E \to X \to 0,$$

in $\mathrm{Ind}\,\mathcal{T}$ *splits.*

Proof. Since φ^* is exact, $\varphi_*\mathbf{1}$ is injective. □

Lemma 3.5 implies that U has a weaker variant of this property, given by the following proposition:

Proposition 3.8 (Splitting property). *Every short exact sequence of the form*

$$0 \to U \to E \to \sigma^*V \to 0,$$

in $\mathrm{Ind}\,\mathcal{T}$, *where V is a finite-dimensional vector space, splits.*

Proof. Without loss of generality, we may replace σ^*V by $\mathbf{1}$. We have that E is the filtered colimit of its subobjects lying in $\mathcal{T}$. Let L be a subobject of E, in $\mathcal{T}$, which is not contained in (the image of) U (it is clearly possible to find such an L). Then, let $L' := L \cap U$, and denote the cokernel of $L' \to L$ by Q. We obtain an inclusion of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & \mathbf{1} & \longrightarrow & 0 \end{array}$$

Thus Q has to be either 0 or $\mathbf{1}$, and it can't be 0 since $L \neq L'$ by assumption. We therefore have our sequence as a pushout along $L' \to U$ of this subsequence lying in $\mathcal{T}$. The map $L' \to U$ factors through $U_n \to U_{n+1}$ for some n, and we conclude by Lemma 3.5. □

Consider the map $u : \varphi^*U \to \mathbf{1}$, obtained from $b^*\alpha_n$:

$$\begin{array}{ccc} b^*f^*W_n & \xrightarrow{\sim} & \varphi^*U_n \\ \downarrow{\scriptstyle b^*\alpha_n} & \swarrow_{u_n} & \\ \mathbf{1} & & \end{array}$$

By definition, $u_0 : \varphi^*\mathbf{1} = \varphi^*U_0 \to \mathbf{1}$ is the map coming from the monoidal structure of φ^*. Using the splitting property (Proposition 3.8) and the fact that $u_0 : \varphi^*U_0 \to \mathbf{1}$ is an isomorphism, we can now prove that $U \simeq \varphi_*\mathbf{1}$.

Proposition 3.9. *Assume that $\mathcal{T}$ is unipotent. Then, the natural transformation*

$$\nu : \mathrm{Hom}(-, U) \to \mathrm{Hom}(\varphi^*(-), \mathbf{1}), \quad \psi \mapsto u \circ \varphi^*(\psi),$$

is an isomorphism. In particular, the maps $U \to \varphi_*\mathbf{1}$ and $\varphi_*\mathbf{1} \to U$ induced by $u : \varphi^*U \to \mathbf{1}$ and $\varphi^*\varphi_*\mathbf{1} \to \mathbf{1}$, respectively, are mutually inverse isomorphisms.

Proof. We need to prove that ν_L is bijective for every $L \in \mathcal{T}$. The method of proof is induction on the unipotency index n of L. More precisely, in the induction step we assume that for an L' with unipotency index strictly lower than n, the map $\mathrm{Hom}(L', U) \to \mathrm{Hom}(\varphi^*L', \mathbf{1})$, sending ψ' to $u \circ \varphi^*(\psi')$, is a bijection.

Surjectivity: *Base case*: L is isomorphic to some σ^*V. We suppose we're given a map $\bar{\psi} : V \simeq \varphi^*\sigma^*V \to \mathbf{1}$. Applying σ^*, we get a map $\psi : \sigma^*V \to \sigma^*\mathbf{1} = U_0 \to U$. Then $\nu(\psi) = u \circ \varphi^*(\psi) = \bar{\psi}$.

Induction step: Now suppose that L has unipotency index $n > 0$ and that

$$0 \to L' \xrightarrow{a} L \xrightarrow{b} \sigma^*V \to 0, \tag{10}$$

where L' has a unipotency index of at most $n - 1$. We suppose we're given a map $\bar{\psi} : \varphi^*L \to \mathbf{1}$, which we need to lift to a map $\psi : L \to U$ such that $\nu(\psi) = \bar{\psi}$.

Composing $\varphi^*(a)$ with $\bar{\psi}$ yields $\bar{\psi}' : \varphi^*L' \to \mathbf{1}$. By the induction hypothesis, $\bar{\psi}'$ can be lifted to $\psi' : L' \to U$, such that $\nu(\psi') = \bar{\psi}'$. We push out the short exact sequence (10) via ψ':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \xrightarrow{a} & L & \xrightarrow{b} & \sigma^*V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & U & \xrightarrow[\ulcorner]{} & E & \longrightarrow & \sigma^*V & \longrightarrow & 0 \end{array}$$

By Proposition 3.8, the lower sequence must split: choose a splitting. Denote the composed map $L \to U$ by ψ_0. This is not quite the lift we're looking for (it depends on the choice of splitting). We have $\psi_0 a = \psi'$, so that

$$u \circ \varphi^*(\psi_0) \circ \varphi^*(a) = u \circ \varphi^*(\psi') = \bar{\psi}' = \bar{\psi} \circ \varphi^*(a)$$

This implies that $u \circ \varphi^*(\psi_0) - \bar{\psi}$ factors through $\varphi^*(b)$, i.e., that there exists $\bar{\psi}'' : V \to \mathbf{1}$ such that

$$u \circ \varphi^*(\psi_0) - \bar{\psi} = \bar{\psi}'' \circ \varphi^*(b).$$

By induction, $\bar{\psi}''$ lifts to $\psi'' : \sigma^*V \to U$, and we define $\psi := \psi_0 - \psi'' \circ b$. This is the lift of $\bar{\psi}$ that we are looking for:

$$\nu(\psi) = u \circ \varphi^*(\psi) = u \circ \varphi^*(\psi_0 - \psi'' \circ b) = u \circ \varphi^*(\psi_0) - \bar{\psi}'' \circ \varphi^*(b) = \bar{\psi}.$$

Injectivity: *Base case*: φ^* is fully faithful on the essential image of σ^*, and u_0 is an isomorphism.

Induction step: Let $\psi : L \to U$ be such that $\nu(\psi) := u \circ \varphi^*(\psi) = 0$. Then, with L' as before, the composition $L' \to L \to U$ is zero, by induction. Therefore, $\psi = \psi'' \circ b$. Since b is epic and $\nu(\psi) = 0$, we get $\nu(\psi'') = 0$, and by induction $\psi'' = 0$, implying $\psi = 0$. □

Remark 3.10. The above proof is inspired by the proof of [27, Proposition 1.17], which Lazda writes he copied from [19, Proposition 2.1.6].

We have the following useful property of $\varphi_*\mathbf{1}$, which is also enjoyed by U, given the previous result:

Lemma 3.11. $$\sigma_*(\varphi_*\mathbf{1}\otimes -)\simeq \varphi^*(-).$$

Proof. Apply Lemma 2.18. □

3.3.3. *Characterization of W.* In this section, we prove that $W\simeq f_*\mathbf{1}$ under the assumptions of Theorem 3.1. We are able to do this just using the fact that W realizes to U (Proposition 3.6), and the characterization of U (Proposition 3.9).

Lemma 3.12. $$s_*(f_*\mathbf{1}\otimes -)\simeq f^*(-).$$

Proof. Apply Lemma 2.18. □

By the universal property of $f_*\mathbf{1}$, the map $\alpha : f^*W\to\mathbf{1}$ induces a map $W\to f_*\mathbf{1}$. This then yields a natural transformation

$$s_*(W\otimes -)\to s_*(f_*\mathbf{1}\otimes -). \tag{11}$$

Lemma 3.13. *Assume that $\mathcal{T}$ is unipotent and that the canonical maps $b^*R^is_*\to R^i\sigma_*B^*$, for $i\le 1$, are isomorphisms. Then the natural transformation* (11) *is an isomorphism.*

Proof. After applying b^* we may view the transformation as the composition

$$b^*s_*(W\otimes -)\simeq\sigma_*(U\otimes B^*(-))\simeq\sigma_*(\varphi_*\mathbf{1}\otimes B^*(-))\simeq\varphi^*B^*\simeq b^*f^*\simeq b^*s_*(f_*\mathbf{1}\otimes -),$$

using $b^*s_*\simeq\sigma_*B^*$, Propositions 3.6 and 3.9, and Lemmas 3.11 and 3.12. We conclude by the conservativity of b^*. □

Proposition 3.14. *Assume that $\mathcal{T}$ is unipotent and that the canonical maps $b^*R^is_*\to R^i\sigma_*B^*$, for $i\le 1$, are isomorphisms. Then $\alpha : f^*W\to\mathbf{1}$ induces an isomorphism $W\simeq f_*\mathbf{1}$.*

Proof. For every $A\in\mathcal{M}$, we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}(A,W)&\simeq\mathrm{Hom}(\mathbf{1},W\otimes A^\vee)\\&\simeq\mathrm{Hom}(\mathbf{1},s_*(W\otimes A^\vee))\\&\simeq\mathrm{Hom}(\mathbf{1},s_*(f_*\mathbf{1}\otimes A^\vee))\quad\text{(by Lemma 3.13)}\\&\simeq\mathrm{Hom}(\mathbf{1},f_*\mathbf{1}\otimes A^\vee)\\&\simeq\mathrm{Hom}(A,f_*\mathbf{1}),\end{aligned}$$

and we conclude by the Yoneda lemma. □

Remark 3.15 (On choices and canonicity). Lots of choices are made in the constructions of W and U. Namely, each time we take an extension corresponding to an extension class, this involves a choice. Moreover, the successive lifts α_{n+1} of α_n giving the map $\alpha : f^*W\to\mathbf{1}$ all involve choices. However, the

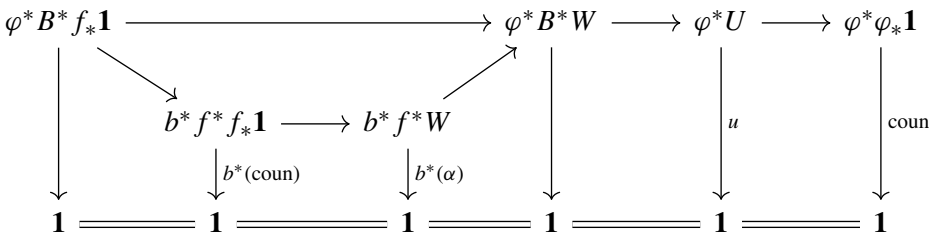
choice of α is in a sense the only one that matters: after this, the comparison isomorphism $B^*W \simeq U$ is uniquely determined, in view of the universal properties Proposition 3.9 and Proposition 3.14.

3.4. ***Proof of Theorem 3.1.*** With the properties above established for U and W, we can now prove the main result of this section.

Proof of Theorem 3.1. We have

$$B^* f_* \mathbf{1} \simeq B^* W \simeq U \simeq \varphi_* \mathbf{1},$$

by Propositions 3.14, 3.6 and 3.9. We claim that this isomorphism is the natural algebra morphism $B^* f_* \mathbf{1} \to \varphi_* \mathbf{1}$ induced by the counit $b^* f^* f_* \mathbf{1} \to \mathbf{1}$. This follows from the commutativity of the diagram



□

4. Malcev completions

Let $\varrho : P \to K$ be a homomorphism of pro-algebraic groups.

Definition 4.1. Let $\mathrm{Un}(\varrho^*)$ be the smallest Tannakian subcategory (see Definition 2.12) of $\mathrm{Rep}\, P$ that contains the image of ϱ^* and is stable under extensions. The Tannaka dual of $\mathrm{Un}(\varrho^*)$ is called the *Malcev completion* of P with respect to $\varrho : P \to K$.

Lemma 4.2. $\mathrm{Un}(\varrho^*)$ *is the category of* $\mathrm{Res}_P^{\mathrm{im}(\varrho)}$*-unipotent objects in* $\mathrm{Rep}\, P$.

Proof. By Lemma 2.13, $\mathrm{Un}(\varrho^*)$ contains the image of $\mathrm{Res}_P^{\mathrm{im}(\varrho)}$. Thus, it also contains all $\mathrm{Res}_P^{\mathrm{im}(\varrho)}$-unipotent objects, since it's stable under extensions. To conclude, note that the class of $\mathrm{Res}_P^{\mathrm{im}(\varrho)}$-unipotent objects is stable under extension (Lemma 2.4), and apply Lemma 4.3. □

Lemma 4.3 (Extension-closure of a Tannakian subcategory). *Let $\mathcal{T}$ be a neutral Tannakian category and $\mathcal{S} \subseteq \mathcal{T}$ a Tannakian subcategory. The smallest subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and is closed under extensions is also a Tannakian subcategory.* □

Remark 4.4. We spell things out more explicitly: by Lemma 4.2, $\mathrm{Un}(\varrho^*)$ is the full subcategory of $\mathrm{Rep}\, P$ with objects those $X \in \mathrm{Rep}\, P$ that admit a filtration

$$0 \subseteq X_0 \subseteq \ldots \subseteq X_n = X$$

such that X_{i+1}/X_i lies in the image of $\mathrm{Res}_P^{\mathrm{im}(\varrho)}$. Equivalently, the X_{i+1}/X_i are subquotients of objects in the image of $\varrho^* = \mathrm{Res}_P^K$.

Lemma 4.2 implies that, replacing K with $\mathrm{im}(\varrho)$, we may assume that ϱ is surjective, without changing the Malcev completion.

Remark 4.5 (Universal property of the Malcev completion). Let $\mathcal{G}$ be the Malcev completion of ϱ. We get an exact sequence of groups

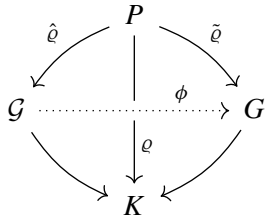
$$1 \to \mathcal{U} \to \mathcal{G} \to K,$$

and the kernel $\mathcal{U}$ is pro-unipotent (Lemma 2.14 and Lemma 4.2). The map $\mathcal{G} \to K$ is given dually, by sending a K-representation X to $\varrho^* X$, which is an object in the full subcategory $\mathrm{Rep}\,\mathcal{G}$ of $\mathrm{Rep}\,P$. There is a canonical map $\hat{\varrho} : P \to \mathcal{G}$ which by definition factors ϱ. It's defined via $\hat{\varrho}^*$, which is just the full inclusion of $\mathrm{Rep}\,\mathcal{G}$ into $\mathrm{Rep}\,P$.

The Malcev completion $\mathcal{G}$ is universal among groups factoring ϱ in this way. Let G be another pro-algebraic group equipped with a map $G \to K$, which has a pro-unipotent kernel U, and a map $\tilde{\varrho} : P \to G$ factoring ϱ.

$$\begin{array}{ccccc} 1 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & K \\ & & & & \uparrow{\scriptstyle \tilde{\varrho}} & \nearrow{\scriptstyle \varrho} & \\ & & & & P & & \end{array}$$

There is then a unique map $\phi : \mathcal{G} \to G$ such that $\phi \circ \hat{\varrho} = \tilde{\varrho}$. To see this, we need to check that $\tilde{\varrho}^*$ lands in the full subcategory $\mathrm{Rep}\,\mathcal{G}$ of $\mathrm{Rep}\,P$. Since $\mathrm{Rep}\,\mathcal{G}$ is stable under extensions and taking subquotients in $\mathrm{Rep}\,P$, we only need to check that $\tilde{\varrho}^*$ lands in $\mathrm{Rep}\,\mathcal{G}$ when restricted to those representations coming from K, by Proposition 2.11.(2) and Lemma 2.14. But on those representations, $\tilde{\varrho}^*$ coincides with ϱ^*, which we know lands in $\mathrm{Rep}\,\mathcal{G}$, and we are done.



Lemma 4.6. *Given a homomorphism $\varrho : P \to K$, its Malcev completion $\hat{\varrho} : P \to \mathcal{G}$ is surjective.*

Proof. It follows immediately from Proposition 2.11.(2). □

Definition 4.7. Let $P \to K$ be a homomorphism of pro-algebraic groups. We say that it's *Malcev complete* if the homomorphism $\mathcal{G} \to K$ from the Malcev completion to K is an isomorphism.

Example 4.8. As a trivial example, note that $\mathrm{id} : P \to P$, and more generally any automorphism of P, is always Malcev complete. More generally, the Malcev completion of an injective homomorphism is simply the image of the homomorphism.

Lemma 4.9. *A homomorphism $P \to K$ is Malcev complete if and only if it's surjective and the essential image of* $\mathrm{Rep}\,K$ *in* $\mathrm{Rep}\,P$ *is closed under extensions.* □

Lemma 4.10. *Let $P \to G \to K$ be homomorphisms such that $P \to G$ is Malcev complete. Then, the Malcev completions $\mathcal{G}$ and $\mathcal{G}'$ of $P \to K$ and $G \to K$, respectively, coincide.*

Proof. Note that $\operatorname{Rep} G$ is a full Tannakian subcategory of $\operatorname{Rep} P$, and that $\operatorname{Rep} \mathcal{G}$ and $\operatorname{Rep} \mathcal{G}'$ are full Tannakian subcategories of P and G, respectively. We check that $\operatorname{Rep} \mathcal{G}$ and $\operatorname{Rep} \mathcal{G}'$ coincide as collections of objects in $\operatorname{Rep} P$. The former is constructed by successive extensions in $\operatorname{Rep} P$ of objects coming from K. The latter is constructed by successive extensions in $\operatorname{Rep} G$ of objects coming from K. But since $\operatorname{Rep} G$ is closed under extensions in $\operatorname{Rep} P$ by the previous lemma, we are done. □

Definition 4.11. Let P be a pro-algebraic group. The *unipotent completion* of P is the Malcev completion of $P \to 1$.

The unipotent completion of P is the initial pro-unipotent group equipped with a homomorphism from P. Given a homomorphism $P \to K$, the unipotent completion of $\ker(P \to K)$ is exactly the kernel $\mathcal{U}$ of the Malcev completion $\mathcal{G} \to K$.

Remark 4.12. Our context for defining Malcev completions differs a bit from what is often found in the literature. Usually, P is taken to be a discrete group π, K is taken to be (pro-)reductive, and $\varrho : \pi \to K(F)$ is assumed to have Zariski dense image. We round out this section by discussing these three differences, as well as functoriality.

Definition 4.13. Given a discrete group π, the Tannaka dual of $\operatorname{Rep} \pi$ is a pro-algebraic group. We denote it by $\hat{\pi}$ and call it the *pro-algebraic completion* of π. (Here, $\operatorname{Rep} \pi$ is the category of finite-dimensional F-linear π-representations.)

The pro-algebraic completion $\hat{\pi}$ comes equipped with a homomorphism $\pi \to \hat{\pi}(F)$. Given a pro-algebraic group K, composing homomorphisms $\hat{\pi} \to K$ with $\pi \to \hat{\pi}(F)$ gives a bijection $\operatorname{Hom}(\hat{\pi}, K) \simeq \operatorname{Hom}(\pi, K(F))$, which is natural in K. More precisely:

Lemma 4.14. *The functor $\pi \mapsto \hat{\pi}$ from discrete groups to pro-algebraic groups given by pro-algebraic completion is left adjoint to the functor $P \mapsto P(F)$.* □

Given a map from a discrete group π to a pro-algebraic group K, our definition of the Malcev completion of the corresponding map $\hat{\pi} \to K$, coincides with definition of the Malcev completion of $\pi \to K$ found in the literature. This takes care of the first difference in Remark 4.12.

Next, let $\varrho : P \to K$ be a homomorphism as before. As we've seen, replacing K by $\operatorname{im} \varrho$, we still get the same Malcev completion. Therefore, we can assume that ϱ is surjective (or that the image is Zariski dense, when P is discrete).

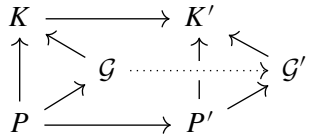
Finally, replacing K with a quotient by a pro-unipotent normal subgroup, still gives the same Malcev completion (the most important example of this being the maximal (pro-)reductive quotient of K):

Lemma 4.15. *Let $\varrho : P \to K$ be a homomorphism and $q : K \twoheadrightarrow S$ a quotient by a pro-unipotent normal subgroup. Then the Malcev completions of $P \to K$ and $P \to S$ coincide.*

Proof. We check it on the side of the Tannakian categories. Trivially, $\mathrm{Un}(\varrho^*)$ contains $\mathrm{Un}(\varrho^* q^*)$ (both are full subcategories of Rep P). For the other direction, note that every object in Rep K is obtained by successive extension from objects coming from q^* (Lemma 2.14). Thus, everything in the image of ϱ^* is obtained by successive extension from objects coming from $\varrho^* q^*$. This gives the reverse inclusion, using the extension-stability of $\mathrm{Un}(\varrho^* q^*)$ in Rep P. □

Remark 4.16. By Lemma 4.6, if $\varrho : P \to K$ is Malcev complete, then ϱ is surjective. The group K and the homomorphism $P \to K$ are determined by the maximal pro-reductive quotient S of K and the homomorphism $P \to S$. We recover K as the Malcev completion of $P \to S$.

Proposition 4.17 (Functoriality). *Consider a commutative diagram of pro-algebraic groups,*



where $\mathcal{G}$ and $\mathcal{G}'$ are the Malcev completions of $P \to K$ and $P' \to K'$, respectively. Then, there is a unique group homomorphism $\mathcal{G} \to \mathcal{G}'$ making everything commute. This construction is functorial.

Proof. To give such a map is the same as to give a map $\mathcal{G} \to \mathcal{G}' \times_{K'} K$, which when composed with the projection to K gives the canonical map $\mathcal{G} \to K$, and which when precomposed with $P \to \mathcal{G}$ gives the obvious arrow $P \to \mathcal{G}' \times_{K'} K$. But the kernel

$$\ker\left(\mathcal{G}' \times_{K'} K \to K\right) = \ker\left(\mathcal{G}' \to K'\right)$$

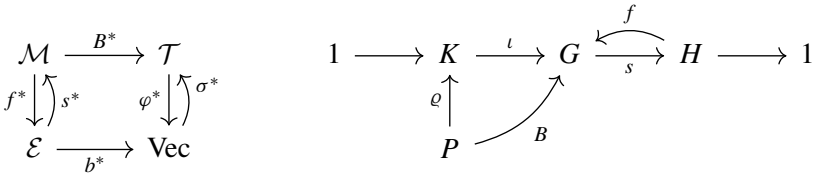
is pro-unipotent, so the universal property of $\mathcal{G}$ gives us a unique map of the form we wanted. Functoriality follows easily from the uniqueness. □

Proposition 4.18. *Malcev completion gives a functor from the category of homomorphisms of pro-algebraic groups, to the category of homomorphisms of pro-algebraic groups with pro-unipotent kernels. This functor is left adjoint to the inclusion functor.*

Proof. We already proved the functoriality, what is left to show is the adjunction. This follows easily from the universal property of Malcev completion, and the functoriality above (including the uniqueness statement). □

5. The Malcev completeness criterion

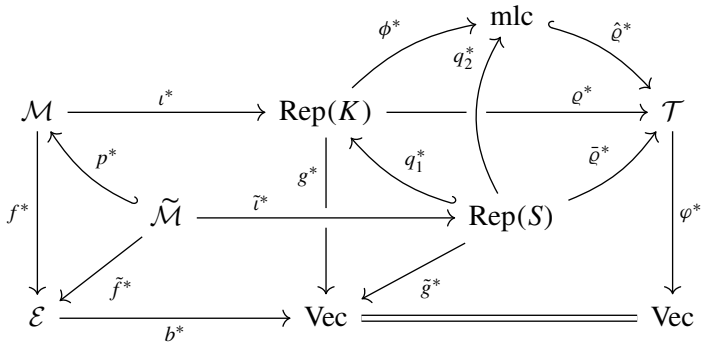
We once again place ourselves in the Standard Situation, keeping all the notation from Section 2.6:



Let us introduce some more notation that will be helpful in discussing the Malcev completeness of ϱ.

Fix a quotient $q_1 : K \twoheadrightarrow S$ with a pro-unipotent kernel. Denote the composition of ϱ with q_1 by $\bar{\varrho} : P \to S$. We've seen, in Lemma 4.15, that the Malcev completions of ϱ and $\bar{\varrho}$ coincide.

Let mlc be the full subcategory $\mathrm{Un}(\bar{\varrho}^*)$ of $\mathcal{T}$, as in Definition 4.1. Its objects are those built by successive extension from objects in the image of $\bar{\varrho}^*$ (see Remark 4.4). In the following diagram, I omit the (systematically named) sections (s^*, t^*, σ^* of f^*, g^*, φ^*, plus variants with tilde) in order to reduce clutter. Let $\widetilde{\mathcal{M}}$ be the (full) preimage of $\mathrm{Rep}\, S$ in $\mathcal{M}$.



Writing $\mathcal{K}$ for the Malcev completion of $\bar{\varrho} : P \to S$, i.e., the Tannaka dual $\mathscr{G}(\mathrm{mlc})$ of mlc, we can formulate our goal as the following theorem:

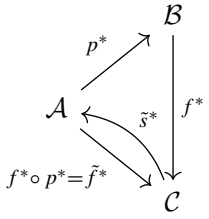
Theorem 5.1 (Malcev completeness criterion). *Assume that $\bar{\varrho}$ is surjective and that the canonical maps $b^* R^i s_* \to R^i \sigma_* B^*$, for $i = 0, 1$, are isomorphisms. Then $\phi : \mathcal{K} \to K$ is an isomorphism* (*or equivalently*, *$\varrho : P \to K$ is Malcev complete*).

Remark 5.2. This is the main technical result of this paper, and it has two key assumptions: the surjectivity of $\bar{\varrho}$ we refer to as the *surjectivity assumption*, and the other assumption we call the *cohomological assumption*. Note that the conclusion of the theorem implies the surjectivity of ϱ.

Our general strategy is, as it was for Theorem 3.3, to utilize Proposition 2.20. That means we construct models of the regular representations of the groups and show that the natural map between them is an isomorphism.

Since both K and $\mathcal{K}$ are extensions of S by unipotent groups, it's not too surprising that the unipotent theorem (Theorem 3.3) becomes useful. In Section 5.1, we show how, given a relatively unipotent neutral Tannakian category, one can produce a new one, which is unipotent over a deeper base (Proposition 5.4). In the rest of Section 5, we use this result to apply the unipotent theorem, and finally prove Theorem 5.1.

5.1. ***On certain tensor products of Tannakian categories.*** Consider a 2-commutative diagram



in nTan. We assume that $\tilde{s}^*$ is a monoidal 2-section of $\tilde{f}^*$. This gives a monoidal 2-section $s^* := p^*\tilde{s}^*$ of f^*. Before we move on to the main topic of this subsection, we state a basic lemma which becomes useful in the sequel.

As in Section 2.4, we have an algebra morphism $p_{\mathrm{alg}} : p^*\tilde{f}_*\mathbf{1} \to f_*\mathbf{1}$. It is uniquely characterized by a compatibility with Hopf algebra counits after applying f^*. Under adjunction, it corresponds to a morphism $\tilde{f}_*\mathbf{1} \to p_*f_*\mathbf{1}$, and this morphism is also a morphism of algebras. On the other hand, there is an isomorphism $\tilde{f}_*\mathbf{1} \simeq p_*f_*\mathbf{1}$ of objects in $\operatorname{Ind}\mathcal{A}$, induced by the 2-commutativity of the above diagram.

Lemma 5.3. *The two morphisms $\tilde{f}_*\mathbf{1} \to p_*f_*\mathbf{1}$, described above, coincide.* □

Let $A := p^*(\tilde{f}_*\mathbf{1})$: it's an algebra object in $\operatorname{Ind}\mathcal{B}$. We consider the tensor category $\mathrm{Mod}^{\mathrm{fg}}(A)$ of finitely generated A-modules in $\operatorname{Ind}\mathcal{B}$, as in Section 2.5, sitting as a full subcategory in the abelian tensor category $\mathrm{Mod}(A)$ of A-modules in $\operatorname{Ind}\mathcal{B}$.

As in Section 2.6, it's a fact that $f^*A = \tilde{f}^*\tilde{f}_*\mathbf{1}$ is a Hopf algebra object in $\mathcal{C}$, which thus has a counit map $f^*A \to \mathbf{1}$. We may therefore define a monoidal, right-exact functor $w^* : \mathrm{Mod}^{\mathrm{fg}}(A) \to \mathcal{C}$ by $w^*(-) := \mathbf{1} \otimes_{f^*A} f^*(-)$. (A finitely generated $\mathbf{1}$-module in $\operatorname{Ind}\mathcal{C}$ is just an object of $\mathcal{C}$.) It has a monoidal[4] 2-section e^* defined by $e^*(-) := A \otimes s^*(-)$.

Proposition 5.4. *Assume $\mathcal{B}$ is p^*-unipotent, that p^* is fully faithful, and that the essential image of p^* is closed under taking subobjects. Then* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *is* **(*i*)** *e^*-unipotent and* **(*ii*)** *neutral Tannakian. Moreover, w^* is faithful and exact.*

Remark 5.5. In fact, $\mathrm{Mod}^{\mathrm{fg}}(A)$ is the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$ of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$, once one has defined what that means. Group theoretically, it means that $\mathscr{G}(\mathrm{Mod}^{\mathrm{fg}}(A))$ is isomorphic to $\mathscr{G}(\mathcal{B}) \times_{\mathscr{G}(\mathcal{A})} \mathscr{G}(\mathcal{C})$, and in particular, that $\mathscr{G}(\mathcal{B}) \to \mathscr{G}(\mathcal{A})$ and $\mathscr{G}(\mathrm{Mod}^{\mathrm{fg}}(A)) \to \mathscr{G}(\mathcal{C})$ have the same (pro-unipotent) kernel.

We'll prove this proposition using the following result due to Deligne:

Lemma 5.6 [15, Corollary 2.10]. *Let $\mathcal{T}$ be a rigid abelian tensor category satisfying* $\operatorname{End}\mathbf{1} = F$, *and let $\mathcal{M}$ be a nonzero abelian monoidal category with a right exact tensor product. Then any right exact functor $\mathcal{T} \to \mathcal{M}$ is left exact and faithful.*

The above lemma gives, in particular, that both $e^* : \mathcal{C} \to \mathrm{Mod}^{\mathrm{fg}}(A)$ and $A \otimes (-) : \mathcal{B} \to \mathrm{Mod}^{\mathrm{fg}}(A)$ are exact, which is useful. We now spend some time establishing the assumptions of Deligne's result for the functor w^*. More precisely, we prove that $\mathrm{Mod}^{\mathrm{fg}}(A)$ is rigid and abelian, and that $\operatorname{End}\mathbf{1} = F$. To prove Proposition 5.4, it is then essentially enough to show the e^*-unipotency.

Lemma 5.7. *Assume that $\mathcal{B}$ is p^*-unipotent, that p^* is fully faithful, and that the essential image of p^* is closed under taking subobjects. Then* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *is Noetherian, i.e., any ascending chain of subobjects of an object in* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *must stabilize.*

Proof. Since a quotient of a Noetherian object is Noetherian, it's enough to consider finitely generated free modules $A \otimes Y$. We proceed by induction on the p^*-unipotency index of Y.

[4] In particular, the isomorphism $w^*e^* \simeq \mathrm{id}$ is monoidal.

Base case: We have that $A \otimes p^*X \simeq p^*(\tilde{f}_*\mathbf{1} \otimes X)$, so we reduce to studying sub-$\tilde{f}_*\mathbf{1}$-modules of $\tilde{f}_*\mathbf{1} \otimes X$, by the assumption on the essential image of p^*. But $\mathrm{Mod}(\tilde{f}_*\mathbf{1})$ is equivalent to $\mathcal{C}$ (e.g., by the fundamental theorem of Hopf modules and Tannakian duality), and $\mathcal{C}$ is Noetherian (since it's Tannakian).

Induction step: We assume that Y is an extension of p^*X by Y' such that $A \otimes Y'$ is a Noetherian A-module. By the exactness of $A \otimes (-)$, we get $A \otimes Y$ as an extension of $A \otimes p^*A$ by $A \otimes Y'$. A submodule N of $A \otimes Y$ gives rise to a submodule N' of $A \otimes Y'$ by intersecting with $A \otimes Y'$ (pullback), and to a submodule N'' of $A \otimes p^*X$ by taking the quotient by N'. If a larger submodule $\tilde{N} \supseteq N$ gives rise, in this way, to the same N' and N'', then it must coincide with N. Since $A \otimes Y'$ and $A \otimes p^*X$ are Noetherian, we are done. □

Lemma 5.8. *Assume that all finitely generated free A-modules are Noetherian. Then,* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *is closed under taking subobjects in* $\mathrm{Mod}(A)$. *In particular,* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *is abelian.*

Proof. Let M be an A-module which is finitely generated by $Y \in \mathcal{B}$, and let N be a submodule of M. Then N is a quotient of a submodule N' of $A \otimes Y$ (namely the pullback of N in $A \otimes Y$), and we can thus replace M by $A \otimes Y$, without loss of generality.

We now show that for every $Y \in \mathcal{B}$, every sub-A-module of $A \otimes Y$ is finitely generated. Let N be such a submodule, generated by $\tilde{Y}$ in $\operatorname{Ind}\mathcal{B}$, i.e., we have a map $A \otimes \tilde{Y} \twoheadrightarrow N$. Let the kernel of this map be denoted by K, and let $(\tilde{Y}_\alpha)_{\alpha \in I}$ be a filtered system of $\tilde{Y}_\alpha \in \mathcal{B}$ such that $\tilde{Y} = \operatorname{colim}_\alpha \tilde{Y}_\alpha$. We may assume that the $\tilde{Y}_\alpha$ are subobjects of $\tilde{Y}$ and that all transition maps are monomorphisms [14, Lemme 4.2.1]. Note that $A \otimes \tilde{Y}$ is then isomorphic to $\operatorname{colim}_\alpha A \otimes \tilde{Y}_\alpha$, a filtered colimit of sub-A-modules, by the exactness of $A \otimes (-)$.

Let K_α be the intersection of K and $A \otimes \tilde{Y}_\alpha$ in $A \otimes \tilde{Y}$, and let N_α be the cokernel of $K_\alpha \to A \otimes \tilde{Y}_\alpha$. We have short exact sequences

$$0 \to K_\alpha \to A \otimes \tilde{Y}_\alpha \to N_\alpha \to 0,$$

and since colimits commute with colimits, we get that $\operatorname{colim}_\alpha N_\alpha$ is the cokernel of $K \to A \otimes \tilde{Y}$, i.e., that it's N. The Noetherianity of $A \otimes Y$ implies that the partially ordered system N_α of subobjects of N has a maximal element N_β (since they are also subobjects of $A \otimes Y$). The fact that it's filtered implies that there is a unique such maximal subobject, and it must be equal to all of N, since the colimit equals N. All in all, we get

$$N = N_\beta = (A \otimes \tilde{Y}_\beta)/K_\beta,$$

so that N is finitely generated by $\tilde{Y}_\beta$, and we're done. □

Lemma 5.9. *We have that* $A \otimes p^*X \simeq e^*\tilde{f}^*X$.

Proof. This follows from the projection formula, since

$$A \otimes p^*X = p^*\tilde{f}_*\mathbf{1} \otimes p^*X \simeq p^*(\tilde{f}_*\mathbf{1} \otimes X),$$

and then, by Lemma 2.19,

$$A \otimes p^*X \simeq p^*(\tilde{f}_*\mathbf{1} \otimes \tilde{s}^*\tilde{f}^*X) \simeq A \otimes s^*\tilde{f}^*X \simeq e^*\tilde{f}^*X. \qquad \square$$

Lemma 5.10. *Assume that $\mathcal{B}$ is p^*-unipotent, that p^* is fully faithful, and that the essential image of p^* is closed under taking subobjects. Let M be a quotient of e^*Z in* $\mathrm{Mod}^{\mathrm{fg}}(A)$. *Then M is in the essential image of e^*.*

Proof. First, replace Z by Z/Z_0, where Z_0 is the largest subobject of Z such that $e^*Z_0 \to M$ is zero. By Lemmas 5.7 and 5.8, the kernel K of $e^*Z \to M$ is finitely generated, i.e., it's the quotient of some $A \otimes Y$. If Y is nonzero, the p^*-unipotency of $\mathcal{B}$ implies that Y has a nonzero subobject of the form p^*X. But $A \otimes p^*X \simeq e^*\tilde{f}^*X =: e^*Z'$, by Lemma 5.9. We thus have a map $e^*Z' \to e^*Z$, and its composition with $e^*Z \to M$ is zero. By the fullness of e^*, we have an underlying map $Z' \to Z$, and it must be zero, because of the simplification of Z we started with. Therefore, we may replace Y by Y/e^*Z, and in the end assume Y is zero, so that K is zero and $e^*Z \simeq M$. $\square$

Lemma 5.11. $\mathrm{End}_{\mathrm{Mod}(A)}(A) \simeq F$.

Proof. Since p^* is fully faithful, we have $\mathrm{End}_{\mathrm{Mod}(A)}(A) \simeq \mathrm{End}_{\mathrm{Mod}(\tilde{f}_*\mathbf{1})}(\tilde{f}_*\mathbf{1})$, and the latter is F:

$$\mathrm{Hom}_{\mathrm{Mod}(\tilde{f}_*\mathbf{1})}(\tilde{f}_*\mathbf{1} \otimes \mathbf{1},\, \tilde{f}_*\mathbf{1}) \simeq \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{A}}(\mathbf{1},\, \tilde{f}_*\mathbf{1}) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}) \simeq F,$$

under the adjunction $\tilde{f}_*\mathbf{1} \otimes (-)\colon\ \mathrm{Ind}\,\mathcal{A} \rightleftarrows \mathrm{Mod}(\tilde{f}_*\mathbf{1})\ :\mathrm{Forget}$. $\square$

We're now ready to prove the main result of this section.

Proof of Proposition 5.4. (i) Take N in $\mathrm{Mod}^{\mathrm{fg}}(A)$, a quotient of $A \otimes Y$ for some Y in $\mathcal{B}$. We want to establish that N is an extension of e^*Z by something e^*-unipotent, for some $Z \in \mathcal{C}$. This is done by induction on the p^*-unipotency index of Y.

Base case: Assume $Y = p^*X$. Then $A \otimes Y = e^*(\tilde{f}^*X)$, by Lemma 5.9. Then Lemma 5.10, says that N must be of the form $A \otimes s^*(Z) = e^*(Z)$, for some $Z \in \mathcal{C}$. In particular, it's e^*-unipotent.

Induction step: We write Y as an extension

$$0 \to Y' \to Y \to p^*X \to 0,$$

where Y' is of a lower p^*-unipotency index. The induction hypothesis tells us that any quotient of $A \otimes Y'$ in $\mathrm{Mod}^{\mathrm{fg}}(A)$ is e^*-unipotent. Thus,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes Y' & \longrightarrow & A \otimes Y & \longrightarrow & A \otimes p^*(X) & \longrightarrow & 0 \\
 & & \vdots & & \downarrow & & \vdots & & \\
 & & \twoheadrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & & \\
0 & \longrightarrow & U & \longrightarrow & N & \longrightarrow & e^*(Z) & \longrightarrow & 0
\end{array}$$

where U is e^*-unipotent, and we're done.

(ii) With Lemmas 5.8 and 5.11 in mind, it's enough to show that (a) $\mathrm{Mod}^{\mathrm{fg}}(A)$ is rigid, and (b) w^* is faithful and exact. The existence of a fiber functor then follows from (b) and the neutrality of $\mathcal{C}$. (See Section 2.2.)

(a) Rigidity is stable under extension, and everything in the image of e^* is rigid, since $\mathcal{C}$ is rigid and e^* is monoidal.

(b) Clearly, w^* is right exact. By Lemma 5.6, it is therefore exact and faithful. □

5.2. ***Proof of Theorem 5.1.*** We now go back to the situation described in the introduction to Section 5.

5.2.1. *Applying the unipotent theorem.* We write $\hat{\varphi}^* := \varphi^* \hat{\varrho}^* : \mathrm{mlc} \to \mathrm{Vec}$. We also write $\hat{\sigma}^*$ for a section to $\hat{\varphi}^*$ such that $\hat{\varrho}^* \hat{\sigma}^* = \sigma^*$.

Apply Proposition 5.4 to (Rep S, mlc, Vec) as $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, and write T for $q_2^* \tilde{g}_* \mathbf{1}$ (playing the role of the algebra A). The unipotency assumption needed follows from the pro-unipotency of the kernel of $\mathcal{K} \to S$ and the surjectivity of $\bar{\varrho}$ (by Lemma 2.14). The proposition then gives us a unipotent Tannakian category

$$\begin{array}{c} \mathrm{Mod}^{\mathrm{fg}}(T) \\ {\scriptstyle \omega^*}\big\downarrow \big\uparrow {\scriptstyle \varepsilon^*} \\ \mathrm{Vec} \end{array}$$

with a forgetful functor to Ind mlc.

Remark 5.12. In fact, $\mathrm{Mod}^{\mathrm{fg}}(T)$ is nothing but the category of representations of the (pro-unipotent) kernel of $\mathcal{K} \twoheadrightarrow S$.

It is not hard to see that $\mathcal{M}$ is unipotent over $\widetilde{\mathcal{M}}$. In fact, $\mathscr{G}(\widetilde{\mathcal{M}})$ is the pushout of $G \leftarrow K \to S$, so that $G \to \mathscr{G}(\widetilde{\mathcal{M}})$ is surjective with the same pro-unipotent kernel as $K \to S$. We can thus apply Proposition 5.4 to $(\widetilde{\mathcal{M}}, \mathcal{M}, \mathcal{E})$ as $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, with $M := p^* \tilde{f}_* \mathbf{1}$ playing the role of A. We get a relatively unipotent Tannakian category

$$\begin{array}{c} \mathrm{Mod}^{\mathrm{fg}}(M) \\ {\scriptstyle w^*}\big\downarrow \big\uparrow {\scriptstyle e^*} \\ \mathcal{E} \end{array}$$

over $\mathcal{E}$, which has a forgetful functor to Ind $\mathcal{M}$.

Remark 5.13. In fact, $\mathrm{Mod}^{\mathrm{fg}}(M)$ is nothing but $\mathrm{Rep}(\ker(K \to S) \ltimes H)$.

The composition of ι^* and ϕ^* induces a functor $\mathrm{Mod}^{\mathrm{fg}}(M) \to \mathrm{Mod}^{\mathrm{fg}}(T)$, since

$$\phi^* \iota^* M = \phi^* \iota^* p^* \tilde{f}_* \mathbf{1} \simeq q_2^* \tilde{\iota}^* \tilde{f}_* \mathbf{1} \simeq T.$$

Here, we've used the fact that $\tilde{\iota}^* \tilde{f}_* \simeq \tilde{g}_* \mathbf{1}$, which is true by definition. To summarize the situation, we have a diagram

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{fg}}(M) & \xrightarrow{\phi^*\iota^*} & \mathrm{Mod}^{\mathrm{fg}}(T) \\ {\scriptstyle w^*}\downarrow\uparrow{\scriptstyle e^*} & & {\scriptstyle \omega^*}\downarrow\uparrow{\scriptstyle \varepsilon^*} \\ \mathcal{E} & \xrightarrow{b^*} & \mathrm{Vec} \end{array} \tag{12}$$

in nTan, where $\mathrm{Mod}^{\mathrm{fg}}(M)$ is relatively unipotent over $\mathcal{E}$ and $\mathrm{Mod}^{\mathrm{fg}}(T)$ is unipotent. In order to apply the unipotent theorem, we need the following result:

Lemma 5.14. *If $b^* \circ R^i s_* \simeq R^i \sigma_* \circ B^*$ for $i \leq 1$, then $b^* \circ R^i e_* \simeq R^i \varepsilon_* \circ (\phi^* \iota^*)$ for $i \leq 1$.*

Proof. Note that $R\varepsilon_*$ is the right adjoint of ε^* (on the derived level, of course). By definition, ε^* is the composition $(T \otimes -) \circ \hat{\sigma}^*$. The (ind-)right adjoints of $(T \otimes -)$ and $\hat{\sigma}^*$ are $\mathrm{Forget} : \mathrm{Mod}^{\mathrm{fg}}(T) \to \mathrm{Ind}\,\mathrm{mlc}$, and $\hat{\sigma}_*$, respectively. The former is exact and takes injectives to injectives (since $(T \otimes -)$ is exact), and we get $R\varepsilon_* = (R\hat{\sigma}_*) \circ \mathrm{Forget}$. In the same way, $Re_* = (Rs_*) \circ \mathrm{Forget}'$, where $\mathrm{Forget}' : \mathrm{Mod}^{\mathrm{fg}}(M) \to \mathrm{Ind}\,\mathcal{M}$.

Thus, for $i \leq 1$,

$$\begin{aligned} b^* \circ (R^i e_*) &= b^* \circ (R^i s_*) \circ \mathrm{Forget}' = (R^i \sigma_*) \circ B^* \circ \mathrm{Forget}' \\ &= (R^i \sigma_*) \circ (\hat{\varrho}^* \phi^* \iota^*) \circ \mathrm{Forget}' = (R^i \sigma_*) \circ \hat{\varrho}^* \circ \mathrm{Forget} \circ (\phi^* \iota^*), \end{aligned}$$

and we've reduced to showing $(R^i \sigma_*) \circ \hat{\varrho}^* = R^i \hat{\sigma}_*$. For $i = 0$, this follows from the surjectivity of $\hat{\varrho}$. For $i = 1$, it follows from the stability of mlc in $\mathcal{T}$ under extensions. □

We may thus apply Theorem 3.1 to the diagram (12), yielding the following theorem:

Theorem 5.15. *Assume that $\bar{\varrho}$ is surjective and that $b^* R^i s_* \to R^i \sigma_* B^*$ is an isomorphism for $i = 0, 1$. Then the natural algebra morphism $\phi^* \iota^* w_* \mathbf{1} \to \omega_* \mathbf{1}$ in $\mathrm{Mod}^{\mathrm{fg}}(T)$ is an isomorphism.* □

5.2.2. *The final argument.*

Proof of Theorem 5.1. From the previous section, we get that

$$\mathrm{Forget}(\phi^* \iota^* w_* \mathbf{1}) \to \mathrm{Forget}(\omega_* \mathbf{1})$$

in Ind mlc is an isomorphism of algebras (Theorem 5.15). Note that, writing $\mathrm{Forget}'' : \mathrm{Mod}^{\mathrm{fg}}(\iota^* M) \to \mathrm{Ind}\,\mathrm{Rep}\,K$, we have

$$\mathrm{Forget}(\phi^* \iota^* w_* \mathbf{1}) = \phi^* \circ \mathrm{Forget}'' \circ \iota^* w_* \mathbf{1} = \phi^* \iota^* \circ \mathrm{Forget}' \circ w_* \mathbf{1}$$

Lemma 5.3 gives us algebra isomorphisms $f_* \mathbf{1} \simeq \mathrm{Forget}' \circ w_* \mathbf{1}$ (note that $M \otimes (-)$ is a morphism in nTan, by Lemma 5.6, and that Forget' is its ind-right adjoint), and $\hat{\varphi}_* \mathbf{1} \simeq \mathrm{Forget} \circ \omega_* \mathbf{1}$. Finally, $\iota^* f_* \mathbf{1} = g_* \mathbf{1}$, by definition. Putting everything together we thus get that the natural algebra morphism

$$\phi^* g_* \mathbf{1} \to \hat{\varphi}_* \mathbf{1}$$

is an isomorphism. By Proposition 2.20, we are done. □

Part II. Fundamental groups

6. Enriched local systems

In this section, we work out, in an axiomatic setting, the common arguments in our two main applications of the Malcev completeness criterion (Theorem 5.1). The applications themselves are found in Section 7. The main results of this section are Propositions 6.13, 6.15, and 6.16.

Let k be a fixed subfield of $\mathbb{C}$. We usually write pt for $\operatorname{Spec} k$. For a k-variety X, let X^{an} denote $X(\mathbb{C})$ equipped with the complex analytic topology, and let $\mathrm{LS}(X)$ denote the category of locally constant sheaves of finite-dimensional F-vector spaces (*local systems*) on X^{an}. Given a k-point x of X, denote by $\pi_1(X, x)$ the (topological) fundamental group of X^{an}. Recall that $\mathrm{LS}(X)$ is equivalent to the category of (finite-dimensional) F-representations of $\pi_1(X, x)$. Lastly, SmVar_k is the category of smooth, separated and geometrically connected schemes of finite type over k.

Definition 6.1. Following [2], we say that a *weak theory of enriched local systems* is a contravariant 2-functor

$$\begin{aligned} \mathcal{E} : (\mathrm{SmVar}_k)^{\mathrm{op}} &\to \mathrm{nTan}, \\ X &\mapsto \mathcal{E}(X), \\ (f : Y \to X) &\mapsto (f^*_{\mathcal{E}} : \mathcal{E}(X) \to \mathcal{E}(Y)), \end{aligned}$$

together with a monoidal natural transformation $b^* = b^*_{\mathcal{E}} : \mathcal{E}(-) \to \mathrm{LS}(-)$. (Recall that the morphisms in nTan are faithful and exact functors.)

For any X in SmVar_k with a k-point x, we then get an instance of the Standard Situation (Section 2.6), and with that in mind, we denote by $K = K(X, x)$ is the kernel of $s : \mathscr{G}(\mathcal{E}(X)) \to \mathscr{G}(\mathcal{E}(\mathrm{pt}))$. As in Section 2.6, we get a commutative diagram:

$$\begin{array}{ccccc} & & \overset{b^*_X}{\frown} & & \\ \mathcal{E}(X) & \xrightarrow{\iota^*} & \mathrm{Rep}\, K & \xrightarrow{\varrho^*} & \mathrm{LS}(X) \\ {\scriptstyle f^*}\downarrow\uparrow{\scriptstyle s^*} & & {\scriptstyle g^*}\downarrow\uparrow{\scriptstyle t^*} & & {\scriptstyle \varphi^*}\downarrow\uparrow{\scriptstyle \sigma^*} \\ \mathcal{E}(\mathrm{pt}) & \xrightarrow[b^*]{} & \mathrm{Vec} & = & \mathrm{Vec} \end{array}$$

Here, $s^* = p^*_{\mathcal{E}}$ and $\sigma^* = p^*_{\mathrm{LS}}$ are induced by the structure morphism $p : X \to \mathrm{pt}$, while $f^* = x^*_{\mathcal{E}}$ and $\varphi^* = x^*_{\mathrm{LS}}$ are induced by the point $x : \mathrm{pt} \to X$. We briefly translate the current setting to group theory, starting with the following definitions:

Definition 6.2. We denote by $\mathrm{LS}_{\mathcal{E}}(X)$ the smallest Tannakian subcategory of $\mathrm{LS}(X)$ containing the image of b^*. We define the *$\mathcal{E}$-fundamental group* of (X, x) to be the Tannaka dual group

$$\pi_1^{\mathcal{E}}(X, x) := \mathscr{G}(\mathrm{LS}_{\mathcal{E}}(X), x^*)$$

of $\mathrm{LS}_{\mathcal{E}}(X)$, with fiber functor $x^* = \varphi^*$. The objects of $\mathrm{LS}_{\mathcal{E}}(X)$ are local systems *coming from* $\mathcal{E}$, or alternatively, *of* $\mathcal{E}$*-origin*. Lastly, let $G(X,x) := \mathscr{G}(\mathcal{E}(X), x^* \circ b^*)$ and $G(k) := \mathscr{G}(\mathcal{E}(\mathrm{pt}), b^*)$, and let $S(X,x)$ be the maximal pro-reductive quotient of $K(X,x)$.

Note that the map $\pi_1(X,x) \to G(X,x)$ induced by $b^* : \mathrm{Rep}\, G(X,x) \to \mathrm{Rep}\, \pi_1(X,x)$ factors through a map $\varrho : \pi_1(X,x) \to K(X,x)$, as above. By definition, $\varrho : \pi_1(X,x) \to K(X,x)$ factors further through $\pi_1^{\mathcal{E}}(X,x)$, and in fact the latter is the Zariski closure of the image $\mathrm{im}(\varrho)$ of the former (by Lemma 2.13). In the end, we get a diagram

$$\begin{array}{ccccccccc} & & & & S(X,x) & & & & \\ & & & & \uparrow\!\!\uparrow & & & & \\ 1 & \longrightarrow & K(X,x) & \longrightarrow & G(X,x) & \twoheadrightarrow & G(k) & \longrightarrow & 1 \\ & & & \nearrow & \uparrow & & & & \\ \pi_1(X,x) & \longrightarrow & \pi_1^{\mathcal{E}}(X,x) & & & & & & \end{array}$$

where the middle row is (split) exact.

Above, whenever we have a map from the discrete group $\pi_1(X,x)$ to a pro-algebraic group G, we mean a map to the F-points $G(F)$ of G (or equivalently a map from the pro-algebraic completion of $\pi_1(X,x)$ to G).

The goal of this section is to develop tools to show that $\pi_1^{\mathcal{E}}(X,x) = K(X,x)$ and that $\pi_1(X,x) \to \pi_1^{\mathcal{E}}(X,x)$ is Malcev complete, both of which follow from simply showing that $\pi_1(X,x) \to K(X,x)$ is Malcev complete. To show this, we utilize the Malcev completeness criterion (Theorem 5.1), which transforms our task into checking the two assumptions of the theorem.

6.1. ***Checking the cohomological assumption.*** Our ultimate goal is to apply Theorem 5.1, and we thus need a way to establish the cohomological assumption, namely, that the canonical maps $b^* R^i s_* \to R^i \sigma_* B^*$, for $i = 0, 1$, are isomorphisms. The idea is that a theory of enriched local systems usually lives inside some ambient theory of (perverse or constructible) sheaves, where an analogue of the cohomological assumption holds. A motivating example to have in mind is the way admissible variations of mixed Hodge structures sit in the larger category of mixed Hodge modules.

Notation 6.3. Let $\mathcal{D}(X)$ be the derived category $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X, F)$ of bounded complexes of sheaves of F-vector spaces on X^{an} with constructible cohomology.

Remark 6.4. Given a morphism $f : Y \to X$, we write Rf_* and f^* for the pushforward and pullback on $\mathcal{D}(-)$. Note that f^* takes local systems to local systems, and so restricts to a functor $f^*_{\mathrm{LS}} : \mathrm{LS}(X) \to \mathrm{LS}(Y)$. When f is such that the restriction of $R^0 f_*$ to local systems lands in (ind-)local systems (e.g., when $X = \mathrm{pt}$), then this restriction gives the right adjoint f_*^{LS} of f^*_{LS}. *However*, for $i > 0$, the restriction of $R^i f_*$ will generally *not* agree with $R^i f_*^{\mathrm{LS}}$, even when it takes values in (ind-)local systems.

Definition 6.5. Let $\mathcal{E}$ be a weak theory of enriched local systems, and $X \in \mathrm{SmVar}_k$, with structure morphism $p : X \to \mathrm{pt}$. An *ambient theory* $\mathcal{A}(X)$ for $\mathcal{E}$ over X is a triangulated category $\mathcal{A}(X)$, together with

(a) a triangulated functor $\mathrm{inc}_{\mathcal{A}} : \mathcal{D}^{\mathrm{b}}(\mathcal{E}(X)) \to \mathcal{A}(X)$;

(b) a triangulated *realization* functor $b^*_{\mathcal{A}} : \mathcal{A}(X) \to \mathcal{D}(X)$;

(c) an isomorphism $b^*_{\mathcal{A}} \circ \mathrm{inc}_{\mathcal{A}} \simeq \mathrm{inc} \circ b^*_{\mathcal{E}}$ (where $b^*_{\mathcal{E}}$ is viewed as a functor $\mathcal{D}^{\mathrm{b}}(\mathcal{E}(X)) \to \mathcal{D}^{\mathrm{b}}(\mathrm{LS}(X))$).

Let $p^*_{\mathcal{A}} := \mathrm{inc}_{\mathcal{A}} \circ p^*_{\mathcal{E}}$, where $p^*_{\mathcal{E}}$ is viewed as a functor $\mathcal{D}^{\mathrm{b}}(\mathcal{E}(\mathrm{pt})) \to \mathcal{D}^{\mathrm{b}}(\mathcal{E}(X))$. We moreover require the following conditions to hold:

(i) The pullback functor $p^*_{\mathcal{A}}$ has a right adjoint $p^{\mathcal{A}}_* : \mathcal{A}(X) \to \mathcal{D}^{\mathrm{b}}(\mathcal{E}(\mathrm{pt}))$.

(ii) The composition $\mathcal{E}(X) \to \mathcal{D}^{\mathrm{b}}(\mathcal{E}(X)) \to \mathcal{A}(X)$ is fully faithful.

(iii) An object M in $\mathcal{A}(X)$ lies in the full subcategory $\mathcal{E}(X)$ if and only if $b^*_{\mathcal{A}} M$ lies in $\mathrm{LS}(X)$.

(iv) The canonical map $b^* p^{\mathcal{A}}_* \to R p_* b^*_{\mathcal{A}}$ is an isomorphism.

(v) For every M in $\mathcal{E}(X)$, the canonical map $p^{\mathcal{A}}_* M \to \tau^{\geq 0} p^{\mathcal{A}}_* M$ is an isomorphism.

Remark 6.6. We have two natural transformations:

$$R p^{\mathcal{E}}_* \to p^{\mathcal{A}}_* \circ \mathrm{inc}_{\mathcal{A}}, \quad b^* \circ p^{\mathcal{A}}_* \to R p_* \circ b^*_{\mathcal{A}}.$$

The former corresponds under adjunction to

$$p^*_{\mathcal{A}} \circ R p^{\mathcal{E}}_* = \mathrm{inc}_{\mathcal{A}} \circ p^*_{\mathcal{E}} \circ R p^*_{\mathcal{E}} \to \mathrm{inc}_{\mathcal{A}}.$$

The latter is the "canonical map" in (iv) and corresponds under adjunction to

$$p^* \circ b^* \circ p^{\mathcal{A}}_* \simeq \mathrm{inc} \circ b^*_{\mathcal{E}} \circ p^*_{\mathcal{E}} \circ p^{\mathcal{A}}_* \xrightarrow{\text{(c)}} b^*_{\mathcal{A}} \circ \mathrm{inc}_{\mathcal{A}} \circ p^*_{\mathcal{E}} \circ p^{\mathcal{A}}_* = b^*_{\mathcal{A}} \circ p^*_{\mathcal{A}} \circ p^{\mathcal{A}}_* \to b^*_{\mathcal{A}}.$$

Remark 6.7. Some care has to be taken with ind-categories at this point. We have that $R p^{\mathcal{E}}_*$ goes from $\mathcal{D}^{\mathrm{b}}(\mathrm{Ind}\,\mathcal{E}(X))$ to $\mathcal{D}^{\mathrm{b}}(\mathrm{Ind}\,\mathcal{E})$, whereas $p^{\mathcal{A}}_* \circ \mathrm{inc}_{\mathcal{A}}$ goes between $\mathcal{D}^{\mathrm{b}}(\mathcal{E}(X))$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{E})$. The latter two categories are full triangulated subcategories of the former two (see, e.g., [23, Proposition 2.2]). In particular, when writing the first natural transformation in the previous remark, we are suppressing the fact that we are actually extending the right-hand side functor to the ind-categories (ind-category formation is functorial). This is quite harmless and doesn't affect Proposition 6.11, for example.

Examples 6.8.

(1) The trivial example is given by $\mathcal{E} := \mathrm{LS}(-)$ and $\mathcal{A}(X) := \mathcal{D}(X)$ (see Notation 6.3).

(2) Also of interest is $\mathcal{E} := \mathrm{MHS}(-)$ (admissible variations of mixed Hodge structure) and $\mathcal{A}(X) = \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ (the bounded derived category of mixed Hodge modules). See Lemma 7.1.

For the rest of the section, we fix a weak theory $\mathcal{E}$ of enriched local systems, $X \in \mathrm{SmVar}_k$, and an ambient theory $\mathcal{A}(X)$ for $\mathcal{E}$ over X.

Lemma 6.9. *The full subcategory $\mathcal{E}(X)$ is closed under extensions in $\mathcal{A}(X)$.*

Proof. Let E be an extension in $\mathcal{A}(X)$ of M and N, both lying in the full subcategory $\mathcal{E}(X)$ (see 6.5(ii)). Then b^*E is an extension in $\mathcal{D}(X)$ of local systems b^*M and b^*N, so b^*E is a local system. We conclude using (iii) in Definition 6.5. □

Lemma 6.10. *Given M and N in $\mathcal{E}(X)$, the map* $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{E}(X))}(M, N[1]) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{inc}(M), \mathrm{inc}(N)[1])$ *induced by* inc *is bijective.*

Proof. Surjectivity follows immediately from Lemma 6.9. For injectivity, let δ be an element in the left-hand side. Then δ and $\mathrm{inc}(\delta)$ correspond to extensions

$$1 \to N \to E \to M \to 1, \quad \mathrm{inc}(N) \to \mathrm{inc}(E) \to \mathrm{inc}(M) \to,$$

in $\mathcal{E}(X)$ and $\mathcal{A}(X)$, respectively. Suppose the latter splits, i.e., $\mathrm{inc}(E) \to \mathrm{inc}(M)$ has a section. By the fullness in (ii), we can lift this section back to $\mathcal{E}(X)$, thereby splitting the extension corresponding to δ, and we're done. □

Proposition 6.11 (cf. [12, Lemma 4.3]). *The natural transformation $R^i p_*^{\mathcal{E}} \to H^i p_*^{\mathcal{A}} \circ \mathrm{inc}_{\mathcal{A}}$ is an isomorphism for $i = 0, 1$.* (See Remark 6.6.)

Proof. To reduce clutter, let us suppress the functor $\mathrm{inc}_{\mathcal{A}}$ from the notation for the duration of the proof; it can easily be reinserted. We begin with the case $i = 0$. Note that, naturally in $E \in \mathcal{E}$ and $M \in \mathcal{E}(X)$, we have

$$\mathrm{Hom}(E, p_*^{\mathcal{E}} M) = \mathrm{Hom}(p_{\mathcal{E}}^* E, M) = \mathrm{Hom}(p_{\mathcal{A}}^* E, M) = \mathrm{Hom}(E, p_*^{\mathcal{A}} M) = \mathrm{Hom}(E, H^0 p_*^{\mathcal{A}} M),$$

using property (ii) and property (v), which implies that $p_*^{\mathcal{E}} \simeq H^0 p_*^{\mathcal{A}}$.

Next, we deal with the more interesting case of $i = 1$. Applying $\mathrm{Hom}(-, -)$ (in the second argument) to the morphism of triangles

$$\begin{array}{ccccccc}
R^0 p_*^{\mathcal{E}} M & \longrightarrow & R p_*^{\mathcal{E}} M & \longrightarrow & \tau^{\geq 1} R p_*^{\mathcal{E}} M & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
H^0 p_*^{\mathcal{A}} M & \longrightarrow & p_*^{\mathcal{A}} M & \longrightarrow & \tau^{\geq 1} p_*^{\mathcal{A}} M & \longrightarrow
\end{array}$$

for $M \in \mathcal{E}(X)$ (keeping property (v) in mind), yields a morphism of (vertical) exact sequences

$$\begin{array}{ccc}
\mathrm{Hom}(-, R^0 p_*^{\mathcal{E}} M[1]) & \longrightarrow & \mathrm{Hom}(-, (H^0 p_*^{\mathcal{A}} M)[1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(-, R p_*^{\mathcal{E}} M[1]) & \longrightarrow & \mathrm{Hom}(-, p_*^{\mathcal{A}} M[1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(-, (\tau^{\geq 1} R p_*^{\mathcal{E}} M)[1]) & \longrightarrow & \mathrm{Hom}(-, (\tau^{\geq 1} p_*^{\mathcal{A}} M)[1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(-, R^0 p_*^{\mathcal{E}} M[2]) & \longrightarrow & \mathrm{Hom}(-, (H^0 p_*^{\mathcal{A}} M)[2])
\end{array}$$

Denote the regular representation $b_*\mathbf{1}$ in $\operatorname{Ind}\mathcal{E}$ by $\mathcal{O}$. After evaluating these long exact sequences at the pro-object $\mathcal{O}^\vee$, the topmost and bottommost terms vanish, since $\mathcal{O}^\vee$ doesn't have any nontrivial extensions (cf. Proposition 3.7). This lack of extensions implies, in fact, that there are no morphisms from $\mathcal{O}^\vee$ to $\mathcal{D}^{\leq -1}(\mathcal{E})$, and we thus have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}(p_{\mathcal{E}}^*\mathcal{O}^\vee, M[1]) & \longrightarrow & \operatorname{Hom}(p_{\mathcal{A}}^*\mathcal{O}^\vee, M[1]) \\
\| & & \| \\
\operatorname{Hom}(\mathcal{O}^\vee, Rp_*^{\mathcal{E}}M[1]) & \longrightarrow & \operatorname{Hom}(\mathcal{O}^\vee, p_*^{\mathcal{A}}M[1]) \\
\downarrow\sim & & \downarrow\sim \\
\operatorname{Hom}(\mathcal{O}^\vee, (\tau^{\geq 1}Rp_*^{\mathcal{E}}M)[1]) & \longrightarrow & \operatorname{Hom}(\mathcal{O}^\vee, (\tau^{\geq 1}p_*^{\mathcal{A}}M)[1]) \\
\| & & \| \\
\operatorname{Hom}(\mathcal{O}^\vee, R^1p_*^{\mathcal{E}}M) & \longrightarrow & \operatorname{Hom}(\mathcal{O}^\vee, H^1p_*^{\mathcal{A}}M)
\end{array}$$

Using the definition of $p_{\mathcal{A}}^*$ and Lemma 6.10, the leftmost vertical arrow is an isomorphism. Thus, applying $\operatorname{Hom}(\mathcal{O}^\vee, -)$ to the morphism $R^1p_*^{\mathcal{E}}M \to H^1p_*^{\mathcal{A}}M$, yields an isomorphism. We conclude by the conservativity of $\operatorname{Hom}(\mathcal{O}^\vee, -) = \operatorname{Hom}(-, \mathcal{O}) \circ (-)^\vee \simeq b^*$. □

Remark 6.12. In the above proof, we are implicitly viewing categories as full subcategories of their respective pro-categories in a number of places after the introduction of the pro-object $\mathcal{O}^\vee$.

Here, finally, is the point of this subsection:

Proposition 6.13. *Let X be a variety in SmVar_k with a k-point x. If $\mathcal{E}$ admits an ambient theory over X, then the cohomological assumption is satisfied, i.e., $b^*R^ip_*^{\mathcal{E}} \to R^ip_*b_{\mathcal{E}}^*$ is an isomorphism for $i = 0, 1$.*

Proof. This follows from property (iv), Proposition 6.11, and the commutativity of the following diagram:

$$\begin{array}{ccc}
b^*Rp_*^{\mathcal{E}} & \longrightarrow & Rp_*b_{\mathcal{E}}^* \\
\downarrow & & \uparrow{\scriptstyle (c)} \\
b^*p_*^{\mathcal{A}}\mathrm{inc}_{\mathcal{A}} & \longrightarrow & Rp_*b_{\mathcal{A}}^*\mathrm{inc}_{\mathcal{A}}
\end{array}$$

(see Remark 6.6). □

6.2. ***Checking the surjectivity assumption.*** We begin with a definition:

Definition 6.14. Say a weak theory $\mathcal{E}$ of enriched local systems *respects semisimplicity* if for every X in SmVar_k, the map $b^* : \mathcal{E}(X) \to \mathrm{LS}(X)$ sends semisimple objects to semisimple objects.

It turns out that this property is all we need for the surjectivity assumption:

Proposition 6.15. *Assume the characteristic of F is zero. Suppose $\mathcal{E}$ respects semisimplicity. Then, for any X in SmVar_k admitting an ambient theory and any k-point x, the map $\bar{\varrho} : \pi_1(X, x) \to S(X, x)$ has Zariski dense image.*

Proof. We show it on the Tannakian side, i.e., we work with

$$\bar{\varrho}^* : \operatorname{Rep} S(X, x) \to \operatorname{LS}(X).$$

By Proposition 2.11, we need to prove that the image of $\bar{\varrho}^*$ is stable under taking subobjects in $\operatorname{LS}(X)$. Recall that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(X) & \xrightarrow{\quad b^* \quad} & \operatorname{LS}(X) \\ {\scriptstyle \iota^*}\downarrow & \overset{\varrho^*}{\nearrow} & \uparrow{\scriptstyle \bar{\varrho}^*} \\ \operatorname{Rep} K(X, x) & \hookleftarrow & \operatorname{Rep} S(X, x). \end{array}$$

Step 1. Let $\mathcal{E}(X)_{\mathrm{ss}}$ denote the full subcategory of semisimple objects in $\mathcal{E}(X)$. We note that under ι^*, $\mathcal{E}(X)_{\mathrm{ss}}$ lands in the full subcategory $\operatorname{Rep} S(X, x) \subseteq \operatorname{Rep} K(X, x)$. Let $G(X, x)_{\mathrm{red}}$ be the Tannaka dual of $\mathcal{E}(X)_{\mathrm{ss}}$. It's the maximal pro-reductive quotient of $G(X, x)$, as F is of characteristic zero. As the pro-unipotent radical $R_{\mathrm{u}}K(X, x)$ is a characteristic subgroup of $K(X, x)$, it is normal in the larger group $G(X, x)$. Since $R_{\mathrm{u}}G(X, x)$ is the largest connected, pro-unipotent, normal subgroup of $G(X, x)$, it must contain $R_{\mathrm{u}}K(X, x)$. Thus, $K(X, x) \to G(X, x)_{\mathrm{red}}$ factors uniquely through $K(X, x) \to S(X, x)$.

Step 2. Next, we note that the map $S(X, x) \to G(X, x)_{\mathrm{red}}$ is injective. Indeed, this follows from $K(X, x) \cap R_{\mathrm{u}}G(X, x) = R_{\mathrm{u}}K(X, x)$, which holds true by the previous step and the definition of the right-hand side. In particular, for every object $A \in \operatorname{Rep} S(X, x)$, there exists $M \in \mathcal{E}(X)_{\mathrm{ss}}$ such that A is a subquotient of ι^*M, in $\operatorname{Rep} S(X, x)$. By the semisimplicity of $\operatorname{Rep} S(X, x)$, A can actually be taken to be a subobject of ι^*M.

Step 3. We now show that $\bar{\varrho}^*$ is full. Take two objects M and N in $\mathcal{E}(X)$. Then, writing $\iota^* : \mathcal{E}(X) \to \operatorname{Rep} K$, as before, we have

$$\operatorname{Hom}(\iota^*M, \iota^*N) = b^*s_* \operatorname{\underline{Hom}}(M, N) = \sigma_* \operatorname{\underline{Hom}}(b_X^*M, b_X^*N) = \operatorname{Hom}(\varrho^*\iota^*M, \varrho^*\iota^*N),$$

using the $i = 0$ part of Proposition 6.11; so we have fullness (for ϱ^* and hence also $\bar{\varrho}^*$) on objects coming from $\mathcal{E}(X)$.

By the previous step, every object in $\operatorname{Rep} S(X, x)$ is a direct summand of an object coming from $\mathcal{E}(X)_{\mathrm{ss}}$. Let two $S(X, x)$-representations A and B be direct summands of ι^*M and ι^*N, and cut out by projectors e and f, respectively. Then,

$$\operatorname{Hom}(A, B) = f \circ \operatorname{Hom}(\iota^*M, \iota^*N) \circ e = \varrho^*(f) \circ \operatorname{Hom}(\varrho^*\iota^*M, \varrho^*\iota^*N) \circ \varrho^*(e) = \operatorname{Hom}(\varrho^*A, \varrho^*B).$$

Note finally that the fullness of $\bar{\varrho}^*$ implies that its image is stable under taking direct summands (as fullness lets us lift projectors).

Step 4. Now we are ready to conclude. Let $A \in \operatorname{Rep} S(X, x)$ and $L \subseteq \varrho^*A$. By Step 2, there is $M \in \mathcal{E}(X)_{\mathrm{ss}}$ such that $A \subseteq \iota^*M$. Thus, $L \subseteq \bar{\varrho}^*\iota^*M = b^*M$, whence L is a direct factor of the right-hand side, as $\mathcal{E}$ respects semisimplicity, and we are done by Step 3. □

6.3. ***A corollary to Malcev completeness.*** Fix an enriched local system M in $\mathcal{E}(X)$ on some $X \in \mathrm{SmVar}_k$ with a rational point x. Denote by M_x the vector space x^*b^*M. We may consider the associated monodromy representation $\pi_1(X,x) \to \mathrm{GL}(M_x)$. Take the Malcev completion of this morphism, and denote it by $\hat{\pi}_1^M(X,x)$.

Proposition 6.16. *Suppose that $\pi_1(X,x) \to K(X,x)$ is Malcev complete. Then the group $\hat{\pi}_1^M(X,x)$ is canonically enriched in $\mathcal{E}(\mathrm{pt})$, in the sense that $\mathcal{O}(\hat{\pi}_1^M(X,x))$ is naturally a Hopf algebra object in* $\mathrm{Ind}\,\mathcal{E}(\mathrm{pt})$, *or equivalently that there is a natural action by $G(k)$ on $\hat{\pi}_1^M(X,x)$.*

Proof. Consider the full subcategory $\langle M, \mathcal{E}(\mathrm{pt})\rangle^{\otimes}$ tensor-generated by M and $\mathcal{E}(\mathrm{pt})$ in $\mathcal{E}(X)$, i.e., consisting of subquotients of sums of tensor products of M, $M^\vee$ and constant objects (i.e., objects coming from $\mathcal{E}$). We obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{G}(\langle \iota^* M\rangle^{\otimes}) & \longrightarrow & \mathscr{G}(\langle M, \mathcal{E}(\mathrm{pt})\rangle^{\otimes}) & \longrightarrow & G(k) & \longrightarrow & 1 \\ & & \uparrow\uparrow & & \uparrow\uparrow & & \| & & \\ 1 & \longrightarrow & \pi_1^{\mathcal{E}}(X,x) & \longrightarrow & G(X,x) & \longrightarrow & G(k) & \longrightarrow & 1. \end{array}$$

We have seen that the bottom row is (split) exact, and the (split) exactness of the top row follows.

We have a factorization

$$\pi_1(X,x) \to \mathscr{G}(\langle \iota^* M\rangle^{\otimes}) \hookrightarrow \mathrm{GL}(M_x),$$

of the monodromy representation. By Lemma 4.2, the Malcev completion of $\pi_1(X,x) \to \mathrm{GL}(M_x)$ only depends on the image in $\mathrm{GL}(M_x)$, which by this factorization lies in the subgroup $\mathscr{G}(\langle \iota^* M\rangle^{\otimes})$. Thus, $\hat{\pi}_1^M(X,x)$ is the Malcev completion of $\pi_1(X,x) \to \mathscr{G}(\langle \iota^* M\rangle^{\otimes})$.

Next, $\hat{\pi}_1^M(X,x)$ can be obtained as the Malcev completion of $\pi_1^{\mathcal{E}}(X,x) \to \mathscr{G}(\langle \iota^* M\rangle^{\otimes})$, by Lemma 4.10. Note that $G(k)$ acts on this morphism, and by functoriality of Malcev completions (Proposition 4.17), it acts on the Malcev completion, $\hat{\pi}_1^M(X,x)$. This finishes the proof. □

7. Hodge theory and motives

In this section, we present our two main applications of the Malcev completeness criterion (Theorem 5.1). By the previous section, for each choice of theory of enriched local systems, we simply need to check the existence of an ambient theory (Definition 6.5) and that semisimplicity is satisfied (Definition 6.14). For the entirety of this section, the field of coefficients F will be $\mathbb{Q}$. We extensively use the notation and terminology introduced in the previous section.

7.1. ***Variations of Hodge structures.*** We start with weak theory of enriched local systems $\mathcal{E}(-) = \mathrm{MHS}(-)$ given by *admissible variations of mixed Hodge structures*. (See, for example, [29, Ch. 10,14].) Let $X \in \mathrm{SmVar}_k$ and let $x \in X(\mathbb{C})$.[5] In this context, we denote the corresponding fundamental group (previously $\pi_1^{\mathcal{E}}(X,x)$) by $\pi_1^{\mathrm{Hdg}}(X,x)$ and call it the *Hodge fundamental group*. The representations of

[5] In this setting, x doesn't need to be rational, as $\mathcal{E}(\mathrm{Spec}\,k) = \mathcal{E}(\mathrm{Spec}\,\mathbb{C})$.

the Hodge fundamental group are called *local systems of Hodge origin*, and form a full subcategory of the category $\mathrm{LS}(X)$ of local systems, see Definition 6.2. They are exactly those local systems that arise as subquotients of local systems underlying admissible variations of mixed Hodge structures.

Over the point, we get the category MHS of graded-polarizable mixed Hodge structures. We call the Tannaka dual group H of MHS the *Hodge group*.

Lemma 7.1. $\mathrm{MHS}(-)$ *is a weak theory of enriched local systems, and that* (*the derived category of*) *mixed Hodge modules provides an ambient theory in the sense of Definition 6.5.*

Proof. Let $X \in \mathrm{SmVar}_k$. We write $\mathrm{MHM}(X)$ for the category of mixed Hodge modules on X, and $\mathcal{A}(X)$ for the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$. The functor $\mathrm{inc}_{\mathcal{A}}$ is the composition of the usual inclusion $\mathrm{MHS}(X) \subseteq \mathrm{MHM}(X)$, followed by the inclusion of the heart $\mathrm{MHM}(X) \subseteq \mathcal{A}(X)$, and then a shift by $-\dim(X)$. (This is because local systems embed via $\mathcal{D}(X)$ into $\mathscr{P}(X)$ after a shift by $\dim(X)$.) Condition (ii) is then clear, and condition (i) is in [30, Theorem 0.1]. For the realization functor $b^*_{\mathcal{A}}$, we consider the (exact) forgetful functor $\mathrm{MHM}(X) \to \mathscr{P}(X)$ from mixed Hodge modules to perverse sheaves. Deriving it, and combining with Beilinson's equivalence $\mathcal{D}^{\mathrm{b}}(\mathscr{P}(X)) \simeq \mathcal{D}(X)$ ([7, Theorem 1.3]), we get the functor $b^*_{\mathcal{A}} : \mathcal{A}(X) \to \mathcal{D}(X)$ (denoted by rat in [30]). The isomorphism in (c) follows.

Condition (iii) follows from [30, Theorem 0.2] and the definition of smooth mixed Hodge modules that precedes it. Condition (iv) is part of [30, Theorem 0.1]. Finally, in this case, condition (v) can be checked after realization, where it follows from the fact that Rp_* has perverse cohomological amplitude $\geq -\dim(X)$ ([8, 4.2.4]). □

Note that $\mathcal{E}(-) = \mathrm{MHS}(-)$ respects semisimplicity by Deligne's semisimplicity theorem (Theorem 4.2.6 of [13]). Thus, plugging Propositions 6.13 and 6.15 into Theorem 5.1 and combining with Proposition 6.16, we deduce the following:

Theorem 7.2. *Let X be a smooth, geometrically connected k-variety and $x \in X(\mathbb{C})$ a complex point.*

1. *The map $\pi_1(X, x) \to K^{\mathrm{Hdg}}(X, x)$ is Malcev complete. Equivalently, $\pi_1^{\mathrm{Hdg}}(X, x) = K^{\mathrm{Hdg}}(X, x)$, and the full subcategory of local systems on X of Hodge origin is stable under extension in the category of local systems on X. In particular, every unipotent local system on X is of Hodge origin.*
2. *For V an admissible variation of mixed Hodge structures, the Malcev completion $\hat{\pi}_1^V(X, x)$ of the associated monodromy representation $\pi_1(X, x) \to \mathrm{GL}(V_x)$ is canonically equipped with a mixed Hodge structure, in the sense that it admits a canonical action by the Hodge group.* □

Remark 7.3. Part 1 of Theorem 7.2 has been previously shown by D'Addezio and Esnault in [12, Theorem 4.4], by a different method. Part 2 is a generalization of [21, Theorem 13.1], where V is assumed to be pure and polarizable, and $\pi_1(X, x)$ is assumed to map Zariski densely onto $\mathrm{Aut}(V_x, \langle , \rangle)$. In particular, we prove Conjecture 5.5 from [2].

7.2. ***Motivic local systems.*** Ivorra and Morel have constructed and studied categories of *perverse* (*Nori*) *motives* in [24], and we will briefly review their construction. Given an additive category $\mathcal{Q}$ and an additive

functor $T : \mathcal{Q} \to \mathcal{A}$ to an abelian category $\mathcal{A}$, they construct a universal abelian category $\mathbf{A}^{\mathrm{ad}}(\mathcal{Q}, T)$ factoring this functor along an exact and faithful functor to $\mathcal{A}$. Below, this universal category $\mathbf{A}^{\mathrm{ad}}(\mathcal{Q}, T)$ will be called the *Nori category* of $T : \mathcal{Q} \to \mathcal{A}$.

Given a k-variety (separated and of finite type) X, they consider the additive functor

$$ {}^{p}H^0 \circ \mathrm{Bti}^* : \mathrm{DA}_{\mathrm{ct}}(X) \to \mathcal{P}(X), $$

from the (triangulated) category of constructible étale motivic sheaves (with $\mathbb{Q}$-coefficients) on X to ($\mathbb{Q}$-linear) perverse sheaves on X. We denote the Nori category of this functor by $\mathrm{PM}(X)$: it is the category of *perverse motives* considered in [24] (where it is denoted by $\mathscr{M}(X)$). Here, Bti^* is the Betti realization functor constructed by Ayoub in [3]. We note that the category $\mathrm{PM}(k)$ of perverse motives over the field k is equivalent to the classical category $\mathrm{NM}(k)$ of Nori motives ([24, Proposition 2.11]).

Note that $\mathrm{NM}(k)$ is neutral Tannakian, and that its Tannaka dual $\mathscr{G}^{\mathrm{mot}}(k)$ is the *motivic Galois group* of k. It's a pro-algebraic group (over $\mathbb{Q}$).

Perverse motives come equipped with a realization functor $\mathrm{PM}(X) \to \mathcal{P}(X)$ to perverse sheaves which we denote by b_X^*. Ivorra and Morel develop a four-functor formalism for perverse motives. In [32], this is extended to a full six-functor formalism, with the construction of a tensor product on $\mathcal{D}^{\mathrm{b}}\,\mathrm{PM}(X)$.

Definition 7.4. Let X be a smooth, geometrically connected k-variety. Denote the full subcategory of $\mathrm{PM}(X)$ on objects that are sent into $\mathrm{LS}(X)[d]$ under b_X^* by $\mathrm{MLS}(X)$. It's the category of *motivic local systems* on X.

By [32, Theorem 6.2], these are neutral Tannakian categories over $\mathbb{Q}$, neutralized by $x^* \circ b_X^*$ for any complex point $x \in X(\mathbb{C})$. Since pullbacks preserve local systems, the same is true for motivic local systems. The Betti realization induces faithful exact functors $b^* : \mathrm{MLS}(X) \to \mathrm{LS}(X)$. In conclusion, we have a weak theory of enriched local systems $\mathcal{E} = \mathrm{MLS}$ (Definition 6.1).

We denote the associated categories $\mathrm{LS}_{\mathcal{E}}(X)$ (see Definition 6.2) by $\mathrm{LS}_{\mathrm{geo}}(X)$ and call its objects local systems *of geometric origin*. Their dual groups we denote by $\pi_1^{\mathrm{mot}}(X, x)$ (for a complex point x). We moreover denote the Tannaka dual of $\mathrm{MLS}(X)$ relative the fiber functor $x^* \circ \mathrm{Bti}^*$ by $G(X, x)$. As in Section 6, we denote the kernel of $G(X, x) \to \mathscr{G}^{\mathrm{mot}}(k)$ by $K(X, x)$, and its maximal pro-reductive quotient by $S(X, x)$. Finally, note that if x is a rational point, then we get a section of the homomorphism $G(X, x) \to \mathscr{G}^{\mathrm{mot}}(k)$.

In summary, for X a geometrically connected, smooth k-variety with a k-point x, we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & S(X,x) & & & & \\ & & & & \uparrow\!\!\uparrow & & & & \\ 1 & \longrightarrow & K(X,x) & \longrightarrow & G(X,x) & \hookleftarrow & \mathscr{G}^{\mathrm{mot}}(k) & \longrightarrow & 1 \\ & & & \nearrow & \uparrow & & & & \\ & & \pi_1(X,x) & \twoheadrightarrow & \pi_1^{\mathrm{mot}}(X,x) & & & & \end{array}$$

whose middle row is (split) exact (just as in Section 6).

Lemma 7.5. *Let $X \in \mathrm{SmVar}_k$. Then $\mathcal{D}^{\mathrm{b}}(\mathrm{PM}(X))$ is an ambient theory for* MLS *over X (Definition 6.5).*

Proof. We follow the notation in Definition 6.5 and let $\mathcal{A}(X) := D^{\mathrm{b}}(\mathrm{PM}(X))$. For $\mathrm{inc}_{\mathcal{A}}$, we take the composition $\mathrm{MLS}(X) \hookrightarrow \mathrm{PM}(X) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathrm{PM}(X))$. For $b^*_{\mathcal{A}}$, we take the Betti realization. Lastly, the isomorphism asked for in (c) is actually an equality in this case. We now quickly check that the conditions for an ambient theory hold in this case.

Condition (i) is true by [24], and conditions (ii) and (iii) are true by definition. Next, (iv) is true by [24] (compatibility of the four operations with the Betti realization). Finally, condition (v) can (by conservativity) be checked under realization, where it follows from [8, 4.2.4]. □

In [24, Section 6], it is shown that their perverse motives admit a robust theory of weights. This lets us show the following:

Lemma 7.6. *The theory $\mathcal{E}(-) = \mathrm{MLS}(-)$ respects semisimplicity* (Definition 6.14).

Proof. We need to show that $b^* : \mathrm{MLS}(X) \to \mathrm{LS}(X)$ sends semisimple objects to semisimple objects. But thanks to [33, Theorem 1.1], perverse motives admit a Hodge realization, and we can factor the Betti realization through the Hodge realization as

$$\mathrm{MLS}(X) \to \mathrm{MHS}(X) \to \mathrm{LS}(X).$$

In the proof of [33, Proposition 6.9] it is explained how [24, Corollary 6.27] implies that the Hodge realization sends pure objects to pure objects. In both categories, the semisimple objects are exactly the ones obtained as direct sums of pure ones. We can thus conclude using Deligne's semisimplicity theorem. □

Thus, as in the Hodge theoretic case, we get the following theorem:

Theorem 7.7. *Let $X \in \mathrm{SmVar}_k$ with a rational point x.*

1. *The homomorphism $\pi_1(X, x) \to K(X, x)$ is Malcev complete. Equivalently,*

$$\pi_1^{\mathrm{mot}}(X, x) = K(X, x) := \ker(G(X, x) \to \mathscr{G}^{\mathrm{mot}}(k)),$$

   and the full subcategory of local systems on X of geometric origin is stable under extension. In particular, every unipotent local system on X is of geometric origin.

2. *For a motivic local system M on X, the Malcev completion $\hat{\pi}_1^M(X, x)$ of the associated monodromy representation $\pi_1(X, x) \to \mathrm{GL}(M_x)$, is motivic, in the sense that it carries a canonical action by the motivic Galois group.* □

The second part is a (Nori) motivic refinement of Hain's theorem.

8. De Rham fundamental groups

We round out the paper with a slightly different kind of application. The goal of this section is to use the unipotent isomorphism criterion (Theorem 3.3) to give a quick proof of Lazda's theorem. The theorem says that, for a nice enough family, the relative unipotent de Rham fundamental group of the family and the unipotent fundamental group of the special fiber are isomorphic.

Let $k = F$ be a field of characteristic zero, let S be a smooth k-scheme, let $f : X \to S$ be a smooth S-scheme with an S-point $x : S \to X$, let $s : \mathrm{pt} \to S$ be a closed point of S, and write x also for the closed point of X given by $x \circ s$. Denote the fiber of X over s by X_s, and the canonical inclusion by $i : X_s \to X$. For Y a smooth k-scheme, let $\mathrm{IC}(Y)$ denote the category of vector bundles on Y with regular integrable connection: in short, the category of *regular connections* (see for example [22, Definitions 1.2.2 and 5.3.2]). If Y is moreover geometrically connected, then $\mathrm{IC}(Y)$ is a k-linear Tannakian category.

Given a morphism of smooth k-schemes $g : Y \to Z$, we have an induced functor $g^* : \mathrm{IC}(Z) \to \mathrm{IC}(Y)$. And in particular, if $y : \mathrm{pt} \to Y$ is a closed point of Y, and Y is geometrically connected, then $y^* : \mathrm{IC}(Y) \to \mathrm{Vec}$ is a neutral fiber functor. In this case, we denote the Tannaka dual of $\mathrm{IC}(Y)$ by $\pi_1^{\mathrm{dR}}(Y, y)$, and call it the *de Rham fundamental group* of (Y, y).

Suppose that both Y and Z as above are geometrically connected and $g : Y \to Z$, then, following the terminology of Section 2.1, we denote the full subcategory of g^*-unipotent objects in $\mathrm{IC}(Y)$ by $\mathrm{IC}_g^{\mathrm{un}}(Y)$. If $p : Y \to \mathrm{pt}$ is the structure map of Y, then we simply write $\mathrm{IC}^{\mathrm{un}}(Y) := \mathrm{IC}_p^{\mathrm{un}}(Y)$: this is the full subcategory of (absolutely) unipotent objects in $\mathrm{IC}(Y)$. We denote the Tannaka dual of $\mathrm{IC}^{\mathrm{un}}(Y)$ by $\pi_1^{\mathrm{udR}}(Y, y)$, and call it the *unipotent de Rham fundamental group* of (Y, y) (it is the unipotent completion of $\pi_1^{\mathrm{dR}}(Y, y)$). We don't have a good symbol for the Tannaka dual of $\mathrm{IC}_g^{\mathrm{un}}(Y)$, and we simply denote it by $\mathscr{G}(\mathrm{IC}_g^{\mathrm{un}}(Y), y)$, following our general conventions.

Remark 8.1. In the literature, it is often the case that only the unipotent de Rham fundamental group is considered. Usually, the word "unipotent" is then omitted from the terminology.

Assuming that X and S are geometrically connected, and that f has geometrically connected fibers, we obtain a diagram of exact and faithful functors

$$\begin{array}{ccc} \mathrm{IC}_f^{\mathrm{un}}(X) & \xrightarrow{i^*} & \mathrm{IC}^{\mathrm{un}}(X_s) \\ {\scriptstyle x^*}\downarrow\uparrow{\scriptstyle f^*} & & {\scriptstyle x^*}\downarrow\uparrow{\scriptstyle f_s^*} \\ \mathrm{IC}(S) & \xrightarrow{s^*} & \mathrm{Vec} \end{array}$$

which is an instance of the Standard Situation (Section 2.6). We get, in particular, a split exact sequence of pro-algebraic groups

$$1 \longrightarrow K \longrightarrow \mathscr{G}(\mathrm{IC}_f^{\mathrm{un}}(X), x) \longrightarrow \pi_1^{\mathrm{dR}}(S, s) \longrightarrow 1$$

$$\uparrow$$

$$\pi_1^{\mathrm{udR}}(X_s, x)$$

and a morphism from $\pi_1^{\mathrm{udR}}(X_s, x)$ to the kernel K. We call the kernel the *relative unipotent de Rham fundamental group* of X/S and denote it by $\pi_1^{\mathrm{udR}}(X/S, x)$. If we could apply Theorem 3.3 to this situation, we would conclude that $\pi_1^{\mathrm{udR}}(X_s, x) \to \pi_1^{\mathrm{udR}}(X/S, x)$ is an isomorphism.

In order to apply Theorem 3.3, we have two conditions that need to hold: the natural maps

$$s^* \circ R^k f_* \to R^k f_{s,*} \circ i^*$$

are isomorphisms, for $k = 0, 1$ (we use subscript-asterisks to denote ind-right adjoints, as usual).

Proposition 8.2. *Assume that k is algebraically closed. Let $f : X \to S$ be a smooth family with geometrically connected fibers, let $s : S \to X$ be a section, and keep all the notation introduced above. Assume moreover that f admits a good compactification.*[6] *Then the natural maps*

$$s^* \circ R^k f_* \to R^k f_{s,*} \circ i^*$$

are isomorphisms for $k = 0, 1$.

Proof. Following Lazda, we set $f_*^{\mathrm{dR}} = \mathscr{H}^{-d}(f_+(-))$, where d is the relative dimension of $f : X \to S$, and f_+ is the pushforward of regular holonomic complexes of $\mathscr{D}$-modules. By [27, Lemma 1.9], f_*^{dR} takes f-unipotent regular connections to regular connections, and, viewing it as a functor $\mathrm{IC}_f^{\mathrm{un}}(X) \to \mathrm{IC}(S)$, it is equal to f_*. Moreover, pullback (f^+) of regular holonomic complexes becomes $f^*[-d]$ when restricted to the abelian category of regular connections.

Next, note that regular connections are stable under extension in the category of regular holonomic $\mathscr{D}$-modules. For the $k = 1$ part, the same strategy as in the proof of Proposition 6.11 works, and the details are left to the reader. The input required from the theory of $\mathscr{D}$-modules to make the argument work is that base change holds: $f_{s,+} i^+ \simeq s^+ f_+$ ([22, Theorem 1.7.3 and Remark 1.7.6]). □

Combining this result and Theorem 3.3, we get the following:

Theorem 8.3 ([27, Corollary 1.20]). *Under the assumptions of the previous proposition, the map $\pi_1^{\mathrm{udR}}(X_s, x) \to \pi_1^{\mathrm{udR}}(X/S, x)$ is an isomorphism.*

Remark 8.4. The assumption that k is algebraically closed is not expected to be necessary (cf. [27, Remark 1.21]), and the only place we use it is in the invocation of [27, Lemma 1.9].

Remark 8.5. A referee produced an example showing that if the family $f : X \to S$ fails to admit a section, then the two groups in the theorem need not be isomorphic.

In work in progress, I am applying the Malcev completeness criterion (Theorem 5.1) to this setting of de Rham fundamental groups. It yields, under appropriate hypotheses, a result on the stability under extension of regular connections over X_s that come from regular connections over X.

[6] I.e., it factors as an open immersion $X \to \bar{X}$, followed by a smooth and proper map $\bar{X} \to S$, such that the complement of X in $\bar{X}$ is a relative normal crossing divisor.

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