Algebra & Number Theory

Volume 20
2026
No. 2

The Alperin weight conjecture and the Glauberman correspondence via character triples

J. Miquel Martínez, Noelia Rizo and Damiano Rossi

msp

ALGEBRA AND NUMBER THEORY 20:2 (2026)
https://doi.org/10.2140/ant.2026.20.333

The Alperin weight conjecture and the Glauberman correspondence via character triples

J. Miquel Martínez, Noelia Rizo and Damiano Rossi

In 2017, G. Navarro introduced a new conjecture that unifies the Alperin weight conjecture and the Glauberman correspondence into a single statement. In this paper, we reduce this problem to simple groups and prove it for several classes of groups and blocks. Our reduction can be divided into two steps. First, we show that by assuming the so-called *inductive* (*blockwise*) *Alperin weight condition* for finite simple groups, we obtain an analogous statement for arbitrary finite groups, that is, an automorphism-equivariant version of the Alperin weight conjecture inducing isomorphisms of modular character triples. Then, we show that the latter implies Navarro's conjecture for each finite group.

1. Introduction

The local-global principle in representation theory of finite groups asserts that a large part of the information on the representation theory of a finite group at a prime p can be recovered by studying the p-local subgroups of the group. In this context, a p-local subgroup of a finite group G should be understood as a normalizer of a radical p-subgroup: that is a p-subgroup P of G that coincides with the largest normal p-subgroup of the normalizer $\boldsymbol{N}_G(P)$. The simplest examples of radical p-subgroups are given by the Sylow p-subgroups of G and the p-core, defined as the largest normal p-subgroup of G. These are in fact the extremal cases: each radical p-subgroup contains the p-core and is contained in some Sylow p-subgroup.

The local-global principle is based on a large amount of evidence and can be explained by several conjectural statements known as the local-global counting conjectures. These include the McKay conjecture [McK72] (whose solution recently appeared in [CS]), the Alperin–McKay conjecture [Alp76],

Martínez and Rizo are partially supported by the Spanish Ministerio de Ciencia e Innovación Grant PID2022-137612NB-I00 (funded by MCIN/AEI/ 10.13039/501100011033 and "ERDF A way of making Europe"). Martínez is also supported by a fellowship UV-INV-PREDOC20-1356056 from Universitat de València and by the European Union – Next Generation EU, Missione 4 Componente 1, PRIN 2022- 2022PSTWLB – Group Theory and Applications, CUP B53D23009410006. Rizo is supported by a CDEIGENT grant CIDEIG/2022/29 funded by Generalitat Valenciana. Rossi is supported by the EPSRC grant EP/W028794/1, by a Leibniz Fellowship of the Mathematisches Forschungsinstitut Oberwolfach, and by the Walter Benjamin Programme of the DFG – Project number 525464727. The authors also acknowledge support of the Institut Henri Poincaré (UAR 839 CNRS-Sorbonne Université) and LabEx CARMIN (ANR-10-LABX-59-01).
MSC2020: primary 20C20; secondary 20C15.
Keywords: Alperin weight conjecture, Glauberman correspondence, block theory, isomorphisms of character triples.

the Alperin weight conjecture, and Dade's conjecture [Dad92]. These statements are among the main research questions in representation theory of finite groups and, in recent years, their study has seen remarkable progress thanks to the collective efforts of many researchers.

In this paper, we focus on the Alperin weight conjecture, introduced in [Alp87]. This statement provides a way to determine the number of isomorphism classes of irreducible p-modular representations of a finite group G in terms of local data called p-weights. More precisely, for a prime number p, we define a p-weight to be a pair (Q, ψ) where Q is a radical p-subgroup of G and ψ is a p-Brauer character of $\mathbf{N}_G(Q)$ whose deflation to $\mathbf{N}_G(Q)/Q$ belongs to a p-block of defect zero. The set $\mathrm{Alp}(G)$ of p-weights of G is equipped with an action of G by conjugation, whose corresponding set of orbits is denoted by $\mathrm{Alp}(G)/G$. The Alperin weight conjecture states that

$$|\mathrm{IBr}(G)| = |\mathrm{Alp}(G)/G|$$

where $\mathrm{IBr}(G)$ is the set of irreducible p-Brauer characters of G. In this paper, we work with a refinement of this statement to the framework of block theory. Given a p-block B of G, we say that a p-weight (Q, ψ) is a B-weight if $\mathrm{bl}(\psi)^G = B$ and where $\mathrm{bl}(\psi)$ denotes the unique p-block of $\mathbf{N}_G(Q)$ to which ψ belongs. The set $\mathrm{Alp}(B)$ of B-weights is also equipped with an action of G by conjugation and we denote its set of orbits by $\mathrm{Alp}(B)/G$. Then, the blockwise version of the Alperin weight conjecture posits that

$$|\mathrm{IBr}(B)| = |\mathrm{Alp}(B)/G|$$

where $\mathrm{IBr}(B)$ is the set of irreducible p-Brauer characters belonging to the p-block B.

The Alperin weight conjecture and other local-global counting conjectures are intimately connected with the existence of natural correspondences of characters and blocks. One of the most useful such statements is the Glauberman correspondence and its blockwise version, known as the Dade–Glauberman–Nagao correspondence. In its most basic form, this asserts that whenever a p-group A acts via automorphisms on a group G of order prime to p there exists a canonical bijection

$$\mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A)),$$

where we denote by $\mathrm{Irr}_A(G)$ the set of A-invariant irreducible characters of G. In this case the above bijection is equivalent to the Brauer–Glauberman correspondence introduced in [NST17, Conjecture A]. We refer the reader to Section 5 and [NT11, Section 4] for further information on the Dade–Glauberman–Nagao correspondence.

Navarro suggested in [Nav17] a new surprising statement that unifies the Alperin weight conjecture and the Dade–Glauberman–Nagao correspondence into a single statement. Let $G \trianglelefteq \Gamma$ be finite groups and consider a p-block B of G. For every radical p-subgroup Q of Γ, we denote by $\mathrm{dz}(\mathbf{N}_\Gamma(Q)/Q \mid B)$ the set of irreducible characters $\bar{\vartheta}$ of $\mathbf{N}_\Gamma(Q)/Q$ of p-defect zero such that $\mathrm{bl}(\vartheta)^\Gamma$ covers B and where $\vartheta \in \mathrm{Irr}(\mathbf{N}_G(Q))$ corresponds to $\bar{\vartheta}$ via inflation of characters. Navarro's conjecture can then be stated as follows.

Conjecture A (Navarro). *Let $G \trianglelefteq \Gamma$ be finite groups and consider a p-block B of G. If Γ/G is a p-group and B is Γ-invariant, then*

$$|\mathrm{IBr}_\Gamma(B)| = \sum_Q \left|\mathrm{dz}(N_\Gamma(Q)/Q \mid B)\right|,$$

where Q runs over a set of representatives for the action of Γ on the set of radical p-subgroups of Γ such that $\Gamma = GQ$ and $Q \cap G$ is contained in some defect group of the p-block B.

The blockwise version of the Alperin weight conjecture can be recovered from Conjecture A by choosing $\Gamma = G$. Furthermore, from Conjecture A we also recover the Glauberman correspondence, by considering the case where G has order prime to p, and more generally the Dade–Glauberman–Nagao correspondence (see Lemma 7.2). We point out that the above statement admits a more general version in which the quotient Γ/G need not be a p-group (see Conjecture 7.3).

At the end of [Nav17] it was asked whether Conjecture A could be obtained as a consequence of the so-called inductive (blockwise) Alperin weight condition introduced in [Spä13b] to reduce the (blockwise) Alperin weight conjecture to simple groups (see also [NT11] for the original reduction of the block-free version of Alperin's conjecture). In this paper, we show that this is indeed the case and therefore obtain a reduction of Conjecture A to simple groups.

Before stating our reduction theorem for Conjecture A, we remind the reader of a (perhaps not so well-known) phenomenon that has been observed in relation to the reduction theorems for the local-global counting conjectures. Originally, going back to the work of E. C. Dade [Dad92; Dad94; Dad97], it was expected that for each of the local-global conjectures there would be a refinement of such a statement that would be strong enough to hold for every finite group if proved for all nonabelian finite simple groups. Dade's project remained open long after its formulation and no such reduction was found for several years. The first breakthrough in this direction was achieved by Isaacs, Malle, and Navarro in [IMN07] where a reduction for the McKay conjecture was proved. This seminal work was then followed by several other reduction theorems [NT11; Spä13a; Spä13b; Spä17b]. Contrary to what Dade expected, all these theorems reduce a given local-global conjecture to a much stronger statement usually referred to as its *inductive condition*. However, the full strength of such inductive conditions was at first not recovered for all finite groups. This was first accomplished in [NS14b] where it was shown that assuming the inductive Alperin–McKay condition for all finite simple groups then, not only would the Alperin–McKay conjecture hold for every finite group, but even a refinement analogous to its inductive condition, as was (ideologically) expected by Dade. Following [NS14b], an analogous result was obtained for the McKay conjecture in [Ros23c].

In this paper, we prove a similar reduction theorem in the context of the (blockwise) Alperin weight conjecture. First, we state a version of the inductive condition for arbitrary finite groups by using the notion of block isomorphism of modular character triples, denoted by $\succeq_b$, as defined in Section 3.

Conjecture B (inductive blockwise Alperin weight condition). *Let $G \trianglelefteq A$ be finite groups and consider a p-block B of G. If A_B denotes the stabilizer of B in A, then there exists an A_B-equivariant bijection*

$$\Omega : \mathrm{IBr}(B) \to \mathrm{Alp}(B)/G$$

such that

$$(A_\vartheta, G, \vartheta) \succeq_b \left(\mathbf{N}_A(Q)_\psi, \mathbf{N}_G(Q), \psi\right)$$

for every $\vartheta \in \mathrm{IBr}(B)$ and $(Q, \psi) \in \Omega(\vartheta)$.

In what follows, we say that Conjecture B holds for G at the prime p if it holds for every p-block B of G and every choice of $G \trianglelefteq A$. Recall, furthermore that a simple group S is involved in G if there exists $K \trianglelefteq H \leq G$ such that $S \simeq H/K$. We can now state our first main result.

Theorem C. *Let G be a finite group and p a prime number. If Conjecture B holds at the prime p for every covering group of any nonabelian finite simple group involved in G, then Conjecture B holds for G.*

These enhanced reduction theorems have been shown to have important implications. For instance, the reduction of the (inductive) Alperin–McKay conjecture obtained in [NS14b] was used to deduce a reduction theorem for Brauer's Height Zero Conjecture, which led ultimately to a final solution of Brauer's conjecture for the prime $p = 2$ thanks to work of Ruhstorfer [Ruh22a]. The latter was in turn used in the final proof recently obtained by Malle, Navarro, Schaeffer-Fry, and Tiep in [MNSFT22] while relying on a different argument for odd primes. On the other hand, the reduction of the (inductive) McKay conjecture from [Ros23c] is used in the verification of the inductive McKay condition for finite simple groups of Lie type (in type D) in the work of Cabanes and Späth [CS] and ultimately contributes to the final proof of the McKay conjecture itself. Similarly, the inductive condition for Dade's conjecture, also known as the Character Triple Conjecture (see [Spä17b, Conjecture 6.3]), has been shown to impact the construction of certain character bijections needed to even verify the original version of Dade's conjecture (see [Ros22b, Section 6] and [Ros23b, Section 4.2]).

Following the path described in the above paragraph, we prove yet another application of these stronger reduction theorems. In fact, we use Theorem C to obtain a reduction theorem for Conjecture A. This will follow as a consequence of the following result.

Theorem D. *Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a p-block B of G. If Conjecture B holds for the p-block B with respect to $G \trianglelefteq \Gamma$, then Conjecture A holds for the p-block B with respect to $G \trianglelefteq \Gamma$.*

Combining Theorem D and Theorem C we finally obtain the following reduction to finite simple groups for Conjecture A.

Corollary E. *Let G be a finite group and p a prime number. If Conjecture B holds at the prime p for every covering group of any nonabelian finite simple group involved in G, then Conjecture A holds for every p-block of G.*

As mentioned above, Conjecture A can be extended to arbitrary quotients Γ/G (see Conjecture 7.3). In Section 7, we show that this more general statement is also a consequence of Conjecture B and can therefore be reduced to finite simple groups (see Proposition 7.5). Furthermore, we obtain block-free versions of these results (see Section 7.2).

As an application of our reduction theorems, and using the fact that Conjecture B has been verified for several classes of finite simple groups (see, for instance, [FZ22] and the references therein), we show that Conjecture A and Conjecture B hold for several classes of groups and blocks, including groups with abelian Sylow 2-subgroups or abelian Sylow 3-subgroups, groups with odd Sylow automizer, p-blocks with cyclic defect groups, nilpotent p-blocks, and 2-blocks with abelian defect groups.

Outline. After introducing the relevant notation and preliminary results on block isomorphisms of modular character triples in Section 2 and Section 3, we prove certain consequences of the inductive blockwise Alperin weight condition in Section 4. Next, in Section 5 we prove a modular version of the Dade–Glauberman–Nagao correspondence compatible with block isomorphisms of modular character triples (see Theorem 5.6). Using this result, we then prove Theorem C in Section 6 (see also Section 6.1 for a block-free version of the reduction). Section 7 is devoted to Navarro's conjecture and to the proofs of Theorem D and Corollary E, as well as analogous block-free results (see Section 7.2). In Section 7.1, we use one further result from Section 5 (see Corollary 5.12) to obtain a version of Conjecture A compatible with isomorphisms of modular character triples that extends the main result of [Tur08]. Finally, in Section 8 we prove Conjectures A and B for the above-mentioned classes of groups and blocks.

2. Notation and preliminary results

Throughout the paper we use standard notation from ordinary and modular character theory. We refer the reader to [Nav18] and [Nav98] for a detailed introduction to the subject.

Let p be a prime, $\boldsymbol{R}$ the ring of algebraic integers in $\overline{\mathbb{Q}}$, and fix a maximal ideal $\boldsymbol{M}$ of $\boldsymbol{R}$ containing p. Then the quotient $\mathbb{F} = \boldsymbol{R}/\boldsymbol{M}$ is an algebraically closed field of characteristic p. Furthermore, if $\boldsymbol{S}$ is the localization of $\boldsymbol{R}$ at $\boldsymbol{M}$ then we denote by

$$^{*} : \boldsymbol{S} \to \mathbb{F}$$

the epimorphism from [Nav98, Chapter 2].

We denote by $\mathrm{Bl}(G)$ the set of p-blocks (or simply blocks) of a finite group G and by $\lambda_B : \boldsymbol{Z}(\mathbb{F}G) \to \mathbb{F}$ the central function associated to each $B \in \mathrm{Bl}(G)$. Whenever $\chi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$, the central function $\lambda_\chi : \boldsymbol{Z}(\mathbb{F}G) \to \mathbb{F}$ coincides with λ_B if and only if the block $\mathrm{bl}(\chi)$ of G containing χ coincides with B. In this case, we write $\chi \in \mathrm{Irr}(B) \cup \mathrm{IBr}(B)$. If $H \leq G$ and $b \in \mathrm{Bl}(H)$, then the induced block b^G is defined if the linear map λ_b^G defined in [Nav98, p. 87] is an algebra homomorphism. In this case, there is a unique $B \in \mathrm{Bl}(G)$ such that $\lambda_b^G = \lambda_B$ and we write $b^G = B$. The central functions considered here are determined by their values on a basis of $\boldsymbol{Z}(\mathbb{F}G)$. One such basis is provided by the conjugacy class sums

$$\mathfrak{Cl}_G(x)^+ = \sum_{y \in \mathfrak{Cl}_G(x)} y,$$

considered as an element of the group algebra of G and where $\mathfrak{Cl}_G(x)$ denotes the conjugacy class of x in G.

Recall that the set of characters $\mathrm{Irr}(G/N)$ can be identified with the set of irreducible characters of G containing N in their kernel. A similar remark holds for Brauer characters. This identification is often referred to as *inflation* of characters.

Now consider an ordinary character $\chi \in \mathrm{Irr}(G)$ and let G^0 be the set of p-regular elements of G, that is, the set of elements of G whose order is prime to p. Then, the restriction χ^0 of χ to G^0 is a Brauer character that decomposes as

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi,\varphi}\varphi$$

for some integers $d_{\chi,\varphi}$, called decomposition numbers. Using decomposition numbers, we can then define projective indecomposable characters. More precisely, for any Brauer character $\varphi \in \mathrm{IBr}(G)$, the projective indecomposable character associated to φ is the ordinary character of G defined by

$$\Phi_\varphi := \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi,\varphi}\chi.$$

We finally introduce the notion of character triple that will be fundamental in the rest of this work. If $N \triangleleft G$ and $\vartheta \in \mathrm{Irr}(N) \cup \mathrm{IBr}(N)$ is G-invariant then we say (G, N, ϑ) is a *character triple*. We say that (G, N, ϑ) is an *ordinary* or *modular* character triple if $\vartheta \in \mathrm{Irr}(N)$ or $\vartheta \in \mathrm{IBr}(N)$ respectively. In the particular situation where $\vartheta \in \mathrm{Irr}(N)$ and $\vartheta^0 \in \mathrm{IBr}(N)$, we say that (G, N, ϑ) is an *ordinary-modular* character triple. This is the case, for instance, when $\vartheta \in \mathrm{Irr}(N)$ has *p-defect zero*, that is, when ϑ satisfies $\vartheta(1)_p = |N|_p$ (see [Nav98, Theorem 3.18]). The set of irreducible ordinary characters of defect zero of a finite group G is denoted by $\mathrm{dz}(G)$.

Lemma 2.1. *Let (G, N, ϑ) be an ordinary-modular character triple with $\vartheta \in \mathrm{dz}(N)$. Then ϑ extends to G if and only if ϑ^0 extends to G. Furthermore, restriction to p-regular elements is a surjection from the set of extensions of ϑ in $\mathrm{Irr}(G)$ onto the set of extensions of ϑ^0 in $\mathrm{IBr}(G)$.*

Proof. To start, recall that by [Nav98, Problem 8.13] there exists an ordinary-modular character triple (G^*, N^*, ϑ^*) that is isomorphic, as ordinary-modular triple, to (G, N, ϑ) and where N^* is a p'-group (that is, of order prime to p). In particular, ϑ (resp. ϑ^0) extends to G if and only if ϑ^* (resp. $(\vartheta^*)^0$) extends to G^*. Therefore, it is no loss of generality to assume that N has order prime to p. Suppose now that ϑ^0 extends to G and let Q/N be a Sylow q-subgroup of G/N for a prime q. Suppose first that $q \neq p$. By assumption, we know that ϑ^0 has an extension $\psi \in \mathrm{IBr}(G)$ and therefore $\psi_Q \in \mathrm{IBr}(Q)$ is an extension of ϑ^0. However, since $q \neq p$ and p does not divide the order of N, we conclude that $\psi_Q \in \mathrm{IBr}(Q) = \mathrm{Irr}(Q)$ is an ordinary extension of ϑ. On the other hand, if $q = p$, then $|Q : N|$ and $|N|$ are coprime and so ϑ extends to an irreducible character in $\mathrm{Irr}(Q)$ according to [Nav18, Theorem 6.2]. Then [Nav18, Theorem 5.10] implies that ϑ extends to an irreducible ordinary character in $\mathrm{Irr}(G)$. Assume then that ϑ extends to G. Then, we deduce that ϑ^0 extends to an irreducible Brauer character of G by arguing as before but using [Nav98, Theorems 8.11 and 8.29]. □

Lemma 2.2. *Let $N \lhd G$ and let $\vartheta \in \mathrm{Irr}(N)$ with $\vartheta^0 \in \mathrm{IBr}(N)$. Suppose that ϑ extends to $\tilde{\vartheta} \in \mathrm{Irr}(G)$ and that $\tilde{\vartheta}^0 \in \mathrm{IBr}(G)$. Consider $\chi \in \mathrm{Irr}(G \mid \vartheta)$ and set $\chi := \eta\tilde{\vartheta}$ for some $\eta \in \mathrm{Irr}(G/N)$. Then $\chi^0 \in \mathrm{IBr}(G)$ if and only if $\eta^0 \in \mathrm{IBr}(G/N)$.*

Proof. By Lemma 2.1 we know that $\tilde{\vartheta}^\circ \in \mathrm{IBr}(G)$ is an extension of ϑ^0. Now, if $\eta^0 \in \mathrm{IBr}(G/N)$, we deduce from [Nav98, Corollary 8.20] that $\chi^0 = \eta^0\tilde{\vartheta}^0 \in \mathrm{IBr}(G)$. Assume conversely that $\chi^0 \in \mathrm{IBr}(G)$ and write

$$\eta^0 = \sum_{\varphi \in \mathrm{IBr}(G/N)} d_{\eta,\varphi}\varphi.$$

Multiplying this equality by $\tilde{\vartheta}^0$ and recalling that $\eta^0\tilde{\vartheta}^0 = \chi^0 \in \mathrm{IBr}(G)$ and $\varphi\tilde{\vartheta}^0 \in \mathrm{IBr}(G)$ for every $\varphi \in \mathrm{IBr}(G/N)$ (again by using [Nav98, Corollary 8.20]), we deduce that there is a unique $\varphi \in \mathrm{IBr}(G/N)$ such that $d_{\eta,\varphi} = 1$ and that $d_{\eta,\varphi'} = 0$ for all $\varphi' \neq \varphi$. Hence $\eta^0 = \varphi \in \mathrm{IBr}(G/N)$ and we are done. □

Lemma 2.3. *Let (G, N, ϑ) be an ordinary-modular character triple and assume there is an extension $\tilde{\vartheta} \in \mathrm{Irr}(G)$ of ϑ such that $\tilde{\vartheta}^0 \in \mathrm{IBr}(G)$. Let $\eta \in \mathrm{Irr}(G/N)$ and $\varphi \in \mathrm{IBr}(G/N)$. If $\mathrm{bl}(\eta) = \mathrm{bl}(\varphi)$, then $\mathrm{bl}(\eta\tilde{\vartheta}) = \mathrm{bl}(\varphi\tilde{\vartheta}^0)$.*

Proof. We need to prove that $\lambda_{\eta\tilde{\vartheta}}(\mathfrak{Cl}_G(x)^+) = \lambda_{\varphi\tilde{\vartheta}^0}(\mathfrak{Cl}_G(x)^+)$ for all $x \in G$. By applying [Spä13b, Lemma 2.5], we get

$$\begin{aligned}\lambda_{\eta\tilde{\vartheta}}(\mathfrak{Cl}_G(x)^+) &= \lambda_\eta(\mathfrak{Cl}_{G/N}(xN)^+)\lambda_{\tilde{\vartheta}_L}(\mathfrak{Cl}_L(x)^+)\\ &= \lambda_\varphi(\mathfrak{Cl}_{G/N}(xN)^+)\lambda_{\tilde{\vartheta}^0_L}(\mathfrak{Cl}_L(x)^+)\\ &= \lambda_{\varphi\tilde{\vartheta}^0}(\mathfrak{Cl}_G(x)^+),\end{aligned}$$

where $L/N = \boldsymbol{C}_{G/N}(xN)$, as desired. □

We conclude this section by showing that multiplication by a linear Brauer character preserves blocks of defect zero.

Lemma 2.4. *Let B be a block of defect zero of a finite group G and consider its unique Brauer character $\varphi \in \mathrm{IBr}(B)$. If $\lambda \in \mathrm{IBr}(G)$ is linear, then $\lambda\varphi$ belongs to a block of defect zero.*

Proof. It suffices to show that there exists $\psi \in \mathrm{Irr}(G)$ such that $\psi^0 = \lambda\varphi$. In fact, this would imply that $\psi(1)_p = \lambda(1)_p\varphi(1)_p = |G|_p$ and the result would follow from [Nav98, Theorem 3.18]. To prove our claim, let $\chi \in \mathrm{Irr}(B)$ so that $\varphi = \chi^0$. By [Nav98, Problem 2.13] we know that $\lambda\Phi^0_\varphi = \Phi^0_{\lambda\varphi}$ while by the definition of Φ_φ, and recalling that $\chi^0 = \varphi$, we have $\Phi^0_\varphi = \varphi$. From this we deduce that

$$\lambda\varphi = \Phi^0_{\lambda\varphi} = \sum_{\psi \in \mathrm{Irr}(G)} d_{\psi,\lambda\varphi}\psi^0 = \sum_{\psi \in \mathrm{Irr}(G)} d_{\psi,\lambda\varphi}\Bigg(\sum_{\xi \in \mathrm{IBr}(G)} d_{\psi,\xi}\xi\Bigg).$$

Since $\lambda\varphi$ is an irreducible Brauer character, and because decomposition numbers are nonnegative integers, the above equality forces $d_{\psi,\lambda\varphi}d_{\psi,\xi} \neq 0$ for a unique choice of $\psi \in \mathrm{Irr}(G)$ and $\xi \in \mathrm{IBr}(G)$. Then, we must have $\xi = \lambda\varphi$, $d_{\psi,\lambda\varphi} = 1$, and $d_{\psi,\nu} = 0$ for every $\nu \in \mathrm{IBr}(G)$ with $\nu \neq \lambda\varphi$. This shows that $\psi^0 = \lambda\varphi$ and the result follows. □

3. Central and block isomorphisms of modular character triples

In this section, we collect the relevant results on isomorphisms of modular character triples that will be used in the rest of this paper. We refer the reader to [Nav98, Section 8] and [Spä17a, Section 3] for an overview of this theory. In particular, we will make use of the notion of *central isomorphism* and *block isomorphism* of modular character triples that can be found in [Spä17a, Definition 3.1 and Definition 3.2] (see also [NS14b, Section 3]).

Recall that given a modular character triple (G, N, ϑ) there is a projective $\mathbb{F}$-representation of G such that the restriction $\mathcal{P}_N$ affords the Brauer character ϑ. Furthermore, we can always choose $\mathcal{P}$ such that its factor set $\alpha : G \times G \to \mathbb{F}^\times$ satisfies $\alpha(g, n) = 1 = \alpha(n, g)$ for every $g \in G$ and $n \in N$ (see [SV16, Section 3] and [Nav98, Section 8]). In this case, we say that $\mathcal{P}$ is a projective $\mathbb{F}$-representation associated with (G, N, ϑ). We will often refer to $\mathcal{P}$ simply as a projective representation, instead of a projective $\mathbb{F}$-representation, when it is clear from the context that it is associated to a modular character triple. The next result allows us to construct well behaved strong isomorphisms of modular character triples (by *strong* we mean that the maps σ_J that appear in the statement of Theorem 3.1 are compatible with the conjugation action of the groups involved, see [SV16, Section 3]).

Theorem 3.1. [SV16, Theorem 3.1]. *Let (G, N, ϑ) and (H, M, φ) be modular character triples and assume that $G = NH$, $M = H \cap N$ and that there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ associated with (G, N, ϑ) and (H, M, φ) respectively and whose factor sets α and α' coincide via the natural isomorphism $\tau : G/N \to H/M$. Then, for any $N \leq J \leq G$ there exists a bijection*

$$\sigma_J : \mathrm{IBr}(J \mid \vartheta) \to \mathrm{IBr}(J \cap H \mid \varphi), \quad \mathrm{tr}(\mathcal{Q} \otimes \mathcal{P}_J) \mapsto \mathrm{tr}(\mathcal{Q}_{J\cap H} \otimes \mathcal{P}'_{J\cap H}),$$

for any irreducible projective representation $\mathcal{Q}$ of J/N with factor set $\alpha^{-1}_{J\times J}$ and

$$(\sigma, \tau) : (G, N, \vartheta) \to (H, M, \varphi)$$

is a strong isomorphism of modular character triples as defined in [SV16, p. 281].

In the situation considered in the above theorem, we say that (σ, τ) is an isomorphism of character triples given by the projective representations $\mathcal{P}$ and $\mathcal{P}'$. By imposing additional requirements on the isomorphism of modular character triple (σ, τ) constructed above, we can define the notion of central and block isomorphisms of modular character triples. For this we follow [SV16, Section 3] (see also [NS14b] and [Spä17b]). In what follows, if φ is a (possibly reducible) Brauer character, then we denote by $\mathrm{IBr}(\varphi)$ the set of its irreducible constituents.

Definition 3.2 (central isomorphism of modular character triples). Let $(\sigma, \tau) : (G, N, \vartheta) \to (H, M, \varphi)$ be as in Theorem 3.1. Suppose furthermore that $\mathbf{C}_G(N) \leq H$ and that

$$\mathrm{IBr}\left(\psi_{\mathbf{C}_J(N)}\right) = \mathrm{IBr}\left(\sigma_J(\psi)_{\mathbf{C}_J(N)}\right)$$

for any $N \leq J \leq G$ and $\psi \in \mathrm{IBr}(J \mid \vartheta)$. Then, we say that (σ, τ) is a central isomorphism of modular character triples and write $(G, N, \vartheta) \succeq_c (H, M, \varphi)$.

As is the case for ordinary modular character triples (see [NS14b, Lemma 3.3]), the condition required in Definition 3.2 can be reformulated in terms of scalar functions. More precisely, if $\mathcal{P}$ is a projective representation associated with the modular character triple (G, N, ϑ) then by Schur's lemma $\mathcal{P}(x)$ is a scalar matrix $\zeta(x)I_{\vartheta(1)}$ for every $x \in \boldsymbol{C}_G(N)$. This yields a scalar function $\zeta : \boldsymbol{C}_G(N) \to \mathbb{F}^\times$.

Lemma 3.3. *Let $(\sigma, \tau) : (G, N, \vartheta) \to (H, M, \psi)$ be an isomorphism given by projective representations $\mathcal{P}$ and $\mathcal{P}'$ and assume $\boldsymbol{C}_G(N) \leq H$. Then the following are equivalent*:

(i) *For every $x \in \boldsymbol{C}_G(N)$ the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are associated with the same scalar.*

(ii) $\mathrm{IBr}(\psi_{\boldsymbol{C}_J(N)}) = \mathrm{IBr}(\sigma_J(\psi)_{\boldsymbol{C}_J(N)})$ *for every $N \leq J \leq G$ and $\psi \in \mathrm{IBr}(J \mid \vartheta)$.*

Proof. See the comment following [SV16, Definition 3.3]. □

We can additionally require some compatibility conditions for block induction. The following is a modular version of [NS14b, Definition 3.6]. Notice however that we are considering a slightly different setting that can be found, for instance, in [Spä18, Definition 4.2].

Definition 3.4 (block isomorphism of modular character triples). Let $(\sigma, \tau) : (G, N, \vartheta) \to (H, M, \varphi)$ be as in Definition 3.2. Assume that there exists a defect group D of $\mathrm{bl}(\varphi)$ such that $\boldsymbol{C}_G(D) \leq H$ and that

$$\mathrm{bl}(\psi) = \mathrm{bl}(\sigma_J(\psi))^J$$

for any $N \leq J \leq G$ and $\psi \in \mathrm{IBr}(J \mid \vartheta)$. Then we say that (σ, τ) is a block isomorphism of modular character triples and write $(G, N, \vartheta) \succeq_b (H, M, \varphi)$.

As done in Lemma 3.3, we can reformulate the condition on block induction required in the above definition in terms of certain scalars induced by the projective representations $\mathcal{P}$ and $\mathcal{P}'$. More precisely, notice that by [Nav98, Theorem 8.16] the representations $\mathcal{P}_{\langle N,x\rangle}$ and $\mathcal{P}'_{\langle N,x\rangle\cap H}$ afford extensions $\tilde{\vartheta}$ and $\tilde{\psi}$ of ϑ and ψ. It then follows that $\mathcal{P}(\mathfrak{Cl}_{\langle N,x\rangle}(x)^+) = \xi I_{\vartheta(1)}$ and $\mathcal{P}'((\mathfrak{Cl}_{\langle N,x\rangle}(x) \cap H)^+) = \xi' I_{\varphi(1)}$ for scalars ξ and ξ' in $\mathbb{F}$.

Lemma 3.5. *Let $(\sigma, \tau) : (G, N, \vartheta) \to (H, M, \psi)$ be the isomorphism of modular character triples given by Theorem 3.1 for a choice of projective representations $\mathcal{P}$ and $\mathcal{P}'$. If $(G, N, \vartheta) \succeq_c (H, M, \psi)$, then the following are equivalent*:

(i) $(G, N, \vartheta) \succeq_b (H, M, \psi)$.

(ii) *For every $x \in G$ the matrices $\mathcal{P}(\mathfrak{Cl}_{\langle N,x\rangle}(x)^+)$ and $\mathcal{P}'((\mathfrak{Cl}_{\langle N,x\rangle}(x) \cap H)^+)$ are associated with the same scalar.*

Proof. This is [Spä17a, Proposition 3.7 (b)]. □

Next, we collect some basic properties of block isomorphisms of modular character triples.

Lemma 3.6. *Let (G, N, ϑ) and (H, M, ψ) be modular character triples and assume that $(G, N, \vartheta) \succeq_b (H, M, \psi)$. Then*:

(i) $(J, N, \vartheta) \succeq_b (H \cap J, M, \psi)$, *for every $N \leq J \leq G$.*

(ii) $(G^a, N^a, \vartheta^a) \succeq_b (H^a, M^a, \psi^a)$, *for every* $a \in \mathrm{Aut}(G)$.

(iii) *if* $(H, M, \psi) \succeq_b (K, L, \rho)$ *for some* (K, L, ρ), *then* $(G, N, \vartheta) \succeq_b (K, L, \rho)$.

Proof. These properties follow directly from Definition 3.4. See, for instance, the argument used to prove [NS14b, Lemma 3.8]. □

Another important feature of (central and) block isomorphisms of (ordinary and) modular character triples is the fact that the bijections σ_J considered in Theorem 3.1 in turn give analogous isomorphisms.

Lemma 3.7. *Suppose that* $(\sigma, \tau) : (G, N, \vartheta) \to (H, M, \psi)$ *is a block isomorphism of modular character triples. If* $N \leq J \leq G$, *then the map*

$$\sigma_J : \mathrm{IBr}(J \mid \vartheta) \to \mathrm{IBr}(J \cap H \mid \psi)$$

is $\mathbf{N}_H(J)$*-equivariant and*

$$\big(\mathbf{N}_G(J)_\mu, J, \mu\big) \succeq_b \big(\mathbf{N}_H(J)_\mu, J \cap H, \sigma_J(\mu)\big)$$

for every $\mu \in \mathrm{IBr}(J \mid \vartheta)$.

Proof. This statement can be found in [FLZ22, Lemma 3.3] and is based on [NS14b, Proposition 3.9]. □

3.1. ***Construction of block isomorphisms of modular character triples.*** Several standard constructions that are used in representation theory are well behaved with respect to block isomorphisms of modular character triples. Here we collect the results needed in the subsequent sections. To start, we consider irreducible induction, which appears for instance when using the Clifford correspondence as well as the Fong–Reynolds correspondence.

Lemma 3.8 [FLZ22, Proposition 3.4]. *Let* $N \lhd G$, $H \leq G$, $G = NH$ *and* $M = N \cap H$. *Consider* $G_1 \leq G$, $H_1 = G_1 \cap H$, $N_1 = G_1 \cap N$, $M_1 = G_1 \cap M$ *such that* $G_1 N = G$ *and* $H_1 M = H$. *Assume furthermore that induction gives bijections*

$$\mathrm{IBr}(J \cap G_1 \mid \vartheta) \to \mathrm{IBr}(J \mid \vartheta^N)$$

and

$$\mathrm{IBr}(J \cap H_1 \mid \psi) \to \mathrm{IBr}(J \cap H \mid \psi^M)$$

for every $N \leq J \leq G$. *If* $(G_1, N_1, \vartheta) \succeq_b (H_1, M_1, \psi)$ *and* $\mathbf{C}_G(D) \leq H$ *for some defect group* D *of* $\mathrm{bl}(\psi^M)$, *then* $(G, N, \vartheta^N) \succeq_b (H, M, \psi^M)$.

Next, we show that $\succeq_b$ is compatible with direct products.

Lemma 3.9. *Suppose that* $(G_i, N_i, \vartheta_i) \succeq_b (H_i, M_i, \psi_i)$ *for* $i = 1, 2$. *Then*

$$(G_1 \times G_2, N_1 \times N_2, \vartheta_1 \times \vartheta_2) \succeq_b (H_1 \times H_2, N_1 \times N_2, \psi_1 \times \psi_2).$$

Proof. For $i = 1, 2$, let (σ_i, τ_i) be a block isomorphism associated with a choice of projective representations $\mathcal{P}_i$ and $\mathcal{P}'_i$ with factor sets α_i and α'_i. Set $\mathcal{P}(x, y) = \mathcal{P}_1(x) \otimes \mathcal{P}_2(y)$ and $\mathcal{P}'(x', y') = \mathcal{P}'_1(x') \otimes \mathcal{P}'_2(y')$ for every $x \in G_1$, $y \in G_2$, $x' \in H_1$ and $y' \in H_2$. Now $\mathcal{P}$ and $\mathcal{P}'$ are projective representations of $G_1 \times G_2$ and $H_1 \times H_2$ associated with $\vartheta_1 \times \vartheta_2$ and $\psi_1 \times \psi_2$ respectively. By the properties of the Kronecker product, if $x, g \in G_1$ and $y, t \in G_2$, we have

$$P((xg, yt)) = \big(\alpha_1(x, g)\mathcal{P}_1(x)\mathcal{P}_1(g)\big)\otimes\big(\alpha_2(y, t)(\mathcal{P}_2(y)\mathcal{P}_2(t))\big) = \alpha_1(x, g)\alpha_2(y, t)\mathcal{P}((x, y))\mathcal{P}((g, t))$$

and therefore the factor set α of $\mathcal{P}$ satisfies

$$\alpha((x, y), (g, t)) = \alpha_1(x, g)\alpha_2(y, t).$$

The same argument applies for the factor set α' of $\mathcal{P}'$ and hence we deduce that the factor sets α and α' coincide via the isomorphism

$$\tau : (G_1 \times G_2)/(N_1 \times N_2) \to (H_1 \times H_2)/(M_1 \times M_2).$$

We then obtain a strong isomorphism of modular character triples

$$(\sigma, \tau) : (G_1 \times G_2, N_1 \times N_2, \vartheta_1 \times \vartheta_2) \to (H_1 \times H_2, N_1 \times N_2, \psi_1 \times \psi_2)$$

according to Theorem 3.1. We show that (σ, τ) is a block isomorphism by using Lemma 3.3 and Lemma 3.5. First, notice that $\boldsymbol{C}_{G_1\times G_2}(N_1\times N_2) = \boldsymbol{C}_{G_1}(N_1)\times\boldsymbol{C}_{G_2}(N_2) \leq H_1\times H_2$ and let $x \in \boldsymbol{C}_{G_1}(N_1)$ and $y \in \boldsymbol{C}_{G_2}(N_2)$. Then $\mathcal{P}((x, y)) = \mathcal{P}_1(x) \otimes \mathcal{P}_2(y)$ is associated with the same scalar as $\mathcal{P}'((x, y)) = \mathcal{P}'_1(x) \otimes \mathcal{P}'_2(y)$. Consider now $x \in G_1$ and $y \in G_2$ and observe that $\langle N_1 \times N_2, (x, y)\rangle = \langle N_1, x\rangle \times \langle N_2, y\rangle$ and that $\mathfrak{Cl}_{\langle N_1\times N_2,(x,y)\rangle}((x, y)) = \mathfrak{Cl}_{\langle N_1,x\rangle}(x) \times \mathfrak{Cl}_{\langle N_2,y\rangle}(y)$. As a consequence

$$\mathfrak{Cl}_{\langle N_1\times N_2,(x,y)\rangle}((x, y))^+ = \sum_{\substack{u\in\mathfrak{Cl}_{\langle N_1,x\rangle}(x)\\ v\in\mathfrak{Cl}_{\langle N_2,y\rangle}(y)}} (u, v)$$

and hence

$$\mathcal{P}\left(\mathfrak{Cl}_{\langle N_1\times N_2,(x,y)\rangle}(x, y)^+\right) = \mathcal{P}_1\left(\mathfrak{Cl}_{\langle N_1,x\rangle}(x)^+\right) \otimes \mathcal{P}_2\left(\mathfrak{Cl}_{\langle N_2,y\rangle}(y)^+\right).$$

Similarly, we have

$$\mathcal{P}\big((\mathfrak{Cl}_{\langle N_1\times N_2,(x,y)\rangle}(x, y) \cap (H_1 \times H_2))^+\big) = \mathcal{P}_1\big((\mathfrak{Cl}_{\langle N_1,x\rangle}(x) \cap H_1)^+\big) \otimes \mathcal{P}_2\big((\mathfrak{Cl}_{\langle N_2,y\rangle}(y) \cap H_2)^+\big)$$

and we conclude that $\mathcal{P}(\mathfrak{Cl}_{\langle N_1\times N_2,(x,y)\rangle}(x, y)^+)$ and $\mathcal{P}'(\mathfrak{Cl}_{\langle N_1\times N_2,(x,y)\rangle}(x, y)\cap(H_1 \times H_2))^+)$ are associated with the same scalar as desired. □

Using Lemma 3.9 we can show that block isomorphisms of modular character triples are compatible with wreath products. First, recall that, if $N \triangleleft G$ and b is a block of N, then Dade's ramification group of b, denoted by $G[b]$, coincides with the subgroup of G generated by N and the elements $x \in G$ such that $\lambda_{\bar{b}}(\mathfrak{Cl}_{\langle N,x\rangle}(x)^+) \neq 0$ for some block $\bar{b}$ of $\langle N, x\rangle$ covering b (see [Spä17b, Section 2.4]).

Lemma 3.10. *Suppose that* $(G, N, \vartheta) \geq_b (H, M, \psi)$ *and let* n *be a positive integer. Write* $\widetilde{G} = G \wr S_n$, $\widetilde{N} = N^n$, $\tilde{\vartheta} = \vartheta^n$ *and similarly* $\widetilde{H} = H \wr S_n$, $\widetilde{M} = M^n$, $\tilde{\psi} = \psi^n$. *Then*

$$\left(\widetilde{G}, \widetilde{N}, \tilde{\vartheta}\right) \geq_b \left(\widetilde{H}, \widetilde{M}, \tilde{\psi}\right).$$

Proof. To start, we show that $(\widetilde{G}, \widetilde{N}, \tilde{\vartheta}) \geq_c (\widetilde{H}, \widetilde{M}, \tilde{\psi})$. Let $\mathcal{R} : S_n \to \mathrm{GL}_n(\mathbb{F})$ be the representation from the proof of [Val16, Theorem 4.24], and consider projective representations $\mathcal{P}$ and $\mathcal{P}'$ with factor sets α and α' giving the isomorphism $(G, N, \vartheta) \geq_b (H, M, \psi)$. Let $\widetilde{\mathcal{P}}$ be the projective representation of $\widetilde{G}$ given by

$$\widetilde{\mathcal{P}}((g_1, \dots, g_n)\sigma) = (\mathcal{P}(g_1) \otimes \cdots \otimes \mathcal{P}(g_n))\mathcal{R}(\sigma)$$

for $g_i \in G$ and define analogously the projective a representation $\widetilde{\mathcal{P}}'$ of $\widetilde{H}$. The factor sets $\tilde{\alpha}$ of $\widetilde{\mathcal{P}}$ and $\tilde{\alpha}'$ of $\widetilde{\mathcal{P}}'$ satisfy

$$\tilde{\alpha}\big((g_1, \dots, g_n)\sigma, (x_1, \dots, x_n)\tau\big) = \prod_{i=1}^{n} \alpha\left(g_i, x_{\sigma(i)}\right),$$

$$\tilde{\alpha}'\big((h_1, \dots, h_n)\sigma, (y_1, \dots, y_n)\tau\big) = \prod_{i=1}^{n} \alpha'\left(h_i, y_{\sigma(i)}\right),$$

for every $g_i, x_i \in G$ and $h_i, y_i \in H$. Then, noticing that $\mathbf{C}_{\widetilde{G}}(\widetilde{N}) = \mathbf{C}_G(N)^n \leq H^n$ and arguing as in the final part of the proof of [Nav18, Theorem 10.21], by using Theorem 3.1 and Lemma 3.3 we conclude that $(\widetilde{G}, \widetilde{N}, \tilde{\vartheta}) \geq_c (\widetilde{H}, \widetilde{M}, \tilde{\psi})$. By Lemma 3.5 it remains to check that $\widetilde{\mathcal{P}}(\mathfrak{Cl}_{\langle \widetilde{N}, \bar{x}\rangle}(\bar{x})^+)$ and $\widetilde{P}'(\mathfrak{Cl}_{\langle \widetilde{N}, \bar{x}\rangle}(\bar{x}) \cap \widetilde{H})^+)$ are associated with the same scalar for every $\bar{x} = (g_1, \dots, g_n)\sigma \in \widetilde{G}$. For this, arguing as in [Spä17b, Lemma 4.2(b)] and noticing that $\mathrm{bl}(\tilde{\psi})^{\widetilde{N}} = \mathrm{bl}(\tilde{\vartheta})$, we may assume that $\bar{x}$ belongs to Dade's ramification group $\widetilde{G}[\mathrm{bl}(\tilde{\vartheta})]$. In this case, [Spä17a, Proposition 2.5(a)] implies that $\bar{x} \in \widetilde{N}\mathbf{C}_{\widetilde{G}}(\widetilde{D}) \leq G^n$. Since $\widetilde{\mathcal{P}}_{G^n} = \mathcal{P} \otimes \cdots \otimes \mathcal{P}$, arguing as in the proof of Lemma 3.9 we can show that the matrices above are associated with the same scalar and the result follows. □

The next result, often referred to as the Butterfly theorem, shows that when considering block isomorphisms of (ordinary or) modular character triples we can replace the ambient group with any other group inducing the same automorphisms on the normal subgroups of the triples. This was formally stated for the first time in [Spä17b, Theorem 5.3] building on earlier similar ideas used in several other reduction theorems.

Lemma 3.11 (Butterfly theorem, [Spä17a, Theorem 3.5]). *Let* (G, N, ϑ) *and* (H, M, ψ) *be modular character triples with* $(G, N, \vartheta) \geq_b (H, M, \psi)$. *Suppose that* $N \triangleleft K$ *and that* $\epsilon_G(G) = \epsilon_K(K)$ *where* $\epsilon_G : G \to \mathrm{Aut}(N)$ *and* $\epsilon_K : K \to \mathrm{Aut}(N)$ *are the homomorphisms defined by conjugation. If* $L := \epsilon_K^{-1}(\epsilon_G(H))$, *then* $(K, N, \vartheta) \geq_b (L, M, \psi)$.

Let (G, N, ϑ) be a modular character triple. For any choice of projective representation $\mathcal{P}$ associated with (G, N, ϑ) we can construct a central extension of G containing an isomorphic copy of N and where the character corresponding to ϑ extends. This can often be used to reduce questions about (G, N, ϑ) to the case where ϑ extends to G. The next lemma shows that this construction is compatible with block isomorphisms of modular character triples. The analogous result of ordinary character triple can be found in [NS14b, Theorem 4.1].

Lemma 3.12. *Let (G, K, ϑ) be a modular character triple. Let $\mathcal{P}$ be a projective representation of G associated with ϑ. Then $\mathcal{P}$ defines a group $\widehat{G}$ together with a surjective homomorphism $\epsilon : \widehat{G} \to G$ with finite cyclic central kernel Z of order not divisible by p with the following properties.*

(i) *$\widehat{K} = K_0 \times Z$ where $\widehat{K} = \epsilon^{-1}(K)$, $K_0 \cong K$ via the restriction ϵ_{K_0} and $K_0 \lhd \widehat{G}$. Further, the action of $\widehat{G}$ on K_0 coincides with the action of G on K via ϵ.*

(ii) *The character $\vartheta_0 \in \mathrm{IBr}(K_0)$ associated to ϑ via the isomorphism ϵ_{K_0} extends to $\widehat{G}$.*

(iii) *If $K \leq J \leq G$ and $\hat{J} = \epsilon^{-1}(J)$ then $\epsilon(\boldsymbol{C}_{\widehat{G}}(\hat{J})) = \boldsymbol{C}_G(J)$.*

(iv) *Let (H, M, ϑ') be a modular character triple with $(G, K, \vartheta) \succeq_c (H, M, \vartheta')$, and denote by M_0 the subgroup of K_0 corresponding to $M \leq K$ under ϵ_{K_0} and by ϑ_0' the character corresponding to ϑ'. If $(\widehat{G}, K_0, \vartheta_0) \succeq_b (\widehat{H}, M_0, \vartheta_0')$ then $(G, K, \vartheta) \succeq_b (H, M, \vartheta')$.*

Proof. This follows arguing as in the proof of [NS14b, Theorem 4.1]. □

Before proceeding further, we consider some additional compatibility properties of the construction given above.

Remark 3.13. Consider the setting of Lemma 3.12 and let $K \leq J \leq G$. If $\sigma_J : \mathrm{IBr}(J \mid \vartheta) \to \mathrm{IBr}(J \cap H \mid \vartheta')$ and $\sigma_{\hat{J}} : \mathrm{IBr}(\hat{J} \mid \vartheta_0 \times 1_Z) \to \mathrm{IBr}(\hat{J} \cap \widehat{H} \mid \vartheta_0' \times 1_Z)$ are the character bijections induced by the isomorphisms of character triples considered in Lemma 3.12 (iv), then

$$\sigma_J(\chi) = \epsilon_{\widehat{J \cap H}}(\sigma_{\hat{J}}(\widehat{\chi}))$$

for every $\widehat{\chi} \in \mathrm{IBr}(\hat{J} \mid \vartheta_0 \times 1_Z)$ and where $\chi \in \mathrm{IBr}(J \mid \vartheta)$ corresponds to $\widehat{\chi}$ via the isomorphism $\hat{J}/Z \simeq J$ induced by $\epsilon_{\hat{J}}$.

Next, we consider the behavior of block isomorphisms of modular character triples with respect to inflation of Brauer characters.

Lemma 3.14. *Let G be a finite group and consider subgroups $H \leq G$ and $Z, N \trianglelefteq G$ such that $Z \leq M := N \cap H$. Set $\overline{G} := G/Z$, $\overline{H} := H/Z$, $\overline{N} := N/Z$, and $\overline{M} := M/Z$ and suppose that $(\overline{G}, \overline{N}, \bar{\vartheta}) \succeq_b (\overline{H}, \overline{M}, \bar{\varphi})$ for some irreducible Brauer characters $\bar{\vartheta} \in \mathrm{IBr}(\overline{N})$ and $\bar{\varphi} \in \mathrm{IBr}(\overline{M})$. If $\vartheta \in \mathrm{IBr}(N)$ and $\varphi \in \mathrm{IBr}(M)$ are the inflations of $\bar{\vartheta}$ and $\bar{\varphi}$, then $(G, N, \vartheta) \succeq_b (H, M, \varphi)$.*

Proof. To verify the group theoretical conditions, observe that by hypothesis there is a defect group Q of $\mathrm{bl}(\bar{\varphi})$ such that $\boldsymbol{C}_{\overline{G}}(Q) \leq \overline{H}$ and that according to [Nav98, Theorem 9.9] we can find a defect group D of $\mathrm{bl}(\varphi)$ satisfying $Q \leq DZ/Z$ and hence $\boldsymbol{C}_G(D)Z/Z \leq \boldsymbol{C}_{\overline{G}}(Q) \leq \overline{H}$. It follows that $\boldsymbol{C}_G(D) \leq H$ as required. We can now conclude arguing as in the proof of [NS14b, Lemma 3.12]. □

The converse of Lemma 3.14 does not hold in general. However, we can still prove an analogous statement under additional structural assumptions.

Lemma 3.15. *Let (G, N, ϑ) and (H, M, φ) be modular character triples and assume that $(G, N, \vartheta) \succeq_b (H, M, \varphi)$. Consider $Z \le \ker(\vartheta) \cap \ker(\varphi)$ and set $\bar{J} := JZ/Z$ for every $J \le G$. Denote by $\bar{\vartheta} \in \mathrm{IBr}(\bar{N})$ and $\bar{\varphi} \in \mathrm{IBr}(\bar{M})$ the characters corresponding to ϑ and φ respectively via inflation. If $\boldsymbol{C}_G(N)/Z = \boldsymbol{C}_{G/Z}(N/Z)$ and p does not divide the order of Z, then $(\bar{G}, \bar{N}, \bar{\vartheta}) \succeq_b (\bar{H}, \bar{M}, \bar{\varphi})$.*

Proof. By [Spä18, Lemma 2.17] we know that $(\bar{G}, \bar{N}, \bar{\vartheta}) \succeq_c (\bar{H}, \bar{M}, \bar{\varphi})$. Next, observe that under the above assumptions we have $\boldsymbol{C}_G(D)/Z = \boldsymbol{C}_{G/Z}(D/Z)$ for some defect group D of $\mathrm{bl}(\varphi)$. Then, we conclude $(\bar{G}, \bar{N}, \bar{\vartheta}) \succeq_b (\bar{H}, \bar{M}, \bar{\varphi})$ by [NS14b, Proposition 2.4(b)]. □

We conclude this section by considering the compatibility of block isomorphisms of modular character triples with respect to multiplication of characters. This situation appears, for instance, when applying Gallagher's theorem and was described in [NS14b, Theorem 4.6] in the ordinary case.

Lemma 3.16. *Let $K \lhd G$, $H \le G$, $M = K \cap H$ and $Z \le M$ such that $Z \lhd G$. Consider $\chi \in \mathrm{IBr}(G)$ and suppose that χ_Z is irreducible and there exists $\beta \in \mathrm{Irr}(Z)$ such that $\beta^0 = \chi_Z$, $\boldsymbol{O}_p(Z) \le \ker(\beta)$ and where the inflation of β to $Z/\boldsymbol{O}_p(Z)$ has defect zero. Set $\bar{J} := JZ/Z$ for every $J \le G$ and let $(\bar{G}, \bar{K}, \bar{\rho})$ and $(\bar{H}, \bar{M}, \bar{\rho}')$ be modular character triples with $(\bar{G}, \bar{K}, \bar{\rho}) \succeq_b (\bar{H}, \bar{M}, \bar{\rho}')$. Denote by ρ and ρ' the inflations of $\bar{\rho}$ and $\bar{\rho}'$ to K and M respectively and define $\tau := \rho\chi_K$ and $\tau' := \rho'\chi_M$. If $\boldsymbol{C}_G(D) \le H$ for some defect group D of $\mathrm{bl}(\tau')$, then $(G, K, \tau) \succeq_b (H, M, \tau')$.*

Proof. Suppose that $(\bar{G}, \bar{N}, \bar{\rho}) \succeq_b (\bar{H}, \bar{M}, \bar{\rho}')$ is given by a choice of projective representations $\bar{P}$ and $\bar{P}'$ and consider the inflations $\mathcal{P}$ and $\mathcal{P}'$ of $\bar{P}$ and $\bar{P}'$ respectively. Let $\mathcal{Q}$ be a representation of G affording χ and define $\mathcal{R} = \mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{R}' = \mathcal{Q}_H \otimes \mathcal{P}'$. Then it follows that $(G, K, \tau) \succeq_c (H, M, \tau')$ via $(\mathcal{R}, \mathcal{R}')$. Consider now $N \le J \le G$ and let $\sigma_J : \mathrm{IBr}(J \mid \tau) \to \mathrm{IBr}(J \cap H \mid \tau')$ be the map given by Theorem 3.1 with respect to the projective representations $\mathcal{R}$ and $\mathcal{R}'$. Similarly, let $\bar{\sigma}_{\bar{J}} : \mathrm{IBr}(\bar{J} \mid \bar{\rho}) \to \mathrm{IBr}(\overline{J \cap H} \mid \bar{\rho}')$ be the map given by Theorem 3.1 with respect to the projective representations $\bar{P}$ and $\bar{P}'$. If $\psi \in \mathrm{IBr}(J \mid \tau)$ we may write $\psi = \chi_J \vartheta$, where $\bar{\vartheta} \in \mathrm{IBr}(\bar{J} \mid \bar{\rho})$, and then

$$\sigma_J(\chi_J \vartheta) = \chi_{J \cap H} \vartheta'$$

where $\vartheta' \in \mathrm{IBr}(J \cap H)$ corresponds to $\bar{\vartheta}' := \bar{\sigma}_{\bar{J}}(\bar{\vartheta})$ via inflation. Consider $\bar{\gamma} \in \mathrm{Irr}(\mathrm{bl}(\bar{\vartheta}))$ and $\bar{\gamma}' \in \mathrm{Irr}(\mathrm{bl}(\bar{\vartheta}'))$ with inflations γ and γ' to J and $J \cap H$ respectively. By Lemma 2.1, we can find some $\xi \in \mathrm{Irr}(G)$ such that $\xi^0 = \chi$ so that $\xi_Z = \beta$. Then, according to Lemma 2.3, we get $\mathrm{bl}(\gamma\xi_J) = \mathrm{bl}(\vartheta\chi_J)$ and similarly $\mathrm{bl}(\gamma'\xi_{J\cap H}) = \mathrm{bl}(\vartheta'\chi_{J\cap H})$. Therefore to show that $\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi)$ it suffices to show that $\mathrm{bl}(\gamma\xi_J) = \mathrm{bl}(\gamma'\xi_{J\cap H})^J$, or equivalently, that

$$\lambda_{\gamma\xi_J}(\mathfrak{Cl}_J(x)^+) = \lambda^J_{\gamma'\xi_{J\cap H}}(\mathfrak{Cl}_J(x)^+)$$

for all $x \in J$. Write $L/Z := \boldsymbol{C}_{\bar{J}}(xZ)$ and notice that

$$\lambda_{\gamma\xi_J}(\mathfrak{Cl}_J(x)^+) = \lambda_{\xi_L}(\mathfrak{Cl}_L(x)^+)\lambda_{\bar{\gamma}}(\mathfrak{Cl}_{\bar{J}}(xZ)^+)$$

by [Spä13b, Lemma 2.5]. Since $\mathrm{bl}(\bar{\gamma}')^{\bar{J}} = \mathrm{bl}(\bar{\gamma})$, we deduce that

$$\lambda_{\bar{\gamma}}(\mathfrak{Cl}_{\bar{J}}(xZ)^+) = \lambda^{\bar{J}}_{\bar{\gamma}'}(\mathfrak{Cl}_{\bar{J}}(xZ)^+).$$

Finally, using [NT89, Lemma 5.3.1(i)] we conclude that

$$\begin{aligned}\lambda_{\gamma\xi_J}(\mathfrak{Cl}_J(x)^+) &= \lambda_{\xi_L}(\mathfrak{Cl}_L(x)^+)\lambda^{\bar J}_{\bar\gamma'}(\mathfrak{Cl}_{\bar J}(xZ)^+)\\ &= \left(\frac{|\mathfrak{Cl}_L(x)|\xi(x)}{\xi(1)}\right)^*\left(\frac{|\mathfrak{Cl}_{\bar J}(xZ)|(\bar\gamma')^{\bar J}(xZ)}{(\bar\gamma')^{\bar J}(1)}\right)^*\\ &= \left(\frac{|\mathfrak{Cl}_J(x)|\xi(x)(\bar\gamma')^{\bar J}(xZ)}{\xi(1)(\bar\gamma')^{\bar J}(1)}\right)^*\\ &= \left(\frac{|\mathfrak{Cl}_J(x)|(\gamma'\xi_{J\cap H})^J(x)}{(\gamma'\xi_{J\cap H})^J(1)}\right)^* = \lambda^J_{\gamma'\xi_{J\cap H}}(\mathfrak{Cl}_J(x)^+),\end{aligned}$$

as desired. □

4. inductive blockwise Alperin weight condition and consequences

The *inductive Alperin weight condition* (iAWC) first appeared in the reduction theorem of Navarro and Tiep for the Alperin weight conjecture [NT11]. More precisely, in [NT11, Theorem A] it was shown that the Alperin weight conjecture holds for a finite group G provided that every finite nonabelian simple group involved in G satisfies the iAWC. Later, in [Spä13b] Späth introduced the *inductive blockwise Alperin weight condition* (iBAWC) and proved a similar reduction theorem for the blockwise version of the Alperin weight conjecture. Both the iAWC and the iBAWC were later reformulated by Cabanes [Cab13] and Späth [Spä17a] in terms of isomorphisms of modular character triples. Observe that, while originally tailored to finite quasisimple groups, these conditions can be formulated more generally for arbitrary finite groups. Before introducing a precise statement, we introduce some further notation.

Let G be a finite group and denote by $\mathrm{dz}^\circ(G)$ the set of irreducible Brauer characters ψ of G whose corresponding character $\bar\psi$ of the quotient $G/\boldsymbol{O}_p(G)$ belongs to a p-block of defect zero. Here, we are using the fact that $\boldsymbol{O}_p(G)$ is contained in the kernel of any irreducible Brauer character of G thanks to [Nav98, Lemma 2.32]. Moreover, notice that in this case $\bar\psi = \bar\varphi^0$ for some uniquely defined $\bar\varphi \in \mathrm{dz}(G/\boldsymbol{O}_p(G))$ according to [Nav98, Theorem 3.18]. Now, we define a *p-weight* of G to be a pair (Q,ψ) where Q is a radical p-subgroup of G, that is $Q = \boldsymbol{O}_p(\boldsymbol{N}_G(Q))$, and $\psi \in \mathrm{dz}^\circ(\boldsymbol{N}_G(Q))$. Let $\mathrm{Alp}(G)$ be the set of p-weights of G. We also denote by $\mathrm{Rad}(G)$ the set of radical p-subgroups of G and write $\mathrm{Rad}^\circ(G)$ for the subset consisting of those radical p-subgroups Q of G such that $(Q,\psi)\in\mathrm{Alp}(G)$ for some $\psi \in \mathrm{dz}^\circ(\boldsymbol{N}_G(Q))$. Notice that the group G acts by conjugation on $\mathrm{Alp}(G)$ and denote by $\mathrm{Alp}(G)/G$ the corresponding set of G-orbits and by $\overline{(Q,\psi)}$ the G-orbit of a p-weight (Q,ψ). We can now state the inductive blockwise Alperin weight condition for arbitrary finite groups.

Conjecture 4.1 (inductive blockwise Alperin weight condition). *Let $G \trianglelefteq A$ be finite groups and consider a prime number p. Then there exists an A-equivariant bijection*

$$\Omega : \mathrm{IBr}(G) \to \mathrm{Alp}(G)/G$$

such that

$$(A_\vartheta, G, \vartheta) \succeq_b \left(\mathbf{N}_A(Q)_\psi, \mathbf{N}_G(Q), \psi\right)$$

for every $\vartheta \in \mathrm{IBr}(G)$ *and* $(Q, \psi) \in \Omega(\vartheta)$.

Throughout the rest of this paper we say that Conjecture 4.1 holds for a finite group G at the prime p if it holds with respect to the prime p and for every choice of $G \trianglelefteq A$. We will often avoid mentioning the choice of the prime p when this is clear from the context. We now collect some important properties and consequence of Conjecture 4.1. First, using Lemma 3.9 and Lemma 3.10 we show that Conjecture 4.1 extends to direct and wreath products.

Proposition 4.2. *Let $G \trianglelefteq A$ be finite groups, n a positive integer, and set $\widetilde{G} := G^n$ and $\tilde{A} := A \wr S_n$. If Conjecture 4.1 holds for $G \trianglelefteq A$, then it holds for $\widetilde{G} \trianglelefteq \tilde{A}$.*

Proof. By hypothesis we have an A-equivariant bijection $\Omega : \mathrm{IBr}(G) \to \mathrm{Alp}(G)/G$ such that for every $\chi \in \mathrm{IBr}(G)$ and $(Q, \psi) \in \Omega(\chi)$ we have

$$(A_\chi, G, \chi) \succeq_b \left(\mathbf{N}_A(Q)_\chi, \mathbf{N}_G(Q), \psi\right).$$

By [Nav98, Theorem 8.21] we know that $\mathrm{IBr}(\widetilde{G})$ consists of Brauer character of the form $\chi_1 \times \cdots \times \chi_n$ with $\chi_i \in \mathrm{IBr}(G)$. Similarly, by [NT11, Lemma 2.3(b)] the radical p-subgroups of $\widetilde{G}$ can be written as $Q_1 \times \cdots \times Q_n$ for some $Q_i \in \mathrm{Rad}(G)$ and hence each $\tilde{\psi} \in \mathrm{dz}^\circ(\mathbf{N}_{\widetilde{G}}(\widetilde{Q}))$ can be written as a product $\tilde{\psi} = \psi_1 \times \cdots \times \psi_n$ with $\psi_i \in \mathrm{dz}^\circ(\mathbf{N}_G(Q_i))$. We can then define a bijection $\widetilde{\Omega} : \mathrm{IBr}(\widetilde{G}) \to \mathrm{Alp}(\widetilde{G})/\widetilde{G}$ by setting

$$\widetilde{\Omega}\left(\chi_1 \times \cdots \times \chi_n\right) := \overline{(Q_1 \times \cdots \times Q_n, \psi_1 \times \cdots \times \psi_n)}$$

for every $\chi_1, \ldots, \chi_n \in \mathrm{IBr}(G)$ and $(Q_1, \psi_1), \ldots, (Q_n, \psi_n) \in \mathrm{Alp}(G)$ such that $(Q_i, \psi_i) \in \Omega(\chi_i)$. It follows from this definition, and using the fact that Ω is A-invariant, we also deduce that $\widetilde{\Omega}$ is an $\tilde{A}$-equivariant bijection. To conclude, we fix $\tilde{\chi} \in \mathrm{IBr}(\widetilde{G})$ and $(\widetilde{Q}, \tilde{\psi}) \in \widetilde{\Omega}(\tilde{\chi})$ and show that the corresponding modular character triples are block isomorphic. By Lemma 3.6 it is no loss of generality to assume that $\widetilde{\chi} = \chi_1 \times \cdots \times \chi_n$, $\widetilde{Q} = Q_1 \times \cdots \times Q_n$, and $\tilde{\psi} = \psi_1 \times \cdots \times \psi_n$ with $(Q_i, \psi_i) \in \Omega(\chi_i)$ and where χ_i and χ_j are either equal or not A-conjugate and $(Q_i, \psi_i) = (Q_j, \psi_j)$ whenever $\chi_i = \chi_j$. In this case the stabilizer $\tilde{A}_{\tilde{\chi}}$ is a direct product of groups of the form $A_{\chi_i} \wr S_{m_i}$ where m_i is the number of factors equal to χ_i appearing in $\tilde{\chi}$. Similarly, $\mathbf{N}_{\tilde{A}}(\widetilde{Q})_{\tilde{\psi}}$ is the direct product of groups of the form $\mathbf{N}_A(Q_i)_{\psi_i} \wr S_{m_i}$ and we then obtain

$$(\tilde{A}_{\tilde{\chi}}, \widetilde{G}, \tilde{\chi}) \succeq_b \left(\mathbf{N}_{\tilde{A}}(\widetilde{Q})_{\tilde{\psi}}, \mathbf{N}_{\widetilde{G}}(\widetilde{Q}), \tilde{\psi}\right)$$

by applying Lemmas 3.9 and 3.10. This completes the proof. □

The following corollary allows us to control how simple groups embed in a larger group and is an important ingredient in the proof of Theorem C. The following argument is somewhat standard and already appeared in the reduction theorems for other local-global conjectures (see, for instance, [Nav18, Theorem 10.25]).

Corollary 4.3. *Let $K \trianglelefteq A$ be finite groups with K perfect and assume that $p \nmid |\mathbf{Z}(K)|$ and that $K/\mathbf{Z}(K)$ is a direct product of isomorphic, say to S, nonabelian simple groups of order divisible by p. If Conjecture 4.1 holds for $X \trianglelefteq X \rtimes \mathrm{Aut}(X)$ where X is the universal p'-covering group of S, then Conjecture 4.1 holds for $K \trianglelefteq A$.*

Proof. Assume that $K/\mathbf{Z}(K)$ is isomorphic to r copies of S and write $H = X^r$, so that H is a perfect central extension of K, and $\tilde{A} = \mathrm{Aut}(H)$. By Proposition 4.2 we know that Conjecture 4.1 holds for $H \triangleleft H \rtimes \tilde{A}$ and so there exists an $\tilde{A}$-equivariant bijection

$$\widetilde{\Omega} : \mathrm{IBr}(H) \to \mathrm{Alp}(H)/H$$

such that

$$\left(H \rtimes \tilde{A}_\chi, H, \chi\right) \succeq_b \left(\mathbf{N}_H(D) \rtimes \mathbf{N}_{\tilde{A}}(D)_\varphi, \mathbf{N}_H(D), \varphi\right)$$

for all $\chi \in \mathrm{IBr}(H)$ and $(D, \varphi) \in \widetilde{\Omega}(\chi)$. Now let $\pi : H \to K$ be the canonical epimorphism and set $Z := \ker(\pi) \leq \mathbf{Z}(H)$ and $\bar{J} := JZ/Z$ for every $J \leq H$. By the definition of central isomorphism, for all $\chi \in \mathrm{IBr}(H)$ and $(D, \varphi) \in \widetilde{\Omega}(\chi)$, we have $\mathrm{IBr}(\chi_Z) = \mathrm{IBr}(\varphi_Z)$. In particular, $Z \leq \ker(\chi)$ if and only if $Z \leq \ker(\varphi)$ and so, if $\tilde{A}_Z$ denotes the stabilizer of Z under the action of $\tilde{A}$, then it follows that the $\widetilde{\Omega}$ induces an $\tilde{A}_Z$-equivariant bijection

$$\widetilde{\Omega}_Z : \mathrm{IBr}(\overline{H}) \to \mathrm{Alp}(\overline{H})/\overline{H}.$$

Moreover, by applying Lemma 3.15 together with [Nav18, Theorem 10.24(c)], we deduce that

$$\left(\overline{H} \rtimes \tilde{A}_Z, \overline{H}, \bar{\chi}\right) \succeq_b \left(\mathbf{N}_{\overline{H}}(\overline{D}) \rtimes \mathbf{N}_{\tilde{A}_Z}(\overline{D}), \mathbf{N}_{\overline{H}}(\overline{D}), \bar{\varphi}\right)$$

for all $\bar{\chi} \in \mathrm{IBr}(\overline{H})$ and $(\overline{D}, \bar{\varphi}) \in \widetilde{\Omega}_Z(\bar{\chi})$. Since $\overline{H} \simeq K$ and $\overline{H} \rtimes \tilde{A}_Z \simeq K \rtimes \mathrm{Aut}(K)$ (see, for instance, the proof of [Nav18, Theorem 10.25]) this proves that Conjecture 4.1 holds for $K \triangleleft K \rtimes \mathrm{Aut}(K)$. Therefore there exists an $\mathrm{Aut}(K)$-equivariant bijection

$$\Omega : \mathrm{IBr}(K) \to \mathrm{Alp}(K)/K$$

such that

$$(K \rtimes \mathrm{Aut}(K), K, \psi) \succeq_b \left(\mathbf{N}_K(Q) \rtimes \mathbf{N}_{\mathrm{Aut}(K)}(Q)_\vartheta, \mathbf{N}_K(Q), \vartheta\right)$$

for every $\psi \in \mathrm{IBr}(K)$, $(Q, \vartheta) \in \Omega(\psi)$. We can now conclude by applying Späth's Butterfly theorem. More precisely, let $\epsilon : A \to \mathrm{Aut}(K)$ and $\hat{\epsilon} : K \rtimes \mathrm{Aut}(K) \to \mathrm{Aut}(K)$ be the homomorphisms induced by conjugation on K via elements of A and $K \rtimes \mathrm{Aut}(K)$ respectively. Set $Y := \epsilon(A) \leq \mathrm{Aut}(K)$ and notice that Ω is Y-equivariant since it is $\mathrm{Aut}(K)$-equivariant. Now, let $\psi \in \mathrm{IBr}(K)$, $(Q, \vartheta) \in \Omega(\psi)$ and write $U = \hat{\epsilon}^{-1}(Y_\psi)$. Observe that $Y_\psi = \epsilon(A_\psi)$ and that $\hat{\epsilon}(U) = Y_\psi = \epsilon(A_\psi)$. Since $U \leq K \rtimes \mathrm{Aut}(K)$ we get

$$(U, K, \psi) \succeq_b \left(\mathbf{N}_U(Q), \mathbf{N}_K(Q), \vartheta\right)$$

by Lemma 3.6. Then, if we prove that $\mathbf{N}_A(Q)_\psi = \epsilon^{-1}(\hat{\epsilon}(\mathbf{N}_U(Q)))$, applying Lemma 3.11 we finally obtain

$$(A_\chi, K, \psi) \succeq_b \left(\mathbf{N}_A(Q)_\vartheta, \mathbf{N}_K(Q), \vartheta\right).$$

To prove the claimed equality, let $x \in N_A(Q)_\psi$ so that $\epsilon(x) \in Y_\psi$ and hence $\epsilon(x) = \epsilon(g)$ for some $g \in U$. Since x and g induce the same action on K, it follows that g normalizes Q and so $g \in N_U(Q)$. Conversely, if $g \in N_U(Q)$ then $\hat{\epsilon}(g) \in Y_\psi$ and there is some $x \in A_\psi$ with $\epsilon(x) = \hat{\epsilon}(g)$. Then x must normalize Q and the claim follows. This finally completes the proof. □

Following the proofs of [Ros22a, Proposition 2.10] and [Ros23c, Proposition 2.4], we now show how to lift the bijections given by Corollary 4.3 from the group K to any intermediate subgroup $K \leq J \leq A$. We first introduced one more definition.

Definition 4.4. For any finite group G with a normal subgroup $K \trianglelefteq G$, we define the set $\mathrm{Alp}(G \mid K)$ consisting of pairs (Q, η) where $Q \in \mathrm{Rad}(K)$ and $\eta \in \mathrm{IBr}(N_G(Q))$ lies above some Brauer character in the set $\mathrm{dz}^\circ(N_K(Q))$. Observe that the group G acts by conjugation on the $\mathrm{Alp}(G \mid K)$ and denote by $\mathrm{Alp}(G \mid K)/G$ the corresponding set of G-orbits.

We can now prove the main result of this section.

Theorem 4.5. *Suppose that Conjecture 4.1 holds with respect to the finite groups $K \trianglelefteq A$. If $K \leq J \leq A$, then there exists an $N_A(J)$-equivariant bijection*

$$\Omega_K^J : \mathrm{IBr}(J) \to \mathrm{Alp}(J \mid K)/J$$

such that

$$\left(N_A(J)_\chi, J, \chi\right) \succeq_b \left(N_A(J, Q)_\eta, N_J(Q), \eta\right) \tag{1}$$

for every $\chi \in \mathrm{IBr}(J)$ and $(Q, \eta) \in \Omega_K^J(\chi)$.

Proof. To start, replacing A with $N_A(J)$, observe that it is no loss of generality to assume that J is normal in A. Let Ω_K be the A-equivariant bijection given by Conjecture 4.1 and, for every $\vartheta \in \mathrm{IBr}(K)$ and $(Q, \psi) \in \Omega_K(\vartheta)$, fix a pair of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}^{(Q,\psi)})$ inducing the block isomorphism of modular character triples

$$(A_\vartheta, K, \vartheta) \succeq_b \left(N_A(Q)_\psi, N_K(Q), \psi\right). \tag{2}$$

Let $\mathcal{S}$ be an A-transversal in $\mathrm{IBr}(K)$ and denote by § the set consisting of the K-orbits $\Omega_K(\vartheta)$ for $\vartheta \in \mathcal{S}$. The equivariance properties of the bijection Ω_K imply that § is an A-transversal in $\mathrm{Alp}(K)/K$. Observe that each irreducible Brauer character χ of J lies above an irreducible Brauer character ϑ' of K that is A-conjugate to a unique $\vartheta \in \mathcal{S}$. In particular, there exists an A-transversal $\mathcal{T}$ in $\mathrm{IBr}(J)$ such that each element $\chi \in \mathcal{T}$ lies above some element $\vartheta \in \mathcal{S}$. Furthermore, if χ lies above another $\vartheta' \in \mathcal{S}$ then there exists an element $x \in J$ such that $\vartheta' = \vartheta^x$. Since $J \leq A$, the choice of $\mathcal{S}$ yields $\vartheta = \vartheta'$. Therefore, every element $\chi \in \mathcal{T}$ lies over a unique $\vartheta \in \mathcal{S}$. Consider now the Clifford correspondent $\varphi \in \mathrm{IBr}(J_\vartheta \mid \vartheta)$ of χ (see [Nav98, Theorem 8.9]) and remember that the choice of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}^{(Q,\psi)})$ associated with (2) induces an $N_A(Q)_\vartheta$-equivariant bijection

$$\sigma_{J_\vartheta} : \mathrm{IBr}(J_\vartheta \mid \vartheta) \to \mathrm{IBr}(N_J(Q)_\psi \mid \psi)$$

where we are using the fact that $N_A(Q)_\vartheta = N_A(Q)_\psi$. Again using [Nav98, Theorem 8.9] it follows that $\sigma_{J_\vartheta}(\varphi)^{N_J(Q)}$ is an irreducible Brauer character of $N_J(Q)$ for each $\varphi \in \mathrm{IBr}(J_\vartheta \mid \vartheta)$. Then, the set $\mathbb{T}$ consisting of J-orbits of pairs $(Q, \sigma_{J_\vartheta}(\varphi)^{N_J(Q)})$ is an A-transversal in $\mathrm{Alp}(J \mid K)/J$ and there exists a bijection

$$\Phi : \mathcal{T} \to \mathbb{T}$$

given by sending χ to the J-orbit of $(Q, \sigma_{J_\vartheta}(\varphi)^{N_J(Q)})$ where φ is the Clifford correspondent of χ over ϑ as above. We can finally define an A-equivariant bijection by setting

$$\Omega_K^J(\chi^x) := \Phi(\chi)^x$$

for every $\chi \in \mathcal{T}$ and $x \in A$. It remains to show that the isomorphism (2) implies (1). For this purpose, let $\chi \in \mathcal{T}$ and $(Q, \eta) \in \Phi(\chi)$ so that $\eta = \sigma_{J_\vartheta}(\varphi)^{N_J(Q)}$ where ϑ is the unique character of $\mathcal{S}$ lying below χ and $\varphi \in \mathrm{IBr}(J_\vartheta)$ is the Clifford correspondent of χ over ϑ. By Lemma 3.6 (ii) it is enough to show that the condition on modular character triples (1) is satisfied for this specific choice of χ and (Q, η). Observe that because σ_{J_ϑ} is $N_A(Q)_\vartheta$-equivariant and $N_A(Q)_\vartheta = N_A(Q)_\psi$, the stabilizer $N_A(Q)_{\vartheta,\varphi}$ coincides with $N_A(Q)_{\psi,\sigma_{J_\vartheta}(\varphi)}$. Then, by applying Lemma 3.7 to the block isomorphism (2), we get

$$(A_{\vartheta,\varphi}, J_\vartheta, \varphi) \succeq_b \big(N_A(Q)_{\psi,\sigma_{J_\vartheta}(\varphi)}, N_J(Q)_\psi, \sigma_{J_\vartheta}(\varphi)\big),$$

from which we deduce

$$(A_\chi, J, \chi) \succeq_b \big(N_A(Q)_\eta, N_J(Q), \eta\big)$$

according to Lemma 3.8. Observe that the latter result can be applied because $A_{\vartheta,\varphi} = A_{\vartheta,\chi}$ by the Clifford correspondence while $A_\chi = JA_{\vartheta,\chi}$ and $A_\chi = JN_A(Q)_\eta$ by the Frattini argument applied together with Clifford's theorem and the equivariance properties of Ω_K^J respectively. □

5. The Dade–Glauberman–Nagao correspondence and modular character triples

The aim of this section is to obtain a bijection for Brauer characters compatible with the Dade–Glauberman–Nagao correspondence and inducing block isomorphisms of modular character triples. Our Theorem 5.6 below extends [NT11, Theorem 4.2], [Spä13b, Theorem 3.8], [FLZ23b, Proposition 3.11], and provides a modular version of [NS14b, Theorem 5.13].

5.1. *Relative defect zero Brauer characters.* Let $N \trianglelefteq G$ be finite groups. For every irreducible character $\chi \in \mathrm{Irr}(G)$ and $\vartheta \in \mathrm{Irr}(N)$ with χ lying above ϑ, recall that $\chi(1)/\vartheta(1)$ divides the index $|G : N|$ according to [Nav18, Theorem 5.12]. Then, we define the N-*relative defect* of χ to be the nonnegative integer $d_N(\chi)$ such that

$$p^{d_N(\chi)} = \frac{|G : N|_p}{\chi(1)_p/\vartheta(1)_p}.$$

Observe that $d_N(\chi)$ does not depend on the choice of $\vartheta \in \mathrm{Irr}(N)$ lying below χ. For a given $\vartheta \in \mathrm{Irr}(N)$, we denote by $\mathrm{rdz}(G \mid \vartheta)$ the set of irreducible characters of G with N-relative defect zero and lying above ϑ.

Our first aim is to define a notion of relative defect zero Brauer character. Unfortunately, for a Brauer character $\chi \in \mathrm{IBr}(G)$ it is not true in general that $\chi(1)_p$ divides $|G|_p$ (see the example preceding [Nav98, Theorem 3.18]) and therefore the obvious definition in terms of character degrees will not work in this context. To circumvent this problem, we show that (under suitable assumptions) relative defect zero characters remain irreducible under reduction modulo p. More precisely, we prove the following result.

Lemma 5.1. *Let $K \leq M$ be normal subgroups of G with M/K a p-group and consider a G-invariant character $\vartheta \in \mathrm{dz}(K)$. Let $\widehat{\vartheta} \in \mathrm{Irr}(M)$ be a G-invariant extension of ϑ* (which exists according to [NT11, Theorem 2.4]).

(i) *If $\chi \in \mathrm{rdz}(G \mid \widehat{\vartheta})$, then $\chi^0 \in \mathrm{IBr}(G)$.*

(ii) *The map $\mathrm{rdz}(G \mid \widehat{\vartheta}) \to \mathrm{IBr}(G)$ given by sending χ to χ^0 is injective.*

(iii) *The image of $\mathrm{rdz}(G \mid \widehat{\vartheta})$ in $\mathrm{IBr}(G)$ under the above map does not depend on the choice of the extension $\widehat{\vartheta}$.*

Proof. By [Nav98, Problem 8.13] we can find an ordinary-modular character triple (H, Z, λ) with Z a central subgroup of H with order prime to p and an isomorphism of ordinary-modular character triples $(\sigma, \tau) : (G, K, \vartheta) \to (H, Z, \lambda)$. Let $Z \leq N \leq H$ such that $\tau(M/K) = N/Z$ and set $\widehat{\lambda} := \sigma_M(\widehat{\vartheta})$. Observe that $\lambda \in \mathrm{dz}(Z)$, that $\widehat{\lambda}$ is an H-invariant extension of λ, and that N/Z is a p-group. Next, let $\varphi := \sigma_G(\chi)$ and notice that φ lies above $\widehat{\lambda}$. Furthermore, $\chi(1)/\widehat{\vartheta}(1) = \varphi(1)/\widehat{\lambda}(1)$ and therefore $\varphi \in \mathrm{rdz}(H \mid \widehat{\lambda})$. Furthermore, since (σ, τ) is an isomorphism of ordinary-modular character triples, it follows that $\chi^0 \in \mathrm{IBr}(G)$ if and only if $\varphi^0 \in \mathrm{IBr}(H)$. Hence, it is no loss of generality to assume that K is a central subgroup of G of order prime to p.

Now, we can write $M = K \times D$ for a Sylow p-subgroup D of M and $\widehat{\vartheta} = \vartheta \times \mu$ for some G-invariant linear character $\mu \in \mathrm{Irr}(D)$. In this situation [Nav04, Theorem 4.1] yields a canonical bijection

$$\mathrm{dz}(G/D) \to \mathrm{rdz}(G \mid \mu), \quad \psi \mapsto \psi_\mu,$$

and where $\psi_\mu^0 = \widehat{\mu} \cdot \psi^0$ for some linear Brauer character $\widehat{\mu}$ of G. Then, recalling that ψ^0 is an irreducible Brauer character for every $\psi \in \mathrm{dz}(G/D)$, we deduce that $\psi_\mu^0 \in \mathrm{IBr}(G)$ for every $\psi_\mu \in \mathrm{rdz}(G \mid \mu)$. To prove the first statement, it now suffices to show that each character $\chi \in \mathrm{rdz}(G \mid \widehat{\vartheta})$ belongs to $\mathrm{rdz}(G \mid \mu)$. To see this recall that K is a p'-group and μ is linear so that $\chi(1)_p = \widehat{\vartheta}(1)_p |G : M|_p = \mu(1)_p |G : D|$ for every $\chi \in \mathrm{rdz}(G \mid \widehat{\vartheta})$. This shows that $\chi \in \mathrm{rdz}(G \mid \mu)$ as claimed. Furthermore, if $\chi_i \in \mathrm{rdz}(G \mid \widehat{\vartheta})$ for $i = 1, 2$, then we can find $\psi_i \in \mathrm{dz}(G/D)$ such that $\chi_i = \psi_{i,\mu}$. If $\chi_1^0 = \chi_2^0$, then we get $\widehat{\mu} \cdot \psi_1^0 = \widehat{\mu} \cdot \psi_2^0$ and therefore $\psi_1^0 = \psi_2^0$. This implies that $\psi_1 = \psi_2$ and therefore that $\chi_1 = \chi_2$ which implies the second sentence of the statement.

We now show that the set of characters of the form χ^0 for $\chi \in \mathrm{rdz}(G \mid \widehat{\vartheta})$ does not depend on the choice of the extension $\widehat{\vartheta}$. Suppose that $\widehat{\vartheta}'$ is another G-invariant extension of ϑ to M. Arguing as in the previous paragraph, this determines a unique G-invariant character $\mu' \in \mathrm{Irr}(D)$ and a linear Brauer character $\widehat{\mu}' \in \mathrm{IBr}(G)$ such that $\psi'^0_{\mu'} = \widehat{\mu}' \cdot \psi'^0$ for every $\psi' \in \mathrm{dz}(G/D)$. Now let $\chi' \in \mathrm{rdz}(G \mid \mu')$ and write $\chi' = \psi'_{\mu'}$ for some $\psi' \in \mathrm{dz}(G/D)$. In order to prove (iii), we need to find $\chi \in \mathrm{rdz}(G \mid \mu)$ such that $\chi'^0 = \chi^0$. As explained before, we can write $\chi'^0 = \widehat{\mu}' \cdot \psi'^0$ and hence $\chi'^0 = \widehat{\mu} \cdot (\lambda \cdot \psi'^0)$ for $\lambda := \widehat{\mu}^{-1} \cdot \widehat{\mu}'$. Moreover, since λ is linear, Lemma 2.4 implies that $\lambda \cdot \psi'^0$ belongs to a block of defect zero of G/D. In particular, there exists some $\psi \in \mathrm{dz}(G/D)$ such that $\psi^0 = \lambda \cdot \psi'^0$. This implies that $\chi'^0 = \widehat{\mu} \cdot \psi^0 = \psi^0_\mu$ and our claim follows by setting $\chi = \psi_\mu \in \mathrm{rdz}(G \mid \mu)$. □

We can now define the set of relative defect zero Brauer characters.

Definition 5.2. Let $K \leq M$ be normal subgroups of G with M/K a p-group and consider a G-invariant $\varphi \in \mathrm{dz}^\circ(K)$. Set $\bar{H} := H\boldsymbol{O}_p(K)/\boldsymbol{O}_p(K)$ for every $H \leq G$. By definition $\bar{\varphi}$ belongs to a block of defect zero of $\bar{K}$ and we can find a unique $\bar{\vartheta} \in \mathrm{Irr}(\bar{K})$ such that $\bar{\vartheta}^0 = \bar{\varphi}$ according to [Nav98, Theorem 3.18]. By applying Lemma 5.1 to $\bar{G}$, for any $\bar{G}$-invariant extension $\widehat{\vartheta} \in \mathrm{Irr}(\bar{M})$ of $\bar{\vartheta}$, the set of Brauer characters $\bar{\chi}^0$ for $\bar{\chi} \in \mathrm{rdz}(\bar{G} \mid \widehat{\vartheta})$ is a well defined subset of $\mathrm{IBr}(\bar{G})$ which does not depend on the choice of the extension $\widehat{\vartheta}$. We define the set $\mathrm{rdz}^\circ(G \mid M, \varphi)$ to be the set of inflations to G of such Brauer characters $\bar{\chi}^0$. More generally, if $\varphi \in \mathrm{dz}^\circ(K)$ is M-invariant, but not necessarily G-invariant, then we denote by $\mathrm{rdz}^\circ(G \mid M, \varphi)$ the set of Brauer characters of G whose Clifford correspondent over φ (see [Nav98, Theorem 8.9]) belongs to $\mathrm{rdz}^\circ(G_\varphi \mid M, \varphi)$.

Recall that the set $\mathrm{dz}(G)$ of defect zero characters of a finite group G can be recovered as the set of 1-relative defect zero characters, i.e., $\mathrm{dz}(G) = \mathrm{rdz}(G \mid 1)$, and where we denote by 1 the trivial character of the identity group. Similarly, we observe that the set $\mathrm{dz}^\circ(G)$ can be recovered as a particular case of Definition 5.2.

Remark 5.3. Consider $K \leq M \leq G$ and ϑ as in Definition 5.2. If $K = 1$, $M = \boldsymbol{O}_p(G)$ and $\vartheta = 1$, then [Nav98, Theorem 3.18] implies that $\mathrm{dz}^\circ(G)$ coincides with $\mathrm{rdz}^\circ(G \mid \boldsymbol{O}_p(G), 1)$ and where we denote by 1 the trivial Brauer character of the identity group.

5.2. ***A bijection above the Dade–Glauberman–Nagao correspondence.*** We now come to the main result of this section. In order to introduce this statement, we quickly recall the definition of the Dade–Glauberman–Nagao correspondence as defined in [NT11, Section 4]. Assume that $K \trianglelefteq M$ with M/K a p-group and let $\vartheta \in \mathrm{dz}(K)$ be M-invariant. By [Nav98, Corollary 9.6] there is a unique block, say b, of M covering the block of ϑ. Furthermore, if D is a defect group of b, then [Nav98, Theorem 9.17] implies that D is a complement of K in M, that is, $M = KD$ and $1 = K \cap D$. Now, notice that $\boldsymbol{N}_M(D) = D \times \boldsymbol{C}_K(D)$ and that, if C is the Brauer correspondent of b in $\boldsymbol{N}_M(D)$, then C covers a unique block c of $\boldsymbol{C}_K(D)$ with defect zero. We denote by $\Pi_D(\vartheta) \in \mathrm{dz}(\boldsymbol{C}_K(D))$ the unique ordinary character belonging to c, called the *Dade–Glauberman–Nagao correspondent* (DGN correspondent for short) of ϑ with respect to D. We refer the reader to [NT11, Section 4] and [NS14a] for further information.

Next, we define a version of the DGN correspondence for Brauer characters. To start, and for future reference, we consider the following hypothesis.

Hypothesis 5.4. Suppose that $K \leq M$ with M/K a p-group and let $\varphi \in \mathrm{dz}^\circ(K)$ be M-invariant. Set $L := \boldsymbol{O}_p(K)$ and denote by $\bar{\varphi}$ the Brauer character of $\overline{K} := K/L$ corresponding to φ. By [Nav98, Theorem 3.18] we can find a unique character $\bar{\vartheta} \in \mathrm{dz}(\overline{K})$ such that $\bar{\vartheta}^0 = \bar{\varphi}$. Denote by $\vartheta \in \mathrm{Irr}(K)$ the inflation of $\bar{\vartheta}$. Let D be a p-subgroup of M such that $\overline{D} := D/L$ is a defect group of the unique block of $\overline{M} := M/L$ covering the block of $\bar{\varphi}$ and notice that $M = KD$ and $K \cap D = L$ by [Nav98, Theorem 9.17]. Observe that the uniqueness of $\bar{\vartheta}$ implies that $\bar{\vartheta}$ is $\overline{M}$-invariant.

We can now define a version of the DGN correspondence for Brauer characters as follows.

Definition 5.5. Assume Hypothesis 5.4. By the above paragraph we can define the DGN correspondent $\bar{\vartheta}' := \Pi_{\overline{D}}(\bar{\vartheta}) \in \mathrm{dz}(\boldsymbol{C}_{\overline{K}}(\overline{D}))$ of $\bar{\vartheta}$ with respect to $\overline{D}$. Now, let ϑ' be the ordinary character of $\boldsymbol{N}_K(D)$ corresponding to $\bar{\vartheta}'$ via inflation and define the *Dade–Glauberman–Nagao correspondence* (DGN correspondent for short) of φ with respect to D by setting

$$\pi_D(\varphi) := \vartheta'^0 \in \mathrm{dz}^\circ(\boldsymbol{N}_K(D)).$$

We are now ready to state the main result of this section.

Theorem 5.6. *Let $K \leq M \leq A$ be finite groups with K and M normal in A and M/K a p-group. Let $\varphi \in \mathrm{dz}^\circ(K)$ be A-invariant and D a p-subgroup of M such that $D/\boldsymbol{O}_p(K)$ is a defect group of the unique block of $M/\boldsymbol{O}_p(K)$ covering the block of $\bar{\varphi}$ in $K/\boldsymbol{O}_p(K)$. Consider the DGN correspondent $\pi_D(\varphi) \in \mathrm{dz}^\circ(\boldsymbol{N}_K(D))$ as in Definition 5.5. If $M \leq G \trianglelefteq A$ and M/K is a radical p-subgroup of G/K, then there exists an $\boldsymbol{N}_A(D)$-equivariant bijection*

$$\Delta^G_{D,\varphi} : \mathrm{rdz}^\circ(G \mid M, \varphi) \to \mathrm{dz}^\circ(\boldsymbol{N}_G(D) \mid \pi_D(\varphi))$$

such that

$$\left(A_\chi, G, \chi\right) \succeq_b \left(\boldsymbol{N}_A(D)_\chi, \boldsymbol{N}_G(D), \Delta^G_{D,\varphi}(\chi)\right)$$

for every $\chi \in \mathrm{rdz}^\circ(G \mid M, \varphi)$.

Remark 5.7. Observe that if D is the p-subgroup of M considered in the above theorem, then D is a radical p-subgroup of G. In fact, notice first that D is a radical p-subgroup of M and recall furthermore that M/K is a radical p-subgroup of G/K by hypothesis. Since G/K is isomorphic to $\boldsymbol{N}_G(D)/\boldsymbol{N}_K(D)$ we deduce that $\boldsymbol{N}_M(D)/\boldsymbol{N}_K(D)$ is a radical p-subgroup of $\boldsymbol{N}_G(D)/\boldsymbol{N}_K(D)$. Then, since $\boldsymbol{N}_M(D)/\boldsymbol{N}_K(D)$ is normal in $\boldsymbol{N}_G(D)/\boldsymbol{N}_K(D)$, it follows that $\boldsymbol{O}_p(\boldsymbol{N}_G(D)/\boldsymbol{N}_M(D)) = 1$. This implies that $\boldsymbol{O}_p(\boldsymbol{N}_G(D)/D)$ is contained in $\boldsymbol{N}_M(D)/D$. Recalling that D is a radical p-subgroup of M, we get $\boldsymbol{O}_p(\boldsymbol{N}_M(D)/D) = 1$ and therefore $\boldsymbol{O}_p(\boldsymbol{N}_G(D)/D) \leq \boldsymbol{O}_p(\boldsymbol{N}_M(D)/D) = 1$, which implies that $D = \boldsymbol{O}_p(\boldsymbol{N}_G(D))$ as claimed.

Our proof of Theorem 5.6 is inspired by the argument developed in [NS14b, Section 5]. We start with the following lemma (see [NS14b, Proposition 5.12]).

Lemma 5.8. *Assume Hypothesis 5.4 with $L = 1$ and suppose $M, K \lhd A$ and that ϑ extends to $\tilde{\vartheta} \in \mathrm{Irr}(A)$. Set $\vartheta' := \Pi_D(\vartheta)$, $H := \boldsymbol{N}_A(D)$, and $N := \boldsymbol{C}_K(D)$. Then there exists an extension $\tilde{\vartheta}' \in \mathrm{Irr}(H)$ of ϑ' such* *that*

$$\tilde{\vartheta}(x)^* = e^* \tilde{\vartheta}'(x)^*$$

whenever x is a p-regular element of G with $D \in \mathrm{Syl}_p(\mathbf{C}_M(x))$ and where $e = [\vartheta_N, \vartheta']$. Furthermore,

$$\mathrm{IBr}\big((\tilde{\vartheta}^0)_{\mathbf{C}_A(M)}\big) = \mathrm{IBr}\big((\tilde{\vartheta}'^0)_{\mathbf{C}_A(M)}\big).$$

Proof. Let $\tilde{\vartheta}'$ be the extension of ϑ given by [NS14b, Proposition 5.12] and observe that the first part of the statement is satisfied and that in addition

$$\mathrm{Irr}(\tilde{\vartheta}_{\mathbf{C}_A(M)}) = \mathrm{Irr}(\tilde{\vartheta}'_{\mathbf{C}_A(M)}).$$

But then the latter equality implies that $\mathrm{IBr}(\tilde{\vartheta}^0_C) = \mathrm{IBr}((\tilde{\vartheta}')^0_C)$ as required. □

We now use Lemma 5.8 to construct the following block isomorphisms of modular character triples.

Proposition 5.9. *Assume Hypothesis 5.4 with $L = 1$ and suppose that $M, K \lhd A$ and that φ extends to $\widetilde{\varphi} \in \mathrm{IBr}(A)$. Consider $\vartheta' = \Pi_D(\vartheta)$ and $\varphi' = \pi_D(\varphi) = (\vartheta')^0$, and let $\psi \in \mathrm{IBr}(M)$ and $\psi' \in \mathrm{IBr}(\mathbf{N}_M(D))$ be the unique characters lying above φ and φ' respectively. Then*

$$(A, M, \psi) \succeq_b (\mathbf{N}_A(D), \mathbf{N}_M(D), \psi').$$

Furthermore, if $M \leq J \leq A$ and D is a radical p-subgroup of J then the bijection

$$\sigma_J : \mathrm{IBr}(J \mid \psi) \to \mathrm{IBr}\big(\mathbf{N}_J(D) \mid \psi'\big)$$

given by Theorem 3.1 maps $\mathrm{rdz}^\circ(J \mid M, \varphi)$ *onto* $\mathrm{dz}^\circ(\mathbf{N}_J(D) \mid \varphi')$.

Proof. To start, we construct the block isomorphism of modular character triples stated above. Straightforward calculations show that the group theoretical conditions from Definition 3.4 are satisfied. By Lemma 2.1, there exists an extension $\tilde{\vartheta} \in \mathrm{Irr}(A)$ of ϑ such that $\tilde{\vartheta}^0 = \widetilde{\varphi}$. Set $H := \mathbf{N}_A(D)$ and let $\tilde{\vartheta}' \in \mathrm{Irr}(H)$ be the extension of ϑ' given by Lemma 5.8. Now, if we define $\widetilde{\varphi}' := (\tilde{\vartheta}')^0$, then we deduce from Lemma 2.1 that $\widetilde{\varphi}' \in \mathrm{IBr}(H)$ is an extension of φ'. By [Nav98, Theorem 8.11] it follows that $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ are actually extensions of ψ and ψ' respectively. Let $\mathcal{P}$ and $\mathcal{P}'$ be modular representations of A and H affording $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ respectively and consider the corresponding strong isomorphism of modular character triples $(\sigma, \tau) : (A, M, \psi) \to (H, \mathbf{N}_M(D), \psi')$ given by Theorem 3.1. Notice that if $M \leq J \leq A$ and $\chi \in \mathrm{IBr}(J \mid \psi)$, then [Nav98, Corollary 8.20] implies that $\chi = \beta\widetilde{\varphi}_J$ for some $\beta \in \mathrm{IBr}(J/M)$ and then we also have $\sigma_J(\chi) = \beta_{J\cap H}\widetilde{\varphi}'_{J\cap H}$ with $\beta_{J\cap H} \in \mathrm{IBr}(J \cap H/\mathbf{N}_M(D))$.

Next, by the definition of $\widetilde{\varphi}'$ and according to Lemma 5.8, we deduce that $\mathrm{IBr}(\widetilde{\varphi}_{\mathbf{C}_A(M)}) = \mathrm{IBr}(\widetilde{\varphi}'_{\mathbf{C}_A(M)})$ and therefore that $\mathrm{IBr}(\widetilde{\varphi}_{\mathbf{C}_J(M)}) = \mathrm{IBr}(\widetilde{\varphi}'_{\mathbf{C}_J(M)})$ for every $M \leq J \leq A$. Furthermore, if $\beta \in \mathrm{IBr}(J/M)$, then $\mathrm{IBr}((\beta\widetilde{\varphi})_{\mathbf{C}_J(M)})$ coincides with the set of irreducible constituents of the characters of the form $\eta\nu_{\mathbf{C}_J(M)}$ with $\eta \in \mathrm{IBr}(\beta_{\mathbf{C}_J(M)})$ and $\nu \in \mathrm{IBr}(\widetilde{\varphi}_{\mathbf{C}_G(M)})$. This shows that

$$\mathrm{IBr}((\beta\widetilde{\varphi})_{\mathbf{C}_J(M)}) = \mathrm{IBr}((\beta_H\widetilde{\varphi}')_{\mathbf{C}_J(M)}) = \mathrm{IBr}(\sigma_J(\beta\widetilde{\varphi})_{\mathbf{C}_J(M)})$$

and hence that $(A, M, \psi) \succeq_c (H, \mathbf{N}_M(D), \psi')$.

We now show that the condition on block induction from Definition 3.4 is satisfied. For this purpose, we first show that $\mathrm{bl}(\widetilde{\varphi}_J) = \mathrm{bl}(\widetilde{\varphi}'_{J\cap H})^J$, or equivalently that $\mathrm{bl}(\tilde{\vartheta}_J) = \mathrm{bl}(\tilde{\vartheta}'_{J\cap H})^J$, for every $M \leq J \leq A$. Let $e := [\vartheta_{\mathbf{C}_K(D)}, \vartheta'] \neq 0 \mod p$ and recall that, according to Lemma 5.8, the characters $\tilde{\vartheta}$ and $\tilde{\vartheta}'$ satisfy

$$\tilde{\vartheta}(x)^* = e^* \tilde{\vartheta}'(x)^* \tag{3}$$

for all $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathbf{C}_M(x))$. By [NS14b, Theorem 5.2] we get

$$\tilde{\vartheta}(1)_{p'} \equiv e|K : \mathbf{C}_K(D)|_{p'} \tilde{\vartheta}'(1)_{p'} \mod p$$

and, by using the fact that $|K : \mathbf{C}_K(D)| = |M : \mathbf{N}_M(D)|$ and applying (3), we obtain

$$\left(\frac{|\mathbf{N}_M(D)|_{p'} \tilde{\vartheta}'(x)}{\tilde{\vartheta}'(1)_{p'}}\right)^* = \left(\frac{|M|_{p'} \tilde{\vartheta}(x)}{\tilde{\vartheta}(1)_{p'}}\right)^*,$$

for all $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathbf{C}_M(x))$. Now, observe that D is a common defect group of $\mathrm{bl}(\tilde{\vartheta}_M)$ and $\mathrm{bl}(\tilde{\vartheta}'_{\mathbf{N}_M(D)})$. In fact, by definition D is a defect group of $\mathrm{bl}(\psi)$ and $\mathrm{bl}(\psi) = \mathrm{bl}(\tilde{\vartheta}_M)$. Moreover, if E is a defect group of $\mathrm{bl}(\tilde{\vartheta}'_{\mathbf{N}_M(D)})$, then $D \leq \mathbf{O}_p(\mathbf{N}_M(D)) \leq E$. On the other hand, by [Nav98, Theorem 9.17] we know that $E \cap \mathbf{C}_K(D)$ is a defect group of $\mathrm{bl}(\vartheta')$ and hence $E \cap K = 1$. This shows that E is a complement of K in M. But so is D (see [Nav98, Theorem 9.17]) and therefore $D = E$ is a common defect group of $\mathrm{bl}(\tilde{\vartheta}_M)$ and $\mathrm{bl}(\tilde{\vartheta}'_{\mathbf{N}_M(D)})$. It now follows from [NS14b, Lemma 4.2] that $\mathrm{bl}(\tilde{\vartheta}_J) = \mathrm{bl}(\tilde{\vartheta}'_{J\cap H})^J$ as required. Finally, by [Spä13b, Proposition 3.6] we know that

$$\mathrm{bl}\left(\beta\widetilde{\varphi}_J\right) = \mathrm{bl}\left(\beta_{J\cap H}\widetilde{\varphi}'_{J\cap H}\right)^J = \mathrm{bl}\left(\sigma_J(\beta\widetilde{\varphi}_J)\right)^J$$

which proves that $(A, M, \psi) \succeq_b (H, \mathbf{N}_M(D), \psi')$. This concludes the first part of the proof.

Consider now $M \leq J \leq A$ such that D is a radical p-subgroup of J and let $\gamma \in \mathrm{IBr}(J \mid \psi)$. We wish to prove that $\gamma \in \mathrm{rdz}^\circ(J \mid M, \varphi)$ if and only if $\gamma' := \sigma_J(\gamma) \in \mathrm{dz}^\circ(\mathbf{N}_J(D) \mid \varphi')$. As in the previous paragraph, we may write $\gamma = \beta\widetilde{\varphi}_J$ and $\gamma' = \beta_{J\cap H}\widetilde{\varphi}'_{J\cap H}$ for some $\beta \in \mathrm{IBr}(J/M) = \mathrm{IBr}(J/K)$. Assume first that $\gamma' \in \mathrm{dz}^\circ(\mathbf{N}_J(D) \mid \varphi')$ and recall that D is a radical p-subgroup of J. Then, there exists $\chi \in \mathrm{Irr}(\mathbf{N}_J(D))$ with $D \leq \ker(\chi)$ and such that $\chi^0 = \gamma'$ and $\chi(1)_p = |\mathbf{N}_J(D) : D|_p$. Now, by Lemma 2.2 we can find some $\eta' \in \mathrm{Irr}(J \cap H/\mathbf{N}_M(D))$ such that $\eta'^0 = \beta_{J\cap H}$. Furthermore via the isomorphism $J/M \simeq \mathbf{N}_J(D)/\mathbf{N}_M(D)$ we can write $\eta' = \eta_{J\cap H}$ for a unique $\eta \in \mathrm{Irr}(J/M)$. Observe then that $\eta^0 = \beta$. Furthermore,

$$\beta(1)_p = \frac{\sigma_J(1)_p}{\widetilde{\varphi}'(1)_p} = \frac{\left|\mathbf{N}_J(D) : D\right|_p}{|\mathbf{C}_K(D)|_p} = |J : M|_p.$$

By applying Lemma 2.2 once again, we deduce now that there is some $\rho \in \mathrm{Irr}(J)$ with $\rho^0 = \gamma$. Moreover, ρ lies over some extension $\tilde{\vartheta} \in \mathrm{Irr}(M)$ of ϑ and satisfies

$$\rho(1)_p = \gamma(1)_p = \beta(1)_p \widetilde{\varphi}(1)_p = |J : M|_p \tilde{\vartheta}(1)_p$$

which implies that $\rho \in \mathrm{rdz}(J \mid \tilde{\vartheta})$, so $\gamma \in \mathrm{rdz}^\circ(J \mid M, \varphi)$. A similar argument shows that if $\gamma \in \mathrm{rdz}^\circ(J \mid M, \varphi)$ then $\gamma' \in \mathrm{dz}^\circ(\mathbf{N}_J(D) \mid \varphi')$ and the proof is now complete. □

We now extend Proposition 5.9 to the case where φ is A-invariant but does not necessarily extend to A. Before proving this result, we show that the central extension constructed in Lemma 3.12 preserves (relative) defect zero Brauer characters.

Lemma 5.10. *Consider the setting of Lemma 3.12 .*

(i) *For every $K \leq J \leq G$ the restriction $\epsilon_{\hat{J}} : \hat{J} \to J$ maps $\mathrm{dz}^\circ(\hat{J} \mid \vartheta_0 \times 1_Z)$ onto $\mathrm{dz}^\circ(J \mid \vartheta)$.*

(ii) *Let $K \leq U \leq J \leq G$ such that $U \lhd G$ and U/K is a p-group. Then the restriction $\epsilon_{\hat{J}} : \hat{J} \to J$ maps $\mathrm{rdz}^\circ(\hat{J} \mid \widehat{U}, \vartheta_0 \times 1_Z)$ onto $\mathrm{rdz}^\circ(J \mid U, \vartheta)$.*

Proof. Let $\varphi \in \mathrm{IBr}(\hat{J} \mid \vartheta_0 \times 1_Z)$ and assume there is some $\chi \in \mathrm{Irr}(\hat{J})$ with $\chi^0 = \varphi$. Then $\epsilon_{\hat{J}}(\chi) \in \mathrm{Irr}(J)$ satisfies $\epsilon_{\hat{J}}(\chi)^0 = \epsilon_{\hat{J}}(\varphi)$. Furthermore, using that Z is a central p'-group we obtain that $\chi(1)_p = |\hat{J} : \boldsymbol{O}_p(\hat{J})|_p$ if and only if $\epsilon_{\hat{J}}(\chi)(1)_p = |J : \boldsymbol{O}_p(J)|_p$, and then the first part of the statement follows. Similarly, noticing that $\chi(1)_p = |\hat{J} : \widehat{U}|_p$ if and only if $\epsilon_{\hat{J}}(\chi)(1)_p = |J : U|_p$, we obtain the second part of the statement. □

Using the central extension constructed in Lemma 3.12 we obtain the following corollary as a direct consequence of Proposition 5.9 and Lemma 5.10.

Corollary 5.11. *Assume Hypothesis 5.4 with $L = 1$ and suppose that $M, K \lhd A$ and that φ is A-invariant. Consider $\vartheta' = \Pi_D(\vartheta)$ and $\varphi' = \pi_D(\varphi) = (\vartheta')^0$, and let $\psi \in \mathrm{IBr}(M)$ and $\psi' \in \mathrm{IBr}(\boldsymbol{N}_M(D))$ be the unique characters lying above φ and φ' respectively. Then*

$$(A, M, \psi) \succeq_b (\boldsymbol{N}_A(D), \boldsymbol{N}_M(D), \psi').$$

Furthermore, if $M \leq J \leq A$ and D is a radical p-subgroup of J then the bijection

$$\sigma_J : \mathrm{IBr}(J \mid \psi) \to \mathrm{IBr}(\boldsymbol{N}_J(D) \mid \psi')$$

given by Theorem 3.1 maps $\mathrm{rdz}^\circ(J \mid M, \varphi)$ onto $\mathrm{dz}^\circ(\boldsymbol{N}_J(D) \mid \varphi')$.

Proof. Let $\mathcal{P}$ be a projective representation of A associated with (A, M, ψ) and consider the central extension $\epsilon : \hat{A} \to A$ from Lemma 3.12. Let $Z = \ker(\epsilon)$ and set $\widehat{M} := \epsilon^{-1}(M)$, so that $\widehat{M} = M_0 \times Z$, $M_0 \cong M$ via the restriction ϵ_{M_0}, and $\psi_0 = \psi \circ \epsilon_{M_0} \in \mathrm{IBr}(M_0)$ extends to $\hat{A}$. By first applying Proposition 5.9 in the group $\hat{A}$ and then using Lemma 3.12 (iv), we deduce that $(A, M, \psi) \succeq_b (\boldsymbol{N}_A(D), \boldsymbol{N}_M(D), \psi')$. Moreover, using the second half of Proposition 5.9 and applying Lemma 5.10 (see also Remark 3.13), we deduce that the bijection σ_J maps $\mathrm{rdz}^\circ(J \mid M, \varphi)$ onto $\mathrm{dz}^\circ(\boldsymbol{N}_J(D) \mid \varphi')$ as required. □

We can finally prove Theorem 5.6.

Proof of Theorem 5.6. Let $L := \boldsymbol{O}_p(K)$ and define $\overline{H} = H/L$ for every $L \leq H \leq A$. Notice that L is contained in D and therefore $L \leq \boldsymbol{N}_K(D)$ so that $\overline{3}2\boldsymbol{N}_K(D) = \boldsymbol{N}_{\overline{K}}(\overline{D}) = \boldsymbol{C}_{\overline{K}}(\overline{D})$. Now we can consider the Brauer characters $\bar{\varphi} \in \mathrm{dz}^\circ(\overline{K})$ and $\bar{\varphi}' \in \mathrm{dz}^\circ(\boldsymbol{C}_{\overline{K}}(\overline{D}))$ corresponding to $\varphi \in \mathrm{dz}^\circ(K)$ and $\varphi' := \pi_D(\varphi)$

respectively via inflation of characters. Noticing that $\bar{\varphi}' := \pi_{\bar{D}}(\bar{\varphi})$ we can apply Corollary 5.11, with $J = \overline{G}$, to obtain an $\mathbf{N}_{\bar{A}}(\bar{D})$-equivariant bijection

$$\bar{\Delta}^{\overline{G}}_{\bar{D},\bar{\varphi}} : \mathrm{rdz}^{\circ}(\overline{G} \mid \overline{M}, \bar{\varphi}) \to \mathrm{dz}^{\circ}(\mathbf{N}_{\overline{G}}(\bar{D}) \mid \bar{\varphi}')$$

that satisfies

$$(\bar{A}_{\bar{\chi}}, \overline{G}, \bar{\chi}) \succeq_b \left(\mathbf{N}_{\bar{A}}(\bar{D})_{\bar{\chi}}, \mathbf{N}_{\overline{G}}(\bar{D}), \Delta^{\overline{G}}_{\bar{D},\bar{\varphi}}(\bar{\chi})\right)$$

for every $\bar{\chi} \in \mathrm{rdz}^{\circ}(\overline{G} \mid \overline{M}, \bar{\varphi})$ thanks to the block isomorphism of modular character triples given by Corollary 5.11 and applying Lemma 3.7. Then, since L is contained in the kernel of every character belonging either to $\mathrm{rdz}^{\circ}(G \mid M, \varphi)$ or to $\mathrm{dz}^{\circ}(\mathbf{N}_G(D) \mid \varphi')$, we deduce that by inflation of characters the bijection $\bar{\Delta}^{\overline{G}}_{\bar{D},\bar{\varphi}}$ induces a bijection $\Delta^{G}_{D,\varphi}$ with the required properties. To obtain the block isomorphism of modular character triples from the statement, we apply Lemma 3.14. This completes the proof. □

We conclude this section with a result of independent interest. This can be seen as a modular version of [Lad10, Corollary 11.3].

Corollary 5.12. *Assume Hypothesis 5.4 and suppose that $M, K \lhd A$ and that φ is A-invariant. Then the modular character triples (A, K, φ) and $(\mathbf{N}_A(D), \mathbf{N}_K(D), \pi_D(\varphi))$ are strongly isomorphic in the sense of Theorem 3.1.*

Proof. First, notice that since $M\mathbf{N}_A(D) = A$ and $M = KD \leq K\mathbf{N}_A(D)$ then we have $K\mathbf{N}_A(D) = M\mathbf{N}_A(D) = A$ so the group theoretical conditions of Theorem 3.1 are satisfied. Let $\mathcal{P}$ and $\mathcal{P}'$ be projective representations giving the isomorphism

$$(A, M, \psi) \succeq_b \left(\mathbf{N}_A(D), \mathbf{N}_M(D), \psi'\right)$$

from Theorem 5.6 and where ψ and ψ' are the extensions of φ and $\pi_D(\varphi)$ to M and $\mathbf{N}_M(D)$ respectively. Notice that $\mathcal{P}$ and $\mathcal{P}'$ are actually projective representations associated with (A, K, φ) and $\left(\mathbf{N}_A(D), \mathbf{N}_K(D), \pi_D(\varphi)\right)$. Now, using [Nav98, Theorem 8.14] it follows that the factor sets of $\mathcal{P}$ and $\mathcal{P}'$ coincide via the natural isomorphism $A/K \to \mathbf{N}_A(D)/\mathbf{N}_K(D)$. We can then conclude by applying Theorem 3.1. □

6. The reduction

In this section, we finally show that Conjecture B reduces to quasisimple groups and hence prove Theorem C. For the reader's convenience, we restate Theorem C below. Recall that a simple group S is said to be *involved* in a finite group G if there exist subgroups $N \trianglelefteq H \leq G$ such that H/N is isomorphic to S.

Theorem 6.1. *Let G be a finite group and suppose that Conjecture 4.1 holds for every covering group of any nonabelian finite simple group of order divisible by p involved in G. Then Conjecture 4.1 holds for G.*

We prove Theorem 6.1 arguing inductively by considering a minimal counterexample. Suppose that G is a finite group satisfying the assumptions of Theorem 6.1 and for which the conclusion fails. More precisely, suppose that Conjecture 4.1 fails with respect to $G \trianglelefteq A$ and that G and A have been minimized with respect to $|G : \mathbf{Z}(G)|$ first and then $|A|$. In this case, the pair (G, A) satisfies the following hypothesis.

Hypothesis 6.2. Let $G \trianglelefteq A$ be finite groups, with G satisfying the requirements of Theorem 6.1, and suppose that Conjecture 4.1 holds for every $X \trianglelefteq Y$ such that every nonabelian finite simple group of order divisible by p involved in X is also involved in G, and at least one of the following conditions is satisfied:

(i) $|X : \mathbf{Z}(X)| < |G : \mathbf{Z}(G)|$;

(ii) $|X : \mathbf{Z}(X)| = |G : \mathbf{Z}(G)|$ and $|Y| < |A|$.

We now describe the structure of the minimal counterexample G. To start, we show that G does not have nontrivial normal p-subgroups.

Lemma 6.3. *If Conjecture 4.1 holds for $G/\mathbf{O}_p(G) \trianglelefteq A/\mathbf{O}_p(G)$, then it holds for $G \trianglelefteq A$.*

Proof. Set $\bar{A} := A/\mathbf{O}_p(G)$ and $\bar{G} := G/\mathbf{O}_p(G)$ and suppose that $\bar{\Omega} : \mathrm{IBr}(\bar{G}) \to \mathrm{Alp}(\bar{G})/\bar{G}$ is the map given by Conjecture 4.1 applied to $\bar{G} \trianglelefteq \bar{A}$. By [Nav98, Lemma 2.32] we know that $\mathbf{O}_p(G)$ is contained in the kernel of every irreducible Brauer character and therefore we can identify $\mathrm{IBr}(\bar{G})$ with $\mathrm{IBr}(G)$ via inflation of characters. On the other hand, if (Q, ψ) is a p-weight of G then Q is a radical p-subgroup and hence $\mathbf{O}_p(G)$ is contained in Q (see, for instance, [Dad92, Lemma 1.3]). Then, applying [Nav98, Lemma 2.32] we deduce that $\psi \in \mathrm{dz}^\circ(\mathbf{N}_G(Q))$ can be identified with the corresponding character $\bar{\psi} \in \mathrm{dz}^\circ(\mathbf{N}_{\bar{G}}(\bar{Q}))$ where $\bar{Q} := Q/\mathbf{O}_p(G)$. It follows that the map $\bar{\Omega}$ induces an A-equivariant bijection $\Omega : \mathrm{IBr}(G) \to \mathrm{Alp}(G)/G$. To conclude, notice that the isomorphisms of modular character triples induced by $\bar{\Omega}$ can be lifted to analogous block isomorphisms for the map Ω thanks to Lemma 3.14. □

Corollary 6.4. *Suppose that Hypothesis 6.2 holds for the pair (G, A) while Conjecture 4.1 fails with respect to $G \trianglelefteq A$. Then $\mathbf{O}_p(G) = 1$.*

Proof. Assume that $\mathbf{O}_p(G) \neq 1$, set $\bar{A} := A/\mathbf{O}_p(G)$ and $\bar{G} := G/\mathbf{O}_p(G)$. Then $|\bar{G} : \mathbf{Z}(\bar{G})| \leq |G : \mathbf{Z}(G)|$ and $|\bar{A}| < |A|$ so that Conjecture 4.1 holds for $\bar{G} \trianglelefteq \bar{A}$ thanks to Hypothesis 6.2. Lemma 6.3 now implies that Conjecture 4.1 holds for $G \trianglelefteq A$, a contradiction. □

By applying Corollary 6.4 together with Corollary 4.3, we can give a description of the structure of G.

Proposition 6.5. *Suppose that Hypothesis 6.2 holds for the pair (G, A) while Conjecture 4.1 fails with respect to $G \trianglelefteq A$. Then, there exists a subgroup K of G with $K \trianglelefteq A$ and $K \not\leq \mathbf{Z}(G)$ such that Conjecture 4.1 holds with respect to $K \trianglelefteq A$.*

Proof. By Corollary 6.4 we have $\mathbf{O}_p(G) = 1$ and thus $\mathbf{Z}(G) \leq \mathbf{O}_{p'}(G)$. If $\mathbf{Z}(G) < \mathbf{O}_{p'}(G)$, then we define $K := \mathbf{O}_{p'}(G)$ and observe that Conjecture 4.1 trivially holds for $K \trianglelefteq A$. We may therefore assume that $\mathbf{O}_{p'}(G) = \mathbf{Z}(G)$. Now, if G is p-solvable it must be abelian and Conjecture 4.1 holds for G, against our assumptions. We conclude that G has a nonabelian composition factor of order divisible by p. More precisely, $\mathbf{Z}(G) < \mathbb{F}^*(G)$ and we can find a perfect subgroup $K \leq \mathbb{F}^*(G)$ with K characteristic in G, hence normal in A, and such that $K/\mathbf{Z}(K)$ is isomorphic to a direct product of copies of S for some nonabelian simple group S of order divisible by p. Observe also that p does not divide the order of $\mathbf{Z}(K)$

since $\boldsymbol{O}_p(G) = 1$. Then, $K \not\leq \boldsymbol{Z}(G)$ and Conjecture 4.1 holds with respect to $K \trianglelefteq A$ by our assumption and thanks to Corollary 4.3. □

If K is the group given by Proposition 6.5, then by applying Theorem 4.5 with $J = G$ we obtain an A-equivariant bijection between the sets $\mathrm{IBr}(G)$ and $\mathrm{Alp}(G \mid K)/G$ (see Definition 4.4) inducing block isomorphisms of modular character triples. Therefore, to conclude the proof of Theorem 6.1 we now need to construct a bijection with similar properties between the sets $\mathrm{Alp}(G \mid K)/G$ and $\mathrm{Alp}(G)/G$. We start by introducing some further notation.

Definition 6.6. Let $K \trianglelefteq G$ be finite groups and $\vartheta \in \mathrm{dz}^\circ(K)$. We denote by $\mathrm{Alp}_{\mathrm{r}}(G \mid \vartheta)$ the set of pairs (R, χ) such that $R/K \in \mathrm{Rad}(G_\vartheta/K)$ and $\chi \in \mathrm{rdz}^\circ(\boldsymbol{N}_G(R) \mid R, \vartheta)$. The group G_ϑ acts by conjugation on $\mathrm{Alp}_{\mathrm{r}}(G \mid \vartheta)$ and we let $\mathrm{Alp}_{\mathrm{r}}(G \mid \vartheta)/G_\vartheta$ denote the corresponding set of G_ϑ-orbits.

The argument used to prove the following lemma is inspired by [NS14b, Lemma 7.3].

Lemma 6.7. *Suppose that Hypothesis 6.2 holds for the pair (G, A) and let $K \leq G$ with $K \trianglelefteq A$ and with an A-invariant $\vartheta \in \mathrm{dz}^\circ(K)$. If $|G : K\boldsymbol{Z}(G)| < |G : \boldsymbol{Z}(G)|$, then there exists an A-equivariant bijection*

$$\Upsilon_\vartheta^G : \mathrm{IBr}(G \mid \vartheta) \to \mathrm{Alp}_{\mathrm{r}}(G \mid \vartheta)/G$$

such that

$$(A_\eta, G, \eta) \succeq_b (\boldsymbol{N}_A(R)_\chi, \boldsymbol{N}_G(R), \chi)$$

for every $\eta \in \mathrm{IBr}(G \mid \vartheta)$ and $(R, \chi) \in \Upsilon_\vartheta^G(\eta)$.

Proof. Let $\mathcal{P}$ be a projective representation associated with the modular character triple (A, K, ϑ) and consider the central extension $\hat{A}$ of A by the p'-subgroup Z constructed in Lemma 3.12. Let $\epsilon : \hat{A} \to A$ be the epimorphism given by $\epsilon(x, z) = x$ for each $x \in A$ and $z \in Z$ and set $\widehat{L} := \epsilon^{-1}(L)$ for every $L \leq A$. Recall then that $\widehat{K} = K_0 \times Z$ for a subgroup $K_0 \leq \hat{A}$ isomorphic to K via the restriction ϵ_{K_0}. Let $\vartheta_0 := \vartheta \circ \epsilon_{K_0}$ and observe that $\vartheta_0 \in \mathrm{dz}^\circ(K_0)$ and that, by Lemma 3.12, ϑ_0 has an extension $\tilde{\vartheta} \in \mathrm{IBr}(\hat{A})$. Notice that $\widehat{G}/\widehat{K}$ is isomorphic to G/K and, recalling that $\widehat{K}/K_0 \simeq Z$ is a p'-group, it follows that every nonabelian simple group of order divisible by p involved in $\widehat{G}/K_0$ is also involved in G. Furthermore, if we set $\overline{X} := XK_0/K_0$ for every $X \leq \hat{A}$, then $|\overline{G} : \boldsymbol{Z}(\overline{G})| \leq |G : K\boldsymbol{Z}(G)| < |G : \boldsymbol{Z}(G)|$ and therefore, by Hypothesis 6.2, Conjecture 4.1 holds for the pair $(\overline{G}, \overline{A})$. Thus, there exists an $\overline{A}$-equivariant bijection

$$\Omega_{\overline{G}} : \mathrm{IBr}(\overline{G}) \to \mathrm{Alp}(\overline{G})/\overline{G}$$

such that

$$\left(\overline{A}_{\bar{\rho}}, \overline{G}, \bar{\rho}\right) \succeq_b \left(\boldsymbol{N}_{\overline{A}}(\overline{R})_{\bar{\psi}}, \boldsymbol{N}_{\overline{G}}(\overline{R}), \bar{\psi}\right) \tag{4}$$

for all $\bar{\rho} \in \mathrm{IBr}(\overline{G})$ and $(\overline{R}, \bar{\psi}) \in \Omega_{\overline{G}}(\bar{\rho})$. Consider now $\tau \in \mathrm{IBr}(\widehat{G} \mid \vartheta_0)$. Since ϑ_0 extends to $\tilde{\vartheta} \in \mathrm{IBr}(\hat{A})$, applying [Nav98, Corollary 8.20], we can find a unique $\bar{\rho} \in \mathrm{IBr}(\overline{G})$ such that $\tau = \rho\tilde{\vartheta}_{\widehat{G}}$ and where ρ is the inflation of $\bar{\rho}$ to $\widehat{G}$. Similarly, given any pair $(\widehat{R}, \varphi) \in \mathrm{Alp}_r(\widehat{G} \mid \vartheta_0)$, we can find a unique $\bar{\psi} \in \mathrm{IBr}(\boldsymbol{N}_{\widehat{G}}(\widehat{R})/K_0)$, with inflation $\psi \in \mathrm{IBr}(\boldsymbol{N}_{\widehat{G}}(\widehat{R}))$, such that $\varphi = \psi\tilde{\vartheta}_{\boldsymbol{N}_{\widehat{G}}(\widehat{R})}$. Since ϑ_0 has defect zero (recall here that it is no loss of generality to assume that $\boldsymbol{O}_p(G) = 1$ thanks to Lemma 6.3), it lifts to an ordinary character of K_0. Then Lemma 2.2 implies that ψ lifts to some ordinary character since so does φ. Moreover,

$$\psi(1)_p = \varphi(1)_p/\vartheta_0(1)_p = \left|\boldsymbol{N}_{\widehat{G}}(\widehat{R}) : \widehat{R}\right|_p = \left|\boldsymbol{N}_{\overline{G}}(\bar{R}) : \bar{R}\right|_p$$

and hence we conclude that $(\bar{R}, \bar{\psi})$ belongs to $\mathrm{Alp}(\overline{G})$. This shows that the map $\Omega_{\overline{G}}$ induces a bijection

$$\Upsilon^{\widehat{G}}_{\vartheta_0} : \mathrm{IBr}(\widehat{G} \mid \vartheta_0) \to \mathrm{Alp}_r(\widehat{G} \mid \vartheta_0)/\widehat{G}$$

given by sending the Brauer character $\rho\tilde{\vartheta}_{\widehat{G}}$ to the $\widehat{G}$-orbit of the pair $(\widehat{R}, \psi\tilde{\vartheta}_{\boldsymbol{N}_{\widehat{G}}(\widehat{R})})$ whenever $(\bar{R}, \bar{\psi})$ belongs to the $\overline{G}$-orbit $\Omega_{\overline{G}}(\bar{\psi})$ as described above. Now, according to [Mur96, Corollary 1.5 (i.b)], we can find a defect group $\overline{U}$ of $\mathrm{bl}(\bar{\psi})$ and a defect group D of $\mathrm{bl}(\varphi)$ such that $\overline{U} \leq DK_0/K_0$. On the other hand, since $\widehat{R}/K_0$ is a radical p-subgroup of $\widehat{G}/K_0$, we deduce that $\widehat{R}/K_0 \leq \overline{U}$ thanks to [Nav98, Theorem 4.8]. It follows that $\boldsymbol{C}_{\hat{A}}(D)K_0/K_0 \leq \boldsymbol{N}_{\hat{A}}(\widehat{R})/K_0$ and therefore $\boldsymbol{C}_{\hat{A}_\rho}(D) \leq \boldsymbol{N}_{\hat{A}}(\widehat{R})_\rho = \boldsymbol{N}_{\hat{A}}(\widehat{R})_\psi$. We can now apply Lemma 3.16 to the block isomorphisms (4) to get

$$(\hat{A}_\tau, \widehat{G}, \tau) \succeq_b \left(\boldsymbol{N}_{\hat{A}_\varphi}(\widehat{R}), \boldsymbol{N}_{\widehat{G}}(\widehat{R}), \varphi\right), \tag{5}$$

where recall that $\tau = \rho\tilde{\vartheta}_{\widehat{G}}$ and $\varphi = \psi\tilde{\vartheta}_{\boldsymbol{N}_{\widehat{G}}(\widehat{R})}$. Observe also that the bijection $\Upsilon^{\widehat{G}}_{\vartheta_0}$ is $\hat{A}$-equivariant because the Brauer character $\tilde{\vartheta}_{\widehat{G}}$ is $\hat{A}$-invariant and using the equivariance properties of $\Omega_{\overline{G}}$. Next, for every $\tau \in \mathrm{IBr}(\widehat{G} \mid \vartheta_0)$ and $(\widehat{R}, \varphi) \in \Upsilon^{\widehat{G}}_{\vartheta_0}(\tau)$, we know by Definition 3.2 that $Z \leq \ker(\tau)$ if and only if $Z \leq \ker(\varphi)$. As a consequence, $\Upsilon^{\widehat{G}}_{\vartheta_0}$ restricts to a bijection

$$\mathrm{IBr}\big(\widehat{G} \mid \vartheta_0 \times 1_Z\big) \to \mathrm{Alp}_r\big(\widehat{G} \mid \vartheta_0 \times 1_Z\big)/\widehat{G}.$$

Finally, observe that $\boldsymbol{N}_{\widehat{G}}(\widehat{R}) = \widehat{\boldsymbol{N}_G(R)}$ and hence $\boldsymbol{N}_G(R)$ is isomorphic to $\boldsymbol{N}_{\widehat{G}}(\widehat{R})/Z$. Therefore the epimorphism ϵ maps the character sets $\mathrm{IBr}(\widehat{G} \mid \vartheta_0 \times 1_Z)$ and $\mathrm{Alp}_r(\widehat{G} \mid \vartheta_0 \times 1_Z)/\widehat{G}$ onto $\mathrm{IBr}(G \mid \vartheta)$ and $\mathrm{Alp}_r(G \mid \vartheta)/G$ respectively. In this way we can then construct an A-equivariant bijection

$$\Upsilon^G_\vartheta : \mathrm{IBr}(G \mid \vartheta) \to \mathrm{Alp}_r(G \mid \vartheta)/G$$

as required in the statement. To prove the condition on block isomorphisms let $\eta \in \mathrm{IBr}(G \mid \vartheta)$ and $(R, \chi) \in \Upsilon^G_\vartheta(\eta)$. Since $\boldsymbol{C}_{\hat{A}}(\widehat{G})/Z = \boldsymbol{C}_A(G)$ by Lemma 3.12, we can apply Lemma 3.15 to the block isomorphism (5) in order to get

$$(A_\eta, G, \eta) \succeq_b \left(\boldsymbol{N}_A(R)_\chi, \boldsymbol{N}_G(R), \chi\right),$$

as desired. □

We now combine the bijections obtain in the lemma above for the various $\vartheta \in \mathrm{dz}^\circ(K)$. Denote by $\mathrm{Alp}_\mathrm{r}(G \mid \mathrm{dz}^\circ(K))$ and $\mathrm{IBr}(G \mid \mathrm{dz}^\circ(K))$ the union of the sets $\mathrm{Alp}_\mathrm{r}(G \mid \vartheta)$ and $\mathrm{IBr}(G \mid \vartheta)$ respectively, for ϑ running in the set $\mathrm{dz}^\circ(K)$. Since the set $\mathrm{dz}^\circ(K)$ is stable under G-conjugation, we deduce that G acts on the set $\mathrm{Alp}_\mathrm{r}(G \mid \mathrm{dz}^\circ(K))$ and denote by $\mathrm{Alp}_\mathrm{r}(G \mid \mathrm{dz}^\circ(K))/G$ the corresponding set of G-orbits.

Proposition 6.8. *Suppose that Hypothesis 6.2 holds for the pair* (G, A) *and let* $K \leq G$ *with* $K \trianglelefteq A$. *If* $|G : K\mathbf{Z}(G)| < |G : \mathbf{Z}(G)|$, *then there exists an A-equivariant bijection*

$$\Upsilon_K^G : \mathrm{IBr}(G \mid \mathrm{dz}^\circ(K)) \to \mathrm{Alp_r}(G \mid \mathrm{dz}^\circ(K))/G$$

such that

$$\left(A_\eta, G, \eta\right) \succeq_b \left(\mathbf{N}_A(R)_\chi, \mathbf{N}_G(R), \chi\right)$$

for every $\eta \in \mathrm{IBr}(G \mid \vartheta)$ *and* $(R, \chi) \in \Upsilon_K^G(\eta)$.

Proof. Let $\mathcal{U}$ be an A-transversal in the set $\mathrm{dz}^\circ(K)$ and observe that for each $\vartheta \in \mathcal{U}$, by applying Lemma 6.7 to $G_\vartheta \trianglelefteq A_\vartheta$, there exists an A_ϑ-equivariant bijection

$$\Upsilon_\vartheta^{G_\vartheta} : \mathrm{IBr}(G_\vartheta \mid \vartheta) \to \mathrm{Alp_r}(G_\vartheta \mid \vartheta)/G_\vartheta$$

such that

$$\left(A_{\vartheta,\nu}, G_\vartheta, \nu\right) \succeq_b \left(\mathbf{N}_{A_\vartheta}(R)_\psi, \mathbf{N}_{G_\vartheta}(R), \psi\right) \tag{6}$$

for every $\nu \in \mathrm{IBr}(G_\vartheta \mid \vartheta)$ and $(R, \psi) \in \Upsilon_\vartheta^{G_\vartheta}(\nu)$. Next, choose an A_ϑ-transversal $\S_\vartheta$ in $\mathrm{IBr}(G_\vartheta \mid \vartheta)$ and observe that the equivariance properties of $\Upsilon_\vartheta^{G_\vartheta}$ imply that the image $\mathbb{T}_\vartheta := \Upsilon_\vartheta^{G_\vartheta}(\S_\vartheta)$ is an A_ϑ-transversal in $\mathrm{Alp_r}(G_\vartheta \mid \vartheta)/G_\vartheta$. By using [Nav98, Theorem 8.9] we deduce that the set $\mathcal{S}_\vartheta$ of characters of the form $\eta = \nu^G$ with $\nu \in \S_\vartheta$ is an A_ϑ-transversal in the set $\mathrm{IBr}(G \mid \vartheta)$. Similarly, if $\mathcal{T}_\vartheta$ denotes the set of G-orbits of pairs of the form (R, χ) with $\chi = \psi^{\mathbf{N}_G(R)}$ and where the G_ϑ-orbit of (R, ψ) belongs to $\mathbb{T}_\vartheta$, then $\mathcal{T}_\vartheta$ is an A_ϑ-transversal in $\mathrm{Alp_r}(G \mid \vartheta)/G$. Finally, [Nav98, Corollary 8.7] shows that the set $\mathcal{S}$ consisting of characters η belonging to $\mathcal{S}_\vartheta$ for some $\vartheta \in \mathcal{U}$ is an A-transversal in $\mathrm{IBr}(G \mid \mathrm{dz}^\circ(K))$. Likewise, the set $\mathcal{T}$ consisting of G-orbits $\overline{(R, \chi)} \in \mathcal{T}_\vartheta$ for some $\vartheta \in \mathcal{U}$ is an A-transversal in $\mathrm{Alp_r}(G \mid \mathrm{dz}^\circ(K))/G$. We then obtain an A-equivariant bijection Υ_K^G by setting

$$\Upsilon_K^G(\eta^x) := \overline{(R, \chi)}^x$$

for all $x \in A$ and every $\eta \in \mathcal{S}$ and $\overline{(R, \chi)} \in \mathcal{T}$ such that there exists some $\vartheta \in \mathcal{U}$ for which $\eta \in \mathcal{S}_\vartheta$, $\overline{(R, \chi)} \in \mathcal{T}_\vartheta$ and $(R, \psi) \in \Upsilon_\vartheta^{G_\vartheta}(\nu)$, where $\nu \in \mathrm{IBr}(G_\vartheta \mid \vartheta)$ and $\psi \in \mathrm{IBr}(\mathbf{N}_G(R)_\vartheta \mid \vartheta)$ are the Clifford correspondents of η and χ respectively.

It remains to prove the condition on block isomorphisms of modular character triples from the statement. Let η, ν, χ, and ψ be as in the previous paragraph and observe that by Lemma 3.6 (ii) it is enough to show that

$$\left(A_\eta, G, \eta\right) \succeq_b \left(\mathbf{N}_A(R)_\chi, \mathbf{N}_G(R), \chi\right)$$

for our choice of η and (R, χ). This condition follows by applying Lemma 3.8 since the block isomorphism of modular character triples from (6) is satisfied with respect to our choice of ν and ψ. This completes the proof. □

Using Proposition 6.8 we can now construct a bijection from $\mathrm{Alp}(G \mid K)/G$ to an intermediate set $\mathcal{W}_{\mathrm{r}}(G \mid K)/G$ that we now define. Recall that for any finite group H we denote by $\mathrm{Rad}^{\circ}(H)$ the set of radical p-subgroups Q of H such that $(Q, \psi) \in \mathrm{Alp}(H)$ for some $\psi \in \mathrm{dz}^{\circ}(\mathbf{N}_H(Q))$.

Definition 6.9. Let $K \trianglelefteq G$ be finite groups. We denote by $\mathcal{W}_{\mathrm{r}}(G \mid K)$ the set of triples (Q, R, χ) where Q is a radical p-subgroup in $\mathrm{Rad}^{\circ}(K)$ and the pair (R, χ) belongs to the set $\mathrm{Alp}_{\mathrm{r}}(\mathbf{N}_G(Q) \mid \mathrm{dz}^{\circ}(\mathbf{N}_K(Q)))$ introduced in Definition 6.6 (see also the comment before Proposition 6.8). Once again the group G acts by conjugation on $\mathcal{W}_{\mathrm{r}}(G \mid K)$ and we denote by $\mathcal{W}_{\mathrm{r}}(G \mid K)/G$ the set of G-orbits.

We now construct a bijection between the set $\mathrm{Alp}(G \mid K)/G$ (from Definition 4.4) and the set $\mathcal{W}_{\mathrm{r}}(G \mid K)/G$.

Theorem 6.10. *Suppose that Hypothesis 6.2 holds for the pair (G, A) and let K be a subgroup of G with $K \trianglelefteq A$ and $K \not\leq \mathbf{Z}(G)$. Then there exists an A-equivariant bijection*

$$\Psi_K^G : \mathrm{Alp}(G \mid K)/G \to \mathcal{W}_{\mathrm{r}}(G \mid K)/G$$

such that

$$\big(\mathbf{N}_A(Q)_\eta, \mathbf{N}_G(Q), \eta\big) \succeq_b \big(\mathbf{N}_A(Q, R)_\chi, \mathbf{N}_G(Q, R), \chi\big) \tag{7}$$

for every $(Q, \eta) \in \mathrm{Alp}(G \mid K)$ and every $(Q, R, \chi) \in \mathcal{W}_{\mathrm{r}}(G \mid K)$ whose G-orbits correspond via the bijection Ψ_K^G.

Proof. To start, let $\mathbb{T}$ be an A-transversal in $\mathrm{Rad}^{\circ}(K)$ and observe that, for every $Q \in \mathbb{T}$, we have $|\mathbf{N}_G(Q) : \mathbf{N}_K(Q)\mathbf{Z}(\mathbf{N}_G(Q))| \leq |\mathbf{N}_G(Q) : \mathbf{N}_K(Q)\mathbf{Z}(G)| \leq |G : K\mathbf{Z}(G)| < |G : \mathbf{Z}(G)|$. We can then apply (the argument of) Proposition 6.8 to $\mathbf{N}_G(Q) \trianglelefteq \mathbf{N}_A(Q)$ to construct an $\mathbf{N}_A(Q)$-equivariant bijection

$$\Upsilon_{\mathbf{N}_K(Q)}^{\mathbf{N}_G(Q)} : \mathrm{IBr}\big(\mathbf{N}_G(Q) \mid \mathrm{dz}^{\circ}(\mathbf{N}_K(Q))\big) \to \mathrm{Alp}_{\mathrm{r}}(\mathbf{N}_G(Q) \mid \mathrm{dz}^{\circ}(\mathbf{N}_K(Q)))/\mathbf{N}_G(Q)$$

such that

$$\big(\mathbf{N}_A(Q)_\eta, \mathbf{N}_G(Q), \eta\big) \succeq_b \big(\mathbf{N}_A(Q, R)_\chi, \mathbf{N}_G(Q, R)_\chi, \chi\big)$$

for every $\eta \in \mathrm{IBr}(\mathbf{N}_G(Q) \mid \mathrm{dz}^{\circ}(\mathbf{N}_K(Q)))$ and $(R, \chi) \in \Upsilon_{\mathbf{N}_K(Q)}^{\mathbf{N}_G(Q)}(\eta)$. Next, let $\S_Q$ be an $\mathbf{N}_A(Q)$-transversal in $\mathrm{IBr}(\mathbf{N}_G(Q) \mid \mathrm{dz}^{\circ}(\mathbf{N}_K(Q)))$ and consider its image $\mathcal{S}_Q$ under the bijection $\Upsilon_{\mathbf{N}_K(Q)}^{\mathbf{N}_G(Q)}$. Since the latter is $\mathbf{N}_A(Q)$-equivariant, it follows that $\mathcal{S}_Q$ is an $\mathbf{N}_A(Q)$-transversal in $\mathrm{Alp}_{\mathrm{r}}(\mathbf{N}_G(Q) \mid \mathrm{dz}^{\circ}(\mathbf{N}_K(Q)))/\mathbf{N}_G(Q)$. Observe that the set $\S$ consisting of G-orbits of pairs (Q, η) with $Q \in \mathbb{T}$ and $\eta \in \S_Q$ is an A-transversal in $\mathrm{Alp}(G \mid K)/G$. Similarly, the set $\mathcal{S}$ consisting of G-orbits of triples (Q, R, χ) with $Q \in \mathbb{T}$ and where the $\mathbf{N}_G(Q)$-orbit of (R, χ) belongs to $\mathcal{S}_Q$, more precisely $(R, \chi) \in \Upsilon_{\mathbf{N}_K(Q)}^{\mathbf{N}_G(Q)}(\eta)$, is an A-transversal in $\mathcal{W}_{\mathrm{r}}(G \mid K)/G$. Our construction shows that there is a bijection $\widehat{\Psi}_K^G : \S \to \mathcal{S}$ and we can therefore define an A-equivariant bijection Ψ_K^G as required in the statement above by setting

$$\Psi_K^G\big(\overline{(Q, \eta)}^x\big) := \widehat{\Psi}_K^G\big(\overline{(Q, \eta)}\big)^x$$

for every G-orbit $\overline{(Q, \eta)} \in \S$ and every $x \in A$. Observe that the desired block isomorphisms of modular character triples are given directly by the properties of the bijections $\Upsilon_{\mathbf{N}_K(Q)}^{\mathbf{N}_G(Q)}$. □

In our final step we use the results on the Dade–Glauberman–Nagao correspondence obtained in Section 5, and in particular Theorem 5.6, to construct a bijection between $\mathcal{W}_{\rm r}(G \mid K)/G$ and $\mathrm{Alp}(G)/G$. Observe that this step of our proof is independent on the inductive hypothesis and holds in full generality.

Theorem 6.11. *Let $K \leq G \leq A$ be finite groups with $K, G \trianglelefteq A$. Then, there exists an A-equivariant bijection*

$$\Lambda_K^G : \mathcal{W}_{\rm r}(G \mid K)/G \to \mathrm{Alp}(G)/G$$

such that

$$\big(\mathbf{N}_A(Q, R)_\chi, \mathbf{N}_G(Q, R), \chi\big) \succeq_b (\mathbf{N}_A(D)_\nu, \mathbf{N}_G(D), \nu)$$

for every $(Q, R, \chi) \in \mathcal{W}_{\rm r}(G \mid K)$ and every $(D, \nu) \in \Lambda_K^G(\overline{(Q, R, \chi)})$.

Proof. To start fix $(Q, R, \chi) \in \mathcal{W}_{\rm r}(G \mid K)$. Recall that this means that Q is a radical p-subgroup of K and there is some $\psi \in \mathrm{dz}^\circ(\mathbf{N}_K(Q))$ such that $R/\mathbf{N}_K(Q)$ is a radical p-subgroup of $\mathbf{N}_G(Q)_\psi/\mathbf{N}_K(Q)$ and $\chi \in \mathrm{rdz}^\circ(\mathbf{N}_G(Q, R) \mid R, \psi)$. Observe that ψ is uniquely determined by (Q, R, χ) up to $\mathbf{N}_G(Q, R)$-conjugation. Now let D/Q be a defect group of the unique block of R/Q that covers $\mathrm{bl}(\bar{\psi})$ and where $\bar{\psi}$ is the Brauer character of $\mathbf{N}_K(Q)/Q$ corresponding to ψ via inflation. By [Nav98, Theorem 9.17] we know that $R = D\mathbf{N}_K(Q)$ and $Q = D \cap \mathbf{N}_K(Q) = D \cap K$. In particular $\mathbf{N}_K(D) = \mathbf{N}_{\mathbf{N}_K(Q)}(D)$ and, because ψ is D-invariant, we can define the Brauer character $\xi := \pi_D(\psi) \in \mathrm{dz}^\circ(\mathbf{N}_K(D))$ as in Definition 5.5. Next, notice that the Brauer character χ determines a unique $\chi_\psi \in \mathrm{IBr}(\mathbf{N}_G(Q, R)_\psi)$ lying above ψ according to [Nav98, Theorem 8.9]. Moreover, using the fact that $\chi \in \mathrm{rdz}^\circ(\mathbf{N}_G(Q, R) \mid R, \psi)$, we deduce that χ_ψ belongs to the set $\mathrm{rdz}^\circ(\mathbf{N}_G(Q, R)_\psi \mid R, \psi)$. Notice furthermore that $\mathbf{N}_{\mathbf{N}_G(Q,R)_\psi}(D) = \mathbf{N}_G(D)_\xi$ and that a Frattini argument yields $\mathbf{N}_G(Q, R)_\psi = R\mathbf{N}_G(D)_\xi = \mathbf{N}_K(Q)\mathbf{N}_G(D)_\xi$. Now, if

$$\Delta_{D,\psi}^{\mathbf{N}_G(Q,R)_\psi} : \mathrm{rdz}^\circ(\mathbf{N}_G(Q, R)_\psi \mid R, \psi) \to \mathrm{dz}^\circ(\mathbf{N}_G(D)_\xi \mid \xi)$$

denotes the bijection given by Theorem 5.6, then we define $\nu_\xi := \Delta_{D,\psi}^{\mathbf{N}_G(Q,R)_\psi}(\chi_\psi)$ and set $\nu := (\nu_\xi)^{\mathbf{N}_G(D)}$ which is an irreducible Brauer character belonging to $\mathrm{dz}^\circ(\mathbf{N}_G(D))$ (see [Nav98, Theorem 8.9]). We now use the above argument to construct an equivariant map Λ_K^G with the properties required in the statement above. Observe that because $\Delta_{D,\psi}^{\mathbf{N}_G(Q,R)_\psi}$ is not canonical, our construction will have to depend on some choices. We now make these choices explicit.

Pick an A-transversal $\mathbb{T}$ in the set of radical p-subgroups of K and, for each $Q \in \mathbb{T}$, choose an $\mathbf{N}_A(Q)$-transversal $\mathbb{T}^{(Q)}$ in the set of Brauer characters $\mathrm{dz}^\circ(\mathbf{N}_K(Q))$. Now, for each $\psi \in \mathbb{T}^{(Q)}$ we fix an $\mathbf{N}_A(Q)_\psi$-transversal $\mathbb{T}^{(Q,\psi)}$ in the set of radical p-subgroups $\mathrm{Rad}(\mathbf{N}_G(Q)_\psi/\mathbf{N}_K(Q))$. Each element of $\mathbb{T}^{(Q,\psi)}$ can be written as $R/\mathbf{N}_K(Q)$ for some $\mathbf{N}_K(Q) \leq R \leq \mathbf{N}_G(Q)_\psi$. Moreover, if D/Q is a defect group of the unique block of R/Q covering $\mathrm{bl}(\bar{\psi})$ with $\bar{\psi}$ the Brauer character of $\mathbf{N}_K(Q)/Q$ corresponding to ψ via inflation, then [Nav98, Theorem 9.17] tells us that $R = D\mathbf{N}_K(Q)$ and $Q = D \cap K$. Observe that D is uniquely determined up to $\mathbf{N}_K(Q)$-conjugation. In order to fix a choice of D, for every $R/\mathbf{N}_K(Q) \in \mathbb{T}^{(Q,\psi)}$ we pick an $\mathbf{N}_A(Q, R)_\psi$-transversal $\mathbb{T}^{(Q,\psi,R)}$ in the set of defect groups of the unique block of R/Q covering the block of $\bar{\psi}$. Finally, for each $D \in \mathbb{T}^{(Q,\psi,R)}$, observe that $\mathbf{N}_A(D)_\psi = \mathbf{N}_A(Q, R, D)_\psi$ and choose an $\mathbf{N}_A(D)_\psi$-transversal $\mathbb{T}^{(Q,\psi,R,D)}$ in $\mathrm{rdz}^\circ(\mathbf{N}_G(Q, R) \mid R, \psi)$. Then, the set $\mathcal{T}$ of G-orbits

$\overline{(Q,R,\chi)}$ with $Q\in\mathbb{T}$ and where $R/N_K(Q)\in\mathbb{T}^{(Q,\psi)}$ and $\chi\in\mathbb{T}^{(Q,\psi,R,D)}$, for some $\psi\in\mathbb{T}^{(Q)}$ and $D\in\mathbb{T}^{(Q,\psi,R)}$, is an A-transversal in $\mathcal{W}_{\rm r}(G\mid K)/G$.

We claim that the set $\S$ of groups D belonging to $\mathbb{T}^{(Q,\psi,R)}$ for some $Q\in\mathbb{T}$, $\psi\in\mathbb{T}^{(Q)}$, and $R\in\mathbb{T}^{(Q,\psi)}$ is an A-transversal in $\mathrm{Rad}^\circ(G)$. To see this, observe first that two distinct elements of $\S$ cannot be A-conjugate. On the other hand, let $D'\in\mathrm{Rad}^\circ(G)$ and notice that $Q':=D'\cap K$ is a radical p-subgroup of K by [NT11, Lemma 2.3(a)]. Then there exist $Q\in\mathbb{T}$ and $x\in A$ such that $Q'^a=Q$. Recall that by the definition of $\mathrm{Rad}^\circ(G)$ there exists $\nu'\in\mathrm{dz}^\circ(N_G(D'))$ and that we can see $\mathrm{bl}(\nu')$ as a block of defect zero in the quotient $N_G(D')/D'$. Let c be a block of $N_{KD'}(D')/D'$ covered by $\mathrm{bl}(\nu')$. By [Nav98, Lemma 9.27] it follows that c is a block of defect zero of $N_{KD'}(D')/D'$. Moreover, let $\widehat{\xi}\in\mathrm{IBr}(N_{KD'}(D'))$ correspond to the unique Brauer character of $N_{KD'}(D')/D'$ belonging to c (see [Nav98, Theorem 3.18]). Since $N_{KD'}(D')/D'\simeq N_K(D')/Q'$ we deduce that the character $\widehat{\xi}$ restricts irreducibly to $\xi\in\mathrm{IBr}(N_K(D'))$ and its block $\mathrm{bl}(\xi)$ has defect zero when regarded as a block of $N_K(D')/Q'$. Thus ξ belongs to $\mathrm{dz}^\circ(N_K(D'))$ and is D'-invariant. Since $N_K(D')=N_K(Q',D')$, we can then write $\xi=\pi_{D'}(\psi')$ for a unique $\psi'\in\mathrm{dz}^\circ(N_K(Q'))$. It follows that ψ'^a belongs to $\mathrm{dz}^\circ(N_K(Q))$ and there exist $\psi\in\mathbb{T}^{(Q)}$ and $x\in N_A(Q)$ such that $\psi'^{ax}=\psi$. Define $R':=N_{KD'}(Q')=N_K(Q')D'$ and observe that $R'/N_K(Q')$ is a radical p-subgroup of $N_G(Q')_{\psi'}/N_K(Q')$ and that D'/Q' is a defect group of the unique block of R'/Q' covering $\mathrm{bl}(\psi')$, where the latter is regarded as a block of $N_K(Q')/Q'$. As a consequence $R'^{ax}/N_K(Q)$ is a radical p-subgroup of $N_G(Q)_\psi/N_K(Q)$ and we can find $R/N_K(Q)\in\mathbb{T}^{(Q,\psi)}$ and $y\in N_A(Q)_\psi$ such that $R'^{axy}=R$. Moreover, D'^{axy}/Q is a defect group of the unique block of R/Q covering $\mathrm{bl}(\psi)$, viewed as a block of $N_K(Q)/Q$, and hence there is $D\in\mathbb{T}^{(Q,\psi,R)}$ and $z\in N_A(Q,R)_\psi$ such that $D'^{axyz}=D$. This shows that D' is A-conjugate to $D\in\S$ and so $\S$ is an A-transversal in $\mathrm{Rad}^\circ(G)$ as claimed.

We now want to construct an A-transversal in $\mathrm{Alp}(G)/G$ in bijection with $\mathcal{T}$. By [Nav98, Theorem 8.9] each $\chi\in\mathbb{T}^{(Q,\psi,R,D)}$ is induced by a unique Brauer character $\chi_\psi\in\mathrm{IBr}(N_G(Q,R)_\psi\mid\psi)$. As explained in the previous paragraph, we have that $\chi_\psi\in\mathrm{rdz}^\circ(N_G(Q,R)_\psi\mid R,\psi)$ and it follows that the set $\widehat{\mathbb{T}}^{(Q,\psi,R,D)}$ of all Brauer characters χ_ψ for $\chi\in\mathbb{T}^{(Q,\psi,R,D)}$ is an $N_A(D)_\psi$-transversal in $\mathrm{rdz}^\circ(N_G(Q,R)_\psi\mid R,\psi)$. Since the map $\Delta_{D,\psi}^{N_G(Q,R)_\psi}$ from Theorem 5.6 is $N_A(D)_\psi$-equivariant and $N_A(D)_\psi=N_A(D)_{\pi_D(\psi)}$, it follows that the image $\widehat{\S}^{(Q,\psi,R,D)}$ of $\widehat{\mathbb{T}}^{(Q,\psi,R,D)}$ under $\Delta_{D,\psi}^{N_G(Q,R)_\psi}$ is an $N_A(D)_\psi$-transversal in $\mathrm{dz}^\circ(N_G(D)_{\pi_D(\psi)}\mid\pi_D(\psi))$. As before, by applying [Nav98, Theorem 9.14] we deduce that the set $\S^{(Q,\psi,R,D)}$ of Brauer characters $\nu:=\vartheta^{N_G(D)}$ for $\vartheta\in\widehat{\S}^{(Q,\psi,R,D)}$ is an $N_A(D)_\psi$-transversal in the set $\mathrm{dz}^\circ(N_G(D)\mid\pi_D(\psi))$. We can now conclude that the set $\mathcal{S}$ of G-orbits $\overline{(D,\nu)}$ with $D\in\mathbb{T}^{(Q,\psi,R)}$ and $\nu\in\S^{(Q,\psi,R,D)}$, for some $Q\in\mathbb{T}$, $\psi\in\mathbb{T}^{(Q)}$, and $R\in\mathbb{T}^{(Q,\psi)}$, is an A-transversal in $\mathrm{Alp}(G)/G$. In addition, there is a bijection between $\mathcal{T}$ and $\mathcal{S}$ given by mapping the G-orbit of (Q,R,χ) to that of (D,ν) whenever $Q\in\mathbb{T}$ and there is some $\psi\in\mathbb{T}^{(Q)}$ such that $R/N_K(Q)\in\mathbb{T}^{(Q,\psi)}$, $D\in\mathbb{T}^{(Q,\psi,R)}$, $\chi\in\mathbb{T}^{(Q,\psi,R,D)}$ and $\nu\in\S^{(Q,\psi,R,D)}$ with χ corresponding to ν as described above. We define the map Λ_K^G by setting

$$\Lambda_K^G\big(\overline{(Q,R,\chi)}^x\big):=\overline{(D,\nu)}^x$$

for every $\overline{(Q,R,\chi)}\in\mathcal{T}$ corresponding to $\overline{(D,\nu)}\in\mathcal{S}$ and every $x\in A$. This defines an A-equivariant bijection between $\mathcal{W}_{\rm r}(G\mid K)/G$ and $\mathrm{Alp}(G)/G$. To conclude, we need to show that

$$\big(N_A(Q,R)_\chi, N_G(Q,R), \chi\big) \succeq_b \big(N_A(D)_\nu, N_G(D), \nu\big). \tag{8}$$

First, let $\chi_\psi \in \widehat{\mathbb{T}}^{(Q,\psi,R,D)}$ and $\nu_\psi \in \widehat{\mathbb{S}}^{(Q,\psi,R,D)}$ such that $\chi = (\chi_\psi)^{N_G(Q,R)}$ and $\nu = (\nu_\psi)^{N_G(D)}$. By construction, we know that ν_ψ is the image of χ_ψ under the bijection $\Delta_{D,\psi}^{N_G(Q,R)_\psi}$ and hence Theorem 5.6 yields

$$\Big(N_A(Q,R)_{\psi,\chi_\psi}, N_G(Q,R)_\psi, \chi_\psi\Big) \succeq_b \big(N_A(D)_{\psi,\nu_\psi}, N_G(D)_\psi, \nu_\psi\big),$$

from which (8) follows thanks to Lemma 3.8. Observe that the latter can be applied since $N_A(Q,R)_\chi = N_G(Q,R)N_A(Q,R)_{\psi,\chi}$ by a Frattini argument and using Clifford's theorem [Nav98, Corollary 8.7]. This concludes the proof. □

Finally, we can prove Theorem 6.1 as a consequence of Theorem 4.5, Proposition 6.5, Theorem 6.10, and Theorem 6.11.

Proof of Theorem 6.1. We consider a counterexample G to Theorem 6.1 and assume that Conjecture 4.1 fails to hold for a choice $G \trianglelefteq A$. We further assume that G and A have been minimized with respect to $|G : \mathbf{Z}(G)|$ first and then $|A|$. As explained at the beginning of this section, it follows that Hypothesis 6.2 holds for the pair (G, A). By Proposition 6.5 there exists a subgroup K of G with $K \trianglelefteq A$ and $K \not\leq \mathbf{Z}(G)$ such that Conjecture 4.1 holds for $K \trianglelefteq A$. Now, we can apply Theorem 4.5 with $J = G$ to obtain an A-equivariant bijection

$$\Omega_K^G : \mathrm{IBr}(G) \to \mathrm{Alp}(G \mid K)/G$$

such that

$$\big(A_\varphi, G, \varphi\big) \succeq_b \big(N_A(Q)_\eta, N_G(Q), \eta\big) \tag{9}$$

for every $\varphi \in \mathrm{IBr}(G)$ and $(Q,\eta) \in \Omega_K^G(\varphi)$. On the other hand, combining the bijections Ψ_K^G and Λ_K^G given by Theorem 6.10 and Theorem 6.11 respectively, we obtain an A-equivariant bijection

$$\Lambda_K^G \circ \Psi_K^G : \mathrm{Alp}(G \mid K)/G \to \mathcal{W}_{\mathrm{r}}(G \mid K)/G \to \mathrm{Alp}(G)/G$$

such that

$$\big(N_A(Q)_\eta, N_G(Q), \eta\big) \succeq_b \big(N_A(Q,R)_\chi, N_G(Q,R)_\chi, \chi\big) \succeq_b (N_A(D)_\nu, N_G(D)_\nu, \nu) \tag{10}$$

whenever $(Q,R,\chi) \in \Psi_K^G(\overline{(Q,\eta)})$ and $(D,\nu) \in \Lambda_K^G(\overline{(Q,R,\chi)})$. We conclude that the map $\Omega := \Lambda_K^G \circ \Psi_K^G \circ \Omega_K^G$ satisfies the requirements of Conjecture 4.1 by (9) and (10) thanks to the transitivity of the relation $\succeq_b$. This contradicts the choice of G and A and the proof is now complete. □

6.1. ***A reduction in the block-free case.*** When proving that two modular character triple isomorphisms are block isomorphic it is necessary, in particular, to show that they are also central isomorphic. With this in mind, an inspection of the proofs of the lemmas in Section 3 shows that all those statements admit a version where the block isomorphisms are replaced by central isomorphisms. Similarly, we can state a block-free version of Conjecture B as follows.

Conjecture 6.12. *Let $G \trianglelefteq A$ be finite groups. Then there exists an A-equivariant bijection*

$$\Omega : \mathrm{IBr}(G) \to \mathrm{Alp}(G)/G$$

such that

$$(A_\vartheta, G, \vartheta) \succeq_c \big(\mathbf{N}_A(Q)_\psi, \mathbf{N}_G(Q), \psi\big)$$

for every $\vartheta \in \mathrm{IBr}(G)$ and $(Q, \psi) \in \Omega(\vartheta)$.

Proceeding as in Section 6 we can then obtain the following block-free version of Theorem C.

Theorem 6.13. *Let G be a finite group and p a prime number. Suppose that Conjecture 6.12 holds at the prime p for every covering group of any nonabelian finite simple group of order divisible by p involved in G. Then Conjecture 6.12 holds for G at the prime p.*

Proof. The proofs of Lemma 6.3, Corollary 6.4, and Proposition 6.5 show that in a minimal counterexample $G \trianglelefteq A$ we can find a normal subgroup K of A contained in G with $K \not\leq \mathbf{Z}(G)$ and such that Conjecture 6.12 holds for $K \trianglelefteq A$. The argument used to prove Theorem 4.5 now yields an A-equivariant bijection

$$\Omega_K^G : \mathrm{IBr}(G) \to \mathrm{Alp}(G \mid K)/G$$

such that

$$\big(A_\chi, G, \chi\big) \succeq_c \big(\mathbf{N}_A(Q)_\eta, \mathbf{N}_G(Q), \eta\big)$$

for every $\chi \in \mathrm{IBr}(G)$ and $(Q, \eta) \in \Omega_K^G(\chi)$. Next, proceeding as in Lemma 6.7, Proposition 6.8, and Theorem 6.10 we construct an A-equivariant bijection

$$\Psi_K^G : \mathrm{Alp}(G \mid K)/G \to \mathcal{W}_\mathrm{r}(G \mid K)/G$$

such that

$$\big(\mathbf{N}_A(Q)_\eta, \mathbf{N}_G(Q), \eta\big) \succeq_c \big(\mathbf{N}_A(Q, R)_\chi, \mathbf{N}_G(Q, R), \chi\big)$$

for every $(Q, \eta) \in \mathrm{Alp}(G \mid K)$ and every $(Q, R, \chi) \in \mathcal{W}_\mathrm{r}(G \mid K)$ whose G-orbits correspond via the bijection Ψ_K^G. Finally, recalling that block isomorphisms of modular character triples are central isomorphisms, we conclude by combining the bijections Ω_K^G and Ψ_K^G and applying Theorem 6.11. □

7. Application to Navarro's conjecture

In [Nav17], Navarro introduced a new conjecture (see [Nav17, Conjecture E]) that unifies the Alperin weight conjecture and the Glauberman correspondence into a single statement. In our paper we are mainly interested in the blockwise version of this statement that was introduced in Conjecture A and which we recall below. Recall that if $G \trianglelefteq \Gamma$ and B is a block of G, then we denote by $\mathrm{IBr}_\Gamma(B)$ the set of Γ-invariant irreducible Brauer characters of G that belongs to B.

Conjecture 7.1 (Navarro). *Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group. For every Γ-invariant p-block B of G, we have*

$$|\mathrm{IBr}_\Gamma(B)| = \sum_{Q \in \Theta_B/\Gamma} \big|\mathrm{dz}\,(\mathbf{N}_\Gamma(Q)/Q \mid B)\big|, \tag{11}$$

where Θ_B is the set of p-subgroups Q of Γ such that $\Gamma = GQ$ and $Q \cap G$ is contained in some defect group of the block B, and $\mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q \mid B)$ is the set of irreducible characters $\bar{\vartheta} \in \mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q)$ such that $\mathrm{bl}(\vartheta)^\Gamma$ covers B and where $\vartheta \in \mathrm{Irr}(\boldsymbol{N}_G(Q))$ corresponds to $\bar{\vartheta}$ via inflation of characters.

As in the case of [Nav17, Conjecture E], this statement unifies into a single statement both the blockwise Alperin weight conjecture and the Dade–Glauberman–Nagao correspondence. In fact, Conjecture 7.1 becomes the blockwise Alperin weight conjecture when $G = \Gamma$, and implies the count of the Dade–Glauberman–Nagao correspondence (see [NT11, Theorem 4.1]) when considering blocks of defect zero. We prove the latter implication in the following lemma.

Lemma 7.2. *Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a (possibly empty) set $\mathcal{S}$ of representatives for the Γ-conjugacy classes of complements of G in Γ. If Conjecture 7.1 holds for every Γ-invariant block of defect zero of G, then*

$$|\mathrm{dz}_\Gamma(G)| = \sum_{Q \in \mathcal{S}} \left|\mathrm{dz}(\boldsymbol{C}_G(Q))\right|,$$

where we denote by $\mathrm{dz}_\Gamma(G)$ the set of Γ-invariant characters in $\mathrm{dz}(G)$.

Proof. First, notice that the number of Γ-invariant defect zero characters of G coincides with the number of Γ-invariant irreducible Brauer characters belonging to some block B of defect zero of G, that is

$$|\mathrm{dz}_\Gamma(G)| = \sum_{B} \left|\mathrm{IBr}_\Gamma(B)\right| \tag{12}$$

where the sum runs over all Γ-invariant blocks B of defect zero of G. On the other hand, for each such block B, observe that $\mathcal{S}$ is a representative set for the Γ-orbits on Θ_B as defined in Conjecture 7.1. In particular, if $Q \in \Theta_B$, then we have $\boldsymbol{N}_\Gamma(Q) = \boldsymbol{C}_G(Q) \times Q$ and it follows that $\mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q)$ is in bijection with $\mathrm{dz}(\boldsymbol{C}_G(Q))$. Next, for $Q \in \mathcal{S}$, let $\bar{\vartheta} \in \mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q)$, consider its inflation ϑ to $\boldsymbol{N}_\Gamma(Q)$, and set $C := \mathrm{bl}(\vartheta)^\Gamma$. Since $\boldsymbol{N}_\Gamma(Q) = \boldsymbol{C}_G(Q) \times Q$ we can write $\vartheta = \varphi \times 1_Q$ for some $\varphi \in \mathrm{dz}(\boldsymbol{C}_G(Q))$ and hence Q is a defect group of $\mathrm{bl}(\vartheta)$. We deduce from [NS14a, Lemma 2.1] that C covers a Γ-invariant block B of G with defect $G \cap Q = 1$. This shows that for every $Q \in \mathcal{S}$ each character of $\mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q)$ belongs to some set $\mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q \mid B)$ for some Γ-invariant block B of defect zero of G. Now, Conjecture 7.1 implies that the right hand side of (12) coincides with

$$\sum_{B} \sum_{Q \in \mathcal{S}} \left|\mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q \mid B)\right| = \sum_{Q \in \mathcal{S}} \left|\mathrm{dz}(\boldsymbol{N}_\Gamma(Q)/Q)\right| = \sum_{Q \in \mathcal{S}} \left|\mathrm{dz}(\boldsymbol{C}_G(Q))\right| \tag{13}$$

where B runs over all Γ-invariant blocks of defect zero of G. Combining (12) and (13) we obtain the desired equality. □

We now want to prove that Conjecture 7.1 follows from the inductive blockwise Alperin weight condition and hence obtain Theorem D. Together with our Theorem C, this will also yield Corollary E. In this section, we obtain all these results as consequences of a stronger theorem. In fact, we can show that the inductive blockwise Alperin weight condition implies a more general version of Conjecture 7.1 which does not require the quotient Γ/G to be a p-group. To introduce this new statement we first collect some further notation.

Let $G \trianglelefteq \Gamma$ be finite groups and consider a union of p-blocks $\mathcal{B}$ of G. We denote by $\mathrm{EBr}(\Gamma \mid \mathcal{B})$ the set of those $\chi \in \mathrm{IBr}(\Gamma)$ that are extensions of some Brauer character belonging to some block contained in $\mathcal{B}$, that is, such that $\chi_G \in \mathrm{IBr}(B)$ for some $B \in \mathcal{B}$. Similarly, we denote by $\mathrm{EBr}(\Gamma \mid \mathrm{dz}^\circ(\mathcal{B}))$ the set of those $\chi \in \mathrm{IBr}(\Gamma)$ such that $\chi_G \in \mathrm{IBr}(B) \cap \mathrm{dz}^\circ(G)$ for some $B \in \mathcal{B}$ (recall that, as defined in Section 4, a character $\psi \in \mathrm{IBr}(G)$ belongs to $\mathrm{dz}^\circ(G)$ if the corresponding character $\bar{\psi} \in \mathrm{IBr}(G/\boldsymbol{O}_p(G))$ belongs to a block of defect zero). Finally, given a p-subgroup Q of G and a p-block B of G, we denote by B_Q the union of all p-blocks b of $\boldsymbol{N}_G(Q)$ such that $b^G = B$. We can now generalize Conjecture 7.1 to arbitrary quotients Γ/G as follows.

Conjecture 7.3. *Let $G \trianglelefteq \Gamma$ be finite groups and consider a block B of G. Then*

$$|\mathrm{EBr}(\Gamma \mid B)| = \sum_{Q} \left|\mathrm{EBr}\left(\boldsymbol{N}_\Gamma(Q) \mid \mathrm{dz}^\circ\left(B_Q\right)\right)\right|$$

where Q runs over a set of representatives for the Γ-orbits of radical p-subgroups of G such that $\Gamma = G\boldsymbol{N}_\Gamma(Q)$.

As mentioned above, Conjecture 7.1 can be recovered from Conjecture 7.3 in the case where the quotient Γ/G is a p-group. We prove this fact in the following lemma.

Lemma 7.4. *Let $G \trianglelefteq \Gamma$ be finite groups and consider a p-block B of G. If Γ/G is a p-group, then*:

(i) $|\mathrm{EBr}(\Gamma \mid B)| = |\mathrm{IBr}_\Gamma(B)|$.

(ii) *If Q is a radical p-subgroup of G such that $\Gamma = G\boldsymbol{N}_\Gamma(Q)$ and* $\mathrm{EBr}(\boldsymbol{N}_\Gamma(Q) \mid \mathrm{dz}^\circ(B_Q))$ *is nonempty, then there exists some $D \in \Theta_B$, unique up to $\boldsymbol{N}_\Gamma(Q)$-conjugation, such that $Q = D \cap G$.*

(iii) *If Q and D are the p-subgroups considered in* (*ii*), *then*

$$\left|\mathrm{EBr}\left(\boldsymbol{N}_\Gamma(Q) \mid \mathrm{dz}^\circ(B_Q)\right)\right| = \left|\mathrm{dz}(\boldsymbol{N}_\Gamma(D)/D \mid B)\right|.$$

In particular, if Conjecture 7.3 holds for the block B, then so does Conjecture 7.1.

Proof. By [Nav98, Theorem 8.11] every Γ-invariant irreducible Brauer character of G admits a unique extension to Γ and therefore (i) follows. Let now Q be a radical p-subgroup of G such that $\Gamma = G\boldsymbol{N}_\Gamma(Q)$ and consider $\psi \in \mathrm{EBr}(\boldsymbol{N}_\Gamma(Q) \mid \mathrm{dz}^\circ(B_Q))$. Set $\vartheta := \psi_{\boldsymbol{N}_G(Q)}$ and observe that $\vartheta \in \mathrm{dz}^\circ(\boldsymbol{N}_G(Q))$ is $\boldsymbol{N}_\Gamma(Q)$-invariant and satisfies $\mathrm{bl}(\vartheta)^G = B$. Let $\bar{\psi}$ and $\bar{\vartheta}$ be the Brauer characters of $\boldsymbol{N}_\Gamma(Q)/Q$ and of $\boldsymbol{N}_G(Q)/Q$ corresponding to ψ and ϑ respectively via inflation. Now, if D/Q is a defect group of $\mathrm{bl}(\bar{\psi})$, then [Nav98, Theorem 9.17] implies that $(D \cap \boldsymbol{N}_G(Q))/Q$ is a defect group of $\mathrm{bl}(\bar{\vartheta})$ and that $D\boldsymbol{N}_G(Q)/Q = \boldsymbol{N}_\Gamma(Q)/Q$. In particular, $\Gamma = G\boldsymbol{N}_\Gamma(Q) = GD$. Furthermore, since $\mathrm{bl}(\bar{\vartheta})$ has defect zero, we deduce that $Q = D \cap \boldsymbol{N}_G(Q) = D \cap G$ and, recalling that Q is contained in every defect group of $\mathrm{bl}(\vartheta)$ and that $\mathrm{bl}(\vartheta)^G = B$, we conclude that $D \in \Theta_B$ thanks to [Nav98, Lemma 4.13]. This proves (ii).

We keep D, Q, ϑ, and ψ as in the previous paragraph and prove (iii). By [Nav98, Theorem 3.18] there exists a unique $\varphi \in \mathrm{Irr}(\boldsymbol{N}_G(Q)/Q)$ such that $\varphi^0 = \vartheta$. Furthermore, according to [NT11, Theorem

2.4] there exists an extension $\chi \in \operatorname{Irr}(\boldsymbol{N}_\Gamma(Q)/Q)$ of φ. Since χ belongs to $\operatorname{rdz}(\boldsymbol{N}_\Gamma(Q) \mid \chi)$, we can apply Lemma 5.1 to show that $\chi^0 \in \operatorname{IBr}(\boldsymbol{N}_\Gamma(Q))$. Then χ^0 is an extension of $\vartheta = \varphi^0$ and [Nav98, Theorem 8.11] yields $\chi^0 = \psi$. This shows that $\psi \in \operatorname{rdz}^\circ(\boldsymbol{N}_\Gamma(Q) \mid \boldsymbol{N}_\Gamma(Q), \vartheta)$. Conversely, each character of $\operatorname{rdz}^\circ(\boldsymbol{N}_\Gamma(Q) \mid \boldsymbol{N}_\Gamma(Q), \vartheta)$ is an extension of ϑ and we conclude that

$$\operatorname{EBr}(\boldsymbol{N}_\Gamma(Q) \mid \vartheta) = \operatorname{rdz}^\circ(\boldsymbol{N}_\Gamma(Q) \mid \boldsymbol{N}_\Gamma(Q), \vartheta). \tag{14}$$

Next, applying Theorem 5.6 with $A = M = G = \boldsymbol{N}_\Gamma(Q)$ and $K = \boldsymbol{N}_G(Q)$, we get a bijection

$$\Delta_{D,\vartheta}^{\boldsymbol{N}_\Gamma(Q)} : \operatorname{rdz}^\circ(\boldsymbol{N}_\Gamma(Q) \mid \boldsymbol{N}_\Gamma(Q), \vartheta) \to \operatorname{dz}^\circ(\boldsymbol{N}_\Gamma(D) \mid \pi_D(\vartheta))$$

such that if $\zeta := \Delta_{D,\vartheta}^{\boldsymbol{N}_\Gamma(Q)}(\psi)$ then $\operatorname{bl}(\zeta)^{\boldsymbol{N}_\Gamma(Q)} = \operatorname{bl}(\psi)$. On the other hand, since $\vartheta \in \operatorname{dz}^\circ(B_Q)$, applying [KS15, Theorem B] we deduce that $\operatorname{bl}(\psi)^\Gamma$ covers B and by the transitivity of block induction the same holds for $\operatorname{bl}(\zeta)^\Gamma$. In other words, if $\xi \in \operatorname{dz}(\boldsymbol{N}_\Gamma(D)/D)$ is the ordinary character such that $\xi^0 = \zeta$, then ξ belongs to $\operatorname{dz}(\boldsymbol{N}_\Gamma(D)/D \mid B)$ and the assignment $\psi \mapsto \xi$ is one-to-one. We then obtain (iii) by arguing as above and consider any $\vartheta \in \operatorname{dz}^\circ(B_Q)$. □

Next, we show that our Conjecture 7.3 follows from the inductive blockwise Alperin weight condition.

Proposition 7.5. *Let $G \trianglelefteq \Gamma$ be finite groups and consider a block B of G. If Conjecture 4.1 holds for $G \trianglelefteq \Gamma$, then Conjecture 7.3 holds for B.*

Proof. By assumption there exists a Γ-equivariant bijection Ω from $\operatorname{IBr}(G)$ to $\operatorname{Alp}(G)/G$ inducing block isomorphisms of character triples. Consider $\varphi \in \operatorname{IBr}(G)$, $(Q, \vartheta) \in \Omega(\varphi)$ and let

$$\sigma_\Gamma : \operatorname{IBr}(\Gamma \mid \varphi) \to \operatorname{IBr}(\boldsymbol{N}_\Gamma(Q) \mid \vartheta)$$

be the bijection given by Theorem 3.1 applied with $J = \Gamma$. Then, for every $\chi \in \operatorname{IBr}(\Gamma \mid \varphi)$, we know that $\operatorname{bl}(\sigma_\Gamma(\chi))^\Gamma = \operatorname{bl}(\chi)$ and that $\chi_G = \varphi$ if and only if $\sigma_\Gamma(\chi)_{\boldsymbol{N}_G(Q)} = \vartheta$. With this in mind, proceeding as in the proof of Theorem 4.5 with $A = J = \Gamma$ and $K = G$ we can construct a Γ-equivariant bijection Ω_G^Γ between the set of extensions $\operatorname{EBr}(\Gamma \mid B)$ and the set of Γ-orbits of pairs (Q, ψ) with Q a radical p-subgroup of G and $\psi \in \operatorname{EBr}(\boldsymbol{N}_\Gamma(Q) \mid \operatorname{dz}^\circ(B_Q))$. Next, we claim that if (Q, ψ) is any such pair, then $\Gamma = G\boldsymbol{N}_\Gamma(Q)$. Observe that then the constructed bijection would imply the equality of Conjecture 7.3. To prove the claim, let $\chi \in \operatorname{EBr}(\Gamma \mid B)$ and $(Q, \psi) \in \Omega_G^\Gamma(\chi)$ so that $\varphi := \chi_G \in \operatorname{IBr}(G)$, $\vartheta := \psi_{\boldsymbol{N}_G(Q)} \in \operatorname{IBr}(\boldsymbol{N}_G(Q))$ and $(Q, \vartheta) \in \Omega(\varphi)$. Then, since φ is Γ-invariant and Ω is Γ-equivariant, we deduce that Γ fixes the G-orbit of (Q, ϑ), that is, $\Gamma = G\boldsymbol{N}_\Gamma(Q)_\vartheta$. On the other hand ϑ is $\boldsymbol{N}_\Gamma(Q)$-invariant and we conclude that $\Gamma = G\boldsymbol{N}_\Gamma(Q)$ as claimed. □

We can finally prove Theorem D and Corollary E.

Proof of Theorem D. Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a block B of G. Since by assumption Conjecture 4.1 holds with respect to $G \trianglelefteq \Gamma$, we can apply Proposition 7.5 to show that Conjecture 7.3 holds for the block B. Then Lemma 7.4 implies that Conjecture A holds for the block B. □

Proof of Corollary E. Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a block B of G. By assumption Conjecture 4.1 holds at the prime p for every covering group of any nonabelian finite simple group of order divisible by p involved in G. Then Conjecture 4.1 holds with respect to $G \trianglelefteq \Gamma$ thanks to Theorem 6.1. We can then apply Theorem D to show that Conjecture A holds for the block B. □

7.1. *Navarro's conjecture and isomorphisms of character triples.* In the previous section we have introduced a generalization of [Nav17, Conjecture E] to arbitrary quotients Γ/G. On the other hand, in this section we show how [Nav17, Conjecture E] can be strengthened in a different direction, namely by showing that it is compatible with isomorphisms of character triples.

Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a Γ-invariant block B of G. We denote by $\mathrm{nav}(\Gamma \mid B)$ the subset of $\mathrm{Alp}(\Gamma)$ consisting of those pairs (Q, ψ) where Q is a radical p-subgroup of Γ such that $\Gamma = GQ$ and $\mathrm{bl}(\psi)^\Gamma$ covers B. Since B is Γ-invariant, we deduce that $\mathrm{nav}(\Gamma \mid B)$ is a Γ-stable subset of $\mathrm{Alp}(\Gamma)$ and we denote by $\mathrm{nav}(\Gamma \mid B)/\Gamma$ the corresponding set of Γ-orbits. Furthermore, observe that if $(Q, \psi) \in \mathrm{nav}(\Gamma \mid B)$ then the restriction $\psi_{N_G(Q)}$ is irreducible. This follows, for instance, by considering ψ as a Brauer character of the quotient $N_\Gamma(Q)/Q$ and noticing that, since $\Gamma = GQ$, the quotient $N_\Gamma(Q)/Q$ is isomorphic to $N_G(Q)/(Q \cap N_G(Q))$.

Conjecture 7.6. *Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a Γ-invariant block B of G. If $G, \Gamma \trianglelefteq A$, then there exists an A_B-equivariant bijection*

$$\Omega_B^\Gamma : \mathrm{IBr}_\Gamma(B) \to \mathrm{nav}(\Gamma \mid B)/\Gamma$$

such that the character triples (A_χ, G, χ) and $(N_A(Q)_\vartheta, N_G(Q), \vartheta)$ are strongly isomorphic for every $\chi \in \mathrm{IBr}_\Gamma(B)$, $(Q, \psi) \in \Omega_B^\Gamma(\chi)$ and where $\vartheta = \psi_{N_G(Q)}$.

Since the number of Γ-orbits on the set $\mathrm{nav}(\Gamma \mid B)$ coincides with the right hand side of (11) it follows that our Conjecture 7.6 implies Conjecture 7.1. Next, we show that even this strengthened form of the conjecture is a consequence of the inductive blockwise Alperin weight condition.

Theorem 7.7. *Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group and consider a Γ-invariant block B of G. Suppose in addition that $G, \Gamma \trianglelefteq A$ and that Conjecture 4.1 holds with respect to $G \trianglelefteq A$. Then Conjecture 7.6 holds for the block B with respect to $G, \Gamma \trianglelefteq A$.*

Proof. To start, we define the set $\mathcal{A}_\Gamma(B)$ consisting of those pairs $(Q, \varphi) \in \mathrm{Alp}(G)$ such that $\Gamma = GN_\Gamma(Q)_\varphi$ and $\mathrm{bl}(\varphi)^G = B$. Observe that the first condition is equivalent to requiring that the G-orbit of (Q, φ) is invariant under the action of Γ on $\mathrm{Alp}(G)/G$. Then, if we denote by $\mathcal{A}_\Gamma(B)/G$ the set of G-orbits on $\mathcal{A}_\Gamma(B)$, applying Conjecture 4.1 we obtain a bijection

$$\Omega : \mathrm{IBr}_\Gamma(B) \to \mathcal{A}_\Gamma(B)/G$$

inducing block isomorphisms of character triples. Fix now a pair $(Q, \varphi) \in \mathcal{A}_\Gamma(B)$ and observe that $N_\Gamma(Q) = N_\Gamma(Q)_\varphi$ and hence φ is $N_\Gamma(Q)$-invariant. Then, by [Nav98, Theorem 8.11], there exists a unique extension $\tau \in \mathrm{IBr}(N_\Gamma(Q))$ of φ. Let D/Q be a defect group of the block $\mathrm{bl}(\overline{\tau})$ and where $\overline{\tau}$ is the Brauer character of $N_\Gamma(Q)/Q$ corresponding to τ via inflation. By [Nav98, Theorem 9.17] we have $\Gamma = GD$ and $G \cap D = Q$, while applying [KS15, Theorem B] we deduce that $\mathrm{bl}(\tau)^\Gamma$ covers B. Consider now the unique ordinary character $\varphi' \in \mathrm{dz}(N_G(Q)/Q)$ such that $\varphi'^0 = \varphi$ and let $\tau' \in \mathrm{Irr}(N_\Gamma(Q)/Q)$ be an extension of φ' (this exists according to [NT11, Theorem 2.4]). Then, Lemma 5.1 implies that $\tau'^0 \in \mathrm{IBr}(N_\Gamma(Q))$ and the uniqueness part of [Nav98, Theorem 8.11] yields $\tau'^0 = \tau$. This shows that $\tau \in \mathrm{rdz}^\circ(N_\Gamma(Q) \mid N_\Gamma(Q), \varphi)$. We can now apply Theorem 5.6 to show that τ corresponds to a unique $\psi \in \mathrm{dz}^\circ(N_\Gamma(D) \mid \pi_D(\varphi))$. Furthermore, using the block isomorphisms given by Theorem 5.6, we also get $\mathrm{bl}(\psi)^{N_\Gamma(Q)} = \mathrm{bl}(\tau)$ and that $\psi_{N_G(D)}$ is irreducible. We then deduce that the pair (D, ψ) belongs to $\mathrm{nav}(\Gamma \mid B)$ and can define an A_B-equivariant bijection

$$\Delta : \mathcal{A}_\Gamma(B)/G \to \mathrm{nav}(\Gamma \mid B)/\Gamma$$

by sending the G-orbit of (Q, φ) to the Γ-orbit of (D, ψ) as described above. Finally, we define Ω_B^Γ to be the composition of Ω and Δ. To complete the proof it remains to show that Ω_B^Γ induces strong isomorphisms of character triples. More precisely, let $\chi \in \mathrm{IBr}_\Gamma(B)$, $(Q, \varphi) \in \Omega(\chi)$ and (D, ψ) as constructed above. First, by Conjecture 4.1 we know that (A_χ, G, χ) and $(N_A(Q)_\varphi, N_G(Q), \varphi)$ are block isomorphic and therefore strongly isomorphic. Next, we notice set $\vartheta := \psi_{N_G(D)}$ and notice that $\vartheta = \pi_D(\varphi)$. Then, applying Corollary 5.12 we get a strong isomorphism between $(N_A(Q)_\varphi, N_G(Q), \varphi)$ and $(N_A(D)_\vartheta, N_G(D), \vartheta)$ as required. □

We conclude this section with a remark on the isomorphisms of character triples considered above.

Remark 7.8. It is natural to ask whether the strong isomorphisms considered in Conjecture 7.6 are actually block isomorphisms. Unfortunately, this is not necessarily the case. The reason for this is that the condition on defect groups required by Definition 3.4 might fail in this case. Consider, for instance, the case where $A = \Gamma$ and G is a p'-group.

7.2. *Block-free version of Navarro's conjecture.* In this section, using the results of Section 6.1, we obtain block-free analogues of Theorem D and Corollary E. For the reader's convenience, we first state the block-free version of Conjecture 7.1. The following is basically the statement of [Nav17, Conjecture E], although notice that we do not require the group Γ to split over G. Recall that if $G \trianglelefteq \Gamma$ then $\mathrm{IBr}_\Gamma(G)$ denotes the set of irreducible Brauer characters of G that are Γ-invariant.

Conjecture 7.9 (Navarro). *Let $G \trianglelefteq \Gamma$ be finite groups with Γ/G a p-group. Then*

$$\left|\mathrm{IBr}_\Gamma(G)\right| = \sum_{Q \in \Theta/\Gamma} \left|\mathrm{dz}(N_\Gamma(Q)/Q)\right|,$$

where Θ is the set of p-subgroups Q of Γ such that $\Gamma = GQ$.

Next, as done in the blockwise setting, we introduce a generalization of this conjecture to arbitrary quotients Γ/G. For $G \trianglelefteq \Gamma$, let $\mathrm{EBr}(\Gamma \mid G)$ be the set of irreducible Brauer characters of Γ that restrict irreducibly to G. Similarly, we denote by $\mathrm{EBr}(\Gamma \mid \mathrm{dz}^\circ(G))$ the subset of those characters $\chi \in \mathrm{EBr}(\Gamma \mid G)$

whose restriction χ_G belongs to $\mathrm{dz}^\circ(G)$. We can then state a block-free version of Conjecture 7.3 as follows.

Conjecture 7.10. *Let $G \trianglelefteq \Gamma$ be finite groups. Then*

$$\left|\mathrm{EBr}(\Gamma \mid G)\right| = \sum_{Q} \left|\mathrm{EBr}\big(\boldsymbol{N}_\Gamma(Q) \mid \mathrm{dz}^\circ(\boldsymbol{N}_G(Q))\big)\right|$$

where Q runs over a set of representatives for the Γ-orbits of radical p-subgroups of G such that $\Gamma = G\boldsymbol{N}_\Gamma(Q)$.

Arguing as in the proofs of Lemma 7.4 and Proposition 7.5 we can then prove the following proposition.

Proposition 7.11. *Let $G \trianglelefteq \Gamma$ be finite groups.*

(i) *If Conjecture 6.12 holds with respect to $G \trianglelefteq \Gamma$, then Conjecture 7.10 holds with respect to $G \trianglelefteq \Gamma$.*

(ii) *Suppose that Γ/G is a p-group. If Conjecture 7.10 holds with respect to $G \trianglelefteq \Gamma$, then Conjecture 7.9 holds with respect to $G \trianglelefteq \Gamma$.*

As an immediate consequence we obtain a block-free version of Theorem D.

Theorem 7.12. *Let $G \trianglelefteq \Gamma$ be finite groups such that Γ/G is a p-group. If Conjecture 6.12 holds with respect to $G \trianglelefteq \Gamma$, then Conjecture 7.9 holds with respect to $G \trianglelefteq \Gamma$.*

Proof. This follows immediately by combining the two parts of Proposition 7.11. □

Then, using this theorem together with the reduction obtained in Theorem 6.13, we can prove the following block-free analogue of Corollary E.

Corollary 7.13. *Let G be a finite group and p a prime number. If Conjecture 6.12 holds at the prime p for every covering group of any nonabelian finite simple group of order divisible by p involved in G, then Conjecture 7.9 holds at the prime p with respect to any $G \trianglelefteq \Gamma$ such that Γ/G is a p-group.*

8. Verification of Conjectures A and B

Since their introduction in [NT11] and [Spä13b] the inductive Alperin weight condition (Conjecture 6.12) and the inductive blockwise Alperin weight condition (Conjecture 4.1) have been verified for many classes of blocks of quasisimple groups. We refer the reader to the survey [FZ22] and the references therein (see also the most recently published [FLZ23a] and [FLZ23b]). Using these results, and introducing some new arguments, we can then verify Conjecture A and Conjecture B for several classes of groups and blocks.

8.1. ***Groups with abelian Sylow p-subgroups and odd Sylow automizer.*** The inductive blockwise Alperin weight condition has been verified for all quasisimple groups whose simple quotient has an abelian Sylow 2-subgroup [Spä13b, Corollary 6.6], or an abelian Sylow 3-subgroup [FLZ23a, Section 5], or is involved in a group with odd Sylow normalizer [GNT16] (see also [XZ19] for the case of odd Sylow automizers). Thanks to Theorem D and Theorem C, we then obtain new evidence for Conjecture A and Conjecture B.

Proposition 8.1. *Let G be a finite group and consider any prime number p. Then Conjecture A and Conjecture B both hold for G at the prime p it at least one of the following conditions is satisfied*:

(i) *Every simple group involved in G has either abelian Sylow 2-subgroups or abelian Sylow 3-subgroups or both.*

(ii) *p is an odd prime and the automizer $\mathbf{N}_G(P)/P\mathbf{C}_G(P)$ has odd order for a Sylow p-subgroup P of G.*

Proof. Recall that Conjecture A follows from Conjecture B thanks to Theorem D. Moreover, in order to prove Conjecture B for the group G it suffices to show that it holds for every nonabelian simple group of order divisible by p involved in G according to Theorem C. We then obtain the first part of the statement by applying the results obtained in [Spä13b, Theorem 6.6], [FLZ23a, Section 5] and the second part by applying the results in [XZ19] (see also [GNT16]). □

8.2. ***Blocks with cyclic defect groups.*** It was shown in [KS16a] and [KS16b] that the inductive blockwise Alperin weight condition holds for every block with cyclic defect groups of any quasisimple group. In order to apply these results to verify Conjecture A and Conjecture B, we first need to prove a version of Theorem C compatible with certain families of defect groups as done in [Spä13b, Theorem C]. To start, we need to restate Conjecture B.

Let B be a p-block of a finite group G and consider a p-weight (Q, ψ) of G as defined in Section 4. Observe that the block $\mathrm{bl}(\psi)^G$ of G obtained via Brauer induction of blocks is well defined according to [Nav98, Theorem 4.14]. Then we say that (Q, ψ) is a *p-weight of B* provided that $\mathrm{bl}(\psi)^G = B$. We denote by $\mathrm{Alp}(B)$ the set of all p-weights of B and by $\mathrm{Alp}(B)/G$ the set of G-orbits of such p-weights. Now, using the properties of block isomorphisms of character triples (see Definition 3.4) we deduce that Conjecture 4.1 can be reformulated as follows.

Conjecture 8.2 (inductive blockwise Alperin weight condition). *Let $G \trianglelefteq A$ be finite groups and consider a p-block B of G. Then there exists an A_B-equivariant bijection*

$$\Omega_B : \mathrm{IBr}(B) \to \mathrm{Alp}(B)/G$$

such that

$$(A_\vartheta, G, \vartheta) \succeq_b \left(\mathbf{N}_A(Q)_\psi, \mathbf{N}_G(Q), \psi\right)$$

for every $\vartheta \in \mathrm{IBr}(B)$ and $(Q, \psi) \in \Omega_B(\vartheta)$.

Remark 8.3. Let $G \trianglelefteq A$ be finite groups and consider a block B of G. Arguing as in the proof of Proposition 7.5 we can show that if Conjecture 8.2 holds for the block B with respect to $G \trianglelefteq A$, then so does Conjecture 7.3. In particular, if Γ/G is a p-group, we deduce that Conjecture A holds for the block B as a consequence of Lemma 7.4. This observation will be used in the subsequent sections to obtain Conjecture A for certain classes of blocks.

By considering the above version of the conjecture we can obtain a refined version of our Theorem 6.1 and prove a reduction theorem compatible with any fixed class of defect groups closed under taking quotients and subgroups as done in [Spä13b, Theorem C]. Given a family of p-subgroups $\mathcal{R}$, we say that Conjecture 8.2 holds for a finite group G with respect to $\mathcal{R}$ if it holds for every choice of $G \trianglelefteq A$ and every p-block B of G with defect groups contained in $\mathcal{R}$.

Theorem 8.4. *Let $\mathcal{R}$ be a family of finite p-groups closed under taking quotients and subgroups. Let G be a finite group and suppose that Conjecture 8.2 holds with respect to $\mathcal{R}$ for every covering group of any nonabelian finite simple group of order divisible by p involved in G. Then Conjecture 8.2 holds for G with respect to $\mathcal{R}$.*

Proof. As for [Spä13b, Theorem 5.20], this follows by an inspection of the proof of Theorem 6.1. For this, fix a block B with defect groups contained in $\mathcal{R}$. Then, for every normal subgroup $K \trianglelefteq A$ with $K \leq G$ and every A-invariant $\vartheta \in \mathrm{dz}^\circ(K)$ with B covering the block of ϑ, the construction used in the proof of Lemma 6.7 restricted to the set of Brauer characters $\mathrm{IBr}(B \mid \vartheta)$ will produce a block $\overline{B}$ of $\overline{G} = \widehat{G}/K_0$ with defect group contained in $\mathcal{R}$ by recalling that the central subgroup Z of $\widehat{G}$ has order prime to p. We can then apply the inductive hypothesis to the block $\overline{B}$ and finish the proof proceeding as in the remaining part of the proof of Theorem 6.1. □

Using this theorem, we can then apply the results of [KS16a] and [KS16b] to obtain Conjecture A, Conjecture 7.3, and Conjecture 8.2 for every block with cyclic defect groups of any finite group, and where we further assume Γ/G to be a p-group in the case of Conjecture A.

Proposition 8.5. *Let $G \trianglelefteq A$ be finite groups and consider a p-block B of G with cyclic defect groups. Then Conjecture 7.3 and Conjecture 8.2 hold for the block B. Furthermore, if Γ/G is a p-group, then Conjecture A holds for B.*

Proof. By Remark 8.3 it suffices to show that Conjecture 8.2 holds for the block B. The latter follows by applying Theorem 8.4 since Conjecture 8.2 has been verified for all p-blocks with cyclic defect of quasisimple groups (see [KS16a] and [KS16b]). □

8.3. ***Nilpotent blocks.*** The results obtained in [KS16a] also imply that the inductive blockwise Alperin weight condition holds for every nilpotent block of a quasisimple group. Their argument heavily relies on the fact that such blocks are known to have abelian defect groups thanks to [AE11] and [AE13]. In order to obtain Conjecture A, Conjecture 7.3, and Conjecture 8.2 for nilpotent blocks of all finite groups, we therefore need to develop a new argument. Our proof does not depend on the classification of finite simple groups and makes use of some deep results on graded Morita equivalences obtained in [PZ12].

As a first step, we give a reformulation of Conjecture 8.2 in the spirit of [KR89]. For this, denote by $\mathcal{P}(G)$ the set of p-chains of the form $\sigma = \{1 = Q_0 < Q_1 < \cdots < Q_n\}$ where each Q_i is a p-subgroup of G. We denote by $|\sigma|$ the integer n called the *length* of σ. Observe that G acts by conjugation on the set of p-chains and denote by $G_\sigma = \bigcap_i N_G(Q_i)$ the stabilizer of σ in G. Next, for any block B of G, let B_σ denote the union of all blocks of G_σ that correspond to B via Brauer induction of blocks and let $\mathrm{IBr}(B_\sigma)$ be the union of all Brauer characters belonging to some block in B_σ. Similarly, for a p-subgroup Q of G we denote by B_Q the union of blocks $B_{\{1<Q\}}$. We then consider the set $\mathcal{C}^\circ(B)$ of pairs (σ, ψ)

where $\sigma \in \mathcal{P}(G)$ and $\psi \in \mathrm{IBr}(B_\sigma)$. The notion of length yields a partition of $\mathcal{C}^\circ(B)$ into the sets $\mathcal{C}^\circ(B)_\pm$ consisting of those pairs (σ, ψ) such that $(-1)^{|\sigma|} = \pm 1$. The group G now acts by conjugation on $\mathcal{C}^\circ(B)_\pm$ and we denote by $\mathcal{C}^\circ(B)_\pm/G$ the corresponding set of G-orbits and by $\overline{(\sigma, \psi)}$ the G-orbit of an element $(\sigma, \psi) \in \mathcal{C}^\circ(B)_\pm$. We can now restate the inductive blockwise Alperin weight condition as follows.

Lemma 8.6. *The following statements are equivalent*:

(i) *Conjecture 8.2 holds for every p-block of any finite group G and any choice of $G \trianglelefteq A$.*

(ii) *For all finite groups $G \trianglelefteq A$ and every p-block B of G with nontrivial defect, there exists an A_B-equivariant bijection*

$$\Xi : \mathcal{C}^\circ(B)_-/G \to \mathcal{C}^\circ(B)_+/G$$

such that

$$(A_{\sigma,\psi}, G_\sigma, \psi) \succeq_b (A_{\rho,\varphi}, G_\rho, \varphi)$$

for every $(\sigma, \psi) \in \mathcal{C}^\circ(B)_-$ and any $(\rho, \varphi) \in \Xi(\overline{(\sigma, \psi)})$.

Proof. Assume that (ii) holds for $N_G(Q)/Q$ for every nontrivial p-subgroup Q of G. We now fix a G-transversal $\mathcal{T}$ in the set of nontrivial p-subgroups of G and, for each $Q \in \mathcal{T}$, denote by $\mathcal{C}^\circ_Q(B)$ the subset of $\mathcal{C}^\circ(B)$ consisting of pairs (σ, ψ) with $\sigma = \{1 = Q_0 < Q_1 < \cdots < Q_n\}$ where $n \geq 1$ and Q_1 is G-conjugate to Q. Observe that there is a one-to-one correspondence between the set $\mathcal{C}^\circ_Q(B)$ and the set $\mathcal{C}^\circ(\overline{B_Q})$ where $\overline{B_Q}$ is the set of blocks of $N_G(Q)/Q$ dominated by some block in B_Q. Denote by $\overline{b_Q}$ the union of those blocks in $\overline{B_Q}$ with positive defect and by $\overline{c_Q}$ the union of the blocks in $\overline{B_Q}$ of defect zero. Now, applying [Nav18, Lemma 9.15 (a)] we deduce that a pair $(\bar{\sigma}, \bar{\psi}) \in \mathcal{C}^\circ(\overline{c_Q})$ must have $\bar{\sigma} = \{1\}$ and $\mathrm{bl}(\bar{\psi})$ a block of defect zero in $N_G(Q)/Q$. Hence, the set $\mathcal{C}^\circ(\overline{c_Q})$ corresponds to the set of pairs $(\sigma, \psi) \in \mathcal{C}^\circ_Q(B)$ with $\sigma = \{1 = Q_0 < Q_1 = Q\}$ and $\psi \in \mathrm{dz}^\circ(N_G(Q))$ such that $\mathrm{bl}(\psi)^G = B$. We denote by $\mathcal{A}_Q(B)$ the set of such pairs and set $\mathcal{B}_Q(B) := \mathcal{C}^\circ_Q(B) \setminus \mathcal{A}_Q(B)$. If we now consider the remaining blocks $\overline{b_Q}$ of positive defect, then arguing as in [Ros25b, Lemma 2.2, Corollary 2.3, and Proposition 2.4] and using the fact that (ii) holds for all blocks with nontrivial defect of $N_G(Q)/Q$, we obtain a bijection

$$\Xi_Q : \mathcal{B}_Q(B)_+/G \to \mathcal{B}_Q(B)_-/G$$

inducing block isomorphisms of modular character triples. Now, arguing by induction on the order of G, it follows from the preceding argument, and combining the bijections Ξ_Q for $Q \in \mathcal{T}$, that a bijection Ξ as in (ii) exists if and only if we can find a bijection

$$\Omega : \mathrm{IBr}(B) \to \mathrm{Alp}(B)/G$$

satisfying the statement of Conjecture 8.2, which is exactly the statement of (i). This completes the proof. □

The reformulation of Conjecture 4.1 given in Lemma 8.6 is inspired by [KR89, Theorem 3.8] and can be used to prove Conjecture 4.1 in certain situations. While it is not true that the two statements in Lemma 8.6 are equivalent block-by-block, they can still be shown to be equivalent for certain classes of blocks. This is the case for nilpotent blocks. For this, let B be a nilpotent block of a finite group G and consider a p-subgroup Q of G. If b is a block of $\mathbf{N}_G(Q)$ and $b^G = B$, then it follows that b is nilpotent (see the first paragraph of the proof of [Rob02, Theorem 3.2.2]) Moreover, if Q is normal in G and $\bar{B}$ is a block of G/Q dominated by B, then it follows that $\bar{B}$ is nilpotent (see, for instance, [CT20, Theorem 1.1 (i)] for a precise reference).

Corollary 8.7. *Statements* (*i*) *and* (*ii*) *in Lemma 8.6 are equivalent for nilpotent blocks. More precisely, Conjecture 8.2 holds for every nilpotent block of every finite group* G*, and any choice* $G \trianglelefteq A$*, if and only if a bijection* Ξ *with the properties described in* (*ii*) *exists for every nilpotent block of every finite group* G *and with respect to any* $G \trianglelefteq A$.

Proof. This follows from the fact that the argument used in the proof of Lemma 8.6 is compatible with nilpotent blocks. In fact, suppose that B is a nilpotent block of a finite group G and let Q be a p-subgroup of G. If b is a block in the union B_Q, then b is a block of $\mathbf{N}_G(Q)$ such that $b^G = B$ and it follows that b is nilpotent. If $\bar{b}$ is dominated by b, it follows also that $\bar{b}$ is nilpotent. Hence every block in $\overline{B_Q}$ is nilpotent so that we can apply the inductive hypothesis in the $\mathbf{N}_G(Q)/Q$ and proceed as in the proof of Lemma 8.6. □

Next, we need a modular version of [Ros25a, Corollary 3.2]. For this, we will use the main result of [PZ12] together with a modular version of [MM21, Proposition 5.6] that was kindly provided to us by A. Marcus. We state this latter result below for the reader's convenience. Let $N \trianglelefteq G$ be finite groups and consider a p-modular system $(K, \mathcal{O}, k)$ such that k is algebraically closed and K contains a primitive $|G|$-th root of unity. Following [MM21, Section 4.1], let C be a G-invariant block of N and set $\mathcal{A} := \mathcal{O}GC$ and $\mathcal{A}' := \mathcal{O}\mathbf{N}_G(D)c$ where c is the Brauer correspondent of C in $\mathbf{N}_N(D)$ with respect to a defect group D of C. Denote by $\mathcal{B} := \mathcal{O}NC$ and by $\mathcal{B}' := \mathcal{O}\mathbf{N}_N(D)c$ the 1-component of $\mathcal{A}$ and $\mathcal{A}'$ respectively. Set $k\mathcal{B} := k \otimes_{\mathcal{O}} \mathcal{B}$ and $k\mathcal{B}' := k \otimes_{\mathcal{O}} \mathcal{B}'$. Proceeding as in [MM21, Definition 4.2 and Definition 5.1] (but considering modules over k), for every G-invariant $k\mathcal{B}$-module V and every $\mathbf{N}_G(D)$-invariant $k\mathcal{B}'$-module V' we define the order relation $(\mathcal{A}, \mathcal{B}, V) \succeq_b (\mathcal{A}', \mathcal{B}', V')$ which in turns implies that $(G, N, \vartheta) \succeq_b (\mathbf{N}_G(D), \mathbf{N}_N(D), \vartheta')$ for the Brauer characters ϑ and φ corresponding to V and V' respectively. We then have the following criterion for establishing $(\mathcal{A}, \mathcal{B}, V) \succeq_b (\mathcal{A}', \mathcal{B}', V')$.

Lemma 8.8. *Let* $N \trianglelefteq G$ *be finite groups and consider a* G*-invariant block* C *of* N *with defect group* D *and Brauer correspondent* c *in* $\mathbf{N}_N(D)$*. Set* $\mathcal{A} := \mathcal{O}GC$, $\mathcal{A}' := \mathcal{O}\mathbf{N}_G(D)c$, $\mathcal{B} := \mathcal{O}NC$*, and* $\mathcal{B}' := \mathcal{O}\mathbf{N}_N(D)c$ *as above. Suppose that* $\mathcal{A}$ *and* $\mathcal{A}'$ *are* G/N*-graded basic Morita equivalent and that this equivalence sends the* $k\mathcal{B}$*-module* V *to the* $k\mathcal{B}'$*-module* V'*. Then* $(\mathcal{A}, \mathcal{B}, V) \succeq_b (\mathcal{A}', \mathcal{B}', V')$.

Proof. This follows by arguing as in the proof of [MM21, Proposition 5.6] and noticing that, according to [MM21, Remark 5.7], the Morita equivalence in the statement is compatible with the Brauer map as defined in [MM21, Definition 5.5]. □

We can now obtain the above mentioned modular version of [Ros25a, Corollary 3.2].

Lemma 8.9. *Let C be a p-block of a finite group H with defect group D and consider its Brauer correspondent c in $\mathbf{N}_H(D)$. Assume that C is nilpotent and let ψ denote its unique Brauer character. Observe that c must be nilpotent and hence contains a unique Brauer character φ. If $H \trianglelefteq A$, then*

$$(A_\psi, H, \psi) \succeq_b \left(\mathbf{N}_A(D)_\varphi, \mathbf{N}_H(D), \varphi\right).$$

Proof. We proceed as in the proof of [Ros25a, Corollary 3.2]. First notice that it is no loss of generality to assume that ψ is A-invariant in which case φ is $\mathbf{N}_A(D)$-invariant. Since ψ and φ are the unique Brauer characters of C and c respectively it also follows that C is A-invariant and c is $\mathbf{N}_A(D)$-invariant. We now set $\bar{A} := A/H$, $\mathcal{A} := \mathcal{O}AC$, and $\mathcal{A}' := \mathcal{O}\mathbf{N}_A(D)c$. A Frattini argument yields $A = \mathbf{N}_A(D)H$ and hence $\bar{A} = \mathbf{N}_A(D)/\mathbf{N}_H(D)$. Since C is nilpotent, [PZ12, Theorem 3.14 and Corollary 3.15] implies that there exists an $\bar{A}$-graded $(\mathcal{A}, \mathcal{A}')$-bimodule M inducing an $\bar{A}$-graded basic Morita equivalence between $\mathcal{A}$ and $\mathcal{A}'$. Consider the identity components $\mathcal{B} := \mathcal{A}_1 = \mathcal{O}HC$ and $\mathcal{B}' := \mathcal{A}'_1 = \mathcal{O}\mathbf{N}_H(D)c$ and notice that M_1 induces a basic Morita equivalence between $\mathcal{B}$ and $\mathcal{B}'$ that sends the Brauer character ψ of C to the Brauer character φ of c. Then, by applying Lemma 8.8 and considering modules over $k = \mathcal{O}/J(\mathcal{O})$, we obtain the required block isomorphism of modular character triples. □

Using Lemma 8.9, we can finally show that Conjectures A, 7.3, and 8.2 hold for every nilpotent block of a finite group. This will follow from our next result thanks to Corollary 8.7 (see also Remark 8.3).

Proposition 8.10. *Let $G \trianglelefteq A$ be finite groups and consider a nilpotent block B of G. If B has positive defect, then there exists an A_B-equivariant bijection*

$$\Xi : \mathcal{C}^\circ(B)_-/G \to \mathcal{C}^\circ(B)_+/G$$

such that

$$(A_{\sigma,\psi}, G_\sigma, \psi) \succeq_b (A_{\rho,\varphi}, G_\rho, \varphi)$$

for every $(\sigma, \psi) \in \mathcal{C}^\circ(B)_-$ and any $(\rho, \varphi) \in \Xi(\overline{(\sigma, \psi)})$.

Proof. Our argument is inspired by the cancellation theorem introduced in [KR89, Proposition 5.5]. First, as remarked in the proof of Corollary 8.7, for every p-chain $\rho \in \mathcal{P}(G)$ and every block c contained in the union B_ρ the nilpotency of B forces c to be nilpotent as well. Thus, for each pair $(\rho, \varphi) \in \mathcal{C}^\circ(B)$ we know that φ is the unique Brauer character in its block. Write $\rho = \{1 = Q_0 < Q_1 < \cdots < Q_n\}$ and consider a defect group D of $\mathrm{bl}(\varphi)$. Observe that Q_n is a normal p-subgroup of G_ρ and so is contained in D by [Nav98, Theorem 4.8]. Assume first that $Q_n = D$. If $Q_n = 1$, then $\mathrm{bl}(\varphi) = B$ has defect $D = 1$, a contradiction. Thus we must have $Q_n \neq 1$ and we can defined the p-chain $\sigma \in \mathcal{P}(G)$ given by deleting the last term Q_n from ρ. Noticing that $G_\rho = \mathbf{N}_{G_\sigma}(D)$, by Brauer's first main theorem there exists a unique block $C = \mathrm{bl}(\varphi)^{G_\sigma}$ of G_σ with defect group D. Once again C must be nilpotent and it contains a unique Brauer character, say ψ. Then, by applying Lemma 8.9 with $A := A_\sigma$ and $H := G_\sigma$, we get $\left(A_{\sigma,\psi}, G_\sigma, \psi\right) \succeq_b \left(A_{\rho,\varphi}, G_\rho, \varphi\right)$. We then map the G-orbit of (ρ, φ) to that of (σ, ψ). On the other hand, if the last term Q_n of ρ is strictly contained in D, then we define σ to be the p-chain obtained by adding D at the end of ρ and let ψ be the unique Brauer character belonging to the nilpotent block of G_σ that induces to $\mathrm{bl}(\varphi)$. In this case, we argue as before by switching the role of (σ, ψ) and (ρ, φ). This yields a bijection Ξ with the properties required in the statement. □

8.4. ***2-blocks with abelian defect groups.*** The blockwise Alperin weight conjecture has recently been proved for 2-blocks with abelian defect groups by Ruhstorfer in [Ruh22b]. This result was obtained as a consequence of the validity of the Alperin–McKay conjecture for this class of blocks and applying [KR89, Proposition 5.6]. Unfortunately, this strategy does not imply the full inductive blockwise Alperin weight condition. The latter has instead been verified by Zhang and Zhou [ZZ21] and Hu and Zhou [HZ] and relies on the classification of 2-blocks with abelian defect groups obtained in [EKKS14]. Thanks to these results we can then prove Conjecture A, Conjecture 7.3, and Conjecture 8.2 for 2-blocks with abelian defect groups in any finite group.

Proposition 8.11. *Conjecture A, Conjecture 7.3, and Conjecture 8.2 hold for every 2-block with abelian defect groups of any finite group.*

Proof. Let B be a 2-block with abelian defect groups of a finite group G and consider $G \trianglelefteq A$. By Remark 8.3 it suffices to show that Conjecture 8.2 holds for B with respect to $G \trianglelefteq A$. Furthermore, by Theorem 8.4 it is no loss of generality to assume that G is quasisimple. In this case the structure of B has been determined in [EKKS14, Theorem 6.1] and, using this description, the results of [ZZ21] and [HZ] show that Conjecture 8.2 holds for B as desired. □

Acknowledgments

We thank Gabriel Navarro for an insightful conversation on the content of Section 5 during the workshop 2316 Representations of Finite groups at Mathematisches Forschungsinstitut Oberwolfach, where part of this research was conducted. We are thankful to Marc Cabanes and Gunter Malle for providing comments and corrections to an earlier version of this paper. The third author would also like to thank Andrei Marcus for a discussion about the results of [MM21] — in particular, for pointing out at the validity of a modular version of [MM21, Proposition 5.6], Yuanyang Zhou for providing us with a copy of the preprint [HZ], and Julian Brough for helpful conversations about the inductive Alperin weight condition during this author's doctoral studies at the Bergische Universität Wuppertal. We extend our gratitude to the Institut Henri Poincaré for the hospitality during our visit as part of the Research in Paris program. Finally, we are grateful to the anonymous referee for a thorough reading and valuable comments that helped improve the paper.

References

[AE11] J. An and C. W. Eaton, "Nilpotent blocks of quasisimple groups for odd primes", *J. Reine Angew. Math.* **656** (2011), 131–177. MR

[AE13] J. An and C. W. Eaton, "Nilpotent blocks of quasisimple groups for the prime two", *Algebr. Represent. Theory* **16**:1 (2013), 1–28. MR

[Alp76] J. L. Alperin, "The main problem of block theory", pp. 341–356 in *Proceedings of the Conference on Finite Groups* (Park City, UT, 1975), edited by W. R. Scott and F. Gross, Academic Press, New York, 1976. MR

[Alp87] J. L. Alperin, "Weights for finite groups", pp. 369–379 in *The Arcata Conference on Representations of Finite Groups*, *I* (Arcata, CA, 1986), edited by P. Fong, Proc. Sympos. Pure Math. **47**, Amer. Math. Soc., Providence, RI, 1987. MR

[Cab13] M. Cabanes, "Two remarks on the reduction of Alperin's weight conjecture", *Bull. Lond. Math. Soc.* **45**:5 (2013), 895–906. MR

[CS] M. Cabanes and B. Späth, "The McKay conjecture on character degrees", preprint, 2024. To appear in *Ann. of Math.* (2). arXiv 2410.20392

[CT20] T. Coconeț and C.-C. Todea, "Domination of blocks, fusion systems and hyperfocal subgroups", *J. Algebra* **549** (2020), 30–44. MR

[Dad92] E. C. Dade, "Counting characters in blocks, I", *Invent. Math.* **109**:1 (1992), 187–210. MR

[Dad94] E. C. Dade, "Counting characters in blocks, II", *J. Reine Angew. Math.* **448** (1994), 97–190. MR

[Dad97] E. C. Dade, "Counting characters in blocks, II.9", pp. 45–59 in *Representation theory of finite groups* (Columbus, OH, 1995), edited by R. Solomon, Ohio State Univ. Math. Res. Inst. Publ. **6**, de Gruyter, Berlin, 1997. MR

[EKKS14] C. W. Eaton, R. Kessar, B. Külshammer, and B. Sambale, "2-blocks with abelian defect groups", *Adv. Math.* **254** (2014), 706–735. MR

[FLZ22] Z. Feng, Z. Li, and J. Zhang, "Jordan decomposition for weights and the blockwise Alperin weight conjecture", *Adv. Math.* **408** (2022), art. id. 108594. MR

[FLZ23a] Z. Feng, C. Li, and J. Zhang, "On the inductive blockwise Alperin weight condition for type $\mathbb{A}$", *J. Algebra* **631** (2023), 287–305. MR

[FLZ23b] Z. Feng, Z. Li, and J. Zhang, "Morita equivalences and the inductive blockwise Alperin weight condition for type $\mathbb{A}$", *Trans. Amer. Math. Soc.* **376**:9 (2023), 6497–6520. MR

[FZ22] Z. Feng and J. Zhang, "Alperin weight conjecture and related developments", *Bull. Math. Sci.* **12**:3 (2022), art. id. 2230005. MR

[GNT16] R. M. Guralnick, G. Navarro, and P. H. Tiep, "Finite groups with odd Sylow normalizers", *Proc. Amer. Math. Soc.* **144**:12 (2016), 5129–5139. MR

[HZ] X. Hu and Y. Zhou, "The blockwise Alperin weight conjecture and inertial blocks", *J. Algebra* **672** (2025), 379–399.

[IMN07] I. M. Isaacs, G. Malle, and G. Navarro, "A reduction theorem for the McKay conjecture", *Invent. Math.* **170**:1 (2007), 33–101. MR

[KR89] R. Knörr and G. R. Robinson, "Some remarks on a conjecture of Alperin", *J. Lond. Math. Soc.* (2) **39**:1 (1989), 48–60. MR

[KS15] S. Koshitani and B. Späth, "Clifford theory of characters in induced blocks", *Proc. Amer. Math. Soc.* **143**:9 (2015), 3687–3702. MR

[KS16a] S. Koshitani and B. Späth, "The inductive Alperin–McKay and blockwise Alperin weight conditions for blocks with cyclic defect groups and odd primes", *J. Group Theory* **19**:5 (2016), 777–813. MR

[KS16b] S. Koshitani and B. Späth, "The inductive Alperin–McKay condition for 2-blocks with cyclic defect groups", *Arch. Math.* (*Basel*) **106**:2 (2016), 107–116. MR

[Lad10] F. Ladisch, "Character correspondences induced by magic representations", preprint, 2010. arXiv 1004.4538

[McK72] J. McKay, "Irreducible representations of odd degree", *J. Algebra* **20** (1972), 416–418. MR

[MM21] A. Marcus and V.-A. Minuță, "Blockwise relations between triples, and derived equivalences for wreath products", *Comm. Algebra* **49**:7 (2021), 2941–2951. MR

[MNSFT22] G. Malle, G. Navarro, A. A. Schaeffer Fry, and P. H. Tiep, "Brauer's height zero conjecture", *Ann. of Math.* (2) **200**:2 (2024), 557–608. MR

[Mur96] M. Murai, "Normal subgroups and heights of characters", *J. Math. Kyoto Univ.* **36**:1 (1996), 31–43. MR

[Nav04] G. Navarro, "Actions and characters in blocks", *J. Algebra* **275**:2 (2004), 471–480. MR

[Nav17] G. Navarro, "Some remarks on global/local conjectures", pp. 151–158 in *Finite simple groups*: *thirty years of the atlas and beyond* (Princeton, NJ, 2015), edited by M. Bhargava et al., Contemp. Math. **694**, Amer. Math. Soc., Providence, RI, 2017. MR

[Nav18] G. Navarro, *Character theory and the McKay conjecture*, Cambridge Stud. Adv. Math. **175**, Cambridge Univ. Press, 2018. MR

[Nav98] G. Navarro, *Characters and blocks of finite groups*, Lond. Math. Soc. Lect. Note Ser. **250**, Cambridge Univ. Press, 1998. MR

[NS14a] G. Navarro and B. Späth, "Character correspondences in blocks with normal defect groups", *J. Algebra* **398** (2014), 396–406. MR

[NS14b] G. Navarro and B. Späth, "On Brauer's height zero conjecture", *J. Eur. Math. Soc.* **16**:4 (2014), 695–747. MR

[NST17] G. Navarro, B. Späth, and P. H. Tiep, "Coprime actions and correspondences of Brauer characters", *Proc. Lond. Math. Soc.* (3) **114**:4 (2017), 589–613. MR

[NT11] G. Navarro and P. H. Tiep, "A reduction theorem for the Alperin weight conjecture", *Invent. Math.* **184**:3 (2011), 529–565. MR

[NT89] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press, Boston, MA, 1989. MR

[PZ12] L. Puig and Y. Zhou, "Glauberman correspondents and extensions of nilpotent block algebras", *J. Lond. Math. Soc.* (2) **85**:3 (2012), 809–837. MR

[Rob02] G. R. Robinson, "Cancellation theorems related to conjectures of Alperin and Dade", *J. Algebra* **249**:1 (2002), 196–219. MR

[Ros22a] D. Rossi, "Character triple conjecture for p-solvable groups", *J. Algebra* **595** (2022), 165–193. MR

[Ros22b] D. Rossi, "Counting conjectures and e-local structures in finite reductive groups", *Adv. Math.* **436** (2024), art. id. 109403. MR

[Ros23b] D. Rossi, "A local-global principle for unipotent characters", *Forum Math. Sigma* **12** (2024), art. id. e125. MR

[Ros23c] D. Rossi, "The McKay conjecture and central isomorphic character triples", *J. Algebra* **618** (2023), 42–55. MR

[Ros25a] D. Rossi, "A reduction theorem for the character triple conjecture", *Proc. London Math. Soc.* (3) **131**:4 (2025), art. id. e70090.

[Ros25b] D. Rossi, "The character triple conjecture for height zero characters and the prime 2", *Rev. Mat. Iberoam.* **41**:6 (2025), 2357–2378. arXiv 2307.13809

[Ruh22a] L. Ruhstorfer, "The Alperin–McKay and Brauer's height zero conjecture for the prime 2", *Ann. of Math.* (2) **201**:2 (2025), 379–457. MR

[Ruh22b] L. Ruhstorfer, "Quasi-isolated blocks and the Alperin–McKay conjecture", *Forum Math. Sigma* **10** (2022), art. id. e48. MR

[Spä13a] B. Späth, "A reduction theorem for the Alperin–McKay conjecture", *J. Reine Angew. Math.* **680** (2013), 153–189. MR

[Spä13b] B. Späth, "A reduction theorem for the blockwise Alperin weight conjecture", *J. Group Theory* **16**:2 (2013), 159–220. MR

[Spä17a] B. Späth, "Inductive conditions for counting conjectures via character triples", pp. 665–680 in *Representation theory: current trends and perspectives*, edited by H. Krause et al., Eur. Math. Soc., Zürich, 2017. MR

[Spä17b] B. Späth, "A reduction theorem for Dade's projective conjecture", *J. Eur. Math. Soc.* **19**:4 (2017), 1071–1126. MR

[Spä18] B. Späth, "Reduction theorems for some global-local conjectures", pp. 23–61 in *Local representation theory and simple groups* (Lausanne, Switzerland, 2016), edited by R. Kessar et al., Eur. Math. Soc., Zürich, 2018. MR

[SV16] B. Späth and C. Vallejo Rodríguez, "Brauer characters and coprime action", *J. Algebra* **457** (2016), 276–311. MR

[Tur08] A. Turull, "Above the Glauberman correspondence", *Adv. Math.* **217**:5 (2008), 2170–2205. MR

[Val16] C. Vallejo Rodríguez, *Characters, correspondences and fields of values of finite groups*, Ph.D. thesis, Universitat de València, 2016, available at http://hdl.handle.net/10550/53934.

[XZ19] C. Xu and Y. Zhou, "Finite groups with odd Sylow automizers", *Arch. Math. (Basel)* **112**:6 (2019), 567–579. MR

[ZZ21] K. Zhang and Y. Zhou, "The inductive blockwise Alperin weight condition for certain 2-blocks with abelian defect groups", *J. Algebra* **579** (2021), 174–194. MR

Communicated by Pham Huu Tiep
Received 2024-05-30 Revised 2024-11-25 Accepted 2025-01-20

josep.m.martinez@uv.es *Departament de Matemàtiques, Universitat de València, 46100 Burjassot (Valencia), Spain*

Dipartamento di Matematica e Informatica "Ulisse Dini", 50134 Firenze, Italy

noelia.rizo@uv.es *Departament de Matemàtiques, Universitat de València, 46100 Burjassot (Valencia), Spain*

damiano.rossi@rutgers.edu *Department of Mathematics, Rutgers University, Piscataway, NJ 08854, United States*

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2026 is US $590/year for the electronic version, and $865/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 20 No. 2 2026