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The Mahler measure of exact polynomials in three variables

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We prove that under certain explicit conditions, the Mahler measure of a three-variable polynomial can be expressed in terms of elliptic curve L-values and Bloch–Wigner dilogarithmic values, conditionally on Beilinson's conjecture. In some cases, these dilogarithmic values simplify to Dirichlet L-values. The proof involves a construction of an element in $K_4^{(3)}$ of a smooth projective curve over a number field. This generalizes a result of Lalín (2015) for the polynomial $z+(x+1)(y+1)$. We apply our method to several other Mahler measure identities conjectured by Boyd and Brunault.

Introduction

Let $P(x_1,\dots,x_n) \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be a nonzero Laurent polynomial. The (logarithmic) Mahler measure of P is defined by

$$\mathrm{m}(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1,\dots,x_n)| \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}, \tag{0-1}$$

where $\mathbb{T}^n : |x_1| = \cdots = |x_n| = 1$ is the n-dimensional torus. This quantity was firstly introduced by Mahler [27] in 1962.

In 1997, Deninger [12] linked the Mahler measure of polynomials $P(x_1,\dots,x_n)$ under certain conditions to the motivic cohomology of V_P, where V_P is the zero locus of P in $\mathbb{C}^n$. This allowed him to place the Mahler measure in the very general framework of Beilinson's conjectures on special values of L-functions. More precisely, Deninger defined the chain

$$\Gamma = \big\{(x_1,\dots,x_n) \in \mathbb{C}^n : P(x_1,\dots,x_n) = 0, |x_1| = \cdots = |x_{n-1}| = 1, |x_n| \geq 1\big\}.$$

He showed that if Γ is contained in the regular locus V_P^{reg} of V_P, then there is a differential $(n-1)$-form $\eta(x_1,\dots,x_n)$ on $\mathbb{G}_m^n$ such that its restriction to V_P represents the regulator of the Milnor symbol $\{x_1,\dots,x_n\}$, and we have

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + \frac{(-1)^{n-1}}{(2\pi i)^{n-1}} \int_\Gamma \eta(x_1,\dots,x_n), \tag{0-2}$$

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where $\tilde{P}$ is the leading coefficient of P seen as a polynomial in x_n.

From now on we assume that P has rational coefficients and Γ is contained in V_P^{reg}. If $\partial\Gamma = \varnothing$, then Γ is a cycle. Then Deninger found out that in certain situations, identity (0-2) together with Beilinson's conjecture imply that $\mathrm{m}(P)$ can be expressed in terms of the L-function of the motive $H^{n-1}(\overline{V}_P)$, where $\overline{V}_P$ is a smooth compactification of V_P. As an example, he showed that under the Beilinson conjecture,

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \overset{?}{=} L'(E_{15}, 0), \tag{0-3}$$

where E_{15} is the elliptic curve (of conductor 15) defined by $x + 1/x + y + 1/y + 1 = 0$. In this example, $\partial\Gamma \neq \varnothing$ but a symmetry argument reduces this to the case $\partial\Gamma = \varnothing$. It was completely shown (without assuming the Beilinson conjecture) by Rogers and Zudilin [33] in 2014.

Boyd [3] conjectured, based on numerical evidence, that

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \overset{?}{=} r_k\, L'(E_{N(k)}, 0), \tag{0-4}$$

where $k \in \mathbb{Z} \setminus \{0, \pm 4\}$, $r_k \in \mathbb{Q}^\times$ and $E_{N(k)}$ is the elliptic curve (of conductor $N(k)$) obtained as a smooth compactification of the zero set of $P_k = x + 1/x + y + 1/y + k$. Until now, identity (0-4) is only proved for a finite number of k:

$$k \in \{-4\sqrt{2}, -2\sqrt{2}, 1, 2, 3, 2\sqrt{2}, 3\sqrt{2}, 5, 8, 12, 16, i, 2i, 3i, 4i, \sqrt{2}i\},$$

by the works of Brunault, Lalín, Rodriguez-Villegas, Rogers, Samart, and Zudilin (see [5; 22; 26; 25; 36; 32; 33; 39]).

The case $\partial\Gamma \neq \varnothing$ is more difficult, Maillot [28] suggested we should look at the variety

$$W_P := V_P \cap V_{P^*}, \tag{0-5}$$

where $P^*(x_1, \ldots, x_n) = \bar{P}(x_1^{-1}, \ldots, x_n^{-1})$. We call W_P the *Maillot variety*. If P is an *exact* polynomial, i.e., $\eta = d\omega$, where ω is a differential form on V_P^{reg}, then Stokes' theorem gives

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + \frac{(-1)^{n-1}}{(2\pi i)^{n-1}} \int_{\partial\Gamma} \omega.$$

Moreover, $\partial\Gamma$ is contained in W_P, hence we can hope that $\mathrm{m}(P)$ is related to the cohomology of W_P. In this direction, Lalín showed the following result.

Theorem 0.1 [23, Theorem 2]. *Assume that $P \in \mathbb{Q}[x, y, z]$ is irreducible and satisfies the following conditions*:

(i) *W_P is birationally equivalent to an elliptic curve E over $\mathbb{Q}$.*

(ii) *$\partial\Gamma$ defines an element of $H_1(E(\mathbb{C}), \mathbb{Z})^+$, the invariant part of $H_1(E(\mathbb{C}), \mathbb{Z})$ under the action induced by the complex conjugation on $E(\mathbb{C})$.*

(iii) *$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j$ in $\bigwedge^3 \mathbb{Q}(V_P)^\times$, for some functions $f_j, g_j \in \mathbb{Q}(V_P)^\times$.*

(iv) $x \wedge y \wedge z = 0$ *in* $\bigwedge^3 \mathbb{Q}(E)^\times$.

(v) $\sum_j v_p(g_j)\{f_j(p)\}_2 = 0$ *in* $\mathbb{Z}[\mathbb{P}^1_{\bar{\mathbb{Q}}}]/R_2(\bar{\mathbb{Q}})$ *for all* $p \in E(\bar{\mathbb{Q}})$.

Here $R_2(\bar{\mathbb{Q}})$ *is the subgroup generated by the five-term relations* (2-1), *and* $v_p(g_j)$ *is the vanishing order at* p *of* g_j *seen as a function on* E. *Then under Beilinson's conjecture*,

$$\mathrm{m}(P) = m(\tilde{P}) + a \cdot L'(E, -1), \ a \in \mathbb{Q}. \tag{0-6}$$

Condition (3) implies that P is exact (see Remark 4.3). In this article, we relax Lalín's conditions in order to deal with Mahler measure identities which are more general than (0-6), for example, containing also Dirichlet L-values. We only assume that W_P is of genus 1 and we do not require the conditions (iv)-(v) above. Recall that the Bloch–Wigner dilogarithm function $D : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ is defined by

$$D(z) = \begin{cases} \mathrm{Im}\big(\sum_{k=1}^{\infty} z^k/k^2\big) + \arg(1-z)\log|z| & (|z| \le 1), \\ -D(1/z) & (|z| \ge 1). \end{cases} \tag{0-7}$$

For any field F, we denote by $\mathcal{B}(F)$ the Bloch group of F tensored with $\mathbb{Q}$ (see [38, Section 2]). Let τ be the involution of $\mathbb{G}_m^3$ given by $(x, y, z) \mapsto (1/x, 1/y, 1/z)$. Since P has rational coefficients, τ induces an involution of W_P. For A an abelian group, we write $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. Let us state our main theorem here.

Theorem 0.2. *Assume* $P \in \mathbb{Q}[x, y, z]$ *is irreducible and that* W_P *is a curve of genus* 1. *Let* C *be the normalization of* W_P. *Suppose that*

$$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j \quad \text{in } \bigwedge^3 \mathbb{Q}(V_P)^\times_{\mathbb{Q}}, \tag{0-8}$$

for some functions f_j, g_j *on* V_P. *Let* S *be the closed subscheme of* C *consisting of the zeros and poles of the functions* g_j *and* $g_j \circ \tau$ *on* C *for all* j. *Then, for* $p \in S$,

$$u_p := \sum_j v_p(g_j)\{f_j(p)\}_2 + v_p(g_j \circ \tau)\{f_j \circ \tau(p)\}_2$$

define elements in the Bloch group $\mathcal{B}(\mathbb{Q}(p))$, *where* $\mathbb{Q}(p)$ *is the residue field of* C *at* p.

Assume that the Deninger chain Γ *is contained in* V_P^{reg} *and that its boundary* $\partial\Gamma$ *is contained in* W_P^{reg}, *then* $\partial\Gamma$ *defines an element in* $H_1(C(\mathbb{C}), \mathbb{Z})^+$. *If* $u_p = 0$ *for all* $p \in S$, *then under Beilinson's conjecture for the curve* C (Conjecture 1.11), *we have*

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + a \cdot L'(E, -1) \qquad (a \in \mathbb{Q}),$$

where E *is the Jacobian of* C. *Otherwise, let* S' *be the subset of* S *consisting of the points* p *such that* $u_p \neq 0$. *Let* K *be the splitting field of* S' *in* $\mathbb{C}$. *Let* $\mathcal{O}$ *be any fixed point in* $S'(K)$. *Assume that the difference of any two geometric points in* $S'(K)$ *has finite order dividing* N *in* $E(K)$, *then under Beilinson's conjecture for the curve* C (Conjecture 1.11),

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + a \cdot L'(E, -1) + \frac{1}{4N\pi} \sum_{q \in S'(K) \setminus \{\mathcal{O}\}} b_q \cdot D(u_q) \qquad (a \in \mathbb{Q}, b_q \in \mathbb{Z}), \tag{0-9}$$

where for $q \in S'(K)$ supported on a closed point $p \in S'$, we define u_q to be the image of u_p under the map $\mathcal{B}(\mathbb{Q}(p)) \hookrightarrow \mathcal{B}(K)$ induced by the embedding $\mathbb{Q}(p) \hookrightarrow^q K$.

The rational number a comes from Beilinson's conjecture, and does not depend on the choice of $\mathcal{O}$, but the integer numbers b_q actually do. However, the D-values in identity (0-9) are independent of the choice of $\mathcal{O}$. Indeed, when we remove $\mathcal{O}$ from the sum, the complex conjugation of $\mathcal{O}$ maintains the D-values in identity (0-9). Now let us use Theorem 0.2 to investigate several conjectural Mahler measure identities of the following types:

(a) *Pure identities*: $\mathrm{m}(P) \stackrel{?}{=} a \cdot L'(E,-1)$ for some $a \in \mathbb{Q}^\times$. Table 1 consists of pure identities that we prove up to a rational factor and conditionally on Beilinson's conjecture. Identity 3 in Table 1 was studied by Lalín [23]. By contrast, Theorem 0.1 of Lalín does not apply to identity 5 in Table 1 as it violates condition (v). Identities 6, 7 and 8 are conjectured by Lalín and Nair in [24], in fact, they showed that by some changes of variables, the Mahler measure of polynomials 5, 6, 7 and 8 are equal. Moreover, from Table 1, we have the following relations (under the Beilinson conjecture):

$$\mathrm{m}((1+x)(1+y)(x+y)+z) \stackrel{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}(1+x+y+z+xy+xz+yz),$$
$$\mathrm{m}(1+(x+1)y+(x-1)z) \stackrel{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}((x+1)^2(y+1)+z).$$

We also give in Section 5.1(g) an example showing that Theorem 0.2 does not imply that

$$\mathrm{m}((1+x)(1+y)+(1-x-y)z) = \tfrac{1}{288} L'(E_{450c1},-1). \tag{0-10}$$

	P	E	a	reference
1	$(1+x)(1+y)(x+y)+z$	$14a4$	-3	[9, p. 81]
2	$1+x+y+z+xy+xz+yz$	$14a4$	$-\frac{5}{2}$	[6]
3	$(x+1)(y+1)+z$	15a8	-2	[4]
4	$(x+1)^2+(1-x)(y+z)$	$20a1$	-2	[4], [9, p. 81]
5	$1+(x+1)y+(x-1)z$	$21a1$	$-\frac{5}{4}$	[4]
6	$\frac{1}{2}(x+2)+(x^2+x+1)y+(x^2-1)z$			[24]
7	$\frac{1}{2}(x^2-2x+2)+(x^4-x^3+x^2-x+1)y+(x^4-x^3+x-1)z$			[24]
8	$\frac{1}{2}(x^4+x+2)+(x^5+x^4+x+1)y+(x^5-1)z$			[24]
9	$(x+1)^2(y+1)+z$	$21a4$	$-\frac{3}{2}$	[4], [9, p. 81]
10	$(1+x)^2+y+z$	$24a4$	-1	[4]
11	$1+x+y+z+xy+xz+yz-xyz$	$36a1$	$-\frac{1}{2}$	[6]
12	$1+xy+(1+x+y)z$	$90b1$	$-\frac{1}{20}$	[6]
13	$(x+1)^2+(x-1)^2y+z$	$225c2$	$-\frac{1}{48}$	[4; 6]

Table 1. Pure identities $\mathrm{m}(P) \stackrel{?}{=} a \cdot L'(E,-1)$.

	P	E	a	b_1	b_2
1	$1+(x^2-x+1)y+(x^2+x+1)z$	$45a2$	$-\frac{1}{6}$	1	0
2	$x^2+1+(x+1)^2y+(x^2-1)z$	$48a1$	$-\frac{1}{10}$	0	1

Table 2. Conjectural identities $\mathrm{m}(P) \stackrel{?}{=} a\cdot L'(E,-1)+b_1\cdot L'(\chi_{-3},-1)+b_2\cdot L'(\chi_{-4},-1)$. The reference for all entries is [6].

(b) *Identities with Dirichlet L-values*:

$$\mathrm{m}(P(x,y,z)) \stackrel{?}{=} a\cdot L'(E,-1)+\sum_{\chi} b_\chi \cdot L'(\chi,-1) \qquad (a\in\mathbb{Q}, b_\chi \in \mathbb{Q}^\times), \tag{0-11}$$

where χ are odd quadratic Dirichlet characters. In cases where the coefficient a is nonzero, such identities are referred to as *mixed identities*. Table 2 consists of mixed identities we prove (up to rational factors) under Beilinson's conjecture for genus 1 curves (see Sections 5.2(a) and 5.2(e)). The first polynomial in Table 2 does not satisfy conditions (iv)–(v) of Theorem 0.1 of Lalín. Moreover, the Maillot variety W_P is a curve of genus 1 and does not have any rational point, hence violates condition (i) also. For the second polynomial in Table 2, the Maillot variety W_P is a union of a line and a nonsingular curve of genus 1. We also give an example to which our theorem does not apply (see Section 5.2(d)):

$$\mathrm{m}(x^2+x+1+(x^2+x+1)y+(x-1)^2z) = -\tfrac{1}{12}L'(E_{72a1},-1)+\tfrac{3}{2}L'(\chi_{-3},-1). \tag{0-12}$$

This was conjectured by Brunault [6].

By a method of Lalín in [23, Example 4.2], we prove without assuming Beilinson's conjecture the Mahler measure identities in Table 3; they involve only Dirichlet L-values (see Section 5.2(b)–(c)).

Outline. The article contains five sections. In the first three, we recall some tools that needed for our constructions. In Sections 1.1–1.4, we recall the definitions of the Deligne cohomology and some facts about motivic cohomology. We give an explicit isomorphism between Chow motives of a genus 1 smooth

	P	b_1	b_2
1	$1+(x+1)(x^2+x+1)y+(x+1)^3z$	3	0
2	$x^2+1+(x^2+x+1)y+(x+1)^3z$	$\frac{7}{2}$	0
3	$x^2+1+(x+1)(x^2+x+1)y+(x+1)^3z$	$\frac{7}{2}$	0
4	$x^2+1+(x+1)(x^2+x+1)y+(x-1)(x^2-x+1)z$	0	$\frac{7}{3}$
5	$(x+1)(x^2+1)+(x+1)(x^2+x+1)y+(x-1)(x^2-x+1)z$	0	$\frac{7}{3}$
6	$x^2+1+(x+1)^2y+(x-1)^2z$	0	2
7	$x^2+1+(x+1)^3y+(x-1)^3z$	0	3
8	$(x+1)(x^2+1)+(x+1)^3y+(x-1)^3z$	0	3

Table 3. $\mathrm{m}(P) = b_1\cdot L'(\chi_{-3},-1)+b_2\cdot L'(\chi_{-4},-1)$. The reference for all entries is [6].

projective curve and its Jacobian. We then recall briefly the regulator maps and Beilinson's conjecture for smooth projective curves of genus 1. In Section 2, we bring back Goncharov's polylogarithmic complexes and his regulator maps on the cohomology of these complexes. In Section 3, we recall de Jeu's polylogarithmic complexes and discuss his maps connecting the cohomology of Goncharov's polylogarithmic complexes to the motivic cohomology. In particular, in Section 3.4, given a 2-cocycle in weight 3 Goncharov's polylogarithmic complex, we compare its Goncharov regulator and the Beilinson regulator of its image under the map defined in Section 3.2. In Section 4.1, given an exact polynomial P in $\mathbb{Q}[x, y, z]$, we construct an element in the Deligne cohomology of an open subset of the normalization C of the Maillot variety W_P (0-5). We then relate this element to the Mahler measure of P in Section 4.2. In Section 4.3, we construct explicitly an element in $K_4^{(3)}(C)$ satisfying that its Beilinson regulator has connection with the Deligne cohomology element constructed in Section 4.1. We then prove the main theorem in Section 4.4. We end the article with Section 5, where we study the conjectural Mahler measure identities mentioned above.

1. The Beilinson regulator map

1.1. ***Deligne cohomology.*** Deligne cohomology of a smooth complex algebraic variety X is firstly introduced by Deligne in 1972, it is given by the hypercohomology of

$$0 \to \mathbb{Z}(n) \to \mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^{n-1} \to 0, \tag{1-1}$$

where the constant sheaf $\mathbb{Z}(n) := (2\pi i)^n\mathbb{Z}$ is placed in degree 0 and Ω_X^j is the sheaf of holomorphic j-forms on X placed in degree $j+1$. Burgos [11] then showed that this hypercohomology can be the cohomology of a single complex. Let us recall briefly Burgos' construction (see [11; 9]).

Let X be a smooth complex algebraic variety of dimension d. Let $(\bar{X}, \iota)$ be a good compactification of X, i.e., that $\bar{X}$ is a smooth proper variety and $\iota : X \hookrightarrow \bar{X}$ is an open immersion such that $D := \bar{X} - \iota(X)$ is locally given by $z_1 \ldots z_m = 0$ for some analytic local coordinates $z_1, \ldots, z_d$ on $\bar{X}$ and $m \leq d$.

Definition 1.1 [11, Proposition 1.1]. A complex smooth differential form ω on X *has logarithmic singularities along D* if locally ω belongs to the algebra generated by the smooth forms on $\bar{X}$ and $\log|z_i|$, dz_i/z_i, $d\bar{z}_i/\bar{z}_i$, for $1 \leq i \leq m$, where $z_1 \ldots z_m = 0$ is a local equation of D. For $\Lambda \in \{\mathbb{R}, \mathbb{C}\}$, $E^n_{X,\Lambda}(\log D)$ denotes the space of such Λ-valued smooth differential forms of degree n on X.

We have $E^n_{X,\mathbb{C}}(\log D) = \bigoplus_{p+q=n} E^{p,q}_{X,\mathbb{C}}(\log D)$, where $E^{p,q}$ is the subspace of type (p, q)-forms. We denote by $\bar{\partial} : E^{p,q} \to E^{p,q+1}$ and $\partial : E^{p,q} \to E^{p+1,q}$ as the usual operators and $d = \partial + \bar{\partial}$. Burgos defined

$$E^*_{\log,\Lambda}(X) = \varinjlim_{(\bar{X},\iota)\in\mathcal{I}^{\mathrm{opp}}} E^*_{X,\Lambda}(\log D),$$

where $\mathcal{I}$ is the category of good compactification of X. He then introduced the following complex.

Definition 1.2 [11, Theorem 2.6]. For any integers $j, n \geq 0$, set

$$E_j(X)^n := \begin{cases} (2\pi i)^{j-1} E^{n-1}_{\log,\mathbb{R}}(X) \cap \left(\bigoplus_{p+q=n-1;\, p,q<j} E^{p,q}_{\log,\mathbb{C}}(X)\right) & \text{if } n \leq 2j-1, \\ (2\pi i)^{j} E^{n}_{\log,\mathbb{R}}(X) \cap \left(\bigoplus_{p+q=n;\, p,q\geq j} E^{p,q}_{\log,\mathbb{C}}(X)\right) & \text{if } n \geq 2j, \end{cases}$$

$$d^n\omega := \begin{cases} -\mathrm{pr}_j(d\omega) & \text{if } n < 2j-1, \\ -2\partial\bar{\partial}\omega & \text{if } n = 2j-1, \\ d\omega & \text{if } n \geq 2j, \end{cases}$$

where pr_j is the projection $\bigoplus_{p,q} \to \bigoplus_{p,q<j}$. Denote by $E_j(X)$ the complex $(E_j(X)^n, d^n)_{n\geq 0}$.

Definition 1.3 (Deligne cohomology [11, Corollary 2.7]). Let X be a smooth complex algebraic variety. The Deligne cohomology of X is the cohomology of the complex $E_j(X)$, that is,

$$H^n_{\mathcal{D}}(X, \mathbb{R}(j)) = H^n(E_j(X)) \quad \text{for } j, n \geq 0.$$

As the canonical map $E^*_{X,\mathbb{C}}(\log D) \to E^*_{\log,\mathbb{C}}(X)$ is a quasi-isomorphism by [10, Theorem 1.2], we can use $E^*_{X,\Lambda}(\log D)$ for some good compactification of X instead of $E^*_{\log,\Lambda}(X)$ in Definition 1.2.

Remark 1.4. For the case $j > \dim X \geq 1$ or $j > n$, $H^n_{\mathcal{D}}(X, \mathbb{R}(j))$ is canonically isomorphic to the de Rham cohomology $H^{n-1}(X, (2\pi i)^{j-1}\mathbb{R})$ by the canonical map sending a Deligne cohomology class to its de Rham cohomology class (see [9, Section 8.1]). For $j > 1$, we thus have $H^1_{\mathcal{D}}(\mathrm{Spec}\,(\mathbb{C}), \mathbb{R}(j)) \simeq \mathbb{R}(j-1)$.

Let X be a smooth variety over $\mathbb{R}$. Let $X(\mathbb{C})$ denote the set of complex points of $X \times_{\mathbb{R}} \mathbb{C}$. Denote by c the complex conjugation on $X(\mathbb{C})$. For a complex differential form ω on $X(\mathbb{C})$, we define an involution $F_{\mathrm{dR}}(\omega) := c^*(\bar{\omega})$. It acts on the complex $E_j(X(\mathbb{C}))$, hence acts on the Deligne cohomology.

Definition 1.5 [11, Remark 6.5]. Let X be a smooth variety over $\mathbb{R}$. The Deligne cohomology of X is defined by

$$H^n_{\mathcal{D}}(X, \mathbb{R}(j)) := H^n_{\mathcal{D}}(X \times_{\mathbb{R}} \mathbb{C}, \mathbb{R}(j))^+,$$

where "+" denotes the invariant part under the action of the involution F_{dR}.

Let X be a smooth real or complex variety, there is a cup-product in Deligne cohomology

$$\cup : H^n_{\mathcal{D}}(X, \mathbb{R}(j)) \otimes H^m_{\mathcal{D}}(X, \mathbb{R}(k)) \to H^{n+m}_{\mathcal{D}}(X, \mathbb{R}(j+k)), \tag{1-2}$$

(see [11, Theorem 3.3]). It is graded commutative (i.e., $\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha$), and associative. In the case $n < 2j$, $m < 2k$, for $\alpha \in H^n_{\mathcal{D}}(X, \mathbb{R}(j))$ and $\beta \in H^m_{\mathcal{D}}(X, \mathbb{R}(k))$, we have that $\alpha \cup \beta$ is represented by

$$(-1)^n r_j(\alpha) \wedge \beta + \alpha \wedge r_k(\beta), \tag{1-3}$$

where $r_\ell(\alpha) := \partial(\alpha^{\ell-1, n-\ell}) - \bar{\partial}(\alpha^{n-\ell, \ell-1})$.

1.2. ***Chow motives.*** In this section, we discuss the Chow motives of smooth projective curves. For more details, we refer to [29] or [34]. Recall that the Chow groups of a variety X are $\mathrm{CH}^n(X) := Z^n(X)/\sim$, where $Z^n(X)$ is the free abelian group generated by irreducible subvarieties of X of codimension n, and $\sim$ is the rational equivalence (see [29, Section 1.2]). If $\phi : X \to Y$ is a morphism of varieties, one has the homomorphisms

$$\phi^* : \mathrm{CH}^n(Y) \to \mathrm{CH}^n(X), \qquad \phi_* : \mathrm{CH}^n(X) \to \mathrm{CH}^{n+\dim Y - \dim X}(Y).$$

Let k be a field and $r \geq 0$. Let $X, Y \in \mathrm{SmProj}(k)$. If X is of pure dimension d, the group of *correspondences of degree r* is given by $\mathrm{Corr}^r(X, Y) := \mathrm{CH}^{d+r}(X \times Y) \otimes \mathbb{Q}$. If $X = \coprod X_d$ is a decomposition of subschemes with X_d is of pure dimension d, then $\mathrm{Corr}^r(X, Y) := \bigoplus \mathrm{Corr}^r(X_d, Y)$. Let $X, Y, Z \in \mathrm{SmProj}(k)$ and $f \in \mathrm{Corr}^r(X, Y)$, $g \in \mathrm{Corr}^s(Y, Z)$, the composition of correspondences

$$\mathrm{Corr}^r(X, Y) \times \mathrm{Corr}^s(Y, Z) \to \mathrm{Corr}^{r+s}(X, Z)$$

is defined by

$$(f, g) \mapsto g \circ f := \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* f \cdot \mathrm{pr}_{23}^* g) = \mathrm{pr}_{13*}(f \times Z \cdot X \times g),$$

where pr is the canonical projection and $\cdot$ is the intersection product. Let $\phi : X \to Y$ in $\mathrm{SmProj}(k)$, with X and Y are of pure dimensions d and e, respectively. Let Γ_ϕ denote the image of the closed immersion $(\mathrm{id}_X, \phi) : X \to X \times Y$. We have the correspondences

$$\phi_* = [\Gamma_\phi] \in \mathrm{Corr}^{e-d}(X, Y), \qquad \phi^* := [{}^t\Gamma_\phi] \in \mathrm{Corr}^0(Y, X).$$

Definition 1.6 (Chow motives). Objects of the category of Chow motives $\mathrm{CHM}_{\mathbb{Q}}(k)$ are triples (X, p, m), where $X \in \mathrm{SmProj}(k)$, $p \in \mathrm{Corr}^0(X, X)$ is an idempotent, i.e., $p \circ p = p$, and m is an integer. If (X, p, m) and (Y, q, n) are Chow motives, then

$$\mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(k)}((X, p, m), (Y, q, n)) = p \circ \mathrm{Corr}^{n-m}(X, Y) \circ q \subset \mathrm{Corr}^{n-m}(X, Y).$$

There is a contravariant functor

$$h : \mathrm{SmProj}(k) \to \mathrm{CHM}_{\mathbb{Q}}(k), \quad X \mapsto h(X) := (X, [\Delta_X], 0), \tag{1-4}$$

where Δ_X is the graph of the diagonal map. If $\phi : X \to Y$ is a morphism, $h(\phi) := \phi^* = [{}^t\Gamma_\phi] \in \mathrm{Corr}^0(Y, X) = \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(k)}(h(Y), h(X))$. One calls $h(X)$ the Chow motive of X.

Let C be a smooth projective curve over k (not necessarily having a k-rational point). Pick any zero cycle Z on C of positive degree N, one defines projectors $p_0(C) := \frac{1}{N}[Z \times C]$, $p_2(C) := \frac{1}{N}[C \times Z]$, and Chow motives $h^i(C) := (C, p_i(C), 0) \in \mathrm{CHM}_{\mathbb{Q}}(k)$ for $i = 0, 2$. These motives do not depend on the choice of Z, in fact, $h^0(C) \simeq h(\mathrm{Spec}\, k')$ and $h^2(C) \simeq h(\mathrm{Spec}\, k') \otimes \mathbb{L}$, where $k' = \Gamma(C, \mathcal{O}_C)$ and $\mathbb{L} = (\mathrm{Spec}\, k, \mathrm{id}, -1)$ is the Lefschetz motive. One sets $p_1(C) := \Delta_C - p_0(C) - p_2(C)$ and $h^1(C) := (C, p_1(C), 0) \in \mathrm{CHM}_{\mathbb{Q}}(k)$. We have the direct sum decomposition

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C),$$

which depends on the choice of Z, but $h^1(C)$ is well-defined up to unique isomorphism (see [29, Section 2.3] or [34, Section 3]). If C is further of genus 1, one can show that $h^1(C) \simeq h^1(E)$, where E is the Jacobian of C, by using the following equivalence of categories (see the proof of [29, Theorem 2.7.2(b)]):

$$M''_{\mathbb{Q}} \xrightarrow{\simeq} \{\text{category of Jacobian of curves}\} \otimes \mathbb{Q},$$

where $M''_{\mathbb{Q}}$ is the full subcategory of $\mathrm{CHM}_{\mathbb{Q}}(k)$ of motives isomorphic to $h^1(Y)$ for some smooth projective curve Y. Let us give explicitly the isomorphism.

Proposition 1.7. *Let C be a smooth projective curve of genus 1 over a number field k and E be its Jacobian, then $h(C) \simeq h(E)$ and $h^1(C) \simeq h^1(E)$.*

Proof. Let $\bar{k}$ be the algebraic closure of k and let us fix a point $x_0 \in C(\bar{k})$. We consider the morphism $\phi : C_{\bar{k}} \to E_{\bar{k}}$, which maps $x \in C(\bar{k})$ to the divisor $N(x) - \sum_\sigma (\sigma(x_0))$, where σ runs through all the embeddings $k(x_0) \hookrightarrow \bar{k}$ and N is the number of these embeddings. This map is well-defined as $N(x) - \sum_\sigma (\sigma(x_0))$ is a divisor of degree 0. We have $\phi_* \in \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(\bar{k})}(h(C_{\bar{k}}), h(E_{\bar{k}}))$ and $\phi^* \in \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Q}}(\bar{k})}(h(E_{\bar{k}}), h(C_{\bar{k}}))$. By [29, Section 2.3], we have $\phi_* \circ \phi^* = \deg(\phi)[\Delta_{E_{\bar{k}}}] = N^2[\Delta_{E_{\bar{k}}}]$. Conversely, we have

$$\phi^* \circ \phi_* \stackrel{\text{def}}{=} \mathrm{pr}_{13*}((\Gamma_\phi \times C_{\bar{k}}) \cdot (C_{\bar{k}} \times {}^t\Gamma_\phi)).$$

As sets, we observe that

$$\begin{aligned}(\Gamma_\phi \times C_{\bar{k}}) \cap (C_{\bar{k}} \times {}^t\Gamma_\phi) &= \{(x, \phi(x), y) \mid x, y \in C(\bar{k})\} \cap \{(z, \phi(t), t) \mid z, t \in C(\bar{k})\}\\ &= \{(x, \phi(x), y) \mid x, y \in C(\bar{k}), \phi(x) = \phi(y)\}\\ &= \{(x, \phi(x), y) \mid x, y \in C(\bar{k}), N(x) - N(y) = 0 \text{ in } E(\bar{k})\}\\ &= \{(x, \phi(x), x + p) \mid x \in C(\bar{k}), p \in E_{\bar{k}}[N]\},\end{aligned}$$

where $E_{\bar{k}}[N]$ is the set of N-torsion points of $E(\bar{k})$ and "+" is the canonical action of $E_{\bar{k}}$ on $C_{\bar{k}}$. So

$$\phi^* \circ \phi_* = \sum_{p \in E_{\bar{k}}[N]} [\Gamma_{\varphi_p}] = N^2[\Delta_{C_{\bar{k}}}],$$

where $\varphi_p : C_{\bar{k}} \to C_{\bar{k}},\ x \mapsto x + p$, and the last equality follows from the fact that Γ_{φ_p} is rationally equivalent to $\Delta_{C_{\bar{k}}}$ for $p \in E_{\bar{k}}[N]$. We thus obtain that $\phi_* : h(C_{\bar{k}}) \to h(E_{\bar{k}})$ is an isomorphism in the category $\mathrm{CHM}_{\mathbb{Q}}(\bar{k})$. For $\alpha \in \mathrm{Gal}(\bar{k}/k)$ and $x \in C(\bar{k})$,

$$(\alpha \circ \phi)(x) = \alpha(N(x)) - \sum_\sigma (\alpha \circ \sigma(x_0)) = N(\alpha(x)) - \sum_\sigma (\sigma(x_0)) = (\phi \circ \alpha)(x).$$

This implies that Γ_ϕ and ${}^t\Gamma_\phi$ are $\mathrm{Gal}(\bar{k}/k)$-invariant. Hence by Galois descent (see, e.g., Theorem 1.3(6) of [13])

$$\mathrm{CH}^1(C_{\bar{k}} \times_{\bar{k}} J_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \simeq \mathrm{CH}^1(C \times_k E), \quad \mathrm{CH}^1(E_{\bar{k}} \times_{\bar{k}} C_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \simeq \mathrm{CH}^1(E \times_k C),$$

ϕ_* defines an isomorphism from $h(C)$ to $h(E)$ in the category CHM_k with inverse ϕ^*.

Denote by A the positive zero-cycle of degree N corresponding to x_0. We set $p_0(C) := \frac{1}{N}[A \times C]$, $p_2(C) := \frac{1}{N}[C \times A]$, and $p_1(C) := \Delta_C - p_0(C) - p_2(C)$. Let $\mathcal{O}$ be the trivial element in $E(k)$, we set $p_0(E) := \mathcal{O} \times E$, $p_2(E) = E \times \mathcal{O}$, and $p_1(E) := \Delta_E - p_0(E) - p_2(E)$. By explicit computations, we have

$$\phi^* \circ p_0(E) \circ \phi_* = N^2 p_0(C) \quad \text{and} \quad \phi^* \circ p_2(E) \circ \phi_* = N^2 p_2(C).$$

Now we show that $p_1(E) \circ \phi_* \circ p_1(C)$ and $p_1(C) \circ \phi^* \circ p_1(E)$ define isomorphisms from $h^1(C)$ to $h^1(E)$ and inverse, respectively. We have

$$\begin{aligned}\phi^* \circ p_1(E) \circ \phi_* = \phi^* \circ (\phi_* - p_0(E) \circ \phi_* - p_2(E) \circ \phi_*) &= N^2[\Delta_C] - \phi^* \circ p_0(E) \circ \phi_* - \phi^* \circ p_2(E) \circ \phi_* \\ &= N^2[\Delta_C] - N^2 p_0(C) - N^2 p_2(C) \\ &= N^2 p_1(C).\end{aligned}$$

We thus have $p_1(C) \circ \phi^* \circ p_1(E) \circ p_1(E) \circ \phi_* \circ p_1(C) = p_1(C) \circ \phi^* \circ p_1(E) \circ \phi_* \circ p_1(C) = N^2 p_1(C)$. Similarly, we have $p_1(E) \circ \phi_* \circ p_1(C) \circ p_1(C) \circ \phi^* \circ p_1(E) = N^2 p_1(E)$. □

1.3. ***Motivic cohomology.*** Let k be an arbitrary field of characteristic 0. Let us recall briefly the definition and some facts of motivic cohomology. For more details, we refer to [14, Chapter 5, Section 2]. Voevodsky constructed a triangulated category, called *geometrical motives*, denoted by $\mathrm{DM}_{\mathrm{gm}}(k)$ and a covariant functor

$$M_{\mathrm{gm}} : \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$$

(see [14, p. 192]). The motivic cohomology of X with coefficients in $\mathbb{Q}$ is defined by

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) := \mathrm{Hom}_{\mathrm{Dm}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(X), \mathbb{Q}(j)[n]), \qquad \text{for } n, j \in \mathbb{Z}, \tag{1-5}$$

where $\mathbb{Q}(1) \in \mathrm{DM}_{\mathrm{gm}}(k)$ is the *Tate motive* (see also [14, p. 192]) and $\mathbb{Q}(j) = \mathbb{Q}(1)^{\otimes j}$. It is known that the motivic cohomology $H^n_{\mathcal{M}}(X, \mathbb{Q}(j))$ is isomorphic to the j-eigenspace of Quillen's K-group $K_{2j-n}(X)_{\mathbb{Q}}$ with respect to Adams operation (see [37, Chapter II.4] for the definition), namely,

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \simeq K^{(j)}_{2j-n}(X) \tag{1-6}$$

(see [14, p. 197]). By the functorial property of M_{gm}, for any morphism $f : X \to Y$, we have a $\mathbb{Q}$-linear map $f^* : H^n_{\mathcal{M}}(Y, \mathbb{Q}(j)) \to H^n_{\mathcal{M}}(X, \mathbb{Q}(j))$, called pull back of f. Moreover, for proper maps $f : X \to Y$ of pure codimension $c = \dim Y - \dim X$, we have a $\mathbb{Q}$-linear map, called push-forward of f (see [13, Theorem 1.3])

$$f_* : H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^{n+2c}_{\mathcal{M}}(Y, \mathbb{Q}(j+c)).$$

Let X and X' be smooth quasiprojective varieties over k and $\pi : X' \to X$ be a finite Galois covering with group G. We have Galois descent for motivic cohomology, i.e.,

$$\pi^* : H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \xrightarrow{\simeq} H^n_{\mathcal{M}}(X', \mathbb{Q}(j))^G \tag{1-7}$$

is an isomorphism (see [13, Theorem 1.3]).

Let $X \in \mathrm{Sm}(k)$ and $\iota : Z \hookrightarrow X$ be a closed immersion of smooth varieties of codimension c with open complement $\jmath : X - Z \hookrightarrow X$. We have a *localization sequence* for motivic cohomology (see [14, p. 196] or [13, Theorem 1.3])

$$\cdots \to H^{i-2c}_{\mathcal{M}}(Z, \mathbb{Q}(j-c)) \xrightarrow{\iota_*} H^{i}_{\mathcal{M}}(X, \mathbb{Q}(j)) \xrightarrow{\jmath^*} H^{i}(X - Z, \mathbb{Q}(j)) \to H^{i+1-2c}_{\mathcal{M}}(Z, \mathbb{Q}(j-c)) \to \cdots . \tag{1-8}$$

Let C be a smooth curve over a number field k. Denote by $F = k(C)$ the function field of C. Then

$$H^n_{\mathcal{M}}(F, \mathbb{Q}(j)) = \varinjlim_{U \subset C \text{ open}} H^n_{\mathcal{M}}(U, \mathbb{Q}(j)).$$

We have the following version of localization sequence:

$$0 \to H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \xrightarrow{\mathrm{Res}^{\mathcal{M}}} \bigoplus_{x \in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)), \tag{1-9}$$

where C^1 is the set of closed points of C. This follows from the localization sequence of Quillen's K-groups (see [37, V.6.12]). The left exactness follows from Borel's theorem (see, e.g., [37, IV.1.18]), which states that K_4 of a number field is torsion, hence $H^0_{\mathcal{M}}(K, \mathbb{Q}(2)) \simeq K_4^{(2)}(K) = 0$.

1.4. ***The Beilinson regulator map.*** Let X be a smooth variety over $\mathbb{R}$ or $\mathbb{C}$. The Beilinson regulator map, as defined in [30], is a $\mathbb{Q}$-linear map

$$\mathrm{reg}_X : H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^n_{\mathcal{D}}(X, \mathbb{R}(j)). \tag{1-10}$$

For $n = j = 1$, we have $H^1_{\mathcal{M}}(X, \mathbb{Q}(1)) = \mathcal{O}(X)^\times_{\mathbb{Q}}$ and the regulator map sends an invertible function f to the class of $\log|f|$, by [9, Appendix A.3]. As the regulator map is compatible with taking cup products, we observe that the regulator map sends the Milnor symbol $\{f_1, \dots, f_n\} \in H^n_{\mathcal{M}}(X, \mathbb{Q}(n))$ to the class of $\log|f_1| \cup \cdots \cup \log|f_n|$ in $H^n_{\mathcal{D}}(X, \mathbb{R}(n))$. When X is defined over a number field k, we write $X_{\mathbb{R}} := X \times_{\mathbb{Q}} \mathbb{R}$, and the Beilinson regulator map is defined as the composition

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(j)) \xrightarrow{\text{base change}} H^n_{\mathcal{M}}(X_{\mathbb{R}}, \mathbb{Q}(j)) \xrightarrow{\mathrm{reg}_{X_{\mathbb{R}}}} H^n_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j)). \tag{1-11}$$

Now let C be a smooth curve over a number field k. Let $F = k(C)$ be the function field of C. We define Deligne cohomology of F by the direct limit

$$H^n_{\mathcal{D}}(F, \mathbb{R}(j)) := \varinjlim_{U \subset C \text{ open}} H^n_{\mathcal{D}}(U_{\mathbb{R}}, \mathbb{R}(j)). \tag{1-12}$$

And the regulator map for the function field is defined by $\mathrm{reg}_F := \varinjlim_{U \subset C \text{ open}} \mathrm{reg}_U$

$$\mathrm{reg}_F : H^n_{\mathcal{M}}(F, \mathbb{Q}(j)) \to H^n_{\mathcal{D}}(F, \mathbb{Q}(j)). \tag{1-13}$$

Now let us recall the *regulator integral* for smooth projective curve (see [20, Section 3] for more details). Let C be a smooth projective curve over a number field k. Denote by $C(\mathbb{C})$ the set of complex points of $C \times_{\mathbb{Q}} \mathbb{C}$. If ω is a holomorphic 1-form on $C(\mathbb{C})$ such that $F_{\mathrm{dR}}(\omega) = \omega$, where F_{dR} is defined in Section 1.1, then we have a map

$$H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \to \mathbb{R}(1), \qquad \eta \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega} \tag{1-14}$$

(see [20, Remark 3.1]). This integral depends on the choice of the orientation of $C(\mathbb{C})$. Recall that there is a canonical isomorphism $H^2_{\mathcal{D}}(C_{\mathbb{R}}, \mathbb{R}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+$ (see Remark 1.4). We thus write the Beilinson regulator map on C as the composition

$$\mathrm{reg}_C : H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \to H^2_{\mathcal{D}}(C_{\mathbb{R}}, \mathbb{R}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+. \tag{1-15}$$

The composition map

$$H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \xrightarrow{\mathrm{reg}_C} H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \xrightarrow{\eta \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega}} \mathbb{R}(1) \tag{1-16}$$

is called the *regulator integral*. Similarly, we have the regulator integral for the function field

$$H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \xrightarrow{\mathrm{reg}_F} H^1(F, \mathbb{R}(2))^+ \xrightarrow{\eta \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega}} \mathbb{R}(1), \tag{1-17}$$

where $H^1(F, \mathbb{R}(2))^+ := \varinjlim_{U \subset C \text{ open}} H^1(U(\mathbb{C}), \mathbb{R}(2))^+$.

1.5. ***Beilinson's conjecture for genus 1 curves.*** In this section, we recall Beilinson's conjecture for smooth projective curves of genus 1 (see [30, Section 6] or [19, Section 4] for more details). Let us recall the definition of L-function attached to the pure motive $h^i(X)$, for X is a smooth projective variety over $\mathbb{Q}$ of dimension n.

Definition 1.8 [30, Section 1.4]. Let p be a prime number. For $0 \leq i \leq 2n$, we set

$$L_p(h^i(X), s) = \det(1 - \mathrm{Frob}_p p^{-s} \,|\, h^i_\ell(X)^{I_p})^{-1},$$

where $\ell \neq p$ is a prime number, $\mathrm{Frob}_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a Frobenius element at p, acting on the étale realization

$$h^i_\ell(X) := H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

and I_p is the inertia group at p.

Remark 1.9. If X has good reduction at p, then $L_p(h^i(X), s)$ does not depend on the choice of ℓ [30, Section 1.4]. And it is conjectured by Serre that if X has bad reduction at p, then $L_p(h^i(X), s)$ is independent of the choice of ℓ and has integer coefficients (compare [21, Conjecture 5.45]). This conjecture holds if $i \in \{0, 1, 2n-1, 2n\}$, by [21, Theorem 5.46]. In particular, it holds when X is a curve.

Definition 1.10 (L-function [30, Section 1.5]). The L-function associated to $h^i(X)$ is defined by

$$L(h^i(X), s) = \prod_{p \text{ prime}} L_p(h^i(X), s).$$

Let $C/\mathbb{Q}$ be a smooth projective curve of genus 1 and E be its Jacobian. By Proposition 1.7, we have $L(h^1(C), s) = L(h^1(E), s)$. Hence $L(h^1(C), s) = L(E, s)$ the Hasse–Weil L-function. We then have the following version of Beilinson's conjecture for smooth projective curve of genus 1.

Conjecture 1.11 ([30, Section 6], [19, Section 4]). *Let C be a smooth projective curve over $\mathbb{Q}$ of genus* 1 *and E be its Jacobian. For any nontrivial element $\alpha \in H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$, we have*

$$\frac{1}{(2\pi i)^2}\left\langle \gamma_C^+, \mathrm{reg}_C(\alpha)\right\rangle_{C(\mathbb{C})} = a \cdot L'(E, -1) \qquad (a \in \mathbb{Q}^\times),$$

where γ_C^+ is a generator of $H_1(C(\mathbb{C}), \mathbb{Q})^+$, reg_C is the Beilinson regulator map (1-15), *and $\langle\,\cdot\,,\cdot\,\rangle$ is then the pairing in de Rham cohomology.*

2. Goncharov's polylogarithmic complexes

In 1990s, Goncharov introduced polylogarithmic complexes and regulator maps from the cohomology of these complexes to Deligne cohomology. They have connections with motivic cohomology and the Beilinson regulator map (see [15; 16; 17]). In this section, we recall briefly these constructions of Goncharov, which will be used in Section 4.3 to construct elements in motivic cohomology.

2.1. *Goncharov's complexes.* For any field F of characteristic 0 and an integer $n \geq 1$, Goncharov defined $\mathcal{B}_n(F)$ to be the quotient of the $\mathbb{Q}$-vector space $\mathbb{Q}[\mathbb{P}^1_F]$ by a certain (inductively defined) subspace $\mathcal{R}_n(F)$. For $x \in F \cup \{\infty\}$, we denote by $\{x\}_n$ the class of $\{x\} \in \mathbb{Q}[\mathbb{P}^1_F]$ in $\mathcal{B}_n(F)$. Goncharov then constructed a *weight n polylogarithmic complex*, in degree 1 to n (see [15, Section 1])

$$\Gamma_F(n) : \mathcal{B}_n(F) \to \mathcal{B}_{n-1}(F) \otimes F^\times_{\mathbb{Q}} \to \mathcal{B}_{n-2}(F) \otimes \bigwedge^2 F^\times_{\mathbb{Q}} \to \cdots \to \mathcal{B}_2(F) \otimes \bigwedge^{n-2} F^\times_{\mathbb{Q}} \to \bigwedge^n F^\times_{\mathbb{Q}},$$

where the maps are given by

$$\begin{aligned} \{x\}_{n-p} \otimes y_1 \wedge \cdots \wedge y_p &\mapsto \{x\}_{n-p-1} \otimes x \wedge y_1 \wedge \cdots \wedge y_p \qquad \text{if } 0 \leq p < n-2, \\ \{x\}_2 \otimes y_1 \wedge \cdots \wedge y_{n-2} &\mapsto (1-x) \wedge x \wedge y_1 \wedge \cdots \wedge y_{n-2}. \end{aligned}$$

It is conjectured that the cohomology of this complex computes the motivic cohomology.

Conjecture 2.1 [15, Conjecture A, p. 222]. *$H^p(\Gamma_F(n)) \simeq H^p_{\mathcal{M}}(F, \mathbb{Q}(n))$ for $p, n \geq 1$.*

Goncharov also defined the $\mathbb{Q}$-vector space $B_n(F)$ $(1 \leq n \leq 3)$ to be the quotient of $\mathbb{Q}[\mathbb{P}^1_F]$ by a certain subspace $R_n(F)$, generated by explicit relations as follows.

$$\begin{aligned} R_1(F) &:= \left\langle \{x\} + \{y\} - \{xy\},\, x, y \in F^\times; \{0\}; \{\infty\}\right\rangle, \\ R_2(F) &:= \left\langle \{x\} + \{y\} + \{1-xy\} + \left\{\frac{1-x}{1-xy}\right\} + \left\{\frac{1-y}{1-xy}\right\},\, x, y \in F^\times \setminus \{1\}; \{0\}; \{\infty\}\right\rangle, \end{aligned} \tag{2-1}$$

and $R_3(F)$ is generated by explicit relations corresponding to the functional equations for the trilogarithm (see [15, p. 214]) that we do not mention here. We still denote by $\{x\}_k$ the class of $\{x\} \in \mathbb{Q}[\mathbb{P}^1_F]$ in $B_k(F)$. As Goncharov's constructions, $B_1(F) = \mathcal{B}_1(F) = F^\times_{\mathbb{Q}}$ (see [15, Sections 1.8 and 1.9]). And it was proved

by de Jeu that $B_2(F)=\mathcal{B}_2(F)$ (see [20, Remark 5.3]). Goncharov showed that $R_3(F)\subset\mathcal{R}_3(F)$ and conjectured that they are equal (see [15, p. 225]).
Goncharov then constructed the polylogarithmic complexes $\Gamma(F,n)$ for $n=2,3$ with the same shape as $\Gamma_F(n)$ but $\mathcal{B}_n(F)$ are replaced by $B_n(F)$ (notice that these complexes are denoted by $B_F(n)$ in [15, Section 1.8]). In this article, we only use the vector spaces with explicit relations $B_n(F)$ and the corresponding polylogarithmic complexes $\Gamma(F,n)$ for $n=2,3$. For $n=2$, it is given as follows, in degree 1 and 2:

$$\Gamma(F,2): \qquad B_2(F)\xrightarrow{\alpha_2(1)}\bigwedge^2 F_{\mathbb{Q}}^{\times}, \quad \{x\}_2\mapsto(1-x)\wedge x.$$

We have $H^2(\Gamma(F,2))\simeq H^2_{\mathcal{M}}(F,\mathbb{Q}(2))$ by Matsumoto's theorem. And $H^1(\Gamma(F,2))\simeq H^1_{\mathcal{M}}(F,\mathbb{Q}(2))$ by Suslin's work [35]. The group $H^1(\Gamma(F,2))$ is also called *Bloch group*, denoted by $\mathcal{B}(F)$ (see [38, Section 2]).

For $n=3$, we have the following polylogarithmic complex in degree 1 to 3:

$$\begin{array}{llllll} \Gamma(F,3): & B_3(F) & \xrightarrow{\alpha_3(1)} & B_2(F)\otimes F_{\mathbb{Q}}^{\times} & \xrightarrow{\alpha_3(2)} & \bigwedge^3 F_{\mathbb{Q}}^{\times}, \\ & \{x\}_3 & \longmapsto & \{x\}_2\otimes x & & \\ & & & \{x\}_2\otimes y & \longmapsto & (1-x)\wedge x\wedge y. \end{array} \tag{2-2}$$

Then $H^3(\Gamma(F,3))\simeq H^3_{\mathcal{M}}(F,\mathbb{Q}(3))$. In degree 2, Goncharov constructed a map $K_4(F)_{\mathbb{Q}}\to H^2(\Gamma(F,3))$ and conjectured that this induces an isomorphism $H^2_{\mathcal{M}}(F,\mathbb{Q}(3))\simeq K_4^{(3)}(F)\to H^2(\Gamma(F,3))$. De Jeu constructed a map in other direction $H^2(\Gamma(F,3))\to H^2_{\mathcal{M}}(F,\mathbb{Q}(3))$. We discuss the later map in Section 3.3.

2.2. ***The residue homomorphism of complexes.*** Let K be a field with a discrete valuation v. Denote by $\mathcal{O}_K$, k_v, π_v the ring of integers, the residue field, and a uniformizer, respectively. Goncharov defined a residue homomorphism on his polylogarithmic complexes (see [15, Section 1.14])

$$\partial_v:\Gamma(K,3)\to\Gamma(k_v,2)[-1]. \tag{2-3}$$

More precisely, it is given by

$$\begin{tikzcd} B_3(K) \arrow[r, "\alpha_3(1)"] & B_2(K)\otimes K_{\mathbb{Q}}^{\times} \arrow[r, "\alpha_3(2)"] \arrow[d, "\partial_v^{(2)}"] & \bigwedge^3 K_{\mathbb{Q}}^{\times} \arrow[d, "\partial_v^{(3)}"] \\ & B_2(k_v) \arrow[r, "\alpha_2(1)"'] & \bigwedge^2 (k_v^{\times})_{\mathbb{Q}}, \end{tikzcd} \tag{2-4}$$

where the vertical maps are defined as follows. For $f\in K^{\times}$, we denote by f_v the image of $f\pi_v^{-\mathrm{ord}_v(f)}$ under the canonical map $\mathcal{O}_K^{\times}\to k_v^{\times}$. We have

$$\partial_v^{(2)}:\{f\}_2\otimes g\mapsto \mathrm{ord}_v(g)\{f(v)\}_2, \text{ with the convention } \{0\}_2=\{1\}_2=\{\infty\}_2=0 \text{ in } B_2(k_v), \tag{2-5}$$

$$\partial_v^{(3)}: f\wedge g\wedge h\mapsto \mathrm{ord}_v(f)g_v\wedge h_v-\mathrm{ord}_v(g)f_v\wedge h_v+\mathrm{ord}_v(h)f_v\wedge g_v. \tag{2-6}$$

Now let C be a smooth connected curve over a number field k and let F be its function field. Denote by C^1 the set of closed points of C, and $k(x)$ the residue field of $x \in C^1$. We have the morphism of complexes

$$\partial := \oplus_{x\in C^1}\partial_x : \Gamma(F,3) \to \bigoplus_{x\in C^1} \Gamma(k(x),2)[-1]. \tag{2-7}$$

Goncharov defined the polylogarithmic complex $\Gamma(C,3)$ as the mapping cone of (2-7). We then have the exact sequence

$$0 \to H^2(\Gamma(C,3)) \to H^2(\Gamma(F,3)) \xrightarrow{\partial} \bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)), \tag{2-8}$$

where the left exactness is due to the fact that the cohomology group of degree 0 of polylogarithmic complexes vanishes. This should be isomorphic to the localization sequence of motivic cohomology (1-9). We will construct a morphism from complexes (2-8) to (1-9) in (3-16), using work of de Jeu.

2.3. *Goncharov's regulator maps.* In this section, we recall Goncharov's regulator maps. Let X be a regular variety over a number field. Denote by F the function field of X. For a nonempty open subscheme $U \subset X$, $U(\mathbb{C})$ denotes the set of complex points of $U \times_{\mathbb{Q}} \mathbb{C}$. Let $\Omega^j(\eta_X) := \varinjlim_{U\subset X \text{ open}} \Omega^j(U)$ and $\Omega^j(U)$ is the space of real smooth j-forms on $U(\mathbb{C})$. Goncharov gave explicitly a homomorphism of complexes (see [17, Theorem 2.2]):

$$\begin{CD}
B_3(F) @>{\alpha_3(1)}>> B_2(F)\otimes F^\times_{\mathbb{Q}} @>{\alpha_3(2)}>> \bigwedge^3 F^\times_{\mathbb{Q}}\\
@V{r_3(1)}VV @V{r_3(2)}VV @V{r_3(3)}VV\\
\Omega^0(\eta_X) @>{d}>> \Omega^1(\eta_X) @>{d}>> \Omega^2(\eta_X).
\end{CD} \tag{2-9}$$

For $f \in F^\times \setminus \{1\}$, $g, h \in F^\times$, the vertical maps in degrees 2 and 3 are given respectively by

$$r_3(2) : \{f\}_2 \otimes g \mapsto \rho(f,g), \qquad r_3(3) : f\wedge g\wedge h \mapsto -\eta(f,g,h),$$

where

$$\begin{aligned}\eta(f,g,h) := {} & \log|f|\left(\tfrac{1}{3} d\log|g| \wedge d\log|h| - d\arg(g)\wedge d\arg(h)\right)\\ & + \log|g|\left(\tfrac{1}{3} d\log|h| \wedge d\log|f| - d\arg(h)\wedge d\arg(f)\right)\\ & + \log|h|\left(\tfrac{1}{3} d\log|f| \wedge d\log|g| - d\arg(f)\wedge d\arg(g)\right),\end{aligned} \tag{2-10}$$

and

$$\rho(f,g) := -D(f)\, d\arg g + \tfrac{1}{3}\log|g|\theta(1-f,f), \tag{2-11}$$

where

$$\theta(h,f) = \log|h| d\log|f| - \log|f| d\log|h|, \tag{2-12}$$

and $D : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ is the Bloch–Wigner dilogarithm function (0-7). In particular, we have

$$d\rho(f,g) = -\eta(1-f,f,g) = \eta(f,1-f,g) \text{ for } f \in F^\times \setminus \{1\},\ g \in F^\times.$$

Let C be a smooth connected curve over a number field and let F be its function field. Goncharov showed that the map $r_3(2)$ gives rise to a regulator map on the cohomology of his complexes of the function field

$$r_3(2)_F : H^2(\Gamma(F,3)) \to H^2_{\mathcal{D}}(F, \mathbb{R}(3)) \simeq H^1(F, \mathbb{R}(2))^+ \tag{2-13}$$

(see [17, Section 2.7]). Moreover, the map $r_3(2)_F$ is compatible with taking residues (2-7) (see [17, Theorem 2.6]), hence it extends to a homomorphism

$$r_3(2)_C : H^2(\Gamma(C,3)) \to H^2_{\mathcal{D}}(C \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+. \tag{2-14}$$

Now let us compute the residues of the differential form representing $r_3(2)_F(\alpha)$ for $\alpha \in H^2(\Gamma(F,3))$. First let us recall the definition of the residues of a differential form.

Definition 2.2 ([9, Definition 7.3]). Let C be a smooth connected curve over a number field and let Z be a subset of closed points of C. We set $Y := C \setminus Z$. Let $\eta \in H^1(Y(\mathbb{C}), \mathbb{R})$. The residue of η at $p \in C(\mathbb{C})$ is

$$\operatorname{Res}_p(\eta) = \int_{\gamma_p} \eta, \tag{2-15}$$

where γ_p is the boundary of any small disc containing p and avoiding $Z(\mathbb{C}) \setminus \{p\}$.

Lemma 2.3. *Let $\alpha = \sum_j c_j \{f_j\}_2 \otimes g_j \in H^2(\Gamma(F,3))$ with $c_j \in \mathbb{Q}$, and $f_j \in F^\times \setminus \{1\}$, $g_j \in F^\times$. Denote by Z the closed subset of C consisting of zeros and poles of f_j, $1 - f_j$, g_j for all j. Set $Y = C \setminus Z$. Then $r_3(2)_F(\alpha)$ is represented by the differential form $\rho := \sum_j c_j \rho(f_j, g_j) \in H^1(Y(\mathbb{C}), \mathbb{R}(2))^+$ and its residue at $p \in C(\mathbb{C})$ is given by*

$$\operatorname{Res}_p(\rho) = -2\pi \sum_j c_j v_p(g_j) D(f_j(p)),$$

where D is the Bloch–Wigner dilogarithm function (0-7).

Proof. The first statement follows directly from the definition of the map $r_3(2)_F$ (2-13). Now we compute the residues. Let $f, g \in \mathbb{C}(C)^\times$ such that all the zeros and poles of f, $1 - f$, g are contained in Z. Let $p \in C(\mathbb{C})$ and γ_p be a sufficiently small loop around p and does not surround any point of $Z \setminus \{p\}$. Using the local coordinate $z = re^{is}$, for $r > 0$ small and $s \in [0, 2\pi]$, we have $f(z) = (re^{is})^{v_p(f)} F(re^{is})$ and $g(z) = (re^{is})^{v_p(g)} G(re^{is})$, where F and G are holomorphic functions such that $F(0), G(0) \neq 0$. Then

$$\begin{aligned}\int_{\gamma_p} D(f) d\arg(g) &= \int_0^{2\pi} D(f(re^{is}))\, d\arg\big((re^{is})^{v_p(g)} G(re^{is})\big) \\ &= \int_0^{2\pi} D(f(re^{is})) v_p(g) ds + \int_0^{2\pi} D(f(re^{is}))\, d\arg G(re^{is}).\end{aligned} \tag{2-16}$$

As

$$d\arg G(z) = \frac{1}{2i}\left(\frac{dG}{G} - \frac{d\overline{G}}{\overline{G}}\right) = \frac{1}{2i}\left(\frac{1}{G}\frac{\partial G}{\partial z} dz - \frac{1}{\overline{G}}\frac{\partial \overline{G}}{\partial \bar{z}} d\bar{z}\right),$$

we have

$$d\arg G(re^{is}) = \frac{1}{2i}\left(\frac{G_z}{G} rie^{is}ds - \frac{\overline{G}_{\bar{z}}}{\overline{G}} r(-i)e^{-is}ds\right) = O(r)ds,$$

where G_z is the derivative of G with respect to z. Then by taking $r \to 0$ in (2-16), the limit of $\int_{\gamma_p} D(f)d\arg(g)$ as γ_p shrinks to p is

$$\int_0^{2\pi} D(f(p))v_p(g)ds = 2\pi v_p(g)D(f(p)). \tag{2-17}$$

Moreover, we have

$$\log|f| = \log|F(re^{is})| + v_p(f)\log r,$$

and

$$d\log|f| = d\log|F| = \frac{1}{2}\left(\frac{dF}{F} + \frac{d\overline{F}}{\overline{F}}\right) = O(r)ds.$$

Therefore, $\theta(1-f, f) = O(r\log r)\ ds \ \to 0$ as $r \to 0$. Hence

$$\mathrm{Res}_p(\rho) = -2\pi\sum_j c_j v_p(g_j)D(f_j(p)),\ \text{ for } p \in C(\mathbb{C}). \qquad \square$$

3. De Jeu's polylogarithmic complexes

In this section, we recall de Jeu's polylogarithmic complexes and his maps from the cohomology of these complexes to the motivic cohomology. In particular, they give rise to maps from the cohomology of Goncharov's polylogarithmic complexes to the motivic cohomology. We then compare the images of Goncharov's regulator and Beilinson's regulator composed with these maps. These results are used in the construction of the motivic cohomology classes in Section 4.3. In this article, we consider only the cases of the polylogarithmic complexes of weight 2 and weight 3. The references for this section are [18; 19; 20].

3.1. ***De Jeu's polylogarithmic complexes.*** Let F be a field of characteristic 0. De Jeu defined $\widetilde{M}_{(j)}(F)$ to be a certain $\mathbb{Q}$-vector space generated by symbols $[f]_j$ with $f \in F^\times \setminus \{1\}$ and constructed the following complex in degree 1 to 2:

$$\widetilde{\mathcal{M}}^\bullet_{(2)}(F): \qquad \widetilde{M}_{(2)}(F) \to \textstyle\bigwedge^2 F^\times_{\mathbb{Q}}, \quad [f]_2 \mapsto (1-f)\wedge f, \tag{3-1}$$

and this complex in degree 1 to 3:

$$\begin{array}{llll} \widetilde{\mathcal{M}}^\bullet_{(3)}(F): & \widetilde{M}_{(3)}(F) \longrightarrow & \widetilde{M}_{(2)}(F)\otimes F^\times_{\mathbb{Q}} \longrightarrow & \bigwedge^3 F^\times_{\mathbb{Q}}, \\ & [f]_3 \longmapsto & [f]_2 \otimes f & \\ & & [f]_2\otimes g \longmapsto & (1-f)\wedge f\wedge g \end{array} \tag{3-2}$$

(see [18, Corollary 3.22, Example 3.24] or [20, Section 2]). We have $H^n(\widetilde{\mathcal{M}}^\bullet_{(n)}(F)) \simeq H^n_{\mathcal{M}}(F, \mathbb{Q}(n))$ for $n \in \{2, 3\}$. Let k be a number field. By Suslin's work, we have the following isomorphism (up to a universal choice of sign) (see [18, Theorem 5.3] or [20, Theorem 2.3])

$$\varphi^1_{(2)} : H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \xrightarrow{\simeq} H^1_{\mathcal{M}}(k, \mathbb{Q}(2)). \tag{3-3}$$

Let $\sigma : k \hookrightarrow \mathbb{C}$ be any embedding of k into $\mathbb{C}$. We consider the composition map

$$H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \xrightarrow{\varphi^1_{(2)}} H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_k} \mathbb{R}(1), \tag{3-4}$$

where reg_k is the composition $H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \xrightarrow{\sigma^*} H^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_{\mathbb{C}}} H^1_{\mathcal{D}}(\mathbb{C}, \mathbb{R}(2)) \simeq \mathbb{R}(1)$; the last isomorphism here is the canonical isomorphism mentioned in Remark 1.4. It is shown that the map (3-4) is given by $[z]_2$ to $\pm i D(\sigma(z))$, where D is the Bloch–Wigner dilogarithm (see [18, Proposition 4.1]). So we can fix the sign of $\varphi^1_{(2)}$ such that it is induced by $[z]_2 \mapsto i D(\sigma(z))$.

Moreover, de Jeu ([19, p. 529]) constructed a map (up to a universal choice of sign)

$$\varphi^2_{(3)} : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)). \tag{3-5}$$

We discuss more about this map when F is the function field of a curve in the following section.

3.2. ***De Jeu's maps.*** Let C be a smooth geometrically connected curve over a number field k. Denote by F the function field of C and $k(x)$ the residue field of a closed point $x \in C^1$. De Jeu [19, Proposition 5.1] also defined the residue map

$$\delta : \widetilde{\mathcal{M}}^\bullet_{(3)}(F) \to \bigoplus_{x \in C^1} \widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))[-1] \tag{3-6}$$

similarly to Goncharov's residue map (2-7). The complex $\widetilde{\mathcal{M}}^\bullet_{(3)}(C)$ is also defined to be the mapping cone of (3-6). As the maps $\varphi^2_{(3)}$ (3-5) and $\varphi^1_{(2)}$ (3-3) are defined universally up to sign, we have the (possibly noncommutative) diagram

$$\begin{tikzcd}
H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi^2_{(3)}"] \arrow[d, "\pm 2\delta"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\
\bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x \in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd}$$

Recall that we have the following cup product of K-groups:

$$\cup : K_3^{(2)}(k) \otimes K_1^{(1)}(F) \to K_4^{(3)}(F).$$

Since $K^{(j)}_{2j-i}(X) \simeq H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$ and $K_1^{(1)}(F) \simeq F^\times_{\mathbb{Q}}$, we have

$$H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}} \subset H^2_{\mathcal{M}}(F, \mathbb{Q}(3)).$$

Denote by $H = H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}}$. De Jeu showed that $(\mathrm{Res}^{\mathcal{M}} \circ \varphi^2_{(3)}) \pm 2(\varphi^1_{(2)} \circ \delta)$ has image in $\mathrm{Res}^{\mathcal{M}}|_H(H)$ (see [19, Theorem 5.2]). With the fixed choice of sign of $\varphi^1_{(2)}$ in Section 3.1, we can choose the sign of $\varphi^2_{(3)}$ such that $(\mathrm{Res}^{\mathcal{M}} \circ \varphi^2_{(3)}) - 2(\varphi^1_{(2)} \circ \delta)$ has image in $\mathrm{Res}^{\mathcal{M}}|_H(H)$ (see [19, diagram (15)]).

$$\begin{tikzcd} H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi^2_{(3)}"] \arrow[d, "2\delta"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \supset H \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\ \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)). \end{tikzcd}$$

Let $\xi \in H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$. As $\mathrm{Res}^{\mathcal{M}}|_H$ is injective (see [20, Remark 4.4], for example), there exists a unique $h \in H$ such that

$$\mathrm{Res}^{\mathcal{M}}|_H(h) = ((\mathrm{Res}^{\mathcal{M}} \circ \varphi^2_{(3)}) - 2(\varphi^1_{(2)} \circ \delta))(\xi).$$

Then we define a map

$$\varphi_F : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)), \tag{3-7}$$

by setting $\varphi_F(\xi) := \varphi^2_{(3)}(\xi) - h$. It is a $\mathbb{Q}$-linear map making the following diagram commute:

$$\begin{tikzcd} H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "\varphi_F"] \arrow[d, "2\delta"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\ \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)). \end{tikzcd} \tag{3-8}$$

This modification was mentioned briefly by de Jeu in [19, Remark 5.3]. From diagram (3-8), φ_F induces a map

$$\varphi_C : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \to H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$$

such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \arrow[r] \arrow[d, "\varphi_C"'] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r, "2\delta"] \arrow[d, "\varphi_F"'] & \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[d, "\simeq"] \\ 0 \arrow[r] & H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \arrow[r] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[r, "\mathrm{Res}^{\mathcal{M}}"] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)), \end{tikzcd} \tag{3-9}$$

where the lower horizontal sequence is the localization sequence of motivic cohomology from (1-9).

3.3. ***Relation to Goncharov's complexes.*** Let F be an arbitrary field of characteristic 0. De Jeu showed that there is a map $B_2(F) \to \widetilde{M}_{(2)}(F)$ given by $\{x\}_2 \mapsto [x]_2$ (see [20, Lemma 5.2]). This map fits into the commutative diagram

$$\begin{tikzcd} \Gamma(F,2): & B_2(F) \arrow[r] \arrow[d] & \bigwedge^2 F_{\mathbb{Q}}^{\times} \arrow[d, equal] \\ \widetilde{\mathcal{M}}_{(2)}^{\bullet}(F): & \widetilde{M}_{(2)}(F) \arrow[r] & \bigwedge^2 F_{\mathbb{Q}}^{\times}. \end{tikzcd} \tag{3-10}$$

This diagram gives rise to a map $\psi_{(2)}^1 : H^1(\Gamma(F,2)) \to H^1(\widetilde{\mathcal{M}}_{(2)}^{\bullet}(F))$. In particular, if k is a number field, we have $\psi_{(2)}^1 : H^1(\Gamma(k,2)) \xrightarrow{\simeq} H^1(\widetilde{\mathcal{M}}_{(2)}^{\bullet}(k))$ is an isomorphism (see [18, Section 5]). We then define $\beta_{(2)}^1 : H^1(\Gamma(k,2)) \to H^1_{\mathcal{M}}(k, \mathbb{Q}(2))$ to be the composition of $\varphi_{(2)}^1$ and $\psi_{(2)}^1$

$$\begin{tikzcd} H^1(\Gamma(k,2)) \arrow[r, "\beta_{(2)}^1", "\simeq"'] \arrow[dr, "\simeq", "\psi_{(2)}^1"'] & H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \\ & H^1(\widetilde{\mathcal{M}}_{(2)}^{\bullet}(k)). \arrow[u, "\simeq", "\varphi_{(2)}^1"'] \end{tikzcd} \tag{3-11}$$

The map $B_2(F) \to \widetilde{M}_{(2)}(F)$, $\{x\}_2 \mapsto [x]_2$ also fits into the commutative diagram

$$\begin{tikzcd} \Gamma(F,3): & B_3(F) \arrow[r] & B_2(F) \otimes F_{\mathbb{Q}}^{\times} \arrow[r] \arrow[d] & \bigwedge^3 F_{\mathbb{Q}}^{\times} \arrow[d, equal] \\ \widetilde{\mathcal{M}}_{(3)}^{\bullet}(F): & \widetilde{M}_{(3)}(F) \arrow[r] & \widetilde{M}_{(2)}(F) \otimes F_{\mathbb{Q}}^{\times} \arrow[r] & \bigwedge^3 F_{\mathbb{Q}}^{\times}. \end{tikzcd} \tag{3-12}$$

The middle vertical arrow in the diagram (3-12) sends objects of the form $\{x\}_x \otimes x$ to $[x]_2 \otimes x$, so that it maps the image of $B_3(F)$ to the image of $\widetilde{M}_{(3)}(F)$. It then induces a map

$$\psi_F : H^2(\Gamma(F,3)) \to H^2(\widetilde{\mathcal{M}}_{(3)}^{\bullet}(F)). \tag{3-13}$$

Now let C be a smooth geometrically connected curve over a number field k. Let F be its function field and for any $x \in C^1$, we denote by $k(x)$ the residue of C at x. Since the residue maps of Goncharov and de Jeu are defined similarly, we have the commutative diagram

$$\begin{tikzcd} H^2(\Gamma(F,3)) \arrow[r, "\psi_F"] \arrow[d, "2\partial"'] & H^2(\widetilde{\mathcal{M}}_{(3)}^{\bullet}(F)) \arrow[d, "2\delta"] \\ \bigoplus_{x \in C^1} H^1(\Gamma(k(x),2)) \arrow[r, "\simeq", "\psi_{(2)}^1"'] & \bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}_{(2)}^{\bullet}(k(x))). \end{tikzcd}$$

Then ψ_F induces a map $\psi_C : H^2(\Gamma(C,3)) \to H^2(\widetilde{\mathcal{M}}_{(3)}^{\bullet}(C))$ that makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Gamma(C,3)) \arrow[r] \arrow[d, "\psi_C"] & H^2(\Gamma(F,3)) \arrow[r, "2\partial"] \arrow[d, "\psi_F"] & \bigoplus_{x \in C^1} H^1(\Gamma(k(x),2)) \arrow[d] \\ 0 \arrow[r] & H^2(\widetilde{\mathcal{M}}_{(3)}^{\bullet}(C)) \arrow[r] & H^2(\widetilde{\mathcal{M}}_{(3)}^{\bullet}(F)) \arrow[r, "2\delta"] & \bigoplus_{x \in C^1} H^1(\widetilde{\mathcal{M}}_{(2)}^{\bullet}(k(x))). \end{tikzcd}$$

Putting $\beta_F := \varphi_F \circ \psi_F$, we have the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma(F,3)) \arrow[r, "\psi_F"] \arrow[d, "2\partial"'] \arrow[rr, bend left, "\beta_F"] & H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)) \arrow[r, "\varphi_F"] \arrow[d, "2\delta"] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[d, "\mathrm{Res}^{\mathcal{M}}"] \\
\bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)) \arrow[r, "\simeq", "\psi^1_{(2)}"'] \arrow[rr, bend right, "\simeq", "\beta^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k(x)) \arrow[r, "\simeq", "\varphi^1_{(2)}"'] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd} \tag{3-14}$$

Again, the map

$$\beta_F : H^2(\Gamma(F,3)) \to H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \tag{3-15}$$

induces a map $\beta_C : H^2(\Gamma(C,3)) \to H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^2(\Gamma(C,3)) \arrow[r] \arrow[d, "\beta_C"'] & H^2(\Gamma(F,3)) \arrow[r, "2\partial"] \arrow[d, "\beta_F"] & \bigoplus_{x\in C^1} H^1(\Gamma(k(x),2)) \arrow[d, "\simeq"', "\beta^1_{(2)}"] \\
0 \arrow[r] & H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \arrow[r] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[r, "\mathrm{Res}^{\mathcal{M}}"] & \bigoplus_{x\in C^1} H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)).
\end{tikzcd} \tag{3-16}$$

In particular, we have $\beta_C = \varphi_C \circ \psi_C$.

3.4. ***Regulator maps.*** Let C be a smooth proper geometrically connected curve over a number field k and let F be its function field. We have the following lemma, which is a consequence of de Jeu's theorem [20, Theorem 3.5] and Goncharov's theorem [16, Theorem 3.3].

Lemma 3.1 (de Jeu). *Let ω be a holomorphic* 1*-form on $C(\mathbb{C})$ such that $F_{\mathrm{dR}}(\omega) = \omega$, where F_{dR} is the action defined in Section 1.1. Let $\alpha = \sum_j c_j\{f_j\}_2 \otimes g_j \in H^2(\Gamma(F,3))$ with $c_j \in \mathbb{Q}$ and $f_j \in F^\times \setminus \{1\}$, $g_j \in F^\times$. With the fixed sign of $\varphi^2_{(3)}$ as in Section 3.2, we have*

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\beta_F(\alpha)) \wedge \bar{\omega} = 2\int_{C(\mathbb{C})} r_3(2)_F(\alpha) \wedge \bar{\omega}, \tag{3-17}$$

where β_F is the map defined in (3-15), reg_F *is Beilinson's regulator map* (1-13), *and $r_3(2)_F$ is Goncharov's regulator map* (2-13).

Proof. We consider the regulator integral (1-17)

$$H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \xrightarrow{\mathrm{reg}_F} H^1(F, \mathbb{R}(2))^+ \xrightarrow{\eta \mapsto \int_{C(\mathbb{C})} \eta\wedge\bar{\omega}} \mathbb{R}(1).$$

The image of $H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}}$ under the regulator integral is trivial (see [19, Theorem 4.2]). This can be seen by noting that

$$\int_{C(\mathbb{C})} d\arg g \wedge \omega = 2\pi \int_{g^{-1}(\mathbb{R}_{>0})} \omega = 2\pi \int_0^\infty g_*\omega = 0,$$

where g_* is the pushforward by the correspondence from C to $\mathbb{P}^1$, and the fact that $g_*\omega = 0$ since $\mathbb{P}^1$ has no holomorphic forms. Hence, for $\xi \in H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$, we have

$$\begin{aligned}\int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi_F(\xi)) \wedge \bar{\omega} &= \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi) - h) \wedge \bar{\omega} \quad \text{for some } h \in H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) \cup F^\times_{\mathbb{Q}} \text{ (see (3-7))} \\ &= \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega} - \int_{C(\mathbb{C})} \mathrm{reg}_F(h) \wedge \bar{\omega} \\ &= \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega}. \end{aligned} \tag{3-18}$$

Now let $\alpha = \sum_j c_j \{f_j\}_2 \otimes g_j \in H^2(\Gamma(F, 3))$. Then $\psi_F(\alpha) = \sum_j c_j [f_j]_2 \otimes g_j \in H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$. Using (3-18) with $\xi = \psi_F(\alpha)$, we have

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\beta_F(\alpha)) \wedge \bar{\omega} = \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi_F(\xi)) \wedge \bar{\omega} = \int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega}.$$

With the fixed sign of $\varphi^2_{(3)}$ in Section 3.2, one can show that

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\varphi^2_{(3)}(\xi)) \wedge \bar{\omega} = \frac{8}{3} \sum_j c_j \int_{C(\mathbb{C})} \log|g_j| \theta(1 - f_j, f_j) \wedge \bar{\omega}$$

(see [20, Theorem 3.5]). On the other hand, by some computations, one obtains the formula

$$\int_{C(\mathbb{C})} r_3(2)_F(\alpha) \wedge \bar{\omega} = \frac{4}{3} \sum_j c_j \int_{C(\mathbb{C})} \log|g_j| \theta(1 - f_j, f_j) \wedge \bar{\omega}$$

(see [16, Theorem 3.3]). Therefore, we have

$$\int_{C(\mathbb{C})} \mathrm{reg}_F(\beta_F(\alpha)) \wedge \bar{\omega} = 2 \int_{C(\mathbb{C})} r_3(2)_F(\alpha) \wedge \bar{\omega}. \qquad \square$$

As C is proper, the map

$$H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \to \mathrm{Hom}(H^0(C(\mathbb{C}), \Omega^1)^+, \mathbb{R}(1)), \qquad \eta \mapsto \Big(\omega \mapsto \int_{C(\mathbb{C})} \eta \wedge \bar{\omega}\Big) \tag{3-19}$$

is injective (see [20, Remark 3.1]). By Lemma 3.1, for $\alpha \in H^2(\Gamma(C, 3))$, we have

$$\int_{C(\mathbb{C})} \mathrm{reg}_C(\beta_C(\alpha)) \wedge \bar{\omega} = 2 \int_{C(\mathbb{C})} r_3(2)_C(\alpha) \wedge \bar{\omega},$$

where $r_3(2)_C$ is Goncharov's regulator map (2-14). Therefore, we have the commutative diagram

$$\begin{tikzcd} H^2(\Gamma(C,3)) \arrow[r, "\psi_C"] \arrow[rr, bend left, "\beta_C"] \arrow[dr, "r_3(2)_C"'] & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \arrow[r, "\varphi_C"] \arrow[d] & H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \arrow[dl, "\frac{1}{2}\mathrm{reg}_C"] \\ & H^1(C(\mathbb{C}), \mathbb{R}(2))^+, & \end{tikzcd} \tag{3-20}$$

where the middle vertical map is just the composition $\frac{1}{2}\mathrm{reg}_C \circ \varphi_C$. In [20, Corollary 5.5], de Jeu showed that the images of the $r_3(2)_C$ and reg_C, as vector spaces, are the same.

Lemma 3.2. *Let $\alpha = \sum_j c_j \{f_j\}_2 \otimes g_j \in H^2(\Gamma(F, 3))$ with $c_j \in \mathbb{Q}$ and $f_j \in F^\times \setminus \{1\}$, $g_j \in F^\times$. Denote by $Y = C \setminus Z$ where Z is the closed subscheme of C consisting of the zeros and poles of $f_j, 1 - f_j, g_j$ for all j. With the fixed choice of signs of $\varphi^1_{(2)}$ and $\varphi^2_{(3)}$, we have*

$$\int_\gamma \mathrm{reg}_F(\beta_F(\alpha)) = 2 \int_\gamma r_3(2)_F(\alpha) \qquad \text{for any loop } \gamma \in H_1(Y(\mathbb{C}), \mathbb{Z}).$$

Proof. First notice that $\beta_F(\alpha) \in H^2_{\mathcal{M}}(F, \mathbb{Q}(3))$ actually belongs to the subgroup $H^2_{\mathcal{M}}(Y, \mathbb{Q}(3))$ (see [7, Theorem 5.4]). Then $\mathrm{reg}_F(\beta_F(\alpha))$ belongs to $H^1(Y(\mathbb{C}), \mathbb{R}(2))^+$. In particular, the integral $\int_\gamma \mathrm{reg}_F(\beta_F(\alpha))$ is well-defined for any loop $\gamma \in H_1(Y(\mathbb{C}), \mathbb{Z})$. Since $r_3(2)_F(\alpha)$ is represented by the form $\sum_j c_j \rho(f_j, g_j)$ which defines an element in $H^1(Y(\mathbb{C}), \mathbb{R})^+$, the integral $\int_\gamma r_3(2)_F(\alpha)$ is also well-defined.

In particular, we have $\mathrm{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha) \in H^1(Y(\mathbb{C}), \mathbb{R}(2))^+$. We consider the Mayer–Vietoris sequence

$$0 \longrightarrow H^1(C(\mathbb{C}), \mathbb{R}(2))^+ \longrightarrow H^1(Y(\mathbb{C}), \mathbb{R}(2))^+ \xrightarrow{\oplus(2\pi i)^{-1}\mathrm{Res}_p} \oplus_{p \in Z(\mathbb{C})} \mathbb{R}(1), \tag{3-21}$$

where Res_p is the residue map defined in Definition 2.2. We are going to show that $\mathrm{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)$ extends to $H^1(C(\mathbb{C}), \mathbb{R}(2))^+$. Let $p \in Z(\mathbb{C})$ supported on a closed point $x \in Z^1$ with the embedding $\sigma : k(x) \hookrightarrow \mathbb{C}$, i.e.,

$$\begin{tikzcd}
\mathrm{Spec}\, \mathbb{C} \arrow[r] \arrow[d, "\sigma"'] \arrow[dr, "p"] \arrow[rr, bend left, "\mathrm{id}"] & Z \times_{\mathbb{Q}} \mathbb{C} \arrow[r] \arrow[d] & \mathrm{Spec}\, \mathbb{C} \arrow[d] \\
\mathrm{Spec}\, k(x) \arrow[r, "x"'] & Z \arrow[r] & \mathrm{Spec}\, \mathbb{Q}.
\end{tikzcd}$$

With the fixed signs of $\varphi^1_{(2)}$ and $\varphi^2_{(3)}$ as before, we have the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma(F, 3)) \arrow[r, "\beta_F"] \arrow[d, "2\partial_x"'] & H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \arrow[r, "\mathrm{reg}_F"] \arrow[d, "\mathrm{Res}^{\mathcal{M}}_x"] & H^1(F, \mathbb{R}(2))^+ \arrow[d, "(2\pi i)^{-1}\mathrm{Res}_p"] \\
H^1(\Gamma(k(x), 2)) \arrow[r, "\beta^1_{(2)}"'] & H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)) \arrow[r, "\mathrm{reg}_{k(x)}"'] & \mathbb{R}(1),
\end{tikzcd} \tag{3-22}$$

where $\mathrm{reg}_{k(x)}$ is the composition $H^1_{\mathcal{M}}(k(x), \mathbb{Q}(2)) \xrightarrow{\sigma^*} H^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_{\mathbb{C}}} H^1_{\mathcal{D}}(\mathbb{C}, \mathbb{R}(2)) \simeq \mathbb{R}(1)$ as mentioned in (3-4). The commutativity of the right square follows from the compatibility of the Beilinson regulators and residues maps. We then have

$$\mathrm{Res}_p(\mathrm{reg}_F(\beta_F(\alpha))) = (4\pi i)\, \mathrm{reg}_{k(x)}(\beta^1_{(2)}(\partial_x(\alpha))) = (4\pi i)(\mathrm{reg}_{k(x)} \circ \beta^1_{(2)})\Big(\sum_j c_j v_x(g_j)\{f_j(x)\}_2\Big),$$

where the last equality follows from the definition of Goncharov's residues map (2-5). As mentioned in (3-4), the sign of $\varphi^1_{(2)}$ is chosen such that the lower composition map in the diagram (3-22) is induced by the map $\{z\}_2 \mapsto i\,D(\sigma(z))$. Therefore, we have

$$\begin{aligned}\operatorname{Res}_p(\operatorname{reg}_F(\beta_F(\alpha))) &= (4\pi i)i\ \sum_j c_j v_x(g_j) D(\sigma(f_j(x)))\\ &= -4\pi \sum_j c_j v_{\sigma(x)}(g_j) D(f_j(\sigma(x))) \quad (\text{as } f_j, g_j \in k(C))\\ &= 2\operatorname{Res}_p(r_3(2)_F(\alpha)) \quad (\text{by Lemma 2.3}).\end{aligned}$$

So $\operatorname{Res}_p(\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)) = 0$ for all $p \in Z(\mathbb{C})$, hence $\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)$ extends to $H^1(C(\mathbb{C}), \mathbb{R}(2))^+$. Therefore, the class $\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)(\alpha)$ is represented by $\eta + dt$, where η is a F_{dR}-invariant closed differential 1-form on $C(\mathbb{C})$ and t is a logarithmic growth function on $Y(\mathbb{C})$. Now let ω be a holomorphic 1-form on $C(\mathbb{C})$ such that $F_{\mathrm{dR}}(\omega) = \omega$. Since t is a logarithmic growth function on $Y(\mathbb{C})$, we have

$$\int_{C(\mathbb{C})} dt \wedge \bar{\omega} = \int_{C(\mathbb{C})} d(t\bar{\omega}) = 0$$

by using Stokes' theorem (see the proof of [19, Theorem 4.6]). We then have

$$\int_{C(\mathbb{C})} \eta \wedge \bar{\omega} = \int_{C(\mathbb{C})} (\eta + dt) \wedge \bar{\omega} = \int_{C(\mathbb{C})} (\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha)) \wedge \bar{\omega} = 0,$$

where the last equality is by Lemma 3.1. Since ω is an arbitrary F_{dR}-invariant holomorphic 1-form and such forms span a real vector space dual to $H^1(C(\mathbb{C}), \mathbb{R}(2))^+$ ([20, Remark 3.1]), we obtain that $\eta = ds$ for some function s on $C(\mathbb{C})$. So $\operatorname{reg}_F(\beta_F(\alpha)) - 2r_3(2)_F(\alpha))$ is represented by $d(s+t)$ for some logarithmic growth function $s + t$ on $Y(\mathbb{C})$. Hence

$$\int_\gamma \operatorname{reg}_F(\beta_F(\alpha)) = 2\int_\gamma r_3(2)_F(\alpha) \qquad \text{for any loop } \gamma \in H_1(Y(\mathbb{C}), \mathbb{Z}). \qquad \square$$

4. Main result

In Section 4.1, we construct an element in Deligne cohomology and in Section 4.2, we connect it to the Mahler measure. In Section 4.3, we construct an element in $K_4^{(3)}$ of a curve such that its regulator is related to the Deligne cohomology class constructed in Section 4.1. We prove Theorem 0.2 in Section 4.

4.1. ***Constructing an element in Deligne cohomology.*** Let

$$P(x, y, z) \in \mathbb{Q}[x, y, z]$$

be an irreducible polynomial. We denote by V_P the zero locus of P in $(\mathbb{C}^\times)^3$ and V_P^{reg} the smooth part of V_P. For $f, g, h \in \mathbb{C}(V_P^{\mathrm{reg}})^\times$, we recall the differential form mentioned in (2-10)

$$\begin{aligned}\eta(f, g, h) = \log|f| \left(\tfrac{1}{3} d\log|g| \wedge d\log|h| - d\arg(g) \wedge d\arg(h)\right)\\ + \log|g| \left(\tfrac{1}{3} d\log|h| \wedge d\log|f| - d\arg(h) \wedge d\arg(f)\right)\\ + \log|h| \left(\tfrac{1}{3} d\log|f| \wedge d\log|g| - d\arg(f) \wedge d\arg(g)\right).\end{aligned} \tag{4-1}$$

The form is bilinear and antisymmetric in f, g, h. It is defined on $V_P^{\mathrm{reg}} \setminus S_{f,g,h}$, where $S_{f,g,h}$ is the set of zeros and poles of f, g and h. Moreover, $\eta(f,g,h)$ is a closed form on $V_P^{\mathrm{reg}} \setminus S_{f,g,h}$ since

$$d\eta(f,g,h) = \mathrm{Re}\left(\frac{df}{f} \wedge \frac{dh}{h} \wedge \frac{dg}{g}\right),$$

which is zero in $V_P^{\mathrm{reg}} \setminus S_{f,g,h}$.

Lemma 4.1. *The differential form $\eta(x,y,z)$ defines an element in the Deligne cohomology $H^3_{\mathcal{D}}(\mathbb{G}_m^3, \mathbb{R}(3))$. Moreover, it represents the class $\mathrm{reg}_{\mathbb{G}_m^3}(\{x,y,z\})$, where $\mathrm{reg}_{\mathbb{G}_m^3} : H^3_{\mathcal{M}}(\mathbb{G}_m^3, \mathbb{Q}(3)) \to H^3_{\mathcal{D}}(\mathbb{G}_m^3, \mathbb{R}(3))$ is the Beilinson regulator map and $\{x,y,z\} \in H^3_{\mathcal{M}}(\mathbb{G}_m^3, \mathbb{Q}(3))$ is the Milnor symbol.*

Proof. By definition, $\eta(x,y,z) \in E^2_{\log,\mathbb{R}}(\mathbb{G}_m^3)$, and defines an element in $H^3_{\mathcal{D}}(\mathbb{G}_m^3, \mathbb{R}(3))$. By an observation in Section 1.4, $\mathrm{reg}_{\mathbb{G}_m^3}(\{x,y,z\})$ is represented by the cup product $\log|x| \cup \log|y| \cup \log|z|$ in Deligne cohomology. By the cup product formula (1-3), we have

$$\begin{aligned}
\log|x| \cup \log|y| \cup \log|z| &= (\log|x| \cup \log|y|) \cup \log|z| \\
&= (-1)^2 r_2(\log|x| \cup \log|y|) \log|z| + (\log|x| \cup \log|y|)\, r_1(\log|z|) \\
&= \left(\partial\left(\frac{1}{2}\log|x|\,\frac{dy}{y} - \frac{1}{2}\log|y|\,\frac{dx}{x}\right) - \bar{\partial}\left(\frac{1}{2}\log|y|\,\frac{d\bar{x}}{\bar{x}} - \frac{1}{2}\log|x|\,\frac{d\bar{y}}{\bar{y}}\right)\right)\log|z| \\
&\qquad + i \cdot (\log|x| d\arg y - \log|y| d\arg(x)) \wedge (\partial \log|z| - \bar{\partial}\log|z|) \\
&= \left(\frac{1}{2}\frac{dx}{x} \wedge \frac{dy}{y} + \frac{1}{2}\frac{d\bar{x}}{\bar{x}} \wedge \frac{d\bar{y}}{\bar{y}}\right)\log|z| - (\log|x|\, d\arg y - \log|y|\, d\arg x) \wedge d\arg z \\
&= \log|z|(d\log|x| \wedge d\log|y| - d\arg(x) \wedge d\arg y) \\
&\qquad - \log|y|\, d\arg(z) \wedge d\arg x - \log|x|\, d\arg(y) \wedge d\arg z.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\eta(x,y,z) - \log|x| \cup \log|y| \cup \log|z| \\
&= \tfrac{1}{3}\log|x| d\log|y| \wedge d\log|z| + \tfrac{1}{3}\log|y| d\log|z| \wedge d\log|x| - \tfrac{2}{3}\log|z| d\log|x| \wedge d\log|y| \\
&= -\tfrac{1}{3} d(\log|x| \log|z|\, d\log|y|) + \tfrac{1}{3} d(\log|y| \log|z|\, d\log|x|),
\end{aligned}$$

which is an exact form, hence $\mathrm{reg}_{\mathbb{G}_m^3}(\{x,y,z\})$ is represented by $\eta(x,y,z)$. □

Consequently, pulling back $\eta(x,y,z)$ by the embedding $V_P^{\mathrm{reg}} \overset{i}{\hookrightarrow} \mathbb{G}_m^3$, we see that $\eta(x,y,z)|_{V_P^{\mathrm{reg}}}$ is a representative of $\mathrm{reg}_{V_P^{\mathrm{reg}}}(\{x,y,z\})$ in $H^3_{\mathcal{D}}(V_P^{\mathrm{reg}}, \mathbb{R}(3))$. We come to the definition of *exact polynomials*.

Definition 4.2 (exact polynomial). A polynomial $P(x,y,z)$ is called *exact* if $\mathrm{reg}_{V_P^{\mathrm{reg}}}(\{x,y,z\})$ is trivial, i.e., $\eta(x,y,z)$ is an exact form on V_P^{reg}.

Remark 4.3. If P satisfies Lalín's condition (see Theorem 0.1(iii)), namely

$$x \wedge y \wedge z = \sum_j f_j \wedge (1-f_j) \wedge g_j \quad \text{in } \textstyle\bigwedge^3 \mathbb{Q}(V_P)^\times_{\mathbb{Q}} \tag{4-2}$$

for some functions $f_j \in \mathbb{Q}(V_P)^\times \setminus \{1\}$ and $g_j \in \mathbb{Q}(V_P)^\times$, then P is exact because

$$\eta(x,y,z)|_{V_P^{\mathrm{reg}}} = \sum_j \eta(f_j, 1-f_j, g_j) = \sum_j d\rho(f_j, g_j) = d\sum_j \rho(f_j, g_j),$$

where $\rho(f, g)$ is the differential form defined in (2-11). In particular, the polynomials of the form $A(x) + B(x)y + C(x)z$, where $A(x), B(x), C(x)$ are products of cyclotomic polynomials, are exact. Indeed, we have

$$\begin{aligned} x \wedge y \wedge z &= x \wedge y \wedge \frac{A(x)+B(x)y}{C(x)} \\ &= x \wedge y \wedge \Big(\frac{A(x)}{C(x)} \cdot \frac{A(x)+B(x)y}{A(x)}\Big) \\ &= x \wedge y \wedge \frac{A(x)}{C(x)} + x \wedge y \wedge \Big(1 + \frac{B(x)y}{A(x)}\Big) \\ &= x \wedge y \wedge \frac{A(x)}{C(x)} + x \wedge \frac{B(x)y}{A(x)} \wedge \Big(1 + \frac{B(x)y}{A(x)}\Big) - x \wedge \frac{B(x)}{A(x)} \wedge \Big(1 + \frac{B(x)y}{A(x)}\Big). \end{aligned} \tag{4-3}$$

For cyclotomic polynomials $\Phi(x)$, we have

$$\begin{aligned} x \wedge y \wedge \Phi_n(x) = x \wedge y \wedge \frac{x^n - 1}{\prod_{d \mid n, d \neq n} \Phi_d(x)} &= x \wedge y \wedge (x^n - 1) - \sum_{d \mid n, d \neq n} x \wedge y \wedge \Phi_d(x) \\ &= -\frac{1}{n} x^n \wedge (1 - x^n) \wedge y - \sum_{d \mid n, d \neq n} x \wedge y \wedge \Phi_d. \end{aligned}$$

For $n = 1$, $x \wedge y \wedge (x + 1) = -x \wedge (1 + x) \wedge y$. So we get (4-2) by induction on n.

From now on, let us assume our polynomial P satisfies the condition (4-2). We consider the involution

$$\tau : \mathbb{G}_m^3 \to \mathbb{G}_m^3, (x, y, z) \mapsto (1/x, 1/y, 1/z), \tag{4-4}$$

which maps V_P to V_{P^*}, where $P^*(x, y, z) := \bar{P}(1/x, 1/y, 1/z) = P(1/x, 1/y, 1/z)$. Let W_P be the curve defined by

$$\begin{cases} P(x, y, z) = 0, \\ P(1/x, 1/y, 1/z) = 0. \end{cases} \tag{4-5}$$

We call W_P the *Maillot variety*. The restriction $\tau|_{W_P} : W_P \to W_P$ is an automorphism. We view W_P as a curve over $\mathbb{Q}$. Then let C be the normalization of W_P. The condition (4-2) implies that

$$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j \quad \text{in } \textstyle\bigwedge^3 \mathbb{Q}(C)^\times_{\mathbb{Q}} \tag{4-6}$$

for some functions $f_j \in \mathbb{Q}(C)^\times \setminus \{1\}$ and $g_j \in \mathbb{Q}(C)^\times$.

Definition 4.4. Let $F = \mathbb{Q}(C)$ be the function field of C. We write

$$\xi := \sum_j \{f_j\}_2 \otimes g_j, \qquad \xi^* := \sum_j \{f_j \circ \tau\}_2 \otimes (g_j \circ \tau), \qquad \lambda := \xi + \xi^*, \tag{4-7}$$

which are elements in $B_2(F) \otimes F^\times_{\mathbb{Q}}$. Let us consider the following closed subschemes of V_P and V_{P^*}:

$$Z_1 = \{\text{zeros and poles of } f_j, 1 - f_j, g_j \text{ on } V_P \text{ for all } j\}, \tag{4-8}$$

$$Z_2 = \{\text{zeros and poles of } f_j \circ \tau, 1 - f_j \circ \tau, g_j \circ \tau \text{ on } V_{P^*} \text{ for all } j\}. \tag{4-9}$$

We define the following differential 1-forms on $V_P^{\mathrm{reg}} \setminus Z_1$ and $V_{P^*}^{\mathrm{reg}} \setminus Z_2$, respectively:

$$\rho(\xi) := \sum_j \rho(f_j, g_j), \qquad \rho(\xi^*) := \sum_j \rho(f_j \circ \tau, g_j \circ \tau), \tag{4-10}$$

where $\rho(f, g)$ is mentioned in (2-11). Denote by Z the closed subscheme of C

$$Z = \{\text{zeros and poles of } f_j, 1 - f_j, g_j, f_j \circ \tau, 1 - f_j \circ \tau, g_j \circ \tau \text{ on } C \text{ for all } j\}, \tag{4-11}$$

and set $Y = \iota(W_P^{\mathrm{reg}}) \setminus Z$, where $\iota : W_P^{\mathrm{reg}} \hookrightarrow C$ is the canonical embedding. Using the canonical embeddings of $Y(\mathbb{C})$ into V_P and V_{P^*}, we define the following differential 1-form on $Y(\mathbb{C})$:

$$\rho(\lambda) = \rho(\xi)|_{Y(\mathbb{C})} + \rho(\xi^*)|_{Y(\mathbb{C})}. \tag{4-12}$$

Lemma 4.5. *The element λ defines a class in $H^2(\Gamma(F, 3))$, and also an element in $H^2(\Gamma(Y, 3))$.*

Proof. We recall the polylogarithmic Goncharov complex

$$\begin{array}{rcccl} \Gamma(F,3): & B_3(F) \longrightarrow & B_2(F) \otimes F_{\mathbb{Q}}^\times & \xrightarrow{\alpha_3(2)} & \bigwedge^3 F_{\mathbb{Q}}^\times \\ & & \{f\}_2 \otimes g & \longmapsto & (1-f) \wedge f \wedge g. \end{array}$$

We have

$$\alpha_3(2)(\xi) = \sum_j \alpha_3(2)(\{f_j\}_2 \otimes g_j) = \sum_j (1 - f_j) \wedge f_j \wedge g_j = -x \wedge y \wedge z,$$

and

$$\begin{aligned} \alpha_3(2)(\xi^*) &= \sum_j \alpha_3(2)\big(\{f_j \circ \tau\}_2 \otimes (g_j \circ \tau)\big) = \sum_j (1 - f \circ \tau) \wedge (f_j \circ \tau) \wedge (g_j \circ \tau) \\ &= \tau^*\Big(\sum_j (1 - f_j) \wedge f_j \wedge g_j\Big) = \tau^*(-x \wedge y \wedge z) \\ &= -\frac{1}{x} \wedge \frac{1}{y} \wedge \frac{1}{z} = x \wedge y \wedge z, \end{aligned}$$

so $\alpha_3(2)(\lambda) = \alpha_3(2)(\xi) + \alpha_3(2)(\xi^*) = 0$. Then λ defines a class in $H^2(\Gamma(F, 3))$. Now we consider the following exact sequence (see Section 2.2):

$$0 \to H^2(\Gamma(Y, 3)) \to H^2(\Gamma(F, 3)) \xrightarrow{\oplus \partial_p} \bigoplus_{p \in Y^1} H^1(\Gamma(\mathbb{Q}(p), 2)), \tag{4-13}$$

where Y^1 is the set of closed points of Y. The residue of λ at $p \in Y^1$ is given by

$$\partial_p(\lambda) = \sum_j v_p(g_j)\{f_j(p)\}_2 + v_p(g_j \circ \tau)\{f_j \circ \tau(p)\}_2 \in H^1(\Gamma(\mathbb{Q}(p), 2), \tag{4-14}$$

which is trivial for every point $p \notin S$, where S is the closed subscheme of C

$$S = \{\text{zeros and poles of } g_j, g_j \circ \tau \text{ on } C \text{ for all } j\}. \tag{4-15}$$

We have $\partial_p(\lambda) = 0$ for all $p \in Y^1$, hence λ defines an element in $H^2(\Gamma(Y, 3))$. $\square$

Lemma 4.6. *The differential* 1*-form* $\rho(\lambda)$ *defines an element in* $H^2_{\mathcal{D}}(Y_{\mathbb{R}}, \mathbb{R}(3)) \simeq H^1(Y(\mathbb{C}), \mathbb{R}(2))^+$. *For any point* $p \in C(\mathbb{C})$, *the residue of* $\rho(\lambda)$ *at* p *is given by*

$$\mathrm{Res}_p(\rho(\lambda)) = -2\pi\left(\sum_j v_p(g_j)D(f_j(p)) + v_p(g_j \circ \tau)D(f_j \circ \tau)(p)\right), \tag{4-16}$$

where D *is the Bloch–Wigner dilogarithm function* (0-7).

Proof. The first statement follows from the fact that $\rho(\lambda)$ represents $r_3(2)_Y(\lambda)$, where

$$r_3(2)_Y : H^2(\Gamma(Y, 3)) \to H^2_{\mathcal{D}}(Y, R(3)) \simeq H^1(Y(\mathbb{C}), R(2))^+$$

is Goncharov's regulator map (2-14), or by checking directly that $\rho(\lambda)$ is closed and fixed under the action of the involution F_{dR}. The formula (4-16) follows directly from Lemma 2.3. □

Remark 4.7. If all the residues $u_p := \partial_p(\lambda)$ are trivial for all $p \in S$ (see (4-15)), then λ defines a unique class λ_C in $H^2(\Gamma(C, 3))$ and $\rho(\lambda)$ represents the class $r_3(2)_C(\lambda_C) \in H^2_{\mathcal{D}}(C, \mathbb{Q}(3)) \simeq H^1(C(\mathbb{C}), \mathbb{R}(2))^+$.

4.2. ***Relating the Mahler measure to the Deligne cohomology.*** We retain the notation from the previous section. In this section, we connect $\rho(\lambda)$ to the Mahler measure of P. Recall that the Deninger chain associated to P is defined by

$$\Gamma = \{(x, y, z) \in (\mathbb{C}^\times)^3 : P(x, y, z) = 0,\ |x| = |y| = 1,\ |z| \geq 1\}. \tag{4-17}$$

Its orientation is induced from $\mathbb{T}^2$: for each $(x_0, y_0) \in \mathbb{T}^2$, there are finitely many $z \in \mathbb{C}$ such that $|z| \geq 1$ and $P(x_0, y_0, z) = 0$, then by letting (x_0, y_0) runs on the torus $\mathbb{T}^2_{(x,y)}$ along the usual orientation, we get the orientation of Γ. Its boundary is given by

$$\partial\Gamma = \{(x, y, z) \in (\mathbb{C}^\times)^3 : P(x, y, z) = 0,\ |x| = |y| = |z| = 1\}.$$

Deninger [12, Proposition 3.3] showed that if Γ is contained in V_P^{reg}, then we get the formula

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{4\pi^2}\int_\Gamma \eta(x, y, z), \tag{4-18}$$

where $\tilde{P}(x, y)$ is the leading coefficient of $P(x, y, z)$ considered as a polynomial in z. If furthermore, $\partial\Gamma = \varnothing$, then $[\Gamma] \in H_2(V_P^{\mathrm{reg}}, \mathbb{Z})$ and the Mahler measure is written as a pairing in Deligne cohomology

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{4\pi^2}\langle[\Gamma], \mathrm{reg}_{V_P^{\mathrm{reg}}}(\{x, y, z\})\rangle_{V_P^{\mathrm{reg}}}.$$

Since $P(x, y, z)$ has rational coefficients, we can write

$$\partial\Gamma = \{P(x, y, z) = P(1/x, 1/y, 1/z) = 0\} \cap \{|x| = |y| = |z| = 1\},$$

which is contained in W_P, and may contain some singularities of W_P. We have the following lemma.

Lemma 4.8. *We assume that*

$$x \wedge y \wedge z = \sum_j f_j \wedge (1 - f_j) \wedge g_j \quad \text{in } \textstyle\bigwedge^3 \mathbb{Q}(V_P)^\times_{\mathbb{Q}}. \tag{4-19}$$

Suppose that Γ is contained in V_P^{reg} and that $\partial\Gamma$ is contained in $Y(\mathbb{C})$ (see Definition 4.4). Then $\partial\Gamma$ defines an element in the singular homology group $H_1(Y(\mathbb{C}), \mathbb{Z})^+$, where "+" denotes the invariant part by the complex conjugation, and we can write the Mahler measure as a pairing in Deligne cohomology of $Y_{\mathbb{R}}$:

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{8\pi^2} \langle [\partial\Gamma], [\rho(\lambda)] \rangle_Y \,, \tag{4-20}$$

where the pairing is given by

$$\langle \cdot\,, \cdot \rangle_Y : H_1(Y(\mathbb{C}), \mathbb{Z})^+ \times H^1(Y(\mathbb{C}), \mathbb{R}(2))^+ \to \mathbb{R}(2). \tag{4-21}$$

Proof. Since $\Gamma \subset V_P^{\mathrm{reg}}$ and $\partial\Gamma \subset Y(\mathbb{C})$, $\partial\Gamma$ defines an element in $H_1(Y(\mathbb{C}), \mathbb{Z})$ by considering the sequence

$$\begin{array}{ccccc} H_2(V_P^{\mathrm{reg}}, \partial\Gamma, \mathbb{Z}) & \longrightarrow & H_1(\partial\Gamma, \mathbb{Z}) & \longrightarrow & H_1(Y(\mathbb{C}), \mathbb{Z}) \\ [\Gamma] & \longmapsto & [\partial\Gamma] & \longmapsto & [\partial\Gamma]. \end{array}$$

Now we show that $\partial\Gamma$ is invariant under the complex conjugation. Notice that the action of the complex conjugation on $\partial\Gamma$ is the same as the action of the involution τ (4-4) on $\partial\Gamma$ because $\bar{x} = x^{-1}$ for $x \in \mathbb{T}^1$. So it suffices to show that τ fixes $\partial\Gamma$. Clearly, $\tau(\partial\Gamma) = \partial\Gamma$ as sets. We claim that τ preserves the orientation of $\partial\Gamma$. Notice that the orientation of $\partial\Gamma$ is induced from the orientation of Γ, and the orientation of Γ comes from the orientation of $\mathbb{T}^2$. The map

$$\tau|_{\mathbb{T}^2} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (1/x, 1/y) \tag{4-22}$$

preserves the orientation of $\mathbb{T}^2$; hence τ preserves the orientation of $\partial\Gamma$. Note that the condition (4-19) implies that $\eta(x, y, z)|_{V_P^{\mathrm{reg}}} = d\rho(\xi)$. Then by applying Stokes' theorem to (4-18), we get

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) - \frac{1}{4\pi^2} \int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})}. \tag{4-23}$$

We have

$$\int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})} = \int_{\tau(\partial\Gamma)} \tau^*(\rho(\xi)|_{Y(\mathbb{C})}) = \int_{\partial\Gamma} \tau^*(\rho(\xi)|_{Y(\mathbb{C})}) = \int_{\partial\Gamma} \rho(\xi^*)|_{Y(\mathbb{C})},$$

where the second equality is because $\tau(\partial\Gamma) = \partial\Gamma$ as sets and τ preserves the orientation of $\partial\Gamma$. Then, by (4-23), we get

$$\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = -\frac{1}{4\pi^2} \int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})} = -\frac{1}{8\pi^2} \int_{\partial\Gamma} \rho(\xi)|_{Y(\mathbb{C})} + \rho(\xi^*)|_{Y(\mathbb{C})} = -\frac{1}{8\pi^2} \int_{\partial\Gamma} \rho(\lambda),$$

which is exactly formula (4-20). $\square$

4.3. ***Constructing an element in the motivic cohomology.*** In Section 4.1, we constructed an element λ that defines a class in $H^2(\Gamma(Y,3))$ and its regulator is represented by the differential 1-form $\rho(\lambda)$. In this section, we construct an element in $H^2(\Gamma(C_K,3))$, where $C_K = C\times_{\mathbb{Q}} K$ for a certain number field K. It gives rise to an element in motivic cohomology $H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$ via de Jeu's map β_{C_K}. Finally, we show that this motivic cohomology class descends to an element in $H^2_{\mathcal{M}}(C,\mathbb{Q}(3))$.

Recall that the residues u_p are trivial for all $p\notin S$, where S is the closed subset of C defined in (4-15). As discussed in Remark 4.7, if u_p vanish for all $p\in S$, λ defines an element in $H^2(\Gamma(C,3))$. When the residues are nontrivial, we modify λ by its residues. This method is inspired by Bloch's trick (see, e.g., [1; 30]). Let S' be the closed subset of S consisting of the points p such that $u_p\neq 0$. Let K be the splitting field of S' in $\mathbb{C}$; this is the smallest Galois extension $K/\mathbb{Q}$ that contains all the residue fields $\mathbb{Q}(p)$ for $p\in S'$. For a geometric point $q:\mathbb{Q}(p)\hookrightarrow K$ over a point p of S', we define u_q as the image of u_p under the embedding $\mathcal{B}(\mathbb{Q}(p))\hookrightarrow^q \mathcal{B}(K)$. Then for $q\in S'(K)$, u_q defines an element in the Bloch group $\mathcal{B}(K)$. It is compatible with the Galois action, i.e., $\sigma(u_q)=u_{\sigma(q)}$ for $q\in S'(K)$ and $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$,

$$\begin{tikzcd}
\mathcal{B}(\mathbb{Q}(p)) \arrow[r, hook, "q"] \arrow[dr, hook] & \mathcal{B}(K) \arrow[d, "\sigma"] \\
 & \mathcal{B}(K).
\end{tikzcd} \tag{4-24}$$

Denote by $K(C)$ the function field of $C\times_{\mathbb{Q}} K$. The inclusion $\mathbb{Q}(p)\hookrightarrow^j K(C)$ induces a map $B_i(\mathbb{Q}(C))\xrightarrow{j} B_i(K(C))$, which is not an inclusion generally. We have the commutative diagram

$$\begin{tikzcd}
B_3(\mathbb{Q}(C)) \arrow[r] \arrow[d] & B_2(\mathbb{Q}(C))\otimes\mathbb{Q}(C)^\times_{\mathbb{Q}} \arrow[r, "\alpha_3(2)"] \arrow[d, "j"] & \bigwedge^3\mathbb{Q}(C)^\times_{\mathbb{Q}} \arrow[d, hook] \\
B_3(K(C)) \arrow[r] & B_2(K(C))\otimes K(C)^\times_{\mathbb{Q}} \arrow[r, "\alpha_3(2)"] & \bigwedge^3 K(C)^\times_{\mathbb{Q}}.
\end{tikzcd}$$

This implies a map $j: H^2(\Gamma(\mathbb{Q}(C),3))\to H^2(\Gamma(K(C),3))$. By Lemma 4.5, λ defines a class in $H^2(\Gamma(\mathbb{Q}(C),3))$, hence $j(\lambda)$ defines a class in $H^2(\Gamma(K(C),3))$. We have the exact sequence

$$0\to H^2(\Gamma(C_K,3))\longrightarrow H^2(\Gamma(K(C),3))\xrightarrow{\oplus\partial_q}\bigoplus_{q\in(C_K)^1} H^1(\Gamma(K,2)),$$

where $(C_K)^1$ is the set of closed points of C_K. We have $\partial_q(j(\lambda))=u_q$ for $q\in S'_K=S'(K)$ and trivial otherwise.

We assume that the difference of any two geometric points $q_1,q_2\in S'(K)$ in the Jacobian of C is torsion of order dividing a fixed integer N. Fix $\mathcal{O}\in S'(K)$. Then for any point $q\in S'(K)-\{\mathcal{O}\}$, there is a rational function $f_q\in K(C)^\times$ such that

$$\mathrm{div}(f_q)=N(\mathcal{O})-N(q) \tag{4-25}$$

in C_K. We set $f_{\mathcal{O}}=1$.

Definition 4.9. We set

$$\lambda' := j(\lambda) + \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}(u_q \otimes f_q), \tag{4-26}$$

which defines an element in $B_2(K(C)) \otimes K(C)^\times_{\mathbb{Q}}$.

Lemma 4.10. *The element λ' defines a class in $H^2(\Gamma(K(C),3))$.*

Proof. For $q \in S'(K)$, we recall the following Goncharov's complex (2-4):

$$\begin{tikzcd} B_3(K(C)) \arrow[r] & B_2(K(C)) \otimes K(C)^\times_{\mathbb{Q}} \arrow[r, "\alpha_3(2)"] \arrow[d, "\partial_q"'] & \bigwedge^3 K(C)^\times_{\mathbb{Q}} \arrow[d] \\ & B_2(K) \arrow[r, "\alpha_2(1)"] & \bigwedge^2 K^\times_{\mathbb{Q}}. \end{tikzcd} \tag{4-27}$$

As discussed above, $j(\lambda)$ defines a class in $H^2(\Gamma(K(C),3))$, hence $\alpha_3(2)(j(\lambda)) = 0$. For $q \in S'(K)$, we have $\alpha_2(1)(u_q) = 0$ because $u_q \in \mathcal{B}(K)$. We thus have $\alpha_3(2)\left(u_q \otimes f_q\right) = (\alpha_2(1)(u_q)) \wedge f_q = 0$. This implies that

$$\alpha_3(2)(\lambda') = \alpha_3(2)(j(\lambda)) + \frac{1}{N} \sum_{q\in S'(K)-\{\mathcal{O}\}} \alpha_3(2)(u_q \otimes f_q) = 0,$$

hence λ' defines an element in $H^2(\Gamma(K(C),3))$. □

Notice that λ' depends on the choice of rational function $f_q \in K(C)^\times$. However, the following lemma is sufficient for us.

Lemma 4.11. *The image of λ' under de Jeu's map* (3-15)

$$\beta_{K(C)} : H^2(\Gamma(K(C),3)) \to H^2_{\mathcal{M}}(K(C), \mathbb{Q}(3)), \tag{4-28}$$

does not depend on the choice of $f_q \in K(C)^\times$.

Proof. Let $q \in S'(K)$. Let $f'_q \in K(C)^\times$ be another rational function such that $\operatorname{div}(f'_q) = N(\mathcal{O}) - N(q)$. Then $\operatorname{div}(f_q/f'_q) = 0$, hence f_q/f'_q defines an element in a finite field extension of K, denoted by L. Then $u_q \otimes (f_q/f'_q)$ defines an element in $B_2(L) \otimes L^\times$. In the proof of Lemma 4.10, we showed that $\alpha_3(2)(u_q \otimes f_q) = 0$, this implies that

$$\alpha_3(2)(u_q \otimes (f_q/f'_q)) = \alpha_3(2)(u_q \otimes f_q) - \alpha_3(2)(u_q \otimes f'_q) = 0,$$

hence $u_q \otimes (f_q/f'_q)$ defines a class in $H^2(\Gamma(L,3))$. We consider de Jeu's map

$$\beta_L : H^2(\Gamma(L,3)) \to K_4(L)_{\mathbb{Q}}.$$

By Borel's theorem, K_4 group of a number field is torsion, so $K_4(L)_{\mathbb{Q}} = 0$. This implies that the images of $u_q \otimes (f_q/f'_q)$ under the map β_L in $K_4(L)_{\mathbb{Q}}$ all vanish. Hence the image of λ' under de Jeu's map does not depend on the choice of f_q. □

Lemma 4.12. *All the residues of λ' in the following localization sequence vanish*:

$$0 \to H^2(\Gamma(C_K,3)) \to H^2(\Gamma(K(C),3)) \xrightarrow{2\partial} \bigoplus_{q\in (C_K)^1} H^1(\Gamma(K,2)),$$

and thus λ' defines a unique element $\lambda'_{C_K} \in H^2(\Gamma(C_K,3))$.

Proof. We have $\partial_q(\lambda')=0$ for all $q \notin S'(K)$. For $q \in S'(K)$, we have

$$\begin{aligned}
\partial_q(\lambda') &= u_q + \sum_{q'\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\,\partial_q(u_{q'}\otimes f_{q'}) \\
&= u_q + \sum_{q'\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\cdot v_q(f_{q'})\cdot u_{q'} \\
&= \begin{cases} u_q + \dfrac{1}{N}\cdot v_q(f_q)\cdot u_q = u_q - u_q = 0 & \text{if } q \neq \mathcal{O}, \\ u_{\mathcal{O}} + \sum_{q'\in S'(K)-\{\mathcal{O}\}} \dfrac{1}{N}\cdot v_{\mathcal{O}}(f_{q'})\cdot u_{q'} = \sum_{q'\in S'(K)} u_{q'} & \text{if } q = \mathcal{O}. \end{cases}
\end{aligned}$$

Now let $\pi_K : C_K \to \operatorname{Spec} K$ be the structure morphism and $i_K : (C_K)^1 \hookrightarrow C_K$ be the canonical embedding. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(\Gamma(C_K,3)) \arrow[r] \arrow[d, "\beta_{C_K}"'] & H^2(\Gamma(K(C),3)) \arrow[r, "2\partial"] \arrow[d, "\beta_{K(C)}"'] & \bigoplus_{q\in (C_K)^1} H^1(\Gamma(K,2)) \arrow[d, "\simeq"', "\beta^1_{(2)}"] & \\
0 \arrow[r] & H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[r] & H^2_{\mathcal{M}}(K(C),\mathbb{Q}(3)) \arrow[r, "\mathrm{Res}^{\mathcal{M}}"] & \bigoplus_{q\in (C_K)^1} H^1_{\mathcal{M}}(K,\mathbb{Q}(2)) \arrow[r, "(i_K)_*"] \arrow[dr, "\Sigma"'] & H^3_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[d, "(\pi_K)_*"] \\
 & & & & H^1_{\mathcal{M}}(K,\mathbb{Q}(2)),
\end{tikzcd}$$

(see diagram (3-16)), where the two horizontal sequences are exact and Σ is the trace map, which sends $(u_q)_{q\in S'(K)}$ to $\sum_{q\in S'(K)} u_q$. Then we have $\sum_{q\in S'(K)} u_q = 0$ by the commutativity of the bottom triangle. This shows that $\partial_q(\lambda')=0$ for all $q \in S'(K)$, then λ' defines a unique element in $H^2(\Gamma(C_K,3))$. □

By the previous lemma, we constructed an element $\lambda'_{C_K} \in H^2(\Gamma(C_K,3))$. Via the map

$$\beta_{K_C} : H^2(\Gamma(C_K,3)) \to H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)),$$

we obtain a class $\beta_{C_K}(\lambda'_{C_K}) \in H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$.

Lemma 4.13. *The class $\beta_{C_K}(\lambda'_{C_K}) \in H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$ is* $\mathrm{Gal}(K/\mathbb{Q})$*-invariant.*

Proof. The Galois action of $\mathrm{Gal}(K/\mathbb{Q})$ on $H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))$ is induced from the action on the function field, hence it is sufficient to check that $\beta_{K(C)}(\lambda') \in H^2(\Gamma(K(C),3))$ is $\mathrm{Gal}(K/\mathbb{Q})$-invariant. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we have

$$\sigma(\lambda') = \sigma(j(\lambda)) + \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\,\sigma(u_q\otimes f_q) = j(\lambda) + \sum_{q\in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\,u_{\sigma(q)}\otimes\sigma(f_q),$$

because $\lambda \in B_2(\mathbb{Q}(C)) \otimes \mathbb{Q}(C)^\times$ and $\sigma(u_q) = u_{\sigma(q)}$ (see diagram (4-24)). Since $\operatorname{div}(f_q) = N(\mathcal{O}) - N(q)$ for $q \in S'(K) - \{\mathcal{O}\}$, we have $\operatorname{div}(\sigma(f_q)) = N(\sigma(\mathcal{O})) - N(\sigma(q))$. And by definition of $f_{\sigma(q)}$, we have $\operatorname{div}(f_{\sigma(q)}) = N(\mathcal{O}) - N(\sigma(q))$. Hence

$$\operatorname{div}(\sigma(f_q)) = \operatorname{div}(f_{\sigma(q)}) - N(\mathcal{O}) + N(\sigma(\mathcal{O})) = \operatorname{div}(f_{\sigma(q)}) - \operatorname{div}(f_{\sigma(\mathcal{O})}) = \operatorname{div}(f_{\sigma(q)}/f_{\sigma(\mathcal{O})}). \tag{4-29}$$

Write $\mathcal{O}' = \sigma^{-1}(\mathcal{O}) \in S'(K)$, we have

$$\begin{aligned}
\lambda + \sum_{q \in S'(K)-\{\mathcal{O}\}} \frac{1}{N} u_{\sigma(q)} \otimes \frac{f_{\sigma(q)}}{f_{\sigma(\mathcal{O})}}
&= \lambda + \sum_{q \in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes \frac{f_{\sigma(q)}}{f_{\sigma(\mathcal{O})}} + \frac{1}{N}\, u_{\mathcal{O}} \otimes \frac{f_{\mathcal{O}}}{f_{\sigma(\mathcal{O})}} \\
&= \lambda + \sum_{q \in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes \frac{f_{\sigma(q)}}{f_{\sigma(\mathcal{O})}} - \frac{1}{N}\, u_{\mathcal{O}} \otimes f_{\sigma(\mathcal{O})} \qquad \text{(since } f_{\mathcal{O}} = 1) \\
&= \lambda + \sum_{q \in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(q)} - \sum_{q \in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(\mathcal{O})} - \frac{1}{N}\, u_{\mathcal{O}} \otimes f_{\sigma(\mathcal{O})} \\
&= \lambda + \sum_{q \in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(q)} - \sum_{q \in S'(K)-\{\mathcal{O}\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(\mathcal{O})} \\
&= \lambda + \sum_{q \in S'(K)-\{\mathcal{O},\mathcal{O}'\}} \frac{1}{N}\, u_{\sigma(q)} \otimes f_{\sigma(q)} + \frac{1}{N}\, u_{\sigma(\mathcal{O})} \otimes f_{\sigma(\mathcal{O})} \qquad \Bigl(\text{since } \sum_{q \in S'(K)} u_q = 0\Bigr) \\
&= \lambda + \sum_{q \in S'(K)-\{\mathcal{O}'\}} \frac{1}{N} u_{\sigma(q)} \otimes f_{\sigma(q)} \\
&= \lambda'.
\end{aligned}$$

By Lemma 4.11, then, $\beta_{K(C)}(\sigma(\lambda')) = \beta_{K(C)}(\lambda')$ for all $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$. Since de Jeu's map β is functorial, it is compatible with the Galois action, so $\sigma(\beta_{K(C)}(\lambda')) = \beta_{K(C)}(\lambda')$ for all $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$. $\square$

Thus $\beta_{C_K}(\lambda'_{C_K})$ defines a class in $H^2_{\mathcal{M}}(C_K, \mathbb{Q}(3))^{\operatorname{Gal}(K/\mathbb{Q})}$. Setting $\pi : C_K \to C$, we have the Galois descent of motivic cohomology as mentioned in (1-7):

$$\pi^* : H^2_{\mathcal{M}}(C, \mathbb{Q}(3)) \xrightarrow{\simeq} H^2_{\mathcal{M}}(C_K, \mathbb{Q}(3))^{\operatorname{Gal}(K/\mathbb{Q})}. \tag{4-30}$$

Hence $(\pi^*)^{-1}(\beta_{C_K}(\lambda'_{K_C}))$ is an element in $H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$.

4.4. ***Proof of Theorem 0.2.*** In this section, we keep the notations as in Section 4.3. To prove Theorem 0.2, the main idea is that we relate the regulator of the motivic cohomology class constructed in Section 4.3 to the Deligne cohomology class constructed in Section 4.1, hence to the Mahler measure of the polynomial P by Section 4.2.

First, as mentioned in (4-14), u_p defines an element in $\mathcal{B}(\mathbb{Q}(p))$ for $p \in S$. By Remark 4.7, if $u_p = 0$ for all $p \in S$, then $\beta_C(\lambda_C) \in H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$ and $\rho(\lambda)$ represents the class $r_3(2)_C(\lambda_C) \in H^1(C(\mathbb{C}), \mathbb{R}(2))^+$. Denote by $i : Y(\mathbb{C}) \hookrightarrow C(\mathbb{C})$ the canonical embedding. Then by Lemma 4.8, we have

$$\begin{aligned}\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = -\frac{1}{8\pi^2}\langle[\partial\Gamma], [\rho(\lambda)]\rangle_{Y(\mathbb{C})} &= -\frac{1}{8\pi^2}\left\langle[\partial\Gamma], i^* r_3(2)_C(\lambda_C)\right\rangle_{Y(\mathbb{C})} \\ &= -\frac{1}{8\pi^2}\langle i_*[\partial\Gamma], r_3(2)_C(\lambda_C)\rangle_{C(\mathbb{C})} \\ &= -\frac{1}{16\pi^2}\left\langle i_*[\partial\Gamma], \mathrm{reg}_C(\beta_C(\lambda_C))\right\rangle_{C(\mathbb{C}),}\end{aligned}$$

where the last equality follows from the diagram (3-20). Apply Beilinson's conjecture 1.11 to $\beta_C(\lambda_C) \in H^2_{\mathcal{M}}(C, \mathbb{Q}(3))$ and $i_*[\partial\Gamma] \in H_1(C(\mathbb{C}), \mathbb{Q})^+$ (see Lemma 4.8), we thus have

$$\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = a \cdot L'(E, -1) \qquad (a \in \mathbb{Q}).$$

When the residues $u_p \in \mathcal{B}(\mathbb{Q}(p))$ are nontrivial for some $p \in S' \subset S$, we define $u_q \in \mathcal{B}(K)$ for $q \in S'(K)$, where K is the splitting field of S' in $\mathbb{C}$ (see the beginning of Section 4.3). Let $K(C)$ denote the function field of C_K. Fix a point $\mathcal{O} \in S'(K)$. Recall from Definition 4.9 the element of $B_2(K(C)) \otimes K(C)^\times$ defined by

$$\lambda' = j(\lambda) + \sum_{q \in S'(K) - \{\mathcal{O}\}} \frac{1}{N}(u_q \otimes f_q),$$

where $f_q \in K(C)^\times$ is defined just before Definition 4.9. We prove that λ' defines a class in $H^2(\Gamma(K(C), 3))$ (see Lemma 4.10). The differential form $\rho(\lambda)$ represents the class $r_3(2)_{\mathbb{Q}(C)}(\lambda)$, we then have

$$\begin{aligned}\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = -\frac{1}{8\pi^2}\int_{\partial\Gamma}\rho(\lambda) &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{\mathbb{Q}(C)}(\lambda) \\ &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{K(C)}(j(\lambda)),\end{aligned}$$

where $j : \mathbb{Q}(C) \hookrightarrow K(C)$. Hence

$$\begin{aligned}\mathrm{m}(P) - \mathrm{m}(\tilde{P}) &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{K(C)}\Big(\lambda' - \sum_{q \in S'(K) - \{\mathcal{O}\}} \frac{1}{N} u_q \otimes f_q\Big) \\ &= -\frac{1}{8\pi^2}\int_{\partial\Gamma} r_3(2)_{K(C)}(\lambda') + \frac{1}{8N\pi^2}\sum_{q \in S'(K) - \{\mathcal{O}\}}\int_{\partial\Gamma} r_3(2)_{K(C)}(u_q \otimes f_q) \\ &= -\frac{1}{16\pi^2}\int_{\partial\Gamma}\mathrm{reg}_{K(C)}(\beta_{K(C)}(\lambda')) + \frac{1}{8N\pi^2}\sum_{q \in S'(K) - \{\mathcal{O}\}} D(u_q)\int_{\partial\Gamma} d\arg(f_q), \qquad (4\text{-}31)\end{aligned}$$

where the last equality follows from Lemma 3.2 and the fact that $\partial\Gamma$ is assumed to be contained in $Y(\mathbb{C})$

(see Lemma 4.8). We have the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma(C_K,3)) \arrow[r, "\beta_{C_K}"] \arrow[d, hook] & H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[r, "\mathrm{reg}_{C_K}"] \arrow[d, hook] & H^1(C_K(\mathbb{C}),(\mathbb{R}(2))^+) \arrow[d, hook] \\
H^2(\Gamma(K(C),3)) \arrow[r, "\beta_{K(C)}"'] & H^2_{\mathcal{M}}(K(C),\mathbb{Q}(3)) \arrow[r, "\mathrm{reg}_{K(C)}"'] & H^1(K(C),\mathbb{R}(2))^+).
\end{tikzcd}$$

As $\lambda' \in H^2(\Gamma(K(C),3))$ defines a unique a class $\lambda'_{C_K} \in H^2(\Gamma(C_K,3))$ (see Lemma 4.12), we have

$$\mathrm{reg}_{K(C)}(\beta_{K(C)}(\lambda')) = \mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'_{C_K})).$$

Hence the first integral in (4-31) can be written as the following pairing in de Rham cohomology:

$$\langle [\partial\Gamma], (i_K)^*\mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'))\rangle_{Y_K(\mathbb{C})} = \langle (i_K)_*[\partial\Gamma], \mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'_{C_K}))\rangle_{C_K(\mathbb{C})}, \tag{4-32}$$

where $i_K : Y_K(\mathbb{C}) \hookrightarrow C_K(\mathbb{C})$ is the canonical embedding. Moreover, by the functorial property of the Beilinson regulator map, we have the commutative diagram

$$\begin{tikzcd}
H^2(\Gamma(C_K,3)) \arrow[r, "\beta_{C_K}"] & H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3)) \arrow[d, "\mathrm{reg}_{C_K}"] & H^2_{\mathcal{M}}(C,\mathbb{Q}(3)) \arrow[l, hook', "\pi^*"'] \arrow[d, "\mathrm{reg}_C"] \\
 & H^1(C_K(\mathbb{C}),\mathbb{R}(2))^+ \arrow[r, equal] & H^1(C(\mathbb{C}),\mathbb{R}(2))^+,
\end{tikzcd}$$

where π^* is induced from the isomorphism $\pi^* : H^2_{\mathcal{M}}(C,\mathbb{Q}(3)) \xrightarrow{\simeq} H^2_{\mathcal{M}}(C_K,\mathbb{Q}(3))^{\mathrm{Gal}(K/\mathbb{Q})}$ mentioned in (4-30). Hence by identifying de Rham cohomology as well as singular homology of $C_K(\mathbb{C})$ and $C(\mathbb{C})$, we have

$$\langle (i_K)_*[\partial\Gamma], \mathrm{reg}_{C_K}(\beta_{C_K}(\lambda'_{C_K}))\rangle_{C_K(\mathbb{C})} = \langle i_*[\partial\Gamma], \mathrm{reg}_C((\pi^*)^{-1}(\beta_{C_K}(\lambda'_{C_K})))\rangle_{C(\mathbb{C})}.$$

Applying Beilinson's conjecture to $(\pi^*)^{-1}(\beta_{C_K}(\lambda'_{C_K})) \in H^2_{\mathcal{M}}(C,\mathbb{Q}(3))$, we obtain that

$$\frac{1}{(2\pi i)^2}\langle i_*[\partial\Gamma], \mathrm{reg}_C((\pi^*)^{-1}(\beta_{C_K}(\lambda'_{C_K})))\rangle_{C(\mathbb{C})} = a\cdot L'(E,-1), \quad a \in \mathbb{Q},$$

where E is the Jacobian of C. From (4-31), by setting $b_q = \frac{1}{2\pi}\int_{\partial\Gamma} d\arg f_q$, we have

$$\mathrm{m}(P) - \mathrm{m}(\tilde{P}) = a\cdot L'(E,-1) + \frac{1}{4N\pi}\sum_{q\in S'(K)\setminus\{\mathcal{O}\}} b_q\cdot D(u_q), \quad a\in\mathbb{Q}. \tag{4-33}$$

We will show that for $f \in \bar{\mathbb{Q}}(C)^\times$ and $\gamma : [0,1] \to C(\mathbb{C})$ is a loop, $\int_\gamma d\arg f$ is a integral multiple of 2π. In fact, we can always find a partition

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1,$$

such that γ is the union of $\gamma_j : [a_j, a_{j+1}] \to C(\mathbb{C})$ for $j = 0,\ldots,n-1$ and $\gamma_j([a_j,a_{j+1}])$ is contained in a local coordinate chart of $C(\mathbb{C})$. Then

$$\begin{aligned}\int_\gamma d\arg f &= \sum_{j=0}^{n-1}\int_{\gamma_j} d\arg f = \sum_{j=0}^{n-1}\arg f(\gamma_j(a_{j+1})) - \arg f(\gamma_j(a_j))\\ &= -\arg f(\gamma_0(0)) + \arg f(\gamma_{n-1}(1)) + \sum_{j=0}^{n-2}\arg f(\gamma_j(a_{j+1})) - \arg f(\gamma_{j+1}(a_{j+1}))\\ &= 2\pi k,\end{aligned}$$

for some integer k, since $\gamma_0(0) = \gamma(0) = \gamma(1) = \gamma_{n-1}(1)$ and $\gamma_j(a_{j+1}) = \gamma_{j+1}(a_{j+1})$ for $j = 0, \dots, n-2$. In particular, we get $\int_{\partial\Gamma} d\arg f = 2\pi k$, for some $k \in \mathbb{Z}$. This implies that $b_q \in \mathbb{Z}$ for all $q \in S'(K)$. Although the coefficients b_q depend on the choice of $\mathcal{O}$, the D-values in identity (4-33) do not. Indeed, if we remove $\mathcal{O}$ from $S'(K)$, its complex conjugation $c(\mathcal{O}) \in S'(K)$ maintains the D-values in identity (4-33) because $D(u_{c(\mathcal{O})}) = D(c(u_{\mathcal{O}})) = -D(u_{\mathcal{O}})$, where c is the complex conjugation. □

Remark 4.14. (a) By Lemma 4.6, we have

$$\mathrm{m}(P) = \mathrm{m}(\tilde{P}) + a \cdot L'(E, -1) - \frac{1}{8N\pi^2}\sum_{q\in S'(K)-\{\mathcal{O}\}} b_q \cdot \mathrm{Res}_q(\rho(\lambda)).$$

(b) In some cases, D-values on Bloch group elements can relate to Dirichlet L-values. Let χ be a primitive Dirichlet character of conductor f, we have

$$L(\chi, 2) = \frac{1}{G(\bar{\chi})}\sum_{k=1}^{f} \bar{\chi}(k)\ Li_2(e^{2\pi i k/f}),$$

where $G(\bar{\chi}) = \sum_{k=1}^{f} \bar{\chi}(k)e^{2\pi i k/f}$ is the Gauss sum of χ. Thus, when χ is odd quadratic, then

$$L(\chi, 2) = \frac{1}{\sqrt{f}}\sum_{k=1}^{f} \chi(k)\ D(e^{2\pi i k/f}).$$

Then

$$L'(\chi, -1) = \frac{f^{3/2}}{4\pi}L(\chi, 2) = \frac{f}{4\pi}\sum_{k=1}^{f} \chi(k)\ D(e^{2\pi i k/f}).$$

In particular, if χ_{-3} and χ_{-4} denote the nontrivial characters of modulo 3 and 4, respectively, we have

$$L'(\chi_{-3}, -1) = \frac{3}{2\pi}D(e^{2\pi i/3}) = \frac{1}{\pi}D(e^{i\pi/3}), \quad L'(\chi_{-4}, -1) = \frac{2}{\pi}D(e^{i\pi/2}).$$

5. Examples

In this section, we apply Theorem 0.2 to several Mahler measure's identities. We also describe some polynomials to which our main theorem fails to apply. They are numerically conjectured by Boyd and Brunault.

5.1. ***Pure identities.*** In this section, we apply Theorem 0.2 to study pure identities of Mahler's measure

$$\mathrm{m}(P) \sim_{\mathbb{Q}^\times} L'(E, -1),$$

where the notation $a \sim_{\mathbb{Q}^\times} b$ means $a/b \in \mathbb{Q}^\times$. Notice that most of polynomials in this section are of the form considered in Remark 4.3,

$$P(x, y, z) = A(x) + B(x)y + C(x)z, \tag{5-1}$$

where A, B, C are products of cyclotomic polynomials. In this case, $\mathrm{m}(\tilde{P}) = 0$ and $\mathrm{m}(P) \neq 0$. A typical example of pure identity is the Mahler measure of $z + (x+1)(y+1)$, which is conjectured by D. Boyd to be

$$\mathrm{m}(z + (x+1)(y+1)) = -2L'(E_{15}, -1).$$

It was proved under Beilinson's conjecture and up to a rational factor by Lalín [23, Section 4.1]. It is then completely proven by Brunault [8]. This polynomial also satisfies our main theorem, we do not discuss it here but focus on other examples. All figures in this section are generated using Maple.

(a) We prove under Beilinson's conjecture the pure identity

$$\mathrm{m}((1+x)(1+y)(x+y)+z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{14}, -1), \tag{5-2}$$

which is the first identity mentioned in Table 1. In this case, P is not of the form (5-1), but we still have $\mathrm{m}(\tilde{P}) = 0$ and the decomposition

$$x \wedge y \wedge z = -x \wedge (1+x) \wedge y \;+\; y \wedge (1+y) \wedge x \;+\; \frac{y}{x} \wedge \Big(1 + \frac{y}{x}\Big) \wedge x.$$

Hence

$$f_1 = -g_2 = -g_3 = -x, \quad f_2 = -g_1 = -y, \quad f_3 = -y/x.$$

We have that W_P is given by

$$(xy + x + y)(1 + x + y)((x+1)y^2 + (x^2 + x + 1)y + x^2 + x) = 0,$$

which is the union of lines $L_1 : xy + x + y = 0$, $L_2 : 1 + x + y = 0$ and the curve

$$C : (x+1)y^2 + (x^2 + x + 1)y + x^2 + x = 0,$$

which is a nonsingular curve of genus 1. By the change of variables

$$x = -\frac{Y + X^2 + 1}{X(X-1)}, \quad y = -\frac{Y}{X(X+1)} - \frac{1}{X},$$

we obtain that the Jacobian of C is given by

$$E/\mathbb{Q} : Y^2 + XY + Y = X^3 - X,$$

which is an elliptic curve of type $14a4$. Its torsion subgroup is $\mathbb{Z}/6\mathbb{Z} = \langle A \rangle$ with $A = (1, -2)$. With the help of Magma [2], we have

$$\mathrm{div}(x) = -(5A) + (A) - (4A) + (2A), \quad \mathrm{div}(y) = (\mathcal{O}) + (A) - (4A) - (3A).$$

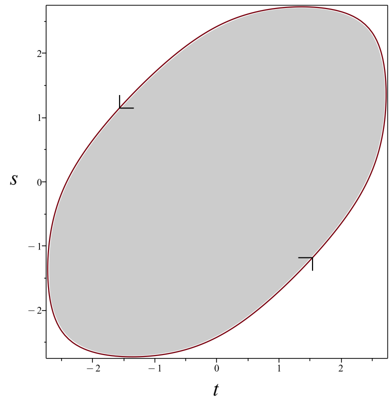

Figure 1. The Deninger chain Γ for the proof of identity 1 in Table 1.

Denote by S the closed subscheme of E consisting of all points in supports of above divisors. The values of f_j and $f_j \circ \tau$ at $p \in S$ are either 0, 1 or ∞ for all j, then the elements $v_p(g_j)\{f_j(p)\}_2$ and $v_p(g_j \circ \tau)\{f_j \circ \tau(p)\}_2$ are all trivial in $B_2(\mathbb{Q})$ for all j and $p \in S$. Figure 1 describes the Deninger chain

$$\Gamma : \{|x| = |y| = |(1+x)(1+y)(x+y)| \geq 1\},$$

and its boundary in polar coordinates $x = e^{it}$ and $y = e^{is}$ for $t, s \in [-\pi, \pi]$. We obtain that $\partial\Gamma$ is contained completely in C and $\partial\Gamma$ does not contain any zeros and poles of $f_j, 1 - f_j, g_j$ for all j. Then by Theorem 0.2, we have identity (5-2) under Beilinson's conjecture.

(b) We study the pure identity 2 in Table 1:

$$\mathrm{m}(1+x+y+z+xy+xz+yz) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{14}, -1). \tag{5-3}$$

First we notice that

$$\mathrm{m}(1+x+y+xy+z(1+x+y)) = \mathrm{m}(1+x+y+z(1+x+y+xy)),$$

so it suffices to prove the identity

$$\mathrm{m}(1+x+y+z(1+x+y+xy)) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{14}, -1). \tag{5-4}$$

We have $\mathrm{m}(\tilde{P}) = \mathrm{m}(1+x+y+xy) = \mathrm{m}(x+1)\mathrm{m}(y+1) = 0$. We have the decomposition

$$\begin{aligned} x\wedge y\wedge z = x\wedge(1+x)\wedge y - y\wedge(1+y)\wedge x + (x+y)\wedge(1+x+y)\wedge x \\ - (x+y)\wedge(1+x+y)\wedge y - \frac{x}{y}\wedge\left(1+\frac{x}{y}\right)\wedge(1+x+y), \end{aligned}$$

leading to

$$f_1 = -x, \quad f_2 = -y, \quad f_3 = f_4 = -(x+y), \quad f_5 = -x/y, \quad g_1 = g_4 = y, \quad g_2 = g_3 = x, \quad g_5 = 1+x+y.$$

W_P is given by $x(x+1)y^2 + (x^2+x+1)y + x + 1 = 0$, which is an irreducible nonsingular curve of genus 1. Using the change of variables

$$x = -\frac{Y + X^2 + 1}{X(X-1)}, \quad y = \frac{Y}{X(X+1)},$$

we obtain that the Jacobian of W_P is given by $E/\mathbb{Q} : Y^2 + XY + Y = X^3 - X$, which is the same elliptic curve in item (a). We have

$$\operatorname{div} x = -(5A) + (A) - (4A) + (2A), \quad \operatorname{div}(1+x+y) = 2(\mathcal{O}) - (5A) + 2(A) - (4A) - (2A) - (3A),$$

$$\operatorname{div} y = (\mathcal{O}) + (5A) - (2A) - (3A), \quad \operatorname{div}(1+1/x+1/y) = -(\mathcal{O}) + 2(4A) - (2A) + 2(3A) - (5A) - (A).$$

With the same reasoning as in item (a), we get identity (5-4) conditionally on Beilinson's conjecture. Moreover, as mentioned in the introduction, we have

$$\mathrm{m}((1+x)(1+y)(x+y)+z) \overset{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}(1+x+y+z+xy+xz+yz),$$

because they are rational multiples of the same elliptic curve L-value $L'(E_{14}, -1)$.

(c) Similarly, one gets identity 11 in Table 1:

$$\mathrm{m}(1+x+y+z+xy+xz+yz-xyz) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{36}, -1).$$

This identity is interesting because this is the only example that has been found with CM elliptic curves.

(d) The same arguments apply to all pure identities in Table 1, except for identities 5, 6, 7, 8, and 12. In this section, we study the first four of these identities. It suffices to consider identity 5 since Lalín and Nair showed in [24] that the polynomials 5, 6, 7, and 8 share the same Mahler measure. Let us recall identity 5:

$$\mathrm{m}(1+(x+1)y+(x-1)z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{21}, -1). \tag{5-5}$$

The polynomial $P = 1 + (x+1)y + (x-1)z$ is of the form (5-1). We have the decomposition

$$x \wedge y \wedge z = x \wedge (1-x) \wedge y + (x+1)y \wedge (1+(x+1)y) \wedge x - x \wedge (1+x) \wedge (1+(x+1)y),$$

so

$$f_1 = -f_3 = g_2 = x, \quad f_2 = -(x+1)y, \quad g_1 = y, \quad g_3 = 1+(x+1)y, \quad f_2 \circ \tau = -\frac{x+1}{xy}, \quad g_3 \circ \tau = \frac{xy+x+1}{xy}.$$

W_P is given by $x(x+1)y^2 + (2x^2+x+2)y + 1 + x = 0$, which is a nonsingular curve of genus 1. Using the change of variables

$$x = -\frac{X^2 - 6X + 3Y}{X(X-6)}, \quad y = \frac{Y - 3X - 3}{X(X+1)},$$

we get for the Jacobian of W_P the equation

$$E/\mathbb{Q} : Y^2 - 3XY - 3Y = X^3 - 5X^2 - 6X,$$

which is an elliptic curve of type $21a1$. Its torsion subgroup is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \langle A \rangle \times \langle B \rangle$ with $A = (-1, 0)$, and $B = (0, 0)$. With the help of Magma, we have

$$\operatorname{div} x = -(A+B) + (A+3B) - (3B) + (B), \quad \operatorname{div}(1+(x+1)y) = 2(2B) - (3B) - (B),$$

$$\operatorname{div} y = (\mathcal{O}) + (A+B) - (B) - (A), \quad \operatorname{div}(xy+x+1) = (\mathcal{O}) + 2(A+2B) - (A+B) - (3B) - (A).$$

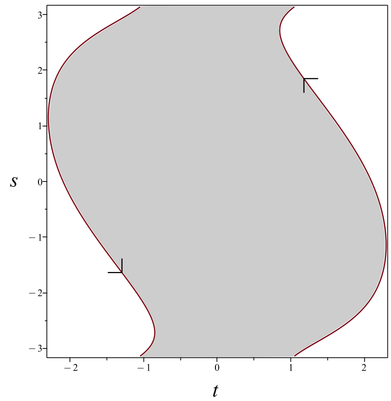

Figure 2. The Deninger chain Γ for the proof of identity 5 in Table 1.

Let S be the closed subscheme of E consisting all the points in the supports of the above divisors. We have

$$\begin{aligned}\sum_j v_B(g_j)\{f_j(B)\}_2 + v_B(g_j\circ\tau)\{f_j\circ\tau(B)\}_2 &= v_B(g_2)\{f_2(B)\}_2 + v_B(g_2\circ\tau)\{f_2\circ\tau(B)\}_2\\ &= \{\infty\}_2 - \{1/2\}_2 = \{2\}_2,\end{aligned}$$

which is nontrivial in $B_2(\mathbb{Q})$. Therefore, the theorem of Lalín mentioned in the introduction does not apply to this example. As S consists of points in E_{tors}, we can choose N in Theorem 0.2 equal to $\#E_{\mathrm{tors}} = 8$. Since the points of S have rational coordinates and the f_i have rational coefficients, then f_i take rational values on S. Therefore, the Bloch–Wigner dilogarithmic values in identity (4-33) all vanish. Figure 2 describes the Deninger chain and its boundary in polar coordinates $x = e^{it}$, $y = e^{it}$ for $s, t \in [-\pi, \pi]$. The boundary $\partial\Gamma$ consists of 2 loops and does not contain any zeros and poles of f_j, $1 - f_j$, g_j, $f_j\circ\tau$, $1 - f_j\circ\tau$, $g_j\circ\tau$ for all j. Hence by Theorem 0.2, we get identity (5-5) conditionally on Beilinson's conjecture. In particular, under Beilinson's conjecture, we have

$$\mathrm{m}(1 + (x+1)y + (x-1)z) \overset{?}{\sim}_{\mathbb{Q}^\times} \mathrm{m}((x+1)^2(y+1) + z),$$

as they are rational multiples of $L'(E_{21}, -1)$.

(e) There is an interesting remark on identities 4 and 10 of Table 1. By some trivial change of variables, we obtain

$$\mathrm{m}((x+1)^2 + (1-x)(y+z)) = \mathrm{m}((x+1)(y+1) + (x-1)^2 z).$$

Theorem 0.2 applies to $P = (x+1)^2 + (1-x)(y+z)$ but not to $P = (x+1)(y+1) + (x-1)^2 z$. Indeed, in the latter case, W_P is given by

$$(-x^3 - 2x^2 - x)y^2 + (x^4 - 6x^3 + 2x^2 - 6x + 1)y - x^3 - 2x^2 - x = 0,$$

which is a curve having a singularity at $(1, -1)$, and the boundary $\partial\Gamma$ passes this singular point. Figure 3 describes the Deninger chain Γ and its boundary $\partial\Gamma$ in polar coordinates $x = e^{it}$, $y = e^{is}$ for $t, s \in [-\pi, \pi]$, where the marked points indicate the singular point $(1, -1)$. Using Magma, one can check that $\partial\Gamma$ is no longer a loop on the normalization of W_P.

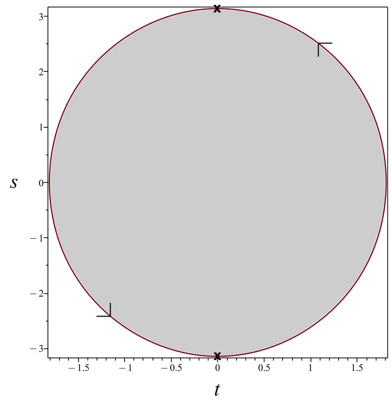

Figure 3. The Deninger chain Γ of the polynomial $P = (x+1)(y+1)+(x-1)^2 z$.

The same situation happens with identity 10 in Table 1. By some trivial changes of variables, we have

$$\mathrm{m}((1+x)^2+y+z) = \mathrm{m}(1+y+(1+x)^2 z),$$

where Theorem 0.2 applies to the first polynomial but not the second one.

(f) We study identity 12 in Table 1:

$$\mathrm{m}(1+xy+(1+x+y)z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{90},-1). \tag{5-6}$$

We have the decomposition

$$\begin{aligned} &x\wedge y\wedge z\\ &\quad = xy\wedge(1+xy)\wedge x-(x+y)\wedge(1+x+y)\wedge x+(x+y)\wedge(1+x+y)\wedge y+\frac{-x}{y}\wedge\left(1+\frac{x}{y}\right)\wedge(1+x+y), \end{aligned}$$

so

$$f_1=-xy,\quad f_2=f_3=-(x+y),\quad f_4=-x/y,\quad g_1=g_2=x,\quad g_3=y,\quad g_4=1+x+y.$$

The curve W_P is given by

$$(-x^2+x+1)y^2+(x^2+x+1)y+x^2+x-1=0,$$

which is an irreducible curve of genus 1 and does not contain any rational points. Figure 4 describes the Deninger chain and its boundary in polar coordinates. We find that $\partial\Gamma$ does not contain any singular points of W_P.

Using Magma, we obtain that the Jacobian of C is given by

$$E/\mathbb{Q}: Y^2+XY+Y = X^3-X^2-8X+11,$$

which is an elliptic curve of type $90b1$. Its torsion subgroup is $\mathbb{Z}/6\mathbb{Z}=\langle A\rangle$, with $A=(3,1)$. Denote by K the real quadratic field $\mathbb{Q}(\alpha)$, where $\alpha\in\mathbb{R}$ is such that $\alpha^2+\alpha-1=0$. Let $B_1=(6\alpha+9,-24\alpha-35)$, $B_2=(-4\alpha+1,12\alpha-3)$, $B_3=\left(\frac{9}{5},\frac{1}{25}(24\alpha-23)\right)$, $B_4=(2,-\alpha-2)$, $B_5=(-6\alpha+3,-18\alpha+7)$,

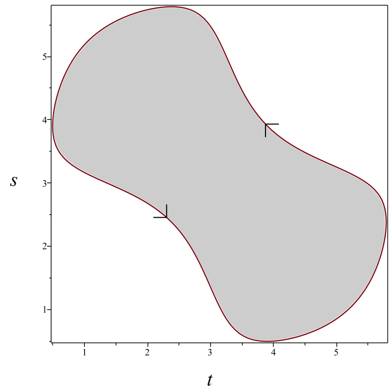

Figure 4. The Deninger chain Γ corresponding to (5-6).

$B_6 = (4\alpha + 5, 8\alpha + 9)$ and we denote by (B_i) the divisor in E corresponding to the point B_i. We have the following divisors in E/K:

$$\begin{aligned}
\operatorname{div}(x) &= (4A) + (B_1) - (A) - (B_2),\\
\operatorname{div}(y) &= (\mathcal{O}) + (B_3) - (B_4) - (3A),\\
\operatorname{div}(1+x+y) &= 2(2A) + 2(B_5) - (A) - (B_4) - (3A) - (B_2),\\
\operatorname{div}(1+1/x+1/y) &= -(\mathcal{O}) + 2(5A) + 2(B_6) - (B_3) - (4A) - (B_1).
\end{aligned}$$

Note that the Bloch group of real quadratic fields is trivial after tensoring with $\mathbb{Q}$. Therefore, the residues u_{B_i} are trivial because they are elements of $\mathcal{B}(\mathbb{Q}(\alpha))$, the Bloch group (tensored with $\mathbb{Q}$) of the real quadratic field $\mathbb{Q}(\alpha)$. The remaining points in the supports of the above divisors are of the form mA for $m = 1, \dots, 6$. Hence we can choose N in Theorem 0.2 as the order of A which equals 6. As the points mA have rational coordinates and the functions f_i have rational coefficients, the Bloch–Wigner dilogarithmic values at u_{mA} in (4-33) all vanish. We then get identity (5-6) under Beilinson's conjecture for genus 1 curves 1.11.

(g) We show that Theorem 0.2 does not imply identity (0-10):

$$\mathrm{m}((1+x)(1+y) + (1-x-y)z) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_{450}, -1).$$

We have the decomposition

$$x \wedge y \wedge z = -x \wedge (1+x) \wedge y + y \wedge (1+y) \wedge x + (x+y) \wedge (1-x-y) \wedge x \\ - (x+y) \wedge (1-x-y) \wedge y - \frac{x}{y} \wedge \left(1 + \frac{x}{y}\right) \wedge (1-x-y).$$

Therefore, we have

$$f_1 = -x, \quad f_2 = -y, \quad f_3 = f_4 = x+y, \quad f_5 = -x/y, g_1 = g_4 = y, \quad g_2 = g_3 = x, \quad g_5 = 1-x-y.$$

The Maillot variety is given by

$$(x^2 + 3x)y^2 + (3x^2 + x + 3)y + 3x + 1 = 0.$$

By the change of variables

$$x = -\frac{3}{(X^2-9X)Y} + \frac{-X^2+3X-27}{X^2-9X}, \quad y = \frac{3}{(X^2-27X+162)Y} + \frac{2X^2-21X-27}{X^2-27X+162},$$

we get the elliptic curve

$$E/\mathbb{Q}: \quad Y^2 + XY = X^3 - X^2 - 27X + 81,$$

which is of conductor 450. Its torsion group is isomorphic to $\mathbb{Z}/2$ and generated by $A = (-6, 3)$. We have the following divisors in E:

$$\begin{aligned}
&\mathrm{div}(x) = -(P_1) + (P_2) - (P_3) + (P_4),\\
&\mathrm{div}(y) = (P_3) + (P_5) - (P_6) - (P_2),\\
&\mathrm{div}(1-x-y) = 2(\mathcal{O}) + 2(P_7) - (P_6) - (P_1) - (P_2) - (P_3),\\
&\mathrm{div}(1-1/x-1/y) = 2(P_8) + 2(A) - (P_2) - (P_3) - (P_4) - (P_5),
\end{aligned}$$

where

$$\begin{aligned}
&P_1 := (9, 18), && P_2 := (9, -27), && P_3 := (0, 9), && P_4 := (0, -9),\\
&P_5 := \left(-\tfrac{9}{4}, -\tfrac{81}{8}\right), && P_6 := (18, 63) && P_7 := (4, 3), && P_8 := (3, -6).
\end{aligned}$$

The residue

$$u_{P_1} := \sum_{j=1}^{5} v_{P_1}(g_j)\{f_j(P_1)\}_2 + v_{P_1}(g_j \circ \tau)\{f_j \circ \tau(P_1)\}_2 = -3\{3\}_2$$

is nontrivial in the Bloch group $\mathcal{B}(\mathbb{Q})$. Since P_1 is torsion-free, it violates the finite order condition of Theorem 0.2.

5.2. ***Identities with Dirichlet characters.*** In this section, we study Mahler measure identities of the form

$$\mathrm{m}(P) \stackrel{?}{=} a \cdot L'(E, -1) + \sum_{\chi} b_\chi \cdot L'(\chi, -1),$$

where $a \in \mathbb{Q}$, $b_\chi \in \mathbb{Q}^\times$, E is an elliptic curve and the χ are odd quadratic Dirichlet characters.

(a) We prove the first identity of Table 2 conditionally on Beilinson's conjecture. The polynomial $P = 1 + (x^2 - x + 1)y + (x^2 + x + 1)z$ is of the form (5-1). We have on V_P the decomposition

$$\begin{aligned}
x \wedge y \wedge z = -\tfrac{1}{3}x^3 \wedge (1-x^3) \wedge y \,+\, x \wedge (1-x) \wedge y \,+\, (x^2-x+1)y \wedge (1+(x^2-x+1)y) \wedge x\\
-\tfrac{1}{3}x^3 \wedge (1+x^3) \wedge (1+(x^2-x+1)y) \,+\, x \wedge (1+x) \wedge (1+(x^2-x+1)y).
\end{aligned}$$

We have

$$\begin{gathered}
f_1 = x^3, \quad f_2 = x, \quad f_3 = -(x^2-x+1)y, \quad f_4 = -x^3,\ f_5 = -x,\\
g_1 = g_2 = y, \quad g_3 = x, \quad g_4 = g_5 = 1 + (x^2-x+1)y.
\end{gathered}$$

The curve W_P is given by $x^2(x^2-x+1)y^2 - x(4x^2-x+4)y + x^2-x+1 = 0$, which is a nonsingular curve of genus 1 and does not contain any rational point. By the change of variables $x = X$, $y = Y/X$, we get the new equation

$$(X^2-X+1)Y^2 - (4X^2-X+4)Y + X^2-X+1 = 0.$$

Using Pari/GP [31], one gets the following Weierstrass form for the Jacobian of W_P:

$$E/\mathbb{Q} : v^2 + uv = u^3 - u^2 - 45u - 104,$$

which is an elliptic of type $45a2$. We set $k = \mathbb{Q}(\alpha)$ with $\alpha^2 - \alpha + 1 = 0$. A base change of E over k can be given by

$$E_k : V^2 + 3UV + 3V = U^3 - U^2 - 9U,$$

by using the change of variables

$$x = \frac{(2-\alpha)V + \alpha U^2 - 3(\alpha-1)U + 3}{U^2 + (4\alpha-2)U - 3\alpha},$$

$$y = \frac{((1-\alpha)U^2 - (\alpha+4)U + (\alpha+4))V + 2\alpha U^3 - (8\alpha-3)U^2 + (3\alpha-12)U + 12}{U^4 - (4\alpha+1)U^3 + (15\alpha-7)U^2 - (17\alpha-13)U + 6(\alpha-1)}.$$

The torsion subgroup of E_k is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \langle A \rangle \times \langle B \rangle$ with $A = (-3, 3)$ and $B = (0, 0)$. Let K be the number field $\mathbb{Q}(\alpha, r, s)$ with

$$r^2 - 2(2\alpha-1)r + 3(\alpha-1) = 0, \text{ and } s^2 + 2(2\alpha-1)s - 3\alpha = 0.$$

We set $P_1 = (r, \alpha r - 2\alpha - 2)$, $P_2 = (s, (1-\alpha)s + 2\alpha - 4)$, which are points in $E(K)$. We denote by (P_i) the divisor corresponding to P_i in E_k. Note that the divisors (P_i) have degree 2 on E_k. Using Magma, one obtains the following divisors in E_k:

$$\begin{aligned}
&\operatorname{div}(g_3) = \operatorname{div}(x) = (P_1) - (P_2),\\
&\operatorname{div}(g_1) = \operatorname{div}(g_2) = \operatorname{div}(y) = (\mathcal{O}) + (A+3B) - (A+B) + (P_2) - (2B) - (P_1),\\
&\operatorname{div}(g_4) = \operatorname{div}(g_5) = \operatorname{div}(1 + (x^2-x+1)y) = 2(3B) + 2(A) - (P_1) - (P_2),\\
&\operatorname{div}(g_4 \circ \tau) = \operatorname{div}(g_5 \circ \tau) = \operatorname{div}(1 + (1/x^2 - 1/x + 1)(1/y)) = 2(B) + 2(A+2B) - (P_1) - (P_2).
\end{aligned}$$

The values of f_j and $f_j \circ \tau$ at P_1, P_2 and their conjugates are either 0 or ∞, so we are only concerned with the other points. We obtain the equalities

$$\begin{aligned}
u_A &= v_A(g_4)\{f_4(A)\}_2 + v_A(g_5)\{f_5(A)\}_2 = \{-1\}_2 + \{1/\alpha\}_2 = -\{\alpha\}_2,\\
u_B &= v_B(g_4 \circ \tau)\{f_4 \circ \tau(B)\}_2 + v_B(g_5 \circ \tau)\{f_5 \circ \tau(B)\}_2 = 2\{-1\}_2 + 2\{\alpha\}_2 = 2\{\alpha\}_2,\\
u_{2B} &= v_{2B}(g_1)\{f_1(2B)\}_2 + v_{2B}(g_1 \circ \tau)\{f_1 \circ \tau(2B)\}_2 + v_{2B}(g_2)\{f_2(2B)\}_2 + v_{2B}(g_2 \circ \tau)\{f_2 \circ \tau(2B)\}_2\\
&= -\{-1\}_2 + \{-1\}_2 - \{1/\alpha\}_2 + \{\alpha\}_2 = 2\{\alpha\}_2,
\end{aligned}$$

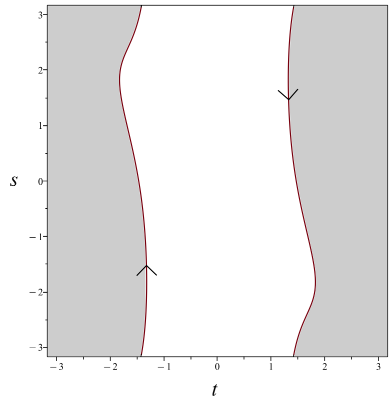

Figure 5. The Deninger chain Γ for the proof of identity 1 in Table 2.

$$\begin{aligned}
u_{3B} &= v_{3B}(g_4)\{f_4(3B)\}_2 + v_{3B}(g_5)\{f_5(3B)\}_2 = 2\{-1\}_2 + 2\{\alpha\}_2 = 2\{\alpha\}_2,\\
u_{A+B} &= v_{A+B}(g_1)\{f_1(A+B)\}_2 + v_{A+B}(g_1\circ\tau)\{f_1\circ\tau(A+B)\}_2\\
&\quad + v_{A+B}(g_2)\{f_2(A+B)\}_2 + v_{A+B}(g_2\circ\tau)\{f_2\circ\tau(A+B)\}_2\\
&= -\{-1\}_2 + \{-1\}_2 - \{\alpha\}_2 + \{1/\alpha\}_2 = -2\{\alpha\}_2,\\
u_{A+2B} &= v_{A+2B}(g_4\circ\tau)\{f_4\circ\tau(A+2B)\}_2 = 2\{-1\}_2 + 2\{1/\alpha\}_2 = -2\{\alpha\}_2,\\
u_{A+3B} &= v_{A+3B}(g_1)\{f_1(A+3B)\}_2 + v_{A+3B}(g_1\circ\tau)\{f_1\circ\tau(A+3B)\}_2\\
&\quad + v_{A+3B}(g_2)\{f_2(A+3B)\}_2 + v_{A+3B}(g_2\circ\tau)\{f_2\circ\tau(A+3B)\}_2\\
&= \{-1\}_2 - \{-1\}_2 + \{1/\alpha\}_2 - \{\alpha\}_2 = -2\{\alpha\}_2,
\end{aligned}$$

which are all nontrivial in $B_2(K)$. Notice that P_1, P_2 have order 8 in $E(K)$ and all the other points belong to the torsion subgroup of E_k, whose cardinality equals 8, hence we choose N in Theorem 0.2 equal to 8. Figure 5 indicates the Deninger chain (the shaded region) and its boundary in polar coordinates $x = e^{it}$, $y = e^{is}$ for $s, t \in [-\pi, \pi]$. The boundary $\partial\Gamma$ consists of 2 loops, which do not contain any points in the supports of the above divisors. By (4-20), we have $\mathrm{m}(P) = -\frac{1}{8\pi^2}\int_{\partial\Gamma}\rho(\lambda)$, so $\partial\Gamma$ must be nontrivial as otherwise $m(P)$ vanishes. Hence $\partial\Gamma$ defines a generator of $H_1(C(\mathbb{C}), \mathbb{Z})^+$. Then by Theorem 0.2, under Beilinson's conjecture, we have

$$\mathrm{m}(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) \stackrel{?}{=} a \cdot L'(E_{45}, -1) + \frac{b}{32\pi} \cdot D(\alpha), \qquad a \in \mathbb{Q}^\times, b \in \mathbb{Z}\setminus\{0\}.$$

We are unable to determine the coefficient b as computing the integrals $\int_{\partial\Gamma} d\arg f_p$ for $p \in S$ is difficult. By Remark 4.14, we have

$$D(\alpha) = \frac{3\sqrt{3}}{4} L(\chi_{-3}, 2) = \pi L'(\chi_{-3}, -1).$$

Finally, we get

$$\mathrm{m}(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) \stackrel{?}{=} a \cdot L'(E_{45}, -1) + \tfrac{1}{32} b \cdot L'(\chi_{-3}, -1), \;\; a \in \mathbb{Q}^\times, b \in \mathbb{Z}\setminus\{0\}.$$

(b) Using a method of Lalín [23, Section 4.2], we prove without assuming Beilinson's conjecture identity 6 of Table 3, which involves only the L-function of the Dirichlet character χ_{-4}:

$$\mathrm{m}(x^2+1+(x+1)^2y+(x-1)^2z)=2L'(\chi_{-4},-1).$$

We have $\mathrm{m}(\tilde{P})=0$. We have the following decomposition on V_P:

$$\begin{aligned}x\wedge y\wedge z=&-\tfrac{1}{2}x^2\wedge(1+x^2)\wedge y+2x\wedge(1-x)\wedge y+x\wedge\frac{(x+1)^2y}{x^2+1}\wedge\Big(1+\frac{(x+1)^2y}{x^2+1}\Big)\\&-2x\wedge(1+x)\wedge\Big(1+\frac{(x+1)^2y}{x^2+1}\Big)+\tfrac{1}{2}x^2\wedge(1+x^2)\wedge\Big(1+\frac{(x+1)^2y}{x^2+1}\Big).\end{aligned}$$

We have

$$\rho(\xi)=-\tfrac{1}{2}\rho(-x^2,y)+2\rho(x,y)+\rho\Big(\frac{-(x+1)^2y}{x^2+1},x\Big)-2\rho\Big(-x,1+\frac{(x+1)^2y}{x^2+1}\Big)+\tfrac{1}{2}\rho\Big(-x^2,1+\frac{(x+1)^2y}{x^2+1}\Big),$$

where

$$\rho(f,g)=-D(f)d\arg g+\tfrac{1}{3}\log|g|\big(\log|1-f|\,d\log|f|-\log|f|\,d\log|1-f|\big).$$

W_P is given by

$$(x^2+1)((x+1)^2y^2+(x^2+8x+1)y+(x+1)^2)=0,$$

which is the union of $L:x^2+1=0$ and the curve $C:(x+1)^2y^2+(x^2+8x+1)y+(x+1)^2=0$. Figure 6 describes the Deninger chain Γ in polar coordinates:

$$\Gamma:\left|\frac{x^2+1+(x+1)^2y}{(x-1)^2}\right|\geq 1,\quad x=e^{it},\quad y=e^{is},\quad s,t\in[-\pi,\pi].$$

Its boundary $\partial\Gamma$ consists of 2 loops $\gamma=\{t=\pi/2,-\pi\leq s\leq\pi\}$ and $\delta=\{t=-\pi/2,-\pi\leq s\leq\pi\}$ (with orientations as shown in the figure), which are contained in L. As $\partial\Gamma$ contains poles of $\rho(\xi)$, we do not

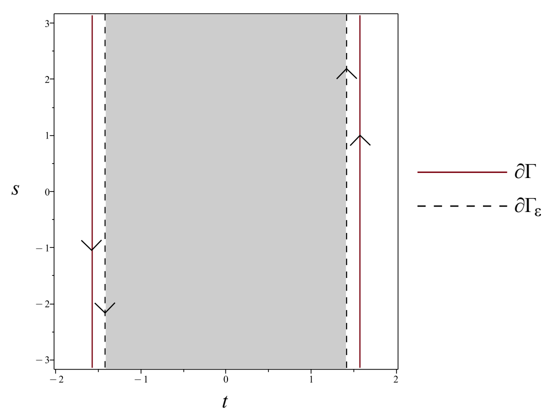

Figure 6. The Deninger chain Γ for the proof of identity 6 in Table 3 and the modified path Γ_ϵ used as the integration domain.

have (4-23) directly. We adjust the Deninger chain as follows, for $\varepsilon > 0$:

$$\Gamma_\varepsilon : \left|\frac{x^2+1+(x+1)^2y}{(x-1)^2}\right| \geq 1, x = e^{i(1+\varepsilon)t}, \quad y = e^{is}, \quad \text{for } s, t \in [-\pi, \pi],$$

This is the shaded region in Figure 6 with the boundary $\partial\Gamma_\varepsilon = \gamma_\varepsilon \cup \delta_\varepsilon$, where

$$\gamma_\varepsilon = \left\{t = \frac{\pi}{2(1+\varepsilon)}, -\pi \leq s \leq \pi\right\}, \quad \delta_\varepsilon = \left\{t = -\frac{\pi}{2(1+\varepsilon)}, -\pi \leq s \leq \pi\right\}.$$

We consider the differential forms η and $\rho(\lambda)$ defined in (4-1) and Definition 4.4, respectively. We have

$$\int_{\Gamma_\varepsilon} \eta = \int_{\partial\Gamma_\varepsilon} \rho(\xi) = \frac{1}{2}\int_{\partial\Gamma_\varepsilon} \rho(\lambda), \tag{5-7}$$

where the first equality is obtained by using Stokes's theorem and the second equality can be proved similarly as the proof of Lemma 4.8. Since $\rho(\lambda)$ is a closed differential form, we can take the limit of (5-7) as $\varepsilon \to 0$ without changing the value of the integration, so that

$$\mathrm{m}(P) = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\partial\Gamma_\varepsilon} \rho(\xi).$$

We have

$$\begin{aligned}
\int_{\partial\Gamma_\varepsilon} \rho(\xi) &= \int_{\partial\Gamma_\varepsilon} -\tfrac{1}{2}\rho(-x^2, y) + 2\rho(x, y) + \rho\Big(\frac{-(x+1)^2y}{x^2+1}, x\Big) \\
&\qquad\qquad - 2\rho\Big(-x, 1 + \frac{(x+1)^2y}{x^2+1}\Big) + \tfrac{1}{2}\rho\Big(-x^2, 1 + \frac{(x+1)^2y}{x^2+1}\Big) \\
&= \int_{\gamma_\varepsilon\cup\delta_\varepsilon} 2\rho(x, y) - 2\rho\Big(-x, 1 + \frac{(x+1)^2y}{x^2+1}\Big) \\
&= \int_{\gamma_\varepsilon\cup\delta_\varepsilon} -2D(x)d\arg(y) + 2D(-x)d\arg\Big(1 + \frac{(x+1)^2y}{x^2+1}\Big) \\
&= \Big(-2D\big(e^{\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\gamma_\varepsilon} d\arg(y) - 2D\big(e^{-\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\delta_\varepsilon} d\arg(y)\Big) \\
&\quad + \Big(2D\big(-e^{\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\gamma_\varepsilon} d\arg\Big(1 + \frac{(x+1)^2y}{x^2+1}\Big)\Big) + 2D\big(-e^{-\frac{i\pi}{2(1+\varepsilon)}}\big)\int_{\delta_\varepsilon} d\arg\Big(1 + \frac{(x+1)^2y}{x^2+1}\Big).
\end{aligned}$$

We have

$$\int_{\gamma_\varepsilon} d\arg(y) = \int_{-\pi}^{\pi} ds = 2\pi, \quad \int_{\delta_\varepsilon} d\arg y = \int_{\pi}^{-\pi} ds = -2\pi.$$

We also get

$$\int_{\gamma_\varepsilon} d\arg\left(1 + \frac{(x+1)^2y}{x^2+1}\right) = 2\pi, \quad \int_{\delta_\varepsilon} d\arg\left(1 + \frac{(x+1)^2y}{x^2+1}\right) = -2\pi,$$

by looking at Figure 7, left, and the inequality $\left|(x+1)^2/(x^2+1)\right| > 1$. Then $\lim_{\varepsilon\to 0}\int_{\partial\Gamma_\varepsilon} \rho(\xi) = -16\pi D(e^{i\pi/2})$. It follows that

$$\mathrm{m}(P) = \frac{4}{\pi}D(e^{i\pi/2}) = 2L'(\chi_{-4}, -1).$$

The same arguments apply to identities 4, 5, 7, and 8 of Table 3.

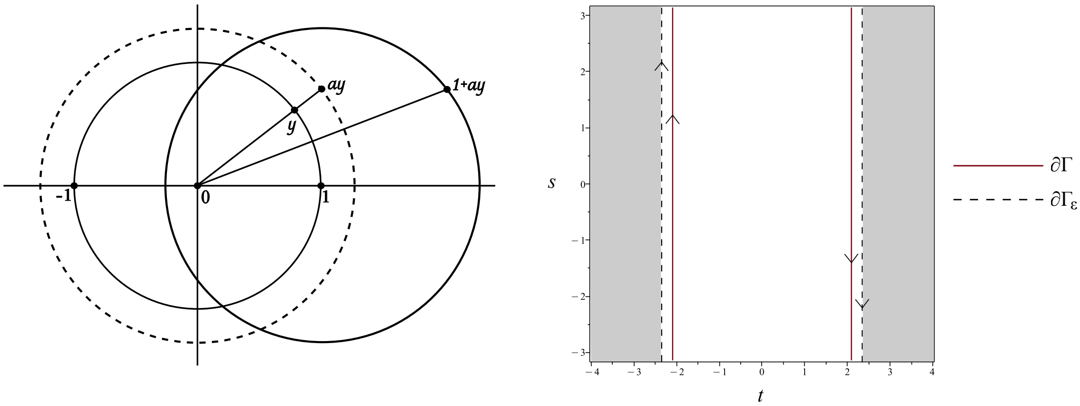

Figure 7. Left: the argument of $1+ay$ with $|a|>1$. Right: the integration domain.

(c) Let us study identity 1 of Table 3, which involves only the L-function of the Dirichlet character χ_{-3}

$$\mathrm{m}(1+(x+1)(x^2+x+1)y+(x+1)^3z)=3L'(\chi_{-3},-1).$$

We have that W_P is given by $(x^2+x+1)((x^4+x^3)y^2+(-2x^3-5x^2-2x)y+x+1)=0$, which is the union of the line $L: x^2+x+1=0$ and the curve $C:(x^4+x^3)y^2+(-2x^3-5x^2-2x)y+x+1=0$. Figure 7, right, describes the Deninger chain in local coordinates $x=e^{it}$ and $y=e^{is}$ for $s,t\in[-\pi,\pi]$. The boundary $\partial\Gamma=\gamma\cup\delta$, where

$$\gamma=\{t=2\pi/3,\ -\pi\le s\le\pi\} \text{ and } \delta=\{t=-2\pi/3,\ -\pi\le s\le\pi\},$$

which are both contained in L.

The differential form $\rho(\xi)$ is again not well-defined on $\partial\Gamma$. So we adjust the Deninger chain to get Γ_ε (see the shaded region). By a similar computation as in item (b), we have

$$\mathrm{m}(P)=-\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\partial\Gamma_\varepsilon}\rho(\xi)=3L'(\chi_{-3},-1).$$

One can do similarly with identities 2 and 3 of Table 3.

(d) Theorem 0.2 does not apply to identity (0-12),

$$\mathrm{m}(x^2+x+1+(x^2+x+1)y+(x-1)^2)\stackrel{?}{=}-\tfrac{1}{12}L'(E_{72},-1)+\tfrac{3}{2}L'(\chi_3,-1),$$

because the boundary $\partial\Gamma$ passes the singular point $(1,-1)$ of W_P (see Figure 8) and $\partial\Gamma$ is no longer a loop in the normalization of W_P.

(e) We prove the second identity of Table 2, under Beilinson's conjecture for genus 1 curves:

$$\mathrm{m}(x^2+1+(x+1)^2y+(x^2-1)z)\stackrel{?}{=}-\tfrac{1}{10}L'(E_{48},-1)+L'(\chi_{-4},-1). \tag{5-8}$$

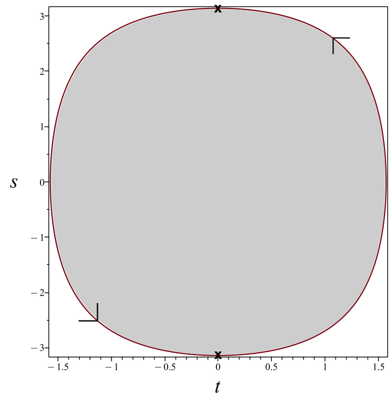

Figure 8. The Deninger chain Γ for (0-12).

We have

$$x \wedge y \wedge z = -\tfrac{1}{2}x^2 \wedge (1+x^2) \wedge y \;+\; \tfrac{1}{2}x^2 \wedge (1-x^2) \wedge y \;+\; \frac{(x+1)^2 y}{x^2+1} \wedge \Big(1+\frac{(x+1)^2 y}{x^2+1}\Big) \wedge x$$
$$- 2x \wedge (1+x) \wedge \Big(1+\frac{(x+1)^2 y}{x^2+1}\Big) \;+\; \tfrac{1}{2}x^2 \wedge (1+x^2) \wedge \Big(1+\frac{(x+1)^2 y}{x^2+1}\Big).$$

Then

$$\rho(\xi) =$$
$$-\tfrac{1}{2}\rho(-x^2, y) + \tfrac{1}{2}\rho(x^2, y) + \rho\Big(-\frac{(x+1)^2 y}{x^2+1}, x\Big) - 2\rho\Big(-x, 1+\frac{(x+1)^2 y}{x^2+1}\Big) + \tfrac{1}{2}\rho\Big(-x^2, 1+\frac{(x+1)^2 y}{x^2+1}\Big).$$

W_P is given by

$$(x^2+1)((x+1)^2 y^2 + (3x^2+4x+3)y + (x+1)^2) = 0,$$

which is the union of $L : x^2+1=0$ and the curve $C : (x+1)^2 y^2 + (3x^2+4x+3)y + (x+1)^2 = 0$, which is a nonsingular curve of genus 1. Figure 9 describes the Deninger chain Γ and its boundary $\partial\Gamma$ in polar coordinates $x = e^{it}$ and $y = e^{is}$ for $t, s \in [-\pi, \pi]$. We have $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the shaded region in the center with the boundary

$$\partial\Gamma_1 = \{t = -\pi/2, -\pi \le s \le \pi\} \cup \{t = \pi/2, -\pi \le s \le \pi\},$$

and Γ_2 is the shaded region with the boundary $\partial\Gamma_2$ as in the figure. We observe that $\partial\Gamma_1$ is contained in L and $\partial\Gamma_2$ is contained in C. We have

$$\mathrm{m}(P) = m_1 + m_2,$$

where m_1 can be computed by the same method as the example (b). Let $\Gamma_{1,\epsilon}$ be the adjustment of Γ_1 shown in Figure 9. We have

$$m_1 = -\frac{1}{4\pi^2}\int_{\Gamma_1} \eta = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\Gamma_{1,\varepsilon}} \eta = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{\partial\Gamma_{1,\varepsilon}} \rho(\xi) = L'(\chi_{-4}, -1),$$

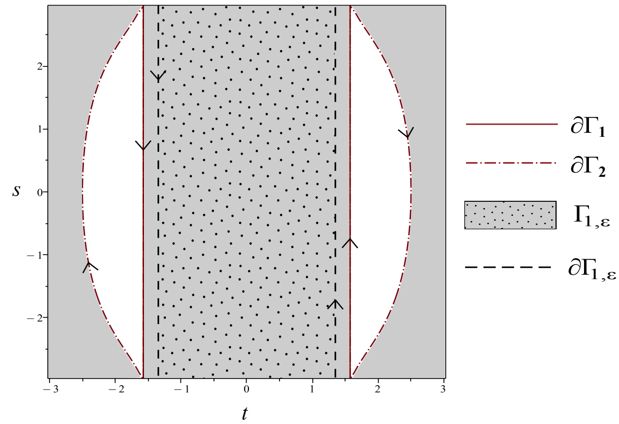

Figure 9. The Deninger chain Γ corresponding to (5-8).

and

$$m_2 = -\frac{1}{4\pi^2}\int_{\Gamma_2}\eta = -\frac{1}{4\pi^2}\int_{\partial\Gamma_2}\rho(\xi).$$

By the change of variables

$$x = -\frac{2Y+X^2}{X^2-2X-4}, \quad y = -\frac{2}{X+2},$$

the Jacobian of C is given by

$$E/\mathbb{Q} : Y^2 = X^3 + X^2 - 4X - 4,$$

which the elliptic curve of type $48a1$. Its torsion subgroup is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} = \langle A\rangle\times\langle B\rangle$, where $A = (2, 0)$ and $B = (-1, 0)$. Set $K = \mathbb{Q}(\alpha, \beta)$ where $\alpha^2 - 2\alpha - 4 = 0$ and $\beta^2 + 4 = 0$. Let us write

$$P_1 = (\alpha, \alpha+2),\ P_2 = (\alpha, -\alpha-2),\ P_3 = (0, s, 1).$$

We have

$$\begin{aligned} \operatorname{div}(x) &= -(P_1) + (P_2), & \operatorname{div}\Big(\frac{1+(1+x)^2y}{x^2+1}\Big) &= 2(A) + 2(A+B) - 2(P_3),\\ \operatorname{div}(y) &= 2(\mathcal{O}) - 2(A+B), & \operatorname{div}\Big(\frac{x^2y+x^2+2x+y+1}{y(x^2+1)}\Big) &= 2(\mathcal{O}) + 2(B) - 2(P_3). \end{aligned}$$

We have $u_A = u_B = u_{A+B} = 0$ and $u_{P_3} = -2\{-\beta/2\}_2 - 2\{\beta/2\}_2 = 0$. The residues u_{P_i} for $i = 1, 2$ are trivial because they belong to $\mathcal{B}(\mathbb{Q}(\alpha))$, the Bloch group (tensored with $\mathbb{Q}$) of the real quadratic field $\mathbb{Q}(\alpha)$. Therefore, we have the following identity under Beilinson's conjecture:

$$m_2 = a\cdot L'(E_{48}, -1), \quad a \in \mathbb{Q}^\times.$$

In conclusion, we obtain the following identity under Beilinson's conjecture:

$$\mathrm{m}(P) = L'(\chi_{-4}, -1) + m_2 = a\cdot L'(E_{48}, -1) + L'(\chi_{-4}, -1), \quad a \in \mathbb{Q}^\times.$$

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