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Julian Lawrence Demeio

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Ramified descent and transcendental Brauer–Manin obstruction

Julian Lawrence Demeio

Given a smooth geometrically connected variety X defined over a number field K and an étale torsor $V \to U$ over a Zariski-open U of X, we investigate the problem of which adelic points of X can be approximated by adelic points that lift to a (twist of a) V. The question has long been investigated in the literature when $U = X$, but less so in the general case. We introduce a Brauer–Manin obstruction to the problem, and provide an example where this obstruction is nontrivial and purely transcendental. This answers in the negative a question posed by Harari at a 2019 workshop. Our example is also an explicit example of a nontrivial transcendental Brauer–Manin obstruction on a smooth compactification of a quotient SL_n/G, with G constant metabelian.

1. Introduction

Descent theory has long been used to understand how rational points $X(K)$ of a smooth complete variety X defined over a number field K are distributed in the adelic points $X(\mathbb{A}_K)$. It was first developed for proper varieties by Colliot-Thélène and Sansuc [5; 26], and it was later extended to open varieties by Harari and Skorobogatov [15, Chapter 6]. We investigate the matter of "ramified descent", i.e., the behavior of open descent theory under compactification, and answer a question of Harari on this topic.

For a torsor $\lambda : V \to U$ under a group of multiplicative type M/K, the *descent set* is defined to be the set of those adelic points of U that lift to adelic points of a K-twist of V:

$$U(\mathbb{A}_K)^{\lambda} = \bigcup_{\sigma \in H^1(K,M)} \lambda_\sigma(V_\sigma(\mathbb{A}_K)).$$

It follows from open descent theory [15, Proposition 3.1] that this set may be described in terms of a(n algebraic) Brauer–Manin obstruction: i.e., there exists a subgroup $\mathrm{Br}_\lambda U \subseteq \mathrm{Br}_1 U$ such that

$$U(\mathbb{A}_K)^{\lambda} = U(\mathbb{A}_K)^{\mathrm{Br}_\lambda U}. \tag{1-1}$$

Now let X be a smooth compactification of U, and $X(\mathbb{A}_K)^{\lambda}$ be the adelic closure of $U(\mathbb{A}_K)^{\lambda}$ in $X(\mathbb{A}_K)$. The first result of this paper is that $X(\mathbb{A}_K)^{\lambda}$ provides an obstruction to the Hasse principle and weak approximation for X:

Theorem 1.1. *The inclusion $\overline{X(K)} \subseteq X(\mathbb{A}_K)^{\lambda}$ holds.*

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The proof we present, quite compact, is due to Olivier Wittenberg, whom the author thanks profoundly. (The original proof that the author had in mind was much more involved.)

We are mainly interested in this paper in the case where M is finite, in which case we call $X(\mathbb{A}_K)^\lambda$ the *ramified descent set*, the adjective "ramified" indicating that the relative normalization of U in V is allowed to be ramified. (See Section 3 for an alternative definition of the ramified descent set.)

In a 2019 workshop, Harari formulated a question: to investigate how (1-1) behaves under "compactification". We present his question as Question 4.1, but a special, yet illustrative, case may be reformulated as follows (see Proposition 4.2):

Question 1.2 (Harari). Assume that $\operatorname{Br} X/\operatorname{Br}_0 X$ and M are finite. Does $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^{\operatorname{Br}_\lambda U \cap \operatorname{Br} X}$ always?

Note that the inclusion $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\operatorname{Br}_\lambda U \cap \operatorname{Br} X}$ follows from (1-1). Harari's question was motivated by the fact that the analog of Question 1.2 has a positive answer when M is a torus (see [3, Proposition 3.1]). Moreover, when $M = \mu_n$ is cyclic, and some mild ramification assumptions are satisfied, then a positive answer follows from a result of Colliot-Thélène and Skorobogatov [7, Theorem 14.2.25] (see Appendix B for details).

We answer Question 1.2 negatively. To do so, we introduce in Section 5 a new Brauer subgroup $\operatorname{Br}_\lambda^{\mathrm{ram}} X \subseteq \operatorname{Br} X$, defined as the intersection $\operatorname{Br}_\lambda^{\mathrm{ram}} U \cap \operatorname{Br} X$, where $\operatorname{Br}_\lambda^{\mathrm{ram}} U$ is the image of the composition

$$H^2(\Gamma_M, \overline{K}^*) \to H^2(\Gamma_M, \overline{K}[V]^*) \xrightarrow{\check{C}_\mathfrak{l}} H^2(U, \mathbb{G}_m) = \operatorname{Br} U,$$

where $\check{C}_\mathfrak{l}$ is the Čech-to-étale map associated to $V_{\overline{K}} \to U$, which is a profinite torsor under the constant profinite group $\Gamma_M := M(\overline{K}) \rtimes \Gamma_K$ (see Section 5 for details). In the special case where both $\operatorname{Pic} V$ and $\overline{K}[V]^*/\overline{K}^*$ vanish, one has $\operatorname{Br}_\lambda^{\mathrm{ram}} X = \operatorname{Ker}(\operatorname{Br} X \to \operatorname{Br} V_{\overline{K}})$; see Remark 5.1.4. (In Examples 5.1.5 we provide some examples where this vanishing happens.) One verifies that

Proposition 1.3. *The inclusion $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\operatorname{Br}_\lambda^{\mathrm{ram}} X}$ holds.*

The group $\operatorname{Br}_\lambda^{\mathrm{ram}} X$ contains $\operatorname{Br}_\lambda U \cap \operatorname{Br} X$; see Proposition 5.1.6. However, the former may in general be a bigger group and provide a bigger Brauer–Manin obstruction. We provide some explicit families where this is indeed the case in Section 6.1, where we prove:

Proposition 1.4. *Let K be a number field and e be a natural number such that $\mu_e \subset K^*$. If H is a constant metabelian finite subgroup of SL_n of exponent e, H^{ab} is its abelianization, and λ is the H^{ab}-torsor $\operatorname{SL}_n/[H, H] \to \operatorname{SL}_n/H$, then $\operatorname{Br}_\lambda^{\mathrm{ram}} X = \operatorname{Br} X$.*

Theorem 1.5. *For every number field K and every prime $p \geq 5$ such that $\mu_p \subset K^*$, there exists a constant nilpotent metabelian finite group H of exponent p such that, for any embedding $H \hookrightarrow \operatorname{SL}_{n,K}$, letting X be a smooth compactification of $\operatorname{SL}_{n,K}/H$, we have $\operatorname{Br}_a X = 0$ and*

$$X(\mathbb{A}_K)^{\operatorname{Br} X} \neq X(\mathbb{A}_K).$$

Here $\mathrm{Br}_a X$ denotes the algebraic Brauer group of X modulo constants. Combining Proposition 1.4, Theorem 1.5, and Proposition 1.3 we obtain the sought negative answer to Harari's question: indeed, for X as in the theorem, $\mathrm{Br}_\lambda U \cap \mathrm{Br} X \subseteq \mathrm{Br}_1 X$ is constant as $\mathrm{Br}_a X = 0$, while $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X} = X(\mathbb{A}_K)^{\mathrm{Br} X} \neq X(\mathbb{A}_K)$.

Incidentally, Theorem 1.5 appears to be only the second known example of transcendental obstruction to weak approximation for quotients SL_n / H, or, in other words (see [13, Section 1.2]) to the Grunwald problem for a finite group H. The first such example was obtained by Demarche, Lucchini and Neftin in [9, Theorem 1.2]. But in contrast with loc.cit., where the existence of a transcendental obstruction is proven nonconstructively, we show the nontriviality of the transcendental Brauer–Manin pairing by computing it explicitly.

Structure of the paper. In Section 2 we settle our notation. In Section 3 we prove some basic facts about the "descent set" $X(\mathbb{A}_K)^\lambda$ for a finite M. Since it requires no further effort, we replace M here with a general finite group scheme G/K (not necessarily commutative). In the same section, we prove that $X(\mathbb{A}_K)^\lambda$ provides an obstruction to Hasse principle and weak approximation on the whole X (see Corollary 3.2.2).

In Section 4, we formulate Harari's question precisely and show its relation with Question 1.2.

In Section 5, we introduce the Brauer subgroup $\mathrm{Br}_\lambda^{\mathrm{ram}} X$, prove that it contains $\mathrm{Br}_\lambda U \cap \mathrm{Br} X$ and that it obstructs $X(\mathbb{A}_K)^\lambda$.

In Section 6, we prove Proposition 1.4 and Theorem 1.5. We do so by explicitly computing the unramified Brauer–Manin pairing on SL_n / H for H nilpotent metabelian of odd prime exponent. This explicit computation extends earlier work of Bogomolov [1, Section 5].

Appendix A contains some elementary lemmas that are used in Section 6.1. Appendix B talks briefly about other already existing works containing the idea of "ramified descent".

2. Notation

Fields. Unless specified otherwise, k will always denote a field of characteristic 0 and K a number field.

M_K (resp. M_K^f, M_K^∞) denotes the set of (nonarchimedean, archimedean) places of K.

For a place $v \in M_K$ (resp. $v \in M_K^f$), K_v (resp. O_v) denotes the v-adic completion of K (resp. the v-adic integers).

$\mathbb{A}_K$ (resp. $\mathbb{A}_K^S$, for a subset $S \subset M_K$) denotes the topological ring of adeles of K (resp. S-adeles), i.e., the topological ring $\prod'_{v\in M_K} K_v$ (resp. $\prod'_{v\in M_K\setminus S} K_v$), the restricted product being on $O_v \subseteq K_v$.

For a finite subset $S \subseteq M_K$, K_S denotes the product $\prod_{v\in S} K_v$. We let K_Ω denote the product $\prod_{v\in M_K} K_v$.

For a Galois extension L/K, $\mathrm{Gal}(L/K)$ denotes the Galois group of the extension. For a field k with algebraic closure $\overline{k}$, $\Gamma_k := \mathrm{Gal}(\overline{k}/k)$.

Duals. For a group M of multiplicative type over a field k (i.e., a commutative group scheme which is an extension of a finite group scheme by a torus), $M' := \mathcal{H}om(M, \mathbb{G}_m)$ denotes its Cartier dual.

For a torsion abelian group A, A^D denotes the profinite abelian group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology. If A is a profinite abelian group, A^D denotes the torsion group $\mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ is endowed with its discrete topology. By Pontryagin duality, if A is torsion or profinite, there is a canonical isomorphism $A \cong (A^D)^D$.

Geometry. All schemes we consider are separated. We tacitly assume this throughout the paper.

A *variety* X over a field k is an integral scheme of finite type over a field k.

For a k-scheme X, we denote the residue field of a point $\xi \in X$ by $k(\xi)$. We denote the base change $X_{\bar{k}}$ by $\overline{X}$.

Groups and torsors. Group actions are assumed to be right actions unless specified otherwise.

Let S be a scheme, G be a group scheme over S and X be an S-scheme. A right G-*torsor* over X is an X-scheme $Y \to X$, endowed with a G-action $m : Y \times_S G \to Y$ that is X-equivariant (i.e., such that the composition $Y \times_S G \xrightarrow{m} Y \to X$ is equal to the composition $Y \times_S G \xrightarrow{\mathrm{pr}_1} Y \to X$) and such that there exists an étale covering $X' \to X$ and an X'-isomorphism $X' \times_X Y \cong X' \times_X G$ that is G-equivariant.

For an abstract group N, and a scheme S (resp. a field F), we denote by N_S (resp. N_F) the S-scheme (resp. F-scheme) $\sqcup_{n \in N} S$, endowed with its natural S(resp. F)-group scheme structure. If X is an S-scheme, a torsor $Y \to X$ under an abstract group G is a torsor under the constant group G_S.

If G/k is an algebraic group, and $k \subseteq F$ is a field extension, we use the notation $H^i(F, G)$ (with $i \in \mathbb{N}$ and $i = 0, 1$ if G is not commutative) to denote the cohomology group/set $H^i(\Gamma_F, G(\overline{F})) = Z^i(\Gamma_F, G(\overline{F}))/B^i(\Gamma_F, G(\overline{F}))$ (where $B^i(\Gamma_F, G(\overline{F}))$ is a subgroup when G is commutative and is just an equivalence relation otherwise).

If G is not commutative the set of cocycles $Z^1(\Gamma_F, G(\overline{F}))$ is the one of nonabelian (1-)cocycles, i.e., those functions $g_\sigma : \Gamma_F \to G(\overline{F})$ that satisfy $g_{\sigma\tau} = g_\sigma\, {}^\sigma g_\tau$. The set $H^1(\Gamma_F, G(\overline{F}))$ is the quotient of 1-cocycles by the equivalence relation $B^1(\Gamma_F, G(\overline{F})) : g_\sigma \sim g'_\sigma$ if there exists $g \in G(\overline{F})$ such that $g'_\sigma = g^{-1} g_\sigma\, {}^\sigma g$. Note that these cocycles correspond to (left) G-torsors through the standard correspondence [26, p. 18, 2.10].

If $\xi \in Z^1(K, G)$, we use the notation G^ξ to denote the inner twist of G by ξ, and G_ξ to denote the left principal homogeneous space of G obtained by twisting G by the cocycle ξ. This twist is naturally endowed with a right action of G^ξ. See [26, pp. 12–13] for details on these constructions.

If X is a quasi-projective k-scheme endowed with a G-action, and $\xi \in Z^1(k, G)$, we use the notation X_ξ to denote the twisted quasi-projective k-scheme $(X \times_k^G G_\xi)$. (We refer the reader to [[26, p. 20]; [25, Section I.5.3]; [25, Section III.1.3]] for the existence of the twist and immediate properties of the twisting operation). The k-scheme X_ξ is naturally endowed with a G^ξ-action. We recall that there always exists a $G \times_k \bar{k}$-equivariant isomorphism $X_\xi \times_k \bar{k} \cong X \times_k \bar{k}$. If X' is another k-scheme and $\psi : X \to X'$ is a G-invariant morphism (i.e., G-equivariant when we endow X' with the trivial action), we denote by $\psi_\xi : X_\xi \to X'$ the twisted form of ψ by ξ.

If X is endowed with a left G-action we may still do the twisting operations, by taking the corresponding right action, using the canonical isomorphism $G \cong G^{op}$, $g \mapsto g^{-1}$.

Equivariant commutative diagrams. Let S be a scheme. For S-group schemes G_1, G_2, a (usually implicit) homomorphism $G_1 \to G_2$, and torsors $Z_1 \xrightarrow{G_1} W_1$, $Z_2 \xrightarrow{G_2} W_2$, a diagram

$$\begin{array}{ccc} Z_1 & \longrightarrow & Z_2 \\ \downarrow{\scriptstyle G_1} & & \downarrow{\scriptstyle G_2} \\ W_1 & \longrightarrow & W_2 \end{array} \tag{2-1}$$

commutes if the underlying diagram is commutative and $Z_1 \to Z_2$ is $(G_1 \to G_2)$-equivariant.

Category of torsors. Let S be a scheme, and X an S-scheme. The category of torsors over X with base-scheme S is the category whose objects are pairs (Y, G), where G is an S-group scheme, and Y is a G-torsor over X, and whose morphisms are pairs $(Y_1 \to Y_2, G_1 \to G_2)$, where $G_1 \to G_2$ is a homomorphism, and $Y_1 \to Y_2$ is a $(G_1 \to G_2)$-equivariant morphism.

Profinite (étale) torsors under constant profinite groups. Let S be a scheme, X an S-scheme, and G be a(n abstract) profinite group. A *profinite torsor* over X under G is an X-scheme $Y \to X$, endowed with a G-action $m : Y \times G \to Y$ that is X-equivariant, and such that there exists an inverse system $(Y_i, G_i), i \in I$ in the category of torsors over X with each G_i finite, and such that there exist an isomorphism $\psi : G \xrightarrow{\sim} \operatorname{proj\,lim} G_i$ and a ψ-equivariant isomorphism $Y \cong \operatorname{proj\,lim} Y_i$.

(The inverse limit of the Y_i exists in the category of schemes, as each Y_i is the relative spectrum of a finite étale $\mathcal{O}_X$-algebra $\mathcal{O}_{Y_i}$, and their inverse limit may be realized as the relative spectrum of the finite étale ind-algebra $\operatorname{inj\,lim} \mathcal{O}_{Y_i}$ [28, Tag 01YV].)

When in addition Y is connected, we say that $Y \to X$ is a *profinite étale Galois cover*; see [21, Remark 2.21(b)]. See also [31, Section 2] for details and an alternative definition.

Brauer group. Recall that the Brauer group of a scheme X is defined to be the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$. When X is a variety defined over a number field K, this provides an obstruction, known as *Brauer–Manin obstruction*, to local-global principles, in the following sense. There is a *Brauer–Manin* pairing

$$X(\mathbb{A}_K) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z},$$

sending $((P_v)_{v \in M_K}, B)$ to $((P_v)_{v \in M_K}, B)_{BM} := \sum_v \operatorname{inv}_v B(P_v)$, where $\operatorname{inv}_v : H^2(\Gamma_{K_v}, \overline{K_v}^*) \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map (see, e.g., [14, Theorem 8.9] for a definition). Whenever $B \in \operatorname{Im} \operatorname{Br} K$ or $(P_v)_{v \in M_K} \in X(K)$ (diagonally embedded in $X(\mathbb{A}_K)$), $((P_v)_{v \in M_K}, B)_{BM} = 0$ by the Albert–Brauer–Hasse–Noether theorem (see [26, Section 5]). It follows that $X(K)$ is a subset of

$$X(\mathbb{A}_K)^{\operatorname{Br} X} := \{(P_v)_{v \in M_K} \in X(\mathbb{A}_K) \mid ((P_v)_{v \in M_K}, B)_{BM} = 0 \text{ for all } B \in \operatorname{Br} X\}.$$

For a geometrically integral scheme X over a field F, we notate $\operatorname{Br}_1 X := \operatorname{Ker}(\operatorname{Br} X \to \operatorname{Br} X\overline{F})$ and $\operatorname{Br}_0 X := \operatorname{Im}(p^* : \operatorname{Br} F \to \operatorname{Br} X)$ as usual, where $p : X \to \operatorname{Spec} F$ denotes the structural morphism.

When X is smooth and integral over F, and $U \subseteq X$ is an open subscheme, we identify, with a slight abuse of notation, the injective [7, Theorem 3.5.5] pullback $\operatorname{Br} X \to \operatorname{Br} U$ with an inclusion

$\operatorname{Br} X \subseteq \operatorname{Br} U \subseteq \operatorname{Br} k(X)$. We say β is *unramified* if $\beta \in \operatorname{Br} X^c$ for one (equivalently, by [7, Proposition 3.7.10], every) smooth compactification X^c of X. We denote the subgroup of unramified elements by $\operatorname{Br}_{\mathrm{ur}}(k(X))$ or $\operatorname{Br}_{\mathrm{ur}} X$.

Cohomology. For a scheme X and an étale abelian sheaf $\mathcal{F}$ on X, $H^n(X, \mathcal{F}), n \geq 0$ denotes the étale cohomology group $H^n_{\text{ét}}(X, \mathcal{F})$.

Map from Čech cohomology to étale cohomology. Let U be a scheme, $\mathcal{F}$ an étale sheaf on U, and $\phi : V \to U$ an étale cover. We may think of ϕ as an étale covering $\mathcal{U}$ of U made of a single cover: $\mathcal{U} = \{V \to U\}$. Recall that to such a covering we may naturally associate its Čech cohomology groups $\check{H}^n(V/U, \mathcal{F}), n \geq 0$ [21, Section III.2]. There are natural Čech-to-étale morphisms

$$\check{C}_\phi : \check{H}^n(V/U, \mathcal{F}) \to H^n(U, \mathcal{F}) \tag{2-2}$$

for each $n \geq 0$ (as edge maps of the first spectral sequence in [21, Proposition III.2.7]).

When $\phi : V \to U$ is an étale torsor under the constant finite group G, there are natural identifications [21, Example III.2.6] (technically loc.cit. is formulated for Galois covers, or equivalently for *connected* torsors under constant finite groups, but the connectedness assumption is never used):

$$\check{H}^n(V/U, \mathcal{F}) = H^n(G, \mathcal{F}(V)), \tag{2-3}$$

where the latter denotes group cohomology. Under these identifications, (2-2) becomes

$$\check{C}_\phi : H^n(G, \mathcal{F}(V)) \to H^n(U, \mathcal{F}). \tag{2-4}$$

If $\phi : V \to U$ is a torsor under a profinite constant group G, then one may define natural maps $\check{C}_\phi : H^n(G, \mathcal{F}(V)) \to H^n(U, \mathcal{F}), n \geq 0$, as the colimit of (2-4) on all quotients $V/H \to U$ by open subgroups H of G. See [21, Remark III.2.21(b)] for details.

Remark 2.1. When $Y = X \times G$ (i.e., ϕ is the trivial torsor) with G finite, then the trivial covering $\mathcal{U}' = \{U \to U\}$ refines $\mathcal{U} = \{V \to U\}$ via the morphism $U = U \times \{e\} \hookrightarrow U \times G$. Thus $\check{C}_\phi : H^n(G, \mathcal{F}(V)) \to H^n(U, \mathcal{F})$ factors through $H^n(U/U, \mathcal{F}(U))$. The latter is 0 for $n \geq 1$, and thus $\check{C}_\phi = 0$ for $n \geq 1$.

3. Descent set

3.1. *Definitions.*

Descent set for torsors. Let K be a number field, G/K a finite group scheme, $p : U \to \operatorname{Spec} K$ a smooth geometrically connected variety over K, and $\lambda : V \to U$ a G-torsor.

For every $\xi \in H^1(K, G)$, there exists a twisted form $\lambda_\xi : V_\xi \to U$ of the torsor λ. This is a torsor under the twisted form G^ξ of G. The class $[\lambda_\xi] \in H^1(U, G^\xi)$ is given by the image of $[\lambda] \in H^1(U, G)$ under the well-known isomorphism [26, pp. 20–21]

$$H^1(U, G) \to H^1(U, G^\xi), [V] \mapsto [V_\xi].$$

When G is commutative, we have $G^\xi = G$, and the morphism $H^1(U, G) \to H^1(U, G)$, $[V] \mapsto [V_\xi]$ becomes $[V] \mapsto [V] - p^*[\xi]$.

Recall that the descent set $U(\mathbb{A}_K)^\lambda$ associated to λ is defined as

$$U(\mathbb{A}_K)^\lambda := \bigcup_{\xi \in H^1(K,G)} \lambda_\xi(V_\xi(\mathbb{A}_K)) \subseteq U(\mathbb{A}_K). \tag{3-1}$$

This is adelically closed in $U(\mathbb{A}_K)$ [4, Proposition 6.4] and contains $U(K)$ [26, Section 5.3].

Compactifying the descent set. Let X be a smooth compactification of U. Recall from the introduction:

Definition 3.1.1. The *ramified descent set* for λ is

$$X(\mathbb{A}_K)^\lambda := \overline{\bigcup_{\xi \in H^1(K,G)} \lambda_\xi(V_\xi(\mathbb{A}_K))}.$$

The closure denotes the adelic closure in $X(\mathbb{A}_K)$. Let $\psi : Y \to X$ be the relative normalization of X in V. By the universal property of the relative normalization, the G-action on V extends to a G-action on Y. Let $\nu : Y^{\mathrm{sm}} \to Y$ be a G-equivariant desingularization of Y [11], and let ψ^{sm} be the composition $\psi \circ \nu : Y^{\mathrm{sm}} \to X$. The following lemma provides alternative descriptions of the ramified descent set.

Lemma 3.1.2. *We denote by ψ_ξ (resp. ψ_ξ^{sm}) the twisted forms of ψ (resp. ψ^{sm}) by $\xi \in H^1(K, G)$. The following sets coincide*:

(i) $\overline{U(\mathbb{A}_K)^\lambda}$,

(ii) $\overline{\bigcup_{\xi \in H^1(K,G)} \psi_\xi^{\mathrm{sm}}(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))}$,

(iii) $\overline{\bigcup_{\xi \in H^1(K,G)} \psi_\xi(Y_\xi^{reg}(\mathbb{A}_K))}$, *where Y^{reg} is the open subscheme of regular points of Y.*

The closures denote adelic closures in $X(\mathbb{A}_K)$.

Proof. We first prove that (i) and (ii) coincide. Note that $V' := \nu^{-1}(V) \xrightarrow{\nu} V$ is an isomorphism since V is regular. We have

$$\overline{\bigcup_\xi \lambda_\xi(V_\xi(\mathbb{A}_K))} = \overline{\bigcup_\xi \psi_\xi^{\mathrm{sm}}(V'_\xi(\mathbb{A}_K))} = \overline{\bigcup_\xi \overline{\psi_\xi^{\mathrm{sm}}(V'_\xi(\mathbb{A}_K))}} = \overline{\bigcup_\xi \psi_\xi^{\mathrm{sm}}(\overline{V'_\xi(\mathbb{A}_K)})} = \overline{\bigcup_\xi \psi_\xi^{\mathrm{sm}}(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))}, \tag{3-2}$$

where the union is over $\xi \in H^1(K, G)$ everywhere, and in the third term, $\overline{V'_\xi(\mathbb{A}_K)}$ denotes the closure in $Y_\xi^{\mathrm{sm}}(\mathbb{A}_K)$. The first two identities are immediate, the third follows from the properness of ψ_ξ^{sm}, and the fourth holds because $V'_\xi(\mathbb{A}_K)$ is dense in $Y_\xi^{\mathrm{sm}}(\mathbb{A}_K)$ (this follows from [7, Theorem 10.5.1] since Y_ξ^{sm} is smooth). This proves that (i) and (ii) coincide. They also coincide with (iii), since this is contained between the left- and right-hand sides of (3-2). □

Remark 3.1.3.
- Since (iii) is independent of the choice of U and Y^{sm}, the lemma shows that (ii) is as well, and (i) is too in the sense that $X(\mathbb{A}_K)^\lambda$ only depends on the generic fiber $\lambda|_{\operatorname{Spec} K(X)} : \operatorname{Spec} K(V) \to \operatorname{Spec} K(X)$, viewed as a G-torsor, of the torsor λ.
- No conflict of notation on $X(\mathbb{A}_K)^\lambda$ arises with (3-1) when $U = X$, as in this case $U(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^\lambda$ is closed in $X(\mathbb{A}_K)$ by [4, Proposition 6.4].

Warning. As the continuous map $U(\mathbb{A}_K) \hookrightarrow X(\mathbb{A}_K)$ is not a topological immersion, the set $X(\mathbb{A}_K)^\lambda \cap U(\mathbb{A}_K)$ might in general very well be bigger than $U(\mathbb{A}_K)^\lambda$. The reader may verify that in the example given in Section 6.1 this is exactly the case.

Setting. From now on we fix, through Section 5, a number field K, a finite group scheme G/K, a G-torsor $\lambda : V \to U$ over a geometrically integral smooth K-variety $p : U \to \operatorname{Spec} K$, and a smooth compactification X of U.

3.2. ***Obstruction to adelic density of rational points on X.*** Let, as above, $\psi : Y \to X$ be the relative normalization of X in V, $\nu : Y^{\mathrm{sm}} \to Y$ be a G-equivariant desingularization of Y, and r be the composition $\psi \circ \nu : Y^{\mathrm{sm}} \to X$.

Theorem 3.2.1. *The inclusion* $X(K) \subseteq \bigcup_{\xi \in H^1(K,G)} r_\xi(Y_\xi(K))$ *holds.*

Combining Theorem 3.2.1 with Lemma 3.1.2(ii), we deduce:

Corollary 3.2.2. *The inclusion* $\overline{X(K)} \subseteq X(\mathbb{A}_K)^\lambda$ *holds.*

The following proof of Theorem 3.2.1 is due to Olivier Wittenberg, who kindly suggested a proof that is much simpler than the previous one the author had.

Proof of Theorem 3.2.1 (Olivier Wittenberg). Let $d = \dim X$, $P \in X(K)$ be a rational point, and $u_1, \dots, u_d \in \mathcal{O}_{X,P}$ be a regular system of parameters at P. Let $C \subseteq X$ be the Zariski-closure of the curve $u_2 = \cdots = u_d = 0$. Since K is infinite, after a linear change of coordinates of $u_1, \dots, u_d$, we may assume that C is not contained in $D := X \setminus U$.

Note that C is smooth at P. Choosing a local parameter t for C at P, we get a morphism $\operatorname{Spec} K[[t]] \to C$ that sends the special point $t = 0$ to P. This morphism induces a morphism $\operatorname{Spec} K((t)) \to X$, whose set-theoretic image is the generic point of C. In particular, by construction of C, it belongs to U. Hence the G-torsor $V \to U$ gives a class in $H^1(K((t)), G)$, which we may push to $H^1(K((t^{\frac{1}{\infty}})), G)$.

The inclusion $K \subseteq K((t^{\frac{1}{\infty}}))$ induces an identification $\Gamma_{K((t^{\frac{1}{\infty}}))} = \Gamma_K$ (this follows from the algebraic-closedness of $\overline{K}((t^{\frac{1}{\infty}}))$ [24, Chapter IV, Proposition 8]), and hence an identification $H^1(K((t^{\frac{1}{\infty}})), G) = H^1(K, G)$. Hence, after replacing Y with a K-twist, we may assume that the class in $H^1(K((t^{\frac{1}{\infty}})), G)$ is trivial. Therefore it has to be trivial already in $H^1(K((t^{\frac{1}{n}})), G)$ for some $n \geq 1$. In other words, the G-torsor

$$\operatorname{Spec} K((t^{\frac{1}{n}})) \times_U V \to \operatorname{Spec} K((t^{\frac{1}{n}}))$$

has a section. This section induces a commutative diagram

$$\begin{array}{ccc} & \nearrow & V \\ & & \downarrow \\ \operatorname{Spec} K((t^{\frac{1}{n}})) & \longrightarrow & U. \end{array}$$

By the valuative criterion of properness (applied to $Y^{\mathrm{sm}} \to X$), we may extend the diagram above to

$$\begin{array}{ccc} & \nearrow & Y^{\mathrm{sm}} \\ & & \downarrow \\ \operatorname{Spec} K[\![t^{\frac{1}{n}}]\!] & \longrightarrow & X. \end{array}$$

Since the lower morphism specializes to P, the specialization of the diagonal morphism provides the sought lift of P. $\square$

4. Harari's question

Setup. Recall that $p : U \to \operatorname{Spec} K$ is a geometrically integral smooth variety over a number field K, X is a smooth compactification of U, and $\lambda : V \to U$ is a torsor under a finite group scheme G/K. We assume here that $G = A$ is commutative, and let $A' = \operatorname{Hom}(A, \mathbb{G}_{m,K})$ be its Cartier dual. For every v, define $E_v := \operatorname{Im}\big(U(K_v) \to H^1(K_v, A),\ P_v \mapsto [V|_{P_v}]\big)$ (this is not a subgroup in general). Let S be a finite set of places of K, let $(P_v)_{v\in S} \in \prod_{v\in S} U(K_v)$ and $f_v := [V|_{P_v}],\ v \in S$.

Let $\operatorname{Br}_\lambda U < \operatorname{Br}_1(U)$ be the subgroup generated by the cup products $p^*b \cup [V]$, as b varies in $H^1(K, A')$, and let $B := \operatorname{Br}_\lambda U \cap \operatorname{Br}(X)$.

Question 4.1 (Harari). Assume that there is no Brauer–Manin obstruction for $(P_v)_{v\in S}$ with respect to B. Does there exist then an $a \in H^1(K, A)$ such that $a_v = f_v$ for all $v \in S$ and $a_v \in E_v$ for $v \notin S$?

Using Poitou–Tate duality, one may obtain a positive answer to Question 4.1 by replacing E_v with the subgroup $\langle E_v\rangle$ of $H^1(K_v, A)$ generated by it (we leave this as an exercise to the interested reader, or see [10]). However, as originally remarked by Harari, there is a big difference between E_v and $\langle E_v\rangle$ in general!

The following proposition relates Question 4.1 with Question 1.2:

Proposition 4.2. *Assume that* $\operatorname{Br} X/\operatorname{Br}_0 X$ *is finite and* $X(\mathbb{A}_K) \neq \varnothing$. *Then the identity*

$$X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^B$$

holds if and only if there exists a finite $S_0 \subseteq M_K$ *such that, for all* $S \supseteq S_0$, *Question 4.1 has a positive answer.*

Proof. The assumption implies that B is finite.

We prove the forward implication first. Let S_0 be a set of places such that the Brauer–Manin pairing associated to B is trivial outside S_0. Let $S \subseteq M_K$ be a finite set containing S_0, and $(P_v)_{v\in S} \in \big(\prod_{v\in S} U(K_v)\big)^B$. We wish to find an $a \in H^1(K, A)$ such that $a_v = f_v$ for all $v \in S$ and $a_v \in E_v$ for all $v \notin S$. Let $P_v,\ v \notin S$ be any point of $U(K_v)$. Note that $(P_v)_{M_K} \in X(\mathbb{A}_K)^B$. Since $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^B$, we may approximate arbitrarily well $(P_v)_{M_K}$ with an adelic point $(Q_v)_{v\in M_K}$ such that there exists $a \in H^1(K, A)$ for which $(Q_v)_{v\in M_K} \in \lambda_a(V_a(\mathbb{A}_K))$. In particular, $Q_v \in \lambda_a(V_a(K_v))$ for each v, or in other words the torsor $[V_a|_{Q_v}]$ over K_v contains a K_v-point and is thus trivial. It follows that $0 = [V_a|_{Q_v}] = [V|_{Q_v}] - a_v \in H^1(K_v, A)$ for all v, and hence $a_v \in E_v$ for all v. Moreover, the map $[V|_{-}] : U(K_v) \to H^1(K_v, A)$ is locally constant, and thus $[V|_{Q_v}] = [V|_{P_v}] = f_v$ for $v \in S$. The class a is now the sought class.

For the other direction, assume there is an S_0 as in the statement. We need to show that $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^B$. We may assume after enlarging S_0 that its Brauer–Manin pairing on X is trivial for all places $v \notin S_0$.

Let $(P_v)_{M_K} \in X(\mathbb{A}_K)^B$. Since B is finite, $X(\mathbb{A}_K)^B \subseteq X(\mathbb{A}_K)$ is open. In particular, after an arbitrarily small approximation, we may assume that $(P_v)_{M_K} \in U(K_\Omega)^B$. Then, for all $S \supseteq S_0$, our assumption that the answer to Question 4.1 is "yes" shows that there exists an $a \in H^1(K, A)$ such that $a_v = [V|_{P_v}]$ for $v \in S$ and $a_v = [V|_{Q_v}]$ for some $Q_v \in U(K_v)$ for $v \notin S$. Thus the A-torsor $V_a \to X$ specializes to the trivial torsor over P_v, $v \in S$ and over Q_v, $v \notin S$, and hence there exists a point $(R_v) \in V_a(K_\Omega)$ whose image is P_v for $v \in S$ and Q_v for $v \notin S$. In particular, $V_a(K_\Omega) \neq \varnothing$ and therefore $V_a(\mathbb{A}_K) \neq \varnothing$, and we may thus modify R_v so that $(R_v) \in V_a(\mathbb{A}_K)$ and so that the image of R_v is P_v for $v \in S$. We now modify Q_v to $\lambda_a(R_v)$ for $v \notin S$, and by construction we have $((P_v)_{v\in S}, (Q_v)_{v\notin S}) \in U(\mathbb{A}_K)^\lambda$. Enlarging S, the points $((P_v)_{v\in S}, (Q_v)_{v\notin S})$ approximate arbitrarily well $(P_v)_{M_K}$, and thus we deduce that $(P_v)_{M_K} \in X(\mathbb{A}_K)^\lambda$. □

5. A Brauer–Manin obstruction to ramified descent

We recall that $\lambda : V \to U$ is a torsor under a finite group scheme G/K, that U is smooth and geometrically integral over K, that X is a compactification of U, and that $Y \to X$ is the relative normalization of X in V. We defined

$$X(\mathbb{A}_K)^\lambda = \overline{\bigcup_{\xi\in H^1(K,G)} \lambda_\xi(V_\xi(\mathbb{A}_K))}$$

We define in this section a subgroup $\mathrm{Br}_\lambda^{\mathrm{ram}} X \subseteq \mathrm{Br}\, X$ such that $X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X}$. The group $\mathrm{Br}_\lambda^{\mathrm{ram}} X$ may be transcendental, as we show in Section 6.1.

5.1. *Definition of* $\mathbf{Br}_\lambda^{\mathbf{ram}}$ *X*. Let $\mathfrak{l}$ be the composition $V_{\overline{K}} \to V \xrightarrow{\lambda} U$. There are natural $G(\overline{K})$- and Γ_K-actions on $V_{\overline{K}}$, the first induced by the G-action on V, and the second via the second factor of $V_{\overline{K}} = V \times_K \operatorname{Spec} \overline{K}$.

Lemma 5.1.1. *The $G(\overline{K})$- and the Γ_K- actions generate a $(G(\overline{K}) \rtimes \Gamma_K)$-action on $V_{\overline{K}}$. The profinite étale cover $\mathfrak{l} : V_{\overline{K}} \to U$ is a profinite torsor under this $(G(\overline{K}) \rtimes \Gamma_K)$-action.*

Proof. Let L/K be a field extension. The points of V_L with values in a K-algebra R are described by

$$V_L(R) = \bigsqcup_{\iota : L \hookrightarrow R} V(R), \tag{5-1}$$

where ι ranges among all K-embeddings of L in R. Consider the following natural actions:

(i) the right action of $G(L)$ on V_L defined by letting $G(L)$ act via the map $G(L) \to G(R)$ on each disjoint set appearing in (5-1);

(ii) when L/K is Galois, the right $\mathrm{Gal}(L/K)$-action defined by letting $\gamma \in \mathrm{Gal}(L/K)$ act via $\iota \mapsto \iota \circ \gamma^{-1}$.

For $g \in G(L)$ and $\gamma \in \mathrm{Gal}(L/K)$, we have ${}^\gamma g \cdot \gamma \cdot x = \gamma \cdot g \cdot x$, where x is an R-point of V_L. By this relation, the two actions above generate a $(G(L) \rtimes \mathrm{Gal}(L/K))$-action on V_L. This action is fixed-point-free and commutes with the projection $V_L \to U$. In addition, when L/K is finite and splits G (i.e., when $G(L) = G(\overline{K})$), letting $\Gamma := G(L) \rtimes \operatorname{Gal}(L/K)$, we have $|\Gamma| = |G(\overline{K})| \cdot [L : K] = \deg(V_L \to U)$. Thus the morphism $(\mu, id) : \Gamma \times_U V_L \to V_L \times_U V_L$, where μ denotes the Γ-action on V_L, is an injective morphism of finite étale covers of U of the same degree, and hence an isomorphism. In other words, $V_L \to U$ is a torsor under Γ.

When $L = \overline{K}$, the two actions (i) and (ii) are the ones described before the statement of the lemma, and taking an inverse limit over all finite Galois subextensions $L \subset \overline{K}$ that split G (these form a cofinal subset) finishes the proof. □

Let $\Gamma_G := G(\overline{K}) \rtimes \Gamma_K$. Through the construction of Section 2, the profinite étale torsor $V_{\overline{K}} \to U$ under Γ_G gives rise to a Čech-to-étale map on cohomology:

$$\check{C}_{\mathfrak{l}} : H^2(\Gamma_G, \overline{K}[V]^*) \to H^2(U, \mathbb{G}_m), \tag{5-2}$$

where Γ_G acts on $\mathbb{G}_m(V_{\overline{K}}) = \overline{K}[V]^*$ by pullback. The restriction of this Γ_G-action on $\overline{K}^* \subseteq \overline{K}[V]^*$ is equal to the pullback of the natural Γ_K-action along the projection $\Gamma_G \to \Gamma_K$. Hence we have a natural morphism

$$H^2(\Gamma_G, \overline{K}^*) \to H^2(\Gamma_G, \overline{K}[V]^*) = H^2(\Gamma_G, \mathbb{G}_m(V_{\overline{K}})),$$

where the implied action on the LHS is the pullback described above.

Definition 5.1.2. We define the subgroup $\operatorname{Br}^{\mathrm{ram}}_{\lambda}(U)$ of $\operatorname{Br} U$ as the image of the composition

$$H^2(\Gamma_G, \overline{K}^*) \to H^2(\Gamma_G, \overline{K}[V]^*) \xrightarrow{\check{C}_{\mathfrak{l}}} H^2(U, \mathbb{G}_m) = \operatorname{Br} U.$$

We define $\operatorname{Br}^{\mathrm{ram}}_{\lambda} X \subseteq \operatorname{Br}(X)$ as the intersection $\operatorname{Br}(X) \cap \operatorname{Br}^{\mathrm{ram}}_{\lambda}(U)$.

Remark 5.1.3. The fact that U does not appear in the notation "$\operatorname{Br}^{\mathrm{ram}}_{\lambda} X$" is justified by the fact that $\operatorname{Br}^{\mathrm{ram}}_{\lambda} X$ may be defined purely in terms of the ramified G-cover $Y \to X$ (and, in fact, only in terms of the "generic fiber" G-torsor $\operatorname{Spec} K(Y) \to \operatorname{Spec} K(X)$). Indeed $\operatorname{Br}^{\mathrm{ram}}_{\lambda} X = \operatorname{Br}(X) \cap \operatorname{Br}^{\mathrm{ram}}_{\lambda}(K(X))$, where $\operatorname{Br}^{\mathrm{ram}}_{\lambda}(K(X)) \subseteq \operatorname{Br}(K(X))$ is defined as the image of $H^2(\Gamma_G, \overline{K}^*)$ in $H^2(K(X), \mathbb{G}_m)$ through the morphism

$$H^2(\Gamma_G, \overline{K}^*) \to H^2(\Gamma_G, \overline{K}(Y)^*) \xrightarrow{\check{C}_{\mathfrak{l}|K(X)}} H^2(K(X), \mathbb{G}_m).$$

Remark 5.1.4. When $\overline{K}[V]^*/\overline{K}^* = \operatorname{Pic} V_{\overline{K}} = 0$, the Hochschild–Serre spectral sequence

$$H^i(\Gamma_G, H^j(V_{\overline{K}}, \mathbb{G}_m)) \Rightarrow H^{i+j}(U, \mathbb{G}_m)$$

yields the short exact sequence $0 \to H^2(\Gamma_G, \overline{K}^*) \to \operatorname{Br} U \to \operatorname{Br} V_{\overline{K}}$. Hence, in this case, $\operatorname{Br}^{\mathrm{ram}}_{\lambda}(U) = \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br} V_{\overline{K}})$ and

$$\operatorname{Br}^{\mathrm{ram}}_{\lambda} X = \operatorname{Br}^{\mathrm{ram}}_{\lambda}(U) \cap \operatorname{Br} X = \operatorname{Ker}(\operatorname{Br} X \to \operatorname{Br} V_{\overline{K}}).$$

Although the condition $\overline{K}[V]^*/\overline{K}^* = \operatorname{Pic} V_{\overline{K}} = 0$ is rarely satisfied in practice, we give some examples below.

Examples 5.1.5. (I) If V is a simply connected semi-simple algebraic group, then $\overline{K}[V]^*/\overline{K}^* = \text{Pic } V_{\overline{K}} = 0$. To get an example of a G-torsor $V \to U$, we may take as G any finite subgroup(-scheme) of V, and define $U := V/G$.

(II) If V is a universal torsor of a smooth proper rationally connected variety (as defined in [5]), one has $\overline{K}[V]^*/\overline{K}^* = \text{Pic } V_{\overline{K}} = 0$ (see (2.1.1) of loc.cit.). We then let G be any finite subgroup of the Néron–Severi torus T (i.e., the one under which V is a torsor), and let $U = V/G$.

(III) Finally, we give an example where both V and U are open K3 surfaces. Let $\mathcal{E} \to \mathbb{P}^1_K$ be an elliptic K3 surface with Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$, with (exactly) two reducible fibers of Kodaira types I_{10}^* and I_2, and such that the 0-section O and the 2-torsion section τ intersect different components of the I_2-fiber. Such an $\mathcal{E}$ exists, and it may be defined over $\mathbb{Q}$; see 5.2 and 5.3 of [19], for example, where $\mathcal{E}$ is realized as the double-cover of the Kummer surface associated to the Jacobian of a curve of genus 2. Then the Picard group of $\mathcal{E}\overline{K}$ is of rank 16, freely generated by the fourteen components of the I_{10}^*-fiber, by O and by τ. In particular, letting V be the complement of these divisors in $\mathcal{E}$, we have $\overline{K}[V]^* = \overline{K}^*$ and $\text{Pic } \bar{V} = 0$. The involution on $\mathcal{E}$ induced by τ restricts to an involution of V, let $G \cong \mathbb{Z}/2\mathbb{Z}$ be the group generated by it. A smooth minimal compactification X of the quotient $U = V/G$ is an elliptic surface isogenous to the original K3 elliptic surface $\mathcal{E} \to \mathbb{P}^1$, and is thus K3. (In fact, if $\mathcal{E}$ is as in [19], the quotient $U = V/G$ is a Kummer surface; see loc.cit.).

Proposition 5.1.6. *Assume that $G = A$ is commutative. Then* $\text{Br}(X) \cap \text{Br}_\lambda U$ *is contained in* $\text{Br}_\lambda^{\text{ram}} X$.

Proof. We prove the stronger inclusion $\text{Br}_\lambda U \subseteq \text{Br}_\lambda^{\text{ram}} U$. Recall that $\text{Br}_\lambda U$ is generated by the cup products $p^*b \cup [V]$, as b varies in $H^1(K, A')$. Fix a $b \in H^1(K, A')$. Both classes p^*b and $[V]$ trivialize when pulled back along the pro-étale cover $V_{\overline{K}} \to U$. Hence, following, e.g., [26, p. 18], there exist Čech cocycles $b_V \in \check{H}^1(V_{\overline{K}}/U, A')$, $\alpha_V \in \check{H}^1(V_{\overline{K}}/U, A)$ that represent p^*b and $[V]$, i.e., such that $\check{C}_\mathfrak{l}(b_V) = p^*b$ and $\check{C}_\mathfrak{l}(\alpha_V) = [V]$.

We have a commutative diagram of cup products [29, Corollary 3.10]

$$\begin{array}{ccccc}
\check{H}^1(V_{\overline{K}}/U, A') & \times & \check{H}^1(V_{\overline{K}}/U, A) & \xrightarrow{\cup} & \check{H}^2(V_{\overline{K}}/U, \overline{K}^*) \\
\Big\| = & & \Big\| = & & \Big\downarrow \\
\check{H}^1(V_{\overline{K}}/U, A') & \times & \check{H}^1(V_{\overline{K}}/U, A) & \xrightarrow{\cup} & \check{H}^2(V_{\overline{K}}/U, \overline{K}[V]^*) \\
\Big\downarrow \check{C}_\mathfrak{l} & & \Big\downarrow \check{C}_\mathfrak{l} & & \Big\downarrow \check{C}_\mathfrak{l} \\
H^1(U, A') & \times & H^1(U, A) & \longrightarrow & H^2(U, \mathbb{G}_m),
\end{array}$$

see, e.g., [29] for the definition of the Čech cup product on the first two rows. In particular, we have $\check{C}_\mathfrak{l}(b_V \cup \alpha_V) = \check{C}_\mathfrak{l}(b_V) \cup \check{C}_\mathfrak{l}(\alpha_V) = p^*b \cup [V]$, and $p^*b \cup [V]$ belongs to the image $\text{Br}_\lambda^{\text{ram}} U$ of the composition $\check{H}^2(V_{\overline{K}}/U, \overline{K}^*) \to \check{H}^2(V_{\overline{K}}/U, \overline{K}[V]^*) \xrightarrow{\check{C}_\mathfrak{l}} H^2(U, \mathbb{G}_m)$. □

We go on to prove Proposition 1.3, i.e., that $X(\mathbb{A}_K)^\lambda$ is contained in $X(\mathbb{A}_K)^{\text{Br}_\lambda^{\text{ram}} X}$. I profoundly thank one of the anonymous referees, who provided the following proof, which simplifies a lot the previous one the author had in mind.

Proof of Proposition 1.3. We shall prove that $U(\mathbb{A}_K)^\lambda \subseteq U(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} U}$, from which the statement follows by taking adelic closures in $X(\mathbb{A}_K)$ and noting that $U(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} U} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X}$.

Let $(P_v)_{v\in M_K} \in U(\mathbb{A}_K)^\lambda$ and let $\xi \in Z^1(K, G)$ be such that $P_v = \lambda_\xi(Q_v)$, $v \in M_K$ for some $(Q_v)_{v\in M_K} \in V_\xi(\mathbb{A}_K)$. The cocycle ξ defines a section $s_\xi = (\xi, id)$ of the surjection

$$\Gamma_G = G(\overline{K}) \rtimes \Gamma_K \to \Gamma_K \to 1.$$

Recall that $V_{\overline{K}}$ is naturally endowed with a Γ_G-action that makes $V_{\overline{K}} \to U$ a profinite Γ_G-torsor. One easily checks from the definition of twist [26, p.12] that

$$V_\xi = V_{\overline{K}}/s_\xi(\Gamma_K).$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \overline{V} & \xleftarrow{=} & \overline{V} \\ \big\downarrow{\scriptstyle \Gamma_G} & & \big\downarrow{\scriptstyle \Gamma_K} \\ U & \longleftarrow & V_\xi \end{array},$$

equivariant with respect to $s_\xi : \Gamma_K \to \Gamma_G$, which induces by functoriality of $\check{C}$ the commutative diagram

$$\begin{array}{ccccc} H^2(\Gamma_G, \overline{K}^*) & \longrightarrow & \check{H}^2(\overline{V}/U, \mathbb{G}_m) & \xrightarrow{\check{C}_\Gamma} & \mathrm{Br}\, U \\ \big\downarrow{\scriptstyle res^{\Gamma_G}_{s_\xi(\Gamma_K)}} & & \big\downarrow{\scriptstyle \lambda_\xi^*} & & \big\downarrow{\scriptstyle \lambda_\xi^*} \\ \mathrm{Br}\, K = H^2(\Gamma_K, \overline{K}^*) & \longrightarrow & \check{H}^2(\overline{V}/V_\xi, \mathbb{G}_m) & \xrightarrow{\check{C}_{\overline{V}/V_\xi}} & \mathrm{Br}\, V_\xi \end{array}.$$

Recalling that $\mathrm{Br}_\lambda^{\mathrm{ram}} U$ is defined to be the image of the upper composition, we deduce by the commutativity that $\lambda_\xi^* \mathrm{Br}_\lambda^{\mathrm{ram}} U \subseteq \mathrm{Br}_0 V_\xi$. Thus, for any $B \in \mathrm{Br}_\lambda^{\mathrm{ram}} U$,

$$((P_v), B)_{BM} = ((Q_v), \lambda_\xi^* B)_{BM} = 0,$$

as wished. □

In summary, we have the series of inclusions

$$X(\mathbb{A}_K)^\lambda \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}\, X \cap \mathrm{Br}_\lambda U}. \tag{5-3}$$

However, in contrast to what happens on U (where the inclusions $U(\mathbb{A}_K)^\lambda \subseteq U(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} U} \subseteq U(\mathbb{A}_K)^{\mathrm{Br}_\lambda(U)}$ are actually identities by [15]), the last inclusion in (5-3) may well be strict! (See Section 6.1).

6. Unramified Brauer groups of SL_n/H, H metabelian

For Section 6 fix a finite group scheme H over a field k of characteristic 0. Let $B = [H, H]$, and $A = H/[H, H]$. We assume that H is *metabelian*, i.e., that B is commutative.

Let $U = \mathrm{SL}_{n,k}/H$, and $V = \mathrm{SL}_{n,k}/B$. As B is normal in H with quotient A, the morphism $\lambda : V \to U$ is an A-torsor, where the A-action on V is the one that on $\bar{k}$-points is given by $(\mathrm{SL}_{n,K}/B) \times A \to \mathrm{SL}_{n,K}/B$, $(xB, a) \mapsto xBa = xaB$.

Recall that $\mathfrak{l}: \overline{V} \to U$ induces a Čech-to-étale map on cohomology:

$$\check{C}_{\mathfrak{l}}: H^2(\Gamma_A, \overline{K}^*) = H^2(\Gamma_A, \overline{K}[V]^*) \to \operatorname{Br} U.$$

The author thanks Olivier Wittenberg for making him notice the following:

Theorem 6.0.1. *If* $Ш^1_\omega(K, B') = 0$, *then* $\operatorname{Br} X = \operatorname{Br}^{\mathrm{ram}}_\lambda X$.

(Recall that X denotes a smooth compactification of U.)

Proof. It suffices to prove that $\operatorname{Br}_{\mathrm{ur}} U \subseteq \operatorname{Im} H^2(\Gamma_A, \overline{K}^*)$. Consider the Hochschild–Serre spectral sequence of $\overline{V} \xrightarrow{\Gamma_G} U$, keeping in mind that $\overline{K}[V]^* = \overline{K}^*$ and $\operatorname{Pic} \overline{V} = B'$, we get the exact sequence

$$H^2(\Gamma_A, \overline{K}^*) \to \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br} \overline{V}) \to H^1(\Gamma_A, B') \tag{6-1}$$

Since $Ш^1_\omega(K, B') = 0$, we have $\operatorname{Br}_{\mathrm{ur}} V = \operatorname{Br} K$ by [13, Proposition 4]. Thus any element of $\operatorname{Br}_{\mathrm{ur}} U$ lies in the kernel of $\operatorname{Br} U \to \operatorname{Br} \overline{V}$. In addition, we know that after base-changing to $\overline{K}$ every element of $\operatorname{Br}_{\mathrm{ur}} U$ comes from $H^2(A, \overline{K}^*)$ (see [1, Lemma 5.1]). In particular, $\operatorname{Br}_{\mathrm{ur}} U$ maps to $\operatorname{Ker}(H^1(\Gamma_A, B') \xrightarrow{\mathrm{res}} H^1(A, B'))$ in the sequence above. By the inflation-restriction five-term sequence of $A \trianglelefteq \Gamma_A$, we have $\operatorname{Ker}(H^1(\Gamma_A, B') \to H^1(A, B')) = H^1(\Gamma_K, B')$. For $\beta \in \operatorname{Br}_{\mathrm{ur}} U$, we denote by $\delta(\beta)$ its image in $H^1(\Gamma_K, B')$.

Finally, by functoriality of the Hochschild–Serre spectral sequence, we get the commutative diagram

$$\begin{array}{ccccc} H^2(\Gamma_A, \overline{K}^*) & \longrightarrow & \operatorname{Ker}(\operatorname{Br} U \to \operatorname{Br} \overline{V}) & \longrightarrow & H^1(\Gamma_A, B') \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\Gamma_K, \overline{K}^*) & \longrightarrow & \operatorname{Ker}(\operatorname{Br} V \to \operatorname{Br} \overline{V}) & \longrightarrow & H^1(\Gamma_K, B'), \end{array}$$

where the second row is just (6-1) with $A = 0$. Hence (again because $\operatorname{Br}_{\mathrm{ur}} V = \operatorname{Br} K$) every element of $\operatorname{Br}_{\mathrm{ur}} U$ has to map to 0 in $H^1(\Gamma_K, B')$. This implies that $\delta(\beta) = 0$ for every β. Hence $\operatorname{Br}_{\mathrm{ur}} U \subseteq \operatorname{Im} H^2(\Gamma_A, \overline{K}^*) = \operatorname{Br}^{\mathrm{ram}}_\lambda U$, therefore $\operatorname{Br} X = \operatorname{Br}_{\mathrm{ur}} U = \operatorname{Br}^{\mathrm{ram}}_\lambda U \cap \operatorname{Br} X = \operatorname{Br}^{\mathrm{ram}}_\lambda X$, as wished. □

By Chebotarev's theorem, $Ш^1_\omega(K, B') = 0$ when B' is constant. We thus get:

Corollary 6.0.2. *If B is constant of exponent e and $\mu_e \subseteq K^*$, then* $\operatorname{Br} X = \operatorname{Br}^{\mathrm{ram}}_\lambda X$.

Proposition 1.4 is a consequence of the above corollary.

6.1. *Nilpotent H*. For the rest of the section we assume that B is central in H and that H is constant of prime exponent p, where $p \neq 2$ and $\mu_p \subseteq k^*$. We choose a primitive p-th root of unity, and use it to identify throughout this subsection μ_p with $\mathbb{Z}/p\mathbb{Z}$.

In particular, we shall always tacitly identify μ_p with $\mathbb{Z}/p\mathbb{Z}$, after making the implicit choice of a p-th root of unity.

Our aim in this subsection is to explicitly describe $\operatorname{Br}_{\mathrm{ur}} U$; see Theorem 6.1.3 below. We may naturally associate to the central extension

$$1 \to B \to H \to A \to 1$$

a class $[H] \in H^2(A, B)$ (see [2, Section IV.3]). We denote by $[-,-] : A \times A \to B$ the map that sends $a_1, a_2 \in A$ to the commutator of (any two) lifts $\bar{a}_1, \bar{a}_2$ of them in H. Let $\mathfrak{c} : \Lambda^2 A \to B$ be the natural homomorphism $a_1 \wedge a_2 \mapsto [a_1, a_2]$.

Recall that $A^D = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$. We shall make frequent use of the identification

$$\Lambda^2(A^D) = (\Lambda^2 A)^D, \alpha_1 \wedge \alpha_2 \mapsto (a_1 \wedge a_2 \mapsto \alpha_1(a_1)\alpha_2(a_2) - \alpha_1(a_2)\alpha_2(a_1)) .$$

We denote $\Lambda^2(A^D) = (\Lambda^2 A)^D$ with $\Lambda^2 A^D$. Let $(\Lambda^2 A^D)_{\text{bic}} \subseteq \Lambda^2 A^D$ be the subgroup of elements $\beta \in \Lambda^2 A^D$ such that $\beta(a_1 \wedge a_2) = 0$ for any $a_1, a_2 \in A$ with $[a_1, a_2] = 0$.

Let $\xi_U : \Lambda^2 A^D = \Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) \to \operatorname{Br} U$ be the composition of the three maps

$$\Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cup} H^2(A, \mathbb{Z}/p\mathbb{Z}) \qquad \text{(cup product)} \qquad (6\text{-}2)$$

$$H^2(A, \mathbb{Z}/p\mathbb{Z}) = H^2(A, \mu_p) \xrightarrow{\check{C}_\lambda} H^2(U, \mu_p) \qquad \text{(Čech-to-étale map)} \qquad (6\text{-}3)$$

$$H^2(U, \mu_p) \to H^2(U, \mathbb{G}_m) = \operatorname{Br} U \qquad \text{(changing coefficients)} \qquad (6\text{-}4)$$

We also define analogously a map $\xi\overline{U} : \Lambda^2 A^D \to \operatorname{Br} \overline{U}$. The following is a reformulation of a result of Bogomolov (see Lemma 5.1 of [1]):

Theorem 6.1.1 (Bogomolov, reformulated). *Assume that k is algebraically closed. The image of $(\Lambda^2 A^D)_{\text{bic}}$ under ξ_U is unramified, and the following sequence is exact*:

$$B^D \xrightarrow{\mathfrak{c}^D} (\Lambda^2 A^D)_{\text{bic}} \xrightarrow{\xi_U} \operatorname{Br}_{\text{ur}} U \to 1. \qquad (6\text{-}5)$$

We introduce the following notation used in the proof. Let $\gamma \in \Lambda^2 A^D = \operatorname{Hom}(\Lambda^2 A, \mathbb{Q}/\mathbb{Z})$ be a cocycle. Define H_γ as the central extension of A by $\mathbb{Q}/\mathbb{Z}$, characterized by the following property: for any $a_1, a_2 \in A$ the commutator of their corresponding lifts $\bar{a}_1, \bar{a}_2$ in H_γ is

$$\bar{a}_1 \bar{a}_2 \bar{a}_1^{-1} \bar{a}_2^{-1} = \gamma(a_1 \wedge a_2) \in \mathbb{Q}/\mathbb{Z}.$$

Proof of Theorem 6.1.1. By Lemma 5.1 of [1], $\operatorname{Br}_{\text{ur}} U$ is isomorphic to $\operatorname{Coker}(B^D \to (\Lambda^2 A^D)_{\text{bic}})$ (the group S/S_Λ appearing in loc.cit. is the dual of $\operatorname{Coker}(B^D \to (\Lambda^2 A^D)_{\text{bic}})$). It only remains to show that the induced homomorphism $\xi_{\text{Bog}} : (\Lambda^2 A^D)_{\text{bic}} \to \operatorname{Br}_{\text{ur}} U$ is ξ_U. This follows from the proof of [1, Lemma 5.1], but we include a detailed proof for completeness.

One may read from pp. 462 and 469 in [1] that the induced morphism $\xi_{\text{Bog}} : (\Lambda^2 A^D)_{\text{bic}} \to \operatorname{Br}_{\text{ur}} U$ is the restriction of the composition

$$\Lambda^2 A^D \cong H^2(A, \mathbb{Q}/\mathbb{Z}) = H^2(A, \mu_\infty) \to H^2(A, k^*) \xrightarrow{\check{C}_\lambda} \operatorname{Br} U, \qquad (6\text{-}6)$$

where the last map is defined via the Hochschild–Serre spectral sequence, and the first isomorphism is defined by sending an element $\gamma \in \Lambda^2 A^D = \operatorname{Hom}(\Lambda^2 A, \mathbb{Q}/\mathbb{Z})$ to the class in $H^2(A, \mathbb{Q}/\mathbb{Z})$ of the central extension H_γ of A. The following lemma then shows that this composition is ξ_U. □

Lemma 6.1.2. *The isomorphism $\Lambda^2 A^D \cong H^2(A, \mathbb{Q}/\mathbb{Z}), \gamma \mapsto [H_\gamma]$ coincides with the composition $\Lambda^2 A^D \xrightarrow{\cup} H^2(A, \mathbb{Z}/p\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})$.*

Proof. For a monomial $\gamma = \gamma_1 \wedge \gamma_2 \in \Lambda^2 A^D$, the extension H_γ may be realized as the set $\mathbb{Q}/\mathbb{Z} \times A$ with multiplication defined by $(b_1, a_1) \cdot (b_2, a_2) = (b_1 + b_2 + \gamma_1(a_1)\gamma_2(a_2), a_1 + a_2)$. The class $[H_\gamma] \in H^2(A, \mathbb{Q}/\mathbb{Z})$ is then represented by the 2-cocycle $a_1, a_2 \mapsto \gamma_1(a_1)\gamma_2(a_2)$ [2, p.92]. This is precisely the cup product $\gamma_1 \cup \gamma_2$ [22, Proposition 1.4.8]. Thus the two maps coincide on monomials $\gamma_1 \wedge \gamma_2$. Since these span $\Lambda^2 A^D$, the two maps coincide, as wished. □

Let $\operatorname{Br}_e U = \operatorname{Ker}(e^* : \operatorname{Br} U \to \operatorname{Br} K)$, where e is the K-point corresponding to the equivalence class of the identity in SL_n / H. Recall that we have a direct sum decomposition $\operatorname{Br} U = \operatorname{Br}_e U \oplus \operatorname{Br}_0 U$, where $\operatorname{Br}_0 U \cong \operatorname{Br} K$ denotes the constant Brauer elements. Let $\operatorname{Br}_{\mathrm{e,ur}} U = \operatorname{Br}_e U \cap \operatorname{Br}_{\mathrm{ur}} U$. We now remove the algebraically closed assumption from Theorem 6.1.1:

Theorem 6.1.3. *The image of* $(\Lambda^2 A^D)_{\mathrm{bic}}$ *under* ξ_U *is contained in* $\operatorname{Br}_{\mathrm{e,ur}} U$, *and the following sequence is exact*:

$$B^D \xrightarrow{\mathfrak{c}^D} (\Lambda^2 A^D)_{\mathrm{bic}} \xrightarrow{\xi_U} \operatorname{Br}_{\mathrm{e,ur}} U \to 1. \tag{6-7}$$

Proof. The A-torsor λ restricts to the trivial torsor over $e \in U(K)$, which implies that the image of ξ_U is contained in $\operatorname{Br}_e U$ (see Remark 2.1).

By Bogomolov's Theorem 6.1.1 and the key Lemma 6.1.4 below, we have $\xi_U((\Lambda^2 A^D)_{\mathrm{bic}}) \subseteq \operatorname{Br}_{\mathrm{ur}} U$. Finally, the composition

$$\xi\overline{U} : (\Lambda^2 A^D)_{\mathrm{bic}} \xrightarrow{\xi_U} \operatorname{Br}_{\mathrm{e,ur}} U \hookrightarrow \operatorname{Br}_{\mathrm{ur}} \overline{U}.$$

is surjective with kernel $\operatorname{Im}(\mathfrak{c}^D : B^D \to \Lambda^2 A^D)$ by Bogomolov's Theorem, while the second morphism is injective by [20, Proposition 5.9]. Hence the last map is an isomorphism, and the statement follows. □

Lemma 6.1.4. *For* $\alpha \in \Lambda^2 A^D$, $\xi_U(\alpha) \in \operatorname{Br}_e U$ *is unramified if and only if* $\xi\overline{U}(\alpha) \in \operatorname{Br} \overline{U}$ *is unramified.*

Proof. The forward implication is clear; we prove the converse. Let $\alpha \in \Lambda^2 A^D$ be an element whose image in $\operatorname{Br} \overline{U}$ is unramified. By Bogomolov's Theorem 6.1.1, this unramifiedness is equivalent to α lying in $(\Lambda^2 A^D)_{\mathrm{bic}}$.

We first assume that $k = K$ is a number field. Let v be a finite place of k coprime with p. Let $P \in U(K_v)$. Let ψ be the projection $\mathrm{SL}_n \to \mathrm{SL}_n / H$. Choose a geometric point $\bar{P} \in \psi^{-1}(P)(\overline{K_v})$, and let $f : \Gamma_v \to H$ be the homomorphism defined by $\gamma \cdot \bar{P} = \bar{P} \cdot f(\gamma)$. Since $v \nmid p$ the homomorphism f factors though the tame decomposition group Γ_v^{tame}, which is topologically generated by elements ι and ϕ satisfying $\phi\iota\phi^{-1} = \iota^{Nv}$. Since $\mu_p \subseteq K^*$, we have $Nv \equiv 1 \bmod p$. In particular, since the exponent of H is p, the images of ι and ϕ in H commute. The group $A' = \langle f(\iota), f(\phi) \rangle$ is then bicyclic. The point P lifts via $(\mathrm{SL}_n / A')(K_v) \to (\mathrm{SL}_n / H)(K_v)$.

Let $g : A' \to A$ be the projection, and consider the commutative diagram

$$\begin{array}{ccc} \Lambda^2 (A')^D & \xleftarrow{\quad g^* \quad} & \Lambda^2 A^D \\ \downarrow \xi_{\mathrm{SL}_n / A'} & & \downarrow \xi_U = \xi_{\mathrm{SL}_n / H} \\ \operatorname{Br}(\mathrm{SL}_n / A') & \longleftarrow & \operatorname{Br}(\mathrm{SL}_n / H) \end{array} .$$

Let $\bar{a}_1 = f(\iota), \bar{a}_2 = f(\phi)$, a_i be the image of $\bar{a}_i$ in A. Note that $\Lambda^2 A'$ is generated by $\bar{a}_1 \wedge \bar{a}_2$, and $g^*\alpha(\bar{a}_1 \wedge \bar{a}_2) = \alpha(a_1 \wedge a_2)$. Since $\bar{a}_1$ and $\bar{a}_2$ commute and $\alpha \in (\Lambda^2 A^D)_{\mathrm{bic}}$, $g^*\alpha$ is 0. It follows that $\xi_U(\alpha)$ maps to 0 in $\operatorname{Br}(\mathrm{SL}_n / A')$, and thus specializes to 0 at all points lying in the image of $(\mathrm{SL}_n / A')(K_v) \to (\mathrm{SL}_n / H)(K_v)$. In particular, $\xi_U(\alpha)(P) = 0 \in \operatorname{Br} K_v$. Since P and $v \nmid p$ were arbitrary, this means that $\xi_U(\alpha)$ is unramified by a well-known consequence of Harari's formal lemma [12, théorèm 2.1.1].

The case of a general k follows from the number field case and a "no-name lemma" argument. Namely, let $L := \mathbb{Q}(\mu_p) \subseteq k$, $\rho : H \hookrightarrow \mathrm{SL}_n(k)$ be the representation defining the quotient $\mathrm{SL}_{n,k} / H$, and $\rho' : H \hookrightarrow \mathrm{SL}_{n'}(L)$ be another faithful representation. Let $U'_k := \mathrm{SL}_{n',k} / H$, $U'_L := \mathrm{SL}_{n',L} / H$ (note that we have a natural map $U'_k \to U'_L$), and U''_k be the diagonal quotient $(\mathrm{SL}_{n,k} \times \mathrm{SL}_{n',k})/H$. The variety U''_k is both an $\mathrm{SL}_{n,k}$-torsor over U'_k and a $\mathrm{SL}_{n',k}$-torsor over $U_k = U$:

$$U_k \xleftarrow{\mathrm{SL}_{n',k}} U''_k \xrightarrow{\mathrm{SL}_{n,k}} U'_k$$

We get a commutative diagram

$$\begin{array}{ccccc} & & \Lambda^2 A^D & & \\ & \swarrow^{\xi_{U'_k}} & \downarrow^{\xi_{U''_k}} & \searrow^{\xi_{U_k}} & \\ \operatorname{Br} U'_k & \longrightarrow & \operatorname{Br} U''_k & \longleftarrow & \operatorname{Br} U_k \end{array}$$

where the horizontal maps are pullbacks. Moreover, the morphism $\xi_{U'_k}$ factors as $\Lambda^2 A^D \xrightarrow{\xi_{U'_L}} \operatorname{Br} U'_L \to \operatorname{Br} U'_k$, where the last morphism is an extension of scalars. We know by the number field case that $\xi_{U'_L}(\alpha) \in \operatorname{Br}_{\mathrm{ur}} U'_L$, and thus $\xi_{U'_k}(\alpha) \in \operatorname{Br}_{\mathrm{ur}} U'_k$ and $\xi_{U''_k}(\alpha) \in \operatorname{Br}_{\mathrm{ur}} U''_k$. Therefore, if we let $\beta := \xi_{U_k}(\alpha)$ and denote by π the projection $U''_k \to U_k$, then $\pi^*\beta$ belongs to $\operatorname{Br}_{\mathrm{ur}} U''_k$. But $U''_k \to U'_k$ is an $\mathrm{SL}_{n,k}$-torsor, and in particular possesses a rational section $s : U_k \dashrightarrow U''_k$ by Hilbert 90. It follows that $\beta = (\pi s)^*\beta = s^*(\pi^*\beta)$ is unramified as well, as wished. □

Remark 6.1.5. The composition map ξ_U is easily seen to be equal to the composition

$$\Lambda^2 A^D \xrightarrow{\cup} H^2(A, \mathbb{Z}/p\mathbb{Z}) = H^2(A, \mu_p) \xrightarrow{\inf} H^2(\Gamma_A, \mu_p) \to H^2(\Gamma_A, \bar{K}^*) \xrightarrow{\check{C}_\mathfrak{l}} \operatorname{Br} U;$$

thus $\operatorname{Im} \xi_U \subseteq \operatorname{Br}^{\mathrm{ram}}_{\lambda} U$.

6.2. ***A decomposition of $H^2(A, B)$.*** For general abstract commutative groups A and B, we have a split exact sequence (see [2, V.6.5])

$$0 \to \operatorname{Ext}(A, B) \to H^2(A, B) \xrightarrow{\omega_B} \operatorname{Hom}(\Lambda^2 A, B) \to 0. \tag{6-8}$$

The splitting is noncanonical in general, but it is canonical when $\#B$ is odd: in this case a section s_B of ω_B is given by $s_B : \gamma \mapsto \big(a_1, a_2 \mapsto \frac{1}{2}\gamma(a_1 \wedge a_2)\big)$. If $B = \mathbb{Z}/n\mathbb{Z}$, then $\operatorname{Hom}(\Lambda^2 A, \mathbb{Z}/n\mathbb{Z}) = \Lambda^2 \operatorname{Hom}(A, \mathbb{Z}/n\mathbb{Z})$ and $2 \cdot s_B$ is the cup product:

$$2 \cdot s_B = \cup : \Lambda^2 \operatorname{Hom}(A, \mathbb{Z}/n\mathbb{Z}) \to H^2(A, \mathbb{Z}/n\mathbb{Z}). \tag{6-9}$$

The splitting s_B of (6-8) induces a canonical decomposition $H^2(A, B) = \operatorname{Ext}(A, B) \oplus \operatorname{Hom}(\Lambda^2 A, B)$.

Lemma 6.2.1. *Let p be an odd prime number, and assume that A and B have exponent p. Let $1 \to B \to H \xrightarrow{\pi} A \to 1$ be a nilpotent central extension of exponent p. Then the class $[H] \in H^2(A, B)$ lies in the image of s_B.*

Proof. Equivalently, we have to prove that $[H]$ maps to 0 under the projection $H^2(A, B) \to \operatorname{Ext}(A, B)$. This projection is natural in A (and B as well, but we do not need this). Therefore, since the functor $\operatorname{Ext}(A, B)$ is additive in A, and every abelian group decomposes into a direct sum of cyclic groups, it suffices to prove the result for a cyclic A. If A is cyclic and generated by $a \in A$, every $h \in \pi^{-1}(a)$ must be of order p because H has exponent p. The element h thus provides a section $a \mapsto h$ of π, and $H = B \oplus A$. Hence $[H] = 0 \in H^2(A, B)$, and in particular H maps to 0 in $\operatorname{Ext}(A, B)$ as wished. □

6.3. ***Formula for the Brauer–Manin pairing.*** We work in the setting of 6.1, but we also assume that $k = K$ is a number field. All cohomology and homomorphism groups appearing in this subsection have a natural $\mathbb{F}_p$-vector space structure, so whenever we take tensor products, alternating products, etc, we mean that these operations are performed as $\mathbb{F}_p$-vector spaces.

Let v be a finite place of K, $\Gamma_v = \operatorname{Gal}(\overline{K}_v/K_v)$, and $G_v = \Gamma_v^{\mathrm{ab}}/p\Gamma_v^{\mathrm{ab}}$. Remembering that we fixed an isomorphism $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$, and using the identifications $H^1(\Gamma_v, \mathbb{F}_p) = \operatorname{Hom}(\Gamma_v, \mathbb{F}_p) = \operatorname{Hom}(G_v, \mathbb{F}_p)$, we may identify the perfect local duality pairing

$$\cup : H^1(\Gamma_v, \mathbb{F}_p)^{\otimes 2} \to H^2(\Gamma_v, \mathbb{F}_p) \cong \mathbb{F}_p$$

with a pairing

$$H_v : \Lambda^2 \operatorname{Hom}(G_v, \mathbb{F}_p) \to \mathbb{F}_p.$$

(Note that this is alternating because p is odd.)

We choose a basis $g_1, \dots, g_r$ of G_v, and a basis $t_1, \dots, t_a$ of A. These induce an identification

$$\operatorname{Hom}(G_v, A) = \operatorname{Mat}_{a\times r}(\mathbb{F}_p) \tag{6-10}$$

For a finite-dimensional vector space $V/\mathbb{F}_p$, a basis $v_1, \dots, v_l$ of V induces an identification

$$\Lambda^2(V^D) \cong \operatorname{Mat}_{ant,l\times l}(\mathbb{F}_p), \quad v_i^D \wedge v_j^D \mapsto e_i e_j^T - e_j e_i^T,$$

where $e_1, \dots, e_l$ denotes the standard basis of $\mathbb{F}_p^l$, and Mat_{ant} denotes antisymmetric matrices. Specializing to $V = A$ and $V = G_v$, we get identifications

$$\Lambda^2 A^D = \operatorname{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p), \quad \Lambda^2 G_v^D = \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p). \tag{6-11}$$

Let $\phi \in \operatorname{Hom}(G_v, A)$, $\beta \in \Lambda^2 A^D$, $\gamma \in \Lambda^2 G_v^D$, and let $M_\phi \in \operatorname{Mat}_{a\times r}(\mathbb{F}_p)$, $M_\beta \in \operatorname{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$, $M_\gamma \in \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p)$ be their corresponding matrices under the identifications above. Let $\tilde{H}_v \in \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p)$ be the matrix defined by $(\tilde{H}_v)_{i,j} = \frac{1}{2} H_v(g_j \wedge g_i)$. One easily verifies that

$$\begin{array}{lcccl} \phi^*\beta \in \Lambda^2 G_v^D & \text{corresponds to} & {M_\phi}^T M_\beta M_\phi & \in \operatorname{Mat}_{r\times r,\mathrm{ant}}(\mathbb{F}_p), \\ \text{and} \quad H_v(\gamma) & = & \operatorname{tr}(\tilde{H}_v M_\gamma) & \in \mathbb{F}_p. \end{array} \tag{6-12}$$

Recall that we have a homomorphism

$$\xi_U : \Lambda^2 \operatorname{Hom}(A, \mathbb{F}_p) \xrightarrow{\cup} H^2(A, \mathbb{F}_p) \to \operatorname{Br} U.$$

We may compute the local Brauer pairing between $\operatorname{Im} \xi_U$ and $U(K_v)$ as follows:

Lemma 6.3.1. *Let $\beta \in \Lambda^2 \operatorname{Hom}(A, \mathbb{F}_p)$, $b := \xi_U(\beta) \in \operatorname{Br} U$, $P \in U(K_v)$, and $\phi \in \operatorname{Hom}(G_v, A)$ be induced by the torsor type $[\lambda|_P] \in H^1(\Gamma_v, A) = \operatorname{Hom}(\Gamma_v, A) \cong \operatorname{Hom}(G_v, A)$. Then*

$$(b, P)_v = \operatorname{tr}(\tilde{H}_v \tilde{M}_\phi^{\,T} M_\beta M_\phi).$$

Proof. This follows by just unraveling the notation. Namely, we have $(b, P)_v = \operatorname{inv}_v(P^* b)$, and the latter is equal to $H_v(\phi^* \beta) = \operatorname{inv}_v(P^* b)$ by the commutativity of the diagram

$$\begin{array}{llllllll}
\xi_U : & \Lambda^2 A^D & \xrightarrow{\cup} & H^2(A, \mathbb{F}_p) & \xrightarrow{\check{C}_\lambda} & H^2(U, \mathbb{F}_p) & \longrightarrow & (\operatorname{Br} U)[p] \\
 & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle P^*} & & \downarrow{\scriptstyle P^*} \\
H_v : & \Lambda^2 G_v^D & \xrightarrow{\cup} & H^2(G_v, \mathbb{F}_p) & \xrightarrow{\inf} & H^2(\Gamma_v, \mathbb{F}_p) = H^2(K_v, \mathbb{F}_p) & \longrightarrow & (\operatorname{Br} K_v)[p] \cong \mathbb{F}_p.
\end{array}$$

The formulas (6-12) now give the statement. ☐

6.4. ***Proof of Theorem 1.5.*** Let $\Theta_v \subseteq \operatorname{Hom}(\Gamma_v, A)$ be the set of torsor types to which λ specializes, i.e., the image of $[\lambda|_-] : U(K_v) \to \operatorname{Hom}(\Gamma_v, A)$.

Lemma 6.4.1. *The set Θ_v is the inverse image of* 0 *under*

$$\operatorname{Hom}(\Gamma_v, A) \to H^2(\Gamma_v, B), \quad \xi \mapsto \xi^*([H]), \tag{6-13}$$

where $[H] \in H^2(A, B)$ is the class representing H.

(Note that this inverse image is not a subspace in general!)

Proof. The commutative diagram

$$\begin{array}{ccccc}
\mathrm{SL}_n(K_v) \to (\mathrm{SL}_n/H)(K_v) = U(K_v) & \longrightarrow & H^1(K_v, H) & \longrightarrow & H^1(K_v, \mathrm{SL}_n) = 0 \\
 & {\scriptstyle [\lambda_-]} \searrow & \downarrow & & \\
 & & H^1(K_v, A) = \operatorname{Hom}(\Gamma_v, A) & &
\end{array}$$

with exact first row, shows Θ_v is the image of $H^1(K_v, H) = \operatorname{Hom}(\Gamma_v, H) \to \operatorname{Hom}(\Gamma_v, A)$. Equivalently, Θ_v consists of those homomorphisms that lift from A to the central extension H. The statement now follows from the theory of central extensions [2, Section IV.3]. ☐

Proposition 6.4.2. *For every $p \geq 5$, every number field K with $\mu_p \subseteq K^*$, and place v of K dividing p, there exists a constant $c(v) \geq 0$ such that for any $a \geq c(v)$, there exists a metabelian nilpotent H of exponent p with H^{ab} of rank a such that, letting $A = H^{\mathrm{ab}}$:*

(i) $(\Lambda^2 A^D)_{\mathrm{bic}} = \Lambda^2 A^D$;

(ii) *there exists* $\alpha \in \Lambda^2 A^D$ *such that the local pairing* $(-, \xi_U(\alpha)) : U(K_v) \to \mathbb{Z}/p\mathbb{Z}$ *attains at least two values.*

Proof. Let a be a natural number and $A := (\mathbb{Z}/p\mathbb{Z})^a$. Lemma A.2 of Appendix A guarantees the existence of a metabelian nilpotent H of exponent p with $H^{\mathrm{ab}} = A$ such that $B = [H, H]$ has rank $b := 2a - 3$ and $(\Lambda^2 A^D)_{\mathrm{bic}} = \Lambda^2 A^D$.

Recall that Θ_v is the image of $[\lambda|\_] : U(K_v) \to \mathrm{Hom}(\Gamma_v, A)$. Let $\Xi_v \subseteq \Theta_v$ be the image under $[\lambda|\_]$ of the left kernel of $U(K_v) \times \mathrm{Br}_{\mathrm{e,ur}} U \to \mathbb{Q}/\mathbb{Z}$, $(P, b) \mapsto \mathrm{inv}_v(P^* b)$. Since $e \in \Xi_v$, Ξ_v is nonempty and to prove point (ii) it suffices to prove that $\Xi_v \neq \Theta_v$. We do so by proving that the cardinality of these two sets have different p-adic valuation for $a \geq c(v)$. We start by computing $v_p(\#\Theta_v)$.

Recall the identifications

$$\mathrm{Hom}(G_v, A) = \mathrm{Mat}_{a\times r}(\mathbb{F}_p),\ \Lambda^2 A^D = Mat_{a\times a,ant}(\mathbb{F}_p),\ \Lambda^2 G_v^D = Mat_{r\times r,ant}(\mathbb{F}_p).$$

Identifying B with $\mathbb{F}_p^b$ we get identifications $H^2(A, B) = H^2(A, \mathbb{F}_p)^b$ and $H^2(\Gamma_v, B) = H^2(\Gamma_v, \mathbb{F}_p)^b \xrightarrow{\sim} \mathrm{inv}_v \mathbb{F}_p^b$. By Lemma 6.2.1 and (6-9), there exist $h_1, \dots, h_b \in \Lambda^2 \mathrm{Hom}(A, \mathbb{F}_p) = \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$ such that $[H] = (\mathfrak{h}_1, \dots, \mathfrak{h}_b) \in H^2(A, \mathbb{F}_p)^b$, $\mathfrak{h}_i := \cup(h_i)$, where $\cup$ indicates the cup product $\Lambda^2 \mathrm{Hom}(A, \mathbb{F}_p) = \Lambda^2 H^1(A, \mathbb{F}_p) \to H^2(A, \mathbb{F}_p)$. Then, by (6-13),

$$\Theta_v = \{M \in \mathrm{Hom}(\Gamma_v, \mathbb{F}_p) \mid \mathrm{inv}_v(M^* \mathfrak{h}_i) = 0,\ i = 1, \dots, b\}. \tag{6-14}$$

By Lemma 6.3.1, under the identification $\mathrm{Hom}(\Gamma_v, \mathbb{F}_p) = \mathrm{Hom}(G_v, \mathbb{F}_p) = \mathrm{Mat}_{a\times r}(\mathbb{F}_p)$, Θ_v corresponds to

$$\Theta_v = \{M \in \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(\tilde{H}_v \tilde{M}^T \tilde{h}_i \tilde{M}) = 0,\ i = 1, \dots, b\}. \tag{6-15}$$

A special case of the Ax–Katz theorem [18] states that the p-adic valuation of the number of solutions of a system of m polynomial equations of degree $\leq d$ in n (affine) variables in $\mathbb{F}_p$ is at least $\lceil \frac{n-dm}{d} \rceil$. Hence, since (6-15) describes Θ_v as the solution set of b quadratic equations in ra variables:

$$v_p(\#\Theta_v) \geq \left\lceil \frac{ra - 2b}{2} \right\rceil = \left\lceil \frac{(r-4)a + 6}{2} \right\rceil \geq \left\lceil \frac{2a + 6}{2} \right\rceil$$

as $r \geq p + 1 \geq 6$ by [22, Theorem 7.5.11].

We now compute $v_p(\#\Xi_v)$, using Lemma A.3 from Appendix A. Recall that $(\Lambda^2 A^D)_{\mathrm{bic}} = \Lambda^2 A^D$ by our choice of H, and thus $\xi_U(\Lambda^2 A^D) = \mathrm{Br}_{\mathrm{ur}} U$ by Theorem 6.1.3. Hence Lemma 6.3.1 implies that P lies in the left kernel of the local Brauer pairing $U(K_v) \times \mathrm{Br}_{\mathrm{ur}} U \to \mathbb{Q}/\mathbb{Z}$ if and only if $H_v(M_P^* \alpha) = (P, \xi_U(\alpha))_v = 0$ for all $\alpha \in \Lambda^2 A^D$. Therefore, (6-12) implies that the image of Ξ_v under the identification $\mathrm{Hom}(\Gamma_v, \mathbb{F}_p) = \mathrm{Hom}(G_v, \mathbb{F}_p) = \mathrm{Mat}_{a\times r}(\mathbb{F}_p)$ is

$$\begin{aligned}
\Xi_v &= \{M \in \Theta_v \subseteq \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(\tilde{H}_v M^T N M) = 0,\ \text{for all } N \in \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)\} \\
&= \{M \in \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(\tilde{H}_v M^T N M) = 0,\ \text{for all } N \in \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)\}, \\
&= \{M \in \mathrm{Mat}_{a\times r}(\mathbb{F}_p) \mid \mathrm{tr}(M \tilde{H}_v M^T N) = 0,\ \text{for all } N \in \mathrm{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)\},
\end{aligned}$$

where the second identity holds because the b quadratic equations describing Θ_v are redundant in the description of Ξ_v (see (6-15)). Since $\operatorname{tr}(X^T Y)$ induces a perfect pairing on $\operatorname{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$ and $M\tilde{H}_v M^T$ lies in $\operatorname{Mat}_{a\times a,\mathrm{ant}}(\mathbb{F}_p)$, we obtain from the above

$$\Xi_v^T = \{M \in \operatorname{Mat}_{r\times a}(\mathbb{F}_p) \mid M\tilde{H}_v^T M^T = 0\}.$$

The matrix $\tilde{H}_v^T = -\tilde{H}_v$ is invertible by local duality, and so Lemma A.3 shows that there exists a function $C : \mathbb{N} \to \mathbb{Z}$ (depending only on p) such that $\#\Xi_v^T \equiv C(r) \mod p^a$ if $a \geq 2r$. Recall that $r = r(v) = [K_v : \mathbb{Q}_p] + 2$, and let $c(v) = \max\{2r, v_p(C(r)) + 1\}$. If $a \geq c(v)$, then $v_p(\#\Xi_v^T) = v_p(C(r))$ by the ultrametric triangle inequality. Now

$$v_p(\#\Xi_v) = v_p(\#\Xi_v^T) = v_p(C(r)) < a < \left\lceil \frac{2a+6}{2} \right\rceil \leq v_p(\#\Theta_v)$$

as wished. □

We refer the reader to the proof of Lemma A.3 for a formula for $C(r)$. As a consequence of Proposition 6.4.2, we may now prove Theorem 1.5:

Proof of Theorem 1.5. Let H be any group as in Lemma A.3, and let $U = \mathrm{SL}_{n,K}/H$. By point (ii) of the lemma there is a place v, an element $\beta \in \Lambda^2 A^D$, and points $P_v \neq Q_v \in U(K_v)$ such that

$$(P_v, b)_v \neq (Q_v, b)_v \in \mathbb{Q}/\mathbb{Z}, \quad b := \xi_U(\beta) \in \operatorname{Br} X.$$

Consider then the adelic point $\underline{P} \in U(\mathbb{A}_K)$ (resp. $\underline{Q}$) that is equal to e at all places $\neq v$ and is equal to P_v (resp. Q_v) at v. Then $(\underline{P}, b)_{BM} = \operatorname{inv}_v(b(P_v)) \neq \operatorname{inv}_v(b(Q_v)) = (\underline{Q}, b)_{BM}$. Hence (at least) one between $\underline{P}$ and $\underline{Q}$ does not lie in $X(\mathbb{A}_K)^{\operatorname{Br} X}$, concluding the proof. □

Appendix A. Elementary counting facts

Lemma A.1. *Let $n \leq N$ be positive integers and $X \subseteq \mathbb{P}^N(\mathbb{F}_p)$ a subset of cardinality $< \#\mathbb{P}^n(\mathbb{F}_p)$. There exists then an n-codimensional subspace $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ such that $X \cap L = \varnothing$.*

Proof. Let $k \geq 0$ be the smallest integer such that X intersects every k-dimensional subspace in $\mathbb{P}^N_{\mathbb{F}_p}$. If $k = 0$, there is nothing to prove. Otherwise, let $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ be a $(k-1)$-dimensional subspace such that $L \cap X = \varnothing$. Let $\pi_L : \mathbb{P}^N \setminus l \to \mathbb{P}^{N-k}$ be a projection outside of L. We know by assumption that $\pi_L(X(\mathbb{F}_p)) = \mathbb{P}^{N-k}(\mathbb{F}_p)$, hence $\#X(\mathbb{F}_p) \geq \#\mathbb{P}^{N-k}(\mathbb{F}_p) \Rightarrow N - k < n$, i.e., $k \geq N - n + 1$. Hence the dimension of L is $\geq N - n$ and it is the sought subspace. □

The following lemma is inspired by [1, Section 5]; see also [6, p. 37].

Lemma A.2. *Let $p \neq 2$ be a prime. For every $\mathbb{F}_p$-vector space A of dimension $4 \leq a < \infty$, there exists an $\mathbb{F}_p$-vector space B, of dimension $b = 2a - 3$, and a (surjective) morphism*

$$\mathfrak{c} : \Lambda^2 A \to B, \tag{A-1}$$

such that, if $1 \to B \to H \xrightarrow{\pi} A \to 1$ is the extension whose commutator map is $\mathfrak{c}$, there are no pure wedges $0 \neq a_1 \wedge a_2 \in \Lambda^2 A$ lying in the kernel of $\mathfrak{c}$.

Proof. Let $X \subseteq \mathbb{P}_{\mathbb{F}_p}(\Lambda^2 A)$ be the image of the "alternating Segre morphism"

$$- \wedge - : \mathbb{P}(A) \times \mathbb{P}(A) \setminus \Delta \to \mathbb{P}(\Lambda^2 A),$$

which is isomorphic to the Grassmannian variety $\mathrm{Gr}_{\mathbb{F}_p}(2, A)$. Since $X(\mathbb{F}_p)$ parametrizes two-dimensional $\mathbb{F}_p$-subspaces of A,

$$\# X(\mathbb{F}_p) = \frac{(p^a-1)(p^{a-1}-1)}{(p^2-1)(p-1)}. \tag{A-2}$$

It suffices to show that there exists a $(2a-3)$-codimensional subspace L in $\mathbb{P}(\Lambda^2 A)$ such that $L \cap X(\mathbb{F}_p) = \varnothing$, and choose $\mathfrak{c}$ such that $\Lambda^2 A \supseteq \mathbb{F}_p \cdot L(\mathbb{F}_p) = \operatorname{Ker} \mathfrak{c}$. Noting that

$$\frac{(p^a-1)(p^{a-1}-1)}{(p^2-1)(p-1)} < \frac{(p^a-1)(p^{a-1}-1)}{(p+1)(p-1)} \le \frac{(p^{2a-2}-1)(p+1)}{(p+1)(p-1)} = \#\mathbb{P}^{2a-3}(\mathbb{F}_p),$$

such a subspace always exists by Lemma A.1. □

Lemma A.3. *Let A, V be $\mathbb{F}_p$-vector spaces with $p \neq 2$, and let $a := \dim A$, $r := \dim V$. Assume that $a \ge 2r$, and that V is endowed with an alternating nondegenerate bilinear form $b : V \times V \to \mathbb{F}_p$. Then*

$$\Xi(A, V) := \#\{\xi \in \operatorname{Hom}(A, V) \mid \xi^* b = 0\} \equiv C(r) \mod p^a,$$

where $C(r)$ is a nonzero integer depending only on r.

Proof. Let

$$M_d := \#\{\text{isotropic } d\text{-dimensional subspaces in } V\},$$
$$I_d := \#\{\text{surjective homomorphisms from } A \text{ to a } d\text{-dimensional } \mathbb{F}_p\text{-vector space}\}.$$

Then

$$\Xi(A, V) = \sum_{d=0}^{\min(a, r/2)} I_d M_d$$

(the fact that V is endowed with a nondegenerate alternating linear form and $p \neq 2$ implies that r is even). One can easily see that

$$I_d = (p^a-1)\cdot(p^a-p)\cdots(p^a-p^{d-1}) \qquad \text{for every } d \le a,$$
$$M_d = \frac{(p^r-1)\cdot(p^{r-1}-p)\cdots(p^{r-d+1}-p^{d-1})}{(p^d-1)\cdot(p^d-p)\cdots(p^d-p^{d-1})} \qquad \text{for every } d \le r/2.$$

In particular, $\Xi(A, V) = \Xi'(a, r)$ depends only on a and r. Note that, for a fixed r, $\Xi'(a, r)$ converges p-adically, as $a \to \infty$, to the following sum, which happens to be an integer number:

$$C(r) := \Xi'(\infty, r) := \sum_{d=0}^{r/2} (-1)^d \frac{(p^r-1)\cdot(p^{r-1}-p)\cdots(p^{r-d+1}-p^{d-1})}{(p-1)\cdot(p^2-1)\cdots(p^d-1)}$$
$$= \sum_{d=0}^{r/2} (-1)^d \binom{r/2}{d}_{p^2} (p+1)\cdots(p^d+1),$$

where the subscript in the binomial denotes a Gaussian binomial coefficient (an integer number). Moreover, $\Xi'(a, r) \equiv \Xi'(\infty, r) \mod p^a$ if $a \ge 2r$. Denoting by $a(d)$ the term multiplying the $(-1)^d$ appearing above, we notice that the sequence $a(0), \dots, a(r/2)$ is strictly increasing, as follows by induction from the fact that $(p^{r-d+1} - p^{d-1})/(p^d - 1) > 1$ for all $d \in \{0, \dots, r/2\}$. In particular, a standard elementary calculus argument (à la Leibniz' rule) shows that $\Xi'(\infty, r) \neq 0$. □

Appendix B. Other works where ramified descent appears

Let me mention other works where the idea of "ramified descent" has already appeared. One is [16] by Harpaz and Skorobogatov (successor to Skorobogatov and Swinnerton-Dyer's work [27] [30]), where the authors use the cyclic ramified covers of some specific Kummer surfaces to prove that, under certain technical assumptions, these satisfy the Hasse principle.

Another work is Corvin and Schlank's paper [8], where the authors build upon Poonen's example [23] to show (employing one specific ramified S_4-cover) that the following obstruction is *stronger* than étale Brauer–Manin obstruction:

$$X(\mathbb{A}_K)^{\mathrm{Br,ram,sol}} = \bigcap_{\substack{\psi:Y\to X \text{ a } G\text{-cover} \\ G \text{ solvable}}} \overline{\bigcup_{\xi\in H^1(K,G)} \psi'_\xi(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))^{\mathrm{Br}\, Y_\xi^{\mathrm{sm}}}},$$

where the ψ'_ξ is the composition $Y_\xi^{\mathrm{sm}} \to Y_\xi \xrightarrow{\psi_\xi} X$.

Lastly, we mention Sections 11.5 and 14.2.5 of Colliot-Thélène and Skorobogatov's book [7], where ramified descent is investigated for μ_n-covers. In Theorem 14.2.25 of loc.cit., the authors prove a result which translates in our language to saying that, if $\lambda : V \to U$ is a μ_n-torsor such that there is a divisor on X over which the "compactification" $\psi : Y \to X$ of λ (notation as in Section 3) is totally ramified, then $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X}$. Their result and our Proposition 1.3 naturally lead us to the question:

Question B.1. Let $\lambda : V \to U$ and $\psi : Y \to X$ be as in Section 3. Assume that the cover $Y \to X$ is totally ramified, i.e., Y is geometrically integral and $Y \to X$ does not have any unramified subcovers. Does one then have $X(\mathbb{A}_K)^\lambda = X(\mathbb{A}_K)^{\mathrm{Br}_\lambda^{\mathrm{ram}} X}$?

Note that a positive answer to the question above would guarantee that, for instance, if Y is a variety all of whose G-twists satisfy the Hasse principle, then X satisfies the Hasse principle up to Brauer–Manin obstruction.

Let us mention that, when G is supersolvable and Y is rationally connected, Harpaz and Wittenberg [17, Theorem 1.4] proved that

$$X(\mathbb{A}_K)^{\mathrm{Br}\, X} = \overline{\bigcup_{\xi\in H^1(K,G)} \psi_\xi^{\mathrm{sm}}\big(Y_\xi(\mathbb{A}_K)^{\mathrm{Br}\, Y_\xi^{\mathrm{sm}}}\big)}$$

(using our notation). It follows that $X(\mathbb{A}_K)^\lambda \supset X(\mathbb{A}_K)^{\mathrm{Br}\, X}$, i.e., Brauer–Manin obstruction is finer than ramified descent obstruction, but it also seems likely that their methods could be in fact used to give a positive answer to Question B.1 in this case. For instance, when, in addition to the conditions above, $\overline{K}[V]^*/\overline{K}^* = 0$ and $\mathrm{Pic}\, \overline{V} = 0$ (e.g., if $V = \mathrm{SL}_n$), then $\mathrm{Br}_\lambda^{\mathrm{ram}} X = \mathrm{Br}\, X$ by Remark 5.1.4, and a positive answer to the question follows.

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demeiojulian@yahoo.it Department of Mathematics, University of Bath, Bath, United Kingdom

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