Algebra & Number Theory

Volume 20
2026
No. 3

msp

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2026 is US $590/year for the electronic version, and $865/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.419

Ramified descent and transcendental Brauer–Manin obstruction

Julian Lawrence Demeio

Given a smooth geometrically connected variety X defined over a number field K and an étale torsor $V \to U$ over a Zariski-open U of X, we investigate the problem of which adelic points of X can be approximated by adelic points that lift to a (twist of a) V. The question has long been investigated in the literature when $U = X$, but less so in the general case. We introduce a Brauer–Manin obstruction to the problem, and provide an example where this obstruction is nontrivial and purely transcendental. This answers in the negative a question posed by Harari at a 2019 workshop. Our example is also an explicit example of a nontrivial transcendental Brauer–Manin obstruction on a smooth compactification of a quotient SL_n/G, with G constant metabelian.

1. Introduction

Descent theory has long been used to understand how rational points $X(K)$ of a smooth complete variety X defined over a number field K are distributed in the adelic points $X(\mathbb{A}_K)$. It was first developed for proper varieties by Colliot-Thélène and Sansuc [5; 26], and it was later extended to open varieties by Harari and Skorobogatov [15, Chapter 6]. We investigate the matter of "ramified descent", i.e., the behavior of open descent theory under compactification, and answer a question of Harari on this topic.

For a torsor $\lambda : V \to U$ under a group of multiplicative type M/K, the *descent set* is defined to be the set of those adelic points of U that lift to adelic points of a K-twist of V:

$$U(\mathbb{A}_K)^{\lambda} = \bigcup_{\sigma \in H^1(K,M)} \lambda_{\sigma}(V_{\sigma}(\mathbb{A}_K)).$$

It follows from open descent theory [15, Proposition 3.1] that this set may be described in terms of a(n algebraic) Brauer–Manin obstruction: i.e., there exists a subgroup $\mathrm{Br}_{\lambda} U \subseteq \mathrm{Br}_1 U$ such that

$$U(\mathbb{A}_K)^{\lambda} = U(\mathbb{A}_K)^{\mathrm{Br}_{\lambda} U}. \tag{1-1}$$

Now let X be a smooth compactification of U, and $X(\mathbb{A}_K)^{\lambda}$ be the adelic closure of $U(\mathbb{A}_K)^{\lambda}$ in $X(\mathbb{A}_K)$. The first result of this paper is that $X(\mathbb{A}_K)^{\lambda}$ provides an obstruction to the Hasse principle and weak approximation for X:

Theorem 1.1. *The inclusion $\overline{X(K)} \subseteq X(\mathbb{A}_K)^{\lambda}$ holds.*

MSC2020: primary 14G05, 14G12; secondary 11G35.
Keywords: rational point, Brauer–Manin obstruction, ramified cover.

The proof we present, quite compact, is due to Olivier Wittenberg, whom the author thanks profoundly. (The original proof that the author had in mind was much more involved.)

We are mainly interested in this paper in the case where M is finite, in which case we call $X(\mathbb{A}_K)^\lambda$ the *ramified descent set*, the adjective "ramified" indicating that the relative normalization of U in V is allowed to be ramified. (See Section 3 for an alternative definition of the ramified descent set.)

In a 2019 workshop, Harari formulated a question: to investigate how (1-1) behaves under "compactification". We present his question as Question 4.1, but a special, yet illustrative, case may be reformulated as follows (see Proposition 4.2):

Question 1.2 (Harari). Assume that $\operatorname{Br} X/\operatorname{Br}_0 X$ and M are finite. Does $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^{\operatorname{Br}_\lambda U \cap \operatorname{Br} X}$ always?

Note that the inclusion $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\operatorname{Br}_\lambda U \cap \operatorname{Br} X}$ follows from (1-1). Harari's question was motivated by the fact that the analog of Question 1.2 has a positive answer when M is a torus (see [3, Proposition 3.1]). Moreover, when $M = \mu_n$ is cyclic, and some mild ramification assumptions are satisfied, then a positive answer follows from a result of Colliot-Thélène and Skorobogatov [7, Theorem 14.2.25] (see Appendix B for details).

We answer Question 1.2 negatively. To do so, we introduce in Section 5 a new Brauer subgroup $\operatorname{Br}^{\mathrm{ram}}_\lambda X \subseteq \operatorname{Br} X$, defined as the intersection $\operatorname{Br}^{\mathrm{ram}}_\lambda U \cap \operatorname{Br} X$, where $\operatorname{Br}^{\mathrm{ram}}_\lambda U$ is the image of the composition

$$H^2(\Gamma_M, \overline{K}^*) \to H^2(\Gamma_M, \overline{K}[V]^*) \xrightarrow{\check{C}_{\mathfrak{l}}} H^2(U, \mathbb{G}_m) = \operatorname{Br} U,$$

where $\check{C}_{\mathfrak{l}}$ is the Čech-to-étale map associated to $V_{\overline{K}} \to U$, which is a profinite torsor under the constant profinite group $\Gamma_M := M(\overline{K}) \rtimes \Gamma_K$ (see Section 5 for details). In the special case where both $\operatorname{Pic} V$ and $\overline{K}[V]^*/\overline{K}^*$ vanish, one has $\operatorname{Br}^{\mathrm{ram}}_\lambda X = \operatorname{Ker}(\operatorname{Br} X \to \operatorname{Br} V_{\overline{K}})$; see Remark 5.1.4. (In Examples 5.1.5 we provide some examples where this vanishing happens.) One verifies that

Proposition 1.3. *The inclusion $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\operatorname{Br}^{\mathrm{ram}}_\lambda X}$ holds.*

The group $\operatorname{Br}^{\mathrm{ram}}_\lambda X$ contains $\operatorname{Br}_\lambda U \cap \operatorname{Br} X$; see Proposition 5.1.6. However, the former may in general be a bigger group and provide a bigger Brauer–Manin obstruction. We provide some explicit families where this is indeed the case in Section 6.1, where we prove:

Proposition 1.4. *Let K be a number field and e be a natural number such that $\mu_e \subset K^*$. If H is a constant metabelian finite subgroup of SL_n of exponent e, H^{ab} is its abelianization, and λ is the H^{ab}-torsor $\operatorname{SL}_n/[H, H] \to \operatorname{SL}_n/H$, then $\operatorname{Br}^{\mathrm{ram}}_\lambda X = \operatorname{Br} X$.*

Theorem 1.5. *For every number field K and every prime $p \geq 5$ such that $\mu_p \subset K^*$, there exists a constant nilpotent metabelian finite group H of exponent p such that, for any embedding $H \hookrightarrow \operatorname{SL}_{n,K}$, letting X be a smooth compactification of $\operatorname{SL}_{n,K}/H$, we have $\operatorname{Br}_a X = 0$ and*

$$X(\mathbb{A}_K)^{\operatorname{Br} X} \neq X(\mathbb{A}_K).$$

Here $\mathrm{Br}_a X$ denotes the algebraic Brauer group of X modulo constants. Combining Proposition 1.4, Theorem 1.5, and Proposition 1.3 we obtain the sought negative answer to Harari's question: indeed, for X as in the theorem, $\mathrm{Br}_\lambda U \cap \mathrm{Br}\, X \subseteq \mathrm{Br}_1 X$ is constant as $\mathrm{Br}_a X = 0$, while $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X} = X(\mathbb{A}_K)^{\mathrm{Br}\, X} \neq X(\mathbb{A}_K)$.

Incidentally, Theorem 1.5 appears to be only the second known example of transcendental obstruction to weak approximation for quotients SL_n / H, or, in other words (see [13, Section 1.2]) to the Grunwald problem for a finite group H. The first such example was obtained by Demarche, Lucchini and Neftin in [9, Theorem 1.2]. But in contrast with loc.cit., where the existence of a transcendental obstruction is proven nonconstructively, we show the nontriviality of the transcendental Brauer–Manin pairing by computing it explicitly.

Structure of the paper. In Section 2 we settle our notation. In Section 3 we prove some basic facts about the "descent set" $X(\mathbb{A}_K)^\lambda$ for a finite M. Since it requires no further effort, we replace M here with a general finite group scheme G/K (not necessarily commutative). In the same section, we prove that $X(\mathbb{A}_K)^\lambda$ provides an obstruction to Hasse principle and weak approximation on the whole X (see Corollary 3.2.2).

In Section 4, we formulate Harari's question precisely and show its relation with Question 1.2.

In Section 5, we introduce the Brauer subgroup $\mathrm{Br}_\lambda^{\mathrm{ram}} X$, prove that it contains $\mathrm{Br}_\lambda U \cap \mathrm{Br}\, X$ and that it obstructs $X(\mathbb{A}_K)^\lambda$.

In Section 6, we prove Proposition 1.4 and Theorem 1.5. We do so by explicitly computing the unramified Brauer–Manin pairing on SL_n / H for H nilpotent metabelian of odd prime exponent. This explicit computation extends earlier work of Bogomolov [1, Section 5].

Appendix A contains some elementary lemmas that are used in Section 6.1. Appendix B talks briefly about other already existing works containing the idea of "ramified descent".

2. Notation

Fields. Unless specified otherwise, k will always denote a field of characteristic 0 and K a number field.

M_K (resp. M_K^f, M_K^∞) denotes the set of (nonarchimedean, archimedean) places of K.

For a place $v \in M_K$ (resp. $v \in M_K^f$), K_v (resp. O_v) denotes the v-adic completion of K (resp. the v-adic integers).

$\mathbb{A}_K$ (resp. $\mathbb{A}_K^S$, for a subset $S \subset M_K$) denotes the topological ring of adeles of K (resp. S-adeles), i.e., the topological ring $\prod'_{v \in M_K} K_v$ (resp. $\prod'_{v \in M_K \setminus S} K_v$), the restricted product being on $O_v \subseteq K_v$.

For a finite subset $S \subseteq M_K$, K_S denotes the product $\prod_{v \in S} K_v$. We let K_Ω denote the product $\prod_{v \in M_K} K_v$.

For a Galois extension L/K, $\mathrm{Gal}(L/K)$ denotes the Galois group of the extension. For a field k with algebraic closure $\bar{k}$, $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$.

Duals. For a group M of multiplicative type over a field k (i.e., a commutative group scheme which is an extension of a finite group scheme by a torus), $M' := \mathcal{H}om(M, \mathbb{G}_m)$ denotes its Cartier dual.

For a torsion abelian group A, A^D denotes the profinite abelian group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology. If A is a profinite abelian group, A^D denotes the torsion group $\mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ is endowed with its discrete topology. By Pontryagin duality, if A is torsion or profinite, there is a canonical isomorphism $A \cong (A^D)^D$.

Geometry. All schemes we consider are separated. We tacitly assume this throughout the paper.

A *variety* X over a field k is an integral scheme of finite type over a field k.

For a k-scheme X, we denote the residue field of a point $\xi \in X$ by $k(\xi)$. We denote the base change $X_{\bar{k}}$ by $\overline{X}$.

Groups and torsors. Group actions are assumed to be right actions unless specified otherwise.

Let S be a scheme, G be a group scheme over S and X be an S-scheme. A right G-*torsor* over X is an X-scheme $Y \to X$, endowed with a G-action $m : Y \times_S G \to Y$ that is X-equivariant (i.e., such that the composition $Y \times_S G \xrightarrow{m} Y \to X$ is equal to the composition $Y \times_S G \xrightarrow{\mathrm{pr}_1} Y \to X$) and such that there exists an étale covering $X' \to X$ and an X'-isomorphism $X' \times_X Y \cong X' \times_X G$ that is G-equivariant.

For an abstract group N, and a scheme S (resp. a field F), we denote by N_S (resp. N_F) the S-scheme (resp. F-scheme) $\sqcup_{n \in N} S$, endowed with its natural S(resp. F)-group scheme structure. If X is an S-scheme, a torsor $Y \to X$ under an abstract group G is a torsor under the constant group G_S.

If G/k is an algebraic group, and $k \subseteq F$ is a field extension, we use the notation $H^i(F, G)$ (with $i \in \mathbb{N}$ and $i = 0, 1$ if G is not commutative) to denote the cohomology group/set $H^i(\Gamma_F, G(\overline{F})) = Z^i(\Gamma_F, G(\overline{F}))/B^i(\Gamma_F, G(\overline{F}))$ (where $B^i(\Gamma_F, G(\overline{F}))$ is a subgroup when G is commutative and is just an equivalence relation otherwise).

If G is not commutative the set of cocycles $Z^1(\Gamma_F, G(\overline{F}))$ is the one of nonabelian (1-)cocycles, i.e., those functions $g_\sigma : \Gamma_F \to G(\overline{F})$ that satisfy $g_{\sigma\tau} = g_\sigma \, {}^\sigma g_\tau$. The set $H^1(\Gamma_F, G(\overline{F}))$ is the quotient of 1-cocycles by the equivalence relation $B^1(\Gamma_F, G(\overline{F})) : g_\sigma \sim g'_\sigma$ if there exists $g \in G(\overline{F})$ such that $g'_\sigma = g^{-1} g_\sigma \, {}^\sigma g$. Note that these cocycles correspond to (left) G-torsors through the standard correspondence [26, p. 18, 2.10].

If $\xi \in Z^1(K, G)$, we use the notation G^ξ to denote the inner twist of G by ξ, and G_ξ to denote the left principal homogeneous space of G obtained by twisting G by the cocycle ξ. This twist is naturally endowed with a right action of G^ξ. See [26, pp. 12–13] for details on these constructions.

If X is a quasi-projective k-scheme endowed with a G-action, and $\xi \in Z^1(k, G)$, we use the notation X_ξ to denote the twisted quasi-projective k-scheme $(X \times_k^G G_\xi)$. (We refer the reader to [[26, p. 20]; [25, Section I.5.3]; [25, Section III.1.3]] for the existence of the twist and immediate properties of the twisting operation). The k-scheme X_ξ is naturally endowed with a G^ξ-action. We recall that there always exists a $G \times_k \bar{k}$-equivariant isomorphism $X_\xi \times_k \bar{k} \cong X \times_k \bar{k}$. If X' is another k-scheme and $\psi : X \to X'$ is a G-invariant morphism (i.e., G-equivariant when we endow X' with the trivial action), we denote by $\psi_\xi : X_\xi \to X'$ the twisted form of ψ by ξ.

If X is endowed with a left G-action we may still do the twisting operations, by taking the corresponding right action, using the canonical isomorphism $G \cong G^{op}$, $g \mapsto g^{-1}$.

Equivariant commutative diagrams. Let S be a scheme. For S-group schemes G_1, G_2, a (usually implicit) homomorphism $G_1 \to G_2$, and torsors $Z_1 \xrightarrow{G_1} W_1$, $Z_2 \xrightarrow{G_2} W_2$, a diagram

$$\begin{array}{ccc} Z_1 & \longrightarrow & Z_2 \\ \big\downarrow{\scriptstyle G_1} & & \big\downarrow{\scriptstyle G_2} \\ W_1 & \longrightarrow & W_2 \end{array} \tag{2-1}$$

commutes if the underlying diagram is commutative and $Z_1 \to Z_2$ is $(G_1 \to G_2)$-equivariant.

Category of torsors. Let S be a scheme, and X an S-scheme. The category of torsors over X with base-scheme S is the category whose objects are pairs (Y, G), where G is an S-group scheme, and Y is a G-torsor over X, and whose morphisms are pairs $(Y_1 \to Y_2, G_1 \to G_2)$, where $G_1 \to G_2$ is a homomorphism, and $Y_1 \to Y_2$ is a $(G_1 \to G_2)$-equivariant morphism.

Profinite (étale) torsors under constant profinite groups. Let S be a scheme, X an S-scheme, and G be a(n abstract) profinite group. A *profinite torsor* over X under G is an X-scheme $Y \to X$, endowed with a G-action $m : Y \times G \to Y$ that is X-equivariant, and such that there exists an inverse system (Y_i, G_i), $i \in I$ in the category of torsors over X with each G_i finite, and such that there exist an isomorphism $\psi : G \xrightarrow{\sim} \operatorname{proj\,lim} G_i$ and a ψ-equivariant isomorphism $Y \cong \operatorname{proj\,lim} Y_i$.

(The inverse limit of the Y_i exists in the category of schemes, as each Y_i is the relative spectrum of a finite étale $\mathcal{O}_X$-algebra $\mathcal{O}_{Y_i}$, and their inverse limit may be realized as the relative spectrum of the finite étale ind-algebra $\operatorname{inj\,lim} \mathcal{O}_{Y_i}$ [28, Tag 01YV].)

When in addition Y is connected, we say that $Y \to X$ is a *profinite étale Galois cover*; see [21, Remark 2.21(b)]. See also [31, Section 2] for details and an alternative definition.

Brauer group. Recall that the Brauer group of a scheme X is defined to be the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$. When X is a variety defined over a number field K, this provides an obstruction, known as *Brauer–Manin obstruction*, to local-global principles, in the following sense. There is a *Brauer–Manin* pairing

$$X(\mathbb{A}_K) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z},$$

sending $((P_v)_{v \in M_K}, B)$ to $((P_v)_{v \in M_K}, B)_{BM} := \sum_v \operatorname{inv}_v B(P_v)$, where $\operatorname{inv}_v : H^2(\Gamma_{K_v}, \overline{K_v}^*) \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map (see, e.g., [14, Theorem 8.9] for a definition). Whenever $B \in \operatorname{Im} \operatorname{Br} K$ or $(P_v)_{v \in M_K} \in X(K)$ (diagonally embedded in $X(\mathbb{A}_K)$), $((P_v)_{v \in M_K}, B)_{BM} = 0$ by the Albert–Brauer–Hasse–Noether theorem (see [26, Section 5]). It follows that $X(K)$ is a subset of

$$X(\mathbb{A}_K)^{\operatorname{Br} X} := \{(P_v)_{v \in M_K} \in X(\mathbb{A}_K) \mid ((P_v)_{v \in M_K}, B)_{BM} = 0 \text{ for all } B \in \operatorname{Br} X\}.$$

For a geometrically integral scheme X over a field F, we notate $\operatorname{Br}_1 X := \operatorname{Ker}(\operatorname{Br} X \to \operatorname{Br} X\overline{F})$ and $\operatorname{Br}_0 X := \operatorname{Im}(p^* : \operatorname{Br} F \to \operatorname{Br} X)$ as usual, where $p : X \to \operatorname{Spec} F$ denotes the structural morphism.

When X is smooth and integral over F, and $U \subseteq X$ is an open subscheme, we identify, with a slight abuse of notation, the injective [7, Theorem 3.5.5] pullback $\operatorname{Br} X \to \operatorname{Br} U$ with an inclusion

$\operatorname{Br} X \subseteq \operatorname{Br} U \subseteq \operatorname{Br} k(X)$. We say β is *unramified* if $\beta \in \operatorname{Br} X^c$ for one (equivalently, by [7, Proposition 3.7.10], every) smooth compactification X^c of X. We denote the subgroup of unramified elements by $\operatorname{Br}_{\mathrm{ur}}(k(X))$ or $\operatorname{Br}_{\mathrm{ur}} X$.

Cohomology. For a scheme X and an étale abelian sheaf $\mathcal{F}$ on X, $H^n(X, \mathcal{F})$, $n \geq 0$ denotes the étale cohomology group $H^n_{\text{ét}}(X, \mathcal{F})$.

Map from Čech cohomology to étale cohomology. Let U be a scheme, $\mathcal{F}$ an étale sheaf on U, and $\phi : V \to U$ an étale cover. We may think of ϕ as an étale covering $\mathcal{U}$ of U made of a single cover: $\mathcal{U} = \{V \to U\}$. Recall that to such a covering we may naturally associate its Čech cohomology groups $\check{H}^n(V/U, \mathcal{F})$, $n \geq 0$ [21, Section III.2]. There are natural Čech-to-étale morphisms

$$\check{C}_\phi : \check{H}^n(V/U, \mathcal{F}) \to H^n(U, \mathcal{F}) \tag{2-2}$$

for each $n \geq 0$ (as edge maps of the first spectral sequence in [21, Proposition III.2.7]).

When $\phi : V \to U$ is an étale torsor under the constant finite group G, there are natural identifications [21, Example III.2.6] (technically loc.cit. is formulated for Galois covers, or equivalently for *connected* torsors under constant finite groups, but the connectedness assumption is never used):

$$\check{H}^n(V/U, \mathcal{F}) = H^n(G, \mathcal{F}(V)), \tag{2-3}$$

where the latter denotes group cohomology. Under these identifications, (2-2) becomes

$$\check{C}_\phi : H^n(G, \mathcal{F}(V)) \to H^n(U, \mathcal{F}). \tag{2-4}$$

If $\phi : V \to U$ is a torsor under a profinite constant group G, then one may define natural maps $\check{C}_\phi : H^n(G, \mathcal{F}(V)) \to H^n(U, \mathcal{F})$, $n \geq 0$, as the colimit of (2-4) on all quotients $V/H \to U$ by open subgroups H of G. See [21, Remark III.2.21(b)] for details.

Remark 2.1. When $Y = X \times G$ (i.e., ϕ is the trivial torsor) with G finite, then the trivial covering $\mathcal{U}' = \{U \to U\}$ refines $\mathcal{U} = \{V \to U\}$ via the morphism $U = U \times \{e\} \hookrightarrow U \times G$. Thus $\check{C}_\phi : H^n(G, \mathcal{F}(V)) \to H^n(U, \mathcal{F})$ factors through $H^n(U/U, \mathcal{F}(U))$. The latter is 0 for $n \geq 1$, and thus $\check{C}_\phi = 0$ for $n \geq 1$.

3. Descent set

3.1. *Definitions.*

Descent set for torsors. Let K be a number field, G/K a finite group scheme, $p : U \to \operatorname{Spec} K$ a smooth geometrically connected variety over K, and $\lambda : V \to U$ a G-torsor.

For every $\xi \in H^1(K, G)$, there exists a twisted form $\lambda_\xi : V_\xi \to U$ of the torsor λ. This is a torsor under the twisted form G^ξ of G. The class $[\lambda_\xi] \in H^1(U, G^\xi)$ is given by the image of $[\lambda] \in H^1(U, G)$ under the well-known isomorphism [26, pp. 20–21]

$$H^1(U, G) \to H^1(U, G^\xi), [V] \mapsto [V_\xi].$$

When G is commutative, we have $G^\xi = G$, and the morphism $H^1(U, G) \to H^1(U, G)$, $[V] \mapsto [V_\xi]$ becomes $[V] \mapsto [V] - p^*[\xi]$.

Recall that the descent set $U(\mathbb{A}_K)^\lambda$ associated to λ is defined as

$$U(\mathbb{A}_K)^\lambda := \bigcup_{\xi \in H^1(K,G)} \lambda_\xi(V_\xi(\mathbb{A}_K)) \subseteq U(\mathbb{A}_K). \tag{3-1}$$

This is adelically closed in $U(\mathbb{A}_K)$ [4, Proposition 6.4] and contains $U(K)$ [26, Section 5.3].

Compactifying the descent set. Let X be a smooth compactification of U. Recall from the introduction:

Definition 3.1.1. The *ramified descent set* for λ is

$$X(\mathbb{A}_K)^\lambda := \overline{\bigcup_{\xi \in H^1(K,G)} \lambda_\xi(V_\xi(\mathbb{A}_K))}.$$

The closure denotes the adelic closure in $X(\mathbb{A}_K)$. Let $\psi : Y \to X$ be the relative normalization of X in V. By the universal property of the relative normalization, the G-action on V extends to a G-action on Y. Let $\nu : Y^{\mathrm{sm}} \to Y$ be a G-equivariant desingularization of Y [11], and let ψ^{sm} be the composition $\psi \circ \nu : Y^{\mathrm{sm}} \to X$. The following lemma provides alternative descriptions of the ramified descent set.

Lemma 3.1.2. *We denote by ψ_ξ (resp. ψ_ξ^{sm}) the twisted forms of ψ (resp. ψ^{sm}) by $\xi \in H^1(K, G)$. The following sets coincide*:

(i) $\overline{U(\mathbb{A}_K)^\lambda}$,

(ii) $\overline{\bigcup_{\xi \in H^1(K,G)} \psi_\xi^{\mathrm{sm}}(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))}$,

(iii) $\overline{\bigcup_{\xi \in H^1(K,G)} \psi_\xi(Y_\xi^{reg}(\mathbb{A}_K))}$, *where Y^{reg} is the open subscheme of regular points of Y.*

The closures denote adelic closures in $X(\mathbb{A}_K)$.

Proof. We first prove that (i) and (ii) coincide. Note that $V' := \nu^{-1}(V) \xrightarrow{\nu} V$ is an isomorphism since V is regular. We have

$$\overline{\bigcup_\xi \lambda_\xi(V_\xi(\mathbb{A}_K))} = \overline{\bigcup_\xi \psi_\xi^{\mathrm{sm}}(V'_\xi(\mathbb{A}_K))} = \overline{\bigcup_\xi \overline{\psi_\xi^{\mathrm{sm}}(V'_\xi(\mathbb{A}_K))}} = \overline{\bigcup_\xi \psi_\xi^{\mathrm{sm}}(\overline{V'_\xi(\mathbb{A}_K)})} = \overline{\bigcup_\xi \psi_\xi^{\mathrm{sm}}(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))}, \tag{3-2}$$

where the union is over $\xi \in H^1(K, G)$ everywhere, and in the third term, $\overline{V'_\xi(\mathbb{A}_K)}$ denotes the closure in $Y_\xi^{\mathrm{sm}}(\mathbb{A}_K)$. The first two identities are immediate, the third follows from the properness of ψ_ξ^{sm}, and the fourth holds because $V'_\xi(\mathbb{A}_K)$ is dense in $Y_\xi^{\mathrm{sm}}(\mathbb{A}_K)$ (this follows from [7, Theorem 10.5.1] since Y_ξ^{sm} is smooth). This proves that (i) and (ii) coincide. They also coincide with (iii), since this is contained between the left- and right-hand sides of (3-2). ☐

Remark 3.1.3.
- Since (iii) is independent of the choice of U and Y^{sm}, the lemma shows that (ii) is as well, and (i) is too in the sense that $X(\mathbb{A}_K)^\lambda$ only depends on the generic fiber $\lambda|_{\operatorname{Spec} K(X)} : \operatorname{Spec} K(V) \to \operatorname{Spec} K(X)$, viewed as a G-torsor, of the torsor λ.
- No conflict of notation on $X(\mathbb{A}_K)^\lambda$ arises with (3-1) when $U = X$, as in this case $U(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^\lambda$ is closed in $X(\mathbb{A}_K)$ by [4, Proposition 6.4].

Warning. As the continuous map $U(\mathbb{A}_K) \hookrightarrow X(\mathbb{A}_K)$ is not a topological immersion, the set $X(\mathbb{A}_K)^\lambda \cap U(\mathbb{A}_K)$ might in general very well be bigger than $U(\mathbb{A}_K)^\lambda$. The reader may verify that in the example given in Section 6.1 this is exactly the case.

Setting. From now on we fix, through Section 5, a number field K, a finite group scheme G/K, a G-torsor $\lambda : V \to U$ over a geometrically integral smooth K-variety $p : U \to \operatorname{Spec} K$, and a smooth compactification X of U.

3.2. ***Obstruction to adelic density of rational points on X.*** Let, as above, $\psi : Y \to X$ be the relative normalization of X in V, $\nu : Y^{\mathrm{sm}} \to Y$ be a G-equivariant desingularization of Y, and r be the composition $\psi \circ \nu : Y^{\mathrm{sm}} \to X$.

Theorem 3.2.1. *The inclusion* $X(K) \subseteq \bigcup_{\xi \in H^1(K,G)} r_\xi(Y_\xi(K))$ *holds.*

Combining Theorem 3.2.1 with Lemma 3.1.2(ii), we deduce:

Corollary 3.2.2. *The inclusion* $\overline{X(K)} \subseteq X(\mathbb{A}_K)^\lambda$ *holds.*

The following proof of Theorem 3.2.1 is due to Olivier Wittenberg, who kindly suggested a proof that is much simpler than the previous one the author had.

Proof of Theorem 3.2.1 (Olivier Wittenberg). Let $d = \dim X$, $P \in X(K)$ be a rational point, and $u_1, \dots, u_d \in \mathcal{O}_{X,P}$ be a regular system of parameters at P. Let $C \subseteq X$ be the Zariski-closure of the curve $u_2 = \cdots = u_d = 0$. Since K is infinite, after a linear change of coordinates of $u_1, \dots, u_d$, we may assume that C is not contained in $D := X \setminus U$.

Note that C is smooth at P. Choosing a local parameter t for C at P, we get a morphism $\operatorname{Spec} K[\![t]\!] \to C$ that sends the special point $t = 0$ to P. This morphism induces a morphism $\operatorname{Spec} K((t)) \to X$, whose set-theoretic image is the generic point of C. In particular, by construction of C, it belongs to U. Hence the G-torsor $V \to U$ gives a class in $H^1(K((t)), G)$, which we may push to $H^1(K((t^{\frac{1}{\infty}})), G)$.

The inclusion $K \subseteq K((t^{\frac{1}{\infty}}))$ induces an identification $\Gamma_{K((t^{\frac{1}{\infty}}))} = \Gamma_K$ (this follows from the algebraic-closedness of $\overline{K}((t^{\frac{1}{\infty}}))$ [24, Chapter IV, Proposition 8]), and hence an identification $H^1(K((t^{\frac{1}{\infty}})), G) = H^1(K, G)$. Hence, after replacing Y with a K-twist, we may assume that the class in $H^1(K((t^{\frac{1}{\infty}})), G)$ is trivial. Therefore it has to be trivial already in $H^1(K((t^{\frac{1}{n}})), G)$ for some $n \geq 1$. In other words, the G-torsor

$$\operatorname{Spec} K((t^{\frac{1}{n}})) \times_U V \to \operatorname{Spec} K((t^{\frac{1}{n}}))$$

has a section. This section induces a commutative diagram

$$\begin{tikzcd} & V \arrow[d] \\ \operatorname{Spec} K((t^{\frac{1}{n}})) \arrow[ur] \arrow[r] & U. \end{tikzcd}$$

By the valuative criterion of properness (applied to $Y^{\mathrm{sm}} \to X$), we may extend the diagram above to

$$\begin{tikzcd} & Y^{\mathrm{sm}} \arrow[d] \\ \operatorname{Spec} K[\![t^{\frac{1}{n}}]\!] \arrow[r] \arrow[ur] & X. \end{tikzcd}$$

Since the lower morphism specializes to P, the specialization of the diagonal morphism provides the sought lift of P. □

4. Harari's question

Setup. Recall that $p : U \to \operatorname{Spec} K$ is a geometrically integral smooth variety over a number field K, X is a smooth compactification of U, and $\lambda : V \to U$ is a torsor under a finite group scheme G/K. We assume here that $G = A$ is commutative, and let $A' = \operatorname{Hom}(A, \mathbb{G}_{m,K})$ be its Cartier dual. For every v, define $E_v := \operatorname{Im}\big(U(K_v) \to H^1(K_v, A),\ P_v \mapsto [V|_{P_v}]\big)$ (this is not a subgroup in general). Let S be a finite set of places of K, let $(P_v)_{v\in S} \in \prod_{v\in S} U(K_v)$ and $f_v := [V|_{P_v}],\ v \in S$.

Let $\operatorname{Br}_\lambda U < \operatorname{Br}_1(U)$ be the subgroup generated by the cup products $p^*b \cup [V]$, as b varies in $H^1(K, A')$, and let $B := \operatorname{Br}_\lambda U \cap \operatorname{Br}(X)$.

Question 4.1 (Harari). Assume that there is no Brauer–Manin obstruction for $(P_v)_{v\in S}$ with respect to B. Does there exist then an $a \in H^1(K, A)$ such that $a_v = f_v$ for all $v \in S$ and $a_v \in E_v$ for $v \notin S$?

Using Poitou–Tate duality, one may obtain a positive answer to Question 4.1 by replacing E_v with the subgroup $\langle E_v\rangle$ of $H^1(K_v, A)$ generated by it (we leave this as an exercise to the interested reader, or see [10]). However, as originally remarked by Harari, there is a big difference between E_v and $\langle E_v\rangle$ in general!

The following proposition relates Question 4.1 with Question 1.2:

Proposition 4.2. *Assume that* $\operatorname{Br} X/\operatorname{Br}_0 X$ *is finite and* $X(\mathbb{A}_K) \neq \varnothing$. *Then the identity*

$$X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^B$$

holds if and only if there exists a finite $S_0 \subseteq M_K$ *such that, for all* $S \supseteq S_0$, *Question 4.1 has a positive answer.*

Proof. The assumption implies that B is finite.

We prove the forward implication first. Let S_0 be a set of places such that the Brauer–Manin pairing associated to B is trivial outside S_0. Let $S \subseteq M_K$ be a finite set containing S_0, and $(P_v)_{v\in S} \in \big(\prod_{v\in S} U(K_v)\big)^B$. We wish to find an $a \in H^1(K, A)$ such that $a_v = f_v$ for all $v \in S$ and $a_v \in E_v$ for all $v \notin S$. Let $P_v,\ v \notin S$ be any point of $U(K_v)$. Note that $(P_v)_{M_K} \in X(\mathbb{A}_K)^B$. Since $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^B$, we may approximate arbitrarily well $(P_v)_{M_K}$ with an adelic point $(Q_v)_{v\in M_K}$ such that there exists $a \in H^1(K, A)$ for which $(Q_v)_{v\in M_K} \in \lambda_a(V_a(\mathbb{A}_K))$. In particular, $Q_v \in \lambda_a(V_a(K_v))$ for each v, or in other words the torsor $[V_a|_{Q_v}]$ over K_v contains a K_v-point and is thus trivial. It follows that $0 = [V_a|_{Q_v}] = [V|_{Q_v}] - a_v \in H^1(K_v, A)$ for all v, and hence $a_v \in E_v$ for all v. Moreover, the map $[V|_{-}] : U(K_v) \to H^1(K_v, A)$ is locally constant, and thus $[V|_{Q_v}] = [V|_{P_v}] = f_v$ for $v \in S$. The class a is now the sought class.

For the other direction, assume there is an S_0 as in the statement. We need to show that $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^B$. We may assume after enlarging S_0 that its Brauer–Manin pairing on X is trivial for all places $v \notin S_0$.

Let $(P_v)_{M_K} \in X(\mathbb{A}_K)^B$. Since B is finite, $X(\mathbb{A}_K)^B \subseteq X(\mathbb{A}_K)$ is open. In particular, after an arbitrarily small approximation, we may assume that $(P_v)_{M_K} \in U(K_\Omega)^B$. Then, for all $S \supseteq S_0$, our assumption that the answer to Question 4.1 is "yes" shows that there exists an $a \in H^1(K, A)$ such that $a_v = [V|_{P_v}]$ for $v \in S$ and $a_v = [V|_{Q_v}]$ for some $Q_v \in U(K_v)$ for $v \notin S$. Thus the A-torsor $V_a \to X$ specializes to the trivial torsor over P_v, $v \in S$ and over Q_v, $v \notin S$, and hence there exists a point $(R_v) \in V_a(K_\Omega)$ whose image is P_v for $v \in S$ and Q_v for $v \notin S$. In particular, $V_a(K_\Omega) \neq \varnothing$ and therefore $V_a(\mathbb{A}_K) \neq \varnothing$, and we may thus modify R_v so that $(R_v) \in V_a(\mathbb{A}_K)$ and so that the image of R_v is P_v for $v \in S$. We now modify Q_v to $\lambda_a(R_v)$ for $v \notin S$, and by construction we have $((P_v)_{v\in S}, (Q_v)_{v\notin S}) \in U(\mathbb{A}_K)^\lambda$. Enlarging S, the points $((P_v)_{v\in S}, (Q_v)_{v\notin S})$ approximate arbitrarily well $(P_v)_{M_K}$, and thus we deduce that $(P_v)_{M_K} \in X(\mathbb{A}_K)^\lambda$. $\square$

5. A Brauer–Manin obstruction to ramified descent

We recall that $\lambda : V \to U$ is a torsor under a finite group scheme G/K, that U is smooth and geometrically integral over K, that X is a compactification of U, and that $Y \to X$ is the relative normalization of X in V. We defined

$$X(\mathbb{A}_K)^\lambda = \overline{\bigcup_{\xi \in H^1(K,G)} \lambda_\xi(V_\xi(\mathbb{A}_K))}$$

We define in this section a subgroup $\mathrm{Br}^{\mathrm{ram}}_\lambda X \subseteq \mathrm{Br}\, X$ such that $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\mathrm{Br}^{\mathrm{ram}}_\lambda X}$. The group $\mathrm{Br}^{\mathrm{ram}}_\lambda X$ may be transcendental, as we show in Section 6.1.

5.1. *Definition of* $\mathbf{Br}^{\mathbf{ram}}_\lambda$ *X*. Let $\mathfrak{l}$ be the composition $V_{\overline{K}} \to V \xrightarrow{\lambda} U$. There are natural $G(\overline{K})$- and Γ_K-actions on $V_{\overline{K}}$, the first induced by the G-action on V, and the second via the second factor of $V_{\overline{K}} = V \times_K \operatorname{Spec} \overline{K}$.

Lemma 5.1.1. *The $G(\overline{K})$- and the Γ_K- actions generate a $(G(\overline{K}) \rtimes \Gamma_K)$-action on $V_{\overline{K}}$. The profinite étale cover $\mathfrak{l} : V_{\overline{K}} \to U$ is a profinite torsor under this $(G(\overline{K}) \rtimes \Gamma_K)$-action.*

Proof. Let L/K be a field extension. The points of V_L with values in a K-algebra R are described by

$$V_L(R) = \bigsqcup_{\iota : L \hookrightarrow R} V(R), \tag{5-1}$$

where ι ranges among all K-embeddings of L in R. Consider the following natural actions:

(i) the right action of $G(L)$ on V_L defined by letting $G(L)$ act via the map $G(L) \to G(R)$ on each disjoint set appearing in (5-1);

(ii) when L/K is Galois, the right $\mathrm{Gal}(L/K)$-action defined by letting $\gamma \in \mathrm{Gal}(L/K)$ act via $\iota \mapsto \iota \circ \gamma^{-1}$.

For $g \in G(L)$ and $\gamma \in \mathrm{Gal}(L/K)$, we have ${}^\gamma g \cdot \gamma \cdot x = \gamma \cdot g \cdot x$, where x is an R-point of V_L. By this relation, the two actions above generate a $(G(L) \rtimes \mathrm{Gal}(L/K))$-action on V_L. This action is fixed-point-free and commutes with the projection $V_L \to U$. In addition, when L/K is finite and splits G (i.e., when $G(L)=G(\overline{K})$), letting $\Gamma := G(L)\rtimes \mathrm{Gal}(L/K)$, we have $|\Gamma| = |G(\overline{K})|\cdot[L:K] = \deg(V_L\to U)$. Thus the morphism $(\mu, id):\Gamma\times_U V_L\to V_L\times_U V_L$, where μ denotes the Γ-action on V_L, is an injective morphism of finite étale covers of U of the same degree, and hence an isomorphism. In other words, $V_L\to U$ is a torsor under Γ.

When $L=\overline{K}$, the two actions (i) and (ii) are the ones described before the statement of the lemma, and taking an inverse limit over all finite Galois subextensions $L\subset\overline{K}$ that split G (these form a cofinal subset) finishes the proof. □

Let $\Gamma_G := G(\overline{K})\rtimes\Gamma_K$. Through the construction of Section 2, the profinite étale torsor $V_{\overline{K}}\to U$ under Γ_G gives rise to a Čech-to-étale map on cohomology:

$$\check{C}_{\mathfrak{l}} : H^2(\Gamma_G, \overline{K}[V]^*)\to H^2(U,\mathbb{G}_m), \tag{5-2}$$

where Γ_G acts on $\mathbb{G}_m(V_{\overline{K}})=\overline{K}[V]^*$ by pullback. The restriction of this Γ_G-action on $\overline{K}^*\subseteq\overline{K}[V]^*$ is equal to the pullback of the natural Γ_K-action along the projection $\Gamma_G\to\Gamma_K$. Hence we have a natural morphism

$$H^2(\Gamma_G,\overline{K}^*)\to H^2(\Gamma_G,\overline{K}[V]^*) = H^2(\Gamma_G,\mathbb{G}_m(V_{\overline{K}})),$$

where the implied action on the LHS is the pullback described above.

Definition 5.1.2. We define the subgroup $\mathrm{Br}^{\mathrm{ram}}_{\lambda}(U)$ of $\mathrm{Br}\,U$ as the image of the composition

$$H^2(\Gamma_G,\overline{K}^*)\to H^2(\Gamma_G,\overline{K}[V]^*)\xrightarrow{\check{C}_{\mathfrak{l}}} H^2(U,\mathbb{G}_m)=\mathrm{Br}\,U.$$

We define $\mathrm{Br}^{\mathrm{ram}}_{\lambda} X\subseteq \mathrm{Br}(X)$ as the intersection $\mathrm{Br}(X)\cap\mathrm{Br}^{\mathrm{ram}}_{\lambda}(U)$.

Remark 5.1.3. The fact that U does not appear in the notation "$\mathrm{Br}^{\mathrm{ram}}_{\lambda} X$" is justified by the fact that $\mathrm{Br}^{\mathrm{ram}}_{\lambda} X$ may be defined purely in terms of the ramified G-cover $Y\to X$ (and, in fact, only in terms of the "generic fiber" G-torsor $\mathrm{Spec}\,K(Y)\to\mathrm{Spec}\,K(X)$). Indeed $\mathrm{Br}^{\mathrm{ram}}_{\lambda} X = \mathrm{Br}(X)\cap\mathrm{Br}^{\mathrm{ram}}_{\lambda}(K(X))$, where $\mathrm{Br}^{\mathrm{ram}}_{\lambda}(K(X))\subseteq\mathrm{Br}(K(X))$ is defined as the image of $H^2(\Gamma_G,\overline{K}^*)$ in $H^2(K(X),\mathbb{G}_m)$ through the morphism

$$H^2(\Gamma_G,\overline{K}^*)\to H^2(\Gamma_G,\overline{K}(Y)^*)\xrightarrow{\check{C}_{\mathfrak{l}|K(X)}} H^2(K(X),\mathbb{G}_m).$$

Remark 5.1.4. When $\overline{K}[V]^*/\overline{K}^* = \mathrm{Pic}\,V_{\overline{K}}=0$, the Hochschild–Serre spectral sequence

$$H^i(\Gamma_G, H^j(V_{\overline{K}},\mathbb{G}_m))\Rightarrow H^{i+j}(U,\mathbb{G}_m)$$

yields the short exact sequence $0\to H^2(\Gamma_G,\overline{K}^*)\to\mathrm{Br}\,U\to\mathrm{Br}\,V_{\overline{K}}$. Hence, in this case, $\mathrm{Br}^{\mathrm{ram}}_{\lambda}(U) = \mathrm{Ker}(\mathrm{Br}\,U\to\mathrm{Br}\,V_{\overline{K}})$ and

$$\mathrm{Br}^{\mathrm{ram}}_{\lambda} X = \mathrm{Br}^{\mathrm{ram}}_{\lambda}(U)\cap\mathrm{Br}\,X = \mathrm{Ker}(\mathrm{Br}\,X\to\mathrm{Br}\,V_{\overline{K}}).$$

Although the condition $\overline{K}[V]^*/\overline{K}^* = \mathrm{Pic}\,V_{\overline{K}}=0$ is rarely satisfied in practice, we give some examples below.

Examples 5.1.5. (I) If V is a simply connected semi-simple algebraic group, then $\overline{K}[V]^*/\overline{K}^* = \operatorname{Pic} V_{\overline{K}} = 0$. To get an example of a G-torsor $V \to U$, we may take as G any finite subgroup(-scheme) of V, and define $U := V/G$.

(II) If V is a universal torsor of a smooth proper rationally connected variety (as defined in [5]), one has $\overline{K}[V]^*/\overline{K}^* = \operatorname{Pic} V_{\overline{K}} = 0$ (see (2.1.1) of loc.cit.). We then let G be any finite subgroup of the Néron–Severi torus T (i.e., the one under which V is a torsor), and let $U = V/G$.

(III) Finally, we give an example where both V and U are open K3 surfaces. Let $\mathcal{E} \to \mathbb{P}^1_K$ be an elliptic K3 surface with Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$, with (exactly) two reducible fibers of Kodaira types I_{10}^* and I_2, and such that the 0-section O and the 2-torsion section τ intersect different components of the I_2-fiber. Such an $\mathcal{E}$ exists, and it may be defined over $\mathbb{Q}$; see 5.2 and 5.3 of [19], for example, where $\mathcal{E}$ is realized as the double-cover of the Kummer surface associated to the Jacobian of a curve of genus 2. Then the Picard group of $\mathcal{E}\overline{K}$ is of rank 16, freely generated by the fourteen components of the I_{10}^*-fiber, by O and by τ. In particular, letting V be the complement of these divisors in $\mathcal{E}$, we have $\overline{K}[V]^* = \overline{K}^*$ and $\operatorname{Pic} \bar{V} = 0$. The involution on $\mathcal{E}$ induced by τ restricts to an involution of V, let $G \cong \mathbb{Z}/2\mathbb{Z}$ be the group generated by it. A smooth minimal compactification X of the quotient $U = V/G$ is an elliptic surface isogenous to the original K3 elliptic surface $\mathcal{E} \to \mathbb{P}^1$, and is thus K3. (In fact, if $\mathcal{E}$ is as in [19], the quotient $U = V/G$ is a Kummer surface; see loc.cit.).

Proposition 5.1.6. *Assume that $G = A$ is commutative. Then $\operatorname{Br}(X) \cap \operatorname{Br}_\lambda U$ is contained in $\operatorname{Br}_\lambda^{\mathrm{ram}} X$.*

Proof. We prove the stronger inclusion $\operatorname{Br}_\lambda U \subseteq \operatorname{Br}_\lambda^{\mathrm{ram}} U$. Recall that $\operatorname{Br}_\lambda U$ is generated by the cup products $p^*b \cup [V]$, as b varies in $H^1(K, A')$. Fix a $b \in H^1(K, A')$. Both classes p^*b and $[V]$ trivialize when pulled back along the pro-étale cover $V_{\overline{K}} \to U$. Hence, following, e.g., [26, p. 18], there exist Čech cocycles $b_V \in \check{H}^1(V_{\overline{K}}/U, A'), \alpha_V \in \check{H}^1(V_{\overline{K}}/U, A)$ that represent p^*b and $[V]$, i.e., such that $\check{C}_{\mathfrak{l}}(b_V) = p^*b$ and $\check{C}_{\mathfrak{l}}(\alpha_V) = [V]$.

We have a commutative diagram of cup products [29, Corollary 3.10]

$$\begin{array}{ccccc}
\check{H}^1(V_{\overline{K}}/U, A') & \times & \check{H}^1(V_{\overline{K}}/U, A) & \xrightarrow{\cup} & \check{H}^2(V_{\overline{K}}/U, \overline{K}^*) \\
\Big\| {\scriptstyle =} & & \Big\| {\scriptstyle =} & & \Big\downarrow \\
\check{H}^1(V_{\overline{K}}/U, A') & \times & \check{H}^1(V_{\overline{K}}/U, A) & \xrightarrow{\cup} & \check{H}^2(V_{\overline{K}}/U, \overline{K}[V]^*) \\
\Big\downarrow {\scriptstyle \check{C}_{\mathfrak{l}}} & & \Big\downarrow {\scriptstyle \check{C}_{\mathfrak{l}}} & & \Big\downarrow {\scriptstyle \check{C}_{\mathfrak{l}}} \\
H^1(U, A') & \times & H^1(U, A) & \longrightarrow & H^2(U, \mathbb{G}_m),
\end{array}$$

see, e.g., [29] for the definition of the Čech cup product on the first two rows. In particular, we have $\check{C}_{\mathfrak{l}}(b_V \cup \alpha_V) = \check{C}_{\mathfrak{l}}(b_V) \cup \check{C}_{\mathfrak{l}}(\alpha_V) = p^*b \cup [V]$, and $p^*b \cup [V]$ belongs to the image $\operatorname{Br}_\lambda^{\mathrm{ram}} U$ of the composition $\check{H}^2(V_{\overline{K}}/U, \overline{K}^*) \to \check{H}^2(V_{\overline{K}}/U, \overline{K}[V]^*) \xrightarrow{\check{C}_{\mathfrak{l}}} H^2(U, \mathbb{G}_m)$. □

We go on to prove Proposition 1.3, i.e., that $X(\mathbb{A}_K)^\lambda$ is contained in $X(\mathbb{A}_K)^{\operatorname{Br}_\lambda^{\mathrm{ram}} X}$. I profoundly thank one of the anonymous referees, who provided the following proof, which simplifies a lot the previous one the author had in mind.

Proof of Proposition 1.3. We shall prove that $U(\mathbb{A}_K)^\lambda \subseteq U(\mathbb{A}_K)^{\mathrm{Br}^{\mathrm{ram}}_\lambda U}$, from which the statement follows by taking adelic closures in $X(\mathbb{A}_K)$ and noting that $U(\mathbb{A}_K)^{\mathrm{Br}^{\mathrm{ram}}_\lambda U} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}^{\mathrm{ram}}_\lambda X}$.

Let $(P_v)_{v\in M_K} \in U(\mathbb{A}_K)^\lambda$ and let $\xi \in Z^1(K, G)$ be such that $P_v = \lambda_\xi(Q_v)$, $v \in M_K$ for some $(Q_v)_{v\in M_K} \in V_\xi(\mathbb{A}_K)$. The cocycle ξ defines a section $s_\xi = (\xi, id)$ of the surjection

$$\Gamma_G = G(\overline{K}) \rtimes \Gamma_K \to \Gamma_K \to 1.$$

Recall that $V_{\overline{K}}$ is naturally endowed with a Γ_G-action that makes $V_{\overline{K}} \to U$ a profinite Γ_G-torsor. One easily checks from the definition of twist [26, p.12] that

$$V_\xi = V_{\overline{K}}/s_\xi(\Gamma_K).$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \overline{V} & \xleftarrow{=} & \overline{V} \\ \big\downarrow{\scriptstyle \Gamma_G} & & \big\downarrow{\scriptstyle \Gamma_K} \\ U & \longleftarrow & V_\xi \end{array},$$

equivariant with respect to $s_\xi : \Gamma_K \to \Gamma_G$, which induces by functoriality of $\check{C}$ the commutative diagram

$$\begin{array}{ccccc} H^2(\Gamma_G, \overline{K}^*) & \longrightarrow & \check{H}^2(\overline{V}/U, \mathbb{G}_m) & \xrightarrow{\check{C}_{\mathrm{I}}} & \mathrm{Br}\, U \\ \big\downarrow{\scriptstyle res^{\Gamma_G}_{s_\xi(\Gamma_K)}} & & \big\downarrow{\scriptstyle \lambda_\xi^*} & & \big\downarrow{\scriptstyle \lambda_\xi^*} \\ \mathrm{Br}\, K = H^2(\Gamma_K, \overline{K}^*) & \longrightarrow & \check{H}^2(\overline{V}/V_\xi, \mathbb{G}_m) & \xrightarrow{\check{C}_{\overline{V}/V_\xi}} & \mathrm{Br}\, V_\xi \end{array}.$$

Recalling that $\mathrm{Br}^{\mathrm{ram}}_\lambda U$ is defined to be the image of the upper composition, we deduce by the commutativity that $\lambda_\xi^* \mathrm{Br}^{\mathrm{ram}}_\lambda U \subseteq \mathrm{Br}_0 V_\xi$. Thus, for any $B \in \mathrm{Br}^{\mathrm{ram}}_\lambda U$,

$$((P_v), B)_{BM} = ((Q_v), \lambda_\xi^* B)_{BM} = 0,$$

as wished. □

In summary, we have the series of inclusions

$$X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\mathrm{Br}^{\mathrm{ram}}_\lambda X} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}\, X \cap \mathrm{Br}_\lambda U}. \tag{5-3}$$

However, in contrast to what happens on U (where the inclusions $U(\mathbb{A}_K)^\lambda \subseteq U(\mathbb{A}_K)^{\mathrm{Br}^{\mathrm{ram}}_\lambda U} \subseteq U(\mathbb{A}_K)^{\mathrm{Br}_\lambda(U)}$ are actually identities by [15]), the last inclusion in (5-3) may well be strict! (See Section 6.1).

6. Unramified Brauer groups of SL_n/H, H metabelian

For Section 6 fix a finite group scheme H over a field k of characteristic 0. Let $B = [H, H]$, and $A = H/[H, H]$. We assume that H is *metabelian*, i.e., that B is commutative.

Let $U = \mathrm{SL}_{n,k}/H$, and $V = \mathrm{SL}_{n,k}/B$. As B is normal in H with quotient A, the morphism $\lambda : V \to U$ is an A-torsor, where the A-action on V is the one that on $\bar{k}$-points is given by $(\mathrm{SL}_{n,K}/B) \times A \to \mathrm{SL}_{n,K}/B$, $(xB, a) \mapsto xBa = xaB$.

Recall that $\mathfrak{l} : \overline{V} \to U$ induces a Čech-to-étale map on cohomology:

$$\check{C}_{\mathfrak{l}} : H^2(\Gamma_A, \overline{K}^*) = H^2(\Gamma_A, \overline{K}[V]^*) \to \operatorname{Br} U.$$

The author thanks Olivier Wittenberg for making him notice the following:

Theorem 6.0.1. *If* $Ш^1_\omega(K, B') = 0$, *then* $\operatorname{Br} X = \operatorname{Br}^{\mathrm{ram}}_\lambda X$.

(Recall that X denotes a smooth compactification of U.)

Proof. It suffices to prove that $\operatorname{Br}_{\mathrm{ur}} U \subseteq \operatorname{Im} H^2(\Gamma_A, \overline{K}^*)$. Consider the Hochschild–Serre spectral sequence of $\overline{V} \xrightarrow{\Gamma_G} U$, keeping in mind that $\overline{K}[V]^* = \overline{K}^*$ and $\operatorname{Pic} \overline{V} = B'$, we get the exact sequence

$$H^2(\Gamma_A, \overline{K}^*) \to \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br} \overline{V}) \to H^1(\Gamma_A, B') \tag{6-1}$$

Since $Ш^1_\omega(K, B') = 0$, we have $\operatorname{Br}_{\mathrm{ur}} V = \operatorname{Br} K$ by [13, Proposition 4]. Thus any element of $\operatorname{Br}_{\mathrm{ur}} U$ lies in the kernel of $\operatorname{Br} U \to \operatorname{Br} \overline{V}$. In addition, we know that after base-changing to $\overline{K}$ every element of $\operatorname{Br}_{\mathrm{ur}} U$ comes from $H^2(A, \overline{K}^*)$ (see [1, Lemma 5.1]). In particular, $\operatorname{Br}_{\mathrm{ur}} U$ maps to $\operatorname{Ker}(H^1(\Gamma_A, B') \xrightarrow{\mathrm{res}} H^1(A, B'))$ in the sequence above. By the inflation-restriction five-term sequence of $A \trianglelefteq \Gamma_A$, we have $\operatorname{Ker}(H^1(\Gamma_A, B') \to H^1(A, B')) = H^1(\Gamma_K, B')$. For $\beta \in \operatorname{Br}_{\mathrm{ur}} U$, we denote by $\delta(\beta)$ its image in $H^1(\Gamma_K, B')$.

Finally, by functoriality of the Hochschild–Serre spectral sequence, we get the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma_A, \overline{K}^*) \arrow[r] \arrow[d] & \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br} \overline{V}) \arrow[r] \arrow[d] & H^1(\Gamma_A, B') \arrow[d] \\
H^2(\Gamma_K, \overline{K}^*) \arrow[r] & \operatorname{Ker}(\operatorname{Br} V \to \operatorname{Br} \overline{V}) \arrow[r] & H^1(\Gamma_K, B'),
\end{tikzcd}$$

where the second row is just (6-1) with $A = 0$. Hence (again because $\operatorname{Br}_{\mathrm{ur}} V = \operatorname{Br} K$) every element of $\operatorname{Br}_{\mathrm{ur}} U$ has to map to 0 in $H^1(\Gamma_K, B')$. This implies that $\delta(\beta) = 0$ for every β. Hence $\operatorname{Br}_{\mathrm{ur}} U \subseteq \operatorname{Im} H^2(\Gamma_A, \overline{K}^*) = \operatorname{Br}^{\mathrm{ram}}_\lambda U$, therefore $\operatorname{Br} X = \operatorname{Br}_{\mathrm{ur}} U = \operatorname{Br}^{\mathrm{ram}}_\lambda U \cap \operatorname{Br} X = \operatorname{Br}^{\mathrm{ram}}_\lambda X$, as wished. □

By Chebotarev's theorem, $Ш^1_\omega(K, B') = 0$ when B' is constant. We thus get:

Corollary 6.0.2. *If B is constant of exponent e and $\mu_e \subseteq K^*$, then* $\operatorname{Br} X = \operatorname{Br}^{\mathrm{ram}}_\lambda X$.

Proposition 1.4 is a consequence of the above corollary.

6.1. *Nilpotent H*. For the rest of the section we assume that B is central in H and that H is constant of prime exponent p, where $p \neq 2$ and $\mu_p \subseteq k^*$. We choose a primitive p-th root of unity, and use it to identify throughout this subsection μ_p with $\mathbb{Z}/p\mathbb{Z}$.

In particular, we shall always tacitly identify μ_p with $\mathbb{Z}/p\mathbb{Z}$, after making the implicit choice of a p-th root of unity.

Our aim in this subsection is to explicitly describe $\operatorname{Br}_{\mathrm{ur}} U$; see Theorem 6.1.3 below. We may naturally associate to the central extension

$$1 \to B \to H \to A \to 1$$

a class $[H] \in H^2(A, B)$ (see [2, Section IV.3]). We denote by $[-,-] : A \times A \to B$ the map that sends $a_1, a_2 \in A$ to the commutator of (any two) lifts $\bar{a}_1, \bar{a}_2$ of them in H. Let $\mathfrak{c} : \Lambda^2 A \to B$ be the natural homomorphism $a_1 \wedge a_2 \mapsto [a_1, a_2]$.

Recall that $A^D = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. We shall make frequent use of the identification

$$\Lambda^2(A^D) = (\Lambda^2 A)^D, \alpha_1 \wedge \alpha_2 \mapsto (a_1 \wedge a_2 \mapsto \alpha_1(a_1)\alpha_2(a_2) - \alpha_1(a_2)\alpha_2(a_1))\,.$$

We denote $\Lambda^2(A^D) = (\Lambda^2 A)^D$ with $\Lambda^2 A^D$. Let $(\Lambda^2 A^D)_{\mathrm{bic}} \subseteq \Lambda^2 A^D$ be the subgroup of elements $\beta \in \Lambda^2 A^D$ such that $\beta(a_1 \wedge a_2) = 0$ for any $a_1, a_2 \in A$ with $[a_1, a_2] = 0$.

Let $\xi_U : \Lambda^2 A^D = \Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Br}\, U$ be the composition of the three maps

$$\Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cup} H^2(A, \mathbb{Z}/p\mathbb{Z}) \qquad \text{(cup product)} \tag{6-2}$$

$$H^2(A, \mathbb{Z}/p\mathbb{Z}) = H^2(A, \mu_p) \xrightarrow{\check{C}_\lambda} H^2(U, \mu_p) \qquad \text{(Čech-to-étale map)} \tag{6-3}$$

$$H^2(U, \mu_p) \to H^2(U, \mathbb{G}_m) = \mathrm{Br}\, U \qquad \text{(changing coefficients)} \tag{6-4}$$

We also define analogously a map $\xi\overline{U} : \Lambda^2 A^D \to \mathrm{Br}\, \overline{U}$. The following is a reformulation of a result of Bogomolov (see Lemma 5.1 of [1]):

Theorem 6.1.1 (Bogomolov, reformulated). *Assume that k is algebraically closed. The image of $(\Lambda^2 A^D)_{\mathrm{bic}}$ under ξ_U is unramified, and the following sequence is exact*:

$$B^D \xrightarrow{\mathfrak{c}^D} (\Lambda^2 A^D)_{\mathrm{bic}} \xrightarrow{\xi_U} \mathrm{Br}_{\mathrm{ur}}\, U \to 1. \tag{6-5}$$

We introduce the following notation used in the proof. Let $\gamma \in \Lambda^2 A^D = \mathrm{Hom}(\Lambda^2 A, \mathbb{Q}/\mathbb{Z})$ be a cocycle. Define H_γ as the central extension of A by $\mathbb{Q}/\mathbb{Z}$, characterized by the following property: for any $a_1, a_2 \in A$ the commutator of their corresponding lifts $\bar{a}_1, \bar{a}_2$ in H_γ is

$$\bar{a}_1 \bar{a}_2 \bar{a}_1^{-1} \bar{a}_2^{-1} = \gamma(a_1 \wedge a_2) \in \mathbb{Q}/\mathbb{Z}.$$

Proof of Theorem 6.1.1. By Lemma 5.1 of [1], $\mathrm{Br}_{\mathrm{ur}}\, U$ is isomorphic to $\mathrm{Coker}(B^D \to (\Lambda^2 A^D)_{\mathrm{bic}})$ (the group S/S_Λ appearing in loc.cit. is the dual of $\mathrm{Coker}(B^D \to (\Lambda^2 A^D)_{\mathrm{bic}})$). It only remains to show that the induced homomorphism $\xi_{\mathrm{Bog}} : (\Lambda^2 A^D)_{\mathrm{bic}} \to \mathrm{Br}_{\mathrm{ur}}\, U$ is ξ_U. This follows from the proof of [1, Lemma 5.1], but we include a detailed proof for completeness.

One may read from pp. 462 and 469 in [1] that the induced morphism $\xi_{\mathrm{Bog}} : (\Lambda^2 A^D)_{\mathrm{bic}} \to \mathrm{Br}_{\mathrm{ur}}\, U$ is the restriction of the composition

$$\Lambda^2 A^D \cong H^2(A, \mathbb{Q}/\mathbb{Z}) = H^2(A, \mu_\infty) \to H^2(A, k^*) \xrightarrow{\check{C}_\lambda} \mathrm{Br}\, U, \tag{6-6}$$

where the last map is defined via the Hochschild–Serre spectral sequence, and the first isomorphism is defined by sending an element $\gamma \in \Lambda^2 A^D = \mathrm{Hom}(\Lambda^2 A, \mathbb{Q}/\mathbb{Z})$ to the class in $H^2(A, \mathbb{Q}/\mathbb{Z})$ of the central extension H_γ of A. The following lemma then shows that this composition is ξ_U. □

Lemma 6.1.2. *The isomorphism $\Lambda^2 A^D \cong H^2(A, \mathbb{Q}/\mathbb{Z}), \gamma \mapsto [H_\gamma]$ coincides with the composition $\Lambda^2 A^D \xrightarrow{\cup} H^2(A, \mathbb{Z}/p\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})$.*

Proof. For a monomial $\gamma = \gamma_1 \wedge \gamma_2 \in \Lambda^2 A^D$, the extension H_γ may be realized as the set $\mathbb{Q}/\mathbb{Z} \times A$ with multiplication defined by $(b_1, a_1) \cdot (b_2, a_2) = (b_1 + b_2 + \gamma_1(a_1)\gamma_2(a_2), a_1 + a_2)$. The class $[H_\gamma] \in H^2(A, \mathbb{Q}/\mathbb{Z})$ is then represented by the 2-cocycle $a_1, a_2 \mapsto \gamma_1(a_1)\gamma_2(a_2)$ [2, p.92]. This is precisely the cup product $\gamma_1 \cup \gamma_2$ [22, Proposition 1.4.8]. Thus the two maps coincide on monomials $\gamma_1 \wedge \gamma_2$. Since these span $\Lambda^2 A^D$, the two maps coincide, as wished. □

Let $\mathrm{Br}_e U = \mathrm{Ker}(e^* : \mathrm{Br}\, U \to \mathrm{Br}\, K)$, where e is the K-point corresponding to the equivalence class of the identity in SL_n / H. Recall that we have a direct sum decomposition $\mathrm{Br}\, U = \mathrm{Br}_e U \oplus \mathrm{Br}_0 U$, where $\mathrm{Br}_0 U \cong \mathrm{Br}\, K$ denotes the constant Brauer elements. Let $\mathrm{Br}_{\mathrm{e,ur}} U = \mathrm{Br}_e U \cap \mathrm{Br}_{\mathrm{ur}} U$. We now remove the algebraically closed assumption from Theorem 6.1.1:

Theorem 6.1.3. *The image of* $(\Lambda^2 A^D)_{\mathrm{bic}}$ *under* ξ_U *is contained in* $\mathrm{Br}_{\mathrm{e,ur}} U$, *and the following sequence is exact*:

$$B^D \xrightarrow{\mathfrak{c}^D} (\Lambda^2 A^D)_{\mathrm{bic}} \xrightarrow{\xi_U} \mathrm{Br}_{\mathrm{e,ur}} U \to 1. \tag{6-7}$$

Proof. The A-torsor λ restricts to the trivial torsor over $e \in U(K)$, which implies that the image of ξ_U is contained in $\mathrm{Br}_e U$ (see Remark 2.1).

By Bogomolov's Theorem 6.1.1 and the key Lemma 6.1.4 below, we have $\xi_U((\Lambda^2 A^D)_{\mathrm{bic}}) \subseteq \mathrm{Br}_{\mathrm{ur}} U$. Finally, the composition

$$\xi \bar{U} : (\Lambda^2 A^D)_{\mathrm{bic}} \xrightarrow{\xi_U} \mathrm{Br}_{\mathrm{e,ur}} U \hookrightarrow \mathrm{Br}_{\mathrm{ur}} \bar{U}.$$

is surjective with kernel $\mathrm{Im}(\mathfrak{c}^D : B^D \to \Lambda^2 A^D)$ by Bogomolov's Theorem, while the second morphism is injective by [20, Proposition 5.9]. Hence the last map is an isomorphism, and the statement follows. □

Lemma 6.1.4. *For* $\alpha \in \Lambda^2 A^D$, $\xi_U(\alpha) \in \mathrm{Br}_e U$ *is unramified if and only if* $\xi \bar{U}(\alpha) \in \mathrm{Br}\, \bar{U}$ *is unramified.*

Proof. The forward implication is clear; we prove the converse. Let $\alpha \in \Lambda^2 A^D$ be an element whose image in $\mathrm{Br}\, \bar{U}$ is unramified. By Bogomolov's Theorem 6.1.1, this unramifiedness is equivalent to α lying in $(\Lambda^2 A^D)_{\mathrm{bic}}$.

We first assume that $k = K$ is a number field. Let v be a finite place of k coprime with p. Let $P \in U(K_v)$. Let ψ be the projection $\mathrm{SL}_n \to \mathrm{SL}_n / H$. Choose a geometric point $\bar{P} \in \psi^{-1}(P)(\overline{K_v})$, and let $f : \Gamma_v \to H$ be the homomorphism defined by $\gamma \cdot \bar{P} = \bar{P} \cdot f(\gamma)$. Since $v \nmid p$ the homomorphism f factors though the tame decomposition group Γ_v^{tame}, which is topologically generated by elements ι and ϕ satisfying $\phi \iota \phi^{-1} = \iota^{Nv}$. Since $\mu_p \subseteq K^*$, we have $Nv \equiv 1 \bmod p$. In particular, since the exponent of H is p, the images of ι and ϕ in H commute. The group $A' = \langle f(\iota), f(\phi) \rangle$ is then bicyclic. The point P lifts via $(\mathrm{SL}_n / A')(K_v) \to (\mathrm{SL}_n / H)(K_v)$.

Let $g : A' \to A$ be the projection, and consider the commutative diagram

$$\begin{tikzcd} \Lambda^2 (A')^D \arrow[d, "\xi_{\mathrm{SL}_n / A'}"] & \Lambda^2 A^D \arrow[l, "g^*"'] \arrow[d, "\xi_U = \xi_{\mathrm{SL}_n / H}"] \\ \mathrm{Br}(\mathrm{SL}_n / A') & \mathrm{Br}(\mathrm{SL}_n / H) \arrow[l] \end{tikzcd} .$$

Let $\bar{a}_1 = f(\iota), \bar{a}_2 = f(\phi)$, a_i be the image of $\bar{a}_i$ in A. Note that $\Lambda^2 A'$ is generated by $\bar{a}_1 \wedge \bar{a}_2$, and $g^*\alpha(\bar{a}_1 \wedge \bar{a}_2) = \alpha(a_1 \wedge a_2)$. Since $\bar{a}_1$ and $\bar{a}_2$ commute and $\alpha \in (\Lambda^2 A^D)_{\mathrm{bic}}$, $g^*\alpha$ is 0. It follows that $\xi_U(\alpha)$ maps to 0 in $\operatorname{Br}(\mathrm{SL}_n / A')$, and thus specializes to 0 at all points lying in the image of $(\mathrm{SL}_n / A')(K_v) \to (\mathrm{SL}_n / H)(K_v)$. In particular, $\xi_U(\alpha)(P) = 0 \in \operatorname{Br} K_v$. Since P and $v \nmid p$ were arbitrary, this means that $\xi_U(\alpha)$ is unramified by a well-known consequence of Harari's formal lemma [12, théorèm 2.1.1].

The case of a general k follows from the number field case and a "no-name lemma" argument. Namely, let $L := \mathbb{Q}(\mu_p) \subseteq k$, $\rho : H \hookrightarrow \mathrm{SL}_n(k)$ be the representation defining the quotient $\mathrm{SL}_{n,k} / H$, and $\rho' : H \hookrightarrow \mathrm{SL}_{n'}(L)$ be another faithful representation. Let $U'_k := \mathrm{SL}_{n',k} / H$, $U'_L := \mathrm{SL}_{n',L} / H$ (note that we have a natural map $U'_k \to U'_L$), and U''_k be the diagonal quotient $(\mathrm{SL}_{n,k} \times \mathrm{SL}_{n',k})/H$. The variety U''_k is both an $\mathrm{SL}_{n,k}$-torsor over U'_k and a $\mathrm{SL}_{n',k}$-torsor over $U_k = U$:

$$U_k \xleftarrow{\mathrm{SL}_{n',k}} U''_k \xrightarrow{\mathrm{SL}_{n,k}} U'_k$$

We get a commutative diagram

$$\begin{tikzcd}
 & \Lambda^2 A^D \arrow[dl, dashed, "\xi_{U'_k}"'] \arrow[d, dashed, "\xi_{U''_k}"] \arrow[dr, "\xi_{U_k}"] & \\
\operatorname{Br} U'_k \arrow[r] & \operatorname{Br} U''_k & \operatorname{Br} U_k \arrow[l]
\end{tikzcd}$$

where the horizontal maps are pullbacks. Moreover, the morphism $\xi_{U'_k}$ factors as $\Lambda^2 A^D \xrightarrow{\xi_{U'_L}} \operatorname{Br} U'_L \to \operatorname{Br} U'_k$, where the last morphism is an extension of scalars. We know by the number field case that $\xi_{U'_L}(\alpha) \in \operatorname{Br}_{\mathrm{ur}} U'_L$, and thus $\xi_{U'_k}(\alpha) \in \operatorname{Br}_{\mathrm{ur}} U'_k$ and $\xi_{U''_k}(\alpha) \in \operatorname{Br}_{\mathrm{ur}} U''_k$. Therefore, if we let $\beta := \xi_{U_k}(\alpha)$ and denote by π the projection $U''_k \to U_k$, then $\pi^*\beta$ belongs to $\operatorname{Br}_{\mathrm{ur}} U''_k$. But $U''_k \to U'_k$ is an $\mathrm{SL}_{n,k}$-torsor, and in particular possesses a rational section $s : U_k \dashrightarrow U''_k$ by Hilbert 90. It follows that $\beta = (\pi s)^*\beta = s^*(\pi^*\beta)$ is unramified as well, as wished. □

Remark 6.1.5. The composition map ξ_U is easily seen to be equal to the composition

$$\Lambda^2 A^D \xrightarrow{\cup} H^2(A, \mathbb{Z}/p\mathbb{Z}) = H^2(A, \mu_p) \xrightarrow{\inf} H^2(\Gamma_A, \mu_p) \to H^2(\Gamma_A, \bar{K}^*) \xrightarrow{\check{C}_\dagger} \operatorname{Br} U;$$

thus $\operatorname{Im} \xi_U \subseteq \operatorname{Br}^{\mathrm{ram}}_\lambda U$.

6.2. ***A decomposition of $H^2(A, B)$.*** For general abstract commutative groups A and B, we have a split exact sequence (see [2, V.6.5])

$$0 \to \operatorname{Ext}(A, B) \to H^2(A, B) \xrightarrow{\omega_B} \operatorname{Hom}(\Lambda^2 A, B) \to 0. \tag{6-8}$$

The splitting is noncanonical in general, but it is canonical when $\#B$ is odd: in this case a section s_B of ω_B is given by $s_B : \gamma \mapsto \big(a_1, a_2 \mapsto \frac{1}{2}\gamma(a_1 \wedge a_2)\big)$. If $B = \mathbb{Z}/n\mathbb{Z}$, then $\operatorname{Hom}(\Lambda^2 A, \mathbb{Z}/n\mathbb{Z}) = \Lambda^2 \operatorname{Hom}(A, \mathbb{Z}/n\mathbb{Z})$ and $2 \cdot s_B$ is the cup product:

$$2 \cdot s_B = \cup : \Lambda^2 \operatorname{Hom}(A, \mathbb{Z}/n\mathbb{Z}) \to H^2(A, \mathbb{Z}/n\mathbb{Z}). \tag{6-9}$$

The splitting s_B of (6-8) induces a canonical decomposition $H^2(A, B) = \operatorname{Ext}(A, B) \oplus \operatorname{Hom}(\Lambda^2 A, B)$.

Lemma 6.2.1. *Let p be an odd prime number, and assume that A and B have exponent p. Let $1 \to B \to H \xrightarrow{\pi} A \to 1$ be a nilpotent central extension of exponent p. Then the class $[H] \in H^2(A, B)$ lies in the image of s_B.*

Proof. Equivalently, we have to prove that $[H]$ maps to 0 under the projection $H^2(A, B) \to \operatorname{Ext}(A, B)$. This projection is natural in A (and B as well, but we do not need this). Therefore, since the functor $\operatorname{Ext}(A, B)$ is additive in A, and every abelian group decomposes into a direct sum of cyclic groups, it suffices to prove the result for a cyclic A. If A is cyclic and generated by $a \in A$, every $h \in \pi^{-1}(a)$ must be of order p because H has exponent p. The element h thus provides a section $a \mapsto h$ of π, and $H = B \oplus A$. Hence $[H] = 0 \in H^2(A, B)$, and in particular H maps to 0 in $\operatorname{Ext}(A, B)$ as wished. □

6.3. ***Formula for the Brauer–Manin pairing.*** We work in the setting of 6.1, but we also assume that $k = K$ is a number field. All cohomology and homomorphism groups appearing in this subsection have a natural $\mathbb{F}_p$-vector space structure, so whenever we take tensor products, alternating products, etc, we mean that these operations are performed as $\mathbb{F}_p$-vector spaces.

Let v be a finite place of K, $\Gamma_v = \operatorname{Gal}(\overline{K}_v/K_v)$, and $G_v = \Gamma_v^{\mathrm{ab}}/p\Gamma_v^{\mathrm{ab}}$. Remembering that we fixed an isomorphism $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$, and using the identifications $H^1(\Gamma_v, \mathbb{F}_p) = \operatorname{Hom}(\Gamma_v, \mathbb{F}_p) = \operatorname{Hom}(G_v, \mathbb{F}_p)$, we may identify the perfect local duality pairing

$$\cup : H^1(\Gamma_v, \mathbb{F}_p)^{\otimes 2} \to H^2(\Gamma_v, \mathbb{F}_p) \cong \mathbb{F}_p$$

with a pairing

$$H_v : \Lambda^2 \operatorname{Hom}(G_v, \mathbb{F}_p) \to \mathbb{F}_p.$$

(Note that this is alternating because p is odd.)

We choose a basis $g_1, \dots, g_r$ of G_v, and a basis $t_1, \dots, t_a$ of A. These induce an identification

$$\operatorname{Hom}(G_v, A) = \operatorname{Mat}_{a\times r}(\mathbb{F}_p) \tag{6-10}$$

For a finite-dimensional vector space $V/\mathbb{F}_p$, a basis $v_1, \dots, v_l$ of V induces an identification

$$\Lambda^2(V^D) \cong \operatorname{Mat}_{ant,l\times l}(\mathbb{F}_p), \quad v_i^D \wedge v_j^D \mapsto e_i e_j^T - e_j e_i^T,$$

where $e_1, \dots, e_l$ denotes the standard basis of $\mathbb{F}_p^l$, and Mat_{ant} denotes antisymmetric matrices. Specializing to $V = A$ and $V = G_v$, we get identifications

$$\Lambda^2 A^D = \operatorname{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p), \quad \Lambda^2 G_v^D = \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p). \tag{6-11}$$

Let $\phi \in \operatorname{Hom}(G_v, A)$, $\beta \in \Lambda^2 A^D$, $\gamma \in \Lambda^2 G_v^D$, and let $M_\phi \in \operatorname{Mat}_{a\times r}(\mathbb{F}_p)$, $M_\beta \in \operatorname{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$, $M_\gamma \in \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p)$ be their corresponding matrices under the identifications above. Let $\tilde{H}_v \in \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p)$ be the matrix defined by $(\tilde{H}_v)_{i,j} = \frac{1}{2} H_v(g_j \wedge g_i)$. One easily verifies that

$$\begin{array}{lccc} \phi^*\beta \in \Lambda^2 G_v^D & \text{corresponds to} & {M_\phi}^T M_\beta M_\phi & \in \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p), \\ \text{and} \quad H_v(\gamma) & = & \operatorname{tr}(\tilde{H}_v M_\gamma) & \in \mathbb{F}_p. \end{array} \tag{6-12}$$

Recall that we have a homomorphism

$$\xi_U : \Lambda^2 \operatorname{Hom}(A, \mathbb{F}_p) \xrightarrow{\cup} H^2(A, \mathbb{F}_p) \to \operatorname{Br} U.$$

We may compute the local Brauer pairing between $\operatorname{Im} \xi_U$ and $U(K_v)$ as follows:

Lemma 6.3.1. *Let $\beta \in \Lambda^2 \operatorname{Hom}(A, \mathbb{F}_p)$, $b := \xi_U(\beta) \in \operatorname{Br} U$, $P \in U(K_v)$, and $\phi \in \operatorname{Hom}(G_v, A)$ be induced by the torsor type $[\lambda|_P] \in H^1(\Gamma_v, A) = \operatorname{Hom}(\Gamma_v, A) \cong \operatorname{Hom}(G_v, A)$. Then*

$$(b, P)_v = \operatorname{tr}(\tilde{H}_v \tilde{M}_\phi{}^T M_\beta M_\phi).$$

Proof. This follows by just unraveling the notation. Namely, we have $(b, P)_v = \operatorname{inv}_v(P^* b)$, and the latter is equal to $H_v(\phi^* \beta) = \operatorname{inv}_v(P^* b)$ by the commutativity of the diagram

$$\begin{tikzcd}
\xi_U : & \Lambda^2 A^D \arrow[r, "\cup"] \arrow[d, "\phi^*"] & H^2(A, \mathbb{F}_p) \arrow[r, "\check{C}_\lambda"] \arrow[d, "\phi^*"] & H^2(U, \mathbb{F}_p) \arrow[r] \arrow[d, "P^*"] & (\operatorname{Br} U)[p] \arrow[d, "P^*"] \\
H_v : & \Lambda^2 G_v^D \arrow[r, "\cup"] & H^2(G_v, \mathbb{F}_p) \arrow[r, "\inf"] & H^2(\Gamma_v, \mathbb{F}_p) = H^2(K_v, \mathbb{F}_p) \arrow[r] & (\operatorname{Br} K_v)[p] \cong \mathbb{F}_p.
\end{tikzcd}$$

The formulas (6-12) now give the statement. ☐

6.4. ***Proof of Theorem 1.5.*** Let $\Theta_v \subseteq \operatorname{Hom}(\Gamma_v, A)$ be the set of torsor types to which λ specializes, i.e., the image of $[\lambda|\_] : U(K_v) \to \operatorname{Hom}(\Gamma_v, A)$.

Lemma 6.4.1. *The set Θ_v is the inverse image of* 0 *under*

$$\operatorname{Hom}(\Gamma_v, A) \to H^2(\Gamma_v, B), \quad \xi \mapsto \xi^*([H]), \tag{6-13}$$

where $[H] \in H^2(A, B)$ is the class representing H.

(Note that this inverse image is not a subspace in general!)

Proof. The commutative diagram

$$\begin{tikzcd}
\mathrm{SL}_n(K_v) \arrow[r] & (\mathrm{SL}_n / H)(K_v) = U(K_v) \arrow[r] \arrow[dr, "{[\lambda_]}"'] & H^1(K_v, H) \arrow[r] \arrow[d] & H^1(K_v, \mathrm{SL}_n) = 0 \\
& & H^1(K_v, A) = \operatorname{Hom}(\Gamma_v, A) &
\end{tikzcd}$$

with exact first row, shows Θ_v is the image of $H^1(K_v, H) = \operatorname{Hom}(\Gamma_v, H) \to \operatorname{Hom}(\Gamma_v, A)$. Equivalently, Θ_v consists of those homomorphisms that lift from A to the central extension H. The statement now follows from the theory of central extensions [2, Section IV.3]. ☐

Proposition 6.4.2. *For every $p \geq 5$, every number field K with $\mu_p \subseteq K^*$, and place v of K dividing p, there exists a constant $c(v) \geq 0$ such that for any $a \geq c(v)$, there exists a metabelian nilpotent H of exponent p with H^{ab} of rank a such that, letting $A = H^{\mathrm{ab}}$:*

(i) $(\Lambda^2 A^D)_{\mathrm{bic}} = \Lambda^2 A^D$;

(ii) *there exists* $\alpha \in \Lambda^2 A^D$ *such that the local pairing* $(-, \xi_U(\alpha)) : U(K_v) \to \mathbb{Z}/p\mathbb{Z}$ *attains at least two values.*

Proof. Let a be a natural number and $A := (\mathbb{Z}/p\mathbb{Z})^a$. Lemma A.2 of Appendix A guarantees the existence of a metabelian nilpotent H of exponent p with $H^{\mathrm{ab}} = A$ such that $B = [H, H]$ has rank $b := 2a - 3$ and $(\Lambda^2 A^D)_{\mathrm{bic}} = \Lambda^2 A^D$.

Recall that Θ_v is the image of $[\lambda|\_] : U(K_v) \to \mathrm{Hom}(\Gamma_v, A)$. Let $\Xi_v \subseteq \Theta_v$ be the image under $[\lambda|\_]$ of the left kernel of $U(K_v) \times \mathrm{Br}_{\mathrm{e,ur}}\, U \to \mathbb{Q}/\mathbb{Z}$, $(P, b) \mapsto \mathrm{inv}_v(P^*b)$. Since $e \in \Xi_v$, Ξ_v is nonempty and to prove point (ii) it suffices to prove that $\Xi_v \neq \Theta_v$. We do so by proving that the cardinality of these two sets have different p-adic valuation for $a \geq c(v)$. We start by computing $v_p(\#\Theta_v)$.

Recall the identifications

$$\mathrm{Hom}(G_v, A) = \mathrm{Mat}_{a\times r}(\mathbb{F}_p),\ \Lambda^2 A^D = Mat_{a\times a,ant}(\mathbb{F}_p),\ \Lambda^2 G_v^D = Mat_{r\times r,ant}(\mathbb{F}_p).$$

Identifying B with $\mathbb{F}_p^b$ we get identifications $H^2(A, B) = H^2(A, \mathbb{F}_p)^b$ and $H^2(\Gamma_v, B) = H^2(\Gamma_v, \mathbb{F}_p)^b \xrightarrow{\sim} \mathrm{inv}_v \mathbb{F}_p^b$. By Lemma 6.2.1 and (6-9), there exist $h_1, \dots, h_b \in \Lambda^2 \mathrm{Hom}(A, \mathbb{F}_p) = \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$ such that $[H] = (\mathfrak{h}_1, \dots, \mathfrak{h}_b) \in H^2(A, \mathbb{F}_p)^b$, $\mathfrak{h}_i := \cup(h_i)$, where $\cup$ indicates the cup product $\Lambda^2 \mathrm{Hom}(A, \mathbb{F}_p) = \Lambda^2 H^1(A, \mathbb{F}_p) \to H^2(A, \mathbb{F}_p)$. Then, by (6-13),

$$\Theta_v = \{M \in \mathrm{Hom}(\Gamma_v, \mathbb{F}_p) \mid \mathrm{inv}_v(M^*\mathfrak{h}_i) = 0,\ i = 1, \dots, b\}. \tag{6-14}$$

By Lemma 6.3.1, under the identification $\mathrm{Hom}(\Gamma_v, \mathbb{F}_p) = \mathrm{Hom}(G_v, \mathbb{F}_p) = \mathrm{Mat}_{a\times r}(\mathbb{F}_p)$, Θ_v corresponds to

$$\Theta_v = \{M \in \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(\tilde{H}_v \tilde{M}^T \tilde{h}_i \tilde{M}) = 0,\ i = 1, \dots, b\}. \tag{6-15}$$

A special case of the Ax–Katz theorem [18] states that the p-adic valuation of the number of solutions of a system of m polynomial equations of degree $\leq d$ in n (affine) variables in $\mathbb{F}_p$ is at least $\left\lceil \frac{n-dm}{d} \right\rceil$. Hence, since (6-15) describes Θ_v as the solution set of b quadratic equations in ra variables:

$$v_p(\#\Theta_v) \geq \left\lceil \frac{ra - 2b}{2} \right\rceil = \left\lceil \frac{(r-4)a + 6}{2} \right\rceil \geq \left\lceil \frac{2a + 6}{2} \right\rceil$$

as $r \geq p + 1 \geq 6$ by [22, Theorem 7.5.11].

We now compute $v_p(\#\Xi_v)$, using Lemma A.3 from Appendix A. Recall that $(\Lambda^2 A^D)_{\mathrm{bic}} = \Lambda^2 A^D$ by our choice of H, and thus $\xi_U(\Lambda^2 A^D) = \mathrm{Br}_{\mathrm{ur}}\, U$ by Theorem 6.1.3. Hence Lemma 6.3.1 implies that P lies in the left kernel of the local Brauer pairing $U(K_v) \times \mathrm{Br}_{\mathrm{ur}}\, U \to \mathbb{Q}/\mathbb{Z}$ if and only if $H_v(M_P^*\alpha) = (P, \xi_U(\alpha))_v = 0$ for all $\alpha \in \Lambda^2 A^D$. Therefore, (6-12) implies that the image of Ξ_v under the identification $\mathrm{Hom}(\Gamma_v, \mathbb{F}_p) = \mathrm{Hom}(G_v, \mathbb{F}_p) = \mathrm{Mat}_{a\times r}(\mathbb{F}_p)$ is

$$\begin{aligned}\Xi_v &= \{M \in \Theta_v \subseteq \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(\tilde{H}_v M^T N M) = 0,\ \text{for all } N \in \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)\}\\ &= \{M \in \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(\tilde{H}_v M^T N M) = 0,\ \text{for all } N \in \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)\},\\ &= \{M \in \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(M \tilde{H}_v M^T N) = 0,\ \text{for all } N \in \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)\},\end{aligned}$$

where the second identity holds because the b quadratic equations describing Θ_v are redundant in the description of Ξ_v (see (6-15)). Since $\operatorname{tr}(X^TY)$ induces a perfect pairing on $\mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$ and $M\tilde{H}_vM^T$ lies in $\mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$, we obtain from the above

$$\Xi_v^T = \{M \in \mathrm{Mat}_{r\times a}(\mathbb{F}_p) \mid M\tilde{H}_v^TM^T = 0\}.$$

The matrix $\tilde{H}_v^T = -\tilde{H}_v$ is invertible by local duality, and so Lemma A.3 shows that there exists a function $C : \mathbb{N} \to \mathbb{Z}$ (depending only on p) such that $\#\Xi_v^T \equiv C(r) \mod p^a$ if $a \geq 2r$. Recall that $r = r(v) = [K_v : \mathbb{Q}_p] + 2$, and let $c(v) = \max\{2r, v_p(C(r)) + 1\}$. If $a \geq c(v)$, then $v_p(\#\Xi_v^T) = v_p(C(r))$ by the ultrametric triangle inequality. Now

$$v_p(\#\Xi_v) = v_p(\#\Xi_v^T) = v_p(C(r)) < a < \left\lceil \frac{2a+6}{2} \right\rceil \leq v_p(\#\Theta_v)$$

as wished. □

We refer the reader to the proof of Lemma A.3 for a formula for $C(r)$. As a consequence of Proposition 6.4.2, we may now prove Theorem 1.5:

Proof of Theorem 1.5. Let H be any group as in Lemma A.3, and let $U = \mathrm{SL}_{n,K}/H$. By point (ii) of the lemma there is a place v, an element $\beta \in \Lambda^2 A^D$, and points $P_v \neq Q_v \in U(K_v)$ such that

$$(P_v, b)_v \neq (Q_v, b)_v \in \mathbb{Q}/\mathbb{Z}, \quad b := \xi_U(\beta) \in \operatorname{Br} X.$$

Consider then the adelic point $\underline{P} \in U(\mathbb{A}_K)$ (resp. $\underline{Q}$) that is equal to e at all places $\neq v$ and is equal to P_v (resp. Q_v) at v. Then $(\underline{P}, b)_{BM} = \mathrm{inv}_v(b(P_v)) \neq \mathrm{inv}_v(b(Q_v)) = (\underline{Q}, b)_{BM}$. Hence (at least) one between $\underline{P}$ and $\underline{Q}$ does not lie in $X(\mathbb{A}_K)^{\operatorname{Br} X}$, concluding the proof. □

Appendix A. Elementary counting facts

Lemma A.1. *Let $n \leq N$ be positive integers and $X \subseteq \mathbb{P}^N(\mathbb{F}_p)$ a subset of cardinality $< \#\mathbb{P}^n(\mathbb{F}_p)$. There exists then an n-codimensional subspace $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ such that $X \cap L = \varnothing$.*

Proof. Let $k \geq 0$ be the smallest integer such that X intersects every k-dimensional subspace in $\mathbb{P}^N_{\mathbb{F}_p}$. If $k = 0$, there is nothing to prove. Otherwise, let $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ be a $(k-1)$-dimensional subspace such that $L \cap X = \varnothing$. Let $\pi_L : \mathbb{P}^N \setminus l \to \mathbb{P}^{N-k}$ be a projection outside of L. We know by assumption that $\pi_L(X(\mathbb{F}_p)) = \mathbb{P}^{N-k}(\mathbb{F}_p)$, hence $\#X(\mathbb{F}_p) \geq \#\mathbb{P}^{N-k}(\mathbb{F}_p) \Rightarrow N - k < n$, i.e., $k \geq N - n + 1$. Hence the dimension of L is $\geq N - n$ and it is the sought subspace. □

The following lemma is inspired by [1, Section 5]; see also [6, p. 37].

Lemma A.2. *Let $p \neq 2$ be a prime. For every $\mathbb{F}_p$-vector space A of dimension $4 \leq a < \infty$, there exists an $\mathbb{F}_p$-vector space B, of dimension $b = 2a - 3$, and a (surjective) morphism*

$$\mathfrak{c} : \Lambda^2 A \to B, \tag{A-1}$$

such that, if $1 \to B \to H \xrightarrow{\pi} A \to 1$ is the extension whose commutator map is $\mathfrak{c}$, there are no pure wedges $0 \neq a_1 \wedge a_2 \in \Lambda^2 A$ lying in the kernel of $\mathfrak{c}$.

Proof. Let $X \subseteq \mathbb{P}_{\mathbb{F}_p}(\Lambda^2 A)$ be the image of the "alternating Segre morphism"

$$- \wedge - : \mathbb{P}(A) \times \mathbb{P}(A) \setminus \Delta \to \mathbb{P}(\Lambda^2 A),$$

which is isomorphic to the Grassmannian variety $\mathrm{Gr}_{\mathbb{F}_p}(2, A)$. Since $X(\mathbb{F}_p)$ parametrizes two-dimensional $\mathbb{F}_p$-subspaces of A,

$$\# X(\mathbb{F}_p) = \frac{(p^a - 1)(p^{a-1} - 1)}{(p^2 - 1)(p - 1)}. \tag{A-2}$$

It suffices to show that there exists a $(2a-3)$-codimensional subspace L in $\mathbb{P}(\Lambda^2 A)$ such that $L \cap X(\mathbb{F}_p) = \varnothing$, and choose $\mathfrak{c}$ such that $\Lambda^2 A \supseteq \mathbb{F}_p \cdot L(\mathbb{F}_p) = \operatorname{Ker} \mathfrak{c}$. Noting that

$$\frac{(p^a-1)(p^{a-1}-1)}{(p^2-1)(p-1)} < \frac{(p^a-1)(p^{a-1}-1)}{(p+1)(p-1)} \le \frac{(p^{2a-2}-1)(p+1)}{(p+1)(p-1)} = \#\mathbb{P}^{2a-3}(\mathbb{F}_p),$$

such a subspace always exists by Lemma A.1. □

Lemma A.3. *Let A, V be $\mathbb{F}_p$-vector spaces with $p \neq 2$, and let $a := \dim A$, $r := \dim V$. Assume that $a \geq 2r$, and that V is endowed with an alternating nondegenerate bilinear form $b : V \times V \to \mathbb{F}_p$. Then*

$$\Xi(A, V) := \#\{\xi \in \operatorname{Hom}(A, V) \mid \xi^* b = 0\} \equiv C(r) \mod p^a,$$

where $C(r)$ is a nonzero integer depending only on r.

Proof. Let

$$M_d := \#\{\text{isotropic } d\text{-dimensional subspaces in } V\},$$
$$I_d := \#\{\text{surjective homomorphisms from } A \text{ to a } d\text{-dimensional } \mathbb{F}_p\text{-vector space}\}.$$

Then

$$\Xi(A, V) = \sum_{d=0}^{\min(a, r/2)} I_d M_d$$

(the fact that V is endowed with a nondegenerate alternating linear form and $p \neq 2$ implies that r is even). One can easily see that

$$\begin{aligned} I_d &= (p^a-1)\cdot(p^a-p)\cdots(p^a-p^{d-1}) && \text{for every } d \leq a,\\ M_d &= \frac{(p^r-1)\cdot(p^{r-1}-p)\cdots(p^{r-d+1}-p^{d-1})}{(p^d-1)\cdot(p^d-p)\cdots(p^d-p^{d-1})} && \text{for every } d \leq r/2. \end{aligned}$$

In particular, $\Xi(A, V) = \Xi'(a, r)$ depends only on a and r. Note that, for a fixed r, $\Xi'(a, r)$ converges p-adically, as $a \to \infty$, to the following sum, which happens to be an integer number:

$$\begin{aligned} C(r) := \Xi'(\infty, r) &:= \sum_{d=0}^{r/2} (-1)^d \frac{(p^r-1)\cdot(p^{r-1}-p)\cdots(p^{r-d+1}-p^{d-1})}{(p-1)\cdot(p^2-1)\cdots(p^d-1)} \\ &= \sum_{d=0}^{r/2} (-1)^d \binom{r/2}{d}_{p^2} (p+1)\cdots(p^d+1), \end{aligned}$$

where the subscript in the binomial denotes a Gaussian binomial coefficient (an integer number). Moreover, $\Xi'(a, r) \equiv \Xi'(\infty, r) \mod p^a$ if $a \geq 2r$. Denoting by $a(d)$ the term multiplying the $(-1)^d$ appearing above, we notice that the sequence $a(0), \dots, a(r/2)$ is strictly increasing, as follows by induction from the fact that $(p^{r-d+1} - p^{d-1})/(p^d - 1) > 1$ for all $d \in \{0, \dots, r/2\}$. In particular, a standard elementary calculus argument (à la Leibniz' rule) shows that $\Xi'(\infty, r) \neq 0$. □

Appendix B. Other works where ramified descent appears

Let me mention other works where the idea of "ramified descent" has already appeared. One is [16] by Harpaz and Skorobogatov (successor to Skorobogatov and Swinnerton-Dyer's work [27] [30]), where the authors use the cyclic ramified covers of some specific Kummer surfaces to prove that, under certain technical assumptions, these satisfy the Hasse principle.

Another work is Corvin and Schlank's paper [8], where the authors build upon Poonen's example [23] to show (employing one specific ramified S_4-cover) that the following obstruction is *stronger* than étale Brauer–Manin obstruction:

$$X(\mathbb{A}_K)^{\mathrm{Br,ram,sol}} = \bigcap_{\substack{\psi: Y \to X \text{ a } G\text{-cover} \\ G \text{ solvable}}} \overline{\bigcup_{\xi \in H^1(K,G)} \psi'_\xi(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))^{\mathrm{Br}\, Y_\xi^{\mathrm{sm}}}},$$

where the ψ'_ξ is the composition $Y_\xi^{\mathrm{sm}} \to Y_\xi \xrightarrow{\psi_\xi} X$.

Lastly, we mention Sections 11.5 and 14.2.5 of Colliot-Thélène and Skorobogatov's book [7], where ramified descent is investigated for μ_n-covers. In Theorem 14.2.25 of loc.cit., the authors prove a result which translates in our language to saying that, if $\lambda : V \to U$ is a μ_n-torsor such that there is a divisor on X over which the "compactification" $\psi : Y \to X$ of λ (notation as in Section 3) is totally ramified, then $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X}$. Their result and our Proposition 1.3 naturally lead us to the question:

Question B.1. Let $\lambda : V \to U$ and $\psi : Y \to X$ be as in Section 3. Assume that the cover $Y \to X$ is totally ramified, i.e., Y is geometrically integral and $Y \to X$ does not have any unramified subcovers. Does one then have $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X}$?

Note that a positive answer to the question above would guarantee that, for instance, if Y is a variety all of whose G-twists satisfy the Hasse principle, then X satisfies the Hasse principle up to Brauer–Manin obstruction.

Let us mention that, when G is supersolvable and Y is rationally connected, Harpaz and Wittenberg [17, Theorem 1.4] proved that

$$X(\mathbb{A}_K)^{\mathrm{Br}\, X} = \overline{\bigcup_{\xi \in H^1(K,G)} \psi_\xi^{\mathrm{sm}}\big(Y_\xi(\mathbb{A}_K)^{\mathrm{Br}\, Y_\xi^{\mathrm{sm}}}\big)}$$

(using our notation). It follows that $X(\mathbb{A}_K)^\lambda \supset X(\mathbb{A}_K)^{\mathrm{Br}\, X}$, i.e., Brauer–Manin obstruction is finer than ramified descent obstruction, but it also seems likely that their methods could be in fact used to give a positive answer to Question B.1 in this case. For instance, when, in addition to the conditions above, $\overline{K}[V]^*/\overline{K}^* = 0$ and $\mathrm{Pic}\, \overline{V} = 0$ (e.g., if $V = \mathrm{SL}_n$), then $\mathrm{Br}_\lambda^{\mathrm{ram}} X = \mathrm{Br}\, X$ by Remark 5.1.4, and a positive answer to the question follows.

Acknowledgement

This work was part of the author's thesis. The author thanks his PhD advisor, David Harari, for providing motivation towards the question, for his guidance and for helpful comments on the text.

The author also thanks his thesis examiners, Olivier Wittenberg and Daniel Loughran, for their interesting comments. Wittenberg, especially, pointed out that the counterexamples of the last section provide an interesting example of transcendental obstruction to the Grunwald problem; he also pointed out a simplification in the proof of Theorem 1.1 and brought Theorem 6.0.1 to the author's attention.

The author thanks the referees, who helped greatly improve the exposition of the paper, and provided valuable insights into simplifications of various proofs, in particular what is now Proposition 1.3 and the general proof path of Theorem 1.5 in Section 6.

Part of this work was written at the Max Planck Institute for Mathematics at Bonn, and the author is grateful for their financial support and for the excellent working conditions they provided.

References

[1] F. A. Bogomolov, "The Brauer group of quotient spaces of linear representations", *Izv. Akad. Nauk SSSR Ser. Mat.* **51**:3 (1987), 485–516. In Russian; translated in Math. URSS Izv. **30**:3 (1987), 455-485. MR

[2] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, 1982. Corrected reprint, 1994. MR

[3] T. D. Browning, L. Matthiesen, and A. N. Skorobogatov, "Rational points on pencils of conics and quadrics with many degenerate fibers", *Ann. of Math.* (2) **180**:1 (2014), 381–402. MR

[4] Y. Cao, C. Demarche, and F. Xu, "Comparing descent obstruction and Brauer–Manin obstruction for open varieties", *Trans. Amer. Math. Soc.* **371**:12 (2019), 8625–8650. MR

[5] J.-L. Colliot-Thélène and J.-J. Sansuc, "La descente sur les variétés rationnelles, II", *Duke Math. J.* **54**:2 (1987), 375–492. MR

[6] J.-L. Colliot-Thélène and J.-J. Sansuc, "The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)", pp. 113–186 in *Algebraic groups and homogeneous spaces*, Tata Inst. Fund. Res. Stud. Math. **19**, Tata Inst. Fund. Res., Mumbai, 2007. MR

[7] J.-L. Colliot-Thélène and A. N. Skorobogatov, *The Brauer–Grothendieck group*, Ergebnisse der Math. **71**, Springer, 2021. MR

[8] D. Corwin and T. Schlank, "Brauer and étale homotopy obstructions to rational points on open covers", preprint, 2020. arXiv 2006.11699

[9] C. Demarche, G. Lucchini Arteche, and D. Neftin, "The Grunwald problem and approximation properties for homogeneous spaces", *Ann. Inst. Fourier* (*Grenoble*) **67**:3 (2017), 1009–1033. MR

[10] J. L. Demeio, *Abundance of rational points*, PhD thesis, Université Paris-Saclay and Scuola normale superiore di Pisa, 2021.

[11] S. Encinas and H. Hauser, "Strong resolution of singularities in characteristic zero", *Comment. Math. Helv.* **77**:4 (2002), 821–845. MR

[12] D. Harari, "Méthode des fibrations et obstruction de Manin", *Duke Math. J.* **75**:1 (1994), 221–260. MR

[13] D. Harari, "Quelques propriétés d'approximation reliées à la cohomologie galoisienne d'un groupe algébrique fini", *Bull. Soc. Math. France* **135**:4 (2007), 549–564. MR

[14] D. Harari, *Galois cohomology and class field theory*, Springer, 2020. MR

[15] D. Harari and A. N. Skorobogatov, "Descent theory for open varieties", pp. 250–279 in *Torsors*, *étale homotopy and applications to rational points*, London Math. Soc. Lecture Note Ser. **405**, Cambridge Univ. Press, 2013. MR

[16] Y. Harpaz and A. N. Skorobogatov, “Hasse principle for Kummer varieties”, *Algebra Number Theory* **10**:4 (2016), 813–841. MR

[17] Y. Harpaz and O. Wittenberg, “Supersolvable descent for rational points”, *Algebra Number Theory* **18**:4 (2024), 787–814. MR

[18] N. M. Katz, “On a theorem of Ax”, *Amer. J. Math.* **93** (1971), 485–499. MR

[19] A. Kumar, “K3 surfaces associated with curves of genus two”, *Int. Math. Res. Not.* **2008**:6 (2008), art. id. rnm165, 26 pp. MR

[20] G. Lucchini Arteche, “Groupe de Brauer non ramifié des espaces homogènes à stabilisateur fini”, *J. Algebra* **411** (2014), 129–181. MR

[21] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University Press, 1980. MR

[22] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundl. Math. Wissen. **323**, Springer, 2008. MR

[23] B. Poonen, “Insufficiency of the Brauer–Manin obstruction applied to étale covers”, *Ann. of Math.* (2) **171**:3 (2010), 2157–2169. MR

[24] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, 1979. MR

[25] J.-P. Serre, *Cohomologie galoisienne*, 5th ed., Lecture Notes in Mathematics **5**, Springer, 1994. MR

[26] A. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics **144**, Cambridge University Press, 2001. MR

[27] A. Skorobogatov and P. Swinnerton-Dyer, “2-descent on elliptic curves and rational points on certain Kummer surfaces”, *Adv. Math.* **198**:2 (2005), 448–483. MR

[28] “The Stacks project”, electronic resource, 2022, available at https://stacks.math.columbia.edu.

[29] R. G. Swan, “Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions”, *J. Pure Appl. Algebra* **144**:2 (1999), 169–211. MR

[30] P. Swinnerton-Dyer, “The solubility of diagonal cubic surfaces”, *Ann. Sci. École Norm. Sup.* (4) **34**:6 (2001), 891–912. MR

[31] R. Vakil and K. Wickelgren, “Universal covering spaces and fundamental groups in algebraic geometry as schemes”, *J. Théor. Nombres Bordeaux* **23**:2 (2011), 489–526. MR

Communicated by Jean-Louis Colliot-Thélène
Received 2023-02-11 Revised 2025-03-19 Accepted 2025-05-02

demeiojulian@yahoo.it *Department of Mathematics, University of Bath, Bath, United Kingdom*

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.445

Logarithmic base change theorem and smooth descent of positivity of log canonical divisor

Sung Gi Park

We prove a logarithmic base change theorem for pushforwards of pluricanonical bundles and use it to deduce that positivity properties of log canonical divisors descend via smooth projective morphisms.

As an application, for a surjective morphism $f : X \to Y$ with $\kappa(X) \geq 0$ and $-K_Y$ big, we prove $Y \setminus \Delta(f)$ is of log general type, where $\Delta(f)$ is the discriminant locus. In particular, when $Y = \mathbb{P}^n$ we have $\dim \Delta(f) = n-1$ and $\deg \Delta(f) \geq n+2$, generalizing the case $n = 1$ proved by Viehweg and Zuo. We also prove Popa's conjecture on the superadditivity of the logarithmic Kodaira dimension of smooth algebraic fiber spaces over bases of dimension at most three and analyze related problems.

1. Introduction

In this paper, we prove a number of results on the behavior of positivity, hyperbolicity, and Kodaira dimension under smooth morphisms of quasiprojective varieties, based on a technical tool we study under the name of logarithmic base change theorem. Throughout the text, a variety is a reduced separated scheme of finite type over $\mathbb{C}$. A pair (Y, D) consists of a variety Y and a formal $\mathbb{Q}$-linear combination of divisors D, and we call it a log smooth pair if Y is smooth and D is a reduced simple normal crossing divisor.

Given a surjective morphism $f : X \to Y$ of smooth projective varieties, the study of the discriminant locus $\Delta(f) \subset Y$, defined as the set of points $y \in Y$ with singular scheme-theoretic fibers $X_y := f^*(y)$, has been an active area of research. For example, Catanese and Schneider [7, Question 4.1] asked the question whether $f : X \to \mathbb{P}^1$ has at least 3 singular fibers if X is of general type. Kovács [23] gave a

MSC2020: 14D06, 14E30, 14F10, 14J10.
Keywords: base change theorem, positivity, discriminant locus, Kodaira dimension.

partial answer when X is canonically polarized and Viehweg and Zuo [41] answered affirmatively under the weaker assumption that X has nonnegative Kodaira dimension. The question is alternatively phrased as follows: the effectivity of the canonical divisor of X imposes the bigness of the log canonical divisor of $\mathbb{P}^1 \setminus \Delta(f)$.

We refine and generalize this phenomenon to higher dimensional bases; given a smooth projective morphism $f : X \to Y$ of smooth quasiprojective varieties, the positivity of the log canonical divisor of X descends to the positivity of the log canonical divisor of Y. For example, bigness (resp. effectivity) descends to bigness (resp. pseudoeffectivity). The precise statements are obtained as consequences of our main technical result, which we call the *logarithmic base change theorem* for pushforwards of logarithmic pluricanonical bundles.

A priori, Viehweg introduced the base change theorem for pushforwards of pluricanonical bundles to study the subadditivity of the Kodaira dimension for algebraic fiber spaces. Here is a version given by Mori [27, (4.10)].

Theorem 1.1 (base change theorem [27; 39]). *Let X, Y, Y' be smooth quasiprojective varieties and $f : X \to Y$ be a surjective projective morphism and $g : Y' \to Y$ be a flat projective morphism. Let $\mu : X'' \to X' = X \times_Y Y'$ be a resolution of singularities. Let N be a positive integer, and consider the commutative diagram*

$$\begin{tikzcd} X & X' \arrow[l, "g'"'] \arrow[d, "f'"] & X'' \arrow[l, "\mu"'] \arrow[dl, "f''"] \\ Y \arrow[from=u, "f"] & Y' \arrow[l, "g"] & \end{tikzcd}$$

(1) *There is an inclusion*

$$f''_* \omega^N_{X''/Y'} \subset g^* f_* \omega^N_{X/Y}.$$

It is an equality at a point $y' \in Y'$ if $g(y')$ is a codimension-1 point of Y and f is semistable in the neighborhood of $g(y')$.

(2) *There is an inclusion*

$$g_* f''_* \omega^N_{X''/Y} \subset (f_* \omega^N_{X/Y} \otimes g_* \omega^N_{Y'/Y})^{**}.$$

It is an equality at a codimension-1 point $y \in Y$ if f or g is semistable in the neighborhood of y, where ** denotes the double dual.

Note that the morphism f is semistable over a codimension-1 point $y \in Y$ if the fiber X_y has reduced normal crossing singularities. Outside of a codimension-2 subvariety, Viehweg's base change theorem gives an inclusion of pushforwards of pluricanonical bundles of the fiber product to the tensor products of pushforwards of pluricanonical bundles of each morphism.

We start by pointing out a striking new phenomenon: by suitably introducing poles in the discriminant loci of morphisms, the inclusion in the base change theorem is reversed. This represents a distinct departure from previously observed phenomena.

Theorem 1.2 (logarithmic base change theorem). *Let $(X, E), (Y, D), (Y', D')$ be quasiprojective log smooth pairs, and let $f : (X, E) \to (Y, D)$, $g : (Y', D') \to (Y, D)$ be surjective projective morphisms of pairs such that $E = f^{-1}(D)$, $D' = g^{-1}(D)$ and $g|_{Y' \setminus D'} : Y' \setminus D' \to Y \setminus D$ is smooth. Let X' be the union of the irreducible components of $X \times_Y Y'$ dominating Y, and $E' = g'^{-1}(E)$. Consider the commutative diagram*

$$\begin{tikzcd}
(X, E) \arrow[d, "f"'] & (X', E') \arrow[l, "g'"'] \arrow[d, "f'"'] & (X'', E'') \arrow[l, "\mu"'] \arrow[dl, "f''"] \\
(Y, D) & (Y', D') \arrow[l, "g"]
\end{tikzcd} \tag{1.3}$$

where $\mu : (X'', E'') \to (X', E')$ is a log resolution of pairs with $X'' \setminus E'' \cong X' \setminus E'$, so that $(g \circ f'')^{-1} D = E''$. Then there exists an inclusion for a positive integer N:

$$\big[f_*(\omega_X(E)/\omega_Y(D)^{\otimes N}) \otimes g_*(\omega_{Y'}(D')/\omega_Y(D)^{\otimes N})\big]^{**} \subset \big[h_*(\omega_{X''}(E'')/\omega_Y(D)^{\otimes N})\big]^{**}$$

where $h = g \circ f''$.

Here, $\omega_X(E)/\omega_Y(D) := \omega_X(E) \otimes f^*\omega_Y(D)^{-1}$, and $\omega_X(E)/\omega_Y(D)^{\otimes N}$ is the N-tensor power of this line bundle. By $f^{-1}(\bullet)$ we denote the set-theoretic preimage of $\bullet$ with the reduced scheme structure. While the equality condition in Viehweg's base change theorem follows from the observation that the fiber product X' has normal toric singularities, the fiber product X' in the logarithmic base change theorem has binomial hypersurface singularities. The proof of Theorem 1.2 follows from a careful analysis of Bierstone and Milman's resolution of binomial hypersurface singularities [1] applied to singularities of pairs.

Viehweg applied the base change theorem inductively on the iterated fiber products to study Iitaka's $C^+_{n,m}$ conjecture on the subadditivity of the Kodaira dimension; this technique is the so-called Viehweg's fiber product trick. Likewise, we employ a logarithmic analogue of this fiber product trick, which is described in the following paragraph.

Let $f : (X, E) \to (Y, D)$ be a projective morphism of quasiprojective log smooth pairs such that $E = f^{-1}(D)$ and $f|_{X \setminus E}$ is smooth. Define $X^s := X \times_Y \cdots \times_Y X$ as the s-fold fiber product of f, with the induced morphism $f^s : X^s \to Y$, and define $E^s = (f^s)^{-1}(D)$. Then $f^s|_{X^s \setminus E^s}$ is a smooth projective morphism. Choose a log resolution $\mu^s : (X^{(s)}, E^{(s)}) \to (X^s, E^s)$ of the union of the irreducible components of X^s dominating Y, with the induced morphism $f^{(s)} : (X^{(s)}, E^{(s)}) \to (Y, D)$ of log smooth pairs satisfying $E^{(s)} = (f^{(s)})^{-1}(D)$ and $f^{(s)}|_{X^{(s)} \setminus E^{(s)}}$ smooth. Then we have:

Corollary 1.4 (logarithmic fiber product trick). *With the notation in the previous paragraph, we have the inclusion*

$$\Big[\bigotimes^s f_*(\omega_X(E)/\omega_Y(D)^{\otimes N})\Big]^{**} \hookrightarrow \big[f_*^{(s)}(\omega_{X^{(s)}}(E^{(s)})/\omega_Y(D)^{\otimes N})\big]^{**}$$

for $N, s > 0$.

The right side of this inclusion is independent of the choice of a log resolution of the pair (X^s, E^s). The proof is immediate from Theorem 1.2 iterated s-times.

In contrast to Viehweg's base change theorem and fiber product trick, the logarithmic analogues have a distinctive feature: the inclusion goes in the opposite direction, which is consistent with Popa's conjectures [30] on the superadditivity of the logarithmic Kodaira dimension. Specifically, the (log) Kodaira dimension of the source imposes a lower bound on the (log) Kodaira dimension of the base. This phenomenon, namely a *smooth projective descent of positivity of log canonical divisor*, is opposite in nature to Iitaka's $C^+_{n,m}$ conjecture. As a consequence of the logarithmic base change theorem, we derive a number of results on the (log) Kodaira dimension and the discriminant loci of morphisms.

To begin with, we recall the definition of the logarithmic Kodaira dimension of a quasiprojective variety. For a $\mathbb{Q}$-Cartier divisor L on a normal projective variety X, $\kappa(X, L)$ denotes the Iitaka dimension of L. For instance, $\kappa(X) := \kappa(X, \omega_X)$ is the Kodaira dimension of X when X is a smooth projective variety. For a smooth quasiprojective variety X, the logarithmic Kodaira dimension is defined as

$$\bar{\kappa}(X) := \kappa(\overline{X}, K_{\overline{X}} + D),$$

where $\overline{X}$ is a compactification of X with boundary a reduced simple normal crossing divisor D. We say X is of log general type if $\bar{\kappa}(X) = \dim X$. It is well known that the logarithmic Kodaira dimension is well-defined, independent of the choice of a compactification.

As a first application of the logarithmic base change theorem, we prove a logarithmic analogue of Popa and Schnell's Theorem A [32].

Theorem 1.5. *Let $f : X \to Y$ be a smooth projective morphism of smooth quasiprojective varieties whose general fiber F is connected and $\kappa(F) \geq 0$. Then Y is of log general type if and only if $\bar{\kappa}(X) = \kappa(F) + \dim Y$.*

The "only if" part of the theorem is Iitaka's logarithmic $C_{n,m}$-conjecture when the base is of log general type, proven by Maehara [26, Corollary 2]. Our contribution is the "if" part, which can alternatively be phrased as follows: If the source has the maximal log Kodaira dimension attained by the equality of the Easy addition formula [27, Corollary 1.7], $\bar{\kappa}(X) \leq \kappa(F) + \dim Y$, then the base is of log general type.

In particular, the bigness of the log canonical divisor descends via smooth projective morphisms.

Corollary 1.6. *Let $f : X \to Y$ be a smooth projective morphism of smooth quasiprojective varieties. If X is of log general type, then Y is of log general type.*

Analogously, the next theorem states that the effectivity of the log canonical divisor descends to the pseudoeffectivity of the log canonical divisor via smooth projective morphisms.

Theorem 1.7. *Let $f : (X, E) \to (Y, D)$ be a surjective morphism of projective log smooth pairs with $E = f^{-1}(D)$, and $f|_{X \setminus E}$ smooth.*

(1) *If $K_X + (1 - \epsilon)E$ is $\mathbb{Q}$-effective for some $\epsilon > 0$, then $K_Y + (1 - \delta)D$ is pseudoeffective for some $\delta > 0$. In particular, if additionally $-K_Y + ND$ is big for some nonnegative integer N, then D and $K_Y + D$ are big.*

(2) *If $K_X + E$ is $\mathbb{Q}$-effective, then $K_Y + D$ is pseudoeffective.*

Here, we say a divisor is $\mathbb{Q}$-effective if it is $\mathbb{Q}$-linearly equivalent to an effective divisor. To avoid repetition, we will refer to $\mathbb{Q}$-effective divisors simply as effective divisors from now on. The effectivity of $K_X+(1-\epsilon)E$ (resp. pseudoeffectivity of $K_Y+(1-\delta)D$) for small enough ϵ (resp. δ) is independent of the choice of a compactification of $X\setminus E$ (resp. $Y\setminus D$). This is also true for the effectivity of K_X+E (resp. pseudoeffectivity of K_Y+D). Therefore, Theorem 1.7 can be stated purely in terms of a smooth projective morphism $f|_{X\setminus E}: X\setminus E\to Y\setminus D$ of smooth quasiprojective varieties.

The proof of (2) is treated separately at the end of Section 3B, since this follows from the results in the literature without the use of the logarithmic base change theorem. This extends [32, Proposition G] to the logarithmic setting.

Remark 1.8. Theorem 1.7(1) and the nonvanishing conjecture for klt pairs imply the following statement: if $K_X+(1-\epsilon)E$ is (pseudo)effective for some $\epsilon>0$, then $K_Y+(1-\delta)D$ is (pseudo)effective for some $\delta>0$. When $X\setminus E\to Y\setminus D$ is finite étale, this is easily proven. Indeed, the descent of the effectivity follows from the same technique used to prove the invariance of the logarithmic Kodaira dimension under étale covers by Iitaka [14, Theorem 3]. The descent of the pseudoeffectivity is also immediate from Lemma 3.4 (5), explained later. Note that when we replace "(pseudo)effective" by "big" in the statement, we obtain Corollary 1.6.

The next theorem is an immediate consequence of Theorems 1.5 and 1.7(1). As explained at the beginning, the discriminant locus $\Delta(f)\subset Y$ of a morphism $f: X\to Y$ between smooth varieties is the complement of the locus in Y over which f is smooth.

Theorem 1.9. *Let $f: X\to Y$ be a surjective morphism of smooth projective varieties with $\kappa(X)\geq 0$, and let D be the divisorial component of the discriminant locus $\Delta(f)$ in Y. Suppose either*

(i) $\kappa(X)=\kappa(F)+\dim Y$ *where F is the general fiber of f, or*

(ii) $-K_Y$ *is big.*

Then, $Y\setminus\Delta(f)$ is of log general type. In particular, K_Y+D is big.

This extends to a normal variety Y when we extend the notion of bigness to a rank 1 reflexive sheaf. Part (i) extends a result from [32, Remark 5] that further assumed that Y is not uniruled. Part (ii) can be seen as a vast generalization of Catanese and Schneider's question [7, Question 4.1] and Viehweg and Zuo's result [41, Theorem 0.2] stating that a surjective morphism $f: X\to\mathbb{P}^1$ with $\kappa(X)\geq 0$ has at least three singular fibers. For instance, when $Y=\mathbb{P}^n$ we have:

Corollary 1.10. *Let $f: X\to\mathbb{P}^n$ be a surjective morphism from a smooth projective variety X of nonnegative Kodaira dimension. Then* $\dim\Delta(f)=n-1$ *and* $\deg\Delta(f)\geq n+2$.

Remark 1.11 (hyper-Kähler manifolds). This applies to a Lagrangian fibration $f: X\to Y$ of a projective hyper-Kähler manifold X of dimension $2n$. When Y is smooth, it is known that $Y=\mathbb{P}^n$ and the discriminant locus $\Delta(f)$ is a divisor (see [12, Theorem 1.2] and [13, Proposition 3.1]). Corollary 1.10 implies that its degree is at least $n+2$ in $\mathbb{P}^n$.

More generally, when $\mathbb{P}^n$ is replaced by the product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $\dim \Delta(f) = \sum_{i=1}^{k} n_i - 1$ and the multidegree $\deg \Delta(f) = (a_1, \ldots, a_k)$ satisfies $a_i \geq n_i + 2$ for all i. Examples 7.1 and 7.3 illustrate that the inequality in Corollary 1.10 is sharp.

In our next main application, we prove Popa's conjecture on the superadditivity of the (log) Kodaira dimension for smooth projective morphisms [30, Conjecture 3.1], under an additional assumption on the base. This assumption is implied by the conjectures of the log minimal model program.

Theorem 1.12. *Let $f : X \to Y$ be a smooth projective morphism with connected fibers between smooth quasiprojective varieties. Assume $\bar{\kappa}(Y) \geq 0$, and that the very general fiber of the log Iitaka fibration of Y has a good minimal model. Then*

$$\bar{\kappa}(X) \leq \bar{\kappa}(Y) + \kappa(F)$$

where F is the general fiber of f.

Recall that the log Iitaka fibration of Y is the Iitaka fibration associated to the log canonical bundle $K_{\overline{Y}} + D$ of a log smooth pair $(\overline{Y}, D)$. Here, $\overline{Y}$ is a compactification of Y with boundary a reduced simple normal crossing divisor D. By taking a log resolution $(\widetilde{Y}, \widetilde{D})$ of the pair $(\overline{Y}, D)$, we have the log Iitaka fibration $\eta : (\widetilde{Y}, \widetilde{D}) \to I$, where the very general fiber $(G, \widetilde{D}|_G)$ has log Kodaira dimension zero.

When $\bar{\kappa}(Y) = -\infty$, the nonvanishing conjecture for log canonical pairs and Theorem 1.7(2) imply $\bar{\kappa}(X) = -\infty$. This conjecture and the existence of good minimal models for log canonical pairs are well known to hold in dimension at most three (see e.g. [20; 36]). Therefore, we have:

Corollary 1.13. *Let $f : X \to Y$ be a smooth projective morphism with connected fibers between smooth quasiprojective varieties. If* $\dim Y \leq 3$, *then*

$$\bar{\kappa}(X) \leq \bar{\kappa}(Y) + \kappa(F)$$

and the equality holds when $\dim Y = 1$.

This extends [32, Corollary E and F] which assumed that X and Y are projective. On a related note, the logarithmic Iitaka conjecture suggests subadditivity:

$$\bar{\kappa}(X) \geq \bar{\kappa}(Y) + \kappa(F),$$

even without the smoothness assumption. This is known when Y is a quasiprojective curve (hence, the equality in Corollary 1.13), but not when Y is an arbitrary quasiprojective surface or a threefold. Theorem 1.12 and Corollary 1.13 prove that this inequality is reversed when the morphism is smooth.

Remark 1.14. Campana [4] stated the same superadditivity result when the fibers have semiample canonical bundles. The logarithmic Iitaka conjecture is known to hold when the general fiber F has a good minimal model [11, Theorem 1.2].

On a different note, in light of the smooth descent of the positivity of the log canonical divisor, we additionally investigate the uniruledness of X when Y is a rational curve.

Theorem 1.15. *Let $f : X \to \mathbb{P}^1$ be a surjective morphism with connected fibers from a smooth projective variety X. Suppose f has at most 2 singular fibers. If the general fiber F has a good minimal model, then X is uniruled. In particular, this holds when $\kappa(F) \geq \dim F - 3$.*[1]

Indeed, the nonvanishing conjecture and [41] suggest X is uniruled. Using symplectic geometry, Pieloch [29] recently proved that X is uniruled when f has at most one singular fiber without the assumption on F, and made some progress when f has two singular fibers. In general, we do not have an unconditional algebraic proof of these statements.

What is new. Viehweg's base change theorem has been one of the keys to studying Iitaka's conjecture and hyperbolicity problems like the Viehweg's conjecture on families with maximal variation; see, for instance, [5; 18; 19; 31; 41; 42].

The main technical contribution of this paper is the use of the logarithmic base change theorem instead, as a more appropriate tool to answer questions in a logarithmic setting. For instance, the logarithmic fiber product trick allows us to overcome obstructions to generalize Viehweg and Zuo's result as in Corollary 1.10. Specifically, Viehweg and Zuo's proof for a surjective morphism to $\mathbb{P}^1$ uses the base change theorem together with semistable reduction (Kempf et al. [21]). This essentially reduces to the case where every fiber is either smooth or a simple normal crossing divisor, so that the equality holds when we apply the base change theorem (Theorem 1.1). For higher dimensional bases, like $\mathbb{P}^n$, this reduction procedure does not work properly. However, all technical issues are resolved by the fact that the logarithmic base change theorem reverses inclusions as described above; it relies on a resolution algorithm for binomial hypersurface singularities. For the same reason, the smooth projective descent of the positivity of the log canonical divisor can be verified as mentioned after Corollary 1.4.

Overview. To derive various geometric consequences, we rely on three main technical components:

(1) the logarithmic base change theorem,

(2) the construction of logarithmic Higgs sheaves, and

(3) Campana and Păun's pseudoeffectivity result on the log cotangent bundle.

Section 2 is devoted to the proof of the logarithmic base change theorem. Section 3A explains the construction of logarithmic Higgs sheaves, slightly modifying that of Popa and Schnell [31]. Section 3B explains the results of Campana and Păun [5].

The rest of the paper is mainly devoted to the applications of those three main components. Section 4 gives proofs of the theorems on the smooth projective descent of the positivity of the log canonical divisor. Section 5 explains the conjecture of Popa on the superadditivity of the logarithmic Kodaira dimension and its proof assuming some conjectures of the minimal model program. Section 6 proves some uniruledness of fibrations over projective spaces. Section 7 discusses some interesting boundary examples regarding the lower bound of the degree of the discriminant locus in Corollary 1.10.

[1] Here, we use the fact that a smooth projective variety has a good minimal model if the general fiber of its Iitaka fibration has a good minimal model [25].

2. Logarithmic base change theorem

The main idea for the logarithmic base change theorem comes from the equality condition in Viehweg's base change theorem.

2A. ***Equality in Viehweg's base change theorem.*** Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of smooth quasiprojective varieties. Under the assumptions of Viehweg's base change theorem (Theorem 1.1), it is easy to show that the equality conditions of the inclusions are obtained when the fiber product $X' = X \times_Y Y'$ has canonical singularities. This turns out to be the case when the morphism f or g is semistable in a neighborhood of a codimension-1 point of Y after further birational modifications.

We briefly sketch the proof of the equality conditions in Theorem 1.1 following Mori [27, (4.10)] in this paragraph. Over a neighborhood of a codimension-1 point $y \in Y$, assume f is semistable, or equivalently the fiber has reduced normal crossing singularities. Let D be the divisor associated with y. Since a birational modification of the source does not change the pushforward of the pluricanonical bundle, we replace Y' with a log resolution of pair $(Y', g^{-1}(D))$. Accordingly, we assume $g^{-1}(D)$ to be a normal crossing divisor. Then, the fiber product X' is locally analytically isomorphic to the hypersurface defined by the equation

$$x_1 \cdots x_n = y_1^{f_1} \cdots y_m^{f_m}$$

for some positive integers $f_1, \ldots, f_m$. This singularity is normal and toric, and thus is canonical, attaining equality of the inclusions in Theorem 1.1.

We will see later that the singularities of the fiber product X' in the logarithmic setting are locally analytically isomorphic to binomial hypersurface singularities, defined by the equation

$$x_1^{e_1} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m}$$

for some positive integers $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$. We analyze these singularities using extra resolution techniques and prove Theorem 1.2 as a consequence.

2B. ***Fiber product of morphisms of log smooth pairs.*** A morphism $f : (X, E) \to (Y, D)$ of pairs is *strict* if $E = f^{-1}(D)$. Let $E = \bigcup_{i \in \{1,\ldots,k\}} E_i$ be a simple normal crossing divisor of X with smooth irreducible components E_i. Then the stratum E_I of E for a nonempty subset $I \subset \{1, \ldots, k\}$ is defined by $E_I = \bigcap_{i \in I} E_i$.

Definition 2.1. A *strict* morphism of pairs $f : (X, E) \to (Y, D)$ is *strictly smooth* if D is a smooth divisor and $E = f^{-1}(D)$ is a simple normal crossing divisor, such that $f|_{X \setminus E} : X \setminus E \to Y \setminus D$ is smooth and $f|_{E_I} : E_I \to D$ is smooth for every stratum E_I of E.

In other words, a morphism of pairs $f : (X, E) \to (Y, D)$ is *strictly smooth* if the morphism is locally analytically (or étale locally) equivalent to

$$(\mathbb{C}^{a+k}, (x_{a+1} \cdots x_{a+k})) \to (\mathbb{C}^{b+1}, (t)), \quad a \geq b,$$
$$(x_1, \ldots, x_a, x_{a+1}, \ldots, x_{a+k}) \mapsto (x_1, \ldots, x_b, x_{a+1}^{e_1} \cdots x_{a+k}^{e_k}),$$

where $x_1, \dots, x_{a+k}$ are local coordinates of X with divisor $E = V(x_{a+1} \cdots x_{a+k})$ and $x_1, \dots, x_b, t$ are local coordinates of Y with divisor $D = V(t)$. The local equation for E_i is x_{a+i} and $f^* D = \sum_{i=1}^k e_i E_i$. It is easy to check that a strictly smooth morphism of pairs is a flat morphism.

Lemma 2.2. *Let (X, E), (Y, D), (Y', D') be quasiprojective log smooth pairs. Let $f : (X, E) \to (Y, D)$ be a strict morphism of pairs and $g : (Y', D') \to (Y, D)$ be a strictly smooth morphism of pairs. Consider the commutative diagram* (1.3) *in Theorem 1.2*:

$$\begin{tikzcd}
(X, E) \arrow[d, "f"'] & (X', E') \arrow[l, "g'"'] \arrow[d, "f'"'] & (X'', E'') \arrow[l, "\mu"'] \arrow[dl, "f''"] \\
(Y, D) & (Y', D') \arrow[l, "g"] &
\end{tikzcd}$$

where $X' = X \times_Y Y'$, $E' = g'^{-1}(E)$ and $\mu : (X'', E'') \to (X', E')$ is a log resolution of pairs with $X'' \setminus E'' \cong X' \setminus E'$. Then there exists a natural inclusion

$$\omega_{X'}(E + D' - D)^{\otimes N} \subset \mu_*\big(\omega_{X''}(E'')^{\otimes N}\big),$$

where $\omega_{X'}(E + D' - D) := \omega_{X'} \otimes g'^* \mathcal{O}_X(E) \otimes f'^* \mathcal{O}_{Y'}(D') \otimes (g \circ f')^* \mathcal{O}_Y(-D)$.

The proof of Lemma 2.2 will be given in Section 2D. We first give a locally analytic description of X'. Let $x' \in X'$ be a point. Suppose E is locally $x_1 \cdots x_n = 0$ where x_i's are local coordinates at $g'(x') \in X$, and D' is locally $y_1 \cdots y_m = 0$ where y_j's are local coordinates at $f'(x') \in Y'$. Assume $t = 0$ is a local equation of D at $(g \circ f')(x')$ and $f^* t = x_1^{e_1} \cdots x_n^{e_n}$, $g^* t = y_1^{f_1} \cdots y_m^{f_m}$. From the above locally analytic description of a strictly smooth morphism, g is locally analytically isomorphic to

$$\mathbb{C}^m \times \mathbb{C}^b \times \mathbb{C}^{b'} \to \mathbb{C} \times \mathbb{C}^b, \quad (y_1, \dots, y_m) \times p \times q \mapsto (y_1^{f_1} \cdots y_m^{f_m}) \times p.$$

Therefore, X' is locally analytically isomorphic to the binomial hypersurface H in $\mathbb{C}^{n+m+r}$ defined by

$$H : x_1^{e_1} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m} \tag{2.3}$$

for some $r \geq 0$ and positive integers $e_1, \dots, e_n$, $f_1, \dots, f_m$. As the local equation suggests, X' is not normal in general.

Conventionally, singularities of pairs (X, Δ) are defined when X is a normal variety and Δ is a formal $\mathbb{Q}$-linear combination of divisors. However, the description of semilog canonical pairs in [22, Definition 5.10] and the description of a canonical sheaf for a G_1 and S_2-variety in [24, Section 5] suitably generalize the definition of singularities of pairs on a G_1 and S_2-variety. In particular, we make precise what it means for the pair $(X', E + D' - D)$ to be log canonical, and reduce the proof of Lemma 2.2 to proving that the pair $(X', E + D' - D)$ is indeed log canonical.

Readers comfortable with singularities of pairs for nonnormal Gorenstein varieties may skip the following section.

2C. ***Singularities of pairs: S_2-varieties, Gorenstein over codimension 1.*** Let (X, Δ) be a pair consisting of an S_2-variety X, Gorenstein over codimension 1, and a formal $\mathbb{Q}$-linear sum Δ of generically nonzero (rational) sections of line bundles on a Zariski open $j : U \hookrightarrow X$ satisfying $\operatorname{codim}_X X \setminus U \geq 2$ and $\omega_U = \omega_X|_U$ invertible:

$$\Delta = \sum_{i\in I} a_i(\mathcal{O}_U \dashrightarrow \mathcal{L}_i).$$

Here, I is a finite set, a_i is a rational number, $\mathcal{L}_i$ is a line bundle on U, and a rational section $\mathcal{O}_U \dashrightarrow \mathcal{L}_i$ is a section defined on a Zariski dense open subset of X. The following remark gives a sheaf theoretic description of Δ as a formal $\mathbb{Q}$-linear sum of global sections. This contains a formal $\mathbb{Q}$-linear sum of Cartier divisors.

Remark 2.4. Let $\mathcal{O}_X^*$ be the sheaf of invertible elements in $\mathcal{O}_X$, $\mathcal{R}_X$ be the sheaf of rational sections of $\mathcal{O}_X$, and $\mathcal{R}_X^*$ be the sheaf of generically nonzero rational sections of $\mathcal{O}_X$. Precisely, for an open subvariety $U \subset X$, $\mathcal{R}_X(U)$ is the product of the rational function fields at the generic points of U and $\mathcal{R}_X^*(U)$ is the subset of elements of $\mathcal{R}_X(U)$, nonzero at every generic point of U.

Up to multiplication by a global section of $\mathcal{O}_U^*$, a generically nonzero rational section $\mathcal{O}_U \dashrightarrow \mathcal{L}$ is equivalent to the global section of $\mathcal{R}_U^*/\mathcal{O}_U^*$. Therefore, Δ is a formal $\mathbb{Q}$-linear sum of global sections of $\mathcal{R}^*/\mathcal{O}^*$ on a Zariski open $U \subset X$ with $\operatorname{codim}_X X \setminus U \geq 2$. This includes a formal $\mathbb{Q}$-linear sum of Cartier divisors on U, or equivalently, a formal $\mathbb{Q}$-linear sum of the global sections of $\mathcal{K}^*/\mathcal{O}^*$ on U. Here, $\mathcal{K}_X^*$ is the sheaf of invertible elements in $\mathcal{K}_X$, which is the sheaf of total quotient rings of $\mathcal{O}_X$. See Hartshorne [10, p.141] for the description of Cartier divisors. In particular, $\mathcal{K}_X^*/\mathcal{O}_X^* \subset \mathcal{R}_X^*/\mathcal{O}_X^*$.

Unifying the formal sum by a common denominator of a_i's, the sum is alternatively expressed as a single term $\Delta = a(\mathcal{O}_U \dashrightarrow \mathcal{L})$, for some rational number a and some rational section of a line bundle $\mathcal{L} = \bigotimes_{i\in I} \mathcal{L}_i^{\otimes Na_i}$ with sufficiently divisible N. In particular, $(\mathcal{O}_U \dashrightarrow \mathcal{L}_i)$ is equivalent to $-(\mathcal{O}_U \dashrightarrow \mathcal{L}_i^{-1})$, where the latter rational section is the reciprocal of the former rational section.

Assume that $\omega_U^{\otimes N}(N\Delta) := \omega_U^{\otimes N} \otimes \bigotimes_{i\in I} \mathcal{L}_i^{\otimes Na_i}$ extends to a line bundle on X for some $N > 0$. Since X is an S_2-variety, this line bundle is uniquely determined by

$$\omega_X^{\otimes N}(N\Delta) := j_*\omega_U^{\otimes N}(N\Delta).$$

Let $\mu : \widetilde{X} \to X$ be a resolution of singularities of X. Over a Zariski dense open subset of X, we have a rational morphism of line bundles $\mu^*\omega_X^{\otimes N} \dashrightarrow \omega_{\widetilde{X}}^{\otimes N}$, and pullbacks of generically nonzero rational sections $(\mathcal{O}_U \dashrightarrow \mathcal{L}_i)$. Therefore, there exists a canonically defined rational morphism of line bundles

$$\mu^*\omega_X^{\otimes N}(N\Delta) \dashrightarrow \omega_{\widetilde{X}}^{\otimes N},$$

which is an isomorphism on a Zariski dense open subset of $\widetilde{X}$. As a result, we obtain a generically nonzero rational section $s : \mathcal{O}_{\widetilde{X}} \dashrightarrow \omega_{\widetilde{X}}^{\otimes N} \otimes [\mu^*\omega_X^{\otimes N}(N\Delta)]^{-1}$ on $\widetilde{X}$.

Let E be a divisor on $\widetilde{X}$. The discrepancy of E with respect to the pair (X, Δ) is defined as

$$a(E, X, \Delta) := \frac{\mathrm{ord}_E(s)}{N}.$$

It is easy to verify that $a(E, X, \Delta)$ is well defined, independent of the choice of N or a resolution of singularities. In conclusion, we define singularities of pairs (log canonical, log terminal, and so on) as in the normal case.

Under the setting of Lemma 2.2, it suffices to prove that the pair $(X', E + D' - D)$ is log canonical. More specifically, the pair $(X', E + D' - D)$ consists of a Gorenstein variety X' and a formal sum of sections of line bundles

$$(\mathcal{O}_{X'} \to g'^*\mathcal{O}_X(E)) + (\mathcal{O}_{X'} \to f'^*\mathcal{O}_{Y'}(D')) - (\mathcal{O}_{X'} \to (g \circ f')^*\mathcal{O}_Y(D)),$$

where the section $(\mathcal{O}_{X'} \to g'^*\mathcal{O}_X(E))$ is induced by the pullback of the section $\mathcal{O}_X \to \mathcal{O}_X(E)$. Therefore, $(X', E + D' - D)$ is log canonical if and only if there exists a natural inclusion

$$\mu^*\big(\omega_{X'}(E + D' - D)^{\otimes N}\big) \subset \omega_{X''}(E'')^{\otimes N},$$

which is equivalent to the conclusion of Lemma 2.2 due to the adjointness of the pair (μ^*, μ_*).

2D. *Singularities of binomial hypersurfaces: Proof of Lemma 2.2.* We prove $(X', E + D' - D)$ is log canonical, which implies Lemma 2.2. From equation (2.3), the pair $(X', E + D' - D)$ is locally analytically equivalent to

$$\big(H : x_1^{e_1} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m},\ \ D(x_1 \cdots x_n) + D(y_1 \cdots y_m) - D(t)\big),$$

where D is to indicate sections of line bundles defined by multiplications of $x_1 \cdots x_n$, $y_1 \cdots y_m$, or $t = x_1^{e_1} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m}$. The singularities of pairs can be checked locally analytically (or étale locally), so it suffices to prove that

$$\big(H,\ \ D(x_1 \cdots x_n) + D(y_1 \cdots y_m) - D(t)\big)$$

is log canonical, which is implied by the following:

Proposition 2.5. *For a binomial hypersurface $H : \{t = x_1^{e_1} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m}\} \subset \mathbb{A}^{n+m+r}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots z_r$, the pair*

$$\big(H : x_1^{e_1} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m},\ \ D(x_1 \cdots x_n y_1 \cdots y_m z_1 \cdots z_r) - D(t)\big)$$

is log canonical.

Let $w = x_1 \cdots x_n y_1 \cdots y_m z_1 \cdots z_r$ be the multiplication of all the coordinates. The reason for adding $D(z_1 \cdots z_r)$ into the pair is to apply induction. In the remaining section, we resolve the singularity of H through a sequence of blow-ups along the ideals cut out by coordinate sections, such as

$(x_1,\dots,x_k,y_1\dots,y_l)$. Write $Z := V(x_1,\dots,x_k,y_1\dots,y_l)$. Recall that

$$Bl_Z\mathbb{A}^{n+m+r} = \operatorname{Proj} \frac{k[x_1,\dots,x_n,y_1,\dots,y_m,z_1\dots z_r][X_1,\dots,X_k,Y_1,\dots,Y_l]}{\left(\frac{X_1}{x_1}=\cdots=\frac{X_k}{x_k}=\frac{Y_1}{y_1}=\cdots=\frac{Y_l}{y_l}\right)}$$

with the induced morphism $\beta : Bl_{(x_1,\dots,x_k,y_1\dots,y_l)}\mathbb{A}^{n+m+r} \to \mathbb{A}^{n+m+r}$. Restricted to the affine chart $\mathbb{A}^{n+m+r} : X_1 \neq 0$, we obtain the transformation

$$\beta : \mathbb{A}^{n+m+r} \to \mathbb{A}^{n+m+r}$$

given by

$$x_1 \mapsto x_1, x_2 \mapsto x_1x_2,\dots,x_k \mapsto x_1x_k, \quad y_1 \mapsto x_1y_1, y_2 \mapsto x_1y_2,\dots,y_l \mapsto x_1y_l, \tag{2.6}$$

with all other coordinates remaining the same. Therefore, the pullback β^*H of H in the chart $X_1 \neq 0$ is the binomial hypersurface

$$x_1^{e_1+\cdots+e_k}x_2^{e_2}\cdots x_n^{e_n} = x_1^{f_1+\cdots+f_l}y_1^{f_1}\cdots y_m^{f_m}.$$

The strict transform $\widetilde{H}$ of H in the chart $X_1 \neq 0$ is the binomial hypersurface above divided by $x_1^{\min(e_1+\cdots+e_k,f_1+\cdots+f_l)}$.

Therefore, for each affine chart, we may consider a pair $(\widetilde{H}, D(\tilde{w})-D(\tilde{t}))$, where $\tilde{w}$ is the multiplication of the coordinates and $\tilde{t}$ is the monomial appearing in the binomial equation.

Lemma 2.7. *In the above notation, let $\beta : \widetilde{H} \to H$. We work in the affine chart $X_1 \neq 0$.*

(1) *If $e_1+\cdots+e_k \geq f_1+\cdots+f_l$, then $\widetilde{H}$ is a binomial hypersurface*

$$x_1^{(e_1+\cdots+e_k)-(f_1+\cdots+f_l)}x_2^{e_2}\cdots x_n^{e_n} = y_1^{f_1}\cdots y_m^{f_m}.$$

(2) *If $e_1+\cdots+e_k \leq f_1+\cdots+f_l$, then $\widetilde{H}$ is a binomial hypersurface*

$$x_2^{e_2}\cdots x_n^{e_n} = x_1^{-(e_1+\cdots+e_k)+(f_1+\cdots+f_l)}y_1^{f_1}\cdots y_m^{f_m}.$$

In either case, there exists a canonical isomorphism

$$\beta^*[\omega_H(D(w)-D(t))] \simeq \omega_{\widetilde{H}}(D(\tilde{w})-D(\tilde{t}))$$

In particular, $(H, D(w)-D(t))$ is log canonical if and only if $(\widetilde{H}, D(\tilde{w})-D(\tilde{t}))$ is log canonical on every affine chart.

More specifically, the canonical morphism indicates that

$$a(E,\widetilde{H},D(\tilde{w})-D(\tilde{t})) = a(E,H,D(w)-D(t)),$$

for all divisors E appearing in a resolution of $\widetilde{H}$.

Proof. We write $Bl\mathbb{A} := Bl_{(x_1,\dots,x_k,y_1\dots,y_l)}\mathbb{A}^{n+m+r}$ and $\mathbb{A} := \mathbb{A}^{n+m+r}$ with the blow-up morphism $\beta : Bl\mathbb{A} \to \mathbb{A}$, in short. On the chart $X_1 \neq 0$, the exceptional divisor is given by $x_1 = 0$. Therefore,

$$\omega_{Bl\mathbb{A}} = \beta^*\omega_{\mathbb{A}} + (k+l-1)D(x_1).$$

By the adjunction formula, there exist natural isomorphisms $\omega_{\widetilde{H}} = \omega_{Bl\mathbb{A}}(\widetilde{H})|_{\widetilde{H}}$ and $\omega_H = \omega_{\mathbb{A}}(H)|_H$. Since

$$\beta^*(H) = (\widetilde{H}) + \min(e_1 + \cdots + e_k, f_1 + \cdots + f_l) D(x_1),$$

we have a natural isomorphism

$$\omega_{\widetilde{H}} = \beta^*\omega_H + (k + l - 1 - \min(e_1 + \cdots + e_k, f_1 + \cdots + f_l))\, D(x_1).$$

From the transformation relation (2.6), we have

$$\beta^* D(x_1 \cdots x_n y_1 \cdots y_m z_1 \cdots z_r) = D(x_1 \cdots x_n y_1 \cdots y_m z_1 \cdots z_r) + (k + l - 1) D(x_1)$$

and in either case (1) or (2), it is easy to check that

$$\beta^* D(t) = D(\tilde{t}) + \min(e_1 + \cdots + e_k, f_1 + \cdots + f_l) D(x_1).$$

Combining all of the above, we obtain a canonical isomorphism

$$\beta^*[\omega_H(D(w) - D(t))] \simeq \omega_{\widetilde{H}}(D(\tilde{w}) - D(\tilde{t})),$$

which completes the proof. □

By symmetry, Lemma 2.7 holds for every affine chart. We finish the proof of Proposition 2.5 using Bierstone and Milman's algorithm [1, Section 5] for resolution of singularities of binomial varieties along with Lemma 2.7. Define

$$m(H) := \min(e_1 + \cdots + e_n, f_1 + \cdots + f_m), \quad M(H) := \max(e_1 + \cdots + e_n, f_1 + \cdots + f_m).$$

Without loss of generality, assume $e_1 + \cdots + e_n \geq f_1 + \cdots + f_m$. Choose a subset $S \subset \{1, \ldots, n\}$ such that

$$0 \leq \sum_{s \in S} e_s - m(H) < \min_{s \in S} \{e_s\}.$$

For convenience, assume $S = \{1, \ldots, k\}$ for some $1 \leq k \leq n$. Let a subvariety $Z : x_1 = \cdots = x_k = y_1 = \cdots = y_m = 0$, and we now perform an algorithm of blowing up along Z: $Bl_Z \mathbb{A}^{n+m+r} \to \mathbb{A}^{n+m+r}$. The strict transform $\widetilde{H}$ of H of the blow-up is the binomial hypersurface

$$x_1^{e_1 + \cdots + e_k - m(H)} x_2^{e_2} \cdots x_n^{e_n} = y_1^{f_1} \cdots y_m^{f_m} \quad \text{on the chart } X_1 \neq 0,$$
$$y_1^{e_1 + \cdots + e_k - m(H)} x_1^{e_1} \cdots x_n^{e_n} = y_2^{f_2} \cdots y_m^{f_m} \quad \text{on the chart } Y_1 \neq 0.$$

On every affine chart of the blow-up, the lexicographic ordering of the pair

$$(m(H), M(H))$$

decreases. Therefore, the algorithm terminates on each affine chart, in which case, we obtain a hypersurface of the form

$$H : \left\{x_1^{e_1} \cdots x_n^{e_n} = 1\right\} \subset \mathbb{A}^{n+m+r}.$$

It is now obvious that the pair $(H, D(w) - D(t))$ is log canonical. Indeed, H is smooth, $D(t)$ is a zero divisor, and $D(w)$ is a simple normal crossing divisor defined by the product of the coordinates except for x_i's. Therefore, by Lemma 2.7, the original $(H, D(w) - D(t))$ is log canonical by induction on $(m(H), M(H)) \subset \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ endowed with the lexicographic ordering. This completes the proof of Proposition 2.5, and thus Lemma 2.2.

2E. ***Proof of the logarithmic base change theorem.*** Here is an immediate consequence of Lemma 2.2.

Corollary 2.8. *In the setting of Lemma 2.2, we have the inclusion*

$$g^* f_*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big) \subset f''_*\big(\omega_{X''}(E'')/\omega_{Y'}(D')^{\otimes N}\big).$$

Proof. Since g is flat, $g'^*\omega_{X/Y} = \omega_{X'/Y'}$, which implies that

$$\begin{aligned} g'^*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big) &= \big(\omega_{X'}(E - D)/\omega_{Y'}\big)^{\otimes N} \\ &= \big(\omega_{X'}(E + D' - D)/\omega_{Y'}(D')\big)^{\otimes N} \\ &\subset \mu_*\big(\omega_{X''}(E'')/\omega_{Y'}(D')^{\otimes N}\big). \end{aligned}$$

Taking pushforward f'_* on each side, we obtain the inclusion, since $g^* f_* = f'_* g'^*$ by the flatness of g. □

Proof of Theorem 1.2. As given in the assumption, $g : (Y', D') \to (Y, D)$ is a strict morphism of log smooth pairs such that $g|_{Y' \setminus D'} : Y' \setminus D' \to Y \setminus D$ is smooth. Then the set of points on D, such that either D is singular or $g|_{D'_I} : D'_I \to D$ is singular for some stratum D'_I of D', is a Zariski closed subset of Y with codimension at least 2. Therefore, there exists an open subvariety $Y_0 \subset Y$ with $\operatorname{codim}_Y Y \setminus Y_0 \geq 2$, such that $g|_{g^{-1}(Y_0)} : (Y'_0, D'_0) \to (Y_0, D_0)$ is strictly smooth. Since we are taking the double dual at the end, we need only prove the inclusion in the statement of the theorem on Y_0. Therefore, it suffices to prove the theorem when g is a strictly smooth morphism of log smooth pairs.

Then by Corollary 2.8, we have

$$g_*\big[g^* f_*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big) \otimes \big(\omega_{Y'}(D')/\omega_Y(D)^{\otimes N}\big)\big] \subset h_*\big(\omega_{X''}(E'')/\omega_Y(D)^{\otimes N}\big).$$

Further shrinking an open subvariety $Y_0 \subset Y$ with $\operatorname{codim}_Y Y \setminus Y_0 \geq 2$ where $f_*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big)$ is locally free, we have the inclusion

$$f_*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big) \otimes g_*\big(\omega_{Y'}(D')/\omega_Y(D)^{\otimes N}\big) \subset h_*\big(\omega_{X''}(E'')/\omega_Y(D)^{\otimes N}\big)$$

from which we obtain the conclusion. □

3. Logarithmic Higgs sheaves and Campana–Păun criterion

In Sections 3A and 3B we review some important constructions from the literature, with small modifications needed here in the first one. In the end, we prove Theorem 1.7(2) as an immediate application.

3A. ***Construction of logarithmic Higgs sheaves.*** Given a morphism from a smooth projective variety to a curve, Viehweg and Zuo [41] studied a lower bound of the number of singular fibers via the construction of Higgs bundles. This construction was later generalized in [42] to a morphism of higher dimensional varieties with positivity conditions on the fibers. Using the theory of Hodge modules, Popa and Schnell [31] refined the construction of logarithmic Higgs sheaves with poles along the discriminant divisor in general. Along with the results of Campana–Păun [5], these provide a powerful method to study the hyperbolicity of the base of a smooth projective morphism. In this section, we summarize and explain the above, following Popa and Schnell's treatment with some simplifications. As before, (Y, D) is a log smooth pair.

Definition 3.1. A graded $\mathcal{O}_Y$-module $\mathcal{F}_\bullet = \oplus_k \mathcal{F}_k$ is a *graded logarithmic Higgs sheaf* with poles along D if there exists a logarithmic Higgs structure

$$\phi_\bullet : \mathcal{F}_\bullet \to \mathcal{F}_{\bullet+1} \otimes \Omega_Y(\log D)$$

such that $\phi \wedge \phi : \mathcal{F}_\bullet \to \mathcal{F}_{\bullet+2} \otimes \Omega_Y^2(\log D)$ is the zero morphism. Unless otherwise stated, $\mathcal{F}_k = 0$ for $k \ll 0$. Define

$$\mathcal{K}_k(\phi) := \ker\bigl(\phi_k : \mathcal{F}_k \to \mathcal{F}_{k+1} \otimes \Omega_Y(\log D)\bigr)$$

to be the kernel of the Higgs field for each k.

When a surjective morphism $f : X \to Y$ with connected fibers satisfies some additional effectivity condition for a particular line bundle, Popa and Schnell [31, Theorem 2.3] constructed a graded logarithmic Higgs sheaf with poles along the discriminant locus. For our later use, we modify [31, Theorem 2.3] for a logarithmic setting, and drop the assumption on the connectedness of the fibers.

Theorem 3.2. *Let $f : (X, E) \to (Y, D)$ be a surjective projective morphism of quasiprojective log smooth pairs with $E = f^{-1}(D)$, and $f|_{X \setminus E}$ is smooth. For a line bundle $\mathcal{L}$ on Y, assume that some power of $\omega_X(E)/\omega_Y(D) \otimes f^*\mathcal{L}^{-1}$ is effective (i.e., has a nonzero section). Then there exists a graded logarithmic Higgs sheaf $\mathcal{F}_\bullet$ with poles along D satisfying the following properties*:

(a) *One has $\mathcal{L} \subset \mathcal{F}_0$, and $\mathcal{F}_k = 0$ for $k < 0$.*

(b) *There exists d such that $\mathcal{F}_k = 0$ for all $k > d$.*

(c) *Each $\mathcal{F}_k$ is a reflexive coherent sheaf on Y.*

(d) *Each dual $\mathcal{K}_k(\phi)^*$ of the kernel of the Higgs field is weakly positive if $\mathcal{K}_k(\phi) \neq 0$.*

In [42], Viehweg and Zuo initially constructed similar Higgs sheaves, under the slightly stronger assumption that the canonical bundle of the general fiber is semiample.

We recall the definitions and properties of weakly positive sheaves and big sheaves, introduced by Viehweg [39; 40]; see also [27, Section 5]. In what follows, $\hat{S}^\alpha F$ denotes the double dual of the α-symmetric product of F, and $\widehat{\det} F$ denotes the double dual of the determinant of F.

Definition 3.3. A torsion-free coherent sheaf F on a normal quasiprojective variety W is *weakly positive* if, for every positive integer α and every ample line bundle H, there exists a positive integer β such that

$\hat{S}^{\alpha\beta} F \otimes H^{\beta}$ is generically generated by global sections. We say F is *big* if, for every ample line bundle H, there exists a positive integer α such that $\hat{S}^{\alpha} F \otimes H^{-1}$ is weakly positive.

If F is a line bundle and W is a normal projective variety, then F is weakly positive (resp. big) if and only if F is pseudoeffective (resp. big). This is immediate from the definition. We recall additional important properties of weakly positive and big sheaves.

Lemma 3.4 [16; 27; 39; 40]. *Let F be a nonzero torsion-free coherent sheaf on a normal quasiprojective variety W.*

(1) *Let $W_0 \subset W$ be a Zariski open subset. If F is weakly positive* (*resp. big*), *then $F|_{W_0}$ is weakly positive* (*resp. big*). *If $\mathrm{codim}_W W \setminus W_0 \geq 2$, then the converse is true.*

(2) *Let $F \to G$ be a generically surjective morphism of nonzero torsion-free coherent sheaves. If F is weakly positive* (*resp. big*), *then G is weakly positive* (*resp. big*).

(3) *Let A be a big line bundle. If F is weakly positive*, *then $F \otimes A$ is big.*

(4) *If F is weakly positive* (*resp. big*), *then $\widehat{\det} F$ is weakly positive* (*resp. big*).

(5) *If $\tau : W' \to W$ is a finite surjective morphism of normal varieties*, *then F is weakly positive* (*resp. big*) *if and only if $\tau^* F$ is weakly positive* (*resp. big*).

Proof of Theorem 3.2. It suffices to construct $\mathcal{F}_\bullet$ on the complement of a codimension-2 subvariety in Y, satisfying properties (a)–(d). Indeed, taking the reflexive hull of $\mathcal{F}_\bullet$ over Y, we obtain a graded logarithmic Higgs sheaf, still satisfying the properties of Theorem 3.2. (See Lemma 3.4(1) for property (d).)

Take a resolution $\psi : Z \to X$ of the cyclic cover of X induced by the section of some power of $B := \omega_X(E)/\omega_Y(D) \otimes f^*\mathcal{L}^{-1}$. In particular, there exists a natural inclusion $\psi^* B^{-1} \to \mathcal{O}_Z$. Denote $h := f \circ \psi : Z \to Y$, and let D_h be the union of the discriminant locus $\Delta(h)$ and D in Y. After removing a codimension-2 subvariety in Y and taking a suitable resolution Z of the cyclic cover, we may assume that D_h is a smooth divisor such that $E_h := h^{-1} D_h$ is a simple normal crossing divisor of Z.

Let $\Omega_{X/Y}(\log E)$ (resp. $\Omega_{Z/Y}(\log E_h)$) be the cokernel of the natural inclusion of the logarithmic cotangent bundles

$$f^*\Omega_Y(\log D) \to \Omega_X(\log E) \quad (\text{resp. } h^*\Omega_Y(\log D_h) \to \Omega_Z(\log E_h)).$$

Further removing a codimension-2 subvariety in Y so that $f : (X, E) \to (Y, D)$ and $h : (Z, E_h) \to (Y, D_h)$ are strictly smooth (see the proof of Theorem 1.2, for example), we may assume that $\Omega_{X/Y}(\log E)$ and $\Omega_{Z/Y}(\log E_h)$ are locally free from the locally analytic description. As a consequence, we have the Koszul filtration

$$\mathrm{Koz}^q\, \Omega^i_X(\log E) := \mathrm{image}\big[f^*\Omega^q_Y(\log D) \otimes \Omega^{i-q}_X(\log E) \to \Omega^i_X(\log E)\big],$$

where $\Omega^i_X(\log E) := \bigwedge^i \Omega_X(\log E)$ is the i-th exterior power, for all $0 \leq i \leq \dim X$. Hence, we have the natural isomorphism

$$\mathrm{Koz}^q/\mathrm{Koz}^{q+1}\, \Omega^i_X(\log E) \cong f^*\Omega^q_Y(\log D) \otimes \Omega^{i-q}_{X/Y}(\log E)$$

and the tautological short exact sequence

$$0 \to f^*\Omega_Y(\log D) \otimes \Omega^{i-1}_{X/Y}(\log E) \to \mathrm{Koz}^0/\mathrm{Koz}^2\, \Omega^i_X(\log E) \to \Omega^i_{X/Y}(\log E) \to 0,$$

which we denote by $\mathcal{C}^i_{X/Y}(\log E)$. Likewise, we have the tautological short exact sequence $\mathcal{C}^i_{Z/Y}(\log E_h)$.

From the natural morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & h^*\Omega_Y(\log D) \arrow[r] \arrow[d] & \psi^*\Omega_X(\log E) \arrow[r] \arrow[d] & \psi^*\Omega_{X/Y}(\log E) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & h^*\Omega_Y(\log D_h) \arrow[r] & \Omega_Z(\log E_h) \arrow[r] & \Omega_{Z/Y}(\log E_h) \arrow[r] & 0 \end{tikzcd}$$

we have the morphism $\psi^*\mathcal{C}^i_{X/Y}(\log E) \to \mathcal{C}^i_{Z/Y}(\log E_h)$ of the tautological short exact sequences, for all i. Tensored with the natural injection $\psi^*B^{-1} \to \mathcal{O}_Z$, we obtain the following morphism of short exact sequences:

$$\psi^*\big(\mathcal{C}^i_{X/Y}(\log E) \otimes B^{-1}\big) \to \mathcal{C}^i_{Z/Y}(\log E_h). \tag{3.5}$$

Let $d = \dim X - \dim Y$. From the Leray spectral sequence associated with $Rh_* \cong Rf_* \circ R\psi_*$ and the adjointness of the pair (ψ^*, ψ_*), we have the natural morphism

$$R^{d-i} f_*\big(\Omega^i_{X/Y}(\log E) \otimes B^{-1}\big) \to R^{d-i} h_*\big(\psi^*\big(\Omega^i_{X/Y}(\log E) \otimes B^{-1}\big)\big),$$

which induces the following commutative diagram of the connecting homomorphisms via the derived pushforward of (3.5):

$$\begin{tikzcd} R^{d-i} f_*\big(\Omega^i_{X/Y}(\log E) \otimes B^{-1}\big) \arrow[r] \arrow[d, "\rho_{d-i}"'] & R^{d-i+1} f_*\big(\Omega^{i-1}_{X/Y}(\log E) \otimes B^{-1}\big) \otimes \Omega_Y(\log D) \arrow[d, "\rho_{d-i+1}\otimes\iota"] \\ R^{d-i} h_*\big(\Omega^i_{Z/Y}(\log E_h)\big) \arrow[r, "\phi'_{d-i}"] & R^{d-i+1} h_*\big(\Omega^{i-1}_{Z/Y}(\log E_h)\big) \otimes \Omega_Y(\log D_h) \end{tikzcd}$$

Due to Steenbrink (see for example [38; 43]), it is well known that $\phi'_\bullet$ is the logarithmic Higgs structure of the graded logarithmic Higgs bundle

$$\mathrm{gr}^F_\bullet \mathcal{V}_0 \cong \bigoplus_{k=0}^{d} R^k h_*\big(\Omega^{d-k}_{Z/Y}(\log E_h)\big),$$

where $\mathcal{V}_0$ is Deligne's lower canonical extension along D_h on Y, with the induced filtration F, associated to the polarized variation of Hodge structure given by the middle cohomologies of the smooth fibers of h. The lower canonical extension means that the eigenvalues of the residues of the logarithmic connection lie in $[0, 1)$. Define

$$\mathcal{F}_k := \big(\mathrm{image}\big[\rho_k : R^k f_*\big(\Omega^{d-k}_{X/Y}(\log E) \otimes B^{-1}\big) \to R^k h_*\big(\Omega^{d-k}_{Z/Y}(\log E_h)\big)\big]\big)^{**}$$

for $0 \le k \le d$, and $\mathcal{F}_k := 0$ otherwise. Then, $\mathcal{F}_\bullet$ is a graded logarithmic Higgs sheaf with poles along D. Its logarithmic Higgs structure $\phi_\bullet$ is induced by $\phi'_\bullet$ of $\mathrm{gr}^F_\bullet \mathcal{V}_0$. Notice that ρ_0 is the pushforward f_* of the inclusion

$$\omega_X(E)/\omega_Y(D) \otimes B^{-1} \to \psi_*(\omega_Z(E_h)/\omega_Y(D_h)),$$

which is the adjoint of $\psi^*(\omega_X(E)/\omega_Y(D) \otimes B^{-1}) \to \omega_Z(E_h)/\omega_Y(D_h)$. Accordingly, $\mathcal{F}_0 = (\mathcal{L} \otimes f_*\mathcal{O}_X)^{**}$, which implies $\mathcal{L} \subset \mathcal{F}_0$. Therefore, properties (a)–(c) are immediate from the construction.

Observe that $\mathcal{F}_\bullet \subset \mathrm{gr}^F_\bullet \mathcal{V}_0$, which implies $\mathcal{K}_k(\phi) \subset \mathcal{K}_k(\phi')$. Hence, we have a generically surjective morphism $\mathcal{K}_k(\phi')^* \to \mathcal{K}_k(\phi)^*$. It is well known that $\mathcal{K}_k(\phi')^*$ is weakly positive if not zero [33; 3; 44]. From Lemma 3.4(2), $\mathcal{K}_k(\phi)^*$ is weakly positive if not zero, which verifies the property (d). ☐

Remark 3.6. The conclusion of Theorem 3.2 continues to hold when we replace the assumption $\kappa(X, \omega_X(E)/\omega_Y(D) \otimes f^*\mathcal{L}^{-1}) \geq 0$ by the existence of a nonzero morphism

$$\mathcal{L}^{\otimes N} \to \big[f_*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big)\big]^{**}$$

for some positive integer N. By the left-right adjointness of (f^*, f_*), this morphism implies the existence of a nonzero section of N-th power of

$$\omega_X(E)/\omega_Y(D) \otimes f^*\mathcal{L}^{-1}$$

over the complement of a codimension-2 subvariety in Y. Therefore, we obtain a graded logarithmic Higgs sheaf $\mathcal{F}_\bullet$, on the complement of a codimension-2 subvariety in Y, satisfying the properties of Theorem 3.2. As at the beginning of the proof of Theorem 3.2, we obtain the desired graded logarithmic Higgs sheaf via taking the reflexive hull of $\mathcal{F}_\bullet$ over Y. This refined assumption turns out to be more convenient for the applications of the logarithmic base change theorem.

The following lemma is obtained from the standard manipulation of logarithmic Higgs sheaves by Viehweg and Zuo. This allows us to apply Campana and Păun's results explained in the next subsection to deduce results on hyperbolicity.

Lemma 3.7. *Let (Y, D) be a log smooth pair. Let $\mathcal{F}_\bullet$ be a graded logarithmic Higgs sheaf with poles along D satisfying the properties (a)-(d) of Theorem 3.2. Then there exists a pseudoeffective line bundle P and a nonzero morphism*

$$\mathcal{L}^{\otimes r} \otimes P \to (\Omega_Y(\log D))^{\otimes kr}$$

for some $r > 0$, $k \geq 0$.

Proof. From the logarithmic Higgs structure $\phi_\bullet$ of $\mathcal{F}_\bullet$, we have a sequence of morphisms

$$\phi_k \otimes \mathrm{id} : \mathcal{F}_k \otimes (\Omega_Y(\log D))^{\otimes k} \to \mathcal{F}_{k+1} \otimes (\Omega_Y(\log D))^{\otimes k+1}.$$

Notice that $\mathcal{F}_k = 0$ for large enough k. Therefore, the line bundle $\mathcal{L}$ is contained in the kernel of $\phi_k \otimes \mathrm{id}$ for some $k \geq 0$:

$$\mathcal{L} \subset \mathcal{K}_k(\phi) \otimes (\Omega_Y(\log D))^{\otimes k}.$$

This implies the existence of a nonzero morphism

$$\mathcal{K}_k(\phi)^* \to (\Omega_Y(\log D))^{\otimes k} \otimes \mathcal{L}^{-1}.$$

Let $\mathcal{K}^*$ be the image of the morphism and r be the generic rank of $\mathcal{K}^*$. From the split surjection $\mathcal{K}^{*\otimes r} \to \det \mathcal{K}^*$, where $\mathcal{K}^*$ is a vector bundle outside of a codimension-2 locus in Y, we obtain a nonzero morphism

$$\widehat{\det}\,\mathcal{K}^* \to \left[(\Omega_Y(\log D))^{\otimes k} \otimes \mathcal{L}^{-1}\right]^{\otimes r}.$$

Note that $P := \widehat{\det}\,\mathcal{K}^*$ is a pseudoeffective line bundle by Lemma 3.4(4). Therefore, we complete the proof of the lemma. □

3B. ***Positivity properties of the logarithmic cotangent bundle.*** We highlight the results of Campana and Păun [5]. Investigating foliations on orbifold tangent bundles, they studied the positivity properties of their tensor powers. Together with the logarithmic base change theorem and the construction of logarithmic Higgs sheaves, this provides the machinery to study the positivity of the base of a smooth projective morphism.

Theorem 3.8 [5, Theorem 1.3]. *Let (Y, D) be a projective log smooth pair such that $K_Y + D$ is pseudoeffective. For every quotient Q of a tensor power of the logarithmic cotangent bundle $(\Omega_Y(\log D))^{\otimes N}$ with $N \geq 1$, the first Chern class $c_1(Q)$ is pseudoeffective.*

Theorem 3.9 [5, Theorem 7.6]. *Let (Y, D) be a projective log smooth pair, and let L be a pseudoeffective line bundle on Y. If there exists a nonzero morphism*

$$L \to (\Omega_Y(\log D))^{\otimes k} \otimes (\omega_Y(D))^{\otimes r}$$

for some integers $k \geq 0$ and $r \geq 1$, then $K_Y + D$ is pseudoeffective.

Originally, Campana and Păun stated their results in terms of an orbifold pair (Y, D), but we restrict the statements to a log smooth pair since it suffices for our purpose.

As a quick application of the results in this section, we prove Theorem 1.7(2).

Proof of Theorem 1.7(2). Since $\omega_X(E)/\omega_Y(D) \otimes f^*\omega_Y(D)$ is effective, there exists a graded logarithmic Higgs sheaf $\mathcal{F}_\bullet$ satisfying the properties of Theorem 3.2, with $\mathcal{L} = \omega_Y(D)^{-1}$. From Lemma 3.7, we have a pseudoeffective line bundle P and a nonzero morphism

$$(\omega_Y(D))^{\otimes -r} \otimes P \to (\Omega_Y(\log D))^{\otimes kr}$$

for $r > 0$, $k \geq 0$. Therefore, $K_Y + D$ is pseudoeffective by Theorem 3.9. □

Suppose $f : X \to \mathbb{P}^n$ is a surjective morphism, with X a smooth projective variety of nonnegative Kodaira dimension. By Theorem 1.7(2), it is easy to see that $\dim \Delta(f) = n - 1$ and $\deg \Delta(f) \geq n + 1$. However, when $n = 1$, Viehweg and Zuo [41, Theorem 0.2] suggest this inequality is not sharp. Theorem 1.9 and Corollary 1.10 give a sharp inequality whose proofs are provided in the next section, which primarily use the logarithmic base change theorem.

4. Proofs

4A. ***Proof of Theorem 1.5.*** Theorem 1.5 is a generalization of Popa and Schnell [32, Theorem A], applying the logarithmic fiber product trick to their proof.

To begin with, we compactify X and Y using Hironaka's resolution of singularities. Therefore, we may assume $f:(X,E)\to(Y,D)$ to be a morphism of log smooth pairs with $E=f^{-1}(D)$ such that $f|_{X\setminus E}:X\setminus E\to Y\setminus D$ is the initial smooth morphism of quasiprojective varieties.

The forward implication is immediate due to Maehara [26, Corollary 2]: when $Y\setminus D$ is of log general type, then $\bar{\kappa}(X\setminus E)=\kappa(F)+\dim Y$. It remains to prove the converse implication, assuming $\bar{\kappa}(X\setminus E)=\kappa(F)+\dim Y$. Recall the Easy Addition formula [27, Corollary 1.7], applied to the Cartier divisor K_X+E:

$$\bar{\kappa}(X\setminus E)=\kappa(X,K_X+E)\le\kappa(F)+\dim Y.$$

The equality holds if and only if there exists a positive integer N and an ample divisor A on Y such that

$$f^*A\subset\omega_X(E)^{\otimes N},$$

by [27, Proposition 1.14]. This is an adjoint of the inclusion $A\subset f_*(\omega_X(E)^{\otimes N})$. Applying Corollary 1.4, we obtain the following inclusions:

$$\big(A\otimes(\omega_Y(D)^{\otimes -N})\big)^{\otimes N}\subset\Big[\bigotimes^N f_*\big(\omega_X(E)/\omega_Y(D)^{\otimes N}\big)\Big]^{**}\subset\big[f_*^{(N)}\big(\omega_{X^{(N)}}(E^{(N)})/\omega_Y(D)^{\otimes N}\big)\big]^{**}.$$

Therefore, taking $\mathcal{L}=A\otimes(\omega_Y(D)^{\otimes -N})$, there exists a logarithmic Higgs sheaf $\mathcal{F}_\bullet$ with poles along D which satisfies the properties of Theorem 3.2, by Remark 3.6. From Lemma 3.7, we have a pseudoeffective line bundle P and a nonzero morphism

$$A^{\otimes r}\otimes P\to(\Omega_Y(\log D))^{\otimes kr}\otimes(\omega_Y(D))^{\otimes Nr}$$

for some $r>0$, $k\ge 0$. By Theorem 3.9, K_Y+D is pseudoeffective. Note that $\omega_Y(D)$ is a line subbundle in $(\Omega_Y(\log D))^{\otimes\dim Y}$. Hence, there exists some positive integer N' such that

$$A^{\otimes r}\otimes P\to(\Omega_Y(\log D))^{\otimes N'}$$

is a nonzero morphism. Thus, Theorem 3.8 implies that K_Y+D is the sum of a big divisor and a pseudoeffective divisor, so it is big as desired.

4B. ***Proof of Theorem 1.7(1).*** We use a similar technique as in the proof of Theorem 1.5. Since $K_X+(1-\epsilon)E$ is effective, there exists a sufficiently divisible positive integer N such that

$$f^*\mathcal{O}_Y(D)\subset\omega_X(E)^{\otimes N}.$$

This is an adjoint of the inclusion $\mathcal{O}_Y(D)\subset f_*(\omega_X(E)^{\otimes N})$. Applying Corollary 1.4 as in the proof of Theorem 1.5, we get

$$\big(\mathcal{O}_Y(D)\otimes(\omega_Y(D)^{\otimes -N})\big)^{\otimes N}\subset\big[f_*^{(N)}\big(\omega_{X^{(N)}}(E^{(N)})/\omega_Y(D)^{\otimes N}\big)\big]^{**},$$

and obtain a nonzero morphism

$$\mathcal{O}_Y(D)^{\otimes r} \otimes P \to (\Omega_Y(\log D))^{\otimes kr} \otimes (\omega_Y(D))^{\otimes Nr}$$

induced by the logarithmic Higgs sheaf construction with poles along D. Here, $r > 0$, $k \geq 0$, and P is a pseudoeffective line bundle. By Theorem 3.9, $K_Y + D$ is pseudoeffective. From the split inclusion $\omega_Y(D) \subset (\Omega_Y(\log D))^{\otimes \dim Y}$, there exists some positive integer N' such that

$$\mathcal{O}_Y(D)^{\otimes r} \otimes P \to (\Omega_Y(\log D))^{\otimes N'}$$

is a nonzero morphism. If Q is its cokernel, then $c_1(Q)$ is pseudoeffective by Theorem 3.8. Since

$$c_1\big((\Omega_Y(\log D))^{\otimes N'}\big) = N'(\dim Y)^{N'-1} c_1(K_Y + D) = r c_1(D) + c_1(P) + c_1(Q)$$

and P, Q are pseudoeffective, there exists $\delta > 0$ such that $K_Y + (1-\delta)D$ is pseudoeffective.

The sum of a big divisor and a pseudoeffective divisor is big. If $-K_Y + ND$ is big for some nonnegative integer N, then the sum

$$(N + 1 - \delta)D = (-K_Y + ND) + (K_Y + (1-\delta)D)$$

is big. Hence D is big, and therefore $K_Y + D = (K_Y + (1-\delta)D) + \delta D$ is also big.

4C. *Proof of Theorem 1.9.* Let $\mu : (\widetilde{Y}, \widetilde{D}) \to (Y, \Delta(f))$ be a log resolution of $(Y, \Delta(f))$. By taking a suitable resolution $\widetilde{X}$ of the main component of $X \times_Y \widetilde{Y}$, we obtain an induced morphism $\tilde{f} : (\widetilde{X}, \widetilde{E}) \to (\widetilde{Y}, \widetilde{D})$ of projective log smooth pairs with $\widetilde{E} := \tilde{f}^{-1}(\widetilde{D})$, and $\tilde{f}|_{\widetilde{X} \setminus \widetilde{E}}$ is smooth. By Theorems 1.5 and 1.7(1), $K_{\widetilde{Y}} + \widetilde{D}$ is big in all cases, which implies that $Y \setminus \Delta(f)$ is of log general type. Therefore, from Lemma 3.4(1), $K_Y + D$ is big.

5. Popa's superadditivity of logarithmic Kodaira dimension

Popa recently conjectured that for a smooth projective morphism with connected fibers, the (logarithmic) Kodaira dimension is additive.

Theorem 5.1 [30, Conjecture 3.1]. *Let $f : X \to Y$ be a smooth projective morphism with connected fibers between smooth quasiprojective varieties. Then*

$$\bar{\kappa}(X) = \bar{\kappa}(Y) + \kappa(F)$$

where F is the general fiber of f.

As mentioned after Corollary 1.13, the subadditivity $\bar{\kappa}(X) \geq \bar{\kappa}(Y) + \kappa(F)$ is implied by the logarithmic Iitaka conjecture. Hence, Popa's conjecture suggests its counterpart, namely the superadditivity $\bar{\kappa}(X) \leq \bar{\kappa}(Y) + \kappa(F)$, when the morphism is smooth.

As a preparation, recall that the numerical log Kodaira dimension of Y is defined as

$$\bar{\kappa}_\sigma(Y) := \kappa_\sigma(\overline{Y}, K_{\overline{Y}} + D),$$

where $\overline{Y}$ is a compactification of Y with boundary a reduced simple normal crossing divisor D and κ_σ is Nakayama's numerical dimension [28, V.2.5 Definition]. This is well defined, independent of the choice of a compactification. Additionally, it is well known that if $\bar{\kappa}_\sigma(Y) = 0$, then $\bar{\kappa}(Y) = 0$ (see Kawamata's [17, Theorem 1] combined with Nakayama's [28, V.2.7 Proposition (8)]).

We first prove superadditivity under the additional assumption that the numerical log Kodaira dimension of the base Y is equal to zero.

Proposition 5.2. *With the notation in Theorem 5.1, assume that $\bar{\kappa}_\sigma(Y) = 0$. Then*

$$\bar{\kappa}(X) \le \kappa(F).$$

The proof of this proposition uses the logarithmic base change theorem as in the proofs in Section 4, with the following lemma. This is essentially [32, Lemma 14], modified for our use in the case of torsion-free sheaves. We include its proof for completeness.

Lemma 5.3. *Let $\mathcal{F}$ be a torsion-free sheaf on a normal projective variety Z, globally generated on the complement of a codimension-2 locus. If $h^0(Z, \mathcal{F}) > \operatorname{rank} \mathcal{F}$, then $h^0(Z, \widehat{\det} \mathcal{F}) \ge 2$.*

Proof. Denote $s := \operatorname{rank} \mathcal{F}$. Via taking the double dual, we may assume that $\mathcal{F}$ is reflexive. Let $V \subset Z$ be the complement of a codimension-2 subvariety in Z, over which $\mathcal{F}$ is a globally generated locally free sheaf of rank s. Consequently, there exists a generically isomorphic morphism

$$\alpha_1 : \mathcal{O}_Z^{\oplus s} \to \mathcal{F}.$$

If α_1 is an isomorphism on V, then the reflexive hull of α_1 is an isomorphism on Z, which contradicts $h^0(Z, \mathcal{F}) > s$. Hence, there exists a point $z \in V$ such that $\alpha_1|_z$ is not surjective as a linear map of vector spaces. Since $\mathcal{F}$ is globally generated at z, there exists a morphism

$$\alpha_2 : \mathcal{O}_Z^{\oplus s} \to \mathcal{F}$$

such that $\alpha_2|_z$ is an isomorphism of vector spaces. Then we have the nonzero global sections

$$\widehat{\det} \alpha_1 : \mathcal{O}_Z \to \widehat{\det} \mathcal{F}, \quad \widehat{\det} \alpha_2 : \mathcal{O}_Z \to \widehat{\det} \mathcal{F},$$

such that $\widehat{\det} \alpha_1$ vanishes at z and $\widehat{\det} \alpha_2$ is an isomorphism at z. Therefore, we have the conclusion. □

Proof of Proposition 5.2. We argue by contradiction: assume that $\bar{\kappa}(X) > \kappa(F)$. As at the beginning of the proof of Theorem 1.5, let $f : (\overline{X}, E) \to (\overline{Y}, D)$ be a morphism of log smooth pairs with $E = f^{-1}(D)$ such that $f|_{\overline{X}\setminus E} : \overline{X} \setminus E \to \overline{Y} \setminus D$ is the initial smooth morphism. Notice that

$$\operatorname{rank}\big(f_*(\omega_{\overline{X}}(E)^{\otimes N})\big) = P_N(F)$$

where $P_N(F)$ is the N-plurigenus of F. Therefore, there exists a positive integer N such that

$$h^0\big(\overline{Y}, f_*(\omega_{\overline{X}}(E)^{\otimes N})\big) > \operatorname{rank}\big(f_*(\omega_{\overline{X}}(E)^{\otimes N})\big).$$

Let $\mathcal{F}$ be the subsheaf of $f_*\big(\omega_{\overline{X}}(E)^{\otimes N}\big)$, generated by its global sections. Therefore, $\mathcal{F}$ is torsion-free and $h^0(\overline{Y}, \mathcal{F}) > s := \operatorname{rank} \mathcal{F}$. From the inclusions

$$\widehat{\det}\,\mathcal{F} \subset \big[\textstyle\bigwedge^s f_*\big(\omega_{\overline{X}}(E)^{\otimes N}\big)\big]^{**} \subset \Big[\bigotimes^s f_*\big(\omega_{\overline{X}}(E)^{\otimes N}\big)\Big]^{**},$$

we have

$$\begin{aligned}\big(\widehat{\det}\,\mathcal{F} \otimes \big(\omega_{\overline{Y}}(D)^{\otimes -Ns}\big)\big)^{\otimes N} &\subset \Big[\bigotimes^{Ns} f_*\big(\omega_{\overline{X}}(E)/\omega_{\overline{Y}}(D)^{\otimes N}\big)\Big]^{**}\\ &\subset \big[f_*^{(Ns)}\big(\omega_{\overline{X}^{(Ns)}}(E^{(Ns)})/\omega_{\bar{Y}}(D)^{\otimes N}\big)\big]^{**}\end{aligned}$$

by Corollary 1.4. As in the proofs of Theorems 1.5 and 1.7(1), we obtain a nonzero morphism

$$A^{\otimes r} \otimes P \to (\Omega_{\overline{Y}}(\log D))^{\otimes kr} \otimes (\omega_{\overline{Y}}(D))^{\otimes Nsr},$$

where $A := \widehat{\det}\,\mathcal{F}$, $r > 0$, $k \geq 0$, and P is a pseudoeffective line bundle. Hence, there exists some positive integer N' and an injection

$$A^{\otimes r} \otimes P \to (\Omega_{\overline{Y}}(\log D))^{\otimes N'}$$

with the quotient Q, which implies that

$$N'(\dim \overline{Y})^{N'-1} c_1(K_{\overline{Y}} + D) = r c_1(A) + c_1(P) + c_1(Q).$$

We now take Nakayama's numerical dimension on both sides. Due to [28, V.2.7 Proposition (1)] and Theorem 3.8, we have

$$\bar{\kappa}_\sigma(Y) = \kappa_\sigma(\overline{Y}, K_{\overline{Y}} + D) \geq \kappa_\sigma(\overline{Y}, A).$$

However by Lemma 5.3, we have $\kappa_\sigma(\bar{Y}, A) \geq 1$, which contradicts $\bar{\kappa}_\sigma(Y) = 0$. □

Theorem 1.12 states the superadditivity of the log Kodaira dimension when the very general fiber of the log Iitaka fibration of the base has a good minimal model. This is obtained as a consequence of Proposition 5.2 via the Easy addition formula.

Proof of Theorem 1.12. Let $\mu : (\widetilde{Y}, \widetilde{D}) \to (\overline{Y}, D)$ be a log resolution with the log Iitaka fibration $\eta : (\widetilde{Y}, \widetilde{D}) \to I$. Since the log Kodaira dimension is invariant under birational modifications (see e.g. [8, Lemma 2.3.34]), we have $\bar{\kappa}(\widetilde{Y} \setminus \widetilde{D}) = \bar{\kappa}(Y)$ and $\bar{\kappa}(X \times_Y \widetilde{Y} \setminus \widetilde{D}) = \bar{\kappa}(X)$. Hence, taking the base change of f via μ, we may assume $(\overline{Y}, D) = (\widetilde{Y}, \widetilde{D})$.

Let $f : (\overline{X}, E) \to (\overline{Y}, D)$ be a morphism of log smooth pairs with $E = f^{-1}(D)$ such that $f|_{\overline{X}\setminus E} : \overline{X} \setminus E \to \overline{Y} \setminus D$ is the initial smooth morphism. Then we have the composition of morphisms

$$(\overline{X}, E) \xrightarrow{f} (\overline{Y}, D) \xrightarrow{\eta} I.$$

Denote by $(G, D|_G)$ (resp. $(H, E|_H)$) the very general fiber of η (resp. $\eta \circ f$). The restriction map

$$f|_H : (H, E|_H) \to (G, D|_G)$$

is a morphism of log smooth pairs with $E|_H = f|_H^{-1}(D|_G)$, and $f|_{H\setminus E|_H}$ is smooth. From the assumption, the pair $(G, D|_G)$ has a good minimal model of log Kodaira dimension zero, which implies that

$$\kappa_\sigma(G, K_G + D|_G) = 0.$$

By Proposition 5.2, we have

$$\kappa(H, K_H + E|_H) \leq \kappa(F),$$

hence, by the Easy addition formula applied to $\eta \circ f$, we obtain

$$\bar{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + E) \leq \kappa(H, K_H + E|_H) + \dim I \leq \kappa(F) + \bar{\kappa}(Y),$$

which concludes the proof. □

In the remaining section, we state the logarithmic Iitaka conjecture for projective morphisms over quasiprojective curves. This result is well known to experts but is not explicitly stated in the literature, so we provide a brief proof.

Proposition 5.4. *Let $f : X \to Y$ be a projective morphism with connected fibers between smooth quasiprojective varieties. If* $\dim Y = 1$, *then*

$$\bar{\kappa}(X) \geq \bar{\kappa}(Y) + \kappa(F).$$

Proof. When Y is of log general type, the conclusion follows from Maehara [26, Corollary 2]. When Y is a projective curve of genus 1, it follows from Kawamata [15, Theorem 2]. When $Y = \mathbb{P}^1$ or $Y = \mathbb{P}^1 - \{\infty\}$, we have $\bar{\kappa}(Y) = -\infty$.

It suffices to prove $\bar{\kappa}(X) \geq \kappa(F)$ when $Y = \mathbb{P}^1 - \{0, \infty\}$. To begin with, we compactify the morphism f as $\bar{f} : (\overline{X}, E) \to (\mathbb{P}^1, D)$ so that $D = 0 + \infty$, $E = \bar{f}^{-1}(D)$, and $\bar{f}|_{\bar{X}\setminus E} = f$. As in the proof by Viehweg and Zuo [41], there exists a cyclic cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d, étale over $\mathbb{C}^* = \mathbb{P}^1 - \{0, \infty\}$, which induces a semistable reduction at $\{0, \infty\}$ for a sufficiently large and divisible d (Kempf et al. [21]). Let the resulting semistable reduction at $\{0, \infty\}$ be the commutative diagram

$$\begin{tikzcd} \overline{X}' \arrow[r, "\nu"] \arrow[d, "\bar{f}'"'] & \overline{X} \arrow[d, "\bar{f}"'] \\ \mathbb{P}^1 \arrow[r, "z \mapsto z^d"] & \mathbb{P}^1 \end{tikzcd}$$

Write $X' := \nu^{-1}(X)$. Since $\nu|_{X'} : X' \to X$ is a finite étale morphism, we have $\bar{\kappa}(X') = \bar{\kappa}(X)$ by Iitaka's [14, Theorem 3]. Therefore, we reduce to the case where the fibers of $\bar{f}$ over $\{0, \infty\}$ are reduced simple normal crossing divisors.

Since $\mathcal{O}_{\overline{X}}(E) \cong \bar{f}^*\mathcal{O}_{\mathbb{P}^1}(2) \cong \bar{f}^*\omega_{\mathbb{P}^1}^{-1}$, we have $\omega_{\overline{X}}(E) \cong \omega_{\overline{X}/\mathbb{P}^1}$. By Viehweg's [39, Theorem III], $\bar{f}_*\omega^k_{\overline{X}/\mathbb{P}^1}$ is weakly positive for all $k > 0$. Therefore, the vector bundle $\bar{f}_*\omega^k_{\overline{X}/\mathbb{P}^1}$ decomposes into a direct sum of line bundles of nonnegative degree on $\mathbb{P}^1$, which implies that

$$h^0(\mathbb{P}^1, \bar{f}_*\omega^k_{\overline{X}/\mathbb{P}^1}) \geq \operatorname{rank}\big(\bar{f}_*\omega^k_{\overline{X}/\mathbb{P}^1}\big) = P_k(F)$$

where $P_k(F)$ is the k-plurigenus of F. Hence, $h^0(\overline{X}, k(K_{\bar{X}} + E)) \geq P_k(F)$ for all $k > 0$, so we have $\bar{\kappa}(X) \geq \kappa(F)$. □

In particular, Theorem 1.12 and Proposition 5.4 imply Corollary 1.13.

6. Uniruledness of fibrations over projective spaces

Assume we have a surjective projective morphism $f : X \to \mathbb{P}^1$ with at most two singular fibers. From [41], we have $\kappa(X) = -\infty$, and the nonvanishing conjecture implies that X is uniruled. In other words, putting these two together, we have:

Conjecture 6.1. *Let X be a smooth projective variety, and $f : X \to \mathbb{P}^1$ be a surjective morphism with at most 2 singular fibers. Then X is uniruled.*

As mentioned in the introduction after Theorem 1.15, Pieloch [29] proved the conjecture when f has at most one singular fiber, and obtained a partial result when f has two singular fibers. The proofs are symplectic; the algebro-geometric proofs were previously unknown.

In what follows, we give an algebraic proof of this conjecture under the additional assumption that the general fiber has a good minimal model. The key idea is the invariance of the canonical rings of smooth fibers stated in the following theorem. This contains Theorem 1.15. We also state an analogous conjecture over a higher dimensional base below.

Theorem 6.2. *In the setting of Conjecture 6.1, assume that the fibers of f are connected. Then, the canonical rings of all smooth fibers are isomorphic. Moreover, if the general fiber F has a good minimal model, then X is uniruled. In particular, if the fibers are of general type, then f is birationally isotrivial, and X is uniruled.*

Proof. As in the proof of Proposition 5.4, it suffices to consider the case when f is semistable with at most 2 singular fibers over $\{0, \infty\}$. Let $D = 0 + \infty$ and $E = f^{-1}(D)$. Since $K_{\mathbb{P}^1} + (1-\delta)D$ is not pseudoeffective for all $\delta > 0$, $K_X + (1-\epsilon)E$ is not effective for all $\epsilon > 0$ by Theorem 1.7(1). Therefore, there is no nonzero morphism

$$\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega^k_{X/\mathbb{P}^1},$$

which implies that

$$f_*\omega^k_{X/\mathbb{P}^1} \cong \bigoplus^{P_k(F)} \mathcal{O}_{\mathbb{P}^1}. \tag{6.3}$$

Alternatively, [41, Proposition 4.2] proves that the degree of $f_*\omega^k_{X/\mathbb{P}^1}$ is equal to zero, which implies (6.3). As a consequence, the multiplication map

$$f_*\omega^k_{X/\mathbb{P}^1} \otimes f_*\omega^l_{X/\mathbb{P}^1} \to f_*\omega^{k+l}_{X/\mathbb{P}^1}$$

is constant on the fibers, and hence is isomorphic to

$$\left[H^0(F, \omega^k_F) \otimes_{\mathbb{C}} H^0(F, \omega^l_F) \to H^0(F, \omega^{k+l}_F)\right] \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}.$$

Therefore, the canonical ring of every smooth fiber is isomorphic to the canonical ring $R(F,\omega_F)$ of F, which is isomorphic to $R(X,\omega_{X/\mathbb{P}^1})$, establishing the first assertion. Accordingly, the Iitaka model I of $(X,\omega_{X/\mathbb{P}^1})$ is the Iitaka model of (F,ω_F). Resolving the indeterminacy of the Iitaka morphism, we obtain the diagram

$$\begin{tikzcd} X' \arrow[d, "\mu"] \arrow[dr, "\eta"] \arrow[dd, bend right, "f'"'] & \\ X \arrow[r, dashed] \arrow[d, "f"] & I \\ \mathbb{P}^1 & \end{tikzcd}$$

Notice that η is also the Iitaka fibration of $(X',\omega_{X'/\mathbb{P}^1})$, and the restriction map $\eta|_{f'^{-1}(y)} : f'^{-1}(y) \to I$ for the general point $y \in \mathbb{P}^1$ is the Iitaka fibration of F.

Suppose F has a good minimal model. Then, it is well known that every fiber in the smooth locus of the Iitaka fibration of F has a good minimal model (see e.g. [9, Theorem 1.2]). Now, it suffices to prove that X' is uniruled, hence that the very general fiber Z' of η is uniruled. We have $\kappa(Z',\omega_{X'/\mathbb{P}^1}|_{Z'}) = 0$ by Iitaka's theory of D-dimension [27, (1.11)], which implies that $\kappa(Z',\omega_{Z'/\mathbb{P}^1}) = 0$. Consider $f' : Z' \to \mathbb{P}^1$, then its general fiber G' is the smooth fiber of the Iitaka fibration of F. By Viehweg's weak positivity theorem, we have

$$f'_*\omega^k_{Z'/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1},$$

whenever $\omega^k_{G'}$ admits sections. Since G' admits a good minimal model from the assumption, it is well known that Z' has a relative good minimal model Z'' over $\mathbb{P}^1$. If we denote $f'' : Z'' \to \mathbb{P}^1$, then $\omega_{Z''}$ is f''-semiample. From $f''_*\omega^k_{Z''/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}$, we have $\omega^k_{Z''} \cong f''^*\omega^k_{\mathbb{P}^1}$. Accordingly, the canonical divisor $K_{Z''}$ is not pseudoeffective, which implies that any resolution of Z'' is uniruled [2]. Therefore, Z' is uniruled as desired. $\square$

Remark 6.4. In fact, Theorem 6.2 extends to the case where f has disconnected fibers. Indeed, the Stein factorization of $f : X \to \mathbb{P}^1$ is $\mathbb{P}^1$, and the induced morphism again has at most two singular fibers. This is because a finite covering of $\mathbb{P}^1$ branched over at most two points is either an identity map or a cyclic map, i.e., $z \mapsto z^d$ for $d > 1$ with an appropriate choice of the coordinate z.

Combining Corollary 1.10 and the nonvanishing conjecture, we obtain the following higher-dimensional analogue of Conjecture 6.1. In the remaining section, we prove this conjecture for $n = 2$ under the same additional assumption as in Theorem 6.2, that the general fiber has a good minimal model.

Conjecture 6.5. *Let X be a smooth projective variety, and $f : X \to \mathbb{P}^n$ be a surjective morphism with either* $\dim \Delta(f) \le n-2$ *or* $\deg \Delta(f) \le n+1$. *Then X is uniruled.*

Corollary 6.6. *In the setting of Conjecture 6.5, assume that $n = 2$ and the fibers of f are connected. If the general fiber F has a good minimal model, then X is uniruled.*

Proof. The discriminant locus $\Delta(f)$ is the union of a plane curve C (possibly empty) of degree at most 3 and a finite set S of points.

<u>Case $\deg C \leqslant 2$ or $C = \phi$:</u> The general projective line in $\mathbb{P}^2$ intersects $\Delta(f)$ at most 2 points. Then the preimage of the line is uniruled by Theorem 6.2. Therefore, X is uniruled.

<u>Case $\deg C = 3$:</u> If C is a smooth cubic plane curve, $\mathbb{P}^2$ is covered by projective lines tangent to C, and only finitely many such lines pass through S. If C is a singular cubic plane curve, pick a singular point p and consider all projective lines through that point. Likewise, only finitely many such lines pass through S. In all cases, $\mathbb{P}^2$ is generically covered by lines that intersect $\Delta(f)$ at most 2 points. Therefore, we conclude that X is uniruled. □

Remark 6.7. For $n \geq 3$, the possible configurations for the discriminant locus in Conjecture 6.5 are very complicated; the case by case analysis in the proof of Corollary 6.6 is no longer valid. Still, if the degree of the divisorial component of the discriminant locus is at most 3 and F has a good minimal model, then X is uniruled. This is because when we slice $\mathbb{P}^n$ by a general projective plane $\mathbb{P}^2 \subset \mathbb{P}^n$, the preimage is uniruled by Corollary 6.6.

7. Boundary examples and further remarks

We present boundary examples showing that the inequality in Corollary 1.10 is sharp. Specifically, we demonstrate morphisms to $\mathbb{P}^n$ such that the domains have the Kodaira dimension equal to zero and the discriminant loci are divisors of degree $n+2$ in $\mathbb{P}^n$.

Example 7.1 (isotrivial example with discriminant locus of minimal degree). First, we construct a finite cover of $\mathbb{P}^n$ with trivial canonical bundle branched over a hypersurface of degree $n+2$. Let $D \subset \mathbb{P}^n$ be a degree $n+2$ hypersurface with simple normal crossing singularities. Consider a degree $n+2$ cyclic cover $X \to \mathbb{P}^n$ associated to D. Then X has Gorenstein rational singularities and $\omega_X \cong \mathcal{O}_X$. We resolve the singularities of X by a series of smooth blow-ups over the branch locus, $\mu : \widetilde{X} \to X$. Then the composition $f : \widetilde{X} \to \mathbb{P}^n$ is an example with $h^0(\omega_{\widetilde{X}}) = 1$ and the discriminant locus $\Delta(f) = D$ is the hypersurface of degree $n+2$.

Now, we use the construction above to build an example with connected fibers over $\mathbb{P}^n$, with $n \geq 2$. To begin with, when D is a smooth degree $n+2$ hypersurface, the cover $f : X \to \mathbb{P}^n$ we constructed above is a smooth Galois cover with a Galois group $G = \mathbb{Z}/(n+2)\mathbb{Z}$. Consider a G-action on $X \times X$:

$$\sigma : G \times X \times X \to X \times X, \quad \sigma(g, x_1, x_2) := (g^{-1}x_1, gx_2).$$

Since G is abelian, the action is well defined. Hence, the projection onto the second factor $\mathrm{pr}_2 : X \times X \to X$ is a G-equivariant morphism, from which we obtain an induced morphism of quotients by the G-action:

$$\overline{\mathrm{pr}}_2 : (X \times X)/G \to X/G \cong \mathbb{P}^n.$$

All the fibers of $\overline{\mathrm{pr}}_2$ over $\mathbb{P}^n \setminus D$ are isomorphic to X. Therefore, if we denote by $\mu : W \to (X \times X)/G$ a resolution of singularities, which is an isomorphism away from the singular locus, then we have $\kappa(W) = 0$ from the following lemma, and the induced morphism $h : W \to \mathbb{P}^n$ satisfies $\Delta(h) = D$.

Lemma 7.2. *Under the above notation, $(X \times X)/G$ has canonical singularities and the Kodaira dimension of a desingularization of $(X \times X)/G$ is equal to zero.*

Proof. Let $E = f^{-1}(D)$ be the ramification divisor of f on X. The G-action is free outside of $E \times E \subset X \times X$; hence, it suffices to prove that the quotient singularities at points $(x_1, x_2) \in E \times E$ are canonical. Let ζ be a primitive $(n+2)$-th root of unity. Locally analytically, the G-action near $x \in E \subset X$ is equivalent to

$$(z_1, \dots, z_{n-1}, z_n) \mapsto (z_1, \dots, z_{n-1}, \zeta^m z_n), \quad m \in \mathbb{Z}/(n+2)\mathbb{Z}$$

where $z_1, \dots, z_{n-1}, z_n$ are local coordinates and $z_n = 0$ is a local equation of E. Therefore, at each point $(x_1, x_2) \in E \times E$, we obtain a singularity, which is analytically isomorphic to

$$\mathbb{C}^{2n-2} \times \left(\mathbb{C}^2 / \tfrac{1}{n+2}(1, q)\right)$$

where $\mathbb{C}^2/\frac{1}{n+2}(1,q)$ is the standard abbreviation of the surface cyclic quotient singularity of type $\frac{1}{n+2}(1,q)$. This singularity is well known to be canonical if and only if $q = -1$ (by the Reid–Tai criterion [34, (4.11)], for example). Therefore, the locus in $E \times E$ where the quotient singularities are canonical is both open and closed in the analytic topology. Notice that $q = -1$ on the diagonal $E \subset E \times E$ from the definition of the G-action. Thus, $(X \times X)/G$ has canonical singularities.

Since the quotient morphism $X \times X \to (X \times X)/G$ is étale away from a codimension-2 subvariety, we have an equality

$$\kappa(X \times X, \omega_{X \times X}) = \kappa\left((X \times X)/G, \omega_{(X \times X)/G}\right) = 0,$$

which completes the proof. ☐

In particular, when $n = 1$, we have a triple cyclic cover $f : C \to \mathbb{P}^1$, with C an elliptic curve branched over three points. Then the minimal resolution of $(C \times C)/G$ turns out to be an elliptic K3 surface fibered over $\mathbb{P}^1$ with three singular fibers sitting above the branch points. See [35, Section 2] for details.

The next example is a threefold with zero Kodaira dimension, mapping to $\mathbb{P}^1$ with three singular fibers, obtained as an application of Viehweg's base change theorem. In this example, the fibers are connected and the family is birationally nonisotrivial, unlike in the previous one. We say that the family is birationally isotrivial if the general fibers are pairwise birationally isomorphic.

Example 7.3 (nonisotrivial example with discriminant locus of minimal degree). Let $f : S \to \mathbb{P}^1$ be a family of elliptic curves parametrized by

$$y^2 = x(x-1)(x-\lambda).$$

To be specific, let $S \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a hypersurface of type $(3,1)$ defined by $y^2 z = x(x-z)(x-\lambda z)$, where x, y, z are the coordinates of $\mathbb{P}^2$ and λ is the coordinate of $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. By adjunction,

$\omega_S \cong f^*\mathcal{O}_{\mathbb{P}^1}(-1)$, so that

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1).$$

Notice that f has three singular fibers at $\{0, 1, \infty\}$; over $\{0, 1\}$ we have a nodal cubic curve, and over $\{\infty\}$ we have a union of three concurrent lines. We can verify by computation that S has A_1-singularities at the nodes over $\{0, 1\}$ and has an A_4-singularity at the planar triple point over $\{\infty\}$. Let $\mu : \widetilde{S} \to S$ be the minimal resolution of singularities and $\tilde{f} : \widetilde{S} \to \mathbb{P}^1$ be the induced morphism. Then $\tilde{f}$ has semistable fibers over $\{0, 1\}$ and a nonsemistable fiber over $\{\infty\}$, with

$$\tilde{f}_*\omega_{\widetilde{S}/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1).$$

Take an automorphism of $\mathbb{P}^1$, fixing 0 and flipping 1 and ∞. Let $\tilde{f}' : \widetilde{S} \to \mathbb{P}^1$ be the induced family, which now has semistable fibers over $\{0, \infty\}$ and a nonsemistable fiber over $\{1\}$. Let $X := \widetilde{S} \times_{\mathbb{P}^1} \widetilde{S}$ be the fiber product of $\tilde{f}$ and $\tilde{f}'$, and let $\widetilde{X}$ be the resolution of X, which is an isomorphism over the smooth locus of X. Let $h : \widetilde{X} \to \mathbb{P}^1$ be the induced morphism. By Viehweg's base change theorem (Theorem 1.1),

$$h_*\omega_{\widetilde{X}/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2),$$

which implies that $h^0(\omega_{\widetilde{X}}) = 1$. In fact, we have $\kappa(\widetilde{X}) = 0$, analyzing the pushforwards of pluricanonical bundles. Therefore, there exists a threefold $\widetilde{X}$ such that $\kappa(\widetilde{X}) = 0$, and $\widetilde{X} \to \mathbb{P}^1$ is birationally nonisotrivial with exactly three singular fibers.

We close by addressing a question, raised by Kovács [23, Question 0.6], whether imposing a stronger condition on the Kodaira dimension of the total space increases the degree of the discriminant locus. We observe that this is not the case, by demonstrating morphisms $f : X \to \mathbb{P}^n$ such that X is of general type and the degree of the discriminant locus $\Delta(f)$ is either $n + 2$ or $n + 3$.

Example 7.4 (general type example with discriminant locus of small degree). We first explain a general strategy to construct an explicit example. Suppose we have a smooth Galois cover $g : Y \to \mathbb{P}^n$, with a Galois group G, branched over a smooth hypersurface D as in Example 7.1. For a positive integer N, the projection onto the N-th factor,

$$\mathrm{pr} : Y^N \to Y,$$

is a G-equivariant morphism, where Y^N is endowed with the diagonal G-action. Then the induced morphism

$$\overline{\mathrm{pr}} : Y^N/G \to Y/G \cong \mathbb{P}^n$$

has $\Delta(\overline{\mathrm{pr}}) = D$, and the general fiber is connected. Furthermore, if Y is of general type, then a desingularization of Y^N/G is of general type for $N \gg 0$, by a result of Caporaso, Harris and Mazur [6, Corollary 4.1]. Therefore, we obtain a morphism $f : X \to \mathbb{P}^n$ such that X is of general type and $\Delta(f) = D$.

When $n = 1$, we start with a Galois Belyĭ map $C \to \mathbb{P}^1$ such that C is a curve of genus at least 2 (see e.g. [37, Section 6]). Then the construction above shows that the minimal degree of $\Delta(f)$ is indeed 3.

When $n \geq 2$, take a cyclic cover $g : Y \to \mathbb{P}^n$ branched over a smooth hypersurface of degree $n + 3$. Then Y is of general type; consequently, we have an example with deg $\Delta(f) = n + 3$ via the construction above. We expect there to be an example with deg $\Delta(f) = n + 2$, which we do not know yet.

Acknowledgements

I am deeply grateful to my advisor, Mihnea Popa, for his constant support, invaluable advice, and suggestions. I am also indebted to Joe Harris for his support. I thank Jungkai Chen, Yoon-Joo Kim, Fanjun Meng, and Justin Sawon for helpful discussions, and Dori Bejleri for answering my questions and providing references regarding Section 7. I also thank the anonymous referee for helpful comments and suggestions. I was supported by the 17th Kwanjeong Study Abroad Scholarship.

References

[1] E. Bierstone and P. D. Milman, "Desingularization of toric and binomial varieties", *J. Algebraic Geom.* **15**:3 (2006), 443–486. MR

[2] S. Boucksom, J.-P. Demailly, M. Pˇ aun, and T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", *J. Algebraic Geom.* **22**:2 (2013), 201–248. MR

[3] Y. Brunebarbe, "Symmetric differentials and variations of Hodge structures", *J. Reine Angew. Math.* **743** (2018), 133–161. MR

[4] F. Campana, "Kodaira additivity, birational isotriviality, and specialness", *Mosc. Math. J.* **23**:3 (2023), 319–330. MR

[5] F. Campana and M. Pˇ aun, "Foliations with positive slopes and birational stability of orbifold cotangent bundles", *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 1–49. MR

[6] L. Caporaso, J. Harris, and B. Mazur, "Uniformity of rational points", *J. Amer. Math. Soc.* **10**:1 (1997), 1–35. MR

[7] F. Catanese and M. Schneider, "Polynomial bounds for abelian groups of automorphisms", *Compositio Math.* **97**:1-2 (1995), 1–15. MR

[8] O. Fujino, *Iitaka conjecture—an introduction*, Springer, 2020. MR

[9] C. D. Hacon, J. McKernan, and C. Xu, "Boundedness of moduli of varieties of general type", *J. Eur. Math. Soc.* **20**:4 (2018), 865–901. MR

[10] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, 1977. MR

[11] K. Hashizume, "Log Iitaka conjecture for abundant log canonical fibrations", *Proc. Japan Acad. Ser. A Math. Sci.* **96**:10 (2020), 87–92. MR

[12] J.-M. Hwang, "Base manifolds for fibrations of projective irreducible symplectic manifolds", *Invent. Math.* **174**:3 (2008), 625–644. MR

[13] J.-M. Hwang and K. Oguiso, "Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration", *Amer. J. Math.* **131**:4 (2009), 981–1007. MR

[14] S. Iitaka, "On logarithmic Kodaira dimension of algebraic varieties", pp. 175–189 in *Complex analysis and algebraic geometry*, Iwanami Shoten Publishers, Tokyo, 1977. MR

[15] Y. Kawamata, "Kodaira dimension of algebraic fiber spaces over curves", *Invent. Math.* **66**:1 (1982), 57–71. MR

[16] Y. Kawamata, "Minimal models and the Kodaira dimension of algebraic fiber spaces", *J. Reine Angew. Math.* **363** (1985), 1–46. MR

[17] Y. Kawamata, "On the abundance theorem in the case of numerical Kodaira dimension zero", *Amer. J. Math.* **135**:1 (2013), 115–124. MR

[18] S. Kebekus and S. J. Kovács, “Families of canonically polarized varieties over surfaces”, *Invent. Math.* **172**:3 (2008), 657–682. MR

[19] S. Kebekus and S. J. Kovács, “The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties”, *Duke Math. J.* **155**:1 (2010), 1–33. MR

[20] S. Keel, K. Matsuki, and J. McKernan, “Log abundance theorem for threefolds”, *Duke Math. J.* **75**:1 (1994), 99–119. MR

[21] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings*, *I*, Lecture Notes in Mathematics **339**, Springer, 1973. MR

[22] J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics **200**, Cambridge University Press, 2013. MR

[23] S. J. Kovács, “Algebraic hyperbolicity of fine moduli spaces”, *J. Algebraic Geom.* **9**:1 (2000), 165–174. MR

[24] S. J. Kovács, “Singularities of stable varieties”, pp. 159–203 in *Handbook of moduli*, Adv. Lect. Math. **25**, International Press, Somerville, MA, 2013. MR

[25] C.-J. Lai, “Varieties fibered by good minimal models”, *Math. Ann.* **350**:3 (2011), 533–547. MR

[26] K. Maehara, “The weak 1-positivity of direct image sheaves”, *J. Reine Angew. Math.* **364** (1986), 112–129. MR

[27] S. Mori, “Classification of higher-dimensional varieties”, pp. 269–331 in *Algebraic geometry* (Brunswick, ME, 1985), vol. 1, Proc. Sympos. Pure Math. **46**:1, Amer. Math. Soc., 1987. MR

[28] N. Nakayama, *Zariski-decomposition and abundance*, Math. Soc. Japan Memoirs **14**, Mathematical Society of Japan, 2004. MR

[29] A. J. Pieloch, *Sections and unirulings of families over the projective line*, Ph.D. thesis, Columbia University, 2022, available at https://www.proquest.com/docview/2653329705. MR

[30] M. Popa, “Conjectures on the Kodaira dimension”, pp. 19–28 in *Higher dimensional algebraic geometry*: *a volume in honor of V. V. Shokurov*, London Math. Soc. Lecture Note Ser. **489**, Cambridge Univ. Press, 2025. MR

[31] M. Popa and C. Schnell, “Viehweg’s hyperbolicity conjecture for families with maximal variation”, *Invent. Math.* **208**:3 (2017), 677–713. MR

[32] M. Popa and C. Schnell, “On the behavior of the Kodaira dimension under smooth morphisms”, *Algebr. Geom.* **10**:5 (2023), 607–619. MR

[33] M. Popa and L. Wu, “Weak positivity for Hodge modules”, *Math. Res. Lett.* **23**:4 (2016), 1139–1155. MR

[34] M. Reid, “Young person’s guide to canonical singularities”, pp. 345–414 in *Algebraic geometry* (Brunswick, ME, 1985), vol. 1, Proc. Sympos. Pure Math. **46**:1, Amer. Math. Soc., 1987. MR

[35] J. Sawon, “Isotrivial elliptic K3 surfaces and Lagrangian fibrations”, preprint, 2014. arXiv 1406.1233

[36] V. V. Shokurov, “3-fold log models”, *J. Math. Sci.* **81**:3 (1996), 2667–2699. MR

[37] J. Sijsling and J. Voight, “On computing Belyi maps”, pp. 73–131 in *Numéro consacré au trimestre “Méthodes arithmétiques et applications”*, *automne* 2013, Publ. Math. Besançon Algèbre Théorie Nr. **2014/1**, Presses Univ. Franche-Comté, Besançon, 2014. MR

[38] J. Steenbrink, “Limits of Hodge structures”, *Invent. Math.* **31**:3 (1975/76), 229–257. MR

[39] E. Viehweg, “Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces”, pp. 329–353 in *Algebraic varieties and analytic varieties* (Tokyo, 1981), Adv. Stud. Pure Math. **1**, North-Holland, Amsterdam, 1983. MR

[40] E. Viehweg, “Weak positivity and the additivity of the Kodaira dimension, II: The local Torelli map”, pp. 567–589 in *Classification of algebraic and analytic manifolds* (Katata, 1982), Progr. Math. **39**, Birkhäuser, Boston, 1983. MR

[41] E. Viehweg and K. Zuo, “On the isotriviality of families of projective manifolds over curves”, *J. Algebraic Geom.* **10**:4 (2001), 781–799. MR

[42] E. Viehweg and K. Zuo, “Base spaces of non-isotrivial families of smooth minimal models”, pp. 279–328 in *Complex geometry* (Göttingen, 2000), Springer, 2002. MR

[43] S. Zucker, “Degeneration of Hodge bundles (after Steenbrink)”, pp. 121–141 in *Topics in transcendental algebraic geometry* (Princeton, NJ, 1981/1982), Ann. of Math. Stud. **106**, Princeton Univ. Press, 1984. MR

[44] K. Zuo, "On the negativity of kernels of Kodaira–Spencer maps on Hodge bundles and applications", *Asian J. Math.* **4**:1 (2000), 279–301. Kodaira's issue. MR

Communicated by Christopher Hacon
Received 2023-10-03 Revised 2025-02-02 Accepted 2025-04-02

sp6631@princeton.edu *Department of Mathematics, Princeton University, Princeton, NJ, United States*
sgpark@ias.edu

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.477

Chevalley formulae for motivic Chern classes of Schubert cells and for stable envelopes

Leonardo C. Mihalcea, Hiroshi Naruse and Changjian Su

We prove a Chevalley formula to multiply the motivic Chern classes of Schubert cells in a generalized flag manifold G/P by the class of any line bundle $\mathcal{L}_\lambda$. Our formula is given in terms of the λ-chains of Lenart and Postnikov. Its proof relies on a change of basis formula in the affine Hecke algebra due to Ram, and on the Hecke algebra action on torus-equivariant K-theory of the complete flag manifold G/B via left Demazure–Lusztig operators. We revisit some wall-crossing formulae for the stable envelopes in $T^*(G/B)$. We use our Chevalley formula, and the equivalence between motivic Chern classes of Schubert cells and K-theoretic stable envelopes in $T^*(G/B)$, to give formulae for the change of polarization, and for the change of slope for stable envelopes. We prove several additional applications, including Serre, star, and Dynkin, dualities of the Chevalley coefficients, new formulae for the Whittaker functions, and for the Hall–Littlewood polynomials. We also discuss positivity properties of Chevalley coefficients, and properties of the coefficients arising from multiplication by minuscule weights.

L. C. Mihalcea was supported in part by the NSF grant DMS-2152294 and a Simons Collaboration Grant. H. Naruse was supported in part by JSPS KAKENHI Grant Number 16H03921. C. Su is supported by the National Key R&D Program of China (No. 2024YFA1014700).

MSC2020: primary 14C17, 14M15; secondary 14N15, 17B10, 33D80.

Keywords: motivic Chern classes of Schubert cells, stable envelopes, λ-chains, Hecke algebra, Demazure–Lusztig operators.

1. Introduction

Let G be a complex, semisimple, Lie group and $T \subset B \subset P \subset G$ be a parabolic subgroup containing a Borel subgroup and the (standard) maximal torus. Let W be the Weyl group determined by (G, T). In the study of cohomology and K-theory rings of (generalized) flag manifolds G/P, the Chevalley formula expresses the multiplication of a Schubert class by the class of a line bundle, or a Schubert divisor in G/P. If one works equivariantly, this formula determines completely the multiplication in the equivariant K ring. In this paper we prove a Chevalley formula for the coefficients $C^w_{u,\lambda}(y) \in K_T(pt)[y]$ arising in the multiplication

$$MC_y(X(w)^\circ) \cdot \mathcal{L}_\lambda = \sum C^w_{u,\lambda}(y) MC_y(X(u)^\circ) \tag{1}$$

in the equivariant K-theory ring $K_T(G/B)[y]$; see (27) and Theorem 5.2 below. Here $MC_y(X(w)^\circ) \in K_T(G/B)[y]$ is the *motivic Chern class* of a Schubert cell $X(w)^\circ \subset G/B$, and $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ is the line bundle on G/B associated to the one-dimensional B-module of weight λ.

The motivic Chern classes $MC_y(X(w)^\circ) \in K_T(G/B)[y]$ have been defined by Brasselet, Schürmann, and Yokura [BSY10] more generally for elements $[Y \to X]$ in the Grothendieck group $G_0(var/X)$ of varieties over X. They are the unique classes which are functorial with respect to proper morphisms $f : X_1 \to X_2$, and which satisfy the normalization condition $MC[\mathrm{id}_X : X \to X] = \lambda_y(T^*_X)$ for X smooth, where $\lambda_y(T^*_X) = \sum y^i \left[\bigwedge^i T^*_X\right]$ is the Hirzebruch λ_y class; see Section 4 below. They may be thought of as the K-theoretic generalizations of Chern–Schwartz–MacPherson classes defined by MacPherson [Mac74].

The motivic Chern classes of Schubert cells generalize well studied classes from Schubert calculus. If $y = 0$, the motivic class $MC_y(X(w)^\circ)$ is equal to the class of the ideal sheaf $[\mathcal{O}_{X(w)}(-\partial X(w))]$ of the boundary $\partial X(w) = X(w) \setminus X(w)^\circ$, where $X(w) = \overline{X(w)^\circ}$ is the Schubert variety. If $y = -1$, then $MC_y(X(w)^\circ)$ is equal to the class of the unique T-fixed point in $X(w)^\circ$; see [AMSS24b]. The Poincaré duals of the classes $MC_y(X(w)^\circ)$, the *Segre motivic classes* [AMSS24a; MS22], specialize when $y = 0$ to the Grothendieck classes of the structure sheaves of the opposite Schubert varieties. Our Chevalley formula (1), and its analogous formula for the Segre motivic classes, specialize to known Chevalley formulae for K-theoretic Schubert classes and ideal sheaves from [GR04; LP07].

The formula for the coefficients $C^w_{u,\lambda}(y) \in K_T(pt)[y]$ from (1) follows from a formula of Ram [Ram06] in the affine Hecke algebra $\mathbb{H}$, calculating transition coefficients between two bases $\{T_w X^\lambda\}$ and $\{X^\lambda T_w\}$ of the affine Hecke algebra:

$$T_w X^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} (-q)^{\ell(w)-\ell(u)} c^{w,\lambda}_{u,\mu} X^{-\mu} T_u. \tag{2}$$

Here T_w is an element in the standard basis of $\mathbb{H}$, $X^{-\lambda}$ is an affine element in $\mathbb{H}$, and $X^*(T)$ denotes the weight lattice of T. Ram's formula is stated in terms of a combinatorial model utilizing alcove walks, and it is convenient for our purposes to rewrite it utilizing in terms of λ-chains, a model introduced and studied by Lenart and Postnikov [LP07; LP08] in relation to equivariant K theory of flag manifolds. We refer to Theorem 3.9 and Theorem 3.10 for the precise statements in the Hecke algebra in terms of

λ-chains, and to Section 5.1 for the formulae involving motivic Chern classes. We also note that (affine) Hecke algebras have long been used to obtain Chevalley formulas in various contexts, for example in [PR99; LP07]. We state next our main result.

Assume λ is an integral weight and fix a reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$. The chain corresponds to an alcove walk from the fundamental alcove $A_\circ$ to $A_\circ - \lambda$, with separating hyperplanes $H_{-\beta_j, d_j}$. Denote by s_β the reflection determined by the root β. We refer the reader to Section 3 below for full definitions. The following is our main result; cf. Theorem 5.5.

Theorem 1.1. *The following Chevalley formula holds in* $K_T(G/B)[y]$:

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ) = \sum_{\substack{\mu \in X^*(T) \\ u \in W}} C^w_{u,-\lambda} MC_y(X(u)^\circ),$$

where the Chevalley coefficients are given by

$$C^w_{u,-\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (1+y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\tilde{r}_{J_>}(\lambda)}, \tag{3}$$

and the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ *such that* $u < us_{\beta_{j_1}} < us_{\beta_{j_1}} s_{\beta_{j_2}} < \cdots < us_{\beta_{j_1}} s_{\beta_{j_2}} \cdot \cdots \cdot s_{\beta_{j_t}} = w$; *the Weyl group element* $\tilde{r}_{J_>}$ *is defined in* (19). *For the multiplication* $\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)$, *the Chevalley coefficients are given by*

$$C^w_{u,\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (-1-y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\hat{r}_{J_<}(-\lambda)}, \tag{4}$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ *such that* $u < us_{\beta_{j_t}} < us_{\beta_{j_t}} s_{\beta_{j_{t-1}}} < \cdots < us_{\beta_{j_t}} \cdot \cdots \cdot s_{\beta_{j_1}} = w$, *and with* $\hat{r}_{J_<}$ *defined in* (16).

The connection between the Hecke algebra coefficients from (2) and the Chevalley coefficients above is given by

$$C^w_{u,-\lambda} = \sum_{\mu \in X^*(T)} y^{\ell(w)-\ell(u)} e^{-\mu} c^{w,\lambda}_{u,\mu}|_{q=-y}. \tag{5}$$

The coefficients $c^{w,\lambda}_{u,\mu}$ are in general Laurent polynomials in y, while $C^w_{u,-\lambda}$ are polynomials in $K_T(pt)[y]$. In fact, the power $y^{\ell(w)-\ell(u)}$ from the formula (5) is absorbed into $c^{w,\lambda}_{u,\mu}$ so it becomes polynomial in y.

As mentioned above, our Chevalley formula generalizes to the motivic situation the classical Chevalley multiplication in $K_T(G/B)$. It also generalizes the Chevalley multiplication by (equivariant) Chern–Schwartz–MacPherson classes of Schubert cells from [AMSS23]; a short, self-contained, proof of this is given in Appendix A.

All these specializations are appropriately positive, in the sense of [Buc02; Bri02; AGM11]. In Section 5.3 below we investigate some positivity results for the general formula. Notably, our formula for the multiplication by $\mathcal{L}_\lambda$ with λ dominant (i.e., when $\mathcal{L}_{-\lambda}$ is globally generated) may be written as a positive combination of products $q^a(q-1)^b$, with $q = -y$; see Proposition 5.7. This positivity is similar to the one satisfied by R-polynomials in Kazhdan–Lusztig theory. In an earlier arχiv version of this paper, we conjectured different positivity properties for special cases of the Chevalley coefficients, regarded as polynomials in y. As we explain in Remark 5.10, we since found examples in Lie types D_6, E_6, A_7 where the conjectured positivity fails.

We now give a rough idea on the proof of Theorem 1.1. The key connection between the Chevalley formula in the Hecke algebra to motivic Chern classes, proved in [MNS22b], and ultimately based on results from [AMSS24a], is that the motivic Chern classes are recursively obtained by certain *left* Demazure–Lusztig operators $\mathcal{T}_w^L$ acting on $K_T(G/B)[y]$:

$$MC_y(X(w)^\circ) = \mathcal{T}_w^L[\mathcal{O}_{1.B}].$$

These operators commute with elements in $K_G(G/B)[y]$ (i.e., the Weyl-group invariants of $K_T(G/B)$), and an argument based on equivariant localization shows that

$$MC_y(X(w)^\circ) \cdot \mathcal{L}_\lambda = \mathcal{T}_w^L[\mathcal{O}_{1.B}] \cdot \mathcal{L}_\lambda = \mathcal{T}_w^L(\mathcal{L}_\lambda \cdot [\mathcal{O}_{1.B}]) = \mathcal{T}_w^L(e^\lambda \cdot [\mathcal{O}_{1.B}]).$$

Therefore, the knowledge of the expansion from (2) implies the Chevalley formula in the geometric case. This argument may be generalized to any homogeneous bundle; see Remark 5.3. In cohomology (i.e., for the Chern–Schwartz–MacPherson classes), and for $G = \mathrm{SL}_n$, this argument is implicitly utilized in the paper [FGX22] to obtain a Murnaghan–Nakayama formula.

We briefly survey next other results from this note. Having established a formula to calculate the Chevalley coefficients, in Section 6 we utilize several dualities with geometric origin (the Serre duality, the star duality, and the Dynkin automorphism duality) to obtain several symmetries of the coefficients $C_{v,\lambda}^w(y)$. See Proposition 6.5, for example. Combining these dualities shows that the polynomials $C_{v,\lambda}^w(y)$ are palindromic.

A remarkable property of the motivic Chern classes of Schubert cells, proved in [AMSS24a; FRW21], is that they are equivalent to the K-theoretic version of Maulik and Okounkov's stable envelopes; see [MO19; AO21]. The stable envelopes are elements in the $T \times \mathbb{C}^*$-equivariant K-theory of the cotangent bundle, $K_{T\times\mathbb{C}^*}(T^*(G/B))$. In this context, the formal variable y may be identified to the (inverse) of the character given by the $\mathbb{C}^*$ fiber dilation on the cotangent bundle. If $\iota : G/B \hookrightarrow T^*(G/B)$ is the inclusion of the zero section, then $\iota^*(\mathrm{stab}(w))$ is a multiple of the motivic Chern class of the (opposite) Schubert cell for w, where stab is a stable envelope, appropriately normalized.

The stable envelopes depend on three parameters: a chamber, a polarization, and a slope, and the precise normalizations are essential for this paper. A variation in the chamber results in conjugating by the Borel subgroup [AMSS24a], and it is encoded in the left Weyl group action [MNS22b] and certain R-matrix operators [RTV15; RTV19]. Varying the polarization, or the slope, results in the multiplication of $\mathrm{stab}(w)$ by a line bundle $\mathcal{L}_\lambda$ pulled back from G/B; cf. [AMSS24a; Oko27], and see also Section 7 below. In particular, the coefficients $C_{v,\lambda}^w(y)$ from (1) give "wall-crossing" formulae, recording the change of stable envelopes when its defining parameters are varied. While these wall crossing formulae have been worked out in [Oko27; SZZ20; SZZ21] (see also [KW25]), in Section 7 we revisit some of these from the point of view of Theorem 1.1. In particular, we utilize the Chevalley formula to give an explicit combinatorial rule relating the stable envelope for the fundamental alcove $\mathrm{A}_\circ$ to the one corresponding to any translation $\mathrm{A}_\circ + \lambda$; see Theorem 7.8.

In addition to our application mentioned above to wall crossing formulae for stable envelopes, in Section 8 we utilize known relations between motivic Chern classes of Schubert cells, Whittaker functions, and Hall–Littlewood polynomials, to obtain new formulae for the latter.

Finally, in Appendix A we obtain an analogue of the Chevalley formula (1) for the homological analogue of the motivic Chern classes, the Chern–Schwartz–MacPherson classes. While this formula may be obtained by a specialization argument as in [AMSS24b], the degenerate affine Hecke algebra is much simpler in this case, and a direct proof of the Chevalley formula may be obtained rather quickly.

Notation. We fix the notation utilized throughout the paper. Let G be a simply connected complex Lie group with Borel subgroup B and maximal torus $T \subset B$. Denote by $\mathfrak{g} = \mathrm{Lie}(G)$ and by $\mathfrak{h} = \mathrm{Lie}(T)$ be corresponding Lie algebras. Let $R^+ \subset \mathfrak{h}^* := \mathfrak{h}^*_{\mathbb{Q}}$ denote the positive roots, which by convention are the roots in B, and by $\Sigma = \{\alpha_i : i \in I\}$ the set of simple roots. The set of all roots is $R := R^+ \sqcup -R^+$. We use $\alpha > 0$ (resp. $\alpha < 0$) to denote $\alpha \in R^+$ (resp. $\alpha \in -R^+$). For any root $\alpha \in R$, let $\alpha^\vee \subset \mathfrak{h}$ denote the corresponding coroot. Denote by

$$\langle\cdot,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Q}$$

the evaluation pairing, and let $X^*(T) \subset \mathfrak{h}^*$ be the weight lattice. For any weight $\lambda \in X^*(T)$, let $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$ be the line bundle on G/B associated to λ. The Weyl group is $W = N_G(T)/T$ and it is generated by simple reflections $s_i = s_{\alpha_i}$ $(i \in I)$. It is equipped with a length function $\ell : W \to \mathbb{Z}_{\geq 0}$ defined as the length of a minimal expression of w in terms of the simple reflections; we denote by w_0 the longest element. The Bruhat order on W is a partial order determined by the covering relations $u \leq us_\alpha$ where $\alpha \in R$ and $\ell(us_\alpha) = \ell(u) + 1$.

For any $w \in W$, let $X(w)^\circ := BwB/B \subset G/B$ and $Y(w)^\circ := B^- wB/B \subset G/B$ be Schubert cells, where B^- is the opposite Borel subgroup. Let $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ be the respective Schubert varieties. Let $P(\supseteq B)$ be a parabolic subgroup with simple roots $\Sigma_P \subset \Sigma$. Let R_P^+ denote the positive roots spanned by Σ_P. Let W_P be the Weyl group generated by the simple reflections s_α, $\alpha \in \Sigma_P$. Let $W^P \simeq W/W_P$ denote the set of minimal length representatives. For any $w \in W^P$, let $X(wW_P)^\circ := BwP/P \subset G/P$ (resp. $Y(wW_P)^\circ := B^- wP/P \subset G/P$) denote the Schubert cell with closure $X(wW_P)$ (resp. $Y(wW_P)$). Let $X^*(T)_P := \{\lambda \in X^*(T) \mid \langle\lambda, \gamma^\vee\rangle = 0 \text{ for all } \gamma \in R_P^+\}$ be the set of integral weights which vanish on $(R_P^+)^\vee$. For any $\lambda \in X^*(T)_P$, we still use $\mathcal{L}_\lambda$ to denote the line bundle $G \times^P \mathbb{C}_\lambda \in \mathrm{Pic}(G/P)$, which has fiber over $1.P$ the one-dimensional T-module of weight λ.

2. Affine Hecke algebra via alcove walk algebra

In this section we introduce the alcove walk algebra, and a formula of Ram [Ram06] describing a change of bases matrix for the affine Hecke algebra.

2.1. ***Affine Hecke algebra.*** The affine Hecke algebra $\mathbb{H}$ is a free $\mathbb{Z}[q, q^{-1}]$ module with basis $\{T_w X^\lambda | w \in W, \lambda \in X^*(T)\}$, such that

- For any $\lambda, \mu \in X^*(T)$, $X^\lambda X^\mu = X^{\lambda+\mu}$.
- For any simple root α, $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0$.
- For any $w, y \in W$, such that $\ell(wy) = \ell(w) + \ell(y)$, $T_w T_y = T_{wy}$
- For any simple root α and $\lambda \in X^*(T)$,

$$T_{s_\alpha} X^\lambda - X^{s_\alpha \lambda} T_{s_\alpha} = (1-q) \frac{X^{s_\alpha\lambda} - X^\lambda}{1 - X^{-\alpha}}.$$

For our geometric application we will need two other bases of the affine Hecke algebra $\mathbb{H}$: $\{T_{w^{-1}}^{-1} X^\lambda \mid w \in W, \lambda \in X^*(T)\}$ and $\{X^\lambda T_{w^{-1}}^{-1} \mid w \in W, \lambda \in X^*(T)\}$. Define the transition matrix coefficients $c_{u,\mu}^{w,\lambda} \in \mathbb{Z}[q, q^{-1}]$ by

$$T_{w^{-1}}^{-1} X^\lambda = \sum_{\substack{\mu \in X^*(T)\\ u \in W}} c_{u,\mu}^{w,\lambda} X^\mu T_{u^{-1}}^{-1}. \tag{6}$$

The main result of this section is a formula for $c_{u,\mu}^{w,\lambda}$ obtained by Ram [Ram06]; see Theorem 2.4.

For the later application to the motivic Chern classes, we also introduce the Iwahori–Matsumoto $\mathbb{Z}[q, q^{-1}]$-algebra involution Θ on $\mathbb{H}$ defined by

$$\Theta(T_{s_\alpha}) = -q T_{s_\alpha}^{-1} \quad \text{and} \quad \Theta(X^\lambda) = X^{-\lambda},$$

where s_α is a simple reflection; see [EM97, Section 5.1]. Hence, $\Theta(T_{w^{-1}}^{-1}) = (-q)^{-\ell(w)} T_w$. Applying Θ to (6), we obtain

$$T_w X^{-\lambda} = \sum_{\substack{\mu \in X^*(T)\\ u \in W}} (-q)^{\ell(w)-\ell(u)} c_{u,\mu}^{w,\lambda} X^{-\mu} T_u \tag{7}$$

2.2. ***Alcove walk algebra.*** In this section we review Ram's definition of the alcove walk algebra, and state his formula for the matrix coefficients $c_{u,\mu}^{w,\lambda}$. We refer the reader to [Ram06] for a more detailed account of the alcove walk algebras.

2.2.1. *Alcoves.* Let $\mathfrak{t}_\mathbb{R}^*$ be the dual of the Lie algebra of the maximal torus T. For any root α and $j \in \mathbb{Z}$, define

$$H_{\alpha,j} := \{\lambda \in \mathfrak{t}_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle = j\}.$$

Notice that $H_{\alpha,j} = H_{-\alpha,-j}$. The connected components of $\mathfrak{t}_\mathbb{R}^* \setminus \bigcup_{\alpha>0, j\in\mathbb{Z}} H_{\alpha,j}$ are called *alcoves*. The codimension 1 faces of any alcove are called the walls of that alcove. The *fundamental alcove* $\mathrm{A}_\circ$ is defined by

$$\mathrm{A}_\circ = \{\lambda \in \mathfrak{t}_\mathbb{R}^* \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \text{ for any positive root } \alpha\}.$$

If $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the simple roots, and $\theta^\vee$ is the highest coroot, then the walls of the fundamental alcove $\mathrm{A}_\circ$ are $H_{\theta,1}$ and $H_{\alpha_i,0}$ $(1 \le i \le r)$. We label these walls of $\mathrm{A}_\circ$ by $0, 1, \ldots, r$ respectively.

The affine Weyl group for the dual root system is defined by $W_{\mathrm{aff}} := Q \rtimes W$, where Q is the root lattice. Then W_{aff} acts simply transitively on the set of alcoves, and this action is determined by the reflections across the hyperplanes $h = H_{\alpha,j}$, given by

$$s_{\alpha,j}(\mu) = \hat{r}_h(\mu) := s_\alpha(\mu) + j\alpha \quad \text{ for } \mu \in X^*(T). \tag{8}$$

The affine Weyl group is a Coxeter group generated by the reflections $s_0 := s_{\theta,1}$ and s_i $(1 \le i \le r)$ along the walls of $\mathrm{A}_\circ$. In fact, $\mathrm{A}_\circ$ is a fundamental domain for the action of W_{aff} on the set of alcoves, in the sense that any element in $\mathfrak{t}^*_{\mathbb{R}} \setminus \bigcup_{\alpha>0,\, j\in\mathbb{Z}} H_{\alpha,j}$ is sent to exactly one element in $\mathrm{A}_\circ$. See [Ram06; Hum90] for details.

The extended affine Weyl group for the dual root system is $W^{\mathrm{ext}}_{\mathrm{aff}} = X^*(T) \rtimes W$, where $X^*(T)$ is the weight lattice. For any $\lambda \in X^*(T)$, let t_λ denote the corresponding element in $W^{\mathrm{ext}}_{\mathrm{aff}}$. There is a length function ℓ on $W^{\mathrm{ext}}_{\mathrm{aff}}$ defined by the following formula (see [Mac96, equation (2.8)]):

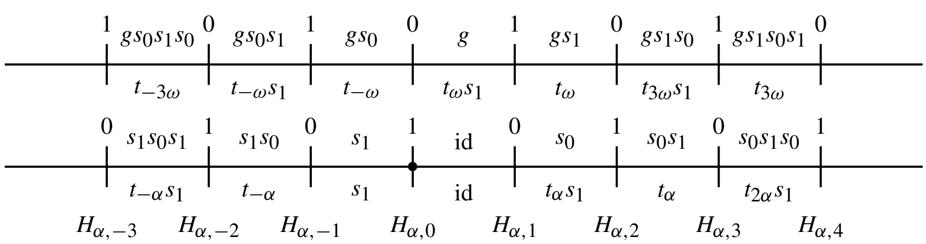
$$\ell(t_\mu w) = \sum_{\alpha\in R^+} |\langle \mu, w(\alpha^\vee)\rangle + \chi(w(\alpha))|, \qquad \text{where } \chi(\alpha) = \begin{cases} 0 & \text{if } \alpha \in R^+, \\ 1 & \text{if } \alpha \in R^-. \end{cases}$$

Let $\Omega \subset W^{\mathrm{ext}}_{\mathrm{aff}}$ be the subgroup of length zero elements in $W^{\mathrm{ext}}_{\mathrm{aff}}$. Then $W^{\mathrm{ext}}_{\mathrm{aff}} \simeq W_{\mathrm{aff}} \rtimes \Omega$, see [Mac96, equation (2.10)], and $\ell(wg) = \ell(w)$ for any $w \in W_{\mathrm{aff}}$ and $g \in \Omega$. The elements in Ω preserve the fundamental alcove $\mathrm{A}_\circ$ and act as automorphisms.

Using a bijection between W_{aff} and the alcoves in $\mathfrak{t}^*_{\mathbb{R}}$, one defines a bijection between $W^{\mathrm{ext}}_{\mathrm{aff}} \simeq W_{\mathrm{aff}} \rtimes \Omega$ and the alcoves in $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$ ($|\Omega|$ copies of $\mathfrak{t}^*_{\mathbb{R}}$, each tiled by alcoves). We label the walls of every alcove in $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$ in an $W^{\mathrm{ext}}_{\mathrm{aff}}$-equivariant way: for each $w \in W^{\mathrm{ext}}_{\mathrm{aff}}$ the walls of $w\mathrm{A}_\circ$ are $wH_{\alpha_i,0}$ $(1 \le i \le n)$ and $wH_{\theta,1}$, and they are labeled by i and 0, respectively. In particular, if two adjacent alcoves A_1 and A_2 are separated by a wall labeled by i (in both A_1 and A_2), and $A_1 = w\mathrm{A}_\circ$ for some $w \in W^{\mathrm{ext}}_{\mathrm{aff}}$, then $A_2 = ws_i\mathrm{A}_\circ$. Equivalently, in terms of the wall crossings, if $w = gs_{i_1}s_{i_2}\cdot \cdots \cdot s_{i_\ell} \in W^{\mathrm{ext}}_{\mathrm{aff}}$, with $g \in \Omega$ and $0 \le i_j \le r$, then the alcove $w\mathrm{A}_\circ$ in $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$ is the alcove obtained from rotating the fundamental alcove $\mathrm{A}_\circ$ according to the automorphism g, then reflecting along the walls labeled (in order) by $i_1, \ldots, i_\ell$. See also Lemma 3.3.

Example 2.1. We consider the example of the root system of type A_1. Let α be the positive root, and $\omega = \alpha/2$ be the fundamental weight. The weight lattice $X^*(T)$ is $\mathbb{Z}\omega$, the root lattice Q is $\mathbb{Z}\alpha$, and the finite Weyl group W is $\{\mathrm{id}, s_1 = s_\alpha\}$. The affine Weyl group $W_{\mathrm{aff}} = Q \rtimes W$ has Coxeter generators s_1 and $s_0 = t_\alpha s_1$. The subgroup of length zero elements in W^{ext}_{aff} is $\Omega = \{\mathrm{id}, g = t_\omega s_1\} \simeq \mathbb{Z}/2\mathbb{Z}$.

In the following picture for $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$, the lower sheet is the identity sheet, while the upper sheet is the



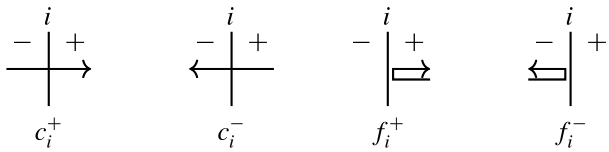
sheet $g \times \mathfrak{t}_{\mathbb{R}}^*$. Each alcove $w\mathrm{A}_\circ$ is labeled by the corresponding $w \in W_{\mathrm{aff}}^{ext}$, both in the Coxeter presentation and the translation presentation. In the lower sheet, the walls $H_{\alpha,n}$ are labeled by 1 if n is even, and 0 if n is odd. On the upper sheet, the labelings are in the opposite way.

2.2.2. *Alcove walk algebra.* We recall a realization of the Hecke algebra in terms of alcove walks; see [Ram06]. For each positive root α and hyperplane $H_{\alpha,j}$, set the positive side of it to be $\{\lambda \in \mathfrak{t}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle > j\}$.

Definition 2.2. The *alcove walk algebra* is generated over $\mathbb{Z}[q, q^{-1}]$ by elements $g \in \Omega$, and for $0 \le i \le r$, the elements c_i^+ (positive i-crossing), c_i^- (negative i-crossing), f_i^+ (positive i-fold) and f_i^- (negative i-fold), subject to the following relations, sometimes called straightening laws:

$$c_i^+ = c_i^- + f_i^+, \quad c_i^- = c_i^+ + f_i^-, \quad gc_i^\pm = c_{g(i)}^\pm g, \quad gf_i^\pm = f_{g(i)}^\pm g.$$

In terms of pictures, these generators can be drawn as follows:



$(0 \le i \le r)$

Here, c_i^+ represents a crossing of a wall labeled by i from its negative to its positive side, and similarly for the other generators. The product is given as concatenation. An *alcove walk* is a word in the generators such that

- the tail of the first step is in the fundamental alcove $\mathrm{A}_\circ$, and
- at every step, either we change the sheet according to an element in Ω (thus rotating the alcove according to this elements), or the head of each arrow is in the same alcove as the tail of the next arrow.

An alcove walk p is called *nonfolded* if there is no $f_i^\pm$ in its word. The *length* of an alcove walk is the number of letters $c_i^\pm$, $f_i^\pm$ in an alcove walk. (In particular, rotation with respect to an element of Ω does not contribute to the length.) For a *minimal* alcove walk between two alcoves, one can show that the walk is nonfolded, thus its length is the number of $c_i^\pm$ in the walk [Ram06]. From this it follows that if $w \in W_{\mathrm{aff}}^{\mathrm{ext}}$, then $\ell(w)$=length of a minimal length walk from $\mathrm{A}_\circ$ to $w\mathrm{A}_\circ$.

Pick a square root $q^{\frac{1}{2}}$ of q. The following was proved by A. Ram.

Proposition 2.3 [Ram06, §3.2]. (a) *The affine Hecke algebra $\mathbb{H}$ is isomorphic as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra to the quotient of the alcove walk algebra by the relations*

$$c_i^+ = (c_i^-)^{-1}, \quad f_i^+ = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad f_i^- = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \tag{9}$$

and

$$p = p' \textit{ if } p \textit{ and } p' \textit{ are nonfolded alcove walks with } \mathrm{end}(p) = \mathrm{end}(p'), \tag{10}$$

where $\mathrm{end}(p)$ *means the final alcove of* p.

(b) *Under the previous isomorphism, for any $w \in W$ and $\lambda \in X^*(T)$:* [1]

[1] The q^2 and T_{s_i} in [Ram06, Proposition 3.2(b)] are our q and $q^{-\frac{1}{2}}T_i$, respectively.

- *a minimal length alcove walk from* $\mathrm{A}_\circ$ *to* $w\mathrm{A}_\circ$ *is sent to* $q^{\ell(w)/2}T_{w^{-1}}^{-1}$, *and,*
- *a minimal length alcove walk from* $\mathrm{A}_\circ$ *to* $t_\lambda \mathrm{A}_\circ$ *is sent to* X^λ.

For an alcove walk p, define the functions *weight* $wt(p) \in X^*(T)$, and *final direction* $\varphi(p) \in W$ of p by the condition that p ends in the alcove $t_{wt(p)}\varphi(p)\mathrm{A}_\circ$. Let

$$\begin{aligned} f^-(p) &= \text{number of negative folds of } p,\\ f^+(p) &= \text{number of positive folds of } p,\\ f(p) &= f^+(p) + f^-(p). \end{aligned}$$

Now we can state the formula for the matrix coefficients $c_{u,\mu}^{w,\lambda}$ defined in (6); see also (7).

Theorem 2.4 [Ram06, Theorem 3.3]. *Let* $\lambda \in X^*(T)$ *and* $w \in W$. *Fix a minimal length walk* $p_w = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^-$ *from* $\mathrm{A}_\circ$ *to* $w\mathrm{A}_\circ$ *and a minimal length walk* $p_\lambda = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \cdots c_{j_s}^{\epsilon_s} g$ *from* $\mathrm{A}_\circ$ *to* $t_\lambda \mathrm{A}_\circ$, *where* $g \in W_{\mathrm{aff}}^{ext}$ *is defined by the condition* $gW_{\mathrm{aff}} = t_\lambda W_{\mathrm{aff}}$ [2], *and* $\epsilon_i = \pm$ *for each* i. *Then*

$$T_{w^{-1}}^{-1} X^\lambda = \sum_p (-1)^{f^-(p)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{f(p)} q^{\frac{1}{2}(\ell(\varphi(p)) - \ell(w))} X^{wt(p)} T_{\varphi(p)^{-1}}^{-1}, \tag{11}$$

where the sum is over all alcove walks p *of the form*

$$p = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^- p_{j_1} p_{j_2} \cdots p_{j_s} g \text{ such that } p_{j_k} \in \{c_{j_k}^{\pm}, f_{j_k}^{\epsilon_k}\}. \tag{12}$$

Therefore the matrix coefficients $c_{u,\mu}^{w,\lambda}$ *in* (6) *are given by*

$$c_{u,\mu}^{w,\lambda} = \sum_{\substack{p \text{ of the form (12)} \\ \varphi(p) = u,\, wt(p) = \mu}} (-1)^{f^+(p)} (1-q)^{f(p)} q^{\frac{1}{2}(\ell(u) - \ell(w) - f(p))}. \tag{13}$$

Example 2.5. Let $G = \mathrm{SL}(2, \mathbb{C})$. We use the same notation as in Example 2.1. We check the above theorem for $w = s_1$ and $\lambda = \omega$. From the alcove picture in Example 2.1, $T_{s_1}^{-1}$ is represented by the minimal length walk $q^{-1/2}c_1^-$, while X^ω is represented by the walk $gc_1^+ = c_0^+ g$. Thus, the sum in the right hand side of the Theorem is over the alcove walks $c_1^- c_0^- g$ and $c_1^- f_0^+ g$, which end at the alcoves $t_{-\omega}s_1\mathrm{A}_\circ$ and $t_{-\omega}\mathrm{A}_\circ$, respectively. (Note that $c_1^- c_0^+ g$, which represents $T_{s_1}^{-1}X^\omega$, is not an alcove walk.) Therefore, the identity in the theorem is

$$T_{s_1}^{-1} X^\omega = X^{-\omega} T_{s_1}^{-1} - q^{-1}(1-q) X^{-\omega}.$$

On the other hand, it is easy to check the above equation using the definition of the affine Hecke algebra in Section 2.1.

Remark 2.6. In Theorem 2.4, one may relax the hypotheses about the alcove walks p_w and p_λ to be of nonminimal length. This follows from analyzing the proof of Ram's result in [Ram06]. We do not use this, but it is consistent with our use of nonreduced λ-chains in Section 3 below.

[2] We need to add this extra $g \in \Omega$ since $t_\lambda \mathrm{A}_\circ$ may not on the same sheet as $\mathrm{A}_\circ$, see Example 2.5. The stated conditions determine g uniquely.

Consider an ordered collection of hyperplanes $\mathcal{H} = \{H_{\beta_1,k_1}, \dots, H_{\beta_s,k_s}\}$ and set $h_i := H_{\beta_i,k_i}$. Associated to this sequence we define the elements

$$\hat{r}_{\mathcal{H}} = s_{\beta_1,k_1} \cdot \cdots \cdot s_{\beta_s,k_s} \in W_{\mathrm{aff}}; \quad r_{\mathcal{H}} = s_{\beta_1} \cdot \cdots \cdot s_{\beta_s} \in W.$$

(These depend on the order of $\mathcal{H}$.) We also define an $\mathcal{H}$-restricted version of the Bruhat order on W by

$$w \xrightarrow{\mathcal{H}} u \iff w > ws_{\beta_1} > ws_{\beta_1}s_{\beta_2} > \cdots > ws_{\beta_1}s_{\beta_2} \cdot \cdots \cdot s_{\beta_s} = u. \tag{14}$$

Lemma 2.7 (compare [Len11, Proposition 2.5]). *Let $w \in W$ and $\lambda \in X^*(T)$. Fix:*

- *an alcove walk p_w from $\mathrm{A}_\circ$ to $w\mathrm{A}_\circ$;*
- *an alcove walk $p_\lambda = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \cdots c_{j_s}^{\epsilon_s} g$ from $\mathrm{A}_\circ$ to $t_\lambda \mathrm{A}_\circ$.*

Let $h_i = H_{\beta_i,k_i}$, for $1 \le i \le s$ be the sequence of hyperplanes defined by the walls of alcoves crossed by p_λ, with $\beta_i \in R^{\epsilon_i}$ and $k_i \in \mathbb{Z}$.

Then there is a bijection between

(1) *the set of alcove walks of the form $\bar{p} = p_w p_{j_1} \dots p_{j_s}$ such that $p_{j_k} \in \{c_{j_k}^{\pm}, f_{j_k}^{\epsilon_k}\}$, and*

(2) *the set of subsets $\mathcal{M} = \{h_{m_1}, \dots, h_{m_t}\} \subset \mathcal{H} = \{h_1, \dots, h_s\}$ with $m_1 < m_2 < \cdots < m_t$, s.t. $w \overset{\mathcal{M}}{\Longrightarrow} wr_{\mathcal{M}}$.*

Under this bijection, the indices m_i correspond to the positions of foldings $f_{m_i}^{\epsilon_{m_i}}$. Furthermore,

$$\varphi(\bar{p}g) = wr_{\mathcal{M}} \text{ and } \mathrm{wt}(\bar{p}g) = w\hat{r}_{\mathcal{M}}(\lambda).$$

Remark 2.8. This statement is a slight generalization of Lenart's result. We do not require λ to be dominant, and therefore we need to allow $\beta_i \in R^{\epsilon_i}$ to be a negative root.

Proof. The proof follows the same outline as that of [Len11, Proposition 2.5], so we will be brief. To start, consider the *unique* unfolded alcove walk $p_0 = p_w p_{j_1} \dots p_{j_s}$ such that $p_{j_k} \in \{c_{j_k}^{\pm}\}$. Any other alcove walk $\bar{p}$ as in the statement is of the form

$$\bar{p} = \phi_{m_t} \cdots \phi_{m_1}(p_0),$$

where ϕ_{m_j} is the folding operation at position m_j (cf. [Len11]) and the m_i's with $m_{i-1} < m_i$ are the folding positions of $\bar{p}$. This alcove walk has the property that if $k \notin \{m_1, \dots, m_t\}$ then $p_{j_k} \in \{c_{j_k}^{\pm}\}$ and $p_{j_{m_i}} = f_{j_{m_i}}^{\epsilon'_{m_i}}$ $(1 \le i \le t)$. In addition $\varphi(\bar{p}g) = wr_{\mathcal{M}}$ and $\mathrm{wt}(\bar{p}g) = w\hat{r}_{\mathcal{M}}(\lambda)$. Let $\mathcal{M}_i = \{h_{m_1}, \dots, h_{m_i}\} \subset \mathcal{H} = \{h_1, \dots, h_s\}$ for $1 \le i \le t$ with the convention that $\mathcal{M}_0 = \varnothing$ and $r_\varnothing = \mathrm{id}$. A key point is that $wr_{\mathcal{M}_{i-1}} > wr_{\mathcal{M}_i}$ if and only if the folding orientations satisfy $\epsilon_{m_i} = \epsilon'_{m_i}$. (The proof uses the condition that if $\beta > 0$, then $ws_\beta > w$ if and only if $w(\beta) > 0$.) All this put together implies that $\{h_{m_1}, \dots, h_{m_i}\} \leftrightarrow \phi_{m_i} \cdots \phi_{m_1}(p_0)$ gives the bijection in the statement. ☐

For a subset $\mathcal{M} \subset \mathcal{H}$ as in Lemma 2.7, set $p(\mathcal{M})$ to be the alcove walk $\bar{p}g$ associated to $\mathcal{M}$, and set $f^+(\mathcal{M}) = f^+(p(\mathcal{M}))$.

We next reformulate Ram's result from Theorem 2.4 in terms of paths in the Bruhat order.

Corollary 2.9. *Let $u, w \in W$ and $\lambda, \mu \in X^*(T)$. Assume the same hypotheses and notation as in Lemma 2.7. Then*

$$c_{u,\mu}^{w,\lambda} = \sum_{\mathcal{M}} (-1)^{f^+(\mathcal{M})}(1-q)^{|\mathcal{M}|} q^{\frac{1}{2}(\ell(u)-\ell(w)-|\mathcal{M}|)},$$

where the sum is over subsets $\mathcal{M} \subset \mathcal{H}$ which satisfy $w \overset{\mathcal{M}}{\Longrightarrow} u = wr_{\mathcal{M}}$ and $\mu = w\hat{r}_{\mathcal{M}}(\lambda)$.

Remark 2.10. *A priori*, the coefficients $c_{u,\mu}^{w,\lambda}$ appearing in the walk algebra are in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. However, after matching these with the initial definition of Hecke algebras, it turns out that $c_{u,\mu}^{w,\lambda} \in \mathbb{Z}[q^{\pm 1}]$. This can also be seen directly in the formula above, by observing that $\ell(u) - \ell(w) - |\mathcal{M}|$ is even. (This uses that a reflection has odd length, and the cancellation property of nonreduced expressions.)

3. A λ-chain formula for the transition coefficients $c_{u,\mu}^{w,\lambda}$

In this section we reformulate the alcove walk formula from Corollary 2.9 in terms of the notion of λ-chains introduced in [LP07]. This notion was utilized to obtain a K-theory Chevalley formula for the structure sheaves of Schubert varieties. The main results of this section are Theorem 3.9 and Theorem 3.10. Specializing $y \mapsto 0$, these recover [LP07, Theorem 6.1]. Throughout this section, we only consider alcoves on the identity sheet $\mathfrak{t}_{\mathbb{R}}^*$. Recall that $\mathrm{A}_\circ + \lambda := \{x + \lambda \mid x \in \mathrm{A}_\circ\}$ is the alcove on $\mathfrak{t}_{\mathbb{R}}^*$. (If λ is not a root, the alcove $t_\lambda(\mathrm{A}_\circ)$ is not the alcove $\mathrm{A}_\circ + \lambda$.)

3.1. ***Alcove paths and λ-chains.*** For any two alcoves A and B, which are separated by a common wall lying on a hyperplane $H_{\beta,k}$, write $A \xrightarrow{\beta} B$ if the root β points from A to B.

Definition 3.1 [LP07, Definition 5.2]. An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_l)$ such that A_{j-1} and A_j are adjacent, $1 \le j \le l$. We denote this by

$$A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l.$$

When the length l is minimal among all alcove paths from A_0 to A_l, it is called a *reduced alcove path*.

Remark 3.2. By definition, there is a one-to-one correspondence between alcove paths

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l = \mathrm{A}_\circ - \lambda \tag{15}$$

from $\mathrm{A}_\circ$ to $A_l = v_{-\lambda}(\mathrm{A}_\circ)$ and alcove walks from A_0 to A_l of the form $c_{i_1}^{\epsilon_1} c_{i_2}^{\epsilon_2} \cdots c_{i_l}^{\epsilon_l}$, where $-\beta_j \in R^{\epsilon_j}$ for $1 \le j \le l$.

Recall that $s_0 = s_{\theta,1}$ is the affine reflection along the hyperplane $H_{\theta,1}$ with $\theta^\vee$ the highest coroot. Define $\alpha_0 = -\theta$, $\bar{s}_0 = s_\theta$ and $\bar{s_i} = s_i$ for $1 \le i \le r$.

Lemma 3.3 [LP07, Lemma 5.3]. *For any $v \in W_{\mathrm{aff}}$, there is a one-to-one correspondence between decompositions of v in the simple reflections $s_i \in W_{\mathrm{aff}}$ $(0 \le i \le r)$ and alcove paths from $\mathrm{A}_\circ$ to $v(\mathrm{A}_\circ)$ as follows.*

For any decomposition $v = s_{i_1} s_{i_2} \cdots s_{i_l}$, *define*

$$\beta_j := \bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_{j-1}}(\alpha_{i_j}), \quad 1 \le j \le l.$$

Then

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l = v(\mathrm{A}_\circ)$$

is an alcove path from $\mathrm{A}_\circ$ *to* $v(\mathrm{A}_\circ)$.

The affine reflection along the j*-th hyperplane separating* A_{j-1} *and* A_j *is* $s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_2} s_{i_1}$; *in particular, the separating wall is labeled by* i_j. *Under this correspondence, a reduced alcove path corresponds to a reduced decomposition for* v.

Let λ be an integral weight and let $v_{-\lambda} \in W_{\mathrm{aff}}$ be the affine Weyl group element which satisfies $v_{-\lambda}(\mathrm{A}_\circ) = \mathrm{A}_\circ - \lambda$, i.e., $t_{-\lambda} = v_{-\lambda} g$ with $g \in \Omega$. Choose a (possibly nonreduced) decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ and let β_j be defined by

$$\beta_j := \bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_{j-1}}(\alpha_{i_j}), \quad 1 \le j \le l,$$

with the convention that $\alpha_0 = -\theta$ (see Lemma 3.3). Then

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} \mathrm{A}_\circ - \lambda$$

is an alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$.

Definition 3.4 [LP07, Definition 5.4]. The sequence of roots $(\beta_1, \ldots, \beta_l)$ is a λ*-chain* of roots associated to the decomposition of $v_{-\lambda}$. A λ-chain is called reduced if the decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ is reduced.

Let $H_{-\beta_j, d_j}$ be the hyperplane separating the alcoves A_j and A_{j+1}. The sequence of integers d_j are determined by the sequence of roots β_j, but we occasionally keep the information of d's in the notation, and we refer to the sequence of pairs $(\beta_1, d_1), (\beta_2, d_2), \ldots, (\beta_l, d_l)$ as a λ-chain [3]. Following [LP07, Prop. 6.8] we recall a combinatorial construction of a λ-chain for an integral weight λ.

Fix a linear order on the index of Dynkin nodes (for example $1 < 2 < \cdots < r$ in $I = \{1, 2, \ldots, r\}$). The set $\mathcal{R}_\lambda \subset W_{\mathrm{aff}}$ of affine reflections $s_{\alpha,k}$ for the hyperplanes $H_{\alpha,k}$ separating the fundamental alcove $\mathrm{A}_\circ$ and $\mathrm{A}_\circ - \lambda$ is given by

$$\mathcal{R}_\lambda = \bigcup_{\alpha \in R^+} \begin{cases} \{s_{\alpha,k} : 0 \ge k > -\langle \lambda, \alpha^\vee \rangle\} & \text{if } \langle \lambda, \alpha^\vee \rangle > 0, \\ \{s_{\alpha,k} : 0 < k \le -\langle \lambda, \alpha^\vee \rangle\} & \text{if } \langle \lambda, \alpha^\vee \rangle < 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

One defines a "height function"

$$h : \mathcal{R}_\lambda \to \mathbb{R}^{r+1}; \quad h(s_{\alpha,k}) = \frac{1}{\langle \lambda, \alpha^\vee \rangle}(-k, \langle \varpi_1, \alpha^\vee \rangle, \ldots, \langle \varpi_r, \alpha^\vee \rangle).$$

[3] If λ is dominant, this definition was extended to the Kac–Moody situation in [LP08].

It turns out that this function is injective. Now order the images of h in lexicographic order, so we obtain $h(s_{\alpha_1,k_1}) < \cdots < h(s_{\alpha_l,k_l})$. Define another function $b : \mathcal{R}_\lambda \to R^+ \cup R^-$ by

$$b(s_{\alpha,k}) = \begin{cases} \alpha & \text{if } k \le 0,\ \alpha \in R^+, \\ -\alpha & \text{if } k > 0,\ \alpha \in R^+. \end{cases}$$

Then $b(s_{\alpha_1,k_1}), \dots, b(s_{\alpha_l,k_l})$ is a (reduced) λ-chain of roots.

Remark 3.5. A particularly nice situation occurs for a minuscule fundamental weight ϖ_i, i.e., when $\langle \varpi_i, \alpha^\vee \rangle \in \{0, 1\}$ for any positive root α. In this case all $k_i = 0$ and $v_{-\varpi_i} = (w^{P_i})^{-1} \in W$, with w^{P_i} being the longest minimal length representative for cosets of W/W_{P_i}, where $W_{P_i} = \mathrm{Stab}_W(\varpi_i)$; equivalently, the Schubert variety $X(w^{P_i} W_{P_i}) = G/P_i$. A reduced decomposition and the associated ϖ_i-chain may be read from the associated Young poset of G/P_i; see [Pro99; Ste01] and also [BCMP18, §3.1] and Example 3.6 below. It may also be obtained as a reverse linear extension of the heap $H(w^{P_i})$, and this gives a one-to-one correspondence between reduced ϖ_i-chains and reverse linear extensions of the heap $H(w^{P_i})$. We refer the readers to [MNS22a; NO19] for the heap perspective.

Example 3.6. Consider $G = \mathrm{SL}_5$ and the fundamental weight ϖ_2. The stabilizer is the maximal parabolic P_2 so that G/P_2 is the Grassmannian Gr(2, 5). The inversion set and a reduced decomposition of $v_{-\varpi_2}$ may be read from the Young diagrams below, with the notation $(i - j) = \alpha_i + \cdots + \alpha_{j-1}$.

(2−3)	(2−4)	(2−5)
(1−3)	(1−4)	(1−5)

α_2	α_3	α_4
α_1	α_2	α_3

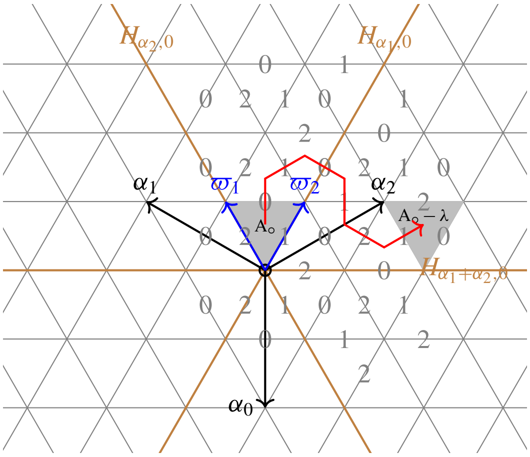
Then $v_{-\varpi_2} = s_2 s_3 s_4 s_1 s_2 s_3$ and a ϖ_2-chain of roots is given by $\{\alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$.

Example 3.7. Let $G = \mathrm{SL}_3$, with the Weyl group $W = S_3$, and consider $\lambda = 2\varpi_1 - 2\varpi_2$. An example of alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$ is

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} A_2 \xrightarrow{-\beta_3} A_3 \xrightarrow{-\beta_4} A_4 \xrightarrow{-\beta_5} A_5 \xrightarrow{-\beta_6} A_6 = \mathrm{A}_\circ - \lambda\ (A_i = r_i A_{i-1},\ 1 \le i \le 6),$$

which is the red path below, and it corresponds to the decomposition $v_{-\lambda} = s_0 s_1 s_0 s_1 s_2 s_1$. The corresponding alcove walk is $\bar{p} = c_0^+ c_1^+ c_0^- c_1^- c_2^- c_1^+$, and the corresponding λ-chain of roots is $(\beta_1, 1)$, $(\beta_2, 1)$, $(\beta_3, 0)$, $(\beta_4, -1)$, $(\beta_5, 1)$, $(\beta_6, 2)$, as calculated in Figure 1. Here we included the d's in the notation. We can choose another λ-chain for $\lambda = 2\varpi_1 - 2\varpi_2$ by using the reduced decomposition $v_{-\lambda} = s_1 s_0 s_2 s_1$. The corresponding reduced λ-chain is $(\alpha_1, 0)$, $(-\alpha_2, 1)$, $(\alpha_1, 1)$, $(-\alpha_2, 2)$.

Lemma 3.8 [LP07, Remark 5.5]. *Let $L = (\beta_1, \dots, \beta_l)$ be a λ-chain. Then $\bar{L} := (-\beta_l, \dots, -\beta_1)$ is a $(-\lambda)$-chain. If $H_{-\beta_j, d_j}$ is the j-th hyperplane of the alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$ determined by L, then the j-th hyperplane of alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ + \lambda$ determined by $\bar{L}$ is $H_{\beta_{l+1-j}, \langle \lambda, \beta_{l+1-j}^\vee \rangle - d_{l+1-j}}$. If L is reduced, then $\bar{L}$ is also reduced.*



$$\begin{aligned}
i_1 &= 0, \quad \beta_1 = \alpha_0 = -(\alpha_1+\alpha_2)\\
i_2 &= 1, \quad \beta_2 = \bar{s}_0(\alpha_1) = -\alpha_2\\
i_3 &= 0, \quad \beta_3 = \bar{s}_0\bar{s}_1(\alpha_0) = \alpha_1\\
i_4 &= 1, \quad \beta_4 = \bar{s}_0\bar{s}_1\bar{s}_0(\alpha_1) = \alpha_1+\alpha_2\\
i_5 &= 2, \quad \beta_5 = \bar{s}_0\bar{s}_1\bar{s}_0\bar{s}_1(\alpha_2) = \alpha_1\\
i_6 &= 1, \quad \beta_6 = \bar{s}_0\bar{s}_1\bar{s}_0\bar{s}_1\bar{s}_2(\alpha_1) = -\alpha_2
\end{aligned}$$

$$\begin{aligned}
&s_0 = \hat{r}_{H_{-\beta_1,1}}\\
&s_0s_1s_0 = \hat{r}_{H_{-\beta_2,1}}\\
&(s_0s_1)s_0(s_1s_0) = s_1 = \hat{r}_{H_{-\beta_3,0}}\\
&(s_0s_1s_0)s_1(s_0s_1s_0) = s_0 = \hat{r}_{H_{-\beta_4,-1}}\\
&(s_0s_1s_0s_1)s_2(s_1s_0s_1s_0) = \hat{r}_{H_{-\beta_5,1}}\\
&(s_0s_1s_0s_1s_2)s_1(s_2s_1s_0s_1s_0) = \hat{r}_{H_{-\beta_6,2}}
\end{aligned}$$

Figure 1. To Example 3.7.

3.2. ***λ-chain formulae.*** Next we state the main theorem of this section. For any root hyperplane $h = H_{\beta,k}$, let r_h denote the reflection along the hyperplane $H_{\beta,0}$, and $\hat{r}_h$ be the reflection along h.

Assume λ is an integral weight and fix a reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$, which corresponds to an alcove walk from $A_\circ$ to $A_\circ - \lambda$, with separating hyperplanes $H_{-\beta_j,d_j} =: h_j$.

For a subset $J = \{j_1 < j_2 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$, define the relation

$$u \xrightarrow{J_>} w \quad \overset{\text{def.}}{\Longleftrightarrow} \quad u < ur_{h_{j_t}} < ur_{h_{j_t}} r_{h_{j_{t-1}}} < \cdots < ur_{h_{j_t}} \cdots r_{h_{j_1}} = w,$$

and let

$$\hat{r}_{J_<} := \hat{r}_{h_{j_1}} \cdots \hat{r}_{h_{j_t}}. \tag{16}$$

Hence, $w \overset{J}{\Longrightarrow} u$ from (14) is equivalent to $u \xrightarrow{J_>} w$. Therefore, Corollary 2.9 can be restated as follows.

Theorem 3.9. *In the above setting,*

$$c_{u,\mu}^{w,-\lambda} = \sum (-1)^{n(J)} (1-q)^{|J|} q^{\frac{1}{2}(\ell(u)-\ell(w)-|J|)}, \tag{17}$$

where the sum is over subsets $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J_>} w$ and $w\hat{r}_{J_<}(-\lambda) = \mu$, and where $n(J) := \#\{j \in J \mid \beta_j < 0\}$.

Proof. In the summation in Corollary 2.9, we let $J \subset \{1, 2, \dots, l\}$ be the indices of the hyperplanes in $\mathcal{M}$. Then by Remark 3.2, $f^+(\mathcal{M}) = n(J)$. This finishes the proof. □

Using the transformation between a λ-chain and a $(-\lambda)$-chain, we can get rid of the negative sign in front of λ in Theorem 3.9 as follows.

First of all, $(-\beta_l, -\beta_{l-1}, \dots, -\beta_1)$ is a $(-\lambda)$-chain, which corresponds to an alcove walk from $A_\circ$ to $A_\circ + \lambda$, with the j-th separating hyperplane being

$$h'_j := H_{\beta_{l+1-j}, \langle \lambda, \beta_{l+1-j}^\vee \rangle - d_{l+1-j}}. \tag{18}$$

Then $r_{h'_j} = r_{h_{l+1-j}}$. Let $\tilde{r}_{h_{l+1-j}} := \hat{r}_{h'_j}$ be the reflection along h'_j. Define

$$u \xrightarrow{J_<} w \quad \overset{\text{def.}}{\Longleftrightarrow} \quad u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} \cdots r_{h_{j_t}} = w$$

and

$$\tilde{r}_{J_>} := \tilde{r}_{h_{j_t}} \cdots \tilde{r}_{h_{j_1}}. \tag{19}$$

Theorem 3.10. $$c^{w,\lambda}_{u,\mu} = \sum (-1)^{n(J)} (q-1)^{|J|} q^{\frac{1}{2}(\ell(u)-\ell(w)-|J|)}, \tag{20}$$

where the sum is over subsets $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J_<} w$ and $w\tilde{r}_{J_>}(\lambda) = \mu$, and where $n(J) := \#\{j \in J \mid \beta_j < 0\}$.

Proof. Applying Theorem 3.9 to the $(-\lambda)$-chain $(-\beta_l, -\beta_{l-1}, \dots, -\beta_1)$, we get

$$c^{w,\lambda}_{u,\mu} = \sum (-1)^{\#\{j_i \mid -\beta_{l+1-j_i} < 0\}} (1-q)^t q^{\frac{1}{2}(\ell(u)-\ell(w)-t)},$$

where the summation is over subsets $J' := \{1 \le j_1 < j_2 < \cdots < j_t \le l\}$ such that

$$u < ur_{h'_{j_t}} < \cdots < ur_{h'_{j_t}} \cdots r_{h'_{j_1}} = w \tag{21}$$

and

$$w\hat{r}_{h'_{j_1}} \cdots \hat{r}_{h'_{j_t}}(\lambda) = \mu. \tag{22}$$

Let

$$J := \{1 \le l+1-j_t < l+1-j_{t-1} < \cdots < l+1-j_1 \le l\}.$$

Then

$$(-1)^{\#\{j_i \mid -\beta_{l+1-j_i} < 0\}} (1-q)^t = (-1)^{n(J)} (q-1)^t,$$

where $n(J)$ is defined in Theorem 3.9. Since $r_{h'_j} = r_{h_{l+1-j}}$, condition (21) is equivalent to $u \xrightarrow{J_<} w$. On the other hand, condition (22) is equivalent to $w\tilde{r}_{J_>}(\lambda) = \mu$ as $\tilde{r}_{h_{l+1-j}} = \hat{r}_{h'_j}$. This finishes the proof. □

For further use, we also record the following technical result, which will allow us to rewrite the elements $w\hat{r}_{J_<}(-\lambda)$ from Theorem 3.9 and $w\tilde{r}_{J_<}$ from Theorem 3.10.

Proposition 3.11. *Consider a λ-chain $\beta_1, \dots, \beta_l$. For any subsequence $J = \{j_1 < j_2 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$, we have*

$$-\hat{r}_{J_<}(-\lambda) = r_J\, \tilde{r}_{J_>}(\lambda), \quad \textit{where } r_J = r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}.$$

Proof. Let $m_j = \langle \lambda, \beta_j^\vee \rangle$ $(1 \le j \le l)$. By induction on $|J|$, it is easy to show that

$$\begin{aligned}
\hat{r}_{h_{j_1}} \hat{r}_{h_{j_2}} \cdots \hat{r}_{h_{j_t}}(-\lambda) &= -\lambda + (m_{j_1} - d_{j_1})\beta_{j_1} + (m_{j_2} - d_{j_2}) r_{h_{j_1}} \beta_{j_2} + \cdots + (m_{j_t} - d_{j_t}) r_{j_1} r_{h_{j_2}} \cdots r_{h_{j_{t-1}}} \beta_{j_t},\\
\tilde{r}_{h_{j_t}} \tilde{r}_{h_{j_{t-1}}} \cdots \tilde{r}_{h_{j_1}}(\lambda) &= \lambda - d_{j_t}\beta_{j_t} - d_{j_{t-1}} r_{h_{j_t}} \beta_{j_{t-1}} - \cdots - d_{j_1} r_{h_{j_t}} r_{h_{j_{t-1}}} \cdots r_{h_{j_2}} \beta_{j_1},\\
r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}(\lambda) &= \lambda - m_{j_1}\beta_{j_1} - m_{j_2} r_{h_{j_1}} \beta_{j_2} - \cdots - m_{j_t} r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_{t-1}}} \beta_{j_t}.
\end{aligned}$$

As $r_J = r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}$ is a linear transformation, we get

$$r_J \tilde{r}_{J_>}(\lambda) = r_J(\lambda) + d_{j_t} r_{h_{j_1}} \cdots r_{h_{j_{t-1}}} \beta_{j_t} + d_{j_{t-1}} r_{h_{j_1}} \cdots r_{h_{j_{t-2}}} \beta_{j_{t-1}} + \cdots + d_{j_1}\beta_{j_1} = -\hat{r}_{J_<}(-\lambda). \qquad \square$$

Remark 3.12. With the notation as above, the condition that $u \xrightarrow{J_>} w$ in Theorem 3.9 implies that $ur_J^{-1} = w$, thus by Proposition 3.11,

$$w\hat{r}_{J_<}(-\lambda) = -u\tilde{r}_{J_>}(\lambda).$$

Similarly, in Theorem 3.10, we have that $w = ur_J$, thus

$$w\tilde{r}_{J_>}(\lambda) = -u\hat{r}_{J_<}(-\lambda).$$

This leads to alternative formulae in Theorems 3.9 and 3.10.

Appendix B includes a fully worked out example illustrating calculations of some coefficients $c_{u,\mu}^{w,\lambda}$ utilizing Theorem 3.9 and Theorem 3.10.

4. Motivic Chern classes of Schubert cells

We recall basic properties of the motivic Chern classes of the Schubert cells in the (partial) flag varieties.

4.1. ***Definition of the Motivic Chern classes.*** Let X be a quasiprojective complex algebraic variety, and let $G_0(var/X)$ be the (relative) Grothendieck group of varieties over X. It consists of isomorphism classes of morphisms $[f : Z \to X]$ modulo the scissor relations. Brasselet, Schürmann and Yokura [BSY10] defined the *motivic Chern transformation* $MC_y : G_0(var/X) \to K(X)[y]$ with values in the K-theory group of coherent sheaves in X to which one adjoins a formal variable y. The transformation MC_y is a group homomorphism, it is functorial with respect to proper push-forwards, and if X is smooth it satisfies the normalization condition

$$MC_y[\mathrm{id}_X : X \to X] = \sum [\textstyle\bigwedge^j T_X^*] y^j.$$

Here $\left[\bigwedge^j T_X^*\right]$ is the K-theory class of the bundle of degree j differential forms on X. As explained in [BSY10], the motivic Chern class is related by a Hirzebruch–Riemann–Roch type statement to the Chern–Schwartz–MacPherson (CSM) class in the homology of X; see Section A.2.

There is also an equivariant version of the motivic Chern class transformation, which uses equivariant varieties and morphisms, and has values in the suitable equivariant K-theory group. Its definition was given in [AMSS24a; FRW21], following closely the approach of [BSY10].

Assume X is smooth and there is a torus T acting on X. Let $K_T(X)$ denote the equivariant K-theory of X, see [CG10]. If X is a point, $K_T(pt) = K^0(\mathrm{Rep}(T)) = \mathbb{Z}[T]$. For any $\mathcal{F} \in K_T(X)$, let

$$\chi_T(X, \mathcal{F}) := \sum_i (-1)^i H^i(X, \mathcal{F}) \in K_T(pt).$$

Let $\langle -, - \rangle$ be the nondegenerate pairing on $K_T(X)$ defined by

$$\langle \mathcal{F}, \mathcal{G} \rangle = \chi_T(X, \mathcal{F} \otimes \mathcal{G}) \in K_T(pt),$$

where $\mathcal{F}, \mathcal{G} \in K_T(X)$. For a vector bundle E, the λ_y-class of E is the class

$$\lambda_y(E) := \sum_k [\textstyle\bigwedge^k E] y^k \in K_T(X)[y].$$

The λ_y-class is multiplicative; i.e., for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles there is an equality $\lambda_y(E) = \lambda_y(E_1)\lambda_y(E_2)$ as elements in $K_T(X)[y]$.

Recall the (relative) motivic Grothendieck group $G_0^T(var/X)$ of varieties over X is the free abelian group generated by the isomorphism classes $[f : Z \to X]$ where Z is a quasiprojective T-variety and $f : Z \to X$ is a T-equivariant morphism modulo the usual additivity relations

$$[f : Z \to X] = [f : U \to X] + [f : Z \setminus U \to X]$$

for $U \subset Z$ an open T-invariant subvariety. For any equivariant morphism $f : X \to Y$ of quasiprojective T-varieties there are well defined push-forwards $f_! : G_0^T(var/X) \to G_0^T(var/Y)$ given by composition. The equivariant motivic Chern class is defined by the following theorem.

Theorem 4.1 [AMSS24a; FRW21]. *Let X be a quasiprojective, nonsingular, complex algebraic variety with an action of the torus T. There exists a unique natural transformation $MC_y : G_0^T(var/X) \to K_T(X)[y]$ satisfying the following properties:*

(1) *It is functorial with respect to T-equivariant proper morphisms of nonsingular, quasiprojective varieties.*

(2) *It satisfies the normalization condition*

$$MC_y[\mathrm{id}_X : X \to X] = \lambda_y(T_X^*) = \sum y^i [\textstyle\bigwedge^i T_X^*]_T \in K_T(X)[y].$$

The transformation MC_y satisfies a Verdier–Riemann–Roch (VRR) formula: for any smooth, T-equivariant morphism $\pi : X \to Y$ of quasiprojective and nonsingular algebraic varieties, and any $[f : Z \to Y] \in G_0^T(var/Y)$, the following holds:

$$\lambda_y(T_\pi^*) \cap \pi^* MC_y[f : Z \to Y] = MC_y[\pi^* f : Z \times_Y X \to X].$$

Define the Grothendieck–Serre dual operator $\mathcal{D} : K_T(X) \to K_T(X)$ by setting, for any $[\mathcal{F}] \in K_T(X)$,

$$\mathcal{D}[\mathcal{F}] := [RHom(\mathcal{F}, \omega_X^\bullet)] := [\omega_X^\bullet \otimes \mathcal{F}^\vee] \in K_T(X), \tag{23}$$

where $\omega_X^\bullet \simeq \omega_X[\dim X]$ is the (equivariant) dualizing complex of X (the canonical bundle ω_X shifted by dimension). The class $[\mathcal{F}^\vee]$ is obtained by taking an equivariant resolution of $\mathcal{F}$ by vector bundles, and then taking duals. Extend the operators $\mathcal{D}$ and $(-)^\vee$ to $K_T(X)[y^{\pm 1}]$ by sending $y \mapsto y^{-1}$. We also let $(-)^\vee$ denote the operator on $K_T(pt)[y^{\pm 1}]$, which sends e^λ to $e^{-\lambda}$ and y to y^{-1}.

Definition 4.2. Assume $\Omega \hookrightarrow X$ is a T-stable subvariety.

(1) The *motivic Chern* (MC) class of Ω is

$$MC_y(\Omega) := MC_y[\Omega \hookrightarrow X] \in K_T(X)[y].$$

(2) If Ω is pure-dimensional, the *Segre motivic Chern class* $SMC_y(\Omega) \in K_T(X)[\![y]\!]$ is the class

$$SMC_y(\Omega) := (-y)^{\dim \Omega} \frac{\mathcal{D}(MC_y(\Omega))}{\lambda_y(T_X^*)} \in K_T(X)[\![y]\!].$$

4.2. *The complete flag variety case.* Both $\{MC_y(X(w)^\circ) \mid w \in W\}$ and $\{SMC_y(Y(w)^\circ) \mid w \in W\}$ are bases for the localized equivariant K-theory

$$K_T(G/B)[\![y]\!]_{\mathrm{loc}} := K_T(G/B)[\![y]\!] \otimes_{K_T(pt)} \mathrm{Frac}\, K_T(pt).$$

These classes can be calculated recursively using the Demazure–Lusztig operators as follows.

The left multiplication action of G on G/B induces a left Weyl group action on $K_T(G/B)$. For any $w \in W$, let w^L denote this action.

Definition 4.3 [MNS22b, Section 5.3]. For any simple reflection s_i, we define the following left Demazure–Lusztig operator on $K_T(G/B)_{\mathrm{loc}}$:

$$\mathcal{T}_i^L := \frac{1 + ye^{-\alpha_i}}{1 - e^{-\alpha_i}} s_i^L - \frac{1+y}{1 - e^{-\alpha_i}}. \tag{24}$$

Lemma 4.4 [MNS22b]. *The operators* $\mathcal{T}_i^L$ *satisfy the braid relation and the quadratic relation*

$$(\mathcal{T}_i^L + 1)(\mathcal{T}_i^L + y) = 0.$$

Moreover, they commute with the operators of tensoring by elements in $K_G(G/B)$.

The following lemma is easy to verify.

Lemma 4.5. *The map* Ψ *defined by*

$$\Psi(T_i) = \mathcal{T}_i^L \quad \textit{and} \quad \Psi(X^\lambda) = e^\lambda,$$

where $e^\lambda \in K_T(\mathrm{pt})$ *acts on* $K_T(G/B)_{\mathrm{loc}}$ *by multiplication, induces an action of the affine Hecke algebra* $\mathbb{H}$ *(with* $q = -y$*) on* $K_T(G/B)_{\mathrm{loc}}$.

Theorem 4.6. (1) [MNS22b, Theorem 7.6] *For* $w \in W$ *and a simple root* α_i,

$$\mathcal{T}_i^L(MC_y(X(w)^\circ)) = \begin{cases} MC_y(X(s_iw)^\circ) & \textit{if } s_iw > w, \\ -(y+1)MC_y(X(w)^\circ) - yMC_y(X(s_iw)^\circ) & \textit{if } s_iw < w. \end{cases}$$

In particular,

$$MC_y(X(w)^\circ) = \mathcal{T}_w^L([\mathcal{O}_{X(\mathrm{id})}]).$$

(2) [MNS22b, Theorem 7.1] *For any* $w, u \in W$,

$$\langle MC_y(X(w)^\circ), SMC_y(Y(u)^\circ)\rangle = \delta_{u,w}.$$

Remark 4.7. (1) By [AMSS24b, Theorem 5.1 and Corollary 5.3],

$$MC_0(X(w)^\circ) = [\mathcal{O}_{X(w)}(-\partial X(w))] \quad \text{and} \quad SMC_0(Y(w)^\circ) = [\mathcal{O}_{Y(w)}],$$

where $\partial X(w) := X(w) \setminus X(w)^\circ$. Thus, the duality in the second part of the theorem reduces to the classical fact

$$\langle [\mathcal{O}_{X(w)}(-\partial X(w))], [\mathcal{O}_{Y(u)}] \rangle = \delta_{u,w}.$$

(2) By definition, for any $w \in W$,

$$MC_y(Y(w)) = \sum_{u \geq w} MC_y(Y(u)^\circ).$$

Besides, by the linearity of the Grothendieck–Serre dual operator $\mathcal{D}$,

$$SMC_y(Y(w)) = \sum_{u \geq w} (-y)^{\ell(u)-\ell(w)} SMC_y(Y(u)^\circ).$$

Therefore,

$$SMC_0(Y(w)) = [\mathcal{O}_{Y(w)}].$$

4.3. ***The partial flag variety case.*** For any $w \in W$, let $\ell(wW_P)$ denote the length of the minimal length representative in wW_P. Let $\pi : G/B \to G/P$ be the natural projection. It is proved in [AMSS24a, Remark 5.7] that

$$\pi_* MC_y(X(w)^\circ) = (-y)^{\ell(w)-\ell(wW_P)} MC_y(X(wW_P)^\circ). \tag{25}$$

In particular, if w is a minimal length representative, then $\pi_* MC_y(X(w)^\circ) = MC_y(X(wW_P)^\circ)$; this also follows directly from the functoriality property of the motivic classes.

Proposition 4.8. (1) [MNS22b, Proposition 6.3] *Let $\Omega \subset G/P$ be a T-stable subvariety of G/P and $\pi : G/B \to G/P$ be the projection. Then*

$$\pi^* SMC_y(\Omega) = SMC_y(\pi^{-1}\Omega) \in K_T(G/B)[\![y]\!].$$

(2) [MNS22b, Proposition 7.2] *Let $u, w \in W^P$. The Segre motivic classes are dual to the motivic Chern classes for any G/P; i.e.,*

$$\langle MC_y(X(wW_P)^\circ), SMC_y(Y(uW_P)^\circ) \rangle = \delta_{u,w}.$$

Remark 4.9. By the first property in the proposition, for any $w \in W^P$,

$$\pi^*(SMC_y(Y(wW_P))) = SMC_y(Y(w)).$$

Letting $y = 0$, we get

$$\pi^*(SMC_0(Y(wW_P))) = SMC_0(Y(w)) = [\mathcal{O}_{Y(w)}],$$

where the second equality follows from Remark 4.7. By the definition of the SMC class,

$$SMC_y(Y(wW_P)) = \sum_{u \in W^P, u \geq w} (-y)^{\ell(u)-\ell(w)} SMC_y(Y(uW_P)^\circ). \tag{26}$$

Hence,

$$\pi^*(SMC_0(Y(wW_P)^\circ)) = \pi^*(SMC_0(Y(wW_P))) = [\mathcal{O}_{Y(w)}].$$

Since $\pi^*([\mathcal{O}_{Y(wW_P)}]) = [\mathcal{O}_{Y(w)}]$ and π^* is injective, we get

$$SMC_0(Y(wW_P)^\circ) = SMC_0(Y(wW_P)) = [\mathcal{O}_{Y(wW_P)}].$$

Combining with the second part of the proposition, we get

$$MC_0(X(wW_P)^\circ) = [\mathcal{O}_{X(wW_P)}(-\partial X(wW_P))].$$

This can also be proved using Remark 4.7(1), equation (25), and the fact that the pushforward of an ideal sheaf is an ideal sheaf, by [Bri02].

5. Chevalley formulae for the motivic Chern classes

In this section we obtain several Chevalley formulae for the motivic Chern classes, in terms of alcove walks, λ-chains, and certain operators. The main technique is to reinterpret formulae from Hecke algebras such as Theorem 3.9 in terms of multiplications of motivic Chern classes by line bundles. Our main results are Theorems 5.5 and 5.12. In Section 5.3 we discuss several positivity properties and conjectures of the Chevalley coefficients. Finally, in Section 5.5 we discuss parabolic Chevalley formulae.

5.1. ***Chevalley coefficients.*** Consider a torus weight $\lambda \in X^*(T)$ and $u, w \in W$. The Chevalley coefficient $C^w_{u,\lambda}$ is defined by the following formula:

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = \sum_{u \le w} C^w_{u,\lambda} MC_y(X(u)^\circ). \tag{27}$$

Note that for any simple reflection s_i there is a short exact sequence of equivariant sheaves

$$0 \to \mathcal{L}_{\varpi_i} \otimes \mathbb{C}_{-w_0(\varpi_i)} \to \mathcal{O}_{G/B} \to \mathcal{O}_{X(w_0 s_i)} \to 0$$

with ϖ_i the fundamental weight, see [BM15, §8], for example. Therefore,

$$[\mathcal{O}_{X(w_0 s_i)}] = 1 - e^{-w_0(\omega_i)} \mathcal{L}_{\varpi_i} \in K_T(G/B),$$

and the coefficients from (27) for $\lambda = \varpi_i$ also recover the multiplication of $[\mathcal{O}_{X(w_0 s_i)}]$ with the MC classes of the Schubert cells.

The coefficients $C^w_{u,\lambda}$ also arise from Chevalley formulae involving Segre motivic classes:

Lemma 5.1.
$$\mathcal{L}_\lambda \otimes SMC_y(Y(u)^\circ) = \sum_{w \ge u} C^w_{u,\lambda} SMC_y(Y(w)^\circ). \tag{28}$$

Proof. This follows from the (Poincaré) duality in Theorem 4.6(2), as the Chevalley coefficients in (27) and (28) are given by

$$C^w_{u,\lambda} = \langle \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ), SMC_y(Y(u)^\circ)\rangle. \qquad \square$$

We now relate the Chevalley coefficients above to the coefficients $c^{w,\lambda}_{u,\mu}$ from (7) in the Hecke algebra.

Theorem 5.2. *Let λ be any weight in $X^*(T)$. The following Chevalley formula holds in $K_T(G/B)[y]$:*

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ) = \sum_{\mu \in X^*(T), u \in W} y^{\ell(w)-\ell(u)} e^{-\mu} c_{u,\mu}^{w,\lambda}|_{q=-y} MC_y(X(u)^\circ).$$

In particular, the following equation holds for the Chevalley coefficients in (27):

$$C_{u,-\lambda}^{w} = \sum_{\mu \in X^*(T)} y^{\ell(w)-\ell(u)} e^{-\mu} c_{u,\mu}^{w,\lambda}|_{q=-y}.$$

From (27), the Chevalley coefficients $C_{u,\lambda}^{v}$ belong to a localization $K_T(pt)[y]$ which allows division by $1+e^{\alpha}y$ for any root α. However, the expansion from (7) implies that the coefficients $(-q)^{\ell(w)-\ell(v)} c_{u,\mu}^{w,\lambda}$ are *polynomials* in $\mathbb{Z}[q]$. Then it follows that $C_{u,\lambda}^{v}$ are in fact polynomials in $K_T(pt)[y]$. This will be seen explicitly in Theorem 5.4.

Proof of Theorem 5.2. Applying the map Ψ in Lemma 4.5 to (7), we get

$$\mathcal{T}_w^L e^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} y^{\ell(w)-\ell(u)} e^{-\mu} c_{u,\mu}^{w,\lambda}|_{q=-y} \mathcal{T}_u^L \in \mathrm{End}_{\mathbb{C}}\, K_T(G/B)[y].$$

The theorem follows by applying both sides to $[\mathcal{O}_{X(\mathrm{id})}]$, and utilizing that

$$\mathcal{T}_w^L e^{-\lambda}([\mathcal{O}_{X(\mathrm{id})}]) = \mathcal{T}_w^L(\mathcal{L}_{-\lambda} \otimes [\mathcal{O}_{X(\mathrm{id})}]) = \mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ).$$

Here the first equality follows from $\mathcal{L}_{-\lambda} \otimes [\mathcal{O}_{X(\mathrm{id})}] = e^{-\lambda}[\mathcal{O}_{X(\mathrm{id})}]$, while the second one follows from Lemma 4.4 and Theorem 4.6. □

Remark 5.3. This argument can be generalized to the case when the line bundle $\mathcal{L}_\lambda$ is replaced by any homogeneous bundle $\mathcal{V} = G \times^B V \to G/B$ associated to a B-representation of V. If the character of V is $ch(V) = \sum_\lambda a_\lambda e^\lambda$, then a localization argument shows that the class of $\mathcal{V}$ in $K_T(G/B)$ is equal to

$$[\mathcal{V}] = \sum a_\lambda \mathcal{L}_\lambda.$$

It follows that, for any $w \in W$,

$$\mathcal{V} \otimes MC(X(w)^\circ) = \sum_\lambda a_\lambda \mathcal{L}_\lambda \otimes MC(X(w)^\circ) = \sum_u \sum_\lambda a_\lambda C_{u,\lambda}^{w} MC(X(u)^\circ).$$

We illustrate this for $G/B = \mathrm{Fl}(n)$, the complete flag manifold. This is equipped with the tautological sequence $\mathcal{F}_0 = 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathbb{C}^n$. For $1 \le i \le n-1$ define $X_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ regarded in $K_T(G/B)$. Then

$$\textstyle\bigwedge^j \mathcal{F}_i = e_j(X_1, \ldots, X_i), \quad Sym^j \mathcal{F}_i = h_j(X_1, \ldots, X_i)$$

where e_j and h_j denote the elementary symmetric function, respectively the complete homogeneous symmetric function. Note that if ϖ_i denotes the ith fundamental weight, then $X_i = \mathcal{L}_{\varpi_i - \varpi_{i-1}}$ for $1 \le i \le n-1$, with the convention that $\varpi_0 = 0$. Theorem 5.2 gives a formula for the multiplication by monomials $X_1^{a_1} \cdot \cdots \cdot X_{n-1}^{a_{n-1}}$, which in turn gives formulae to multiply by $e_j(X_1, \ldots, X_i)$ and $h_j(X_1, \ldots, X_i)$.

In the next section, we give explicit formulae for the Chevalley coefficients $C^u_{\lambda,w}$ based on the formulae for the Hecke algebra coefficients $c^{w,\lambda}_{u,\mu}$.

5.2. *Chevalley formulae via alcove walks and the λ-chains.* Let us recall the setting of Corollary 2.9. For $\lambda \in X^*(T)$, choose a minimal length alcove walk $p_{v_\lambda} = c^{\epsilon_1}_{j_1} c^{\epsilon_2}_{j_2} \cdots c^{\epsilon_s}_{j_s}$ from $A_\circ$ to $A_\circ + \lambda$, and let $\mathcal{H} = \{h_1, h_2, \dots, h_s\}$ be the ordered sequence of hyperplanes defined by the walls of alcoves crossed by p_λ. Define $\beta_i \in R^{\epsilon_i}$ and $k_i \in \mathbb{Z}$ $(1 \le i \le s)$ by the condition $h_i = H_{\beta_i,k_i}$. Combining Corollary 2.9 and Theorem 5.2, we get the following formula.

Theorem 5.4 (alcove walk formula for the Chevalley coefficients).

$$C^w_{u,-\lambda} = \sum_{\mathcal{M}} (-1)^{f^+(\mathcal{M})} (-1-y)^{|\mathcal{M}|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|\mathcal{M}|)} e^{-w\hat{r}_{\mathcal{M}}(\lambda)}$$

where the sum is over ordered subsets $\mathcal{M} \subset \mathcal{H}$ such that $w \overset{\mathcal{M}}{\Longrightarrow} u = wr_{\mathcal{M}}$.

We now recall the setup of Theorem 3.9. Assume λ is an integral weight and fix a reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$, which corresponds to an alcove walk from $A_\circ$ to $A_\circ - \lambda$, with separating hyperplanes $h_j := H_{-\beta_j,d_j}$.

Combining Theorems 3.9, 3.10, and 5.2, we get the following formulae.

Theorem 5.5 (λ-chain formula for the Chevalley coefficients).

$$C^w_{u,-\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (1+y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\tilde{r}_{J_>}(\lambda)}, \tag{29}$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} r_{h_{j_2}} \cdot \cdots \cdot r_{h_{j_t}} = w$ and $\tilde{r}_{J_>}$ is defined in (19). *Furthermore,*

$$C^w_{u,\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (-1-y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\hat{r}_{J_<}(-\lambda)}, \tag{30}$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < ur_{h_{j_t}} < ur_{h_{j_t}} r_{h_{j_{t-1}}} < \cdots < ur_{h_{j_t}} \cdot \cdots \cdot r_{h_{j_1}} = w$ and $\hat{r}_{J_<}$ is defined in (16).

Remark 5.6. (1) It follows from Remark 3.12 that $-w\tilde{r}_{J_>}(\lambda) = u\hat{r}_{J_<}(-\lambda)$ in (29), and that $-w\hat{r}_{J_<}(-\lambda) = u\tilde{r}_{J_>}(\lambda)$ in (30), giving alternative ways to calculate these.

(2) By Remark 4.7(1), the MC and SMC classes specialize at $y = 0$ to the ideal sheaves and the structure sheaves, respectively. Under this specialization, and using that $-w\tilde{r}_{J_>}(\lambda) = u\hat{r}_{J_<}(-\lambda)$, equation (29) reduces to the equivariant K-theory Chevalley formula of Lenart–Postnikov [LP07, Theorem 6.1].

One can also consider the more general situation of Kac–Moody flag varieties defined in, say, [Kum02]. These are ind-varieties, and one can define the motivic Chern classes of the finite-dimensional Schubert cells using the ind-structure. There are analogues of the (left and right) Demazure operators, and the notion of λ-chains extends to this setting, by results of Lenart and Postnikov [LP08]. Note that for an infinite Weyl group W, a λ-chain may be an infinite sequence, i.e., $l = \infty$. But, for given $u \le w$, $t = |J|$

is finite, and the number of J which satisfies the condition in Theorem 5.5 is also finite. Based on these similarities to the finite case, we expect the following conjecture to hold.

Conjecture 1. *For a Kac–Moody Weyl group W and a dominant integral weight λ, the analogues of the equations* (29) *and* (30) *hold.*

5.3. *Miscellaneous.* In this section we discuss positivity properties of the coefficients from the Chevalley formula, and some special properties of the multiplication by line bundles given by minuscule weights. We start with the following consequence of Theorem 5.5.

Proposition 5.7. *Let λ be a dominant weight and set $q = -y$. Then*:

(1) *$C^w_{u,\lambda}$ may be written as a combination of terms of the form $e^\mu q^a (q-1)^b$ with nonnegative integer coefficients.*

(2) *$C^w_{u,-\lambda}$ may be written as a combination of terms of the form $(-1)^b e^\mu q^a (q-1)^b$ with nonnegative integer coefficients.*

In both situations, b has the same parity as $\ell(w) - \ell(u)$.

Proof. Both statements follow from Theorem 5.5, using that for a reduced λ-chain with λ dominant we have $n(J) = 0$, and that $\ell(w) - \ell(u) - |J|$ is an even integer. □

Example 5.8. Consider type A_2, with $\lambda = 2\varpi_1 + \varpi_2$, and with the λ-chain from Appendix B. Take $w = s_2 s_1$. Then from Theorem 5.5 we get, with $q = -y$,

$$\begin{aligned}\mathcal{L}_\lambda \otimes MC_{-q}(X(s_2s_1)^\circ) = e^{\varpi_1 - 3\varpi_2} MC_{-q}(X(s_2s_1)^\circ)\\ + (q-1)(e^{-\varpi_2} + e^{-\varpi_1+\varpi_2} + e^{-2\varpi_1+3\varpi_2}) MC_{-q}(X(s_1)^\circ)\\ + (q-1)(e^{2\varpi_1-2\varpi_2} + e^{3\varpi_1-\varpi_2}) MC_{-q}(X(s_2)^\circ)\\ + (q-1)^2(e^{2\varpi_1+\varpi_2} + e^{2\varpi_2} + e^{\varpi_1}) MC_{-q}(X(\mathrm{id})^\circ),\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{-\lambda} \otimes MC_{-q}(X(s_2s_1)^\circ) = e^{-\varpi_1 + 3\varpi_2} MC_{-q}(X(s_2s_1)^\circ)\\ - (q-1)(e^{\varpi_1-\varpi_2} + e^{\varpi_2} + e^{-\varpi_1+3\varpi_2}) MC_{-q}(X(s_1)^\circ)\\ - (q-1)(e^{-2\varpi_1+2\varpi_2} + e^{-\varpi_1+3\varpi_2}) MC_{-q}(X(s_2)^\circ)\\ + (q-1)^2(e^{-\varpi_1} + e^{\varpi_1-\varpi_2} + e^{-2\varpi_1+2\varpi_2} + e^{\varpi_2} + e^{-\varpi_1+3\varpi_2}) MC_{-q}(X(\mathrm{id})^\circ).\end{aligned}$$

We now investigate the special multiplication by $\mathcal{L}_{\pm\varpi_i}$ in the case of minuscule fundamental weights. In this case the coefficients have a particularly pleasing factorization.

Lemma 5.9. *If $\lambda = \varpi_i$ is a minuscule weight, then for any $u, w \in W$,*

$$C^w_{u,\lambda} = e^{u(\lambda)} P^w_{u,\lambda}(y) \quad \textit{and} \quad C^w_{u,-\lambda} = (-1)^{\ell(w)-\ell(u)} e^{-w(\lambda)} P^{w_0 u}_{w_0 w,\lambda}(y),$$

where $P^w_{u,\lambda}(y) \in \mathbb{Z}[y]$. These polynomials are palindromic, that is, they satisfy

$$P^w_{u,\lambda}(y^{-1}) \cdot y^{\ell(w)-\ell(u)} = P^w_{u,\lambda}(y).$$

Proof. Since $\lambda = \varpi_i$ is minuscule, for any reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$, the separating hyperplanes must be of the form $h_j := H_{-\beta_j,0}$, thus $\hat{r}_{h_j} = r_{h_{j_1}}$. Therefore the first equality follows from (30), since $-w\hat{r}_{J_<}(-\lambda) = -wr_{J_<}(-\lambda) = u(\lambda)$. The second equality follows this and from the star duality in Proposition 6.2 below. The palindromic property follows from Proposition 6.5(a) below. □

Remark 5.10. In an earlier arXiv version of this article, we conjectured that if $\lambda = \varpi_i$ is a minuscule weight, then the coefficients $C^w_{u,\lambda}(y)$ in the multiplication

$$MC(X(w)^\circ) \cdot \mathcal{L}_\lambda = \sum C^w_{u,\lambda}(y) MC(X(u)^\circ)$$

are polynomials with nonnegative coefficients. Since then, we have found counterexamples to this conjecture in Lie types D_6, E_6 and A_7.

The next example shows that even in the cases when a coefficient $C^w_{u,\lambda}(y)$ happens to have positive coefficients, cancellations may still occur in the formula (30), which calculates it.

Example 5.11. Consider type A_3, $\lambda = \varpi_2$, $w = s_1s_2s_3s_1s_2s_1$ and $u = s_3s_1$. An ϖ_2-chain is $\beta_1 = \alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$, $\beta_3 = \alpha_1 + \alpha_2$, $\beta_4 = \alpha_1 + \alpha_2 + \alpha_3$; see Example 3.6. We have two paths from u to w in (30):

- $J_1 = \{2, 3\}$ which gives $u < us_{\beta_3} < us_{\beta_3}s_{\beta_2} = w$, where $\ell(us_{\beta_3}) + 3 = \ell(w)$;
- $J_2 = \{1, 2, 3, 4\}$ which gives $u < us_{\beta_4} < us_{\beta_4}s_{\beta_3} < us_{\beta_4}s_{\beta_3}s_{\beta_2} < us_{\beta_4}s_{\beta_3}s_{\beta_2}s_{\beta_1} = w$.

The path J_1 gives coefficient $(-1-y) \times (-1-y)(-y)e^{u(\lambda)} = -y(y+1)^2 e^{u(\lambda)}$, and the path J_2 gives coefficient $(-1-y)^4 e^{u(\lambda)} = (y+1)^4 e^{u(\lambda)}$. Therefore

$$C^w_{u,\lambda} = e^{u(\lambda)}((y+1)^4 - y(y+1)^2) = e^{u(\lambda)}(y^2 + y + 1)(y+1)^2.$$

5.4. ***An operator formula.*** In this section we reformulate the λ-chain Chevalley formula from Theorem 5.5 via operators generalizing to motivic Chern classes similar ones from [LP07].

Let $h := (\rho, \theta^\vee) + 1$ be the Coxeter number, where $\rho := \sum_{i=1}^r \varpi_i$ and $\theta^\vee$ is the highest coroot. Let $\tilde{R}(T) := \mathbb{Z}[e^{\pm\varpi_1/h}, \dots, e^{\pm\varpi_r/h}]$, and let

$$\tilde{K}_T(G/B) := K_T(G/B)[y] \otimes_{K_T(\mathrm{pt})[y]} \mathrm{Frac}(\tilde{R}(T)[y]).$$

Then $\tilde{K}_T(G/B)$ has a basis over $\mathrm{Frac}(\tilde{R}(T)[y])$ given by the motivic Chern classes of the Schubert cells. Define $\mathrm{Frac}(\tilde{R}(T)[y])$-linear operators B_β ($\beta \in R^+$) and E^μ ($\mu \in X^*(T)$) on $\tilde{K}_T(G/B)$ by

$$B_\beta(MC_y(X(w)^\circ)) := \begin{cases} (-1-y)(-y)^{\frac{1}{2}(\ell(w)-\ell(ws_\beta)-1)} MC_y(X(ws_\beta)^\circ) & \text{if } ws_\beta < w, \\ 0 & \text{otherwise,} \end{cases}$$
$$E^\mu(MC_y(X(w)^\circ)) := e^{w(\mu)/h} MC_y(X(w)^\circ).$$

If $\beta \in R^-$, define $B_\beta := -B_{-\beta}$. Then

$$E^\mu E^{\mu'} = E^{\mu+\mu'} \quad \text{and} \quad B_\beta E^{s_\beta \mu} = E^\mu B_\beta.$$

Given a λ-chain $(\beta_1, \dots, \beta_l)$, define

$$R^{[\lambda]} := R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_1}, \quad \text{where} \quad R_\beta := E^\rho (E^\beta + B_\beta) E^{-\rho} = E^\beta + E^{(\rho,\beta^\vee)\beta} B_\beta.$$

Theorem 5.12 (operator Chevalley formula). *For any integral weight $\lambda \in X^*(T)$,*

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = R^{[\lambda]}(MC_y(X(w)^\circ)). \tag{31}$$

Remark 5.13. (1) The theorem implies that the definition of $R^{[\lambda]}$ does not depend on the choice of the λ-chain, which is equivalent to the Yang–Baxter equations [LP07, Definition 9.1] satisfied by the operators R_β.

(2) The formula analogous to (31) involving SMC is obtained by replacing the operator $R^{[\lambda]}$ by an operator defined via the adjoint operators of B_β and E^μ.

(3) Recall from Remark 4.7(1) that $MC_0(X(w)^\circ) = [\mathcal{O}_{X(w)}(-\partial X(w))]$. Specializing $y = 0$ in the theorem, we get a dual version of Lenart–Postnikov's formula [LP07, Theorem 13.1].

For the proof of the theorem, we need a result from [LP07]. Recall that $(\beta_1, \dots, \beta_l)$ is a λ-chain. The hyperplane $h_j := H_{-\beta_j, d_j}$ separates the alcoves A_{j-1} and A_j in the corresponding alcove path, and $\hat{r}_{h_j}$ is the reflection along h_j.

Lemma 5.14 [LP07, proof of Proposition 14.5]. *For any $1 \le j_1 < j_2 < \cdots < j_t \le l$,*

$$-\rho+\beta_1+\cdots+\beta_{j_1-1}+s_{\beta_{j_1}}(\beta_{j_1+1}+\cdots\beta_{j_2-1})+\cdots+s_{\beta_{j_1}}\cdots s_{\beta_{j_t}}(\beta_{j_t+1}+\cdots\beta_l+\rho) = -h\hat{r}_{j_1}\cdots\hat{r}_{j_t}(-\lambda).$$

Proof of Theorem 5.12. By definition,

$$R^{[\lambda]} = R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_1} = E^\rho (E^{\beta_l} + B_{\beta_l}) \cdots (E^{\beta_2} + B_{\beta_2})(E^{\beta_1} + B_{\beta_1}) E^{-\rho} = \sum_J R_J^{[\lambda]}.$$

Here $J = \{j_1 < j_2 < \cdots < j_t\}$ is a subset of $\{1, 2, \dots, l\}$, and $R_J^{[\lambda]}$ is the term that contains B_{β_j} if $j \in J$, and E^{β_j}, otherwise. Moving all the B-operators to the left, and using the relation $B_\beta E^{s_\beta \mu} = E^\mu B_\beta$ and Lemma 5.14, we get $R_J^{[\lambda]} = B_{\beta_{j_t}} \cdots B_{\beta_{j_1}} E^{-h\hat{r}_{j_1}\cdots\hat{r}_{j_t}(-\lambda)} = B_{\beta_{j_t}} \cdots B_{\beta_{j_1}} E^{-h\hat{r}_{J^<}(-\lambda)}$. The theorem follows from this and Theorem 5.5. ☐

5.5. *Parabolic case.* We now extend the Chevalley formula to the partial flag variety case G/P.

Theorem 5.15 (Chevalley formula for the G/P case). *Let $w \in W^P$ and $\lambda \in X^*(T)_P$. Then we have*

$$\mathcal{L}_\lambda \otimes MC_y(X(wW_P)^\circ) = \sum_{u \in W^P} \Bigl(\sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C^w_{v,\lambda} \Bigr) MC_y(X(uW_P)^\circ),$$

and

$$\mathcal{L}_\lambda \otimes SMC_y(Y(uW_P)^\circ) = \sum_{w \in W^P} \Bigl(\sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C^w_{v,\lambda} \Bigr) SMC_y(Y(wW_P)^\circ),$$

where $C^v_{u,\lambda}$ are the Chevalley coefficients for full flag G/B given explicitly in Theorem 5.5.

Proof. The first equality follows from (25) and (27), while the second equality follows from the first one and the duality in Proposition 4.8(2). □

Remark 5.16. In a paper in preparation we use this theorem to give a combinatorial Chevalley formula for minuscule flag varieties and a K-theoretic generalization of Nakada's colored hook formula. See also [FGSX24] for a Pieri formula for the motivic Chern classes of Schubert cells in the equivariant K-theory of Grassmannians.

6. Dualities of Chevalley coefficients

Recall the expansion from (27):

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = \sum_{u \le w} C^w_{u,\lambda} MC_y(X(u)^\circ).$$

In this section we state and prove several duality properties of the Chevalley coefficients $C^w_{u,\lambda}$. All these dualities have a geometric origin (Serre duality, star duality, Dynkin automorphisms) and we name the Chevalley dualities correspondingly.

Recall the Grothendieck–Serre operator $\mathcal{D}$ from (23), and the duality functor $(-)^\vee$ on $K_T(G/B)[y^{\pm 1}]$ and $K_T(\mathrm{pt})[y^{\pm 1}]$, which sends a vector bundle to its dual and y^i to y^{-i}. This induces the following property of the Chevalley coefficients in (27).

Proposition 6.1 (Serre duality). $C^w_{u,\lambda} = (-y)^{\ell(w)-\ell(u)} w_0(C^{w_0 u}_{w_0 w, -\lambda})^\vee.$

Proof. Since

$$SMC_y(Y(w)^\circ) = (-y)^{\dim Y(w)} \frac{\mathcal{D}(MC_y(Y(w)^\circ))}{\lambda_y(T^*(G/B))},$$

equation (28) can be written as

$$\mathcal{L}_\lambda \otimes \mathcal{D}(MC_y(Y(u)^\circ)) = \sum_{w \in W} C^w_{u,\lambda} (-y)^{\ell(u)-\ell(w)} \mathcal{D}(MC_y(Y(w)^\circ)).$$

Taking the dual $\mathcal{D}$ on both sides, we have

$$\mathcal{L}_{-\lambda} \otimes MC_y(Y(u)^\circ) = \sum_{w \in W} (C^w_{u,\lambda})^\vee (-y)^{\ell(w)-\ell(u)} MC_y(Y(w)^\circ).$$

Finally, applying the left w_0-action to this identity, we get

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w_0 u)^\circ) = \sum_{w \in W} w_0(C^w_{u,\lambda})^\vee (-y)^{\ell(w)-\ell(u)} MC_y(X(w_0 w)^\circ).$$

This finishes the proof of the theorem by comparison with (27). □

The star duality $*$ acts on $K_T(G/B)[y^{\pm 1}]$ and $K_T(pt)[y^{\pm 1}]$ by sending a vector bundle to its dual, and leaving y^i unchanged. Consider the composition

$$\iota := w_0 * : K_T(\mathrm{pt})[y^{\pm 1}] \to K_T(\mathrm{pt})[y^{\pm 1}].$$

Combining Theorem 9.1(a) and Remark 4.7 of [AMSS24b], we have

$$\mathbb{C}_{-\rho} \otimes \mathcal{L}_{-\rho} \otimes MC_y(X(w)^\circ) = (-1)^{\dim G/B - \ell(w)} \prod_{\alpha>0} (1 + ye^{-\alpha}) * \big(SMC_y(X(w)^\circ)\big). \tag{32}$$

Proposition 6.2 (star duality). $C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} \iota(C^{w_0u}_{w_0w,-\lambda})$.

Proof. Apply the star duality functor $*$ to (27), then use (32) to get

$$\mathcal{L}_{-\lambda} \otimes SMC_y(X(w)^\circ) = \sum_u (-1)^{\ell(w)-\ell(u)} * (C^w_{u,\lambda}) SMC_y(X(u)^\circ).$$

Applying the left w_0-action to the above equation and comparing with (28), we get the result. □

The map sending $\alpha_i \mapsto -w_0(\alpha_i)$ for every simple root α_i induces an automorphism on the Dynkin diagram, hence also on the flag variety G/B. This automorphism maps $X(w)^\circ$ to $X(w_0ww_0)^\circ$, and it induces a ring automorphism ϕ on $K_T(G/B)$, which sends $\mathcal{L}_\lambda$ to $\mathcal{L}_{-w_0\lambda}$, and twists the base ring $K_T(\mathrm{pt})$ by the map ι above.

Proposition 6.3 (Dynkin duality). $C^w_{u,\lambda} = \iota(C^{w_0ww_0}_{w_0uw_0,-w_0\lambda})$.

Proof. Applying the Dynkin automorphism ϕ to (27), we obtain

$$\mathcal{L}_{-w_0\lambda} \otimes MC_y(X(w_0ww_0)^\circ) = \sum_{u\in W} \iota(C^w_{u,\lambda}) MC_y(X(w_0uw_0)^\circ).$$

The claim follows from the definition of the coefficients $C^w_{u,\lambda}$. □

Combining Propositions 6.1 and 6.2 in one instance, and Propositions 6.2 and 6.3 in a second instance, we obtain:

Proposition 6.4 (1) (Serre duality + star duality).

$$(C^w_{u,\lambda})|_{y\to y^{-1}} \times y^{\ell(w)-\ell(u)} = C^w_{u,\lambda}.$$

(2) (star duality + Dynkin duality).

$$C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} C^{uw_0}_{ww_0,w_0\lambda}.$$

Using the λ-chain formula in Theorem 5.5, we can also give a direct combinatorial proof for the various duality identities in this section. The proofs are similar to [LP07, Theorem 8.6] and [LP07, Theorem 8.7].

Proposition 6.5. *Let λ be any integral weight and let and $w, u \in W$. Then*:

(a) $C^w_{u,\lambda} \in K_T(pt)[y]$ *and* $(C^w_{u,\lambda})|_{y\to y^{-1}} \times y^{\ell(w)-\ell(u)} = C^w_{u,\lambda}$; *in other words*, $C^w_{u,\lambda}$ *is palindromic as a polynomial in* y. (Compare Proposition 6.4(1).)

(b) $C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} C^{uw_0}_{ww_0,w_0\lambda}$. (Compare Proposition 6.4(2).)

(c) $C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} \iota\left(C^{w_ou}_{w_ow,-\lambda}\right)$. (Compare Proposition 6.2.)

(d) $C^w_{u,\lambda} = \iota\left(C^{w_0ww_0}_{w_0uw_0,-w_0\lambda}\right)$. (Compare Proposition 6.3).

Proof. Part (a) is straightforward from (30); part (d) follows from (b) and (c). Parts (b) and (c) follow from known properties of λ-chain, i.e., if $(\beta_1, \dots, \beta_l)$ is a λ-chain, then $(w_0\beta_l, \dots, w_0\beta_1)$ is a $(w_0\lambda)$-chain and $(-\beta_l, \dots, -\beta_1)$ is a $(-\lambda)$-chain. $\square$

Remark 6.6. Serre duality extends to any G/P. For $w, u \in W^P$ and $\lambda \in X^*(T)_P$ recall the expansion

$$\mathcal{L}_\lambda \otimes MC_y(X(wW_P)^\circ) = \sum_{u \in W^P} C^{w,P}_{u,\lambda} MC_y(X(uW_P)^\circ).$$

Then Theorem 5.15 can be rewritten as $C^{w,P}_{u,\lambda} = \sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C^w_{v,\lambda}$. Arguing as in the proof of Proposition 6.1, we obtain:

Proposition 6.7 (Serre duality on parabolic Chevalley coefficients).

$$C^{w,P}_{u,\lambda} = (-y)^{\ell(w)-\ell(u)} w_0 (C^{\overline{w_0u},P}_{\overline{w_0w},-\lambda})^\vee, \tag{33}$$

where $\overline{w_0w}$ and $\overline{w_0u} \in W^P$ are minimal coset representatives of w_0wW_P and w_0uW_P.

7. K-theoretic stable envelopes for $T^*(G/B)$

In this section we apply the Chevalley formula for motivic classes to calculate the transformation of stable envelopes in $T^*(G/B)$ under the change of arbitrary alcoves. For alcoves adjacent to the fundamental alcove, a formula for this transformation was obtained in [SZZ20, Theorem 5.4], see also [KW25].

7.1. ***Definition of the stable envelopes.*** The stable envelopes were defined by Maulik and Okounkov in their seminal work on quantum cohomology of Nakajima quiver varieties [MO19]. Later, this was generalized by Okounkov and his collaborators to K-theory and elliptic cohomology [Oko27; AO21]. We recall next the definition of the stable envelopes for $T^*(G/B)$.

The torus T acts by left multiplication on G/B. Hence, it induces a natural action on $T^*(G/B)$. There is also a natural dilation $\mathbb{C}^*$-action on the cotangent fibers by a character of q^{-1}. Throughout this section, we use $q^{1/2}$ to denote the standard representation of $\mathbb{C}^*$, so that $K_{T\times\mathbb{C}^*}(\mathrm{pt}) = K_T(\mathrm{pt})[q^{\pm 1/2}]$.

The definition of the stable envelopes depends on three parameters:

- a *chamber* $\mathcal{C}$ in the Lie algebra of the maximal torus T;
- a *polarization* $T^{\frac{1}{2}} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ of the tangent bundle $T(T^*(G/B))$, i.e., a solution of the equation
$$T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^\vee = T(T^*(G/B))$$
in the ring $K_{T\times\mathbb{C}^*}(T^*(G/B))_{\mathrm{loc}}$;
- an *alcove* A in $\mathfrak{t}^*_{\mathbb{R}}$, which is also called a *slope* in [Oko27].

Given a polarization $T^{\frac{1}{2}}$, there is an opposite polarization defined by $T^{\frac{1}{2}}_{\mathrm{opp}} := q^{-1}(T^{\frac{1}{2}})^\vee$. A typical example of a polarization is $T(G/B)$ (pulled back from G/B to $T^*(G/B)$). Its opposite is equal to $T^*(G/B)$. Because we will work with fibers of polarizations over fixed points, in what follows we will assume that the polarization is given by a subbundle of $T(T^*(G/B))$, or possibly a virtual vector subbundle, i.e., a formal difference of such subbundles.

The torus fixed point set $(T^*(G/B))^T = (G/B)^T$ is in one-to-one correspondence with the Weyl group W. For every $w \in W$, recall that e_w denotes the corresponding fixed point. For a chosen Weyl chamber $\mathfrak{C}$ in Lie T, pick any cocharacter $\sigma \in \mathfrak{C}$. The attracting set of the fixed point e_w, also called the Białynicki-Birula cell in the literature, is defined by

$$\mathrm{Attr}_{\mathfrak{C}}(w) = \Big\{x \in T^*(G/B) \mid \lim_{z\to 0} \sigma(z)\cdot x = e_w\Big\}.$$

By analyzing the (signs of the roots in the) weight space decomposition of $T_w(T^*(G/B))$, one may show that $\mathrm{Attr}_{\mathfrak{C}}(w)$ is the conormal bundle of the attracting variety of w in G/B; i.e., the conormal bundle of the Schubert cell stable under the Borel subgroup associated to the chamber $\mathfrak{C}$.[4] Define a partial order on the fixed point set W to be the (transitive closure of the) following relation:

$$e_w \preceq_{\mathfrak{C}} e_v \quad \text{if } \overline{\mathrm{Attr}_{\mathfrak{C}}(v)} \cap e_w \neq \varnothing.$$

Then the order determined by the positive (resp., negative) chamber is the same as the Bruhat order (resp., the opposite Bruhat order).

Any chamber $\mathfrak{C}$ determines a decomposition of the tangent space $N_w := T_w(T^*(G/B))$ as $N_w = N_{w,+} \oplus N_{w,-}$ into T-weight spaces which are positive and negative with respect to $\mathfrak{C}$ respectively. For every polarization $T^{\frac{1}{2}}$, denote $N_w \cap T^{\frac{1}{2}}|_w$ by $N_w^{\frac{1}{2}}$. Similarly, we have $N_{w,+}^{\frac{1}{2}}$ and $N_{w,-}^{\frac{1}{2}}$. In particular, $N_{w,-} = N_{w,-}^{\frac{1}{2}} \oplus q^{-1}(N_{w,+}^{\frac{1}{2}})^\vee$. Consequently, we have

$$N_{w,-} - N_w^{\frac{1}{2}} = q^{-1}(N_{w,+}^{\frac{1}{2}})^\vee - N_{w,+}^{\frac{1}{2}}$$

as virtual vector bundles. The determinant bundle of the virtual bundle $N_{w,-} - N_w^{\frac{1}{2}}$ is a complete square and its square root will be denoted by

$$\left(\frac{\det N_{w,-}}{\det N_w^{1/2}}\right)^{\frac{1}{2}};$$

cf. [Oko27, §9.1.5]. For instance, if we choose the polarization $T^{1/2} = T(G/B)$, the positive chamber, and $w = \mathrm{id}$ then both $N_{\mathrm{id}}^{\frac{1}{2}}$ and $N_{\mathrm{id},-}$ have weights $-\alpha$, where α varies in the set of positive roots; in this case the virtual bundle $N_{\mathrm{id},-} - N_{\mathrm{id}}^{\frac{1}{2}}$ is 0.

Let $f := \sum_\mu f_\mu e^\mu \in K_{T\times\mathbb{C}^*}(\mathrm{pt})$ be a Laurent polynomial, where $e^\mu \in K_T(\mathrm{pt})$ and $f_\mu \in \mathbb{Q}[q^{1/2}, q^{-1/2}]$. The *Newton polytope* of f, denoted by $\deg_T f$, is

$$\deg_T f = \text{convex hull } (\{\mu | f_\mu \neq 0\}) \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $X^*(T)$ denotes the character lattice of T. The following theorem defines the K-theoretic stable envelopes.

[4] $T^*(G/B)$ is not compact, so not all points have well defined limits at 0. For example, if $\mathfrak{C}$ is the positive chamber, then the points in the open set $T^*(X(w_0)^\circ) \setminus X(w_0)^\circ \subset T^*(G/B)$ do not have limits at 0.

Theorem 7.1 [Oko27, §9.1; OS22, Theorem 1]. *For every chamber $\mathfrak{C}$, polarization $T^{\frac{1}{2}}$, and alcove A, there exists a unique map of $K_{T\times\mathbb{C}^*}(\mathrm{pt})$-modules*

$$\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A} : K_{T\times\mathbb{C}^*}((T^*(G/B))^T) \to K_{T\times\mathbb{C}^*}(T^*(G/B))$$

such that for every $w \in W$, the class $\Gamma := \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w)$ satisfies the following conditions:

(1) (*support*) $$\operatorname{Supp}\Gamma \subseteq \bigcup_{z\preceq_{\mathfrak{C}} w} \overline{\mathrm{Attr}_{\mathfrak{C}}(z)}.$$

(2) (*normalization*) $$\Gamma|_w = (-1)^{\operatorname{rk} N^{\frac{1}{2}}_{w,+}} \left(\frac{\det N_{w,-}}{\det N_w^{1/2}}\right)^{\frac{1}{2}} \mathcal{O}_{\mathrm{Attr}_{\mathfrak{C}}(w)}|_w.$$

(3) (*degree*) *For every $e_v \prec_{\mathfrak{C}} e_w$,*

$$\deg_T \Gamma|_v \subseteq \deg_T \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(v)|_v + v\lambda - w\lambda,$$

where $\lambda \in (X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q}) \cap A$ is any rational weight in the alcove A.

Strictly speaking, $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w) \in K_{T\times\mathbb{C}^*}(G/B)$ denotes the image of $1 \in K_{T\times\mathbb{C}^*}(e_w)$ under the map $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}$. From the definition, it is easy to see that $\{\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w) \mid w \in W\}$ forms a basis for the localized equivariant K-theory $K_{T\times\mathbb{C}^*}(T^*(G/B))_{\mathrm{loc}}$, called the *stable basis*. Explicit combinatorial formulae and recursions for the localizations of stable envelopes in $T^*(G/B)$ may be found in [AMSS24a, §8.3].

7.2. ***Changing the polarizations.*** A natural question is to study the change of the stable envelopes when we vary the above three parameters. To start, the change of chambers is encoded in the left Weyl group action. More precisely, the group G acts on $T^*(G/B)$ by left multiplication, which induces a left Weyl group action on $K_{T\times\mathbb{C}^*}(T^*(G/B))$, see [MNS22b]. If we change the chamber $\mathfrak{C}$ to another chamber $w(\mathfrak{C})$ ($w \in W$), we have the following formula (see [AMSS24a, Lemma 8.2(a)]):

$$w \cdot (\mathrm{stab}_{\mathcal{C},T^{\frac{1}{2}},A}(u)) = \mathrm{stab}_{w(\mathcal{C}),w(T^{\frac{1}{2}}),A}(wu).$$

Next we consider the change of polarizations. In this case the results are stated, e.g., in [Oko27] in the more general setting of symplectic resolutions; for the convenience of the reader we include proofs for $T^*(G/B)$.

Lemma 7.2 [Oko27, Section 7.5.8]. *For any two polarizations $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$, there exists a class $\mathcal{F} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ such that $T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$.*

Remark 7.3. Classes of the form $\mathcal{F} - q^{-1}\mathcal{F}^\vee$ are called *balanced classes* in *loc. cit.*

Proof. It suffices to prove that any solution of the equation $\mathcal{G} + q^{-1}\mathcal{G}^\vee = 0 \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ is of the form $\mathcal{G} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$ for some $\mathcal{F} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$. Since $\mathcal{G} \in K_{T\times\mathbb{C}^*}(T^*(G/B)) \simeq K_T(G/B)[q,q^{-1}]$, we can write $\mathcal{G} = \mathcal{F} + q^{-1}\mathcal{F}'$ for some $\mathcal{F} \in K_T(G/B)[q]$ and $\mathcal{F}' \in q^{-1}K_T(G/B)[q^{-1}]$. Thus,

$$\mathcal{F} + q^{-1}\mathcal{F}' + q^{-1}\mathcal{F}^\vee + (\mathcal{F}')^\vee = 0.$$

Since $\mathcal{F}$, $(\mathcal{F}')^\vee \in K_T(G/B)[q]$ and $q^{-1}\mathcal{F}', q^{-1}\mathcal{F}^\vee \in q^{-1}K_T(G/B)[q^{-1}]$, we get

$$\mathcal{F} + (\mathcal{F}')^\vee = 0 \quad \textit{and} \quad q^{-1}\mathcal{F}' + q^{-1}\mathcal{F}^\vee = 0.$$

Therefore, $\mathcal{F}' = -\mathcal{F}^\vee$, and $\mathcal{G} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$. □

For any two polarizations $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$ let $\mathcal{F}$ be defined by $T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$ as in Lemma 7.2. Define a $T \times \mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ on $T^*(G/B)$ by

$$\mathcal{L} := \det \mathcal{F}.$$

Here for a virtual bundle $\mathcal{F} = \mathcal{V}_1 - \mathcal{V}_2$, $\det \mathcal{F} := \det \mathcal{V}_1 / \det \mathcal{V}_2$.

Example 7.4. Consider the polarization $T_1^{\frac{1}{2}} := T(G/B)$ and the opposite polarization $T_2^{\frac{1}{2}} = q^{-1}(T_1^{\frac{1}{2}})^\vee = T^*(G/B)$. Then the element $\mathcal{F}$ in Lemma 7.2 can be taken to be $T(G/B)$, therefore $\mathcal{L} = \det \mathcal{F} = \mathcal{L}_{-2\rho}$.

The next proposition shows that the change of polarizations results in a multiplication by a line bundle, therefore it is encoded in the Chevalley formula for the stable envelopes.

Proposition 7.5 [Oko27, Exercise 9.1.12].

$$\mathrm{stab}_{\mathfrak{C}, T_2^{\frac{1}{2}}, A}(w) = (-1)^{\mathrm{rk}\, N_{w,+,2}^{\frac{1}{2}} - \mathrm{rk}\, N_{w,+,1}^{\frac{1}{2}}} q^{\frac{1}{2}\mathrm{rk}\,\mathcal{F}|_w} \mathcal{L} \otimes \mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w),$$

where $N_{w,+,i}^{\frac{1}{2}} := N_{w,+,i} \cap T_i^{\frac{1}{2}}$ *for* $i = 1, 2$, *and if* $\mathcal{F} = T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{V}_1 - \mathcal{V}_2$ *is the virtual bundle from Lemma 7.2, then* $\mathrm{rk}\,\mathcal{F}|_w := \mathrm{rk}\,\mathcal{V}_1|_w - \mathrm{rk}\,\mathcal{V}_2|_w$.

Proof. From the characterization Theorem 7.1, it suffices to show that the right hand side satisfies the defining properties of the stable envelope on the left hand side. The support condition is immediate. For the degree condition, we need to check that for every $e_v \prec_{\mathfrak{C}} e_w$,

$$\deg_T(\mathcal{L} \otimes \mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w))|_v \subseteq \deg_T(\mathcal{L} \otimes \mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(v))|_v + v\lambda - w\lambda,$$

for some $\lambda \in A$. The terms $\mathcal{L}|_v$ on both sides cancel, and the above inclusion reduces to the degree condition for $\mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w)$. The normalization condition follows from the following calculation:

$$\begin{aligned}
(-1)^{\mathrm{rk}\, N_{w,+,2}^{\frac{1}{2}} - \mathrm{rk}\, N_{w,+,1}^{\frac{1}{2}}} \frac{\mathrm{stab}_{\mathfrak{C}, T_2^{\frac{1}{2}}, A}(w)|_w}{\mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w)|_w} &= \left(\frac{\det N_{w,-}}{\det N_{w,2}^{1/2}}\right)^{\frac{1}{2}} \left(\frac{\det N_{w,-}}{\det N_{w,1}^{1/2}}\right)^{-\frac{1}{2}} = \left(\frac{\det N_{w,1}^{1/2}}{\det N_{w,2}^{1/2}}\right)^{\frac{1}{2}} \\
&= \left(\det N_w \cap (\mathcal{F}|_w - q^{-1}\mathcal{F}^\vee|_w)\right)^{\frac{1}{2}} \\
&= \left(\frac{\det(N_w \cap \mathcal{F}|_w)}{\det(N_w \cap q^{-1}(\mathcal{F}|_w)^\vee)}\right)^{\frac{1}{2}} = q^{\frac{1}{2}\mathrm{rk}\,\mathcal{F}|_w} \mathcal{L}|_w.
\end{aligned}$$

Here we have used our assumption that $\mathcal{F}$ is represented by a subbundle of $T(T^*(G/B))$, and also that the weights of $N_w = T_w(T^*(G/B))$ are distinct. The reason for the last equality in the display is as follows. We have $N_w \cap \mathcal{F}|_w = \mathcal{F}|_w$; suppose e^λ is a torus weight of it. Since N_w is a symplectic vector space, $q^{-1}e^{-\lambda}$ is a weight of N_w. Hence $q^{-1}e^{-\lambda}$ is also a weight of the intersection $N_w \cap q^{-1}(\mathcal{F}|_w)^\vee$

and in fact $N_w \cap q^{-1}(\mathcal{F}|_w)^\vee = q^{-1}(\mathcal{F}|_w)^\vee$, giving the equality. The case of $\mathcal{F} = \mathcal{V}_1 - \mathcal{V}_2$ being a virtual bundle follows from linearity of the constructions. □

7.3. *Changing the alcoves.* We now turn to what happens under the change of alcoves. This can be answered using recursive formulae from [SZZ20; SZZ21], reported in Theorem 7.6 below. Our subsequent Theorem 7.8 provides a nonrecursive answer, expressing the Chevalley formula in terms of λ-chains, thus relating the stable envelope for the fundamental alcove $\mathrm{A}_\circ$ to the stable envelope for a translate $\mathrm{A}_\circ + \lambda$.

The alcoves in $\mathfrak{t}^*_{\mathbb{R}}$ are of the form $x(\mathrm{A}_\circ) + \lambda$ for some $x \in W$ and some λ in the root lattice. It was proved in [AMSS24a, Lemma 8.2] and [SZZ20, Remark 2.3] that

$$\mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A+\lambda}(w) = e^{-w\lambda}\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}(w), \tag{34}$$

where $\mathcal{L}_\lambda$ is the pullback of $G \times^B \mathbb{C}_\lambda$ from G/B to $T^*(G/B)$. Fix the chamber $\mathcal{C}$ to be the antidominant Weyl chamber

$$\mathcal{C} := \{\lambda \in \mathfrak{t}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle < 0 \ \text{ for any positive root } \alpha\}$$

and the polarization $T^{\frac{1}{2}} = T^*(G/B)$. To simplify notation, we denote $\mathrm{stab}_{\mathcal{C}, T^*(G/B), A}(w)$ by $\mathrm{stab}_A(w)$.

Let $Z = T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$ be the Steinberg variety where $\mathcal{N} \subset \mathfrak{g}$ denotes the nilpotent cone. There is an algebra isomorphism

$$\mathbb{H} \simeq K_{G \times \mathbb{C}^*}(Z),$$

where $K_{G \times \mathbb{C}^*}(Z)$ has an algebra structure given by convolution; this was proved by Kazhdan and Lusztig [KL87] and Ginzburg [CG10]. The convolution induces an action of the Hecke algebra $\mathbb{H}$ on $K_{T \times \mathbb{C}^*}(T^*(G/B))$ which we recall next. For a simple root α_i, and the corresponding minimal parabolic subgroup $P_i \supset B$, define the operator T_i on $K_{T \times \mathbb{C}^*}(T^*(G/B))$ by the following formula:

$$T_i(\mathcal{F}) := -\mathcal{F} - \pi_{1*}(\pi_2^* \mathcal{F} \otimes \pi_2^* \mathcal{L}_{\alpha_i}).$$

Here $\mathcal{F} \in K_{A \times \mathbb{C}^*}(T^*(G/B))$, $Y_i := G/B \times_{G/P_i} G/B \subset G/B \times G/B$, $T^*_{Y_i}$ is the conormal bundle of Y_i inside $G/B \times G/B$, and $\pi_j : T^*_{Y_i} \to T^*(G/B)$ $(j = 1, 2)$ are the two projections. These operators satisfy the quadratic relations and the braid relations in $\mathbb{H}$. In particular, T_w is well defined for any $w \in W$. Recall that $\mathrm{A}_\circ$ denotes the fundamental alcove.

Theorem 7.6. (a) [SZZ20, Theorem 4.5]

$$T_i(\mathrm{stab}_{\mathrm{A}_\circ}(w)) = \begin{cases} (q-1)\,\mathrm{stab}_{\mathrm{A}_\circ}(w) + q^{1/2}\,\mathrm{stab}_{\mathrm{A}_\circ}(ws_i), & \text{if} \quad ws_i < w, \\ q^{1/2}\,\mathrm{stab}_{\mathrm{A}_\circ}(ws_i), & \text{if} \quad ws_i > w. \end{cases}$$

(b) [SZZ21, Theorem 5.4] *Let $x \in W$. Then*

$$\mathrm{stab}_{x(\mathrm{A}_\circ)}(w) = q^{-\ell(x)/2} T_x(\mathrm{stab}_{\mathrm{A}_\circ}(wx)).$$

(c) [SZZ21, Lemma 3.5 and Corollary 5.3] *Assume A_1 and A_2 are two adjacent alcoves separated by a wall of the form $H_{\alpha,n}$, where $\alpha > 0$. Assume A_2 is on the positive side of $H_{\alpha,n}$, i.e., for any $\mu \in A_2$,*

$(\mu, \alpha^\vee) > n$. Then

$$\mathrm{stab}_{A_1}(w) = \begin{cases} \mathrm{stab}_{A_2}(w) + e^{-nw\alpha}(q^{1/2} - q^{-1/2})\,\mathrm{stab}_{A_2}(ws_\alpha) & \text{if} \quad ws_\alpha > w, \\ \mathrm{stab}_{A_2}(w) & \text{if} \quad ws_\alpha < w. \end{cases}$$

Using statement (a), one can calculate $T_x(\mathrm{stab}_{\mathrm{A}_\circ}(wx))$ recursively in terms of $\{\mathrm{stab}_{\mathrm{A}_\circ}(w) \mid w \in W\}$. Part (b) implies that the same is true for $\mathrm{stab}_{x(\mathrm{A}_\circ)}(w)$. Finally, part (c) may be used to relate directly the stable bases for any two adjacent alcoves, and therefore recursively relate the stable bases for two arbitrary alcoves.

We focus next on relating (34) to our Chevalley formulae obtained earlier in this paper. Together with (a) and (b) from Theorem 7.6 above, this gives an alternative recursion to (c), calculating the stable envelope for an arbitrary alcove $x\mathrm{A}_\circ + \lambda$ starting from the stable envelope for $\mathrm{A}_\circ$.

Fix λ an integral weight. By Theorem 7.6(a), $\mathrm{stab}_{x(\mathrm{A}_\circ)}(w)$ may be written as a linear combination of $\{\mathrm{stab}_{\mathrm{A}_\circ}(w) \mid w \in W\}$. By (34), to determine $\mathrm{stab}_{x(\mathrm{A}_\circ)+\lambda}(w)$, it suffices to find a formula for $\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(w)$. This can be achieved by the Chevalley formula for the motivic Chern classes. The key is the following result.

Lemma 7.7 [AMSS24a, Theorem 8.6]. *Let $\iota : G/B \hookrightarrow T^*(G/B)$ denote the inclusion of the zero section. For any $w \in W$,*

$$\iota^*(\mathrm{stab}_{\mathrm{A}_\circ}(w)) = (-1)^{\dim G/B} q^{\dim G/B - \ell(w)/2} MC_{-q^{-1}}(Y(w)^\circ) \otimes \mathcal{L}_{-2\rho}.$$

Recall the operator $(-)^\vee$ on $K_T(pt)[y^{\pm 1}]$, which sends e^μ to $e^{-\mu}$ and y to y^{-1}. We have the following Chevalley formula for the stable bases.

Theorem 7.8. *Let $\lambda \in X^*(T)$ be a weight and fix $\beta_1, \ldots, \beta_l$ a reduced λ-chain corresponding to an alcove walk from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$. Then*

$$\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(u) = \sum_w q^{\frac{1}{2}(\ell(u) - \ell(w))} (C^w_{u,-\lambda})^\vee|_{y=-q^{-1}}\, \mathrm{stab}_{\mathrm{A}_\circ}(w),$$

where $C^w_{u,\lambda}$ are the coefficients defined in (27).

In the notation of Theorem 5.5, one can write this in terms of λ-chains as

$$\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(u) = \sum_{J \subset \{1,2,\ldots,l\}} (-1)^{n(J)} (q^{-1/2} - q^{1/2})^{|J|} e^{w\tilde{r}_{J_>}(\lambda)}\, \mathrm{stab}_{\mathrm{A}_\circ}(ur_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}),$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \ldots, l\}$ such that $u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} r_{h_{j_2}} \cdot \cdots \cdot r_{h_{j_t}}$.

Remark 7.9. Recall we have the following duality between the stable bases (see [OS22, Proposition 1]):

$$\langle \mathrm{stab}_{\mathcal{C}, T^{\frac{1}{2}}, A}(u), \mathrm{stab}_{-\mathcal{C}, T^{\frac{1}{2}}_{\mathrm{opp}}, -A}(w) \rangle = \delta_{u,w}.$$

Therefore, similar arguments as in the proof of Lemma 5.1 will give a Chevalley formula for the dual stale basis $\mathrm{stab}_{-\mathcal{C}, T(G/B)), -\mathrm{A}_\circ}(u)$.

Proof. By the definition of the Segre motivic classes from Definition 4.2(2), the expression in (28) becomes

$$\mathcal{L}_\lambda \otimes \mathcal{D}(MC_y(Y(u)^\circ)) = \sum_{w\geq u}(-y)^{\ell(u)-\ell(w)}C^w_{u,\lambda}\mathcal{D}(MC_y(Y(w)^\circ)).$$

Taking $(-)^\vee$ on both sides of the equation, we get

$$\mathcal{L}_{-\lambda} \otimes MC_y(Y(u)^\circ) = \sum_{w\geq u}(-y)^{\ell(w)-\ell(u)}(C^w_{u,\lambda})^\vee MC_y(Y(w)^\circ).$$

The first equation of the theorem follows from this and Lemma 7.7. The second equation is a consequence of the first and of (29) in Theorem 5.5. □

Example 7.10. In type A_2, set $u = s_2s_1$ and $\lambda = 2\varpi_1 + \varpi_2$. A λ-chain of roots is given by $\beta_1 = \alpha_2,\ \beta_2 = \alpha_1 + \alpha_2,\ \beta_3 = \alpha_1,\ \beta_4 = \alpha_1 + \alpha_2,\ \beta_5 = \alpha_1,\ \beta_6 = \alpha_1 + \alpha_2$; see Appendix B. From Theorem 7.8, we have

$$\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(s_2s_1) \ = \ e^{\varpi_1-3\varpi_2}\,\mathrm{stab}_{\mathrm{A}_\circ}(s_2s_1) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})e^{-\varpi_1-2\varpi_2}\,\mathrm{stab}_{\mathrm{A}_\circ}(s_1s_2s_1).$$

Therefore, by (34), using that $-u(\lambda) = -\varpi_1 + 3\varpi_2$, we have

$$\mathrm{stab}_{\mathrm{A}_\circ+\lambda}(s_2s_1) \ = \ \mathrm{stab}_{\mathrm{A}_\circ}(s_2s_1) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})e^{-\alpha_1}\,\mathrm{stab}_{\mathrm{A}_\circ}(s_1s_2s_1).$$

Example 7.11. Consider $u = s_2$, $\lambda = 2\varpi_1 + \varpi_2$, $w_0u = s_2s_1$. We can use Serre duality (Proposition 6.1) and Example 5.8 to get

$$\begin{aligned}\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(s_2) = {} & e^{3\varpi_1-\varpi_2}\,\mathrm{stab}_{\mathrm{A}_\circ}(s_2)\\ & + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{\varpi_1} + e^{-\varpi_1+\varpi_2} + e^{-3\varpi_1+2\varpi_2})\,\mathrm{stab}_{\mathrm{A}_\circ}(s_1s_2)\\ & + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{2\varpi_1-2\varpi_2} + e^{\varpi_1-3\varpi_2})stab_{\mathrm{A}_\circ}(s_2s_1)\\ & + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2(e^{-\varpi_1-2\varpi_2} + e^{-2\varpi_1} + e^{-\varpi_2})\,\mathrm{stab}_{\mathrm{A}_\circ}(s_1s_2s_1).\end{aligned}$$

Since $-s_2(\lambda) = -3\varpi_1 + \varpi_2$, we have

$$\begin{aligned}\mathrm{stab}_{\mathrm{A}_\circ+\lambda}(s_2) = {} & \mathrm{stab}_{\mathrm{A}_\circ}(s_2)\\ & + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{-\alpha_1} + e^{-2\alpha_1} + e^{-3\alpha_1})\,\mathrm{stab}_{\mathrm{A}_\circ}(s_1s_2)\\ & + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{-\alpha_1-\alpha_2} + e^{-2\alpha_1-2\alpha_2})\,\mathrm{stab}_{\mathrm{A}_\circ}(s_2s_1)\\ & + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2(e^{-3\alpha_1-2\alpha_2} + e^{-3\alpha_1-\alpha_2} + e^{-2\alpha_1-\alpha_2})\,\mathrm{stab}_{\mathrm{A}_\circ}(s_1s_2s_1).\end{aligned}$$

8. Whittaker functions and Hall–Littlewood polynomials

In this section we apply the Chevalley formula to obtain combinatorial expressions for Whittaker functions and Hall–Littlewood polynomials. Variants of the formulae we obtain were already available in the literature, and our approach based on the cohomological calculations adds a geometric perspective to this.

8.1. ***Whittaker functions.*** In this section we study Whittaker functions. These appear in p-adic representation theory, and in this note we utilize a cohomological construction of these functions from [AMSS24a], see also [MS22]. Recall the definition of the Demazure–Lusztig operators on $K_T(pt)[y]$:

$$\widetilde{T}_i(e^\lambda) = -e^\lambda \frac{1+y}{1-e^{-\alpha_i}} + e^{s_i\lambda}\frac{1+ye^{\alpha_i}}{1-e^{-\alpha_i}}, \quad \widetilde{T}_i^\vee(e^\lambda) = -e^\lambda \frac{1+y}{1-e^{-\alpha_i}} + e^{s_i\lambda}\frac{1+ye^{-\alpha_i}}{1-e^{-\alpha_i}}.$$

The operators satisfy the usual quadratic and braid relations in the Hecke algebra, therefore for any $w \in W$ there are operators $\widetilde{T}_w$ and $\widetilde{T}_w^\vee$ acting on $K_T(pt)[y]$, and defined using any reduced decomposition of w. The following has been proved in [MS22, Theorem 1.1]:

Proposition 8.1. *Let* $\lambda_y(\mathrm{id}) := \prod_{\alpha>0}(1+ye^\alpha)$, *and set*

$$MC'_y(X(w)^\circ) := \lambda_y(\mathrm{id})\frac{MC_y(X(w)^\circ)}{\lambda_y(T^*_{G/B})}.$$

Then, for any $\lambda \in X^*(T)$,

(1) $\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)\big) = \widetilde{T}_w^\vee(e^\lambda)$, *and*

(2) $\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)\big) = \widetilde{T}_w(e^\lambda)$.

We note that for any antidominant weight λ and $w \in W$,

$$\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)\big) = \widetilde{T}_w(e^\lambda) = \mathcal{W}_{\lambda,w}, \tag{35}$$

where $\mathcal{W}_{\lambda,w}$ is the Iwahori–Whittaker function for the Langlands dual group over a nonarchimedean local field; we refer to [MS22] for more details, including the number theoretic definition of $\mathcal{W}_{\lambda,w}$. Here we identified y with $-q^{-1}$, where q is the number of elements in the residue field. As explained in [MS22], from the fact that

$$\sum_{w\in W} MC'_y(X(w)^\circ) = 1$$

by the additivity motivic Chern classes, one recovers the Casselman–Shalika formula for the spherical Whittaker function [CS80]:

$$\sum_{w\in W} \mathcal{W}_{\lambda,w} = \prod_{\alpha>0}(1+ye^\alpha)\chi_T(G/B, \mathcal{L}_\lambda) = \prod_{\alpha>0}(1+ye^\alpha)\chi_{w_0\lambda}. \tag{36}$$

Here $\chi_{w_0\lambda}$ denotes the character for the irreducible representation of G of highest weight $w_0(\lambda)$. We also note that, in type A, an interpretation of the Iwahori–Whittaker function in terms of the partition function of the Iwahori lattice model has been obtained in [BBBG24].

Using the Chevalley coefficients in (27), we obtain the following formula for the Iwahori–Whittaker function $\mathcal{W}_{\lambda,w}$. Let ρ denote the half sum of the positive roots.

Theorem 8.2. *For any antidominant weight* λ *and* $w \in W$,

$$\mathcal{W}_{\lambda,w} = e^\rho \sum_u (-1)^{\ell(u)} C^w_{\lambda-\rho,u}|_{y\mapsto y^{-1}} y^{\ell(w)-\ell(u)}.$$

Proof. For any $u \in W$,

$$\chi_T(G/B, MC_y(X(u)^\circ)) = MC_y[X(u)^\circ \to \mathrm{pt}] = MC_y[\mathbb{A}^1 \to \mathrm{pt}]^{\ell(u)} = (-y)^{\ell(u)}, \tag{37}$$

where the second equality follows from [AMSS24a, Theorem 4.2(3)]. Therefore, taking the equivariant Euler characteristics of both sides of (27), we get

$$\widetilde{T}_w^\vee(e^\lambda) = \chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)\big) = \sum_u C^w_{\lambda,u}(-y)^{\ell(u)}.$$

On the other hand, it is immediate to check the following relation between the two Demazure–Lusztig operators:

$$\widetilde{T}_w = e^\rho \widetilde{T}_w^\vee|_{y\mapsto y^{-1}} e^{-\rho} y^{\ell(w)}. \tag{38}$$

Hence,

$$\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)\big) = \widetilde{T}_w(e^\lambda) = e^\rho \sum_u (-1)^{\ell(u)} C^w_{\lambda-\rho,u}|_{y\mapsto y^{-1}} y^{\ell(w)-\ell(u)}.$$

The proof ends by applying Proposition 8.1. □

Remark 8.3. Notice that $C^w_{0,u} = \delta_{u,w}$. The above proof shows

$$\chi_T\big(G/B, \mathcal{L}_\rho \otimes MC'_y(X(w)^\circ)\big) = (-1)^{\ell(w)} e^\rho.$$

As a corollary we prove a variant of the Casselman–Shalika formula (36), obtained by Li [Li92]; see also [BBBG24, Proposition 9.4]. First define

$$R_\lambda(y) := \chi_T(G/B, \lambda_y(T^*_{G/B}) \otimes \mathcal{L}_\lambda) \in K_T(pt)[y]. \tag{39}$$

Corollary 8.4. *Let λ be an antidominant integral weight. Then*

$$\sum_w y^{-\ell(w)} \mathcal{W}_{\lambda,w} = e^\rho R_{\lambda-\rho}(y^{-1}).$$

Proof. By the additivity of the motivic Chern classes and Proposition 8.1(1),

$$R_\lambda(y) = \sum \chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)\big) = \sum_w \widetilde{T}_w^\vee(e^\lambda). \tag{40}$$

On the other hand, for an antidominant weight λ, we have

$$\begin{aligned}\sum_w y^{-\ell(w)} \mathcal{W}_{\lambda,w} &= \sum_w y^{-\ell(w)} \widetilde{T}_w(e^\lambda) = e^\rho \sum_w y^{-\ell(w)} e^{-\rho} \widetilde{T}_w e^\rho (e^{\lambda-\rho}) \\ &= e^\rho \sum_w \widetilde{T}_w^\vee|_{y\mapsto y^{-1}}(e^{\lambda-\rho}) = e^\rho R_{\lambda-\rho}(y^{-1}).\end{aligned}$$

Here, the first equality follows from Proposition 8.1(2), the third one follows from (38), and the last one follows from (40). □

8.2. *Hall–Littlewood polynomials.* In this section, we assume either λ or $-\lambda$ to be a dominant integral weight. Set $\Sigma_\lambda := \{\alpha \in \Sigma \mid \langle \lambda, \alpha^\vee \rangle = 0\}$, and let R^+_λ be the set of positive roots which are linear combinations of the simple roots in Σ_λ. Denote by $W_\lambda \subset W$ the subgroup generated by the simple reflections s_α, where $\alpha \in \Sigma_\lambda$. Let W^λ be the set of minimal length representatives for the cosets W/W_λ. Finally, let P_λ be the parabolic subgroup containing the Borel subgroup B defined by the condition that $W_{P_\lambda} = W_\lambda$.

Definition 8.5. (1) Define $H_\lambda(y) := \chi_T(G/P_\lambda, \lambda_y(T^*_{G/P_\lambda}) \otimes \mathcal{L}_\lambda) \in K_T(pt)[y]$.

(2) (Hall–Littlewood polynomial; cf. [Mac98, p. 208, (2.2)]) For a dominant weight λ, define

$$HL_\lambda(\boldsymbol{x}; t) := \sum_{w \in W^\lambda} w\Bigg(\boldsymbol{x}^\lambda \prod_{\alpha \in R^+ \setminus R^+_\lambda} \frac{1 - t\boldsymbol{x}^{-\alpha}}{1 - \boldsymbol{x}^{-\alpha}}\Bigg),$$

where $\boldsymbol{x}^\lambda$ denotes e^λ.

Let $\pi_\lambda : G/B \to G/P_\lambda$ be the natural projection. Then

$$\lambda_y(G/B) = \pi^*_\lambda(\lambda_y(G/P_\lambda)) \cdot \lambda_y(P_\lambda/B).$$

By the projection formula, and using that $\pi^*_\lambda(\mathcal{L}_\lambda) = \mathcal{L}_\lambda$, we have

$$H_\lambda(y) = \chi_T(G/P_\lambda, \lambda_y(T^*_{G/P_\lambda}) \otimes \mathcal{L}_\lambda) = \frac{\chi_T(G/B, \lambda_y(T^*_{G/B}) \otimes \mathcal{L}_\lambda)}{\chi_T(P_\lambda/B, \lambda_y(T^*_{P_\lambda/B}))} = \frac{R_\lambda(y)}{\sum_{w \in W_\lambda} (-y)^{\ell(w)}}$$

Here the last equality follows from (39), and the fact that

$$\chi_T(P_\lambda/B, \lambda_y(T^*_{P_\lambda/B})) = \sum_{w \in W_\lambda} \chi(MC_y(X(w)^\circ)) = \sum_{w \in W_\lambda} (-y)^{\ell(w)}.$$

The relation between H_λ and the Hall–Littlewood polynomial, summarized next, was obtained in a related (upcoming) collaboration with B. Ion.

Lemma 8.6. *We have the following formulae for $H_\lambda(y)$:*

$$H_\lambda(y) = \sum_{w \in W^P} \sum_{u \in W} C^w_{u,\lambda} (-y)^{\ell(u)}, \tag{41}$$

$$H_\lambda(y) = \sum_{w \in W^\lambda} w\Bigg(e^\lambda \prod_{\alpha \in R^+ \setminus R^+_\lambda} \frac{1 + ye^\alpha}{1 - e^\alpha}\Bigg). \tag{42}$$

Proof. Equation (41) follows from $\lambda_y(T^*_{G/P}) = MC_y(G/P) = \sum_{w \in W^P} MC_y(X(wW_P)^\circ)$, Theorem 5.15, and $\chi_T(MC_y(X(uW_P)^\circ)) = (-y)^{\ell(u)}$ for any $u \in W^P$. Equation (42) follows from the localization formula [Nie74; MS22, Theorem 2.1 (c)]. To be more specific, the torus fixed points in G/P_λ are $\{wP_\lambda \mid w \in W^P\}$, and the torus weights of the tangent space at the fixed point wP_λ are $\{-w\alpha \mid \alpha \in R^+ \setminus R^+_\lambda\}$. $\square$

Corollary 8.7. *For a dominant integral* λ, *the Hall–Littlewood polynomial* $HL_\lambda(x;t)$ *can be expressed using* $H_{-\lambda}(y)$ *or* $H_\lambda(y)$ *as follows*:

$$HL_\lambda(\boldsymbol{x};t) = H_{-\lambda}(y)\Big|_{e^\alpha \mapsto \boldsymbol{x}^{-\alpha},\, y\mapsto -t}, \tag{43}$$

$$HL_\lambda(\boldsymbol{x};t) = \left(\frac{1}{(-y)^{\dim G/P_\lambda}} H_\lambda(y)\right)\Big|_{e^\alpha \mapsto \boldsymbol{x}^{\alpha},\, y\mapsto -t^{-1}}. \tag{44}$$

We next assume that λ is a dominant integral weight. Fix a reduced $(-\lambda)$-chain $\Gamma = (-\beta_1, -\beta_2, \dots, -\beta_l)$ and the sequence of hyperplanes $H_{\beta_1,d_1}, H_{\beta_2,d_2}, \dots, H_{\beta_l,d_l}$. This corresponds to an alcove path from $A_\circ$ to $A_\circ + \lambda$. Since λ is dominant, we have $\beta_j > 0$, $d_j > 0$.

We recover the following known formula for the Hall–Littlewood polynomial:

Proposition 8.8 [Sch06; Ram06; Len11, Theorem 2.7].

$$HL_\lambda(\boldsymbol{x};t) = \sum_{(w,J,u)\in\mathcal{A}(\Gamma)} t^{\frac{1}{2}(\ell(w)+\ell(u)-|J|)}(1-t)^{|J|}\boldsymbol{x}^{w\hat{r}_{J_<}(\lambda)}, \tag{45}$$

where (with notation as in Section 3.2),

$$\mathcal{A}(\Gamma) = \big\{(w,J,u) \mid w\in W^P, u\in W, J\subset\{1,2,\dots,l\}, u \xrightarrow{J_>} w\big\}.$$

Proof. Apply equations (43), (41), and (30) to the $(-\lambda)$-chain $(-\beta_1, -\beta_2, \dots, -\beta_l)$. □

We also get a new formula for $HL_\lambda(\boldsymbol{x};t)$:

Proposition 8.9.

$$HL_\lambda(\boldsymbol{x};t) = \sum_{(u,J,w)\in\mathcal{A}^{\mathrm{op}}(\Gamma)} t^{\frac{1}{2}(2\dim G/P_\lambda-\ell(w)-\ell(u)-|J|)}(1-t)^{|J|}\boldsymbol{x}^{u\hat{r}_{J_<}(\lambda))}, \tag{46}$$

where $\mathcal{A}^{\mathrm{op}}(\Gamma) = \big\{(u,J,w) \mid w\in W^P, u\in W, J\subset\{1,2,\dots,l\}, u \xrightarrow{J_<} w\big\}$.

Proof. Apply equations (44), (41), and (29) to the $(-\lambda)$-chain, together with Remark 5.6(1). □

Remark 8.10. When $P_\lambda = B$, equations (45) and (46) give the same formula. The correspondence may be seen using the Serre duality in Proposition 6.7. However, the formulae are in general different, as shown by the examples below.

Example 8.11 type A_2. Let $G = \mathrm{GL}_3(\mathbb{C})$, $T = (\mathbb{C}^*)^3$, and $x_i = e^{\varepsilon_i}$, for $i = 1,2,3$. Let $\lambda = \varpi_1 = \varepsilon_1$; then $W_\lambda = \langle s_2\rangle \subset W = \langle s_1, s_2\rangle$, and $W^\lambda = \{\mathrm{id}, s_1, s_2s_1\}$. Fix a reduced $(-\lambda)$-chain $(-\beta_1 = -\alpha_1-\alpha_2, -\beta_2 = -\alpha_1)$.

Then Proposition 8.8 sums over the seven terms shown on the left in Table 1, while Proposition 8.9 sums over those on the right. The respective developments are

$$HL_\lambda(\boldsymbol{x};t) = (x_1) + (tx_2 + (1-t)x_2) + (t^2x_3 + t(1-t)x_3 + t(1-t)x_3 + (1-t)^2x_3) = x_1+x_2+x_3,$$
$$HL_\lambda(\boldsymbol{x};t) = (t^2x_1) + (tx_2 + t(1-t)x_1) + (x_3 + (1-t)x_2 + (1-t)x_1)) = x_1+x_2+x_3.$$

w	J	u	
id	$\varnothing$	id	x_1
s_1	$\varnothing$	s_1	tx_2
	$\{2\}$	id	$(1-t)x_2$
s_2s_1	$\varnothing$	s_2s_1	t^2x_3
	$\{1\}$	s_1	$t(1-t)x_3$
	$\{2\}$	s_2	$t(1-t)x_3$
	$\{1,2\}$	id	$(1-t)^2x_3$

w	J	u	
id	$\varnothing$	id	t^2x_1
s_1	$\varnothing$	s_1	tx_2
	$\{2\}$	id	$t(1-t)x_1$
s_2s_1	$\varnothing$	s_2s_1	x_3
	$\{1\}$	s_1	$(1-t)x_2$
	$\{2\}$	s_2	$(1-t)x_1$

Table 1. To Example 3.7.

w	J	u	
id	$\varnothing$	id	$x_1^2x_2^2$
s_2	$\varnothing$	s_1	$tx_1^2x_3^2$
	$\{2\}$	id	$(1-t)x_1^2x_2x_3$
	$\{4\}$	id	$(1-t)x_1^2x_3^2$
$w=s_1s_2$	$\varnothing$	s_1s_2	$t^2x_2^2x_3^2$
	$\{1\}$	s_2	$t(1-t)x_1x_2x_3^2$
	$\{2\}$	s_1	$t(1-t)x_1x_2^2x_3$
	$\{3\}$	s_2	$t(1-t)x_2^2x_3^2$
	$\{4\}$	s_1	$t(1-t)x_2^2x_3^2$
	$\{1,2\}$	id	$(1-t)^2x_1x_2^2x_3$
	$\{1,4\}$	id	$(1-t)^2x_1x_2x_3^2$
	$\{3,4\}$	id	$(1-t)^2x_2^2x_3^2$

w	J	u	
id	$\varnothing$	id	$t^2x_1^2x_2^2$
s_2	$\varnothing$	s_2	$tx_1^2x_3^2$
	$\{2\}$	id	$t(1-t)x_1^2x_2x_3$
	$\{4\}$	id	$t(1-t)x_1^2x_2^2$
s_1s_2	$\varnothing$	s_1s_2	$x_2^2x_3^2$
	$\{1\}$	s_2	$(1-t)x_1x_2x_3^2$
	$\{2\}$	s_1	$(1-t)x_1x_2^2x_3$
	$\{3\}$	s_2	$(1-t)x_1^2x_3^2$
	$\{4\}$	s_1	$(1-t)x_1^2x_2^2$
	$\{2,3\}$	id	$(1-t)^2x_1^2x_2x_3$

Table 2. To Example 8.12.

Example 8.12. Same setup as in Example 3.7, but with $\lambda=2\varpi_2$. Then $W_\lambda=\langle s_1\rangle$, $W^\lambda=\{\mathrm{id}, s_2, s_1s_2\}$, and $d=\dim G/P_\lambda=2$. A $(-\lambda)$-chain is $(-\beta_1=-(\alpha_1+\alpha_2), -\beta_2=-\alpha_2, -\beta_3=-(\alpha_1+\alpha_2), -\beta_4=-\alpha_2)$, and $d_1=1, d_2=1, d_3=2, d_4=2$.

Proposition 8.8 sums over the twelve terms shown on the left in Table 2, and Proposition 8.9 sums over the ten terms on the right. The respective developments are

$$HL_\lambda(\boldsymbol{x};t)=\underbrace{x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2}_{s_{22}}-t\,\underbrace{(x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2)}_{s_{211}},$$

$$HL_\lambda(\boldsymbol{x};t)=(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)+(1-t)(x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2)=s_{22}-ts_{211}.$$

Appendix A. Chevalley formulae for Chern–Schwartz–MacPherson classes of Schubert cells

We give a short proof of the Chevalley formulae for the equivariant Chern–Schwartz–MacPherson (CSM) and Segre–MacPherson (SM) classes of the Schubert cells in the (partial) flag varieties; see [AMSS23, Theorem 9.10] and Theorem A.4 below. Our proof again relies on the action of the appropriate Hecke algebra, this time on the equivariant cohomology of G/P. Besides the intrinsic interest in these Chevalley formulae, we mention that recursions based on it were recently utilized to obtain proofs of Nakada's colored hook formula for finite Weyl groups [MNS22a] and more general Coxeter groups [MNS24].

A.1. ***Degenerate affine Hecke algebras.***

A.1.1. *A change of bases formula.* Recall that the degenerate affine Hecke algebra $\mathcal{H}$ is generated by T_w, $w \in W$ and x_λ, $\lambda \in X^*(T)$, such that

- $T_w T_u = T_{wu}$ for any $w, u \in W$;
- $x_\lambda x_\mu = x_\mu x_\lambda$ for any $\lambda, \mu \in X^*(T)$;
- $x_{\lambda+\mu} = x_\lambda + x_\mu$ for any $\lambda, \mu \in X^*(T)$;
- for any simple root α_i, $T_{s_i} x_\lambda - x_{s_i\lambda} T_{s_i} = -\langle \lambda, \alpha_i^\vee \rangle$.

Lemma A.1. *For any $w \in W$ and $\lambda \in X^*(T)$, the following commutation relation holds in $\mathcal{H}$:*

$$T_w x_\lambda = x_{w\lambda} T_w - \sum_{\substack{\alpha>0 \\ ws_\alpha<w}} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha}.$$

Proof. We utilize induction on $\ell(w)$. The claim is clear when $\ell(w) = 0$ or 1. Now assume $\ell(w) > 1$, and that the claim holds for any Weyl group element with length smaller than $\ell(w)$. Pick a simple root α_i, such that $w > ws_i$. By induction,

$$T_{ws_i} x_{s_i\lambda} = x_{w\lambda} T_{ws_i} - \sum_{\substack{\alpha>0 \\ ws_i s_\alpha<ws_i}} \langle s_i\lambda, \alpha^\vee \rangle T_{ws_i s_\alpha}.$$

Using the commutation of T_{s_i} and x_λ and the induction hypothesis we obtain

$$\begin{aligned} T_w x_\lambda - x_{w\lambda} T_w = {} & T_{ws_i} x_{s_i\lambda} T_{s_i} - \langle \lambda, \alpha_i^\vee \rangle T_{ws_i} - x_{w\lambda} T_w \\ = {} & - \sum_{\substack{\alpha>0 \\ ws_i s_\alpha<ws_i}} \langle \lambda, s_i\alpha^\vee \rangle T_{ws_i s_\alpha} T_{s_i} - \langle \lambda, \alpha_i^\vee \rangle T_{ws_i} = - \sum_{\substack{\alpha>0 \\ ws_\alpha<w}} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha}. \end{aligned}$$

In the last equality, we have used that if $ws_i < w$ then

$$\{\alpha > 0 \mid ws_\alpha < w\} = \{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_i\} \sqcup \{s_i\alpha \mid \alpha > 0,\, ws_i s_\alpha < ws_i\},$$

and that $T_{ws_i s_\alpha} T_{s_i} = T_{ws_i s_\alpha s_i} = T_{ws_{s_i(\alpha)}}$. □

A.1.2. *The Hecke action on the equivariant cohomology.* Since G acts on G/P by left multiplication, there is a natural left Weyl group action on $H_T^*(G/P)$ for any partial flag variety G/P and which acts on the base ring $H_T^*(\mathrm{pt})$ by the usual Weyl group action; see [MNS22b], for example. For any $w \in W$, we use w^L to denote this action.

For any simple root α_i, define the left Demazure–Lusztig operator on $H_T^*(G/P)$ by the following formula (see [MNS22b, Section 3.2]):

$$\mathcal{T}_i^L := \frac{\alpha_i + 1}{\alpha_i} s_i^L - \frac{1}{\alpha_i}.$$

It is proved in *loc. cit.* that these operators satisfy the braid relations and $(\mathcal{T}_i^L)^2 = \mathrm{id}$. The following lemma is easily proved.

Lemma A.2. *There is an action Ψ of the degenerate affine Hecke algebra $\mathcal{H}$ on $H_T^*(G/P)$, sending T_i to $\mathcal{T}_i^L$ and x_λ to $\lambda \in H_T^*(\mathrm{pt})$.*

A.2. ***Definition of the CSM/SM classes.*** Next we recall the basic definitions and properties of CSM classes for the Schubert cells in G/P, we will be brief. We refer the reader to [Ohm06; AMSS23] for details, including a construction of these classes in the equivariant setting and for general varieties.

The (additive) group of constructible functions $\mathcal{F}(X)$ consists of functions $\varphi = \sum_Z c_Z \mathbb{1}_Z$, where the sum is over a finite set of constructible subsets $W \subset X$, $c_Z \in \mathbb{Z}$ are integers, and $\mathbb{1}_Z$ is the characteristic function of Z. For a proper morphism $f : Y \to X$, there is a linear map $f_* : \mathcal{F}(Y) \to \mathcal{F}(X)$, such that for any constructible subset $Z \subset Y$, $f_*(\mathbb{1}_Z)(x) = \chi_{top}(f^{-1}(x) \cap Z)$, where $x \in X$ and χ_{top} denotes the topological Euler characteristic. A conjecture attributed to Deligne and Grothendieck states that there is a unique natural transformation $c_* : \mathcal{F} \to H_*$ from the functor of constructible functions on a complex algebraic variety X to the homology functor, where all morphisms are proper, such that if X is smooth then $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$. This conjecture was proved by MacPherson [Mac74]; the class $c_*(\mathbb{1}_X)$ for possibly singular X was shown to coincide with a class defined earlier by M.-H. Schwartz [Sch65a; Sch65b; BS81].

There is an equivariant version of MacPherson's transformation defined by Ohmoto [Ohm06]. In this case one starts with a variety X with a T-action, and the equivariant version $\mathcal{F}^T(X)$ of the group of constructible functions $\mathcal{F}(X)$ contains the characteristic functions $\mathbb{1}_Z$ for Z stable under the T-action. If $f : X \to Y$ is a proper T-equivariant morphism of algebraic varieties the induced homomorphism and $Z \subset X$ is constructible and T-stable then one defines $f_*^T : \mathcal{F}^T(X) \to \mathcal{F}^T(Y)$ with the property that $f_*^T(\mathbb{1}_Z) = f_*(\mathbb{1}_Z)$. Ohmoto proves [Ohm06, Theorem 1.1] that there is an equivariant version of MacPherson transformation $c_*^T : \mathcal{F}^T(X) \to H_*^T(X)$ that satisfies $c_*^T(\mathbb{1}_X) = c^T(T_X) \cap [X]_T$ if X is a nonsingular T-variety, which and is functorial with respect to proper push-forwards. The last statement means that for all proper T-equivariant morphisms $Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^T(Y) & \xrightarrow{c_*^T} & H_*^T(Y) \\ {\scriptstyle f_*^T}\Big\downarrow & & \Big\downarrow{\scriptstyle f_*^T} \\ \mathcal{F}^T(X) & \xrightarrow{c_*^T} & H_*^T(X). \end{array}$$

Definition A.3. Let Z be a T-invariant constructible subvariety of X.

(1) We denote by $c_{\mathrm{SM}}(Z) := c_*^T(\mathbb{1}_Z) \in H_*^T(X)$ the *equivariant Chern–Schwartz–MacPherson* (*CSM*) *class* of Z.

(2) If X is smooth, we denote by $s_{\mathrm{M}}(Z \subset X) := c_*^T(\mathbb{1}_Z)/c(T_X) \in H_*^T(X)_{\mathrm{loc}}$ the *equivariant Segre–MacPherson* (*SM*) *class* of Z, where $H_*^T(X)_{\mathrm{loc}} := H_*^T(X) \otimes_{H_*^T(pt)} \mathrm{Frac}\, H_*^T(pt)$ denotes the localization of $H_*^T(X)$, and $\mathrm{Frac}\, H_*^T(pt)$ is the fraction field of $H_*^T(pt)$.

A.3. ***The Chevalley formula in cohomology.*** We now specialize to $X = G/P$ with the usual T-action. For simplicity we will denote by $s_{\mathrm{M}}(Z \subset G/P)$ simply by $s_{\mathrm{M}}(Z)$. We will identify the equivariant (Borel–Moore) homology group $H^T_*(G/P)$ with the equivariant cohomology $H^*_T(G/P)$, using the Poincaré duality. The sets of CSM classes of Schubert cells $\{c_{\mathrm{SM}}(X(wW_P)^\circ) \mid w \in W^P\}$ and of the SM classes $\{s_{\mathrm{M}}(X(wW_P)^\circ) \mid w \in W^P\}$ form bases for $H^*_T(G/P)_{\mathrm{loc}} := H^*_T(G/P) \otimes_{H^*_T(pt)} \operatorname{Frac} H^*_T(pt)$. Moreover, if one takes the opposite Schubert cells in any of these sets, then the two bases are dual under the usual intersection pairing, see [AMSS23, Theorem 9.4]:

$$\langle c_{\mathrm{SM}}(X(wW_P)^\circ), s_{\mathrm{M}}(Y(uW_P)^\circ)\rangle_{G/P} = \delta_{w,u} \quad \text{for any } w, u \in W^P. \tag{47}$$

The left Demazure–Lusztig operator acts on the CSM classes by the following formula (see [MNS22b, Theorem 4.3])

$$\mathcal{T}_i^L(c_{\mathrm{SM}}(X(wW_P)^\circ)) = c_{\mathrm{SM}}(X(s_i wW_P)^\circ).$$

Hence, for any $w \in W$,

$$c_{\mathrm{SM}}(X(wW_P)^\circ) = \mathcal{T}_w^L([X(\mathrm{id})]). \tag{48}$$

Recall for any $\lambda \in X^*(T)_P$, $\mathcal{L}_\lambda := G \times^P \mathbb{C}_\lambda \in \operatorname{Pic}_T(G/P)$. The following is our main result in this appendix, and it has also been proved in [AMSS23, Theorem 9.10] using the Chevalley formula for the cohomological stable envelopes from [Su16, Theorem 3.7]. Here we give a direct proof based on the action of the degenerate affine Hecke algebra.

Theorem A.4. *For any $w \in W^P$ and $\lambda \in X^*(T)_P$, the following holds in $H^*_T(G/P)$:*

$$c_1^T(\mathcal{L}_\lambda) \cup c_{\mathrm{SM}}(X(wW_P)^\circ) = w(\lambda) c_{\mathrm{SM}}(X(wW_P)^\circ) - \sum_{\substack{\alpha>0 \\ ws_\alpha < w}} \langle \lambda, \alpha^\vee\rangle c_{\mathrm{SM}}(X(ws_\alpha W_P)^\circ),$$

$$c_1^T(\mathcal{L}_\lambda) \cup s_{\mathrm{M}}(Y(wW_P)^\circ) = w(\lambda) s_{\mathrm{M}}(Y(wW_P)^\circ) - \sum_{\substack{\alpha>0 \\ ws_\alpha > w}} \langle \lambda, \alpha^\vee\rangle s_{\mathrm{M}}(Y(ws_\alpha W_P)^\circ).$$

Proof. Applying the Hecke action Ψ in Lemma A.2 to the equation in Lemma A.1, and acting on the point class $[X(\mathrm{id})]$, we get

$$\begin{aligned} c_1^T(\mathcal{L}_\lambda) \cup c_{\mathrm{SM}}(X(wW_P)^\circ) &= c_1^T(\mathcal{L}_\lambda) \cup \mathcal{T}_w^L([X(\mathrm{id})]) = \mathcal{T}_w^L(c_1^T(\mathcal{L}_\lambda) \cup [X(\mathrm{id})]) = \mathcal{T}_w^L(\lambda \cdot [X(\mathrm{id})]) \\ &= \Psi(T_w x_\lambda)([X(\mathrm{id})]) = \Psi(x_{w\lambda} T_w - \sum_{\substack{\alpha>0 \\ ws_\alpha < w}} \langle \lambda, \alpha^\vee\rangle T_{ws_\alpha})([X(\mathrm{id})]) \\ &= w(\lambda) c_{\mathrm{SM}}(X(wW_P)^\circ) - \sum_{\substack{\alpha>0 \\ ws_\alpha < w}} \langle \lambda, \alpha^\vee\rangle c_{\mathrm{SM}}(X(ws_\alpha W_P)^\circ). \end{aligned}$$

The second equality follows from the fact that the left operator $\mathcal{T}_w^L$ commutes with $c_1^T(\mathcal{L}_\lambda)$ because the latter is Weyl-group invariant, as $\mathcal{L}_\lambda$ is a G-equivariant line bundle; see [MNS22b]. Finally, the Chevalley formula for the SM classes follows from the one on CSM via the duality in (47), similar to the proof of Lemma 5.1 above. $\square$

Appendix B. An example of the λ-chain formula

We consider Lie type A_2, with the Weyl group $W = S_3$, with $\lambda = 2\varpi_1 + \varpi_2$, and $w = s_2 s_1$. We can find an alcove walk $p_{-\lambda}$ from $A_\circ$ to $A_\circ - \lambda$ as indicated by the red path in Figure 2, right. This gives the corresponding reduced expression $v_{-\lambda} = s_2 s_1 s_2 s_0 s_1 s_2$ and the corresponding alcove path

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} A_2 \xrightarrow{-\beta_3} A_3 \xrightarrow{-\beta_4} A_4 \xrightarrow{-\beta_5} A_5 \xrightarrow{-\beta_6} A_6 = A_\circ - \lambda \; (A_i = r_i A_{i-1}, 1 \le i \le 6),$$

with λ-chain $\beta_1 = \alpha_2,\ \beta_2 = \alpha_1 + \alpha_2,\ \beta_3 = \alpha_1,\ \beta_4 = \alpha_1 + \alpha_2,\ \beta_5 = \alpha_1,\ \beta_6 = \alpha_1 + \alpha_2.$

The associated sequence of hyperplanes is

$$h_1 = H_{\alpha_2,0}, h_2 = H_{\alpha_1+\alpha_2,0}, h_3 = H_{\alpha_1,0}, h_4 = H_{\alpha_1+\alpha_2,-1}, h_5 = H_{\alpha_1,-1}, h_6 = H_{\alpha_1+\alpha_2,-2}.$$

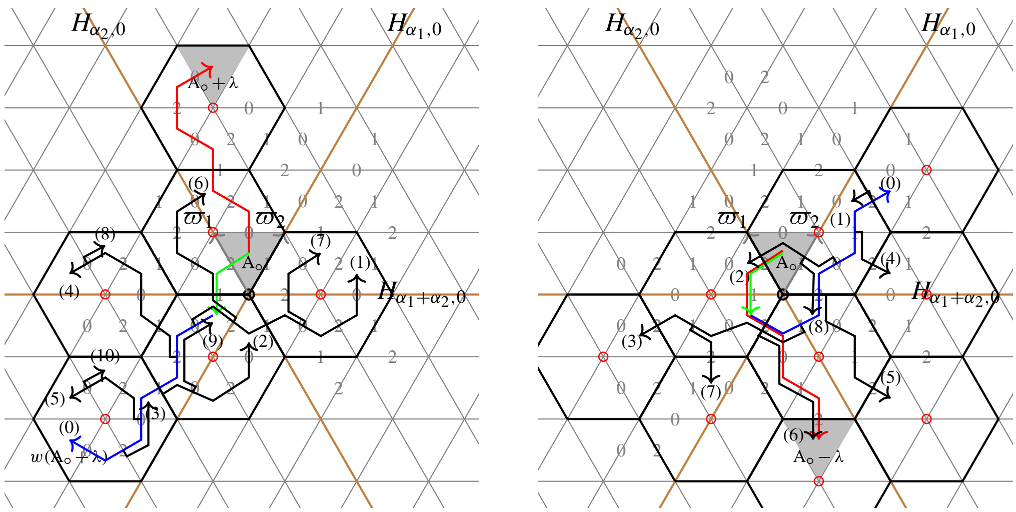


Figure 2. Left: alcove walk p_λ from $A_\circ$ to $A_\circ + \lambda$; $p_w = c_2^- c_1^-$, $p_\lambda = c_0^+ c_2^+ c_1^+ c_0^+ c_2^+ c_0^+$. Right: alcove walk $p_{-\lambda}$ from $A_\circ$ to $A_\circ - \lambda$; $p_w = c_2^- c_1^-$, $p_{-\lambda} = c_2^- c_1^- c_2^- c_0^- c_1^- c_2^-$.

We first calculate $c_{u,\mu}^{w,\lambda}$ according to the formula Theorem 3.10. In the proof we introduced λ-shifted (reversed) hyperplane sequence which can be seen in the diagram on the left. We have

$$h_1' = H_{\alpha_1+\alpha_2,1}, \quad h_2' = H_{\alpha_1,1}, \quad h_3' = H_{\alpha_1+\alpha_2,2}, \quad h_4' = H_{\alpha_1,2}, \quad h_5' = H_{\alpha_1+\alpha_2,3}, \quad h_6' = H_{\alpha_2,1}.$$

According to the formula, we need to choose $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J_<} w$.

If $u = s_1$, $J = \{6\}, \{4\}, \{2\}$ as $s_{\alpha_1+\alpha_2} = s_1 s_2 s_1$. For the weight μ, when $J = \{6\}$, we need to calculate $\mu = w\tilde{r}_{h_6}(\lambda)$. But as is explained in the proof, $\tilde{r}_{h_6} = \hat{r}_{h_1'}$, so, $\mu = w\hat{r}_{h_1'}(\lambda) = w(-\varpi_2) = -\varpi_1 + \varpi_2$. Likewise when $J = \{4\}$, $\mu = w\tilde{r}_{h_4}(\lambda) = w\hat{r}_{h_3'}(\lambda) = -\varpi_2$, and when $J = \{2\}$, $\mu = w\tilde{r}_{h_2}(\lambda) = w\hat{r}_{h_5'}(\lambda) = \varpi_1 - 3\varpi_2$. If $u = \mathrm{id}$, there are two possible Bruhat chains $u = \mathrm{id} < s_1 < s_2 s_1 = w$ and $u = \mathrm{id} < s_2 < s_2 s_1 = w$. For the first case, $J = \{5, 6\}, \{3, 6\}, \{3, 4\}$, and for the second case, $J = \{1, 5\}, \{1, 3\}$. When $J = \{5, 6\}$, $\mu = w\tilde{r}_{h_6}\tilde{r}_{h_5}(\lambda) = w\hat{r}_{h_1'}\hat{r}_{h_2'}(\lambda) = w\hat{r}_{h_1'}(2\varpi_2) = w(-\varpi_1 + \varpi_2) = \varpi_1$. All the possibilities and the corresponding (folded) alcove walks $\tilde{p}_\lambda$ are listed in the table below, followed by the coefficients $c_{u,\mu}^{w,\lambda}$ obtained from it. (We can also see the bijection of Lemma 2.7; see Figure 2, left.)

$p_w\tilde{p}_\lambda$	$\tilde{p}_\lambda$	$\mathcal{M}$	J	$\varphi(p)=u$	$\mathrm{wt}(p)=w\tilde{r}^\lambda_{J_>}(\lambda)$
(0)	$c_0^-c_2^-c_1^-c_0^-c_2^-c_0^+$	$\{\}$	$\{\}$	s_2s_1	$\varpi_1-3\varpi_2$
(1)	$f_0^+c_2^-c_1^+c_0^-c_2^+c_0^+$	$\{h_1\}$	$\{6\}$	s_1	$-\varpi_1+\varpi_2$
(2)	$c_0^-c_2^-f_1^+c_0^-c_2^+c_0^+$	$\{h_3\}$	$\{4\}$	s_1	$-\varpi_2$
(3)	$c_0^-c_2^-c_1^-c_0^-f_2^+c_0^+$	$\{h_5\}$	$\{2\}$	s_1	$\varpi_1-3\varpi_2$
(4)	$c_0^-f_2^+c_1^+c_0^+c_2^+c_0^-$	$\{h_2\}$	$\{5\}$	s_2	$2\varpi_1-2\varpi_2$
(5)	$c_0^-c_2^-c_1^-f_0^+c_2^+c_0^-$	$\{h_4\}$	$\{3\}$	s_2	$\varpi_1-3\varpi_2$
(6)	$f_0^+f_2^+c_1^+c_0^+c_2^+c_0^+$	$\{h_1,h_2\}$	$\{5,6\}$	id	ϖ_1
(7)	$f_0^+c_2^-c_1^+f_0^+c_2^+c_0^+$	$\{h_1,h_4\}$	$\{3,6\}$	id	$-\varpi_1+\varpi_2$
(8)	$c_0^-f_2^+c_1^+c_0^+c_2^+f_0^+$	$\{h_2,h_6\}$	$\{1,5\}$	id	$2\varpi_1-2\varpi_2$
(9)	$c_0^-c_2^-f_1^+f_0^+c_2^+c_0^+$	$\{h_3,h_4\}$	$\{3,4\}$	id	$-\varpi_2$
(10)	$c_0^-c_2^-c_1^-f_0^+c_2^+f_0^+$	$\{h_4,h_6\}$	$\{1,3\}$	id	$\varpi_1-3\varpi_2$

$c^{w,\lambda}_{s_2s_1,\mu}=1$ for $\mu=\varpi_1-3\varpi_2$,
$c^{w,\lambda}_{s_1,\mu}=(q-1)q^{-1}$ for $\mu=-\varpi_1+\varpi_2,-\varpi_2,\varpi_1-3\varpi_2$,
$c^{w,\lambda}_{s_2,\mu}=(q-1)q^{-1}$ for $\mu=2\varpi_1-2\varpi_2,\varpi_1-3\varpi_2$,
$c^{w,\lambda}_{\mathrm{id},\mu}=(q-1)^2q^{-2}$ for $\mu=\varpi_1,-\varpi_1+\varpi_2,2\varpi_1-2\varpi_2,-\varpi_2,\varpi_1-3\varpi_2$.

Next we calculate $c^{w,-\lambda}_{u,\mu}$ according to Theorem 3.9. For this case we need to chose $J\subset\{1,2,\dots,l\}$ such that $u\xrightarrow{J_>}w$.

For example, if $u=\mathrm{id}$, then there are three possible J corresponding to the Bruhat chains $u<us_{\beta_3}<us_{\beta_3}s_{\beta_2}=w$, $u<us_{\beta_5}<us_{\beta_5}s_{\beta_2}=w$, $u<us_{\beta_5}<us_{\beta_5}s_{\beta_4}=w$. For the first case the weight μ can be calculated (see Figure 2, right) as

$$\mu=w\hat{r}_{J_<}(-\lambda)=s_2s_1\hat{r}_{h_2}\hat{r}_{h_3}(-\lambda)=s_2s_1(3\varpi_1-2\varpi_2)=-2\varpi_1-\varpi_2.$$

All the possible J and corresponding alcove walks $\tilde{p}_{-\lambda}$ are listed in the table below.

$p_w\tilde{p}_{-\lambda}$	$\tilde{p}_{-\lambda}$	$\mathcal{M}$	J	$\varphi(p)=u$	$wt(p)=w\hat{r}_{J_<}(-\lambda)$
(0)	$c_2^-c_1^+c_2^+c_0^+c_1^+c_2^+$	$\{\}$	$\{\}$	s_2s_1	$-\varpi_1+3\varpi_2$
(1)	$c_2^-c_1^+c_2^+c_0^+c_1^+f_2^-$	$\{h_6\}$	$\{6\}$	s_1	ϖ_2
(2)	$c_2^-c_1^+c_2^+f_0^-c_1^+c_2^-$	$\{h_4\}$	$\{4\}$	s_1	$\varpi_1-\varpi_2$
(3)	$c_2^-f_1^-c_2^+c_0^-c_1^+c_2^-$	$\{h_2\}$	$\{2\}$	s_1	$2\varpi_1-3\varpi_2$
(4)	$c_2^-c_1^+c_2^+c_0^+f_1^-c_2^-$	$\{h_5\}$	$\{5\}$	s_2	$-2\varpi_1+2\varpi_2$
(5)	$c_2^-c_1^+f_2^-c_0^-c_1^-c_2^-$	$\{h_3\}$	$\{3\}$	s_2	$-3\varpi_1+\varpi_2$
(6)	$c_2^-f_1^-f_2^-c_0^-c_1^-c_2^-$	$\{h_2,h_3\}$	$\{2,3\}$	id	$-2\varpi_1-\varpi_2$
(7)	$c_2^-f_1^-c_2^+c_0^-f_1^-c_2^-$	$\{h_2,h_5\}$	$\{2,5\}$	id	$-2\varpi_2$
(8)	$c_2^-c_1^+c_2^+f_0^-f_1^-c_2^-$	$\{h_4,h_5\}$	$\{4,5\}$	id	$-\varpi_1$

The coefficients $c_{u,\mu}^{w,-\lambda}$ obtained from the table are

$$\begin{aligned}
&c_{s_2 s_1,\mu}^{w,-\lambda} = 1 \text{ for } \mu = -\varpi_1 + 3\varpi_2,\\
&c_{s_1,\mu}^{w,-\lambda} = (1-q)q^{-1} \text{ for } \mu = \varpi_2,\, \varpi_1 - \varpi_2,\, 2\varpi_1 - 3\varpi_2,\\
&c_{s_2,\mu}^{w,-\lambda} = (1-q)q^{-1} \text{ for } \mu = -2\varpi_1 + 2\varpi_2,\, -3\varpi_1 + \varpi_2,\\
&c_{\mathrm{id},\mu}^{w,-\lambda} = (1-q)^2 q^{-2} \text{ for } \mu = -2\varpi_1 - \varpi_2,\, -2\varpi_2,\, -\varpi_1.
\end{aligned}$$

Acknowledgments

H. Naruse and C. Su thank Arun Ram and Andrei Okounkov for answering their questions and providing references; L. C. Mihalcea and Su thank Bogdan Ion for a related collaboration on Hall–Littlewood polynomials, and Paolo Aluffi and Jörg Schürmann for previous collaborations on motivic Chern classes and CSM classes; Mihalcea thanks Andrzej Weber for stimulating discussions.

This project started in Spring 2021 while Mihalcea was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Spring 2021 semester in Combinatorial Algebraic Geometry. He warmly thanks the institute for the excellent working conditions. Mihalcea was also partially supported by NSF grant DMS-2152294 and he gratefully acknowledges the support of Charles Simonyi Endowment, which provided funding for the membership at the Institute of Advanced Study during the 2024–25 Special Year in Algebraic and Geometric Combinatorics. C. Su acknowledges support from National Key R&D Program of China (no. 2024YFA1014700).

References

[AGM11] D. Anderson, S. Griffeth, and E. Miller, "Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces", *J. Eur. Math. Soc.* **13**:1 (2011), 57–84. MR

[AMSS23] P. Aluffi, L. C. Mihalcea, J. Schürmann, and C. Su, "Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells", *Duke Math. J.* **172**:17 (2023), 3257–3320. MR

[AMSS24a] P. Aluffi, L. C. Mihalcea, J. Schürmann, and C. Su, "Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman's problem", *Ann. Sci. Éc. Norm. Supér.* (4) **57**:1 (2024), 87–141. MR

[AMSS24b] P. Aluffi, L. C. Mihalcea, J. C. Schürmann, and C. C. Su, "From motivic Chern classes of Schubert cells to their Hirzebruch and CSM classes", pp. 1–52 in *A glimpse into geometric representation theory*, Contemp. Math. **804**, Amer. Math. Soc., 2024. MR

[AO21] M. Aganagic and A. Okounkov, "Elliptic stable envelopes", *J. Amer. Math. Soc.* **34**:1 (2021), 79–133.

[BBBG24] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson, "Colored vertex models and Iwahori Whittaker functions", *Selecta Math.* (*N.S.*) **30**:4 (2024), Paper No. 78, 58. MR

[BCMP18] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin, "A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties", *Algebr. Geom.* **5**:5 (2018), 568–595. MR

[BM15] A. S. Buch and L. C. Mihalcea, "Curve neighborhoods of Schubert varieties", *J. Differential Geom.* **99**:2 (2015), 255–283. MR

[Bri02] M. Brion, "Positivity in the Grothendieck group of complex flag varieties", *J. Algebra* **258**:1 (2002), 137–159. MR

[BS81] J.-P. Brasselet and M.-H. Schwartz, "Sur les classes de Chern d'un ensemble analytique complexe", pp. 93–147 in *The Euler–Poincaré characteristic* (*French*), Astérisque **83**, Soc. Math. France, Paris, 1981.

[BSY10] J.-P. Brasselet, J. Schürmann, and S. Yokura, "Hirzebruch classes and motivic Chern classes for singular spaces", *J. Topol. Anal.* **2**:1 (2010), 1–55. MR

[Buc02] A. S. Buch, "A Littlewood–Richardson rule for the K-theory of Grassmannians", *Acta Math.* **189**:1 (2002), 37–78. MR

[CG10] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997. MR

[CS80] W. Casselman and J. Shalika, "The unramified principal series of p-adic groups, II: The Whittaker function", *Compos. Math.* **41**:2 (1980), 207–231.

[EM97] S. Evens and I. Mirković, "Fourier transform and the Iwahori–Matsumoto involution", *Duke Math. J.* **86**:3 (1997), 435–464. MR

[FGSX24] N. J. Fan, P. L. Guo, C. Su, and R. Xiong, "A Pieri type formula for motivic Chern classes of Schubert cells in Grassmannians", preprint, 2024. arXiv 2402.04500

[FGX22] N. J. Y. Fan, P. L. Guo, and R. Xiong, "Pieri and Murnaghan–Nakayama type Rules for Chern classes of Schubert Cells", preprint, 2022. arXiv 2211.06802

[FRW21] L. M. Feher, R. Rimanyi, and A. Weber, "Motivic Chern classes and K-theoretic stable envelopes", *Proc. L. Math. Soc.* **122**:1 (2021), 153–189.

[GR04] S. Griffeth and A. Ram, "Affine Hecke algebras and the Schubert calculus", *European J. Combin.* **25**:8 (2004), 1263–1283. MR

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1990. MR

[KL87] D. Kazhdan and G. Lusztig, "Proof of the Deligne–Langlands conjecture for Hecke algebras", *Inv. Math.* **87**:1 (1987), 153–215.

[Kum02] S. Kumar, *Kac–Moody groups, their flag varieties and representation theory*, Progress in Mathematics **204**, Birkhäuser, Boston, 2002. MR

[KW25] J. Koncki and A. Weber, "Hecke algebra action on twisted motivic Chern classes and K-theoretic stable envelopes", *Math. Ann.* **391**:2 (2025), 1899–1964. MR

[Len11] C. Lenart, "Hall–Littlewood polynomials, alcove walks, and fillings of Young diagrams", *Disc. Math.* **311**:4 (2011), 258–275.

[Li92] J.-S. Li, "Some results on the unramified principal series of p-adic groups", *Math. Ann.* **292**:1 (1992), 747–761.

[LP07] C. Lenart and A. Postnikov, "Affine Weyl groups in K-theory and representation theory", *Int. Math. Res. Not.* **2007**:12 (2007), art. id. rnm038, 65 pp. MR

[LP08] C. Lenart and A. Postnikov, "A combinatorial model for crystals of Kac–Moody algebras", *Trans. Amer. Math. Soc.* **360**:8 (2008), 4349–4381.

[Mac74] R. D. MacPherson, "Chern classes for singular algebraic varieties", *Ann. of Math.* (2) **100** (1974), 423–432.

[Mac96] I. G. Macdonald, "Affine Hecke algebras and orthogonal polynomials", pp. exposé 797, pp. 189–207 in *Séminaire Bourbaki*, *volume* 1994/1995, Astérisque **237**, Société mathématique de France, 1996.

[Mac98] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford university press, 1998.

[MNS22a] L. C. Mihalcea, H. Naruse, and C. Su, "Hook formulae from Segre–MacPherson classes", preprint, 2022. To appear in *Alg. Comb.* arXiv 2203.16461

[MNS22b] L. C. Mihalcea, H. Naruse, and C. Su, "Left Demazure–Lusztig operators on equivariant (quantum) cohomology and K-theory", *Int. Math. Res. Not.* **2022**:16 (2022), 12096–12147. MR

[MNS24] L. C. Mihalcea, H. Naruse, and C. Su, "Hook formula for Coxeter groups via the twisted group ring", *Proc. Japan Acad. Ser. A Math. Sci.* **100**:6 (2024), 31–36. MR

[MO19] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, Astérisque **408**, Soc. Math. de France, 2019.

[MS22] L. C. Mihalcea and C. Su, "Whittaker functions from motivic Chern classes", *Transform. Groups* **27**:3 (2022), 1045–1067. MR

[Nie74] H. A. Nielsen, "Diagonalizably linearized coherent sheaves", *Bull. Soc. Math. France* **102** (1974), 85–97. MR

[NO19] H. Naruse and S. Okada, "Skew hook formula for d-complete posets via equivariant K-theory", *Alg. Combin.* **2**:4 (2019), 541–571.

[Ohm06] T. Ohmoto, “Equivariant Chern classes of singular algebraic varieties with group actions”, *Math. Proc. Cambridge Philos. Soc.* **140**:1 (2006), 115–134.

[Oko27] A. Okounkov, “Lectures on K-theoretic computations in enumerative geometry”, pp. 251–380 in *Geometry of moduli spaces and representation theory*, IAS/Park City Math. Ser. **24**, Amer. Math. Soc., 2017. MR

[OS22] A. Okounkov and A. Smirnov, “Quantum difference equation for Nakajima varieties”, *Inv. Math.* **229**:3 (2022), 1203–1299.

[PR99] H. Pittie and A. Ram, “A Pieri–Chevalley formula in the K-theory of a G/B-bundle”, *Electron. Res. Announc. Amer. Math. Soc.* **5** (1999), 102–107. MR

[Pro99] R. A. Proctor, “Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets”, *J. Algebraic Combin.* **9**:1 (1999), 61–94. MR

[Ram06] A. Ram, “Alcove Walks, Hecke Algebras, Spherical Functions, Crystals and Column Strict Tableaux”, *Pure Appl. Math. Q.* **2**:4 (2006), 963–1013.

[RTV15] R. Rimányi, V. Tarasov, and A. Varchenko, “Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety”, *J. Geom. Phys.* **94** (2015), 81–119.

[RTV19] R. Rimányi, V. Tarasov, and A. Varchenko, “Elliptic and K-theoretic stable envelopes and Newton polytopes”, *Selecta Math.* **25**:1 (2019), 16.

[Sch06] C. Schwer, “Galleries, Hall–Littlewood polynomials, and structure constants of the spherical Hecke algebra”, *Int. Math. Res. Notices* **2006**:9 (2006), 75395–75395.

[Sch65a] M.-H. Schwartz, “Classes caractéristiques définies par une stratification d’une variété analytique complexe, I”, *C. R. Math. Acad. Sci. Paris* **260** (1965), 3262–3264.

[Sch65b] M.-H. Schwartz, “Classes caractéristiques définies par une stratification d’une variété analytique complexe, II”, *C. R. Math. Acad. Sci. Paris* **260** (1965), 3535–3537.

[Ste01] J. R. Stembridge, “Minuscule elements of Weyl groups”, *J. Algebra* **235**:2 (2001), 722–743.

[Su16] C. Su, “Equivariant quantum cohomology of cotangent bundle of G/P”, *Adv. Math.* **289** (2016), 362–383. MR

[SZZ20] C. Su, G. Zhao, and C. Zhong, “On the K-theory stable bases of the Springer resolution”, *Ann. Sci. Éc. Norm. Supér.* (4) **53**:3 (2020), 663–711. MR

[SZZ21] C. Su, G. Zhao, and C. Zhong, “Wall-crossings and a categorification of K-theory stable bases of the Springer resolution”, *Compos. Math.* **157**:11 (2021), 2341–2376.

Communicated by Sergey Fomin
Received 2024-02-12 Revised 2024-03-06 Accepted 2025-03-03

lmihalce@vt.edu *Department of Mathematics, Virginia Tech University, Blacksburg, VA, United States*

hnaruse@yamanashi.ac.jp *Graduate School of Education, University of Yamanashi, Kofu, Japan*

changjiansu@mail.tsinghua.edu.cn *Yau Mathematical Sciences Center, Tsinghua University, Beijing, China*

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.525

The Mahler measure of exact polynomials in three variables

Trieu Thu Ha

We prove that under certain explicit conditions, the Mahler measure of a three-variable polynomial can be expressed in terms of elliptic curve L-values and Bloch–Wigner dilogarithmic values, conditionally on Beilinson's conjecture. In some cases, these dilogarithmic values simplify to Dirichlet L-values. The proof involves a construction of an element in $K_4^{(3)}$ of a smooth projective curve over a number field. This generalizes a result of Lalín (2015) for the polynomial $z+(x+1)(y+1)$. We apply our method to several other Mahler measure identities conjectured by Boyd and Brunault.

Introduction

Let $P(x_1,\dots,x_n) \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be a nonzero Laurent polynomial. The (logarithmic) Mahler measure of P is defined by

$$\mathrm{m}(P) = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \log|P(x_1,\dots,x_n)|\,\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}, \tag{0-1}$$

where $\mathbb{T}^n : |x_1| = \cdots = |x_n| = 1$ is the n-dimensional torus. This quantity was firstly introduced by Mahler [27] in 1962.

In 1997, Deninger [12] linked the Mahler measure of polynomials $P(x_1,\dots,x_n)$ under certain conditions to the motivic cohomology of V_P, where V_P is the zero locus of P in $\mathbb{C}^n$. This allowed him to place the Mahler measure in the very general framework of Beilinson's conjectures on special values of L-functions. More precisely, Deninger defined the chain

$$\Gamma = \big\{(x_1,\dots,x_n)\in\mathbb{C}^n : P(x_1,\dots,x_n)=0,\ |x_1|=\cdots=|x_{n-1}|=1,\ |x_n|\ge 1\big\}.$$

He showed that if Γ is contained in the regular locus V_P^{reg} of V_P, then there is a differential $(n-1)$-form $\eta(x_1,\dots,x_n)$ on $\mathbb{G}_m^n$ such that its restriction to V_P represents the regulator of the Milnor symbol $\{x_1,\dots,x_n\}$, and we have

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + \frac{(-1)^{n-1}}{(2\pi i)^{n-1}}\int_\Gamma \eta(x_1,\dots,x_n), \tag{0-2}$$

This work was performed within the framework of the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).
MSC2020: primary 19F27; secondary 11G55, 11R06, 19E15.
Keywords: Mahler's measure, regulator, polylogarithmic complex, motivic cohomology, residue.

where $\tilde{P}$ is the leading coefficient of P seen as a polynomial in x_n.

From now on we assume that P has rational coefficients and Γ is contained in V_P^{reg}. If $\partial\Gamma = \varnothing$, then Γ is a cycle. Then Deninger found out that in certain situations, identity (0-2) together with Beilinson's conjecture imply that $\mathrm{m}(P)$ can be expressed in terms of the L-function of the motive $H^{n-1}(\overline{V}_P)$, where $\overline{V}_P$ is a smooth compactification of V_P. As an example, he showed that under the Beilinson conjecture,

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \stackrel{?}{=} L'(E_{15}, 0), \tag{0-3}$$

where E_{15} is the elliptic curve (of conductor 15) defined by $x + 1/x + y + 1/y + 1 = 0$. In this example, $\partial\Gamma \neq \varnothing$ but a symmetry argument reduces this to the case $\partial\Gamma = \varnothing$. It was completely shown (without assuming the Beilinson conjecture) by Rogers and Zudilin [33] in 2014.

Boyd [3] conjectured, based on numerical evidence, that

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \stackrel{?}{=} r_k\, L'(E_{N(k)}, 0), \tag{0-4}$$

where $k \in \mathbb{Z} \setminus \{0, \pm 4\}$, $r_k \in \mathbb{Q}^\times$ and $E_{N(k)}$ is the elliptic curve (of conductor $N(k)$) obtained as a smooth compactification of the zero set of $P_k = x + 1/x + y + 1/y + k$. Until now, identity (0-4) is only proved for a finite number of k:

$$k \in \{-4\sqrt{2}, -2\sqrt{2}, 1, 2, 3, 2\sqrt{2}, 3\sqrt{2}, 5, 8, 12, 16, i, 2i, 3i, 4i, \sqrt{2}i\},$$

by the works of Brunault, Lalín, Rodriguez-Villegas, Rogers, Samart, and Zudilin (see [5; 22; 26; 25; 36; 32; 33; 39]).

The case $\partial\Gamma \neq \varnothing$ is more difficult, Maillot [28] suggested we should look at the variety

$$W_P := V_P \cap V_{P^*}, \tag{0-5}$$

where $P^*(x_1, \ldots, x_n) = \bar{P}(x_1^{-1}, \ldots, x_n^{-1})$. We call W_P the *Maillot variety*. If P is an *exact* polynomial, i.e., $\eta = d\omega$, where ω is a differential form on V_P^{reg}, then Stokes' theorem gives

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + \frac{(-1)^{n-1}}{(2\pi i)^{n-1}} \int_{\partial\Gamma} \omega.$$

Moreover, $\partial\Gamma$ is contained in W_P, hence we can hope that $\mathrm{m}(P)$ is related to the cohomology of W_P. In this direction, Lalín showed the following result.

Theorem 0.1 [23, Theorem 2]. *Assume that $P \in \mathbb{Q}[x, y, z]$ is irreducible and satisfies the following conditions*:

(i) *W_P is birationally equivalent to an elliptic curve E over $\mathbb{Q}$.*

(ii) *$\partial\Gamma$ defines an element of $H_1(E(\mathbb{C}), \mathbb{Z})^+$, the invariant part of $H_1(E(\mathbb{C}), \mathbb{Z})$ under the action induced by the complex conjugation on $E(\mathbb{C})$.*

(iii) *$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j$ in $\bigwedge^3 \mathbb{Q}(V_P)^\times$, for some functions $f_j, g_j \in \mathbb{Q}(V_P)^\times$.*

(iv) $x \wedge y \wedge z = 0$ *in* $\bigwedge^3 \mathbb{Q}(E)^\times$.

(v) $\sum_j v_p(g_j)\{f_j(p)\}_2 = 0$ *in* $\mathbb{Z}[\mathbb{P}^1_{\bar{\mathbb{Q}}}]/R_2(\bar{\mathbb{Q}})$ *for all* $p \in E(\bar{\mathbb{Q}})$.

Here $R_2(\bar{\mathbb{Q}})$ *is the subgroup generated by the five-term relations* (2-1), *and* $v_p(g_j)$ *is the vanishing order at* p *of* g_j *seen as a function on* E. *Then under Beilinson's conjecture*,

$$\mathrm{m}(P) = m(\tilde{P}) + a \cdot L'(E, -1), \quad a \in \mathbb{Q}. \tag{0-6}$$

Condition (3) implies that P is exact (see Remark 4.3). In this article, we relax Lalín's conditions in order to deal with Mahler measure identities which are more general than (0-6), for example, containing also Dirichlet L-values. We only assume that W_P is of genus 1 and we do not require the conditions (iv)-(v) above. Recall that the Bloch–Wigner dilogarithm function $D : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ is defined by

$$D(z) = \begin{cases} \mathrm{Im}\big(\sum_{k=1}^{\infty} z^k/k^2\big) + \arg(1-z)\log|z| & (|z| \le 1), \\ -D(1/z) & (|z| \ge 1). \end{cases} \tag{0-7}$$

For any field F, we denote by $\mathcal{B}(F)$ the Bloch group of F tensored with $\mathbb{Q}$ (see [38, Section 2]). Let τ be the involution of $\mathbb{G}_m^3$ given by $(x, y, z) \mapsto (1/x, 1/y, 1/z)$. Since P has rational coefficients, τ induces an involution of W_P. For A an abelian group, we write $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. Let us state our main theorem here.

Theorem 0.2. *Assume* $P \in \mathbb{Q}[x, y, z]$ *is irreducible and that* W_P *is a curve of genus* 1. *Let* C *be the normalization of* W_P. *Suppose that*

$$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j \quad \text{in } \bigwedge^3 \mathbb{Q}(V_P)^\times_{\mathbb{Q}}, \tag{0-8}$$

for some functions f_j, g_j *on* V_P. *Let* S *be the closed subscheme of* C *consisting of the zeros and poles of the functions* g_j *and* $g_j \circ \tau$ *on* C *for all* j. *Then, for* $p \in S$,

$$u_p := \sum_j v_p(g_j)\{f_j(p)\}_2 + v_p(g_j \circ \tau)\{f_j \circ \tau(p)\}_2$$

define elements in the Bloch group $\mathcal{B}(\mathbb{Q}(p))$, *where* $\mathbb{Q}(p)$ *is the residue field of* C *at* p.

Assume that the Deninger chain Γ *is contained in* V_P^{reg} *and that its boundary* $\partial\Gamma$ *is contained in* W_P^{reg}, *then* $\partial\Gamma$ *defines an element in* $H_1(C(\mathbb{C}), \mathbb{Z})^+$. *If* $u_p = 0$ *for all* $p \in S$, *then under Beilinson's conjecture for the curve* C (Conjecture 1.11), *we have*

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + a \cdot L'(E, -1) \qquad (a \in \mathbb{Q}),$$

where E *is the Jacobian of* C. *Otherwise, let* S' *be the subset of* S *consisting of the points* p *such that* $u_p \neq 0$. *Let* K *be the splitting field of* S' *in* $\mathbb{C}$. *Let* $\mathcal{O}$ *be any fixed point in* $S'(K)$. *Assume that the difference of any two geometric points in* $S'(K)$ *has finite order dividing* N *in* $E(K)$, *then under Beilinson's conjecture for the curve* C (Conjecture 1.11),

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + a \cdot L'(E, -1) + \frac{1}{4N\pi} \sum_{q \in S'(K) \setminus \{\mathcal{O}\}} b_q \cdot D(u_q) \qquad (a \in \mathbb{Q}, b_q \in \mathbb{Z}), \tag{0-9}$$

where for $q \in S'(K)$ supported on a closed point $p \in S'$, we define u_q to be the image of u_p under the map $\mathcal{B}(\mathbb{Q}(p)) \hookrightarrow \mathcal{B}(K)$ induced by the embedding $\mathbb{Q}(p) \hookrightarrow^q K$.

The rational number a comes from Beilinson's conjecture, and does not depend on the choice of $\mathcal{O}$, but the integer numbers b_q actually do. However, the D-values in identity (0-9) are independent of the choice of $\mathcal{O}$. Indeed, when we remove $\mathcal{O}$ from the sum, the complex conjugation of $\mathcal{O}$ maintains the D-values in identity (0-9). Now let us use Theorem 0.2 to investigate several conjectural Mahler measure identities of the following types:

(a) *Pure identities*: $\mathrm{m}(P) \stackrel{?}{=} a \cdot L'(E,-1)$ for some $a \in \mathbb{Q}^\times$. Table 1 consists of pure identities that we prove up to a rational factor and conditionally on Beilinson's conjecture. Identity 3 in Table 1 was studied by Lalín [23]. By contrast, Theorem 0.1 of Lalín does not apply to identity 5 in Table 1 as it violates condition (v). Identities 6, 7 and 8 are conjectured by Lalín and Nair in [24], in fact, they showed that by some changes of variables, the Mahler measure of polynomials 5, 6, 7 and 8 are equal. Moreover, from Table 1, we have the following relations (under the Beilinson conjecture):

$$\mathrm{m}((1+x)(1+y)(x+y)+z) \stackrel{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}(1+x+y+z+xy+xz+yz),$$
$$\mathrm{m}(1+(x+1)y+(x-1)z) \stackrel{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}((x+1)^2(y+1)+z).$$

We also give in Section 5.1(g) an example showing that Theorem 0.2 does not imply that

$$\mathrm{m}((1+x)(1+y)+(1-x-y)z) = \tfrac{1}{288} L'(E_{450c1},-1). \tag{0-10}$$

	P	E	a	reference
1	$(1+x)(1+y)(x+y)+z$	$14a4$	-3	[9, p. 81]
2	$1+x+y+z+xy+xz+yz$	$14a4$	$-\frac{5}{2}$	[6]
3	$(x+1)(y+1)+z$	15a8	-2	[4]
4	$(x+1)^2+(1-x)(y+z)$	$20a1$	-2	[4], [9, p. 81]
5	$1+(x+1)y+(x-1)z$	$21a1$	$-\frac{5}{4}$	[4]
6	$\frac{1}{2}(x+2)+(x^2+x+1)y+(x^2-1)z$	$21a1$	$-\frac{5}{4}$	[24]
7	$\frac{1}{2}(x^2-2x+2)+(x^4-x^3+x^2-x+1)y+(x^4-x^3+x-1)z$	$21a1$	$-\frac{5}{4}$	[24]
8	$\frac{1}{2}(x^4+x+2)+(x^5+x^4+x+1)y+(x^5-1)z$	$21a1$	$-\frac{5}{4}$	[24]
9	$(x+1)^2(y+1)+z$	$21a4$	$-\frac{3}{2}$	[4], [9, p. 81]
10	$(1+x)^2+y+z$	$24a4$	-1	[4]
11	$1+x+y+z+xy+xz+yz-xyz$	$36a1$	$-\frac{1}{2}$	[6]
12	$1+xy+(1+x+y)z$	$90b1$	$-\frac{1}{20}$	[6]
13	$(x+1)^2+(x-1)^2y+z$	$225c2$	$-\frac{1}{48}$	[4; 6]

Table 1. Pure identities $\mathrm{m}(P) \stackrel{?}{=} a \cdot L'(E,-1)$.

	P	E	a	b_1	b_2
1	$1+(x^2-x+1)y+(x^2+x+1)z$	$45a2$	$-\frac{1}{6}$	1	0
2	$x^2+1+(x+1)^2y+(x^2-1)z$	$48a1$	$-\frac{1}{10}$	0	1

Table 2. Conjectural identities $\mathrm{m}(P) \stackrel{?}{=} a\cdot L'(E,-1)+b_1\cdot L'(\chi_{-3},-1)+b_2\cdot L'(\chi_{-4},-1)$. The reference for all entries is [6].

(b) *Identities with Dirichlet L-values*:

$$\mathrm{m}(P(x,y,z)) \stackrel{?}{=} a\cdot L'(E,-1)+\sum_{\chi} b_\chi\cdot L'(\chi,-1) \qquad (a\in\mathbb{Q}, b_\chi\in\mathbb{Q}^\times), \tag{0-11}$$

where χ are odd quadratic Dirichlet characters. In cases where the coefficient a is nonzero, such identities are referred to as *mixed identities*. Table 2 consists of mixed identities we prove (up to rational factors) under Beilinson's conjecture for genus 1 curves (see Sections 5.2(a) and 5.2(e)). The first polynomial in Table 2 does not satisfy conditions (iv)–(v) of Theorem 0.1 of Lalín. Moreover, the Maillot variety W_P is a curve of genus 1 and does not have any rational point, hence violates condition (i) also. For the second polynomial in Table 2, the Maillot variety W_P is a union of a line and a nonsingular curve of genus 1. We also give an example to which our theorem does not apply (see Section 5.2(d)):

$$\mathrm{m}(x^2+x+1+(x^2+x+1)y+(x-1)^2z) = -\tfrac{1}{12}L'(E_{72a1},-1)+\tfrac{3}{2}L'(\chi_{-3},-1). \tag{0-12}$$

This was conjectured by Brunault [6].

By a method of Lalín in [23, Example 4.2], we prove without assuming Beilinson's conjecture the Mahler measure identities in Table 3; they involve only Dirichlet L-values (see Section 5.2(b)–(c)).

Outline. The article contains five sections. In the first three, we recall some tools that needed for our constructions. In Sections 1.1–1.4, we recall the definitions of the Deligne cohomology and some facts about motivic cohomology. We give an explicit isomorphism between Chow motives of a genus 1 smooth

	P	b_1	b_2
1	$1+(x+1)(x^2+x+1)y+(x+1)^3z$	3	0
2	$x^2+1+(x^2+x+1)y+(x+1)^3z$	$\frac{7}{2}$	0
3	$x^2+1+(x+1)(x^2+x+1)y+(x+1)^3z$	$\frac{7}{2}$	0
4	$x^2+1+(x+1)(x^2+x+1)y+(x-1)(x^2-x+1)z$	0	$\frac{7}{3}$
5	$(x+1)(x^2+1)+(x+1)(x^2+x+1)y+(x-1)(x^2-x+1)z$	0	$\frac{7}{3}$
6	$x^2+1+(x+1)^2y+(x-1)^2z$	0	2
7	$x^2+1+(x+1)^3y+(x-1)^3z$	0	3
8	$(x+1)(x^2+1)+(x+1)^3y+(x-1)^3z$	0	3

Table 3. $\mathrm{m}(P) = b_1\cdot L'(\chi_{-3},-1)+b_2\cdot L'(\chi_{-4},-1)$. The reference for all entries is [6].

projective curve and its Jacobian. We then recall briefly the regulator maps and Beilinson's conjecture for smooth projective curves of genus 1. In Section 2, we bring back Goncharov's polylogarithmic complexes and his regulator maps on the cohomology of these complexes. In Section 3, we recall de Jeu's polylogarithmic complexes and discuss his maps connecting the cohomology of Goncharov's polylogarithmic complexes to the motivic cohomology. In particular, in Section 3.4, given a 2-cocycle in weight 3 Goncharov's polylogarithmic complex, we compare its Goncharov regulator and the Beilinson regulator of its image under the map defined in Section 3.2. In Section 4.1, given an exact polynomial P in $\mathbb{Q}[x, y, z]$, we construct an element in the Deligne cohomology of an open subset of the normalization C of the Maillot variety W_P (0-5). We then relate this element to the Mahler measure of P in Section 4.2. In Section 4.3, we construct explicitly an element in $K_4^{(3)}(C)$ satisfying that its Beilinson regulator has connection with the Deligne cohomology element constructed in Section 4.1. We then prove the main theorem in Section 4.4. We end the article with Section 5, where we study the conjectural Mahler measure identities mentioned above.

1. The Beilinson regulator map

1.1. ***Deligne cohomology.*** Deligne cohomology of a smooth complex algebraic variety X is firstly introduced by Deligne in 1972, it is given by the hypercohomology of

$$0 \to \mathbb{Z}(n) \to \mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^{n-1} \to 0, \tag{1-1}$$

where the constant sheaf $\mathbb{Z}(n) := (2\pi i)^n\mathbb{Z}$ is placed in degree 0 and Ω_X^j is the sheaf of holomorphic j-forms on X placed in degree $j+1$. Burgos [11] then showed that this hypercohomology can be the cohomology of a single complex. Let us recall briefly Burgos' construction (see [11; 9]).

Let X be a smooth complex algebraic variety of dimension d. Let $(\bar{X}, \iota)$ be a good compactification of X, i.e., that $\bar{X}$ is a smooth proper variety and $\iota : X \hookrightarrow \bar{X}$ is an open immersion such that $D := \bar{X} - \iota(X)$ is locally given by $z_1 \dots z_m = 0$ for some analytic local coordinates $z_1, \dots, z_d$ on $\bar{X}$ and $m \le d$.

Definition 1.1 [11, Proposition 1.1]. A complex smooth differential form ω on X *has logarithmic singularities along D* if locally ω belongs to the algebra generated by the smooth forms on $\bar{X}$ and $\log|z_i|$, dz_i/z_i, $d\bar{z}_i/\bar{z}_i$, for $1 \le i \le m$, where $z_1 \dots z_m = 0$ is a local equation of D. For $\Lambda \in \{\mathbb{R}, \mathbb{C}\}$, $E^n_{X,\Lambda}(\log D)$ denotes the space of such Λ-valued smooth differential forms of degree n on X.

We have $E^n_{X,\mathbb{C}}(\log D) = \bigoplus_{p+q=n} E^{p,q}_{X,\mathbb{C}}(\log D)$, where $E^{p,q}$ is the subspace of type (p, q)-forms. We denote by $\bar{\partial} : E^{p,q} \to E^{p,q+1}$ and $\partial : E^{p,q} \to E^{p+1,q}$ as the usual operators and $d = \partial + \bar{\partial}$. Burgos defined

$$E^*_{\log,\Lambda}(X) = \varinjlim_{(\bar{X},\iota)\in\mathcal{I}^{\mathrm{opp}}} E^*_{X,\Lambda}(\log D),$$

where $\mathcal{I}$ is the category of good compactification of X. He then introduced the following complex.

Definition 1.2 [11, Theorem 2.6]. For any integers $j, n \geq 0$, set

$$E_j(X)^n := \begin{cases} (2\pi i)^{j-1} E^{n-1}_{\log,\mathbb{R}}(X) \cap \left(\bigoplus_{p+q=n-1;\, p,q<j} E^{p,q}_{\log,\mathbb{C}}(X)\right) & \text{if } n \leq 2j-1, \\ (2\pi i)^{j} E^{n}_{\log,\mathbb{R}}(X) \cap \left(\bigoplus_{p+q=n;\, p,q\geq j} E^{p,q}_{\log,\mathbb{C}}(X)\right) & \text{if } n \geq 2j, \end{cases}$$

$$d^n\omega := \begin{cases} -\mathrm{pr}_j(d\omega) & \text{if } n < 2j-1, \\ -2\partial\bar{\partial}\omega & \text{if } n = 2j-1, \\ d\omega & \text{if } n \geq 2j, \end{cases}$$

where pr_j is the projection $\bigoplus_{p,q} \to \bigoplus_{p,q<j}$. Denote by $E_j(X)$ the complex $(E_j(X)^n, d^n)_{n\geq 0}$.

Definition 1.3 (Deligne cohomology [11, Corollary 2.7]). Let X be a smooth complex algebraic variety. The Deligne cohomology of X is the cohomology of the complex $E_j(X)$, that is,

$$H^n_{\mathcal{D}}(X, \mathbb{R}(j)) = H^n(E_j(X)) \quad \text{for } j, n \geq 0.$$

As the canonical map $E^*_{X,\mathbb{C}}(\log D) \to E^*_{\log,\mathbb{C}}(X)$ is a quasi-isomorphism by [10, Theorem 1.2], we can use $E^*_{X,\Lambda}(\log D)$ for some good compactification of X instead of $E^*_{\log,\Lambda}(X)$ in Definition 1.2.

Remark 1.4. For the case $j > \dim X \geq 1$ or $j > n$, $H^n_{\mathcal{D}}(X, \mathbb{R}(j))$ is canonically isomorphic to the de Rham cohomology $H^{n-1}(X, (2\pi i)^{j-1}\mathbb{R})$ by the canonical map sending a Deligne cohomology class to its de Rham cohomology class (see [9, Section 8.1]). For $j > 1$, we thus have $H^1_{\mathcal{D}}(\mathrm{Spec}\ (\mathbb{C}), \mathbb{R}(j)) \simeq \mathbb{R}(j-1)$.

Let X be a smooth variety over $\mathbb{R}$. Let $X(\mathbb{C})$ denote the set of complex points of $X \times_{\mathbb{R}} \mathbb{C}$. Denote by c the complex conjugation on $X(\mathbb{C})$. For a complex differential form ω on $X(\mathbb{C})$, we define an involution $F_{\mathrm{dR}}(\omega) := c^*(\bar{\omega})$. It acts on the complex $E_j(X(\mathbb{C}))$, hence acts on the Deligne cohomology.

Definition 1.5 [11, Remark 6.5]. Let X be a smooth variety over $\mathbb{R}$. The Deligne cohomology of X is defined by

$$H^n_{\mathcal{D}}(X, \mathbb{R}(j)) := H^n_{\mathcal{D}}(X \times_{\mathbb{R}} \mathbb{C}, \mathbb{R}(j))^+,$$

where "+" denotes the invariant part under the action of the involution F_{dR}.

Let X be a smooth real or complex variety, there is a cup-product in Deligne cohomology

$$\cup : H^n_{\mathcal{D}}(X, \mathbb{R}(j)) \otimes H^m_{\mathcal{D}}(X, \mathbb{R}(k)) \to H^{n+m}_{\mathcal{D}}(X, \mathbb{R}(j+k)), \tag{1-2}$$

(see [11, Theorem 3.3]). It is graded commutative (i.e., $\alpha \cup \beta = (-1)^{mn}\beta \cup \alpha$), and associative. In the case $n < 2j$, $m < 2k$, for $\alpha \in H^n_{\mathcal{D}}(X, \mathbb{R}(j))$ and $\beta \in H^m_{\mathcal{D}}(X, \mathbb{R}(k))$, we have that $\alpha \cup \beta$ is represented by

$$(-1)^n r_j(\alpha) \wedge \beta + \alpha \wedge r_k(\beta), \tag{1-3}$$

where $r_\ell(\alpha) := \partial(\alpha^{\ell-1,n-\ell}) - \bar{\partial}(\alpha^{n-\ell,\ell-1})$.

1.2. ***Chow motives.*** In this section, we discuss the Chow motives of smooth projective curves. For more details, we refer to [29] or [34]. Recall that the Chow groups of a variety X are $\mathrm{CH}^n(X) := Z^n(X)/\sim$, where $Z^n(X)$ is the free abelian group generated by irreducible subvarieties of X of codimension n, and $\sim$ is the rational equivalence (see [29, Section 1.2]). If $\phi : X \to Y$ is a morphism of varieties, one has the homomorphisms

$$\phi^* : \mathrm{CH}^n(Y) \to \mathrm{CH}^n(X), \qquad \phi_* : \mathrm{CH}^n(X) \to \mathrm{CH}^{n+\dim Y - \dim X}(Y).$$

Let k be a field and $r \geq 0$. Let $X, Y \in \mathrm{SmProj}(k)$. If X is of pure dimension d, the group of *correspondences of degree r* is given by $\mathrm{Corr}^r(X, Y) := \mathrm{CH}^{d+r}(X \times Y) \otimes \mathbb{Q}$. If $X = \coprod X_d$ is a decomposition of subschemes with X_d is of pure dimension d, then $\mathrm{Corr}^r(X, Y) := \bigoplus \mathrm{Corr}^r(X_d, Y)$. Let $X, Y, Z \in \mathrm{SmProj}(k)$ and $f \in \mathrm{Corr}^r(X, Y)$, $g \in \mathrm{Corr}^s(Y, Z)$, the composition of correspondences

$$\mathrm{Corr}^r(X, Y) \times \mathrm{Corr}^s(Y, Z) \to \mathrm{Corr}^{r+s}(X, Z)$$

is defined by

$$(f, g) \mapsto g \circ f := \mathrm{pr}_{13*}\,(\mathrm{pr}_{12}^*\, f \cdot \mathrm{pr}_{23}^*\, g) = \mathrm{pr}_{13*}\,(f \times Z \cdot X \times g),$$

where pr is the canonical projection and $\cdot$ is the intersection product. Let $\phi : X \to Y$ in $\mathrm{SmProj}(k)$, with X and Y are of pure dimensions d and e, respectively. Let Γ_ϕ denote the image of the closed immersion $(\mathrm{id}_X, \phi) : X \to X \times Y$. We have the correspondences

$$\phi_* = [\Gamma_\phi] \in \mathrm{Corr}^{e-d}(X, Y), \qquad \phi^* := [{}^t\Gamma_\phi] \in \mathrm{Corr}^0(Y, X).$$

Definition 1.6 (Chow motives). Objects of the category of Chow motives $\mathrm{CHM}_{\mathbb{Q}}(k)$ are triples (X, p, m), where $X \in \mathrm{SmProj}(k)$, $p \in \mathrm{Corr}^0(X, X)$ is an idempotent, i.e., $p \circ p = p$, and m is an integer. If (X, p, m) and (Y, q, n) are Chow motives, then

$$\mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(k)}((X, p, m), (Y, q, n)) = p \circ \mathrm{Corr}^{n-m}(X, Y) \circ q \subset \mathrm{Corr}^{n-m}(X, Y).$$

There is a contravariant functor

$$h : \mathrm{SmProj}(k) \to \mathrm{CHM}_{\mathbb{Q}}(k), \quad X \mapsto h(X) := (X, [\Delta_X], 0), \tag{1-4}$$

where Δ_X is the graph of the diagonal map. If $\phi : X \to Y$ is a morphism, $h(\phi) := \phi^* = [{}^t\Gamma_\phi] \in \mathrm{Corr}^0(Y, X) = \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(k)}(h(Y), h(X))$. One calls $h(X)$ the Chow motive of X.

Let C be a smooth projective curve over k (not necessarily having a k-rational point). Pick any zero cycle Z on C of positive degree N, one defines projectors $p_0(C) := \frac{1}{N}[Z \times C]$, $p_2(C) := \frac{1}{N}[C \times Z]$, and Chow motives $h^i(C) := (C, p_i(C), 0) \in \mathrm{CHM}_{\mathbb{Q}}(k)$ for $i = 0, 2$. These motives do not depend on the choice of Z, in fact, $h^0(C) \simeq h(\mathrm{Spec}\, k')$ and $h^2(C) \simeq h(\mathrm{Spec}\, k') \otimes \mathbb{L}$, where $k' = \Gamma(C, \mathcal{O}_C)$ and $\mathbb{L} = (\mathrm{Spec}\, k, \mathrm{id}, -1)$ is the Lefschetz motive. One sets $p_1(C) := \Delta_C - p_0(C) - p_2(C)$ and $h^1(C) := (C, p_1(C), 0) \in \mathrm{CHM}_{\mathbb{Q}}(k)$. We have the direct sum decomposition

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C),$$

which depends on the choice of Z, but $h^1(C)$ is well-defined up to unique isomorphism (see [29, Section 2.3] or [34, Section 3]). If C is further of genus 1, one can show that $h^1(C) \simeq h^1(E)$, where E is the Jacobian of C, by using the following equivalence of categories (see the proof of [29, Theorem 2.7.2(b)]):

$$M''_{\mathbb{Q}} \xrightarrow{\simeq} \{\text{category of Jacobian of curves}\} \otimes \mathbb{Q},$$

where $M''_{\mathbb{Q}}$ is the full subcategory of $\mathrm{CHM}_{\mathbb{Q}}(k)$ of motives isomorphic to $h^1(Y)$ for some smooth projective curve Y. Let us give explicitly the isomorphism.

Proposition 1.7. *Let C be a smooth projective curve of genus 1 over a number field k and E be its Jacobian, then $h(C) \simeq h(E)$ and $h^1(C) \simeq h^1(E)$.*

Proof. Let $\bar{k}$ be the algebraic closure of k and let us fix a point $x_0 \in C(\bar{k})$. We consider the morphism $\phi : C_{\bar{k}} \to E_{\bar{k}}$, which maps $x \in C(\bar{k})$ to the divisor $N(x) - \sum_\sigma (\sigma(x_0))$, where σ runs through all the embeddings $k(x_0) \hookrightarrow \bar{k}$ and N is the number of these embeddings. This map is well-defined as $N(x) - \sum_\sigma (\sigma(x_0))$ is a divisor of degree 0. We have $\phi_* \in \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(\bar{k})}(h(C_{\bar{k}}), h(E_{\bar{k}}))$ and $\phi^* \in \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(\bar{k})}(h(E_{\bar{k}}), h(C_{\bar{k}}))$. By [29, Section 2.3], we have $\phi_* \circ \phi^* = \deg(\phi)[\Delta_{E_{\bar{k}}}] = N^2[\Delta_{E_{\bar{k}}}]$. Conversely, we have

$$\phi^* \circ \phi_* \stackrel{\text{def}}{=} \mathrm{pr}_{13*}((\Gamma_\phi \times C_{\bar{k}}) \cdot (C_{\bar{k}} \times {}^t\Gamma_\phi)).$$

As sets, we observe that

$$\begin{aligned}(\Gamma_\phi \times C_{\bar{k}}) \cap (C_{\bar{k}} \times {}^t\Gamma_\phi) &= \{(x, \phi(x), y) \mid x, y \in C(\bar{k})\} \cap \{(z, \phi(t), t) \mid z, t \in C(\bar{k})\}\\ &= \{(x, \phi(x), y) \mid x, y \in C(\bar{k}), \phi(x) = \phi(y)\}\\ &= \{(x, \phi(x), y) \mid x, y \in C(\bar{k}), N(x) - N(y) = 0 \text{ in } E(\bar{k})\}\\ &= \{(x, \phi(x), x + p) \mid x \in C(\bar{k}), p \in E_{\bar{k}}[N]\},\end{aligned}$$

where $E_{\bar{k}}[N]$ is the set of N-torsion points of $E(\bar{k})$ and "+" is the canonical action of $E_{\bar{k}}$ on $C_{\bar{k}}$. So

$$\phi^* \circ \phi_* = \sum_{p \in E_{\bar{k}}[N]} [\Gamma_{\varphi_p}] = N^2[\Delta_{C_{\bar{k}}}],$$

where $\varphi_p : C_{\bar{k}} \to C_{\bar{k}},\ x \mapsto x + p$, and the last equality follows from the fact that Γ_{φ_p} is rationally equivalent to $\Delta_{C_{\bar{k}}}$ for $p \in E_{\bar{k}}[N]$. We thus obtain that $\phi_* : h(C_{\bar{k}}) \to h(E_{\bar{k}})$ is an isomorphism in the category $\mathrm{CHM}_{\mathbb{Q}}(\bar{k})$. For $\alpha \in \mathrm{Gal}(\bar{k}/k)$ and $x \in C(\bar{k})$,

$$(\alpha \circ \phi)(x) = \alpha(N(x)) - \sum_\sigma (\alpha \circ \sigma(x_0)) = N(\alpha(x)) - \sum_\sigma (\sigma(x_0)) = (\phi \circ \alpha)(x).$$

This implies that Γ_ϕ and ${}^t\Gamma_\phi$ are $\mathrm{Gal}(\bar{k}/k)$-invariant. Hence by Galois descent (see, e.g., Theorem 1.3(6) of [13])

$$\mathrm{CH}^1(C_{\bar{k}} \times_{\bar{k}} J_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \simeq \mathrm{CH}^1(C \times_k E), \quad \mathrm{CH}^1(E_{\bar{k}} \times_{\bar{k}} C_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \simeq \mathrm{CH}^1(E \times_k C),$$

ϕ_* defines an isomorphism from $h(C)$ to $h(E)$ in the category CHM_k with inverse ϕ^*.

Denote by A the positive zero-cycle of degree N corresponding to x_0. We set $p_0(C) := \frac{1}{N}[A \times C]$, $p_2(C) := \frac{1}{N}[C \times A]$, and $p_1(C) := \Delta_C - p_0(C) - p_2(C)$. Let $\mathcal{O}$ be the trivial element in $E(k)$, we set $p_0(E) := \mathcal{O} \times E$, $p_2(E) = E \times \mathcal{O}$, and $p_1(E) := \Delta_E - p_0(E) - p_2(E)$. By explicit computations, we have

$$\phi^* \circ p_0(E) \circ \phi_* = N^2 p_0(C) \quad \text{and} \quad \phi^* \circ p_2(E) \circ \phi_* = N^2 p_2(C).$$

Now we show that $p_1(E) \circ \phi_* \circ p_1(C)$ and $p_1(C) \circ \phi^* \circ p_1(E)$ define isomorphisms from $h^1(C)$ to $h^1(E)$ and inverse, respectively. We have

$$\begin{aligned} \phi^* \circ p_1(E) \circ \phi_* = \phi^* \circ (\phi_* - p_0(E) \circ \phi_* - p_2(E) \circ \phi_*) &= N^2[\Delta_C] - \phi^* \circ p_0(E) \circ \phi_* - \phi^* \circ p_2(E) \circ \phi_* \\ &= N^2[\Delta_C] - N^2 p_0(C) - N^2 p_2(C) \\ &= N^2 p_1(C). \end{aligned}$$

We thus have $p_1(C) \circ \phi^* \circ p_1(E) \circ p_1(E) \circ \phi_* \circ p_1(C) = p_1(C) \circ \phi^* \circ p_1(E) \circ \phi_* \circ p_1(C) = N^2 p_1(C)$. Similarly, we have $p_1(E) \circ \phi_* \circ p_1(C) \circ p_1(C) \circ \phi^* \circ p_1(E) = N^2 p_1(E)$. □

1.3. ***Motivic cohomology.*** Let k be an arbitrary field of characteristic 0. Let us recall briefly the definition and some facts of motivic cohomology. For more details, we refer to [14, Chapter 5, Section 2]. Voevodsky constructed a triangulated category, called *geometrical motives*, denoted by $\mathrm{DM}_{\mathrm{gm}}(k)$ and a covariant functor

$$M_{\mathrm{gm}} : \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$$

(see [14, p. 192]). The motivic cohomology of X with coefficients in $\mathbb{Q}$ is defined by

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) := \mathrm{Hom}_{\mathrm{Dm}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(X), \mathbb{Q}(j)[n]), \qquad \text{for } n, j \in \mathbb{Z}, \tag{1-5}$$

where $\mathbb{Q}(1) \in \mathrm{DM}_{\mathrm{gm}}(k)$ is the *Tate motive* (see also [14, p. 192]) and $\mathbb{Q}(j) = \mathbb{Q}(1)^{\otimes j}$. It is known that the motivic cohomology $H^n_{\mathcal{M}}(X, \mathbb{Q}(j))$ is isomorphic to the j-eigenspace of Quillen's K-group $K_{2j-n}(X)_{\mathbb{Q}}$ with respect to Adams operation (see [37, Chapter II.4] for the definition), namely,

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \simeq K^{(j)}_{2j-n}(X) \tag{1-6}$$

(see [14, p. 197]). By the functorial property of M_{gm}, for any morphism $f : X \to Y$, we have a $\mathbb{Q}$-linear map $f^* : H^n_{\mathcal{M}}(Y, \mathbb{Q}(j)) \to H^n_{\mathcal{M}}(X, \mathbb{Q}(j))$, called pull back of f. Moreover, for proper maps $f : X \to Y$ of pure codimension $c = \dim Y - \dim X$, we have a $\mathbb{Q}$-linear map, called push-forward of f (see [13, Theorem 1.3])

$$f_* : H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^{n+2c}_{\mathcal{M}}(Y, \mathbb{Q}(j + c)).$$

Let X and X' be smooth quasiprojective varieties over k and $\pi : X' \to X$ be a finite Galois covering with group G. We have Galois descent for motivic cohomology, i.e.,

$$\pi^* : H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \xrightarrow{\simeq} H^n_{\mathcal{M}}(X', \mathbb{Q}(j))^G \tag{1-7}$$

is an isomorphism (see [13, Theorem 1.3]).

Let $X \in \mathrm{Sm}(k)$ and $\iota : Z \hookrightarrow X$ be a closed immersion of smooth varieties of codimension c with open complement $\jmath : X - Z \hookrightarrow X$. We have a *localization sequence* for motivic cohomology (see [14, p. 196] or [13, Theorem 1.3])

$$\cdots \to H_{\mathcal{M}}^{i-2c}(Z, \mathbb{Q}(j-c)) \xrightarrow{\iota_*} H_{\mathcal{M}}^{i}(X, \mathbb{Q}(j)) \xrightarrow{\jmath^*} H^{i}(X-Z, \mathbb{Q}(j)) \to H_{\mathcal{M}}^{i+1-2c}(Z, \mathbb{Q}(j-c)) \to \cdots . \tag{1-8}$$

Let C be a smooth curve over a number field k. Denote by $F = k(C)$ the function field of C. Then

$$H_{\mathcal{M}}^{n}(F, \mathbb{Q}(j)) = \varinjlim_{U \subset C \text{ open}} H_{\mathcal{M}}^{n}(U, \mathbb{Q}(j)).$$

We have the following version of localization sequence:

$$0 \to H_{\mathcal{M}}^{2}(C, \mathbb{Q}(3)) \to H_{\mathcal{M}}^{2}(F, \mathbb{Q}(3)) \xrightarrow{\mathrm{Res}^{\mathcal{M}}} \bigoplus_{x \in C^1} H_{\mathcal{M}}^{1}(k(x), \mathbb{Q}(2)), \tag{1-9}$$

where C^1 is the set of closed points of C. This follows from the localization sequence of Quillen's K-groups (see [37, V.6.12]). The left exactness follows from Borel's theorem (see, e.g., [37, IV.1.18]), which states that K_4 of a number field is torsion, hence $H_{\mathcal{M}}^{0}(K, \mathbb{Q}(2)) \simeq K_4^{(2)}(K) = 0$.

1.4. ***The Beilinson regulator map.*** Let X be a smooth variety over $\mathbb{R}$ or $\mathbb{C}$. The Beilinson regulator map, as defined in [30], is a $\mathbb{Q}$-linear map

$$\mathrm{reg}_X : H_{\mathcal{M}}^{n}(X, \mathbb{Q}(j)) \to H_{\mathcal{D}}^{n}(X, \mathbb{R}(j)). \tag{1-10}$$

For $n = j = 1$, we have $H_{\mathcal{M}}^{1}(X, \mathbb{Q}(1)) = \mathcal{O}(X)_{\mathbb{Q}}^{\times}$ and the regulator map sends an invertible function f to the class of $\log|f|$, by [9, Appendix A.3]. As the regulator map is compatible with taking cup products, we observe that the regulator map sends the Milnor symbol $\{f_1, \ldots, f_n\} \in H_{\mathcal{M}}^{n}(X, \mathbb{Q}(n))$ to the class of $\log|f_1| \cup \cdots \cup \log|f_n|$ in $H_{\mathcal{D}}^{n}(X, \mathbb{R}(n))$. When X is defined over a number field k, we write $X_{\mathbb{R}} := X \times_{\mathbb{Q}} \mathbb{R}$, and the Beilinson regulator map is defined as the composition

$$H_{\mathcal{M}}^{n}(X, \mathbb{Q}(j)) \xrightarrow{\text{base change}} H_{\mathcal{M}}^{n}(X_{\mathbb{R}}, \mathbb{Q}(j)) \xrightarrow{\mathrm{reg}_{X_{\mathbb{R}}}} H_{\mathcal{D}}^{n}(X_{\mathbb{R}}, \mathbb{R}(j)). \tag{1-11}$$

Now let C be a smooth curve over a number field k. Let $F = k(C)$ be the function field of C. We define Deligne cohomology of F by the direct limit

$$H_{\mathcal{D}}^{n}(F, \mathbb{R}(j)) := \varinjlim_{U \subset C \text{ open}} H_{\mathcal{D}}^{n}(U_{\mathbb{R}}, \mathbb{R}(j)). \tag{1-12}$$

And the regulator map for the function field is defined by $\mathrm{reg}_F := \varinjlim_{U \subset C \text{ open}} \mathrm{reg}_U$

$$\mathrm{reg}_F : H_{\mathcal{M}}^{n}(F, \mathbb{Q}(j)) \to H_{\mathcal{D}}^{n}(F, \mathbb{Q}(j)). \tag{1-13}$$

Now let us recall the *regulator integral* for smooth projective curve (see [20, Section 3] for more details). Let C be a smooth projective curve over a number field k. Denote by $C(\mathbb{C})$ the set of complex points of $C \times_{\mathbb{Q}} \mathbb{C}$. If ω is a holomorphic 1-form on $C(\mathbb{C})$ such that $F_{\mathrm{dR}}(\omega) = \omega$, where F_{dR} is defined in Section 1.1, then we have a map

$$H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \to \mathbb{R}(1), \qquad \eta \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega} \tag{1-14}$$

(see [20, Remark 3.1]). This integral depends on the choice of the orientation of $C(\mathbb{C})$. Recall that there is a canonical isomorphism $H^2_{\mathcal{D}}(C_{\mathbb{R}}, \mathbb{R}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+$ (see Remark 1.4). We thus write the Beilinson regulator map on C as the composition

$$\mathrm{reg}_C : H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \to H^2_{\mathcal{D}}(C_{\mathbb{R}}, \mathbb{R}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+. \tag{1-15}$$

The composition map

$$H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \xrightarrow{\mathrm{reg}_C} H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \xrightarrow{\eta \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega}} \mathbb{R}(1) \tag{1-16}$$

is called the *regulator integral*. Similarly, we have the regulator integral for the function field

$$H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \xrightarrow{\mathrm{reg}_F} H^1(F, \mathbb{R}(2))^+ \xrightarrow{\eta \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega}} \mathbb{R}(1), \tag{1-17}$$

where $H^1(F, \mathbb{R}(2))^+ := \varinjlim_{U \subset C \text{ open}} H^1(U(\mathbb{C}), \mathbb{R}(2))^+$.

1.5. ***Beilinson's conjecture for genus 1 curves.*** In this section, we recall Beilinson's conjecture for smooth projective curves of genus 1 (see [30, Section 6] or [19, Section 4] for more details). Let us recall the definition of L-function attached to the pure motive $h^i(X)$, for X is a smooth projective variety over $\mathbb{Q}$ of dimension n.

Definition 1.8 [30, Section 1.4]. Let p be a prime number. For $0 \le i \le 2n$, we set

$$L_p(h^i(X), s) = \det(1 - \mathrm{Frob}_p p^{-s} \,|\, h^i_\ell(X)^{I_p})^{-1},$$

where $\ell \neq p$ is a prime number, $\mathrm{Frob}_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a Frobenius element at p, acting on the étale realization

$$h^i_\ell(X) := H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

and I_p is the inertia group at p.

Remark 1.9. If X has good reduction at p, then $L_p(h^i(X), s)$ does not depend on the choice of ℓ [30, Section 1.4]. And it is conjectured by Serre that if X has bad reduction at p, then $L_p(h^i(X), s)$ is independent of the choice of ℓ and has integer coefficients (compare [21, Conjecture 5.45]). This conjecture holds if $i \in \{0, 1, 2n-1, 2n\}$, by [21, Theorem 5.46]. In particular, it holds when X is a curve.

Definition 1.10 (L-function [30, Section 1.5]). The L-function associated to $h^i(X)$ is defined by

$$L(h^i(X), s) = \prod_{p \text{ prime}} L_p(h^i(X), s).$$

Let $C/\mathbb{Q}$ be a smooth projective curve of genus 1 and E be its Jacobian. By Proposition 1.7, we have $L(h^1(C), s) = L(h^1(E), s)$. Hence $L(h^1(C), s) = L(E, s)$ the Hasse–Weil L-function. We then have the following version of Beilinson's conjecture for smooth projective curve of genus 1.

Conjecture 1.11 ([30, Section 6], [19, Section 4]). *Let C be a smooth projective curve over $\mathbb{Q}$ of genus 1 and E be its Jacobian. For any nontrivial element $\alpha \in H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$, we have*

$$\frac{1}{(2\pi i)^2}\left\langle \gamma_C^+, \mathrm{reg}_C(\alpha)\right\rangle_{C(\mathbb{C})} = a \cdot L'(E, -1) \qquad (a \in \mathbb{Q}^\times),$$

where γ_C^+ is a generator of $H_1(C(\mathbb{C}), \mathbb{Q})^+$, reg_C is the Beilinson regulator map (1-15), *and $\langle \cdot, \cdot \rangle$ is then the pairing in de Rham cohomology.*

2. Goncharov's polylogarithmic complexes

In 1990s, Goncharov introduced polylogarithmic complexes and regulator maps from the cohomology of these complexes to Deligne cohomology. They have connections with motivic cohomology and the Beilinson regulator map (see [15; 16; 17]). In this section, we recall briefly these constructions of Goncharov, which will be used in Section 4.3 to construct elements in motivic cohomology.

2.1. ***Goncharov's complexes.*** For any field F of characteristic 0 and an integer $n \geq 1$, Goncharov defined $\mathcal{B}_n(F)$ to be the quotient of the $\mathbb{Q}$-vector space $\mathbb{Q}[\mathbb{P}^1_F]$ by a certain (inductively defined) subspace $\mathcal{R}_n(F)$. For $x \in F \cup \{\infty\}$, we denote by $\{x\}_n$ the class of $\{x\} \in \mathbb{Q}[\mathbb{P}^1_F]$ in $\mathcal{B}_n(F)$. Goncharov then constructed a *weight n polylogarithmic complex*, in degree 1 to n (see [15, Section 1])

$$\Gamma_F(n) : \mathcal{B}_n(F) \to \mathcal{B}_{n-1}(F) \otimes F^\times_{\mathbb{Q}} \to \mathcal{B}_{n-2}(F) \otimes \textstyle\bigwedge^2 F^\times_{\mathbb{Q}} \to \cdots \to \mathcal{B}_2(F) \otimes \textstyle\bigwedge^{n-2} F^\times_{\mathbb{Q}} \to \textstyle\bigwedge^n F^\times_{\mathbb{Q}},$$

where the maps are given by

$$\begin{aligned} \{x\}_{n-p} \otimes y_1 \wedge \cdots \wedge y_p &\mapsto \{x\}_{n-p-1} \otimes x \wedge y_1 \wedge \cdots \wedge y_p \qquad \text{if } 0 \leq p < n-2, \\ \{x\}_2 \otimes y_1 \wedge \cdots \wedge y_{n-2} &\mapsto (1-x) \wedge x \wedge y_1 \wedge \cdots \wedge y_{n-2}. \end{aligned}$$

It is conjectured that the cohomology of this complex computes the motivic cohomology.

Conjecture 2.1 [15, Conjecture A, p. 222]. *$H^p(\Gamma_F(n)) \simeq H^p_{\mathcal{M}}(F, \mathbb{Q}(n))$ for $p, n \geq 1$.*

Goncharov also defined the $\mathbb{Q}$-vector space $B_n(F)$ $(1 \leq n \leq 3)$ to be the quotient of $\mathbb{Q}[\mathbb{P}^1_F]$ by a certain subspace $R_n(F)$, generated by explicit relations as follows.

$$\begin{aligned} R_1(F) &:= \left\langle \{x\} + \{y\} - \{xy\},\, x, y \in F^\times;\, \{0\};\, \{\infty\}\right\rangle, \\ R_2(F) &:= \left\langle \{x\} + \{y\} + \{1-xy\} + \left\{\frac{1-x}{1-xy}\right\} + \left\{\frac{1-y}{1-xy}\right\},\, x, y \in F^\times \setminus \{1\};\, \{0\};\, \{\infty\}\right\rangle, \end{aligned} \tag{2-1}$$

and $R_3(F)$ is generated by explicit relations corresponding to the functional equations for the trilogarithm (see [15, p. 214]) that we do not mention here. We still denote by $\{x\}_k$ the class of $\{x\} \in \mathbb{Q}[\mathbb{P}^1_F]$ in $B_k(F)$. As Goncharov's constructions, $B_1(F) = \mathcal{B}_1(F) = F^\times_{\mathbb{Q}}$ (see [15, Sections 1.8 and 1.9]). And it was proved by de Jeu that $B_2(F) = \mathcal{B}_2(F)$ (see [20, Remark 5.3]). Goncharov showed that $R_3(F) \subset \mathcal{R}_3(F)$ and conjectured that they are equal (see [15, p. 225]).

Goncharov then constructed the polylogarithmic complexes $\Gamma(F, n)$ for $n = 2, 3$ with the same shape as $\Gamma_F(n)$ but $\mathcal{B}_n(F)$ are replaced by $B_n(F)$ (notice that these complexes are denoted by $B_F(n)$ in [15, Section 1.8]). In this article, we only use the vector spaces with explicit relations $B_n(F)$ and the corresponding polylogarithmic complexes $\Gamma(F, n)$ for $n = 2, 3$. For $n = 2$, it is given as follows, in degree 1 and 2:

$$\Gamma(F,2): \qquad B_2(F) \xrightarrow{\alpha_2(1)} \bigwedge^2 F_{\mathbb{Q}}^{\times}, \quad \{x\}_2 \mapsto (1-x) \wedge x.$$

We have $H^2(\Gamma(F, 2)) \simeq H^2_{\mathcal{M}}(F, \mathbb{Q}(2))$ by Matsumoto's theorem. And $H^1(\Gamma(F, 2)) \simeq H^1_{\mathcal{M}}(F, \mathbb{Q}(2))$ by Suslin's work [35]. The group $H^1(\Gamma(F, 2))$ is also called *Bloch group*, denoted by $\mathcal{B}(F)$ (see [38, Section 2]).

For $n = 3$, we have the following polylogarithmic complex in degree 1 to 3:

$$\begin{array}{llllll} \Gamma(F,3): & B_3(F) & \xrightarrow{\alpha_3(1)} & B_2(F) \otimes F_{\mathbb{Q}}^{\times} & \xrightarrow{\alpha_3(2)} & \bigwedge^3 F_{\mathbb{Q}}^{\times}, \\ & \{x\}_3 & \longmapsto & \{x\}_2 \otimes x & & \\ & & & \{x\}_2 \otimes y & \longmapsto & (1-x) \wedge x \wedge y. \end{array} \tag{2-2}$$

Then $H^3(\Gamma(F, 3)) \simeq H^3_{\mathcal{M}}(F, \mathbb{Q}(3))$. In degree 2, Goncharov constructed a map $K_4(F)_{\mathbb{Q}} \to H^2(\Gamma(F, 3))$ and conjectured that this induces an isomorphism $H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \simeq K_4^{(3)}(F) \to H^2(\Gamma(F, 3))$. De Jeu constructed a map in other direction $H^2(\Gamma(F, 3)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3))$. We discuss the later map in Section 3.3.

2.2. ***The residue homomorphism of complexes.*** Let K be a field with a discrete valuation v. Denote by $\mathcal{O}_K, k_v, \pi_v$ the ring of integers, the residue field, and a uniformizer, respectively. Goncharov defined a residue homomorphism on his polylogarithmic complexes (see [15, Section 1.14])

$$\partial_v : \Gamma(K, 3) \to \Gamma(k_v, 2)[-1]. \tag{2-3}$$

More precisely, it is given by

$$\begin{array}{ccccc} B_3(K) & \xrightarrow{\alpha_3(1)} & B_2(K) \otimes K_{\mathbb{Q}}^{\times} & \xrightarrow{\alpha_3(2)} & \bigwedge^3 K_{\mathbb{Q}}^{\times} \\ & & \Big\downarrow \partial_v^{(2)} & & \Big\downarrow \partial_v^{(3)} \\ & & B_2(k_v) & \xrightarrow[\alpha_2(1)]{} & \bigwedge^2 (k_v^{\times})_{\mathbb{Q}}, \end{array} \tag{2-4}$$

where the vertical maps are defined as follows. For $f \in K^{\times}$, we denote by f_v the image of $f\pi_v^{-\mathrm{ord}_v(f)}$ under the canonical map $\mathcal{O}_K^{\times} \to k_v^{\times}$. We have

$$\partial_v^{(2)} : \{f\}_2 \otimes g \mapsto \mathrm{ord}_v(g)\{f(v)\}_2, \text{ with the convention } \{0\}_2 = \{1\}_2 = \{\infty\}_2 = 0 \text{ in } B_2(k_v), \tag{2-5}$$

$$\partial_v^{(3)} : f \wedge g \wedge h \mapsto \mathrm{ord}_v(f) g_v \wedge h_v - \mathrm{ord}_v(g) f_v \wedge h_v + \mathrm{ord}_v(h) f_v \wedge g_v. \tag{2-6}$$

Now let C be a smooth connected curve over a number field k and let F be its function field. Denote by C^1 the set of closed points of C, and $k(x)$ the residue field of $x \in C^1$. We have the morphism of complexes

$$\partial := \oplus_{x\in C^1}\partial_x : \Gamma(F,3) \to \bigoplus_{x\in C^1} \Gamma(k(x),2)[-1]. \tag{2-7}$$

Goncharov defined the polylogarithmic complex $\Gamma(C,3)$ as the mapping cone of (2-7). We then have the exact sequence

$$0 \to H^2(\Gamma(C,3)) \to H^2(\Gamma(F,3)) \xrightarrow{\partial} \bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)), \tag{2-8}$$

where the left exactness is due to the fact that the cohomology group of degree 0 of polylogarithmic complexes vanishes. This should be isomorphic to the localization sequence of motivic cohomology (1-9). We will construct a morphism from complexes (2-8) to (1-9) in (3-16), using work of de Jeu.

2.3. *Goncharov's regulator maps.* In this section, we recall Goncharov's regulator maps. Let X be a regular variety over a number field. Denote by F the function field of X. For a nonempty open subscheme $U \subset X$, $U(\mathbb{C})$ denotes the set of complex points of $U \times_{\mathbb{Q}} \mathbb{C}$. Let $\Omega^j(\eta_X) := \varinjlim_{U\subset X \text{ open}} \Omega^j(U)$ and $\Omega^j(U)$ is the space of real smooth j-forms on $U(\mathbb{C})$. Goncharov gave explicitly a homomorphism of complexes (see [17, Theorem 2.2]):

$$\begin{array}{ccccc} B_3(F) & \xrightarrow{\alpha_3(1)} & B_2(F)\otimes F^\times_{\mathbb{Q}} & \xrightarrow{\alpha_3(2)} & \bigwedge^3 F^\times_{\mathbb{Q}} \\ \Big\downarrow{\scriptstyle r_3(1)} & & \Big\downarrow{\scriptstyle r_3(2)} & & \Big\downarrow{\scriptstyle r_3(3)} \\ \Omega^0(\eta_X) & \xrightarrow{d} & \Omega^1(\eta_X) & \xrightarrow{d} & \Omega^2(\eta_X). \end{array} \tag{2-9}$$

For $f \in F^\times \setminus \{1\}$, $g, h \in F^\times$, the vertical maps in degrees 2 and 3 are given respectively by

$$r_3(2) : \{f\}_2 \otimes g \mapsto \rho(f,g), \qquad r_3(3) : f\wedge g\wedge h \mapsto -\eta(f,g,h),$$

where

$$\begin{aligned} \eta(f,g,h) := \log|f| \left(\tfrac{1}{3} d\log|g| \wedge d\log|h| - d\arg(g)\wedge d\arg(h)\right) \\ + \log|g| \left(\tfrac{1}{3} d\log|h| \wedge d\log|f| - d\arg(h)\wedge d\arg(f)\right) \\ + \log|h| \left(\tfrac{1}{3} d\log|f| \wedge d\log|g| - d\arg(f)\wedge d\arg(g)\right), \end{aligned} \tag{2-10}$$

and

$$\rho(f,g) := -D(f)\, d\arg g + \tfrac{1}{3}\log|g|\theta(1-f,f), \tag{2-11}$$

where

$$\theta(h,f) = \log|h| d\log|f| - \log|f| d\log|h|, \tag{2-12}$$

and $D : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ is the Bloch–Wigner dilogarithm function (0-7). In particular, we have

$$d\rho(f,g) = -\eta(1-f,f,g) = \eta(f,1-f,g) \text{ for } f \in F^\times \setminus \{1\},\ g \in F^\times.$$

Let C be a smooth connected curve over a number field and let F be its function field. Goncharov showed that the map $r_3(2)$ gives rise to a regulator map on the cohomology of his complexes of the function field

$$r_3(2)_F : H^2(\Gamma(F,3)) \to H^2_{\mathcal{D}}(F, \mathbb{R}(3)) \simeq H^1(F, \mathbb{R}(2))^+ \tag{2-13}$$

(see [17, Section 2.7]). Moreover, the map $r_3(2)_F$ is compatible with taking residues (2-7) (see [17, Theorem 2.6]), hence it extends to a homomorphism

$$r_3(2)_C : H^2(\Gamma(C,3)) \to H^2_{\mathcal{D}}(C \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+. \tag{2-14}$$

Now let us compute the residues of the differential form representing $r_3(2)_F(\alpha)$ for $\alpha \in H^2(\Gamma(F,3))$. First let us recall the definition of the residues of a differential form.

Definition 2.2 ([9, Definition 7.3]). Let C be a smooth connected curve over a number field and let Z be a subset of closed points of C. We set $Y := C \setminus Z$. Let $\eta \in H^1(Y(\mathbb{C}), \mathbb{R})$. The residue of η at $p \in C(\mathbb{C})$ is

$$\operatorname{Res}_p(\eta) = \int_{\gamma_p} \eta, \tag{2-15}$$

where γ_p is the boundary of any small disc containing p and avoiding $Z(\mathbb{C}) \setminus \{p\}$.

Lemma 2.3. *Let $\alpha = \sum_j c_j \{f_j\}_2 \otimes g_j \in H^2(\Gamma(F,3))$ with $c_j \in \mathbb{Q}$, and $f_j \in F^\times \setminus \{1\}$, $g_j \in F^\times$. Denote by Z the closed subset of C consisting of zeros and poles of f_j, $1-f_j$, g_j for all j. Set $Y = C \setminus Z$. Then $r_3(2)_F(\alpha)$ is represented by the differential form $\rho := \sum_j c_j \rho(f_j, g_j) \in H^1(Y(\mathbb{C}), \mathbb{R}(2))^+$ and its residue at $p \in C(\mathbb{C})$ is given by*

$$\operatorname{Res}_p(\rho) = -2\pi \sum_j c_j v_p(g_j) D(f_j(p)),$$

where D is the Bloch–Wigner dilogarithm function (0-7).

Proof. The first statement follows directly from the definition of the map $r_3(2)_F$ (2-13). Now we compute the residues. Let $f, g \in \mathbb{C}(C)^\times$ such that all the zeros and poles of f, $1-f$, g are contained in Z. Let $p \in C(\mathbb{C})$ and γ_p be a sufficiently small loop around p and does not surround any point of $Z \setminus \{p\}$. Using the local coordinate $z = re^{is}$, for $r > 0$ small and $s \in [0, 2\pi]$, we have $f(z) = (re^{is})^{v_p(f)} F(re^{is})$ and $g(z) = (re^{is})^{v_p(g)} G(re^{is})$, where F and G are holomorphic functions such that $F(0), G(0) \neq 0$. Then

$$\begin{aligned}\int_{\gamma_p} D(f) d\arg(g) &= \int_0^{2\pi} D(f(re^{is}))\, d\arg\big((re^{is})^{v_p(g)} G(re^{is})\big)\\ &= \int_0^{2\pi} D(f(re^{is})) v_p(g) ds + \int_0^{2\pi} D(f(re^{is}))\, d\arg G(re^{is}).\end{aligned} \tag{2-16}$$

As

$$d\arg G(z) = \frac{1}{2i}\left(\frac{dG}{G} - \frac{d\overline{G}}{\overline{G}}\right) = \frac{1}{2i}\left(\frac{1}{G}\frac{\partial G}{\partial z} dz - \frac{1}{\overline{G}}\frac{\partial \overline{G}}{\partial \bar{z}} d\bar{z}\right),$$

we have

$$d\arg G(re^{is}) = \frac{1}{2i}\left(\frac{G_z}{G} rie^{is}ds - \frac{\overline{G}_{\bar z}}{\overline{G}} r(-i)e^{-is}ds\right) = O(r)ds,$$

where G_z is the derivative of G with respect to z. Then by taking $r \to 0$ in (2-16), the limit of $\int_{\gamma_p} D(f)d\arg(g)$ as γ_p shrinks to p is

$$\int_0^{2\pi} D(f(p))v_p(g)ds = 2\pi v_p(g)D(f(p)). \tag{2-17}$$

Moreover, we have

$$\log|f| = \log|F(re^{is})| + v_p(f)\log r,$$

and

$$d\log|f| = d\log|F| = \frac{1}{2}\left(\frac{dF}{F} + \frac{d\overline{F}}{\overline{F}}\right) = O(r)ds.$$

Therefore, $\theta(1-f, f) = O(r\log r)\, ds \;\to 0$ as $r \to 0$. Hence

$$\mathrm{Res}_p(\rho) = -2\pi\sum_j c_j v_p(g_j)D(f_j(p)), \text{ for } p \in C(\mathbb{C}). \qquad \square$$

3. De Jeu's polylogarithmic complexes

In this section, we recall de Jeu's polylogarithmic complexes and his maps from the cohomology of these complexes to the motivic cohomology. In particular, they give rise to maps from the cohomology of Goncharov's polylogarithmic complexes to the motivic cohomology. We then compare the images of Goncharov's regulator and Beilinson's regulator composed with these maps. These results are used in the construction of the motivic cohomology classes in Section 4.3. In this article, we consider only the cases of the polylogarithmic complexes of weight 2 and weight 3. The references for this section are [18; 19; 20].

3.1. ***De Jeu's polylogarithmic complexes.*** Let F be a field of characteristic 0. De Jeu defined $\widetilde{M}_{(j)}(F)$ to be a certain $\mathbb{Q}$-vector space generated by symbols $[f]_j$ with $f \in F^\times \setminus \{1\}$ and constructed the following complex in degree 1 to 2:

$$\widetilde{\mathcal{M}}^\bullet_{(2)}(F): \qquad \widetilde{M}_{(2)}(F) \to \textstyle\bigwedge^2 F^\times_{\mathbb{Q}}, \quad [f]_2 \mapsto (1-f)\wedge f, \tag{3-1}$$

and this complex in degree 1 to 3:

$$\begin{array}{llll} \widetilde{\mathcal{M}}^\bullet_{(3)}(F): & \widetilde{M}_{(3)}(F) \longrightarrow & \widetilde{M}_{(2)}(F)\otimes F^\times_{\mathbb{Q}} \longrightarrow & \bigwedge^3 F^\times_{\mathbb{Q}}, \\ & [f]_3 \longmapsto & [f]_2 \otimes f & \\ & & [f]_2\otimes g \longmapsto & (1-f)\wedge f\wedge g \end{array} \tag{3-2}$$

(see [18, Corollary 3.22, Example 3.24] or [20, Section 2]). We have $H^n(\widetilde{\mathcal{M}}^\bullet_{(n)}(F)) \simeq H^n_{\mathcal{M}}(F, \mathbb{Q}(n))$ for $n \in \{2, 3\}$. Let k be a number field. By Suslin's work, we have the following isomorphism (up to a universal choice of sign) (see [18, Theorem 5.3] or [20, Theorem 2.3])

$$\varphi^1_{(2)} : H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \overset{\simeq}{\to} H^1_{\mathcal{M}}(k, \mathbb{Q}(2)). \tag{3-3}$$

Let $\sigma : k \hookrightarrow \mathbb{C}$ be any embedding of k into $\mathbb{C}$. We consider the composition map

$$H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \xrightarrow{\varphi^1_{(2)}} H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_k} \mathbb{R}(1), \tag{3-4}$$

where reg_k is the composition $H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \xrightarrow{\sigma^*} H^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_{\mathbb{C}}} H^1_{\mathcal{D}}(\mathbb{C}, \mathbb{R}(2)) \simeq \mathbb{R}(1)$; the last isomorphism here is the canonical isomorphism mentioned in Remark 1.4. It is shown that the map (3-4) is given by $[z]_2$ to $\pm i D(\sigma(z))$, where D is the Bloch–Wigner dilogarithm (see [18, Proposition 4.1]). So we can fix the sign of $\varphi^1_{(2)}$ such that it is induced by $[z]_2 \mapsto i D(\sigma(z))$.

Moreover, de Jeu ([19, p. 529]) constructed a map (up to a universal choice of sign)

$$\varphi^2_{(3)} : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)). \tag{3-5}$$

We discuss more about this map when F is the function field of a curve in the following section.

3.2. ***De Jeu's maps.*** Let C be a smooth geometrically connected curve over a number field k. Denote by F the function field of C and $k(x)$ the residue field of a closed point $x \in C^1$. De Jeu [19, Proposition 5.1] also defined the residue map

$$\delta : \widetilde{\mathcal{M}}^\bullet_{(3)}(F) \to \bigoplus_{x \in C^1} \widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))[-1] \tag{3-6}$$

similarly to Goncharov's residue map (2-7). The complex $\widetilde{\mathcal{M}}^\bullet_{(3)}(C)$ is also defined to be the mapping cone of (3-6). As the maps $\varphi^2_{(3)}$ (3-5) and $\varphi^1_{(2)}$ (3-3) are defined universally up to sign, we have the (possibly noncommutative) diagram

$$\begin{tikzcd}
H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi^2_{(3)}"] \arrow[d, "\pm 2\delta"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\
\bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x \in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd}$$

Recall that we have the following cup product of K-groups:

$$\cup : K_3^{(2)}(k) \otimes K_1^{(1)}(F) \to K_4^{(3)}(F).$$

Since $K^{(j)}_{2j-i}(X) \simeq H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$ and $K_1^{(1)}(F) \simeq F^\times_{\mathbb{Q}}$, we have

$$H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}} \subset H^2_{\mathcal{M}}(F, \mathbb{Q}(3)).$$

Denote by $H = H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}}$. De Jeu showed that $(\mathrm{Res}^{\mathcal{M}} \circ \varphi^2_{(3)}) \pm 2(\varphi^1_{(2)} \circ \delta)$ has image in $\mathrm{Res}^{\mathcal{M}}|_H(H)$ (see [19, Theorem 5.2]). With the fixed choice of sign of $\varphi^1_{(2)}$ in Section 3.1, we can choose the sign of $\varphi^2_{(3)}$ such that $(\mathrm{Res}^{\mathcal{M}} \circ \varphi^2_{(3)}) - 2(\varphi^1_{(2)} \circ \delta)$ has image in $\mathrm{Res}^{\mathcal{M}}|_H(H)$ (see [19, diagram (15)]).

$$\begin{tikzcd}
H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi^2_{(3)}"] \arrow[d, "2\delta"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \supset H \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\
\bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x \in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd}$$

Let $\xi \in H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$. As $\mathrm{Res}^{\mathcal{M}}|_H$ is injective (see [20, Remark 4.4], for example), there exists a unique $h \in H$ such that

$$\mathrm{Res}^{\mathcal{M}}|_H(h) = ((\mathrm{Res}^{\mathcal{M}} \circ \varphi^2_{(3)}) - 2(\varphi^1_{(2)} \circ \delta))(\xi).$$

Then we define a map

$$\varphi_F : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)), \tag{3-7}$$

by setting $\varphi_F(\xi) := \varphi^2_{(3)}(\xi) - h$. It is a $\mathbb{Q}$-linear map making the following diagram commute:

$$\begin{tikzcd}
H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi_F"] \arrow[d, "2\delta"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\
\bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x \in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd} \tag{3-8}$$

This modification was mentioned briefly by de Jeu in [19, Remark 5.3]. From diagram (3-8), φ_F induces a map

$$\varphi_C : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \to H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$$

such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \arrow[r] \arrow[d, "\varphi_C"'] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "2\delta"] \arrow[d, "\varphi_F"'] & \bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[d, "\simeq"] \\
0 \arrow[r] & H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \arrow[r] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[r, "\mathrm{Res}^{\mathcal{M}}"] & \bigoplus_{x \in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)),
\end{tikzcd} \tag{3-9}$$

where the lower horizontal sequence is the localization sequence of motivic cohomology from (1-9).

3.3. ***Relation to Goncharov's complexes.*** Let F be an arbitrary field of characteristic 0. De Jeu showed that there is a map $B_2(F) \to \widetilde{M}_{(2)}(F)$ given by $\{x\}_2 \mapsto [x]_2$ (see [20, Lemma 5.2]). This map fits into the commutative diagram

$$\begin{tikzcd} \Gamma(F,2): & B_2(F) \arrow[r] \arrow[d] & \bigwedge^2 F_{\mathbb{Q}}^{\times} \arrow[d, equal] \\ \widetilde{\mathcal{M}}^{\bullet}_{(2)}(F): & \widetilde{M}_{(2)}(F) \arrow[r] & \bigwedge^2 F_{\mathbb{Q}}^{\times}. \end{tikzcd} \tag{3-10}$$

This diagram gives rise to a map $\psi^1_{(2)} : H^1(\Gamma(F,2)) \to H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(F))$. In particular, if k is a number field, we have $\psi^1_{(2)} : H^1(\Gamma(k,2)) \xrightarrow{\simeq} H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k))$ is an isomorphism (see [18, Section 5]). We then define $\beta^1_{(2)} : H^1(\Gamma(k,2)) \to H^1_{\mathcal{M}}(k,\mathbb{Q}(2))$ to be the composition of $\varphi^1_{(2)}$ and $\psi^1_{(2)}$

$$\begin{tikzcd} H^1(\Gamma(k,2)) \arrow[r, "\beta^1_{(2)}", "\simeq"'] \arrow[dr, "\simeq", "\psi^1_{(2)}"'] & H^1_{\mathcal{M}}(k,\mathbb{Q}(2)) \\ & H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k)). \arrow[u, "\simeq"', "\varphi^1_{(2)}"] \end{tikzcd} \tag{3-11}$$

The map $B_2(F) \to \widetilde{M}_{(2)}(F)$, $\{x\}_2 \mapsto [x]_2$ also fits into the commutative diagram

$$\begin{tikzcd} \Gamma(F,3): & B_3(F) \arrow[r] & B_2(F) \otimes F_{\mathbb{Q}}^{\times} \arrow[r] \arrow[d] & \bigwedge^3 F_{\mathbb{Q}}^{\times} \arrow[d, equal] \\ \widetilde{\mathcal{M}}^{\bullet}_{(3)}(F): & \widetilde{M}_{(3)}(F) \arrow[r] & \widetilde{M}_{(2)}(F) \otimes F_{\mathbb{Q}}^{\times} \arrow[r] & \bigwedge^3 F_{\mathbb{Q}}^{\times}. \end{tikzcd} \tag{3-12}$$

The middle vertical arrow in the diagram (3-12) sends objects of the form $\{x\}_x \otimes x$ to $[x]_2 \otimes x$, so that it maps the image of $B_3(F)$ to the image of $\widetilde{M}_{(3)}(F)$. It then induces a map

$$\psi_F : H^2(\Gamma(F,3)) \to H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)). \tag{3-13}$$

Now let C be a smooth geometrically connected curve over a number field k. Let F be its function field and for any $x \in C^1$, we denote by $k(x)$ the residue of C at x. Since the residue maps of Goncharov and de Jeu are defined similarly, we have the commutative diagram

$$\begin{tikzcd} H^2(\Gamma(F,3)) \arrow[r, "\psi_F"] \arrow[d, "2\partial"'] & H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)) \arrow[d, "2\delta"] \\ \bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)) \arrow[r, "\simeq", "\psi^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k(x))). \end{tikzcd}$$

Then ψ_F induces a map $\psi_C : H^2(\Gamma(C,3)) \to H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(C))$ that makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Gamma(C,3)) \arrow[r] \arrow[d, "\psi_C"] & H^2(\Gamma(F,3)) \arrow[r, "2\partial"] \arrow[d, "\psi_F"] & \bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)) \arrow[d] \\ 0 \arrow[r] & H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(C)) \arrow[r] & H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)) \arrow[r, "2\delta"] & \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k(x))). \end{tikzcd}$$

Putting $\beta_F := \varphi_F \circ \psi_F$, we have the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma(F,3)) \arrow[r, "\psi_F"] \arrow[d, "2\partial"'] \arrow[rr, bend left, "\beta_F"] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi_F"] \arrow[d, "2\delta"] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\
\bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)) \arrow[r, "\simeq", "\psi^1_{(2)}"'] \arrow[rr, bend right, "\simeq", "\beta^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd} \tag{3-14}$$

Again, the map

$$\beta_F : H^2(\Gamma(F,3)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \tag{3-15}$$

induces a map $\beta_C : H^2(\Gamma(C,3)) \to H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^2(\Gamma(C,3)) \arrow[r] \arrow[d, "\beta_C"'] & H^2(\Gamma(F,3)) \arrow[r, "2\partial"] \arrow[d, "\beta_F"] & \bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)) \arrow[d, "\simeq"', "\beta^1_{(2)}"] \\
0 \arrow[r] & H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \arrow[r] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[r, "\mathrm{Res}^{\mathcal{M}}"] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd} \tag{3-16}$$

In particular, we have $\beta_C = \varphi_C \circ \psi_C$.

3.4. ***Regulator maps.*** Let C be a smooth proper geometrically connected curve over a number field k and let F be its function field. We have the following lemma, which is a consequence of de Jeu's theorem [20, Theorem 3.5] and Goncharov's theorem [16, Theorem 3.3].

Lemma 3.1 (de Jeu). *Let ω be a holomorphic 1-form on $C(\mathbb{C})$ such that $F_{\mathrm{dR}}(\omega) = \omega$, where F_{dR} is the action defined in Section 1.1. Let $\alpha = \sum_j c_j\{f_j\}_2 \otimes g_j \in H^2(\Gamma(F,3))$ with $c_j \in \mathbb{Q}$ and $f_j \in F^\times \setminus \{1\}$, $g_j \in F^\times$. With the fixed sign of $\varphi^2_{(3)}$ as in Section 3.2, we have*

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\beta_F(\alpha)) \wedge \bar{\omega} = 2\int_{C(\mathbb{C})} r_3(2)_F(\alpha) \wedge \bar{\omega}, \tag{3-17}$$

where β_F is the map defined in (3-15), reg_F *is Beilinson's regulator map* (1-13), *and* $r_3(2)_F$ *is Goncharov's regulator map* (2-13).

Proof. We consider the regulator integral (1-17)

$$H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \xrightarrow{\mathrm{reg}_F} H^1(F, \mathbb{R}(2))^+ \xrightarrow{\eta \mapsto \int_{C(\mathbb{C})} \eta\wedge\bar{\omega}} \mathbb{R}(1).$$

The image of $H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}}$ under the regulator integral is trivial (see [19, Theorem 4.2]). This can be seen by noting that

$$\int_{C(\mathbb{C})} d\arg g \wedge \omega = 2\pi \int_{g^{-1}(\mathbb{R}_{>0})} \omega = 2\pi \int_0^\infty g_*\omega = 0,$$

where g_* is the pushforward by the correspondence from C to $\mathbb{P}^1$, and the fact that $g_*\omega = 0$ since $\mathbb{P}^1$ has no holomorphic forms. Hence, for $\xi \in H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$, we have

$$\begin{aligned}\int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi_F(\xi)) \wedge \bar{\omega} &= \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi) - h) \wedge \bar{\omega} \quad \text{for some } h \in H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}} \text{ (see (3-7))} \\ &= \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega} - \int_{C(\mathbb{C})} \mathrm{reg}_F(h) \wedge \bar{\omega} \\ &= \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega}. \end{aligned} \tag{3-18}$$

Now let $\alpha = \sum_j c_j\{f_j\}_2 \otimes g_j \in H^2(\Gamma(F, 3))$. Then $\psi_F(\alpha) = \sum_j c_j[f_j]_2 \otimes g_j \in H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$. Using (3-18) with $\xi = \psi_F(\alpha)$, we have

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\beta_F(\alpha)) \wedge \bar{\omega} = \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi_F(\xi)) \wedge \bar{\omega} = \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega}.$$

With the fixed sign of $\varphi^2_{(3)}$ in Section 3.2, one can show that

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega} = \frac{8}{3} \sum_j c_j \int_{C(\mathbb{C})} \log|g_j| \theta(1 - f_j, f_j) \wedge \bar{\omega}$$

(see [20, Theorem 3.5]). On the other hand, by some computations, one obtains the formula

$$\int_{C(\mathbb{C})} r_3(2)_F(\alpha) \wedge \bar{\omega} = \frac{4}{3} \sum_j c_j \int_{C(\mathbb{C})} \log|g_j| \theta(1 - f_j, f_j) \wedge \bar{\omega}$$

(see [16, Theorem 3.3]). Therefore, we have

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\beta_F(\alpha)) \wedge \bar{\omega} = 2 \int_{C(\mathbb{C})} r_3(2)_F(\alpha) \wedge \bar{\omega}. \qquad \square$$

As C is proper, the map

$$H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \to \mathrm{Hom}(H^0(C(\mathbb{C}), \Omega^1)^+, \mathbb{R}(1)), \qquad \eta \mapsto \Big(\omega \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega}\Big) \tag{3-19}$$

is injective (see [20, Remark 3.1]). By Lemma 3.1, for $\alpha \in H^2(\Gamma(C, 3))$, we have

$$\int_{C(\mathbb{C})} \mathrm{reg}_C(\beta_C(\alpha)) \wedge \bar{\omega} = 2 \int_{C(\mathbb{C})} r_3(2)_C(\alpha) \wedge \bar{\omega},$$

where $r_3(2)_C$ is Goncharov's regulator map (2-14). Therefore, we have the commutative diagram

$$\begin{tikzcd} H^2(\Gamma(C,3)) \arrow[r, "\psi_C"] \arrow[rr, bend left, "\beta_C"] \arrow[dr, "r_3(2)_C"'] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \arrow[r, "\varphi_C"] \arrow[d] & H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \arrow[dl, "\frac{1}{2}\mathrm{reg}_C"] \\ & H^1(C(\mathbb{C}), \mathbb{R}(2))^+, & \end{tikzcd} \tag{3-20}$$

where the middle vertical map is just the composition $\frac{1}{2}\mathrm{reg}_C \circ \varphi_C$. In [20, Corollary 5.5], de Jeu showed that the images of the $r_3(2)_C$ and reg_C, as vector spaces, are the same.

Lemma 3.2. *Let $\alpha = \sum_j c_j\{f_j\}_2 \otimes g_j \in H^2(\Gamma(F,3))$ with $c_j \in \mathbb{Q}$ and $f_j \in F^\times \setminus \{1\}$, $g_j \in F^\times$. Denote by $Y = C \setminus Z$ where Z is the closed subscheme of C consisting of the zeros and poles of $f_j, 1-f_j, g_j$ for all j. With the fixed choice of signs of $\varphi^1_{(2)}$ and $\varphi^2_{(3)}$, we have*

$$\int_\gamma \mathrm{reg}_F(\beta_F(\alpha)) = 2\int_\gamma r_3(2)_F(\alpha) \qquad \textit{for any loop } \gamma \in H_1(Y(\mathbb{C}),\mathbb{Z}).$$

Proof. First notice that $\beta_F(\alpha) \in H^2_{\mathcal{M}}(F,\mathbb{Q}(3))$ actually belongs to the subgroup $H^2_{\mathcal{M}}(Y,\mathbb{Q}(3))$ (see [7, Theorem 5.4]). Then $\mathrm{reg}_F(\beta_F(\alpha))$ belongs to $H^1(Y(\mathbb{C}),\mathbb{R}(2))^+$. In particular, the integral $\int_\gamma \mathrm{reg}_F(\beta_F(\alpha))$ is well-defined for any loop $\gamma \in H_1(Y(\mathbb{C}),\mathbb{Z})$. Since $r_3(2)_F(\alpha)$ is represented by the form $\sum_j c_j\rho(f_j,g_j)$ which defines an element in $H^1(Y(\mathbb{C}),\mathbb{R})^+$, the integral $\int_\gamma r_3(2)_F(\alpha)$ is also well-defined.

In particular, we have $\mathrm{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha) \in H^1(Y(\mathbb{C}),\mathbb{R}(2))^+$. We consider the Mayer–Vietoris sequence

$$0 \longrightarrow H^1(C(\mathbb{C}),\mathbb{R}(2))^+ \longrightarrow H^1(Y(\mathbb{C}),\mathbb{R}(2))^+ \xrightarrow{\oplus(2\pi i)^{-1}\mathrm{Res}_p} \oplus_{p\in Z(\mathbb{C})}\mathbb{R}(1), \tag{3-21}$$

where Res_p is the residue map defined in Definition 2.2. We are going to show that $\mathrm{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)$ extends to $H^1(C(\mathbb{C}),\mathbb{R}(2))^+$. Let $p \in Z(\mathbb{C})$ supported on a closed point $x \in Z^1$ with the embedding $\sigma : k(x) \hookrightarrow \mathbb{C}$, i.e.,

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C} \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "\sigma"'] \arrow[dr, "p"] & Z\times_{\mathbb{Q}}\mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{C} \arrow[d] \\
\operatorname{Spec} k(x) \arrow[r, "x"'] & Z \arrow[r] & \operatorname{Spec}\mathbb{Q}.
\end{tikzcd}$$

With the fixed signs of $\varphi^1_{(2)}$ and $\varphi^2_{(3)}$ as before, we have the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma(F,3)) \arrow[r, "\beta_F"] \arrow[d, "2\partial_x"'] & H^2_{\mathcal{M}}(F,\mathbb{Q}(3)) \arrow[r, "\mathrm{reg}_F"] \arrow[d, "\mathrm{Res}^{\mathcal{M}}_x"] & H^1(F,\mathbb{R}(2))^+ \arrow[d, "(2\pi i)^{-1}\mathrm{Res}_p"] \\
H^1(\Gamma(k(x),2)) \arrow[r, "\beta^1_{(2)}"'] & H^1_{\mathcal{M}}(k(x),\mathbb{Q}(2)) \arrow[r, "\mathrm{reg}_{k(x)}"'] & \mathbb{R}(1),
\end{tikzcd} \tag{3-22}$$

where $\mathrm{reg}_{k(x)}$ is the composition $H^1_{\mathcal{M}}(k(x),\mathbb{Q}(2)) \xrightarrow{\sigma^*} H^1_{\mathcal{M}}(\mathbb{C},\mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_{\mathbb{C}}} H^1_{\mathcal{D}}(\mathbb{C},\mathbb{R}(2)) \simeq \mathbb{R}(1)$ as mentioned in (3-4). The commutativity of the right square follows from the compatibility of the Beilinson regulators and residues maps. We then have

$$\mathrm{Res}_p(\mathrm{reg}_F(\beta_F(\alpha))) = (4\pi i)\,\mathrm{reg}_{k(x)}(\beta^1_{(2)}(\partial_x(\alpha))) = (4\pi i)(\mathrm{reg}_{k(x)}\circ\beta^1_{(2)})\Big(\sum_j c_j v_x(g_j)\{f_j(x)\}_2\Big),$$

where the last equality follows from the definition of Goncharov's residues map (2-5). As mentioned in (3-4), the sign of $\varphi^1_{(2)}$ is chosen such that the lower composition map in the diagram (3-22) is induced by the map $\{z\}_2 \mapsto i D(\sigma(z))$. Therefore, we have

$$\begin{aligned}\operatorname{Res}_p(\operatorname{reg}_F(\beta_F(\alpha))) &= (4\pi i) i \sum_j c_j v_x(g_j) D(\sigma(f_j(x))) \\ &= -4\pi \sum_j c_j v_{\sigma(x)}(g_j) D(f_j(\sigma(x))) \quad (\text{as } f_j, g_j \in k(C)) \\ &= 2 \operatorname{Res}_p(r_3(2)_F(\alpha)) \quad (\text{by Lemma 2.3}).\end{aligned}$$

So $\operatorname{Res}_p(\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)) = 0$ for all $p \in Z(\mathbb{C})$, hence $\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)$ extends to $H^1(C(\mathbb{C}), \mathbb{R}(2))^+$. Therefore, the class $\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)(\alpha)$ is represented by $\eta + dt$, where η is a F_{dR}-invariant closed differential 1-form on $C(\mathbb{C})$ and t is a logarithmic growth function on $Y(\mathbb{C})$. Now let ω be a holomorphic 1-form on $C(\mathbb{C})$ such that $F_{\mathrm{dR}}(\omega) = \omega$. Since t is a logarithmic growth function on $Y(\mathbb{C})$, we have

$$\int_{C(\mathbb{C})} dt \wedge \bar{\omega} = \int_{C(\mathbb{C})} d(t\bar{\omega}) = 0$$

by using Stokes' theorem (see the proof of [19, Theorem 4.6]). We then have

$$\int_{C(\mathbb{C})} \eta \wedge \bar{\omega} = \int_{C(\mathbb{C})} (\eta + dt) \wedge \bar{\omega} = \int_{C(\mathbb{C})} (\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)) \wedge \bar{\omega} = 0,$$

where the last equality is by Lemma 3.1. Since ω is an arbitrary F_{dR}-invariant holomorphic 1-form and such forms span a real vector space dual to $H^1(C(\mathbb{C}), \mathbb{R}(2))^+$ ([20, Remark 3.1]), we obtain that $\eta = ds$ for some function s on $C(\mathbb{C})$. So $\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha))$ is represented by $d(s+t)$ for some logarithmic growth function $s+t$ on $Y(\mathbb{C})$. Hence

$$\int_\gamma \operatorname{reg}_F(\beta_F(\alpha)) = 2\int_\gamma r_3(2)_F(\alpha) \qquad \text{for any loop } \gamma \in H_1(Y(\mathbb{C}), \mathbb{Z}). \qquad \square$$

4. Main result

In Section 4.1, we construct an element in Deligne cohomology and in Section 4.2, we connect it to the Mahler measure. In Section 4.3, we construct an element in $K_4^{(3)}$ of a curve such that its regulator is related to the Deligne cohomology class constructed in Section 4.1. We prove Theorem 0.2 in Section 4.

4.1. ***Constructing an element in Deligne cohomology.*** Let

$$P(x, y, z) \in \mathbb{Q}[x, y, z]$$

be an irreducible polynomial. We denote by V_P the zero locus of P in $(\mathbb{C}^\times)^3$ and V_P^{reg} the smooth part of V_P. For $f, g, h \in \mathbb{C}(V_P^{\mathrm{reg}})^\times$, we recall the differential form mentioned in (2-10)

$$\begin{aligned}\eta(f, g, h) = \log|f| &\left(\tfrac{1}{3} d\log|g| \wedge d\log|h| - d\arg(g) \wedge d\arg(h)\right) \\ &+ \log|g| \left(\tfrac{1}{3} d\log|h| \wedge d\log|f| - d\arg(h) \wedge d\arg(f)\right) \\ &+ \log|h| \left(\tfrac{1}{3} d\log|f| \wedge d\log|g| - d\arg(f) \wedge d\arg(g)\right). \qquad (4\text{-}1)\end{aligned}$$

The form is bilinear and antisymmetric in f, g, h. It is defined on $V_P^{\mathrm{reg}} \setminus S_{f,g,h}$, where $S_{f,g,h}$ is the set of zeros and poles of f, g and h. Moreover, $\eta(f,g,h)$ is a closed form on $V_P^{\mathrm{reg}} \setminus S_{f,g,h}$ since

$$d\eta(f,g,h) = \mathrm{Re}\left(\frac{df}{f} \wedge \frac{dh}{h} \wedge \frac{dg}{g}\right),$$

which is zero in $V_P^{\mathrm{reg}} \setminus S_{f,g,h}$.

Lemma 4.1. *The differential form $\eta(x,y,z)$ defines an element in the Deligne cohomology $H^3_{\mathcal{D}}(\mathbb{G}_m^3, \mathbb{R}(3))$. Moreover, it represents the class $\mathrm{reg}_{\mathbb{G}_m^3}(\{x,y,z\})$, where $\mathrm{reg}_{\mathbb{G}_m^3} : H^3_{\mathcal{M}}(\mathbb{G}_m^3, \mathbb{Q}(3)) \to H^3_{\mathcal{D}}(\mathbb{G}_m^3, \mathbb{R}(3))$ is the Beilinson regulator map and $\{x,y,z\} \in H^3_{\mathcal{M}}(\mathbb{G}_m^3, \mathbb{Q}(3))$ is the Milnor symbol.*

Proof. By definition, $\eta(x,y,z) \in E^2_{\log,\mathbb{R}}(\mathbb{G}_m^3)$, and defines an element in $H^3_{\mathcal{D}}(\mathbb{G}_m^3, \mathbb{R}(3))$. By an observation in Section 1.4, $\mathrm{reg}_{\mathbb{G}_m^3}(\{x,y,z\})$ is represented by the cup product $\log|x| \cup \log|y| \cup \log|z|$ in Deligne cohomology. By the cup product formula (1-3), we have

$$\begin{aligned}
\log|x| \cup \log|y| \cup \log|z| &= (\log|x| \cup \log|y|) \cup \log|z| \\
&= (-1)^2 r_2(\log|x| \cup \log|y|)\log|z| + (\log|x| \cup \log|y|)\, r_1(\log|z|) \\
&= \left(\partial\left(\frac{1}{2}\log|x|\,\frac{dy}{y} - \frac{1}{2}\log|y|\,\frac{dx}{x}\right) - \bar\partial\left(\frac{1}{2}\log|y|\,\frac{d\bar x}{\bar x} - \frac{1}{2}\log|x|\,\frac{d\bar y}{\bar y}\right)\right)\log|z| \\
&\qquad + i\cdot(\log|x|d\arg y - \log|y|d\arg(x)) \wedge (\partial\log|z| - \bar\partial\log|z|) \\
&= \left(\frac{1}{2}\frac{dx}{x} \wedge \frac{dy}{y} + \frac{1}{2}\frac{d\bar x}{\bar x} \wedge \frac{d\bar y}{\bar y}\right)\log|z| - (\log|x|\,d\arg y - \log|y|\,d\arg x) \wedge d\arg z \\
&= \log|z|(d\log|x| \wedge d\log|y| - d\arg(x) \wedge d\arg y) \\
&\qquad - \log|y|\,d\arg(z) \wedge d\arg x - \log|x|\,d\arg(y) \wedge d\arg z.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\eta(x,y,z) - \log|x| \cup \log|y| \cup \log|z| \\
&= \tfrac{1}{3}\log|x| d\log|y| \wedge d\log|z| + \tfrac{1}{3}\log|y| d\log|z| \wedge d\log|x| - \tfrac{2}{3}\log|z| d\log|x| \wedge d\log|y| \\
&= -\tfrac{1}{3}d(\log|x|\log|z|\,d\log|y|) + \tfrac{1}{3}d(\log|y|\log|z|\,d\log|x|),
\end{aligned}$$

which is an exact form, hence $\mathrm{reg}_{\mathbb{G}_m^3}(\{x,y,z\})$ is represented by $\eta(x,y,z)$. ☐

Consequently, pulling back $\eta(x,y,z)$ by the embedding $V_P^{\mathrm{reg}} \overset{i}{\hookrightarrow} \mathbb{G}_m^3$, we see that $\eta(x,y,z)|_{V_P^{\mathrm{reg}}}$ is a representative of $\mathrm{reg}_{V_P^{\mathrm{reg}}}(\{x,y,z\})$ in $H^3_{\mathcal{D}}(V_P^{\mathrm{reg}}, \mathbb{R}(3))$. We come to the definition of *exact polynomials*.

Definition 4.2 (exact polynomial). A polynomial $P(x,y,z)$ is called *exact* if $\mathrm{reg}_{V_P^{\mathrm{reg}}}(\{x,y,z\})$ is trivial, i.e., $\eta(x,y,z)$ is an exact form on V_P^{reg}.

Remark 4.3. If P satisfies Lalín's condition (see Theorem 0.1(iii)), namely

$$x \wedge y \wedge z = \sum_j f_j \wedge (1-f_j) \wedge g_j \quad \text{in } \textstyle\bigwedge^3 \mathbb{Q}(V_P)^\times_{\mathbb{Q}} \tag{4-2}$$

for some functions $f_j \in \mathbb{Q}(V_P)^\times \setminus \{1\}$ and $g_j \in \mathbb{Q}(V_P)^\times$, then P is exact because

$$\eta(x,y,z)|_{V_P^{\mathrm{reg}}} = \sum_j \eta(f_j, 1-f_j, g_j) = \sum_j d\rho(f_j,g_j) = d\sum_j \rho(f_j,g_j),$$

where $\rho(f, g)$ is the differential form defined in (2-11). In particular, the polynomials of the form $A(x) + B(x)y + C(x)z$, where $A(x), B(x), C(x)$ are products of cyclotomic polynomials, are exact. Indeed, we have

$$\begin{aligned} x \wedge y \wedge z &= x \wedge y \wedge \frac{A(x)+B(x)y}{C(x)} \\ &= x \wedge y \wedge \Big(\frac{A(x)}{C(x)} \cdot \frac{A(x)+B(x)y}{A(x)}\Big) \\ &= x \wedge y \wedge \frac{A(x)}{C(x)} + x \wedge y \wedge \Big(1 + \frac{B(x)y}{A(x)}\Big) \\ &= x \wedge y \wedge \frac{A(x)}{C(x)} + x \wedge \frac{B(x)y}{A(x)} \wedge \Big(1 + \frac{B(x)y}{A(x)}\Big) - x \wedge \frac{B(x)}{A(x)} \wedge \Big(1 + \frac{B(x)y}{A(x)}\Big). \end{aligned} \tag{4-3}$$

For cyclotomic polynomials $\Phi(x)$, we have

$$\begin{aligned} x \wedge y \wedge \Phi_n(x) = x \wedge y \wedge \frac{x^n - 1}{\prod_{d \mid n, d \neq n} \Phi_d(x)} &= x \wedge y \wedge (x^n - 1) - \sum_{d \mid n, d \neq n} x \wedge y \wedge \Phi_d(x) \\ &= -\frac{1}{n} x^n \wedge (1 - x^n) \wedge y - \sum_{d \mid n, d \neq n} x \wedge y \wedge \Phi_d. \end{aligned}$$

For $n = 1$, $x \wedge y \wedge (x + 1) = -x \wedge (1 + x) \wedge y$. So we get (4-2) by induction on n.

From now on, let us assume our polynomial P satisfies the condition (4-2). We consider the involution

$$\tau : \mathbb{G}_m^3 \to \mathbb{G}_m^3, (x, y, z) \mapsto (1/x, 1/y, 1/z), \tag{4-4}$$

which maps V_P to V_{P^*}, where $P^*(x, y, z) := \bar{P}(1/x, 1/y, 1/z) = P(1/x, 1/y, 1/z)$. Let W_P be the curve defined by

$$\begin{cases} P(x, y, z) = 0, \\ P(1/x, 1/y, 1/z) = 0. \end{cases} \tag{4-5}$$

We call W_P the *Maillot variety*. The restriction $\tau|_{W_P} : W_P \to W_P$ is an automorphism. We view W_P as a curve over $\mathbb{Q}$. Then let C be the normalization of W_P. The condition (4-2) implies that

$$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j \quad \text{in } \textstyle\bigwedge^3 \mathbb{Q}(C)^\times_{\mathbb{Q}} \tag{4-6}$$

for some functions $f_j \in \mathbb{Q}(C)^\times \setminus \{1\}$ and $g_j \in \mathbb{Q}(C)^\times$.

Definition 4.4. Let $F = \mathbb{Q}(C)$ be the function field of C. We write

$$\xi := \sum_j \{f_j\}_2 \otimes g_j, \qquad \xi^* := \sum_j \{f_j \circ \tau\}_2 \otimes (g_j \circ \tau), \qquad \lambda := \xi + \xi^*, \tag{4-7}$$

which are elements in $B_2(F) \otimes F^\times_{\mathbb{Q}}$. Let us consider the following closed subschemes of V_P and V_{P^*}:

$$Z_1 = \{\text{zeros and poles of } f_j, 1 - f_j, g_j \text{ on } V_P \text{ for all } j\}, \tag{4-8}$$

$$Z_2 = \{\text{zeros and poles of } f_j \circ \tau, 1 - f_j \circ \tau, g_j \circ \tau \text{ on } V_{P^*} \text{ for all } j\}. \tag{4-9}$$

We define the following differential 1-forms on $V_P^{\mathrm{reg}} \setminus Z_1$ and $V_{P^*}^{\mathrm{reg}} \setminus Z_2$, respectively:

$$\rho(\xi) := \sum_j \rho(f_j, g_j), \qquad \rho(\xi^*) := \sum_j \rho(f_j \circ \tau, g_j \circ \tau), \tag{4-10}$$

where $\rho(f, g)$ is mentioned in (2-11). Denote by Z the closed subscheme of C

$$Z = \{\text{zeros and poles of } f_j, 1 - f_j, g_j, f_j \circ \tau, 1 - f_j \circ \tau, g_j \circ \tau \text{ on } C \text{ for all } j\}, \tag{4-11}$$

and set $Y = \iota(W_P^{\mathrm{reg}}) \setminus Z$, where $\iota : W_P^{\mathrm{reg}} \hookrightarrow C$ is the canonical embedding. Using the canonical embeddings of $Y(\mathbb{C})$ into V_P and V_{P^*}, we define the following differential 1-form on $Y(\mathbb{C})$:

$$\rho(\lambda) = \rho(\xi)|_{Y(\mathbb{C})} + \rho(\xi^*)|_{Y(\mathbb{C})}. \tag{4-12}$$

Lemma 4.5. *The element λ defines a class in $H^2(\Gamma(F, 3))$, and also an element in $H^2(\Gamma(Y, 3))$.*

Proof. We recall the polylogarithmic Goncharov complex

$$\begin{array}{llcl} \Gamma(F,3): & B_3(F) \longrightarrow B_2(F) \otimes F_{\mathbb{Q}}^{\times} & \xrightarrow{\alpha_3(2)} & \bigwedge^3 F_{\mathbb{Q}}^{\times} \\ & \qquad\qquad \{f\}_2 \otimes g & \longmapsto & (1-f) \wedge f \wedge g. \end{array}$$

We have

$$\alpha_3(2)(\xi) = \sum_j \alpha_3(2)(\{f_j\}_2 \otimes g_j) = \sum_j (1 - f_j) \wedge f_j \wedge g_j = -x \wedge y \wedge z,$$

and

$$\begin{aligned} \alpha_3(2)(\xi^*) &= \sum_j \alpha_3(2)\big(\{f_j \circ \tau\}_2 \otimes (g_j \circ \tau)\big) = \sum_j (1 - f \circ \tau) \wedge (f_j \circ \tau) \wedge (g_j \circ \tau) \\ &= \tau^*\Big(\sum_j (1 - f_j) \wedge f_j \wedge g_j\Big) = \tau^*(-x \wedge y \wedge z) \\ &= -\frac{1}{x} \wedge \frac{1}{y} \wedge \frac{1}{z} = x \wedge y \wedge z, \end{aligned}$$

so $\alpha_3(2)(\lambda) = \alpha_3(2)(\xi) + \alpha_3(2)(\xi^*) = 0$. Then λ defines a class in $H^2(\Gamma(F, 3))$. Now we consider the following exact sequence (see Section 2.2):

$$0 \to H^2(\Gamma(Y, 3)) \to H^2(\Gamma(F, 3)) \xrightarrow{\oplus \partial_p} \bigoplus_{p \in Y^1} H^1(\Gamma(\mathbb{Q}(p), 2)), \tag{4-13}$$

where Y^1 is the set of closed points of Y. The residue of λ at $p \in Y^1$ is given by

$$\partial_p(\lambda) = \sum_j v_p(g_j)\{f_j(p)\}_2 + v_p(g_j \circ \tau)\{f_j \circ \tau(p)\}_2 \in H^1(\Gamma(\mathbb{Q}(p), 2), \tag{4-14}$$

which is trivial for every point $p \notin S$, where S is the closed subscheme of C

$$S = \{\text{zeros and poles of } g_j, g_j \circ \tau \text{ on } C \text{ for all } j\}. \tag{4-15}$$

We have $\partial_p(\lambda) = 0$ for all $p \in Y^1$, hence λ defines an element in $H^2(\Gamma(Y, 3))$. $\square$

Lemma 4.6. *The differential* 1*-form* $\rho(\lambda)$ *defines an element in* $H^2_{\mathcal{D}}(Y_{\mathbb{R}}, \mathbb{R}(3)) \simeq H^1(Y(\mathbb{C}), \mathbb{R}(2))^+$. *For any point* $p \in C(\mathbb{C})$, *the residue of* $\rho(\lambda)$ *at* p *is given by*

$$\operatorname{Res}_p(\rho(\lambda)) = -2\pi\left(\sum_j v_p(g_j)D(f_j(p)) + v_p(g_j \circ \tau)D(f_j \circ \tau)(p)\right), \tag{4-16}$$

where D *is the Bloch–Wigner dilogarithm function* (0-7).

Proof. The first statement follows from the fact that $\rho(\lambda)$ represents $r_3(2)_Y(\lambda)$, where

$$r_3(2)_Y : H^2(\Gamma(Y, 3)) \to H^2_{\mathcal{D}}(Y, R(3)) \simeq H^1(Y(\mathbb{C}), R(2))^+$$

is Goncharov's regulator map (2-14), or by checking directly that $\rho(\lambda)$ is closed and fixed under the action of the involution F_{dR}. The formula (4-16) follows directly from Lemma 2.3. □

Remark 4.7. If all the residues $u_p := \partial_p(\lambda)$ are trivial for all $p \in S$ (see (4-15)), then λ defines a unique class λ_C in $H^2(\Gamma(C, 3))$ and $\rho(\lambda)$ represents the class $r_3(2)_C(\lambda_C) \in H^2_{\mathcal{D}}(C, \mathbb{Q}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+$.

4.2. ***Relating the Mahler measure to the Deligne cohomology.*** We retain the notation from the previous section. In this section, we connect $\rho(\lambda)$ to the Mahler measure of P. Recall that the Deninger chain associated to P is defined by

$$\Gamma = \left\{(x, y, z) \in (\mathbb{C}^\times)^3 : P(x, y, z) = 0,\ |x| = |y| = 1,\ |z| \geq 1\right\}. \tag{4-17}$$

Its orientation is induced from $\mathbb{T}^2$: for each $(x_0, y_0) \in \mathbb{T}^2$, there are finitely many $z \in \mathbb{C}$ such that $|z| \geq 1$ and $P(x_0, y_0, z) = 0$, then by letting (x_0, y_0) runs on the torus $\mathbb{T}^2_{(x,y)}$ along the usual orientation, we get the orientation of Γ. Its boundary is given by

$$\partial\Gamma = \left\{(x, y, z) \in (\mathbb{C}^\times)^3 : P(x, y, z) = 0,\ |x| = |y| = |z| = 1\right\}.$$

Deninger [12, Proposition 3.3] showed that if Γ is contained in V_P^{reg}, then we get the formula

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{4\pi^2}\int_\Gamma \eta(x, y, z), \tag{4-18}$$

where $\tilde{P}(x, y)$ is the leading coefficient of $P(x, y, z)$ considered as a polynomial in z. If furthermore, $\partial\Gamma = \varnothing$, then $[\Gamma] \in H_2(V_P^{\mathrm{reg}}, \mathbb{Z})$ and the Mahler measure is written as a pairing in Deligne cohomology

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{4\pi^2}\langle[\Gamma], \mathrm{reg}_{V_P^{\mathrm{reg}}}(\{x, y, z\})\rangle_{V_P^{\mathrm{reg}}}.$$

Since $P(x, y, z)$ has rational coefficients, we can write

$$\partial\Gamma = \left\{P(x, y, z) = P(1/x, 1/y, 1/z) = 0\right\} \cap \left\{|x| = |y| = |z| = 1\right\},$$

which is contained in W_P, and may contain some singularities of W_P. We have the following lemma.

Lemma 4.8. *We assume that*

$$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j \quad \text{in } \textstyle\bigwedge^3 \mathbb{Q}(V_P)^\times_{\mathbb{Q}}. \tag{4-19}$$

Suppose that Γ is contained in V_P^{reg} and that $\partial\Gamma$ is contained in $Y(\mathbb{C})$ (see Definition 4.4). Then $\partial\Gamma$ defines an element in the singular homology group $H_1(Y(\mathbb{C}), \mathbb{Z})^+$, where "$+$" denotes the invariant part by the complex conjugation, and we can write the Mahler measure as a pairing in Deligne cohomology of $Y_{\mathbb{R}}$:

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{8\pi^2} \langle [\partial\Gamma], [\rho(\lambda)] \rangle_Y , \tag{4-20}$$

where the pairing is given by

$$\langle \cdot , \cdot \rangle_Y : H_1(Y(\mathbb{C}), \mathbb{Z})^+ \times H^1(Y(\mathbb{C}), \mathbb{R}(2))^+ \to \mathbb{R}(2). \tag{4-21}$$

Proof. Since $\Gamma \subset V_P^{\text{reg}}$ and $\partial\Gamma \subset Y(\mathbb{C})$, $\partial\Gamma$ defines an element in $H_1(Y(\mathbb{C}), \mathbb{Z})$ by considering the sequence

$$\begin{array}{ccccc} H_2(V_P^{\text{reg}}, \partial\Gamma, \mathbb{Z}) & \longrightarrow & H_1(\partial\Gamma, \mathbb{Z}) & \longrightarrow & H_1(Y(\mathbb{C}), \mathbb{Z}) \\ [\Gamma] & \longmapsto & [\partial\Gamma] & \longmapsto & [\partial\Gamma]. \end{array}$$

Now we show that $\partial\Gamma$ is invariant under the complex conjugation. Notice that the action of the complex conjugation on $\partial\Gamma$ is the same as the action of the involution τ (4-4) on $\partial\Gamma$ because $\bar{x} = x^{-1}$ for $x \in \mathbb{T}^1$. So it suffices to show that τ fixes $\partial\Gamma$. Clearly, $\tau(\partial\Gamma) = \partial\Gamma$ as sets. We claim that τ preserves the orientation of $\partial\Gamma$. Notice that the orientation of $\partial\Gamma$ is induced from the orientation of Γ, and the orientation of Γ comes from the orientation of $\mathbb{T}^2$. The map

$$\tau|_{\mathbb{T}^2} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (1/x, 1/y) \tag{4-22}$$

preserves the orientation of $\mathbb{T}^2$; hence τ preserves the orientation of $\partial\Gamma$. Note that the condition (4-19) implies that $\eta(x, y, z)|_{V_P^{\text{reg}}} = d\rho(\xi)$. Then by applying Stokes' theorem to (4-18), we get

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{4\pi^2} \int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})}. \tag{4-23}$$

We have

$$\int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})} = \int_{\tau(\partial\Gamma)} \tau^*(\rho(\xi)|_{Y(\mathbb{C})}) = \int_{\partial\Gamma} \tau^*(\rho(\xi)|_{Y(\mathbb{C})}) = \int_{\partial\Gamma} \rho(\xi^*)|_{Y(\mathbb{C})},$$

where the second equality is because $\tau(\partial\Gamma) = \partial\Gamma$ as sets and τ preserves the orientation of $\partial\Gamma$. Then, by (4-23), we get

$$\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = -\frac{1}{4\pi^2} \int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})} = -\frac{1}{8\pi^2} \int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})} + \rho(\xi^*)|_{Y(\mathbb{C})} = -\frac{1}{8\pi^2} \int_{\partial\Gamma} \rho(\lambda),$$

which is exactly formula (4-20). $\square$

4.3. ***Constructing an element in the motivic cohomology.*** In Section 4.1, we constructed an element λ that defines a class in $H^2(\Gamma(Y,3))$ and its regulator is represented by the differential 1-form $\rho(\lambda)$. In this section, we construct an element in $H^2(\Gamma(C_K,3))$, where $C_K = C\times_{\mathbb{Q}} K$ for a certain number field K. It gives rise to an element in motivic cohomology $H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$ via de Jeu's map β_{C_K}. Finally, we show that this motivic cohomology class descends to an element in $H^2_{\mathcal{M}}(C,\mathbb{Q}(3))$.

Recall that the residues u_p are trivial for all $p\notin S$, where S is the closed subset of C defined in (4-15). As discussed in Remark 4.7, if u_p vanish for all $p\in S$, λ defines an element in $H^2(\Gamma(C,3))$. When the residues are nontrivial, we modify λ by its residues. This method is inspired by Bloch's trick (see, e.g., [1; 30]). Let S' be the closed subset of S consisting of the points p such that $u_p\neq 0$. Let K be the splitting field of S' in $\mathbb{C}$; this is the smallest Galois extension $K/\mathbb{Q}$ that contains all the residue fields $\mathbb{Q}(p)$ for $p\in S'$. For a geometric point $q:\mathbb{Q}(p)\hookrightarrow K$ over a point p of S', we define u_q as the image of u_p under the embedding $\mathcal{B}(\mathbb{Q}(p))\hookrightarrow^q \mathcal{B}(K)$. Then for $q\in S'(K)$, u_q defines an element in the Bloch group $\mathcal{B}(K)$. It is compatible with the Galois action, i.e., $\sigma(u_q)=u_{\sigma(q)}$ for $q\in S'(K)$ and $\sigma\in \mathrm{Gal}(K/\mathbb{Q})$,

$$\begin{tikzcd} \mathcal{B}(\mathbb{Q}(p)) \arrow[r, hook, "q"] \arrow[dr, hook] & \mathcal{B}(K) \arrow[d, "\sigma"] \\ & \mathcal{B}(K). \end{tikzcd} \tag{4-24}$$

Denote by $K(C)$ the function field of $C\times_{\mathbb{Q}} K$. The inclusion $\mathbb{Q}(p)\hookrightarrow^j K(C)$ induces a map $B_i(\mathbb{Q}(C))\xrightarrow{j} B_i(K(C))$, which is not an inclusion generally. We have the commutative diagram

$$\begin{tikzcd} B_3(\mathbb{Q}(C)) \arrow[r] \arrow[d] & B_2(\mathbb{Q}(C))\otimes \mathbb{Q}(C)^\times_{\mathbb{Q}} \arrow[r, "\alpha_3(2)"] \arrow[d, "j"'] & \bigwedge^3 \mathbb{Q}(C)^\times_{\mathbb{Q}} \arrow[d, hook] \\ B_3(K(C)) \arrow[r] & B_2(K(C))\otimes K(C)^\times_{\mathbb{Q}} \arrow[r, "\alpha_3(2)"] & \bigwedge^3 K(C)^\times_{\mathbb{Q}}. \end{tikzcd}$$

This implies a map $j: H^2(\Gamma(\mathbb{Q}(C),3))\to H^2(\Gamma(K(C),3))$. By Lemma 4.5, λ defines a class in $H^2(\Gamma(\mathbb{Q}(C),3))$, hence $j(\lambda)$ defines a class in $H^2(\Gamma(K(C),3))$. We have the exact sequence

$$0\to H^2(\Gamma(C_K,3))\longrightarrow H^2(\Gamma(K(C),3))\xrightarrow{\oplus\partial_q}\bigoplus_{q\in(C_K)^1} H^1(\Gamma(K,2)),$$

where $(C_K)^1$ is the set of closed points of C_K. We have $\partial_q(j(\lambda))=u_q$ for $q\in S'_K=S'(K)$ and trivial otherwise.

We assume that the difference of any two geometric points $q_1,q_2\in S'(K)$ in the Jacobian of C is torsion of order dividing a fixed integer N. Fix $\mathcal{O}\in S'(K)$. Then for any point $q\in S'(K)-\{\mathcal{O}\}$, there is a rational function $f_q\in K(C)^\times$ such that

$$\mathrm{div}(f_q)=N(\mathcal{O})-N(q) \tag{4-25}$$

in C_K. We set $f_{\mathcal{O}}=1$.

Definition 4.9. We set

$$\lambda' := j(\lambda) + \sum_{q \in S'(K)-\{\mathcal{O}\}} \frac{1}{N}(u_q \otimes f_q), \tag{4-26}$$

which defines an element in $B_2(K(C)) \otimes K(C)^\times_{\mathbb{Q}}$.

Lemma 4.10. *The element λ' defines a class in $H^2(\Gamma(K(C), 3))$.*

Proof. For $q \in S'(K)$, we recall the following Goncharov's complex (2-4):

$$\begin{tikzcd} B_3(K(C)) \arrow[r] & B_2(K(C)) \otimes K(C)^\times_{\mathbb{Q}} \arrow[r, "\alpha_3(2)"] \arrow[d, "\partial_q"'] & \bigwedge^3 K(C)^\times_{\mathbb{Q}} \arrow[d] \\ & B_2(K) \arrow[r, "\alpha_2(1)"] & \bigwedge^2 K^\times_{\mathbb{Q}}. \end{tikzcd} \tag{4-27}$$

As discussed above, $j(\lambda)$ defines a class in $H^2(\Gamma(K(C), 3))$, hence $\alpha_3(2)(j(\lambda)) = 0$. For $q \in S'(K)$, we have $\alpha_2(1)(u_q) = 0$ because $u_q \in \mathcal{B}(K)$. We thus have $\alpha_3(2)\left(u_q \otimes f_q\right) = (\alpha_2(1)(u_q)) \wedge f_q = 0$. This implies that

$$\alpha_3(2)(\lambda') = \alpha_3(2)(j(\lambda)) + \frac{1}{N} \sum_{q \in S'(K)-\{\mathcal{O}\}} \alpha_3(2)(u_q \otimes f_q) = 0,$$

hence λ' defines an element in $H^2(\Gamma(K(C), 3))$. □

Notice that λ' depends on the choice of rational function $f_q \in K(C)^\times$. However, the following lemma is sufficient for us.

Lemma 4.11. *The image of λ' under de Jeu's map* (3-15)

$$\beta_{K(C)} : H^2(\Gamma(K(C), 3)) \to H^2_{\mathcal{M}}(K(C), \mathbb{Q}(3)), \tag{4-28}$$

does not depend on the choice of $f_q \in K(C)^\times$.

Proof. Let $q \in S'(K)$. Let $f'_q \in K(C)^\times$ be another rational function such that $\operatorname{div}(f'_q) = N(\mathcal{O}) - N(q)$. Then $\operatorname{div}(f_q/f'_q) = 0$, hence f_q/f'_q defines an element in a finite field extension of K, denoted by L. Then $u_q \otimes (f_q/f'_q)$ defines an element in $B_2(L) \otimes L^\times$. In the proof of Lemma 4.10, we showed that $\alpha_3(2)(u_q \otimes f_q) = 0$, this implies that

$$\alpha_3(2)(u_q \otimes (f_q/f'_q)) = \alpha_3(2)(u_q \otimes f_q) - \alpha_3(2)(u_q \otimes f'_q) = 0,$$

hence $u_q \otimes (f_q/f'_q)$ defines a class in $H^2(\Gamma(L, 3))$. We consider de Jeu's map

$$\beta_L : H^2(\Gamma(L, 3)) \to K_4(L)_{\mathbb{Q}}.$$

By Borel's theorem, K_4 group of a number field is torsion, so $K_4(L)_{\mathbb{Q}} = 0$. This implies that the images of $u_q \otimes (f_q/f'_q)$ under the map β_L in $K_4(L)_{\mathbb{Q}}$ all vanish. Hence the image of λ' under de Jeu's map does not depend on the choice of f_q. □

Lemma 4.12. *All the residues of λ' in the following localization sequence vanish*:

$$0 \to H^2(\Gamma(C_K,3)) \to H^2(\Gamma(K(C),3)) \xrightarrow{2\partial} \bigoplus_{q\in(C_K)^1} H^1(\Gamma(K,2)),$$

and thus λ' defines a unique element $\lambda'_{C_K} \in H^2(\Gamma(C_K,3))$.

Proof. We have $\partial_q(\lambda') = 0$ for all $q \notin S'(K)$. For $q \in S'(K)$, we have

$$\begin{aligned} \partial_q(\lambda') &= u_q + \sum_{q'\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\,\partial_q(u_{q'}\otimes f_{q'}) \\ &= u_q + \sum_{q'\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\cdot v_q(f_{q'})\cdot u_{q'} \\ &= \begin{cases} u_q + \dfrac{1}{N}\cdot v_q(f_q)\cdot u_q = u_q - u_q = 0 & \text{if } q \neq \mathcal{O}, \\ u_{\mathcal{O}} + \sum_{q'\in S'(K)-\{\mathcal{O}\}} \dfrac{1}{N}\cdot v_{\mathcal{O}}(f_{q'})\cdot u_{q'} = \sum_{q'\in S'(K)} u_{q'} & \text{if } q = \mathcal{O}. \end{cases} \end{aligned}$$

Now let $\pi_K : C_K \to \operatorname{Spec} K$ be the structure morphism and $i_K : (C_K)^1 \hookrightarrow C_K$ be the canonical embedding. We have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Gamma(C_K,3)) \arrow[r] \arrow[d, "\beta_{C_K}"'] & H^2(\Gamma(K(C),3)) \arrow[r, "2\partial"] \arrow[d, "\beta_{K(C)}"'] & \bigoplus_{q\in(C_K)^1} H^1(\Gamma(K,2)) \arrow[d, "\simeq"', "\beta^1_{(2)}"] & \\ 0 \arrow[r] & H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[r] & H^2_{\mathcal{M}}(K(C),\mathbb{Q}(3)) \arrow[r, "\mathrm{Res}^{\mathcal{M}}"] & \bigoplus_{q\in(C_K)^1} H^1_{\mathcal{M}}(K,\mathbb{Q}(2)) \arrow[r, "(i_K)_*"] \arrow[dr, "\Sigma"'] & H^3_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[d, "(\pi_K)_*"] \\ & & & & H^1_{\mathcal{M}}(K,\mathbb{Q}(2)), \end{tikzcd}$$

(see diagram (3-16)), where the two horizontal sequences are exact and Σ is the trace map, which sends $(u_q)_{q\in S'(K)}$ to $\sum_{q\in S'(K)} u_q$. Then we have $\sum_{q\in S'(K)} u_q = 0$ by the commutativity of the bottom triangle. This shows that $\partial_q(\lambda') = 0$ for all $q \in S'(K)$, then λ' defines a unique element in $H^2(\Gamma(C_K,3))$. □

By the previous lemma, we constructed an element $\lambda'_{C_K} \in H^2(\Gamma(C_K,3))$. Via the map

$$\beta_{K_C} : H^2(\Gamma(C_K,3)) \to H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)),$$

we obtain a class $\beta_{C_K}(\lambda'_{C_K}) \in H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$.

Lemma 4.13. *The class $\beta_{C_K}(\lambda'_{C_K}) \in H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$ is* $\mathrm{Gal}(K/\mathbb{Q})$*-invariant.*

Proof. The Galois action of $\mathrm{Gal}(K/\mathbb{Q})$ on $H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$ is induced from the action on the function field, hence it is sufficient to check that $\beta_{K(C)}(\lambda') \in H^2(\Gamma(K(C),3))$ is $\mathrm{Gal}(K/\mathbb{Q})$-invariant. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we have

$$\sigma(\lambda') = \sigma(j(\lambda)) + \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\,\sigma(u_q\otimes f_q) = j(\lambda) + \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\,u_{\sigma(q)}\otimes\sigma(f_q),$$

because $\lambda \in B_2(\mathbb{Q}(C)) \otimes \mathbb{Q}(C)^\times$ and $\sigma(u_q) = u_{\sigma(q)}$ (see diagram (4-24)). Since $\operatorname{div}(f_q) = N(\mathcal{O}) - N(q)$ for $q \in S'(K) - \{\mathcal{O}\}$, we have $\operatorname{div}(\sigma(f_q)) = N(\sigma(\mathcal{O})) - N(\sigma(q))$. And by definition of $f_{\sigma(q)}$, we have $\operatorname{div}(f_{\sigma(q)}) = N(\mathcal{O}) - N(\sigma(q))$. Hence

$$\operatorname{div}(\sigma(f_q)) = \operatorname{div}(f_{\sigma(q)}) - N(\mathcal{O}) + N(\sigma(\mathcal{O})) = \operatorname{div}(f_{\sigma(q)}) - \operatorname{div}(f_{\sigma(\mathcal{O})}) = \operatorname{div}(f_{\sigma(q)}/f_{\sigma(\mathcal{O})}). \tag{4-29}$$

Write $\mathcal{O}' = \sigma^{-1}(\mathcal{O}) \in S'(K)$, we have

$$\begin{aligned}
\lambda + \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N} u_{\sigma(q)} \otimes \frac{f_{\sigma(q)}}{f_{\sigma(\mathcal{O})}}
&= \lambda + \sum_{q\in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes \frac{f_{\sigma(q)}}{f_{\sigma(\mathcal{O})}} + \frac{1}{N}\, u_{\mathcal{O}} \otimes \frac{f_{\mathcal{O}}}{f_{\sigma(\mathcal{O})}} \\
&= \lambda + \sum_{q\in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes \frac{f_{\sigma(q)}}{f_{\sigma(\mathcal{O})}} - \frac{1}{N}\, u_{\mathcal{O}} \otimes f_{\sigma(\mathcal{O})} \qquad \text{(since } f_{\mathcal{O}} = 1\text{)} \\
&= \lambda + \sum_{q\in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(q)} - \sum_{q\in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(\mathcal{O})} - \frac{1}{N}\, u_{\mathcal{O}} \otimes f_{\sigma(\mathcal{O})} \\
&= \lambda + \sum_{q\in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(q)} - \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(\mathcal{O})} \\
&= \lambda + \sum_{q\in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(q)} + \frac{1}{N}\, u_{\sigma(\mathcal{O})} \otimes f_{\sigma(\mathcal{O})} \qquad \Big(\text{since } \sum_{q\in S'(K)} u_q = 0\Big) \\
&= \lambda + \sum_{q\in S'(K)-\{\mathcal{O}'\}} \frac{1}{N} u_{\sigma(q)} \otimes f_{\sigma(q)} \\
&= \lambda'.
\end{aligned}$$

By Lemma 4.11, then, $\beta_{K(C)}(\sigma(\lambda')) = \beta_{K(C)}(\lambda')$ for all $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$. Since de Jeu's map β is functorial, it is compatible with the Galois action, so $\sigma(\beta_{K(C)}(\lambda')) = \beta_{K(C)}(\lambda')$ for all $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$. □

Thus $\beta_{C_K}(\lambda'_{C_K})$ defines a class in $H^2_{\mathcal{M}}(C_K, \mathbb{Q}(3))^{\operatorname{Gal}(K/\mathbb{Q})}$. Setting $\pi : C_K \to C$, we have the Galois descent of motivic cohomology as mentioned in (1-7):

$$\pi^* : H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \xrightarrow{\simeq} H^2_{\mathcal{M}}(C_K, \mathbb{Q}(3))^{\operatorname{Gal}(K/\mathbb{Q})}. \tag{4-30}$$

Hence $(\pi^*)^{-1}(\beta_{C_K}(\lambda'_{K_C}))$ is an element in $H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$.

4.4. ***Proof of Theorem 0.2.*** In this section, we keep the notations as in Section 4.3. To prove Theorem 0.2, the main idea is that we relate the regulator of the motivic cohomology class constructed in Section 4.3 to the Deligne cohomology class constructed in Section 4.1, hence to the Mahler measure of the polynomial P by Section 4.2.

First, as mentioned in (4-14), u_p defines an element in $\mathcal{B}(\mathbb{Q}(p))$ for $p \in S$. By Remark 4.7, if $u_p = 0$ for all $p \in S$, then $\beta_C(\lambda_C) \in H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$ and $\rho(\lambda)$ represents the class $r_3(2)_C(\lambda_C) \in H^1(C(\mathbb{C}), \mathbb{R}(2))^+$. Denote by $i : Y(\mathbb{C}) \hookrightarrow C(\mathbb{C})$ the canonical embedding. Then by Lemma 4.8, we have

$$\begin{aligned}\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = -\frac{1}{8\pi^2}\langle[\partial\Gamma], [\rho(\lambda)]\rangle_{Y(\mathbb{C})} &= -\frac{1}{8\pi^2}\left\langle[\partial\Gamma], i^* r_3(2)_C(\lambda_C)\right\rangle_{Y(\mathbb{C})} \\ &= -\frac{1}{8\pi^2}\langle i_*[\partial\Gamma], r_3(2)_C(\lambda_C)\rangle_{C(\mathbb{C})} \\ &= -\frac{1}{16\pi^2}\left\langle i_*[\partial\Gamma], \mathrm{reg}_C(\beta_C(\lambda_C))\right\rangle_{C(\mathbb{C}),}\end{aligned}$$

where the last equality follows from the diagram (3-20). Apply Beilinson's conjecture 1.11 to $\beta_C(\lambda_C) \in H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$ and $i_*[\partial\Gamma] \in H_1(C(\mathbb{C}), \mathbb{Q})^+$ (see Lemma 4.8), we thus have

$$\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = a \cdot L'(E, -1) \qquad (a \in \mathbb{Q}).$$

When the residues $u_p \in \mathcal{B}(\mathbb{Q}(p))$ are nontrivial for some $p \in S' \subset S$, we define $u_q \in \mathcal{B}(K)$ for $q \in S'(K)$, where K is the splitting field of S' in $\mathbb{C}$ (see the beginning of Section 4.3). Let $K(C)$ denote the function field of C_K. Fix a point $\mathcal{O} \in S'(K)$. Recall from Definition 4.9 the element of $B_2(K(C)) \otimes K(C)^\times$ defined by

$$\lambda' = j(\lambda) + \sum_{q \in S'(K) - \{\mathcal{O}\}} \frac{1}{N}(u_q \otimes f_q),$$

where $f_q \in K(C)^\times$ is defined just before Definition 4.9. We prove that λ' defines a class in $H^2(\Gamma(K(C), 3))$ (see Lemma 4.10). The differential form $\rho(\lambda)$ represents the class $r_3(2)_{\mathbb{Q}(C)}(\lambda)$, we then have

$$\begin{aligned}\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = -\frac{1}{8\pi^2}\int_{\partial\Gamma}\rho(\lambda) &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{\mathbb{Q}(C)}(\lambda) \\ &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{K(C)}(j(\lambda)),\end{aligned}$$

where $j : \mathbb{Q}(C) \hookrightarrow K(C)$. Hence

$$\begin{aligned}\mathrm{m}(P) - \mathrm{m}(\tilde{P}) &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{K(C)}\Bigg(\lambda' - \sum_{q \in S'(K) - \{\mathcal{O}\}} \frac{1}{N} u_q \otimes f_q\Bigg) \\ &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{K(C)}(\lambda') + \frac{1}{8N\pi^2}\sum_{q \in S'(K) - \{\mathcal{O}\}}\int_{\partial\Gamma} r_3(2)_{K(C)}(u_q \otimes f_q) \\ &= -\frac{1}{16\pi^2}\int_{\partial\Gamma}\mathrm{reg}_{K(C)}(\beta_{K(C)}(\lambda')) + \frac{1}{8N\pi^2}\sum_{q \in S'(K) - \{\mathcal{O}\}} D(u_q)\int_{\partial\Gamma} d\arg(f_q), \qquad (4\text{-}31)\end{aligned}$$

where the last equality follows from Lemma 3.2 and the fact that $\partial\Gamma$ is assumed to be contained in $Y(\mathbb{C})$

(see Lemma 4.8). We have the commutative diagram

$$\begin{tikzcd} H^2(\Gamma(C_K,3)) \arrow[r, "\beta_{C_K}"] \arrow[d, hook] & H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[r, "\mathrm{reg}_{C_K}"] \arrow[d, hook] & H^1(C_K(\mathbb{C}),(\mathbb{R}(2))^+) \arrow[d, hook] \\ H^2(\Gamma(K(C),3)) \arrow[r, "\beta_{K(C)}"'] & H^2_{\mathcal{M}}(K(C),\mathbb{Q}(3)) \arrow[r, "\mathrm{reg}_{K(C)}"'] & H^1(K(C),\mathbb{R}(2))^+). \end{tikzcd}$$

As $\lambda' \in H^2(\Gamma(K(C),3))$ defines a unique a class $\lambda'_{C_K} \in H^2(\Gamma(C_K,3))$ (see Lemma 4.12), we have

$$\mathrm{reg}_{K(C)}(\beta_{K(C)}(\lambda')) = \mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'_{C_K})).$$

Hence the first integral in (4-31) can be written as the following pairing in de Rham cohomology:

$$\big\langle[\partial\Gamma],(i_K)^*\mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'))\big\rangle_{Y_K(\mathbb{C})} = \big\langle(i_K)_*[\partial\Gamma],\mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'_{C_K}))\big\rangle_{C_K(\mathbb{C})}, \tag{4-32}$$

where $i_K : Y_K(\mathbb{C}) \hookrightarrow C_K(\mathbb{C})$ is the canonical embedding. Moreover, by the functorial property of the Beilinson regulator map, we have the commutative diagram

$$\begin{tikzcd} H^2(\Gamma(C_K,3)) \arrow[r, "\beta_{C_K}"] & H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[d, "\mathrm{reg}_{C_K}"] & H^2_{\mathcal{M}}(C,\mathbb{Q}(3)) \arrow[l, hook', "\pi^*"'] \arrow[d, "\mathrm{reg}_C"] \\ & H^1(C_K(\mathbb{C}),\mathbb{R}(2))^+ \arrow[r, equal] & H^1(C(\mathbb{C}),\mathbb{R}(2))^+, \end{tikzcd}$$

where π^* is induced from the isomorphism $\pi^* : H^2_{\mathcal{M}}(C,\mathbb{Q}(3)) \xrightarrow{\simeq} H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))^{\mathrm{Gal}(K/\mathbb{Q})}$ mentioned in (4-30). Hence by identifying de Rham cohomology as well as singular homology of $C_K(\mathbb{C})$ and $C(\mathbb{C})$, we have

$$\big\langle(i_K)_*[\partial\Gamma],\mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'_{C_K}))\big\rangle_{C_K(\mathbb{C})} = \big\langle i_*[\partial\Gamma],\mathrm{reg}_C((\pi^*)^{-1}(\beta_{C_K}(\lambda'_{C_K})))\big\rangle_{C(\mathbb{C})}.$$

Applying Beilinson's conjecture to $(\pi^*)^{-1}(\beta_{C_K}(\lambda'_{C_K})) \in H^2_{\mathcal{M}}(C,\mathbb{Q}(3))$, we obtain that

$$\frac{1}{(2\pi i)^2}\big\langle i_*[\partial\Gamma],\mathrm{reg}_C((\pi^*)^{-1}(\beta_{C_K}(\lambda'_{C_K})))\big\rangle_{C(\mathbb{C})} = a\cdot L'(E,-1), \quad a \in \mathbb{Q},$$

where E is the Jacobian of C. From (4-31), by setting $b_q = \frac{1}{2\pi}\int_{\partial\Gamma} d\arg f_q$, we have

$$\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = a\cdot L'(E,-1) + \frac{1}{4N\pi}\sum_{q\in S'(K)\setminus\{\mathcal{O}\}} b_q\cdot D(u_q), \quad a\in\mathbb{Q}. \tag{4-33}$$

We will show that for $f \in \bar{\mathbb{Q}}(C)^\times$ and $\gamma : [0,1] \to C(\mathbb{C})$ is a loop, $\int_\gamma d\arg f$ is a integral multiple of 2π. In fact, we can always find a partition

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1,$$

such that γ is the union of $\gamma_j : [a_j, a_{j+1}] \to C(\mathbb{C})$ for $j = 0,\ldots,n-1$ and $\gamma_j([a_j,a_{j+1}])$ is contained in a local coordinate chart of $C(\mathbb{C})$. Then

$$\begin{aligned}\int_\gamma d\arg f &= \sum_{j=0}^{n-1}\int_{\gamma_j} d\arg f = \sum_{j=0}^{n-1}\arg f(\gamma_j(a_{j+1})) - \arg f(\gamma_j(a_j))\\ &= -\arg f(\gamma_0(0)) + \arg f(\gamma_{n-1}(1)) + \sum_{j=0}^{n-2}\arg f(\gamma_j(a_{j+1})) - \arg f(\gamma_{j+1}(a_{j+1}))\\ &= 2\pi k,\end{aligned}$$

for some integer k, since $\gamma_0(0) = \gamma(0) = \gamma(1) = \gamma_{n-1}(1)$ and $\gamma_j(a_{j+1}) = \gamma_{j+1}(a_{j+1})$ for $j = 0, \ldots, n-2$. In particular, we get $\int_{\partial\Gamma} d\arg f = 2\pi k$, for some $k \in \mathbb{Z}$. This implies that $b_q \in \mathbb{Z}$ for all $q \in S'(K)$. Although the coefficients b_q depend on the choice of $\mathcal{O}$, the D-values in identity (4-33) do not. Indeed, if we remove $\mathcal{O}$ from $S'(K)$, its complex conjugation $c(\mathcal{O}) \in S'(K)$ maintains the D-values in identity (4-33) because $D(u_{c(\mathcal{O})}) = D(c(u_{\mathcal{O}})) = -D(u_{\mathcal{O}})$, where c is the complex conjugation. □

Remark 4.14. (a) By Lemma 4.6, we have

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + a \cdot L'(E, -1) - \frac{1}{8N\pi^2}\sum_{q\in S'(K)-\{\mathcal{O}\}} b_q \cdot \mathrm{Res}_q(\rho(\lambda)).$$

(b) In some cases, D-values on Bloch group elements can relate to Dirichlet L-values. Let χ be a primitive Dirichlet character of conductor f, we have

$$L(\chi, 2) = \frac{1}{G(\bar{\chi})}\sum_{k=1}^{f}\bar{\chi}(k)\; Li_2(e^{2\pi ik/f}),$$

where $G(\bar{\chi}) = \sum_{k=1}^{f}\bar{\chi}(k)e^{2\pi ik/f}$ is the Gauss sum of χ. Thus, when χ is odd quadratic, then

$$L(\chi, 2) = \frac{1}{\sqrt{f}}\sum_{k=1}^{f}\chi(k)\; D(e^{2\pi ik/f}).$$

Then

$$L'(\chi, -1) = \frac{f^{3/2}}{4\pi}L(\chi, 2) = \frac{f}{4\pi}\sum_{k=1}^{f}\chi(k)\; D(e^{2\pi ik/f}).$$

In particular, if χ_{-3} and χ_{-4} denote the nontrivial characters of modulo 3 and 4, respectively, we have

$$L'(\chi_{-3}, -1) = \frac{3}{2\pi}D(e^{2\pi i/3}) = \frac{1}{\pi}D(e^{i\pi/3}), \quad L'(\chi_{-4}, -1) = \frac{2}{\pi}D(e^{i\pi/2}).$$

5. Examples

In this section, we apply Theorem 0.2 to several Mahler measure's identities. We also describe some polynomials to which our main theorem fails to apply. They are numerically conjectured by Boyd and Brunault.

5.1. ***Pure identities.*** In this section, we apply Theorem 0.2 to study pure identities of Mahler's measure

$$\mathrm{m}(P) \sim_{\mathbb{Q}^\times} L'(E, -1),$$

where the notation $a \sim_{\mathbb{Q}^\times} b$ means $a/b \in \mathbb{Q}^\times$. Notice that most of polynomials in this section are of the form considered in Remark 4.3,

$$P(x, y, z) = A(x) + B(x)y + C(x)z, \tag{5-1}$$

where A, B, C are products of cyclotomic polynomials. In this case, $\mathrm{m}(\tilde{P}) = 0$ and $\mathrm{m}(P) \neq 0$. A typical example of pure identity is the Mahler measure of $z + (x+1)(y+1)$, which is conjectured by D. Boyd to be

$$\mathrm{m}(z + (x+1)(y+1)) = -2L'(E_{15}, -1).$$

It was proved under Beilinson's conjecture and up to a rational factor by Lalín [23, Section 4.1]. It is then completely proven by Brunault [8]. This polynomial also satisfies our main theorem, we do not discuss it here but focus on other examples. All figures in this section are generated using Maple.

(a) We prove under Beilinson's conjecture the pure identity

$$\mathrm{m}((1+x)(1+y)(x+y)+z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{14}, -1), \tag{5-2}$$

which is the first identity mentioned in Table 1. In this case, P is not of the form (5-1), but we still have $\mathrm{m}(\tilde{P}) = 0$ and the decomposition

$$x \wedge y \wedge z = -x \wedge (1+x) \wedge y \; + \; y \wedge (1+y) \wedge x \; + \; \frac{y}{x} \wedge \left(1 + \frac{y}{x}\right) \wedge x.$$

Hence

$$f_1 = -g_2 = -g_3 = -x, \quad f_2 = -g_1 = -y, \quad f_3 = -y/x.$$

We have that W_P is given by

$$(xy + x + y)(1 + x + y)((x+1)y^2 + (x^2 + x + 1)y + x^2 + x) = 0,$$

which is the union of lines $L_1 : xy + x + y = 0$, $L_2 : 1 + x + y = 0$ and the curve

$$C : (x+1)y^2 + (x^2 + x + 1)y + x^2 + x = 0,$$

which is a nonsingular curve of genus 1. By the change of variables

$$x = -\frac{Y + X^2 + 1}{X(X-1)}, \quad y = -\frac{Y}{X(X+1)} - \frac{1}{X},$$

we obtain that the Jacobian of C is given by

$$E/\mathbb{Q} : Y^2 + XY + Y = X^3 - X,$$

which is an elliptic curve of type $14a4$. Its torsion subgroup is $\mathbb{Z}/6\mathbb{Z} = \langle A \rangle$ with $A = (1, -2)$. With the help of Magma [2], we have

$$\mathrm{div}(x) = -(5A) + (A) - (4A) + (2A), \quad \mathrm{div}(y) = (\mathcal{O}) + (A) - (4A) - (3A).$$

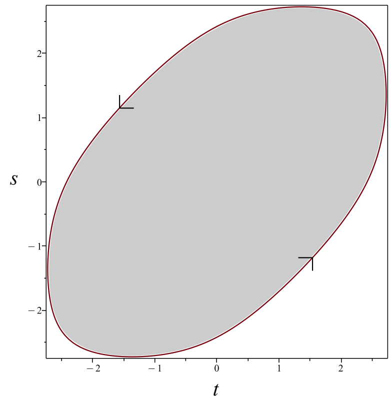


Figure 1. The Deninger chain Γ for the proof of identity 1 in Table 1.

Denote by S the closed subscheme of E consisting of all points in supports of above divisors. The values of f_j and $f_j \circ \tau$ at $p \in S$ are either 0, 1 or ∞ for all j, then the elements $v_p(g_j)\{f_j(p)\}_2$ and $v_p(g_j \circ \tau)\{f_j \circ \tau(p)\}_2$ are all trivial in $B_2(\mathbb{Q})$ for all j and $p \in S$. Figure 1 describes the Deninger chain

$$\Gamma : \{|x| = |y| = |(1+x)(1+y)(x+y)| \geq 1\},$$

and its boundary in polar coordinates $x = e^{it}$ and $y = e^{is}$ for $t, s \in [-\pi, \pi]$. We obtain that $\partial\Gamma$ is contained completely in C and $\partial\Gamma$ does not contain any zeros and poles of $f_j, 1 - f_j, g_j$ for all j. Then by Theorem 0.2, we have identity (5-2) under Beilinson's conjecture.

(b) We study the pure identity 2 in Table 1:

$$\mathrm{m}(1+x+y+z+xy+xz+yz) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{14}, -1). \tag{5-3}$$

First we notice that

$$\mathrm{m}(1+x+y+xy+z(1+x+y)) = \mathrm{m}(1+x+y+z(1+x+y+xy)),$$

so it suffices to prove the identity

$$\mathrm{m}(1+x+y+z(1+x+y+xy)) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{14}, -1). \tag{5-4}$$

We have $\mathrm{m}(\tilde{P}) = \mathrm{m}(1+x+y+xy) = \mathrm{m}(x+1)\mathrm{m}(y+1) = 0$. We have the decomposition

$$x \wedge y \wedge z = x \wedge (1+x) \wedge y \,-\, y \wedge (1+y) \wedge x \,+\, (x+y) \wedge (1+x+y) \wedge x \\ -\, (x+y) \wedge (1+x+y) \wedge y \,-\, \frac{x}{y} \wedge \left(1+\frac{x}{y}\right) \wedge (1+x+y),$$

leading to

$$f_1 = -x, \quad f_2 = -y, \quad f_3 = f_4 = -(x+y), \quad f_5 = -x/y, \quad g_1 = g_4 = y, \quad g_2 = g_3 = x, \quad g_5 = 1+x+y.$$

W_P is given by $x(x+1)y^2 + (x^2+x+1)y + x + 1 = 0$, which is an irreducible nonsingular curve of genus 1. Using the change of variables

$$x=-\frac{Y+X^2+1}{X(X-1)},\qquad y=\frac{Y}{X(X+1)},$$

we obtain that the Jacobian of W_P is given by $E/\mathbb{Q}: Y^2+XY+Y=X^3-X$, which is the same elliptic curve in item (a). We have

$$\operatorname{div} x=-(5A)+(A)-(4A)+(2A),\quad \operatorname{div}(1+x+y)=2(\mathcal{O})-(5A)+2(A)-(4A)-(2A)-(3A),$$
$$\operatorname{div} y=(\mathcal{O})+(5A)-(2A)-(3A),\quad \operatorname{div}(1+1/x+1/y)=-(\mathcal{O})+2(4A)-(2A)+2(3A)-(5A)-(A).$$

With the same reasoning as in item (a), we get identity (5-4) conditionally on Beilinson's conjecture. Moreover, as mentioned in the introduction, we have

$$\mathrm{m}((1+x)(1+y)(x+y)+z)\overset{?}{\sim}_{\mathbb{Q}^\times}\mathrm{m}(1+x+y+z+xy+xz+yz),$$

because they are rational multiples of the same elliptic curve L-value $L'(E_{14},-1)$.

(c) Similarly, one gets identity 11 in Table 1:

$$\mathrm{m}(1+x+y+z+xy+xz+yz-xyz)\overset{?}{\sim}_{\mathbb{Q}^\times}L'(E_{36},-1).$$

This identity is interesting because this is the only example that has been found with CM elliptic curves.

(d) The same arguments apply to all pure identities in Table 1, except for identities 5, 6, 7, 8, and 12. In this section, we study the first four of these identities. It suffices to consider identity 5 since Lalín and Nair showed in [24] that the polynomials 5, 6, 7, and 8 share the same Mahler measure. Let us recall identity 5:

$$\mathrm{m}(1+(x+1)y+(x-1)z)\overset{?}{\sim}_{\mathbb{Q}^\times}L'(E_{21},-1). \tag{5-5}$$

The polynomial $P=1+(x+1)y+(x-1)z$ is of the form (5-1). We have the decomposition

$$x\wedge y\wedge z=x\wedge(1-x)\wedge y+(x+1)y\wedge(1+(x+1)y)\wedge x-x\wedge(1+x)\wedge(1+(x+1)y),$$

so

$$f_1=-f_3=g_2=x,\quad f_2=-(x+1)y,\quad g_1=y,\quad g_3=1+(x+1)y,\quad f_2\circ\tau=-\frac{x+1}{xy},\quad g_3\circ\tau=\frac{xy+x+1}{xy}.$$

W_P is given by $x(x+1)y^2+(2x^2+x+2)y+1+x=0$, which is a nonsingular curve of genus 1. Using the change of variables

$$x=-\frac{X^2-6X+3Y}{X(X-6)},\qquad y=\frac{Y-3X-3}{X(X+1)},$$

we get for the Jacobian of W_P the equation

$$E/\mathbb{Q}: Y^2-3XY-3Y=X^3-5X^2-6X,$$

which is an elliptic curve of type $21a1$. Its torsion subgroup is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}=\langle A\rangle\times\langle B\rangle$ with $A=(-1,0)$, and $B=(0,0)$. With the help of Magma, we have

$$\operatorname{div} x=-(A+B)+(A+3B)-(3B)+(B),\quad \operatorname{div}(1+(x+1)y)=2(2B)-(3B)-(B),$$
$$\operatorname{div} y=(\mathcal{O})+(A+B)-(B)-(A),\quad \operatorname{div}(xy+x+1)=(\mathcal{O})+2(A+2B)-(A+B)-(3B)-(A).$$

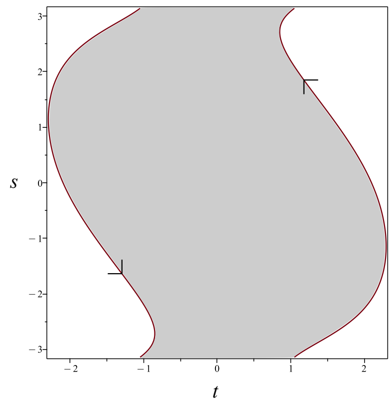


Figure 2. The Deninger chain Γ for the proof of identity 5 in Table 1.

Let S be the closed subscheme of E consisting all the points in the supports of the above divisors. We have

$$\begin{aligned}\sum_j v_B(g_j)\{f_j(B)\}_2 + v_B(g_j\circ\tau)\{f_j\circ\tau(B)\}_2 &= v_B(g_2)\{f_2(B)\}_2 + v_B(g_2\circ\tau)\{f_2\circ\tau(B)\}_2\\ &= \{\infty\}_2 - \{1/2\}_2 = \{2\}_2,\end{aligned}$$

which is nontrivial in $B_2(\mathbb{Q})$. Therefore, the theorem of Lalín mentioned in the introduction does not apply to this example. As S consists of points in E_{tors}, we can choose N in Theorem 0.2 equal to $\#E_{\mathrm{tors}} = 8$. Since the points of S have rational coordinates and the f_i have rational coefficients, then f_i take rational values on S. Therefore, the Bloch–Wigner dilogarithmic values in identity (4-33) all vanish. Figure 2 describes the Deninger chain and its boundary in polar coordinates $x = e^{it}$, $y = e^{it}$ for $s, t \in [-\pi, \pi]$. The boundary $\partial\Gamma$ consists of 2 loops and does not contain any zeros and poles of f_j, $1-f_j$, g_j, $f_j\circ\tau$, $1-f_j\circ\tau$, $g_j\circ\tau$ for all j. Hence by Theorem 0.2, we get identity (5-5) conditionally on Beilinson's conjecture. In particular, under Beilinson's conjecture, we have

$$\mathrm{m}(1+(x+1)y+(x-1)z) \overset{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}((x+1)^2(y+1)+z),$$

as they are rational multiples of $L'(E_{21}, -1)$.

(e) There is an interesting remark on identities 4 and 10 of Table 1. By some trivial change of variables, we obtain

$$\mathrm{m}((x+1)^2+(1-x)(y+z)) = \mathrm{m}((x+1)(y+1)+(x-1)^2 z).$$

Theorem 0.2 applies to $P = (x+1)^2+(1-x)(y+z)$ but not to $P = (x+1)(y+1)+(x-1)^2 z$. Indeed, in the latter case, W_P is given by

$$(-x^3-2x^2-x)y^2 + (x^4-6x^3+2x^2-6x+1)y - x^3 - 2x^2 - x = 0,$$

which is a curve having a singularity at $(1, -1)$, and the boundary $\partial\Gamma$ passes this singular point. Figure 3 describes the Deninger chain Γ and its boundary $\partial\Gamma$ in polar coordinates $x = e^{it}$, $y = e^{is}$ for $t, s \in [-\pi, \pi]$, where the marked points indicate the singular point $(1, -1)$. Using Magma, one can check that $\partial\Gamma$ is no longer a loop on the normalization of W_P.

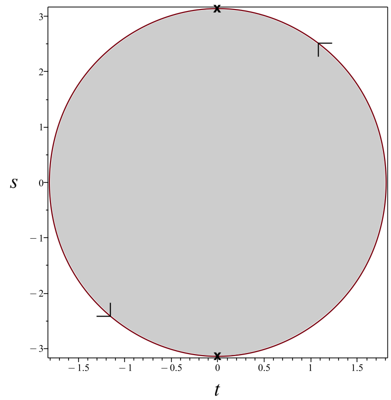


Figure 3. The Deninger chain Γ of the polynomial $P = (x+1)(y+1)+(x-1)^2 z$.

The same situation happens with identity 10 in Table 1. By some trivial changes of variables, we have

$$\mathrm{m}((1+x)^2+y+z) = \mathrm{m}(1+y+(1+x)^2 z),$$

where Theorem 0.2 applies to the first polynomial but not the second one.

(f) We study identity 12 in Table 1:

$$\mathrm{m}(1+xy+(1+x+y)z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{90}, -1). \tag{5-6}$$

We have the decomposition

$$\begin{aligned} &x \wedge y \wedge z \\ &\quad = xy \wedge (1+xy) \wedge x - (x+y) \wedge (1+x+y) \wedge x + (x+y) \wedge (1+x+y) \wedge y + \frac{-x}{y} \wedge \left(1+\frac{x}{y}\right) \wedge (1+x+y), \end{aligned}$$

so

$$f_1 = -xy, \quad f_2 = f_3 = -(x+y), \quad f_4 = -x/y, \quad g_1 = g_2 = x, \quad g_3 = y, \quad g_4 = 1+x+y.$$

The curve W_P is given by

$$(-x^2+x+1)y^2 + (x^2+x+1)y + x^2 + x - 1 = 0,$$

which is an irreducible curve of genus 1 and does not contain any rational points. Figure 4 describes the Deninger chain and its boundary in polar coordinates. We find that $\partial\Gamma$ does not contain any singular points of W_P.

Using Magma, we obtain that the Jacobian of C is given by

$$E/\mathbb{Q} : Y^2 + XY + Y = X^3 - X^2 - 8X + 11,$$

which is an elliptic curve of type $90b1$. Its torsion subgroup is $\mathbb{Z}/6\mathbb{Z} = \langle A \rangle$, with $A = (3, 1)$. Denote by K the real quadratic field $\mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{R}$ is such that $\alpha^2 + \alpha - 1 = 0$. Let $B_1 = (6\alpha+9, -24\alpha-35)$, $B_2 = (-4\alpha+1, 12\alpha-3)$, $B_3 = \left(\frac{9}{5}, \frac{1}{25}(24\alpha-23)\right)$, $B_4 = (2, -\alpha-2)$, $B_5 = (-6\alpha+3, -18\alpha+7)$,

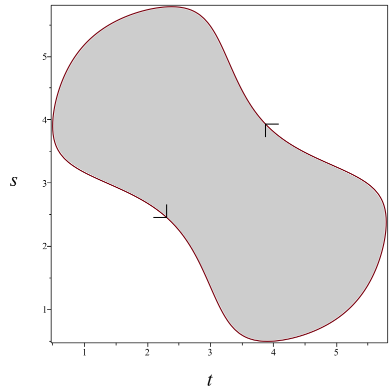


Figure 4. The Deninger chain Γ corresponding to (5-6).

$B_6 = (4\alpha+5, 8\alpha+9)$ and we denote by (B_i) the divisor in E corresponding to the point B_i. We have the following divisors in E/K:

$$\begin{aligned}
\operatorname{div}(x) &= (4A) + (B_1) - (A) - (B_2),\\
\operatorname{div}(y) &= (\mathcal{O}) + (B_3) - (B_4) - (3A),\\
\operatorname{div}(1+x+y) &= 2(2A) + 2(B_5) - (A) - (B_4) - (3A) - (B_2),\\
\operatorname{div}(1+1/x+1/y) &= -(\mathcal{O}) + 2(5A) + 2(B_6) - (B_3) - (4A) - (B_1).
\end{aligned}$$

Note that the Bloch group of real quadratic fields is trivial after tensoring with $\mathbb{Q}$. Therefore, the residues u_{B_i} are trivial because they are elements of $\mathcal{B}(\mathbb{Q}(\alpha))$, the Bloch group (tensored with $\mathbb{Q}$) of the real quadratic field $\mathbb{Q}(\alpha)$. The remaining points in the supports of the above divisors are of the form mA for $m = 1, \dots, 6$. Hence we can choose N in Theorem 0.2 as the order of A which equals 6. As the points mA have rational coordinates and the functions f_i have rational coefficients, the Bloch–Wigner dilogarithmic values at u_{mA} in (4-33) all vanish. We then get identity (5-6) under Beilinson's conjecture for genus 1 curves 1.11.

(g) We show that Theorem 0.2 does not imply identity (0-10):

$$\mathrm{m}((1+x)(1+y)+(1-x-y)z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{450}, -1).$$

We have the decomposition

$$x \wedge y \wedge z = -x \wedge (1+x) \wedge y + y \wedge (1+y) \wedge x + (x+y) \wedge (1-x-y) \wedge x \\ - (x+y) \wedge (1-x-y) \wedge y - \frac{x}{y} \wedge \left(1+\frac{x}{y}\right) \wedge (1-x-y).$$

Therefore, we have

$$f_1 = -x, \quad f_2 = -y, \quad f_3 = f_4 = x+y, \quad f_5 = -x/y, g_1 = g_4 = y, \quad g_2 = g_3 = x, \quad g_5 = 1-x-y.$$

The Maillot variety is given by

$$(x^2+3x)y^2 + (3x^2+x+3)y + 3x + 1 = 0.$$

By the change of variables

$$x=-\frac{3}{(X^2-9X)Y}+\frac{-X^2+3X-27}{X^2-9X},\quad y=\frac{3}{(X^2-27X+162)Y}+\frac{2X^2-21X-27}{X^2-27X+162},$$

we get the elliptic curve

$$E/\mathbb{Q}:\quad Y^2+XY=X^3-X^2-27X+81,$$

which is of conductor 450. Its torsion group is isomorphic to $\mathbb{Z}/2$ and generated by $A=(-6,3)$. We have the following divisors in E:

$$\begin{aligned}
&\operatorname{div}(x)=-(P_1)+(P_2)-(P_3)+(P_4),\\
&\operatorname{div}(y)=(P_3)+(P_5)-(P_6)-(P_2),\\
&\operatorname{div}(1-x-y)=2(\mathcal{O})+2(P_7)-(P_6)-(P_1)-(P_2)-(P_3),\\
&\operatorname{div}(1-1/x-1/y)=2(P_8)+2(A)-(P_2)-(P_3)-(P_4)-(P_5),
\end{aligned}$$

where

$$\begin{aligned}
&P_1:=(9,18), && P_2:=(9,-27), && P_3:=(0,9), && P_4:=(0,-9),\\
&P_5:=\left(-\tfrac{9}{4},-\tfrac{81}{8}\right), && P_6:=(18,63) && P_7:=(4,3), && P_8:=(3,-6).
\end{aligned}$$

The residue

$$u_{P_1}:=\sum_{j=1}^{5}v_{P_1}(g_j)\{f_j(P_1)\}_2+v_{P_1}(g_j\circ\tau)\{f_j\circ\tau(P_1)\}_2=-3\{3\}_2$$

is nontrivial in the Bloch group $\mathcal{B}(\mathbb{Q})$. Since P_1 is torsion-free, it violates the finite order condition of Theorem 0.2.

5.2. ***Identities with Dirichlet characters.*** In this section, we study Mahler measure identities of the form

$$\mathrm{m}(P)\stackrel{?}{=}a\cdot L'(E,-1)+\sum_{\chi}b_\chi\cdot L'(\chi,-1),$$

where $a\in\mathbb{Q}$, $b_\chi\in\mathbb{Q}^\times$, E is an elliptic curve and the χ are odd quadratic Dirichlet characters.

(a) We prove the first identity of Table 2 conditionally on Beilinson's conjecture. The polynomial $P=1+(x^2-x+1)y+(x^2+x+1)z$ is of the form (5-1). We have on V_P the decomposition

$$\begin{aligned}
x\wedge y\wedge z=-\tfrac{1}{3}x^3\wedge(1-x^3)\wedge y\,+\,x\wedge(1-x)\wedge y\,+\,(x^2-x+1)y\wedge(1+(x^2-x+1)y)\wedge x\\
-\tfrac{1}{3}x^3\wedge(1+x^3)\wedge(1+(x^2-x+1)y)\,+\,x\wedge(1+x)\wedge(1+(x^2-x+1)y).
\end{aligned}$$

We have

$$\begin{gathered}
f_1=x^3,\quad f_2=x,\quad f_3=-(x^2-x+1)y,\quad f_4=-x^3,\ f_5=-x,\\
g_1=g_2=y,\quad g_3=x,\quad g_4=g_5=1+(x^2-x+1)y.
\end{gathered}$$

The curve W_P is given by $x^2(x^2-x+1)y^2-x(4x^2-x+4)y+x^2-x+1=0$, which is a nonsingular curve of genus 1 and does not contain any rational point. By the change of variables $x=X$, $y=Y/X$, we get the new equation

$$(X^2-X+1)Y^2-(4X^2-X+4)Y+X^2-X+1=0.$$

Using Pari/GP [31], one gets the following Weierstrass form for the Jacobian of W_P:

$$E/\mathbb{Q}: v^2+uv=u^3-u^2-45u-104,$$

which is an elliptic of type $45a2$. We set $k=\mathbb{Q}(\alpha)$ with $\alpha^2-\alpha+1=0$. A base change of E over k can be given by

$$E_k: V^2+3UV+3V=U^3-U^2-9U,$$

by using the change of variables

$$x=\frac{(2-\alpha)V+\alpha U^2-3(\alpha-1)U+3}{U^2+(4\alpha-2)U-3\alpha},$$
$$y=\frac{((1-\alpha)U^2-(\alpha+4)U+(\alpha+4))V+2\alpha U^3-(8\alpha-3)U^2+(3\alpha-12)U+12}{U^4-(4\alpha+1)U^3+(15\alpha-7)U^2-(17\alpha-13)U+6(\alpha-1)}.$$

The torsion subgroup of E_k is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}=\langle A\rangle\times\langle B\rangle$ with $A=(-3,3)$ and $B=(0,0)$. Let K be the number field $\mathbb{Q}(\alpha,r,s)$ with

$$r^2-2(2\alpha-1)r+3(\alpha-1)=0,\ \text{ and } s^2+2(2\alpha-1)s-3\alpha=0.$$

We set $P_1=(r,\alpha r-2\alpha-2)$, $P_2=(s,(1-\alpha)s+2\alpha-4)$, which are points in $E(K)$. We denote by (P_i) the divisor corresponding to P_i in E_k. Note that the divisors (P_i) have degree 2 on E_k. Using Magma, one obtains the following divisors in E_k:

$$\begin{aligned}
&\mathrm{div}(g_3)=\mathrm{div}(x)=(P_1)-(P_2),\\
&\mathrm{div}(g_1)=\mathrm{div}(g_2)=\mathrm{div}(y)=(\mathcal{O})+(A+3B)-(A+B)+(P_2)-(2B)-(P_1),\\
&\mathrm{div}(g_4)=\mathrm{div}(g_5)=\mathrm{div}(1+(x^2-x+1)y)=2(3B)+2(A)-(P_1)-(P_2),\\
&\mathrm{div}(g_4\circ\tau)=\mathrm{div}(g_5\circ\tau)=\mathrm{div}(1+(1/x^2-1/x+1)(1/y))=2(B)+2(A+2B)-(P_1)-(P_2).
\end{aligned}$$

The values of f_j and $f_j\circ\tau$ at P_1, P_2 and their conjugates are either 0 or ∞, so we are only concerned with the other points. We obtain the equalities

$$\begin{aligned}
u_A &= v_A(g_4)\{f_4(A)\}_2+v_A(g_5)\{f_5(A)\}_2=\{-1\}_2+\{1/\alpha\}_2=-\{\alpha\}_2,\\
u_B &= v_B(g_4\circ\tau)\{f_4\circ\tau(B)\}_2+v_B(g_5\circ\tau)\{f_5\circ\tau(B)\}_2=2\{-1\}_2+2\{\alpha\}_2=2\{\alpha\}_2,\\
u_{2B} &= v_{2B}(g_1)\{f_1(2B)\}_2+v_{2B}(g_1\circ\tau)\{f_1\circ\tau(2B)\}_2+v_{2B}(g_2)\{f_2(2B)\}_2+v_{2B}(g_2\circ\tau)\{f_2\circ\tau(2B)\}_2\\
&=-\{-1\}_2+\{-1\}_2-\{1/\alpha\}_2+\{\alpha\}_2=2\{\alpha\}_2,
\end{aligned}$$

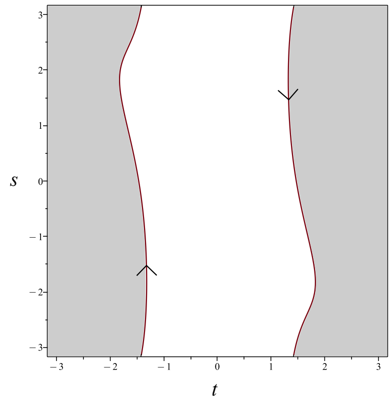


Figure 5. The Deninger chain Γ for the proof of identity 1 in Table 2.

$$\begin{aligned}
u_{3B} &= v_{3B}(g_4)\{f_4(3B)\}_2 + v_{3B}(g_5)\{f_5(3B)\}_2 = 2\{-1\}_2 + 2\{\alpha\}_2 = 2\{\alpha\}_2,\\
u_{A+B} &= v_{A+B}(g_1)\{f_1(A+B)\}_2 + v_{A+B}(g_1\circ\tau)\{f_1\circ\tau(A+B)\}_2\\
&\quad + v_{A+B}(g_2)\{f_2(A+B)\}_2 + v_{A+B}(g_2\circ\tau)\{f_2\circ\tau(A+B)\}_2\\
&= -\{-1\}_2 + \{-1\}_2 - \{\alpha\}_2 + \{1/\alpha\}_2 = -2\{\alpha\}_2,\\
u_{A+2B} &= v_{A+2B}(g_4\circ\tau)\{f_4\circ\tau(A+2B)\}_2 = 2\{-1\}_2 + 2\{1/\alpha\}_2 = -2\{\alpha\}_2,\\
u_{A+3B} &= v_{A+3B}(g_1)\{f_1(A+3B)\}_2 + v_{A+3B}(g_1\circ\tau)\{f_1\circ\tau(A+3B)\}_2\\
&\quad + v_{A+3B}(g_2)\{f_2(A+3B)\}_2 + v_{A+3B}(g_2\circ\tau)\{f_2\circ\tau(A+3B)\}_2\\
&= \{-1\}_2 - \{-1\}_2 + \{1/\alpha\}_2 - \{\alpha\}_2 = -2\{\alpha\}_2,
\end{aligned}$$

which are all nontrivial in $B_2(K)$. Notice that P_1, P_2 have order 8 in $E(K)$ and all the other points belong to the torsion subgroup of E_k, whose cardinality equals 8, hence we choose N in Theorem 0.2 equal to 8. Figure 5 indicates the Deninger chain (the shaded region) and its boundary in polar coordinates $x = e^{it}$, $y = e^{is}$ for $s, t \in [-\pi, \pi]$. The boundary $\partial\Gamma$ consists of 2 loops, which do not contain any points in the supports of the above divisors. By (4-20), we have $\mathrm{m}(P) = -\frac{1}{8\pi^2}\int_{\partial\Gamma}\rho(\lambda)$, so $\partial\Gamma$ must be nontrivial as otherwise $m(P)$ vanishes. Hence $\partial\Gamma$ defines a generator of $H_1(C(\mathbb{C}), \mathbb{Z})^+$. Then by Theorem 0.2, under Beilinson's conjecture, we have

$$\mathrm{m}(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) \stackrel{?}{=} a \cdot L'(E_{45}, -1) + \frac{b}{32\pi}\cdot D(\alpha), \qquad a \in \mathbb{Q}^\times, b \in \mathbb{Z}\setminus\{0\}.$$

We are unable to determine the coefficient b as computing the integrals $\int_{\partial\Gamma} d\arg f_p$ for $p \in S$ is difficult. By Remark 4.14, we have

$$D(\alpha) = \frac{3\sqrt{3}}{4}L(\chi_{-3}, 2) = \pi L'(\chi_{-3}, -1).$$

Finally, we get

$$\mathrm{m}(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) \stackrel{?}{=} a \cdot L'(E_{45}, -1) + \tfrac{1}{32}b \cdot L'(\chi_{-3}, -1),\ \ a \in \mathbb{Q}^\times, b \in \mathbb{Z}\setminus\{0\}.$$

(b) Using a method of Lalín [23, Section 4.2], we prove without assuming Beilinson's conjecture identity 6 of Table 3, which involves only the L-function of the Dirichlet character χ_{-4}:

$$\mathrm{m}(x^2+1+(x+1)^2y+(x-1)^2z) = 2L'(\chi_{-4},-1).$$

We have $\mathrm{m}(\tilde{P})=0$. We have the following decomposition on V_P:

$$x\wedge y\wedge z = -\tfrac{1}{2}x^2\wedge(1+x^2)\wedge y + 2x\wedge(1-x)\wedge y + x\wedge\frac{(x+1)^2y}{x^2+1}\wedge\Bigl(1+\frac{(x+1)^2y}{x^2+1}\Bigr)$$
$$-2x\wedge(1+x)\wedge\Bigl(1+\frac{(x+1)^2y}{x^2+1}\Bigr)+\tfrac{1}{2}x^2\wedge(1+x^2)\wedge\Bigl(1+\frac{(x+1)^2y}{x^2+1}\Bigr).$$

We have

$$\rho(\xi) = -\tfrac{1}{2}\rho(-x^2,y)+2\rho(x,y)+\rho\Bigl(\frac{-(x+1)^2y}{x^2+1},x\Bigr)-2\rho\Bigl(-x,1+\frac{(x+1)^2y}{x^2+1}\Bigr)+\tfrac{1}{2}\rho\Bigl(-x^2,1+\frac{(x+1)^2y}{x^2+1}\Bigr),$$

where

$$\rho(f,g) = -D(f)d\arg g+\tfrac{1}{3}\log|g|\bigl(\log|1-f|\,d\log|f|-\log|f|\,d\log|1-f|\bigr).$$

W_P is given by

$$(x^2+1)((x+1)^2y^2+(x^2+8x+1)y+(x+1)^2)=0,$$

which is the union of $L: x^2+1=0$ and the curve $C: (x+1)^2y^2+(x^2+8x+1)y+(x+1)^2=0$. Figure 6 describes the Deninger chain Γ in polar coordinates:

$$\Gamma: \left|\frac{x^2+1+(x+1)^2y}{(x-1)^2}\right| \geq 1, \quad x=e^{it}, \quad y=e^{is}, \quad s,t\in[-\pi,\pi].$$

Its boundary $\partial\Gamma$ consists of 2 loops $\gamma=\{t=\pi/2, -\pi\leq s\leq\pi\}$ and $\delta=\{t=-\pi/2,-\pi\leq s\leq\pi\}$ (with orientations as shown in the figure), which are contained in L. As $\partial\Gamma$ contains poles of $\rho(\xi)$, we do not

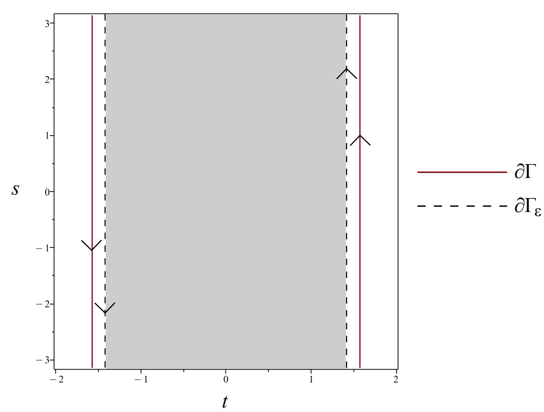


Figure 6. The Deninger chain Γ for the proof of identity 6 in Table 3 and the modified path Γ_ϵ used as the integration domain.

have (4-23) directly. We adjust the Deninger chain as follows, for $\varepsilon > 0$:

$$\Gamma_\varepsilon : \left|\frac{x^2+1+(x+1)^2y}{(x-1)^2}\right| \geq 1, x = e^{i(1+\varepsilon)t}, \quad y = e^{is}, \quad \text{for } s, t \in [-\pi, \pi],$$

This is the shaded region in Figure 6 with the boundary $\partial\Gamma_\varepsilon = \gamma_\varepsilon \cup \delta_\varepsilon$, where

$$\gamma_\varepsilon = \left\{t = \frac{\pi}{2(1+\varepsilon)}, -\pi \leq s \leq \pi\right\}, \quad \delta_\varepsilon = \left\{t = -\frac{\pi}{2(1+\varepsilon)}, -\pi \leq s \leq \pi\right\}.$$

We consider the differential forms η and $\rho(\lambda)$ defined in (4-1) and Definition 4.4, respectively. We have

$$\int_{\Gamma_\varepsilon} \eta = \int_{\partial\Gamma_\varepsilon} \rho(\xi) = \frac{1}{2}\int_{\partial\Gamma_\varepsilon} \rho(\lambda), \tag{5-7}$$

where the first equality is obtained by using Stokes's theorem and the second equality can be proved similarly as the proof of Lemma 4.8. Since $\rho(\lambda)$ is a closed differential form, we can take the limit of (5-7) as $\varepsilon \to 0$ without changing the value of the integration, so that

$$\mathrm{m}(P) = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\partial\Gamma_\varepsilon} \rho(\xi).$$

We have

$$\begin{aligned}\int_{\partial\Gamma_\varepsilon} \rho(\xi) &= \int_{\partial\Gamma_\varepsilon} -\tfrac{1}{2}\rho(-x^2, y) + 2\rho(x, y) + \rho\Big(\frac{-(x+1)^2y}{x^2+1}, x\Big) \\ &\qquad\qquad - 2\rho\Big(-x, 1+\frac{(x+1)^2y}{x^2+1}\Big) + \tfrac{1}{2}\rho\Big(-x^2, 1+\frac{(x+1)^2y}{x^2+1}\Big) \\ &= \int_{\gamma_\varepsilon\cup\delta_\varepsilon} 2\rho(x, y) - 2\rho\Big(-x, 1+\frac{(x+1)^2y}{x^2+1}\Big) \\ &= \int_{\gamma_\varepsilon\cup\delta_\varepsilon} -2D(x)d\arg(y) + 2D(-x)d\arg\Big(1+\frac{(x+1)^2y}{x^2+1}\Big) \\ &= \Big(-2D\big(e^{\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\gamma_\varepsilon} d\arg(y) - 2D\big(e^{-\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\delta_\varepsilon} d\arg(y)\Big) \\ &\quad + \Big(2D\big(-e^{\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\gamma_\varepsilon} d\arg\Big(1+\frac{(x+1)^2y}{x^2+1}\Big)\Big) + 2D\big(-e^{-\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\delta_\varepsilon} d\arg\Big(1+\frac{(x+1)^2y}{x^2+1}\Big).\end{aligned}$$

We have

$$\int_{\gamma_\varepsilon} d\arg(y) = \int_{-\pi}^{\pi} ds = 2\pi, \quad \int_{\delta_\varepsilon} d\arg y = \int_{\pi}^{-\pi} ds = -2\pi.$$

We also get

$$\int_{\gamma_\varepsilon} d\arg\left(1+\frac{(x+1)^2y}{x^2+1}\right) = 2\pi, \quad \int_{\delta_\varepsilon} d\arg\left(1+\frac{(x+1)^2y}{x^2+1}\right) = -2\pi,$$

by looking at Figure 7, left, and the inequality $\left|(x+1)^2/(x^2+1)\right| > 1$. Then $\lim_{\varepsilon\to 0}\int_{\partial\Gamma_\varepsilon}\rho(\xi) = -16\pi D(e^{i\pi/2})$. It follows that

$$\mathrm{m}(P) = \frac{4}{\pi}D(e^{i\pi/2}) = 2L'(\chi_{-4}, -1).$$

The same arguments apply to identities 4, 5, 7, and 8 of Table 3.

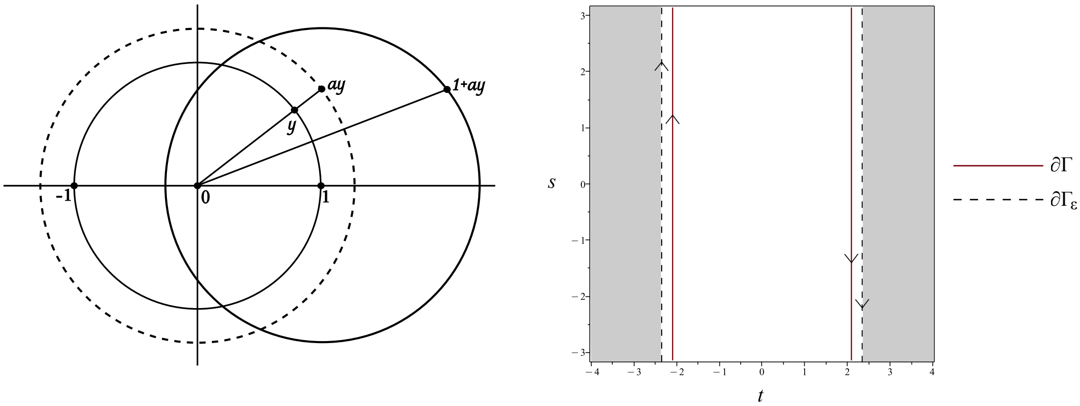


Figure 7. Left: the argument of $1 + ay$ with $|a| > 1$. Right: the integration domain.

(c) Let us study identity 1 of Table 3, which involves only the L-function of the Dirichlet character χ_{-3}

$$\mathrm{m}(1 + (x + 1)(x^2 + x + 1)y + (x + 1)^3 z) = 3L'(\chi_{-3}, -1).$$

We have that W_P is given by $(x^2 + x + 1)((x^4 + x^3)y^2 + (-2x^3 - 5x^2 - 2x)y + x + 1) = 0$, which is the union of the line $L : x^2 + x + 1 = 0$ and the curve $C : (x^4 + x^3)y^2 + (-2x^3 - 5x^2 - 2x)y + x + 1 = 0$. Figure 7, right, describes the Deninger chain in local coordinates $x = e^{it}$ and $y = e^{is}$ for $s, t \in [-\pi, \pi]$. The boundary $\partial\Gamma = \gamma \cup \delta$, where

$$\gamma = \{t = 2\pi/3, -\pi \le s \le \pi\} \text{ and } \delta = \{t = -2\pi/3, -\pi \le s \le \pi\},$$

which are both contained in L.

The differential form $\rho(\xi)$ is again not well-defined on $\partial\Gamma$. So we adjust the Deninger chain to get Γ_ε (see the shaded region). By a similar computation as in item (b), we have

$$\mathrm{m}(P) = -\frac{1}{4\pi^2} \lim_{\varepsilon \to 0} \int_{\partial\Gamma_\varepsilon} \rho(\xi) = 3L'(\chi_{-3}, -1).$$

One can do similarly with identities 2 and 3 of Table 3.

(d) Theorem 0.2 does not apply to identity (0-12),

$$\mathrm{m}(x^2 + x + 1 + (x^2 + x + 1)y + (x - 1)^2) \stackrel{?}{=} -\tfrac{1}{12}L'(E_{72}, -1) + \tfrac{3}{2}L'(\chi_3, -1),$$

because the boundary $\partial\Gamma$ passes the singular point $(1, -1)$ of W_P (see Figure 8) and $\partial\Gamma$ is no longer a loop in the normalization of W_P.

(e) We prove the second identity of Table 2, under Beilinson's conjecture for genus 1 curves:

$$\mathrm{m}(x^2 + 1 + (x + 1)^2 y + (x^2 - 1)z) \stackrel{?}{=} -\tfrac{1}{10}L'(E_{48}, -1) + L'(\chi_{-4}, -1). \tag{5-8}$$

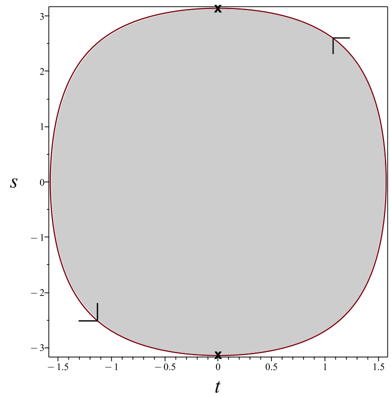


Figure 8. The Deninger chain Γ for (0-12).

We have

$$x \wedge y \wedge z = -\tfrac{1}{2}x^2 \wedge (1+x^2) \wedge y \;+\; \tfrac{1}{2}x^2 \wedge (1-x^2) \wedge y \;+\; \frac{(x+1)^2 y}{x^2+1} \wedge \Big(1+\frac{(x+1)^2 y}{x^2+1}\Big) \wedge x$$
$$-2x \wedge (1+x) \wedge \Big(1+\frac{(x+1)^2 y}{x^2+1}\Big) \;+\; \tfrac{1}{2}x^2 \wedge (1+x^2) \wedge \Big(1+\frac{(x+1)^2 y}{x^2+1}\Big).$$

Then

$$\rho(\xi) =$$
$$-\tfrac{1}{2}\rho(-x^2, y) + \tfrac{1}{2}\rho(x^2, y) + \rho\Big(-\frac{(x+1)^2 y}{x^2+1}, x\Big) - 2\rho\Big(-x, 1+\frac{(x+1)^2 y}{x^2+1}\Big) + \tfrac{1}{2}\rho\Big(-x^2, 1+\frac{(x+1)^2 y}{x^2+1}\Big).$$

W_P is given by

$$(x^2+1)((x+1)^2 y^2 + (3x^2+4x+3)y + (x+1)^2) = 0,$$

which is the union of $L : x^2+1=0$ and the curve $C : (x+1)^2 y^2 + (3x^2+4x+3)y + (x+1)^2 = 0$, which is a nonsingular curve of genus 1. Figure 9 describes the Deninger chain Γ and its boundary $\partial\Gamma$ in polar coordinates $x = e^{it}$ and $y = e^{is}$ for $t, s \in [-\pi, \pi]$. We have $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the shaded region in the center with the boundary

$$\partial\Gamma_1 = \{t = -\pi/2, -\pi \le s \le \pi\} \cup \{t = \pi/2, -\pi \le s \le \pi\},$$

and Γ_2 is the shaded region with the boundary $\partial\Gamma_2$ as in the figure. We observe that $\partial\Gamma_1$ is contained in L and $\partial\Gamma_2$ is contained in C. We have

$$\mathrm{m}(P) = m_1 + m_2,$$

where m_1 can be computed by the same method as the example (b). Let $\Gamma_{1,\epsilon}$ be the adjustment of Γ_1 shown in Figure 9. We have

$$m_1 = -\frac{1}{4\pi^2}\int_{\Gamma_1}\eta = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\Gamma_{1,\varepsilon}}\eta = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\partial\Gamma_{1,\varepsilon}}\rho(\xi) = L'(\chi_{-4}, -1),$$

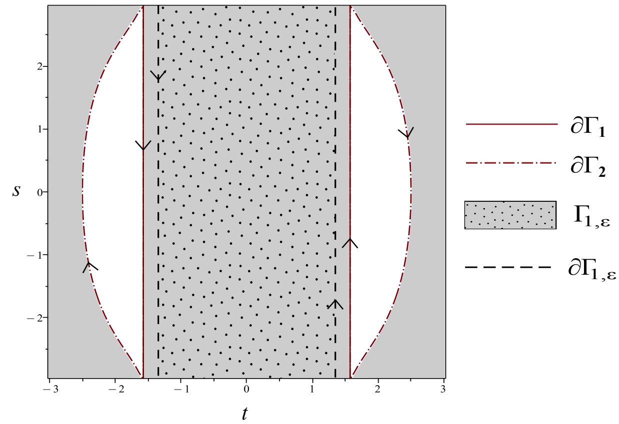


Figure 9. The Deninger chain Γ corresponding to (5-8).

and

$$m_2 = -\frac{1}{4\pi^2}\int_{\Gamma_2}\eta = -\frac{1}{4\pi^2}\int_{\partial\Gamma_2}\rho(\xi).$$

By the change of variables

$$x = -\frac{2Y+X^2}{X^2-2X-4}, \quad y = -\frac{2}{X+2},$$

the Jacobian of C is given by

$$E/\mathbb{Q} : Y^2 = X^3 + X^2 - 4X - 4,$$

which the elliptic curve of type $48a1$. Its torsion subgroup is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} = \langle A\rangle \times \langle B\rangle$, where $A = (2, 0)$ and $B = (-1, 0)$. Set $K = \mathbb{Q}(\alpha, \beta)$ where $\alpha^2 - 2\alpha - 4 = 0$ and $\beta^2 + 4 = 0$. Let us write

$$P_1 = (\alpha, \alpha+2),\ P_2 = (\alpha, -\alpha-2),\ P_3 = (0, s, 1).$$

We have

$$\begin{aligned} \operatorname{div}(x) &= -(P_1) + (P_2), & \operatorname{div}\Big(\frac{1+(1+x)^2y}{x^2+1}\Big) &= 2(A) + 2(A+B) - 2(P_3),\\ \operatorname{div}(y) &= 2(\mathcal{O}) - 2(A+B), & \operatorname{div}\Big(\frac{x^2y+x^2+2x+y+1}{y(x^2+1)}\Big) &= 2(\mathcal{O}) + 2(B) - 2(P_3). \end{aligned}$$

We have $u_A = u_B = u_{A+B} = 0$ and $u_{P_3} = -2\{-\beta/2\}_2 - 2\{\beta/2\}_2 = 0$. The residues u_{P_i} for $i = 1, 2$ are trivial because they belong to $\mathcal{B}(\mathbb{Q}(\alpha))$, the Bloch group (tensored with $\mathbb{Q}$) of the real quadratic field $\mathbb{Q}(\alpha)$. Therefore, we have the following identity under Beilinson's conjecture:

$$m_2 = a\cdot L'(E_{48}, -1), \quad a \in \mathbb{Q}^\times.$$

In conclusion, we obtain the following identity under Beilinson's conjecture:

$$\mathrm{m}(P) = L'(\chi_{-4}, -1) + m_2 = a\cdot L'(E_{48}, -1) + L'(\chi_{-4}, -1), \quad a \in \mathbb{Q}^\times.$$

Acknowledgements

I would like to express my gratitude to my supervisor, François Brunault, who brought me the idea to work on this subject as well as spent an immense amount of time to discuss with me. I also would like to thank for his generosity in sharing his conjectural Mahler measure identities. I would like to thank Rob de Jeu for his enlightening explanation in the construction of his maps and in computing regulators integrals in Section 3. I also would like to thank Nguyen Xuan Bach for several fruitful discussions.

References

[1] S. Bloch, *Lectures on algebraic cycles*, 2nd ed., New Mathematical Monographs **16**, Cambridge Univ. Press, 2010. MR

[2] W. Bosma, J. Cannon, and C. Playoust, "The Magma algebra system, I: The user language", *J. Symbolic Comput.* **24**:3-4 (1997), 235–265. MR

[3] D. W. Boyd, "Mahler's measure and special values of L-functions", *Experiment. Math.* **7**:1 (1998), 37–82. MR

[4] D. W. Boyd, "Conjectural explicit formulas for the Mahler measure of some three variable polynomials", unpublished notes, 2006.

[5] F. Brunault, "Regulators of Siegel units and applications", *J. Number Theory* **163** (2016), 542–569. MR

[6] F. Brunault, unpublished list of conjectural identities for 3-variable Mahler measures, 2020.

[7] F. Brunault, "On the K_4 group of modular curves", 2022. arXiv 2009.07614v2

[8] F. Brunault, "On the Mahler measure of $(1+x)(1+y)+z$", preprint, 2023. arXiv 2305.02992

[9] F. Brunault and W. Zudilin, *Many variations of Mahler measures—a lasting symphony*, Australian Mathematical Society Lecture Series **28**, Cambridge Univ. Press, 2020. MR

[10] J. I. Burgos, "A C^∞ logarithmic Dolbeault complex", *Compositio Math.* **92**:1 (1994), 61–86. MR

[11] J. I. Burgos, "Arithmetic Chow rings and Deligne–Beilinson cohomology", *J. Algebraic Geom.* **6**:2 (1997), 335–377. MR

[12] C. Deninger, "Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbf{Z}^n$-actions", *J. Amer. Math. Soc.* **10**:2 (1997), 259–281. MR

[13] C. Deninger and A. J. Scholl, "The Beĭlinson conjectures", pp. 173–209 in *L-functions and arithmetic* (Durham, 1989), London Math. Soc. Lecture Note Ser. **153**, Cambridge Univ. Press, Cambridge, 1991. MR

[14] E. M. Friedlander, A. Suslin, and V. Voevodsky, "Introduction", pp. 3–9 in *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud. **143**, Princeton Univ. Press, 2000. MR

[15] A. B. Goncharov, "Geometry of configurations, polylogarithms, and motivic cohomology", *Adv. Math.* **114**:2 (1995), 197–318. MR

[16] A. B. Goncharov, "Deninger's conjecture of L-functions of elliptic curves at $s=3$", *J. Math. Sci.* **81**:3 (1996), 2631–2656. MR

[17] A. B. Goncharov, "Explicit regulator maps on polylogarithmic motivic complexes", pp. 245–276 in *Motives, polylogarithms and Hodge theory* (Irvine, CA, 1998), vol. 1, Int. Press Lect. Ser. **3, I**, International Press, Somerville, MA, 2002. MR

[18] R. de Jeu, "Zagier's conjecture and wedge complexes in algebraic K-theory", *Compositio Math.* **96**:2 (1995), 197–247. MR

[19] R. de Jeu, "On $K_4^{(3)}$ of curves over number fields", *Invent. Math.* **125**:3 (1996), 523–556. MR

[20] R. de Jeu, "Towards regulator formulae for the K-theory of curves over number fields", *Compositio Math.* **124**:2 (2000), 137–194. MR

[21] B. Kahn, *Zeta and L-functions of varieties and motives*, London Mathematical Society Lecture Note Series **462**, Cambridge Univ. Press, 2020. MR

[22] M. N. Lalín, "On a conjecture by Boyd", *Int. J. Number Theory* **6**:3 (2010), 705–711. MR

[23] M. Lalín, "Mahler measure and elliptic curve L-functions at $s=3$", *J. Reine Angew. Math.* **709** (2015), 201–218. MR

[24] M. Lalín and S. S. Nair, "An invariant property of Mahler measure", *Bull. Lond. Math. Soc.* **55**:3 (2023), 1129–1142. MR

[25] M. N. Lalín and M. D. Rogers, "Functional equations for Mahler measures of genus-one curves", *Algebra Number Theory* **1**:1 (2007), 87–117. MR

[26] M. Lalín, D. Samart, and W. Zudilin, "Further explorations of Boyd's conjectures and a conductor 21 elliptic curve", *J. Lond. Math. Soc.* (2) **93**:2 (2016), 341–360. MR

[27] K. Mahler, "On some inequalities for polynomials in several variables", *J. London Math. Soc.* **37** (1962), 341–344. MR

[28] V. Maillot, "Mahler measure in Arakelov geometry", workshop lecture at "The many aspects of Mahler's measure", Banff International Research Station, 2003.

[29] J. P. Murre, J. Nagel, and C. A. M. Peters, *Lectures on the theory of pure motives*, Univ. Lecture Series **61**, Amer. Math. Soc., 2013. MR

[30] J. Nekovář, "Beĭlinson's conjectures", pp. 537–570 in *Motives* (Seattle, WA, 1991), vol. 1, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., 1994. MR

[31] "PARI/GP version `2.15.4`", software, Univ. Bordeaux, 2023, available at http://pari.math.u-bordeaux.fr.

[32] M. Rogers and W. Zudilin, "From L-series of elliptic curves to Mahler measures", *Compos. Math.* **148**:2 (2012), 385–414. MR

[33] M. Rogers and W. Zudilin, "On the Mahler measure of $1+X+1/X+Y+1/Y$", *Int. Math. Res. Not.* **2014**:9 (2014), 2305–2326. MR

[34] A. J. Scholl, "Classical motives", pp. 163–187 in *Motives* (Seattle, WA, 1991), vol. 1, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., 1994. MR

[35] A. A. Suslin, "K_3 of a field, and the Bloch group", pp. 180–199, 229 , 1990. In Russian; translated in Proc. Steklov Inst. Math. **1991**, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications. MR

[36] F. R. Villegas, "Modular Mahler measures. I", pp. 17–48 in *Topics in number theory* (University Park, PA, 1997), Math. Appl. **467**, Kluwer, Dordrecht, 1999. MR

[37] C. A. Weibel, *The K-book*, Graduate Studies in Mathematics **145**, Amer. Math. Soc., 2013. An introduction to algebraic K-theory. MR

[38] D. Zagier, "Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields", pp. 391–430 in *Arithmetic algebraic geometry* (Texel, 1989), Progr. Math. **89**, Birkhäuser, Boston, 1991. MR

[39] W. Zudilin, "Regulator of modular units and Mahler measures", *Math. Proc. Cambridge Philos. Soc.* **156**:2 (2014), 313–326. MR

Communicated by Shou-Wu Zhang
Received 2024-04-08 Revised 2024-12-26 Accepted 2025-02-11

ha.trieuthu@hust.edu.vn *Unité de Mathématiques Pures et Appliquées, École normale supérieure de Lyon, 69364 Lyon, France*

thu-ha.trieu@ens-lyon.fr

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.577

On the Frobenius fields of abelian varieties over number fields

Ashay A. Burungale, Haruzo Hida and Shilin Lai

Let A be a non-CM simple abelian variety over a number field K. For a place v of K where A has good reduction, let $F(A, v)$ denote the Frobenius field generated by the corresponding Frobenius eigenvalues. If A has connected monodromy groups, we show that the set of places v such that $F(A, v)$ is isomorphic to a fixed number field has upper Dirichlet density zero. Moreover, assuming the GRH, we give a power saving upper bound for the number of such places.

1. Introduction

For an abelian variety over a number field and a place of good reduction, a basic invariant is the Frobenius field generated by the corresponding Frobenius eigenvalues. In this paper we study its connection with the arithmetic of the abelian variety.

For a CM abelian variety, in view of the CM theory of Shimura, Taniyama and Weil, the Frobenius fields are contained in a fixed number field and equal to it for a set of places of Dirichlet density one. A natural question: to explore the upper Dirichlet density of the set of places at which the Frobenius field of a non-CM abelian variety coincides with a given number field up to an isomorphism. The question has been studied in the literature in various low-dimensional cases, and the primary goal of this paper is to consider the general case via a uniform approach.

We show that the density is zero under a mild connectedness hypothesis (see Theorem 1.1). Moreover, assuming the GRH, we provide a power-saving upper bound on the size of the set of places with bounded norm at which the Frobenius field of a non-CM abelian variety coincides with a given number field (see Theorem 1.3).

Main results. Let K be a number field and Σ_K the set of its finite places. For $\overline{K}$ an algebraic closure, let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the corresponding absolute Galois group. For a place $v \in \Sigma_K$, let Frob_v be an associated geometric Frobenius.

Let A be an abelian variety defined over K of dimension g and conductor $\mathfrak{N}$. For a prime p, let $T_p A$ be the p-adic Tate module of A and

$$\rho_{A,p} : G_K \to \mathrm{Aut}_{\mathbb{Z}_p} T_p A$$

MSC2020: 14K15.
Keywords: abelian varieties, Frobenius fields, Mumford–Tate groups.

the associated p-adic Galois representation. For $v \nmid p\mathfrak{N}$, it is unramified at v. Let $F(A, v)$ be the splitting field of the characteristic polynomial of $\rho_{A,p}(\mathrm{Frob}_v)$. As $(\rho_{A,p})_p$ is a compatible system of Galois representations, $F(A, v)$ is a number field independent of p, referred to as the Frobenius field associated to the pair (A, v).

Let M be a number field and $S_{A,M}$ a subset of finite places Σ_K of K given by

$$S_{A,M} = \{v \in \Sigma_K \mid v \nmid \mathfrak{N}, F(A, v) \cong M\}.$$

When A is a CM abelian variety, the Frobenius fields are a subfield[1] of the corresponding CM field (cf. [35]). Moreover, for a Dirichlet density one subset, the Frobenius fields equal a fixed subfield of the CM field.

In this paper, for non-CM abelian varieties A, we consider dependence of the Frobenius fields $F(A, v)$ on the place v. To state the results, we recall the following notion. For $S \subset \Sigma_K$, the upper Dirichlet density $\mathrm{ud}(S)$ is given by

$$\mathrm{ud}(S) = \limsup_{X\to\infty} \frac{\#\{v \in \Sigma_K : \boldsymbol{N}_{K/\mathbb{Q}} v \le X, v \in S\}}{\#\{v \in \Sigma_K : \boldsymbol{N}_{K/\mathbb{Q}} v \le X\}},$$

where $\boldsymbol{N}_{K/\mathbb{Q}}$ denotes the norm of the extension $K/\mathbb{Q}$.

Our first main result is the following.

Theorem 1.1. *Let A be an absolutely simple non-CM abelian variety defined over a number field K. Suppose that the monodromy groups associated to A over K are connected* (*cf. Hypothesis 2.2*). *Then for any number field M, we have*

$$\mathrm{ud}(S_{A,M}) = 0.$$

The connectivity Hypothesis 2.2 is satisfied by any abelian variety A over some finite extension of its field of definition (cf. Remark 2.3). Moreover, it is satisfied by a class of abelian varieties over their field of definition, for instance: most abelian varieties in a typical family, including hyperelliptic Jacobians [37, Theorem 1.2].

Remark 1.2. Theorem 1.1 gives a criterion for characterization of CM/non-CM abelian varieties. For different criteria in the case of GL_2-type abelian varieties over $\mathbb{Q}$, the reader may refer to [12, §3.1.1].

In view of Theorem 1.1 it is natural to seek to estimate the size of subsets of $S_{A,M}$ with bounded norm. To state the result, we introduce some notation. Let $\boldsymbol{G}$ be the Mumford–Tate group associated to A. Let $\boldsymbol{G}^{\mathrm{ss}}$ be its semisimple quotient. Let

$$d = \dim \boldsymbol{G}^{\mathrm{ss}}, \quad r = \operatorname{rank} \boldsymbol{G}^{\mathrm{ss}}$$

denote the dimension and (absolute) rank of $\boldsymbol{G}^{\mathrm{ss}}$ Then our quantitative result is as follows.

[1] More precisely, the Frobenius fields are a subfield of the CM endomorphism field K given by $\alpha^{\Phi'} \in K$ for some $\alpha \in K'$ for the reflex CM type (K', Φ').

Theorem 1.3. *Let A be an absolutely simple non-CM abelian variety defined over a number field K. Suppose that the monodromy groups associated to A over K are connected (cf. Hypothesis 2.2), and that the generalized Riemann hypothesis (GRH) holds for L-functions as in Remark 1.5. Then for any $\varepsilon > 0$, there exists a constant c depending on (A, K, ε) such that for any number field M and $X \in \mathbb{R}_{>0}$, we have*

$$\left|\{v \in S_{A,M} \mid N_{K/\mathbb{Q}} v \leq X\}\right| \leq c X^{1-\frac{1}{3d-r+2}+\varepsilon}.$$

If A is generic — meaning that the associated adelic Galois representation has open image in $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ — then $\boldsymbol{G}^{\mathrm{ss}} = \mathrm{PGSp}_{2g}$ for an integer $g \geq 1$. In this case $d = 2g^2 + g$ and $r = g$, so we obtain the following.

Corollary 1.4. *In the setting of Theorem 1.3 suppose further that A is generic and let $\boldsymbol{G}^{\mathrm{ss}} = \mathrm{PGSp}_{2g}$ for an integer $g \geq 1$. Then for any $\varepsilon > 0$, there exists a constant c depending on (A, K, ε) such that for any number field M and $X \in \mathbb{R}_{>0}$, we have*

$$\left|\{v \in S_{A,M} \mid N_{K/\mathbb{Q}} v \leq X\}\right| \leq c X^{1-\frac{1}{6g^2+2g+2}+\varepsilon}.$$

Remark 1.5. Our approach to Theorem 1.3 is conditional on the GRH for two types of L-functions. The first is the Dedekind zeta function for a number field M. The second consists of L-functions of finite order Hecke characters appearing in the statement of Lemma 4.11, namely the characters attached to abelian subquotients of Galois extensions of the form $K(A[d])/K$, where d is a square-free product of rational primes.

About the proof. Our approach is based on the compatible system of Galois representations associated to the abelian variety and some group theory. To begin, we recast the problem in terms of (a variant of) Frobenius tori in the algebraic monodromy group of the abelian variety. A volume computation of conjugacy classes arising from these tori and large Galois image results for A are keys of the proof.

We now describe the strategy in more detail. To begin, for primes p which split completely in M, we show that if $v \in S_{A,M}$, then the image of Frob_v in the associated mod p monodromy group lies in a $\mathbb{F}_p$-rational Borel subgroup. Then we give a volume upper bound for the union of all such subgroups (cf. Corollary 3.4), perhaps of independent interest. On the other hand, since A is non-CM, the associated Galois representations have large image, thanks to the work of Wintenberger [36] and Hui and Larsen [13] (cf. Theorem 2.5). In light of the image lower bound and the volume upper bound, the Chebotarev density theorem implies that $\mathrm{ud}(S_{A,M})$ is bounded above by a constant c less than 1, which is independent of the prime p. This deduction relies on basic properties of reductive groups, several of which require the group to be connected.

Next, we synthesise the analysis at different primes via the product Galois representation $\prod \rho_{A,p}$. Ideally, we expect

$$\mathrm{Im}\Big(\prod_p \rho_{A,p}\Big) = \prod_p \mathrm{Im}(\rho_{A,p}), \tag{1-1}$$

which implies: for different choices of p, the events that $\rho_{A,p}(\mathrm{Frob}_v)$ lands in a $\mathbb{F}_p$-rational Borel subgroup are independent. Hence, Theorem 1.1 follows (recall that the density upper bound c is independent of

p). Actually, Serre showed that the independence (1-1) holds upon replacing K with a finite extension (cf. Theorem 2.4(2)). This suffices for our argument.

As for Theorem 1.3, we quantify the above strategy with the aid of the effective Chebotarev theorem of Lagarias and Odlyzko and the Selberg sieve. Some notable features:

- Without the Mumford–Tate conjecture, the p-adic monodromy groups can be different[2] for each p. In order to control their sizes, we introduce a soft argument based on the Weil bound and the classification of reductive groups over algebraically closed fields (cf. the proof of Lemma 4.15).
- Since we are only sieving using primes which split completely in M, a lower bound on the number of such primes - independently of M - is necessary. This is achieved via a simple *a priori* upper bound on the discriminant of a Frobenius field $F(A, v)$ in terms of $\boldsymbol{N}_{K/\mathbb{Q}}v$ (cf. Lemma 4.18).

To refine exponents naturally appearing in the analysis, we make the following improvements:

- Instead of the p-adic monodromy group, we work with its semisimple quotient, reducing the sizes of the relevant Galois groups. This is a generalization of the PGL_2-reduction method used previously.
- Well-known results of M. R. Murty, V. K. Murty and N. Saradha give better error terms in the effective Chebotarev theorem when certain subgroups of the Galois group satisfy the Artin holomorphy conjecture (AHC) (cf. Theorem 4.8). In our case, the natural choice is a Borel subgroup of the monodromy group, and we make an elementary group-theoretic observation that it satisfies the AHC.

The above relies on a fundamental result of Deligne [10]: the Mumford–Tate group "contains" all p-adic monodromy groups (cf. Theorem 4.16).

Prior work and prospects. Our study is inspired by the Lang–Trotter conjecture [20] and a consideration of Serre for Hecke eigenvalues of elliptic newforms [29, §7]. For elliptic curves, the result was first stated (without proof or explicit exponent) by Serre [29, §8.2]. Its various explicit forms have appeared in the work of Cojocaru, Fouvry and Murty [9], Cojocaru and David [6], Zywina [38], and Kulkarni, Patankar and Rajan [17], among others. Cojocaru and David [7] consider the analogous question for Drinfeld modules.

For generic abelian varieties, Theorem 1.1 was first established by Bloom [1]. His method is different and does not seem amenable to quantitative refinements as in Theorem 1.3. As for general abelian varieties, the only known result seems to be that of Khare [16]: for a non-CM simple abelian variety, the set of places whose Frobenius fields equal a fixed number field cannot be the complement of a finite set of places.

Our result applies to any abelian variety up to a base change. In the generic case, no base change is needed, and it gives a better exponent than Bloom [1], assuming only the GRH. The qualitative part of the argument generalizes to function fields, and we expect that a suitable modification yields the quantitative version. Moreover, our method seems to apply to any compatible system of Galois representations for which Frobenius semisimplicity holds (cf. Remark 2.6).

[2] Recall that the Mumford–Tate conjecture is not known in general.

Compared to previous works, our approach is closer in spirit to the square sieve method, which first appeared in [9]. However, by working with abstract reductive groups, we replace all the work in estimating conjugacy class sizes with soft arguments, and we can identify power savings which are not apparent from the explicit matrix-based considerations. It would be interesting to see if our approach can be combined with the mixed Galois representation approach to obtain a further sharpening of Theorem 1.3.

Finally, to remove Hypothesis 2.2, we would need to study the structure of semisimple elements in a disconnected finite reductive group. We plan to return to this question in the near future.

Plan. In Section 2 we describe preliminaries regarding Galois representations, monodromy groups, and large image theorems. In Section 3 we present preliminaries regarding finite reductive groups[3] and introduce the key notion of a bounding set (cf. Definition 3.3). In Section 4 we prove the main theorems.

2. Backdrop

We describe some preliminaries regarding p-adic Galois representations associated to an abelian variety over a number field. The reader may refer to [13; 26; 36] for some details.

Let the setting be as in Section 1. In particular, A is an abelian variety over a number field K of dimension g. Put $\Delta_A = \boldsymbol{N}_{K/\mathbb{Q}}\mathfrak{N}$, so A has good reduction away from places of K which lie above a prime dividing Δ_A.

2.1. ***Galois representations.*** For a prime p, let T_pA be the p-adic Tate module of A. Let

$$\rho_p : G_K \to \mathrm{Aut}_{\mathbb{Z}_p} T_pA$$

be the associated p-adic Galois representation.

Theorem 2.1. *Let v be a finite place of K such that $v \nmid p\Delta_A$.*

(1) (*Serre–Tate*) *The Galois representation ρ_p is unramified at v.*

(2) (*Weil*) *The characteristic polynomial of $\rho_p(\mathrm{Frob}_v)$ has integral coefficients and is independent of p. Moreover*, *all of its roots have complex absolute value $(\boldsymbol{N}_{K/\mathbb{Q}}v)^{\frac{1}{2}}$.*

For a set T of primes, put

$$\rho^T = \prod_{p \notin T} \rho_p, \quad \rho_T = \prod_{p \in T} \rho_p.$$

For N a positive integer, put $\rho^N = \rho^{T(N)}$, $\rho_N = \rho_{T(N)}$, where $T(N)$ is the set of primes dividing N.

2.1.1. *Monodromy groups.* Let

$$\Gamma_p = \rho_p(G_K) \subseteq \mathrm{Aut}_{\mathbb{Z}_p} T_pA,$$

and $\boldsymbol{G}_p$ denote its Zariski closure. This is a linear algebraic group over $\mathbb{Z}_p$. The generic fibre of $\boldsymbol{G}_p$ is the usual p-adic monodromy group. Throughout this paper, we make the following assumption.

[3] As pointed out by the referee, these results may be well-known to specialists (cf. [15; 23]). Due to the lack of a precise reference in the literature (see Remark 3.7), we include details.

Hypothesis 2.2. The group $\boldsymbol{G}_p$ is connected for a prime p.

Remark 2.3. A result of Serre [30, corollaire p. 15] shows that the component group $\boldsymbol{G}_p/\boldsymbol{G}_p^\circ$ is independent of the prime p. So the hypothesis holds for all primes p as soon as it holds for one prime. It is also proven in *op. cit.* that there exists a finite extension K^{conn}/K such that this hypothesis holds for the base change $A_{/K^{\mathrm{conn}}}$.

We will use the following key results on the monodromy group.

Theorem 2.4.

(1) *If p is sufficiently large, then $\boldsymbol{G}_p$ is reductive.*

(2) *There exists a finite Galois extension L/K and an integer N such that*

$$\rho^N(G_L) = \prod_{p \nmid N} \rho_p(G_L)$$

Part (1) is due to Larsen and Pink (cf. the proof of [22, Theorem 3.2]). Part (2) is another result of Serre [31, théorème p. 56] (see also [14; 33]). The reference does not include the Galois requirement, but we can take the Galois closure, which preserves independence away from a finite set of primes.

2.1.2. *Galois image.* This subsection will recall some large image results for non-CM abelian varieties due to Hui and Larsen [13, Theorem 1.3]. An essentially equivalent result was obtained earlier by Wintenberger [36, théorème 2], but this formulation is more convenient for us.

For a prime p such that $\boldsymbol{G}_p$ is reductive, let $\boldsymbol{G}_p^{\mathrm{ss}}$ denote the quotient of $\boldsymbol{G}_p$ by its radical, and Γ_p^{ss} the image of Γ_p in $\boldsymbol{G}_p^{\mathrm{ss}}$. Let $\pi : \boldsymbol{G}_p^{\mathrm{sc}} \to \boldsymbol{G}_p^{\mathrm{ss}}$ be the algebraic universal cover. This is a finite morphism.

Theorem 2.5. *Let A be an abelian variety over a number field K. Then there exists a constant $c_{A,K} \in \mathbb{Z}$ such that for any prime $p > c_{A,K}$, we have*

$$\mathrm{Im}(\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{Z}_p) \to \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{Z}_p)) \subseteq \Gamma_p^{\mathrm{ss}}.$$

Proof. The aforementioned result of Hui and Larsen is that for $p \gg 0$, the group $\boldsymbol{G}_p(\mathbb{Q}_p)$ is unramified, and $\pi^{-1}(\Gamma_p^{\mathrm{ss}})$ is a hyperspecial maximal compact subgroup of $\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{Q}_p)$.

It is not easy to extract the integral model directly from the proof, so we conclude in an indirect way. We have the inclusions

$$\pi^{-1}(\Gamma_p^{\mathrm{ss}}) \subseteq \pi^{-1}(\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{Z}_p)) \subseteq \boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{Z}_p)$$

The first is by definition. The second holds since π is finite, hence proper. The final group is also a hyperspecial maximal compact subgroup of $\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{Q}_p)$, and so all three groups above are equal. □

Remark 2.6. If we have a compatible system of semisimple Galois representations coming from geometry, then Theorem 2.4 still holds. Larsen showed that Theorem 2.5 holds for a density one set of primes p [21, Theorem 3.17]. The exceptional set in that proof is again cut out using Chebotarev sets. Therefore, it's likely that our approach works in such generality, though we have not checked the details carefully.

Define the reduced residual representation

$$\overline{\rho}_p^{\mathrm{ss}} : G_K \to \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p),$$

Its image is also the image of Γ_p^{ss} under the reduction map $\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{Z}_p) \to \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)$. Denote it by $\overline{\Gamma}_p^{\mathrm{ss}}$. Put

$$I_p = \mathrm{Im}(\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{F}_p) \to \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)).$$

A simple consequence of Theorem 2.5 is the following.

Corollary 2.7. *Let A be an abelian variety over a number field K and $c_{A,K} \in \mathbb{Z}$ be as in Theorem 2.5. Then for any prime $p > c_{A,K}$, the image $\overline{\Gamma}_p^{\mathrm{ss}}$ contains I_p.*

Remark 2.8. The lower bound $c_{A,K}$ in the above corollary and Theorem 2.4 depends only on the number field K and the isogeny class of A. It seems likely that $c_{A,K}$ can be expressed as an explicit function of K and the conductor of A following the strategy outlined in [36, Remarque 2.2].

2.2. *A characterization of CM abelian varieties.*

Proposition 2.9. *Let A be a simple abelian variety over a number field K. If the associated p-adic monodromy group $\boldsymbol{G}_{p/\mathbb{Q}_p}$ is abelian for one prime p, then it is so for all primes, and A has CM.*

Proof. Suppose that $\boldsymbol{G}_{p/\mathbb{Q}_p}$ is abelian. The p-adic Galois representation ρ_p is semisimple by Faltings, and rational by construction. Hence, this Galois representation arises from an arithmetic Hecke character over K by a result of Henniart [11] (cf. [28, Theorem 2 in III-13]). Considering infinity type of the Hecke character, it follows that ρ_p corresponds to a CM abelian variety A'. Note that A and A' are isogenous by Faltings' isogeny theorem. Hence A has CM, and the assertion follows. □

3. Results on finite reductive groups

Let $\boldsymbol{G}$ be a connected reductive group defined over $\mathbb{F}_p$. In this section we will give a volume upper bound for the union of certain conjugacy classes.

3.1. ***Tori and Weyl groups.*** We recall some basic results about maximal tori of $\boldsymbol{G}$. A detailed exposition can be found in [4, Section 3.3].

Let $W_{\boldsymbol{G}}$ be the absolute Weyl group of $\boldsymbol{G}$. It carries an action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Let γ be the action corresponding to the Frobenius element. The general theory of forms gives a bijection

$$\{\text{conjugacy class of maximal tori defined over } \mathbb{F}_p\} \longleftrightarrow \{\gamma\text{-conjugacy classes in } W_{\boldsymbol{G}}\}.$$

Let $\boldsymbol{T}$ be a maximal torus in $\boldsymbol{G}$. Define its Weyl group by

$$W(\boldsymbol{G}, \boldsymbol{T}) = N_{\boldsymbol{G}(\mathbb{F}_p)}(\boldsymbol{T})/\boldsymbol{T}(\mathbb{F}_p).$$

It can also be described as the stabilizer of the γ-conjugacy class corresponding to $\boldsymbol{T}$ under the above bijection. In particular, the γ-twisted class equation gives

$$\sum_{\boldsymbol{T}} \frac{1}{|W(\boldsymbol{G}, \boldsymbol{T})|} = 1, \tag{3-1}$$

where the sum is over a set of representatives for the conjugacy classes of maximal tori in $\boldsymbol{G}$.

Lemma 3.1. *If $\boldsymbol{G}$ is not abelian, then $W(\boldsymbol{G}, \boldsymbol{T})$ is nontrivial for any maximal torus $\boldsymbol{T}$.*

Proof. By the class equation (3-1), if the lemma holds for one choice of $\boldsymbol{T}$, then it holds for all choices of $\boldsymbol{T}$. Let $\boldsymbol{B}$ be a Borel subgroup of $\boldsymbol{G}$ defined over $\mathbb{F}_p$. Its existence is guaranteed by Lang's theorem [19]. Let $\boldsymbol{S}$ be a maximal $\mathbb{F}_p$-split torus contained in $\boldsymbol{B}$, and let $\boldsymbol{T}$ be its centralizer. This is a Levi subgroup of $\boldsymbol{B}$, and hence a maximal torus. In addition, $N_{\boldsymbol{G}}(\boldsymbol{S}) \subseteq N_{\boldsymbol{G}}(\boldsymbol{T})$, so the Weyl group $W(\boldsymbol{G}, \boldsymbol{T})$ contains the relative Weyl group $W(\boldsymbol{G}, \boldsymbol{S})$.[4]

Since $\boldsymbol{G}$ is not abelian, its derived subgroup $\boldsymbol{G}^{\mathrm{der}}$ is nontrivial. By Lang's theorem, $\boldsymbol{G}^{\mathrm{der}}$ has a Borel subgroup, which implies that $\boldsymbol{G}^{\mathrm{der}}$ has positive rank [3, corollaire 4.17]. The second paragraph of the proof of [3, théorème 5.3] then shows that the relative Weyl group $W(\boldsymbol{G}, \boldsymbol{S})$ is nontrivial, so $W(\boldsymbol{G}, \boldsymbol{T})$ is also nontrivial for this choice of $\boldsymbol{T}$. $\square$

3.2. *Conjugacy classes.* Given a maximal torus $\boldsymbol{T}$, let $C_{\boldsymbol{T}}$ denote the set of elements in $\boldsymbol{G}(\mathbb{F}_p)$ which are conjugate to an element of $\boldsymbol{T}(\mathbb{F}_p)$. We will also use the superscript "reg" to denote the subset of regular semisimple elements. The volume is the counting measure normalized so that $\mathrm{vol}(\boldsymbol{G}(\mathbb{F}_p)) = 1$.

Proposition 3.2. *There exists a constant C depending only on the (absolute) rank of the group $\boldsymbol{G}$ such that, for all p, the following holds:*

(1) $\mathrm{vol}(\boldsymbol{G}(\mathbb{F}_p)^{\mathrm{reg}}) > 1 - Cp^{-1}$.

(2) *If $\boldsymbol{T}$ is a maximal torus, then*

$$|\mathrm{vol}(C_{\boldsymbol{T}}^{\mathrm{reg}}) - \frac{1}{|W(\boldsymbol{G}, \boldsymbol{T})|}| < Cp^{-1}.$$

Proof. Let $\boldsymbol{T}$ be a maximal torus. The map $\boldsymbol{G}/\boldsymbol{T} \times \boldsymbol{T} \to \boldsymbol{G}$, $(g, t) \mapsto gtg^{-1}$ is finite of degree $|W(\boldsymbol{G}, \boldsymbol{T})|$ above the regular elements in its image. Indeed, if $gtg^{-1} = g't'(g')^{-1}$, then $g^{-1}g'$ conjugates t' to t, so it conjugates $Z_{\boldsymbol{G}}(t')$ to $Z_{\boldsymbol{G}}(t)$. If t is regular, then the centralizer is just $\boldsymbol{T}$, so $g^{-1}g' \in N_{\boldsymbol{G}}(\boldsymbol{T})$. On $\mathbb{F}_p$-points, the image of this map is exactly $C_{\boldsymbol{T}}$, so

$$\mathrm{vol}(C_{\boldsymbol{T}}^{\mathrm{reg}}) = \frac{1}{|W(\boldsymbol{G}, \boldsymbol{T})|} \cdot \frac{|\boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}}|}{|\boldsymbol{T}(\mathbb{F}_p)|}. \tag{3-2}$$

Summing over a set of representatives of conjugacy classes of maximal tori, we get

$$\mathrm{vol}(\boldsymbol{G}(\mathbb{F}_p)^{\mathrm{reg}}) = \sum_{\boldsymbol{T}} \frac{1}{|W(\boldsymbol{G}, \boldsymbol{T})|} \cdot \frac{|\boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}}|}{|\boldsymbol{T}(\mathbb{F}_p)|}.$$

[4] We in fact have equality here, but this will not be needed.

We now give a coarse estimate for the right-hand side.

Let $r = \operatorname{rank} \boldsymbol{T}$. The set of nonregular elements in $\boldsymbol{T}(\mathbb{F}_p)$ is the $\mathbb{F}_p$-points of a variety $\boldsymbol{T}^{\mathrm{nr}}$ of dimension $r-1$. When base changed to $\overline{\mathbb{F}}_p$, it can be described as the subset of $\boldsymbol{G}^r_{m,\overline{\mathbb{F}}_p}$ where at least two of the entries are equal. Let $E_{/\mathbb{Z}}$ be the subscheme of $\boldsymbol{G}^r_{m,\mathbb{Z}}$ defined this way, then we see that $\boldsymbol{T}^{\mathrm{nr}}$ is a form of $E \times_{\mathbb{Z}} \mathbb{F}_p$. It follows from the Weil bound[5] that there exist a constant C (depending only on r) such that

$$\left| \frac{|\boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}}|}{|\boldsymbol{T}(\mathbb{F}_p)|} - 1 \right| \le C p^{-1} \tag{3-3}$$

for all p (cf. the proof of Lemma 4.15).

This estimate together with equation (3-2) immediately gives part (2) of the proposition. Using (3-1), we see that

$$\operatorname{vol}(\boldsymbol{G}(\mathbb{F}_p)^{\mathrm{reg}}) \ge \sum_{\boldsymbol{T}} \frac{1}{|W(\boldsymbol{G},\boldsymbol{T})|} \cdot (1 - Cp^{-1}) = 1 - Cp^{-1}.$$

This proves (1). □

Definition 3.3. Let $\boldsymbol{G}$ be a reductive group over $\mathbb{F}_p$. Its *bounding set* is the set of elements in $\boldsymbol{G}(\mathbb{F}_p)$ which lie in a Borel subgroup defined over $\mathbb{F}_p$.

A consequence of Proposition 3.2 is the following.

Corollary 3.4. *Let $\mathcal{B}$ be the bounding set of $\boldsymbol{G}$. There exists an integer N depending only on the rank of $\boldsymbol{G}$ such that if $\boldsymbol{G}$ is nonabelian and $p > N$, then*

$$\operatorname{vol}(\mathcal{B}) < \tfrac{3}{4}.$$

Proof. Let $\boldsymbol{T}$ be a maximal torus in a Borel subgroup $\boldsymbol{B}$. Let $x \in \mathcal{B}$ be regular semisimple. We will show that x is conjugate to an element of $\boldsymbol{T}$. Indeed, all Borel subgroups are conjugate [3, théorème 4.13(b)], so we may assume $x \in \boldsymbol{B}$. Any maximal torus of $\boldsymbol{B}$ containing x is a Levi component, so it is conjugate to $\boldsymbol{T}$ [3, proposition 4.7]. It follows that $\mathcal{B}$ is contained in the union of nonregular semisimple elements and $C_{\boldsymbol{T}}$. Therefore,

$$\begin{aligned}\operatorname{vol}(\mathcal{B}) &\le \operatorname{vol}(\boldsymbol{G}(\mathbb{F}_p) - \boldsymbol{G}(\mathbb{F}_p)^{\mathrm{reg}}) + \operatorname{vol}(C_{\boldsymbol{T}}^{\mathrm{reg}}) \\ &< 2Cp^{-1} + \frac{1}{|W(\boldsymbol{G},\boldsymbol{T})|}\end{aligned}$$

by Proposition 3.2. Lemma 3.1 shows that $W(\boldsymbol{G},\boldsymbol{T})$ is nontrivial, so its order is at least 2. By taking p sufficiently large, the right-hand side can be made smaller than any real number greater than $\frac{1}{2}$. □

3.3. *Central isogeny.* This subsection refines Proposition 3.2 and Corollary 3.4. This refinement is the key ingredient which allows us to apply the large image results of Theorem 2.5 and Corollary 2.7.

Suppose $\boldsymbol{G}$ is semisimple. Then it has an algebraic universal cover $\pi : \boldsymbol{G}^{\mathrm{sc}} \to \boldsymbol{G}$. Its kernel, denoted by Z, is a finite group scheme over $\mathbb{F}_p$ and contained in the centre of $\boldsymbol{G}^{\mathrm{sc}}$. Let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{G}$, then $\boldsymbol{T}^{\mathrm{sc}} := \pi^{-1}(\boldsymbol{T})$ is a maximal torus in $\boldsymbol{G}^{\mathrm{sc}}$ containing Z.

[5] One may also proceed by an elementary argument: counting $\boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}}$ using inclusion-exclusion, but the details are messy.

We have the following exact sequence of groups:

$$\begin{tikzcd}
1 \arrow[r] & Z(\mathbb{F}_p) \arrow[r] \arrow[d, equal] & \boldsymbol{T}^{\mathrm{sc}}(\mathbb{F}_p) \arrow[r] \arrow[d, hook] & \boldsymbol{T}(\mathbb{F}_p) \arrow[r] \arrow[d, hook] & \mathrm{H}^1(\mathbb{F}_p, Z) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & Z(\mathbb{F}_p) \arrow[r] & \boldsymbol{G}^{\mathrm{sc}}(\mathbb{F}_p) \arrow[r] & \boldsymbol{G}(\mathbb{F}_p) \arrow[r, "\delta"] & \mathrm{H}^1(\mathbb{F}_p, Z) \arrow[r] & 1.
\end{tikzcd}$$

The final 1 in both rows comes from Lang's theorem. A priori, the connecting morphisms are between pointed sets, but since Z is central, they are actually group homomorphisms (cf. [32, §I.5.6]).

Lemma 3.5. *We have* $|\boldsymbol{G}^{\mathrm{sc}}(\mathbb{F}_p)| = |\boldsymbol{G}(\mathbb{F}_p)|$.

Proof. Since $Z(\overline{\mathbb{F}}_p)$ is finite, its Herbrand quotient is 1. Hence the assertion follows by considering the bottom row of the above diagram. This is also a special case of [2, Chapter V, Proposition 16.8]. $\square$

The main result of this section is that the various conjugacy classes we have considered are approximately equally distributed in the fibres of δ. More precisely:

Proposition 3.6. *Let $\boldsymbol{G}$ be a nonabelian semisimple group. Let* $I = \mathrm{Im}(\boldsymbol{G}^{\mathrm{sc}}(\mathbb{F}_p) \to \boldsymbol{G}(\mathbb{F}_p))$.

(1) *There exists a constant C depending only on the rank of $\boldsymbol{G}$ such that for any maximal torus $\boldsymbol{T}$ and $g \in \boldsymbol{G}(\mathbb{F}_p)$, we have*

$$\left| \frac{|C_{\boldsymbol{T}}^{\mathrm{reg}} \cap gI|}{|I|} - \frac{1}{|W(\boldsymbol{G}, \boldsymbol{T})|} \right| < Cp^{-1}.$$

(2) *There exists an integer N depending only on the rank of $\boldsymbol{G}$ such that for all $p > N$, we have*

$$\frac{1}{2|W(\boldsymbol{G}, \boldsymbol{T})|} < \frac{|\mathcal{B} \cap gI|}{|I|} < \frac{3}{4}$$

where $\mathcal{B}$ is the bounding set for $\boldsymbol{G}$.

Proof. Let $n = [\boldsymbol{G}(\mathbb{F}_p) : I] = |\mathrm{H}^1(\mathbb{F}_p, Z)| = |Z(\mathbb{F}_p)|$. The group Z is a subgroup of the centre of a semisimple simply connected group over $\overline{\mathbb{F}}_p$. Since the rank is fixed, there are a finite number of such groups, so n is bounded from above, independently of $\boldsymbol{G}$.

From the above diagram, $I = \ker \delta$ is a normal subgroup with an abelian quotient, so if two elements of $\boldsymbol{G}(\mathbb{F}_p)$ are conjugate, then they lie in the same coset of I. Moreover, δ is still surjective when restricted to $\boldsymbol{T}(\mathbb{F}_p)$, so it identifies $\boldsymbol{T}(\mathbb{F}_p)/(\boldsymbol{T}(\mathbb{F}_p) \cap I)$ with $\mathrm{H}^1(\mathbb{F}_p, Z)$. Fix $g \in \boldsymbol{G}(\mathbb{F}_p)$, then we get

$$|\boldsymbol{T}(\mathbb{F}_p) \cap gI| = \frac{|\boldsymbol{T}(\mathbb{F}_p)|}{|\mathrm{H}^1(\mathbb{F}_p, Z)|}.$$

The conjugation action $\boldsymbol{G}/\boldsymbol{T} \times \boldsymbol{T} \to \boldsymbol{G}$, $(x, t) \mapsto xtx^{-1}$, as in the proof of Proposition 3.2, restricts to an action on the coset gI, so

$$|C_{\boldsymbol{T}}^{\mathrm{reg}} \cap gI| = \frac{1}{|W(\boldsymbol{G}, \boldsymbol{T})|} \cdot \frac{|\boldsymbol{G}(\mathbb{F}_p)|}{|\boldsymbol{T}(\mathbb{F}_p)|} \cdot |\boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}} \cap gI| = \frac{|I|}{|W(\boldsymbol{G}, \boldsymbol{T})|} \cdot \frac{|\boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}} \cap gI|}{|\boldsymbol{T}(\mathbb{F}_p) \cap gI|}.$$

Therefore, we have

$$\left|\frac{|C_{\boldsymbol{T}}^{\mathrm{reg}} \cap gI|}{|I|} - \frac{1}{|W(\boldsymbol{G},\boldsymbol{T})|}\right| = \frac{|(\boldsymbol{T}(\mathbb{F}_p) - \boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}}) \cap gI|}{|\boldsymbol{T}(\mathbb{F}_p) \cap gI|} \leq \frac{|\boldsymbol{T}(\mathbb{F}_p) - \boldsymbol{T}(\mathbb{F}_p)^{\mathrm{reg}}|}{|\boldsymbol{T}(\mathbb{F}_p)|/n} \leq nCp^{-1}$$

where C is the constant from (3-3).

For the second part, take $\boldsymbol{T}$ to be the maximal torus in a Borel subgroup as before. Let C be the constant in Proposition 3.2; then

$$\frac{|\mathcal{B} \cap gI|}{|I|} \leq \frac{|\boldsymbol{G}(\mathbb{F}_p) - \boldsymbol{G}(\mathbb{F}_p)^{\mathrm{reg}}|}{|I|} + \frac{|C_{\boldsymbol{T}}^{\mathrm{reg}} \cap gI|}{|I|} < C(n+1)p^{-1} + \frac{1}{|W(\boldsymbol{G},\boldsymbol{T})|}.$$

On the other hand,

$$\frac{|\mathcal{B} \cap gI|}{|I|} \geq \frac{|C_{\boldsymbol{T}}^{\mathrm{reg}} \cap gI|}{|I|} \geq \frac{1}{|W(\boldsymbol{G},\boldsymbol{T})|} - nCp^{-1}$$

The same argument as in Corollary 3.4 gives the desired bounds. □

Remark 3.7. Some versions of the above counting results for tori have previously appeared in literature (see [15, §3.4] and [23, §3]). For example, our Propositions 3.2(2) and 3.6(1) coincide respectively with Propositions 4.1 and 4.6 of [15]. However, the literature does not seem to contain finer results such as Corollary 3.4 and Proposition 3.6(2), which are necessary for our later arguments.

4. Proof of the main theorem

We now prove the main results. The first subsection sets up some notation. Then the second contains a short proof of the qualitative version (cf. Theorem 1.1). Finally, the third subsection is dedicated to the quantitative version (cf. Theorem 1.3).

4.1. *Preliminaries.*

4.1.1. *Setting.* Let A be a simple abelian variety of dimension g defined over a number field K such that $A_{/\bar{K}}$ does not have CM. Let $\mathfrak{N}$ be the conductor of A. Let M be a number field and

$$S_{A,M} = \{v \in \Sigma_K \mid v \nmid \mathfrak{N},\, F(A,v) \cong M\}.$$

For X a positive real number, define

$$S_{A,M}(X) = \{v \in S_{A,M} \mid \boldsymbol{N}_{K/\mathbb{Q}} v \leq X\}.$$

Our goal is to bound $S_{A,M}(X)$ from above, supposing Hypothesis 2.2.

Let L be a finite Galois extension of K such that the conclusion of Theorem 2.4(2) holds, and let N be the bound given therein. Increase N so that for all $p > N$, the following holds:

(1) The monodromy group $\boldsymbol{G}_p$ is reductive, cf. Theorem 2.4(1).

(2) The large image result in Corollary 2.7 holds for $A_{/L}$.

(3) The density estimate in Proposition 3.6(2) holds for all groups of rank at most $2g$.

In view of Corollary 2.7, this N depends only on A and K. Let $\mathcal{P}$ be the set of all primes greater than N which split completely in M.

4.1.2. *Preliminary lemmas.*

Lemma 4.1. *Pick $p \in \mathcal{P}$ and $v \in S_{A,M}$ which does not lie above p. Then the Frobenius $\bar{\rho}_p^{\mathrm{ss}}(\mathrm{Frob}_v) \in \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)$ lives in a Borel subgroup defined over $\mathbb{F}_p$.*

Proof. Let $F_v = \rho_p^{\mathrm{ss}}(\mathrm{Frob}_v) \in \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{Z}_p)$. By Faltings' theorem, this is a semisimple element in $\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{Q}_p)$. Our assumptions on p and v together imply that F_v lies in a split torus, and hence in a minimal parabolic subgroup of $\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{Q}_p)$, which is necessarily a Borel subgroup since $\boldsymbol{G}_p^{\mathrm{ss}}$ is unramified. The scheme of Borel subgroups over $\mathbb{Z}_p$ is proper [34, XXII, 5.8.3(i)], so we can extend this Borel subgroup to one defined over $\mathbb{Z}_p$. Its fibre over $\mathbb{F}_p$ is a Borel subgroup of $\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)$ containing the reduction of F_v. □

Lemma 4.2. *The image of the Galois representation*

$$\bar{\rho}_{\mathcal{P}}^{\mathrm{ss}} : G_K \to \prod_{p \in \mathcal{P}} \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)$$

is a union of cosets of $\prod_{p\in\mathcal{P}} I_p$, where I_p is defined as in Corollary 2.7.

Proof. By the choice of L, we have

$$\bar{\rho}_{\mathcal{P}}^{\mathrm{ss}}(G_L) = \prod_{p\in\mathcal{P}} \bar{\rho}_p^{\mathrm{ss}}(G_L) \subseteq \prod_{p\in\mathcal{P}} \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p).$$

Item (2) of the choice of $\mathcal{P}$ implies that $\bar{\rho}_p^{\mathrm{ss}}(G_L)$ contains the subgroup I_p for all $p \in \mathcal{P}$. Therefore,

$$\bar{\rho}_{\mathcal{P}}^{\mathrm{ss}}(G_K) \supseteq \bar{\rho}_{\mathcal{P}}^{\mathrm{ss}}(G_L) \supseteq \prod_{p\in\mathcal{P}} I_p.$$

In other words, $\bar{\rho}_{\mathcal{P}}^{\mathrm{ss}}(G_K)$ is a subgroup of $\prod_{p\in\mathcal{P}} \boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)$ which contains the subgroup $\prod_{p\in\mathcal{P}} I_p$. □

4.2. *Proof of Theorem 1.1.* Let d be a square-free product of primes in $\mathcal{P}$. Let $\mathcal{F}_d$ be the union of all conjugacy classes $\bar{\rho}_d^{\mathrm{ss}}(\mathrm{Frob}_v)$ for $v \in S_{A,M}$ not dividing d.

By Lemma 4.1, we have

$$\mathcal{F}_d \subseteq \Big(\prod_{p|d} \mathcal{B}_p\Big) \cap \bar{\rho}_d^{\mathrm{ss}}(G_K),$$

where $\mathcal{B}_p$ is the bounding set for $\boldsymbol{G}_p^{\mathrm{ss}}$ introduced in Definition 3.3. The Chebotarev density theorem implies that

$$\mathrm{ud}(S_{A,M}) \le \frac{|\mathcal{F}_d|}{|\bar{\rho}_d^{\mathrm{ss}}(G_K)|} \le \frac{\big|\big(\prod_{p|d}\mathcal{B}_p\big) \cap \bar{\rho}_d^{\mathrm{ss}}(G_K)\big|}{|\bar{\rho}_d^{\mathrm{ss}}(G_K)|}. \tag{4-1}$$

To estimate the right-hand side, we utilise the following decomposition, given by Lemma 4.2:

$$\bar{\rho}_d^{\mathrm{ss}}(G_K) = \bigsqcup_{i=1}^{s} \Big(g_i \prod_{p|d} I_p\Big) = \bigsqcup_{i=1}^{s} \Big(\prod_{p|d} g_{i,p} I_p\Big).$$

In view of Proposition 2.9, $\boldsymbol{G}_p^{\mathrm{ss}}$ is nonabelian[6]. Therefore, Proposition 3.6 can be applied, giving the upper bound

$$\left|\Big(\prod_{p\mid d}\mathcal{B}_p\Big)\cap\bar\rho_d^{\mathrm{ss}}(G_K)\right| = \sum_{i=1}^{s}\left|\prod_{p\mid d}(\mathcal{B}_p\cap g_{i,p}I_p)\right| \le s\prod_{p\mid d}\big(\tfrac{3}{4}|I_p|\big) = \big(\tfrac{3}{4}\big)^{\omega(d)}|\bar\rho_d^{\mathrm{ss}}(G_K)|, \qquad (4\text{-}2)$$

where $\omega(d)$ is the number of prime divisors of d.

Combining (4-1) and (4-2), we get

$$\mathrm{ud}(S_{A,M}) \le \big(\tfrac{3}{4}\big)^{\omega(d)}$$

for any d which is a square-free product of primes in $\mathcal{P}$. Since $\mathcal{P}$ is infinite, this implies that

$$\mathrm{ud}(S_{A,M}) = 0.$$

4.3. ***Proof of Theorem 1.3.*** We will now use the effective Chebotarev theorem of Lagarias and Odlyzko and the Selberg sieve to obtain a power saving upper bound for the number of places with a given Frobenius field M. This is possibly the approach suggested by Serre in [29, §8.2].

In this section we will write $f(X) = O(g(X))$ or $f(X) \ll g(X)$ to mean the existence of an absolute constant c such that $|f(X)| \le c|g(X)|$ for all X. If the constant is allowed to depend on some other object, we will indicate so in the subscripts. All constants are in fact effectively computable.

Remark 4.3. This section will be conditional on the GRH. Unconditionally, we expect that the standard methods lead to an upper bound of the form $O\big(X/(\log X)^{\alpha}\big)$ for some explicit $\alpha > 1$, but we have not worked out the details.

4.3.1. *Sieving setup.* In this subsection, we recast the problem into a form where the Selberg sieve can be directly applied. There are notational complications since we are only assuming potential independence of the Galois images at different primes p. No important idea is lost if one assumes $L = K$, i.e., the family of Galois representations over K has independent image.

Fix a positive real number X. Let $\Sigma_K(X)$ be the set of all finite places of K of norm at most X where A has good reduction. Recall that in Section 4.1, we have chosen a finite Galois extension L/K and an infinite set of primes $\mathcal{P}$ defined by a splitting condition.

Let $\mathcal{R}$ be the set of square-free products of primes in $\mathcal{P}$. Let $d \in \mathcal{R}\cup\{\infty\}$. In the infinite case, the condition "$p\mid d$" should be interpreted as "$p\in\mathcal{P}$". Let $G_d = \prod_{p\mid d}\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)$, so we have a Galois representation

$$\bar\rho_d^{\mathrm{ss}} : G_K \to G_d.$$

Denote its image by G_d' and the fixed field of its kernel by K_d, so we have an isomorphism

$$\mathrm{Gal}(K_d/K) \simeq G_d',$$

and K_d is the compositum of all K_p for $p\mid d$. Define H_d and L_d similarly using the restriction $\bar\rho_d^{\mathrm{ss}}|_{G_L}$.

[6]The generic fibre is abelian if and if the special fibre is, since $\boldsymbol{G}_p^{\mathrm{ss}}$ is reductive over $\mathbb{Z}_p$.

By the choice of L, we have $H_d = \prod_{p|d} H_p$. In these notations, we have

$$I_d := \prod_{p|d} I_p \subseteq H_d \subseteq G'_d \subseteq G_d,$$

where the first inclusion is due to Corollary 2.7.

Here is a summary of the above definitions:

$$\begin{tikzcd}[arrows=dash]
 & L_d \arrow[dl] \arrow[dr] & & H_d = \prod_{p|d} H_p \\
K_d \arrow[dr] \arrow[ddr, bend right, "G'_d"'] & & L \arrow[dl] \arrow[ddl, bend left, "\mathrm{Gal}(L/K)"] & \\
 & K_d \cap L \arrow[d] & & \\
 & K & &
\end{tikzcd}$$

Let $\mathcal{I} = G'_\infty / H_\infty$. From the above diagram, $\mathcal{I} = \mathrm{Gal}(K_\infty \cap L/K)$ is finite. Fix an integer $\mathtt{d}$ such that $K_{\mathtt{d}} \cap L = K_\infty \cap L$, so if $\mathtt{d}|d$, then $G'_d/H_d \simeq \mathcal{I}$. Write down a coset decomposition

$$G'_\infty = \bigsqcup_{i \in \mathcal{I}} g_i H_\infty = \bigsqcup_{i \in \mathcal{I}} \prod_{p \in \mathcal{P}} g_{i,p} H_p.$$

For any d as above, define $g_{i,d} = (g_{i,p})_{p|d} \in G'_d$, so we have a decomposition $G'_d = \bigcup_{i \in \mathcal{I}} g_{i,d} H_d$. By comparing indices, this is a disjoint union if $\mathtt{d}|d$.

For each $p \in \mathcal{P}$, let $\mathcal{B}_p \subseteq G_p$ be the bounding set as in the previous section (cf. Definition 3.3). Let $\mathcal{C}_p = G_p - \mathcal{B}_p$. For a general $d \in \mathcal{R} \cup \{\infty\}$, define $\mathcal{C}_d$ and $\mathcal{B}_d$ as a product of the corresponding sets for p dividing d. Let $\mathcal{C}'_d = \mathcal{C}_d \cap G'_d$ and $\mathcal{B}'_d = \mathcal{B}_d \cap G'_d$. For an index $i \in \mathcal{I}$, define

$$\mathcal{B}_d^{(i)} = \mathcal{B}_d \cap \prod_{p|d} g_{i,p} H_p.$$

So we have $\mathcal{B}_d^{(i)} = \prod_{p|d} \mathcal{B}_p^{(i)}$. Moreover, $\mathcal{B}'_d = \bigcup_{i \in \mathcal{I}} \mathcal{B}_d^{(i)}$. Define the subset $\mathcal{C}_d^{(i)}$ similarly. If p is a prime, then $g_{i,p} H_p = \mathcal{B}_p^{(i)} \sqcup \mathcal{C}_p^{(i)}$.

Remark 4.4. It could be the case that $\mathcal{B}_d^{(i)} = \mathcal{B}_d^{(i')}$ even though $i \neq i'$ in $\mathcal{I}$. However, this cannot happen if $\mathtt{d}|d$, in which case $\mathcal{B}'_d = \bigsqcup_{i \in \mathcal{I}} \mathcal{B}_d^{(i)}$. In other words, the purpose of the auxiliary $\mathtt{d}$ is to ensure that each element of $\mathcal{B}'_d$ belongs to a unique coset.

Let C be any union of conjugacy classes in G'_d. Define

$$E_d(C) = \left\{ v \in \Sigma_K(X) \,\middle|\, v \text{ does not lie above any } p|d,\ \bar{\rho}_d^{\mathrm{ss}}(\mathrm{Frob}_v) \in C \right\}.$$

The proof of Theorem 1.1 in the previous subsection then implies $S_{A,M}(X) \subseteq E_d(\mathcal{B}'_d)$. This form does not immediately yield a power saving upper bound. Instead, we re-write it in a different way.

Proposition 4.5. *We have an inclusion*

$$S_{A,M}(X) \subseteq \bigsqcup_{i\in\mathcal{I}} \Big(E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)}) - \bigcup_{p\in\mathcal{P}^{(\mathtt{d})}} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) \Big),$$

where $\mathcal{P}^{(\mathtt{d})}$ consists of the primes in $\mathcal{P}$ which do not divide $\mathtt{d}$.

Proof. Implicit in the statement is that $\mathcal{B}_d^{(i)}$ is a union of conjugacy classes in G'_d for all d. To see this, observe that $\mathcal{B}'_d$ is a union of conjugacy classes, and $\mathcal{B}_d^{(i)}$ is its intersection with a coset of H_d in G'_d. It remains to observe that H_d is a normal subgroup of G'_d, and the quotient is abelian, since it is a subquotient of the abelian group G_d/I_d (cf. Section 3.3). The same argument works for the second term.

The containment is apparent: let $v \in S_{A,M}(X)$ and $g = (g_p)_{p\in\mathcal{P}} = \overline{\rho}_\infty^{\mathrm{ss}}(\mathrm{Frob}_v) \in G'_\infty$. Then there exists a unique $i \in \mathcal{I}$ such that $g \in g_i H_\infty$. We know that $g_p \in \mathcal{B}'_p$ for all $p \in \mathcal{P}$, so $g_p \notin \mathcal{C}'_p$. It follows that $(g_p)_{p\mid\mathtt{d}} \in \mathcal{B}_{\mathtt{d}}^{(i)}$, but it is not in $\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}$ for any prime p. □

Each term in the disjoint union in Proposition 4.5 is a sieving problem. We now recall a form of the Selberg sieve.

Theorem 4.6. *Let $\mathscr{P}$ be a set of primes, and let $\mathscr{R}$ be the set of square-free products of primes in $\mathscr{P}$.*

Let $\mathcal{A}$ be a finite set. For each $p \in \mathscr{P}$, let $\mathcal{A}_p \subseteq \mathcal{A}$. For $d \in \mathscr{R}$, define $\mathcal{A}_d = \bigcap_{p\mid d} \mathcal{A}_p$. If $d = 1$, set $\mathcal{A}_1 = \mathcal{A}$. Suppose that, for all d, we can write

$$|\mathcal{A}_d| = \beta_d c \operatorname{Li}(X) + R_d,$$

where R_d is some real number, the function $d \mapsto \beta_d$ is multiplicative, and there exist constants $0 < \underline{\beta} < \overline{\beta} < 1$ such that $\underline{\beta} < \beta_p < \overline{\beta}$ for all $p \in \mathscr{P}$.

Under these assumptions, for any positive real numbers z and ε, we have

$$\Big|\mathcal{A} - \bigcup_{p\in\mathscr{P}} \mathcal{A}_p\Big| \ll_{\underline{\beta},\overline{\beta},\varepsilon} \frac{c\operatorname{Li}(X)}{\pi_{\mathscr{P}}(z)} + \left(\frac{z^{1+\varepsilon}}{\pi_{\mathscr{P}}(z)}\right)^2 \sum_{\substack{d_1,d_2\le z\\ d_1,d_2\in\mathcal{R}}} \frac{R_{[d_1,d_2]}}{d_1 d_2},$$

where $\pi_{\mathscr{P}}(z) = |\{p \in \mathscr{P} \mid p \le z\}|$.

Proof. This follows from a minor refinement to the proof of [8, Theorem 7.2.1], and we borrow their notations, with the obvious modification that their X is $c\operatorname{Li}(X)$ in our case. It is easy to compute that $f(d) = \beta_d^{-1}$, $f_1(d) = \prod_{p\mid d}(1-\beta_p)/\beta_p$, and

$$V(z) = \sum_{\substack{d\le z\\ d\mid P(z)}} \frac{\mu^2(d)}{f_1(d)} \ge \sum_{\substack{p\le z\\ p\in\mathscr{P}}} \frac{1}{f_1(p)} = \sum_{\substack{p\le z\\ p\in\mathscr{P}}} \frac{\beta_p}{1-\beta_p} \ge \frac{\underline{\beta}}{1-\underline{\beta}} \pi_{\mathcal{P}}(z).$$

The first term in the upper bound is the main term of the conclusion in *loc. cit.* For the second term, we start from the third displayed equation on page 122 and improve on the estimate $|\lambda_d| \le 1$. From the second to last displayed equation on page 122, we have

$$|V(z)\lambda_d| \le \prod_{p|d} \frac{1}{1-\beta_p} \cdot \sum_{\substack{t \le \frac{z}{d} \\ t \in \mathscr{R}}} \frac{\mu^2(t)}{f_1(t)} \tag{4-3}$$

Fix an $\varepsilon > 0$. Then, for all $t \in \mathscr{R}$,

$$\frac{1}{f_1(t)} = \prod_{p|t} \frac{\beta_p}{1-\beta_p} \le \left(\frac{\overline{\beta}}{1-\overline{\beta}}\right)^{\omega(t)} \ll_{\overline{\beta},\varepsilon} t^{\varepsilon}.$$

The same argument can be applied to the first term of (4-3). Therefore,

$$|V(z)\lambda_d| \ll_{\overline{\beta},\varepsilon} d^{\varepsilon} \sum_{\substack{t \le \frac{z}{d} \\ t \in \mathscr{R}}} t^{\varepsilon} \le d^{-1} z^{1+\varepsilon}.$$

Combined with the above lower bounds for $V(z)$, we get

$$|\lambda_d| \ll_{\underline{\beta},\overline{\beta},\varepsilon} d^{-1} \frac{z^{1+\varepsilon}}{\pi_{\mathscr{P}}(z)}.$$

This concludes the proof. $\square$

In our applications, we will take $\mathscr{P} = \mathcal{P}^{(\mathtt{d})}$, $\mathcal{A} = E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)})$, and $\mathcal{A}_p = E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)})$. Let $\mathcal{R}^{(\mathtt{d})}$ be the subset of $\mathcal{R}$ which are coprime to $\mathtt{d}$, then for all $d \in \mathcal{R}^{(\mathtt{d})}$, we have

$$\mathcal{A}_d = \bigcap_{p|d} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) = E_{\mathtt{d}d}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_d^{(i)}).$$

The sizes of $\mathcal{A}_d$ will be estimated using versions of the effective Chebotarev theorem, which we now recall.

4.3.2. *Background on effective Chebotarev.* For any number field k, let $n_k = [k : \mathbb{Q}]$ and Δ_k be its absolute discriminant. Given a finite extension l/k, let $\mathcal{D}(l/k)$ denote the set of rational primes that lie below the ramified finite places of l/k. Define

$$M(l/k) = |G| \Delta_k^{\frac{1}{n_k}} \prod_{p \in \mathcal{D}(l/k)} p.$$

Suppose that l/k is Galois with Galois group G. Let $C \subseteq G$ be a union of conjugacy classes. For any positive real number X, define

$$\pi(X, C, l/k) := \left|\{\mathfrak{p} \mid \mathfrak{p} \text{ is unramified in } l,\ \mathrm{Frob}_{\mathfrak{p}} \in C,\ \boldsymbol{N}_{k/\mathbb{Q}}\mathfrak{p} \le X\}\right|.$$

We are interested in the error term

$$R(X, C, l/k) := \pi(X, C, l/k) - \frac{|C|}{|G|}\mathrm{Li}(X).$$

The following result is equation (20_{R}) in [29, §2.4], which is an improvement to Lagarias and Odlyzko's original result [18].

Theorem 4.7. *Assume the GRH holds for the Artin L-functions of irreducible representations of l/k. Then*

$$|R(X, C, l/k)| \ll |C| n_k X^{\frac{1}{2}} \bigl(\log X + \log M(l/k)\bigr).$$

Assuming the truth of the Artin holomorphy conjecture, Murty, Murty and Saradha obtained the following improvement [24, Corollary 3.10].

Theorem 4.8. *Let H be a subgroup of G such that the GRH and AHC hold for the Artin L-functions of the irreducible characters of H. Suppose H meets every conjugacy class contained in C, then*

$$R(X, C, l/k) \ll |C|^{\frac{1}{2}} [G : H]^{\frac{1}{2}} n_k X^{\frac{1}{2}} \bigl(\log X + \log M(l/k)\bigr).$$

If $|C|$ is of roughly the same size as $|G|$, then this estimate saves a factor of $|H|^{\frac{1}{2}}$ compared to Theorem 4.7. We therefore want to take $|H|$ as large as possible.

Remark 4.9. This theorem is a corollary of a precise result [24, Proposition 3.9]. While it suffices for our purpose, we expect that a finer analysis of centralizers in finite reductive groups can give a better exponent.

4.3.3. *Applications.* We now apply the above results to the various terms appearing in Theorem 4.6. Fix $i \in \mathcal{I}$. For each $d \in \mathcal{R}$, define

$$\alpha_d^{(i)} = \frac{|\mathcal{B}_d^{(i)}|}{|H_d|}, \quad \beta_d^{(i)} = \frac{|\mathcal{C}_d^{(i)}|}{|H_d|}.$$

Then we have the relations

$$\alpha_d^{(i)} = \prod_{p \mid d} \alpha_p^{(i)}, \quad \beta_d^{(i)} = \prod_{p \mid d} \beta_p^{(i)}, \quad \alpha_p^{(i)} + \beta_p^{(i)} = 1$$

for all $p \in \mathcal{P}$ and $d \in \mathcal{R}$. If d_1, d_2 are coprime numbers in $\mathcal{R}$, then in the above notations,

$$\bigl|E_{d_1 d_2}(\mathcal{B}_{d_1}^{(i)} \times \mathcal{C}_{d_2}^{(i)})\bigr| = \frac{1}{[G'_{d_1 d_2} : H_{d_1 d_2}]} \alpha_{d_1}^{(i)} \beta_{d_2}^{(i)} \mathrm{Li}(X) + R(X, \mathcal{B}_{d_1}^{(i)} \times \mathcal{C}_{d_2}^{(i)}, K_{d_1 d_2}/K). \tag{4-4}$$

Using Theorem 4.7 alone, this error term is of the order $|G_{d_1 d_2}| X^{\frac{1}{2}+\varepsilon}$.

In this subsection, we show that the error term for $\bigl|E_d(\mathcal{B}_d^{(i)})\bigr|$ can be improved using Theorem 4.8. This yields the required estimate for (4-4) using the principle of inclusion-exclusion. The main result is Proposition 4.13.

For each prime p, fix a Borel subgroup $\boldsymbol{B}_p \subseteq \boldsymbol{G}_p^{\mathrm{ss}}$, and let

$$B_p = \boldsymbol{B}_p(\mathbb{F}_p).$$

Define $B_d = \prod_{p \mid d} B_p$ and $B'_d = B_d \cap G'_d$. This is a subgroup of G'_d. The next two lemmas verify that it satisfies the conditions in Theorem 4.8.

Lemma 4.10. *All conjugacy classes in $\mathcal{B}_d^{(i)}$ intersect B'_d.*

Proof. By the Borel conjugacy theorem [3, théorme 4.13(b)], this result holds in G_d. To conclude, we will show that the conjugating element can be chosen to lie in I_d. It is enough to do this for $d = p$, a prime.

Suppose $b \in \mathcal{B}'_p$ and $gbg^{-1} \in B'_p$, where $g \in G_p$. Let $\boldsymbol{T}_p \subseteq \boldsymbol{B}_p$ be a maximal torus. Then the commutative diagram appearing $Section$ 3.3 shows that $\boldsymbol{T}_p(\mathbb{F}_p)$ intersects all cosets of I_p. Choose $t \in \boldsymbol{T}_p(\mathbb{F}_p)$ so that t and g are in the same coset, then $t^{-1}g$ is an element of I_p conjugating b to B'_p. □

Lemma 4.11. *All irreducible representations of B'_d are induced from abelian characters. Consequently, the AHC holds for $K_d/K_d^{B'_d}$, and the GRH holds provided it holds for all Hecke L-functions.*

Proof. We will show that if $p \in \mathcal{P}$, then B_p is supersolvable. The group B'_d is a subquotient of their product, so it is also supersolvable. The first claim of the lemma is an elementary group theory fact [27, Theorem 12.8.5]. The second part of the lemma is a standard consequence of class field theory.

Let $p \in \mathcal{P}$. We first observe that $\boldsymbol{G}_p^{\mathrm{ss}}$ is actually split over $\mathbb{F}_p$. Indeed, by a result of Serre (cf. [5, Corollary 3.8]), the element $\rho_p(\mathrm{Frob}_v) \in \boldsymbol{G}_p(\mathbb{Q}_p)$ generates a maximal torus for a density one set of places v. On the other hand, p splits completely in M, so this element lies in a split torus. Therefore, we have constructed a split maximal torus of $\boldsymbol{G}_p$ over $\mathbb{Q}_p$, which also implies $\boldsymbol{G}_p^{\mathrm{ss}}$ is split.

Fix a faithful representation $\boldsymbol{G}_p^{\mathrm{ss}} \hookrightarrow \mathrm{GL}_N$ defined over $\mathbb{F}_p$, we see that B_p is isomorphic to a subgroup of R_N, the group of upper triangular matrices in $\mathrm{GL}_N(\mathbb{F}_p)$. Write the derived series of R_N as

$$R_N \triangleright S_N \triangleright S_{N-1} \triangleright \cdots \triangleright S_0 = \{1\}.$$

Then a standard calculation shows that

$$S_k = \{g \in R_N \mid g_{ii} = 1,\ g_{ij} = 0 \text{ if } 0 < j - i \le N - k\}.$$

For $v = 1, \cdots, k-1$, let

$$S_{k,v} = \{g \in S_k \mid g_{i,i+N-k} = 0 \text{ if } i \le v\}.$$

Then we have a normal series $S_k \triangleright S_{k,1} \triangleright \cdots \triangleright S_{k,k-1} = S_{k-1}$ where each quotient is isomorphic to $\mathbb{F}_p$. Conjugation by a diagonal matrix preserves $S_{k,v}$, so all of the groups are normal in R_N. This normal series shows that R_N is a supersolvable group, which completes the proof. □

Proposition 4.12. *For each prime p, let*

$$N_p = |\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)| \cdot |\boldsymbol{B}_p(\mathbb{F}_p)|^{-\frac{1}{2}}. \tag{4-5}$$

Assume the GRH for Hecke L-functions. Then for all $d \in \mathcal{R}$, we have

$$\left|E_d(\mathcal{B}_d^{(i)})\right| = \frac{1}{[G'_d : H_d]}\alpha_d^{(i)}\mathrm{Li}(X) + O_{A,K}\Big(\prod_{p\mid d} N_p \cdot X^{\frac{1}{2}}(\log X + \log d)\Big).$$

Proof. Applying Theorem 4.8 to the extension $l/k = K_d/K$ with the subgroup $H = B'_d$ gives the error term

$$
\begin{aligned}
|R(X, \mathcal{B}_d^{(i)}, K_d/K)| &\ll (|\mathcal{B}_d^{(i)}| \cdot [G'_d : B'_d])^{\frac{1}{2}} n_K X^{\frac{1}{2}} \big(\log X + \log M(K_d/K)\big) \\
&\leq (|G_d| \cdot [G_d : B_d])^{\frac{1}{2}} n_K X^{\frac{1}{2}} \big(\log X + \log M(K_d/K)\big) \\
&= \prod_{p \mid d} N_p \cdot n_K X^{\frac{1}{2}} \big(\log X + \log M(K_d/K)\big).
\end{aligned}
$$

For the terms beside N_p, recall that K_d/K is unramified away from the places of K dividing $d\Delta_A$, so

$$
\begin{aligned}
\log M(K_d/K) &= \log|G'_d| + \frac{\log(\Delta_K)}{n_K} + \sum_{p \mid d} \log p + \sum_{p \mid \Delta_A} \log p \\
&\leq \sum_{p \mid d} \log|\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)| + \log d + O_{A,K}(1) \\
&\ll_{A,K} \log d,
\end{aligned}
$$

where in the final estimate we used the trivial bound $|\boldsymbol{G}_p^{\mathrm{ss}}(\mathbb{F}_p)| \leq |\mathrm{GSp}_{2g}(\mathbb{F}_p)|$. □

We now apply this to calculate the intersection terms in the Selberg sieve.

Proposition 4.13. *Assume the GRH for Hecke L-functions. If $d \in \mathcal{R}$ is coprime to $\mathtt{d}$, then*

$$
\Big|\bigcap_{p \mid d} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)})\Big| = \frac{1}{|\mathcal{I}|} \alpha_{\mathtt{d}}^{(i)} \beta_d^{(i)} \mathrm{Li}(X) + O_{A,K}\Big(2^{\omega(d)} \prod_{p \mid \mathtt{d}d} N_p \cdot X^{\frac{1}{2}} \big(\log X + \log(\mathtt{d}d)\big)\Big).
$$

Proof. We have the equality

$$
\bigcap_{p \mid d} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) = E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)}) - \bigcup_{p \mid d} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}\mathrm{p}}^{(i)}).
$$

To see this, observe that $\mathcal{B}_p^{(i)} \sqcup \mathcal{C}_p^{(i)}$ is the coset of $g_{i,p} H_p$, and the condition defining the set $E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)})$ already forces the Frobenius element to be in this coset (cf. Remark 4.4).

The principle of inclusion-exclusion now gives

$$
\Big|\bigcap_{p \mid d} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)})\Big| = \sum_{d' \mid d} \mu(d') \big|E_{\mathtt{d}d'}(\mathcal{B}_{\mathtt{d}d'}^{(i)})\big|.
$$

Apply the previous proposition to the right-hand side. There are $2^{\omega(d)}$ terms in the above sum, so the error term has the required form by the triangle inequality. In the main term, the coefficient in front of $\mathrm{Li}(X)$ is given by

$$
\sum_{d' \mid d} \mu(d') \cdot \frac{1}{|\mathcal{I}|} \alpha_{\mathtt{d}d'}^{(i)} = \frac{1}{|\mathcal{I}|} \alpha_{\mathtt{d}}^{(i)} \prod_{p \mid d} (1 - \alpha_p^{(i)}) = \frac{1}{|\mathcal{I}|} \alpha_{\mathtt{d}}^{(i)} \beta_d^{(i)}
$$

where we used the observation that $[G'_d : H_d] = [G'_\infty : H_\infty] = |\mathcal{I}|$ for $\mathtt{d} \mid d$. □

4.3.4. *Point counting.* We will now estimate N_p, defined in (4-5).

Proposition 4.14. *Let $\boldsymbol{G}$ be the Mumford–Tate group of $A_{/K}$ and $\boldsymbol{G}^{\mathrm{ss}}$ its semisimple quotient. Put*

$$\gamma = \tfrac{1}{4}(3\dim \boldsymbol{G}^{\mathrm{ss}} - \operatorname{rank} \boldsymbol{G}^{\mathrm{ss}}).$$

There exists a constant C depending only on g such that

$$N_p \le C p^{\gamma}$$

for all primes p.

The key feature of this proposition is that neither C nor γ depend on the prime p, even though the monodromy groups $\boldsymbol{G}_p$ are not assumed to interpolate into a group over $\mathbb{Q}$. The proposition follows by combining Lemma 4.15 and Corollary 4.17 below.

Lemma 4.15. *There exists a constant C depending only on g such that for all prime p,*

$$N_p \le C p^{\dim \boldsymbol{G}_p^{\mathrm{ss}} - \frac{1}{2}\dim \boldsymbol{B}_p}.$$

Proof. In view of Lemma 3.5 we may replace $\boldsymbol{G}_p^{\mathrm{ss}}$ with $\boldsymbol{G}_p^{\mathrm{sc}}$ in the definition of N_p. The base change $(\boldsymbol{G}_p^{\mathrm{sc}})_{/\overline{\mathbb{F}}_p}$ is a simply connected semisimple group, so it is a product of simply connected split simple groups. These are classified by Dynkin diagrams. Since the rank is bounded (say by $2g$), there are only a finite number of possibilities, and each of them is obtained from a Chevalley group over $\mathbb{Z}$ by base change.

Now, $\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{F}_p)$ is the set of fixed points of the Frobenius operator F acting on $\boldsymbol{G}_p^{\mathrm{sc}}(\overline{\mathbb{F}}_p)$. The Grothendieck–Lefschetz trace formula gives

$$|\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{F}_p)| = \sum_{i=0}^{2\dim \boldsymbol{G}_p^{\mathrm{sc}}} (-1)^i \operatorname{Tr}\bigl(F, \mathrm{H}_c^i\bigl((\boldsymbol{G}_p^{\mathrm{sc}})_{/\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell\bigr)\bigr)$$

The top degree term is $p^{\dim \boldsymbol{G}_p^{\mathrm{sc}}}$ since $\boldsymbol{G}_p^{\mathrm{sc}}$ is geometrically connected. Deligne's Weil bound gives an upper bound for the Frobenius eigenvalues on each of the remaining terms. It implies

$$\bigl|p^{-\dim \boldsymbol{G}_p^{\mathrm{sc}}}|\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{F}_p)| - 1\bigr| \le b p^{-\frac{1}{2}},$$

where b is the sum of the dimensions of all lower degree cohomology groups. The argument in the previous paragraph gives a finite list of possibilities for b independently of p. Therefore, we get positive constants $c_1(g), c_2(g)$ depending only on g such that

$$c_1 p^{\dim \boldsymbol{G}_p^{\mathrm{sc}}} \le |\boldsymbol{G}_p^{\mathrm{sc}}(\mathbb{F}_p)| \le c_2 p^{\dim \boldsymbol{G}_p^{\mathrm{sc}}}$$

for all primes p.

By applying the same argument to the term $|\boldsymbol{B}_p(\mathbb{F}_p)|$, the assertion follows. □

Let $\boldsymbol{G}$ be the Mumford–Tate group of A. For our purpose, it suffices to know that this is a reductive group over $\mathbb{Q}$ which "contains" all p-adic monodromy groups, made precise by the following theorem of Deligne [10].

Theorem 4.16. *For all primes p, we have*[7] $\boldsymbol{G}_{p/\mathbb{Q}_p} \subseteq \boldsymbol{G} \times_{\mathbb{Q}} \mathbb{Q}_p$.

Corollary 4.17. *Let $\boldsymbol{G}^{\mathrm{ss}}$ denote the semisimple quotient of $\boldsymbol{G}$, then for all primes p,*

$$\dim \boldsymbol{G}_p^{\mathrm{ss}} - \tfrac{1}{2}\dim \boldsymbol{B}_p \leq \tfrac{1}{4}(3\dim \boldsymbol{G}^{\mathrm{ss}} - \operatorname{rank} \boldsymbol{G}^{\mathrm{ss}}).$$

Proof. Fix a prime p. For simplicity, we base change the pertinent groups to $\overline{\mathbb{Q}}_p$. This does not change its dimension or rank.

Recall that $\boldsymbol{B}_p$ is a Borel subgroup of $\boldsymbol{G}_p^{\mathrm{ss}}$. Let $\boldsymbol{N}_p$ be its unipotent radical. We have the relations

$$\dim \boldsymbol{G}_p^{\mathrm{ss}} = 2\dim \boldsymbol{B}_p - \operatorname{rank} \boldsymbol{G}_p^{\mathrm{ss}} = \dim \boldsymbol{B}_p + \dim \boldsymbol{N}_p.$$

Rearranging this, we get

$$\dim \boldsymbol{G}_p^{\mathrm{ss}} - \tfrac{1}{2}\dim \boldsymbol{B}_p = \tfrac{1}{4}(3\dim \boldsymbol{G}_p^{\mathrm{ss}} - \operatorname{rank} \boldsymbol{G}_p^{\mathrm{ss}}) = \tfrac{1}{2}\dim \boldsymbol{G}_p^{\mathrm{ss}} + \tfrac{1}{2}\dim \boldsymbol{N}_p.$$

It remains to show that $\dim \boldsymbol{G}_p^{\mathrm{ss}} \leq \dim \boldsymbol{G}^{\mathrm{ss}}$ and $\dim \boldsymbol{N}_p \leq \dim \boldsymbol{N}$.

The previous theorem gives an embedding $\boldsymbol{G}_p \subseteq \boldsymbol{G}$ of reductive groups, and hence the inclusion of the derived subgroups $\boldsymbol{G}_p^{\mathrm{der}} \subseteq \boldsymbol{G}^{\mathrm{der}}$. The derived subgroup is isogenous to the semisimple quotient, so they have the same dimension and rank. The first inequality follows. For the second part, the image of $\boldsymbol{N}_p$ is again a unipotent subgroup, so it lies in a conjugate of $\boldsymbol{N}$. □

4.3.5. *Conclusion of the proof.* We now assemble the pieces to conclude the proof.

Lemma 4.18. *If v is a finite place of K not dividing $\mathfrak{N}$, then*

$$\log|\Delta_{F(A,v)}| \ll_g \log \boldsymbol{N}_{K/\mathbb{Q}} v.$$

Proof. Let $P(T) \in \mathbb{Z}[T]$ be the characteristic polynomial of $\rho_p(\mathrm{Frob}_v)$. Let $\alpha_1, \cdots, \alpha_{2g}$ be the roots of $P(T)$, then $F(A,v)$ is the compositum of the fields $\mathbb{Q}(\alpha_i)$. Therefore,

$$\frac{\log|\Delta_{F(A,v)}|}{[F(A,v):\mathbb{Q}]} \leq \sum_{i=1}^{2g} \frac{\log|\Delta_{\mathbb{Q}(\alpha_i)}|}{[\mathbb{Q}(\alpha_i):\mathbb{Q}]}.$$

Let $q = \boldsymbol{N}_{K/\mathbb{Q}} v$. Then all the roots of $P(T)$ have complex absolute value $q^{\frac{1}{2}}$, so

$$\log|\Delta_{\mathbb{Q}(\alpha)}| \leq \log|\operatorname{disc} P(T)| = \sum_{i \neq j} \log|\alpha_i - \alpha_j| \leq ((2g)^2 - 2g)\log(2q^{\frac{1}{2}}).$$

Combining the two inequalities above gives the claim. □

Theorem 4.19. *Assume the GRH. For any $\varepsilon > 0$, we have*

$$|S_{A,M}(X)| \ll_{A,K,\varepsilon} X^{1-\frac{1}{4\gamma+2}+\varepsilon},$$

where $\gamma = \frac{1}{4}(3\dim \boldsymbol{G}^{\mathrm{ss}} - \operatorname{rank} \boldsymbol{G}^{\mathrm{ss}})$ from Proposition 4.14.

[7] Recall that we are assuming all monodromy groups are connected.

Proof. Fix $\varepsilon > 0$; then Proposition 4.5 gives an equality

$$|S_{A,M}(X)| = \sum_{i\in\mathcal{I}} \Big| E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)}) - \bigcup_{p\in\mathcal{P}^{(\mathtt{d})}} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) \Big|.$$

Fix an index $i \in \mathcal{I}$. We will apply Theorem 4.6 to the corresponding term in the above sum. Take $\mathcal{A} = E_\infty(g_i H_\infty)$, sieving primes $\mathcal{P}^{(\mathtt{d})}$, and excluded subsets $\mathcal{A}_p = E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)})$. By combining Propositions 4.13 and 4.14, we get the estimate

$$\begin{aligned} |\mathcal{A}_d| = \Big| \bigcap_{p\mid d} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) \Big| &= \frac{1}{|\mathcal{I}|}\alpha_{\mathtt{d}}^{(i)} \beta_d^{(i)} \mathrm{Li}(X) + O_{A,K}\big((2C)^{\omega(d)} d^\gamma X^{\frac12}(\log X + \log d)\big) \\ &= \frac{1}{|\mathcal{I}|}\alpha_{\mathtt{d}}^{(i)} \beta_d^{(i)} \mathrm{Li}(X) + O_{A,K,\varepsilon}\big(d^{\gamma+\varepsilon} X^{\frac12}(\log X + \log d)\big). \end{aligned}$$

By definition, the function $d \mapsto \beta_d^{(i)}$ is multiplicative. Let $p \in \mathcal{P}^{(\mathtt{d})}$; then

$$\beta_p^{(i)} = \frac{|\mathcal{C}_p \cap g_{i,p} H_p|}{|H_p|} = 1 - \frac{|\mathcal{B}_p \cap g_{i,p} H_p|}{|H_p|}.$$

Recall that H_p contains I_p, so each $g_{i,p} H_p$ is a union of cosets of I_p. Over each such coset, Proposition 3.6 gives uniform upper and lower bounds for its intersection with $\mathcal{B}_p$. It follows that $\beta_p^{(i)}$ is uniformly bounded away from 0 and 1, and the bounds only depend on the dimension g.

Having verified all of the hypotheses, Theorem 4.6 gives the estimate

$$\begin{aligned} &\Big| E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)}) - \bigcup_{p\in\mathcal{P}^{(\mathtt{d})}} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) \Big| \\ &\qquad \ll_{A,K,\varepsilon} \frac{1}{|\mathcal{I}|}\alpha_{\mathtt{d}}^{(i)} \cdot \frac{\mathrm{Li}(X)}{\pi_{\mathcal{P}^{(\mathtt{d})}}(z)} + \left(\frac{z^{1+\varepsilon}}{\pi_{\mathcal{P}^{(\mathtt{d})}}(z)}\right)^2 X^{\frac12} \sum_{\substack{d_1,d_2\le z \\ d_1,d_2\in\mathcal{R}^{(\mathtt{d})}}} \frac{[d_1,d_2]^{\gamma+\varepsilon}}{d_1 d_2}(\log X + \log[d_1,d_2]). \end{aligned}$$

In the sum for the second term, use the trivial bound $[d_1,d_2] \le d_1 d_2$ and extend the sum to all pairs $d_1, d_2 \le z$. In particular, $\log[d_1,d_2] \le \log(z^2)$. Therefore,

$$\begin{aligned} \sum_{\substack{d_1,d_2\le z \\ d_1,d_2\in\mathcal{R}^{(\mathtt{d})}}} \frac{[d_1,d_2]^{\gamma+\varepsilon}}{d_1 d_2}(\log X + \log[d_1,d_2]) &\le \sum_{d_1,d_2\le z} (d_1 d_2)^{\gamma-1+\varepsilon}(\log X + 2\log z) \\ &= (\log X + 2\log z)\Big(\sum_{d\le z} d^{\gamma-1+\varepsilon}\Big)^2 \\ &\ll_\gamma z^{2\gamma+2\varepsilon}(\log X + \log z), \end{aligned}$$

which gives

$$\Big| E_{\mathtt{d}}(\mathcal{B}_{\mathtt{d}}^{(i)}) - \bigcup_{p\in\mathcal{P}^{(\mathtt{d})}} E_{\mathtt{d}p}(\mathcal{B}_{\mathtt{d}}^{(i)} \times \mathcal{C}_p^{(i)}) \Big| \ll_{A,K,\varepsilon} \frac{X}{\pi_{\mathcal{P}^{(\mathtt{d})}}(z)} + \left(\frac{z^{1+\varepsilon}}{\pi_{\mathcal{P}^{(\mathtt{d})}}(z)}\right)^2 X^{\frac12} z^{2\gamma+2\varepsilon}\log(Xz). \tag{4-6}$$

It remains to bound $\pi_{\mathcal{P}^{(\mathrm{d})}}(z)$ from below. By definition,

$$\pi_{\mathcal{P}^{(\mathrm{d})}}(z)=\pi(z,\{1\},M/\mathbb{Q})-O_{A,K}(1)$$

By the effective Chebotarev theorem (Theorem 4.7, also cf. [29, equation (14_{R})]), we have

$$\pi(z,\{1\},M/\mathbb{Q})=\frac{1}{[M:\mathbb{Q}]}\mathrm{Li}(z)+O_g\big(z^{\frac{1}{2}}(\log z+\log|\Delta_M|)\big).$$

We may assume $\log|\Delta_M|\ll_g \log X$, since otherwise $S_{A,M}(X)=\varnothing$ by Lemma 4.18. Now choose $z=X^{\beta}$, where $\beta=\frac{1}{4\gamma+2}$. Then

$$\pi_{\mathcal{P}^{(\mathrm{d})}}(z)=\frac{1}{[M:\mathbb{Q}]}\mathrm{Li}(X^{\beta})+O_{g,\beta}(X^{\frac{\beta}{2}}\log X)-O_{A,K}(1)\gg_{A,K,\varepsilon} X^{\beta-\varepsilon}.$$

Substituting this in (4-6) and summing over all $i\in\mathcal{I}$ gives the desired result. □

Remark 4.20. We only needed the error term in the effective Chebotarev theorem on average. The situation is analogous to the Bombieri–Vinogradov theorem, except that it deals with the family of abelian extensions $\{\mathbb{Q}(\zeta_q)/\mathbb{Q}\}$, and we have a family of nonabelian Lie-extensions $\{K_d/K\}$. There are some works when the Galois group is fixed (cf. [25]), but we are not aware of any work in our setting.

Acknowledgements

We thank Brian Conrad, Chun-Yin Hui, Chandrashekhar Khare, Vijay Patankar, Will Sawin and Victor Wang for helpful communications. We are grateful to the referee for valuable comments and suggestions.

References

[1] S. Bloom, "The square sieve and a Lang–Trotter question for generic abelian varieties", *J. Number Theory* **191** (2018), 119–157. MR

[2] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, 1991. MR

[3] A. Borel and J. Tits, "Groupes réductifs", *Inst. Hautes Études Sci. Publ. Math.* **27** (1965), 55–150. MR

[4] R. W. Carter, *Finite groups of Lie type*: *Conjugacy classes and complex characters*, Wiley, New York, 1985. MR

[5] W. C. Chi, "l-adic and λ-adic representations associated to abelian varieties defined over number fields", *Amer. J. Math.* **114**:2 (1992), 315–353. MR

[6] A. C. Cojocaru and C. David, "Frobenius fields for Drinfeld modules of rank 2", *Compos. Math.* **144**:4 (2008), 827–848. MR

[7] A. C. Cojocaru and C. David, "Frobenius fields for elliptic curves", *Amer. J. Math.* **130**:6 (2008), 1535–1560. MR

[8] A. C. Cojocaru and M. R. Murty, *An introduction to sieve methods and their applications*, London Math. Soc. Student Texts **66**, Cambridge University Press, 2006. MR

[9] A. C. Cojocaru, E. Fouvry, and M. R. Murty, "The square sieve and the Lang–Trotter conjecture", *Canad. J. Math.* **57**:6 (2005), 1155–1177. MR

[10] P. Deligne, "Hodge cycles on abelian varieties", pp. 9–100 in *Hodge cycles*, *motives*, *and Shimura varieties*, Lecture Notes in Math. **900**, Springer, 1982.

[11] G. Henniart, "Représentations l-adiques abéliennes", pp. 107–126 in *Seminar on Number Theory* (Paris, 1980/1981), Progr. Math. **22**, Birkhäuser, Boston, 1982. MR

[12] H. Hida, *Elliptic curves and arithmetic invariants*, Springer, 2013. MR

[13] C. Y. Hui and M. Larsen, "Maximality of Galois actions for abelian and hyper-Kähler varieties", *Duke Math. J.* **169**:6 (2020), 1163–1207. MR

[14] L. Illusie, "Independence of families of l-adic representations and uniform constructibility (after J.-P. Serre)", prenotes for talks at the University of Tokyo, 2010, available at https://tinyurl.com/Illusie-Independence.

[15] J. Jouve, K. Kowalski, and D. Zywina, "Splitting fields of characteristic polynomials of random elements in arithmetic groups", *Israel J. Math.* **193**:1 (2013), 263–307.

[16] C. Khare, "F-split Galois representations are potentially abelian", *Proc. Amer. Math. Soc.* **131**:10 (2003), 3021–3023. MR

[17] M. Kulkarni, V. M. Patankar, and C. S. Rajan, "Locally potentially equivalent two dimensional Galois representations and Frobenius fields of elliptic curves", *J. Number Theory* **164** (2016), 87–102. MR

[18] J. C. Lagarias and A. M. Odlyzko, "Effective versions of the Chebotarev density theorem", pp. 409–464 in *Algebraic number fields: L-functions and Galois properties* (Durham, 1975), Academic Press, 1977. MR

[19] S. Lang, "Algebraic groups over finite fields", *Amer. J. Math.* **78** (1956), 555–563. MR

[20] S. Lang and H. Trotter, *Frobenius distributions in* GL_2*-extensions*, Lecture Notes in Mathematics **504**, Springer, 1976. MR

[21] M. Larsen, "Maximality of Galois actions for compatible systems", *Duke Math. J.* **80**:3 (1995), 601–630. MR

[22] M. Larsen and R. Pink, "Abelian varieties, l-adic representations, and l-independence", *Math. Ann.* **302**:3 (1995), 561–579. MR

[23] A. Lubotzky and L. Rosenzweig, "The Galois group of random elements of linear groups", *Amer. J. Math.* **136**:5 (2014), 1347–1383. MR

[24] M. R. Murty, V. K. Murty, and N. Saradha, "Modular forms and the Chebotarev density theorem", *Amer. J. Math.* **110**:2 (1988), 253–281. MR

[25] L. B. Pierce, C. L. Turnage-Butterbaugh, and M. M. Wood, "An effective Chebotarev density theorem for families of number fields, with an application to ℓ-torsion in class groups", *Invent. Math.* **219**:2 (2020), 701–778. MR

[26] R. Pink, "l-adic algebraic monodromy groups, cocharacters, and the Mumford–Tate conjecture", *J. Reine Angew. Math.* **495** (1998), 187–237. MR

[27] W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, NJ, 1964. MR

[28] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, W. A. Benjamin, 1968. MR

[29] J.-P. Serre, "Quelques applications du théorème de densité de Chebotarev", *Publications Mathématiques de l'IHÉS* **54** (1981), 123–201. MR

[30] J.-P. Serre, "Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981", item 133, pp. 1–20 in *Œuvres = Collected papers*, vol. IV, 1985–1998, Springer, 2000. MR

[31] J.-P. Serre, "Lettre à Ken Ribet du 7/3/1986", item 138, pp. 56–65 in *Œuvres = Collected papers*, vol. IV, 1985–1998, Springer, 2000. MR

[32] J.-P. Serre, *Galois cohomology*, Springer, 2002. MR

[33] J.-P. Serre, "Un critère d'indépendance pour une famille de représentations ℓ-adiques", *Comment. Math. Helv.* **88**:3 (2013), 541–554. MR

[34] M. Demazure and A. Grothendieck, *Schémas en groupes, II: Groupes de type multiplicatif, et structure des schémas en groupes généraux* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **152**, Springer, 1970. MR

[35] G. Shimura, *Abelian varieties with complex multiplication and modular functions*, Princeton Mathematical Series **46**, Princeton University Press, 1998. MR

[36] J.-P. Wintenberger, "Démonstration d'une conjecture de Lang dans des cas particuliers", *J. Reine Angew. Math.* **553** (2002), 1–16. MR

[37] Y. Zarhin, "Two-dimensional families of hyperelliptic Jacobians with big monodromy", *Trans. Amer. Math. Soc.* **368**:5 (2016), 3651–3672. MR

[38] D. Zywina, “Bounds for the Lang–Trotter conjectures”, pp. 235–256 in *SCHOLAR—a scientific celebration highlighting open lines of arithmetic research*, Contemp. Math. **655**, Amer. Math. Soc., 2015. MR

Communicated by Philippe Michel
Received 2024-06-22 Revised 2025-01-14 Accepted 2025-03-03

ashayburungale@gmail.com *Department of Mathematics, University of Texas at Austin, Austin, TX, United States*

hharuzo@gmail.com *Department of Mathematics, University of California, Los Angeles, Los Angeles, CA, United States*

shilinl@umich.edu *Department of Mathematics, University of Michigan, Ann Arbor, MI, United States*

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.603

Galois groups of reciprocal polynomials and the van der Waerden–Bhargava theorem

Theresa C. Anderson, Adam Bertelli and Evan M. O'Dorney

We study the Galois groups G_f of degree $2n$ reciprocal (a.k.a. palindromic) polynomials f of height at most H, finding that G_f falls short of the maximal possible group $S_2 \wr S_n$ for a proportion of all f bounded above and below by constant multiples of $H^{-1}\log H$, whether or not f is required to be monic. This answers a 1998 question of Davis, Duke and Sun and extends Bhargava's 2023 resolution of van der Waerden's 1936 conjecture on the corresponding question for general polynomials. Unlike in that setting, the dominant contribution comes not from reducible polynomials but from those f for which $(-1)^n f(1) f(-1)$ is a square, causing G_f to lie in an index-2 subgroup.

1. Introduction

For a positive integer n, let $E_n(H)$ denote the number of degree n monic separable polynomials $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients $a_i \in [-H, H]$ whose Galois group is *not* S_n. Hilbert's irreducibility theorem implies that Galois groups not equal to S_n occur 0% of the time, in other words,

$$E_n(H) = o(H^n).$$

In 1936, van der Waerden [23] gave a quantitative upper bound and conjectured that the true order of growth is

$$E_n(H) \asymp H^{n-1}, \tag{1}$$

the lower bound coming from counting multiples of x (or any fixed monic linear polynomial). For the next several decades, progress continued, with many authors making improvements on the bound, including Knobloch (1955) [17], Gallagher (1972) [12], Chow and Dietmann (2020) [10] who proved it for $n \le 4$, and a group including the first author (2023) [1]. The conjecture (1) was finally resolved by Bhargava (2023) ([8]; an abridged version of the paper has appeared in print [7]). Bhargava's method harnesses sophisticated known results (classification of subgroups of S_n, distribution of discriminants of number fields) in combination with innovative recent methods, including Fourier equidistribution and the use of the *double discriminant* $\operatorname{disc}_{a_n} \operatorname{disc}_x f$. For transitive subgroups such as A_n, the frequency of appearance is expected to be much lower [5].

MSC2020: 11R32, 11R45, 11C08, 11N35, 20E22.
Keywords: arithmetic statistics, van der Waerden's conjecture, reciprocal polynomial, geometric sieve.

In this paper, we study the analogous question for the subspace of reciprocal polynomials, which was previously posed by Davis, Duke and Sun [11]. A polynomial f is called *reciprocal* if

$$f(1/x) = \frac{1}{x^{\deg f}} f(x),$$

in other words, if the coefficient list of f is palindromic. We focus on the case where f is of even degree $2n$ (in the odd-degree case, f must be reducible, equal to $x+1$ times an even-degree reciprocal polynomial). The roots of f come in reciprocal pairs $\{\alpha, 1/\alpha\}$; the Galois group must preserve the partition into pairs and thus is a subgroup of the wreath product $S_2 \wr S_n$, the subgroup of S_{2n} of order $2^n n!$ preserving this partition. In this paper, we provide a precise estimate for how often the group is strictly smaller:

Theorem 1.1. *Let $\mathcal{E}_n^{\mathrm{monic}}(H)$ be the number of separable monic reciprocal polynomials f of degree $2n$ with coefficients in $[-H, H]$ whose Galois group is not $S_2 \wr S_n$. Then for each $n \geq 2$,*

$$\mathcal{E}_n^{\mathrm{monic}}(H) \asymp H^{n-1} \log H.$$

Remark 1.2. Here and throughout the paper, if $f(H)$, $g(H)$ are real-valued functions of a sufficiently large real number H, then the usual notations

$$f(H) \ll g(H), \quad g(H) \gg f(H), \quad f(H) = O(g(H))$$

mean that $|f(H)| < c \cdot |g(H)|$ for sufficiently large H and some constant c, while the notations

$$f(H) \asymp g(H), \quad f(H) = \Theta(g(H))$$

mean that $f(H) \ll g(H)$ and $f(H) \gg g(H)$. Finally, $f(H) = o(g(H))$ means that $\lim_{H \to \infty} f(H)/g(H) = 0$. The implied constants may depend on n but not on H.

Theorem 1.1 is significant for several reasons:

- This is a sharp improvement on the work of Davis, Duke and Sun [11], who showed that $\mathcal{E}_n^{\mathrm{monic}}(H) \ll H^{n-1/2} \log H$.
- The correct order of growth is *not* H^{n-1}, as one might expect by counting the reducible polynomials by analogy with van der Waerden's conjecture. Instead, the dominant contribution comes from reciprocal polynomials f such that $(-1)^n f(1) f(-1)$ is a square. This makes $\operatorname{disc} f$ a square and causes the Galois group to lie in an index-2 subgroup, called G_1 in the classification below. Reciprocal f such that $(-1)^n f(1) f(-1)$ is a square (among other conditions) show up as characteristic polynomials of automorphisms of lattices in [13].
- Since the characteristic polynomial of a symplectic matrix is reciprocal, a potential application of this work is to understand the characteristic polynomials of random elements of $\mathrm{Sp}_{2n}(\mathbb{Z})$, extending the work of Rivin [21] for $\mathrm{SL}_n(\mathbb{Z})$ and that of the first author and Lemke Oliver [16] for the symplectic case. Since the Weyl group of Sp_{2n} is $S_2 \wr S_n$, this fits into a wider principle that a random element of a connected

split reductive group $G/\mathbb{Q}$, faithfully embedded, should be almost surely equal to the Weyl group of G [15; 18]. Additionally, there are interesting connections to results in quantum chaos (see below).

Throughout this paper, n is fixed and the height $H \to \infty$ (the "large box model"). Work has also been fruitful for the complementary question on the "restricted coefficient model" where the coefficients are drawn from a fixed set and $n \to \infty$ [9; 3; 4]. In particular, Hokken [14] has recently proved an asymptotic count of the number of ± 1-coefficient reciprocal polynomials where the discriminant is a square, that is, $G_f \subseteq G_1$ in the notation below.

As with Bhargava [8] and many other papers on van der Waerden's conjecture, the methods are largely indifferent to whether we look at monic polynomials or general nonmonic polynomials f with all coefficients ranging through a box $[-H, H]^{n+1}$. The analogue of Theorem 1.1 for the nonmonic setting is:

Theorem 1.3. *Let $\mathcal{E}_n(H)$ be the number of separable reciprocal polynomials f of degree $2n$ with coefficients in $[-H, H]$ whose Galois group is not $S_2 \wr S_n$. Then for all $n \geq 1$,*

$$\mathcal{E}_n(H) \asymp H^n \log H.$$

Remark 1.4. Here the range of applicability is $n \geq 1$. In Theorem 1.1, we must exclude $n = 1$, because a monic reciprocal quadratic polynomial $f(x) = x^2 + a_1 x + 1$ has full Galois group S_2 for all $a_1 \neq \pm 2$.

Note that a degree $2n$ polynomial f is reciprocal if and only if the Cayley-transformed polynomial

$$\widetilde{f}(x) = (1+x)^{2n} f\left(\frac{1-x}{1+x}\right)$$

is even, that is, $\widetilde{f}(x) = \widetilde{g}(x^2)$ for a polynomial $\widetilde{g}$. (This is a straightforward computation and well known, see for instance [19, p. 275].) For an even polynomial, the roots again come in pairs $\{\alpha, -\alpha\}$. So we also have the following corollary:

Corollary 1.5. *The number of degree $2n$ even polynomials $\widetilde{f}$ with coefficients in $[-H, H]$ whose Galois group is not $S_2 \wr S_n$ is $\Theta(H^n \log H)$.*

Note that in this setting, the condition for the Galois group to lie in G_1 is that the product $(-1)^n a_{2n} a_0$ of the first and last coefficients of $\widetilde{f}$ be a square. If we impose $a_{2n} = 1$, the likelihood of this rises to $O(H^{-1/2})$, so the naïve analogue of Corollary 1.5 in the monic setting is false.

As alluded to earlier, there are potential applications of our work to the area of quantum chaos. In particular, studying the mass of eigenfunctions in quantum chaos is an area of great mathematical interest. This area shares exciting connections with reciprocal polynomials via quantum cat maps, a toy model of study in the area, that are given by symplectic matrices A. In particular, an important object to study the mass of eigenfunctions is the semiclassical measure μ associated to A. It turns out that μ has nice support properties if the characteristic polynomial of A^m for all $m \in \mathbb{N}$ (including A itself) is irreducible over the integers. In an appendix to a paper of Elena Kim, the first author and Lemke Oliver show that this irreducibility happens 100% of the time and moreover that the generic Galois group of such polynomials is the wreath product $S_2 \wr S_n$ (see [16] for more precise information and connections).

Based on this recent result, we conjecture that if M is a random $n \times n$ matrix, then the characteristic polynomials not only of M but of its powers M^i, $i \geq 1$, are almost surely S_n, where "almost surely" entails error bounds of van der Waerden type. This can be interpreted as saying that, if f is the characteristic polynomial, then not only its root α but each of its powers α^i, $i \geq 1$, generates an S_n-extension of $\mathbb{Q}$. The present work can be regarded as an extension of this to the fractional power $\sqrt{\alpha}$. It is natural to hope that the higher-order roots $\sqrt[m]{\alpha}$ of polynomials $g(x^m)$ can be handled in a similar way, the generic Galois group now being the semidirect product

$$(\mu_m \wr S_n) \rtimes (\mathbb{Z}/m\mathbb{Z})^\times,$$

where $(\mathbb{Z}/m\mathbb{Z})^\times$ acts naturally on the group μ_m of mth roots of unity.

Our methods are parallel to Bhargava's in [8], with judicious modifications when necessary. In Section 2, we lay out preliminary facts about reciprocal polynomials. In Section 3, we classify maximal subgroups of $S_2 \wr S_n$, and in particular, we show that we can narrow our focus to three groups, which we denote G_1, G_2, and G_3. In Sections 4, 5, and 6, we count the number $\mathcal{E}_n(G_i; H)$ of polynomials having each of those Galois groups (or a subgroup thereof). The G_1-polynomials we can count by direct parametrization. To count the G_3-polynomials f, we take advantage of the fact that f is reducible over a quadratic extension of $\mathbb{Q}$. The heart of the argument is to handle G_2, which plays a challenging role like that of the alternating group A_n in the study of general polynomials. Following Bhargava, we divide up the polynomials into three cases based on the size of the discriminant and its prime divisors. While attacking each case in turn, we are led to apply Fourier analysis for equidistribution and to construct a suitably modified double discriminant. The Fourier analysis is more involved than Bhargava's and involves breaking up the desired count into terms supported on different sublattices of the lattice $V(\mathbb{Z})$ of reciprocal polynomials. This type of decomposition is used in harmonic analysis to split up a function into pieces with different Fourier properties, but our use is perhaps novel in this setting.

Because Theorems 1.1 and 1.3 are so similar, we focus on Theorem 1.3, the nonmonic setting, which is the technically simpler of the two. At the end of each of Sections 4, 5, and 6, we explain how the proof must be adapted to the monic case to prove Theorem 1.1.

2. Reciprocal polynomials

We define the *height* of an integer polynomial

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$$

to be the maximum of the coefficients:

$$\operatorname{Ht} P = \max\bigl\{|c_n|, |c_{n-1}|, \ldots, |c_0|\bigr\} \in \mathbb{Z}_{\geq 0}.$$

Let

$$f(x) = a_0 x^{2n} + a_1 x^{2n-1} + \cdots + a_{n-1}x^{n+1} + a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a reciprocal polynomial of degree $2n$. Note that there is a unique degree n integer-coefficient polynomial

$$g(u) = b_n u^n + b_{n-1} u^{n-1} + \cdots + b_1 u + b_0$$

such that

$$f(x) = x^n g\left(x + \frac{1}{x}\right).$$

The passage between f and g is bijective and linear, and it increases or decreases heights by at most a bounded factor, depending only on n. Hence it is immaterial whether we count f or g of height at most H. For most purposes, it is more convenient to count g.

We denote the roots of f by

$$\alpha_1, \frac{1}{\alpha_1}, \alpha_2, \frac{1}{\alpha_2}, \ldots, \alpha_n, \frac{1}{\alpha_n}.$$

Then the roots of g are $\beta_1, \ldots, \beta_n$, where

$$\beta_i = \alpha_i + \frac{1}{\alpha_i}.$$

We will sometimes write $\alpha = \alpha_1$ and $\beta = \beta_1$ when the choice of root is irrelevant.

In view of the main theorem we would like to prove, we can assume any statement that occurs for all but $O(H^n)$ of the $O(H^{n+1})$ polynomials g of height at most H. For example, we may assume that g is irreducible and that $g(2)$ and $g(-2)$ are nonzero. We have the tower of number fields

$$K_f = \mathbb{Q}(\alpha) \quad \supseteq \quad K_g = \mathbb{Q}(\beta) \quad \supset \quad \mathbb{Q},$$

where K_g is an S_n-extension of degree n, while K_f/K_g is of degree at most 2, given by $K_f = K_g(\sqrt{\beta^2 - 4})$. Let $\widetilde{K}_g$ and $\widetilde{K}_f$, respectively, be the splitting fields of g and f, and let G_g and G_f be their respective Galois groups, which are subgroups of S_n and $S_2 \wr S_n$, with $G_f \twoheadrightarrow G_g$ under the natural projection $S_2 \wr S_n \twoheadrightarrow S_n$. By the main result of Bhargava [8, Theorem 1], we can assume that G_g is the whole S_n. Our aim in this paper is to understand when G_f is not the whole $S_2 \wr S_n$.

By the usual formula for the discriminant of a number field tower, we have

$$(\operatorname{Disc} K_f) = (\operatorname{Disc} K_g)^2 \cdot N_{K_g/\mathbb{Q}} \operatorname{disc}_{K_g} K_f$$

as ideals in $\mathbb{Z}$. The following is a closely related result on the discriminants of the associated polynomials.

Lemma 2.1. $\operatorname{disc} f = (-1)^n g(2) g(-2) (\operatorname{disc} g)^2$.

Proof. We have

$$\begin{aligned}
\operatorname{disc} f &= a_0^{4n-2} \prod_{i=1}^{n} (\alpha_i - \alpha_i^{-1})^2 \prod_{i<j} (\alpha_i - \alpha_j)^2 (\alpha_i - \alpha_j^{-1})^2 (\alpha_i^{-1} - \alpha_j)^2 (\alpha_i^{-1} - \alpha_j^{-1})^2 \\
&= b_n^{4n-2} \prod_{i=1}^{n} (\beta_i^2 - 4) \cdot \prod_{i<j} \alpha_i^{-4} \alpha_j^{-4} (\alpha_i - \alpha_j)^4 (\alpha_i \alpha_j - 1)^4 \\
&= (-1)^n \cdot b_n \prod_{i=1}^{n} (2 - \beta_i) \cdot b_n \prod_{i=1}^{n} (-2 - \beta_i) \cdot b_n^{4n-4} \prod_{i<j} (\beta_i - \beta_j)^4 \\
&= (-1)^n \cdot g(2) \cdot g(-2) \cdot (\operatorname{disc} g)^2.
\end{aligned}$$

□

3. Maximal subgroups of $S_2 \wr S_n$

If G_f is not the full $S_2 \wr S_n$, it is contained in a maximal subgroup. We have $S_2 \wr S_n = \mathbb{F}_2^n \rtimes S_n$. The following elementary lemma classifies the maximal subgroups of a semidirect product whose normal factor is abelian.

Lemma 3.1. *Let $X \rtimes S$ be a semidirect product of groups, with X abelian. The maximal subgroups of $X \rtimes S$ are of two types:*

(a) *$X \rtimes S'$, for $S' < S$ a maximal subgroup, and*

(b) *those groups $G < X \rtimes S$ for which $Y = G \cap X$ is a maximal S-invariant subgroup of X such that the projection $G \to S$ is surjective.*

Proof. Let $G < X \rtimes S$ be a maximal subgroup, and let S' be the projection of G onto S. If $S' \neq S$, then $G \leq X \rtimes S'$, and we must have equality so we get a group of type (a).

So we assume $S' = S$. Then $Y = G \cap X$ is a subgroup of X, not the whole of X since $G \neq X \rtimes S$. The conjugation action of $X \rtimes S$ on X is the S-action. Since G surjects onto S, the fact that G is closed under conjugation by itself implies that Y is closed under the S-action. Suppose that Y is not maximal as an S-invariant subspace, $Y < Y' < X$. Let

$$GY' = \{gy : g \in G,\, y \in Y'\} \subset X \rtimes S.$$

We claim that GY' is a subgroup. Since G and Y' are subgroups, it suffices to show that any product yg, $y \in Y'$, $g \in G$, belongs to GY'. Write $g = sx$, $x \in X$, $s \in S$. Since X is abelian,

$$yg = ysx = sy^s x = sxy^s = gy^s,$$

where y^s denotes the conjugate $s^{-1}ys$. Since Y' is S-invariant, the last product belongs to GY'. So GY' is a subgroup.

It is evident that $GY' \cap X = Y'$. So $G < GY' < X \rtimes S$, contradicting maximality of G. So Y is maximal, and G is a group of type (b). □

By Bhargava's result [8, Theorem 1], the extension $\widetilde{K}_g$ has full Galois group S_n for all but $O(H^n)$ polynomials g (or $O(H^{n-1})$ in the monic case), so Theorems 1.1 and 1.3 are already known for subgroups of type (a). To classify the subgroups of type (b), we must find the maximal S_n-invariant subspaces of $X = \mathbb{F}_2^n$, a vector space equipped with an S_n-action permuting the n basis vectors freely. For convenience we let

$$\mathbf{0} = (0, \dots, 0), \quad \mathbf{1} = (1, \dots, 1) \in X.$$

We use a superscript $\perp$ to denote the orthogonal complement of a space under the S_n-invariant inner product

$$(x_1, \dots, x_n) \bullet (y_1, \dots, y_n) = \sum_{i=1}^{n} x_i y_i.$$

Lemma 3.2. *Let $X = \mathbb{F}_2^n$, with the permutation action of S_n. The S_n-invariant subspaces of X are*

- X,
- $\langle \mathbf{1} \rangle^\perp = \{\mathbf{v} = (v_1, \dots, v_n) : \sum_i v_i = 0\}$,
- $\langle \mathbf{1} \rangle = \{\mathbf{0}, \mathbf{1}\}$,
- $\{\mathbf{0}\}$.

Proof. Let $W \subseteq X$ be an invariant subspace. If W contains a vector $\mathbf{v} = (v_1, \dots, v_n)$ besides $\mathbf{0}$ and $\mathbf{1}$, then upon applying some element of S_n we can assume $v_1 = 1$, $v_2 = 0$. Then $W \ni \mathbf{v} + (12)\mathbf{v} = \langle 1, 1, 0, \dots, 0 \rangle$. Applying further permutations from S_n, we get that W contains every vector with exactly two nonzero coordinates. Then W contains their span, which is $\langle \mathbf{1} \rangle^\perp$. Hence $W = \langle \mathbf{1} \rangle^\perp$ or $W = X$. □

Among these, the maximal subgroups are

- $\langle \mathbf{1} \rangle^\perp$, and
- $\langle \mathbf{1} \rangle$, for n odd (since for n even, $\langle \mathbf{1} \rangle \subseteq \langle \mathbf{1} \rangle^\perp$).

Note that for n odd, we have a direct sum decomposition $X = \langle \mathbf{1} \rangle \oplus \langle \mathbf{1} \rangle^\perp$. We now classify the groups G using group cohomology as follows:

Lemma 3.3. *Let $X \rtimes S$ be a semidirect product of groups, with X abelian, and let $K \subseteq X$ be a subgroup fixed by S. The subgroups of $G \subseteq X \rtimes S$ such that $G \cap X = K$ are parametrized by 1-cocycles $\varepsilon \in Z^1(S, X/K)$, in other words, maps $\varepsilon \colon S \to X/K$ satisfying the cocycle condition*

$$\varepsilon(\sigma\tau) = \varepsilon(\sigma) + \sigma(\varepsilon(\tau)). \tag{2}$$

The map sends each ε to the group

$$G = \{(x, \sigma) : x \equiv \varepsilon(\sigma) \bmod K\}. \tag{3}$$

Moreover, two such subgroups G, G' are conjugate if and only if the corresponding ε, ε' are in the same cohomology class in $H^1(S, X/K)$; in other words, if there is a $y \in X$ such that for all $\sigma \in S$,

$$\varepsilon'(\sigma) = \varepsilon(\sigma) + \sigma(y) - y. \tag{4}$$

Proof. This is a fairly standard use of group cohomology. Indeed, it is easy to check that G must have the form (3), that closure under multiplication enforces (2), and that conjugation by $y \in X$ induces (4). (Since G surjects onto S, conjugation by X is sufficient to produce all the conjugates of G in $X \rtimes S$.) □

Theorem 3.4. *The maximal subgroups of $S_2 \wr S_n$ whose projection onto S_n is the whole group are*

$$\begin{aligned}
G_1 &= \Big\{(\mathbf{v}, \sigma) \in \mathbb{F}_2^n \rtimes S_n : \sum_i v_i = 0\Big\} = \langle \mathbf{1} \rangle^\perp \rtimes S_n && \textit{for } n \geq 1, \\
G_2 &= \Big\{(\mathbf{v}, \sigma) \in \mathbb{F}_2^n \rtimes S_n : \sum_i v_i = \operatorname{sgn} \sigma\Big\} && \textit{for } n \geq 2, \\
G_3 &= \langle \mathbf{1} \rangle \times S_n && \textit{for } n \geq 3 \textit{ odd}.
\end{aligned}$$

Proof. By the preceding lemmas, we are left with computing $H^1(S_n, X/Y)$ for each of the maximal S_n-invariant subspaces Y in Lemma 3.2.

If $Y = \langle \mathbf{1} \rangle^\perp$, then

$$H^1(S_n, X/\langle \mathbf{1} \rangle^\perp) = H^1(S_n, C_2) = \mathrm{Hom}(S_n, C_2) = C_2,$$

the two maps ε being the zero map and the sign map, giving the subgroups G_1 and G_2 claimed.

If $Y = \langle \mathbf{1} \rangle$ for n odd, we must compute

$$H^1(S_n, X/\langle \mathbf{1} \rangle).$$

Since n is odd, we have that $X = \langle \mathbf{1} \rangle \oplus \langle \mathbf{1} \rangle^\perp$ is a direct sum and $X/\langle \mathbf{1} \rangle \cong \langle \mathbf{1} \rangle^\perp$. First consider

$$H^1(S_n, X).$$

With this action, $X = \mathrm{Ind}_{S_{n-1}}^{S_n} C_2$ is an induced module, so by Shapiro's lemma,

$$H^1(S_n, X) = H^1(S_{n-1}, C_2) = \mathrm{Hom}(S_{n-1}, C_2) = C_2.$$

Hence

$$H^1(S_n, X/\langle \mathbf{1} \rangle) = \frac{H^1(S_n, X)}{H^1(S_n, C_2)} = 0.$$

Therefore there is only the trivial extension G_3.

Note that G_1, G_2, and G_3 are normal subgroups of X, so we get no further maximal subgroups via conjugation.

The restrictions on n are provided due to the fact that, for some n, the G_i are nonmaximal or coincident:

- For $n = 1$, $G_2 = G_1$ and $G_3 = S_2 \wr S_n$.
- For n even, $G_3 \subseteq G_1$. □

Remark 3.5. A proof of Theorem 3.4 without group cohomology is possible, but it involves some in-depth case analysis with many computations of products in S_n and $S_2 \wr S_n$. After this paper was circulated in preprint form, we learned of the useful reference [2] where these computations are carried out.

Remark 3.6. With a few more cohomological computations, we can classify *all* the subgroups of $S_2 \wr S_n$ that surject onto S_n. They are the whole $S_2 \wr S_n$, the groups G_1, G_2, and G_3 (without parity restrictions on n), and the following additional groups:

- $\{0\} \times S_n \cong S_n$
- $\{((\mathrm{sgn}\,\sigma)\mathbf{1}, \sigma) : \sigma \in S_n\}$, a twisted copy of S_n
- and, for $n = 4$ only, the group $\mathrm{GL}_2\mathbb{F}_3$, which acts on the 8-element set $\mathbb{F}_3^2 \setminus \{0\}$, preserving the partition into opposite pairs of vectors. In terms of its map to S_4, this is the double cover denoted $2 \cdot S_4^+$ in [24; 20]. For instance, this is the generic Galois group of the even octic minimal polynomial of the y-coordinate of a 3-torsion point on an elliptic curve over $\mathbb{Q}$. It is a proper subgroup of the maximal subgroup G_2.

To prove Theorems 1.1 and 1.3, we must bound the number of polynomials f, equivalently g, for which G_f is (conjugate to) a subgroup of G_1, G_2, or G_3 for the values of n listed in Theorem 3.4.

Definition 3.7. For $G \subseteq S_2 \wr S_n$ and $H \geq 2$, let $\mathcal{E}_n(G;\, H)$ be the number of separable reciprocal polynomials f of degree $2n$ with coefficients in $[-H, H]$ such that $G_g = S_n$ and $G_f \subseteq G$. Let $\mathcal{E}_n^{\mathrm{monic}}(G;\, H)$ count the subset of these that are monic.

The following lemma reduces computing $\mathcal{E}_n(G_1;\, H)$ and $\mathcal{E}_n(G_2;\, H)$ to number-theoretic conditions on g that must hold in order to fulfill these conditions. (For G_3, we will use a different technique: see Section 6.)

Lemma 3.8. *Assume that G_g is the whole of S_n. Then*:

(a) $G_f \subseteq G_1$ *if and only if* $(-1)^n g(2)g(-2)$ *is a square.*

(b) $G_f \subseteq G_2$ *if and only if* $(-1)^n g(2)g(-2)\operatorname{disc} g$ *is a square.*

Proof. For (a), note that G_1 is the preimage of A_{2n} under the usual inclusion $S_2 \wr S_n \hookrightarrow S_{2n}$. Thus $G_f \subseteq G_1$ if and only if $\operatorname{disc} f = (-1)^n g(2)g(-2)(\operatorname{disc} g)^2$ is a square, which happens exactly when $(-1)^n g(2)g(-2)$ is a square.

For (b), consider the embedding of $S_2 \wr S_n$ into S_{3n} given by its action on the disjoint union of the roots of f and of g. Note that G_2 is the preimage of A_{3n} under this embedding. Hence $G_f \subseteq G_2$ if and only if $\operatorname{disc} f \cdot \operatorname{disc} g$ is a square, which happens exactly when $(-1)^n g(2)g(-2)\operatorname{disc} g$ is a square. $\square$

4. Counting G_1-polynomials

We first deal with the case G_1, which yields the main term of Theorems 1.1 and 1.3.

Theorem 4.1. *For $n \geq 1$,*

$$\mathcal{E}_n(G_1;\, H) \asymp H^n \log H \tag{5}$$

and for $n \geq 2$,

$$\mathcal{E}_n^{\mathrm{monic}}(G_1;\, H) \asymp H^{n-1} \log H. \tag{6}$$

By Lemma 3.8(a), it suffices to count g such that $(-1)^n g(2)g(-2)$ is a square z^2.

Lemma 4.2. *The number of solutions to the equation $xy = z^2$, $1 \leq x, y, z \leq H$ $(H \geq 2)$ is $\Theta(H \log H)$.*

Proof. A parametrization of the solutions is given by

$$x = ku^2, \quad y = kv^2, \quad z = kuv$$

where k, u, v are positive integers and $\gcd(u, v) = 1$. For each k, $1 \leq k \leq H$, the pair (u, v) is chosen from the box $1 \leq u, v \leq \sqrt{H/k}$, and the number of coprime pairs in this box is $\Theta(H/k)$ (the lower bound comes from citing the limiting proportion $6/\pi^2 > 0$ of coprime pairs when H/k is large, and noting that there is always at least one solution $u = v = 1$). So the total number N of solutions satisfies $N \asymp \sum_{k=1}^{H} H/k \asymp H \log H$, as desired. □

Proof of Theorem 4.1. For simplicity, we prove the nonmonic case (5). As $g(2), g(-2) \ll H$ and each pair $(x, y) = (|g(2)|, |g(-2)|)$ appears $O(H^{n-1})$ times, we get that $G_f \subseteq G_1$ at most $O(H^n \log H)$ times. Conversely, if we take $|x|, |y| \le cH$ for an appropriate constant c, and with $x \equiv y \bmod 4$ (which can be arranged, for instance by taking $4|k$), we find that there are $\Theta(H^{n-1})$ polynomials g with $g(2) = x$ and $g(2) = y$, and thus $\Theta(H^{n-1} \log H)$ polynomials overall with Galois group $G_f \subseteq G_1$. □

4.1. ***Remarks on the monic case.*** For the monic case, the argument is identical, replacing n by $n-1$. It is only necessary to have at least two free coefficients so that $g(2)$ and $g(-2)$ can be adjusted independently, requiring $n \ge 2$.

5. Counting G_2-polynomials

For G_2, we prove the following bounds, which are stronger than those for G_1 by a factor of $\log H$:

Theorem 5.1. *For $n \ge 2$,*

$$\mathcal{E}_n(G_2; H) \ll H^n, \tag{7}$$

$$\mathcal{E}_n^{\text{monic}}(G_2; H) \ll H^{n-1}. \tag{8}$$

By Lemma 3.8(b), we wish to count g such that $(-1)^n g(2) g(-2) \operatorname{disc} g$ is a square. We use a sieve method adapted from Bhargava [8]. We begin with some analytic preliminaries.

5.1. ***Twisted Poisson summation.*** Let $\Phi \colon \mathbb{R}^n \to \mathbb{C}$ be a Schwartz function. We normalize the Fourier transform by

$$\widehat{\Phi}(y) = \int_{\mathbb{R}^n} e^{-2\pi\sqrt{-1}x \bullet y} \Phi(x)\, dx.$$

The usual Poisson summation formula

$$\sum_{x \in \mathbb{Z}^n} \Phi(x) = \sum_{y \in \mathbb{Z}^n} \widehat{\Phi}(y)$$

can be extended in various ways. If $L \subseteq \mathbb{Z}^n$ is a lattice (a subgroup of finite index), then

$$\sum_{x \in L} \Phi(x) = \frac{1}{[\mathbb{Z}^n : L]} \sum_{y \in L^*} \widehat{\Phi}(y), \tag{9}$$

where $L^* \supseteq \mathbb{Z}^n$ is the dual lattice. More generally:

Proposition 5.2. *Let $L \subseteq \mathbb{Z}^n$ be a lattice, and let $\Psi \colon (\mathbb{Z}/M\mathbb{Z})^n \to \mathbb{C}$ be any function, where the modulus M is coprime to $[\mathbb{Z}^n : L]$. Let $\widehat{\Psi} \colon (\mathbb{Z}/M\mathbb{Z})^n \to \mathbb{C}$ be the Fourier transform*

$$\widehat{\Psi}(y) = \frac{1}{M^n} \sum_{x \in (\mathbb{Z}/M\mathbb{Z})^n} e^{-2\pi\sqrt{-1}x \bullet y/M} \Psi(x).$$

For any Schwartz function Φ,

$$\sum_{x \in L} \Psi(x)\Phi(x) = \frac{1}{[\mathbb{Z}^n : L]} \sum_{y \in L^*} \widehat{\Psi}(y)\widehat{\Phi}\left(\frac{y}{M}\right).$$

Observe that $\widehat{\Psi}$ is well defined on L^* since M is coprime to $[\mathbb{Z}^n : L]$. (In fact, one can do without this hypothesis, but then $\widehat{\Psi}$ becomes a function on L^*/ML^* instead of being independent of L.) To prove this proposition, one may assume by linearity that Ψ is supported on a single point, and the result reduces to Poisson summation.

5.2. ***The index of a polynomial over a field.*** Following Bhargava [8, Proposition 26], we make the following definition. If $P \in \Bbbk[x, y]$ is a nonzero homogeneous polynomial over a field, we factor $P = \prod_i P_i^{e_i}$ into powers of distinct irreducibles and define the *index* of P to be

$$\operatorname{ind}(P) = \sum_i (e_i - 1) \deg P_i.$$

The index is significant for bounding the power of a prime p dividing the discriminant of the polynomial and the extension field it defines. The following lemma is used implicitly in [8]; for completeness, we offer a statement and proof.

Lemma 5.3. *Let $g \in R[x, y]$ be a separable homogeneous binary form of degree n over a Dedekind domain R. Let $F =$* Frac R, *and let $E = F[\beta]/g(\beta, 1)$ be the étale algebra* (*product of separable finite field extensions*) *defined by g. If $\mathfrak{p}$ is a prime ideal of R such that E is tamely ramified at $\mathfrak{p}$* (*e.g.*, $\operatorname{char}(R/\mathfrak{p}) > n$) *and g is not identically zero modulo $\mathfrak{p}$, then*

$$v_{\mathfrak{p}}(\operatorname{Disc} E) \le \operatorname{ind}(g \bmod p) \le v_{\mathfrak{p}}(\operatorname{disc} g). \tag{10}$$

Proof. We have deliberately stated the lemma in greater generality than needed to allow for making some reductions. First, we may replace R by its completion at $\mathfrak{p}$. Let $\Bbbk$ be the residue field of R. Now, if $g \equiv g_1 g_2 \bmod \mathfrak{p}$ with $g_i \in \Bbbk[x, y]$ homogeneous and coprime, then by Hensel's lemma, the factorization lifts to R and induces a splitting $E = E_1 \times E_2$. The inequality (10) can then be deduced by summing the corresponding inequalities for g_1 and g_2. Thus we may assume that

$$g \equiv c \cdot g_1^e \bmod \mathfrak{p},$$

where c is a constant and $g_1 \in \Bbbk[x, y]$ is irreducible of some degree f, with $ef = n$. We may change coordinates so that $g_1 \neq y$ is monic in x. Now

$$E = \prod_{j=1}^{r} E_j$$

is a product of fields with the same inertia index f and possibly different ramification indices e_j. We compute, using the usual formula for the discriminant of a tamely ramified extension:

- First,
$$v_{\mathfrak{p}}(\operatorname{Disc} E) = \sum_{j=1}^{r} v_{\mathfrak{p}}(\operatorname{Disc} E_i) = \sum_{j=1}^{r} (e_j - 1) f = n - rf.$$
- $\operatorname{ind}(g \bmod p) = (e-1)f = n - f$.
- Finally, we need to understand the ring $S = R[\beta]/g(\beta, 1) \subset E$. Note that S is contained in
$$S' = \{(x_1, \dots, x_r) \in \mathcal{O}_E : x_1 \equiv \cdots \equiv x_r \bmod \mathfrak{p}\},$$
a subring of $\mathcal{O}_E$ of index $(r-1)f$. Thus
$$\begin{aligned} v_{\mathfrak{p}}(\operatorname{disc} g) &= v_{\mathfrak{p}}(\operatorname{Disc} S) \\ &\geq v_{\mathfrak{p}}(\operatorname{Disc} S') = v_{\mathfrak{p}}(\operatorname{Disc} E) + 2v_{\mathfrak{p}}([\mathcal{O}_E : S']) \\ &= n - rf + 2(r-1)f = n - f + (r-1)f. \end{aligned}$$

The desired inequality follows immediately. Equality for both parts holds exactly when $r = 1$, or in the original setup, when $\mathfrak{p} \nmid [\mathcal{O}_E : S]$. □

Following Bhargava [8, §5], we define
$$D = \operatorname{Disc} K_g \quad \text{and} \quad C = \prod_{p \mid D} D$$
and divide the counting into three cases based on the sizes of C and D relative to H. Unlike in [8], we do not have that D is squarefull; but by Lemma 3.8(b) we have that $(-1)^n Dg(2)g(-2)$ is a square, which limits the cases.

Let δ be a small constant (such as $1/4n$). For R a ring, denote by $V^{\text{hom}}(R)$ the $(n+1)$-dimensional space of binary n-ic forms $P(x, y)$ over R, and denote by $V(R)$ the space of polynomials $g(u)$ of degree at most n over R. The two R-modules are isomorphic, but we will need to identify them in multiple ways.

5.3. ***Case I:*** $C \leq H^{1+\delta}$, $D \geq H^{2+2\delta}$. In this case, we need to estimate the number of g for which $D = \operatorname{Disc}(K_g)$, $g(2)$, and $g(-2)$ have certain factors. We begin with a short argument that yields the result up to ε.

Lemma 5.4. *Let p be a prime, and let k be an integer. The number of binary forms $g \in V^{\text{hom}}(\mathbb{F}_p)$ such that*

- $g(1, 0) = 0$, *that is*, $y \mid g$, *and*
- $\operatorname{ind}(g) \geq k$

is $O(p^{n-k})$.

Proof. We can immediately dispose of the case $p \leq n$, for here both the number of g and the desired bound are $O(1)$. In [8, Corollary 27], it is shown that the number of degree-n binary forms g such that $\operatorname{ind}(g) \geq k$ is $O(p^{n+1-k})$. To impose the condition $g(1, 0) = 0$, we can consider each g as lying in the family of p translates

$$g(x, y + ax), \quad a \in \mathbb{F}_p.$$

The translates all have the same index, and if $p > n$, the translates are all distinct. Moreover, since g has at most n roots over $\mathbb{F}_p$, at most $n = O(1)$ of the translates satisfy the added condition $g(1, 0) = 0$. Hence the total number of such g is $O(p^{n-k})$, as desired. □

Let $p > n$ be a prime dividing C, and suppose $p^k \parallel D$. By the first inequality in (10), we have $\operatorname{ind}(g) \geq k$, and this occurs for a proportion p^{-k} of g. If k is odd, then we additionally have $p \mid g(2)$ or $p \mid g(-2)$, and altogether there is a proportion p^{-k-1} of g satisfying these conditions. Multiplying over p, the proportion of G_2-polynomials with $\operatorname{Disc} K_g = D$ is bounded by

$$\prod_{p \mid D} O\big(p^{-2\lceil k/2 \rceil}\big) = \frac{O(c^{\omega(D_1)})}{D_1^2},$$

where $D_1^2 = \prod_{p \mid D} p^{2\lceil k/2 \rceil}$ is the least square divisible by D. Observe that $D_1 \gg H$ and that each D_1 occurs for at most $2^{\omega(D_1)}$ values of D. Moreover, these g are cut out by congruence conditions mod C. If $C \leq H$, then we can estimate the number of lattice points very precisely because our modulus is lower than the size of the box. We get that the number of polynomials g is

$$\ll H^{n+1} \sum_{D_1 \geq H} \frac{c^{\omega(D_1)}}{D_1^2} \ll_\varepsilon H^{n+\varepsilon}.$$

Using Fourier analysis we can remove the ε and also extend the validity of this case from $C \leq H$ to $C \leq H^{1+\delta}$.

Recall some definitions from [8, §4.1]. If a binary n-ic form f (over $\mathbb{Z}$, or over $\mathbb{F}_p$) factors modulo p as $\prod_{i=1}^r P_i^{e_i}$, with P_i irreducible and $\deg(P_i) = f_i$, then the *splitting type* (f, p) of f is defined as $(f_1^{e_1} \cdots f_r^{e_r})$, and the *index* $\operatorname{ind}(f)$ of f modulo p (or the *index* of the splitting type (f, p) of f) is defined to be $\sum_{i=1}^r (e_i - 1) f_i$. If $p \mid f$, we put $\operatorname{ind}(f) = \infty$. More abstractly, a *splitting type* is an expression σ of the form $(f_1^{e_1} \cdots f_r^{e_r})$, where the f_i and e_i are positive integers. The *degree* $\deg(\sigma)$ is $\sum_{i=1}^r e_i f_i$, and the *index* $\operatorname{ind}(\sigma)$ is $\sum_{i=1}^r (e_i - 1) f_i$. Finally, $\#\operatorname{Aut}(\sigma)$ is defined to be $\prod_i f_i$ times the number of permutations of the factors $f_i^{e_i}$ that preserve σ.

Later on we will need to deal with splitting types with a distinguished factor of degree 1. If σ has $f_1 = 1$, we let $\#\operatorname{Aut}'(\sigma)$ be $\prod_i f_i$ times the number of permutations of the factors $f_i^{e_i}$, $i \geq 2$, that preserve σ.

We first recall the following lemma of Bhargava:

Lemma 5.5 [8, Proposition 26]. *Let $\sigma = (f_1^{e_1} \cdots f_r^{e_r})$ be a splitting type with* $\deg(\sigma) = d \leq n$ *and* $\operatorname{ind}(\sigma) = k$. *Let $w_{p,\sigma} : V^{\mathrm{hom}}(\mathbb{F}_p) \to \mathbb{C}$ be defined by*

$w_{p,\sigma}(f) :=$ *the number of r-tuples $(P_1, \ldots, P_r)$, up to the action of the group of permutations of $\{1, \ldots, r\}$ preserving σ, such that the P_i are distinct irreducible binary forms where, for each i, we have $P_i(x, y)$ is y or is monic as a polynomial in x,* $\deg P_i = f_i$, *and $P_1^{e_1} \cdots P_r^{e_r} \mid f$.*

Then

$$\widehat{w}_{p,\sigma}(g) = \begin{cases} \dfrac{p^{-k}}{\#\mathrm{Aut}(\sigma)} + O(p^{-(k+1)}) & \text{if } g = 0; \\ O(p^{-(k+1)}) & \text{if } g \neq 0. \end{cases}$$

The significance of this function $w_{p,\sigma}$ is twofold: First, $w_{p,\sigma}(f)$ is nonnegative, and is equal to 1 when f has splitting type σ; second, the definition is arranged so that $w_{p,\sigma}$ is a sum of characteristic functions of subspaces, which makes the Fourier transform nonnegative, small, and easily computable.

Bhargava uses this lemma to bound the number of integer polynomials having high index at a set of primes. In our setting, we also need to bound the number of integer polynomials having high index *and* which vanish at a given point mod p; hence we modify the lemma as follows:

Lemma 5.6. *Let* $\sigma = (f_1^{e_1} \cdots f_r^{e_r})$ *be a splitting type with* $f_1 = 1$. *Let* $\deg(\sigma) = d$ *and* $\mathrm{ind}(\sigma) = k$. *Let* $w'_{p,\sigma} : V^{\mathrm{hom}}(\mathbb{F}_p) \to \mathbb{C}$ *be defined by*

$w'_{p,\sigma}(f) :=$ *the number of r-tuples $(P_2, \dots, P_r)$, up to the action of the group of permutations of $\{2, \dots, r\}$ preserving σ, such that the P_i are distinct irreducible binary forms where, for each i, we have $P_i(x, y)$ is monic as a polynomial in x, $\deg P_i = f_i$, and $y^{e_1} P_2^{e_2} \cdots P_r^{e_r} \mid f$.*

Let $V^{\mathrm{hom}}_{e_1,\mathbb{F}_p}$ *denote the subspace of* $V^{\mathrm{hom}}(\mathbb{F}_p)$ *comprising polynomials divisible by* y^{e_1}, *and let* $(V^{\mathrm{hom}}_{e_1,\mathbb{F}_p})^\perp \subseteq V^{\mathrm{hom}}(\mathbb{F}_p)^*$ *be its dual. Then*

$$\hat{w}'_{p,\sigma}(g) = \begin{cases} \dfrac{p^{-(k+1)}}{\#\mathrm{Aut}'(\sigma)} + O(p^{-(k+2)}) & \text{if } g \in (V^{\mathrm{hom}}_{e_1,\mathbb{F}_p})^\perp; \\ O(p^{-(k+2)}) & \text{if } g \notin (V^{\mathrm{hom}}_{e_1,\mathbb{F}_p})^\perp. \end{cases}$$

Proof. Rather than starting from scratch, we derive this lemma from the preceding one. Indeed, let σ' be the splitting type obtained by deleting the first factor $f_1^{e_1} = 1^{e_1}$ from σ. Then $\mathrm{Aut}(\sigma') \cong \mathrm{Aut}'(\sigma)$, and $\mathrm{ind}(\sigma') = k - e_1 + 1$. We have that $w'_{p,\sigma}(f)$ vanishes unless $f \in V^{\mathrm{hom}}_{e_1,\mathbb{F}_p}$ (implying in particular that $\hat{w}'_{p,\sigma}$ is constant on cosets of $(V^{\mathrm{hom}}_{e_1,\mathbb{F}_p})^\perp$), and

$$w'_{p,\sigma}(f) = \breve{w}_{p,\sigma'}(f/y^{e_1})$$

is *almost* $w_{p,\sigma}(f/y^{e_1})$. We say "almost" because we need to exclude the case that one of the P_i is y; so we define $\breve{w}_{p,\sigma}$ to be just like $w_{p,\sigma}$, except that the $P_i(x, y)$ in the definition are not allowed to equal y. Since $w_{p,\sigma'}$ is a sum of characteristic functions of subspaces corresponding to the various choices of the P_i and $\breve{w}_{p,\sigma}$ is obtained by deleting some of the terms from this sum, we have, by Lemma 5.5, $\breve{w}_{p,\sigma'}(f) \le w_{p,\sigma'}(f)$ and

$$\widehat{\breve{w}}_{p,\sigma'}(g) \le \widehat{w}_{p,\sigma'}(g) = \begin{cases} \dfrac{p^{-(k-e_1+1)}}{\#\mathrm{Aut}'(\sigma) + O(p^{-(k-e_1+2)})} & \text{if } g \in V^\perp_{e_1,\mathbb{F}_p}; \\ O(p^{-(k-e_1+2)}) & \text{if } g \notin V^\perp_{e_1,\mathbb{F}_p}. \end{cases}$$

Re-embedding $V^{\mathrm{hom}}_{e_1,\mathbb{F}_p}$ into $V^{\mathrm{hom}}(\mathbb{F}_p)$, the Fourier transform drops by a factor of p^{e_1}, yielding upper bounds of the claimed order of magnitude.

To obtain that the main term is undiminished by the $w \mapsto \breve{w}$ replacement, we can argue that

$$\breve{w}_{p,\sigma} = w_{p,\sigma} - \sum_{i:f_i=1} \breve{w}_{p,\sigma_i}$$

where σ_i is obtained by deleting $f_i^{e_i}$ from σ. Thus

$$\begin{aligned}\widehat{\breve{w}}_{p,\sigma}(0) &= \widehat{w}_{p,\sigma}(0) - \sum_{i:f_i=1} \widehat{\breve{w}}_{p,\sigma_i}(0) \\ &= \frac{p^{-(k-e_1+1)}}{\#\mathrm{Aut}'(\sigma)} + O(p^{-(k-e_1+2)}) + \sum_{i:f_i=1} O(p^{-(k-e_1-e_i+1)}) \\ &= \frac{p^{-(k-e_1+1)}}{\#\mathrm{Aut}'(\sigma)} + O(p^{-(k-e_1+2)}),\end{aligned}$$

as desired. □

We can now estimate the number of polynomials in Case I.

Lemma 5.7. *Let D be a positive integer. Let $C = \prod_{p|D} p$ be its radical, and let $D'^2 = \prod_{p|D} p^{2\lceil v_p(D)/2\rceil}$ be its smallest square multiple. Assume that $C < H^{1+\delta}$. The number of reciprocal integer polynomials f of height $\le H$ for which $G_f \subseteq G_2$ and $D \mid \mathrm{Disc}\, K_g$ is*

$$\ll \frac{O(1)^{\omega(C)} H^{n+1}}{D'^2}.$$

Remark 5.8. Note that in contrast to the notation in the rest of the paper, we do *not* set $D = \mathrm{Disc}\, K_g$, but rather assume only that $D \mid \mathrm{Disc}\, K_g$. For Case I, we set $D = \mathrm{Disc}\, K_g$, but we will reuse this lemma in Case III, and there we will pick a general divisor D.

Note that Lemma 5.7 implies the bound of Theorem 5.1 on the number of g in Case I because each D' determines D up to $O(1)^{\omega(C)}$ possibilities, and the total number of g is thus

$$\ll O(1)^{\omega(C)} H^{n+1} \sum_{D' \ge H^{1+\delta}} \frac{1}{D'^2} \ll O(1)^{\omega(C)} H^{n-\delta} \ll H^{n-\delta+\varepsilon} \ll H^n.$$

Proof of Lemma 5.7. First of all, we divide out all primes $p \le n$ from D. If there is at least one K_g with $D \mid \mathrm{disc}\, K_g$, this change only affects D by a bounded factor, since $v_p(\mathrm{disc}\, K_g)$ is uniformly bounded. Thus we can assume that every prime $p \mid C$ is at least n.

For each prime $p \mid C$, excluding the degenerate case that $g \equiv 0 \bmod p$ (see below), let $\sigma_p = (f_1^{e_1} \cdots f_r^{e_r})$ be the splitting type of its homogenization $\tilde{g} = y^n g(x/y)$. By the second inequality in (10),

$$v_p(D) \le \mathrm{ind}(\tilde{g} \bmod p).$$

Also, if $v_p(D)$ is odd, the relations

$$1 \equiv v_p(D) \equiv v_p(\mathrm{disc}\, g) \equiv v_p g(2) g(-2) \bmod 2$$

imply that $g(u) \bmod p$ is divisible by either $u-2$ or $u+2$. Let $k_p = v_p(D') = \lceil v_p(D)/2 \rceil$. We thus get one of three cases restricting σ_p and $\tilde{g}$:

(0) $\operatorname{ind}(\sigma_p) \geq 2k_p$;

(2) $\operatorname{ind}(\sigma_p) \geq 2k_p - 1$ and $f_1 = 1$ with $P_1 = x - 2y$;

(−2) $\operatorname{ind}(\sigma_p) \geq 2k_p - 1$ and $f_1 = 1$ with $P_1 = x + 2y$.

We call an *annotated splitting type* a pair (σ, j) of a splitting type $\sigma = \sigma_p$ with a choice of case $j \in \{0, 2, -2\}$. Set

$$w_{p,\sigma,j}(h) = \begin{cases} w_{p,\sigma}\big(y^n h(x/y)\big) & \text{if } j = 0, \\ w'_{p,\sigma}\big(x^n h(y/x - j)\big) & \text{if } j = \pm 2, \end{cases}$$

the coordinate changes in the latter two cases being to send the distinguished linear factors $u - j$ in our situation to the distinguished linear factor y in Lemma 5.6. Likewise, let

$$\#\operatorname{Aut}(\sigma, j) = \begin{cases} \#\operatorname{Aut}(\sigma) & \text{if } j = 0, \\ \#\operatorname{Aut}'(\sigma) & \text{if } j = \pm 2. \end{cases}$$

For (σ_p, j_p) an annotated splitting type, let $\Psi_p = w_{p,\sigma_p,j_p} \colon V_{\mathbb{F}_p} \to \mathbb{R}$, and let $\Psi \colon V_{\mathbb{Z}/C\mathbb{Z}} \to \mathbb{R}$ be the product of the Ψ_p. Observe that there are only $O(1)^{\omega(C)}$ choices of the annotated splitting types for all p, and for each g as in the statement of the lemma, there is a choice for which $\Psi(g) \geq 1$. (In the degenerate case that $g \equiv 0 \bmod p$, we pick σ_p and j_p arbitrarily, because $w_{p,\sigma,j}(0) \geq 1$ for all σ and j.) Thus it suffices to prove, for a fixed choice of σ_p and case (a)–(c) for each $p \mid C$, that

$$\sum_{\operatorname{Ht} g \leq H} \Psi(g) \ll \frac{O(1)^{\omega(C)} H^{n+1}}{D'^2}.$$

We claim that there is a decomposition $\Psi_p = \Lambda_p + \Delta_p$ with the following properties:

- $\Lambda_p = a_p \mathbf{1}_{L_p}$ is a rescaled characteristic function of a sublattice $L_p \subseteq V(\mathbb{Z})$, with $0 < a_p \leq 1$, and $\widehat{\Lambda}_p = \hat{a}_p \mathbf{1}_{L_p^\perp}$ where $0 < \hat{a}_p \leq p^{-2k_p}$;
- Δ_p has uniformly small Fourier transform: $\widehat{\Delta}_p(h) \ll p^{-2k_p - 1}$.

Indeed, we take

$$L_p = \begin{cases} V(\mathbb{Z}) & \text{if } j_p = 0, \\ \big\{g : g(j_p) \equiv g'(j_p) \equiv \cdots \equiv g^{(e_{1,p}-1)}(j_p) \equiv 0 \bmod p\big\} & \text{if } j_p = \pm 2. \end{cases}$$

Note that in the latter case, L_p is the image of $V^{\text{hom}}_{e_1,\mathbb{F}_p}$ under the relevant identification of V^{hom} with V sending $x^i y^{n-i}$ to $(u-j)^{n-i}$. Choose the constant

$$a_p = \frac{[V(\mathbb{Z}) : L_p] \cdot p^{-\operatorname{ind}\sigma - |j|/2}}{\#\operatorname{Aut}(\sigma, j)} = \frac{p^{-\sum_{i > |j|/2} (e_i - 1) f_i}}{\#\operatorname{Aut}(\sigma, j)} \leq 1$$

so that $\widehat{\Lambda}_p$ is the dominant term

$$\hat{a}_p \mathbf{1}_{L_p^\perp}, \quad \hat{a}_p = \frac{p^{-\operatorname{ind}\sigma - |j|/2}}{\#\operatorname{Aut}(\sigma, j)} \le p^{-2k_p}$$

of $\widehat{\Psi}_p$ as computed in Lemmas 5.5 and 5.6, which show the smallness of $\widehat{\Delta}_p$.

Now

$$\Psi = \prod_p \big(\Lambda_p + \Delta_p\big) = \sum_{q \mid C} \Lambda_q \Delta_{C/q},$$

where we set

$$\Lambda_q = \prod_{p \mid q} \Lambda_p = a_q \mathbf{1}_{L_q}, \quad a_q = \prod_{p \mid q} a_p, \quad L_q = \bigcap_{p \mid q} L_p, \quad \text{and} \quad \Delta_q = \prod_{p \mid q} \Delta_p.$$

We now use Fourier analysis, as in Case I of [8], to rewrite the desired count of g. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a Schwartz function on $\mathbb{R}$ with the following properties:

(a) $\phi(u)$ is nonnegative, and $\phi(u) \ge 1$ for $|u| \le 1$;

(b) ϕ is compactly supported;

(c) The Fourier transform $\widehat{\phi}$ is real and nonnegative.

Such a ϕ can be constructed, for instance, by taking a usual even "bump function" and convolving with itself, which squares the Fourier transform, ensuring nonnegativity. Then let $\Phi\colon V(\mathbb{R}) \to \mathbb{R}$ be the product $\Phi(g) = \phi(b_0)\phi(b_1)\cdots\phi(b_n)$. We use this as a smoothing factor:

$$\begin{aligned} X &:= \#\{g \in V(\mathbb{Z}) : G_f \subseteq G_2,\ D \mid \operatorname{disc} K_g, \operatorname{Ht} g \le H\} \\ &\le \sum_{\operatorname{Ht} g \le H} \Psi(g) \le \sum_{g \in V(\mathbb{Z})} \Psi(g) \Phi\left(\frac{g}{H}\right) = \sum_{q \mid C} \sum_{g \in V(\mathbb{Z})} \Lambda_q(g) \Delta_{C/q}(g) \Phi\left(\frac{g}{H}\right). \end{aligned}$$

Since C has $O(1)^{\omega(C)}$ divisors q, it suffices to prove that for all $q \mid C$,

$$X_q := \sum_{g \in V(\mathbb{Z})} \Lambda_q(g) \Delta_{C/q}(g) \Phi\left(\frac{g}{H}\right) \ll \frac{O(1)^{\omega(C)} H^{n+1}}{D'^2}.$$

We apply twisted Poisson summation (Proposition 5.2) with modulus $M = C/q$, which is coprime to $[V(\mathbb{Z}) : L_q] \mid q^n$, to get

$$\begin{aligned} X_q &= a_q \sum_{g \in L_q} \Delta_{C/q}(g) \Phi\left(\frac{g}{H}\right) = \frac{a_q H^{n+1}}{[V(\mathbb{Z}) : L_q]} \sum_{h \in L_q^*} \widehat{\Delta}_{C/q}(h) \widehat{\Phi}\left(\frac{qHh}{C}\right) \\ &\ll \frac{a_q H^{n+1}}{[V(\mathbb{Z}) : L_q]} \prod_{p \mid C/q} O(p^{-2k_p - 1}) \sum_{h \in L_q^*} \widehat{\Phi}\left(\frac{qHh}{C}\right). \end{aligned}$$

We apply Poisson summation again, now untwisted, to get

$$
\begin{aligned}
X_q &\ll \frac{a_q C^{n+1}}{q^{n+1}} \prod_{p \mid C/q} O(p^{-2k_p-1}) \sum_{g \in L_q} \Phi\Big(\frac{Cg}{qH}\Big) \\
&\le \frac{a_q C^{n+1}}{q^{n+1}} \prod_{p \mid C/q} O(p^{-2k_p-1}) \sum_{g \in V(\mathbb{Z})} \Phi\Big(\frac{Cg}{qH}\Big) \\
&\ll a_q \prod_{p \mid C/q} O(p^{-2k_p-1}) \left(\frac{C}{q} \max\left\{\frac{qH}{C}, 1\right\}\right)^{n+1} \\
&\ll O(1)^{\omega(C)} \prod_{p \mid q} p^{-2k_p} \prod_{p \mid C/q} O(p^{-2k_p-1}) \cdot \max\Big\{H, \frac{C}{q}\Big\}^{n+1} \\
&= \frac{O(1)^{\omega(C)} q}{C D'^2} \max\Big\{H, \frac{C}{q}\Big\}^{n+1}.
\end{aligned}
$$

If the first argument to the maximum dominates, we get a bound

$$
X_q \ll \frac{O(1)^{\omega(C)} q}{C D'^2} H^{n+1} \le \frac{O(1)^{\omega(C)} H^{n+1}}{D'^2},
$$

as desired. If instead the second argument dominates, we get a bound

$$
X_q \ll \frac{O(1)^{\omega(C)} q}{C D'^2} \left(\frac{C}{q}\right)^{n+1} = \frac{O(1)^{\omega(C)} C^n}{q^n D'^2} \le \frac{O(1)^{\omega(C)} H^{n+n\delta}}{D'^2},
$$

as desired, since $n\delta \le 1$. □

5.4. ***Case II:*** $\boldsymbol{D \le H^{2+2\delta}}$**.** Here we simply invoke Case II of Bhargava's treatment [8, §5], which shows that the number of irreducible g of height $< H$ defining a number field K_g of primitive Galois group G_g and discriminant $D \le H^{2+2\delta}$ is $O(H^n)$ (or $O(H^{n-1})$ in the monic case). In our setting we are only concerned with the case $G_g = S_n$. We do not need to use the added knowledge that $G_f \subseteq G_2$.

5.5. ***Case III:*** $\boldsymbol{C \ge H^{1+\delta}}$**.** Here we adapt the method of Bhargava using the double discriminant.

Lemma 5.9 [8, Proposition 34]. *Let $p > 2$ be a prime. If $h(x_1, \dots, x_n)$ is an integer polynomial, such that $h(c_1, \dots, c_n)$ is a multiple of p^2 for mod p reasons, that is, $h(c_1 + pd_1, \dots, c_n + pd_n)$ is a multiple of p^2 for all $(d_1, \dots, d_n) \in \mathbb{Z}^n$, then $\frac{\partial}{\partial x_n} h(c_1, \dots, c_n)$ is a multiple of p.*

Let

$$
h = g(2)g(-2)\operatorname{disc} g,
$$

considered as a polynomial in the coefficients $b_0, \dots, b_n$ of g. Let $p \mid C$, $p > n$. We claim that $p^2 \mid h$ for mod p reasons. We have that $p \mid D$, and p^2 divides the square $(-1)^n D g(2) g(-2)$. If $p^2 \mid D$, then by the first inequality in (10), the index of g modulo p is at least 2. So by the second inequality on that same line, we have $p^2 \mid \operatorname{disc} g$ for mod p reasons. Otherwise, we have $p \mid D$, so $p \mid \operatorname{disc} g$, and either $p \mid g(2)$ or $p \mid g(-2)$. Thus in all cases the product h is divisible by p^2 for mod p reasons. By Lemma 5.9, this implies that the derivative $\frac{\partial}{\partial b_0}h$ with respect to the constant term is divisible by p. Hence their resultant

$$R(b_1, \dots, b_n) = \operatorname{Res}_{b_0}\left(h, \frac{\partial}{\partial b_0}h\right) = \pm b_n \operatorname{disc}_{b_0} h$$

is a multiple of p.

The polynomial R is the analogue in our setting of the *double discriminant* DD of [8, Proposition 34]. For brevity, let $G = \operatorname{disc}_u g$. Then

$$\begin{aligned} R(b_1, \dots, b_n) &= \pm b_n \operatorname{disc}_{b_0}(g(2)g(-2)G) \\ &= \pm b_n \operatorname{disc}_{b_0} G \cdot \operatorname{Res}_{b_0}(g(2), g(-2))^2 \operatorname{Res}_{b_0}(g(2), G)^2 \operatorname{Res}_{b_0}(g(-2), G)^2 \\ &= \pm b_n \operatorname{disc}_{b_0} G \cdot (g(2) - g(-2))^2 \big(\operatorname{disc}_u(g - g(2))\big)^2 \big(\operatorname{disc}_u(g - g(-2))\big)^2, \end{aligned} \tag{11}$$

where, in the last step, we took advantage of the linearity of $g(\pm 2)$ in b_0 to use the standard formula

$$\operatorname{Res}_x(x - a, P(x)) = P(a).$$

Thus R is a product of which one factor is the double discriminant $\mathrm{DD}(g) = \operatorname{disc}_{b_0} G = \operatorname{disc}_{b_0} \operatorname{disc}_u g$. One easily sees from the factorization (11) that R is not identically zero as a function of $b_1, \dots, b_n$.

We now proceed as in [8]. The number of $b_1, \dots, b_n \in [-H, H]^n$ such that $R(b_1, \dots, b_n) = 0$ is $O(H^{n-1})$ (by, e.g., [6, Lemma 3.1]), and so the number of g with such $b_1, \dots, b_n$ is $O(H^n)$. We now fix $b_1, \dots, b_n$ such that $R(b_1, \dots, b_n) \neq 0$. Then $R(b_1, \dots, b_n)$ has at most $O_\varepsilon(H^\varepsilon)$ factors $C > H$. Once C is determined by $b_1, \dots, b_n$ (up to $O_\varepsilon(H^\varepsilon)$ possibilities), then the number of solutions for b_0 (mod C) to $h \equiv 0 \bmod C$ is $(\deg_{b_0} h)^{\omega(C)} = O_\varepsilon(H^\varepsilon)$, as the number of possibilities for b_0 modulo p such that $h \equiv 0 \bmod p$ for each $p \mid C$ is at most $\deg_{b_0}(h)$. Since $C > H$, the number of possibilities for $b_0 \in [-H, H]$ is also at most $O_\varepsilon(H^\varepsilon)$, and so the total number of g in this case is $O_\varepsilon(H^{n+\varepsilon})$.

To eliminate the ε, we divide into two subcases, the first of which is reduced to Case I, just as in [8]:

Subcase (i): $A = \prod_{\substack{p \mid C \\ p > H^{\delta/2}}} p \leq H.$

In this subcase, C has a factor C_{I} between $H^{1+\delta/2}$ and $H^{1+\delta}$, with $A \mid C_{\mathrm{I}} \mid C$. Pick C_{I} to be the largest such factor, and let $D_{\mathrm{I}} = \prod_{p \mid C_{\mathrm{I}}} p^{v_p(D)}$. We now appeal to Lemma 5.7 from Case I, with C_{I} in place of C, and D_{I} in place of D. (Note that D determines C, C_{I}, and D_{I}.) We get that the number of g with a given D is at most

$$\ll \frac{O(1)^{\omega(C_{\mathrm{I}})H^{n+1}}}{D_{\mathrm{I}}'^2},$$

where

$$D_{\mathrm{I}}' = \prod_{p \mid C_{\mathrm{I}}} p^{\lceil v_p(D)/2 \rceil} \geq C_{\mathrm{I}} \geq H^{1+\delta/2}.$$

For each D_{I}', there are at most $2^{\omega(D)}$ values of D_{I}, so the total number of reciprocal polynomials in this subcase is

$$\sum_{D_{\mathrm{I}}' > H^{1+\delta/2}} \frac{O(1)^{\omega(C_1)} H^{n+1}}{D_{\mathrm{I}}'^2} = O_\varepsilon(H^{n-\delta/2+\varepsilon}) = O(H^n).$$

Subcase (ii): $A = \prod_{\substack{p \mid C \\ p > H^{\delta/2}}} p > H.$

In this subcase, we carry out the original argument of Case III, with C replaced by A. We have $A \mid R(b_1, \dots, b_n)$.

Fix one of the $O(H^n)$ choices of $b_1, \dots, b_n$ such that $R(b_1, \dots, b_n) \neq 0$. Being bounded above by a fixed power of H, we see that $R(b_1, \dots, b_n)$ can have at most a bounded number of possibilities for the factor A (since all prime factors of A are bounded below by a fixed positive power of H). Once A is determined by $b_1, \dots, b_n$, then the number of solutions for $b_0 \pmod{A}$ to $\operatorname{Disc}(f) \equiv 0 \pmod{A}$ is $O(n^{\omega(A)}) = O(1)$. Since $A > H$, the total number of f in this subcase is also $O(H^n)$, completing the proof of Theorem 5.1 in the nonmonic case.

5.6. ***Remarks on the monic case.*** It would be impractical to replicate the full proof of Theorem 5.1 in the monic case, most of which consists of changing n to $n-1$ and correcting calculations appropriately. The following changes are worthy of note:

- Unlike in the nonmonic case, the space $V(R)$ of monic polynomials over a ring does not have a natural origin. We must fix one, such as $g(u) = u^n$, in order to carry out the Fourier analysis.
- In Case I, we replace Lemma 5.5 with the following lemma from Bhargava:

Lemma 5.10 [8, Proposition 30]. *Let $\sigma = (f_1^{e_1} \cdots f_r^{e_r})$ be a splitting type with* $\deg(\sigma) = d \leq n$ *and* $\operatorname{ind}(\sigma) = k$. *Let $w_{p,\sigma} \colon V(\mathbb{F}_p) \to \mathbb{C}$ be defined by*

> $w_{p,\sigma}(f) :=$ *the number of r-tuples $(P_1, \dots, P_r)$, up to the action of the group of permutations of $\{1, \dots, r\}$ preserving σ, such that the P_i are distinct irreducible monic polynomials with* $\deg P_i = f_i$ *for each i and $P_1^{e_1} \cdots P_r^{e_r} \mid f$.*

Then

$$\widehat{w}_{p,\sigma}(g) = \begin{cases} \dfrac{p^{-k}}{\#\mathrm{Aut}(\sigma)} + O(p^{-(k+1)}) & \text{if } g = 0, \\ O(p^{-(k+1)}) & \text{if } g \neq 0 \text{ and } d < n, \\ O(p^{-(k+1/2)}) & \text{if } g \neq 0 \text{ and } d = n. \end{cases}$$

- We then replace Lemma 5.6 as follows:

Lemma 5.11. *Let $\sigma = (f_1^{e_1} \cdots f_r^{e_r})$ be a splitting type with $f_1 = 1$. Let* $\deg(\sigma) = d$ *and* $\operatorname{ind}(\sigma) = k$. *Let $w'_{p,\sigma} \colon V(\mathbb{F}_p) \to \mathbb{C}$ be defined by*

> $w_{p,\sigma}(f) :=$ *the number of r-tuples $(P_1, \dots, P_r)$, up to the action of the group of permutations of $\{2, \dots, r\}$ preserving σ, such that the P_i are distinct irreducible monic polynomials with* $\deg P_i = f_i$ *for each i, $f_1 = x$, and $P_1^{e_1} \cdots P_r^{e_r} \mid f$.*

Let $V_{e_1,\mathbb{F}_p}$ denote the subspace of $V(\mathbb{F}_p)$ comprising polynomials divisible by x^{e_1}, and let $V_{e_1,\mathbb{F}_p}^{\perp} \subseteq V(\mathbb{F}_p)^*$ be its dual. Then

$$\widehat{w}_{p,\sigma}(g) = \begin{cases} \dfrac{p^{-(k+1)}}{\#\mathrm{Aut}'(\sigma)} + O(p^{-(k+2)}) & \text{if } g \in V_{e_1,\mathbb{F}_p}^{\perp}, \\ O(p^{-(k+2)}) & \text{if } g \notin V_{e_1,\mathbb{F}_p}^{\perp} \text{ and } d < n, \\ O(p^{-(k+3/2)}) & \text{if } g \notin V_{e_1,\mathbb{F}_p}^{\perp} \text{ and } d = n. \end{cases}$$

- Our Ψ_p is then supported on

$$\widetilde{L}_p = \begin{cases} V(\mathbb{Z}) & \text{in case (0) on page 618,} \\ \{g : g(2) \equiv g'(2) \equiv \cdots \equiv g^{(e_{1,p}-1)}(2) \equiv 0 \bmod p\} & \text{in case (2),} \\ \{g : g(-2) \equiv g'(-2) \equiv \cdots \equiv g^{(e_{1,p}-1)}(-2) \equiv 0 \bmod p\} & \text{in case } (-2), \end{cases}$$

which is no longer a lattice but a coset $g_p + L_p$ for some fixed monic polynomial g_p and some lattice L_p of index dividing p^n. The Fourier transform $\widehat{\Psi}$ is small away from $L_p^{\perp}$. The intersection

$$\widetilde{L}_q = \bigcap_{p|q} \widetilde{L}_p = g_q + L_q$$

is likewise a coset of a lattice (if nonempty). When we carry out the twisted Poisson summation, the translation g_q contributes a twist factor $e^{-2\pi\sqrt{-1}g_q \bullet h}$ to the values of $\widehat{\Lambda}_q$. Because we then immediately bound each term by its absolute value, this twist factor drops out.

- The final step of Case I, bounding X_q, proceeds as follows:

$$\begin{aligned} X_q &\ll \prod_{p|q} O(p^{-2k_p}) \prod_{p|\frac{C}{q}} O(p^{-2k_p-1/2}) \cdot \max\Big\{H, \frac{C}{q}\Big\}^n = \frac{O(1)^{\omega(C)}\sqrt{q}}{\sqrt{C}D'^2} \max\Big\{H, \frac{C}{q}\Big\}^n \\ &= \frac{O(1)^{\omega(C)}}{D'^2} \max\left\{\frac{H^n\sqrt{q}}{\sqrt{C}}, \Big(\frac{C}{q}\Big)^{n-1/2}\right\} \le \frac{O(1)^{\omega(C)}}{D'^2} \max\left\{H^n, H^{n-1/2+n\delta}\right\} = \frac{O(1)^{\omega(C)}H^n}{D'^2}. \end{aligned}$$

6. Counting G_3-polynomials

Finally, we count the polynomials f having $G_f \subseteq G_3$ for each odd $n \ge 3$ (the even case is subsumed by G_1, as we noted in stating Theorem 3.4).

Theorem 6.1. *For $n \ge 3$ odd,*

$$\mathcal{E}_n(G_3; H) \ll \begin{cases} H^2 \log^2 H & \text{if } n = 3, \\ H^{(n+1)/2} & \text{if } n \ge 5, \end{cases} \tag{12}$$

$$\mathcal{E}_n^{\mathrm{monic}}(G_3; H) \ll \begin{cases} H^2 & \text{if } n = 3, \\ H^2 \log H \log\log H & \text{if } n = 5, \\ H^{(n-1)/2} \log H & \text{if } n \ge 7. \end{cases} \tag{13}$$

6.1. ***Heights.*** We need a notion of height more general than the naïve height on integer polynomials used in Section 5.

If $P = [x_0 : \cdots : x_N] \in \mathbb{P}^N(K)$ is a point in projective space over a number field K, we define its (exponential, projective, Weil) *height* (denoted $H(P)$ in Silverman [22, §VIII.5]) as

$$\operatorname{Htp} P = \prod_v \max\left\{|x_0|_v, \ldots, |x_N|_v\right\}^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]},$$

where the product is taken over the places v of K, and the norm $|x|_v$ extends the usual v-adic norm on $\mathbb{Q}$. This normalization ensures that the height is unchanged if K is embedded into a larger field. There are two natural ways to define a height on a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over a number field, and we distinguish the *projective* and *affine* heights

$$\operatorname{Htp} f = \operatorname{Htp}\,[a_n : \cdots : a_0], \tag{14}$$

$$\operatorname{Ht} f = \operatorname{Htp}\,[a_n : \cdots : a_0 : 1]. \tag{15}$$

For instance, if $f \in \mathbb{Z}[x]$ is nonzero, then $\operatorname{Ht} f = \max\{|a_0|, \ldots, |a_n|\}$ is the naïve height already introduced in Section 2, while $\operatorname{Htp} f = \operatorname{Ht} f / \operatorname{ct} f$ is smaller by a factor of the *content* $\operatorname{ct}(f) = \gcd(a_0, \ldots, a_n)$. In particular, we define heights of algebraic numbers α by $\operatorname{Ht}\alpha = \operatorname{Htp}\,[\alpha : 1]$. The following properties should be noted:

- If α is an algebraic integer with conjugates $\alpha = \alpha_1, \ldots, \alpha_n$, then

  $$\operatorname{Htp}\alpha = \prod_i \max\{1, \alpha_i\}^{1/n}$$

  is none other than the *Mahler measure* of its minimal polynomial (suitably normalized).

- If f has roots $\alpha_1, \ldots, \alpha_n$, then

  $$2^{-n}\prod_i \operatorname{Ht}\alpha_i \le \operatorname{Htp} f \le 2^{n-1}\prod_i \operatorname{Ht}\alpha_i \tag{16}$$

  [22, Theorem VIII.5.9]. In particular, for any factorization $f = gh$, we have

  $$\operatorname{Htp} f \asymp_{\deg f} \operatorname{Htp} g \cdot \operatorname{Htp} h. \tag{17}$$

6.2. ***Proof of Theorem 6.1.***

Proof. Under the conditions, there are only two possibilities for G_f: either $G_f \cong G_3 \cong S_2 \times S_n$, or $G_f \cong S_n$. In the latter case, f is reducible and factors as a product

$$f(x) = c \cdot h(x) \cdot x^n h(1/x), \quad c \in \mathbb{Z}, \quad h \in \mathbb{Z}[x].$$

By taking $|c| = \operatorname{ct} f$, we may assume that $\operatorname{ct} h = 1$. We have $\operatorname{Htp} f = \operatorname{Ht} f/|c| \le H/|c|$, so by (17),

$$\operatorname{Ht} h = \operatorname{Htp} h \ll \sqrt{\frac{H}{|c|}}.$$

As h has $n+1$ free coefficients, the number of polynomials f we get is

$$\ll \sum_{c=1}^{H} \left(\frac{H}{c}\right)^{(n+1)/2} \ll H^{(n+1)/2}.$$

We now assume that $G_f = G_3$. In this case there is a quadratic field $K_2 = \mathbb{Q}(\sqrt{k})$, where k is a squarefree integer, such that $K_f = K_g \cdot K_2$, so that over K_2, the polynomial f factorizes:

$$f(x) = c' \cdot h(x) \cdot x^n h(1/x), \quad c' \in K_2, \quad h \in K_2[x]. \tag{18}$$

We cannot assume that this factorization holds in $\mathcal{O}_{K_2}[x]$ because $\mathcal{O}_{K_2}$ need not be a UFD. However, we can rescale h so that $h(1) \in \mathbb{Z}$. Then $c' = f(1)/h(1)^2 \in \mathbb{Q}$, and the contents $c = \operatorname{ct}(f)$ and $\mathfrak{a} = \operatorname{ct}(h)$ satisfy

$$c\mathcal{O}_{K_2} = c' \cdot \mathfrak{a}^2.$$

Therefore $\mathfrak{a}^2 = (c/c')$ is principal, generated by a rational number; in particular, $\mathfrak{a} = \bar{\mathfrak{a}}$ is self-conjugate. The self-conjugate ideals in a quadratic field are simply the rational multiples of products of ramified primes. Thus, after rescaling by a rational scalar, we have that there is a positive divisor $d \mid k$ such that

$$\mathfrak{a} = \sqrt{(d)} = \begin{cases} \langle d, \sqrt{k} \rangle & \text{if } k \equiv 2, 3 \bmod 4 \\ \left\langle d, \tfrac{1}{2}(k + \sqrt{k}) \right\rangle & \text{if } k \equiv 1 \bmod 4, \end{cases}$$

and $c' = \pm c/d$. Note that $\bar{h}(x)$ and $x^n h(1/x)$ have the same roots, and comparing values at $x = 1$, they must be equal. Thus the coefficients $\theta_i \in K_2$ of $h(x) = \sum_{i=0}^{n} \theta_i x^i$ satisfy

$$\theta_i = \overline{\theta_{n-i}}. \tag{19}$$

We now bound the height of the θ_i. This requires us to control the projective-to-affine height ratio

$$\frac{\operatorname{Htp} h}{\operatorname{Ht} h} = \prod_v \left(\frac{\max\{|\theta_0|_v, \dots, |\theta_n|_v\}}{\max\{|\theta_0|_v, \dots, |\theta_n|_v, 1\}} \right)^{[(K_2)_v:\mathbb{Q}_v]/[K_2:\mathbb{Q}]}.$$

We consider the contributions of the different places v in turn:

- If $v = p \mid d$ is finite, then the numerator is $|\mathfrak{a}|_p = 1/\sqrt{p}$, while the denominator is 1. Since $[(K_2)_v : \mathbb{Q}_v] = [K_2 : \mathbb{Q}] = 2$, we get a contribution $1/\sqrt{p}$ in this case.
- For all other finite v, the numerator and denominator are both 1.
- If v is infinite, the parenthesized fraction is 1 because, by (19),

$$|\theta_0|_v \cdot |\theta_n|_v = |\theta_0|_v \cdot |\overline{\theta_0}|_v = d \cdot \frac{|f(0)|}{c} \geq 1.$$

Hence

$$\operatorname{Ht} h = \sqrt{d} \operatorname{Htp} h \asymp \sqrt{d \operatorname{Htp} f} = \sqrt{\frac{d \operatorname{Ht} f}{c}} \leq \sqrt{\frac{dH}{c}}.$$

Write

$$\theta_i = \frac{du_i + v_i\sqrt{k}}{2}, \quad u_i, v_i \in \mathbb{Z},$$

and observe that not *all* the u_i nor *all* the v_i can be zero, or f would be reducible. Now some coefficient du_i of $h + \bar{h}$ is nonzero and thus at least d in absolute value. So

$$d \le \mathrm{Ht}(h + \bar{h}) \ll \mathrm{Ht}(h) \ll \sqrt{\frac{dH}{c}},$$

which gives us the bound

$$d \ll \frac{H}{c}.$$

Analogously, analyzing the v_i by considering $h - \bar{h}$ gives us the bound

$$d' := \frac{|k|}{d} \ll \frac{H}{c}.$$

Since du_i and $v_i\sqrt{|k|}$ are bounded by $\mathrm{Ht}\, h$, the numbers of possibilities for each u_i and v_i are $\ll \sqrt{H/cd}$ and $\ll \sqrt{H/cd'}$ respectively, and so the total number of polynomials is

$$\begin{aligned} E_n(G_3; H) &\ll \sum_{c=1}^{H} \sum_{d \ll H/c} \sum_{d' \ll H/c} \left(\sqrt{\frac{H}{cd}}\sqrt{\frac{H}{cd'}}\right)^{\frac{n+1}{2}} \\ &= H^{\frac{n+1}{2}} \sum_{c=1}^{H} c^{-\frac{n+1}{2}} \left(\sum_{d \ll H/c} d^{-\frac{n+1}{4}}\right)^2. \end{aligned}$$

If $n \ge 5$, the two remaining sums are both $O(1)$, and we get a bound of $O(H^{\frac{n+1}{2}})$. If $n = 3$, we get

$$E_n(G_3; H) \ll H^{\frac{n+1}{2}} \sum_{c=1}^{H} c^{-\frac{n+1}{2}} \log^2 H \ll H^{\frac{n+1}{2}} \log^2 H = H^2 \log^2 H. \qquad \square$$

Remark 6.2. The "$\ll$" in Theorem 6.1 can be sharpened to "$\asymp$" in the nonmonic case by checking that a positive proportion of choices of the independent variables c, d, d', u_i, v_i satisfy the needed conditions:

- d and d' are squarefree and coprime;
- u_i and v_i satisfy the conditions mod 2 so that $\theta_i \in \mathfrak{a}$;
- and the θ_i do not all lie in an ideal strictly smaller than $\mathfrak{a}$.

All these can be managed using an appropriate form of the squarefree sieve. We omit the details.

6.3. *Remarks on the monic case.* The monic case is proved similarly, but the possibilities are more restricted because we can factor $f(x) = \pm h_1(x) \cdot x^n h_1(1/x)$ over $\mathcal{O}_{K_2}$, where $h_1(x)$ is monic. This scaling h_1 of h may or may not be compatible with the one used above, but since $\mathrm{ct}(h_1) = (1)$, we derive that $\mathfrak{a} = (\theta)$ is principal. Since $\mathfrak{a}$ is known to be self-conjugate, we have a relation

$$\theta = \varepsilon\bar{\theta}, \tag{20}$$

where $\varepsilon \in \mathcal{O}_{K_2}^\times$ is a unit (necessarily of norm 1). The relation (20) determines θ up to scaling by $\mathbb{Q}^\times$; namely

$$\theta = w(1+\varepsilon) \quad \text{and/or} \quad \theta = w\sqrt{k}(1-\varepsilon), \quad w \in \mathbb{Q}^\times,$$

the expression "and/or" being used because these formulas become invalid if $\varepsilon = -1$, respectively $\varepsilon = 1$. Thus ε determines d. Since θ can be rescaled by a unit, it follows that rescaling ε by a square of a unit does not affect d. Since $|\mathcal{O}_{K_2}^\times/(\mathcal{O}_{K_2}^\times)^2|$ is uniformly bounded (at most 4), there are $O(1)$ possible values of d for each k. Each of the coefficients $\theta_1, \dots, \theta_{(n-1)/2}$ has $O(H/\sqrt{k})$ values, as above. As to θ_0, we have that $\eta = \theta_0^2/d$ is a unit and determines θ_0 up to sign. We have $\operatorname{Ht}\theta_0 \ll \sqrt{H/d}$ so $\operatorname{Ht}\eta \ll H$. Up to the finite group of roots of unity, η is a power η_1^m of the fundamental unit η_1 of K_2. We have $|\eta_1| \gg \sqrt{k}$, so the number of possibilities for η is $\ll \log H/\log k$, for a grand total of

$$\mathcal{E}_n^{\text{monic}}(G_3; H) \ll \sum_{2 \le k \ll H^2} \left(\frac{H}{\sqrt{k}}\right)^{\frac{n-1}{2}} \cdot \frac{\log H}{\log k}.$$

Estimating this sum yields the claimed bounds. In contrast to the nonmonic case, it is unclear whether our bounds are sharp, specifically in the $n = 3$ and $n = 5$ cases. A more accurate count of monic G_3-polynomials demands a delicate understanding of the distribution of sizes of fundamental units in real quadratic fields.

Acknowledgements

Anderson was partially supported by the National Science Foundation (grants 2231990, 2237937). We thank Hongyi (Brian) Hu and the rest of the participants in the Research Topics in Mathematical Sciences semester at CMU in spring 2024 for their contributions to the discussion. We thank Arno Fehm, Benedict Gross, Igor Shparlinski, and the referees for insightful comments. We thank Robert Lemke Oliver for a careful read of an earlier draft.

References

[1] T. C. Anderson, A. Gafni, R. J. Lemke Oliver, D. Lowry-Duda, G. Shakan, and R. Zhang, "Quantitative Hilbert irreducibility and almost prime values of polynomial discriminants", *Int. Math. Res. Not.* **2023**:3 (2023), 2188–2214. MR

[2] L. Bary-Soroker and A. Fehm, "Correlations of sums of two squares and other arithmetic functions in function fields", *Int. Math. Res. Not.* **2019**:14 (2019), 4469–4515. MR

[3] L. Bary-Soroker and G. Kozma, "Irreducible polynomials of bounded height", *Duke Math. J.* **169**:4 (2020), 579–598. MR

[4] L. Bary-Soroker, D. Koukoulopoulos, and G. Kozma, "Irreducibility of random polynomials: general measures", *Invent. Math.* **233**:3 (2023), 1041–1120. MR

[5] L. Bary-Soroker, O. Ben-Porath, and V. Matei, "Probabilistic Galois theory: the square discriminant case", *Bull. Lond. Math. Soc.* **56**:6 (2024), 2162–2177. MR

[6] M. Bhargava, "The geometric sieve and the density of squarefree values of invariant polynomials", preprint, 2014. arXiv 1402.0031

[7] M. Bhargava, "A proof of van der Waerden's conjecture on random Galois groups of polynomials", *Pure Appl. Math. Q.* **19**:1 (2023), 45–60. MR

[8] M. Bhargava, "Galois groups of random integer polynomials and van der Waerden's conjecture", preprint, 2024. To appear in *Ann. of Math.* arXiv 2111.06507

[9] E. Breuillard and P. P. Varjú, "Irreducibility of random polynomials of large degree", *Acta Math.* **223**:2 (2019), 195–249. MR

[10] S. Chow and R. Dietmann, "Enumerative Galois theory for cubics and quartics", *Adv. Math.* **372** (2020), 107282, 37. MR

[11] S. Davis, W. Duke, and X. Sun, "Probabilistic Galois theory of reciprocal polynomials", *Exposition. Math.* **16**:3 (1998), 263–270. MR

[12] P. X. Gallagher, "The large sieve and probabilistic Galois theory", pp. 91–101 in *Analytic number theory* (St. Louis, MO, 1972), Proc. Sympos. Pure Math. **24**, Amer. Math. Soc., 1973. MR

[13] B. H. Gross and C. T. McMullen, "Automorphisms of even unimodular lattices and unramified Salem numbers", *J. Algebra* **257**:2 (2002), 265–290. MR

[14] D. Hokken, "Counting (skew-)reciprocal Littlewood polynomials with square discriminant", preprint, 2023. arXiv 2301.05656

[15] F. Jouve, E. Kowalski, and D. Zywina, "Splitting fields of characteristic polynomials of random elements in arithmetic groups", *Israel J. Math.* **193**:1 (2013), 263–307. MR

[16] E. Kim, T. C. Anderson, and R. J. L. Oliver, "Characterizing the support of semiclassical measures for higher-dimensional cat maps", preprint, 2024. arXiv 2410.13449

[17] H.-W. Knobloch, "Zum Hilbertschen Irreduzibilitätssatz", *Abh. Math. Sem. Univ. Hamburg* **19** (1955), 176–190. MR

[18] A. Lubotzky and L. Rosenzweig, "The Galois group of random elements of linear groups", *Amer. J. Math.* **136**:5 (2014), 1347–1383. MR

[19] S. Mattarei and M. Pizzato, "Generalizations of self-reciprocal polynomials", *Finite Fields Appl.* **48** (2017), 271–288. MR

[20] V. Naik, "Double cover of symmetric group", Groupprops article, 2013, available at https://groupprops.subwiki.org/wiki/Double_cover_of_symmetric_group. Accessed 4 May 2024.

[21] I. Rivin, "Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms", *Duke Math. J.* **142**:2 (2008), 353–379. MR

[22] J. H. Silverman, *The arithmetic of elliptic curves*, 2nd ed., Graduate Texts in Mathematics **106**, Springer, 2009. MR

[23] B. L. van der Waerden, "Die Seltenheit der reduziblen Gleichungen und der Gleichungen mit Affekt", *Monatsh. Math. Phys.* **43**:1 (1936), 133–147. MR

[24] "Covering groups of the alternating and symmetric groups", Wikipedia page, 2024, available at http://en.wikipedia.org/wiki/Covering_groups_of_the_alternating_and_symmetric_groups. Accessed 4 May 2024.

Communicated by Melanie Matchett Wood
Received 2024-07-17 Revised 2025-01-16 Accepted 2025-03-03

tanders2@andrew.cmu.edu *Carnegie Mellon University, Pittsburgh, PA, United States*

abertell@alumni.cmu.edu *Pennsylvania State University, University Park, PA, United States*

eodorney@andrew.cmu.edu *Carnegie Mellon University, Pittsburgh, PA, United States*

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at the ANT website.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in *ANT* are usually in English, but articles written in other languages are welcome.

Length There is no a priori limit on the length of an *ANT* article, but *ANT* considers long articles only if the significance-to-length ratio is appropriate. Very long manuscripts might be more suitable elsewhere as a memoir instead of a journal article.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Algebra & Number Theory

Volume 20 No. 3 2026