Algebra & Number Theory

Volume 20
2026
No. 3

Chevalley formulae for motivic Chern classes of Schubert cells and for stable envelopes

Leonardo C. Mihalcea, Hiroshi Naruse and Changjian Su

msp

ALGEBRA AND NUMBER THEORY 20:3 (2026)
https://doi.org/10.2140/ant.2026.20.477

Chevalley formulae for motivic Chern classes of Schubert cells and for stable envelopes

Leonardo C. Mihalcea, Hiroshi Naruse and Changjian Su

We prove a Chevalley formula to multiply the motivic Chern classes of Schubert cells in a generalized flag manifold G/P by the class of any line bundle $\mathcal{L}_\lambda$. Our formula is given in terms of the λ-chains of Lenart and Postnikov. Its proof relies on a change of basis formula in the affine Hecke algebra due to Ram, and on the Hecke algebra action on torus-equivariant K-theory of the complete flag manifold G/B via left Demazure–Lusztig operators. We revisit some wall-crossing formulae for the stable envelopes in $T^*(G/B)$. We use our Chevalley formula, and the equivalence between motivic Chern classes of Schubert cells and K-theoretic stable envelopes in $T^*(G/B)$, to give formulae for the change of polarization, and for the change of slope for stable envelopes. We prove several additional applications, including Serre, star, and Dynkin, dualities of the Chevalley coefficients, new formulae for the Whittaker functions, and for the Hall–Littlewood polynomials. We also discuss positivity properties of Chevalley coefficients, and properties of the coefficients arising from multiplication by minuscule weights.

L. C. Mihalcea was supported in part by the NSF grant DMS-2152294 and a Simons Collaboration Grant. H. Naruse was supported in part by JSPS KAKENHI Grant Number 16H03921. C. Su is supported by the National Key R&D Program of China (No. 2024YFA1014700).
MSC2020: primary 14C17, 14M15; secondary 14N15, 17B10, 33D80.
Keywords: motivic Chern classes of Schubert cells, stable envelopes, λ-chains, Hecke algebra, Demazure–Lusztig operators.

1. Introduction

Let G be a complex, semisimple, Lie group and $T \subset B \subset P \subset G$ be a parabolic subgroup containing a Borel subgroup and the (standard) maximal torus. Let W be the Weyl group determined by (G, T). In the study of cohomology and K-theory rings of (generalized) flag manifolds G/P, the Chevalley formula expresses the multiplication of a Schubert class by the class of a line bundle, or a Schubert divisor in G/P. If one works equivariantly, this formula determines completely the multiplication in the equivariant K ring. In this paper we prove a Chevalley formula for the coefficients $C^w_{u,\lambda}(y) \in K_T(pt)[y]$ arising in the multiplication

$$MC_y(X(w)^\circ) \cdot \mathcal{L}_\lambda = \sum C^w_{u,\lambda}(y) MC_y(X(u)^\circ) \tag{1}$$

in the equivariant K-theory ring $K_T(G/B)[y]$; see (27) and Theorem 5.2 below. Here $MC_y(X(w)^\circ) \in K_T(G/B)[y]$ is the *motivic Chern class* of a Schubert cell $X(w)^\circ \subset G/B$, and $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ is the line bundle on G/B associated to the one-dimensional B-module of weight λ.

The motivic Chern classes $MC_y(X(w)^\circ) \in K_T(G/B)[y]$ have been defined by Brasselet, Schürmann, and Yokura [BSY10] more generally for elements $[Y \to X]$ in the Grothendieck group $G_0(var/X)$ of varieties over X. They are the unique classes which are functorial with respect to proper morphisms $f : X_1 \to X_2$, and which satisfy the normalization condition $MC[\mathrm{id}_X : X \to X] = \lambda_y(T^*_X)$ for X smooth, where $\lambda_y(T^*_X) = \sum y^i \left[\bigwedge^i T^*_X\right]$ is the Hirzebruch λ_y class; see Section 4 below. They may be thought of as the K-theoretic generalizations of Chern–Schwartz–MacPherson classes defined by MacPherson [Mac74].

The motivic Chern classes of Schubert cells generalize well studied classes from Schubert calculus. If $y = 0$, the motivic class $MC_y(X(w)^\circ)$ is equal to the class of the ideal sheaf $[\mathcal{O}_{X(w)}(-\partial X(w))]$ of the boundary $\partial X(w) = X(w) \setminus X(w)^\circ$, where $X(w) = \overline{X(w)^\circ}$ is the Schubert variety. If $y = -1$, then $MC_y(X(w)^\circ)$ is equal to the class of the unique T-fixed point in $X(w)^\circ$; see [AMSS24b]. The Poincaré duals of the classes $MC_y(X(w)^\circ)$, the *Segre motivic classes* [AMSS24a; MS22], specialize when $y = 0$ to the Grothendieck classes of the structure sheaves of the opposite Schubert varieties. Our Chevalley formula (1), and its analogous formula for the Segre motivic classes, specialize to known Chevalley formulae for K-theoretic Schubert classes and ideal sheaves from [GR04; LP07].

The formula for the coefficients $C^w_{u,\lambda}(y) \in K_T(pt)[y]$ from (1) follows from a formula of Ram [Ram06] in the affine Hecke algebra $\mathbb{H}$, calculating transition coefficients between two bases $\{T_w X^\lambda\}$ and $\{X^\lambda T_w\}$ of the affine Hecke algebra:

$$T_w X^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} (-q)^{\ell(w)-\ell(u)} c^{w,\lambda}_{u,\mu} X^{-\mu} T_u. \tag{2}$$

Here T_w is an element in the standard basis of $\mathbb{H}$, $X^{-\lambda}$ is an affine element in $\mathbb{H}$, and $X^*(T)$ denotes the weight lattice of T. Ram's formula is stated in terms of a combinatorial model utilizing alcove walks, and it is convenient for our purposes to rewrite it utilizing in terms of λ-chains, a model introduced and studied by Lenart and Postnikov [LP07; LP08] in relation to equivariant K theory of flag manifolds. We refer to Theorem 3.9 and Theorem 3.10 for the precise statements in the Hecke algebra in terms of

λ-chains, and to Section 5.1 for the formulae involving motivic Chern classes. We also note that (affine) Hecke algebras have long been used to obtain Chevalley formulas in various contexts, for example in [PR99; LP07]. We state next our main result.

Assume λ is an integral weight and fix a reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$. The chain corresponds to an alcove walk from the fundamental alcove $A_\circ$ to $A_\circ - \lambda$, with separating hyperplanes $H_{-\beta_j, d_j}$. Denote by s_β the reflection determined by the root β. We refer the reader to Section 3 below for full definitions. The following is our main result; cf. Theorem 5.5.

Theorem 1.1. *The following Chevalley formula holds in* $K_T(G/B)[y]$:

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ) = \sum_{\substack{\mu \in X^*(T) \\ u \in W}} C^w_{u,-\lambda} MC_y(X(u)^\circ),$$

where the Chevalley coefficients are given by

$$C^w_{u,-\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (1+y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\tilde{r}_{J_>}(\lambda)}, \tag{3}$$

and the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ *such that* $u < us_{\beta_{j_1}} < us_{\beta_{j_1}} s_{\beta_{j_2}} < \cdots < us_{\beta_{j_1}} s_{\beta_{j_2}} \cdot \cdots \cdot s_{\beta_{j_t}} = w$; *the Weyl group element* $\tilde{r}_{J_>}$ *is defined in* (19). *For the multiplication* $\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)$, *the Chevalley coefficients are given by*

$$C^w_{u,\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (-1-y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\hat{r}_{J_<}(-\lambda)}, \tag{4}$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ *such that* $u < us_{\beta_{j_t}} < us_{\beta_{j_t}} s_{\beta_{j_{t-1}}} < \cdots < us_{\beta_{j_t}} \cdot \cdots \cdot s_{\beta_{j_1}} = w$, *and with* $\hat{r}_{J_<}$ *defined in* (16).

The connection between the Hecke algebra coefficients from (2) and the Chevalley coefficients above is given by

$$C^w_{u,-\lambda} = \sum_{\mu \in X^*(T)} y^{\ell(w)-\ell(u)} e^{-\mu} c^{w,\lambda}_{u,\mu}|_{q=-y}. \tag{5}$$

The coefficients $c^{w,\lambda}_{u,\mu}$ are in general Laurent polynomials in y, while $C^w_{u,-\lambda}$ are polynomials in $K_T(pt)[y]$. In fact, the power $y^{\ell(w)-\ell(u)}$ from the formula (5) is absorbed into $c^{w,\lambda}_{u,\mu}$ so it becomes polynomial in y.

As mentioned above, our Chevalley formula generalizes to the motivic situation the classical Chevalley multiplication in $K_T(G/B)$. It also generalizes the Chevalley multiplication by (equivariant) Chern–Schwartz–MacPherson classes of Schubert cells from [AMSS23]; a short, self-contained, proof of this is given in Appendix A.

All these specializations are appropriately positive, in the sense of [Buc02; Bri02; AGM11]. In Section 5.3 below we investigate some positivity results for the general formula. Notably, our formula for the multiplication by $\mathcal{L}_\lambda$ with λ dominant (i.e., when $\mathcal{L}_{-\lambda}$ is globally generated) may be written as a positive combination of products $q^a(q-1)^b$, with $q = -y$; see Proposition 5.7. This positivity is similar to the one satisfied by R-polynomials in Kazhdan–Lusztig theory. In an earlier arχiv version of this paper, we conjectured different positivity properties for special cases of the Chevalley coefficients, regarded as polynomials in y. As we explain in Remark 5.10, we since found examples in Lie types D_6, E_6, A_7 where the conjectured positivity fails.

We now give a rough idea on the proof of Theorem 1.1. The key connection between the Chevalley formula in the Hecke algebra to motivic Chern classes, proved in [MNS22b], and ultimately based on results from [AMSS24a], is that the motivic Chern classes are recursively obtained by certain *left* Demazure–Lusztig operators $\mathcal{T}_w^L$ acting on $K_T(G/B)[y]$:

$$MC_y(X(w)^\circ) = \mathcal{T}_w^L[\mathcal{O}_{1.B}].$$

These operators commute with elements in $K_G(G/B)[y]$ (i.e., the Weyl-group invariants of $K_T(G/B)$), and an argument based on equivariant localization shows that

$$MC_y(X(w)^\circ) \cdot \mathcal{L}_\lambda = \mathcal{T}_w^L[\mathcal{O}_{1.B}] \cdot \mathcal{L}_\lambda = \mathcal{T}_w^L(\mathcal{L}_\lambda \cdot [\mathcal{O}_{1.B}]) = \mathcal{T}_w^L(e^\lambda \cdot [\mathcal{O}_{1.B}]).$$

Therefore, the knowledge of the expansion from (2) implies the Chevalley formula in the geometric case. This argument may be generalized to any homogeneous bundle; see Remark 5.3. In cohomology (i.e., for the Chern–Schwartz–MacPherson classes), and for $G = \mathrm{SL}_n$, this argument is implicitly utilized in the paper [FGX22] to obtain a Murnaghan–Nakayama formula.

We briefly survey next other results from this note. Having established a formula to calculate the Chevalley coefficients, in Section 6 we utilize several dualities with geometric origin (the Serre duality, the star duality, and the Dynkin automorphism duality) to obtain several symmetries of the coefficients $C_{v,\lambda}^w(y)$. See Proposition 6.5, for example. Combining these dualities shows that the polynomials $C_{v,\lambda}^w(y)$ are palindromic.

A remarkable property of the motivic Chern classes of Schubert cells, proved in [AMSS24a; FRW21], is that they are equivalent to the K-theoretic version of Maulik and Okounkov's stable envelopes; see [MO19; AO21]. The stable envelopes are elements in the $T \times \mathbb{C}^*$-equivariant K-theory of the cotangent bundle, $K_{T\times\mathbb{C}^*}(T^*(G/B))$. In this context, the formal variable y may be identified to the (inverse) of the character given by the $\mathbb{C}^*$ fiber dilation on the cotangent bundle. If $\iota : G/B \hookrightarrow T^*(G/B)$ is the inclusion of the zero section, then $\iota^*(\mathrm{stab}(w))$ is a multiple of the motivic Chern class of the (opposite) Schubert cell for w, where stab is a stable envelope, appropriately normalized.

The stable envelopes depend on three parameters: a chamber, a polarization, and a slope, and the precise normalizations are essential for this paper. A variation in the chamber results in conjugating by the Borel subgroup [AMSS24a], and it is encoded in the left Weyl group action [MNS22b] and certain R-matrix operators [RTV15; RTV19]. Varying the polarization, or the slope, results in the multiplication of $\mathrm{stab}(w)$ by a line bundle $\mathcal{L}_\lambda$ pulled back from G/B; cf. [AMSS24a; Oko27], and see also Section 7 below. In particular, the coefficients $C_{v,\lambda}^w(y)$ from (1) give "wall-crossing" formulae, recording the change of stable envelopes when its defining parameters are varied. While these wall crossing formulae have been worked out in [Oko27; SZZ20; SZZ21] (see also [KW25]), in Section 7 we revisit some of these from the point of view of Theorem 1.1. In particular, we utilize the Chevalley formula to give an explicit combinatorial rule relating the stable envelope for the fundamental alcove $A_\circ$ to the one corresponding to any translation $A_\circ + \lambda$; see Theorem 7.8.

In addition to our application mentioned above to wall crossing formulae for stable envelopes, in Section 8 we utilize known relations between motivic Chern classes of Schubert cells, Whittaker functions, and Hall–Littlewood polynomials, to obtain new formulae for the latter.

Finally, in Appendix A we obtain an analogue of the Chevalley formula (1) for the homological analogue of the motivic Chern classes, the Chern–Schwartz–MacPherson classes. While this formula may be obtained by a specialization argument as in [AMSS24b], the degenerate affine Hecke algebra is much simpler in this case, and a direct proof of the Chevalley formula may be obtained rather quickly.

Notation. We fix the notation utilized throughout the paper. Let G be a simply connected complex Lie group with Borel subgroup B and maximal torus $T \subset B$. Denote by $\mathfrak{g} = \mathrm{Lie}(G)$ and by $\mathfrak{h} = \mathrm{Lie}(T)$ be corresponding Lie algebras. Let $R^+ \subset \mathfrak{h}^* := \mathfrak{h}^*_{\mathbb{Q}}$ denote the positive roots, which by convention are the roots in B, and by $\Sigma = \{\alpha_i : i \in I\}$ the set of simple roots. The set of all roots is $R := R^+ \sqcup -R^+$. We use $\alpha > 0$ (resp. $\alpha < 0$) to denote $\alpha \in R^+$ (resp. $\alpha \in -R^+$). For any root $\alpha \in R$, let $\alpha^\vee \subset \mathfrak{h}$ denote the corresponding coroot. Denote by

$$\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Q}$$

the evaluation pairing, and let $X^*(T) \subset \mathfrak{h}^*$ be the weight lattice. For any weight $\lambda \in X^*(T)$, let $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$ be the line bundle on G/B associated to λ. The Weyl group is $W = N_G(T)/T$ and it is generated by simple reflections $s_i = s_{\alpha_i}$ $(i \in I)$. It is equipped with a length function $\ell : W \to \mathbb{Z}_{\geq 0}$ defined as the length of a minimal expression of w in terms of the simple reflections; we denote by w_0 the longest element. The Bruhat order on W is a partial order determined by the covering relations $u \leq us_\alpha$ where $\alpha \in R$ and $\ell(us_\alpha) = \ell(u) + 1$.

For any $w \in W$, let $X(w)^\circ := BwB/B \subset G/B$ and $Y(w)^\circ := B^- wB/B \subset G/B$ be Schubert cells, where B^- is the opposite Borel subgroup. Let $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ be the respective Schubert varieties. Let $P (\supseteq B)$ be a parabolic subgroup with simple roots $\Sigma_P \subset \Sigma$. Let R_P^+ denote the positive roots spanned by Σ_P. Let W_P be the Weyl group generated by the simple reflections s_α, $\alpha \in \Sigma_P$. Let $W^P \simeq W/W_P$ denote the set of minimal length representatives. For any $w \in W^P$, let $X(wW_P)^\circ := BwP/P \subset G/P$ (resp. $Y(wW_P)^\circ := B^- wP/P \subset G/P$) denote the Schubert cell with closure $X(wW_P)$ (resp. $Y(wW_P)$). Let $X^*(T)_P := \{\lambda \in X^*(T) \mid \langle \lambda, \gamma^\vee \rangle = 0 \text{ for all } \gamma \in R_P^+\}$ be the set of integral weights which vanish on $(R_P^+)^\vee$. For any $\lambda \in X^*(T)_P$, we still use $\mathcal{L}_\lambda$ to denote the line bundle $G \times^P \mathbb{C}_\lambda \in \mathrm{Pic}(G/P)$, which has fiber over $1.P$ the one-dimensional T-module of weight λ.

2. Affine Hecke algebra via alcove walk algebra

In this section we introduce the alcove walk algebra, and a formula of Ram [Ram06] describing a change of bases matrix for the affine Hecke algebra.

2.1. ***Affine Hecke algebra.*** The affine Hecke algebra $\mathbb{H}$ is a free $\mathbb{Z}[q, q^{-1}]$ module with basis $\{T_w X^\lambda | w \in W, \lambda \in X^*(T)\}$, such that

- For any $\lambda, \mu \in X^*(T)$, $X^\lambda X^\mu = X^{\lambda+\mu}$.
- For any simple root α, $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0$.
- For any $w, y \in W$, such that $\ell(wy) = \ell(w) + \ell(y)$, $T_w T_y = T_{wy}$
- For any simple root α and $\lambda \in X^*(T)$,

$$T_{s_\alpha} X^\lambda - X^{s_\alpha \lambda} T_{s_\alpha} = (1-q) \frac{X^{s_\alpha \lambda} - X^\lambda}{1 - X^{-\alpha}}.$$

For our geometric application we will need two other bases of the affine Hecke algebra $\mathbb{H}$: $\{T_{w^{-1}}^{-1} X^\lambda \mid w \in W, \lambda \in X^*(T)\}$ and $\{X^\lambda T_{w^{-1}}^{-1} \mid w \in W, \lambda \in X^*(T)\}$. Define the transition matrix coefficients $c_{u,\mu}^{w,\lambda} \in \mathbb{Z}[q, q^{-1}]$ by

$$T_{w^{-1}}^{-1} X^\lambda = \sum_{\substack{\mu \in X^*(T) \\ u \in W}} c_{u,\mu}^{w,\lambda} X^\mu T_{u^{-1}}^{-1}. \tag{6}$$

The main result of this section is a formula for $c_{u,\mu}^{w,\lambda}$ obtained by Ram [Ram06]; see Theorem 2.4.

For the later application to the motivic Chern classes, we also introduce the Iwahori–Matsumoto $\mathbb{Z}[q, q^{-1}]$-algebra involution Θ on $\mathbb{H}$ defined by

$$\Theta(T_{s_\alpha}) = -q T_{s_\alpha}^{-1} \quad \text{and} \quad \Theta(X^\lambda) = X^{-\lambda},$$

where s_α is a simple reflection; see [EM97, Section 5.1]. Hence, $\Theta(T_{w^{-1}}^{-1}) = (-q)^{-\ell(w)} T_w$. Applying Θ to (6), we obtain

$$T_w X^{-\lambda} = \sum_{\substack{\mu \in X^*(T) \\ u \in W}} (-q)^{\ell(w)-\ell(u)} c_{u,\mu}^{w,\lambda} X^{-\mu} T_u \tag{7}$$

2.2. ***Alcove walk algebra.*** In this section we review Ram's definition of the alcove walk algebra, and state his formula for the matrix coefficients $c_{u,\mu}^{w,\lambda}$. We refer the reader to [Ram06] for a more detailed account of the alcove walk algebras.

2.2.1. *Alcoves.* Let $\mathfrak{t}_\mathbb{R}^*$ be the dual of the Lie algebra of the maximal torus T. For any root α and $j \in \mathbb{Z}$, define

$$H_{\alpha,j} := \{\lambda \in \mathfrak{t}_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle = j\}.$$

Notice that $H_{\alpha,j} = H_{-\alpha,-j}$. The connected components of $\mathfrak{t}_\mathbb{R}^* \setminus \bigcup_{\alpha>0, j\in\mathbb{Z}} H_{\alpha,j}$ are called *alcoves*. The codimension 1 faces of any alcove are called the walls of that alcove. The *fundamental alcove* $\mathrm{A}_\circ$ is defined by

$$\mathrm{A}_\circ = \{\lambda \in \mathfrak{t}_\mathbb{R}^* \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \text{ for any positive root } \alpha\}.$$

If $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the simple roots, and $\theta^\vee$ is the highest coroot, then the walls of the fundamental alcove $\mathrm{A}_\circ$ are $H_{\theta,1}$ and $H_{\alpha_i,0}$ $(1 \le i \le r)$. We label these walls of $\mathrm{A}_\circ$ by $0, 1, \ldots, r$ respectively.

The affine Weyl group for the dual root system is defined by $W_{\mathrm{aff}} := Q \rtimes W$, where Q is the root lattice. Then W_{aff} acts simply transitively on the set of alcoves, and this action is determined by the reflections across the hyperplanes $h = H_{\alpha,j}$, given by

$$s_{\alpha,j}(\mu) = \hat{r}_h(\mu) := s_\alpha(\mu) + j\alpha \quad \text{for } \mu \in X^*(T). \tag{8}$$

The affine Weyl group is a Coxeter group generated by the reflections $s_0 := s_{\theta,1}$ and s_i $(1 \le i \le r)$ along the walls of $\mathrm{A}_\circ$. In fact, $\mathrm{A}_\circ$ is a fundamental domain for the action of W_{aff} on the set of alcoves, in the sense that any element in $\mathfrak{t}^*_{\mathbb{R}} \setminus \bigcup_{\alpha>0,\, j\in\mathbb{Z}} H_{\alpha,j}$ is sent to exactly one element in $\mathrm{A}_\circ$. See [Ram06; Hum90] for details.

The extended affine Weyl group for the dual root system is $W^{\mathrm{ext}}_{\mathrm{aff}} = X^*(T) \rtimes W$, where $X^*(T)$ is the weight lattice. For any $\lambda \in X^*(T)$, let t_λ denote the corresponding element in $W^{\mathrm{ext}}_{\mathrm{aff}}$. There is a length function ℓ on $W^{\mathrm{ext}}_{\mathrm{aff}}$ defined by the following formula (see [Mac96, equation (2.8)]):

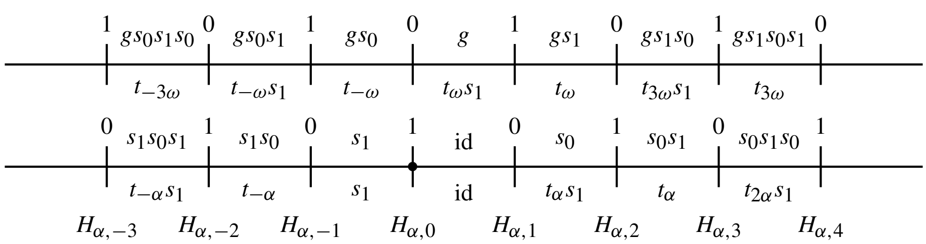
$$\ell(t_\mu w) = \sum_{\alpha\in R^+} |\langle \mu, w(\alpha^\vee)\rangle + \chi(w(\alpha))|, \qquad \text{where } \chi(\alpha) = \begin{cases} 0 & \text{if } \alpha \in R^+, \\ 1 & \text{if } \alpha \in R^-. \end{cases}$$

Let $\Omega \subset W^{\mathrm{ext}}_{\mathrm{aff}}$ be the subgroup of length zero elements in $W^{\mathrm{ext}}_{\mathrm{aff}}$. Then $W^{\mathrm{ext}}_{\mathrm{aff}} \simeq W_{\mathrm{aff}} \rtimes \Omega$, see [Mac96, equation (2.10)], and $\ell(wg) = \ell(w)$ for any $w \in W_{\mathrm{aff}}$ and $g \in \Omega$. The elements in Ω preserve the fundamental alcove $\mathrm{A}_\circ$ and act as automorphisms.

Using a bijection between W_{aff} and the alcoves in $\mathfrak{t}^*_{\mathbb{R}}$, one defines a bijection between $W^{\mathrm{ext}}_{\mathrm{aff}} \simeq W_{\mathrm{aff}} \rtimes \Omega$ and the alcoves in $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$ ($|\Omega|$ copies of $\mathfrak{t}^*_{\mathbb{R}}$, each tiled by alcoves). We label the walls of every alcove in $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$ in an $W^{\mathrm{ext}}_{\mathrm{aff}}$-equivariant way: for each $w \in W^{\mathrm{ext}}_{\mathrm{aff}}$ the walls of $w\mathrm{A}_\circ$ are $wH_{\alpha_i,0}$ $(1 \le i \le n)$ and $wH_{\theta,1}$, and they are labeled by i and 0, respectively. In particular, if two adjacent alcoves A_1 and A_2 are separated by a wall labeled by i (in both A_1 and A_2), and $A_1 = w\mathrm{A}_\circ$ for some $w \in W^{\mathrm{ext}}_{\mathrm{aff}}$, then $A_2 = ws_i\mathrm{A}_\circ$. Equivalently, in terms of the wall crossings, if $w = gs_{i_1}s_{i_2}\cdot\cdots\cdot s_{i_\ell} \in W^{\mathrm{ext}}_{\mathrm{aff}}$, with $g \in \Omega$ and $0 \le i_j \le r$, then the alcove $w\mathrm{A}_\circ$ in $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$ is the alcove obtained from rotating the fundamental alcove $\mathrm{A}_\circ$ according to the automorphism g, then reflecting along the walls labeled (in order) by $i_1, \ldots, i_\ell$. See also Lemma 3.3.

Example 2.1. We consider the example of the root system of type A_1. Let α be the positive root, and $\omega = \alpha/2$ be the fundamental weight. The weight lattice $X^*(T)$ is $\mathbb{Z}\omega$, the root lattice Q is $\mathbb{Z}\alpha$, and the finite Weyl group W is $\{\mathrm{id}, s_1 = s_\alpha\}$. The affine Weyl group $W_{\mathrm{aff}} = Q \rtimes W$ has Coxeter generators s_1 and $s_0 = t_\alpha s_1$. The subgroup of length zero elements in W^{ext}_{aff} is $\Omega = \{\mathrm{id}, g = t_\omega s_1\} \simeq \mathbb{Z}/2\mathbb{Z}$.

In the following picture for $\Omega \times \mathfrak{t}^*_{\mathbb{R}}$, the lower sheet is the identity sheet, while the upper sheet is the



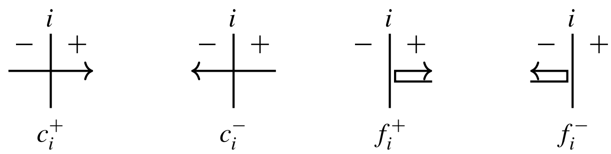
sheet $g \times \mathfrak{t}_{\mathbb{R}}^*$. Each alcove $w\mathrm{A}_\circ$ is labeled by the corresponding $w \in W_{\mathrm{aff}}^{ext}$, both in the Coxeter presentation and the translation presentation. In the lower sheet, the walls $H_{\alpha,n}$ are labeled by 1 if n is even, and 0 if n is odd. On the upper sheet, the labelings are in the opposite way.

2.2.2. *Alcove walk algebra.* We recall a realization of the Hecke algebra in terms of alcove walks; see [Ram06]. For each positive root α and hyperplane $H_{\alpha,j}$, set the positive side of it to be $\{\lambda \in \mathfrak{t}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle > j\}$.

Definition 2.2. The *alcove walk algebra* is generated over $\mathbb{Z}[q, q^{-1}]$ by elements $g \in \Omega$, and for $0 \le i \le r$, the elements c_i^+ (positive i-crossing), c_i^- (negative i-crossing), f_i^+ (positive i-fold) and f_i^- (negative i-fold), subject to the following relations, sometimes called straightening laws:

$$c_i^+ = c_i^- + f_i^+, \quad c_i^- = c_i^+ + f_i^-, \quad gc_i^\pm = c_{g(i)}^\pm g, \quad gf_i^\pm = f_{g(i)}^\pm g.$$

In terms of pictures, these generators can be drawn as follows:



$(0 \le i \le r)$

Here, c_i^+ represents a crossing of a wall labeled by i from its negative to its positive side, and similarly for the other generators. The product is given as concatenation. An *alcove walk* is a word in the generators such that

- the tail of the first step is in the fundamental alcove $\mathrm{A}_\circ$, and
- at every step, either we change the sheet according to an element in Ω (thus rotating the alcove according to this elements), or the head of each arrow is in the same alcove as the tail of the next arrow.

An alcove walk p is called *nonfolded* if there is no $f_i^\pm$ in its word. The *length* of an alcove walk is the number of letters $c_i^\pm$, $f_i^\pm$ in an alcove walk. (In particular, rotation with respect to an element of Ω does not contribute to the length.) For a *minimal* alcove walk between two alcoves, one can show that the walk is nonfolded, thus its length is the number of $c_i^\pm$ in the walk [Ram06]. From this it follows that if $w \in W_{\mathrm{aff}}^{\mathrm{ext}}$, then $\ell(w)$=length of a minimal length walk from $\mathrm{A}_\circ$ to $w\mathrm{A}_\circ$.

Pick a square root $q^{\frac{1}{2}}$ of q. The following was proved by A. Ram.

Proposition 2.3 [Ram06, §3.2]. (a) *The affine Hecke algebra $\mathbb{H}$ is isomorphic as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra to the quotient of the alcove walk algebra by the relations*

$$c_i^+ = (c_i^-)^{-1}, \quad f_i^+ = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad f_i^- = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \tag{9}$$

and

$$p = p' \text{ if } p \text{ and } p' \text{ are nonfolded alcove walks with } \mathrm{end}(p) = \mathrm{end}(p'), \tag{10}$$

where $\mathrm{end}(p)$ *means the final alcove of* p.

(b) *Under the previous isomorphism, for any $w \in W$ and $\lambda \in X^*(T)$:* [1]

[1] The q^2 and T_{s_i} in [Ram06, Proposition 3.2(b)] are our q and $q^{-\frac{1}{2}} T_i$, respectively.

- *a minimal length alcove walk from* $\mathrm{A}_\circ$ *to* $w\mathrm{A}_\circ$ *is sent to* $q^{\ell(w)/2}T_{w^{-1}}^{-1}$, *and,*
- *a minimal length alcove walk from* $\mathrm{A}_\circ$ *to* $t_\lambda\mathrm{A}_\circ$ *is sent to* X^λ.

For an alcove walk p, define the functions *weight* $wt(p) \in X^*(T)$, and *final direction* $\varphi(p) \in W$ of p by the condition that p ends in the alcove $t_{wt(p)}\varphi(p)\mathrm{A}_\circ$. Let

$$\begin{aligned} f^-(p) &= \text{number of negative folds of } p,\\ f^+(p) &= \text{number of positive folds of } p,\\ f(p) &= f^+(p) + f^-(p). \end{aligned}$$

Now we can state the formula for the matrix coefficients $c_{u,\mu}^{w,\lambda}$ defined in (6); see also (7).

Theorem 2.4 [Ram06, Theorem 3.3]. *Let* $\lambda \in X^*(T)$ *and* $w \in W$. *Fix a minimal length walk* $p_w = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^-$ *from* $\mathrm{A}_\circ$ *to* $w\mathrm{A}_\circ$ *and a minimal length walk* $p_\lambda = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \cdots c_{j_s}^{\epsilon_s} g$ *from* $\mathrm{A}_\circ$ *to* $t_\lambda\mathrm{A}_\circ$, *where* $g \in W_{\mathrm{aff}}^{ext}$ *is defined by the condition* $gW_{\mathrm{aff}} = t_\lambda W_{\mathrm{aff}}$ [2], *and* $\epsilon_i = \pm$ *for each* i. *Then*

$$T_{w^{-1}}^{-1} X^\lambda = \sum_p (-1)^{f^-(p)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{f(p)} q^{\frac{1}{2}(\ell(\varphi(p)) - \ell(w))} X^{wt(p)} T_{\varphi(p)^{-1}}^{-1}, \tag{11}$$

where the sum is over all alcove walks p *of the form*

$$p = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^- p_{j_1} p_{j_2} \cdots p_{j_s} g \text{ such that } p_{j_k} \in \{c_{j_k}^{\pm}, f_{j_k}^{\epsilon_k}\}. \tag{12}$$

Therefore the matrix coefficients $c_{u,\mu}^{w,\lambda}$ *in* (6) *are given by*

$$c_{u,\mu}^{w,\lambda} = \sum_{\substack{p \text{ of the form } (12)\\ \varphi(p)=u,\, wt(p)=\mu}} (-1)^{f^+(p)} (1-q)^{f(p)} q^{\frac{1}{2}(\ell(u) - \ell(w) - f(p))}. \tag{13}$$

Example 2.5. Let $G = \mathrm{SL}(2, \mathbb{C})$. We use the same notation as in Example 2.1. We check the above theorem for $w = s_1$ and $\lambda = \omega$. From the alcove picture in Example 2.1, $T_{s_1}^{-1}$ is represented by the minimal length walk $q^{-1/2}c_1^-$, while X^ω is represented by the walk $gc_1^+ = c_0^+ g$. Thus, the sum in the right hand side of the Theorem is over the alcove walks $c_1^- c_0^- g$ and $c_1^- f_0^+ g$, which end at the alcoves $t_{-\omega}s_1\mathrm{A}_\circ$ and $t_{-\omega}\mathrm{A}_\circ$, respectively. (Note that $c_1^- c_0^+ g$, which represents $T_{s_1}^{-1}X^\omega$, is not an alcove walk.) Therefore, the identity in the theorem is

$$T_{s_1}^{-1} X^\omega = X^{-\omega} T_{s_1}^{-1} - q^{-1}(1-q) X^{-\omega}.$$

On the other hand, it is easy to check the above equation using the definition of the affine Hecke algebra in Section 2.1.

Remark 2.6. In Theorem 2.4, one may relax the hypotheses about the alcove walks p_w and p_λ to be of nonminimal length. This follows from analyzing the proof of Ram's result in [Ram06]. We do not use this, but it is consistent with our use of nonreduced λ-chains in Section 3 below.

---

[2] We need to add this extra $g \in \Omega$ since $t_\lambda\mathrm{A}_\circ$ may not on the same sheet as $\mathrm{A}_\circ$, see Example 2.5. The stated conditions determine g uniquely.

Consider an ordered collection of hyperplanes $\mathcal{H} = \{H_{\beta_1,k_1}, \dots, H_{\beta_s,k_s}\}$ and set $h_i := H_{\beta_i,k_i}$. Associated to this sequence we define the elements

$$\hat{r}_{\mathcal{H}} = s_{\beta_1,k_1} \cdot \cdots \cdot s_{\beta_s,k_s} \in W_{\text{aff}}; \quad r_{\mathcal{H}} = s_{\beta_1} \cdot \cdots \cdot s_{\beta_s} \in W.$$

(These depend on the order of $\mathcal{H}$.) We also define an $\mathcal{H}$-restricted version of the Bruhat order on W by

$$w \xrightarrow{\mathcal{H}} u \iff w > ws_{\beta_1} > ws_{\beta_1}s_{\beta_2} > \cdots > ws_{\beta_1}s_{\beta_2} \cdot \cdots \cdot s_{\beta_s} = u. \tag{14}$$

Lemma 2.7 (compare [Len11, Proposition 2.5]). *Let $w \in W$ and $\lambda \in X^*(T)$. Fix:*

- *an alcove walk p_w from $\mathrm{A}_\circ$ to $w\mathrm{A}_\circ$;*
- *an alcove walk $p_\lambda = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \cdots c_{j_s}^{\epsilon_s} g$ from $\mathrm{A}_\circ$ to $t_\lambda \mathrm{A}_\circ$.*

Let $h_i = H_{\beta_i,k_i}$, for $1 \le i \le s$ be the sequence of hyperplanes defined by the walls of alcoves crossed by p_λ, with $\beta_i \in R^{\epsilon_i}$ and $k_i \in \mathbb{Z}$.

Then there is a bijection between

(1) *the set of alcove walks of the form $\bar{p} = p_w p_{j_1} \dots p_{j_s}$ such that $p_{j_k} \in \{c_{j_k}^{\pm}, f_{j_k}^{\epsilon_k}\}$, and*

(2) *the set of subsets $\mathcal{M} = \{h_{m_1}, \dots, h_{m_t}\} \subset \mathcal{H} = \{h_1, \dots, h_s\}$ with $m_1 < m_2 < \cdots < m_t$, s.t. $w \overset{\mathcal{M}}{\Longrightarrow} wr_{\mathcal{M}}$.*

Under this bijection, the indices m_i correspond to the positions of foldings $f_{m_i}^{\epsilon_{m_i}}$. Furthermore,

$$\varphi(\bar{p}g) = wr_{\mathcal{M}} \text{ and } \mathrm{wt}(\bar{p}g) = w\hat{r}_{\mathcal{M}}(\lambda).$$

Remark 2.8. This statement is a slight generalization of Lenart's result. We do not require λ to be dominant, and therefore we need to allow $\beta_i \in R^{\epsilon_i}$ to be a negative root.

Proof. The proof follows the same outline as that of [Len11, Proposition 2.5], so we will be brief. To start, consider the *unique* unfolded alcove walk $p_0 = p_w p_{j_1} \dots p_{j_s}$ such that $p_{j_k} \in \{c_{j_k}^{\pm}\}$. Any other alcove walk $\bar{p}$ as in the statement is of the form

$$\bar{p} = \phi_{m_t} \cdots \phi_{m_1}(p_0),$$

where ϕ_{m_j} is the folding operation at position m_j (cf. [Len11]) and the m_i's with $m_{i-1} < m_i$ are the folding positions of $\bar{p}$. This alcove walk has the property that if $k \notin \{m_1, \dots, m_t\}$ then $p_{j_k} \in \{c_{j_k}^{\pm}\}$ and $p_{j_{m_i}} = f_{j_{m_i}}^{\epsilon'_{m_i}}$ $(1 \le i \le t)$. In addition $\varphi(\bar{p}g) = wr_{\mathcal{M}}$ and $\mathrm{wt}(\bar{p}g) = w\hat{r}_{\mathcal{M}}(\lambda)$. Let $\mathcal{M}_i = \{h_{m_1}, \dots, h_{m_i}\} \subset \mathcal{H} = \{h_1, \dots, h_s\}$ for $1 \le i \le t$ with the convention that $\mathcal{M}_0 = \varnothing$ and $r_\varnothing = \mathrm{id}$. A key point is that $wr_{\mathcal{M}_{i-1}} > wr_{\mathcal{M}_i}$ if and only if the folding orientations satisfy $\epsilon_{m_i} = \epsilon'_{m_i}$. (The proof uses the condition that if $\beta > 0$, then $ws_\beta > w$ if and only if $w(\beta) > 0$.) All this put together implies that $\{h_{m_1}, \dots, h_{m_i}\} \leftrightarrow \phi_{m_i} \cdots \phi_{m_1}(p_0)$ gives the bijection in the statement. □

For a subset $\mathcal{M} \subset \mathcal{H}$ as in Lemma 2.7, set $p(\mathcal{M})$ to be the alcove walk $\bar{p}g$ associated to $\mathcal{M}$, and set $f^+(\mathcal{M}) = f^+(p(\mathcal{M}))$.

We next reformulate Ram's result from Theorem 2.4 in terms of paths in the Bruhat order.

Corollary 2.9. *Let $u, w \in W$ and $\lambda, \mu \in X^*(T)$. Assume the same hypotheses and notation as in Lemma 2.7. Then*

$$c_{u,\mu}^{w,\lambda} = \sum_{\mathcal{M}} (-1)^{f^+(\mathcal{M})}(1-q)^{|\mathcal{M}|} q^{\frac{1}{2}(\ell(u)-\ell(w)-|\mathcal{M}|)},$$

where the sum is over subsets $\mathcal{M} \subset \mathcal{H}$ which satisfy $w \overset{\mathcal{M}}{\Longrightarrow} u = wr_{\mathcal{M}}$ and $\mu = w\hat{r}_{\mathcal{M}}(\lambda)$.

Remark 2.10. *A priori*, the coefficients $c_{u,\mu}^{w,\lambda}$ appearing in the walk algebra are in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. However, after matching these with the initial definition of Hecke algebras, it turns out that $c_{u,\mu}^{w,\lambda} \in \mathbb{Z}[q^{\pm 1}]$. This can also be seen directly in the formula above, by observing that $\ell(u) - \ell(w) - |\mathcal{M}|$ is even. (This uses that a reflection has odd length, and the cancellation property of nonreduced expressions.)

3. A λ-chain formula for the transition coefficients $c_{u,\mu}^{w,\lambda}$

In this section we reformulate the alcove walk formula from Corollary 2.9 in terms of the notion of λ-chains introduced in [LP07]. This notion was utilized to obtain a K-theory Chevalley formula for the structure sheaves of Schubert varieties. The main results of this section are Theorem 3.9 and Theorem 3.10. Specializing $y \mapsto 0$, these recover [LP07, Theorem 6.1]. Throughout this section, we only consider alcoves on the identity sheet $\mathfrak{t}_{\mathbb{R}}^*$. Recall that $\mathrm{A}_\circ + \lambda := \{x + \lambda \mid x \in \mathrm{A}_\circ\}$ is the alcove on $\mathfrak{t}_{\mathbb{R}}^*$. (If λ is not a root, the alcove $t_\lambda(\mathrm{A}_\circ)$ is not the alcove $\mathrm{A}_\circ + \lambda$.)

3.1. *Alcove paths and λ-chains.* For any two alcoves A and B, which are separated by a common wall lying on a hyperplane $H_{\beta,k}$, write $A \xrightarrow{\beta} B$ if the root β points from A to B.

Definition 3.1 [LP07, Definition 5.2]. An *alcove path* is a sequence of alcoves $(A_0, A_1, \dots, A_l)$ such that A_{j-1} and A_j are adjacent, $1 \le j \le l$. We denote this by

$$A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l.$$

When the length l is minimal among all alcove paths from A_0 to A_l, it is called a *reduced alcove path.*

Remark 3.2. By definition, there is a one-to-one correspondence between alcove paths

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l = \mathrm{A}_\circ - \lambda \tag{15}$$

from $\mathrm{A}_\circ$ to $A_l = v_{-\lambda}(\mathrm{A}_\circ)$ and alcove walks from A_0 to A_l of the form $c_{i_1}^{\epsilon_1} c_{i_2}^{\epsilon_2} \cdots c_{i_l}^{\epsilon_l}$, where $-\beta_j \in R^{\epsilon_j}$ for $1 \le j \le l$.

Recall that $s_0 = s_{\theta,1}$ is the affine reflection along the hyperplane $H_{\theta,1}$ with $\theta^\vee$ the highest coroot. Define $\alpha_0 = -\theta$, $\bar{s}_0 = s_\theta$ and $\bar{s_i} = s_i$ for $1 \le i \le r$.

Lemma 3.3 [LP07, Lemma 5.3]. *For any $v \in W_{\mathrm{aff}}$, there is a one-to-one correspondence between decompositions of v in the simple reflections $s_i \in W_{\mathrm{aff}}$ $(0 \le i \le r)$ and alcove paths from* $\mathrm{A}_\circ$ *to* $v(\mathrm{A}_\circ)$ *as follows.*

For any decomposition $v = s_{i_1} s_{i_2} \cdots s_{i_l}$, *define*

$$\beta_j := \bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_{j-1}}(\alpha_{i_j}), \quad 1 \le j \le l.$$

Then

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l = v(\mathrm{A}_\circ)$$

is an alcove path from $\mathrm{A}_\circ$ *to* $v(\mathrm{A}_\circ)$.

The affine reflection along the j*-th hyperplane separating* A_{j-1} *and* A_j *is* $s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_2} s_{i_1}$; *in particular, the separating wall is labeled by* i_j. *Under this correspondence, a reduced alcove path corresponds to a reduced decomposition for* v.

Let λ be an integral weight and let $v_{-\lambda} \in W_{\mathrm{aff}}$ be the affine Weyl group element which satisfies $v_{-\lambda}(\mathrm{A}_\circ) = \mathrm{A}_\circ - \lambda$, i.e., $t_{-\lambda} = v_{-\lambda} g$ with $g \in \Omega$. Choose a (possibly nonreduced) decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ and let β_j be defined by

$$\beta_j := \bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_{j-1}}(\alpha_{i_j}), \quad 1 \le j \le l,$$

with the convention that $\alpha_0 = -\theta$ (see Lemma 3.3). Then

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} \mathrm{A}_\circ - \lambda$$

is an alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$.

Definition 3.4 [LP07, Definition 5.4]. The sequence of roots $(\beta_1, \ldots, \beta_l)$ is a λ*-chain* of roots associated to the decomposition of $v_{-\lambda}$. A λ-chain is called reduced if the decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ is reduced.

Let $H_{-\beta_j, d_j}$ be the hyperplane separating the alcoves A_j and A_{j+1}. The sequence of integers d_j are determined by the sequence of roots β_j, but we occasionally keep the information of d's in the notation, and we refer to the sequence of pairs $(\beta_1, d_1), (\beta_2, d_2), \ldots, (\beta_l, d_l)$ as a λ-chain [3]. Following [LP07, Prop. 6.8] we recall a combinatorial construction of a λ-chain for an integral weight λ.

Fix a linear order on the index of Dynkin nodes (for example $1 < 2 < \cdots < r$ in $I = \{1, 2, \ldots, r\}$). The set $\mathcal{R}_\lambda \subset W_{\mathrm{aff}}$ of affine reflections $s_{\alpha,k}$ for the hyperplanes $H_{\alpha,k}$ separating the fundamental alcove $\mathrm{A}_\circ$ and $\mathrm{A}_\circ - \lambda$ is given by

$$\mathcal{R}_\lambda = \bigcup_{\alpha \in R^+} \begin{cases} \{s_{\alpha,k} : 0 \ge k > -\langle \lambda, \alpha^\vee \rangle\} & \text{if } \langle \lambda, \alpha^\vee \rangle > 0, \\ \{s_{\alpha,k} : 0 < k \le -\langle \lambda, \alpha^\vee \rangle\} & \text{if } \langle \lambda, \alpha^\vee \rangle < 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

One defines a "height function"

$$h : \mathcal{R}_\lambda \to \mathbb{R}^{r+1}; \quad h(s_{\alpha,k}) = \frac{1}{\langle \lambda, \alpha^\vee \rangle}(-k, \langle \varpi_1, \alpha^\vee \rangle, \ldots, \langle \varpi_r, \alpha^\vee \rangle).$$

[3] If λ is dominant, this definition was extended to the Kac–Moody situation in [LP08].

It turns out that this function is injective. Now order the images of h in lexicographic order, so we obtain $h(s_{\alpha_1,k_1}) < \cdots < h(s_{\alpha_l,k_l})$. Define another function $b : \mathcal{R}_\lambda \to R^+ \cup R^-$ by

$$b(s_{\alpha,k}) = \begin{cases} \alpha & \text{if } k \le 0,\ \alpha \in R^+, \\ -\alpha & \text{if } k > 0,\ \alpha \in R^+. \end{cases}$$

Then $b(s_{\alpha_1,k_1}), \dots, b(s_{\alpha_l,k_l})$ is a (reduced) λ-chain of roots.

Remark 3.5. A particularly nice situation occurs for a minuscule fundamental weight ϖ_i, i.e., when $\langle \varpi_i, \alpha^\vee \rangle \in \{0, 1\}$ for any positive root α. In this case all $k_i = 0$ and $v_{-\varpi_i} = (w^{P_i})^{-1} \in W$, with w^{P_i} being the longest minimal length representative for cosets of W/W_{P_i}, where $W_{P_i} = \mathrm{Stab}_W(\varpi_i)$; equivalently, the Schubert variety $X(w^{P_i} W_{P_i}) = G/P_i$. A reduced decomposition and the associated ϖ_i-chain may be read from the associated Young poset of G/P_i; see [Pro99; Ste01] and also [BCMP18, §3.1] and Example 3.6 below. It may also be obtained as a reverse linear extension of the heap $H(w^{P_i})$, and this gives a one-to-one correspondence between reduced ϖ_i-chains and reverse linear extensions of the heap $H(w^{P_i})$. We refer the readers to [MNS22a; NO19] for the heap perspective.

Example 3.6. Consider $G = \mathrm{SL}_5$ and the fundamental weight ϖ_2. The stabilizer is the maximal parabolic P_2 so that G/P_2 is the Grassmannian Gr(2, 5). The inversion set and a reduced decomposition of $v_{-\varpi_2}$ may be read from the Young diagrams below, with the notation $(i - j) = \alpha_i + \cdots + \alpha_{j-1}$.

(2–3)	(2–4)	(2–5)
(1–3)	(1–4)	(1–5)

α_2	α_3	α_4
α_1	α_2	α_3

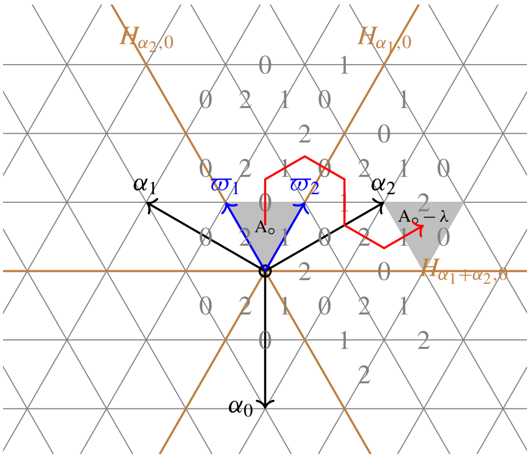
Then $v_{-\varpi_2} = s_2 s_3 s_4 s_1 s_2 s_3$ and a ϖ_2-chain of roots is given by $\{\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4\}$.

Example 3.7. Let $G = \mathrm{SL}_3$, with the Weyl group $W = S_3$, and consider $\lambda = 2\varpi_1 - 2\varpi_2$. An example of alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$ is

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} A_2 \xrightarrow{-\beta_3} A_3 \xrightarrow{-\beta_4} A_4 \xrightarrow{-\beta_5} A_5 \xrightarrow{-\beta_6} A_6 = \mathrm{A}_\circ - \lambda\ (A_i = r_i A_{i-1},\ 1 \le i \le 6),$$

which is the red path below, and it corresponds to the decomposition $v_{-\lambda} = s_0 s_1 s_0 s_1 s_2 s_1$. The corresponding alcove walk is $\bar{p} = c_0^+ c_1^+ c_0^- c_1^- c_2^- c_1^+$, and the corresponding λ-chain of roots is $(\beta_1, 1)$, $(\beta_2, 1)$, $(\beta_3, 0)$, $(\beta_4, -1)$, $(\beta_5, 1)$, $(\beta_6, 2)$, as calculated in Figure 1. Here we included the d's in the notation. We can choose another λ-chain for $\lambda = 2\varpi_1 - 2\varpi_2$ by using the reduced decomposition $v_{-\lambda} = s_1 s_0 s_2 s_1$. The corresponding reduced λ-chain is $(\alpha_1, 0)$, $(-\alpha_2, 1)$, $(\alpha_1, 1)$, $(-\alpha_2, 2)$.

Lemma 3.8 [LP07, Remark 5.5]. *Let $L = (\beta_1, \dots, \beta_l)$ be a λ-chain. Then $\bar{L} := (-\beta_l, \dots, -\beta_1)$ is a $(-\lambda)$-chain. If $H_{-\beta_j, d_j}$ is the j-th hyperplane of the alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$ determined by L, then the j-th hyperplane of alcove path from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ + \lambda$ determined by $\bar{L}$ is $H_{\beta_{l+1-j}, \langle \lambda, \beta_{l+1-j}^\vee \rangle - d_{l+1-j}}$. If L is reduced, then $\bar{L}$ is also reduced.*



$$
\begin{aligned}
&i_1=0,\ \ \beta_1=\alpha_0=-(\alpha_1+\alpha_2)\\
&i_2=1,\ \ \beta_2=\bar{s}_0(\alpha_1)=-\alpha_2\\
&i_3=0,\ \ \beta_3=\bar{s}_0\bar{s}_1(\alpha_0)=\alpha_1\\
&i_4=1,\ \ \beta_4=\bar{s}_0\bar{s}_1\bar{s}_0(\alpha_1)=\alpha_1+\alpha_2\\
&i_5=2,\ \ \beta_5=\bar{s}_0\bar{s}_1\bar{s}_0\bar{s}_1(\alpha_2)=\alpha_1\\
&i_6=1,\ \ \beta_6=\bar{s}_0\bar{s}_1\bar{s}_0\bar{s}_1\bar{s}_2(\alpha_1)=-\alpha_2
\end{aligned}
$$

$$
\begin{aligned}
&s_0=\hat{r}_{H_{-\beta_1,1}}\\
&s_0s_1s_0=\hat{r}_{H_{-\beta_2,1}}\\
&(s_0s_1)s_0(s_1s_0)=s_1=\hat{r}_{H_{-\beta_3,0}}\\
&(s_0s_1s_0)s_1(s_0s_1s_0)=s_0=\hat{r}_{H_{-\beta_4,-1}}\\
&(s_0s_1s_0s_1)s_2(s_1s_0s_1s_0)=\hat{r}_{H_{-\beta_5,1}}\\
&(s_0s_1s_0s_1s_2)s_1(s_2s_1s_0s_1s_0)=\hat{r}_{H_{-\beta_6,2}}
\end{aligned}
$$

Figure 1. To Example 3.7.

3.2. ***λ-chain formulae.*** Next we state the main theorem of this section. For any root hyperplane $h=H_{\beta,k}$, let r_h denote the reflection along the hyperplane $H_{\beta,0}$, and $\hat{r}_h$ be the reflection along h.

Assume λ is an integral weight and fix a reduced λ-chain $(\beta_1,\beta_2,\dots,\beta_l)$, which corresponds to an alcove walk from $A_\circ$ to $A_\circ-\lambda$, with separating hyperplanes $H_{-\beta_j,d_j}=:h_j$.

For a subset $J=\{j_1<j_2<\cdots<j_t\}\subset\{1,2,\dots,l\}$, define the relation

$$u\xrightarrow{J_>}w \quad\overset{\text{def.}}{\Longleftrightarrow}\quad u<ur_{h_{j_t}}<ur_{h_{j_t}}r_{h_{j_{t-1}}}<\cdots<ur_{h_{j_t}}\cdots r_{h_{j_1}}=w,$$

and let

$$\hat{r}_{J_<}:=\hat{r}_{h_{j_1}}\cdots\hat{r}_{h_{j_t}}. \tag{16}$$

Hence, $w\overset{J}{\Longrightarrow}u$ from (14) is equivalent to $u\xrightarrow{J_>}w$. Therefore, Corollary 2.9 can be restated as follows.

Theorem 3.9. *In the above setting,*

$$c_{u,\mu}^{w,-\lambda}=\sum(-1)^{n(J)}(1-q)^{|J|}q^{\frac{1}{2}(\ell(u)-\ell(w)-|J|)}, \tag{17}$$

where the sum is over subsets $J\subset\{1,2,\dots,l\}$ such that $u\xrightarrow{J_>}w$ and $w\hat{r}_{J_<}(-\lambda)=\mu$, and where $n(J):=\#\{j\in J\mid\beta_j<0\}$.

Proof. In the summation in Corollary 2.9, we let $J\subset\{1,2,\dots,l\}$ be the indices of the hyperplanes in $\mathcal{M}$. Then by Remark 3.2, $f^+(\mathcal{M})=n(J)$. This finishes the proof. □

Using the transformation between a λ-chain and a $(-\lambda)$-chain, we can get rid of the negative sign in front of λ in Theorem 3.9 as follows.

First of all, $(-\beta_l,-\beta_{l-1},\dots,-\beta_1)$ is a $(-\lambda)$-chain, which corresponds to an alcove walk from $A_\circ$ to $A_\circ+\lambda$, with the j-th separating hyperplane being

$$h'_j:=H_{\beta_{l+1-j},\langle\lambda,\beta^\vee_{l+1-j}\rangle-d_{l+1-j}}. \tag{18}$$

Then $r_{h'_j} = r_{h_{l+1-j}}$. Let $\tilde{r}_{h_{l+1-j}} := \hat{r}_{h'_j}$ be the reflection along h'_j. Define

$$u \xrightarrow{J_<} w \quad \overset{\text{def.}}{\Longleftrightarrow} \quad u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} \cdots r_{h_{j_t}} = w$$

and

$$\tilde{r}_{J_>} := \tilde{r}_{h_{j_t}} \cdots \tilde{r}_{h_{j_1}}. \tag{19}$$

Theorem 3.10.
$$c^{w,\lambda}_{u,\mu} = \sum (-1)^{n(J)} (q-1)^{|J|} q^{\frac{1}{2}(\ell(u)-\ell(w)-|J|)}, \tag{20}$$
where the sum is over subsets $J \subset \{1, 2, \ldots, l\}$ such that $u \xrightarrow{J_<} w$ and $w\tilde{r}_{J_>}(\lambda) = \mu$, and where $n(J) := \#\{j \in J \mid \beta_j < 0\}$.

Proof. Applying Theorem 3.9 to the $(-\lambda)$-chain $(-\beta_l, -\beta_{l-1}, \ldots, -\beta_1)$, we get

$$c^{w,\lambda}_{u,\mu} = \sum (-1)^{\#\{j_i \mid -\beta_{l+1-j_i} < 0\}} (1-q)^t q^{\frac{1}{2}(\ell(u)-\ell(w)-t)},$$

where the summation is over subsets $J' := \{1 \le j_1 < j_2 < \cdots < j_t \le l\}$ such that

$$u < ur_{h'_{j_t}} < \cdots < ur_{h'_{j_t}} \cdots r_{h'_{j_1}} = w \tag{21}$$

and

$$w\hat{r}_{h'_{j_1}} \cdots \hat{r}_{h'_{j_t}}(\lambda) = \mu. \tag{22}$$

Let

$$J := \{1 \le l+1-j_t < l+1-j_{t-1} < \cdots < l+1-j_1 \le l\}.$$

Then

$$(-1)^{\#\{j_i \mid -\beta_{l+1-j_i} < 0\}} (1-q)^t = (-1)^{n(J)} (q-1)^t,$$

where $n(J)$ is defined in Theorem 3.9. Since $r_{h'_j} = r_{h_{l+1-j}}$, condition (21) is equivalent to $u \xrightarrow{J_<} w$. On the other hand, condition (22) is equivalent to $w\tilde{r}_{J_>}(\lambda) = \mu$ as $\tilde{r}_{h_{l+1-j}} = \hat{r}_{h'_j}$. This finishes the proof. □

For further use, we also record the following technical result, which will allow us to rewrite the elements $w\hat{r}_{J_<}(-\lambda)$ from Theorem 3.9 and $w\tilde{r}_{J_<}$ from Theorem 3.10.

Proposition 3.11. *Consider a λ-chain $\beta_1, \ldots, \beta_l$. For any subsequence $J = \{j_1 < j_2 < \cdots < j_t\} \subset \{1, 2, \ldots, l\}$, we have*

$$-\hat{r}_{J_<}(-\lambda) = r_J\, \tilde{r}_{J_>}(\lambda), \quad \text{where } r_J = r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}.$$

Proof. Let $m_j = \langle \lambda, \beta_j^\vee \rangle$ $(1 \le j \le l)$. By induction on $|J|$, it is easy to show that

$$\begin{aligned}
\hat{r}_{h_{j_1}} \hat{r}_{h_{j_2}} \cdots \hat{r}_{h_{j_t}}(-\lambda) &= -\lambda + (m_{j_1} - d_{j_1})\beta_{j_1} + (m_{j_2} - d_{j_2}) r_{h_{j_1}} \beta_{j_2} + \cdots + (m_{j_t} - d_{j_t}) r_{j_1} r_{h_{j_2}} \cdots r_{h_{j_{t-1}}} \beta_{j_t},\\
\tilde{r}_{h_{j_t}} \tilde{r}_{h_{j_{t-1}}} \cdots \tilde{r}_{h_{j_1}}(\lambda) &= \lambda - d_{j_t}\beta_{j_t} - d_{j_{t-1}} r_{h_{j_t}} \beta_{j_{t-1}} - \cdots - d_{j_1} r_{h_{j_t}} r_{h_{j_{t-1}}} \cdots r_{h_{j_2}} \beta_{j_1},\\
r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}(\lambda) &= \lambda - m_{j_1}\beta_{j_1} - m_{j_2} r_{h_{j_1}} \beta_{j_2} - \cdots - m_{j_t} r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_{t-1}}} \beta_{j_t}.
\end{aligned}$$

As $r_J = r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}$ is a linear transformation, we get

$$r_J \tilde{r}_{J_>}(\lambda) = r_J(\lambda) + d_{j_t} r_{h_{j_1}} \cdots r_{h_{j_{t-1}}} \beta_{j_t} + d_{j_{t-1}} r_{h_{j_1}} \cdots r_{h_{j_{t-2}}} \beta_{j_{t-1}} + \cdots + d_{j_1}\beta_{j_1} = -\hat{r}_{J_<}(-\lambda). \qquad \square$$

Remark 3.12. With the notation as above, the condition that $u \xrightarrow{J_>} w$ in Theorem 3.9 implies that $ur_J^{-1} = w$, thus by Proposition 3.11,

$$w\hat{r}_{J_<}(-\lambda) = -u\tilde{r}_{J_>}(\lambda).$$

Similarly, in Theorem 3.10, we have that $w = ur_J$, thus

$$w\tilde{r}_{J_>}(\lambda) = -u\hat{r}_{J_<}(-\lambda).$$

This leads to alternative formulae in Theorems 3.9 and 3.10.

Appendix B includes a fully worked out example illustrating calculations of some coefficients $c_{u,\mu}^{w,\lambda}$ utilizing Theorem 3.9 and Theorem 3.10.

4. Motivic Chern classes of Schubert cells

We recall basic properties of the motivic Chern classes of the Schubert cells in the (partial) flag varieties.

4.1. ***Definition of the Motivic Chern classes.*** Let X be a quasiprojective complex algebraic variety, and let $G_0(var/X)$ be the (relative) Grothendieck group of varieties over X. It consists of isomorphism classes of morphisms $[f : Z \to X]$ modulo the scissor relations. Brasselet, Schürmann and Yokura [BSY10] defined the *motivic Chern transformation* $MC_y : G_0(var/X) \to K(X)[y]$ with values in the K-theory group of coherent sheaves in X to which one adjoins a formal variable y. The transformation MC_y is a group homomorphism, it is functorial with respect to proper push-forwards, and if X is smooth it satisfies the normalization condition

$$MC_y[\mathrm{id}_X : X \to X] = \sum [\textstyle\bigwedge^j T_X^*] y^j.$$

Here $\left[\bigwedge^j T_X^*\right]$ is the K-theory class of the bundle of degree j differential forms on X. As explained in [BSY10], the motivic Chern class is related by a Hirzebruch–Riemann–Roch type statement to the Chern–Schwartz–MacPherson (CSM) class in the homology of X; see Section A.2.

There is also an equivariant version of the motivic Chern class transformation, which uses equivariant varieties and morphisms, and has values in the suitable equivariant K-theory group. Its definition was given in [AMSS24a; FRW21], following closely the approach of [BSY10].

Assume X is smooth and there is a torus T acting on X. Let $K_T(X)$ denote the equivariant K-theory of X, see [CG10]. If X is a point, $K_T(pt) = K^0(\mathrm{Rep}(T)) = \mathbb{Z}[T]$. For any $\mathcal{F} \in K_T(X)$, let

$$\chi_T(X, \mathcal{F}) := \sum_i (-1)^i H^i(X, \mathcal{F}) \in K_T(pt).$$

Let $\langle -, - \rangle$ be the nondegenerate pairing on $K_T(X)$ defined by

$$\langle \mathcal{F}, \mathcal{G} \rangle = \chi_T(X, \mathcal{F} \otimes \mathcal{G}) \in K_T(pt),$$

where $\mathcal{F}, \mathcal{G} \in K_T(X)$. For a vector bundle E, the λ_y-class of E is the class

$$\lambda_y(E) := \sum_k [\textstyle\bigwedge^k E] y^k \in K_T(X)[y].$$

The λ_y-class is multiplicative; i.e., for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles there is an equality $\lambda_y(E) = \lambda_y(E_1)\lambda_y(E_2)$ as elements in $K_T(X)[y]$.

Recall the (relative) motivic Grothendieck group $G_0^T(var/X)$ of varieties over X is the free abelian group generated by the isomorphism classes $[f : Z \to X]$ where Z is a quasiprojective T-variety and $f : Z \to X$ is a T-equivariant morphism modulo the usual additivity relations

$$[f : Z \to X] = [f : U \to X] + [f : Z \setminus U \to X]$$

for $U \subset Z$ an open T-invariant subvariety. For any equivariant morphism $f : X \to Y$ of quasiprojective T-varieties there are well defined push-forwards $f_! : G_0^T(var/X) \to G_0^T(var/Y)$ given by composition. The equivariant motivic Chern class is defined by the following theorem.

Theorem 4.1 [AMSS24a; FRW21]. *Let X be a quasiprojective, nonsingular, complex algebraic variety with an action of the torus T. There exists a unique natural transformation $MC_y : G_0^T(var/X) \to K_T(X)[y]$ satisfying the following properties*:

(1) *It is functorial with respect to T-equivariant proper morphisms of nonsingular, quasiprojective varieties.*

(2) *It satisfies the normalization condition*

$$MC_y[\mathrm{id}_X : X \to X] = \lambda_y(T_X^*) = \sum y^i [\textstyle\bigwedge^i T_X^*]_T \in K_T(X)[y].$$

The transformation MC_y satisfies a Verdier–Riemann–Roch (VRR) formula: for any smooth, T-equivariant morphism $\pi : X \to Y$ of quasiprojective and nonsingular algebraic varieties, and any $[f : Z \to Y] \in G_0^T(var/Y)$, the following holds:

$$\lambda_y(T_\pi^*) \cap \pi^* MC_y[f : Z \to Y] = MC_y[\pi^* f : Z \times_Y X \to X].$$

Define the Grothendieck–Serre dual operator $\mathcal{D} : K_T(X) \to K_T(X)$ by setting, for any $[\mathcal{F}] \in K_T(X)$,

$$\mathcal{D}[\mathcal{F}] := [RHom(\mathcal{F}, \omega_X^\bullet)] := [\omega_X^\bullet \otimes \mathcal{F}^\vee] \in K_T(X), \tag{23}$$

where $\omega_X^\bullet \simeq \omega_X[\dim X]$ is the (equivariant) dualizing complex of X (the canonical bundle ω_X shifted by dimension). The class $[\mathcal{F}^\vee]$ is obtained by taking an equivariant resolution of $\mathcal{F}$ by vector bundles, and then taking duals. Extend the operators $\mathcal{D}$ and $(-)^\vee$ to $K_T(X)[y^{\pm 1}]$ by sending $y \mapsto y^{-1}$. We also let $(-)^\vee$ denote the operator on $K_T(pt)[y^{\pm 1}]$, which sends e^λ to $e^{-\lambda}$ and y to y^{-1}.

Definition 4.2. Assume $\Omega \hookrightarrow X$ is a T-stable subvariety.

(1) The *motivic Chern* (MC) class of Ω is

$$MC_y(\Omega) := MC_y[\Omega \hookrightarrow X] \in K_T(X)[y].$$

(2) If Ω is pure-dimensional, the *Segre motivic Chern class* $SMC_y(\Omega) \in K_T(X)[\![y]\!]$ is the class

$$SMC_y(\Omega) := (-y)^{\dim \Omega} \frac{\mathcal{D}(MC_y(\Omega))}{\lambda_y(T_X^*)} \in K_T(X)[\![y]\!].$$

4.2. *The complete flag variety case.* Both $\{MC_y(X(w)^\circ) \mid w \in W\}$ and $\{SMC_y(Y(w)^\circ) \mid w \in W\}$ are bases for the localized equivariant K-theory

$$K_T(G/B)[\![y]\!]_{\mathrm{loc}} := K_T(G/B)[\![y]\!] \otimes_{K_T(pt)} \mathrm{Frac}\, K_T(pt).$$

These classes can be calculated recursively using the Demazure–Lusztig operators as follows.

The left multiplication action of G on G/B induces a left Weyl group action on $K_T(G/B)$. For any $w \in W$, let w^L denote this action.

Definition 4.3 [MNS22b, Section 5.3]. For any simple reflection s_i, we define the following left Demazure–Lusztig operator on $K_T(G/B)_{\mathrm{loc}}$:

$$\mathcal{T}_i^L := \frac{1+ye^{-\alpha_i}}{1-e^{-\alpha_i}} s_i^L - \frac{1+y}{1-e^{-\alpha_i}}. \tag{24}$$

Lemma 4.4 [MNS22b]. *The operators $\mathcal{T}_i^L$ satisfy the braid relation and the quadratic relation*

$$(\mathcal{T}_i^L+1)(\mathcal{T}_i^L+y)=0.$$

Moreover, they commute with the operators of tensoring by elements in $K_G(G/B)$.

The following lemma is easy to verify.

Lemma 4.5. *The map Ψ defined by*

$$\Psi(T_i) = \mathcal{T}_i^L \quad \textit{and} \quad \Psi(X^\lambda) = e^\lambda,$$

where $e^\lambda \in K_T(\mathrm{pt})$ acts on $K_T(G/B)_{\mathrm{loc}}$ by multiplication, induces an action of the affine Hecke algebra $\mathbb{H}$ (with $q=-y$) on $K_T(G/B)_{\mathrm{loc}}$.

Theorem 4.6. (1) [MNS22b, Theorem 7.6] *For $w \in W$ and a simple root α_i,*

$$\mathcal{T}_i^L(MC_y(X(w)^\circ)) = \begin{cases} MC_y(X(s_iw)^\circ) & \text{if } s_iw > w, \\ -(y+1)MC_y(X(w)^\circ) - yMC_y(X(s_iw)^\circ) & \text{if } s_iw < w. \end{cases}$$

In particular,

$$MC_y(X(w)^\circ) = \mathcal{T}_w^L([\mathcal{O}_{X(\mathrm{id})}]).$$

(2) [MNS22b, Theorem 7.1] *For any $w, u \in W$,*

$$\langle MC_y(X(w)^\circ), SMC_y(Y(u)^\circ)\rangle = \delta_{u,w}.$$

Remark 4.7. (1) By [AMSS24b, Theorem 5.1 and Corollary 5.3],

$$MC_0(X(w)^\circ) = [\mathcal{O}_{X(w)}(-\partial X(w))] \quad \text{and} \quad SMC_0(Y(w)^\circ) = [\mathcal{O}_{Y(w)}],$$

where $\partial X(w) := X(w) \setminus X(w)^\circ$. Thus, the duality in the second part of the theorem reduces to the classical fact

$$\langle [\mathcal{O}_{X(w)}(-\partial X(w))], [\mathcal{O}_{Y(u)}] \rangle = \delta_{u,w}.$$

(2) By definition, for any $w \in W$,

$$MC_y(Y(w)) = \sum_{u \geq w} MC_y(Y(u)^\circ).$$

Besides, by the linearity of the Grothendieck–Serre dual operator $\mathcal{D}$,

$$SMC_y(Y(w)) = \sum_{u \geq w} (-y)^{\ell(u)-\ell(w)} SMC_y(Y(u)^\circ).$$

Therefore,

$$SMC_0(Y(w)) = [\mathcal{O}_{Y(w)}].$$

4.3. *The partial flag variety case.* For any $w \in W$, let $\ell(wW_P)$ denote the length of the minimal length representative in wW_P. Let $\pi : G/B \to G/P$ be the natural projection. It is proved in [AMSS24a, Remark 5.7] that

$$\pi_* MC_y(X(w)^\circ) = (-y)^{\ell(w)-\ell(wW_P)} MC_y(X(wW_P)^\circ). \tag{25}$$

In particular, if w is a minimal length representative, then $\pi_* MC_y(X(w)^\circ) = MC_y(X(wW_P)^\circ)$; this also follows directly from the functoriality property of the motivic classes.

Proposition 4.8. (1) [MNS22b, Proposition 6.3] *Let $\Omega \subset G/P$ be a T-stable subvariety of G/P and $\pi : G/B \to G/P$ be the projection. Then*

$$\pi^* SMC_y(\Omega) = SMC_y(\pi^{-1}\Omega) \in K_T(G/B)[\![y]\!].$$

(2) [MNS22b, Proposition 7.2] *Let $u, w \in W^P$. The Segre motivic classes are dual to the motivic Chern classes for any G/P; i.e.,*

$$\langle MC_y(X(wW_P)^\circ), SMC_y(Y(uW_P)^\circ) \rangle = \delta_{u,w}.$$

Remark 4.9. By the first property in the proposition, for any $w \in W^P$,

$$\pi^*(SMC_y(Y(wW_P))) = SMC_y(Y(w)).$$

Letting $y = 0$, we get

$$\pi^*(SMC_0(Y(wW_P))) = SMC_0(Y(w)) = [\mathcal{O}_{Y(w)}],$$

where the second equality follows from Remark 4.7. By the definition of the SMC class,

$$SMC_y(Y(wW_P)) = \sum_{u \in W^P, u \geq w} (-y)^{\ell(u)-\ell(w)} SMC_y(Y(uW_P)^\circ). \tag{26}$$

Hence,

$$\pi^*(SMC_0(Y(wW_P)^\circ)) = \pi^*(SMC_0(Y(wW_P))) = [\mathcal{O}_{Y(w)}].$$

Since $\pi^*([\mathcal{O}_{Y(wW_P)}]) = [\mathcal{O}_{Y(w)}]$ and π^* is injective, we get

$$SMC_0(Y(wW_P)^\circ) = SMC_0(Y(wW_P)) = [\mathcal{O}_{Y(wW_P)}].$$

Combining with the second part of the proposition, we get

$$MC_0(X(wW_P)^\circ) = [\mathcal{O}_{X(wW_P)}(-\partial X(wW_P))].$$

This can also be proved using Remark 4.7(1), equation (25), and the fact that the pushforward of an ideal sheaf is an ideal sheaf, by [Bri02].

5. Chevalley formulae for the motivic Chern classes

In this section we obtain several Chevalley formulae for the motivic Chern classes, in terms of alcove walks, λ-chains, and certain operators. The main technique is to reinterpret formulae from Hecke algebras such as Theorem 3.9 in terms of multiplications of motivic Chern classes by line bundles. Our main results are Theorems 5.5 and 5.12. In Section 5.3 we discuss several positivity properties and conjectures of the Chevalley coefficients. Finally, in Section 5.5 we discuss parabolic Chevalley formulae.

5.1. ***Chevalley coefficients.*** Consider a torus weight $\lambda \in X^*(T)$ and $u, w \in W$. The Chevalley coefficient $C^w_{u,\lambda}$ is defined by the following formula:

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = \sum_{u \le w} C^w_{u,\lambda} MC_y(X(u)^\circ). \tag{27}$$

Note that for any simple reflection s_i there is a short exact sequence of equivariant sheaves

$$0 \to \mathcal{L}_{\varpi_i} \otimes \mathbb{C}_{-w_0(\varpi_i)} \to \mathcal{O}_{G/B} \to \mathcal{O}_{X(w_0 s_i)} \to 0$$

with ϖ_i the fundamental weight, see [BM15, §8], for example. Therefore,

$$[\mathcal{O}_{X(w_0 s_i)}] = 1 - e^{-w_0(\omega_i)} \mathcal{L}_{\varpi_i} \in K_T(G/B),$$

and the coefficients from (27) for $\lambda = \varpi_i$ also recover the multiplication of $[\mathcal{O}_{X(w_0 s_i)}]$ with the MC classes of the Schubert cells.

The coefficients $C^w_{u,\lambda}$ also arise from Chevalley formulae involving Segre motivic classes:

Lemma 5.1.
$$\mathcal{L}_\lambda \otimes SMC_y(Y(u)^\circ) = \sum_{w \ge u} C^w_{u,\lambda} SMC_y(Y(w)^\circ). \tag{28}$$

Proof. This follows from the (Poincaré) duality in Theorem 4.6(2), as the Chevalley coefficients in (27) and (28) are given by

$$C^w_{u,\lambda} = \langle \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ), SMC_y(Y(u)^\circ)\rangle. \qquad \square$$

We now relate the Chevalley coefficients above to the coefficients $c^{w,\lambda}_{u,\mu}$ from (7) in the Hecke algebra.

Theorem 5.2. *Let λ be any weight in $X^*(T)$. The following Chevalley formula holds in $K_T(G/B)[y]$:*

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ) = \sum_{\mu \in X^*(T), u \in W} y^{\ell(w)-\ell(u)} e^{-\mu} c^{w,\lambda}_{u,\mu}|_{q=-y} MC_y(X(u)^\circ).$$

In particular, the following equation holds for the Chevalley coefficients in (27):

$$C^w_{u,-\lambda} = \sum_{\mu \in X^*(T)} y^{\ell(w)-\ell(u)} e^{-\mu} c^{w,\lambda}_{u,\mu}|_{q=-y}.$$

From (27), the Chevalley coefficients $C^v_{u,\lambda}$ belong to a localization $K_T(pt)[y]$ which allows division by $1+e^\alpha y$ for any root α. However, the expansion from (7) implies that the coefficients $(-q)^{\ell(w)-\ell(v)} c^{w,\lambda}_{u,\mu}$ are *polynomials* in $\mathbb{Z}[q]$. Then it follows that $C^v_{u,\lambda}$ are in fact polynomials in $K_T(pt)[y]$. This will be seen explicitly in Theorem 5.4.

Proof of Theorem 5.2. Applying the map Ψ in Lemma 4.5 to (7), we get

$$\mathcal{T}^L_w e^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} y^{\ell(w)-\ell(u)} e^{-\mu} c^{w,\lambda}_{u,\mu}|_{q=-y} \mathcal{T}^L_u \in \operatorname{End}_{\mathbb{C}} K_T(G/B)[y].$$

The theorem follows by applying both sides to $[\mathcal{O}_{X(\mathrm{id})}]$, and utilizing that

$$\mathcal{T}^L_w e^{-\lambda}([\mathcal{O}_{X(\mathrm{id})}]) = \mathcal{T}^L_w(\mathcal{L}_{-\lambda} \otimes [\mathcal{O}_{X(\mathrm{id})}]) = \mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ).$$

Here the first equality follows from $\mathcal{L}_{-\lambda} \otimes [\mathcal{O}_{X(\mathrm{id})}] = e^{-\lambda}[\mathcal{O}_{X(\mathrm{id})}]$, while the second one follows from Lemma 4.4 and Theorem 4.6. □

Remark 5.3. This argument can be generalized to the case when the line bundle $\mathcal{L}_\lambda$ is replaced by any homogeneous bundle $\mathcal{V} = G \times^B V \to G/B$ associated to a B-representation of V. If the character of V is $ch(V) = \sum_\lambda a_\lambda e^\lambda$, then a localization argument shows that the class of $\mathcal{V}$ in $K_T(G/B)$ is equal to

$$[\mathcal{V}] = \sum a_\lambda \mathcal{L}_\lambda.$$

It follows that, for any $w \in W$,

$$\mathcal{V} \otimes MC(X(w)^\circ) = \sum_\lambda a_\lambda \mathcal{L}_\lambda \otimes MC(X(w)^\circ) = \sum_u \sum_\lambda a_\lambda C^w_{u,\lambda} MC(X(u)^\circ).$$

We illustrate this for $G/B = \mathrm{Fl}(n)$, the complete flag manifold. This is equipped with the tautological sequence $\mathcal{F}_0 = 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathbb{C}^n$. For $1 \le i \le n-1$ define $X_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ regarded in $K_T(G/B)$. Then

$$\textstyle\bigwedge^j \mathcal{F}_i = e_j(X_1, \dots, X_i), \quad Sym^j \mathcal{F}_i = h_j(X_1, \dots, X_i)$$

where e_j and h_j denote the elementary symmetric function, respectively the complete homogeneous symmetric function. Note that if ϖ_i denotes the ith fundamental weight, then $X_i = \mathcal{L}_{\varpi_i - \varpi_{i-1}}$ for $1 \le i \le n-1$, with the convention that $\varpi_0 = 0$. Theorem 5.2 gives a formula for the multiplication by monomials $X_1^{a_1} \cdot \cdots \cdot X_{n-1}^{a_{n-1}}$, which in turn gives formulae to multiply by $e_j(X_1, \dots, X_i)$ and $h_j(X_1, \dots, X_i)$.

In the next section, we give explicit formulae for the Chevalley coefficients $C^u_{\lambda,w}$ based on the formulae for the Hecke algebra coefficients $c^{w,\lambda}_{u,\mu}$.

5.2. ***Chevalley formulae via alcove walks and the λ-chains.*** Let us recall the setting of Corollary 2.9. For $\lambda \in X^*(T)$, choose a minimal length alcove walk $p_{v_\lambda} = c^{\epsilon_1}_{j_1} c^{\epsilon_2}_{j_2} \cdots c^{\epsilon_s}_{j_s}$ from $A_\circ$ to $A_\circ + \lambda$, and let $\mathcal{H} = \{h_1, h_2, \dots, h_s\}$ be the ordered sequence of hyperplanes defined by the walls of alcoves crossed by p_λ. Define $\beta_i \in R^{\epsilon_i}$ and $k_i \in \mathbb{Z}$ $(1 \le i \le s)$ by the condition $h_i = H_{\beta_i, k_i}$. Combining Corollary 2.9 and Theorem 5.2, we get the following formula.

Theorem 5.4 (alcove walk formula for the Chevalley coefficients).

$$C^w_{u,-\lambda} = \sum_{\mathcal{M}} (-1)^{f^+(\mathcal{M})} (-1-y)^{|\mathcal{M}|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|\mathcal{M}|)} e^{-w\hat{r}_{\mathcal{M}}(\lambda)}$$

where the sum is over ordered subsets $\mathcal{M} \subset \mathcal{H}$ *such that* $w \overset{\mathcal{M}}{\Longrightarrow} u = wr_{\mathcal{M}}$.

We now recall the setup of Theorem 3.9. Assume λ is an integral weight and fix a reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$, which corresponds to an alcove walk from $A_\circ$ to $A_\circ - \lambda$, with separating hyperplanes $h_j := H_{-\beta_j, d_j}$.

Combining Theorems 3.9, 3.10, and 5.2, we get the following formulae.

Theorem 5.5 (λ-chain formula for the Chevalley coefficients).

$$C^w_{u,-\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (1+y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\tilde{r}_{J_>}(\lambda)}, \tag{29}$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ *such that* $u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} r_{h_{j_2}} \cdot \cdots \cdot r_{h_{j_t}} = w$ *and* $\tilde{r}_{J_>}$ *is defined in* (19). *Furthermore,*

$$C^w_{u,\lambda} = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (-1-y)^{|J|} (-y)^{\frac{1}{2}(\ell(w)-\ell(u)-|J|)} e^{-w\hat{r}_{J_<}(-\lambda)}, \tag{30}$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ *such that* $u < ur_{h_{j_t}} < ur_{h_{j_t}} r_{h_{j_{t-1}}} < \cdots < ur_{h_{j_t}} \cdot \cdots \cdot r_{h_{j_1}} = w$ *and* $\hat{r}_{J_<}$ *is defined in* (16).

Remark 5.6. (1) It follows from Remark 3.12 that $-w\tilde{r}_{J_>}(\lambda) = u\hat{r}_{J_<}(-\lambda)$ in (29), and that $-w\hat{r}_{J_<}(-\lambda) = u\tilde{r}_{J_>}(\lambda)$ in (30), giving alternative ways to calculate these.

(2) By Remark 4.7(1), the MC and SMC classes specialize at $y = 0$ to the ideal sheaves and the structure sheaves, respectively. Under this specialization, and using that $-w\tilde{r}_{J_>}(\lambda) = u\hat{r}_{J_<}(-\lambda)$, equation (29) reduces to the equivariant K-theory Chevalley formula of Lenart–Postnikov [LP07, Theorem 6.1].

One can also consider the more general situation of Kac–Moody flag varieties defined in, say, [Kum02]. These are ind-varieties, and one can define the motivic Chern classes of the finite-dimensional Schubert cells using the ind-structure. There are analogues of the (left and right) Demazure operators, and the notion of λ-chains extends to this setting, by results of Lenart and Postnikov [LP08]. Note that for an infinite Weyl group W, a λ-chain may be an infinite sequence, i.e., $l = \infty$. But, for given $u \le w$, $t = |J|$

is finite, and the number of J which satisfies the condition in Theorem 5.5 is also finite. Based on these similarities to the finite case, we expect the following conjecture to hold.

Conjecture 1. *For a Kac–Moody Weyl group W and a dominant integral weight λ, the analogues of the equations* (29) *and* (30) *hold.*

5.3. *Miscellaneous.* In this section we discuss positivity properties of the coefficients from the Chevalley formula, and some special properties of the multiplication by line bundles given by minuscule weights. We start with the following consequence of Theorem 5.5.

Proposition 5.7. *Let λ be a dominant weight and set $q = -y$. Then*:

(1) *$C^w_{u,\lambda}$ may be written as a combination of terms of the form $e^\mu q^a (q-1)^b$ with nonnegative integer coefficients.*

(2) *$C^w_{u,-\lambda}$ may be written as a combination of terms of the form $(-1)^b e^\mu q^a (q-1)^b$ with nonnegative integer coefficients.*

In both situations, b has the same parity as $\ell(w) - \ell(u)$.

Proof. Both statements follow from Theorem 5.5, using that for a reduced λ-chain with λ dominant we have $n(J) = 0$, and that $\ell(w) - \ell(u) - |J|$ is an even integer. □

Example 5.8. Consider type A_2, with $\lambda = 2\varpi_1 + \varpi_2$, and with the λ-chain from Appendix B. Take $w = s_2 s_1$. Then from Theorem 5.5 we get, with $q = -y$,

$$\begin{aligned}
\mathcal{L}_\lambda \otimes MC_{-q}(X(s_2s_1)^\circ) = {}& e^{\varpi_1 - 3\varpi_2} MC_{-q}(X(s_2s_1)^\circ) \\
&+ (q-1)(e^{-\varpi_2} + e^{-\varpi_1+\varpi_2} + e^{-2\varpi_1+3\varpi_2}) MC_{-q}(X(s_1)^\circ) \\
&+ (q-1)(e^{2\varpi_1-2\varpi_2} + e^{3\varpi_1-\varpi_2}) MC_{-q}(X(s_2)^\circ) \\
&+ (q-1)^2(e^{2\varpi_1+\varpi_2} + e^{2\varpi_2} + e^{\varpi_1}) MC_{-q}(X(\mathrm{id})^\circ),
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{-\lambda} \otimes MC_{-q}(X(s_2s_1)^\circ) = {}& e^{-\varpi_1 + 3\varpi_2} MC_{-q}(X(s_2s_1)^\circ) \\
&- (q-1)(e^{\varpi_1-\varpi_2} + e^{\varpi_2} + e^{-\varpi_1+3\varpi_2}) MC_{-q}(X(s_1)^\circ) \\
&- (q-1)(e^{-2\varpi_1+2\varpi_2} + e^{-\varpi_1+3\varpi_2}) MC_{-q}(X(s_2)^\circ) \\
&+ (q-1)^2(e^{-\varpi_1} + e^{\varpi_1-\varpi_2} + e^{-2\varpi_1+2\varpi_2} + e^{\varpi_2} + e^{-\varpi_1+3\varpi_2}) MC_{-q}(X(\mathrm{id})^\circ).
\end{aligned}$$

We now investigate the special multiplication by $\mathcal{L}_{\pm\varpi_i}$ in the case of minuscule fundamental weights. In this case the coefficients have a particularly pleasing factorization.

Lemma 5.9. *If $\lambda = \varpi_i$ is a minuscule weight, then for any $u, w \in W$,*

$$C^w_{u,\lambda} = e^{u(\lambda)} P^w_{u,\lambda}(y) \quad \textit{and} \quad C^w_{u,-\lambda} = (-1)^{\ell(w)-\ell(u)} e^{-w(\lambda)} P^{w_0 u}_{w_0 w,\lambda}(y),$$

where $P^w_{u,\lambda}(y) \in \mathbb{Z}[y]$. These polynomials are palindromic, that is, they satisfy

$$P^w_{u,\lambda}(y^{-1}) \cdot y^{\ell(w)-\ell(u)} = P^w_{u,\lambda}(y).$$

Proof. Since $\lambda = \varpi_i$ is minuscule, for any reduced λ-chain $(\beta_1, \beta_2, \dots, \beta_l)$, the separating hyperplanes must be of the form $h_j := H_{-\beta_j,0}$, thus $\hat{r}_{h_j} = r_{h_{j_1}}$. Therefore the first equality follows from (30), since $-w\hat{r}_{J_<}(-\lambda) = -wr_{J_<}(-\lambda) = u(\lambda)$. The second equality follows this and from the star duality in Proposition 6.2 below. The palindromic property follows from Proposition 6.5(a) below. □

Remark 5.10. In an earlier arXiv version of this article, we conjectured that if $\lambda = \varpi_i$ is a minuscule weight, then the coefficients $C^w_{u,\lambda}(y)$ in the multiplication

$$MC(X(w)^\circ) \cdot \mathcal{L}_\lambda = \sum C^w_{u,\lambda}(y) MC(X(u)^\circ)$$

are polynomials with nonnegative coefficients. Since then, we have found counterexamples to this conjecture in Lie types D_6, E_6 and A_7.

The next example shows that even in the cases when a coefficient $C^w_{u,\lambda}(y)$ happens to have positive coefficients, cancellations may still occur in the formula (30), which calculates it.

Example 5.11. Consider type A_3, $\lambda = \varpi_2$, $w = s_1s_2s_3s_1s_2s_1$ and $u = s_3s_1$. An ϖ_2-chain is $\beta_1 = \alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$, $\beta_3 = \alpha_1 + \alpha_2$, $\beta_4 = \alpha_1 + \alpha_2 + \alpha_3$; see Example 3.6. We have two paths from u to w in (30):

- $J_1 = \{2, 3\}$ which gives $u < us_{\beta_3} < us_{\beta_3}s_{\beta_2} = w$, where $\ell(us_{\beta_3}) + 3 = \ell(w)$;
- $J_2 = \{1, 2, 3, 4\}$ which gives $u < us_{\beta_4} < us_{\beta_4}s_{\beta_3} < us_{\beta_4}s_{\beta_3}s_{\beta_2} < us_{\beta_4}s_{\beta_3}s_{\beta_2}s_{\beta_1} = w$.

The path J_1 gives coefficient $(-1-y) \times (-1-y)(-y)e^{u(\lambda)} = -y(y+1)^2 e^{u(\lambda)}$, and the path J_2 gives coefficient $(-1-y)^4 e^{u(\lambda)} = (y+1)^4 e^{u(\lambda)}$. Therefore

$$C^w_{u,\lambda} = e^{u(\lambda)}((y+1)^4 - y(y+1)^2) = e^{u(\lambda)}(y^2+y+1)(y+1)^2.$$

5.4. ***An operator formula.*** In this section we reformulate the λ-chain Chevalley formula from Theorem 5.5 via operators generalizing to motivic Chern classes similar ones from [LP07].

Let $h := (\rho, \theta^\vee) + 1$ be the Coxeter number, where $\rho := \sum_{i=1}^r \varpi_i$ and $\theta^\vee$ is the highest coroot. Let $\tilde{R}(T) := \mathbb{Z}[e^{\pm\varpi_1/h}, \dots, e^{\pm\varpi_r/h}]$, and let

$$\tilde{K}_T(G/B) := K_T(G/B)[y] \otimes_{K_T(\mathrm{pt})[y]} \mathrm{Frac}(\tilde{R}(T)[y]).$$

Then $\tilde{K}_T(G/B)$ has a basis over $\mathrm{Frac}(\tilde{R}(T)[y])$ given by the motivic Chern classes of the Schubert cells. Define $\mathrm{Frac}(\tilde{R}(T)[y])$-linear operators B_β ($\beta \in R^+$) and E^μ ($\mu \in X^*(T)$) on $\tilde{K}_T(G/B)$ by

$$B_\beta(MC_y(X(w)^\circ)) := \begin{cases} (-1-y)(-y)^{\frac{1}{2}(\ell(w)-\ell(ws_\beta)-1)} MC_y(X(ws_\beta)^\circ) & \text{if } ws_\beta < w, \\ 0 & \text{otherwise,} \end{cases}$$
$$E^\mu(MC_y(X(w)^\circ)) := e^{w(\mu)/h} MC_y(X(w)^\circ).$$

If $\beta \in R^-$, define $B_\beta := -B_{-\beta}$. Then

$$E^\mu E^{\mu'} = E^{\mu+\mu'} \quad \text{and} \quad B_\beta E^{s_\beta\mu} = E^\mu B_\beta.$$

Given a λ-chain $(\beta_1, \dots, \beta_l)$, define

$$R^{[\lambda]} := R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_1}, \quad \text{where } R_\beta := E^\rho(E^\beta + B_\beta)E^{-\rho} = E^\beta + E^{(\rho,\beta^\vee)\beta} B_\beta.$$

Theorem 5.12 (operator Chevalley formula). *For any integral weight $\lambda \in X^*(T)$,*

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = R^{[\lambda]}(MC_y(X(w)^\circ)). \tag{31}$$

Remark 5.13. (1) The theorem implies that the definition of $R^{[\lambda]}$ does not depend on the choice of the λ-chain, which is equivalent to the Yang–Baxter equations [LP07, Definition 9.1] satisfied by the operators R_β.

(2) The formula analogous to (31) involving SMC is obtained by replacing the operator $R^{|\lambda|}$ by an operator defined via the adjoint operators of B_β and E^μ.

(3) Recall from Remark 4.7(1) that $MC_0(X(w)^\circ) = [\mathcal{O}_{X(w)}(-\partial X(w))]$. Specializing $y = 0$ in the theorem, we get a dual version of Lenart–Postnikov's formula [LP07, Theorem 13.1].

For the proof of the theorem, we need a result from [LP07]. Recall that $(\beta_1, \dots, \beta_l)$ is a λ-chain. The hyperplane $h_j := H_{-\beta_j, d_j}$ separates the alcoves A_{j-1} and A_j in the corresponding alcove path, and $\hat{r}_{h_j}$ is the reflection along h_j.

Lemma 5.14 [LP07, proof of Proposition 14.5]. *For any $1 \le j_1 < j_2 < \cdots < j_t \le l$,*

$$-\rho+\beta_1+\cdots+\beta_{j_1-1}+s_{\beta_{j_1}}(\beta_{j_1+1}+\cdots\beta_{j_2-1})+\cdots+s_{\beta_{j_1}}\cdots s_{\beta_{j_t}}(\beta_{j_t+1}+\cdots\beta_l+\rho)=-h\hat{r}_{j_1}\cdots\hat{r}_{j_t}(-\lambda).$$

Proof of Theorem 5.12. By definition,

$$R^{[\lambda]} = R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_1} = E^\rho(E^{\beta_l} + B_{\beta_l}) \cdots (E^{\beta_2} + B_{\beta_2})(E^{\beta_1} + B_{\beta_1})E^{-\rho} = \sum_J R_J^{[\lambda]}.$$

Here $J = \{j_1 < j_2 < \cdots < j_t\}$ is a subset of $\{1, 2, \dots, l\}$, and $R_J^{[\lambda]}$ is the term that contains B_{β_j} if $j \in J$, and E^{β_j}, otherwise. Moving all the B-operators to the left, and using the relation $B_\beta E^{s_\beta \mu} = E^\mu B_\beta$ and Lemma 5.14, we get $R_J^{[\lambda]} = B_{\beta_{j_t}} \cdots B_{\beta_{j_1}} E^{-h\hat{r}_{j_1}\cdots\hat{r}_{j_t}(-\lambda)} = B_{\beta_{j_t}} \cdots B_{\beta_{j_1}} E^{-h\hat{r}_{J_<}(-\lambda)}$. The theorem follows from this and Theorem 5.5. □

5.5. ***Parabolic case.*** We now extend the Chevalley formula to the partial flag variety case G/P.

Theorem 5.15 (Chevalley formula for the G/P case). *Let $w \in W^P$ and $\lambda \in X^*(T)_P$. Then we have*

$$\mathcal{L}_\lambda \otimes MC_y(X(wW_P)^\circ) = \sum_{u\in W^P} \Bigg(\sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C^w_{v,\lambda} \Bigg) MC_y(X(uW_P)^\circ),$$

and

$$\mathcal{L}_\lambda \otimes SMC_y(Y(uW_P)^\circ) = \sum_{w\in W^P} \Bigg(\sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C^w_{v,\lambda} \Bigg) SMC_y(Y(wW_P)^\circ),$$

where $C^v_{u,\lambda}$ are the Chevalley coefficients for full flag G/B given explicitly in Theorem 5.5.

Proof. The first equality follows from (25) and (27), while the second equality follows from the first one and the duality in Proposition 4.8(2). □

Remark 5.16. In a paper in preparation we use this theorem to give a combinatorial Chevalley formula for minuscule flag varieties and a K-theoretic generalization of Nakada's colored hook formula. See also [FGSX24] for a Pieri formula for the motivic Chern classes of Schubert cells in the equivariant K-theory of Grassmannians.

6. Dualities of Chevalley coefficients

Recall the expansion from (27):

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = \sum_{u \le w} C^w_{u,\lambda} MC_y(X(u)^\circ).$$

In this section we state and prove several duality properties of the Chevalley coefficients $C^w_{u,\lambda}$. All these dualities have a geometric origin (Serre duality, star duality, Dynkin automorphisms) and we name the Chevalley dualities correspondingly.

Recall the Grothendieck–Serre operator $\mathcal{D}$ from (23), and the duality functor $(-)^\vee$ on $K_T(G/B)[y^{\pm 1}]$ and $K_T(\mathrm{pt})[y^{\pm 1}]$, which sends a vector bundle to its dual and y^i to y^{-i}. This induces the following property of the Chevalley coefficients in (27).

Proposition 6.1 (Serre duality). $C^w_{u,\lambda} = (-y)^{\ell(w)-\ell(u)} w_0 (C^{w_0 u}_{w_0 w, -\lambda})^\vee.$

Proof. Since

$$SMC_y(Y(w)^\circ) = (-y)^{\dim Y(w)} \frac{\mathcal{D}(MC_y(Y(w)^\circ))}{\lambda_y(T^*(G/B))},$$

equation (28) can be written as

$$\mathcal{L}_\lambda \otimes \mathcal{D}(MC_y(Y(u)^\circ)) = \sum_{w \in W} C^w_{u,\lambda} (-y)^{\ell(u)-\ell(w)} \mathcal{D}(MC_y(Y(w)^\circ)).$$

Taking the dual $\mathcal{D}$ on both sides, we have

$$\mathcal{L}_{-\lambda} \otimes MC_y(Y(u)^\circ) = \sum_{w \in W} (C^w_{u,\lambda})^\vee (-y)^{\ell(w)-\ell(u)} MC_y(Y(w)^\circ).$$

Finally, applying the left w_0-action to this identity, we get

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w_0 u)^\circ) = \sum_{w \in W} w_0 (C^w_{u,\lambda})^\vee (-y)^{\ell(w)-\ell(u)} MC_y(X(w_0 w)^\circ).$$

This finishes the proof of the theorem by comparison with (27). □

The star duality $*$ acts on $K_T(G/B)[y^{\pm 1}]$ and $K_T(pt)[y^{\pm 1}]$ by sending a vector bundle to its dual, and leaving y^i unchanged. Consider the composition

$$\iota := w_0 * : K_T(\mathrm{pt})[y^{\pm 1}] \to K_T(\mathrm{pt})[y^{\pm 1}].$$

Combining Theorem 9.1(a) and Remark 4.7 of [AMSS24b], we have

$$\mathbb{C}_{-\rho} \otimes \mathcal{L}_{-\rho} \otimes MC_y(X(w)^\circ) = (-1)^{\dim G/B - \ell(w)} \prod_{\alpha>0} (1 + ye^{-\alpha}) * \big(SMC_y(X(w)^\circ)\big). \tag{32}$$

Proposition 6.2 (star duality). $C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} \iota(C^{w_0 u}_{w_0 w, -\lambda})$.

Proof. Apply the star duality functor $*$ to (27), then use (32) to get

$$\mathcal{L}_{-\lambda} \otimes SMC_y(X(w)^\circ) = \sum_u (-1)^{\ell(w)-\ell(u)} * (C^w_{u,\lambda}) SMC_y(X(u)^\circ).$$

Applying the left w_0-action to the above equation and comparing with (28), we get the result. □

The map sending $\alpha_i \mapsto -w_0(\alpha_i)$ for every simple root α_i induces an automorphism on the Dynkin diagram, hence also on the flag variety G/B. This automorphism maps $X(w)^\circ$ to $X(w_0 w w_0)^\circ$, and it induces a ring automorphism ϕ on $K_T(G/B)$, which sends $\mathcal{L}_\lambda$ to $\mathcal{L}_{-w_0\lambda}$, and twists the base ring $K_T(\text{pt})$ by the map ι above.

Proposition 6.3 (Dynkin duality). $C^w_{u,\lambda} = \iota(C^{w_0 w w_0}_{w_0 u w_0, -w_0\lambda})$.

Proof. Applying the Dynkin automorphism ϕ to (27), we obtain

$$\mathcal{L}_{-w_0\lambda} \otimes MC_y(X(w_0 w w_0)^\circ) = \sum_{u \in W} \iota(C^w_{u,\lambda}) MC_y(X(w_0 u w_0)^\circ).$$

The claim follows from the definition of the coefficients $C^w_{u,\lambda}$. □

Combining Propositions 6.1 and 6.2 in one instance, and Propositions 6.2 and 6.3 in a second instance, we obtain:

Proposition 6.4 (1) (Serre duality + star duality).

$$(C^w_{u,\lambda})|_{y \to y^{-1}} \times y^{\ell(w)-\ell(u)} = C^w_{u,\lambda}.$$

(2) (star duality + Dynkin duality).

$$C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} C^{u w_0}_{w w_0, w_0\lambda}.$$

Using the λ-chain formula in Theorem 5.5, we can also give a direct combinatorial proof for the various duality identities in this section. The proofs are similar to [LP07, Theorem 8.6] and [LP07, Theorem 8.7].

Proposition 6.5. *Let λ be any integral weight and let and $w, u \in W$. Then*:

(a) $C^w_{u,\lambda} \in K_T(pt)[y]$ *and* $(C^w_{u,\lambda})|_{y \to y^{-1}} \times y^{\ell(w)-\ell(u)} = C^w_{u,\lambda}$; *in other words*, $C^w_{u,\lambda}$ *is palindromic as a polynomial in* y. (Compare Proposition 6.4(1).)

(b) $C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} C^{u w_0}_{w w_0, w_0\lambda}$. (Compare Proposition 6.4(2).)

(c) $C^w_{u,\lambda} = (-1)^{\ell(w)-\ell(u)} \iota\left(C^{w_o u}_{w_o w, -\lambda}\right)$. (Compare Proposition 6.2.)

(d) $C^w_{u,\lambda} = \iota\left(C^{w_0 w w_0}_{w_0 u w_0, -w_0\lambda}\right)$. (Compare Proposition 6.3).

Proof. Part (a) is straightforward from (30); part (d) follows from (b) and (c). Parts (b) and (c) follow from known properties of λ-chain, i.e., if $(\beta_1, \dots, \beta_l)$ is a λ-chain, then $(w_0\beta_l, \dots, w_0\beta_1)$ is a $(w_0\lambda)$-chain and $(-\beta_l, \dots, -\beta_1)$ is a $(-\lambda)$-chain. □

Remark 6.6. Serre duality extends to any G/P. For $w, u \in W^P$ and $\lambda \in X^*(T)_P$ recall the expansion

$$\mathcal{L}_\lambda \otimes MC_y(X(wW_P)^\circ) = \sum_{u \in W^P} C_{u,\lambda}^{w,P} MC_y(X(uW_P)^\circ).$$

Then Theorem 5.15 can be rewritten as $C_{u,\lambda}^{w,P} = \sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C_{v,\lambda}^{w}$. Arguing as in the proof of Proposition 6.1, we obtain:

Proposition 6.7 (Serre duality on parabolic Chevalley coefficients).

$$C_{u,\lambda}^{w,P} = (-y)^{\ell(w)-\ell(u)} w_0 (C_{\overline{w_0 w},-\lambda}^{\overline{w_0 u},P})^\vee, \tag{33}$$

where $\overline{w_0 w}$ and $\overline{w_0 u} \in W^P$ are minimal coset representatives of $w_0 w W_P$ and $w_0 u W_P$.

7. K-theoretic stable envelopes for $T^*(G/B)$

In this section we apply the Chevalley formula for motivic classes to calculate the transformation of stable envelopes in $T^*(G/B)$ under the change of arbitrary alcoves. For alcoves adjacent to the fundamental alcove, a formula for this transformation was obtained in [SZZ20, Theorem 5.4], see also [KW25].

7.1. *Definition of the stable envelopes.* The stable envelopes were defined by Maulik and Okounkov in their seminal work on quantum cohomology of Nakajima quiver varieties [MO19]. Later, this was generalized by Okounkov and his collaborators to K-theory and elliptic cohomology [Oko27; AO21]. We recall next the definition of the stable envelopes for $T^*(G/B)$.

The torus T acts by left multiplication on G/B. Hence, it induces a natural action on $T^*(G/B)$. There is also a natural dilation $\mathbb{C}^*$-action on the cotangent fibers by a character of q^{-1}. Throughout this section, we use $q^{1/2}$ to denote the standard representation of $\mathbb{C}^*$, so that $K_{T\times\mathbb{C}^*}(\mathrm{pt}) = K_T(\mathrm{pt})[q^{\pm 1/2}]$.

The definition of the stable envelopes depends on three parameters:

- a *chamber* $\mathcal{C}$ in the Lie algebra of the maximal torus T;
- a *polarization* $T^{\frac{1}{2}} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ of the tangent bundle $T(T^*(G/B))$, i.e., a solution of the equation
$$T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^\vee = T(T^*(G/B))$$
in the ring $K_{T\times\mathbb{C}^*}(T^*(G/B))_{\mathrm{loc}}$;
- an *alcove* A in $\mathfrak{t}^*_{\mathbb{R}}$, which is also called a *slope* in [Oko27].

Given a polarization $T^{\frac{1}{2}}$, there is an opposite polarization defined by $T^{\frac{1}{2}}_{\mathrm{opp}} := q^{-1}(T^{\frac{1}{2}})^\vee$. A typical example of a polarization is $T(G/B)$ (pulled back from G/B to $T^*(G/B)$). Its opposite is equal to $T^*(G/B)$. Because we will work with fibers of polarizations over fixed points, in what follows we will assume that the polarization is given by a subbundle of $T(T^*(G/B))$, or possibly a virtual vector subbundle, i.e., a formal difference of such subbundles.

The torus fixed point set $(T^*(G/B))^T = (G/B)^T$ is in one-to-one correspondence with the Weyl group W. For every $w \in W$, recall that e_w denotes the corresponding fixed point. For a chosen Weyl chamber $\mathfrak{C}$ in Lie T, pick any cocharacter $\sigma \in \mathfrak{C}$. The attracting set of the fixed point e_w, also called the Białynicki-Birula cell in the literature, is defined by

$$\mathrm{Attr}_{\mathfrak{C}}(w) = \Big\{x \in T^*(G/B) \mid \lim_{z\to 0} \sigma(z)\cdot x = e_w\Big\}.$$

By analyzing the (signs of the roots in the) weight space decomposition of $T_w(T^*(G/B))$, one may show that $\mathrm{Attr}_{\mathfrak{C}}(w)$ is the conormal bundle of the attracting variety of w in G/B; i.e., the conormal bundle of the Schubert cell stable under the Borel subgroup associated to the chamber $\mathfrak{C}$.[4] Define a partial order on the fixed point set W to be the (transitive closure of the) following relation:

$$e_w \preceq_{\mathfrak{C}} e_v \quad \text{if } \overline{\mathrm{Attr}_{\mathfrak{C}}(v)} \cap e_w \neq \varnothing.$$

Then the order determined by the positive (resp., negative) chamber is the same as the Bruhat order (resp., the opposite Bruhat order).

Any chamber $\mathfrak{C}$ determines a decomposition of the tangent space $N_w := T_w(T^*(G/B))$ as $N_w = N_{w,+} \oplus N_{w,-}$ into T-weight spaces which are positive and negative with respect to $\mathfrak{C}$ respectively. For every polarization $T^{\frac12}$, denote $N_w \cap T^{\frac12}|_w$ by $N_w^{\frac12}$. Similarly, we have $N_{w,+}^{\frac12}$ and $N_{w,-}^{\frac12}$. In particular, $N_{w,-} = N_{w,-}^{\frac12} \oplus q^{-1}(N_{w,+}^{\frac12})^\vee$. Consequently, we have

$$N_{w,-} - N_w^{\frac12} = q^{-1}(N_{w,+}^{\frac12})^\vee - N_{w,+}^{\frac12}$$

as virtual vector bundles. The determinant bundle of the virtual bundle $N_{w,-} - N_w^{\frac12}$ is a complete square and its square root will be denoted by

$$\left(\frac{\det N_{w,-}}{\det N_w^{1/2}}\right)^{\frac12};$$

cf. [Oko27, §9.1.5]. For instance, if we choose the polarization $T^{1/2} = T(G/B)$, the positive chamber, and $w = \mathrm{id}$ then both $N_{\mathrm{id}}^{\frac12}$ and $N_{\mathrm{id},-}$ have weights $-\alpha$, where α varies in the set of positive roots; in this case the virtual bundle $N_{\mathrm{id},-} - N_{\mathrm{id}}^{\frac12}$ is 0.

Let $f := \sum_\mu f_\mu e^\mu \in K_{T\times\mathbb{C}^*}(\mathrm{pt})$ be a Laurent polynomial, where $e^\mu \in K_T(\mathrm{pt})$ and $f_\mu \in \mathbb{Q}[q^{1/2}, q^{-1/2}]$. The *Newton polytope* of f, denoted by $\deg_T f$, is

$$\deg_T f = \text{convex hull } (\{\mu | f_\mu \neq 0\}) \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $X^*(T)$ denotes the character lattice of T. The following theorem defines the K-theoretic stable envelopes.

[4] $T^*(G/B)$ is not compact, so not all points have well defined limits at 0. For example, if $\mathfrak{C}$ is the positive chamber, then the points in the open set $T^*(X(w_0)^\circ) \setminus X(w_0)^\circ \subset T^*(G/B)$ do not have limits at 0.

Theorem 7.1 [Oko27, §9.1; OS22, Theorem 1]. *For every chamber $\mathfrak{C}$, polarization $T^{\frac{1}{2}}$, and alcove A, there exists a unique map of $K_{T\times\mathbb{C}^*}(\mathrm{pt})$-modules*

$$\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A} : K_{T\times\mathbb{C}^*}((T^*(G/B))^T) \to K_{T\times\mathbb{C}^*}(T^*(G/B))$$

such that for every $w \in W$, the class $\Gamma := \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w)$ satisfies the following conditions:

(1) (*support*) $$\mathrm{Supp}\,\Gamma \subseteq \bigcup_{z\preceq_{\mathfrak{C}} w} \overline{\mathrm{Attr}_{\mathfrak{C}}(z)}.$$

(2) (*normalization*) $$\Gamma|_w = (-1)^{\mathrm{rk}\, N^{\frac{1}{2}}_{w,+}} \left(\frac{\det N_{w,-}}{\det N_w^{1/2}}\right)^{\frac{1}{2}} \mathcal{O}_{\mathrm{Attr}_{\mathfrak{C}}(w)}|_w.$$

(3) (*degree*) *For every $e_v \prec_{\mathfrak{C}} e_w$,*

$$\deg_T \Gamma|_v \subseteq \deg_T \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(v)|_v + v\lambda - w\lambda,$$

where $\lambda \in (X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap A$ is any rational weight in the alcove A.

Strictly speaking, $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w) \in K_{T\times\mathbb{C}^*}(G/B)$ denotes the image of $1 \in K_{T\times\mathbb{C}^*}(e_w)$ under the map $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}$. From the definition, it is easy to see that $\{\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w) \mid w \in W\}$ forms a basis for the localized equivariant K-theory $K_{T\times\mathbb{C}^*}(T^*(G/B))_{\mathrm{loc}}$, called the *stable basis*. Explicit combinatorial formulae and recursions for the localizations of stable envelopes in $T^*(G/B)$ may be found in [AMSS24a, §8.3].

7.2. ***Changing the polarizations.*** A natural question is to study the change of the stable envelopes when we vary the above three parameters. To start, the change of chambers is encoded in the left Weyl group action. More precisely, the group G acts on $T^*(G/B)$ by left multiplication, which induces a left Weyl group action on $K_{T\times\mathbb{C}^*}(T^*(G/B))$, see [MNS22b]. If we change the chamber $\mathfrak{C}$ to another chamber $w(\mathfrak{C})$ ($w \in W$), we have the following formula (see [AMSS24a, Lemma 8.2(a)]):

$$w \cdot (\mathrm{stab}_{\mathcal{C},T^{\frac{1}{2}},A}(u)) = \mathrm{stab}_{w(\mathcal{C}),w(T^{\frac{1}{2}}),A}(wu).$$

Next we consider the change of polarizations. In this case the results are stated, e.g., in [Oko27] in the more general setting of symplectic resolutions; for the convenience of the reader we include proofs for $T^*(G/B)$.

Lemma 7.2 [Oko27, Section 7.5.8]. *For any two polarizations $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$, there exists a class $\mathcal{F} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ such that $T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$.*

Remark 7.3. Classes of the form $\mathcal{F} - q^{-1}\mathcal{F}^\vee$ are called *balanced classes* in *loc. cit.*

Proof. It suffices to prove that any solution of the equation $\mathcal{G} + q^{-1}\mathcal{G}^\vee = 0 \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ is of the form $\mathcal{G} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$ for some $\mathcal{F} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$. Since $\mathcal{G} \in K_{T\times\mathbb{C}^*}(T^*(G/B)) \simeq K_T(G/B)[q, q^{-1}]$, we can write $\mathcal{G} = \mathcal{F} + q^{-1}\mathcal{F}'$ for some $\mathcal{F} \in K_T(G/B)[q]$ and $\mathcal{F}' \in q^{-1}K_T(G/B)[q^{-1}]$. Thus,

$$\mathcal{F} + q^{-1}\mathcal{F}' + q^{-1}\mathcal{F}^\vee + (\mathcal{F}')^\vee = 0.$$

Since $\mathcal{F}$, $(\mathcal{F}')^\vee \in K_T(G/B)[q]$ and $q^{-1}\mathcal{F}', q^{-1}\mathcal{F}^\vee \in q^{-1}K_T(G/B)[q^{-1}]$, we get

$$\mathcal{F} + (\mathcal{F}')^\vee = 0 \quad \textit{and} \quad q^{-1}\mathcal{F}' + q^{-1}\mathcal{F}^\vee = 0.$$

Therefore, $\mathcal{F}' = -\mathcal{F}^\vee$, and $\mathcal{G} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$. □

For any two polarizations $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$ let $\mathcal{F}$ be defined by $T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$ as in Lemma 7.2. Define a $T \times \mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ on $T^*(G/B)$ by

$$\mathcal{L} := \det \mathcal{F}.$$

Here for a virtual bundle $\mathcal{F} = \mathcal{V}_1 - \mathcal{V}_2$, $\det \mathcal{F} := \det \mathcal{V}_1 / \det \mathcal{V}_2$.

Example 7.4. Consider the polarization $T_1^{\frac{1}{2}} := T(G/B)$ and the opposite polarization $T_2^{\frac{1}{2}} = q^{-1}(T_1^{\frac{1}{2}})^\vee = T^*(G/B)$. Then the element $\mathcal{F}$ in Lemma 7.2 can be taken to be $T(G/B)$, therefore $\mathcal{L} = \det \mathcal{F} = \mathcal{L}_{-2\rho}$.

The next proposition shows that the change of polarizations results in a multiplication by a line bundle, therefore it is encoded in the Chevalley formula for the stable envelopes.

Proposition 7.5 [Oko27, Exercise 9.1.12].

$$\mathrm{stab}_{\mathfrak{C}, T_2^{\frac{1}{2}}, A}(w) = (-1)^{\mathrm{rk}\, N_{w,+,2}^{\frac{1}{2}} - \mathrm{rk}\, N_{w,+,1}^{\frac{1}{2}}} q^{\frac{1}{2}\mathrm{rk}\, \mathcal{F}|_w} \mathcal{L} \otimes \mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w),$$

where $N_{w,+,i}^{\frac{1}{2}} := N_{w,+,i} \cap T_i^{\frac{1}{2}}$ *for* $i = 1, 2$, *and if* $\mathcal{F} = T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{V}_1 - \mathcal{V}_2$ *is the virtual bundle from Lemma 7.2, then* $\mathrm{rk}\, \mathcal{F}|_w := \mathrm{rk}\, \mathcal{V}_1|_w - \mathrm{rk}\, \mathcal{V}_2|_w$.

Proof. From the characterization Theorem 7.1, it suffices to show that the right hand side satisfies the defining properties of the stable envelope on the left hand side. The support condition is immediate. For the degree condition, we need to check that for every $e_v \prec_{\mathfrak{C}} e_w$,

$$\deg_T(\mathcal{L} \otimes \mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w))|_v \subseteq \deg_T(\mathcal{L} \otimes \mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(v))|_v + v\lambda - w\lambda,$$

for some $\lambda \in A$. The terms $\mathcal{L}|_v$ on both sides cancel, and the above inclusion reduces to the degree condition for $\mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w)$. The normalization condition follows from the following calculation:

$$\begin{aligned}(-1)^{\mathrm{rk}\, N_{w,+,2}^{\frac{1}{2}} - \mathrm{rk}\, N_{w,+,1}^{\frac{1}{2}}} \frac{\mathrm{stab}_{\mathfrak{C}, T_2^{\frac{1}{2}}, A}(w)|_w}{\mathrm{stab}_{\mathfrak{C}, T_1^{\frac{1}{2}}, A}(w)|_w} &= \left(\frac{\det N_{w,-}}{\det N_{w,2}^{1/2}}\right)^{\frac{1}{2}} \left(\frac{\det N_{w,-}}{\det N_{w,1}^{1/2}}\right)^{-\frac{1}{2}} = \left(\frac{\det N_{w,1}^{1/2}}{\det N_{w,2}^{1/2}}\right)^{\frac{1}{2}} \\ &= \left(\det N_w \cap (\mathcal{F}|_w - q^{-1}\mathcal{F}^\vee|_w)\right)^{\frac{1}{2}} \\ &= \left(\frac{\det(N_w \cap \mathcal{F}|_w)}{\det(N_w \cap q^{-1}(\mathcal{F}|_w)^\vee)}\right)^{\frac{1}{2}} = q^{\frac{1}{2}\mathrm{rk}\, \mathcal{F}|_w} \mathcal{L}|_w.\end{aligned}$$

Here we have used our assumption that $\mathcal{F}$ is represented by a subbundle of $T(T^*(G/B))$, and also that the weights of $N_w = T_w(T^*(G/B))$ are distinct. The reason for the last equality in the display is as follows. We have $N_w \cap \mathcal{F}|_w = \mathcal{F}|_w$; suppose e^λ is a torus weight of it. Since N_w is a symplectic vector space, $q^{-1}e^{-\lambda}$ is a weight of N_w. Hence $q^{-1}e^{-\lambda}$ is also a weight of the intersection $N_w \cap q^{-1}(\mathcal{F}|_w)^\vee$

and in fact $N_w \cap q^{-1}(\mathcal{F}|_w)^\vee = q^{-1}(\mathcal{F}|_w)^\vee$, giving the equality. The case of $\mathcal{F} = \mathcal{V}_1 - \mathcal{V}_2$ being a virtual bundle follows from linearity of the constructions. □

7.3. ***Changing the alcoves.*** We now turn to what happens under the change of alcoves. This can be answered using recursive formulae from [SZZ20; SZZ21], reported in Theorem 7.6 below. Our subsequent Theorem 7.8 provides a nonrecursive answer, expressing the Chevalley formula in terms of λ-chains, thus relating the stable envelope for the fundamental alcove $\mathrm{A}_\circ$ to the stable envelope for a translate $\mathrm{A}_\circ + \lambda$.

The alcoves in $\mathfrak{t}^*_{\mathbb{R}}$ are of the form $x(\mathrm{A}_\circ) + \lambda$ for some $x \in W$ and some λ in the root lattice. It was proved in [AMSS24a, Lemma 8.2] and [SZZ20, Remark 2.3] that

$$\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A+\lambda}(w) = e^{-w\lambda}\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},A}(w), \tag{34}$$

where $\mathcal{L}_\lambda$ is the pullback of $G \times^B \mathbb{C}_\lambda$ from G/B to $T^*(G/B)$. Fix the chamber $\mathcal{C}$ to be the antidominant Weyl chamber

$$\mathcal{C} := \{\lambda \in \mathfrak{t}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle < 0 \text{ for any positive root } \alpha\}$$

and the polarization $T^{\frac{1}{2}} = T^*(G/B)$. To simplify notation, we denote $\mathrm{stab}_{\mathcal{C},T^*(G/B),A}(w)$ by $\mathrm{stab}_A(w)$.

Let $Z = T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$ be the Steinberg variety where $\mathcal{N} \subset \mathfrak{g}$ denotes the nilpotent cone. There is an algebra isomorphism

$$\mathbb{H} \simeq K_{G\times\mathbb{C}^*}(Z),$$

where $K_{G\times\mathbb{C}^*}(Z)$ has an algebra structure given by convolution; this was proved by Kazhdan and Lusztig [KL87] and Ginzburg [CG10]. The convolution induces an action of the Hecke algebra $\mathbb{H}$ on $K_{T\times\mathbb{C}^*}(T^*(G/B))$ which we recall next. For a simple root α_i, and the corresponding minimal parabolic subgroup $P_i \supset B$, define the operator T_i on $K_{T\times\mathbb{C}^*}(T^*(G/B))$ by the following formula:

$$T_i(\mathcal{F}) := -\mathcal{F} - \pi_{1*}(\pi_2^*\mathcal{F} \otimes \pi_2^*\mathcal{L}_{\alpha_i}).$$

Here $\mathcal{F} \in K_{A\times\mathbb{C}^*}(T^*(G/B))$, $Y_i := G/B \times_{G/P_i} G/B \subset G/B \times G/B$, $T^*_{Y_i}$ is the conormal bundle of Y_i inside $G/B \times G/B$, and $\pi_j : T^*_{Y_i} \to T^*(G/B)$ $(j = 1, 2)$ are the two projections. These operators satisfy the quadratic relations and the braid relations in $\mathbb{H}$. In particular, T_w is well defined for any $w \in W$. Recall that $\mathrm{A}_\circ$ denotes the fundamental alcove.

Theorem 7.6. (a) [SZZ20, Theorem 4.5]

$$T_i(\mathrm{stab}_{\mathrm{A}_\circ}(w)) = \begin{cases} (q-1)\,\mathrm{stab}_{\mathrm{A}_\circ}(w) + q^{1/2}\,\mathrm{stab}_{\mathrm{A}_\circ}(ws_i), & \text{if} \quad ws_i < w, \\ q^{1/2}\,\mathrm{stab}_{\mathrm{A}_\circ}(ws_i), & \text{if} \quad ws_i > w. \end{cases}$$

(b) [SZZ21, Theorem 5.4] *Let $x \in W$. Then*

$$\mathrm{stab}_{x(\mathrm{A}_\circ)}(w) = q^{-\ell(x)/2}T_x(\mathrm{stab}_{\mathrm{A}_\circ}(wx)).$$

(c) [SZZ21, Lemma 3.5 and Corollary 5.3] *Assume A_1 and A_2 are two adjacent alcoves separated by a wall of the form $H_{\alpha,n}$, where $\alpha > 0$. Assume A_2 is on the positive side of $H_{\alpha,n}$, i.e., for any $\mu \in A_2$,*

$(\mu, \alpha^\vee) > n$. *Then*

$$\operatorname{stab}_{A_1}(w) = \begin{cases} \operatorname{stab}_{A_2}(w) + e^{-nw\alpha}(q^{1/2} - q^{-1/2}) \operatorname{stab}_{A_2}(ws_\alpha) & \text{if} \quad ws_\alpha > w, \\ \operatorname{stab}_{A_2}(w) & \text{if} \quad ws_\alpha < w. \end{cases}$$

Using statement (a), one can calculate $T_x(\operatorname{stab}_{A_\circ}(wx))$ recursively in terms of $\{\operatorname{stab}_{A_\circ}(w) \mid w \in W\}$. Part (b) implies that the same is true for $\operatorname{stab}_{x(A_\circ)}(w)$. Finally, part (c) may be used to relate directly the stable bases for any two adjacent alcoves, and therefore recursively relate the stable bases for two arbitrary alcoves.

We focus next on relating (34) to our Chevalley formulae obtained earlier in this paper. Together with (a) and (b) from Theorem 7.6 above, this gives an alternative recursion to (c), calculating the stable envelope for an arbitrary alcove $xA_\circ + \lambda$ starting from the stable envelope for $A_\circ$.

Fix λ an integral weight. By Theorem 7.6(a), $\operatorname{stab}_{x(A_\circ)}(w)$ may be written as a linear combination of $\{\operatorname{stab}_{A_\circ}(w) \mid w \in W\}$. By (34), to determine $\operatorname{stab}_{x(A_\circ)+\lambda}(w)$, it suffices to find a formula for $\mathcal{L}_\lambda \otimes \operatorname{stab}_{A_\circ}(w)$. This can be achieved by the Chevalley formula for the motivic Chern classes. The key is the following result.

Lemma 7.7 [AMSS24a, Theorem 8.6]. *Let $\iota : G/B \hookrightarrow T^*(G/B)$ denote the inclusion of the zero section. For any $w \in W$,*

$$\iota^*(\operatorname{stab}_{A_\circ}(w)) = (-1)^{\dim G/B} q^{\dim G/B - \ell(w)/2} MC_{-q^{-1}}(Y(w)^\circ) \otimes \mathcal{L}_{-2\rho}.$$

Recall the operator $(-)^\vee$ on $K_T(pt)[y^{\pm 1}]$, which sends e^μ to $e^{-\mu}$ and y to y^{-1}. We have the following Chevalley formula for the stable bases.

Theorem 7.8. *Let $\lambda \in X^*(T)$ be a weight and fix $\beta_1, \dots, \beta_l$ a reduced λ-chain corresponding to an alcove walk from $A_\circ$ to $A_\circ - \lambda$. Then*

$$\mathcal{L}_\lambda \otimes \operatorname{stab}_{A_\circ}(u) = \sum_w q^{\frac{1}{2}(\ell(u) - \ell(w))} (C^w_{u,-\lambda})^\vee|_{y=-q^{-1}} \operatorname{stab}_{A_\circ}(w),$$

where $C^w_{u,\lambda}$ are the coefficients defined in (27).

In the notation of Theorem 5.5, one can write this in terms of λ-chains as

$$\mathcal{L}_\lambda \otimes \operatorname{stab}_{A_\circ}(u) = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (q^{-1/2} - q^{1/2})^{|J|} e^{w\tilde{r}_{J>}(\lambda)} \operatorname{stab}_{A_\circ}(ur_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}),$$

where the sum is over subsets $J = \{j_1 < \cdots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} r_{h_{j_2}} \cdot \cdots \cdot r_{h_{j_t}}$.

Remark 7.9. Recall we have the following duality between the stable bases (see [OS22, Proposition 1]):

$$\langle \operatorname{stab}_{\mathcal{C}, T^{\frac{1}{2}}, A}(u), \operatorname{stab}_{-\mathcal{C}, T^{\frac{1}{2}}_{\text{opp}}, -A}(w) \rangle = \delta_{u,w}.$$

Therefore, similar arguments as in the proof of Lemma 5.1 will give a Chevalley formula for the dual stale basis $\operatorname{stab}_{-\mathcal{C}, T(G/B)), -A_\circ}(u)$.

Proof. By the definition of the Segre motivic classes from Definition 4.2(2), the expression in (28) becomes

$$\mathcal{L}_\lambda \otimes \mathcal{D}(MC_y(Y(u)^\circ)) = \sum_{w \geq u} (-y)^{\ell(u)-\ell(w)} C^w_{u,\lambda} \mathcal{D}(MC_y(Y(w)^\circ)).$$

Taking $(-)^\vee$ on both sides of the equation, we get

$$\mathcal{L}_{-\lambda} \otimes MC_y(Y(u)^\circ) = \sum_{w \geq u} (-y)^{\ell(w)-\ell(u)} (C^w_{u,\lambda})^\vee MC_y(Y(w)^\circ).$$

The first equation of the theorem follows from this and Lemma 7.7. The second equation is a consequence of the first and of (29) in Theorem 5.5. □

Example 7.10. In type A_2, set $u = s_2 s_1$ and $\lambda = 2\varpi_1 + \varpi_2$. A λ-chain of roots is given by $\beta_1 = \alpha_2,\ \beta_2 = \alpha_1 + \alpha_2,\ \beta_3 = \alpha_1,\ \beta_4 = \alpha_1 + \alpha_2,\ \beta_5 = \alpha_1,\ \beta_6 = \alpha_1 + \alpha_2$; see Appendix B. From Theorem 7.8, we have

$$\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(s_2 s_1) \ = \ e^{\varpi_1 - 3\varpi_2}\, \mathrm{stab}_{\mathrm{A}_\circ}(s_2 s_1) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) e^{-\varpi_1 - 2\varpi_2}\, \mathrm{stab}_{\mathrm{A}_\circ}(s_1 s_2 s_1).$$

Therefore, by (34), using that $-u(\lambda) = -\varpi_1 + 3\varpi_2$, we have

$$\mathrm{stab}_{\mathrm{A}_\circ + \lambda}(s_2 s_1) \ = \ \mathrm{stab}_{\mathrm{A}_\circ}(s_2 s_1) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) e^{-\alpha_1}\, \mathrm{stab}_{\mathrm{A}_\circ}(s_1 s_2 s_1).$$

Example 7.11. Consider $u = s_2$, $\lambda = 2\varpi_1 + \varpi_2$, $w_0 u = s_2 s_1$. We can use Serre duality (Proposition 6.1) and Example 5.8 to get

$$\begin{aligned}\mathcal{L}_\lambda \otimes \mathrm{stab}_{\mathrm{A}_\circ}(s_2) = e^{3\varpi_1 - \varpi_2}\, &\mathrm{stab}_{\mathrm{A}_\circ}(s_2) \\ &+ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{\varpi_1} + e^{-\varpi_1 + \varpi_2} + e^{-3\varpi_1 + 2\varpi_2})\, \mathrm{stab}_{\mathrm{A}_\circ}(s_1 s_2) \\ &+ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{2\varpi_1 - 2\varpi_2} + e^{\varpi_1 - 3\varpi_2}) stab_{\mathrm{A}_\circ}(s_2 s_1) \\ &+ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (e^{-\varpi_1 - 2\varpi_2} + e^{-2\varpi_1} + e^{-\varpi_2})\, \mathrm{stab}_{\mathrm{A}_\circ}(s_1 s_2 s_1).\end{aligned}$$

Since $-s_2(\lambda) = -3\varpi_1 + \varpi_2$, we have

$$\begin{aligned}\mathrm{stab}_{\mathrm{A}_\circ + \lambda}(s_2) = \ &\mathrm{stab}_{\mathrm{A}_\circ}(s_2) \\ &+ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{-\alpha_1} + e^{-2\alpha_1} + e^{-3\alpha_1})\, \mathrm{stab}_{\mathrm{A}_\circ}(s_1 s_2) \\ &+ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{-\alpha_1 - \alpha_2} + e^{-2\alpha_1 - 2\alpha_2})\, \mathrm{stab}_{\mathrm{A}_\circ}(s_2 s_1) \\ &+ (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (e^{-3\alpha_1 - 2\alpha_2} + e^{-3\alpha_1 - \alpha_2} + e^{-2\alpha_1 - \alpha_2})\, \mathrm{stab}_{\mathrm{A}_\circ}(s_1 s_2 s_1).\end{aligned}$$

8. Whittaker functions and Hall–Littlewood polynomials

In this section we apply the Chevalley formula to obtain combinatorial expressions for Whittaker functions and Hall–Littlewood polynomials. Variants of the formulae we obtain were already available in the literature, and our approach based on the cohomological calculations adds a geometric perspective to this.

8.1. ***Whittaker functions.*** In this section we study Whittaker functions. These appear in p-adic representation theory, and in this note we utilize a cohomological construction of these functions from [AMSS24a], see also [MS22]. Recall the definition of the Demazure–Lusztig operators on $K_T(pt)[y]$:

$$\widetilde{T}_i(e^\lambda) = -e^\lambda \frac{1+y}{1-e^{-\alpha_i}} + e^{s_i\lambda}\frac{1+ye^{\alpha_i}}{1-e^{-\alpha_i}}, \quad \widetilde{T}_i^\vee(e^\lambda) = -e^\lambda \frac{1+y}{1-e^{-\alpha_i}} + e^{s_i\lambda}\frac{1+ye^{-\alpha_i}}{1-e^{-\alpha_i}}.$$

The operators satisfy the usual quadratic and braid relations in the Hecke algebra, therefore for any $w \in W$ there are operators $\widetilde{T}_w$ and $\widetilde{T}_w^\vee$ acting on $K_T(pt)[y]$, and defined using any reduced decomposition of w. The following has been proved in [MS22, Theorem 1.1]:

Proposition 8.1. *Let* $\lambda_y(\mathrm{id}) := \prod_{\alpha>0}(1+ye^\alpha)$, *and set*

$$MC_y'(X(w)^\circ) := \lambda_y(\mathrm{id})\frac{MC_y(X(w)^\circ)}{\lambda_y(T^*_{G/B})}.$$

Then, for any $\lambda \in X^*(T)$,

(1) $\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)\big) = \widetilde{T}_w^\vee(e^\lambda)$, *and*

(2) $\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y'(X(w)^\circ)\big) = \widetilde{T}_w(e^\lambda)$.

We note that for any antidominant weight λ and $w \in W$,

$$\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y'(X(w)^\circ)\big) = \widetilde{T}_w(e^\lambda) = \mathcal{W}_{\lambda,w}, \tag{35}$$

where $\mathcal{W}_{\lambda,w}$ is the Iwahori–Whittaker function for the Langlands dual group over a nonarchimedean local field; we refer to [MS22] for more details, including the number theoretic definition of $\mathcal{W}_{\lambda,w}$. Here we identified y with $-q^{-1}$, where q is the number of elements in the residue field. As explained in [MS22], from the fact that

$$\sum_{w\in W} MC_y'(X(w)^\circ) = 1$$

by the additivity motivic Chern classes, one recovers the Casselman–Shalika formula for the spherical Whittaker function [CS80]:

$$\sum_{w\in W}\mathcal{W}_{\lambda,w} = \prod_{\alpha>0}(1+ye^\alpha)\chi_T(G/B,\mathcal{L}_\lambda) = \prod_{\alpha>0}(1+ye^\alpha)\chi_{w_0\lambda}. \tag{36}$$

Here $\chi_{w_0\lambda}$ denotes the character for the irreducible representation of G of highest weight $w_0(\lambda)$. We also note that, in type A, an interpretation of the Iwahori–Whittaker function in terms of the partition function of the Iwahori lattice model has been obtained in [BBBG24].

Using the Chevalley coefficients in (27), we obtain the following formula for the Iwahori–Whittaker function $\mathcal{W}_{\lambda,w}$. Let ρ denote the half sum of the positive roots.

Theorem 8.2. *For any antidominant weight* λ *and* $w \in W$,

$$\mathcal{W}_{\lambda,w} = e^\rho \sum_u (-1)^{\ell(u)} C^w_{\lambda-\rho,u}|_{y\mapsto y^{-1}}\, y^{\ell(w)-\ell(u)}.$$

Proof. For any $u \in W$,

$$\chi_T(G/B, MC_y(X(u)^\circ)) = MC_y[X(u)^\circ \to \mathrm{pt}] = MC_y[\mathbb{A}^1 \to \mathrm{pt}]^{\ell(u)} = (-y)^{\ell(u)}, \tag{37}$$

where the second equality follows from [AMSS24a, Theorem 4.2(3)]. Therefore, taking the equivariant Euler characteristics of both sides of (27), we get

$$\widetilde{T}_w^\vee(e^\lambda) = \chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)\big) = \sum_u C^w_{\lambda,u}(-y)^{\ell(u)}.$$

On the other hand, it is immediate to check the following relation between the two Demazure–Lusztig operators:

$$\widetilde{T}_w = e^\rho \widetilde{T}_w^\vee|_{y\mapsto y^{-1}} e^{-\rho} y^{\ell(w)}. \tag{38}$$

Hence,

$$\chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)\big) = \widetilde{T}_w(e^\lambda) = e^\rho \sum_u (-1)^{\ell(u)} C^w_{\lambda-\rho,u}|_{y\mapsto y^{-1}} y^{\ell(w)-\ell(u)}.$$

The proof ends by applying Proposition 8.1. □

Remark 8.3. Notice that $C^w_{0,u} = \delta_{u,w}$. The above proof shows

$$\chi_T\big(G/B, \mathcal{L}_\rho \otimes MC'_y(X(w)^\circ)\big) = (-1)^{\ell(w)} e^\rho.$$

As a corollary we prove a variant of the Casselman–Shalika formula (36), obtained by Li [Li92]; see also [BBBG24, Proposition 9.4]. First define

$$R_\lambda(y) := \chi_T(G/B, \lambda_y(T^*_{G/B}) \otimes \mathcal{L}_\lambda) \in K_T(pt)[y]. \tag{39}$$

Corollary 8.4. *Let λ be an antidominant integral weight. Then*

$$\sum_w y^{-\ell(w)} \mathcal{W}_{\lambda,w} = e^\rho R_{\lambda-\rho}(y^{-1}).$$

Proof. By the additivity of the motivic Chern classes and Proposition 8.1(1),

$$R_\lambda(y) = \sum \chi_T\big(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)\big) = \sum_w \widetilde{T}_w^\vee(e^\lambda). \tag{40}$$

On the other hand, for an antidominant weight λ, we have

$$\begin{aligned}\sum_w y^{-\ell(w)} \mathcal{W}_{\lambda,w} &= \sum_w y^{-\ell(w)} \widetilde{T}_w(e^\lambda) = e^\rho \sum_w y^{-\ell(w)} e^{-\rho} \widetilde{T}_w e^\rho (e^{\lambda-\rho}) \\ &= e^\rho \sum_w \widetilde{T}_w^\vee|_{y\mapsto y^{-1}}(e^{\lambda-\rho}) = e^\rho R_{\lambda-\rho}(y^{-1}).\end{aligned}$$

Here, the first equality follows from Proposition 8.1(2), the third one follows from (38), and the last one follows from (40). □

8.2. *Hall–Littlewood polynomials.* In this section, we assume either λ or $-\lambda$ to be a dominant integral weight. Set $\Sigma_\lambda := \{\alpha \in \Sigma \mid \langle \lambda, \alpha^\vee \rangle = 0\}$, and let R^+_λ be the set of positive roots which are linear combinations of the simple roots in Σ_λ. Denote by $W_\lambda \subset W$ the subgroup generated by the simple reflections s_α, where $\alpha \in \Sigma_\lambda$. Let W^λ be the set of minimal length representatives for the cosets W/W_λ. Finally, let P_λ be the parabolic subgroup containing the Borel subgroup B defined by the condition that $W_{P_\lambda} = W_\lambda$.

Definition 8.5. (1) Define $H_\lambda(y) := \chi_T(G/P_\lambda, \lambda_y(T^*_{G/P_\lambda}) \otimes \mathcal{L}_\lambda) \in K_T(pt)[y]$.

(2) (Hall–Littlewood polynomial; cf. [Mac98, p. 208, (2.2)]) For a dominant weight λ, define

$$HL_\lambda(\boldsymbol{x}; t) := \sum_{w \in W^\lambda} w\Bigg(\boldsymbol{x}^\lambda \prod_{\alpha \in R^+ \setminus R^+_\lambda} \frac{1 - t\boldsymbol{x}^{-\alpha}}{1 - \boldsymbol{x}^{-\alpha}}\Bigg),$$

where $\boldsymbol{x}^\lambda$ denotes e^λ.

Let $\pi_\lambda : G/B \to G/P_\lambda$ be the natural projection. Then

$$\lambda_y(G/B) = \pi^*_\lambda(\lambda_y(G/P_\lambda)) \cdot \lambda_y(P_\lambda/B).$$

By the projection formula, and using that $\pi^*_\lambda(\mathcal{L}_\lambda) = \mathcal{L}_\lambda$, we have

$$H_\lambda(y) = \chi_T(G/P_\lambda, \lambda_y(T^*_{G/P_\lambda}) \otimes \mathcal{L}_\lambda) = \frac{\chi_T(G/B, \lambda_y(T^*_{G/B}) \otimes \mathcal{L}_\lambda)}{\chi_T(P_\lambda/B, \lambda_y(T^*_{P_\lambda/B}))} = \frac{R_\lambda(y)}{\sum_{w \in W_\lambda} (-y)^{\ell(w)}}$$

Here the last equality follows from (39), and the fact that

$$\chi_T(P_\lambda/B, \lambda_y(T^*_{P_\lambda/B})) = \sum_{w \in W_\lambda} \chi(MC_y(X(w)^\circ)) = \sum_{w \in W_\lambda} (-y)^{\ell(w)}.$$

The relation between H_λ and the Hall–Littlewood polynomial, summarized next, was obtained in a related (upcoming) collaboration with B. Ion.

Lemma 8.6. *We have the following formulae for* $H_\lambda(y)$:

$$H_\lambda(y) = \sum_{w \in W^P} \sum_{u \in W} C^w_{u,\lambda} (-y)^{\ell(u)}, \tag{41}$$

$$H_\lambda(y) = \sum_{w \in W^\lambda} w\Bigg(e^\lambda \prod_{\alpha \in R^+ \setminus R^+_\lambda} \frac{1 + ye^\alpha}{1 - e^\alpha}\Bigg). \tag{42}$$

Proof. Equation (41) follows from $\lambda_y(T^*_{G/P}) = MC_y(G/P) = \sum_{w \in W^P} MC_y(X(wW_P)^\circ)$, Theorem 5.15, and $\chi_T(MC_y(X(uW_P)^\circ)) = (-y)^{\ell(u)}$ for any $u \in W^P$. Equation (42) follows from the localization formula [Nie74; MS22, Theorem 2.1 (c)]. To be more specific, the torus fixed points in G/P_λ are $\{wP_\lambda \mid w \in W^P\}$, and the torus weights of the tangent space at the fixed point wP_λ are $\{-w\alpha \mid \alpha \in R^+ \setminus R^+_\lambda\}$. □

Corollary 8.7. *For a dominant integral* λ, *the Hall–Littlewood polynomial* $HL_\lambda(x;t)$ *can be expressed using* $H_{-\lambda}(y)$ *or* $H_\lambda(y)$ *as follows*:

$$HL_\lambda(\boldsymbol{x};t) = H_{-\lambda}(y)\big|_{e^\alpha \mapsto \boldsymbol{x}^{-\alpha},\, y \mapsto -t}, \tag{43}$$

$$HL_\lambda(\boldsymbol{x};t) = \left(\frac{1}{(-y)^{\dim G/P_\lambda}} H_\lambda(y)\right)\Big|_{e^\alpha \mapsto \boldsymbol{x}^{\alpha},\, y \mapsto -t^{-1}}. \tag{44}$$

We next assume that λ is a dominant integral weight. Fix a reduced $(-\lambda)$-chain $\Gamma = (-\beta_1, -\beta_2, \ldots, -\beta_l)$ and the sequence of hyperplanes $H_{\beta_1,d_1}, H_{\beta_2,d_2}, \ldots, H_{\beta_l,d_l}$. This corresponds to an alcove path from $A_\circ$ to $A_\circ + \lambda$. Since λ is dominant, we have $\beta_j > 0$, $d_j > 0$.

We recover the following known formula for the Hall–Littlewood polynomial:

Proposition 8.8 [Sch06; Ram06; Len11, Theorem 2.7].

$$HL_\lambda(\boldsymbol{x};t) = \sum_{(w,J,u)\in\mathcal{A}(\Gamma)} t^{\frac{1}{2}(\ell(w)+\ell(u)-|J|)}(1-t)^{|J|}\boldsymbol{x}^{w\hat{r}_{J_<}(\lambda)}, \tag{45}$$

where (with notation as in Section 3.2),

$$\mathcal{A}(\Gamma) = \big\{(w, J, u) \mid w \in W^P, u \in W, J \subset \{1,2,\ldots,l\}, u \xrightarrow{J_>} w\big\}.$$

Proof. Apply equations (43), (41), and (30) to the $(-\lambda)$-chain $(-\beta_1, -\beta_2, \ldots, -\beta_l)$. ☐

We also get a new formula for $HL_\lambda(\boldsymbol{x};t)$:

Proposition 8.9.

$$HL_\lambda(\boldsymbol{x};t) = \sum_{(u,J,w)\in\mathcal{A}^{\mathrm{op}}(\Gamma)} t^{\frac{1}{2}(2\dim G/P_\lambda-\ell(w)-\ell(u)-|J|)}(1-t)^{|J|}\boldsymbol{x}^{u\hat{r}_{J_<}(\lambda))}, \tag{46}$$

where $\mathcal{A}^{\mathrm{op}}(\Gamma) = \big\{(u, J, w) \mid w \in W^P, u \in W, J \subset \{1,2,\ldots,l\}, u \xrightarrow{J_<} w\big\}$.

Proof. Apply equations (44), (41), and (29) to the $(-\lambda)$-chain, together with Remark 5.6(1). ☐

Remark 8.10. When $P_\lambda = B$, equations (45) and (46) give the same formula. The correspondence may be seen using the Serre duality in Proposition 6.7. However, the formulae are in general different, as shown by the examples below.

Example 8.11 type A_2. Let $G = \mathrm{GL}_3(\mathbb{C})$, $T = (\mathbb{C}^*)^3$, and $x_i = e^{\varepsilon_i}$, for $i = 1, 2, 3$. Let $\lambda = \varpi_1 = \varepsilon_1$; then $W_\lambda = \langle s_2\rangle \subset W = \langle s_1, s_2\rangle$, and $W^\lambda = \{\mathrm{id}, s_1, s_2s_1\}$. Fix a reduced $(-\lambda)$-chain $(-\beta_1 = -\alpha_1 - \alpha_2, -\beta_2 = -\alpha_1)$.

Then Proposition 8.8 sums over the seven terms shown on the left in Table 1, while Proposition 8.9 sums over those on the right. The respective developments are

$$HL_\lambda(\boldsymbol{x};t) = (x_1) + (tx_2 + (1-t)x_2) + (t^2x_3 + t(1-t)x_3 + t(1-t)x_3 + (1-t)^2x_3) = x_1 + x_2 + x_3,$$

$$HL_\lambda(\boldsymbol{x};t) = (t^2x_1) + (tx_2 + t(1-t)x_1) + (x_3 + (1-t)x_2 + (1-t)x_1)) = x_1 + x_2 + x_3.$$

w	J	u	
id	$\varnothing$	id	x_1
s_1	$\varnothing$	s_1	tx_2
	$\{2\}$	id	$(1-t)x_2$
s_2s_1	$\varnothing$	s_2s_1	t^2x_3
	$\{1\}$	s_1	$t(1-t)x_3$
	$\{2\}$	s_2	$t(1-t)x_3$
	$\{1,2\}$	id	$(1-t)^2x_3$

w	J	u	
id	$\varnothing$	id	t^2x_1
s_1	$\varnothing$	s_1	tx_2
	$\{2\}$	id	$t(1-t)x_1$
s_2s_1	$\varnothing$	s_2s_1	x_3
	$\{1\}$	s_1	$(1-t)x_2$
	$\{2\}$	s_2	$(1-t)x_1$

Table 1. To Example 3.7.

w	J	u	
id	$\varnothing$	id	$x_1^2x_2^2$
s_2	$\varnothing$	s_1	$tx_1^2x_3^2$
	$\{2\}$	id	$(1-t)x_1^2x_2x_3$
	$\{4\}$	id	$(1-t)x_1^2x_3^2$
$w=s_1s_2$	$\varnothing$	s_1s_2	$t^2x_2^2x_3^2$
	$\{1\}$	s_2	$t(1-t)x_1x_2x_3^2$
	$\{2\}$	s_1	$t(1-t)x_1x_2^2x_3$
	$\{3\}$	s_2	$t(1-t)x_2^2x_3^2$
	$\{4\}$	s_1	$t(1-t)x_2^2x_3^2$
	$\{1,2\}$	id	$(1-t)^2x_1x_2^2x_3$
	$\{1,4\}$	id	$(1-t)^2x_1x_2x_3^2$
	$\{3,4\}$	id	$(1-t)^2x_2^2x_3^2$

w	J	u	
id	$\varnothing$	id	$t^2x_1^2x_2^2$
s_2	$\varnothing$	s_2	$tx_1^2x_3^2$
	$\{2\}$	id	$t(1-t)x_1^2x_2x_3$
	$\{4\}$	id	$t(1-t)x_1^2x_2^2$
s_1s_2	$\varnothing$	s_1s_2	$x_2^2x_3^2$
	$\{1\}$	s_2	$(1-t)x_1x_2x_3^2$
	$\{2\}$	s_1	$(1-t)x_1x_2^2x_3$
	$\{3\}$	s_2	$(1-t)x_1^2x_3^2$
	$\{4\}$	s_1	$(1-t)x_1^2x_2^2$
	$\{2,3\}$	id	$(1-t)^2x_1^2x_2x_3$

Table 2. To Example 8.12.

Example 8.12. Same setup as in Example 3.7, but with $\lambda = 2\varpi_2$. Then $W_\lambda = \langle s_1\rangle$, $W^\lambda = \{\mathrm{id}, s_2, s_1s_2\}$, and $d = \dim G/P_\lambda = 2$. A $(-\lambda)$-chain is $(-\beta_1 = -(\alpha_1+\alpha_2), -\beta_2 = -\alpha_2, -\beta_3 = -(\alpha_1+\alpha_2), -\beta_4 = -\alpha_2)$, and $d_1 = 1, d_2 = 1, d_3 = 2, d_4 = 2$.

Proposition 8.8 sums over the twelve terms shown on the left in Table 2, and Proposition 8.9 sums over the ten terms on the right. The respective developments are

$$HL_\lambda(\boldsymbol{x};t) = \underbrace{x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2}_{s_{22}} - t\,(\underbrace{x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2}_{s_{211}}),$$

$$HL_\lambda(\boldsymbol{x};t) = (x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + (1-t)(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) = s_{22} - ts_{211}.$$

Appendix A. Chevalley formulae for Chern–Schwartz–MacPherson classes of Schubert cells

We give a short proof of the Chevalley formulae for the equivariant Chern–Schwartz–MacPherson (CSM) and Segre–MacPherson (SM) classes of the Schubert cells in the (partial) flag varieties; see [AMSS23, Theorem 9.10] and Theorem A.4 below. Our proof again relies on the action of the appropriate Hecke algebra, this time on the equivariant cohomology of G/P. Besides the intrinsic interest in these Chevalley formulae, we mention that recursions based on it were recently utilized to obtain proofs of Nakada's colored hook formula for finite Weyl groups [MNS22a] and more general Coxeter groups [MNS24].

A.1. ***Degenerate affine Hecke algebras.***

A.1.1. *A change of bases formula.* Recall that the degenerate affine Hecke algebra $\mathcal{H}$ is generated by T_w, $w \in W$ and x_λ, $\lambda \in X^*(T)$, such that

- $T_w T_u = T_{wu}$ for any $w, u \in W$;
- $x_\lambda x_\mu = x_\mu x_\lambda$ for any $\lambda, \mu \in X^*(T)$;
- $x_{\lambda+\mu} = x_\lambda + x_\mu$ for any $\lambda, \mu \in X^*(T)$;
- for any simple root α_i, $T_{s_i} x_\lambda - x_{s_i\lambda} T_{s_i} = -\langle \lambda, \alpha_i^\vee \rangle$.

Lemma A.1. *For any $w \in W$ and $\lambda \in X^*(T)$, the following commutation relation holds in $\mathcal{H}$:*

$$T_w x_\lambda = x_{w\lambda} T_w - \sum_{\substack{\alpha>0\\ ws_\alpha<w}} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha}.$$

Proof. We utilize induction on $\ell(w)$. The claim is clear when $\ell(w) = 0$ or 1. Now assume $\ell(w) > 1$, and that the claim holds for any Weyl group element with length smaller than $\ell(w)$. Pick a simple root α_i, such that $w > ws_i$. By induction,

$$T_{ws_i} x_{s_i\lambda} = x_{w\lambda} T_{ws_i} - \sum_{\substack{\alpha>0\\ ws_is_\alpha<ws_i}} \langle s_i\lambda, \alpha^\vee \rangle T_{ws_is_\alpha}.$$

Using the commutation of T_{s_i} and x_λ and the induction hypothesis we obtain

$$\begin{aligned} T_w x_\lambda - x_{w\lambda} T_w = \quad & T_{ws_i} x_{s_i\lambda} T_{s_i} - \langle \lambda, \alpha_i^\vee \rangle T_{ws_i} - x_{w\lambda} T_w \\ = & - \sum_{\substack{\alpha>0\\ ws_is_\alpha<ws_i}} \langle \lambda, s_i\alpha^\vee \rangle T_{ws_is_\alpha} T_{s_i} - \langle \lambda, \alpha_i^\vee \rangle T_{ws_i} = - \sum_{\substack{\alpha>0\\ ws_\alpha<w}} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha}. \end{aligned}$$

In the last equality, we have used that if $ws_i < w$ then

$$\{\alpha > 0 \mid ws_\alpha < w\} = \{\alpha > 0 \mid w\alpha < 0\} = \{\alpha_i\} \sqcup \{s_i\alpha \mid \alpha > 0, ws_is_\alpha < ws_i\},$$

and that $T_{ws_is_\alpha} T_{s_i} = T_{ws_is_\alpha s_i} = T_{ws_{s_i(\alpha)}}$. □

A.1.2. *The Hecke action on the equivariant cohomology.* Since G acts on G/P by left multiplication, there is a natural left Weyl group action on $H_T^*(G/P)$ for any partial flag variety G/P and which acts on the base ring $H_T^*(\mathrm{pt})$ by the usual Weyl group action; see [MNS22b], for example. For any $w \in W$, we use w^L to denote this action.

For any simple root α_i, define the left Demazure–Lusztig operator on $H_T^*(G/P)$ by the following formula (see [MNS22b, Section 3.2]):

$$\mathcal{T}_i^L := \frac{\alpha_i + 1}{\alpha_i} s_i^L - \frac{1}{\alpha_i}.$$

It is proved in *loc. cit.* that these operators satisfy the braid relations and $(\mathcal{T}_i^L)^2 = \mathrm{id}$. The following lemma is easily proved.

Lemma A.2. *There is an action Ψ of the degenerate affine Hecke algebra $\mathcal{H}$ on $H_T^*(G/P)$, sending T_i to $\mathcal{T}_i^L$ and x_λ to $\lambda \in H_T^*(\mathrm{pt})$.*

A.2. ***Definition of the CSM/SM classes.*** Next we recall the basic definitions and properties of CSM classes for the Schubert cells in G/P, we will be brief. We refer the reader to [Ohm06; AMSS23] for details, including a construction of these classes in the equivariant setting and for general varieties.

The (additive) group of constructible functions $\mathcal{F}(X)$ consists of functions $\varphi = \sum_Z c_Z \mathbb{1}_Z$, where the sum is over a finite set of constructible subsets $W \subset X$, $c_Z \in \mathbb{Z}$ are integers, and $\mathbb{1}_Z$ is the characteristic function of Z. For a proper morphism $f : Y \to X$, there is a linear map $f_* : \mathcal{F}(Y) \to \mathcal{F}(X)$, such that for any constructible subset $Z \subset Y$, $f_*(\mathbb{1}_Z)(x) = \chi_{top}(f^{-1}(x) \cap Z)$, where $x \in X$ and χ_{top} denotes the topological Euler characteristic. A conjecture attributed to Deligne and Grothendieck states that there is a unique natural transformation $c_* : \mathcal{F} \to H_*$ from the functor of constructible functions on a complex algebraic variety X to the homology functor, where all morphisms are proper, such that if X is smooth then $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$. This conjecture was proved by MacPherson [Mac74]; the class $c_*(\mathbb{1}_X)$ for possibly singular X was shown to coincide with a class defined earlier by M.-H. Schwartz [Sch65a; Sch65b; BS81].

There is an equivariant version of MacPherson's transformation defined by Ohmoto [Ohm06]. In this case one starts with a variety X with a T-action, and the equivariant version $\mathcal{F}^T(X)$ of the group of constructible functions $\mathcal{F}(X)$ contains the characteristic functions $\mathbb{1}_Z$ for Z stable under the T-action. If $f : X \to Y$ is a proper T-equivariant morphism of algebraic varieties the induced homomorphism and $Z \subset X$ is constructible and T-stable then one defines $f_*^T : \mathcal{F}^T(X) \to \mathcal{F}^T(Y)$ with the property that $f_*^T(\mathbb{1}_Z) = f_*(\mathbb{1}_Z)$. Ohmoto proves [Ohm06, Theorem 1.1] that there is an equivariant version of MacPherson transformation $c_*^T : \mathcal{F}^T(X) \to H_*^T(X)$ that satisfies $c_*^T(\mathbb{1}_X) = c^T(T_X) \cap [X]_T$ if X is a nonsingular T-variety, which and is functorial with respect to proper push-forwards. The last statement means that for all proper T-equivariant morphisms $Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^T(Y) & \xrightarrow{c_*^T} & H_*^T(Y) \\ \Big\downarrow f_*^T & & \Big\downarrow f_*^T \\ \mathcal{F}^T(X) & \xrightarrow{c_*^T} & H_*^T(X). \end{array}$$

Definition A.3. Let Z be a T-invariant constructible subvariety of X.

(1) We denote by $c_{\mathrm{SM}}(Z) := c_*^T(\mathbb{1}_Z) \in H_*^T(X)$ the *equivariant Chern–Schwartz–MacPherson* (*CSM*) *class* of Z.

(2) If X is smooth, we denote by $s_{\mathrm{M}}(Z \subset X) := c_*^T(\mathbb{1}_Z)/c(T_X) \in H_*^T(X)_{\mathrm{loc}}$ the *equivariant Segre–MacPherson* (*SM*) *class* of Z, where $H_*^T(X)_{\mathrm{loc}} := H_*^T(X) \otimes_{H_*^T(pt)} \mathrm{Frac}\, H_*^T(pt)$ denotes the localization of $H_*^T(X)$, and $\mathrm{Frac}\, H_*^T(pt)$ is the fraction field of $H_*^T(pt)$.

A.3. ***The Chevalley formula in cohomology.*** We now specialize to $X = G/P$ with the usual T-action. For simplicity we will denote by $s_{\mathrm{M}}(Z \subset G/P)$ simply by $s_{\mathrm{M}}(Z)$. We will identify the equivariant (Borel–Moore) homology group $H_*^T(G/P)$ with the equivariant cohomology $H_T^*(G/P)$, using the Poincaré duality. The sets of CSM classes of Schubert cells $\{c_{\mathrm{SM}}(X(wW_P)^\circ) \mid w \in W^P\}$ and of the SM classes $\{s_{\mathrm{M}}(X(wW_P)^\circ) \mid w \in W^P\}$ form bases for $H_T^*(G/P)_{\mathrm{loc}} := H_T^*(G/P) \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt)$. Moreover, if one takes the opposite Schubert cells in any of these sets, then the two bases are dual under the usual intersection pairing, see [AMSS23, Theorem 9.4]:

$$\langle c_{\mathrm{SM}}(X(wW_P)^\circ), s_{\mathrm{M}}(Y(uW_P)^\circ)\rangle_{G/P} = \delta_{w,u} \quad \text{for any } w, u \in W^P. \tag{47}$$

The left Demazure–Lusztig operator acts on the CSM classes by the following formula (see [MNS22b, Theorem 4.3])

$$\mathcal{T}_i^L(c_{\mathrm{SM}}(X(wW_P)^\circ)) = c_{\mathrm{SM}}(X(s_i w W_P)^\circ).$$

Hence, for any $w \in W$,

$$c_{\mathrm{SM}}(X(wW_P)^\circ) = \mathcal{T}_w^L([X(\mathrm{id})]). \tag{48}$$

Recall for any $\lambda \in X^*(T)_P$, $\mathcal{L}_\lambda := G \times^P \mathbb{C}_\lambda \in \operatorname{Pic}_T(G/P)$. The following is our main result in this appendix, and it has also been proved in [AMSS23, Theorem 9.10] using the Chevalley formula for the cohomological stable envelopes from [Su16, Theorem 3.7]. Here we give a direct proof based on the action of the degenerate affine Hecke algebra.

Theorem A.4. *For any $w \in W^P$ and $\lambda \in X^*(T)_P$, the following holds in $H_T^*(G/P)$:*

$$c_1^T(\mathcal{L}_\lambda) \cup c_{\mathrm{SM}}(X(wW_P)^\circ) = w(\lambda) c_{\mathrm{SM}}(X(wW_P)^\circ) - \sum_{\substack{\alpha>0\\ ws_\alpha<w}} \langle \lambda, \alpha^\vee\rangle c_{\mathrm{SM}}(X(ws_\alpha W_P)^\circ),$$

$$c_1^T(\mathcal{L}_\lambda) \cup s_{\mathrm{M}}(Y(wW_P)^\circ) = w(\lambda) s_{\mathrm{M}}(Y(wW_P)^\circ) - \sum_{\substack{\alpha>0\\ ws_\alpha>w}} \langle \lambda, \alpha^\vee\rangle s_{\mathrm{M}}(Y(ws_\alpha W_P)^\circ).$$

Proof. Applying the Hecke action Ψ in Lemma A.2 to the equation in Lemma A.1, and acting on the point class $[X(\mathrm{id})]$, we get

$$\begin{aligned} c_1^T(\mathcal{L}_\lambda) \cup c_{\mathrm{SM}}(X(wW_P)^\circ) &= c_1^T(\mathcal{L}_\lambda) \cup \mathcal{T}_w^L([X(\mathrm{id})]) = \mathcal{T}_w^L(c_1^T(\mathcal{L}_\lambda) \cup [X(\mathrm{id})]) = \mathcal{T}_w^L(\lambda \cdot [X(\mathrm{id})]) \\ &= \Psi(T_w x_\lambda)([X(\mathrm{id})]) = \Psi(x_{w\lambda} T_w - \sum_{\substack{\alpha>0\\ ws_\alpha<w}} \langle \lambda, \alpha^\vee\rangle T_{ws_\alpha})([X(\mathrm{id})]) \\ &= w(\lambda) c_{\mathrm{SM}}(X(wW_P)^\circ) - \sum_{\substack{\alpha>0\\ ws_\alpha<w}} \langle \lambda, \alpha^\vee\rangle c_{\mathrm{SM}}(X(ws_\alpha W_P)^\circ). \end{aligned}$$

The second equality follows from the fact that the left operator $\mathcal{T}_w^L$ commutes with $c_1^T(\mathcal{L}_\lambda)$ because the latter is Weyl-group invariant, as $\mathcal{L}_\lambda$ is a G-equivariant line bundle; see [MNS22b]. Finally, the Chevalley formula for the SM classes follows from the one on CSM via the duality in (47), similar to the proof of Lemma 5.1 above. □

Appendix B. An example of the λ-chain formula

We consider Lie type A_2, with the Weyl group $W = S_3$, with $\lambda = 2\varpi_1 + \varpi_2$, and $w = s_2 s_1$. We can find an alcove walk $p_{-\lambda}$ from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$ as indicated by the red path in Figure 2, right. This gives the corresponding reduced expression $v_{-\lambda} = s_2 s_1 s_2 s_0 s_1 s_2$ and the corresponding alcove path

$$\mathrm{A}_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} A_2 \xrightarrow{-\beta_3} A_3 \xrightarrow{-\beta_4} A_4 \xrightarrow{-\beta_5} A_5 \xrightarrow{-\beta_6} A_6 = \mathrm{A}_\circ - \lambda \ (A_i = r_i A_{i-1},\ 1 \le i \le 6),$$

with λ-chain $\beta_1 = \alpha_2,\ \beta_2 = \alpha_1 + \alpha_2,\ \beta_3 = \alpha_1,\ \beta_4 = \alpha_1 + \alpha_2,\ \beta_5 = \alpha_1,\ \beta_6 = \alpha_1 + \alpha_2$.

The associated sequence of hyperplanes is

$$h_1 = H_{\alpha_2,0}, h_2 = H_{\alpha_1+\alpha_2,0}, h_3 = H_{\alpha_1,0}, h_4 = H_{\alpha_1+\alpha_2,-1}, h_5 = H_{\alpha_1,-1}, h_6 = H_{\alpha_1+\alpha_2,-2}.$$

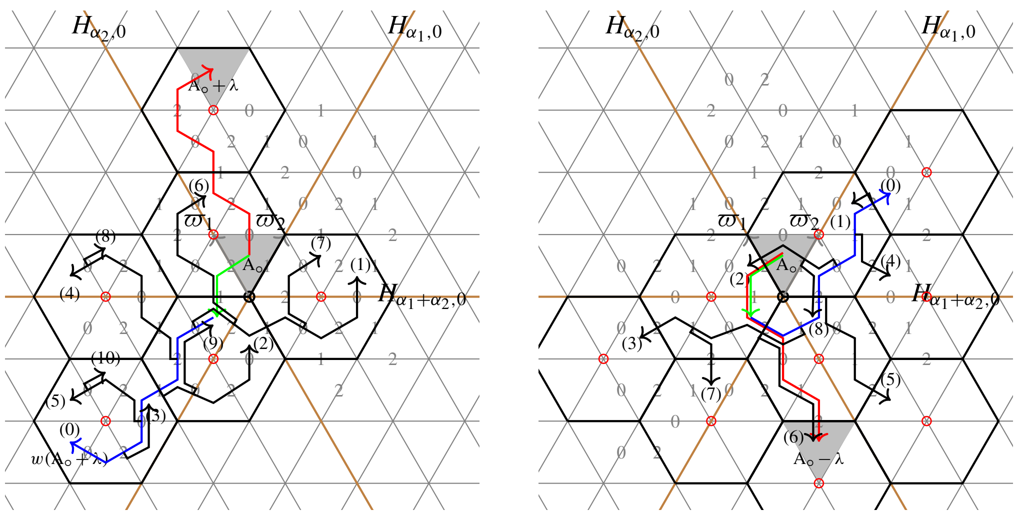


Figure 2. Left: alcove walk p_λ from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ + \lambda$; $p_w = c_2^- c_1^-$, $p_\lambda = c_0^+ c_2^+ c_1^+ c_0^+ c_2^+ c_0^+$. Right: alcove walk $p_{-\lambda}$ from $\mathrm{A}_\circ$ to $\mathrm{A}_\circ - \lambda$; $p_w = c_2^- c_1^-$, $p_{-\lambda} = c_2^- c_1^- c_2^- c_0^- c_1^- c_2^-$.

We first calculate $c_{u,\mu}^{w,\lambda}$ according to the formula Theorem 3.10. In the proof we introduced λ-shifted (reversed) hyperplane sequence which can be seen in the diagram on the left. We have

$$h'_1 = H_{\alpha_1+\alpha_2,1}, \quad h'_2 = H_{\alpha_1,1}, \quad h'_3 = H_{\alpha_1+\alpha_2,2}, \quad h'_4 = H_{\alpha_1,2}, \quad h'_5 = H_{\alpha_1+\alpha_2,3}, \quad h'_6 = H_{\alpha_2,1}.$$

According to the formula, we need to choose $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J_<} w$.

If $u = s_1$, $J = \{6\}, \{4\}, \{2\}$ as $s_{\alpha_1+\alpha_2} = s_1 s_2 s_1$. For the weight μ, when $J = \{6\}$, we need to calculate $\mu = w\tilde{r}_{h_6}(\lambda)$. But as is explained in the proof, $\tilde{r}_{h_6} = \hat{r}_{h'_1}$, so, $\mu = w\hat{r}_{h'_1}(\lambda) = w(-\varpi_2) = -\varpi_1 + \varpi_2$. Likewise when $J = \{4\}$, $\mu = w\tilde{r}_{h_4}(\lambda) = w\hat{r}_{h'_3}(\lambda) = -\varpi_2$, and when $J = \{2\}$, $\mu = w\tilde{r}_{h_2}(\lambda) = w\hat{r}_{h'_5}(\lambda) = \varpi_1 - 3\varpi_2$. If $u = \mathrm{id}$, there are two possible Bruhat chains $u = \mathrm{id} < s_1 < s_2 s_1 = w$ and $u = \mathrm{id} < s_2 < s_2 s_1 = w$. For the first case, $J = \{5, 6\}, \{3, 6\}, \{3, 4\}$, and for the second case, $J = \{1, 5\}, \{1, 3\}$. When $J = \{5, 6\}$, $\mu = w\tilde{r}_{h_6}\tilde{r}_{h_5}(\lambda) = w\hat{r}_{h'_1}\hat{r}_{h'_2}(\lambda) = w\hat{r}_{h'_1}(2\varpi_2) = w(-\varpi_1 + \varpi_2) = \varpi_1$. All the possibilities and the corresponding (folded) alcove walks $\tilde{p}_\lambda$ are listed in the table below, followed by the coefficients $c_{u,\mu}^{w,\lambda}$ obtained from it. (We can also see the bijection of Lemma 2.7; see Figure 2, left.)

$p_w \tilde{p}_\lambda$	$\tilde{p}_\lambda$	$\mathcal{M}$	J	$\varphi(p) = u$	$\mathrm{wt}(p) = w\tilde{r}^{\lambda}_{J_>}(\lambda)$
(0)	$c_0^- c_2^- c_1^- c_0^- c_2^- c_0^+$	$\{\}$	$\{\}$	$s_2 s_1$	$\varpi_1 - 3\varpi_2$
(1)	$f_0^+ c_2^- c_1^+ c_0^- c_2^+ c_0^+$	$\{h_1\}$	$\{6\}$	s_1	$-\varpi_1 + \varpi_2$
(2)	$c_0^- c_2^- f_1^+ c_0^- c_2^+ c_0^+$	$\{h_3\}$	$\{4\}$	s_1	$-\varpi_2$
(3)	$c_0^- c_2^- c_1^- c_0^- f_2^+ c_0^+$	$\{h_5\}$	$\{2\}$	s_1	$\varpi_1 - 3\varpi_2$
(4)	$c_0^- f_2^+ c_1^+ c_0^+ c_2^+ c_0^-$	$\{h_2\}$	$\{5\}$	s_2	$2\varpi_1 - 2\varpi_2$
(5)	$c_0^- c_2^- c_1^- f_0^+ c_2^+ c_0^-$	$\{h_4\}$	$\{3\}$	s_2	$\varpi_1 - 3\varpi_2$
(6)	$f_0^+ f_2^+ c_1^+ c_0^+ c_2^+ c_0^+$	$\{h_1, h_2\}$	$\{5, 6\}$	id	ϖ_1
(7)	$f_0^+ c_2^- c_1^+ f_0^+ c_2^+ c_0^+$	$\{h_1, h_4\}$	$\{3, 6\}$	id	$-\varpi_1 + \varpi_2$
(8)	$c_0^- f_2^+ c_1^+ c_0^+ c_2^+ f_0^+$	$\{h_2, h_6\}$	$\{1, 5\}$	id	$2\varpi_1 - 2\varpi_2$
(9)	$c_0^- c_2^- f_1^+ f_0^+ c_2^+ c_0^+$	$\{h_3, h_4\}$	$\{3, 4\}$	id	$-\varpi_2$
(10)	$c_0^- c_2^- c_1^- f_0^+ c_2^+ f_0^+$	$\{h_4, h_6\}$	$\{1, 3\}$	id	$\varpi_1 - 3\varpi_2$

$c_{s_2 s_1,\mu}^{w,\lambda} = 1$ for $\mu = \varpi_1 - 3\varpi_2$,
$c_{s_1,\mu}^{w,\lambda} = (q-1)q^{-1}$ for $\mu = -\varpi_1 + \varpi_2, -\varpi_2, \varpi_1 - 3\varpi_2$,
$c_{s_2,\mu}^{w,\lambda} = (q-1)q^{-1}$ for $\mu = 2\varpi_1 - 2\varpi_2, \varpi_1 - 3\varpi_2$,
$c_{\mathrm{id},\mu}^{w,\lambda} = (q-1)^2 q^{-2}$ for $\mu = \varpi_1, -\varpi_1 + \varpi_2, 2\varpi_1 - 2\varpi_2, -\varpi_2, \varpi_1 - 3\varpi_2$.

Next we calculate $c_{u,\mu}^{w,-\lambda}$ according to Theorem 3.9. For this case we need to chose $J \subset \{1, 2, \ldots, l\}$ such that $u \xrightarrow{J_>} w$.

For example, if $u = \mathrm{id}$, then there are three possible J corresponding to the Bruhat chains $u < us_{\beta_3} < us_{\beta_3} s_{\beta_2} = w$, $u < us_{\beta_5} < us_{\beta_5} s_{\beta_2} = w$, $u < us_{\beta_5} < us_{\beta_5} s_{\beta_4} = w$. For the first case the weight μ can be calculated (see Figure 2, right) as

$$\mu = w\hat{r}_{J_<}(-\lambda) = s_2 s_1 \hat{r}_{h_2} \hat{r}_{h_3}(-\lambda) = s_2 s_1 (3\varpi_1 - 2\varpi_2) = -2\varpi_1 - \varpi_2.$$

All the possible J and corresponding alcove walks $\tilde{p}_{-\lambda}$ are listed in the table below.

$p_w \tilde{p}_{-\lambda}$	$\tilde{p}_{-\lambda}$	$\mathcal{M}$	J	$\varphi(p) = u$	$wt(p) = w\hat{r}_{J_<}(-\lambda)$
(0)	$c_2^- c_1^+ c_2^+ c_0^+ c_1^+ c_2^+$	$\{\}$	$\{\}$	$s_2 s_1$	$-\varpi_1 + 3\varpi_2$
(1)	$c_2^- c_1^+ c_2^+ c_0^+ c_1^+ f_2^-$	$\{h_6\}$	$\{6\}$	s_1	ϖ_2
(2)	$c_2^- c_1^+ c_2^+ f_0^- c_1^+ c_2^-$	$\{h_4\}$	$\{4\}$	s_1	$\varpi_1 - \varpi_2$
(3)	$c_2^- f_1^- c_2^+ c_0^- c_1^+ c_2^-$	$\{h_2\}$	$\{2\}$	s_1	$2\varpi_1 - 3\varpi_2$
(4)	$c_2^- c_1^+ c_2^+ c_0^+ f_1^- c_2^-$	$\{h_5\}$	$\{5\}$	s_2	$-2\varpi_1 + 2\varpi_2$
(5)	$c_2^- c_1^+ f_2^- c_0^- c_1^- c_2^-$	$\{h_3\}$	$\{3\}$	s_2	$-3\varpi_1 + \varpi_2$
(6)	$c_2^- f_1^- f_2^- c_0^- c_1^- c_2^-$	$\{h_2, h_3\}$	$\{2, 3\}$	id	$-2\varpi_1 - \varpi_2$
(7)	$c_2^- f_1^- c_2^+ c_0^- f_1^- c_2^-$	$\{h_2, h_5\}$	$\{2, 5\}$	id	$-2\varpi_2$
(8)	$c_2^- c_1^+ c_2^+ f_0^- f_1^- c_2^-$	$\{h_4, h_5\}$	$\{4, 5\}$	id	$-\varpi_1$

The coefficients $c_{u,\mu}^{w,-\lambda}$ obtained from the table are

$$\begin{aligned}
&c_{s_2s_1,\mu}^{w,-\lambda} = 1 \text{ for } \mu = -\varpi_1 + 3\varpi_2,\\
&c_{s_1,\mu}^{w,-\lambda} = (1-q)q^{-1} \text{ for } \mu = \varpi_2,\ \varpi_1 - \varpi_2,\ 2\varpi_1 - 3\varpi_2,\\
&c_{s_2,\mu}^{w,-\lambda} = (1-q)q^{-1} \text{ for } \mu = -2\varpi_1 + 2\varpi_2,\ -3\varpi_1 + \varpi_2,\\
&c_{\mathrm{id},\mu}^{w,-\lambda} = (1-q)^2 q^{-2} \text{ for } \mu = -2\varpi_1 - \varpi_2,\ -2\varpi_2,\ -\varpi_1.
\end{aligned}$$

Acknowledgments

H. Naruse and C. Su thank Arun Ram and Andrei Okounkov for answering their questions and providing references; L. C. Mihalcea and Su thank Bogdan Ion for a related collaboration on Hall–Littlewood polynomials, and Paolo Aluffi and Jörg Schürmann for previous collaborations on motivic Chern classes and CSM classes; Mihalcea thanks Andrzej Weber for stimulating discussions.

This project started in Spring 2021 while Mihalcea was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Spring 2021 semester in Combinatorial Algebraic Geometry. He warmly thanks the institute for the excellent working conditions. Mihalcea was also partially supported by NSF grant DMS-2152294 and he gratefully acknowledges the support of Charles Simonyi Endowment, which provided funding for the membership at the Institute of Advanced Study during the 2024–25 Special Year in Algebraic and Geometric Combinatorics. C. Su acknowledges support from National Key R&D Program of China (no. 2024YFA1014700).

References

[AGM11] D. Anderson, S. Griffeth, and E. Miller, "Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces", *J. Eur. Math. Soc.* **13**:1 (2011), 57–84. MR

[AMSS23] P. Aluffi, L. C. Mihalcea, J. Schürmann, and C. Su, "Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells", *Duke Math. J.* **172**:17 (2023), 3257–3320. MR

[AMSS24a] P. Aluffi, L. C. Mihalcea, J. Schürmann, and C. Su, "Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman's problem", *Ann. Sci. Éc. Norm. Supér.* (4) **57**:1 (2024), 87–141. MR

[AMSS24b] P. Aluffi, L. C. Mihalcea, J. C. Schürmann, and C. C. Su, "From motivic Chern classes of Schubert cells to their Hirzebruch and CSM classes", pp. 1–52 in *A glimpse into geometric representation theory*, Contemp. Math. **804**, Amer. Math. Soc., 2024. MR

[AO21] M. Aganagic and A. Okounkov, "Elliptic stable envelopes", *J. Amer. Math. Soc.* **34**:1 (2021), 79–133.

[BBBG24] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson, "Colored vertex models and Iwahori Whittaker functions", *Selecta Math.* (*N.S.*) **30**:4 (2024), Paper No. 78, 58. MR

[BCMP18] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin, "A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties", *Algebr. Geom.* **5**:5 (2018), 568–595. MR

[BM15] A. S. Buch and L. C. Mihalcea, "Curve neighborhoods of Schubert varieties", *J. Differential Geom.* **99**:2 (2015), 255–283. MR

[Bri02] M. Brion, "Positivity in the Grothendieck group of complex flag varieties", *J. Algebra* **258**:1 (2002), 137–159. MR

[BS81] J.-P. Brasselet and M.-H. Schwartz, "Sur les classes de Chern d'un ensemble analytique complexe", pp. 93–147 in *The Euler–Poincaré characteristic* (*French*), Astérisque **83**, Soc. Math. France, Paris, 1981.

[BSY10] J.-P. Brasselet, J. Schürmann, and S. Yokura, "Hirzebruch classes and motivic Chern classes for singular spaces", *J. Topol. Anal.* **2**:1 (2010), 1–55. MR

[Buc02] A. S. Buch, “A Littlewood–Richardson rule for the K-theory of Grassmannians”, *Acta Math.* **189**:1 (2002), 37–78. MR

[CG10] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997. MR

[CS80] W. Casselman and J. Shalika, “The unramified principal series of p-adic groups, II: The Whittaker function”, *Compos. Math.* **41**:2 (1980), 207–231.

[EM97] S. Evens and I. Mirković, “Fourier transform and the Iwahori–Matsumoto involution”, *Duke Math. J.* **86**:3 (1997), 435–464. MR

[FGSX24] N. J. Fan, P. L. Guo, C. Su, and R. Xiong, “A Pieri type formula for motivic Chern classes of Schubert cells in Grassmannians”, preprint, 2024. arXiv 2402.04500

[FGX22] N. J. Y. Fan, P. L. Guo, and R. Xiong, “Pieri and Murnaghan–Nakayama type Rules for Chern classes of Schubert Cells”, preprint, 2022. arXiv 2211.06802

[FRW21] L. M. Feher, R. Rimanyi, and A. Weber, “Motivic Chern classes and K-theoretic stable envelopes”, *Proc. L. Math. Soc.* **122**:1 (2021), 153–189.

[GR04] S. Griffeth and A. Ram, “Affine Hecke algebras and the Schubert calculus”, *European J. Combin.* **25**:8 (2004), 1263–1283. MR

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1990. MR

[KL87] D. Kazhdan and G. Lusztig, “Proof of the Deligne–Langlands conjecture for Hecke algebras”, *Inv. Math.* **87**:1 (1987), 153–215.

[Kum02] S. Kumar, *Kac–Moody groups, their flag varieties and representation theory*, Progress in Mathematics **204**, Birkhäuser, Boston, 2002. MR

[KW25] J. Koncki and A. Weber, “Hecke algebra action on twisted motivic Chern classes and K-theoretic stable envelopes”, *Math. Ann.* **391**:2 (2025), 1899–1964. MR

[Len11] C. Lenart, “Hall–Littlewood polynomials, alcove walks, and fillings of Young diagrams”, *Disc. Math.* **311**:4 (2011), 258–275.

[Li92] J.-S. Li, “Some results on the unramified principal series of p-adic groups”, *Math. Ann.* **292**:1 (1992), 747–761.

[LP07] C. Lenart and A. Postnikov, “Affine Weyl groups in K-theory and representation theory”, *Int. Math. Res. Not.* **2007**:12 (2007), art. id. rnm038, 65 pp. MR

[LP08] C. Lenart and A. Postnikov, “A combinatorial model for crystals of Kac–Moody algebras”, *Trans. Amer. Math. Soc.* **360**:8 (2008), 4349–4381.

[Mac74] R. D. MacPherson, “Chern classes for singular algebraic varieties”, *Ann. of Math.* (2) **100** (1974), 423–432.

[Mac96] I. G. Macdonald, “Affine Hecke algebras and orthogonal polynomials”, pp. exposé 797, pp. 189–207 in *Séminaire Bourbaki*, *volume* 1994/1995, Astérisque **237**, Société mathématique de France, 1996.

[Mac98] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford university press, 1998.

[MNS22a] L. C. Mihalcea, H. Naruse, and C. Su, “Hook formulae from Segre–MacPherson classes”, preprint, 2022. To appear in *Alg. Comb.* arXiv 2203.16461

[MNS22b] L. C. Mihalcea, H. Naruse, and C. Su, “Left Demazure–Lusztig operators on equivariant (quantum) cohomology and K-theory”, *Int. Math. Res. Not.* **2022**:16 (2022), 12096–12147. MR

[MNS24] L. C. Mihalcea, H. Naruse, and C. Su, “Hook formula for Coxeter groups via the twisted group ring”, *Proc. Japan Acad. Ser. A Math. Sci.* **100**:6 (2024), 31–36. MR

[MO19] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, Astérisque **408**, Soc. Math. de France, 2019.

[MS22] L. C. Mihalcea and C. Su, “Whittaker functions from motivic Chern classes”, *Transform. Groups* **27**:3 (2022), 1045–1067. MR

[Nie74] H. A. Nielsen, “Diagonalizably linearized coherent sheaves”, *Bull. Soc. Math. France* **102** (1974), 85–97. MR

[NO19] H. Naruse and S. Okada, “Skew hook formula for d-complete posets via equivariant K-theory”, *Alg. Combin.* **2**:4 (2019), 541–571.

[Ohm06] T. Ohmoto, "Equivariant Chern classes of singular algebraic varieties with group actions", *Math. Proc. Cambridge Philos. Soc.* **140**:1 (2006), 115–134.

[Oko27] A. Okounkov, "Lectures on K-theoretic computations in enumerative geometry", pp. 251–380 in *Geometry of moduli spaces and representation theory*, IAS/Park City Math. Ser. **24**, Amer. Math. Soc., 2017. MR

[OS22] A. Okounkov and A. Smirnov, "Quantum difference equation for Nakajima varieties", *Inv. Math.* **229**:3 (2022), 1203–1299.

[PR99] H. Pittie and A. Ram, "A Pieri–Chevalley formula in the K-theory of a G/B-bundle", *Electron. Res. Announc. Amer. Math. Soc.* **5** (1999), 102–107. MR

[Pro99] R. A. Proctor, "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets", *J. Algebraic Combin.* **9**:1 (1999), 61–94. MR

[Ram06] A. Ram, "Alcove Walks, Hecke Algebras, Spherical Functions, Crystals and Column Strict Tableaux", *Pure Appl. Math. Q.* **2**:4 (2006), 963–1013.

[RTV15] R. Rimányi, V. Tarasov, and A. Varchenko, "Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety", *J. Geom. Phys.* **94** (2015), 81–119.

[RTV19] R. Rimányi, V. Tarasov, and A. Varchenko, "Elliptic and K-theoretic stable envelopes and Newton polytopes", *Selecta Math.* **25**:1 (2019), 16.

[Sch06] C. Schwer, "Galleries, Hall–Littlewood polynomials, and structure constants of the spherical Hecke algebra", *Int. Math. Res. Notices* **2006**:9 (2006), 75395–75395.

[Sch65a] M.-H. Schwartz, "Classes caractéristiques définies par une stratification d'une variété analytique complexe, I", *C. R. Math. Acad. Sci. Paris* **260** (1965), 3262–3264.

[Sch65b] M.-H. Schwartz, "Classes caractéristiques définies par une stratification d'une variété analytique complexe, II", *C. R. Math. Acad. Sci. Paris* **260** (1965), 3535–3537.

[Ste01] J. R. Stembridge, "Minuscule elements of Weyl groups", *J. Algebra* **235**:2 (2001), 722–743.

[Su16] C. Su, "Equivariant quantum cohomology of cotangent bundle of G/P", *Adv. Math.* **289** (2016), 362–383. MR

[SZZ20] C. Su, G. Zhao, and C. Zhong, "On the K-theory stable bases of the Springer resolution", *Ann. Sci. Éc. Norm. Supér.* (4) **53**:3 (2020), 663–711. MR

[SZZ21] C. Su, G. Zhao, and C. Zhong, "Wall-crossings and a categorification of K-theory stable bases of the Springer resolution", *Compos. Math.* **157**:11 (2021), 2341–2376.

Communicated by Sergey Fomin
Received 2024-02-12 Revised 2024-03-06 Accepted 2025-03-03

lmihalce@vt.edu *Department of Mathematics, Virginia Tech University, Blacksburg, VA, United States*

hnaruse@yamanashi.ac.jp *Graduate School of Education, University of Yamanashi, Kofu, Japan*

changjiansu@mail.tsinghua.edu.cn *Yau Mathematical Sciences Center, Tsinghua University, Beijing, China*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2026 is US $590/year for the electronic version, and $865/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 20 No. 3 2026