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A refinement of the Birch and Swinnerton-Dyer conjecture in positive characteristic

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We formulate a refined version of the Birch and Swinnerton-Dyer conjecture for abelian varieties over global function fields. This refinement incorporates both new families of algebraic relations between leading terms (at $s = 1$) of Hasse–Weil–Artin L-series and restrictions on the Galois structure of Selmer complexes, and constitutes a natural analogue for abelian varieties over function fields of the equivariant Tamagawa number conjecture for abelian varieties over number fields. We provide strong supporting evidence for the conjecture including giving a full proof, modulo only the assumed finiteness of Tate–Shafarevich groups, in an important class of examples.

1. Introduction

1.1. Let A be an abelian variety that is defined over a function field K in one variable over a finite field of characteristic p.

In [34] Artin and Tate formulated a precise conjectural formula for the leading term at $s = 1$ of the Hasse–Weil L-series attached to A.

This formula constituted a natural "geometric" analogue of the Birch and Swinnerton-Dyer conjecture for abelian varieties over number fields and was subsequently verified unconditionally by Milne [26] in

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the case that A is constant and by Ulmer [39] in certain other special cases. Further partial results have been obtained by many other authors and these efforts culminate in the main result of the seminal article of Kato and Trihan [22] which shows that the conjecture is valid whenever there exists a prime ℓ such that the ℓ-primary component of the Tate–Shafarevich group of A over K is finite.

In this article we now formulate, and provide strong evidence for, a refined version of this conjecture that also incorporates new families of algebraic relations between the (suitably normalised) leading terms at $s = 1$ of the Hasse–Weil–Artin L-series that are attached to A and to irreducible complex characters (with open kernel) of the absolute Galois group of K. This conjecture is a natural analogue for abelian varieties over function fields of the equivariant Tamagawa number conjecture ("ETNC"), including the p-primary part, for the motive $h^1(A)(1)$ of abelian varieties A over number fields.

To be a little more precise about our results we now fix a finite Galois extension L of K with group G.

Then, as a first step, we shall prove that the leading terms of the Hasse–Weil–Artin L-series that are attached to A and to the irreducible complex characters of G are interpolated by a canonical element of the Whitehead group $K_1(\mathbb{R}[G])$ of the group ring $\mathbb{R}[G]$. (This result is, a priori, far from clear and requires one to prove, in particular, that leading terms at irreducible symplectic characters are strictly positive.)

Our central conjecture is then a precise formula for the image of this element under the connecting homomorphism from $K_1(\mathbb{R}[G])$ to the relative algebraic K_0-group $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ of the ring inclusion $\mathbb{Z}[G] \subset \mathbb{R}[G]$.

The conjectural formula involves a canonical Euler characteristic element that is constructed by combining a natural "Selmer complex" of G-modules together with the classical Néron–Tate height pairing of A over L. This Selmer complex is constructed from the flat cohomology of the torsion subgroup scheme of the Néron model of A over the projective curve X with function field K and, provided that the relevant Tate–Shafarevich groups are finite, is both perfect over $\mathbb{Z}[G]$ and has cohomology groups that are closely related to the classical Mordell–Weil and Selmer groups of A^t and A over L.

The formula also involves the Euler characteristic of an auxiliary perfect complex of G-modules that is constructed directly from the Zariski cohomology of an appropriate line bundle over X and is necessary in order to compensate for certain choices of pro-p subgroups that are made in the definition of the Selmer complex.

If $L = K$, then $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ identifies with the quotient of the multiplicative group $\mathbb{R}^\times$ by $\{\pm 1\}$ and we check that in this case our conjecture recovers the classical Birch and Swinnerton-Dyer conjecture for A over K.

In the general case, the conjecture incorporates both a family of precise algebraic relations between the normalised leading terms of Hasse–Weil–Artin L-series attached to A and to characters of G and also strong restrictions on the Galois structure of Selmer complexes (for more details see the discussion in Section 4.2.3).

To study the conjecture, we adapt — and, in some respects, clarify — certain constructions and arguments from Kato and Trihan's [22] relating to syntomic cohomology complexes. In this way we are able to prove that, whenever there exists a prime ℓ such that the ℓ-primary component of the Tate–Shafarevich group of A over L is finite, then our conjecture is valid modulo a certain finite subgroup $\mathcal{T}_{A,L/K}$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$, the nature of which depends both on the reduction properties of A and the ramification behaviour in L/K.

For example, if A is semistable over K and L/K is tamely ramified, then $\mathcal{T}_{A,L/K}$ vanishes and so we obtain a full verification of our conjecture in this case.

In the worst case the group $\mathcal{T}_{A,L/K}$ coincides with the torsion subgroup of the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ and our result essentially amounts to proving a version of the main result of [22] for the leading terms of the Hasse–Weil–Artin L-series attached to A and to each character of G.

However, even the latter result is new and of interest since, for example, it both establishes the "order of vanishing" part of the Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-series and, in addition, plays a key role in a forthcoming complementary article that deals with, modulo the standard finiteness hypothesis on Tate–Shafarevich groups, the case of abelian varieties A that are generically ordinary.

As a key step in the proof of our main result we shall combine Grothendieck's description of Hasse–Weil–Artin L-series in terms of the action of Frobenius on ℓ-adic cohomology (for some prime $\ell \neq p$) together with a result of Schneider concerning the Néron–Tate height-pairing to show that our conjectural formula naturally decomposes as a sum of "ℓ-primary parts" over all primes ℓ.

It is thus of interest to note that in some related recent work Trihan and Vauclair [36] have adapted the approach of [22] in order to formulate and prove a natural main conjecture of (p-adic) noncommutative Iwasawa theory for A relative to unramified p-adic Lie extensions of K under the assumptions both that A is semistable over K and that certain Iwasawa-theoretic μ-invariants vanish.

In addition, for each prime $\ell \neq p$, Witte [41] has used techniques of Waldhausen K-theory to deduce an analogue of the main conjecture of noncommutative Iwasawa theory for ℓ-adic sheaves over arbitrary p-adic Lie extensions of K from Grothendieck's formula for the Zeta function of such sheaves.

It seems likely that these results can be combined with the descent techniques developed by Venjakob and the first author in [7] and the explicit interpretation of height pairings in terms of Bockstein homomorphisms that we use below to give an alternative, although rather less direct, proof of the ℓ-primary part of our main result for any $\ell \neq p$ and of the p-primary part of our main result in the special case that L/K is unramified and suitable μ-invariants vanish.

However, even now, there are still no ideas as to how one could formulate a main conjecture of (non-commutative) Iwasawa theory for A relative to any general class of ramified p-adic Lie extensions of K.

It is thus one of the main observations of the present article that the techniques developed by Kato and Trihan in [22] are essentially themselves sufficient to prove refined versions of the Birch and Swinnerton-Dyer conjecture without the need to develop an appropriate formalism of noncommutative Iwasawa theory (and hence without the need to assume the vanishing of relevant μ-invariants).

This general philosophy also in fact underpins the complementary work of the first two authors regarding generically ordinary abelian varieties.

Outline of article. In Section 2 we use the leading terms of the Hasse–Weil–Artin L-series attached to complex characters of G to define a canonical element of $K_1(\mathbb{R}[G])$. Then, in Section 3, we define a natural family of "Selmer complexes" of G-modules and establish some of its key properties.

In Section 4 we formulate our main conjecture, describe some of its explicit consequences and state the main supporting evidence for the conjecture that we prove in later sections.

In Section 5 we prove certain useful preliminary results including a purely K-theoretic observation that plays a key role in several subsequent calculations. We also show that our conjecture is consistent in some important respects and use a result of Schneider to give a useful reformulation of the conjecture.

In Section 6 and Section 7 we investigate the syntomic cohomology complexes introduced by Kato and Trihan in [22], with a particular emphasis on understanding conditions under which these complexes can be shown to be perfect.

In Section 8 we analyse when certain morphisms of complexes that arise naturally in the theory are "semisimple" (in the sense of Galois descent) and deduce, modulo the assumed finiteness of Tate–Shafarevich groups, the order of vanishing part of the Birch and Swinnerton-Dyer conjecture for Artin Hasse–Weil L-series.

In Section 9, we combine the results established in earlier sections to prove our main results.

The article has two appendices. In Appendix A, we show that coherent cohomology over a "separated" formal fs log scheme can be computed via the Čech resolution with respect to an affine Kummer-étale covering. (This result plays an important role in the arguments of Section 7 and, whilst it is surely well-known to experts, we have not been able to find a good reference for it.)

In Appendix B, we extend the notion of an overconvergent Λ-F-isocrystal for a finite extension Λ of $\mathbb{Q}_p$ whose residue field is not necessarily contained in the field of constants of the base curve, and also the Lefschetz trace formula for rigid cohomology with such coefficients. (This result is needed to obtain Theorem 8.2 without further restriction, and the proof is a mere repetition of the proof of Etesse and Le Stum in [15].)

Notation and conventions. We fix a prime number p and a function field K in one variable over a finite field of characteristic p. We write X for the proper smooth connected curve over $\mathbb{F}_p$ that has function field K.

Let A be an abelian variety over K. Let U be a dense open subset of X such that A/K has good reduction on U. We write $\mathcal{A}$ for the Néron model of A over X.

Let F be a finite extension of K. Let X_F denote the proper smooth curve over $\mathbb{F}_p$ that has function field F. We will denote the "base extension" of an object $*$ over either K or X to that over F of X_F by a subscript $*_F$. For example A_F and U_F denote $A \times_K F$, $U \times_X X_F$ respectively. If there is no danger of confusion we often omit the subscript F.

If M is an abelian group or complex of abelian groups, we denote its Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ by M^*. If W is a $\mathbb{Q}_\ell$-module or complex of $\mathbb{Q}_\ell$-modules for some prime ℓ, we denote its linear dual $\mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ by $V^\vee$ (regarding ℓ as clear from context). If either M or V has a left action of a group, then we endow M^* and $V^\vee$ with the corresponding left contragredient action.

We fix an algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$ and for every prime ℓ an algebraic closure $\mathbb{Q}_\ell^c$ of $\mathbb{Q}_\ell$ and the ℓ-adic completion $\mathbb{C}_\ell$ of $\mathbb{Q}_\ell^c$. For every prime ℓ, we also fix an embedding $\mathbb{Q}^c \to \mathbb{Q}_\ell^c$.

For each natural number n the n-torsion subgroup of an abelian group M is denoted by $M[n]$. The full torsion subgroup of M is denoted by M_{tor} and, for each prime ℓ, the ℓ-primary part of M_{tor} is denoted by $M\{\ell\}$.

For a finite group G we write $\mathrm{Ir}(G)$ for the set of its irreducible complex valued characters and $\mathrm{Ir}^s(G)$ for the subset of $\mathrm{Ir}(G)$ comprising characters that are symplectic. We write $\check{\chi}$ for the contragredient of each χ in $\mathrm{Ir}(G)$ and $\mathbf{1}_G$ for the trivial character of G.

For any commutative ring R we write $R[G]$ for the group ring of G over R and denote its centre by $\zeta(R[G])$. We identify $\zeta(\mathbb{C}[G])$ with $\prod_{\mathrm{Ir}(G)} \mathbb{C}$ in the standard way.

2. Leading terms of Hasse–Weil–Artin L-series

We fix a finite Galois extension L/K with Galois group G, and choose U not to contain any place that ramifies in L/K. For each χ in $\mathrm{Ir}(G)$ we write $L_U(A, \chi, s)$ for the Hasse–Weil–Artin L-series of the pair (A, χ) that is truncated by removing the Euler factors for all places outside U.

We now show that there exists a canonical element of the Whitehead group $K_1(\mathbb{R}[G])$ that naturally interpolates the leading terms $L_U^*(A, \chi, 1)$ at $s = 1$ in the Taylor expansions of the functions $L_U(A, \chi, s)$ as χ ranges over $\mathrm{Ir}(G)$.

This "K-theoretical leading term" will play an important role in the conjecture that we discuss in subsequent sections (but also see Remark 4.6 in this regard).

To define the element we use the fact that the algebra $\mathbb{R}[G]$ is semisimple and hence that the classical reduced norm construction induces a homomorphism $\mathrm{Nrd}_{\mathbb{R}[G]}$ of abelian groups from $K_1(\mathbb{R}[G])$ to the subgroup $\zeta(\mathbb{R}[G])^\times$ of $\prod_{\mathrm{Ir}(G)} \mathbb{C}^\times$.

Theorem 2.1. *There exists a unique element $L_U^*(A_{L/K}, 1)$ of $K_1(\mathbb{R}[G])$ with the property that*

$$\mathrm{Nrd}_{\mathbb{R}[G]}(L_U^*(A_{L/K}, 1))_\chi = L_U^*(A, \chi, 1)$$

for all χ in $\mathrm{Ir}(G)$.

Proof. Since the natural map $\mathbb{R}[G]^\times \to K_1(\mathbb{R}[G])$ is surjective, the Hasse–Schilling–Maass norm theorem implies both that $\mathrm{Nrd}_{\mathbb{R}[G]}$ is injective and that its image is equal to the subgroup of $\prod_{\mathrm{Ir}(G)} \mathbb{C}^\times$ comprising elements $(x_\chi)_\chi$ that satisfy the conditions

$$\begin{cases} x_{\tau\circ\chi} = \tau(x_\chi) & \text{for all } \chi \text{ in } \mathrm{Ir}(G), \text{ and} \\ x_\chi \in \mathbb{R} \text{ and } x_\chi > 0 & \text{for all } \chi \text{ in } \mathrm{Ir}^s(G), \end{cases}$$

where τ denotes complex conjugation. (For a proof of this result see [12, Theorem (45.3), p. 138].)

The injectivity of $\mathrm{Nrd}_{\mathbb{R}[G]}$ implies that there can be at most one element of $K_1(\mathbb{R}[G])$ with the stated property and to show that such an element exists it is enough to show that the element $(L_U^*(A, \chi, 1))_\chi$ of $\prod_{\mathrm{Ir}(G)} \mathbb{C}^\times$ satisfies the above displayed conditions. This fact is established in Proposition 2.2 below. □

The following result extends an observation of Kato and Trihan from [22, Appendix].

Proposition 2.2. *The following claims are valid for every χ in* $\mathrm{Ir}(G)$.

(i) *For every automorphism ω of $\mathbb{C}$ one has $\tau(L^*_U(A,\chi,1)) = L^*_U(A,\tau\circ\chi,1)$. In particular, one has $L^*_U(A,\chi,1)\in\mathbb{R}$ if χ is real valued.*

(ii) *Write $\mathbb{F}_q$ and $\mathbb{F}_{q'}$ for the total field of constants of K and L respectively. Then if χ is both real valued and not inflated from a nontrivial one dimensional representation of* $\mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$, *one has $L^*_U(A,\chi,1) > 0$.*

Proof. At the outset we fix a prime ℓ with $\ell \neq p$ and write $\mathbb{Q}^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We also fix an isomorphism $\mathbb{C}\cong\mathbb{C}_\ell$ that we suppress from the notation.

In particular, for each ρ in $\mathrm{Ir}(G)$ we fix a realisation V_ρ of ρ over $\mathbb{Q}^c$ and do not distinguish between it and the space $\mathbb{Q}^c_\ell\otimes_{\mathbb{Q}^c} V_\rho$.

Now for every ρ in $\mathrm{Ir}(G)$ Grothendieck [18] (see also the proof of [27, Chapter VI, Theorem 13.3]) has proved that there is an equality of power series

$$L_U(A,\rho,s) = \prod_{i=0}^{i=2} Q_{\rho,i}(q^{-s})^{(-1)^{i+1}}, \tag{1}$$

where each $Q_{\rho,i}(u)$ is a polynomial in u over $\mathbb{Q}^c$ that can be computed as

$$Q_{\rho,i}(u) := \det\big(1-u\cdot\varphi_q \,|\, H^i_{\text{ét},c}\big(U^c, V_\rho\otimes_{\mathbb{Q}} V_\ell(A)\big)(-1)\big)$$

Here U^c denotes $U\times_{\mathbb{F}_q}\mathbb{F}^c_q$, the vector space $V_\ell(A)$ is the $\mathbb{Q}_\ell$-space spanned by the ℓ-adic Tate module of A and φ_q the q-th power Frobenius map. We claim that, for each i and every automorphism ω of $\mathbb{C}$, one has

$$\omega(Q_{\rho,i}(u)) = (Q_{\omega\circ\rho,i}(u)). \tag{2}$$

In fact, since Grothendieck's result implies that the polynomial $Q_{\rho,i}(u)$ has coefficients in $\mathbb{Q}^c$, it is enough to consider automorphisms ω of $\mathbb{Q}^c$. Then, for each such ω, the natural isomorphism of $(\mathbb{Q}_\ell\otimes_{\mathbb{Q}}\mathbb{Q}^c)$-spaces

$$\mathbb{Q}^c\otimes_{\mathbb{Q}^c,\omega}(V_\rho\otimes_{\mathbb{Q}} V_\ell(A)) \cong (\mathbb{Q}^c\otimes_{\mathbb{Q}^c,\omega} V_\rho)\otimes_{\mathbb{Q}} V_\ell(A) \cong V_{\rho^\omega}\otimes_{\mathbb{Q}} V_\ell(A)$$

induces a similar isomorphism of the corresponding sheaves over U^c and hence an isomorphism of cohomology groups

$$\mathbb{Q}^c\otimes_{\mathbb{Q}^c,\omega} H^i_{\text{ét},c}(U^c, V_\rho\otimes_{\mathbb{Q}} V_\ell(A)) \cong H^i_{\text{ét},c}(U^c, V_{\rho^\omega}\otimes_{\mathbb{Q}} V_\ell(A))$$

under which $(1\otimes\varphi_q)$ on the first space corresponds to φ_q on the second space. This proves the claimed equality (2).

The equalities (2) (for each $i \in \{0, 1, 2\}$) can then be combined with (1) to deduce that the orders of vanishing at $s = 1$ of the series $L_U(A, \rho, s)$ and $L_U(A, \omega \circ \rho, s)$ are equal and moreover that

$$\omega(L_U^*(A, \rho, 1)) = L_U^*(A, \omega \circ \rho, 1),$$

as required. The final assertion of claim (i) then follows immediately upon applying this equality with $\omega = \tau$.

To prove claim (ii) we assume ρ is real-valued and hence, by (2) with $\omega = \tau$, that each polynomial $Q_{\rho,i}(u)$ belongs to $\mathbb{R}[u]$.

For each i we set

$$d_{\rho,i} := \dim_{\mathbb{Q}_\ell^c}(H^i_{\text{ét},c}(U^c, V_\rho \otimes_{\mathbb{Q}} V_\ell(A))(-1))$$

and write the eigenvalues, counted with multiplicity, of φ_q on this space as $\{\alpha_{ia}\}_{1 \le a \le d_{\rho,i}}$.

Now, since the weight on U of $(V_\rho \otimes_{\mathbb{Q}} V_\ell(A))(-1)$ is one, Deligne [14] has shown that $|\alpha_{ia}| \le q^{(i+1)/2}$ for all values of i and a.

Further, as the space $H^2_{\text{ét},c}(U^c, V_\rho \otimes_{\mathbb{Q}} V_\ell(A))(-1)$ is dual to $H^0_{\text{ét}}(U^c, V_{\check{\rho}} \otimes_{\mathbb{Q}} V_\ell(A^t))(1)$, and the weight on U of the representation $(V_{\check{\rho}} \otimes_{\mathbb{Q}} V_\ell(A^t))(1)$ is -3, one has $|\alpha_{2a}| = q^{\frac{3}{2}}$ for all a. Therefore neither of the terms $Q_{\rho,0}(q^{-1})$ or $Q_{\rho,2}(q^{-1})$ vanish.

Thus, if m denotes the order of vanishing of $L_U(A, \rho, s)$ at $s = 1$, one has

$$L_U^*(A, \rho, 1) = Q_{\rho,0}(q^{-1})^{-1} Q_{\rho,2}(q^{-1})^{-1} \cdot \lim_{s \to 1} (s-1)^{-m} Q_{\rho,1}(q^{-s}). \tag{3}$$

To prove that this quantity is a strictly positive real number we shall split it into a number of subproducts and show that each separate subproduct is a strictly positive real number.

At the outset we note that if an eigenvalue α_{ia} is not real, then (since $Q_{\rho,i}(u)$ belongs to $\mathbb{R}[u]$) there must exist an index $a \neq a'$ such that $\alpha_{ia'} = \tau(\alpha_{ia})$ and then the product $(1 - \alpha_{ia} q^{-1})(1 - \alpha_{ia'} q^{-1})$ is a strictly positive real number.

We need therefore only consider eigenvalues α_{ia} that are real and to do this we define for each $i \in \{0, 1, 2\}$ sets of indices

$$J_i' := \{a : 1 \le a \le d_{\rho,i} \ \text{ with } \alpha_{ia} = q^{(i+1)/2}\} \subset J_i := \{a : 1 \le a \le d_{\rho,i} \text{ with } \alpha_{ia} \in \mathbb{R}\}.$$

Now if either $i = 0$ and $a \in J_0$ or if $1 \le i \le 2$ and $a \in J_i \setminus J_i'$, then one checks easily that $(1 - \alpha_{ia} q^{-1}) > 0$.

Furthermore, one has $m = |J_1'|$ and

$$\lim_{s \to 1} (s-1)^{-m} \prod_{a \in J_1'} (1 - \alpha_{1a} q^{-s}) = (\lim_{s \to 1} (s-1)^{-1} (1 - q^{1-s}))^m = (\log(q))^m > 0$$

is a strictly positive real number.

To prove the quantity in (3) is strictly positive we are therefore reduced to showing that the product

$$\prod_{a \in J_2'} (1 - \alpha_{2a} q^{-1}) = \prod_{a \in J_2'} (1 - q^{1/2})$$

is strictly positive, or equivalently that $|J_2'|$ is even.

To do this we set $\Delta := \mathrm{Gal}(L\mathbb{F}_q^c/K\mathbb{F}_q^c)$ and recall that $H^2_{\text{ét},c}(U^c, V_\rho \otimes_{\mathbb{Q}} V_\ell(A))$ is dual to the (1)-twist of the space

$$\begin{aligned} H^0_{\text{ét}}(U^c, V_{\check\rho} \otimes_{\mathbb{Q}} V_\ell(A^t)) &= \big(V_{\check\rho} \otimes_{\mathbb{Q}} V_\ell(A^t)\big)^{\mathrm{Gal}(K^c/K\mathbb{F}_q^c)} = \big(V_{\check\rho} \otimes_{\mathbb{Q}} V_\ell(A^t)^{\mathrm{Gal}(K^c/L\mathbb{F}_q^c)}\big)^{\Delta} \\ &\cong \big(V_{\check\rho} \otimes_{\mathbb{Q}} V_\ell(B)\big)^{\Delta} = V_{\check\rho}^{\Delta} \otimes_{\mathbb{Q}} V_\ell(B). \end{aligned}$$

Here the first equality is obvious and the second is true because the restriction of ρ to $\mathrm{Gal}(K^c/L\mathbb{F}_q^c)$ is trivial, B is the $L/\mathbb{F}_{q'}$ trace of A^t (see [24, Chapter VIII, §3, Theorem 6] and note that $L/\mathbb{F}_{q'}$ is primary; i.e., the algebraic closure of $\mathbb{F}_{q'}$ in L is a purely inseparable extension of $\mathbb{F}_{q'}$). The last equality is true because B is defined over $\mathbb{F}_{q'}$.

In particular, if the representation $\check\rho^{\Delta}$ vanishes, then $|J_2'| = d_2 = 0$ is even and we are done.

We claim now that $\check\rho^{\Delta}$ does indeed vanish unless ρ is trivial. To show this we note that Δ identifies with a normal subgroup of G in such a way that the quotient is isomorphic to the cyclic group $H := \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$.

Thus, if η is any irreducible subrepresentation of $\mathrm{res}^G_\Delta(\check\rho)$, then Clifford's theorem (see, for instance, [11, Theorem 11.1(i)]) implies that $\mathrm{res}^G_\Delta(\check\rho)$ is the direct sum of conjugates of η and hence that $\mathrm{res}^G_\Delta(\check\rho)^{\Delta}$ does not vanish if and only if η is trivial.

It follows that $\mathrm{res}^G_\Delta(\check\rho)^{\Delta}$ does not vanish if and only if $\mathrm{res}^G_\Delta(\check\rho)$ is trivial and this happens if and only if $\check\rho$, and hence also ρ itself, is inflated from a representation of H.

Hence, since we have assumed that ρ is both irreducible and not inflated from a nontrivial representation of H, the representation $\mathrm{res}^G_\Delta(\check\rho)^{\Delta}$ does not vanish if and only if ρ is the trivial representation of G.

We have now verified the assertion of claim (ii) for all but the trivial representation of G and in this case the claimed result is proved by Kato and Trihan in [22, Appendix]. □

Remark 2.3. Fix a prime ℓ and an ℓ-adic representation V of G_K. Then the tensor product $\mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V$ is a (left) module over $G \times G_K$ via the rule $(g, \sigma)(x \otimes v) := gx\overline{\sigma}^{-1} \otimes \sigma(v)$ for $g \in G$, $\sigma \in G_K$, $x \in \mathbb{Q}_\ell^c[G]$ and $v \in V$, where $\overline{\sigma}$ is the image of σ under the restriction map $G_K \to G$. In particular, if we fix χ in $\mathrm{Ir}(G)$ and a realisation V_χ over $\mathbb{Q}_\ell^c$, then, with respect to this action, there is a canonical isomorphism

$$V_\chi \otimes_{\mathbb{Q}_\ell} V \cong \mathrm{Hom}_{\mathbb{Q}_\ell^c[G]}(V_{\check\chi}, \mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V)$$

of ℓ-adic representations of G_K, where G_K acts diagonally on $V_\chi \otimes_{\mathbb{Q}_\ell} V$ and on the Hom-group via only $\mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V$. This isomorphism is induced by the canonical composite identification

$$\begin{aligned} H^0(G, \mathrm{Hom}_{\mathbb{Q}_\ell^c}(V_{\check\chi}, \mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V)) &\cong H^0(G, \mathrm{Hom}_{\mathbb{Q}_\ell^c}(V_{\check\chi}, \mathbb{Q}_\ell^c) \otimes_{\mathbb{Q}_\ell^c} (\mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V)) \\ &\cong H_0(G, \mathrm{Hom}_{\mathbb{Q}_\ell^c}(V_{\check\chi}, \mathbb{Q}_\ell^c) \otimes_{\mathbb{Q}_\ell^c} (\mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V)) \\ &\cong \mathrm{Hom}_{\mathbb{Q}_\ell^c}(V_{\check\chi}, \mathbb{Q}_\ell^c) \otimes_{\mathbb{Q}_\ell^c[G]} (\mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Q}_\ell} V) \\ &\cong \mathrm{Hom}_{\mathbb{Q}_\ell^c}(V_{\check\chi}, \mathbb{Q}_\ell^c) \otimes_{\mathbb{Q}_\ell} V. \end{aligned}$$

Here the second isomorphism is induced by the inverse of the canonical norm map (since the order of G is invertible in $\mathbb{Q}_\ell^c$), and all other isomorphisms are clear.

3. Arithmetic complexes

In this section we construct certain canonical complexes of Galois modules whose Euler characteristics will occur in the formulation of our refined Birch and Swinnerton-Dyer conjecture.

In the sequel, for any noetherian ring R we shall write $D^{\mathrm{perf}}(R)$ for the full triangulated subcategory of the derived category $D(R)$ of (left) R-modules comprising complexes that are "perfect" (that is, isomorphic in $D(R)$ to a bounded complex of finitely generated projective R-modules).

3.1. ***Selmer groups.*** The Tate–Shafarevich group and, for any natural number n, the n-torsion Selmer group of A over any finite extension F of K are respectively defined to be the kernels

$$\text{Ш}(A/F) := \ker\bigl(H^1(F, A) \to \bigoplus_v H^1(F_v, A)\bigr),$$
$$\mathrm{Sel}_n(A/F) := \ker\bigl(H^1(F, A[n]) \to \bigoplus_v H^1(F_v, A)\bigr).$$

Here the groups $H^1(F, A)$, $H^1(F, A[n])$ and $H^1(F_v, A)$ denote flat cohomology and in both cases v runs over all places of F and the arrow denotes the natural diagonal restriction map.

One then defines Selmer groups of A over F via the natural limits

$$\mathrm{Sel}_{\mathbb{Q}/\mathbb{Z}}(A/F) := \varinjlim_n \mathrm{Sel}_n(A/F) \quad \text{and} \quad \mathrm{Sel}_{\hat{\mathbb{Z}}}(A/F) := \varprojlim_n \mathrm{Sel}_n(A/F).$$

For convenience, we write $X(A/F)$ for the Pontryagin dual of $\mathrm{Sel}_{\mathbb{Q}/\mathbb{Z}}(A/F)$.

Remark 3.1. We make much use in the sequel of the fact that the above definitions lead naturally to canonical exact sequences

$$0 \to A(F) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \to \mathrm{Sel}_{\hat{\mathbb{Z}}}(A/F) \to \varprojlim_n \text{Ш}(A/F)[n] \to 0,$$
$$0 \to (\text{Ш}(A/F)_{\mathrm{tors}})^\vee \to X(A/F) \to \mathrm{Hom}_{\mathbb{Z}}(A(F), \hat{\mathbb{Z}}) \to 0.$$

3.2. ***Arithmetic cohomology.*** For each place v of F outside U_F we fix a pro-p open subgroup V_v of $A(F_v)$ and denote the family $(V_v)_{v\notin U_F}$ by V_{U_F}, or simply by V_F or V when the context is clear. We then follow Kato and Trihan [22] in defining the "arithmetic cohomology" complex $R\Gamma_{ar,V}(U_F, \mathcal{A}_{\mathrm{tors}})$ to be the mapping fibre of the morphism

$$R\Gamma_{\mathrm{fl}}(U_F, \mathcal{A}_{\mathrm{tors}}) \oplus \Bigl(\bigoplus_{v\notin U_F} (V_v \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})[-1]\Bigr) \xrightarrow{(\kappa_1,\kappa_2)} \bigoplus_{v\notin U_F} R\Gamma_{\mathrm{fl}}(F_v, \mathcal{A}_{\mathrm{tors}}). \tag{4}$$

Here κ_1 denotes the natural diagonal localisation morphism in flat cohomology and κ_2 the restriction of the morphism

$$(A(F_v) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})[-1] \to \bigoplus_{v\notin U_F} R\Gamma_{\mathrm{fl}}(F_v, \mathcal{A}_{\mathrm{tors}})$$

that is obtained by applying $-\otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ to the morphism $A(F_v)[-1] \to R\Gamma_{\mathrm{fl}}(F_v, \varprojlim_n A[n])$ in $D(\mathbb{Z}[G])$ induced by the fact that $H^0_{\mathrm{fl}}(F_v, \varprojlim_n A[n])$ vanishes whilst $A(F_v)$ is canonically isomorphic to a submodule of $H^1_{\mathrm{fl}}(F_v, \varprojlim_n A[n])$.

Proposition 3.2 [22, §2.5]. *The complex $C_V^{\mathrm{ar}} := R\Gamma_{ar,V}(U_F, \mathcal{A}_{\mathrm{tors}})$ is acyclic outside degrees* 0, 1 *and* 2. *In addition, there exists a canonical exact sequence*

$$0 \to H^0(C_V^{\mathrm{ar}}) \to A(F)_{\mathrm{tors}} \to \bigoplus_{v \notin U_F} A(F_v)/V_v \to H^1(C_V^{\mathrm{ar}}) \to \mathrm{Sel}_{\mathbb{Q}/\mathbb{Z}}(A/F) \to 0,$$

and a canonical isomorphism $H^2(C_V^{\mathrm{ar}}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Sel}_{\hat{\mathbb{Z}}}(A^t/F), \mathbb{Q}/\mathbb{Z})$. □

Since $R\Gamma_{ar,V}(U_F, \mathcal{A}_{\mathrm{tors}})$ is a complex of torsion groups it decomposes naturally as a direct sum of ℓ-primary component complexes $R\Gamma_{ar,V}(U_F, \mathcal{A}_{\mathrm{tors}})_\ell$.

Remark 3.3. For any prime $\ell \neq p$ the definition of $R\Gamma_{ar,V}(U_F, \mathcal{A}_{\mathrm{tors}})_\ell$ via the morphism in (4) implies that it identifies with the compactly supported étale cohomology complex $R\Gamma_{\text{ét},c}(U_F, \mathcal{A}\{\ell\})$ of the (étale) sheaf $\mathcal{A}\{\ell\}$ on U_F comprising all ℓ-primary torsion in $\mathcal{A}$.

3.3. ***Pro-p subgroups.*** To make the complex $R\Gamma_{ar,V}(U_F, \mathcal{A}_{\mathrm{tors}})$ constructed above amenable for our purposes we need to make an appropriate choice of the family V. We now explain how to make such a choice following the approach of Kato and Trihan in [22, § 6].

To do this we fix a finite Galois extension L/K and set $G := \mathrm{Gal}(L/K)$. We let X_L be the proper smooth curve with function field L, and let $U_L \subset X_L$ be the preimage of U (and we will later "shrink" U so that L/K is unramified at places in U). For any place w of L we write G_w for its decomposition subgroup in G.

We write $\mathcal{A}'$ for the Néron model of A_L over X_L, and $\mathcal{A}_{X_L}$ for the pull back of $\mathcal{A}$.

Lemma 3.4. *There exists a G-invariant divisor $E = \sum_{w \notin U_L} n(w)w$ on X_L with* $\mathrm{supp}(E) = X_L \setminus U_L$ *and for each place $w \notin U_L$ over $v \notin U$ a G_w-stable pro-p open subgroup V'_w of $A_L(L_w)$ and an open $\mathcal{O}_v[G_w]$-submodule W'_w of* $\mathrm{Lie}(A_L(L_w))$ *that satisfy all of the following properties.*

(1) *For $w \notin U_K$, we have $\mathcal{A}'(\mathfrak{m}_w^{2n(w)}) \subset V'_w \subset \mathcal{A}_{X_L}(\mathfrak{m}_w^{n(w)})$.*

(2) *For $w \notin U_L$, we have* $\mathrm{Lie}(\mathcal{A}')(\mathfrak{m}_w^{2n(w)}) \subset W'_w \subset \mathrm{Lie}(\mathcal{A}_{X_L})(\mathfrak{m}_w^{n(w)})$.

(3) *For $w \notin U_L$, the canonical isomorphism*

$$\mathcal{A}'(\mathfrak{m}_w^{n(w)})/\mathcal{A}'(\mathfrak{m}_w^{2n(w)}) \cong \mathrm{Lie}(\mathcal{A}')(\mathfrak{m}_w^{n(w)})/\mathrm{Lie}(\mathcal{A}')(\mathfrak{m}_w^{2n(w)})$$

sends the image of V'_w to W'_w.

(4) *For each place v outside U the products $\prod_{w|v} V'_w$ and $\prod_{w|v} W'_w$ are stable under the action of G and for each natural number i the associated cohomology groups $H^i(G, \prod_{w|v} V'_w)$ and $H^i(G, \prod_{w|v} W'_w)$ vanish.*

We can furthermore require E to be the pull back of some divisor E_0 of X.

In the application (see Section 7 and following) we need E to be the pull back of a divisor E_0 of X.

Proof. This result is only a slight adaptation of [22, Lemma 6.4] (see Remark 3.5 below). For this reason we only sketch the proof, following closely the argument of [22, §6].

The key point is that it suffices to construct a divisor E and a family of subgroups $\{W'_w\}_w$ with the properties stated in Lemma 3.4, since the family $\{W'_w\}_w$ uniquely determines the family $\{V'_v\}_v$ by property (3) and then the latter family can be shown to satisfy property (4) by repeating the proof of [22, Lemma 6.2(2)].

Now, by the argument of [22, Lemma 6.2(1)], for each place w of L outside U_L there exists a constant $c(w)$ such that for any integer $n \geqslant 0$ there exists a G_w-stable $\mathcal{O}_v$-lattice W'_w of $\mathrm{Lie}(\mathcal{A}_{X_L})(\mathcal{O}_w)$ such that both

$$\mathrm{Lie}(\mathcal{A}_{X_L})(\mathfrak{m}_w^{n+c(w)}) \subset W'_w \subset \mathrm{Lie}(\mathcal{A}_{X_L})(\mathfrak{m}_w^{n}).$$

and the group $H^i(G_w, W'_w)$ vanishes for all $i \geqslant 1$.

By the argument of [22, Lemma 6.3], we may in addition assume that the subgroups W'_w satisfy property (2), at least provided that $n(w)$ is sufficiently large and divisible by the ramification index $e(w|v)$. (We would like $n(w)$ to be divisible by $e(w|v)$ in general since we are only allowed to multiply W'_w by $\mathcal{O}_v$-multiple; recall that W'_w is only an $\mathcal{O}_v$-submodule, not an $\mathcal{O}_w$-submodule.)

To ensure that the product $\prod_{w \notin U_L} W'_w$ is stable under the action of G, we first fix a place w over each $v \notin U$ and a subgroup W'_w that has property (2) and is such that $H^i(G_w, W'_w)$ vanishes for all $i \geqslant 1$.

For each σ in G, we then set $W'_{\sigma(w)} := \sigma(W'_w) \subset \mathrm{Lie}(\mathcal{A}_{X_L})(\mathcal{O}_{\sigma(w)})$ (which, we note, only depends on $\sigma(w)$). Then the collection of subgroups $\{W'_w\}_{w \notin U_L}$ clearly has both of the properties (2) and (4).

To ensure that E is a pull back of some divisor E_0 of X, we may replace E with $\pi^*(\pi_* E)$ and replace $\{W'_w\}_{w|v}$ by some suitable power of uniformiser of $\mathcal{O}_w$. □

Remark 3.5. Lemma 3.4 only differs from [22, Lemma 6.4] in that we require each group W'_w to be an open $\mathcal{O}_v[G_w]$-submodule of $\mathrm{Lie}(\mathcal{A}')(\mathcal{O}_w)$ rather than an open $\mathcal{O}_w$-submodule as in loc. cit. In fact, in [22, Lemma 6.2(1)], it is claimed that W'_w can be chosen as an $\mathcal{O}_w$-sublattice of $\mathrm{Lie}(\mathcal{A}_{X_L})(\mathcal{O}_w)$, but the indicated proof seems only to show that it can be chosen as a G_w-stable $\mathcal{O}_v$-lattice.

Remark 3.6. The proof of Lemma 3.4 shows that for any place v of K that is both unramified in L and of good reduction for A, the subgroup V'_w can be chosen as $\mathcal{A}(\mathfrak{m}_w)$.

3.4. ***Selmer complexes.*** For each place w outside U_L we now fix a choice of subgroups V'_w as in Lemma 3.4. For any subgroup J of G and for any place v outside U_{L^J} we then define a group

$$V_v := \Big(\prod_{w|v} V'_w\Big)^J$$

and we denote the associated families of subgroups $(V'_w)_{w \notin U_L}$ and $(V_v)_{v \notin U_{L^J}}$ by V_L and V_{L^J}, respectively. We may occasionally write V for V_K when $J = G$.

In the next result we use these subgroups to construct a canonical "Selmer complex" $\mathrm{SC}_{V_L}(A, L/K)$ that will play a key role in the formulation of our refined Birch and Swinnerton-Dyer conjecture.

We also use the G-module $X_{\mathbb{Z}}(A/F)$ that is defined as the preimage of $\mathrm{Hom}_{\mathbb{Z}}(A(F), \mathbb{Z})$ under the natural surjective homomorphism $X(A/F) \to \mathrm{Hom}_{\mathbb{Z}}(A(F), \hat{\mathbb{Z}})$ (see Remark 3.1).

Proposition 3.7. *The following claims are valid.*

(i) $R\Gamma_{ar,V_L}(U_L, \mathcal{A}'_{\mathrm{tors}})^*[-2]$ *is an object of* $D^{\mathrm{perf}}(\hat{\mathbb{Z}}[G])$ *that is acyclic outside degrees* 0, 1 *and* 2.

(ii) *If the groups* $Ш(A/L)$ *and* $Ш(A^t/L)$ *are both finite, there exists a complex* $\mathrm{SC}_{V_L} := \mathrm{SC}_{V_L}(A, L/K)$ *in* $D^{\mathrm{perf}}(\mathbb{Z}[G])$ *that is acyclic outside degrees* 0, 1 *and* 2, *is unique up to isomorphisms in* $D^{\mathrm{perf}}(\mathbb{Z}[G])$ *that induce the identity map in all degrees of cohomology and also has the following properties*:

(a) $H^0(\mathrm{SC}_{V_L}) = A^t(L)$, $H^1(\mathrm{SC}_{V_L})$ *contains* $X_{\mathbb{Z}}(A/L)$ *as a submodule of finite index and* $H^2(\mathrm{SC}_{V_L})$ *is finite.*

(b) *There exists a canonical isomorphism in* $D^{\mathrm{perf}}(\hat{\mathbb{Z}}[G])$ *of the form*

$$\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{SC}_{V_L} \cong R\Gamma_{ar,V_L}(U_L, \mathcal{A}_{\mathrm{tors}})^*[-2].$$

Proof. For each subgroup J of G we set $C^{\mathrm{ar}}_{V,J} := R\Gamma_{ar,V_{L^J}}(U_{L^J}, \mathcal{A}_{\mathrm{tors}})$ and we abbreviate $C^{\mathrm{ar}}_{V,J}$ to C^{ar}_V when J is the trivial subgroup.

Then, since $H^i(C^{\mathrm{ar},*}_{V,J}[-2]) = H^{2-i}(C^{\mathrm{ar}}_{V,J})^*$ in all degrees i, the result of Proposition 3.2 implies that each complex $C^{\mathrm{ar},*}_{V,J}[-2]$ is acyclic in all degrees outside 0, 1 and 2 and that its cohomology is finitely generated over $\hat{\mathbb{Z}}$ in all degrees.

By a standard criterion, it follows that $C^{\mathrm{ar},*}_V$, and hence also $C^{\mathrm{ar},*}_V[-2]$, belongs to $D^{\mathrm{perf}}(\hat{\mathbb{Z}}[G])$, and so claim (i) is valid, if for every subgroup J of G there is an isomorphism in $D(\hat{\mathbb{Z}})$ of the form $\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}[J]} C^{\mathrm{ar},*}_V \cong C^{\mathrm{ar},*}_{V,J}$.

In view of the natural isomorphisms $\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}[J]} C^{\mathrm{ar},*}_V \cong R\,\mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, C^{\mathrm{ar}}_V)^*$ we are therefore reduced to showing the existence of isomorphisms in $D(\hat{\mathbb{Z}})$ of the form

$$R\,\mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, R\Gamma_{\mathrm{ar},V_L}(U_L, \mathcal{A}_{\mathrm{tors}})) \cong R\Gamma_{\mathrm{ar},V_{L^J}}(U_{L^J}, \mathcal{A}_{\mathrm{tors}}) \tag{5}$$

and this is proved by Kato and Trihan in [22, Lemma 6.1].

Turning to claim (ii), we note that claim (i) combines with the general result of Lemma 3.8 below to imply it suffices to show that, under the stated hypotheses, the group $H^0(C^{\mathrm{ar}}_V)^*$ is finite, the group $H^2(C^{\mathrm{ar}}_V)^*$ is naturally isomorphic to $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A^t(L)$ and there exists a finitely generated G-module M^1 that contains $X_{\mathbb{Z}}(A/L)$ as a submodule of finite index and is such that there is a canonical isomorphism $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M^1 \cong H^1(C^{\mathrm{ar}}_V)^*$ of $\hat{\mathbb{Z}}[G]$-modules.

In this direction, the exact sequence in Proposition 3.2 implies directly that $H^0(C^{\mathrm{ar}}_V)^*$ is finite.

In addition, since the limit $\varprojlim_n Ш(A^t/L)[n]$ vanishes under the assumption $Ш(A^t/L)$ is finite, the displayed isomorphism in Proposition 3.2 combines with the first exact sequence in Remark 3.1 to give a canonical isomorphism

$$H^2(C^{\mathrm{ar}}_V)^* \cong (\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A^t(L))^{**} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A^t(L)$$

of the required form.

Next we note that, since $Ш(A/L)$ is assumed to be finite, the second exact sequence in Remark 3.1 implies $X_{\mathbb{Z}}(A/L)$ is finitely generated.

Thus, if we write Y for the (finite) cokernel of the map $A(L)_{\mathrm{tors}} \to \bigoplus_{v \notin U_L} A(L_v)/V_v$ that occurs in Proposition 3.2, then the natural map of finite groups

$$\mathrm{Ext}^1_G(Y^*, X_{\mathbb{Z}}(A/L)) = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{Ext}^1_G(Y^*, X_{\mathbb{Z}}(A/L)) \to \mathrm{Ext}^1_{\hat{\mathbb{Z}}[G]}(Y^*, X(A/L))$$

is bijective and so there exists an exact commutative diagram of G-modules

$$\begin{tikzcd} 0 \arrow[r] & X_{\mathbb{Z}}(A/L) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X(A/L) \arrow[r] & H^1(C_V^{\mathrm{ar}})^* \arrow[r] & Y^* \arrow[r] & 0 \end{tikzcd}$$

in which the first vertical arrow is the natural inclusion, and so induces an isomorphism $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} X_{\mathbb{Z}}(A/L) \cong X(A/L)$, and the lower row is induced by the Pontryagin dual of the long exact sequence in Proposition 3.2.

In particular, from the upper row of the above diagram we can deduce that M is finitely generated and hence is a suitable choice for the module M^1 that we seek. □

In the sequel, for a ring Λ and integer a, we write $\tau_{\geq a}$ and $\tau_{\leq a}$ for the truncation functors on $D(\Lambda)$ in degrees at least a and at most a respectively.

We also recall that a G-module is said to be "cohomologically trivial", or c-t for short in the sequel, if for every integer i and every subgroup J of G the Tate cohomology group $\hat{H}^i(J, M)$ vanishes.

Lemma 3.8. *Let $\hat{C}$ be a cohomologically bounded complex of $\hat{\mathbb{Z}}[G]$-modules and assume to be given in each degree j a finitely generated G-module M^j and an isomorphism of $\hat{\mathbb{Z}}[G]$-modules of the form $\iota_j : \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M^j \cong H^j(\hat{C})$.*

Then there exists an object C of $D(\mathbb{Z}[G])$ with all of the following properties:

(i) *$H^j(C) = M^j$ for all j.*

(ii) *There exists an isomorphism $\iota : \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C \cong \hat{C}$ in $D(\hat{\mathbb{Z}}[G])$ for which in each degree j one has $H^j(\iota) = \iota_j$.*

(iii) *C belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ if and only if $\hat{C}$ belongs to $D^{\mathrm{perf}}(\hat{\mathbb{Z}}[G])$.*

Any such complex C is unique to within an isomorphism κ in $D(\mathbb{Z}[G])$ for which $H^j(\kappa)$ is the identity automorphism of M^j in each degree j.

Proof. We prove this by induction on the number of nonzero cohomology groups of $\hat{C}$.

If there is only one nonzero such group, in degree d say, then $\hat{C}$ is isomorphic in $D(\hat{\mathbb{Z}}[G])$ to $(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M^d)[-d]$ and we write C for the complex $M^d[-d]$ in $D(\mathbb{Z}[G])$.

In this case claim (i) is clear and claim (ii) is true with ι the morphism induced by ι_d. Finally, since any finitely generated module over either $\mathbb{Z}[G]$ or $\hat{\mathbb{Z}}[G]$ that is c-t has a finite projective resolution, C belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ if and only if M^d is c-t and $\hat{C}$ belongs to $D^{\mathrm{perf}}(\hat{\mathbb{Z}}[G])$ if and only if $\hat{M}^d$ is c-t. This implies claim (iii) since a finitely generated G-module N is c-t if and only if $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} N$ is c-t as a consequence of the fact that in each degree i and for each subgroup J of G the natural map $\hat{H}^i(J, N) \to \hat{H}^i(J, \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} N)$ is bijective.

To deal with the general case we assume $\hat{C}$ is not acyclic and write d for the unique integer such that $H^d(\hat{C}) \neq 0$ and $H^i(\hat{C}) = 0$ for all $i > d$. We then abbreviate the complexes $\tau_{\leq d-1}\hat{C}$ and $\tau_{\geq d}\hat{C}$ to $\hat{C}_1$ and $\hat{C}_2$ and recall that there is a canonical exact triangle in $D(\hat{\mathbb{Z}}[G])$ of the form

$$\hat{C}_1 \to \hat{C} \to \hat{C}_2 \xrightarrow{\hat{\theta}} \hat{C}_1[1].$$

We note that this triangle induces an isomorphism κ in $D(\hat{\mathbb{Z}}[G])$ between $\hat{C}$ and $\mathrm{Cone}(\hat{\theta})[-1]$, where we write $\mathrm{Cone}(\alpha)$ for the mapping cone of a morphism α.

Now, since $H^j(\hat{C}_1) = H^j(\hat{C})$ for $j < d$ and $H^j(\hat{C}_1) = 0$ for all $j \geq d$, the inductive hypothesis implies the existence of C_1 in $D(\mathbb{Z}[G])$ and an isomorphism $\iota_1 : \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C_1 \cong \hat{C}_1$ in $D(\hat{\mathbb{Z}}[G])$ such that in each degree j with $j < d$ one has $H^j(\iota) = \iota_j$.

In addition, since $\hat{C}_2$ is acyclic outside degree d, the argument given above shows the existence of a complex C_2 in $D(\mathbb{Z}[G])$ and an isomorphism $\iota_2 : \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C_2 \cong \hat{C}_2$ in $D(\hat{\mathbb{Z}}[G])$ with $H^d(\iota) = \iota_d$.

Next we recall that the group $\mathrm{Hom}_{D(\hat{\mathbb{Z}}[G])}(\hat{C}_2, \hat{C}_1[1])$ is equal to $H^0(R\,\mathrm{Hom}_{\hat{\mathbb{Z}}[G]}(\hat{C}_2, \hat{C}_1[1]))$ and so can be computed by using the spectral sequence

$$E_2^{p,q} = \prod_{a \in \mathbb{Z}} \mathrm{Ext}^p_G(H^a(\hat{C}_2), H^{q+a}(\hat{C}_1[1])) \Rightarrow H^{p+q}(R\,\mathrm{Hom}_{\hat{\mathbb{Z}}[G]}(C_2, C_1[1]))$$

constructed by Verdier in [40, III, 4.6.10]. We also note that there is no degree in which the complexes $\hat{C}_2$ and $\hat{C}_1[1]$ have cohomology groups that are both nonzero and that any group of the form $\mathrm{Ext}^p_G(-,-)$ vanishes for $p < 0$ and is torsion for $p > 0$. Given these facts, the above spectral sequence implies that $\mathrm{Hom}_{D(\hat{\mathbb{Z}}[G])}(\hat{C}_2, \hat{C}_1[1])$ is finite and hence that the diagonal localisation map $\mathrm{Hom}_{D(\mathbb{Z}[G])}(C_2, C_1[1]) \to \mathrm{Hom}_{D(\hat{\mathbb{Z}}[G])}(\hat{C}_2, \hat{C}_1[1])$ is bijective.

We now write θ for the preimage of $\hat{\theta}$ under the latter isomorphism and claim that the mapping fibre $C := \mathrm{Cone}(\theta)[-1]$ has all of the required properties.

Firstly, this definition implies directly that $H^j(C)$ is equal to $H^j(C_1)$ if $j < d$ and to $H^j(C_2)$ if $j \geq d$, and so claim (i) follows immediately from the given properties of C_1 and C_2. The definition also implies directly that $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C$ is isomorphic to $\mathrm{Cone}(\hat{\theta})[-1]$ and hence that κ induces an isomorphism in $D(\hat{\mathbb{Z}}[G])$ between $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C$ and $\hat{C}$ with the property described in claim (ii).

To prove claim (iii) it suffices to check that C belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ if $\hat{C}$ belongs to $D^{\mathrm{perf}}(\hat{\mathbb{Z}}[G])$. To do this we can assume, by a standard resolution argument (as described, for example, in [13, rapport, lemme 4.7]), that C is a bounded complex of finitely generated G-modules in which all but the first (nonzero) module, M say, is free. If we then also assume that the complex $\hat{C}$ is isomorphic in $D(\hat{\mathbb{Z}}[G])$ to a bounded complex of finitely generated projective $\hat{\mathbb{Z}}[G]$-modules Q, then there exists a quasi-isomorphism $\pi : Q \to \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C$ of complexes of $\hat{\mathbb{Z}}[G]$-modules.

Now, since all terms of Q and $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C$ are projective $\hat{\mathbb{Z}}[G]$-modules, except possibly for $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M$, the acyclicity of $\mathrm{Cone}(\pi)$ implies that the $\hat{\mathbb{Z}}[G]$-module $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M$ is c-t. This in turn implies that M is c-t and hence has a finite projective resolution. It follows that C belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$, as claimed. $\square$

3.5. ***Coherent cohomology.*** The Selmer complex that is constructed in Proposition 3.7(ii) depends on the choice of subgroups V_L. We shall therefore need to introduce an auxiliary perfect complex that will be used to compensate for this dependence in the formulation of our conjecture.

To do this for each place v outside U we choose a place w of X_L above v and the $\mathcal{O}_v[G_w]$-submodule W'_w of $\mathrm{Lie}(A_L(L_w))$ that corresponds in Lemma 3.4 to the subgroup V'_w fixed at the beginning of Section 3.4. For any other place $w'|v$ of X_L, we choose $\gamma \in G$ such that $w' = \gamma \cdot w$ and set $W'_{w'}$ denote the image of the isomorphism $\mathrm{Lie}(A_L(L_w)) \xrightarrow{\cong} \mathrm{Lie}(A_L(L_{w'}))$ induced by γ. Note that $W'_{w'}$ does not depend on the choice of γ as W'_w is G_w-stable.

For any place v outside U we then set

$$W_v := \Big(\prod_{w|v} W'_w\Big)^G$$

and we denote the associated families of subgroups $(W'_w)_{w\notin U_L}$ and $(W_v)_{v\notin U}$ by W_L and W_K respectively.

We then define $\mathcal{L}$ to be the coherent $\mathcal{O}_X$-submodule of $\mathrm{Lie}(\mathcal{A})$ that extends $\mathrm{Lie}(\mathcal{A}|_U)$ and is such that $\mathcal{L}_v = W_v \subset \mathrm{Lie}(\mathcal{A})(\mathcal{O}_v)$ for each $v \notin U$.

We similarly define $\mathcal{L}_L$ to be the G-equivariant coherent $\mathcal{O}_X$-submodule of $\pi_*\mathrm{Lie}(\mathcal{A}_{X_L})$ with $\mathcal{L}_{L,v} = \prod_{w|v} W'_w$ for each $v \notin U$, where we write $\pi : X_L \to X$ for the natural projection.

Lemma 3.9. *The complex $R\Gamma(X, \mathcal{L}_L)^*$ belongs to $D^{\mathrm{perf}}(\mathbb{F}_p[G])$, and hence to $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$.*

Proof. For each subgroup J of G the complex $R\Gamma(X, (\mathcal{L}_L)^J)^*$ is represented by a complex of finite-dimensional $\mathbb{F}_p$-vector spaces that is acyclic outside degrees 0 and 1.

By the same argument as used to prove Proposition 3.7(i) we are therefore reduced to proving that for each J there is a natural isomorphism in $D(\mathbb{F}_p)$ of the form

$$R\,\mathrm{Hom}_{\mathbb{F}_p[J]}(\mathbb{F}_p, R\Gamma(X, \mathcal{L}_L)) \cong R\Gamma(X, (\mathcal{L}_L)^J) \tag{6}$$

and this is proved by Kato and Trihan in [22, p. 585]. □

Remark 3.10. In view of Remark 3.5, we have here defined $\mathcal{L}_L$ to be a $\mathbb{F}_p[G]$-equivariant vector bundle over X rather than a vector bundle over X_L, as in [22, § 6.5]. This means that various arguments in loc. cit. that rely on the "geometric p-adic cohomology theory" over X_L and will be referred to in later sections must in our case be carried out over X by using the relevant push-forward constructions. This, however, is a routine difference that we do not dwell on.

4. Statements of the conjecture and main results

In this section we formulate our refinement of the Birch and Swinnerton-Dyer conjecture, establish some basic properties of the conjecture and state the main supporting evidence for it that we will obtain in the rest of the article.

4.1. ***Relative K-theory.*** Before stating our conjecture we quickly review relevant aspects of relative algebraic K-theory.

For a Dedekind domain R with field of fractions F, an R-order $\mathfrak{A}$ in a finite dimensional separable F-algebra A and a field extension E of F we set $A_E := E \otimes_F A$.

4.1.1. We use the relative algebraic K_0-group $K_0(\mathfrak{A}, A_E)$ of the ring inclusion $\mathfrak{A} \subset A_E$, as described explicitly in terms of generators and relations by Swan in [33, p. 215].

We recall that for any extension field E' of E there exists an exact commutative diagram

$$\begin{array}{ccccccc} K_1(\mathfrak{A}) & \longrightarrow & K_1(A_{E'}) & \xrightarrow{\partial_{\mathfrak{A},E'}} & K_0(\mathfrak{A}, A_{E'}) & \xrightarrow{\partial'_{\mathfrak{A},E'}} & K_0(\mathfrak{A}) \\ \| & & \uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota'} & & \| \\ K_1(\mathfrak{A}) & \xrightarrow{K_1(A_E)} & & \xrightarrow{\partial_{\mathfrak{A},E}} & K_0(\mathfrak{A}, A_E) & \xrightarrow{\partial'_{\mathfrak{A},E}} & K_0(\mathfrak{A}) \end{array} \tag{7}$$

in which the upper and lower rows are the respective long exact sequences in relative K-theory of the inclusions $\mathfrak{A} \subset A_E$ and $\mathfrak{A} \subset A_{E'}$ and both of the vertical arrows are injective and induced by the inclusion $A_E \subseteq A_{E'}$. (For more details see [33, Theorem 15.5].)

We further recall that the Whitehead group $K_1(A_E)$ comprises (isomorphism classes of) pairs of the form $\langle W, \theta\rangle$ in which θ is an automorphism of the finitely generated projective A_E-module W. In particular, if W is spanned by a (finitely generated) projective $\mathfrak{A}$-module P, then the connecting homomorphism $\partial_{\mathfrak{A},E}$ in (7) sends $\langle W, \theta\rangle$ to the element of $K_0(\mathfrak{A}, A_E)$ that corresponds to the triple (P, θ, P).

If $R = \mathbb{Z}$ and for each prime ℓ we set $\mathfrak{A}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathfrak{A}$ and $A_\ell := \mathbb{Q}_\ell \otimes_{\mathbb{Q}} A$, then we regard each group $K_0(\mathfrak{A}_\ell, A_\ell)$ as a subgroup of $K_0(\mathfrak{A}, A)$ by means of the canonical composite homomorphism

$$\bigoplus_\ell K_0(\mathfrak{A}_\ell, A_\ell) \cong K_0(\mathfrak{A}, A) \subset K_0(\mathfrak{A}, A_{\mathbb{R}}). \tag{8}$$

Here ℓ runs over all primes, the isomorphism is as described in the discussion following [12, (49.12)] and the inclusion is induced by the relevant case of the map ι' in (7).

For each element x of $K_0(\mathfrak{A}, A)$ we write $(x_\ell)_\ell$ for its image in $\bigoplus_\ell K_0(\mathfrak{A}_\ell, A_\ell)$ under the isomorphism in (8).

4.1.2. We shall construct elements of $K_0(\mathfrak{A}, A_E)$ by using the formalism of "nonabelian determinants" described by Fukaya and Kato in [17, §1]. To recall the relevant facts we write Σ for the category $D^{\mathrm{perf}}(\mathfrak{A})$.

Following [17, Definition 1.3.2], one can define a localized K_1-group $K_1(A_E, \Sigma)$. This abelian group is generated by pairs (C, h), where C is an object of Σ for which the Euler characteristic of $E \otimes_R C$ in $K_0(A_E)$ vanishes and h is a morphism $\mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$ in the category $\mathcal{C}_{A_E}$ constructed in [17, §1.2.1]; the relations between these generators are then the obvious analogues of the relations (1), (2) and (3) given as part of [17, Definition 1.3.2]. These relations in turn ensure that there exists a canonical group homomorphism

$$\iota_{\mathfrak{A},E} : K_1(A_E, \Sigma) \to K_0(\mathfrak{A}, A_E).$$

The approach of [17, Theorem 1.3.15] proves the existence of an exact sequence relating $K_1(A_E, \Sigma)$ to $K_1(A_E)$, $K_0(\Sigma) = K_0(\mathfrak{A})$ and $K_0(A_E)$, and by comparing this sequence to (7), one can deduce that $\iota_{\mathcal{A},E}$ is surjective (but we omit details as we make no use of this fact).

For each generator (C, h) of $K_1(A_E, \Sigma)$, we set

$$\chi_{\mathfrak{A},E}(C, h) := \iota_{\mathfrak{A},E}((C, h)) \in K_0(\mathfrak{A}, A_E).$$

If $E \otimes_R C$ is acyclic, then we further set

$$\chi_{\mathfrak{A},E}(C, 0) := \chi_{\mathfrak{A},E}(C, h_{\mathrm{can}}),$$

with h_{can} the canonical morphism $\mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$ in $\mathcal{C}_{A_E}$ (from [17, §1.2.8]).

Example 4.1. Fix a bounded complex $C^\bullet$ of finitely generated projective $\mathfrak{A}$-modules, and set $C^{\mathrm{even}} := \bigoplus_{i\in\mathbb{Z}} C^{2i}$ and $C^{\mathrm{odd}} := \bigoplus_{i\in\mathbb{Z}} C^{2i+1}$. In this case, specifying a morphism $h : \mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$ in $\mathcal{C}_{A_E}$ is equivalent to specifying data as follows: for some finitely generated projective $\mathfrak{A}$-module P, one is given an isomorphism of A_E-modules $\theta : E \otimes_R (C^{\mathrm{even}} \oplus P) \cong E \otimes_R (C^{\mathrm{odd}} \oplus P)$ that is unique up to pre-composition with an automorphism of $E \otimes_R (C^{\mathrm{even}} \oplus P)$ whose image in $K_1(A_E)$ is specified (and so depends only on h). Then, in terms of the standard presentation of $K_0(\mathfrak{A}, A_E)$, the element $\chi_{\mathfrak{A},E}(C^\bullet, h)$ corresponds to the triple $(C^{\mathrm{even}} \oplus P, \theta, C^{\mathrm{odd}} \oplus P)$, with the defining relations of $K_0(\mathfrak{A}, A_E)$ ensuring that this element is indeed independent of both P and the specific choice of θ.

We next record some general properties of the elements $\chi_{\mathfrak{A}}(C, h)$ that will be used frequently in the sequel (often without explicit comment).

Firstly, if $C_1 \to C_2 \to C_3 \to C_1[1]$ is an exact triangle in $D^{\mathrm{perf}}(\mathfrak{A})$ for which the complex $F \otimes_R C_3$ is acyclic, then each morphism $h : \mathrm{Det}_{A_E}(E \otimes_R C_1) \to \mathrm{Det}_{A_E}(0)$ in $\mathcal{C}_{A_E}$ combines with the given triangle to induce a morphism $h' : \mathrm{Det}_{A_E}(E \otimes_R C_2) \to \mathrm{Det}_{A_E}(0)$ in $\mathcal{C}_{A_E}$. The same approach used to prove Lemma 1.3.4 of [17] then shows that

$$\chi_{\mathfrak{A},E}(C_2, h') = \chi_{\mathfrak{A},E}(C_1, h) + \chi_{\mathfrak{A},E}(C_3, 0).$$

Secondly, if h and h' are any two morphisms $\mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$ in $\mathcal{C}_{A_E}$, then the (obvious analogue of the) defining relation (3) in [17, Definition 1.3.2] (with C' taken to be 0) implies that

$$\chi_{\mathfrak{A},E}(C, h') = \chi_{\mathfrak{A},E}(C, h) + \partial_{\mathfrak{A},E}(h' \circ h^{-1}).$$

Here the last term denotes the image under $\partial_{\mathfrak{A},E}$ of the unique element of $K_1(A_E)$ that is determined by the morphism $h' \circ h^{-1} : \mathrm{Det}_{A_E}(0) \to \mathrm{Det}_{A_E}(0)$ in $\mathcal{C}_{A_E}$.

We next assume $\mathfrak{A} = R[G]$ for a finite group G, and write $\iota^{\#}_{R[G]}$ for the involutions on each of the groups $K_1(R[G])$, $K_1(F[G])$ and $K_1(R[G], F[G])$ that are induced by the R-linear anti-involution on

$R[G]$ that inverts elements of G. Then, if M is any finite $R[G]$-module that is c-t, its Pontryagin dual M^* (endowed with contragredient G-action) is also c-t, and

$$\chi_{R[G],F[G]}(M^*[0],0) = \iota^{\#}_{R[G]}\big(\chi_{R[G],F[G]}(M[0],0)\big). \tag{9}$$

(By localisation, the verification of this equality reduces to the case that R is a discrete valuation ring. In the latter case it then follows by explicit computation from the fact that a finite c-t $R[G]$-module has a free resolution of length one.)

Remark 4.2. We often regard E as clear from context and so write $\chi_{\mathfrak{A}}(-,-)$ in place of $\chi_{\mathfrak{A},E}(-,-)$. If $\mathfrak{A} = \mathbb{Z}[G]$, we further abbreviate $\chi_{\mathbb{Z}[G],E}(-,-)$ to $\chi_G(-,-)$, and the maps $\partial_{\mathbb{Z}[G],E}(-)$ and $\partial'_{\mathbb{Z}[G],E}$ to $\partial_G(-)$ and $\partial'_G(-)$ (again regarding E as clear from context).

4.2. *The refined Birch and Swinnerton-Dyer conjecture.*

4.2.1. In the sequel we write

$$h^{\mathrm{NT}}_{A,L} : A(L) \times A^t(L) \to \mathbb{R}$$

for the classical Néron–Tate height-pairing for A over L.

This pairing is nondegenerate and hence, assuming $\text{Ш}(A/L)$ to be finite, combines with the properties of the Selmer complex $\mathrm{SC}_{V_L}(A, L/K)$ established in Proposition 3.7(ii) to induce a canonical isomorphism of $\mathbb{R}[G]$-modules

$$h^{\mathrm{NT,det}}_{A,L} : \mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R}\otimes_{\mathbb{Z}} \mathrm{SC}_{V_L}(A, L/K)) \cong \mathrm{Det}_{\mathbb{R}[G]}(0). \tag{10}$$

In particular, since $\mathrm{SC}_{V_L}(A, L/K)$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$, we obtain an element of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ by setting

$$\chi^{\mathrm{BSD}}_G(A, V_L) := \chi_G(\mathrm{SC}_{V_L}(A, L/K), h^{\mathrm{NT,det}}_{A,L}).$$

Next, since the complex $R\Gamma(X, \mathcal{L}_L)^*$ considered in Lemma 3.9 belongs to $D^{\mathrm{perf}}(\mathbb{F}_p[G])$, it defines an object of $D^{\mathrm{perf}}(\mathbb{Z}[G])$ for which $\mathbb{Q}\otimes_{\mathbb{Z}} R\Gamma(X, \mathcal{L}_L)^*$ is acyclic. The associated element

$$\chi^{\mathrm{coh}}_G(A, V_L) := \chi_G(R\Gamma(X, \mathcal{L}_L)^*, 0)$$

therefore belongs to the image of the natural homomorphism

$$K_0(\mathbb{F}_p[G]) \to K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \subset K_0(\mathbb{Z}[G], \mathbb{R}[G]). \tag{11}$$

Finally, for each prime ℓ, we shall use an explicit computation of Bockstein homomorphisms that arise naturally in arithmetic cohomology to define a canonical, and computable, integer $a_\ell = a_{A,L,\ell}$ in $\{0, 1\}$. We thereby obtain a canonical element of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of order dividing two by setting

$$\chi^{\mathrm{sgn}}_G(A) := \sum_{\ell} \partial_{G,\mathbb{Q}}(\langle \mathbb{Q}\cdot A^t(L), (-1)^{a_\ell}\rangle)_\ell,$$

where ℓ runs over all prime divisors of $|G|$. (Given the relatively minor role that this "sign-term" plays in our conjecture, and the involved nature of the relevant Bockstein homomorphisms, we prefer to delay giving explicit details regarding the integers a_ℓ until the respective computations are made in Proposition 8.1(i) for $\ell = p$ and in (58) for $\ell \neq p$.)

4.2.2. We can now state our refined version of the Birch and Swinnerton-Dyer conjecture for A over L.

For each character ψ in $\mathrm{Ir}(G)$, we fix an associated complex representation V_ψ and write e_ψ for the primitive idempotent $\psi(1)|G|^{-1}\sum_{g\in G}\psi(g^{-1})g$ of $\zeta(\mathbb{C}[G])$. We then set

$$r_{\mathrm{alg}}(\psi) := \psi(1)^{-1}\cdot \dim_{\mathbb{C}}(e_\psi(\mathbb{C}\otimes_{\mathbb{Z}} A^t(L))) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, \mathbb{C}\otimes_{\mathbb{Z}} A^t(L))),$$

and write

$$r_{\mathrm{an}}(\psi) := \mathrm{ord}_{s=1} L_U(A, \psi, s)$$

for the order of vanishing at $s = 1$ of the series $L_U(A, \psi, s)$. We also use the "leading term" element $L_U^*(A_{L/K}, 1)$ of $K_1(\mathbb{R}[G])$ that is defined in Theorem 2.1.

Conjecture 4.3. (i) *For each character ψ in $\mathrm{Ir}(G)$ one has $r_{\mathrm{an}}(\psi) = r_{\mathrm{alg}}(\psi)$.*

(ii) *The group $Ш(A/L)$ is finite.*

(iii) *Let U be a dense open subset of X comprising points at which both L/K is unramified and A/K has good reduction. Then, for every family of groups $V_L = V_{U_L}$ chosen as in Section 3.4, there is an equality*

$$\partial_G(L_U^*(A_{L/K}, 1)) = \chi_G^{\mathrm{BSD}}(A, V_L) - \chi_G^{\mathrm{coh}}(A, V_L) + \chi_G^{\mathrm{sgn}}(A)$$

in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$.

Remark 4.4. If $L = K$, then $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ identifies with the multiplicative group $\mathbb{R}^\times/\{\pm 1\}$ and in Proposition 5.2 below we shall show that this case of Conjecture 4.3 recovers the classical Birch and Swinnerton-Dyer conjecture for A. In Section 5.2 we also show that the validity of the equality in Conjecture 4.3(iii) is independent of the choices of open set U and family of subgroups V_L.

Remark 4.5. Since $\mathbb{C}\otimes_{\mathbb{Z}} A^t(L)$ is the scalar extension of the finitely generated $\mathbb{Q}[G]$-module $\mathbb{Q}\otimes_{\mathbb{Z}} A^t(L)$ one has $r_{\mathrm{alg}}(\psi) = r_{\mathrm{alg}}(\omega\circ\psi)$ for all $\psi\in\mathrm{Ir}(G)$ and all automorphisms ω of $\mathbb{C}$. Conjecture 4.3(i) therefore implies that $r_{\mathrm{an}}(\psi) = r_{\mathrm{an}}(\omega\circ\psi)$ for all such ψ and ω. The validity of these equalities can be derived directly from the equalities (1) and (2) that played the key role in the proof of Proposition 2.2.

Remark 4.6. Theorem 2.1 allows us to formulate Conjecture 4.3 directly in terms of the connecting homomorphism ∂_G. However, without using Theorem 2.1, one could still formulate an analogue of Conjecture 4.3 in terms of the image of the element $(L_U^*(A, \chi, 1))_{\chi\in\mathrm{Ir}(G)}$ of $\zeta(\mathbb{R}[G])^\times$ under the "extended boundary" homomorphism $\zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R}[G])$ constructed by Flach and the first author in [5, §4.2, Lemma 9]. This observation provides the link between the formulation of Conjecture 4.3(iii) in terms of relative algebraic K-theory and the formalism of "equivariant Tamagawa number conjectures" that is discussed in loc. cit. and later refined by Fukaya and Kato in [17].

4.2.3. Conjecture 4.3 entails a variety of explicit consequences concerning the structure of Selmer complexes and relations between leading terms of Hasse–Weil–Artin L-series. To help provide context, we now state two concrete results in this direction (though, for convenience, the proof of these results is deferred to Section 9.4).

We fix a prime p and assume, for simplicity, that G is of p-power order. We write $M_{(p)}$ for the p-localisation of a (complex of) abelian groups M.

The first result concerns the Galois structure of the complex $\mathrm{SC}_{V_L} := \mathrm{SC}_{V_L}(A, L/K)$.

Proposition 4.7. *If G is a group of p-power order, then Conjecture 4.3 implies the following restrictions on the complex* SC_{V_L}.

(i) SC_{V_L} *is isomorphic in $D^{\mathrm{perf}}(\mathbb{Z}[G])$ to a bounded complex of finitely generated free G-modules.*

(ii) *If $A(K)[p]$ and $A^t(K)[p]$ both vanish, then $\mathrm{SC}_{V_L,(p)}$ is isomorphic in $D^{\mathrm{perf}}(\mathbb{Z}_{(p)}[G])$ to a complex $\mathbb{Z}_{(p)}[G]^t \xrightarrow{\phi} \mathbb{Z}_{(p)}[G]^t$, where the first term is placed in degree one.*

The second result we record describes families of algebraic relations between suitable normalisations of the leading terms $L_U^*(A, \psi, 1)$ for varying characters ψ.

To state this result, we assume the hypotheses of Proposition 4.7(ii) and fix a representative of $\mathrm{SC}_{V_L,(p)}$ of the specified form. We then consider the composite isomorphism

$$\iota^{\mathrm{NT}}_{A,L} : \mathbb{R}[G]^t \cong (\mathbb{R}\cdot\ker(\phi)) \oplus (\mathbb{R}\cdot\mathrm{im}(\phi)) \xrightarrow{h^{\mathrm{NT}}_{A,L,*}\oplus\mathrm{id}} (\mathbb{R}\cdot\mathrm{cok}(\phi)) \oplus (\mathbb{R}\cdot\mathrm{im}(\phi)) \cong \mathbb{R}[G]^t$$

of $\mathbb{R}[G]$-modules. Here the first and third maps are induced by a choice of $\mathbb{R}[G]$-equivariant sections to the surjective maps from $\mathbb{R}[G]^t$ to $\mathbb{R}\cdot\mathrm{im}(\phi)$ and $\mathbb{R}\cdot\mathrm{cok}(\phi)$ that are respectively induced by ϕ and by the tautological projection. In addition, $h^{\mathrm{NT}}_{A,L,*}$ denotes the composite

$$\mathbb{R}\cdot\ker(\phi) \cong \mathbb{R}\cdot A^t(L) \cong \mathrm{Hom}_{\mathbb{R}}\big(\mathbb{R}\cdot A(L), \mathbb{R}\big) \cong \mathbb{R}\cdot\mathrm{cok}(\phi),$$

in which the second isomorphism is induced by the nondegenerate pairing $h^{\mathrm{NT}}_{A,L}$ and the first and third by Proposition 3.7(ii)(a) and the fixed identifications of $\ker(\phi)$ and $\mathrm{cok}(\phi)$ with $H^0(\mathrm{SC}_{V_L})_{(p)}$ and $H^1(\mathrm{SC}_{V_L})_{(p)}$.

We write $\chi(\mathcal{L})$ for the integer obtained as the Euler characteristic in $K_0(\mathbb{F}_p) \cong \mathbb{Z}$ of the complex $R\Gamma(X, \mathcal{L})^*$ of $\mathbb{F}_p$-modules.

For each character ψ in $\mathrm{Ir}(G)$, we then normalise the leading term of the associated Hasse–Weil–Artin L-series by setting

$$\mathscr{L}(A, \psi) := \frac{p^{\psi(1)\chi(\mathcal{L})}\cdot L_U^*(A, \psi, 1)}{(-1)^{r_{\mathrm{alg}}(\psi)a_p}\cdot\det(\iota^{\mathrm{NT}}_{A,L,\psi})}, \tag{12}$$

where $\iota^{\mathrm{NT}}_{A,L,\psi}$ is the automorphism of $\mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, \mathbb{C}[G]^t)$ induced by $\iota^{\mathrm{NT}}_{A,L}$. We also write $\mathbb{Q}(\psi)$ for the field generated over $\mathbb{Q}$ by the set $\{\psi(g) : g \in G\}$.

Proposition 4.8. *Assume the hypotheses of Proposition 4.7*(ii) *and the validity of Conjecture 4.3*(i) *and* (ii). *Then the following claims are valid.*

(i) *For all* ψ *in* $\mathrm{Ir}(G)$ *and* α *in* $\mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q})$ *one has*

$$\mathscr{L}(A,\psi)\in\mathbb{Q}(\psi) \quad \textit{and} \quad \alpha(\mathscr{L}(A,\psi))=\mathscr{L}(A,\alpha\circ\psi).$$

In the rest of the result we also assume the validity of the equality in Conjecture 4.3(iii)*.*

(ii) *For every abelian subquotient* $Q=H/J$ *of* G*, there is a containment*

$$\mathscr{L}(A_{L^H},\mathbf{1}_Q)\in\mathbb{Z}_{(p)}^{\times}$$

and, for each γ *in* Q*, a congruence*

$$\sum_{\psi\in\mathrm{Ir}(Q)}\psi(\gamma)^{-1}\mathscr{L}(A_{L^H},\psi)\equiv 0 \pmod{|Q|\cdot\mathbb{Z}_{(p)}}.$$

(iii) *If, for each subgroup* H *of* G*,* ψ_H *is an irreducible character of the abelianisation* H^{ab} *of* H *and* m_H *an integer such that the virtual character* $\sum_{H\le G}m_H\cdot\mathrm{ind}_H^G(\mathrm{inf}_{H^{\mathrm{ab}}}^H(\psi_H))$ *vanishes, then one has*

$$\prod_{H\le G}\mathscr{L}(A_{L^H},\psi_H)^{m_H}=1.$$

Remark 4.9. By developing methods introduced by the second author in [20], the first two authors give an explicit description of the Whitehead group $K_1(\mathbb{Z}_p[G])$ in [6, Theorem 2.1]. In the setting of Proposition 4.8, this result has the following explicit consequence. For every cyclic subgroup C of G, the properties in Proposition 4.8(ii) combine to imply that

$$\mathscr{L}(A,C):=|C|^{-1}\sum_{c\in C}\sum_{\psi\in\mathrm{Ir}(C)}\psi(c)^{-1}\mathscr{L}(A_{L^C},\psi)\cdot c$$

belongs to $\mathbb{Z}_{(p)}[C]^{\times}\subset\mathbb{Z}_p[C]^{\times}\cong K_1(\mathbb{Z}_p[C])$. Fix an embedding $j:\mathbb{R}\to\mathbb{C}_p$ and write j_* for the induced embedding $K_0(\mathbb{Z}[G],\mathbb{R}[G])\to K_0(\mathbb{Z}_p[G],\mathbb{C}_p[G])$ of relative K_0-groups. Then [6, Theorem 2.1] implies that the validity of the image under j_* of the equality in Conjecture 4.3(iii) is equivalent (under the hypotheses of Proposition 4.7(ii)) to the validity of the family of equalities in Proposition 4.8(iii) together with a single explicit congruence relation between the images of the individual elements $\mathscr{L}(A,C)$ under the respective induction maps $K_1(\mathbb{Z}_p[C])\to K_1(\mathbb{Z}_p[G])$.

4.3. ***The main results.*** In order to state our main result we must define the finite subgroup $\mathcal{T}_{A,L/K}$ of $K_0(\mathbb{Z}[G],\mathbb{R}[G])$ that was discussed in the introduction.

If Ξ is a quotient of a subgroup Δ of a finite group Γ, then we consider the composite homomorphism of abelian groups

$$\pi_\Xi^\Gamma: K_0(\mathbb{Z}_p[\Gamma],\mathbb{Q}_p[\Gamma])\to K_0(\mathbb{Z}_p[\Delta],\mathbb{Q}_p[\Delta])\to K_0(\mathbb{Z}_p[\Xi],\mathbb{Q}_p[\Xi]),$$

where the first map is restriction of scalars and the second is the natural coinflation homomorphism.

By the semistable reduction theorem, the set $\Sigma=\Sigma_{A,K}$ of finite Galois extensions of K over which A is semistable is nonempty. For each field K' in Σ we write L' for the composite of L and K', set $G':=\mathrm{Gal}(L'/K)$ and $H':=\mathrm{Gal}(L'/K')$, write P' for the normal subgroup of H' that is generated by the Sylow p-subgroups of the inertia groups in H' of each place in K' and set $\pi_{K'}:=\pi^{G'}_{H'/P'}$. We then define

$$\mathcal{T}_{A,L/K}:=K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])_{\mathrm{tor}}\cap\Big(\bigcap_{K'\in\Sigma_1}\ker(\pi_{K'})\Big)\cap\Big(\bigcap_{K'\in\Sigma_2}\pi_G^{G'}(\ker(\pi_{K'}))\Big), \tag{13}$$

where Σ_1 denotes the possibly empty subset of Σ comprising fields K' that are contained in L (so that $G' = G$) and Σ_2 the possibly empty subset of Σ comprising fields K' that are not contained in L but are such that $\text{Ш}(A/L')$ is finite.

We recall that $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])_{\text{tor}}$ is finite and, in all cases, we use the natural embeddings

$$K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])_{\text{tor}} \subset K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \subset K_0(\mathbb{Z}[G], \mathbb{R}[G])$$

to regard $\mathcal{T}_{A,L/K}$ as a subgroup of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$.

We can now state the main evidence that we shall offer in support of Conjecture 4.3.

Theorem 4.10. *If the ℓ-primary component of $\text{Ш}(A/L)$ is finite for some prime ℓ, then the following claims are also valid.*

(i) *Claims* (i) *and* (ii) *of Conjecture 4.3 are valid.*

(ii) *The equality in Conjecture 4.3*(iii) *is valid modulo the finite subgroup $\mathcal{T}_{A,L/K}$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$.*

Remark 4.11. It is proved by Kato and Trihan in [22] that $\text{Ш}(A/L)$ is finite if and only if at least one of its ℓ-primary components is finite. Thus, under the hypotheses of Theorem 4.10, we can (and do) assume, without further comment, that $\text{Ш}(A/L)$ is finite (and hence that Conjecture 4.3(ii) is valid).

Remark 4.12. The main result that we prove here is, in principle, stronger than Theorem 4.10 but is more technical to state (for more details see Remark 9.4 below). One can also provide further evidence in support of Conjecture 4.3 in the setting of generically ordinary abelian varieties, and we hope to discuss this elsewhere.

Remark 4.13. Assume G is a p-group, that the groups $A(K)[p]$ and $A^t(K)[p]$ both vanish and that some ℓ-primary component of $\text{Ш}(A/L)$ is finite (where ℓ can be different from p). Then Theorem 4.10 combines with Proposition 4.8 to imply the unconditional validity of the relations in Proposition 4.8(i).

In special cases it is possible to describe $\mathcal{T}_{A,L/K}$ explicitly and hence to make Theorem 4.10(ii) much more concrete.

For example, if the sets Σ_1 and Σ_2 that occur in (13) are both empty, then $\mathcal{T}_{A,L/K}$ is equal to $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])_{\text{tor}}$. On the other hand, if A is semistable over K and L/K is tamely ramified, then the field $K' = K$ belongs to Σ_1 and is such that $G = G' = H'$ and P' is trivial and so $\mathcal{T}_{A,L/K}$ vanishes. Hence, in the latter case, Theorem 4.10 has the following more explicit consequence.

Corollary 4.14. *Assume that A is semistable, that L/K is tamely ramified and that some ℓ-primary component of $\text{Ш}(A/L)$ is finite. Then Conjecture 4.3 is unconditionally valid.*

As far as we are aware, this result gives the first verification, modulo only the assumed finiteness of Tate–Shafarevich groups, of a refined version of the Birch–Swinnerton-Dyer conjecture in the context of ramified extensions.

5. Preliminary results

In this section we first prove a purely algebraic result that is important for several subsequent arguments.

We then verify that the statement of Conjecture 4.3 is consistent in certain key respects (as promised in Remark 4.4).

Finally we use a result of Schneider to give a reinterpretation of the conjecture that plays an essential role in the proof of Theorem 4.10.

5.1. ***A result in K-theory.*** The following purely algebraic observation will underpin the proof of several subsequent results.

Proposition 5.1. *Let R be a Dedekind domain with field of fractions F and $\mathfrak{A}$ an R-order in a finite dimensional semisimple F-algebra A.*

We suppose to be given exact triangles in $D^{\mathrm{perf}}(\mathfrak{A})$ of the form

$$C_\theta \to C_1 \xrightarrow{\theta} C_2 \to C_\theta[1] \quad \text{and} \quad C_\phi \to C_1 \xrightarrow{\phi} C_2 \to C_\phi[1] \tag{14}$$

that satisfy all of the following conditions.

(a) *In each degree i there are natural identifications $F \otimes_R H^i(C_1) = F \otimes_R H^i(C_2)$, with respect to which*

(b) *the composite tautological homomorphism of A-modules*

$$F \otimes_R \ker(H^i(\theta)) \subseteq F \otimes_R H^i(C_1) = F \otimes_R H^i(C_2) \to F \otimes_R \operatorname{cok}(H^i(\theta))$$

is bijective, *and*

(c) *the map $H^i(\phi)$ induces the identity homomorphism on $F \otimes_R H^i(C_1) = F \otimes_R H^i(C_2)$.*

Then the following claims are valid.

(i) *The bijectivity of the maps in* @(b) *combines with the first triangle in* (14) *to induce a canonical morphism*

$$\tau_\theta : \mathrm{Det}_A(F \otimes_R C_\theta) \cong \mathrm{Det}_A(0)$$

of (*nonabelian*) *determinants.*

(ii) *In each degree i the homomorphism $H^i(\theta)$ induces an automorphism $H^i(\theta)^\diamond$ of any A-equivariant complement to $F \otimes_R \ker(H^i(\theta))$ in $F \otimes_R H^i(C_1)$ in such a way that* $\mathrm{Nrd}_A(H^i(\theta)^\diamond)$ *is independent of the choice of complement.*

(iii) *The complex $F \otimes_R C_\phi$ is acyclic.*

(iv) *In $K_0(\mathfrak{A}, A)$ one has*

$$\chi_{\mathfrak{A}}(C_\theta, \tau_\theta) - \chi_{\mathfrak{A}}(C_\phi, 0) = \partial_{\mathfrak{A},F}\Big(\prod_{i\in\mathbb{Z}} (H^i(\theta)_F^\diamond)^{(-1)^i}\Big),$$

where we identify each automorphism $H^i(\theta)_F^\diamond$ with the associated element of $K_1(A)$.

Proof. If M denotes either an R-module or a complex of R-modules, then we abbreviate $F \otimes_R M$ to M_F.

To construct a morphism τ_θ as in claim (i) we note first that the long exact cohomology sequence of the left-hand exact triangle in (14) gives in each degree i a short exact sequence of $\mathfrak{A}$-modules

$$0 \to \mathrm{cok}(H^{i-1}(\theta)) \to H^i(C_\theta) \to \ker(H^i(\theta)) \to 0.$$

Then, upon tensoring these exact sequences with F (over R), applying the determinant functor Det_A and then taking account of the isomorphisms given in (b) one obtains isomorphisms of (nonabelian) determinants

$$\begin{aligned}\mathrm{Det}_A(H^i(C_\theta)_F) &\cong \mathrm{Det}_A(\mathrm{cok}(H^{i-1}(\theta))_F)\cdot \mathrm{Det}_A(\ker(H^i(\theta))_F)\\ &\cong \mathrm{Det}_A(\mathrm{cok}(H^{i-1}(\theta))_F)\cdot \mathrm{Det}_A(\mathrm{cok}(H^i(\theta))_F).\end{aligned} \tag{15}$$

We then define the morphism τ_θ in claim (i) to be the composite

$$\begin{aligned}\mathrm{Det}_A((C_\theta)_F) &\cong \prod_{i\in\mathbb{Z}} \mathrm{Det}_A(H^i(C_\theta)_F)^{(-1)^i}\\ &\cong \prod_{i\in\mathbb{Z}} \big(\mathrm{Det}_A(\mathrm{cok}(H^{i-1}(\theta))_F)\cdot \mathrm{Det}_A(\mathrm{cok}(H^i(\theta))_F)\big)^{(-1)^i}\\ &\cong \prod_{i\in\mathbb{Z}} \big(\mathrm{Det}_A(\mathrm{cok}(H^{i}(\theta))_F)^{-1}\cdot \mathrm{Det}_A(\mathrm{cok}(H^i(\theta))_F)\big)^{(-1)^i}\\ &\cong \prod_{i\in\mathbb{Z}} \mathrm{Det}_A(0)^{(-1)^i}\\ &= \mathrm{Det}_A(0).\end{aligned}$$

Here the first map is the canonical "passage to cohomology" map, the second is induced by the maps (15) in each degree i, the third by the obvious rearrangement of terms and the fourth from the canonical morphisms

$$\mathrm{Det}_A(\mathrm{cok}(H^i(\theta))_F)^{-1}\cdot \mathrm{Det}_A(\mathrm{cok}(H^i(\theta))_F) \cong \mathrm{Det}_A(0).$$

Claim (ii) is a straightforward consequence of the condition (b) and claim (iii) follows directly upon combining the long exact cohomology sequence of the second triangle in (14) with the condition (c).

Finally, to prove claim (iv) we fix bounded complexes of finitely generated projective $\mathfrak{A}$-modules P_1 and P_2 that are respectively isomorphic in $D(\mathfrak{A})$ to C_1 and C_2. Then the morphisms θ and ϕ are represented by morphisms of complexes of $\mathfrak{A}$-modules of the form $\theta' : P_1 \to P_2$ and $\phi' : P_1 \to P_2$.

The key to our argument is then to consider the exact triangle

$$C_\theta \oplus \mathrm{Cone}(\phi') \xrightarrow{(\kappa,\mathrm{id})} P_1 \oplus \mathrm{Cone}(\phi') \xrightarrow{(\kappa',0)} \mathrm{Cyl}(\theta') \to (C_\theta \oplus \mathrm{Cone}(\phi'))[1] \tag{16}$$

in $D(\mathfrak{A})$ where κ is the morphism $C_\theta \to P_1$ induced by the first triangle in (14) and κ' the morphism $P_1 \to \mathrm{Cyl}(\theta')$ induced by θ' and the natural quasi-isomorphism $\mathrm{Cyl}(\theta') \cong P_2$.

This triangle satisfies the analogues of conditions (a) and (b) (with C_1, C_2 and θ replaced by $P_1 \oplus \mathrm{Cone}(\phi')$, $\mathrm{Cyl}(\theta')$ and $(\kappa', 0)$) and in each degree i one has $(P_1 \oplus \mathrm{Cone}(\phi'))^i = \mathrm{Cyl}(\theta')^i$.

Further, the acyclicity of $F \otimes_R C_\phi$ implies that

$$\chi_{\mathfrak{A}}(C_\theta \oplus \mathrm{Cone}(\phi'), \tau_\theta) = \chi_{\mathfrak{A}}(C_\theta, \tau_\theta) + \chi_{\mathfrak{A}}(C_\phi[1], 0) = \chi_{\mathfrak{A}}(C_\theta, \tau_\theta) - \chi_{\mathfrak{A}}(C_\phi, 0),$$

where the first equality is true because $\mathrm{Cone}(\phi')$ is isomorphic to $C_\phi[1]$.

In particular, after replacing the first triangle in (14) by (16), we are reduced to proving that if C_1 and C_2 are represented by bounded complexes of finitely generated projective $\mathfrak{A}$-modules P_1 and P_2 with $P_1^i = P_2^i$ in each degree i, then the conditions (a), (b) and (c) combine to imply an equality

$$\chi_{\mathfrak{A}}(C_\theta, \tau_\theta) = \delta_{\mathfrak{A}}\Big(\prod_{i\in\mathbb{Z}} \mathrm{Nrd}_A(H^i(\theta)_F^{\diamond})^{(-1)^i}\Big), \tag{17}$$

where $\delta_{\mathfrak{A}}$ denotes the composite $\partial_{\mathfrak{A},F} \circ (\mathrm{Nrd}_A)^{-1} : \mathrm{im}(\mathrm{Nrd}_A) \to K_0(\mathfrak{A}, A)$.

To do this we note first that, under these conditions, an easy downward induction on i (using hypothesis (c)) implies that in each degree i the F-spaces spanned by the groups of boundaries $B^i(P_1)$ and $B^i(P_2)$ have the same dimension.

If necessary, we can then also change θ by a homotopy (without changing conditions (b)) in order to ensure that, in each degree i, the restriction of θ^{i+1} is injective on $B^i(P_1)$ and hence induces an isomorphism $F \otimes_R B^i(P_1) \cong F \otimes_R B^i(P_2)$ (for details of such an argument see, for example, the proof of [9, Lemma 7.10]).

Having made these constructions, one can then simply mimic the argument of [4, Proposition 3.1] in order to prove the required equality (17) by using induction on the number of nonzero terms in P_1. □

5.2. ***Consistency checks.***

Proposition 5.2. *If $L = K$, then Conjecture 4.3 recovers the classical Birch and Swinnerton-Dyer conjecture for A.*

Proof. We assume $Ш(A/K)$ is finite and abbreviate $\mathrm{SC}_{V_K}(A, K/K)$ to SC_{V_K}.

Now, if $L = K$, then G is the trivial group id and $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ identifies with the multiplicative group $\mathbb{R}^\times/\{\pm 1\}$. In addition, upon unwinding the definition of Euler characteristic one finds that, with respect to the latter identification, there is an equality

$$\chi_{\mathrm{id}}^{\mathrm{BSD}}(A, V_K) \equiv \mathrm{disc}(h_{A,K}^{\mathrm{NT}}) \cdot \prod_{i\in\mathbb{Z}} \#(H^i(\mathrm{SC}_{V_K})_{\mathrm{tor}})^{(-1)^{i+1}} \pmod{\pm 1} \tag{18}$$

where $\mathrm{disc}(h_{A,K}^{\mathrm{NT}})$ denotes the discriminant of the pairing $h_{A,K}^{\mathrm{NT}}$.

To compute the above product we write θ for the natural map $A(K)_{\mathrm{tor}} \to \bigoplus_{v\notin U} A(K_v)/V_v$. Then, from Propositions 3.2 and 3.7, one finds that there are equalities $H^0(\mathrm{SC}_{V_K}) = A^t(K)$ and $H^2(\mathrm{SC}_{V_K}) = \ker(\theta)^*$ and a short exact sequence of the form

$$0 \to X_{\mathbb{Z}}(A/K) \to H^1(\mathrm{SC}_{V_K}) \to \mathrm{cok}(\theta)^* \to 0.$$

Upon combining these observations with the natural exact sequences

$$0 \to \ker(\theta) \to A(K)_{\mathrm{tor}} \to \bigoplus_{v \notin U} A(K_v)/V_v \to \mathrm{cok}(\theta) \to 0$$

and

$$0 \to Ш(A/K)^* \to X_{\mathbb{Z}}(A/K) \to \mathrm{Hom}_{\mathbb{Z}}(A(K), \mathbb{Z}) \to 0$$

one computes that

$$\mathrm{disc}(h^{\mathrm{NT}}_{A,K}) \cdot \prod_{i \in \mathbb{Z}} \#(H^i(\mathrm{SC}_{V_K})_{\mathrm{tor}})^{(-1)^{i+1}} = \frac{\#Ш(A/K)\,\mathrm{disc}(h^{\mathrm{NT}}_{A,K})}{\#A(K)_{\mathrm{tor}}\#A^t(K)_{\mathrm{tor}}} \prod_{v \notin U} [A(K_v) : V_v]. \tag{19}$$

On the other hand, from [22, 3.7.3], one finds that

$$\chi^{\mathrm{coh}}_{\mathrm{id}}(V_K, K/K) \equiv \frac{\#H^1(X, \mathcal{L})}{\#H^0(X, \mathcal{L})} \equiv \frac{\prod_{v \notin U} [A(K_v) : V_v]}{\mathrm{vol}\big(\prod_{v \notin U} A(K_v)\big)} \pmod{\pm 1}, \tag{20}$$

where the "volume term" here is as defined in [22, §1.7].

Thus, since $\chi^{\mathrm{sgn}}_{\mathrm{id}}(A)$ is clearly trivial, the expressions (18), (19) and (20) combine to show the equality in Conjecture 4.3(iii) is equivalent to an equality

$$L^*_U(A, 1) \equiv \pm \frac{\#Ш(A/K)\,\mathrm{disc}(h^{\mathrm{NT}}_{A,K})}{\#A(K)_{\mathrm{tor}}\#A^t(K)_{\mathrm{tor}}} \mathrm{vol}\big(\prod_{v \notin U} A(K_v)\big).$$

Since $L^*_U(A, 1)$ is known to be a strictly positive real number (by Proposition 2.2(ii)), this equality is precisely the form of the Birch and Swinnerton-Dyer conjecture that is discussed in [22, §1.8]. □

Proposition 5.3. *The validity of Conjecture 4.3*(iii) *is independent of the choice of a family of subgroups V_L.*

Proof. It is clearly enough to show that the difference $\chi^{\mathrm{BSD}}_G(A, V_L) - \chi^{\mathrm{coh}}_G(A, V_L)$ is independent of the choice of V_L.

In addition, it suffices to consider replacing V_L by a family of subgroups $V'_L = (V'_w)_{w \notin U_L}$ that satisfies $V'_w \subseteq V_w$ for all $w \notin U_L$.

In this case, the definition of the complexes $\mathrm{SC}_{V'_L}(A, L/K)$ and $\mathrm{SC}_{V_L}(A, L/K)$ via the (dual of the) mapping fibre of the respective morphisms (4) leads naturally to an exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}[G])$ of the form

$$\mathrm{SC}_{V_L}(A, L/K) \to \mathrm{SC}_{V'_L}(A, L/K) \to Q_1^*[-1] \to ,$$

with $Q_1 := \bigoplus_{w \notin U_L} (V_w/V'_w)$, and hence to an equality in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$:

$$\chi^{\mathrm{BSD}}_G(A, V_L) - \chi^{\mathrm{BSD}}_G(A, V'_L) = \chi_G(Q_1^*[0], 0). \tag{21}$$

On the other hand, if $\mathcal{L}'_L$ and $\mathcal{L}_L$ are the coherent sheaves that correspond (as in Section 3.5) to the collections V'_L and V_L respectively, then there is a natural short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{L} \to Q_2 \to 0,$$

with $Q_2 := \bigoplus_{w \notin U_L} W_w / W'_w$. This sequence gives rise to an exact triangle in $D^{\mathrm{perf}}(\mathbb{F}_p[G])$ of the form

$$R\Gamma(X, \mathcal{L}_L)^* \to R\Gamma(X, \mathcal{L}'_L)^* \to Q_2^*[1] \to ,$$

and hence to an equality

$$\chi_G^{\mathrm{coh}}(A, V_L) - \chi_G^{\mathrm{coh}}(A, V'_L) = \chi_{\mathbb{Z}_p[G]}(Q_2^*[0], 0). \tag{22}$$

Now, given the explicit construction of the groups V_w and V'_w from W_w and W'_w, it is straightforward to show that, for both $i = 1$ and $i = 2$ there exists a (finite length) decreasing filtration $(Q_{i,j})_{j \geq 0}$ of the finite $\mathbb{Z}_p[G]$-module Q_i such that each module $Q_{i,j}$ is c-t for G and the graded modules $\mathrm{gr}(Q_i) := \bigoplus_{j \geq 0} (Q_{i,j}/Q_{i,j+1})$ are both c-t for G and mutually isomorphic. This fact in turn implies that

$$\begin{aligned}\chi_{\mathbb{Z}_p[G]}(Q_1^*[0], 0) &= \iota^{\#}_{\mathbb{Z}_p[G]}\big(\chi_{\mathbb{Z}_p[G]}(Q_1[0], 0)\big) = \iota^{\#}_{\mathbb{Z}_p[G]}\big(\chi_{\mathbb{Z}_p[G]}(\mathrm{gr}(Q_1)[0], 0)\big)\\ &= \iota^{\#}_{\mathbb{Z}_p[G]}\big(\chi_{\mathbb{Z}_p[G]}(\mathrm{gr}(Q_2)[0], 0)\big) = \iota^{\#}_{\mathbb{Z}_p[G]}\big(\chi_{\mathbb{Z}_p[G]}(Q_2[0], 0)\big) = \chi_{\mathbb{Z}_p[G]}(Q_2^*[0], 0),\end{aligned}$$

where the first and last equalities follow from the general result (9) and the second and fourth from a standard dévissage argument. These equalities then combine with (21) and (22) to imply the required result. □

Proposition 5.4. *The validity of Conjecture 4.3*(iii) *is independent of the choice of U.*

Proof. It suffices to fix v_0 in U and consider the effect of replacing U by the set $U' := U \setminus \{v_0\}$.

We fix a family $V_L = (V'_w)_{w \notin U'_L}$ of subgroups as in Lemma 3.4 and assume, following Remark 3.6, that for each place w above v_0 one has $V'_w = \mathcal{A}(\mathfrak{m}_w)$. We also write $V_L^{\dagger}$ for the associated family $(V'_w)_{w \notin U_L}$.

Then, setting

$$\mathcal{E}_{v_0} := L_U^*(A_{L/K}, 1) \cdot L_{U'}^*(A_{L/K}, 1)^{-1},$$

it is enough for us to prove that

$$\partial_{G,\mathbb{R}}(\mathcal{E}_{v_0}) = (\chi_G^{\mathrm{BSD}}(A, V_L^{\dagger}) - \chi_G^{\mathrm{BSD}}(A, V_L)) - (\chi_G^{\mathrm{coh}}(A, V_L^{\dagger}) - \chi_G^{\mathrm{coh}}(A, V_L)) \tag{23}$$

in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$. In addition, $\mathcal{E}_{v_0}$ belongs to the subgroup $K_1(\mathbb{Q}[G])$ of $K_1(\mathbb{R}[G])$ and we claim that $\mathrm{Nrd}_{\mathbb{Q}[G]}(\mathcal{E}_{v_0})$ is equal to the evaluation at $u = 1$ of the expression

$$\begin{aligned}&\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}\big(1 - u^{\deg(v_0)}\varphi_p^{\deg(v_0)} : \big(\mathbb{Q}_\ell[G] \otimes_{\mathbb{Z}_\ell} T_\ell(A)\big)^{\vee}\big)\\ &\qquad = \mathrm{Nrd}_{\mathbb{Q}_p[G]}\big(1 - u^{\deg(v_0)}\varphi^{\deg(v_0)} : \big(\mathbb{Q}_p[G] \otimes_{\mathbb{Q}_p} H^0_{\mathrm{crys}}(k(v_0)/\mathbb{Z}_p, D_{v_0})\big)^{\vee}\big).\end{aligned}$$

Here ℓ is any choice of prime different from p and φ_p is the geometric p-th power Frobenius map on $T_\ell(A)$, the endomorphism φ is such that $p\varphi$ is induced by the crystalline Frobenius on the fibre D_{v_0} at v_0 of the covariant Dieudonné crystal D and the above displayed equation follows from the result [23, Theorem 1] of Katz and Messing. To verify this claim about $\mathrm{Nrd}_{\mathbb{Q}[G]}(\mathcal{E}_{v_0})$ it is enough to fix an arbitrary

$\chi \in \mathrm{Ir}(G)$, with a corresponding realisation V_χ over $\mathbb{Q}_\ell^c$ with $\ell \neq p$, and then note that

$$\begin{aligned} e_\chi\big(\mathrm{Nrd}_{\mathbb{Q}[G]}(\mathcal{E}_{v_0})\big) &= \det\big(1-\varphi_p^{\deg(v_0)} : V_\chi \otimes_{\mathbb{Z}_\ell} T_\ell(A)\big) \\ &= \det\big(1-\varphi_p^{\deg(v_0)} : \mathrm{Hom}_{\mathbb{Q}_\ell^c[G]}(V_{\check{\chi}}, \mathbb{Q}_\ell^c[G] \otimes_{\mathbb{Z}_\ell} T_\ell(A))\big) \\ &= e_{\check{\chi}}\big(\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}\big(1-\varphi_p^{\deg(v_0)} : \mathbb{Q}_\ell[G] \otimes_{\mathbb{Z}_\ell} T_\ell(A)\big) \\ &= e_\chi\big(\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}\big(1-\varphi_p^{\deg(v_0)} : (\mathbb{Q}_\ell[G] \otimes_{\mathbb{Z}_\ell} T_\ell(A))^\vee\big)\big). \end{aligned}$$

Here the second equality follows from Remark 2.3 and all others are clear.

In addition, our assumption that $V'_w = \mathcal{A}(\mathfrak{m}_w)$ for places w above v_0 implies there are exact triangles in $D^{\mathrm{perf}}(\mathbb{Z}[G])$ of the form

$$\begin{gathered} \mathrm{SC}_{V_L^\dagger}(A, L/K) \to \mathrm{SC}_{V_L}(A, L/K) \to \bigoplus_{w \mid v_0} A(k(w))^*[-1] \to , \\ R\Gamma(X, \mathcal{L}_L^\dagger)^* \to R\Gamma(X, \mathcal{L}_L)^* \to \bigoplus_{w \mid v_0} \mathrm{Lie}(A)(k(w))^*[-1] \to . \end{gathered}$$

These triangles in turn imply that there are equalities in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$:

$$\begin{gathered} \chi_G^{\mathrm{BSD}}(A, V_L^\dagger) - \chi_G^{\mathrm{BSD}}(A, V_L) = -\chi_G\big(\bigoplus_{w \mid v_0} A(k(w))^*[-1], 0\big), \\ \chi_G^{\mathrm{coh}}(A, V_L^\dagger) - \chi_G^{\mathrm{coh}}(A, V_L) = -\chi_{\mathbb{Z}_p[G]}\big(\bigoplus_{w \mid v_0} \mathrm{Lie}(A)(k(w))^*[-1], 0\big). \end{gathered}$$

To prove (23) it is thus enough to show

$$\delta_{G,\ell}(\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}(1-\varphi_p^{\deg(v_0)} : \big(\mathbb{Q}_\ell[G] \otimes_{\mathbb{Z}_\ell} T_\ell(A)\big)^\vee)) = \chi_{\mathbb{Z}_\ell[G]}\big(\bigoplus_{w \mid v_0} A(k(w))\{\ell\}^*[-1], 0\big) \tag{24}$$

for every prime $\ell \neq p$, and also that

$$\begin{aligned} &\delta_{G,p}(\mathrm{Nrd}_{\mathbb{Q}_p[G]}(1-(p^{-1}\varphi)^{\deg(v_0)} : \big(\mathbb{Q}_p[G] \otimes_{\mathbb{Q}_p} H^0_{\mathrm{crys}}(k(v_0), \overline{D})\big)^\vee)) \\ &\qquad = \chi_{\mathbb{Z}_p[G]}\big(\bigoplus_{w \mid v_0} A(k(w))\{p\}^*[-1], 0\big) - \chi_{\mathbb{Z}_p[G]}\big(\bigoplus_{w \mid v_0} \mathrm{Lie}(A)(k(w))^*[-1], 0\big). \end{aligned} \tag{25}$$

Here, for each prime q, we write $\delta_{G,q}$ for the composite homomorphism

$$\partial_{\mathbb{Z}_q[G],\mathbb{Q}_q} \circ (\mathrm{Nrd}_{\mathbb{Q}_q[G]})^{-1} : \zeta(\mathbb{Q}_q[G])^\times \to K_0(\mathbb{Z}_q[G], \mathbb{Q}_q[G]).$$

Now, if $\ell \neq p$, then the complex

$$R\Gamma(k(v_0), T_\ell(A) \otimes \mathbb{Z}[G]) \cong \bigoplus_{w \mid v_0} R\Gamma(k(w), T_\ell(A))$$

is acyclic outside degree one and has cohomology $\bigoplus_{w \mid v_0} A(k(w))\{\ell\}$ in that degree. This gives rise to a short exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(A), \mathbb{Z}_\ell[G]) \xrightarrow{1-\varphi_p^{\deg(v_0)}} \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(A), \mathbb{Z}_\ell[G]) \to \bigoplus_{w \mid v_0} A(k(w))^*\{\ell\} \to 0,$$

which leads directly to the equality (24).

We next note that, by a result of Kato and Trihan [22, 5.14.6], for each w dividing v_0 the complex $A(k(w))\{p\}[-1]$ identifies with $R\Gamma(k(w), \mathcal{S}_{D_w})$, where $\mathcal{S}_{D_w}$ is the syntomic complex over $k(w)$ (obtained as a fibre of the syntomic complex over U), and hence that there is an exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w|v_0} A(k(w))\{p\}[-1] \to \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} R\Gamma_{\mathrm{crys}}(k(v_0)/\mathbb{Z}_p, D^0_{v_0}) \xrightarrow{1-\varphi^{\deg(v_0)}} \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} R\Gamma_{\mathrm{crys}}(k(v_0)/\mathbb{Z}_p, D_{v_0}) \to .$$

There is also a natural exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$

$$\bigoplus_{w|v_0} \mathrm{Lie}(A)(k(w))[-1] \to \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} R\Gamma_{\mathrm{crys}}(k(v_0)/\mathbb{Z}_p, D^0_{v_0}) \xrightarrow{1} \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} R\Gamma_{\mathrm{crys}}(k(v_0)/\mathbb{Z}_p, D_{v_0}) \to .$$

The required equality (25) now follows directly upon applying Proposition 5.1 with $R = \mathbb{Z}_p[G]$ and the triangles in (14) taken to be the images of the above two triangles under the exact linear duality functor $R\,\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$ on $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$. (These triangles are easily seen to satisfy the hypotheses of Proposition 5.1 since the modules $A(k(w))\{p\}$ and $\mathrm{Lie}(A)(k(w))$ are both finite.) □

Remark 5.5. The results of Propositions 5.3 and 5.4 will play a key role in later arguments. In Proposition 9.2 below we will also establish a further consistency property of Conjecture 4.3 with respect to changes of field extension L/K.

5.3. ***A reformulation.*** We next establish a useful reformulation of the equality in Conjecture 4.3(iii).

In [32, p. 509] Schneider shows that the pairing $h^{\mathrm{NT}}_{A,L}$ can be factored in the form

$$h^{\mathrm{NT}}_{A,L} = \log(p) \cdot h_{A,L} \tag{26}$$

for a certain nondegenerate skew-symmetric bilinear form $h_{A,L} : A(L) \times A^t(L) \to \mathbb{Q}$.

We write

$$h^{\det}_{A,L} : \mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{SC}_{V_L}(A, L/K)) \cong \mathrm{Det}_{\mathbb{Q}[G]}(0).$$

for the isomorphism induced by $h_{A.L}$ and then define an element of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ by setting

$$\chi^{\mathrm{BSD}}_{G,\mathbb{Q}}(A, V_L) := \chi_G(\mathrm{SC}_{V_L}(A, L/K), h^{\det}_{A,L}).$$

For each χ in $\mathrm{Ir}(G)$ we define a function of the $t := p^{-s}$ by setting

$$Z_U(A, \chi, t) := L_U(A, \chi, s)$$

and normalise its leading term at $t = p^{-1}$ as follows

$$Z^*_U(A, \chi, p^{-1}) := \lim_{t \to p^{-1}} (1 - pt)^{-r_{\mathrm{an}}(\chi)} \cdot Z_U(A, \chi, t). \tag{27}$$

Proposition 5.6. (i) *There exists a unique element $Z^*_U(A_{L/K}, p^{-1})$ of $K_1(\mathbb{Q}[G])$ with the property that*

$$\mathrm{Nrd}_{\mathbb{Q}[G]}(Z^*_U(A_{L/K}, p^{-1}))_\chi = Z^*_U(A, \chi, p^{-1})$$

for all χ in $\mathrm{Ir}(G)$.

(ii) *If claims* (i) *and* (ii) *of Conjecture 4.3 are valid, then the equality in claim* (iii) *of Conjecture 4.3 is valid if and only if in* $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ *one has*

$$\partial_G(Z_U^*(A_{L/K}, p^{-1})) = \chi_{G,\mathbb{Q}}^{\mathrm{BSD}}(A, V_L) - \chi_G^{\mathrm{coh}}(A, V_L) + \chi_G^{\mathrm{sgn}}(A).$$

Proof. The argument of Proposition 2.2 implies, via the equalities (1) and (2), that $\omega(Z_U^*(A, \chi, q^{-1})) = Z_U^*(A, \omega\circ\chi, q^{-1})$ for all χ in $\mathrm{Ir}(G)$ and all automorphisms ω of $\mathbb{C}$, and hence that the element $(Z_U^*(A, \chi, p^{-1}))_\chi$ of $\zeta(\mathbb{C}[G])^\times = \prod_{\chi\in\mathrm{Ir}(G)} \mathbb{C}^\times$ belongs to the subgroup $\zeta(\mathbb{Q}[G])^\times$.

Given this, claim (i) follows from the Hasse–Schilling–Maass norm theorem and the fact the same proof also shows $Z_U^*(A, \chi, p^{-1})$ is a strictly positive real number for χ in $\mathrm{Ir}^{\mathrm{s}}(G)$.

To prove claim (ii) we set $r_\chi := r_{\mathrm{an}}(\chi)$ and $r'_\chi := r_{\mathrm{alg}}(\chi)$. Then the order of vanishing of $Z_U(A, \chi, t)$ at $t = p^{-1}$ is equal to r_χ and hence, since the leading term of $(1 - p^{1-s})^{r_\chi}$ at $s = 1$ is equal to $(\log(p))^{r_\chi}$, it follows that

$$Z_U^*(A, \chi, p^{-1}) = (\log(p))^{-r_\chi} \cdot L_U^*(A, \chi, 1).$$

Thus, writing $\varepsilon_{L/K}$ for the unique element of $K_1(\mathbb{R}[G])$ with

$$\mathrm{Nrd}_{\mathbb{R}[G]}(\varepsilon_{L/K})_\chi = (\log(p))^{-r_\chi}$$

for all χ in $\mathrm{Ir}(G)$, one has

$$Z_U^*(A_{L/K}, p^{-1}) = \varepsilon_{L/K} \cdot L_U^*(A_{L/K}, 1).$$

On the other hand, the equality (26) implies that

$$\chi_{G,\mathbb{Q}}^{\mathrm{BSD}}(A, V_L) = \chi_G^{\mathrm{BSD}}(A, V_L) + \partial_G(\varepsilon'_{L/K})$$

where $\varepsilon'_{L/K}$ is the element of $K_1(\mathbb{R}[G])$ that is represented by the automorphism of the $\mathbb{R}[G]$-module $\mathbb{R}\otimes_{\mathbb{Z}} H^0(\mathrm{SC}_{V_L}(A, L/K)) = \mathbb{R}\otimes_{\mathbb{Z}} A^t(L)$ given by multiplication by $\log(p)^{-1}$.

Given the last two displayed formulas, the claimed equivalence will follow if one can show that the assumed validity of Conjecture 4.3(i) implies $\varepsilon'_{L/K} = \varepsilon_{L/K}$. But this is true since, for every χ in $\mathrm{Ir}(G)$, one has

$$\begin{aligned}\mathrm{Nrd}_{\mathbb{R}[G]}(\varepsilon'_{L/K})_\chi &= \det_{\mathbb{C}}\big(\log(p)^{-1} \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_\chi, \mathbb{C}\otimes_{\mathbb{Z}} A^t(L))\big) = (\log(p))^{-r'_\chi}\\ &= (\log(p))^{-r_\chi} = \mathrm{Nrd}_{\mathbb{R}[G]}(\varepsilon_{L/K})_\chi.\end{aligned}$$

Here the first equality follows directly from an explicit computation of reduced norm, the second from the fact r'_χ is (by its definition) equal to $\dim_{\mathbb{C}}\big(\mathrm{Hom}_{\mathbb{C}[G]}(V_\chi, \mathbb{C}\otimes_{\mathbb{Z}} A^t(L))\big)$, the third from the assumption that Conjecture 4.3(i) is valid (and hence $r'_\chi = r_\chi$) and the last directly from the explicit definition of $\varepsilon_{L/K}$ given above. □

6. Syntomic cohomology

In this section we recall relevant facts concerning the complexes of syntomic cohomology with compact supports that are constructed by Kato and Trihan in [22].

At the outset we fix a finite Galois extension K' of K over which $A \otimes_K K'$ is semistable at all places, write L' for the compositum of L and K' and set $G' := \mathrm{Gal}(L'/K)$. Taking advantage of Proposition 5.4 we shrink U (if necessary) in order to assume that no point on U ramifies in L'/K.

We also fix a Galois extension of fields F'/F with

$$K' \subseteq F \subseteq F' \subseteq L'$$

and set $Q := \mathrm{Gal}(F'/F)$. (Whilst the use of this auxiliary extension F'/F adds a degree of notational complexity to the results in this section, it provides results that we can then directly apply in the proof of Theorem 4.10 given in Section 9.)

Then, with N denoting either F' or F we set $A_N := A \otimes_K N$ and write X_N and U_N for the integral closures of X and U in N and $\mathcal{A}_N/X_N$ for the Néron model of A_N over N. Let $\pi_N : X_N \to X$ denote the natural map. Let $X^\sharp_{F'}$ be the log scheme with underlying scheme $X_{F'}$ equipped with the log structure associated to the divisor $X_{F'} - U_{F'}$, and we abbreviate to $\mathcal{O}_{\langle N\rangle}$ the structure sheaf $\mathcal{O}_{(X_N)^\sharp/\mathbb{Z}_p}$ for the small étale log crystalline topos $((X_N)^\sharp/\mathbb{Z}_p)_{\mathrm{crys}}$.

Since A is semistable over N, the construction in [22, §4.8] gives a Dieudonné crystal

$$D_N := D_{\log}(A_N) \tag{28}$$

on $((X_N)^\sharp/\mathbb{Z}_p)_{\mathrm{crys}}$. We then write D^0_N for the kernel of the surjective morphism of sheaves $D_N \to i_{(X_N)^\sharp/\mathbb{Z}_p,*}(\mathrm{Lie}(D_N))$ in $((X_N)^\sharp/\mathbb{Z}_p)_{\mathrm{crys}}$ described at the beginning of [22, §5.5].

We fix a $\mathrm{Gal}(L'/K)$-equivariant $\mathcal{O}_X$-submodule $\mathcal{L}_{L'}$ of $\pi_{L',*}\mathrm{Lie}(D_{L'})$ that is associated to $(W'_w)_{w\notin U_{L'}}$ following Section 3.5, and set $\mathcal{L}_{F'} := (\mathcal{L}_{L'})^{\mathrm{Gal}(L'/F')}$, which is a Q-equivariant $\mathcal{O}_X$-submodule of $\pi_{F',*}\mathrm{Lie}(D_{F'})$. For simplicity, we write

$$\mathcal{L}' := \mathcal{L}_{F'}. \tag{29}$$

(In the intended setting, we will assume that $\mathcal{L}_{L'}$, and hence $\mathcal{L}'$, satisfies the conclusion of Lemma 3.9. As noted in Remark 3.5, it may not be possible to arrange $\mathcal{L}'$ to be the pushforward of a vector bundle on $X_{F'}$ or X_F.)

We also assume that for some positive integers $n(w)$ for each place w of $X_{F'}$ not in $U_{F'}$, we have

$$\mathrm{Lie}(D_{F'})(\mathfrak{m}_w^{2n(w)}) \subset W'_w \subset \mathrm{Lie}(D_{F'})(\mathfrak{m}^{n(w)}).$$

(This can be arranged by shrinking W'_w if necessary.) We set

$$E := \sum_{w\notin U_{F'}} n(w)w,$$

which turns out to be a Q-stable divisor of $X_{F'}$ since $n(w) = n(w')$ if w and w' are above the same place in X_F by construction. Then by the condition on $(W'_w)_w$ we have

$$\pi_{F',*}\mathrm{Lie}(D_{F'}))(-2E) \subset \mathcal{L}' \subset \pi_{F',*}\mathrm{Lie}(D_{F'}))(-E).$$

Let us write $\mathcal{O}_{\langle F'\rangle}(-E)$ for the crystal on $((X_{F'})^\sharp/\mathbb{Z}_p)_{\mathrm{crys}}$ that is obtained as the twist of $\mathcal{O}_{\langle F'\rangle}$ by $-E$ and then set $D(-E)_{F'} := D_{F'} \otimes_{\mathcal{O}_{\langle F'\rangle}} \mathcal{O}_{\langle F'\rangle}(-E)$.

By [22, 5.5.2], we have a distinguished triangle of Q-equivariant (small) étale sheaves on $X_{F'}$:

$$Ru'_*D(-E)^{(0)}_{F'} \to Ru'_*D(-E)_{F'} \to \mathrm{Lie}(D_{F'})(-E) \to \ ,$$

where $u' : ((X_{F'})^\sharp/\mathbb{Z}_p)_{\mathrm{crys}} \to X_{F',\text{ét}}$ is the natural morphism of topoi.

We would like to modify $Ru'_*D(-E)_{F'}$ using the Q-equivariant $\mathcal{O}_X$-submodule $\mathcal{L}'$ of $\pi_{F',*}\mathrm{Lie}(D_{F'})(-E)$, and for this to make sense we need to apply the pushforward $\pi_{F',*}$ to the above distinguished triangle. To alleviate the notation, let us write

$$Ru_{F'/K} := \pi_{F',*}Ru'_*$$

sending a crystalline sheaf of $\mathcal{O}_{\langle F'\rangle}$-modules to a complex of étale sheaves on X (viewed in a suitable derived category).

We can now define a complex $Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'}$ viewed in the derived category of Q-equivariant étale sheaves on X so that it fits in the following distinguished triangle

$$Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'} \to Ru_{F'/K}D(-E)_{F'} \to \pi_{F',*}\mathrm{Lie}(D_{F'})(-E)/\mathcal{L}' \to \ .$$

(For technical reasons, we directly define $Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'}$ via the distinguished triangle above without defining the crystalline subsheaf $D(-E)^{(\mathcal{L}')}_{F'}$ of $D(-E)_F$. Note that $D(-E)^{(\mathcal{L}')}_{F'}$ can be defined if $\mathcal{L}'$ is the pushforward of a vector bundle on $X_{F'}$, in which case the above construction recovers $Ru_{F'/K}(D(-E)^{(\mathcal{L}')}_{F'})$; cf. [22, §5.12].)

Following [22, §5.12] there are canonical morphisms of complexes of étale sheaves on X

$$Ru_{F'/K}D(-E)^{(0)}_{F'} \xrightarrow{\mathbf{1}} Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'} \quad \text{and} \quad Ru_{F'/K}D(-E)^{(0)}_{F'} \xrightarrow{\varphi} Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'},$$

(In fact, since all the above objects can be explicitly represented by choosing *good embeddings* locally, the argument in [22, §5.12] can be directly applied to these complexes of étale sheaves instead of crystalline sheaves on $((X_{F'})^\sharp/\mathbb{Z}_p)_{\mathrm{crys}}$.)

They then define the syntomic complex with compact supports $\mathcal{S}_{D^{(E,\mathcal{L}')}_{F'}}$ to be the mapping fibre of the morphism

$$Ru_{F'/K}D(-E)^{(0)}_{F'} \xrightarrow{\mathbf{1}-\varphi} Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'}, \tag{30}$$

which is an object in the derived category of Q-equivariant étale $\mathbb{Z}_p$-sheaves on X.

If furthermore F'/M is Galois for some intermediate field M of F/K, then by choosing E to be $\mathrm{Gal}(F'/M)$-stable we may give a natural $\mathrm{Gal}(F'/M)$-action on $Ru_{F'/K}D(-E)^{(0)}_{F'}$, $Ru_{F'/K}D(-E)^{(\mathcal{L}')}_{F'}$, and $\mathcal{S}_{D^{(E,\mathcal{L}')}_{F'}}$. Recall that $\mathcal{L}' = (\mathcal{L}_{L'})^{\mathrm{Gal}(L'/F')}$ for some $\mathrm{Gal}(L'/K)$-equivariant $\mathcal{O}$-submodule of $\pi_{L',*}\mathrm{Lie}(D_{L'})$, so the Q-action on $\mathcal{L}'$ naturally extends to the action of $\mathrm{Gal}(F'/M)$.

If we have $\mathcal{L}' = \pi_{F',*}\tilde{\mathcal{L}}'$ for some Q-equivariant $\mathcal{O}_{X_{F'}}$-submodule $\tilde{\mathcal{L}}'$ of $\mathrm{Lie}(\mathcal{A}_{F'})$, then the above constructions can be carried out over X' as in [22, §5.12]. To explain, we can define an $\mathcal{O}_{\langle F'\rangle}$-submodule $D(-E)^{(\tilde{\mathcal{L}}')}_{F'}$ of $D(-E)_{F'}$, and define $\tilde{\mathcal{S}}_{D^{(E,\tilde{\mathcal{L}}')}_{F'}}$ to be the mapping fibre of

$$Ru'_*D(-E)^{(0)}_{F'} \xrightarrow{\mathbf{1}-\varphi} Ru'_*D(-E)^{(\tilde{\mathcal{L}}')}_{F'}. \tag{31}$$

(See loc. cit. for details.) Furthermore, we have a Q-equivariant quasi-isomorphism

$$\mathcal{S}_{D_{F'}^{(E,\mathcal{L}')}} = \pi_{F',*}\tilde{\mathcal{S}}_{D_{F'}^{(E,\tilde{\mathcal{L}}')}}.$$

On the other hand, in the presence of wild ramification it seems difficult to find $\mathcal{L}'$ coming from a Q-equivariant $\mathcal{O}_{X_{F'}}$-submodule that satisfies the conclusion of Lemma 3.9.

Lemma 6.1. *Let E, $\mathcal{L}'$ and $(W'_w)_{w\notin U_{F'}}$ be as above, and write $E = \sum_{w\notin U_{F'}} n(w)w$. For each $w \notin U_{F'}$ write V'_w for the unique subgroup of $\mathcal{A}_{F'}(\mathcal{O}_w)$ with $\mathcal{A}_{F'}(m_w^{2n(w)}) \subset V'_w \subset \mathcal{A}_{F'}(m_w^{n(w)})$ and whose image in $\mathcal{A}_{F'}(m_w^{n(w)})/\mathcal{A}_{F'}(m_w^{2n(w)}) \cong \mathrm{Lie}(\mathcal{A}_{F'})(m_w^{n(w)})/\mathrm{Lie}(\mathcal{A}_{F'}(m_w^{2n(w)}))$ coincides with the image of W'_w. Write $V'_{F'}$ for the family $(V'_w)_w$.*

Then there are natural isomorphisms in $D(\mathbb{Z}_p[Q])$ of the form

$$R\Gamma(X, \mathcal{S}_{D_{F'}^{(E,\mathcal{L}')}} \otimes^{\mathbb{L}} \mathbb{Q}_p/\mathbb{Z}_p) \cong R\Gamma_{\mathrm{ar},V'_{F'}}(U_{F'}, \mathcal{A}_{\mathrm{tor}})_p. \tag{32}$$

In addition, if M is any intermediate field of F/K over which F' is Galois and E is chosen to be $\mathrm{Gal}(F'/M)$-equivariant, then the above isomorphism is well-defined in $D(\mathbb{Z}_p[\mathrm{Gal}(F'/M)])$.

This lemma is a generalisation of [22, Proposition 5.13] in that the isomorphism (32) is proven to be Galois equivariant and $\mathcal{L}'$ is not required to come from a vector bundle over $X_{F'}$.

Proof. Using the definition of $R\Gamma_{\mathrm{ar},V_{F'}}(U_{F'}, \mathcal{A}_{\mathrm{tor}})_p$ (4) and [37, Theorem 1.1], one can reduce the isomorphism (32) to the following local statement: *For any $v \in X \setminus U$, we have a natural isomorphism*

$$R\Gamma(\operatorname{Spec}\mathcal{O}_v, \mathcal{S}_{D_{F'}^{(E,\mathcal{L}')}}) \cong \prod_{w|v} V'_w[-1] \tag{33}$$

equivariant for the Q-action (respectively, for the $\mathrm{Gal}(F'/M)$-action if F'/M is Galois for some intermediate extension M of F/K).

This local claim is a slight generalisation of [22, Lemma 5.14] in that the isomorphism (33) is required to be Galois equivariant and W'_w is not required to be an $\mathcal{O}_w$-module.

It remains to verify the local claim. Observe that for fixed D the restriction of $\mathcal{S}_{D_{F'}^{(E,\mathcal{L}')}}$ to $\operatorname{Spec}\mathcal{O}_v$ only depends on $n(w)$ and W'_w for $w|v$, and $n(w)$ is independent of $w|v$. So let us write

$$\mathcal{S}_{D,v}^{n,(W'_w)} := \mathcal{S}_{D_{F'}^{(E,\mathcal{L}')}}\big|_{\operatorname{Spec} O_v}$$

where $n = n(w)$ for any $w|v$. To simplify the notation, for any positive integer n we write

$$W_w^{(n)\prime} := \mathrm{Lie}(\mathcal{A}_{F'})(\mathfrak{m}_w^n).$$

Note that the choice $(W_w^{(n(w))\prime})_{w\notin U_{F'}}$ corresponds to $\pi_{F',*}\mathrm{Lie}(\mathcal{A}_{F'})(-E)$, which contains $\mathcal{L}'$.

Let us first show that the local claim (33) is implied by the special case for $W'_w = W_w^{(n)\prime}$. For this, we construct a distinguished triangle in the suitable derived category of equivariant étale $\mathbb{Z}_p$-sheaves on $\operatorname{Spec}\mathcal{O}_v$:

$$\mathcal{S}_{D,v}^{n,(W'_w)} \longrightarrow \mathcal{S}_{D,v}^{n,(W_v^{(n)\prime})} \longrightarrow \textstyle\prod_{w|v}(W_w^{(n)\prime}/W'_w)[-1] \longrightarrow. \tag{34}$$

Indeed, this can be obtained from the following commutative diagram where each row is a distinguished triangle

$$\begin{tikzcd}
\mathcal{S}_{D,v}^{n,(W'_w)} \arrow[r] \arrow[d] & Ru_{F'/K}D(-E)^{(0)}|_{\mathcal{O}_v} \arrow[r, "\mathbf{1}-\varphi"] \arrow[d, "="] & Ru_{F'/K}D(-E)^{(\mathcal{L}')}|_{\mathcal{O}_v} \arrow[r] \arrow[d] & {} \\
\mathcal{S}_{D,v}^{n,(W_w^{(n)\prime})} \arrow[r] & Ru_{F'/K}D(-E)^{(0)}|_{\mathcal{O}_v} \arrow[r, "\mathbf{1}-\varphi"] & Ru_{F'/K}D(-E)|_{\mathcal{O}_v} \arrow[r] & {}
\end{tikzcd}$$

together with the fact that the mapping cone of the rightmost vertical arrow is isomorphic to $W_w^{(n)\prime}/W'_w$.

Let $\hat{\mathcal{A}}_{F'}(\mathfrak{m}_v^n) \subset \mathcal{A}_{F'}(\mathcal{O}_w)$ denote the kernel of reduction modulo $\mathfrak{m}_w^n$. Then by (34) and the natural isomorphism

$$W_w^{(n)\prime}/W_w^{(n+1)\prime} \cong \hat{\mathcal{A}}_{F'}(\mathfrak{m}_v^n)/\hat{\mathcal{A}}_{F'}(\mathfrak{m}_v^{n+1}), \tag{35}$$

the general case of the local claim is reduced to obtaining the Galois equivariant isomorphism (33) when $W'_w = W_w^{(n(w))\prime}$ and $V'_w = \hat{\mathcal{A}}_{F'}(\mathfrak{m}_v^{n(w)})$.

We have thus reduced the proof of the lemma to the case when $\mathcal{L}' = \pi_{F',*}\mathrm{Lie}(\mathcal{A}_{F'})(-E)$. We will proceed by induction, for which it is convenient to allow $\mathcal{L}' = \pi_{F',*}\mathrm{Lie}(\mathcal{A}_{F'})(-E')$ where E' is a Q-equivariant divisor such that $E' - E$ and $2E - E'$ are either effective or trivial. Since $\mathcal{L}'$ is the pushforward of a Q-equivariant $\mathcal{O}_{X_{F'}}$-module $\tilde{\mathcal{L}}' := \mathrm{Lie}(\mathcal{A}_{F'})(-E')$, we also have a "syntomic complex" $\tilde{\mathcal{S}}_{D_{F'}^{(E,\tilde{\mathcal{L}}')}}$ over $X_{F'}$, constructed as the mapping fibre of the map (31). For any $w \in X_{F'} \setminus U_{F'}$, let us write

$$\tilde{\mathcal{S}}_{D,w}^{n,W^{(m)\prime}} := \tilde{\mathcal{S}}_{D_{F'}^{(E,\tilde{\mathcal{L}}')}}\big|_{\mathrm{Spec}\,\mathcal{O}_w},$$

where $n = n(w)$ and m are the coefficients of w in E and E', respectively. Since $\mathcal{S}_{D_{F'}^{(E,\mathcal{L}')}} = \pi_{F',*}\tilde{\mathcal{S}}_{D_{F'}^{(E,\tilde{\mathcal{L}}')}}$, the left-hand side of (33) also decomposes in terms of $\tilde{\mathcal{S}}_{D,w}^{n,W^{(m)\prime}}$. Therefore, to complete the proof, it suffices to show that for any positive integer n and $w \in X_{F'} \setminus U_{F'}$ we have a natural isomorphism

$$R\Gamma(\mathrm{Spec}\,\mathcal{O}_w, \tilde{\mathcal{S}}_{D,w}^{n,W^{(n)\prime}}) \cong \hat{\mathcal{A}}(\mathfrak{m}_w^n) \tag{36}$$

equivariant for the Q_w-action (respectively, for the $\mathrm{Gal}(F'_w/M_v)$-action where v is the place under w if F'/M is Galois for some intermediate extension M of F/K).

Let us prove (36) by induction on n. If $n = 1$ then the isomorphism (36) can be deduced by inspecting the distinguished triangle (4) using Theorems 1.1 and 1.2 in [37], where the Galois equivariance follows since the comparison maps in loc. cit. are constructed naturally. Although the results were obtained only for $p > 2$ in loc. cit., there is an alternative proof that works for any p via the prismatic Dieudonné theory [1]. To give further details, [37, Theorem 1.2] holds for $p = 2$ if [37, Theorem 1.1] does, and (the projective limit of) [37, Theorem 1.1] can be deduced from [1, Proposition 4.83 and Remark 4.85] if we show that the prismatic and crystalline constructions of the syntomic complex in [1] and [37] coincide. For this we may (and do) pass to some complete intersection semiperfect ring by p-power root extraction to represent the two syntomic complexes by explicit two-term complexes of modules, and the desired isomorphism follows from the comparison between crystalline and prismatic Dieudonné theory over quasisyntomic bases in characteristic p [1, Theorem 4.44] as well as the comparison of the Hodge and Nygaard filtrations. (The Nygaard filtration for prismatic Dieudonné crystals is defined at the beginning of [1, §4.8], and the claimed compatibility with filtrations can be read off from the proof of [1, Theorem 4.44] using [1, Lemmas 4.40 and 4.43]. Note also that in [1, Remark 4.85] we have $\tilde{\xi} = p$ for characteristic p base rings, so the comparison in [1, Theorem 4.44] is compatible with the divided Frobenius maps.)

Assume that we have a natural isomorphism (36) for n, and we shall deduce (36) for $n+1$. By (34) applied to $W'_w = W_w^{(n+1)\prime}$, we get

$$\tilde{\mathcal{S}}^{n, W^{(n+1)\prime}}_{D,w} \to \tilde{\mathcal{S}}^{n, W^{(n)\prime}}_{D,w} \to \mathrm{Lie}(\mathcal{A}_{F'}) \otimes (\mathfrak{m}_w^n / \mathfrak{m}_w^{n+1})[-1] \to . \tag{37}$$

We now claim that the map

$$\tilde{\mathcal{S}}^{n+1, W_w^{(n+1)\prime}}_{D,w} \xrightarrow{\sim} \tilde{\mathcal{S}}^{n, W_w^{(n+1)\prime}}_{D,w}, \tag{38}$$

induced by the natural map $D(-(n+1)w) \to D(-nw)$, is an isomorphism. To verify this assertion we may ignore the Galois action, which enables us to represent the syntomic complexes explicitly following [22, 5.14.1].

Let $S^\sharp$ be $\operatorname{Spec} \mathcal{O}_w$ equipped with the divisorial log structure for the closed point. Choose an isomorphism $\mathcal{O}_w \cong k_w[\![T]\!]$, and write $P^\sharp$ to be $\operatorname{Spec} W(k_w)[\![T]\!]$ equipped with the divisorial log structure given by the ideal (T). Then the natural closed immersion $S^\sharp \hookrightarrow P^\sharp$ is a *good embedding* in the sense of [22, § 5.6]. Let σ be the lift of Frobenius on $\mathcal{O}_{\widehat{P}} = W(k_w)[\![T]\!]$ given by $\sigma(T) := T^p$.

For any integers n, m with $0 < n \leqslant m < 2n$, we can represent $\tilde{\mathcal{S}}^{n, W_w^{(m)\prime}}_{D,w}$ as the total complex of the double complex

$$\begin{tikzcd}
T^n D^{(0)}_{(S^\sharp, P^\sharp)} \arrow[r, "1-p^{-1}\,\mathrm{Fr}\,\mathbf{1}"] \arrow[d] & T^m D_{(S^\sharp, P^\sharp)} + T^n D^{(0)}_{(S^\sharp, P^\sharp)} \arrow[d] \\
\mathcal{O}_{\widehat{P}} \frac{dt}{t} \otimes T^n D_{(S^\sharp, P^\sharp)} \arrow[r, "1-p^{-1}\sigma\otimes\mathrm{Fr}\,\mathbf{1}"] & \mathcal{O}_{\widehat{P}} \frac{dt}{t} \otimes T^n D_{(S^\sharp, P^\sharp)}
\end{tikzcd}$$

This shows that the mapping cone of (38) is quasi-isomorphic to the total complex of the double complex

$$\begin{tikzcd}
\dfrac{T^n D^{(0)}_{(S^\sharp, P^\sharp)}}{T^{n+1} D^{(0)}_{(S^\sharp, P^\sharp)}} \arrow[r, "1-p^{-1}\,\mathrm{Fr}\,\mathbf{1}"] \arrow[d] & \dfrac{T^{n+1} D_{(S^\sharp, P^\sharp)} + T^n D^{(0)}_{(S^\sharp, P^\sharp)}}{T^{n+1} D_{(S^\sharp, P^\sharp)}} \arrow[d] \\
\mathcal{O}_{\widehat{P}} \dfrac{dt}{t} \otimes \dfrac{T^n D_{(S^\sharp, P^\sharp)}}{T^{n+1} D_{(S^\sharp, P^\sharp)}} \arrow[r, "1-p^{-1}\sigma\otimes\mathrm{Fr}\,\mathbf{1}"] & \mathcal{O}_{\widehat{P}} \dfrac{dt}{t} \otimes \dfrac{T^n D_{(S^\sharp, P^\sharp)}}{T^{n+1} D_{(S^\sharp, P^\sharp)}}
\end{tikzcd}.$$

It suffices to show that both horizontal maps are isomorphisms. Indeed, since $p^{-1}\,\mathrm{Fr}\,\mathbf{1}$ takes $T^n D^{(0)}_{(S^\sharp, P^\sharp)}$ into $T^{np} D_{(S^\sharp, P^\sharp)}$, the top horizontal map coincides with the map induced by the natural inclusion $\mathbf{1}$, which is an isomorphism since $T^{n+1} D^{(0)}_{(S^\sharp, P^\sharp)} = T^{n+1} D_{(S^\sharp, P^\sharp)} \cap T^n D^{(0)}_{(S^\sharp, P^\sharp)}$. Similarly, the bottom horizontal map coincides with the identity map.

Now combining (35), (37) and (38), we verify the desired equivariant isomorphism (36), which the lemma was reduced to. □

Recalling that $Q = \mathrm{Gal}(F'/F)$, we now define objects in $D(\mathbb{Z}_p[Q])$, respectively in $D(\mathbb{Z}_p[\mathrm{Gal}(F'/K)])$ if F'/K is Galois, by setting

$$I_{F'} := R\Gamma(X, Ru_{F'/K} D(-E)^{(0)}_{F'} \otimes^{\mathbb{L}} \mathbb{Q}_p/\mathbb{Z}_p)^*[-2]$$

and

$$P_{F'} := R\Gamma(X, Ru_{F'/K} D(-E)^{(\mathcal{L}')}_{F'} \otimes^{\mathbb{L}} \mathbb{Q}_p/\mathbb{Z}_p)^*[-2],$$

where $\mathcal{L}'$ is as defined in (29). The following result describes the connection between these objects and the constructions made in earlier sections.

Lemma 6.2. *Let E, $V_{F'}$ and M be as in Lemma 6.1. Then there are canonical exact triangles in $D^-(\mathbb{Z}_p[\mathrm{Gal}(F'/M)])$ of the form*

$$P_{F'} \xrightarrow{\mathbf{1}-\varphi} I_{F'} \xrightarrow{\theta} R\Gamma_{\mathrm{ar},V_{F'}}(U_{F'}, \mathcal{A}_{\mathrm{tor}})^*_p[-2] \xrightarrow{\theta'} P_{F'}[1] \tag{39}$$

and

$$P_{F'} \xrightarrow{\mathbf{1}} I_{F'} \to R\Gamma(X, \mathcal{L}')^*[-2] \to P_{F'}[1]. \tag{40}$$

Proof. By the result of Lemma 6.1, the triangle (39) is obtained by applying the exact composite functor $R\Gamma(X, -\otimes^{\mathbb{L}} \mathbb{Q}_p/\mathbb{Z}_p)$ to the exact triangle (of complexes of sheaves) that results from the definition of $\mathcal{S}_{D^{(E,\mathcal{L})}_{F'}}$ as the mapping fibre of (30).

The exact triangle (40) results in a similar way by using the canonical exact triangle

$$\mathcal{L}' \to Ru_{F'/K} D(-E)^{(0)}_{F'} \otimes^{\mathbb{L}} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\mathbf{1}} Ru_{F'/K} D(-E)^{(\mathcal{L}')}_{F'} \otimes^{\mathbb{L}} \mathbb{Q}_p/\mathbb{Z}_p \to$$

described by Kato and Trihan in [22, §6.7]. □

The complexes $I_{F'}$ and $P_{F'}$ are not known, in general, to belong to $D^{\mathrm{perf}}(\mathbb{Z}_p[\mathrm{Gal}(F'/F)])$ and hence, for our purposes, we must adapt the triangles (39) and (40), as per the following result.

Proposition 6.3. *Let M be any extension of K over which F' is Galois. Set $J := \mathrm{Gal}(F'/M)$, and let $\mathfrak{N}$ be an order in $\mathbb{Q}_p[J]$ that contains $\mathbb{Z}_p[J]$ and is such that the complex $\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'})$ can be represented by a bounded complex of projective $\mathfrak{N}$-modules.*

Then the triangles (39) *and* (40) *induce exact triangles in $D^{\mathrm{perf}}(\mathfrak{N})$ of the form*

$$\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} P_{F'}) \xrightarrow{\mathbf{1}-\varphi} \tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'}) \to \mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} R\Gamma_{\mathrm{ar},V_{F'}}(U_{F'}, \mathcal{A}_{\mathrm{tor}})^*_p[-2] \to ,$$

$$\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} P_{F'}) \xrightarrow{\mathbf{1}} \tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'}) \to \mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} R\Gamma(X, \mathcal{L}')^*[-2] \to .$$

Proof. The results of Proposition 3.7(i) and Lemma 3.9 imply that both of the complexes $C_1 := R\Gamma_{\mathrm{ar},V_{F'}}(U_{F'}, \mathcal{A}_{\mathrm{tor}})^*_p[-2]$ and $C_2 := R\Gamma(X, \mathcal{L}')^*[-2]$ belong to $D^{\mathrm{perf}}(\mathbb{Z}_p[J])$ and are acyclic outside degrees 0 and 1 and 2.

In addition, a finitely generated, torsion-free, $\mathbb{Z}_p[J]$-module of finite projective dimension is itself projective (by [3, Theorem 8]). By a standard resolution argument (as in the proof of Lemma 3.8(iii)), it therefore follows that the complexes C_1 and C_2 are both represented by complexes of finitely generated projective $\mathbb{Z}_p[J]$-modules, all terms of which are zero in every degree less than -1 and every degree greater than 2.

This in turn implies that $\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} C_1$ and $\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} C_2$ belong to $D^{\mathrm{perf}}(\mathfrak{N})$ and are both acyclic in all degrees less than -1.

Given this last fact, one obtains exact triangles in $D^-(\mathfrak{N})$ of the stated form by simply applying the exact functor $\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} -$ to the triangles (39) and (40).

To prove that these respective triangles belong to $D^{\mathrm{perf}}(\mathfrak{N})$ (rather than just $D^-(\mathfrak{N})$) it is enough, since $\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} C_1$ and $\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} C_2$ both belong to $D^{\mathrm{perf}}(\mathfrak{N})$, to prove that the complex $C := \tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'})$ also belongs to $D^{\mathrm{perf}}(\mathfrak{N})$.

To do this we note that, by assumption, C is represented by a bounded complex of projective $\mathfrak{N}$-modules and, by [22, Proposition 5.15(i)], all cohomology groups of C are finitely generated over $\mathfrak{N}$. Taken together, these facts combine with a standard construction of resolutions to imply C belongs to $D^{\mathrm{perf}}(\mathfrak{N})$, as required. □

Since $\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'})$ is acyclic outside finitely many degrees the stated condition in Proposition 6.3 is automatically satisfied if the order $\mathfrak{N}$ is hereditary (and hence, by [11, Theorem (26.12)], if it is a maximal order).

With Theorem 4.10 in mind, in the next section we will show that, under suitable conditions on A_M and F'/M the condition in Proposition 6.3 can also be satisfied by orders that are not maximal.

Then, in Section 8, we shall study in greater detail the long exact cohomology sequences of the exact triangles in Proposition 6.3.

7. Crystalline cohomology and tame ramification

We continue to use the general notation of Section 6. We also assume that the extension F'/F is tamely ramified and write $\pi : X_{F'} \to X_F$ for the corresponding cover of smooth projective curves. We fix a log structure on $X_{F'}$ associated to the divisor $X_{F'} - U_{F'}$, write $X^{\sharp}_{F'}$ for the associated log scheme and note that the natural map $\pi^{\sharp} : X^{\sharp}_{F'} \to X^{\sharp}_F$ is Kummer-étale (in the sense of [28, Definition 2.13]).

We write $u : (X^{\sharp}_F/\mathbb{Z}_p)_{\mathrm{crys}} \to X_{F,\text{ét}}$ and $u' : (X^{\sharp}_{F'}/\mathbb{Z}_p)_{\mathrm{crys}} \to (X_{F'})_{\text{ét}}$ for the natural morphism of topoi.

In this section we shall construct certain complexes of Q-equivariant étale $\mathbb{Z}_p$-modules that represent $Ru_*D(-E_F)^{(0)}$ and $Ru'_*D(-E)^{(0)}_{F'}$, where E is the pull back of a suitable divisor E_F of X_F supported exactly at $X_F \setminus U_F$. This construction will play an important role in the proof of Theorem 4.10.

7.1. ***Digression on log de Rham complexes.*** The main result of this section is the following general observation concerning crystalline sheaves.

Proposition 7.1. *Let $\mathcal{E}$ be a locally free crystal of $\mathcal{O}_{\langle F\rangle}$-modules (with $\mathcal{O}_{\langle F\rangle} := \mathcal{O}_{X^{\sharp}_F/\mathbb{Z}_p}$).*

(i) *There exists a bounded below complex $C(\pi^{\sharp,*}\mathcal{E})$ of torsion free $\mathbb{Z}_p[Q]$-modules that has both of the following properties.*

(a) *Each term of $C(\pi^{\sharp,*}\mathcal{E})$ is an* <u>*induced*</u> *$\mathbb{Z}_p[Q]$-module; in other words, in each degree i there is an isomorphism $\mathbb{Z}_p[Q]$-modules*

$$C^i(\pi^{\sharp,*}\mathcal{E}) \cong \mathrm{Ind}_{\{e\}}^{Q}(C^i(\pi^{\sharp,*}\mathcal{E})^Q),$$

where e denotes the identity element of Q.

(b) *For each normal subgroup J of Q there is an isomorphism in $D(\mathbb{Z}_p[Q/J])$*

$$\mathrm{Hom}_{\mathbb{Z}_p[J]}(\mathbb{Z}_p, C(\pi^{\sharp,*}\mathcal{E})) \cong R\Gamma_{\mathrm{crys}}(X^{\sharp}_{F'^J}/\mathbb{Z}_p, \pi_J^{\sharp,*}\mathcal{E}),$$

where $\pi_J^{\sharp} : X^{\sharp}_{F'^J} \to X^{\sharp}_F$ is the natural projection.

(ii) *If there is a short exact sequence of sheaves*

$$0 \to \mathcal{E}^0 \to \mathcal{E} \to i_{X^{\sharp}_F/\mathbb{Z}_p,*}\mathcal{F} \to 0$$

for a vector bundle $\mathcal{F}$ on X_F, then claim (i) *is also true with $\mathcal{E}$ replaced by $\mathcal{E}^0$.*

7.1.1. As preparation for the proof of this proposition we start with the following technical result.

Lemma 7.2. *There exists a formal scheme $\mathfrak{X}^{\sharp}_F$ over $\mathbb{Z}_p$ that is a smooth lift of $X^{\sharp}_F$. Furthermore, for any finite Kummer-étale covering $X^{\sharp}_{F'} \to X^{\sharp}_F$, there exists a finite Kummer-étale covering $\tilde{\pi}^{\sharp} : \mathfrak{X}^{\sharp}_{F'} \to \mathfrak{X}^{\sharp}_F$ that lifts $\pi^{\sharp} : X^{\sharp}_{F'} \to X^{\sharp}_F$.*

Proof. This lemma is obtained from the infinitesimal deformation theory for smooth log schemes (see [21, Proposition 3.14]). More precisely, if $\mathfrak{X}^{\sharp}_{F,n}$ is a (flat) lift of X_F over $\mathbb{Z}_p/p^n$, then it is easy to see that $\mathfrak{X}^{\sharp}_{F,n}$ is log smooth over $\mathbb{Z}_p/p^n$ (where $\mathbb{Z}_p/p^n$ is given the trivial log structure). To see this, one applies Kato's criterion [21, Theorem 3.5]. By [21, Proposition 3.14(4)], the obstruction class for lifting $\mathfrak{X}^{\sharp}_{F,n}$ over $\mathbb{Z}_p/p^{n+1}$ lies in $H^2(X_F, \omega^{\vee}_{X^{\sharp}_F}) = 0$, where $\omega_{X^{\sharp}_F}$ is the sheaf of differentials with log poles at $X_F - U_F$. We write $\mathfrak{X}^{\sharp}_F$ for the natural inverse limit $\varprojlim_n \mathfrak{X}^{\sharp}_{F,n}$.

The sheaf of relative log differentials $\omega_{X^{\sharp}_{F'}/X^{\sharp}}$ being trivial, we conclude that the finite Kummer-étale covering $\pi^{\sharp} : X^{\sharp}_{F'} \to X^{\sharp}_F$ canonically lifts to $\tilde{\pi}^{\sharp} : \mathfrak{X}^{\sharp}_{F',n} \to \mathfrak{X}^{\sharp}_{F,n}$ (cf. [21, Proposition 3.14]). This produces the desired finite Kummer-étale covering $\tilde{\pi}^{\sharp} : \mathfrak{X}^{\sharp}_{F'} \to \mathfrak{X}^{\sharp}_F$. □

We use this lemma to obtain some complexes representing $Ru_*\mathcal{E}$ and $Ru'_*(\pi^{\sharp,*}\mathcal{E})$ for a locally free crystal $\mathcal{E}$ of $\mathcal{O}_{\langle F\rangle}$-modules. Given such $\mathcal{E}$, we obtain a vector bundle $\mathcal{E}_{\mathfrak{X}^{\sharp}_F}$ that is equipped with an integrable connection with log poles $\nabla : \mathcal{E}_{\mathfrak{X}^{\sharp}_F} \to \mathcal{E}_{\mathfrak{X}^{\sharp}_F} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_F}} \widehat{\omega}_{\mathfrak{X}^{\sharp}_F}$.

Since $X^{\sharp}_F \hookrightarrow \mathfrak{X}^{\sharp}_F$ is a good embedding in the sense of [22, § 5.6], it follows that $\mathcal{E}$ is functorially determined by $(\mathcal{E}_{\mathfrak{X}^{\sharp}_F}, \nabla)$ by [21, Theorem 6.2]. The same holds for any locally free crystal $\mathcal{E}'$ of $\mathcal{O}_{\langle F'\rangle}$-modules, and the associated vector bundle with integrable connection with log poles $(\mathcal{E}'_{\mathfrak{X}^{\sharp}_{F'}}, \nabla)$.

Recall that the map $\tilde{\pi}: \mathfrak{X}_{F'} \to \mathfrak{X}_F$ is flat[1] and we have $\tilde{\pi}^*\omega_{\mathfrak{X}^\sharp_F} \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp_{F'}}$ by [21, Proposition 3.12], so we can define pull back and push forward by $\tilde{\pi}$ for vector bundles with connection with log poles (just as the unramified case).

By unwinding the proof of [21, Theorem 6.2], one can see that the construction $\mathcal{E} \rightsquigarrow (\mathcal{E}_{\mathfrak{X}^\sharp_F}, \nabla)$ (and the same construction for $\mathcal{E}'$) respects the pull back and push forward by $\pi^\sharp$ so that one has both $((\pi^{\sharp,*}\mathcal{E})_{\mathfrak{X}^\sharp_{F'}}, \nabla) = \tilde{\pi}^*(\mathcal{E}_{\mathfrak{X}^\sharp_F}, \nabla)$ and $((\pi^\sharp_*\mathcal{E}')_{\mathfrak{X}^\sharp_F}, \nabla) = \tilde{\pi}_*(\mathcal{E}'_{\mathfrak{X}^\sharp_{F'}}, \nabla)$.

In particular, both $(\pi^{\sharp,*}\mathcal{E})_{\mathfrak{X}^\sharp_{F'}}$ and $(\pi^\sharp_*\mathcal{E}')_{\mathfrak{X}^\sharp_F}$ have natural horizontal actions of Q.

Let $\mathfrak{X}_{F,n}$ denote the closed subscheme of $\mathfrak{X}_F$ cut out by the ideal generated by p^n. Then a coherent $\mathcal{O}_{\mathfrak{X}_{F,n}}$-modules $\mathcal{F}_n$ can be seen as a torsion étale sheaf on X_F, where for any étale morphism $f: Y \to \mathfrak{X}_{F,n}$ we have $\mathcal{F}_n(Y) := \Gamma(Y, f^*\mathcal{F}_n)$. Similarly, any coherent $\mathcal{O}_{\mathfrak{X}_F}$-module $\mathcal{F}$ can be viewed as a $\mathbb{Z}_p$-étale sheaf on X_F; namely, the inverse system of torsion étale sheaves $\{\mathcal{F}|_{\mathfrak{X}_n}\}$.

Now, for any locally free crystal $\mathcal{E}$ of $\mathcal{O}_{\langle F\rangle}$-module, the complex $Ru_*\mathcal{E}$ can be computed via the complex of $\mathbb{Z}_p$-étale sheaves on X_F given by $\mathcal{E}_{\mathfrak{X}^\sharp_F} \xrightarrow{\nabla} \mathcal{E}_{\mathfrak{X}^\sharp_F}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_F}}\widehat{\omega}_{\mathfrak{X}^\sharp_F}$, where the first term is placed in degree zero (cf. [22, § 5.6]). One also obtains a similar expression for $Ru'_*(\pi^{\sharp,*}\mathcal{E})$ as a complex of "$\mathbb{Z}_p[Q]$-étale sheaves" on $X_{F'}$.

Given a short exact sequence

$$0 \to \mathcal{E}^0 \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a vector bundle on X_F viewed as a log crystalline sheaf, we have a short exact sequence $0 \to Ru_*\mathcal{E}^0 \to Ru_*\mathcal{E} \to \mathcal{F} \to 0$, where $\mathcal{F}$ is viewed as a torsion étale sheaf on X_F. Therefore, we may express

$$Ru_*\mathcal{E}^0 = [\mathcal{E}^0_{\mathfrak{X}^\sharp_F} \xrightarrow{\nabla} \mathcal{E}_{\mathfrak{X}^\sharp_F}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_F}}\widehat{\omega}_{\mathfrak{X}^\sharp_F}], \tag{41}$$

where $\mathcal{E}^0_{\mathfrak{X}^\sharp_F}$ denotes the kernel of $\mathcal{E}_{\mathfrak{X}^\sharp_F} \twoheadrightarrow i_{X^\sharp_F/\mathbb{Z}_p,*}\mathcal{F}$.

Remark 7.3. Note that $\mathfrak{X}^\sharp_F$ can be obtained as a p-adic completion of a proper smooth log scheme $\widetilde{X}^\sharp_F$ over $\mathbb{Z}_p$, where the underlying scheme $\widetilde{X}_F$ is a smooth lift of X_F and the log structure is given by relative divisor $\widetilde{Z} \subset \widetilde{X}_F$ smoothly lifting $Z := |X_F - U_F|$.

Let us now give some examples of $Ru_*\mathcal{E}$ for some $\mathcal{E}$. When $\mathcal{E} = \mathcal{O}_{\langle F\rangle}$ then $Ru_*\mathcal{O}_{\langle F\rangle}$ is the log de Rham complex of $\mathfrak{X}^\sharp_F$; that is, the p-adic completion of the de Rham complex of $\widetilde{X}_F$ with log poles along $\widetilde{Z}$.

Given any divisor E_F of X_F supported in Z, one obtains a rank one locally free crystal of $\mathcal{O}_{\langle F\rangle}$-modules $\mathcal{E} := \mathcal{O}_{\langle F\rangle}(E_F)$.

Let us now describe $Ru_*\mathcal{O}_{\langle F\rangle}(E_F)$. Viewing $\mathfrak{X}^\sharp_F$ as the p-adic completion of the log scheme $\widetilde{X}^\sharp_F$ with divisorial log structure associated to $\widetilde{Z}$, we can find a relative divisor $\widetilde{E}_F$ of $\widetilde{X}_F$ that lifts E_F and is supported in $\widetilde{Z}$. Then from the definition of $\mathcal{O}_{\langle F\rangle}(E_F)$ (cf. [22, § 5.12]), one can check that

$$Ru_*\mathcal{O}_{\langle F\rangle}(E_F) = [\mathcal{O}_{\widetilde{X}_F}(\widetilde{E}_F) \xrightarrow{\nabla} \mathcal{O}_{\widetilde{X}_F}(\widetilde{E}_F) \otimes \omega_{\widetilde{X}^\sharp_F}] \otimes_{\mathcal{O}_{\widetilde{X}_F}} O_{\mathfrak{X}_F},$$

[1] It suffices to verify the flatness at the formal neighbourhood of any closed point. And by Abhyankar's lemma (cf. [16, A.11]), the map of completed local rings induced by $\tilde{\pi}$ is of the form $W(\mathbb{F}_q)[[t]] \to W(\mathbb{F}_{q'}[[t^{1/e}]])$ for some e not divisible by p.

where ∇ is induced by the universal derivation $d : \mathcal{O}_{\widetilde{X}_F} \to \omega_{\widetilde{X}^\sharp_F} = \Omega_{\widetilde{X}_F}(\log \widetilde{Z})$. (Here, ∇ is well defined since $\widetilde{E}_F$ is supported in $\widetilde{Z}$, where $\omega_{\widetilde{X}^\sharp_F}$ is allowed to have log poles.)

7.1.2. We are now ready to prove Proposition 7.1.

We shall, for brevity, only prove claim (ii) since this is directly relevant to the proof of Theorem 4.10 and claim (i) can be proved by exactly the same argument.

Our strategy is to use Proposition A.7 to construct a complex $C(\pi^{\sharp,*}\mathcal{E}^0)$ of *induced* $\mathbb{Z}_p[Q]$-modules that represents $R\Gamma_{\mathrm{crys}}(X^\sharp_{F'}/\mathbb{Z}_p, \pi^{\sharp,*}\mathcal{E}^0)$ in such a way that $C(\mathcal{E}^0) := C(\pi^{\sharp,*}\mathcal{E}^0)^Q$ is naturally isomorphic in $D(\mathbb{Z}_p)$ to $R\Gamma_{\mathrm{crys}}(X^\sharp_F/\mathbb{Z}_p, \mathcal{E}^0)$. (Since each term of $C(\pi^{\sharp,*}\mathcal{E}^0)$ is an induced $\mathbb{Z}_p[Q]$-module, the complex $C(\mathcal{E}^0)$ of termwise Q-invariants of $C(\pi^{\sharp,*}\mathcal{E}^0)$ represents $R\operatorname{Hom}_{\mathbb{Z}_p[Q]}(\mathbb{Z}_p, C(\pi^{\sharp,*}\mathcal{E}^0))$.)

We recall that $R\Gamma_{\mathrm{crys}}(X^\sharp_F/\mathbb{Z}_p, \mathcal{E}^0)$ identifies with $R\Gamma_{\mathrm{\acute{e}t}}(X_F, Ru_*\mathcal{E}^0)$ and that $Ru_*\mathcal{E}^0$ is equal to the complex $\mathcal{E}^0_{\mathfrak{X}^\sharp_F} \xrightarrow{\nabla} \mathcal{E}_{\mathfrak{X}^\sharp_F} \otimes_{\mathcal{O}_{\mathfrak{X}_F}} \widehat{\omega}_{\mathfrak{X}^\sharp_F}$. In particular, since all the terms of $Ru_*\mathcal{E}^0$ are "coherent $\mathcal{O}_{\mathfrak{X}_F}$-modules", we can compute $R\Gamma_{\mathrm{\acute{e}t}}(X_F, Ru_*\mathcal{E}^0)$ via Zariski topology on $\mathfrak{X}_F$ (viewing $Ru_*\mathcal{E}^0$ as a complex of coherent $\mathcal{O}_{\mathfrak{X}_F}$-modules with additive differential). Note that the same properties hold for $Ru'_*(\pi^{\sharp,*}\mathcal{E}^0)$ as well.

We now choose the disjoint union of some Q-stable finite affine open covering $\mathfrak{U}^\sharp_F$ of $\mathfrak{X}^\sharp_{F'}$ and regard it as a Kummer-étale covering of $\mathfrak{X}^\sharp_F$. We then let $C(\mathcal{E}^0)$ denote the total complex associated to the Čech resolution of $Ru_*\mathcal{E}^0$ with respect to $\mathfrak{U}^\sharp_F$. Similarly, we let $C(\pi^{\sharp,*}\mathcal{E}^0)$ denote the total complex associated to the Čech resolution of $Ru'_*(\pi^{\sharp,*}\mathcal{E}^0)$ with respect to the Kummer-étale covering $\mathfrak{U}^\sharp_F \times_{\mathfrak{X}^\sharp_F} \mathfrak{X}^\sharp_{F'}$ of $\mathfrak{X}^\sharp_{F'}$, which is a complex of $\mathbb{Z}_p[Q]$-modules where the Q-action is induced from the Q-action on $\mathfrak{X}_{F'}$. Then, by Proposition A.7, we know that $C(\mathcal{E}^0)$ is isomorphic in $D(\mathbb{Z}_p)$ to $R\Gamma_{\mathrm{crys}}(X^\sharp_F/\mathbb{Z}_p, \mathcal{E}^0)$ and that $C(\pi^{\sharp,*}\mathcal{E}^0)$ is isomorphic in $D(\mathbb{Z}_p[Q])$ to $R\Gamma_{\mathrm{crys}}(X^\sharp_{F'}/\mathbb{Z}_p, \pi^{\sharp,*}\mathcal{E}^0)$.

In addition, one has $\mathfrak{U}^\sharp_F \times_{\mathfrak{X}^\sharp_F} \mathfrak{X}^\sharp_{F'} \cong \mathfrak{U}^\sharp_F \times Q$ and so in each degree i there is an isomorphism of $\mathbb{Z}_p[Q]$-modules

$$C^i(\pi^{\sharp,*}\mathcal{E}^0) \cong \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[Q], C^i(\mathcal{E}^0)) = \operatorname{Ind}^Q_{\{e\}} C^i(\mathcal{E}^0),$$

where $C^i(\pi^{\sharp,*}\mathcal{E}^0)$ and $C^i(\mathcal{E}^0)$ denote the i-th term of $C(\pi^{\sharp,*}\mathcal{E}^0)$ and $C(\mathcal{E}^0)$, respectively. (Indeed, each term of $Ru'_*(\pi^{\sharp,*}\mathcal{E}^0)$ is obtained by the pull back of the terms of $Ru_*\mathcal{E}^0$ as coherent sheaves, using the isomorphism $\tilde{\pi}^*\widehat{\omega}_{\mathfrak{X}^\sharp_F} \xrightarrow{\sim} \widehat{\omega}^1_{\mathfrak{X}^\sharp_{F'}}$ obtained in [21, Proposition 3.12].) Therefore, we have $C(\mathcal{E}^0) = C(\pi^{\sharp,*}\mathcal{E}^0)^Q$. (To see that the Čech differentials on both sides match, we note that the Čech resolution $C(\pi^{\sharp,*}\mathcal{E}^0)$ is constructed with respect to the pull back $\mathfrak{U}^\sharp_F \times_{\mathfrak{X}_F} \mathfrak{X}_{F'}$ of the Kummer-étale covering $\mathfrak{U}^\sharp_F$ of $\mathfrak{X}_F$, which was used for constructing the Čech resolution $C(\mathcal{E}^0)$.)

We will conclude by showing that for any subgroup J of Q the complex $C(\pi^{\sharp,*}\mathcal{E}^0)^J$ represents $R\Gamma_{\mathrm{crys}}(X^\sharp_{F'^J}/\mathbb{Z}_p, \pi^{\sharp,*}_J\mathcal{E}^0)$. Note that

$$\mathfrak{U}^\sharp_F \times_{\mathfrak{X}^\sharp_F} \mathfrak{X}^\sharp_{F'^J} = \mathfrak{U}^\sharp_F \times_{\mathfrak{X}^\sharp_F} (\mathfrak{X}^\sharp_{F'}/J) \cong \mathfrak{U}^\sharp_F \times (Q/J),$$

So it follows that $C(\pi^{\sharp,*}\mathcal{E}^0)^J$ is the total complex of the Čech resolution of $Ru_{F'^J,*}(\pi^{\sharp,*}_J\mathcal{E}^0)$ with respect to the Kummer-étale covering $\mathfrak{U}^\sharp_F \times_{\mathfrak{X}^\sharp_F} \mathfrak{X}^\sharp_{F'^J}$ of $\mathfrak{X}^\sharp_{F'^J}$, and so $C(\pi^{\sharp,*}\mathcal{E}^0)^J$ represents $R\Gamma_{\mathrm{crys}}(X^\sharp_{F'^J}/\mathbb{Z}_p, \pi^{\sharp,*}_J\mathcal{E}^0)$ by Proposition A.7.

This completes the proof of Proposition 7.1.

7.2. *The complex $I_{F'}$.* The following consequence of Proposition 7.1 regarding the complex $I_{F'}$ constructed in Section 6 will play an important role in the proof of Theorem 4.10.

Proposition 7.4. *If the extension F'/F is tamely ramified, then $I_{F'}$ lies in $D^{\mathrm{perf}}(\mathbb{Z}_p[Q])$ and is acyclic in all degrees outside* 0, 1 *and* 2.

Proof. Throughout this proof we use the notation introduced at the beginning of Section 6 with $K' = F$. By applying Proposition 7.1 to $\mathcal{E}^0 = D(-E_F)^{(0)}$ (so we have $\pi^{\sharp,*}\mathcal{E}^0 = D(-E)^{(0)}_F$ as $E = \pi^* E_F$), we obtain a complex of torsion-free induced $\mathbb{Z}_p[Q]$-modules $C_{F'}$ representing

$$R\Gamma_{\mathrm{crys}}(X^{\sharp}_{F'}/\mathbb{Z}_p, D(-E)^{(0)}_{F'}) \cong R\Gamma_{\mathrm{\acute{e}t}}(X_{F'}, Ru'_*(D(-E)^{(0)}_{F'}))$$

such that for any subgroup J of Q the complex $C^J_{F'}$ represents $R\Gamma_{\mathrm{crys}}(X^{\sharp}_F/\mathbb{Z}_p, D(-E_{F'^J})^{(0)}_{F'^J})$ where $E_{F'^J}$ is the pull back of E_F to $X_{F'^J}$. In particular, in each degree i there is an isomorphism of $\mathbb{Z}_p[Q]$-modules $C^i_{F'} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[Q], (C^i_{F'})^Q)$.

Since $(C^i_{F'})^J$ is $\mathbb{Z}_p$-flat in all degrees i, for any normal subgroup J of Q there is an isomorphism

$$I_{F'^J} \cong (C^J_{F'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*$$

in $D(\mathbb{Z}_p[Q/J])$, where the complexes on the right are defined by the termwise operations.

If we set $I^i_{F'} := (C^i_{F'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*$ and $I^i_F := ((C^i_{F'})^Q \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*$ for any i, then we have

$$I^i_{F'} \cong \mathbb{Z}_p[Q] \otimes_{\mathbb{Z}_p} I^i_F,$$

which is a flat $\mathbb{Z}_p[Q]$-module. Therefore for any subgroup J of Q the derived coinvariants $\mathbb{Z}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'}$ can be represented by the following complex defined by termwise operations:

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_p[J]} (C_{F'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^* \cong ((C_{F'})^J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*.$$

This implies, in particular, that $\mathbb{Z}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'}$ is isomorphic in $D(\mathbb{Z}_p[Q/J])$ to $I_{F'^J}$.

Thus, since each complex $I_{F'^J}$ is acyclic outside degrees 0, 1 and 2 and each cohomology group of $I_{F'}$ is finitely generated over $\mathbb{Z}_p$, a standard argument (as already used at the beginning of the proof of Proposition 3.7) implies that $I_{F'}$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}_p[Q])$, as claimed. □

8. Crystalline cohomology, semisimplicity and vanishing orders

As further preparation for the proof of Theorem 4.10, in this section we establish a link between the long exact cohomology sequences of the exact triangles constructed in Lemma 6.2 and the rational height pairing of Schneider and then use it to study the orders of vanishing of Hasse–Weil–Artin L-series.

Throughout we use the notation of Lemma 6.2. For convenience, we also set

$$Q_M := \mathrm{Gal}(F'/M)$$

and $Y_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y$ for each $\mathbb{Z}_p$-module Y.

8.1. ***Height pairings and semisimplicity.*** At the outset we recall that, by the general discussion given at the beginning of [22, §4.3], for each intermediate field M of L'/K the Dieudonné isocrystal $D(\mathcal{A}_M|_{U_M})\otimes\mathbb{Q}_p$ on $(U_{M/\mathbb{Z}_p})_{\mathrm{crys}}$ comes from an overconvergent F-isocrystal on U_M that we shall denote by

$$D_M^\dagger = D^\dagger(A_M).$$

We further recall that, by [22, §4.9 and Proposition 5.15], there are natural identifications

$$\mathbb{Q}_p\otimes_{\mathbb{Z}_p} I_{F'} = \mathbb{Q}_p\otimes_{\mathbb{Z}_p} P_{F'} = R\,\mathrm{Hom}_{\mathbb{Q}_p}\big(R\Gamma_{\mathrm{rig},c}(U_{F'}, D_{F'}^\dagger), \mathbb{Q}_p\big)[-2] \tag{42}$$

with respect to which the morphism $\mathbf{1}$ in the exact triangle (40) corresponds to the identity endomorphism on $R\Gamma_{\mathrm{rig},c}(U_{F'}, D_{F'}^\dagger)$.

Upon combining these identifications with the long exact cohomology sequence of the exact triangle (39) we obtain a composite homomorphism

$$\beta_{A,F',p} : \mathbb{Q}_p\otimes_{\mathbb{Z}} A^t(F') \xrightarrow{H^0(\theta')} H^1(P_{F'})_{\mathbb{Q}_p} = H^1(I_{F'})_{\mathbb{Q}_p} \xrightarrow{H^1(\theta)} \mathbb{Q}_p\otimes_{\mathbb{Z}}\mathrm{Hom}_{\mathbb{Z}}(A(F'),\mathbb{Z}). \tag{43}$$

We also write

$$h_{A,F',p,*} : \mathbb{Q}_p\otimes_{\mathbb{Z}} A^t(F') \to \mathbb{Q}_p\otimes_{\mathbb{Z}}\mathrm{Hom}_{\mathbb{Z}}(A(F'),\mathbb{Z})$$

for the isomorphism of $\mathbb{Q}_p[Q_M]$-modules that is induced by the algebraic height pairing $h_{A,F'}$ that occurs in Section 5.3.

Proposition 8.1. *If $Ш(A/F')$ is finite, then the following claims are valid.*

(i) *One has $\beta_{A,F',p} = (\pm1)^{a_{A,F',p}}\times h_{A,F',p,*}$ for a computable integer $a_{A,F',p}$ in $\{0,1\}$.*

(ii) *The homomorphisms $H^i(\hat\varphi)_{\mathbb{Q}_p}$ are bijective for all $i\neq 1$, where $\hat\varphi := 1-\varphi$.*

(iii) *The $\mathbb{Q}_p[Q_M]$-module $\ker(H^1(\hat\varphi))_{\mathbb{Q}_p}$ is naturally isomorphic to $\mathbb{Q}_p\otimes A^t(F')$.*

(iv) *The composite map $\ker(H^1(\hat\varphi))_{\mathbb{Q}_p}\subseteq H^1(P_{F'})_{\mathbb{Q}_p} = H^1(I_{F'})_{\mathbb{Q}_p}\to \mathrm{cok}(H^1(\hat\varphi))_{\mathbb{Q}_p}$ is bijective.*

Proof. Write $\mathfrak{C}$ for the quotient of the category of $\mathbb{Z}_p[Q_M]$-modules by the category of finite $\mathbb{Z}_p[Q_M]$-modules.

Then, since $Ш(A/F')$ is assumed to be finite, the (nondegenerate) pairing $h_{A,F'}$ induces an isomorphism in $\mathfrak{C}$ of the form

$$A(F')\otimes_{\mathbb{Z}}\mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Hom}_{\mathbb{Z}}(A^t(F'),\mathbb{Q}_p/\mathbb{Z}_p). \tag{44}$$

Next we set $C' := R\Gamma_{\mathrm{ar},V_{F'}}(U_{F'},\mathcal{A}_{\mathrm{tor}})$. Then, since the kernel of the homomorphism $H^1(C')\to \mathrm{Sel}_{\mathbb{Q}/\mathbb{Z}}(A_{F'})$ in Proposition 3.2 is finite the natural map $A(F')\otimes_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}\to\mathrm{Sel}_{\mathbb{Q}/\mathbb{Z}}(A_{F'})$ factors through a map $A(F')\otimes_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}\to H^1(C')$ in $\mathfrak{C}$. This homomorphism then gives rise to a composite homomorphism in $\mathfrak{C}$ of the form

$$\begin{aligned}A(F')\otimes_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z} &\to H^1(C')_p \to H^1(I_{F'}^*) \xrightarrow{\mathbf{1}} H^1(P_{F'}^*)\\ &\to H^2(C')_p \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Sel}_{\hat{\mathbb{Z}}}(A^t),\mathbb{Q}_p/\mathbb{Z}_p)\to\mathrm{Hom}_{\mathbb{Z}}(A^t(F'),\mathbb{Q}_p/\mathbb{Z}_p),\end{aligned} \tag{45}$$

where the second and fourth maps are induced by the exact triangle (39) and the fifth by Proposition 3.2.

To prove claim (i) it is sufficient, after taking Pontryagin duals, to show that the morphisms (44) and (45) in $\mathfrak{C}$ coincide up to a computable sign and this is precisely what is established by the argument of Kato and Trihan in [22, 3.3.6.2].

To prove the other claims we note that the long exact cohomology sequence of the exact triangle (39) combines with the descriptions in Proposition 3.7(ii) to imply that $H^i(\hat{\varphi})_{\mathbb{Q}_p}$ is bijective for all $i \notin \{0, 1\}$, that $\ker(H^0(\hat{\varphi}))_{\mathbb{Q}_p}$ and $\mathrm{cok}(H^2(\hat{\varphi}))_{\mathbb{Q}_p}$ vanish and that there are exact sequences of $\mathbb{Q}_p[Q_M]$-modules

$$\begin{aligned} &0 \to \mathrm{cok}(H^0(\hat{\varphi}))_{\mathbb{Q}_p} \xrightarrow{H^0(\theta)} \mathbb{Q}_p \otimes A^t(F') \xrightarrow{H^0(\theta')} \ker(H^1(\hat{\varphi}))_{\mathbb{Q}_p} \to 0, \\ &0 \to \mathrm{cok}(H^1(\hat{\varphi}))_{\mathbb{Q}_p} \xrightarrow{H^1(\theta)} \mathbb{Q}_p \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A^t(F'), \mathbb{Z}) \xrightarrow{H^1(\theta')} \ker(H^2(\hat{\varphi}))_{\mathbb{Q}_p} \to 0. \end{aligned} \tag{46}$$

Now, since $h^*_{A,F'}$ is bijective, claim (i) implies the same is true of the map $\beta_{A,F}$ and this fact combines with the above exact sequences to imply that the spaces $\mathrm{cok}(H^0(\hat{\varphi}))_{\mathbb{Q}_p}$ and $\ker(H^2(\hat{\varphi}))_{\mathbb{Q}_p}$ vanish, as required to complete the proof of claim (ii), and hence that the upper sequence in (46) gives an isomorphism of the sort required by claim (iii).

Finally, claim (iv) is true because the bijectivity of $\beta_{A,F'}$ combines with the upper sequence in (46) to imply $\ker(H^1(\hat{\varphi}))_{\mathbb{Q}_p}$ is disjoint from $\ker(H^1(\theta))_{\mathbb{Q}_p}$ whilst the lower sequence in (46) implies that $\ker(H^1(\theta))_{\mathbb{Q}_p}$ is equal to $\mathrm{im}(H^1(\hat{\varphi}))_{\mathbb{Q}_p}$. □

8.2. ***Orders of vanishing and leading terms.*** We now derive from Proposition 8.1 the following result about the order of vanishing $r_{A,M}(\chi)$ at $t = p^{-1}$ of the functions $Z_{U_M}(A_M, \chi, t)$ that are defined in Section 5.3 for each character χ in $\mathrm{Ir}(Q_M)$.

We fix (and do not in the sequel explicitly indicate) an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$ and hence do not distinguish between $\mathrm{Ir}(Q_M)$ and the set of irreducible $\mathbb{C}_p$-valued characters of Q_M.

In particular, for χ in $\mathrm{Ir}(Q_M)$ we may then fix a representation $Q_M \to \mathrm{Aut}_{\mathbb{C}_p}(V_\chi)$ (that we also denote by χ) of character χ, where V_χ is a finite dimensional vector space over $\mathbb{C}_p$.

If R denotes either $\mathbb{Z}_p[Q_M]$ or $\mathbb{Q}_p[Q_M]$, then for each finitely generated R-module W and each χ in $\mathrm{Ir}(Q_M)$ we define a $\mathbb{C}_p$-vector space by setting

$$W^\chi := \mathrm{Hom}_{\mathbb{C}_p[Q_M]}(V_\chi, \mathbb{C}_p[Q_M] \otimes_R W).$$

Theorem 8.2. *For each χ in $\mathrm{Ir}(Q_M)$ the following claims are valid.*

(i) $r_{A,M}(\chi) = \dim_{\mathbb{C}}((\mathbb{C} \otimes_{\mathbb{Z}} A^t(F'))^\chi) = \chi(1)^{-1} \cdot \dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} A^t(F')))$.

(ii) *In each degree i the homomorphism $H^i(\mathbf{1} - \varphi)$ induces an automorphism $H^i(\mathbf{1} - \varphi)^\diamond_\chi$ of any fixed complement to* $\ker(H^i(\mathbf{1} - \varphi))^\chi$ *in* $H^1(P_{F'})^\chi$.

(iii) $Z^*_{U_M}(A_M, \chi, p^{-1}) = \prod_{i=0}^{i=2} \det(H^i(\mathbf{1} - \varphi)^\diamond_\chi)^{(-1)^{i+1}}$, *where the leading term is normalised as in* (27).

Proof. We fix a finite Galois extension Λ of $\mathbb{Q}_p$ such that for any χ in $\mathrm{Ir}(Q_M)$ the $\mathbb{C}_p[Q_M]$-module V_χ descends to a $\Lambda[Q_M]$-module $V_{\chi,\Lambda}$. We write k_Λ for the residue field of Λ and set $q := \#(k_\Lambda)$.

Then, for each χ in $\mathrm{Ir}(Q_M)$, we fix a $\Lambda[Q_M]$-module $V_{\chi,\Lambda}$ such that $\mathbb{C}_p \otimes_\Lambda V_{\chi,\Lambda} \cong V_\chi$ and, for any $\mathbb{Q}_p[Q_M]$-module W, we set

$$W_\Lambda^\chi := \mathrm{Hom}_{\Lambda[Q_M]}(V_{\chi,\Lambda}, \Lambda \otimes_{\mathbb{Q}_p} W).$$

We now give an alternative description of $H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F'})^\chi_\Lambda$ and $H^i(\mathbf{1}-\varphi)^\diamond_{\chi,\Lambda}$ in terms of the rigid cohomology of *overconvergent* Λ-F-*isocrystal*; compare [38, (7.1)]. (In fact, we will work with overconvergent Λ_0-F-isocrystal for some suitable subfield Λ_0 of Λ.) We recall that an Λ-F-isocrystal is, roughly speaking, an isocrystal with scalars in Λ (instead of $\mathbb{Q}_p$) equipped with Λ-linear q-Frobenius operator (denoted by $\varphi^{(\Lambda)}$). In particular, given an overconvergent F-isocrystal $(\mathcal{E}, \varphi)$ over $U_{F'}$, one can "extend scalars" to obtain an overconvergent Λ-F-isocrystal $\mathcal{E}_\Lambda$ in the following way: if F' contains k_Λ and we set $r := [k_\Lambda : \mathbb{F}_p]$, then $(\mathcal{E}, \varphi^r)$ is an overconvergent $\mathbb{Q}_q$-F-isocrystal and so one can set $(\mathcal{E}_\Lambda, \varphi^{(\Lambda)}) := (\mathcal{E} \otimes_{\mathbb{Q}_q} \Lambda, \varphi^r \otimes \Lambda)$. In addition, there is the following base change result (cf. [8, Theorem 11.8.1]): for any overconvergent isocrystal $\mathcal{E}$ over $U_{F'}$, there is in each degree i a natural isomorphism

$$H^i_{\mathrm{rig},c}(U_{F'}, \mathcal{E}) \otimes_{\mathbb{Q}_q} \Lambda \cong H^i_{\mathrm{rig},c}(U_{F'}, \mathcal{E}_\Lambda),$$

and similarly for the rigid cohomology without support condition.

Thus, if we are to construct overconvergent Λ-F-isocrystals, we should assume that the base field contains k_Λ. To do this, we shall, if necessary, replace F' by $F'' := F' \cdot \mathbb{F}_{p^r}$. Then F''/M is a Galois extension and, setting $Q''_M := \mathrm{Gal}(F''/M)$, we regard $\mathrm{Ir}(Q_M)$ as a subset of $\mathrm{Ir}(Q''_M)$ in the natural way. Now, if W'' is a finitely generated module over either $\mathbb{Z}_p[Q''_M]$ or $\mathbb{Q}_p[Q''_M]$ then $(W'')^\chi = W^\chi$ with $W := (W'')^{\mathrm{Gal}(F''/F')}$. Hence, to prove the claimed result, we can assume without loss of generality that $k_\Lambda \subset F'$.

In this case, $H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F'})$ is a Λ_0-vector space and there is a natural isomorphism

$$H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F'}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_q \cong \prod_{\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)} H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F'}),$$

with respect to which the endomorphism $\varphi \otimes \varphi$ of the left-hand side corresponds to the following block matrix on the right-hand side:

$$\begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mathbf{1} \\ \varphi^r & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

One therefore obtains Q_M-equivariant isomorphisms

$$\prod_{\lambda'^r = \lambda^r} H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F'})^{(\lambda')} \otimes_{\mathbb{Q}_p} \mathbb{Q}_q \cong H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F',\mathbb{Q}_q})^{(\lambda^r)},$$

where the left-hand side is the product of generalised λ'-eigenspace for φ and the right-hand side is the generalised λ^r-eigenspace for $\varphi^{(\mathbb{Q}_q)} := \varphi^r$. (Note that the underlying isocrystal for $D^\dagger_{F',\mathbb{Q}_q}$ is $D^\dagger_{F'}$, equipped with the $\mathbb{Q}_q$-linear q-Frobenius φ^r.)

Extending scalars from $\mathbb{Q}_q$ to Λ, we now obtain an isomorphism

$$\prod_{\lambda'^r=\lambda^r} H^i_{\mathrm{rig},c}(U_{F'}, D^\dagger_{F'})^{(\lambda')} \otimes_{\mathbb{Q}_p} \Lambda \cong H^i_{\mathrm{rig},c}(U_{F'}, (D^\dagger_{F'})_\Lambda)^{(\lambda^r)},$$

where the right-hand side is the generalised λ^r-eigenspace for $\varphi^{(\Lambda)} := \varphi^r \otimes \Lambda$, and so

$$\det_{\mathbb{Q}_p}(H^i(\mathbf{1} - t\cdot\varphi)) = \det_\Lambda(H^i(\mathbf{1} - t^{[k_\Lambda:\mathbb{F}_p]}\cdot\varphi^{(\Lambda)})).$$

Now, by the main theorem of Tsuzuki [38, Theorem 7.2.3], there exists an overconvergent unit-root F-isocrystal $\mathcal{O}^\dagger(\chi)$ over U_M with monodromy given by $V_{\chi,\Lambda}$, viewed as a $\mathbb{Q}_p[Q_M]$-module. Furthermore, $\mathcal{O}^\dagger(\chi)$ has a natural action of Λ commuting with the p-Frobenius operator φ and the connection; that is, $\mathcal{O}^\dagger(\chi)$ is a $\mathbb{Q}_p$-F-isocrystal with Λ-action in the sense of Definition B.3 for $\Lambda_0 = \mathbb{Q}_p$. (Indeed, the Λ-action on the level of *convergent* Λ_0-F-isocrystal is clear by construction since the Λ-action on $V_{\chi,\Lambda}$ commutes with the $\mathbb{Q}_p[Q_M]$-action, and the Λ-action extends by the full faithfulness result [29, 5.1.1].)

We then obtain another $\mathbb{Q}_p$-F-isocrystal $D^\dagger_M(\chi) := \mathcal{O}^\dagger(\chi) \otimes_{\mathbb{Q}_p} D^\dagger_M$ with Λ-action and, in each degree i, we regard

$$H^i_M(\chi) := H^i_{\mathrm{rig},c}(U_M, D^\dagger_M(\chi))$$

as a Λ-vector space equipped with Λ-linear p-Frobenius operator φ. We claim that there is an identity of functions

$$Z_{U_M}(A_M, \chi, pt) = \prod_{i=0}^{i=2} \det_\Lambda(1 - pt\cdot\varphi \mid H^i_M(\chi))^{(-1)^{i+1}}. \tag{47}$$

Indeed, this identity is a standard consequence of Lefschetz trace formula for rigid cohomology of $\mathbb{Q}_p$-F-isocrystals with Λ-action; cf. Theorem B.9. (Its proof is a straightforward adaptation of the Lefschetz trace formula for Λ-F-isocrystals in [15, théorème 6.3]. In fact, in the special case that M contains k_Λ, one can directly construct a Λ-F-isocrystal on U_M that computes $Z_{U_M}(A_M, \chi, pt)$ via the more classical Lefschetz trace formula in loc. cit.)

Now, from Proposition 8.1(ii) we know that, for both $i = 0$ and $i = 2$, the endomorphism $H^i(1-\varphi)$ is invertible on the $\mathbb{Q}_p$-linear dual $H^i_M(\chi)^\vee$ of $H^i_M(\chi)$. From (47), we can therefore deduce that

$$\begin{aligned} r_{A,M}(\chi) &= \dim_\Lambda\big(\ker(1-\varphi \mid H^1_M(\chi))\big) \\ &= \dim_\Lambda\big(\ker(1-\varphi \mid H^1_M(\chi)^\vee)\big) \\ &= \dim_\Lambda\big(\ker(1-\varphi \mid (H^1_{\mathrm{rig},c}(U_{F'}, (D^\dagger_{F'})_\Lambda)^\vee)^\chi)\big) \\ &= \dim_\Lambda\big(\ker(1-\varphi \mid H^1(\Lambda \otimes_{\mathbb{Z}_p} P_{F'})^\chi)\big) \\ &= \dim_\mathbb{C}(A^t(F')^\chi). \end{aligned} \tag{48}$$

Here the second equality is clear, the third follows from the isomorphism in Lemma 8.3 below, the fourth from (42) and the first and fifth from Proposition 8.1(iii) and (iv). This proves claim (i).

Claim (ii) follows directly from Proposition 8.1(iv) and the fact (already noted above) that $H^i(1-\varphi)$ is invertible on $H^i_M(\chi)^\vee$ for $i=0$ and $i=2$.

Next, the equality (48) implies that

$$\det\bigl(1-pt\cdot\varphi \mid \ker(1-\varphi \mid H^1(P_{F'})^\chi)\bigr) = (1-pt)^{r_{A,M}(\chi)}.$$

From this and our chosen normalisation of leading terms, the formula in claim (iii) follows directly upon combining claim (ii) with the identity (47). ☐

Lemma 8.3. *For every absolutely irreducible representation $\chi : Q_M \to \mathrm{Aut}_\Lambda(V_{\chi,\Lambda})$ as above, and every degree i, there is a natural Λ-linear, Frobenius equivariant isomorphism*

$$H^i_{\mathrm{rig},c}(U_M, D^\dagger_M(\chi))^\vee \cong (H^i_{\mathrm{rig},c}(U_{F'}, (D^\dagger_{F'})_\Lambda)^\vee)^\chi.$$

Proof. All isomorphisms in the proof below can be checked to be Frobenius equivariant. Poincaré duality identifies the Λ-modules $H^i_{\mathrm{rig},c}(U_M, D^\dagger_M(\chi))^\vee$ and $H^i_{\mathrm{rig},c}(U_{F'}, (D^\dagger_{F'})_\Lambda)^\vee$ with $H^{2-i}_{\mathrm{rig}}(U_M, D^\dagger_M(\chi)^\vee)$ and $H^{2-i}_{\mathrm{rig}}(U_{F'}, (D^{\dagger,\vee}_{F'})_\Lambda)$ respectively, where $D^\dagger_M(\chi)^\vee$ and $(D^{\dagger,\vee}_{F'})_\Lambda$ denote the dual as an overconvergent F-isocrystal and an overconvergent Λ-F-isocrystal respectively. It therefore suffices to prove there exists a natural isomorphism

$$H^i_{\mathrm{rig}}(U_{F'}, (D^{\dagger,\vee}_{F'})_\Lambda)^\chi \cong H^i_{\mathrm{rig}}(U_M, D^\dagger_M(\chi)^\vee). \tag{49}$$

To show this we use the canonical isomorphism $\pi^*_{F'/M}(D^\dagger_M) \cong D^\dagger_{F'}$, where $\pi_{F'/M}$ denotes the natural morphism $X_{F'} \to X_M$. We also note that the proof of [10, Proposition 1.3] implies the overconvergent vector bundle $D^\dagger_{F'}$ has a natural Q_M-action that commutes with its natural Frobenius operator. (To see this, note that the natural Q_M-action and the Frobenius commute on the log Dieudonné crystal $D^{\log}_{F'}$, and so the same must be true on the associated *convergent* isocrystal. Then one need only note that, by [29, 5.1.1], the category of overconvergent F-isocrystals on $U_{F'}$ is naturally a *full subcategory* of the category of convergent F-isocrystals on $U_{F'}$.)

Now, by construction of $D^\dagger_M(\chi)$, there is a natural isomorphism of Q_M-equivariant overconvergent F-isocrystals $\pi^*_{F'/M}(D^\dagger_M(\chi)^\vee) \cong V_{\check\chi,\Lambda} \otimes_{\mathbb{Q}_p} D^{\dagger,\vee}_{F'}$, where $V_{\check\chi,\Lambda}$ is viewed as a $\mathbb{Q}_p[Q_M]$-module and Q_M acts diagonally on the tensor product. This isomorphism also respects the natural Q_M- and p-Frobenius equivariant Λ-actions on both sides. Hence, since the underlying overconvergent isocrystal for $V_{\check\chi,\Lambda} \otimes_{\mathbb{Q}_p} D^{\dagger,\vee}_{F'}$ coincides with that of $V_{\check\chi,\Lambda} \otimes_\Lambda (D^{\dagger,\vee}_{F'})_\Lambda$, one obtains the required isomorphism (49) via the induced composite isomorphisms

$$\begin{aligned} H^i_{\mathrm{rig}}(U_M, D^\dagger_M(\chi)^\vee) &\xrightarrow{\sim} H^i_{\mathrm{rig}}(U_{F'}, \pi^*_{F'/M}(D^\dagger_M(\chi)^\vee))^{Q_M} \\ &\xrightarrow{\sim} H^i_{\mathrm{rig}}(U_{F'}, V_{\check\chi,\Lambda} \otimes_\Lambda (D^{\dagger,\vee}_{F'})_\Lambda)^{Q_M} \\ &\xrightarrow{\sim} H^i_{\mathrm{rig}}(U_{F'}, (D^{\dagger,\vee}_{F'})_\Lambda)^\chi. \end{aligned}$$

Here the first map is induced by Shapiro's lemma; its bijectivity is proved in [35, proposition 4.6]. The change from $\check\chi$ to χ that occurs in the third isomorphism is for the reason outlined in Remark 2.3. ☐

9. Proof of the main result

In this section we use results from earlier sections to obtain a proof of Theorem 4.10. At the outset we note that Theorem 4.10(i) is proved by Theorem 8.2(i) and that Remark 4.11 allows us to assume that $Ш(A/L)$ is finite. We therefore focus on establishing the validity of the equality in Conjecture 4.3(iii).

For convenience, for each Galois extension F'/M (as in Proposition 6.3) we define an element of $K_0(\mathbb{Z}[Q_M], \mathbb{Q}[Q_M])$ by setting

$$\chi(A, F'/M) := \partial_{Q_M}(Z^*_{U_M}((A_M)_{F'/M}, p^{-1})) - \chi^{\mathrm{BSD}}_{Q_M,\mathbb{Q}}(A_M, V_{F'}) + \chi^{\mathrm{coh}}_{Q_M}(A_M, V_{F'}) - \chi^{\mathrm{sgn}}_{Q_M}(A_M),$$

where, we recall, the leading term element is normalised via (27).

9.1. ***A first reduction step.*** For a finite group Γ, a prime number ℓ and an element x of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ we write x_ℓ for the image of x in $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ under the canonical decomposition (8).

Proposition 9.1. *Assume $Ш(A/L)$ is finite. Then the statement of Theorem 4.10 is valid if and only if the following conditions are satisfied.*

(i) *If $\mathfrak{M}_p$ is any given maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ that contains $\mathbb{Z}_p[G]$, then $\chi(A, L/K)_p$ belongs to the kernel of the homomorphism $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G]) \to K_0(\mathfrak{M}_p, \mathbb{Q}_p[G])$.*

(ii) *Assume that the set $\Sigma_1 \cup \Sigma_2$ in* (13) *is nonempty. Fix a field $K' \in \Sigma_1 \cup \Sigma_2$, set $L' = LK'$ and write P' for the normal subgroup of $H' := \mathrm{Gal}(L'/K')$ that is generated by the Sylow p-subgroups of the inertia groups of all places that ramify in L'/K'. Then $\chi(A, (L')^{P'}/K')_p$ vanishes.*

(iii) *For each prime $\ell \neq p$ one has*

$$\partial_{G,\mathbb{Q}}(Z^*_U(A_{L/K}, p^{-1}))_\ell = \chi^{\mathrm{BSD}}_{G,\mathbb{Q}}(A, V_L)_\ell - \chi^{\mathrm{sgn}}_G(A)_\ell.$$

Proof. It suffices to check that the stated conditions are equivalent to the validity of the equality in Conjecture 4.3(iii).

Thus, after taking account of Proposition 5.6, the decomposition (8) combines with the explicit definition of the subgroup $\mathcal{T}_{A,L/K}$ to reduce us to showing that the stated conditions imply the validity of each of the following assertions:

(C_1) $\chi(A, L/K)_p$ has finite order;

(C_2) for every field K' that belongs to either Σ_1 or Σ_2, $\chi(A, L/K)_p$ is the image under $\pi^{G'}_G$ of an element of $K_0(\mathbb{Z}_p[G'], \mathbb{Q}_p[G'])$ that belongs to $\ker(\pi^{G'}_{H'/P'})$;

(C_3) $\chi(A, L/K)_\ell$ vanishes if $\ell \neq p$.

To check this, we first recall (from [5, §4.5, Lemma 11(d)]) that $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])_{\mathrm{tor}}$ is equal to the kernel of the scalar extension homomorphism $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G]) \to K_0(\mathfrak{M}_p, \mathbb{Q}_p[G])$. Given this fact, condition (i) directly implies that $\chi(A, L/K)_p$ has finite order, and hence verifies (C_1).

Next, we note that, for any $K' \in \Sigma_1 \cup \Sigma_2$, the result of Proposition 9.2 below implies (in terms of the notation of condition (ii)) that one has both $\chi(A, L/K)_p = \pi^{G'}_G(\chi(A, L'/K)_p)$ and $\pi^{G'}_{H'/P'}(\chi(A, L'/K)_p) =$

$\chi(A, (L')^{P'}/K')_p$. In particular, in this case, condition (ii) implies $\chi(A, L'/K)_p$ belongs to $\ker(\pi^{G'}_{H'/P'})$, as required to verify (C_2).

Finally, to verify (C_3) we note that if $\ell \neq p$, then $\chi^{\mathrm{coh}}_G(A, V_L)_\ell$ vanishes. Thus, in this case, the vanishing of the image in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ of the equality in Conjecture 4.3(iii) is clearly equivalent to the equality stated in condition (iii). □

Before stating the next result we note that if J is a normal subgroup of a subgroup H of G, and we set $Q := H/J$, then there is a natural commutative diagram

$$\begin{tikzcd} K_1(\mathbb{Q}[G]) \arrow[r, "\theta^1_{G,H}"] \arrow[d, "\partial_{G,\mathbb{Q}}"'] & K_1(\mathbb{Q}[H]) \arrow[r, "\theta^1_{H,Q}"] \arrow[d, "\partial_{H,\mathbb{Q}}"] & K_1(\mathbb{Q}[Q]) \arrow[d, "\partial_{Q,\mathbb{Q}}"] \\ K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \arrow[r, "\theta^0_{G,H}"] & K_0(\mathbb{Z}[H], \mathbb{Q}[H]) \arrow[r, "\theta^0_{H,Q}"] & K_0(\mathbb{Z}[Q], \mathbb{Q}[Q]) \end{tikzcd} \tag{50}$$

where $\theta^i_{G,H}$ and $\theta^i_{H,Q}$ are the natural restriction and coinflation homomorphisms.

Proposition 9.2. *If J is a normal subgroup of a subgroup H of G, with $Q = H/J$, then the composite homomorphism $\theta^0_{H,Q} \circ \theta^0_{G,H}$ sends $\chi(A, L/K)_p$ to $\chi(A_{L^H}, L^J/L^H)_p$.*

Proof. We set $\theta^i_{G,Q} := \theta^i_{H,Q} \circ \theta^i_{G,H}$, $E := L^H$ and $F := L^J$.

At the outset we note that, by a standard argument using the Artin formalism of L-functions, one finds that $\theta^1_{G,Q}(Z^*_U(A_{L/K}, p^{-1})) = Z^*_{U_E}((A_E)_{F/E}, p^{-1})$ and so the commutative diagram (50) implies

$$\theta^0_{G,Q}(\partial_{G,\mathbb{Q}}(Z^*_U(A_{L/K}, p^{-1}))) = \partial_{Q,\mathbb{Q}}(Z^*_{U_E}((A_E)_{F/E}, p^{-1})). \tag{51}$$

It is also clear that $\theta^1_{G,Q}(\langle \mathbb{Q} \cdot A^t(L), -1\rangle) = \langle \mathbb{Q} \cdot A^t(F), -1\rangle$ and, given this, an explicit comparison of the equalities in Proposition 8.1(i) with F' equal to L and F implies

$$\theta^0_{G,Q}(\chi^{\mathrm{sgn}}_G(A)_p) = \chi^{\mathrm{sgn}}_Q(A_E)_p. \tag{52}$$

To proceed we write π, π' and π'' for the natural morphisms $X_L \to X$, $X_L \to X_E$ and $X_E \to X$. We fix families of subgroups V_L and W_L for the extension L/K as in Section 3.4 (the choice of which is, following Proposition 5.3, unimportant) and write $\mathcal{L}_L$ for the associated coherent $\mathcal{O}_X[G]$-submodule of $\pi_*\mathrm{Lie}(\mathcal{A}_{X_L})$. In the same way we fix families of subgroups V'_L and W'_L for the extension L/E and write $\mathcal{L}'_L$ for the associated coherent $\mathcal{O}_{X_E}[H]$-submodule of $\pi'_*\mathrm{Lie}(\mathcal{A}_{X_L})$.

We assume, as we may, that $V_L \subseteq V'_L$, and hence also $W_L \subseteq W'_L$. This implies that there are exact triangles in $D^{\mathrm{perf}}(\mathbb{Z}[H])$ of the form

$$\mathrm{SC}_{V'_L}(A, L/E) \to \mathrm{SC}_{V_L}(A_E, L/K) \to (V'_L/V_L)^*[-1] \to$$

and

$$R\Gamma(X_E, \mathcal{L}'_L)^* \to R\Gamma(X, \mathcal{L}_L)^* \to (W'_L/W_L)^*[1] \to ,$$

where, in the latter case, we have used the fact that the complexes $R\Gamma(X, \pi''_*\mathcal{L}'_L)$ and $R\Gamma(X_E, \mathcal{L}'_L)$ are canonically isomorphic since π''_* is exact. These triangles in turn give rise to equalities in $K_0(\mathbb{Z}[H], \mathbb{Q}[H])$,

$$\begin{aligned}\theta^0_{G,H}(\chi^{\mathrm{BSD}}_{G,\mathbb{Q}}(A, V_L) - \chi^{\mathrm{coh}}_G(A, V_L)) &= \chi^{\mathrm{BSD}}_{H,\mathbb{Q}}(A, V'_L) + \chi_{\mathbb{Z}_p[H]}((V'_L/V_L)^*[-1], 0)\\ &\quad - \big(\chi^{\mathrm{coh}}_H(A, V'_L) + \chi_{\mathbb{Z}_p[H]}((W'_L/W_L)^*[-1], 0)\big)\\ &= \chi^{\mathrm{BSD}}_{H,\mathbb{Q}}(A, V'_L) - \chi^{\mathrm{coh}}_H(A, V'_L),\end{aligned} \tag{53}$$

where the last equality is valid since $\chi_{\mathbb{Z}_p[H]}((V'_L/V_L)^*[-1], 0) = \chi_{\mathbb{Z}_p[H]}((W'_L/W_L)^*[-1], 0)$ (by the same argument as used in the proof of Proposition 5.3).

Upon combining the equalities (51), (52) and (53) one finds that the proof is reduced to showing that there are equalities

$$\begin{aligned}\theta^0_{H,Q}(\chi^{\mathrm{BSD}}_{H,\mathbb{Q}}(A_E, V'_L)) &= \chi^{\mathrm{BSD}}_{Q,\mathbb{Q}}(A_E, (V'_L)^J),\\ \theta^0_{H,Q}(\chi^{\mathrm{coh}}_H(A_E, V'_L)) &= \chi^{\mathrm{coh}}_Q(A_E, (V'_L)^J).\end{aligned}$$

These equalities follow directly from the isomorphisms in $D^{\mathrm{perf}}(\mathbb{Z}[Q])$

$$\begin{aligned}\mathbb{Z}[Q] \otimes^{\mathbb{L}}_{\mathbb{Z}[H]} \mathrm{SC}_{V'_L}(A_E, L/E) &\cong \mathrm{SC}_{(V'_L)^J}(A_E, F/E),\\ \mathbb{Z}[Q] \otimes^{\mathbb{L}}_{\mathbb{Z}[H]} R\Gamma(X_E, \mathcal{L}'_L)^* &\cong R\Gamma(X_E, (\mathcal{L}'_L)^J)^*,\end{aligned} \tag{54}$$

which are respectively used in the proofs of Proposition 3.7 and Lemma 3.9. □

9.2. *The case $\ell = p$.* In this section we verify that the conditions (i) and (ii) in Proposition 9.1 are satisfied.

The key observation we shall use in this regard is provided by the following result. In this result we use the notation and hypotheses of Proposition 9.1(ii).

Lemma 9.3. *Fix a field K' in $\Sigma_1 \cup \Sigma_2$ (so that, by assumption, Ш(A/L') is finite) and a Galois extension of fields M_2/M_1 with $K \subseteq M_1 \subseteq M_2 \subseteq L'$. Set $J := \mathrm{Gal}(M_2K'/M_1)$ and $Q := \mathrm{Gal}(M_2/M_1)$. Also fix a $\mathbb{Z}_p$-order $\mathfrak{N}$ in $\mathbb{Q}_p[J]$ as in Proposition 6.3 with $F' = M_2K'$ and $M = M_1$, and write $\overline{\mathfrak{N}}$ for the image of $\mathfrak{N}$ in $\mathbb{Q}_p[Q]$.*

Then $\chi(A, M_2/M_1)_p$ belongs to the kernel of the natural scalar extension homomorphism

$$K_0(\mathbb{Z}_p[Q], \mathbb{Q}_p[Q]) \to K_0(\overline{\mathfrak{N}}, \mathbb{Q}_p[Q]).$$

Proof. Under the present hypotheses, the exact triangles in Proposition 6.3 lie in $D^{\mathrm{perf}}(\mathfrak{N})$. Hence, after taking account of the relevant cases of the isomorphisms (54), the exact functor $\Delta(-) := \overline{\mathfrak{N}} \otimes^{\mathbb{L}}_{\mathfrak{N}} -$ takes these triangles to exact triangles in $D^{\mathrm{perf}}(\overline{\mathfrak{N}})$ of the form

$$\Delta\big(\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} P_{F'})\big) \xrightarrow{1-\varphi} \Delta\big(\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'})\big) \to \overline{\mathfrak{N}} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[Q]} R\Gamma_{\mathrm{ar}, V_{M_2}}(U_{M_2}, \mathcal{A}_{\mathrm{tor}})^*_p[-2] \to ,$$

$$\Delta\big(\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} P_{F'})\big) \xrightarrow{1} \Delta\big(\tau_{\geq -1}(\mathfrak{N} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[J]} I_{F'})\big) \to \overline{\mathfrak{N}} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[Q]} R\Gamma(X_{M_1}, (\mathcal{L}')^H)^*[-2] \to ,$$

with $H := \mathrm{Gal}(M_2K'/M_2)$. These triangles satisfy conditions (a), (b) and (c) of Proposition 5.1: in fact, the only condition that is not straightforward to check in this case is (b) and this follows from the results of Proposition 8.1(ii) and (iv).

By applying Proposition 5.1 in this context, and taking account of the equality in Proposition 8.1(i), one finds that the image of $\chi(A, M_2/M_1)_p$ in $K_0(\overline{\mathfrak{N}}, \mathbb{Q}_p[Q])$ is equal to the image under the natural connecting homomorphism $K_1(\mathbb{Q}_p[Q]) \to K_0(\overline{\mathfrak{N}}, \mathbb{Q}_p[Q])$ of the product element

$$Z^*_{U_{M_1}}((A_{M_1})_{M_2/M_1}, p^{-1}) \cdot \prod_{i=0}^{i=2} (H^i(\mathbf{1} - \varphi_{M_2/M_1})^{\diamond}_{\mathbb{Q}_p})^{(-1)^i} \in K_1(\mathbb{Q}_p[Q]). \tag{55}$$

Here we write $\mathbf{1} - \varphi_{M_2/M_1}$ for the morphism denoted by $\mathbf{1} - \varphi$ in the first of the exact triangles displayed above, and identify each automorphism $H^i(\mathbf{1} - \varphi_{M_2/M_1})^{\diamond}_{\mathbb{Q}_p}$ with the induced element of $K_1(\mathbb{Q}_p[Q])$.

It is thus enough to prove that the element (55) vanishes, or equivalently, that its image under the (injective) map $\mathrm{Nrd}_{\mathbb{Q}_p[Q]} : K_1(\mathbb{Q}_p[Q]) \to \zeta(\mathbb{Q}_p[Q])^{\times}$ is trivial. In addition, given the characterisation of $Z^*_{U_{M_1}}((A_{M_1})_{M_2/M_1}, p^{-1})$ in Proposition 5.6(i), the required triviality is deduced directly from the formula of Theorem 8.2(iii) (with F'/M replaced by M_2/M_1) for every $\chi \in \mathrm{Ir}(Q)$ and the fact that, in terms of the notation of the corresponding case of Theorem 8.2(ii), for every $i \in \{0, 1, 2\}$ and $\chi \in \mathrm{Ir}(Q)$ one has

$$\mathrm{Nrd}_{\mathbb{Q}_p[Q]}(H^i(\mathbf{1} - \varphi_{M_2/M_1})^{\diamond}_{\mathbb{Q}_p})_{\chi} = \det(H^i(\mathbf{1} - \varphi_{M_2/M_1})^{\diamond}_{\chi}). \qquad \square$$

Turning now to the conditions in Proposition 9.1, we first fix a maximal $\mathbb{Z}_p$-order $\mathfrak{N}$ in $\mathbb{Q}_p[G']$ that contains $\mathbb{Z}_p[G']$. Then $\mathfrak{N}$ is regular and so satisfies the conditions of Proposition 6.3 with $F' = L'$ and $M = K$ (so $J = G'$). From Lemma 9.3 (with $M_2 = L$ and $M_1 = K$, so $Q = G$), it therefore follows that $\chi(A, L/K)_p$ belongs to the kernel of the scalar extension $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G]) \to K_0(\mathfrak{M}, \mathbb{Q}_p[G])$, where $\mathfrak{M}$ denotes the image of $\mathfrak{N}$ in $\mathbb{Q}_p[G]$. In particular, since $\mathfrak{M}$ is a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ that contains $\mathbb{Z}_p[G]$, this shows that the condition of Proposition 9.1(i) is satisfied.

Next we consider condition (ii) of Proposition 9.1. To do this we note that, by our assumption on K', the group $Ш(A/L')$ is finite. In addition, the field $F' := (L')^{P'}$ is a tamely ramified Galois extension of K' and so Proposition 7.4 implies that the conditions of Proposition 6.3 are satisfied by the data $J = \mathrm{Gal}(F'/K')$ and $\mathfrak{N} = \mathbb{Z}_p[J]$. In this case, therefore, Lemma 9.3 implies that $\chi(A, (F')^{P'}/K')_p$ vanishes, and hence that condition (ii) of Proposition 9.1 is satisfied.

Remark 9.4. A close reading of the above argument shows that we actually prove a (possibly) finer version of Theorem 4.10(ii). Specifically, the validity of the equality in Conjecture 4.3(iii) is proved modulo the subgroup of $\mathcal{T}_{A,L/K}$ that is obtained by replacing the group $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])_{\mathrm{tor}}$ in the intersection (13) by its subgroup

$$\ker\big(K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])_{\mathrm{tor}} \xrightarrow{(\lambda_{\mathfrak{N}})_{\mathfrak{N}}} \bigoplus_{\mathfrak{N}} K_0(\overline{\mathfrak{N}}, \mathbb{Q}_p[G])\big).$$

Here in the intersection $\mathfrak{N}$ runs over all $\mathbb{Z}_p$-orders of $\mathbb{Q}_p[G']$ that contain $\mathbb{Z}_p[G']$ and satisfy the hypotheses of Proposition 6.3 (with $F' = L'$ and $M = K$), $\overline{\mathfrak{N}}$ is the image of $\mathfrak{N}$ in $\mathbb{Q}_p[G]$ and each $\lambda_{\mathfrak{N}}$ is the scalar extension map that arises from the inclusion $\mathbb{Z}_p[G] \subseteq \overline{\mathfrak{N}}$. We recall that the hypotheses of Proposition 6.3

are automatically satisfied if the order $\mathfrak{N}$ is hereditary but that, aside from this, finding other interesting, and explicit, examples of such orders (beyond those that are used in the above argument) seems difficult.

9.3. ***The case $\ell \neq p$.*** In this section we verify that condition (iii) in Proposition 9.1 is satisfied, and thereby complete the proof of Theorem 4.10.

Fix a prime $\ell \neq p$, write $T_\ell(\mathcal{A})$ for the ℓ-adic Tate module of $\mathcal{A}$ and set $V_\ell(\mathcal{A}) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(\mathcal{A})$. Write $\mathbb{F}_p^c$ for its algebraic closure of $\mathbb{F}_p$ and φ_p for the Frobenius automorphism at p, and set $U_L^c := U_L \times_{\mathbb{F}_p} \mathbb{F}_p^c$.

For each $\chi \in \mathrm{Ir}(G)$ we fix an associated representation space V_χ over $\mathbb{C}_\ell$. For each finitely generated $\mathbb{Q}_\ell[G]$-module W, we set

$$W^\chi := \mathrm{Hom}_{\mathbb{Q}_\ell[G]}(V_\chi, \mathbb{C}_\ell[G] \otimes_{\mathbb{Q}_\ell[G]} W).$$

Then by repeating the proof of Lemma 8.3 for ℓ-adic cohomology in place of rigid cohomology, we obtain isomorphisms

$$H^i_{\text{ét},c}(U^c, V_\chi \otimes_{\mathbb{Q}_\ell} V_\ell(\mathcal{A}))^\vee \cong \big(H^i_{\text{ét},c}(U_L^c, V_\ell(\mathcal{A}))^\vee\big)^\chi \cong H^i_{\text{ét}}(U_L^c, V_\ell(\mathcal{A}^t))^\chi,$$

where the second isomorphism is induced by the Poincaré duality theorem (as stated, for example, in [27, Chapter VI, Corollary 11.2]). Therefore the identity (1) implies that

$$\begin{aligned} Z_U(A, \chi, p^{-1}t) &= \prod_{i\in\mathbb{Z}} \det\big(1 - \varphi_p \cdot t : H^i_{\text{ét},c}(U^c, V_\chi \otimes_{\mathbb{Q}_\ell} V_\ell(\mathcal{A}))\big)^{(-1)^{i+1}} \\ &= \prod_{i\in\mathbb{Z}} \det\big(1 - \varphi_p \cdot t : H^i_{\text{ét},c}(U^c, V_\chi \otimes_{\mathbb{Q}_\ell} V_\ell(\mathcal{A}))^\vee\big)^{(-1)^{i+1}} \\ &= \prod_{i\in\mathbb{Z}} \det\big(1 - \varphi_p \cdot t : H^i_{\text{ét}}(U_L^c, V_\ell(\mathcal{A}^t))^\chi\big)^{(-1)^{i+1}}. \end{aligned} \tag{56}$$

We now set $\mathrm{SC}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{SC}_{V_L}(A, L/K)$. Then the result of Proposition 3.7(ii)(b) combines with Remark 3.3 and the Artin–Verdier duality theorem to imply there are natural isomorphisms

$$\mathrm{SC}_\ell \cong \mathbb{Z}_\ell \otimes_{\mathbb{Z}} R\Gamma_{\mathrm{ar},V_L}(U_L, \mathcal{A}\{\ell\})^*[2] \cong R\Gamma_{\text{ét},c}(U_L, \mathcal{A}\{\ell\})^*[2] \cong R\Gamma_{\text{ét}}(U_L, T_\ell(\mathcal{A}^t))$$

and hence also a natural exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}_\ell[G])$ of the form

$$\mathrm{SC}_\ell \to R\Gamma_{\text{ét}}(U_L^c, T_\ell(\mathcal{A}^t)) \xrightarrow{1-\varphi_p} R\Gamma_{\text{ét}}(U_L^c, T_\ell(\mathcal{A}^t)) \to \mathrm{SC}_\ell[1]. \tag{57}$$

We consider the composite homomorphism

$$\beta_{A,L,\ell} : \mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^t(L) \cong H^0(\mathrm{SC}_\ell)_{\mathbb{Q}_\ell} \to H^0_{\text{ét}}(U_L^c, V_\ell(\mathcal{A}^t)) \to H^1(\mathrm{SC}_\ell)_{\mathbb{Q}_\ell} \cong \mathbb{Q}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A(L), \mathbb{Z}),$$

where the isomorphisms are from Proposition 3.7(ii)(a) and the other maps are induced by the long exact cohomology sequence of (57).

Then it is shown by Schneider in [32] (and also noted at the beginning of [22, §6.8]) that there exists a computable integer $a_{A,L,\ell} \in \{0, 1\}$ such that

$$\beta_{A,L,\ell} = (-1)^{a_{A,L,\ell}} \cdot h_{A,L,\ell,*} \tag{58}$$

where $h_{A,L,\ell,*}$ is the isomorphism $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^t(L) \cong \mathbb{Q}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A(L), \mathbb{Z})$ induced by the height pairing $h_{A,L}$.

Taken in conjunction with the argument used in Proposition 8.1, this observation implies firstly that the endomorphism $H^i(1-\varphi_p)_{\mathbb{Q}_\ell}$ is bijective for $i \neq 1$, secondly that (57) satisfies all of the hypotheses of Proposition 5.1 (with $\mathfrak{A} = \mathbb{Z}_\ell[G]$) regarding the left-hand triangle in (14), and thirdly (in view of (56)) that

$$\mathrm{ord}_{t=p^{-1}}(Z_U(A, \chi, t)) = \dim_{\mathbb{C}_\ell}\big(\ker\big(H^1(1-\varphi_p) \mid \mathrm{Hom}_{\mathbb{C}_\ell[G]}(V_\chi, \mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} H^i_{\text{ét}}(U^c_L, V_\ell(\mathcal{A}^t)))\big)\big).$$

By applying Proposition 5.1 with the left- and right-hand triangles in (14) taken to be (57) and the zero triangle respectively we can therefore deduce that

$$\begin{aligned}
\iota_{G,\ell}(\chi^{\mathrm{BSD}}_{G,\mathbb{Q}}(A, V_L) + \chi^{\mathrm{sgn}}_G(A)) &= \chi_{\mathbb{Z}_\ell[G]}(\mathrm{SC}_\ell, h^{\det}_{A,L,\ell,*}) + \partial_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}(\beta_{A,L,\ell} \circ h^{-1}_{A,L,\ell,*}) \\
&= \chi_{\mathbb{Z}_\ell[G]}(\mathrm{SC}_\ell, \tau_{1-\varphi_p}) \\
&= \partial_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}(H^1(1-\varphi_p)^{\diamond}_{\mathbb{Q}_\ell}) \\
&= \partial_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}((\mathrm{Nrd}_{\mathbb{Q}_\ell[G]})^{-1}((Z^*_U(A, \chi, p^{-1}))_{\chi \in \mathrm{Ir}(G)})) \\
&= \partial_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}(Z^*_U(A_{L/K}, p^{-1})) \\
&= \iota_{G,\ell}(\partial_{G,\mathbb{Q}}(Z^*_U(A_{L/K}, p^{-1}))).
\end{aligned}$$

Here the first equality follows directly from the definition of $\chi^{\mathrm{sgn}}(A, L/K)_\ell$ in terms of the integer $a_{A,L,\ell}$, the equality (58) and the result of Lemma 9.5. In addition, the fourth equality follows from (56), the fifth directly from the definition of the term $Z^*_U(A_{L/K}, p^{-1})$ and all remaining equalities are clear.

This argument completes the proof that condition (iii) in Proposition 9.1 is satisfied and hence also, when combined with the observations made in Section 9.2, completes the proof of Theorem 4.10.

9.4. ***The proof of Propositions 4.7 and 4.8.*** Throughout this section, we shall use the notation of Section 4.2.3.

Proof of Proposition 4.7. As a first step, we recall that Proposition 3.7(i) implies SC_{V_L} belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ and is acyclic outside degrees 0, 1 and 2. In this case, therefore, the construction of resolutions used in the proofs of Lemma 3.8(iii) and Proposition 6.3 implies SC_{V_L} is isomorphic in $D(\mathbb{Z}[G])$ to a complex

$$P_{-1} \xrightarrow{d^{-1}} P_0 \xrightarrow{d^0} P_1 \xrightarrow{d^1} P_2 \tag{59}$$

in which P_{-1} is a finitely generated projective $\mathbb{Z}[G]$-module that is placed in degree -1 and all other modules P_i are finitely generated and free. By taking the direct sum with complexes of the form $\mathbb{Z}[G] \xrightarrow{1} \mathbb{Z}[G]$, with the first term placed in appropriate degrees, one can also assume that the G-rank $\mathrm{rk}_G(P_i)$ of P_i is greater than 1 for every i.

To prove claim (i) it is therefore enough to show that the G-module P_{-1} is free, or equivalently (by the Bass cancellation theorem [12, Theorem (41.20)], since $\mathrm{rk}_G(P_{-1}) > 1$) that the Euler characteristic $\chi_G(\mathrm{SC}_{V_L})$ of SC_{V_L} in $K_0(\mathbb{Z}[G])$ vanishes. In addition, writing ∂'_G for the connecting homomorphism

$K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}[G])$, the (assumed) equality in Conjecture 4.3(iii) implies that

$$\begin{aligned}\chi_G(\mathrm{SC}_{V_L}) &= \partial'_G\big(\chi_G^{\mathrm{BSD}}(A, V_L)\big)\\ &= \partial'_G\big(\partial_G(L_U^*(A_{L/K}, 1))\big) + \partial'_G\big(\chi_G^{\mathrm{coh}}(A, V_L)\big) - \partial'_G\big(\chi_G^{\mathrm{sgn}}(A)\big)\\ &= \partial'_G\big(\chi_G^{\mathrm{coh}}(A, V_L)\big) - \partial'_G\big(\chi_G^{\mathrm{sgn}}(A)\big),\end{aligned}$$

where the final equality follows directly from the fact that $\partial'_G \circ \partial_G$ is the zero map.

To prove claim (i), we are therefore reduced to showing that if G has p-power order, then the last two terms in the above expression vanish. However, in this case, every finite projective $\mathbb{F}_p[G]$-module is free so that the image of the homomorphism (11) belongs to the kernel of ∂'_G and hence $\partial'_G\big(\chi_G^{\mathrm{coh}}(A, V_L)\big)$ automatically vanishes. In addition, the term $\partial'_G\big(\chi_G^{\mathrm{sgn}}(A)\big)$ vanishes since Lemma 9.5 below implies that $\chi_G^{\mathrm{sgn}}(A)$ is equal to

$$\partial_{G,\mathbb{Q}}(\langle \mathbb{Q}\cdot A^t(L), (-1)^{a_p}\rangle)_p = \partial_{G,\mathbb{Q}}(\langle \mathbb{Q}\cdot A^t(L), (-1)^{a_p}\rangle) = \partial_G(\langle \mathbb{R}\cdot A^t(L), (-1)^{a_p}\rangle). \tag{60}$$

This proves claim (i).

To prove claim (ii) we note that $A(K)[p] = A(L)[p]^G$ and $A^t(K)[p] = A^t(L)[p]^G$. Hence, if $A(K)[p]$ and $A^t(K)[p]$ vanish, then $A(L)[p]$ and $A^t(L)[p]$ also vanish since G is a p-group. In this case, therefore, Proposition 3.7(i) implies $\mathrm{SC}_{V_L,(p)}$ is acyclic outside degrees 0 and 1 and $H^0(\mathrm{SC}_{V_L,(p)})$ is torsion-free. This in turn implies that $\mathrm{SC}_{V_L,(p)}$ is isomorphic in $D(\mathbb{Z}_{(p)}[G])$ to a complex of projective $\mathbb{Z}_{(p)}[G]$-modules of the form (59) in which P_2 vanishes and so there are exact sequences of $\mathbb{Z}_{(p)}[G]$-modules

$$0 \to P_{-1} \to P_0 \to \mathrm{cok}(d^{-1})_{(p)} \to 0$$

and

$$0 \to H^0(\mathrm{SC}_{V_L,(p)}) \to \mathrm{cok}(d^{-1})_{(p)} \xrightarrow{d^0} P_1 \to H^1(\mathrm{SC}_{V_L,(p)}) \to 0.$$

The first of these sequences implies $\mathrm{cok}(d^{-1})_{(p)}$ is a c-t G-module and the second implies it is torsion-free. These two properties combine to imply $\mathrm{cok}(d^{-1})_{(p)}$ is a projective $\mathbb{Z}_{(p)}[G]$-module (by [3, Theorem 8]).

At this stage we therefore know that $\mathrm{SC}_{V_L,(p)}$ is isomorphic in $D(\mathbb{Z}_{(p)}[G])$ to a complex of $\mathbb{Z}_{(p)}[G]$-modules $\mathrm{cok}(d^{-1})_{(p)} \to P_1$ in which the first term is projective and the second is free (and of rank greater than 1). To see that this is a complex of the required form it is then enough to note that, since the Euler characteristic of $\mathrm{SC}_{V_L,(p)}$ in $K_0(\mathbb{Z}_{(p)}[G])$ vanishes, the Bass cancellation theorem implies that the module $\mathrm{cok}(d^{-1})_{(p)}$ is isomorphic to P_1.

This completes the proof of Proposition 4.7. □

Lemma 9.5. *If ℓ is any prime that does not divide $\#G$, then $\partial_{G,\mathbb{Q}}(\langle \mathbb{Q}\cdot A^t(L), -1\rangle)_\ell$ vanishes.*

Proof. If ℓ does not divide $\#G$, then the $\mathbb{Z}_\ell$-order $\mathbb{Z}_\ell[G]$ is maximal and so $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^t(L)$ has a full sublattice that is a projective $\mathbb{Z}_\ell[G]$-module. This implies $\langle \mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^t(L), -1\rangle$ belongs to the image of the natural map $K_1(\mathbb{Z}_\ell[G]) \to K_1(\mathbb{Q}_\ell[G])$ and hence that the element $\partial_{G,\mathbb{Q}}(\langle \mathbb{Q}\cdot A^t(L), -1\rangle)_\ell = \partial_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}(\langle \mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^t(L), -1\rangle)$ vanishes as a consequence of the long exact sequence of relative K-theory (see (7)). □

Proof of Proposition 4.8. We abbreviate the connecting homomorphism $\partial_{\mathbb{Z}_{(p)}[G],\mathbb{R}}$ to ∂_p and use the natural scalar extension map

$$\iota_p = \iota_{G,p} : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}_{(p)}[G], \mathbb{R}[G]).$$

Then, as a first step, we note that there are equalities

$$\iota_p\big(\chi_G^{\mathrm{sgn}}(A)\big) = \partial_p\big(\langle \mathbb{R}\cdot A^t(L), (-1)^{a_p}\rangle\big) \quad \text{and} \quad \iota_p\big(\chi_G^{\mathrm{BSD}}(A, V_L)\big) = \partial_p\big(\langle \mathbb{R}[G]^t, \iota^{\mathrm{NT}}_{A,L}\rangle\big). \tag{61}$$

The first of these follows directly from (60) and the second from a routine comparison of the definition of the automorphism $\iota^{\mathrm{NT}}_{A,L}$ with the explicit computation of $\chi_G^{\mathrm{BSD}}(A, V_L)$ in terms of the nonabelian determinant of the representative of $\mathrm{SC}_{V_L,(p)}$ fixed in Proposition 4.7(ii).

We next claim that

$$\iota_p\big(\chi_G^{\mathrm{coh}}(A, V_L)\big) = \partial_p\big(\langle \mathbb{R}[G], p^{\chi(\mathcal{L})}\rangle\big). \tag{62}$$

To show this we recall from the proof of Lemma 3.9 that the complex $C := R\Gamma(X, \mathcal{L}_L)^*$ belongs to $D^{\mathrm{perf}}(\mathbb{F}_p[G])$ and is acyclic outside degrees 0 and 1. Since G is a p-group, C is therefore isomorphic in $D(\mathbb{F}_p[G])$ to a complex of the form $\mathbb{F}_p[G]^{n_0} \to \mathbb{F}_p[G]^{n_1}$, where the first term is placed in degree 0 and n_0 and n_1 are suitable natural numbers. By using this representative, one computes that

$$\chi_G^{\mathrm{coh}}(A, V_L) := \chi_G(R\Gamma(X, \mathcal{L}_L)^*, 0) = \partial_p\big(\langle \mathbb{R}[G], p^{n_0-n_1}\rangle\big).$$

To deduce (62) we note that a computation of Euler characteristics in $K_0(\mathbb{F}_p) \cong \mathbb{Z}$ implies that

$$\begin{aligned}
\chi(\mathcal{L}) &:= \chi_{\mathbb{F}_p}(R\Gamma(X, \mathcal{L})^*) \\
&= \chi_{\mathbb{F}_p}\big(R\,\mathrm{Hom}_{\mathbb{F}_p[G]}(\mathbb{F}_p, R\Gamma(X, \mathcal{L}_L)^*)\big) \\
&= \chi_{\mathbb{F}_p}\big(R\,\mathrm{Hom}_{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p[G]^{n_0} \to \mathbb{F}_p[G]^{n_1})\big) \\
&= \chi_{\mathbb{F}_p}(\mathbb{F}_p^{n_0} \to \mathbb{F}_p^{n_1}) \\
&= n_0 - n_1,
\end{aligned}$$

where the first aligned equality follows from the isomorphism (6).

Thus, if one defines an element of $K_1(\mathbb{R}[G])$ by setting

$$\mathscr{L} := L_U^*(A_{L/K}, 1) \times \langle \mathbb{R}[G]^t, \iota^{\mathrm{NT}}_{A,L}\rangle^{-1} \times \langle \mathbb{R}[G], p^{\chi(\mathcal{L})}\rangle \times \langle \mathbb{R}\cdot A^t(L), (-1)^{a_p}\rangle^{-1}$$

then the equalities (61) and (62) imply that

$$\partial_p(\mathscr{L}) = \iota_p\big(\partial_G(L_U^*(A_{L/K}, 1)) - \chi_G^{\mathrm{BSD}}(A, V_L) + \chi_G^{\mathrm{coh}}(A, V_L) - \chi_G^{\mathrm{sgn}}(A)\big).$$

In addition, an explicit computation of reduced norm combines with the definition (12) of each term $\mathscr{L}(A, \psi)$ to imply that

$$\mathrm{Nrd}_{\mathbb{R}[G]}(\mathscr{L}) = \sum_{\psi \in \mathrm{Ir}(G)} \mathscr{L}(A, \psi) e_\psi \in \zeta(\mathbb{C}[G])^\times.$$

This equality implies that the conditions stated in Proposition 4.8(i) are equivalent to asserting $\mathrm{Nrd}_{\mathbb{R}[G]}(\mathscr{L})$ belongs to $\zeta(\mathbb{Q}[G])^\times$. Hence, since $K_1(\mathbb{Q}[G])$ is the full preimage under ∂_p of the subgroup

$K_0(\mathbb{Z}_{(p)}[G], \mathbb{Q}[G])$ of $K_0(\mathbb{Z}_{(p)}[G], \mathbb{R}[G])$, these conditions are true if the equality in Conjecture 4.3(iii) is valid modulo $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$. Their validity thus follows directly from the assumed validity of Conjecture 4.3(i) and (ii) and the argument of Section 5.3.

In a similar way, if G is abelian, the above computation shows that Conjecture 4.3(iii) implies $\mathrm{Nrd}_{\mathbb{R}[G]}(\mathscr{L})$ belongs to $\mathbb{Z}_{(p)}[G]^\times$. Since $\mathbb{Z}_{(p)}[G]$ is a local ring (as G is a p-group), the latter containment is valid if and only if $\mathrm{Nrd}_{\mathbb{R}[G]}(\mathscr{L})$ belongs to $\mathbb{Z}_{(p)}[G]$ and its image under the projection $\mathbb{Z}_{(p)}[G] \to \mathbb{Z}_{(p)}$ belongs to $\mathbb{Z}_{(p)}^\times$. These conditions are in turn equivalent to requiring $\mathscr{L}(A, \mathbf{1}_G)$ belongs to $\mathbb{Z}_{(p)}^\times$ and, also, for all $g \in G$ one has

$$\sum_{\psi \in \mathrm{Ir}(G)} \psi(g^{-1})\mathscr{L}(A, \psi) \in |G| \cdot \mathbb{Z}_{(p)}.$$

To deduce the result of Proposition 4.8(ii) it is thus enough to note that, for each abelian subquotient $Q = H/J$ of G, the arguments of Propositions 5.6 and 9.2 combine to imply that the validity of Conjecture 4.3(iii) for the data $(A, L/K)$ implies the validity modulo $\ker(\iota_{Q,p})$ of the equality in Conjecture 4.3(iii) for the data $(A_{L^H}, L^J/L^H)$.

Finally, to prove Proposition 4.8(iii) we assume the validity of Conjecture 4.3 and hence that the element $\mathscr{L}$ belongs to the image of $K_1(\mathbb{Z}_{(p)}[G])$ in $K_1(\mathbb{R}[G])$. Thus, if we fix an embedding of fields $j : \mathbb{R} \to \mathbb{C}_p$, then the image of $\mathscr{L}$ under the induced map $K_1(\mathbb{R}[G]) \to K_1(\mathbb{C}_p[G])$ belongs to the image of the natural map $K_1(\mathbb{Z}_p[G]) \to K_1(\mathbb{C}_p[G])$.

Given this containment, the equalities in claim (iii) follow from the general result of [6, Theorem 2.1] (with $\Lambda = \mathbb{Z}_p$) and the fact that, for each subgroup H of G, the argument of Proposition 9.2 implies $\sum_{\psi \in \mathrm{Ir}(H^{\mathrm{ab}})} \mathscr{L}(A_{L^H}, \psi)e_\psi$ is equal to the image of $\mathrm{Nrd}_{\mathbb{R}[G]}(\mathscr{L})$ under the upper composite map in the diagram

$$\begin{tikzcd}
\zeta(\mathbb{R}[G])^\times \arrow[r, "\subset"] & \zeta(\mathbb{C}[G])^\times \arrow[r, "\varrho_H"] & \zeta(\mathbb{C}[H])^\times \arrow[r, "\varrho'_H"] & \zeta(\mathbb{C}[H^{\mathrm{ab}}])^\times = \mathbb{C}[H^{\mathrm{ab}}]^\times \\
& K_1(\mathbb{C}[G]) \arrow[u, "\mathrm{Nrd}_{\mathbb{C}[G]}", "\cong"'] \arrow[r, "\theta^1_{G,H}"] & K_1(\mathbb{C}[H]) \arrow[u, "\cong", "\mathrm{Nrd}_{\mathbb{C}[H]}A"'] &
\end{tikzcd}$$

Here $\theta^1_{G,H}$ is the natural restriction of scalars map, ϱ_H is defined by the requirement that the square commutes and ϱ'_H is the natural projection map.

This completes the proof of Proposition 4.8. □

Appendix A. Kummer-étale descent for coherent cohomology

In this first appendix, we show that the coherent cohomology over a "separated" formal fs log scheme can be computed via the Čech resolution with respect to an affine *Kummer-étale* covering (not necessarily a Zariski open covering). Whilst this result seems to be well known to experts, we have not been able to locate a good reference for it in the literature.

A.1. ***Fs log schemes and their fibre products.*** We next review the construction of fibre products for fs log schemes, which we need for the sheaf theory on Kummer-étale sites and the construction of Čech complexes. We recall some definitions of monoids and log schemes needed for the construction of fibre products, but refer readers to [21] and [28] for basic definitions in log geometry and to [30] for a more comprehensive reference.

Recall that a monoid P, always assumed commutative, is *fine* if it is finitely generated and the natural map $P \to P^{\mathrm{gp}}$ is injective (where P^{gp} is the commutative group obtained by adjoining the inverse of each element of P). A fine monoid P is said to be *saturated* if for any $\alpha \in P^{\mathrm{gp}}$, we have $\alpha^n \in P$ for some $n > 0$ if and only if $\alpha \in P$. By a *fs* monoid, we mean a fine and saturated monoid.

For each monoid P we define a saturation $P^{\mathrm{sat}} := \{\alpha \in P^{\mathrm{gp}};\ \alpha^n \in P \text{ for some } n > 0\}$.

Lemma A.1. *If P is finitely generated, then the monoid P^{sat} is fs.*

Proof. It suffices to show that P^{sat} is finitely generated, which is a direct consequence of Gordon's lemma (see [30, Chapter I, Theorem 2.3.19]). □

A log scheme $X^\sharp$ is called *fs* (i.e., fine and saturated) if étale locally on the underlying scheme X, the log structure is generated by a map of monoids $P \to \mathcal{O}_X$ where P is a fs monoid. Our log schemes and formal schemes are always assumed to be fs (i.e., fine saturated).

Let $X^\sharp$ and $Y^\sharp$ be fs log schemes over $S^\sharp$ (with underlying schemes denoted as X, Y, and S). We want to construct a fs log scheme $X^\sharp \times_{S^\sharp} Y^\sharp$ satisfying the universal property of fibre products (see [30, Chapter III, Corollary 2.1.6]).

By replacing the formal log schemes with suitable étale coverings, we choose charts $P \to \mathcal{O}_X$, $Q \to \mathcal{O}_Y$ and $M \to \mathcal{O}_S$ defining the log structures (where P, Q and M are fs monoids, viewed as constant sheaves), such that there exist maps $M \to P, Q$ giving rise to the structure morphism $X^\sharp, Y^\sharp \to S^\sharp$. (The existence of such local charts follows from [21, Lemma 2.10].)

The most natural candidate is to endow $X \times_S Y$ with the log structure associated to the chart $P \oplus_M Q \to \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y$, where $P \oplus_M Q$ is the amalgamated sum of monoids. But this may not always work as $P \oplus_M Q$ may not be fine nor saturated.

Writing $P \oplus_M^{\mathrm{sat}} Q$ for the *saturation* of $P \oplus_M Q$ we can define the following fs log scheme

$$X^\sharp \times_{S^\sharp} Y^\sharp := (X \times_S Y) \times_{\operatorname{Spec} \mathbb{Z}[P \oplus_M Q]} \operatorname{Spec} \mathbb{Z}[P \oplus_M^{\mathrm{sat}} Q]$$

with the log structure given by the chart $P \oplus_M^{\mathrm{sat}} Q \to \mathcal{O}_{X^\sharp \times_{S^\sharp} Y^\sharp}$ naturally extending $P \oplus_M Q \to \mathcal{O}_{X \times_S Y}$. By glueing this étale-local construction, we obtain the fibre products for any fs log schemes. We repeat this construction to obtain fibre products of formal fs log schemes.

Note that this notion of fibre product may not be compatible with fibre products of (formal) schemes without log structure, as we can see from the explicit étale-local construction. Instead, we have the following lemma, which is a consequence of Lemma A.1. (See [30, Chapter III, Corollary 2.1.6] for the proof.)

Lemma A.2. *The underlying scheme for $X^\sharp \times_{S^\sharp} Y^\sharp$ is finite over $X \times_S Y$. The same holds for formal fs log schemes.*

Remark A.3. To give a concrete example in which the underlying scheme for $X^\sharp \times_{S^\sharp} Y^\sharp$ differs from $X \times_S Y$ we fix a finite Galois Kummer-étale cover $\pi : X_L^\sharp \xrightarrow{\pi^\sharp} X^\sharp$ of group G. In this case one has $X_L^\sharp \times_{X^\sharp} X_L^\sharp \cong G \times X_L^\sharp$, whereas $X_L \times_X X_L \cong G \times X_L$ only if π is unramified.

Corollary A.4. *Let $X^\sharp$ be a fs log scheme, such that the underlying scheme is separated. Let $U^\sharp$ and $U'^\sharp$ be fs log schemes over $X^\sharp$, such that the underlying schemes U and U' are affine. Then $U^\sharp \times_{X^\sharp} U'^\sharp$ is also affine. The same holds for formal fs log schemes.*

Proof. Under the hypotheses, the scheme $U \times_X U'$ is affine, which follows from the cartesian diagram

$$\begin{tikzcd} U \times_X U' \arrow[r, hook] \arrow[d] & U \times U' \arrow[d] \\ X \arrow[r, hook, "\Delta_X"] & X \times X, \end{tikzcd}$$

Now by Lemma A.2, the underlying scheme of $U^\sharp \times_{X^\sharp} U'^\sharp$ is finite over an affine scheme $U \times_X U'$. This proves the corollary. □

A.2. ***Čech-to-derived functor spectral sequence for Kummer-étale cohomology.*** For a log formal scheme $\mathfrak{X}$, we write $\mathfrak{X}^\sharp_{\text{két}}$ for the associated Kummer-étale site (as per [28, Definition 2.13]).

We quickly recall the definition of Čech complex and Čech-to-derived functor spectral sequences in this setting.

Definition A.5. Let $\mathfrak{U}^\sharp$ be an Kummer-étale covering of $\mathfrak{X}^\sharp$ (i.e., the structure morphism $\mathfrak{U}^\sharp \to \mathfrak{X}^\sharp$ is Kummer-étale and surjective), and let $\mathcal{F}$ be a sheaf of abelian groups on the Kummer-étale site $\mathfrak{X}^\sharp_{\text{két}}$. Then we can form a Čech complex

$$C^\bullet(\mathfrak{U}^\sharp, \mathcal{F}) := [\Gamma(\mathfrak{U}^\sharp, \mathcal{F}) \to \Gamma(\mathfrak{U}^\sharp \times_{\mathfrak{X}^\sharp} \mathfrak{U}^\sharp, \mathcal{F}) \to \Gamma(\mathfrak{U}^\sharp \times_{\mathfrak{X}^\sharp} \mathfrak{U}^\sharp \times_{\mathfrak{X}^\sharp} \mathfrak{U}^\sharp, \mathcal{F}) \to \cdots],$$

with differentials defined in a standard way.

(The usual definition of Cech complexes for the case without log structure, cf. [27, Ch. III, §2], formally goes through.) For any bounded-below complexes $\mathcal{F}^\bullet$, we define the Čech complex $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F}^\bullet)$ as the total complex of the double complex obtained from Čech complex of each term of $\mathcal{F}^\bullet$.

Whilst the Čech complex $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F})$ does not necessarily represent $R\Gamma(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{F})$, there exists a natural "Čech-to-derived functor spectral sequence"

$$E_1^{i,j} : H^j(\mathfrak{U}^\sharp_{i,\text{két}}, \mathcal{F}) \Rightarrow H^{i+j}(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{F}), \tag{63}$$

where $\mathfrak{U}^\sharp_i$ is the $(i+1)$-fold self fibre product of $\mathfrak{U}^\sharp$ over $\mathfrak{X}^\sharp$. One way to read off this spectral sequence from the literature is via the technique of cohomological descent for (simplicial) topoi associated to the Kummer-étale sites $\mathfrak{U}^\sharp_{\text{két}}$ and $\mathfrak{X}^\sharp_{\text{két}}$; see [2, exposé Vbis]. Indeed, since it admits a local section, $\mathfrak{U}^\sharp \to \mathfrak{X}^\sharp$ is a "morphism of universal cohomological descent" by [loc. cit., proposition (3.3.1)] and so the above spectral sequence is just a special case of the descent spectral sequence from [loc. cit., proposition (2.5.5)]).

Remark A.6. The complex $(E_1^{i,0}, d^{i,0})$ coincides with $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F})$ and so the above spectral sequence implies $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F}) = R\Gamma(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{F})$ if $E_1^{i,j}$ vanishes for all $j > 0$ and $i \geqslant 0$.

A.3. ***Coherent cohomology.*** We first recall Kummer-étale descent theory for coherent sheaves on schemes and formal schemes.

Let $\mathfrak{X}^\sharp$ be a log scheme over $\mathbb{Z}/p^n$ for some n and $\mathcal{F}$ a quasicoherent $\mathcal{O}_{\mathfrak{X}}$-module. Then, by Kato's unpublished result (cf. [28, Proposition 2.19]) the presheaf $\mathfrak{U}^\sharp \in \mathfrak{X}^\sharp_{\text{két}} \rightsquigarrow \Gamma(\mathfrak{U}, \mathcal{F}_{\mathfrak{U}})$ is a sheaf on $\mathfrak{X}^\sharp_{\text{két}}$, where $\mathcal{F}_{\mathfrak{U}}$ denotes the pull back of $\mathcal{F}$ via the structure morphism $\mathfrak{U} \to \mathfrak{X}$ of the underlying schemes. We use the same notation $\mathcal{F}$ to denote the Kummer-étale sheaf associated to a quasicoherent sheaf $\mathcal{F}$.

Now, if $\mathfrak{X}^\sharp$ be a locally noetherian formal fs log scheme over $\operatorname{Spf} \mathbb{Z}_p$, we can associate, to a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$, a Kummer-étale $\mathbb{Z}_p$-sheaf $\mathcal{F}$ by extending the construction for coherent sheaves on schemes via projective limit. (We restrict to coherent sheaves to avoid technicalities regarding completion.)

Now, we are interested in $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F})$ when $\mathcal{F}$ is a vector bundle on $\mathfrak{X}$ (viewed as a Kummer-étale sheaf), while $\mathfrak{U}^\sharp$ remains a Kummer-étale covering of $\mathfrak{X}^\sharp$.

Proposition A.7. *Let $\mathfrak{X}^\sharp$ be a noetherian formal fs log scheme over* Spf R (*for some noetherian adic ring* R, *with trivial log structure*), *and assume that $\mathfrak{X}$ is separated. Then for any coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ there is a natural isomorphism $R\Gamma(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{F}) \xrightarrow{\sim} R\Gamma(\mathfrak{X}, \mathcal{F})$.*

Furthermore, for any Kummer-étale covering $\mathfrak{U}^\sharp \to \mathfrak{X}^\sharp$ where $\mathfrak{U}$ is affine, the Čech complex $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F})$ represents $R\Gamma(\mathfrak{X}, \mathcal{F})$.

The same holds if we replace $\mathcal{F}$ with a bounded-below complex $\mathcal{F}^\bullet$ of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules, such that the differential maps $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are additive morphisms of Kummer-étale sheaves.

Proof. By standard argument with hypercohomology spectral sequences, the claim for $\mathcal{F}^\bullet$ can be reduced to $\mathcal{F}$.

Let us first assume that $\mathfrak{X}$ is affine. Then by [28, Proposition 3.27] and the theorem on formal functions, we have $R\Gamma(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{O}_{\mathfrak{X}}) = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Now, by resolving $\mathcal{F}$ with free $\mathcal{O}_{\mathfrak{X}}$-modules, we obtain $R\Gamma(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{F}) = \Gamma(\mathfrak{X}, \mathcal{F})$.

Choose a Kummer-étale covering $\mathfrak{U}^\sharp \to \mathfrak{X}^\sharp$ with $\mathfrak{U}$ affine. Then Corollary A.4 implies

$$\mathfrak{U}_i^\sharp := \underbrace{\mathfrak{U}^\sharp \times_{\mathfrak{X}^\sharp} \cdots \times_{\mathfrak{X}^\sharp} \mathfrak{U}^\sharp}_{i+1 \text{ times}}$$

has an *affine* underlying formal scheme. Therefore, by the Čech-to-derived spectral sequence argument it follows that $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F})$ represents $R\Gamma(\mathfrak{X}^\sharp_{\text{két}}, \mathcal{F})$ (cf. Remark A.6). Now if we choose $\mathfrak{U}^\sharp$ to be the disjoint union of finite affine open covering of $\mathfrak{X}$ (with the natural log structure induced from $\mathfrak{X}^\sharp$), then $C^\bullet(\mathfrak{U}^\sharp, \mathcal{F})$ represents $R\Gamma(\mathfrak{X}, \mathcal{F})$, as claimed. □

Remark A.8. We apply Proposition A.7 to the log de Rham complex $\mathcal{F}^\bullet$, where the maps $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are not $\mathcal{O}_{\mathfrak{X}}$-linear but are additive morphisms of Kummer-étale sheaves.

Appendix B. A Lefschetz trace formula for rigid cohomology

In this second appendix, our aim is to establish a slight extension of the Lefschetz trace formula for rigid cohomology that is proved in [15, théorème 6.3].

As before, we let U be a *smooth affine curve* over a finite field of characteristic $p > 0$. Let Λ_0 be a finite extension of $\mathbb{Q}_p$, and assume that $\mathcal{O}_U$ contains the residue field k_0 of Λ_0. Set $q_0 = p^{r_0} := \#(k_0)$

Let $\mathcal{O}^{\dagger}_{U/\Lambda_0}$ denote the global section of the overconvergent structure with coefficients Λ_0. To explain, let X be a smooth compactification of U, and let $\mathfrak{X}_{\mathcal{O}_{\Lambda_0}}$ be a formal lift of X over the valuation ring $\mathcal{O}_{\Lambda_0}$ of Λ_0. Let $\mathcal{X}_{\Lambda_0}$ denote its rigid generic fibre, which contains the tube $]U[_{\Lambda_0}$ of U as an open subspace. Then $\mathcal{O}^{\dagger}_{U/\Lambda_0}$ is the ring of rigid analytic functions defined on some "strict neighbourhood" of $]U[_{\Lambda_0}$. (We often call such rigid analytic functions *overconvergent* along $X \setminus U$.) Note that if Λ'_0 is a finite extension of Λ_0 whose residue field is contained in $\mathcal{O}_U$, then we have $\mathcal{O}^{\dagger}_{U/\Lambda'_0} \cong \mathcal{O}^{\dagger}_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda'_0$.

Since $\mathcal{O}_U$ contains the residue field k_0 of Λ_0, one can define an overconvergent Λ_0-F-isocrystal $\mathcal{E}$ over U; see Section B.2. (See also [38, §7] for the definition and natural context where overconvergent Λ_0-F-isocrystals appear. In loc. cit. it is called an *overconvergent* Λ_0-F^{r_0}*-isocrystal*, where $r_0 := [k_0 : \mathbb{F}_p]$, but let us suppress r_0 from the notation.) In our intended application however, we would naturally like to remove the assumption that the residue field of the coefficient field Λ_0 is not contained in $\mathcal{O}_U$. (Compare the proof of Theorem 8.2.)

Let Λ be finite extension of $\mathbb{Q}_p$. Unless the residue field of Λ is contained in $\mathcal{O}_U$, the usual definition of overconvergent Λ-F-isocrystals over U does not apply. Instead, let us consider a subextension $\Lambda_0 \subset \Lambda$ whose residue field k_0 is contained in $\mathcal{O}_U$. (We may choose k_0 to be the maximal subfield of the residue field k_Λ of Λ that can be embedded in $\mathcal{O}_U$.) Instead of defining "overconvergent Λ-F-isocrystals" over U, we will work with overconvergent Λ_0-F-isocrystals "equipped with Λ-action"; see Section B.2. The aim of this appendix is to extend the Lefschetz trace formula [15, 6.3] for the rigid cohomology with coefficients in such overconvergent Λ_0-F-isocrystals with Λ-action when $\mathcal{O}_U$ does not contain the residue field of Λ.

B.1. ***Overconvergent modules and duality.*** Let U and Λ_0 be as before and let Λ be a finite extension of Λ_0. We set $U' := U \times_{\operatorname{Spec} k_0} \operatorname{Spec} k_\Lambda$ where k_Λ is the residue field of Λ.

Set $X' := X \times_{\operatorname{Spec} k_0} \operatorname{Spec} k_\Lambda$, which is a smooth compactification of U'. We also choose a formal $\mathcal{O}_\Lambda$-lift $\mathfrak{X}'_\Lambda$ of X', and we subsequently obtain its rigid generic fibre $\mathcal{X}'_\Lambda$ and the tube $]U'[_\Lambda \subset \mathcal{X}'_\Lambda$. With this setting, we define $\mathcal{O}^{\dagger}_{U'/\Lambda}$ to be the ring of rigid analytic functions defined on some strict neighbourhood of $]U'[_\Lambda$. Then we have an isomorphism of Λ-algebras

$$\mathcal{O}^{\dagger}_{U'/\Lambda} \cong \mathcal{O}^{\dagger}_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda. \tag{64}$$

By the very construction, $\mathcal{O}^{\dagger}_{U/\Lambda_0}$ and $\mathcal{O}^{\dagger}_{U'/\Lambda}$ are *Fréchet algebras* over Λ_0 and Λ, respectively, so any finite locally free modules over them are p-adic Fréchet spaces.

Let $\Omega^{\dagger}_{U/\Lambda_0}$ and $\Omega^{\dagger}_{U'/\Lambda}$ denote the modules of overconvergent Kähler differentials. Then we also have

$$\Omega^{\dagger}_{U'/\Lambda} := \Omega^{\dagger}_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda. \tag{65}$$

Let $]U'[_\Lambda \subset \mathcal{V}' \subset \mathcal{X}'_\Lambda$ be a strict neighbourhood, and let $\mathcal{E}_{\mathcal{V}'}$ be a vector bundle defined over $\mathcal{V}'$. We set

$$\mathcal{E} := \varinjlim_{\mathcal{W}'} \Gamma(\mathcal{W}', \mathcal{E}_{\mathcal{V}'}), \tag{66}$$

where the direct limit is taken over all strict neighbourhoods $\mathcal{W}'$ contained in $\mathcal{V}'$. Then $\mathcal{E}$ turns out to be a locally free $\mathcal{O}^\dagger_{U'/\Lambda}$-module; hence, a Fréchet Λ-space. (The local freeness claim can easily reduced to the case when U is an open subscheme of $\mathbb{P}^1$, which is standard; cf. [19, §V, théorème 1].) Note that $\Omega^\dagger_{U'/\Lambda}$ can also be obtained from the line bundle $\mathcal{E}_{\mathcal{V}'} = \Omega_{\mathcal{X}'_\Lambda}$ of Kähler differentials over $\mathcal{V}' = \mathcal{X}'_\Lambda$.

To the "overconvergent vector bundle" $\mathcal{E}_{\mathcal{V}'}$, we associate another $\mathcal{O}^\dagger_{U'/\Lambda}$-module as follows:

Definition B.1. For a sufficiently small strict neighbourhood $\mathcal{V}'$ of $]U'[_\Lambda$, we define

$$\mathcal{E}_c := H^1_{]U'[_\Lambda}(\mathcal{V}', \mathcal{E}_{\mathcal{V}'}),$$

where $H^1_{]U'[_\Lambda}(\mathcal{V}', \mathcal{E}_{\mathcal{V}'})$ is the first cohomology of the mapping fibre of

$$R\Gamma(\mathcal{V}', \mathcal{E}_{\mathcal{V}'}) \to R\Gamma(\mathcal{V}' \cap]X' \setminus U'[_\Lambda, \mathcal{E}_{\mathcal{V}'}).$$

Note that $\mathcal{E}_c$ does not depend on the choice of $\mathcal{V}'$; *cf.* [15, §3.2].

If we set $\mathcal{E}_{\mathcal{V}'} := \mathcal{O}_{\mathcal{V}'}$ (respectively, $\mathcal{E}_{\mathcal{V}'} := \Omega_{\mathcal{V}'}$), then the corresponding $\mathcal{E}_c$ is denoted as $(\mathcal{O}^\dagger_{U'/\Lambda})_c$ (respectively, $(\Omega^\dagger_{U'/\Lambda})_c$).

By shrinking $\mathcal{V}'$ if necessary, we may assume that $\mathcal{V}'$ is affinoid. In that case $\mathcal{V}' \cap]X' \setminus U'[_\Lambda$ is quasi-Stein, so we can deduce the following

- $\mathcal{E}_c$ is a Fréchet Λ-space.
- From the same argument as in [15, §3.2] we can deduce that

$$\mathcal{E}_c \cong \mathcal{E} \otimes_{\mathcal{O}^\dagger_{U'/\Lambda}} (\mathcal{O}^\dagger_{U'/\Lambda})_c. \tag{67}$$

  In particular, $\mathcal{E}_c$ depends only on $\mathcal{E}$, not on the choice of strict neighbourhood $\mathcal{V}'$.

Lemma B.2.

(i) *There is a canonical trace map*

$$\mathrm{tr} : (\Omega^\dagger_{U'/\Lambda})_c \to \Lambda,$$

which factors through an isomorphism $H^2_{\mathrm{rig},c}(U'/\Lambda) \xrightarrow{\sim} \Lambda$.

(ii) *Let $\mathcal{E}_{\mathcal{V}'}$ be a vector bundle on some strict neighbourhood $\mathcal{V}'$ of $]U'[_\Lambda$, and consider $\mathcal{E}$ as above. Then we have the natural Λ-bilinear perfect pairing*

$$\langle -,- \rangle^0 : \mathcal{E}^\vee \times (\mathcal{E} \otimes_{\mathcal{O}^\dagger_{U'/\Lambda}} (\Omega^\dagger_{U'/\Lambda})_c) \longrightarrow \Lambda \quad \langle u, m \otimes \omega_c \rangle^0 := \mathrm{tr}(u(m) \cdot \omega_c);$$
$$\langle -,- \rangle^1 : (\mathcal{E}^\vee \otimes_{\mathcal{O}^\dagger_{U'/\Lambda}} \Omega^\dagger_{U'/\Lambda}) \times \mathcal{E}_c \longrightarrow \Lambda \quad \langle u \otimes \omega, m \otimes f_c \rangle^1 := \mathrm{tr}(u(m) \cdot (\omega \otimes f_c))$$

where $u \in \mathcal{E}^\vee$, $m \in \mathcal{E}$, $f_c \in (\mathcal{O}^\dagger_{U'/\Lambda})_c$, $\omega_c \in (\Omega^\dagger_{U'/\Lambda})_c$ and $\omega \in \Omega^\dagger_{U'/\Lambda}$.

Proof. The first assertion is standard (cf. the proof of [15, lemme 3.4(i)]). If $X' \cong \mathbb{P}^1$ and $U' \cong \mathbb{A}^1$, then the second and third assertions are proved in [15, §3.3]. In general, one can find a finite covering $f : X' \to \mathbb{P}^1_{k_\Lambda}$ so that the preimage of $\mathbb{A}^1_{k_\Lambda}$ is U'. Then although f may not admit a formal lift, one can find a strict neighbourhood $\mathcal{W}'$ of $]U'[_\Lambda$ contained in $\mathcal{V}'$, such that its image in $\mathbb{P}^1_\Lambda$ is a strict neighbourhood of $]\mathbb{A}^1[_\Lambda$; cf. the proof of Proposition 5.2.21 in [25, p. 151]. Since the claim can be checked after push-forward by finite covering, the claim is reduced to the case when $U' = \mathbb{A}^1$, which is already handled. □

B.2. *Overconvergent F-isocrystals.* Let us choose a Λ_0-linear q_0-Frobenius operator

$$\varphi_{\Lambda_0} : \mathcal{O}^\dagger_{U/\Lambda_0} \to \mathcal{O}^\dagger_{U/\Lambda_0}, \tag{68}$$

which is possible by the approximation theorem; see [31, Theorem 2.4.4].

Let us recall the following standard definition.

Definition B.3. An *overconvergent Λ_0-F-isocrystal* over U is a tuple $(\mathcal{E}, \varphi_\mathcal{E}, \nabla_\mathcal{E})$, where

- $\mathcal{E}$ is a finite locally free $\mathcal{O}^\dagger_{U/\Lambda_0}$-module. Note that such $\mathcal{E}$ necessarily comes from a vector bundle on some strict neighbourhood $\mathcal{V}_0$ of $]U[_{\Lambda_0}$ via (66).
- $\nabla_\mathcal{E} : \mathcal{E} \to \mathcal{E} \otimes \Omega^\dagger_{U/\Lambda_0}$ is a continuous integrable connection on $\mathcal{E}$.
- $\varphi_\mathcal{E} : \mathcal{E} \to \mathcal{E}$ is a φ_{Λ_0}-semilinear horizontal endomorphism of $\mathcal{E}$.

If $\nabla_\mathcal{E}$ and $\varphi_\mathcal{E}$ are understood, then we simply use $\mathcal{E}$ to denote an overconvergent Λ_0-F-isocrystal.

For a finite extension Λ of Λ_0, we define a Λ-*action* on the overconvergent Λ_0-F-isocrystal $\mathcal{E}$ to be a Λ-action on the underlying module $\mathcal{E}$ that is compatible with the Λ_0-action and commutes with $\nabla_\mathcal{E}$ and $\varphi_\mathcal{E}$. To be explicit, an *overconvergent Λ_0-F-isocrystal over U with Λ-action* consists of the following data:

- $\mathcal{E}$ is a finite locally free module over $\mathcal{O}^\dagger_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda$.
- $\nabla_\mathcal{E} : \mathcal{E} \to \mathcal{E} \otimes \Omega^\dagger_{U/\Lambda_0}$ is a Λ-*linear* continuous integrable connection on $\mathcal{E}$.
- $\varphi_\mathcal{E} : \mathcal{E} \to \mathcal{E}$ is a Λ-*linear* φ_{Λ_0}-semilinear horizontal endomorphism of $\mathcal{E}$.

Since we have $\mathcal{O}^\dagger_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda \cong \mathcal{O}^\dagger_{U'/\Lambda}$ (64), we may view $\mathcal{E}$ as coming from a vector bundle on some strict neighbourhood $\mathcal{V}'$ of $]U'[_\Lambda$. Also from (65), the connection $\nabla_\mathcal{E}$ is defined over some strict neighbourhood. On the other hand, $\varphi_\mathcal{E}$ can be described more naturally if we view $\mathcal{E}$ as a module over $\mathcal{O}^\dagger_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda$ (instead of $\mathcal{O}^\dagger_{U'/\Lambda}$). Indeed, $\varphi_\mathcal{E}$ is semilinear over

$$\varphi_{\Lambda_0} \otimes \Lambda : \mathcal{O}^\dagger_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda \to \mathcal{O}^\dagger_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda,$$

the Λ-linear extension of φ_{Λ_0}, which cannot be naturally defined for $\mathcal{O}^\dagger_{U'/\Lambda}$ without going through the isomorphism (64).

Lemma B.4. (i) *Let φ_{Λ_0} also denote the endomorphism on $\Omega^\dagger_{U/\Lambda_0}$ induced by φ_{Λ_0}. Then we have*

$$\operatorname{tr}(\varphi_{\Lambda_0}(\omega)) = q_0 \cdot \operatorname{tr}(\omega) \quad \forall \omega \in \Omega^\dagger_{U/\Lambda_0},$$

where $q_0 = \#(k_0)$ and tr *is defined in Lemma B.2.*

(ii) *Let $\mathcal{E}$ be an overconvergent Λ_0-F-isocrystal over U with Λ-action, and let $\langle -,-\rangle^0$ denote the duality pairing in Lemma B.2*(ii). *Then for any $u \in \mathcal{E}^\vee$, $m \in \mathcal{E}$ and $\omega_c \in (\Omega^\dagger_{U/\Lambda_0})_c$, we have*

$$\langle \varphi^\vee_{\mathcal{E}}(u), \varphi_{\mathcal{E}}(m) \otimes \varphi_{\Lambda_0}(\omega_c)\rangle^0 = q_0 \cdot \langle u, m \otimes \omega_c\rangle^0.$$

Proof. As in the proof of Lemma B.2, one can reduce the proof of this lemma to the case when $U \cong \mathbb{A}^1$. In that case, the first claim is proved in [15, proposition 4.2], and the second is proved in [15, §4.3]. □

Definition B.5. Let $\mathcal{E}$ be an overconvergent Λ_0-F-isocrystal over U with Λ-action. Then we define the *Dwork operators* $\psi^i_{\mathcal{E}^\vee}$ for $i = 0, 1$ to be Λ-linear endomorphisms

$$\psi^0_{\mathcal{E}^\vee} : \mathcal{E}^\vee \to \mathcal{E}^\vee$$
$$\psi^1_{\mathcal{E}^\vee} : \mathcal{E}^\vee, \otimes_{\mathcal{O}^\dagger_{U'/\Lambda}} \Omega^\dagger_{U'/\Lambda} \to \mathcal{E}^\vee \otimes_{\mathcal{O}^\dagger_{U'/\Lambda}} \Omega^\dagger_{U'/\Lambda},$$

which are respectively the adjoints of $\varphi_{\mathcal{E}} \otimes \varphi_{\Lambda_0}$ and $\varphi_{\mathcal{E}_c}$ with respect to the duality pairings $\langle -,-\rangle^i$ for $i = 0, 1$; see Lemma B.4(ii).

Clearly, the Dwork operators $\psi^i_{\mathcal{E}^\vee}$ are φ_{Λ_0}-antilinear; i.e., for any $f \in \mathcal{O}^\dagger_{U/\Lambda_0}$ and $u \in \mathcal{E}^\vee$, we have $\psi^0_{\mathcal{E}^\vee}(\varphi_{\Lambda_0}(f) \cdot u) = f \cdot \psi^0_{\mathcal{E}^\vee}(u)$, and similarly for $\psi^1_{\mathcal{E}}$. Furthermore, by Lemma B.4 it follows that

$$\psi^0_{\mathcal{E}^\vee} = q_0 \cdot \varphi^\vee_{\mathcal{E}}. \tag{69}$$

To make $\psi^1_{\mathcal{E}^\vee}$ more explicit, let us consider the case when $\mathcal{E} = \mathcal{O}^\dagger_{U/\Lambda_0}$ (with $\Lambda = \Lambda_0$), equipped with the Frobenius operator φ_{Λ_0} and the usual connection. Then the duality pairing $\langle -,-\rangle^1$ in Lemma B.2(ii) takes the following form

$$\Omega^\dagger_{U/\Lambda_0} \times (\mathcal{O}^\dagger_{U/\Lambda_0})_c \to \Lambda_0.$$

Let

$$\psi_{\Lambda_0} : \Omega^\dagger_{U/\Lambda_0} \to \Omega^\dagger_{U/\Lambda_0} \tag{70}$$

denote the adjoint of $\varphi_{\Lambda_0} : (\mathcal{O}^\dagger_{U/\Lambda_0})_c \to (\mathcal{O}^\dagger_{U/\Lambda_0})_c$. Identifying $\Omega^\dagger_{U/\Lambda_0}$ with Λ_0-linear dual of $(\mathcal{O}^\dagger_{U/\Lambda_0})_c$, it follows that $\psi_{\Lambda_0} = q_0 \cdot \varphi^\vee_{\Lambda_0}$. We then have

$$\psi^1_{\mathcal{E}^\vee} = \psi^0_{\mathcal{E}^\vee} \otimes \psi_{\Lambda_0}.$$

B.3. ***Rigid cohomology with coefficients.*** Let us recall the definition of rigid cohomology with and without compact support with coefficients in Λ_0-F-isocrystals with Λ-actions.

Definition B.6. Let $\mathcal{E}$ be an overconvergent Λ_0-F-isocrystal over U with Λ-action. Suppose that U is affine. Then we set

$$R\Gamma_{\mathrm{rig}}(U/\Lambda_0, \mathcal{E}^\vee) := \Big[\mathcal{E}^\vee \xrightarrow{\nabla_{\mathcal{E}^\vee}} \mathcal{E}^\vee \otimes_{\mathcal{O}^\dagger_{U/\Lambda_0}} \Omega^\dagger_{U/\Lambda_0}\Big],$$

which is a complex of Fréchet Λ-spaces concentrated in degrees [0, 1].

This complex represents the rigid cohomology with coefficients in $\mathcal{E}$ viewed as an overconvergent Λ_0-F-crystal. Furthermore, the Dwork operator $\psi^\bullet_{\mathcal{E}^\vee}$ as in Definition B.5 acts on the complex $R\Gamma_{\mathrm{rig}}(U/\Lambda_0, \mathcal{E}^\vee)$ as a *nuclear* operator on each term; see [15, lemme 5.2].

To define the compactly supported variant, let us recall that we have a derivation

$$d : (\mathcal{O}^\dagger_{U/\Lambda_0})_c = H^1_{]U[_{\Lambda_0}}(\mathcal{X}_{\Lambda_0}, \mathcal{O}_{\mathcal{X}_{\Lambda_0}}) \longrightarrow H^1_{]U[_{\Lambda_0}}(\mathcal{X}_{\Lambda_0}, \Omega_{\mathcal{X}_{\Lambda_0}}) = (\Omega^\dagger_{U/\Lambda_0})_c$$

induced by the universal derivation $d : \mathcal{O}_{\mathcal{X}_{\Lambda_0}} \to \Omega_{\mathcal{X}_{\Lambda_0}}$.

Definition B.7. For $\mathcal{E}$ as before, we set

$$R\Gamma_{\mathrm{rig},c}(U/\Lambda_0, \mathcal{E}) := \Big[0 \to \mathcal{E} \otimes_{\mathcal{O}^\dagger_{U/\Lambda_0}} (\mathcal{O}^\dagger_{U/\Lambda_0})_c \xrightarrow{\nabla_{\mathcal{E}} \otimes d} \mathcal{E} \otimes_{\mathcal{O}^\dagger_{U/\Lambda_0}} (\Omega^\dagger_{U/\Lambda_0})_c\Big],$$

which is a complex of Fréchet Λ-spaces concentrated in degrees $[1, 2]$.

Note that this complex represents the compactly supported rigid cohomology with coefficients in $\mathcal{E}$ viewed as an overconvergent Λ_0-F-crystal. Furthermore, $\varphi_{\mathcal{E}}$ induces a natural $(\varphi_{\Lambda_0} \otimes \Lambda)$-semilinear operator $\varphi^\bullet_{\mathcal{E}_c}$ on the complex $R\Gamma_{\mathrm{rig},c}(U/\Lambda_0, \mathcal{E})$, which is a *nuclear* operator on each term; see [15, lemme 5.2].

Proposition B.8. *The duality pairing $\langle -, - \rangle^\bullet$ defined in Lemma B.2*(ii) *induces a natural Λ-linear isomorphism*

$$R\Gamma_{\mathrm{rig}}(U/\Lambda_0, \mathcal{E}^\vee) \cong R\,\mathrm{Hom}_\Lambda(R\Gamma_{\mathrm{rig},c}(U/\Lambda_0, \mathcal{E}), \Lambda[2]).$$

Furthermore, the Dwork operator $\psi^\bullet_{\mathcal{E}^\vee}$ corresponds to the Λ-linear dual of $\varphi^\bullet_{\mathcal{E}_c}$ via this isomorphism.

Proof. By repeating the proof of Lemma B.2(ii), the first claim can be reduced to the case when $U = \mathbb{A}^1$ and $X = \mathbb{P}^1$, which is proved in [15, §3.3]. The second claim directly follows from Lemma B.4. □

B.4. ***L-functions and Lefschetz trace formula.*** Let $\mathcal{E}$ be an overconvergent Λ_0-F-isocrystal over U with Λ-action as before. Then given any closed point x of U with residue field $k(x)$, we obtain the fibre $\mathcal{E}_x$ at x, which is a finite-rank $W(k(x)) \otimes_{W(k_0)} \Lambda$-module equipped with a Λ-linear q_0-Frobenius operator $\varphi_{\mathcal{E},x}$.

To such $\mathcal{E}$ we associate the L-functions as follows:

$$Z_U(\mathcal{E}, t) := \prod_{x \in |U|} \det_\Lambda\big(1 - (t \cdot \varphi_{\mathcal{E},x})^{[k(x):k_0]} \,|\, \mathcal{E}_x\big)^{-1}. \tag{71}$$

For any positive integer r, let $k_0^{(r)}$ be a degree-r extension of k_0 and set

$$S_r(U, \mathcal{E}) := \sum_{x \in U(k_0^{(r)})} \mathrm{tr}_\Lambda(\varphi^r_{\mathcal{E},x} \,|\, \mathcal{E}_x), \tag{72}$$

which is zero when k is not contained in $k_0^{(r)}$.

One can check that

$$Z_U(\mathcal{E}, t) = \exp\Big(\sum_{r=1}^{\infty} S_r(U, \mathcal{E}) \cdot t^r / r\Big); \tag{73}$$

see [15, 2.3].

The main goal of the appendix is to prove the following slight generalisation of the Lefschetz trace formula [15, théorème 6.3].

Theorem B.9. *Let $\mathcal{E}$ be an overconvergent Λ_0-F-isocrystal over U with Λ-action. Then we have*

$$\begin{aligned} Z_U(\mathcal{E}, t) &= \left(\det_\Lambda(1 - t \cdot \varphi^\bullet_{\mathcal{E}_c} \mid R\Gamma_{\mathrm{rig},c}(U, \mathcal{E}))\right)^{-1} \\ &= \left(\det_\Lambda(1 - t \cdot \psi^\bullet_{\mathcal{E}^\vee} \mid R\Gamma_{\mathrm{rig}}(U, \mathcal{E}^\vee))\right)^{-1}, \end{aligned}$$

where for $\theta^\bullet = \varphi^\bullet_{\mathcal{E}_c}$ or $\psi^\bullet_{\mathcal{E}^\vee}$ we let $\det_\Lambda(1 - t\theta^\bullet) := \prod_i \det_\Lambda(1 - t\theta^i)^{(-1)^i}$ *denote the alternating product Fredholm determinants of Λ-linear nuclear operators on each term.*

Note that the Lefschetz trace formula proved in [15, théorème 6.3] applies only to overconvergent Λ-F-isocrystals. In particular, it does not apply directly to our setting unless $\mathcal{O}_U$ contains the residue field k_Λ of Λ.

Proof. By the Poincaré duality it suffices to show the first equality (i.e., the equality via the compactly supported rigid cohomology). By (73) it suffices to show

$$S_r(U, \mathcal{E}) = \mathrm{tr}_\Lambda((\varphi^\bullet_{\mathcal{E}_c})^r \mid R\Gamma_{\mathrm{rig},c}(U, \mathcal{E})) \tag{74}$$

for any r. And to verify (74) it suffices to show that

$$\mathrm{tr}_\Lambda(\varphi^\bullet_{\mathcal{E}_c} \mid R\Gamma_{\mathrm{rig},c}(U, \mathcal{E})) = 0 \quad \text{if } U(k_0^{(r)}) = \varnothing; \tag{75}$$

i.e., the generalisation of [15, lemme 5.3]. Indeed, if k is not contained in $k_0^{(r)}$ then $S_r(U, \mathcal{E}) = 0$. If k is contained in $k_0^{(r)}$, then we may apply the excision sequence for the compactly supported rigid cohomology to the following setting

$$U(k_0^{(r)})^{\mathrm{Gal}(k_0^{(r)}/k)} \hookrightarrow U \hookleftarrow U \setminus U(k_0^{(r)})^{\mathrm{Gal}(k_0^{(r)}/k)}$$

and conclude that (75) implies (74).

To verify (75) we use the following lemma.

Lemma B.10. *Let R be a Fréchet Λ-algebra equipped with a Λ-linear lift of q_0^r-Frobenius morphism $\varphi : R \to R$. Let M be a Fréchet R-module equipped with a nuclear φ-semilinear endomorphism $\varphi_M : M \to M$. Then for any $f \in R$ we have*

$$\mathrm{tr}_\Lambda\left((f - \varphi(f)) \cdot \varphi_M\right) = 0$$

Proof. (Compare with the proof of [15, lemme 5.3].) Consider the morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r] \arrow[d, "0"'] & M \arrow[r, "f"] \arrow[d, "\varphi(f)\cdot\varphi_M"'] & M \arrow[r] \arrow[d, "f\cdot\varphi_M"] & M/fM \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \ker(f) \arrow[r] & M \arrow[r, "f"] & M \arrow[r] & M/fM \arrow[r] & 0 \end{tikzcd}.$$

It follows that $\mathrm{tr}_\Lambda(\varphi(f) \cdot \varphi_M) = \mathrm{tr}_\Lambda(f \cdot \varphi_M)$, hence the lemma. □

We will apply the above lemma to $R = \mathcal{O}^{\dagger}_{U/\Lambda_0} \otimes_{\Lambda_0} \Lambda$ and r-th iterated Frobenius operators. Since $U(k_0^{(r)}) = \varnothing$, the graph of the q_0^r-Frobenius and the diagonal do not intersect in $U \times U$. Therefore, there exist $f_j, g_j \in R$ such that

$$\sum_j f_j g_j = 1 \quad \text{and} \quad \sum_j \varphi^r(f_j) g_j = 0,$$

so we have $1 = \sum_j (f_j - \varphi^r(f_j)) \cdot g_j$.

Apply the above lemma when M is one of the terms in $R\Gamma_{\mathrm{rig},c}(U/\Lambda_0, \mathcal{E})$ and $\varphi_M = g_j(\varphi^{\bullet}_{\mathcal{E}_c})^r$ for each j, we obtain

$$\mathrm{tr}_\Lambda\big(\varphi_{\mathcal{E}_c}\big) = \mathrm{tr}_\Lambda\Big(\sum_j (f_j - \varphi^r(f_j)) g_j (\varphi^{\bullet}_{\mathcal{E}_c})^r\Big) = \sum_j \mathrm{tr}_\Lambda\big((f_j - \varphi^r(f_j)) g_j (\varphi^{\bullet}_{\mathcal{E}_c})^r\big) = 0,$$

which proves (75), hence the theorem. □

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Deformation rings and images of Galois representations

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Let $\mathscr{G}$ be a connected reductive almost simple group over the Witt ring $W(\mathbb{F})$ for $\mathbb{F}$ a finite field of characteristic p. Let R and R' be complete noetherian local $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. Under a mild condition on p in relation to structural constants of $\mathscr{G}$, we show the following results: (1) Every closed subgroup H of $\mathscr{G}(R)$ with full residual image $\mathscr{G}(\mathbb{F})$ is a conjugate of a group $\mathscr{G}(A)$ for $A \subset R$ a closed subring that is local and has residue field $\mathbb{F}$. (2) Every surjective homomorphism $\mathscr{G}(R) \to \mathscr{G}(R')$ is, up to conjugation, induced from a ring homomorphism $R \to R'$. (3) The identity map on $\mathscr{G}(R)$ represents the universal deformation of the representation of the profinite group $\mathscr{G}(R)$ given by the reduction map $\mathscr{G}(R) \to \mathscr{G}(\mathbb{F})$. This generalizes results of Dorobisz, Eardley and Manoharmayum, and in addition provides an abstract classification result for closed subgroups of $\mathscr{G}(R)$ with residually full image.

We provide an axiomatic framework to study this type of question, also for slightly more general $\mathscr{G}$, and we study in the case at hand in great detail what conditions on $\mathbb{F}$ or on p in relation to $\mathscr{G}$ are necessary for the above results to hold.

1. Introduction

Let R be a complete noetherian local ring with finite residue field $\mathbb{F}$. Let $\bar{\rho} : SL_n(R) \to \mathrm{SL}_n(\mathbb{F})$ be the induced group homomorphism. It is shown in [Dor16; Man15; EM16] that for all but finitely many pairs $(n, \mathbb{F})$ the universal deformation of $\bar{\rho}$ for representations into GL_n, in the sense of [Maz89], is represented by $\mathrm{id} : \mathrm{SL}_n(R) \to \mathrm{SL}_n(R)$. The proofs were based in part on quite long and explicit computations. Our first observation was that these computations could be avoided almost entirely by a systematic use of methods from deformation theory and of largely well-known results on reductive groups over finite fields, their Lie algebras and some related cohomology groups. For a precise list of conditions, we refer the reader to the axiomatic framework introduced in Section 2. This axiomatic viewpoint, allows us to obtain the same result for an arbitrary absolutely simple connected reductive group $\mathscr{G}$ over the ring of Witt vectors $W(\mathbb{F})$ in place of SL_n. In fact, we have extensions to certain nonconnected $\mathscr{G}$ with $\mathscr{G}^o$ as in the previous line, or to $\mathscr{G} \subset \mathscr{G}'$ modeling $\mathrm{SL}_n \subset \mathrm{GL}_n$; for the relevant deformation theory, we refer to [Til96].

A second insight was that, by extending ideas of Boston from [MW86, Appendix], within our axiomatic framework we can obtain general results on closed subgroups of $\mathrm{SL}_n(R)$ (or $\mathscr{G}(R)$) with full residual image. We recover [Man15, Main Theorem] in nearly all cases, and we generalize it to our setting: given an injection $W(\mathbb{F}) \to R$, closed subgroups of $\mathscr{G}(R)$ with residual image $\mathscr{G}(\mathbb{F})$ typically contain a conjugate

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of $\mathscr{G}(W(\mathbb{F}))$. More generally, we show that certain closed subgroups H of $\mathscr{G}(R)$ with full residual image are simply conjugates of groups $\mathscr{G}(A)$ with A a closed subring of R.

Let us give a concrete theorem that highlights the main results of the present article. Let $\mathscr{G}$ be a connected absolutely simple linear algebraic group scheme over the ring of Witt vectors $W(\mathbb{F})$ of a finite field $\mathbb{F}$ of characteristic p. Assume that $p \geq 5$, that p does not divide $n+1$ if $\mathscr{G}$ is of type A_n, and that $\mathbb{F} \neq \mathbb{F}_5$ if $\mathscr{G}$ is of type A_1 or C_n. Let $\widehat{Ar}_{W(\mathbb{F})}$ be the category of complete local $W(\mathbb{F})$-algebras that are filtered inverse limits of local Artin $W(\mathbb{F})$-algebras R with residue field $\mathbb{F}$. Denote by $\pi : R \to \mathbb{F}$ the residue homomorphism and by $\mathscr{G}(\pi)$ the induced homomorphism $\mathscr{G}(R) \to \mathscr{G}(\mathbb{F})$.

Theorem 1.1. *Let R, R' be in $\widehat{Ar}_{W(\mathbb{F})}$.*

(a) *If H is a closed subgroup of $\mathscr{G}(R)$ that surjects under π onto $\mathscr{G}(\mathbb{F})$ then there exists a closed $W(\mathbb{F})$-subalgebra $A \subset R$ in $\widehat{Ar}_{W(\mathbb{F})}$ such that H is conjugate to $\mathscr{G}(A)$.*

(b) *Let $\phi : \mathscr{G}(R) \to \mathscr{G}(R')$ be a surjective group homomorphism that on residue fields induces the identity of $\mathscr{G}(\mathbb{F})$. Then a conjugate of ϕ is the map $\mathscr{G}(\alpha)$ for some surjective ring homomorphism $\alpha : R \to R'$ in $\widehat{Ar}_{W(\mathbb{F})}$.*

(c) *The functor $\widehat{Ar}_{W(\mathbb{F})} \to$* **Sets**,

$$A \mapsto \{\mathscr{G}\text{-valued deformations of } \pi : \mathscr{G}(R) \to \mathscr{G}(\mathbb{F}) \text{ to } A\}$$

is representable and a universal deformation is given by the class of $\mathrm{id}_{\mathscr{G}(R)}$.

Parts (a) and (b) mean that all closed subgroups with maximal residual image and all homomorphisms between such must be 'linear'. We shall prove (a) and (c), from which (b) follows immediately by the definiton of deformation.

Remark 1.2. (a) Part (a) is an abstract big image theorem under strong residual hypotheses. For GL_2 there are more general results under weaker residual hypotheses in [Bel19] or [CLM23]. One of their aims is a general description of the image of certain p-adic families of automorphic forms as in [Hid15]. We plan to apply our results in a similar way in future work.

(b) Part (a) extends [Man15] by Manoharmayum. There it is proved, again for $\mathscr{G} = \mathrm{SL}_n$ and considered inside GL_n, that under the hypothesis of (a), a conjugate of H contains the subgroup $\mathscr{G}(W(\mathbb{F})_R)$ where $W(\mathbb{F})_R$ is the image under the structure morphism $W(\mathbb{F}) \to R$; it is a quotient ring of $W(\mathbb{F})$. Our part (a) implies this particular case for any $\mathscr{G}$ considered, since clearly $W(\mathbb{F})_R = W(\mathbb{F})_A \subset A$.

(c) Part (c) for $\mathscr{G} = \mathrm{SL}_n$ and for deformations of the given π but into GL_n is a result due to Dorobisz [Dor16] and, independently, Eardly and Manoharmayum; [EM16]. Their results are more complete than what we state above and also include $p = 2, 3$. In [Dor16], Dorobisz also characterizes the finitely many exceptions for small p, by giving counterexamples. In Corollary 5.5 we completely recover their list from our axiomatic framework.

(d) Part (c) implies in particular that any complete noetherian local ring occurs as a universal deformation ring, a question posed in [BCdS13, Question 1.1]. It was the motivation behind [Dor16] and [EM16], and was answered for GL_n in these papers.

(e) For the most general results in the sense of Theorem 1.1(a), we refer to Corollaries 6.3 and 6.4, and for those in the sense of Theorem 1.1(c), to Corollaries 5.4 and 5.5.

We now give an outline of the article. In Section 2, we present our basic set-up to be used throughout the remainder of this article and we formulate a number of axiomatic conditions. They have occurred, at least implicitly, in some form in the deformation theory of Galois representations for GL_n. These axiomatic conditions are admittedly somewhat technical. But they match very well with what is needed in our proofs of the main results in Sections 5 and 6. Some basic facts on affine group schemes over general bases used in our axiomatic presentation and throughout this work are collected in the Appendix. In Section 2, we also formulate technical versions of Theorem 1.1(a) and (c), based on our axiomatic conditions. Moreover the section contains, before Definition 2.11, the technically important assignment $H \mapsto H^c$ for closed subgroups H of $\mathscr{G}(R)$ with certain residual images. The group H^c is a variant of the commutator subgroup that preserves the residual image.

In Section 3 we hope to convince the reader that our axiomatic conditions are natural by proving that they are satisfied for connected absolutely simple reductive groups $\mathscr{G}$ over $W(\mathbb{F})$ with 'few' exceptions. Theorem 1.1 gives a good idea of what 'few' might mean, namely that they hold whenever p is 'large' in comparison to data coming from $\mathscr{G}$. Section 3 gives a thorough investigation of when precisely our axiomatic conditions are satisfied for absolutely simple $\mathscr{G}$, even if p is small! There is a finite list of 'obvious' exceptions. But the validity of our conditions depends, for small p, also on further invariants of $\mathscr{G}$, such as its center or the type of its root system; see Theorem 3.2 for a summary. In addition, in Section 3.7 we also study the validity of our axiomatic conditions for a situation $\mathscr{G} \subset \mathscr{G}'$ that resembles that of $\mathrm{SL}_n \subset \mathrm{GL}_n$; see Theorem 3.52. We hope that our thorough analysis of the small prime situation, which makes up almost half of this article, is also useful to others since, as mentioned earlier, several of the conditions we investigate also occur in deformation theory. Our review of the literature also discovered several untreated cases concerning the 1-cohomology of the adjoint module of some Chevalley groups, see Remark 3.44. We hope to get back to these open cases in the near future. The other conditions that occur in our axiomatic treatment given in Conditions 2.5, when specialized to the setting of Chevalley groups in Section 3 we were able to treat in an essentially complete fashion by combining and completing results from the literature.

Section 4 contains preparations for the proofs of our main results. We lay a number of elementary foundations of a group theoretic nature, and we explain how our axiomatic conditions are used to deduce results on group extensions, commutators etc. Some ideas are taken from [MW86, Appendix] by Boston. The rather elementary proofs of our main results are given in Sections 5 and 6. They make essential use of standard results on universal deformations and build on the preparations from Section 4. The proofs are short and contain next to no explicit computations unlike [Man15; Dor16; EM16]. The reason is our method of proof. In concrete cases such as Theorem 1.1, we can rely on the extensive results gathered in Section 3 that are by now standard results on algebraic groups over finite fields.

Let us end this introduction with a general question in the spirit of [BCdS13, Questions 1.1 and 1.2] but from a slightly different perspective. Suppose that Π is a profinite group, and $\bar\rho:\Pi\to \mathrm{GL}_n(\mathbb{F})$ is a continuous homomorphism with trivial determinant.[1] Consider $\mathbb{F}^n$ as being acted on by Π via $\bar\rho$ and assume that the natural map $\mathbb{F}\to \mathrm{End}_\Pi(\mathbb{F}^n)$ is an isomorphism. Then $\bar\rho$ possesses a universal deformation $\rho^u:\Pi\to \mathrm{GL}_n(R^u)$ with trivial determinant (unique up to unique conjugacy) with $R^u\in \widehat{Ar}_{W(\mathbb{F})}$; essentially by [Maz89] but with needed additions from [dSL97, Theorem 2.3]. Let H be the closed compact subgroup $\rho^u(\Pi)$ of $\mathrm{GL}_n(R^u)$ and consider the diagram

$$\begin{tikzcd} \Pi \arrow[r] & H \arrow[r, hook, "\rho_H"] \arrow[dr, "\bar\rho_H"'] & \mathrm{GL}_n(R^u) \arrow[d] \\ & & \mathrm{GL}_n(\mathbb{F}), \end{tikzcd}$$

where the composed maps from Π to $\mathrm{GL}_n(R^u)$ and to $\mathrm{GL}_n(\mathbb{F})$ are ρ^u and $\bar\rho$, respectively. Using again [dSL97, Theorem 2.3] (or Remark 2.9), it is straightforward to deduce the following.

Fact 1.3. The universal deformation of $\bar\rho_H$ (with trivial determinant) exists and is represented by ρ_H.

In light of Theorem 1.1 the following seems a natural question to us:

Question 1.4. For which subgroups $H_{\mathbb{F}}$ of $\mathrm{SL}_n(\mathbb{F})$ can one classify closed subgroups of $\mathrm{GL}_n(R)$ with $R\in \widehat{Ar}_{W(\mathbb{F})}$ in a way similar to Theorem 1.1(a)?

In cases where the question has a reasonable answer, the image H of a universal deformation ρ^u has a uniform description, in which R^u depends on Π, but the shape of H depends on R^u and $H_{\mathbb{F}}$ only, not on Π in any further way. A result for GL_2, very much in this direction, is [Bel19, Theorem 7.2.3].

Notation and conventions.

- p will denote a prime number, $\mathbb{F}_p$ is the field of cardinality p, $\mathbb{F}$ is a finite field of characteristic p and $W(\mathbb{F})$ its ring of Witt vectors.
- $\widehat{Ar}_{W(\mathbb{F})}$ is the category of complete local $W(\mathbb{F})$-algebras R that are filtered inverse limits of local Artin $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. The maximal ideal of R will be denoted $\mathfrak{m}_R$.
- The ring of dual numbers $\mathbb{F}[X]/(X^2)\in \widehat{Ar}_{W(\mathbb{F})}$ will be denoted by $\mathbb{F}[\varepsilon]$.
- Algebraic groups will always be denoted by capital script letters such as $\mathscr{G}$; abstract groups by roman letters such as H; Lie algebras by small gothic letters such as $\mathfrak{g}$; for the Lie algebra of $\mathscr{G}$ we write $\mathrm{Lie}(\mathscr{G})$ but also often simply $\mathfrak{g}$.

[1]For simplicity we phrase the question for GL_n and assume trivial determinant; the generalization to other $\mathscr{G}$ and more general "determinants" is left to the reader.

- For a smooth group scheme $\mathscr{G}$ over $W(\mathbb{F})$, we denote by $\mathscr{G}^o$ its connected component and $\mathscr{G}^{\mathrm{der}}$ the commutator subgroup $[\mathscr{G}^o, \mathscr{G}^o]$. If $\mathscr{G}$ is connected and semisimple, we denote by $\mathscr{G}^{\mathrm{sc}}$ its simply connected cover and by $\mathscr{G}^{\mathrm{ad}}$ its adjoint group.
- The center of a group H or an algebraic group $\mathscr{G}$ or a Lie algebra $\mathfrak{g}$ will be $Z(H)$ or $Z(\mathscr{G})$ or $Z(\mathfrak{g})$, respectively; see also A.9.
- The socle of a representation is the largest semisimple subrepresentation; the socle of a Lie algebra is the largest semisimple Lie subalgebra. Dually one defines the cosocle as the largest semisimple quotient.
- We follow the standard convention that the types of classical groups are $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(C_n)_{n\geq 3}$ and $(D_n)_{n\geq 4}$.

2. An axiomatic framework

Let us begin by introducing some notation. By $\mathscr{G}$ we denote a smooth group scheme over $W(\mathbb{F})$ whose connected component $\mathscr{G}^o$ is reductive over $W(\mathbb{F})$, and such that $\mathscr{G}/\mathscr{G}^o$ is a constant group of order prime to p, see A.1, A.3, A.6 and A.7 in the Appendix. We write $\mathscr{G}^{\mathrm{der}}$ for the commutator subgroup $[\mathscr{G}^o, \mathscr{G}^o]$, see A.16. The group $\mathscr{G}^{\mathrm{der}}$ is a characteristic subgroup of $\mathscr{G}$ because $\mathscr{G}^o$ is characteristic in $\mathscr{G}$ and $\mathscr{G}^{\mathrm{der}}$ in $\mathscr{G}^o$, and in particular it is normal in $\mathscr{G}$. It is semisimple over $W(\mathbb{F})$ and $\mathscr{G}^o/\mathscr{G}^{\mathrm{der}}$ is a torus, see A.16. Moreover $\mathscr{G}/\mathscr{G}^{\mathrm{der}}$ exists as a smooth group scheme over $W(\mathbb{F})$, and it is an extension of the constant group $\mathscr{G}/\mathscr{G}^{\mathrm{der}}$ by the torus $\mathscr{G}^o/\mathscr{G}^{\mathrm{der}}$, see A.18. We write $\mathscr{G}_{\mathbb{F}}$, $\mathscr{G}^o_{\mathbb{F}}$, $\mathscr{G}^{\mathrm{der}}_{\mathbb{F}}$ for the special fibers of $\mathscr{G}$, $\mathscr{G}^o$ and $\mathscr{G}^{\mathrm{der}}$, respectively, and note that $(\mathscr{G}_{\mathbb{F}})^o = \mathscr{G}^o_{\mathbb{F}}$, that $\mathscr{G}^o_{\mathbb{F}}$ is reductive and that $\mathscr{G}^{\mathrm{der}}_{\mathbb{F}}$ is semisimple; see A.3 and A.8. We denote the Lie algebras of $\mathscr{G}_{\mathbb{F}}$ and $\mathscr{G}^{\mathrm{der}}_{\mathbb{F}}$ by $\mathfrak{g}$ and $\mathfrak{g}^{\mathrm{der}}$, respectively; cf. A.2. Via the adjoint representation $\mathfrak{g}$ and $\mathfrak{g}^{\mathrm{der}}$ carry an action of $\mathscr{G}(\mathbb{F})$ and $\mathscr{G}^{\mathrm{der}}(\mathbb{F})$, respectively. Note also that $\mathfrak{g} = \mathrm{Lie}(\mathscr{G}^o_{\mathbb{F}})$.

Throughout this article, we fix a pair $(H_{\mathbb{F}}, H'_{\mathbb{F}})$ consisting of a subgroup $H_{\mathbb{F}} \subset \mathscr{G}(\mathbb{F})$ and a normal subgroup $H'_{\mathbb{F}}$ of $H_{\mathbb{F}}$ such that $H'_{\mathbb{F}} \subset \mathscr{G}^{\mathrm{der}}(\mathbb{F})$. We make the following

Assumption 2.1 (standard hypothesis). The tuple $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$[2] satisfies:

(a) $H_{\mathbb{F}}$ surjects onto $(\mathscr{G}/\mathscr{G}^o)(\mathbb{F})$ and $H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$ is of order prime to p.

(b) There exists a subgroup $M_{\mathbb{F}}$ of $H_{\mathbb{F}}$ of order prime to p such that $M_{\mathbb{F}} H'_{\mathbb{F}} = H_{\mathbb{F}}$.

Remark 2.2. If $H'_{\mathbb{F}}$ is contained in $[H_{\mathbb{F}}, H_{\mathbb{F}}]$, for instance if $H'_{\mathbb{F}}$ is perfect, then the quotient $H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$ being of order prime to p is implied by (b): By hypothesis, $H'_{\mathbb{F}}$ is normal in $H_{\mathbb{F}}$. If in addition $H'_{\mathbb{F}}$ is contained in $[H_{\mathbb{F}}, H_{\mathbb{F}}]$, then we have a surjective homomorphism $H_{\mathbb{F}}/H'_{\mathbb{F}} \to H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$. Now (b) gives the isomorphism $M_{\mathbb{F}}/(M_{\mathbb{F}} \cap H'_{\mathbb{F}}) \cong H_{\mathbb{F}}/H_{\mathbb{F}}'$, showing that $H_{\mathbb{F}}/H_{\mathbb{F}}'$ and hence $H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$ is of order prime to p, because $M_{\mathbb{F}}$ is so.

The possible presence of $\mathscr{G}/\mathscr{G}^o$ allows for instance that $\mathscr{G} \cong \mathscr{G}^o \rtimes \mathrm{Gal}(L/K)$ where L/K is a finite Galois extension of global fields (of order prime to p).

[2] We often simply refer to the pair $(H_{\mathbb{F}}, H'_{\mathbb{F}})$, the group $\mathscr{G}$ and the field $\mathbb{F}$ being implicitly understood.

Definition 2.3. For $A \in \widehat{Ar}_{W(\mathbb{F})}$ we define $H'_A := \{g \in \mathscr{G}^{\mathrm{der}}(A) \mid g \pmod{\mathfrak{m}_A} \in H'_{\mathbb{F}}\}$.

Let $M^o_{\mathbb{F}} = M_{\mathbb{F}} \cap \mathscr{G}^o(\mathbb{F})$. Since $M_{\mathbb{F}}$ is of order prime to p, by Lemma 4.1 for $A \in \widehat{Ar}_{W(\mathbb{F})}$ there exist subgroups $M^o_A \subset \mathscr{G}(A)$ and $M_A \subset \mathscr{G}(A)$ that modulo $\mathfrak{m}_A$ reduce isomorphically to $M^o_{\mathbb{F}}$ and $M_{\mathbb{F}}$, respectively, they both normalize H'_A, and the group $M^o_A H'_A$ is independent of the choice of M^o_A. Our hypotheses imply $M^o_A \subset \mathscr{G}^o(A)$. Throughout this article we impose the following convention.

Convention 2.4. We fix lifts

$$M^o_{W(\mathbb{F})} \subset M_{W(\mathbb{F})} \subset \mathscr{G}(W(\mathbb{F}))$$

of $M^o_{\mathbb{F}} \subset M_{\mathbb{F}}$, for which $M^o_{W(\mathbb{F})} \to M^o_{\mathbb{F}}$ and $M_{W(\mathbb{F})} \to M_{\mathbb{F}}$ are isomorphisms under reduction. For A in $\widehat{Ar}_{W(\mathbb{F})}$, we define M^o_A (resp. M_A) as the image of $M^o_{W(\mathbb{F})}$ (resp. $M_{W(\mathbb{F})}$) under the structure morphism $W(\mathbb{F}) \to A$.

We set

$$H^o_A := M^o_A H'_A, \qquad H_A := M_A H'_A. \tag{2-1}$$

Recall that a Lie algebra $\mathfrak{h}$ is called perfect if $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ for the commutator Lie subalgebra.

Conditions 2.5. For a tuple $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfying Assumption 2.1, we formulate the following conditions:

- **(pf)** The group $H'_{\mathbb{F}}$ is perfect.
- **(ct)** The natural inclusion Lie $Z(\mathscr{G}^o) \hookrightarrow H^0(H_{\mathbb{F}}, \mathfrak{g})$ is an isomorphism, and moreover the schematic center of $\mathscr{G}$ is smooth.
- **(l-ge)** (i) $\mathfrak{g}^{\mathrm{der}}$ is perfect and the center $Z(\mathfrak{g}^{\mathrm{der}})$ of $\mathfrak{g}^{\mathrm{der}}$ is trivial, (ii) $\mathfrak{g}^{\mathrm{der}}$ is irreducible and nontrivial as an $\mathbb{F}_p[H'_{\mathbb{F}}]$-module, and (iii) the natural map $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}})$ is bijective.[3]
- **(l-un)** (i) As an $\mathbb{F}_p[H'_{\mathbb{F}}]$-module, $[\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}] \subset \mathfrak{g}^{\mathrm{der}}$ is nontrivial, and one of the Jordan–Hölder factors of $[\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}]$ is not a Jordan–Hölder factor of $\mathfrak{g}^{\mathrm{der}}/[\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}]$, and (ii) the natural map $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}})$ is bijective.
- **(l-cl)** (i) $\mathfrak{g}^{\mathrm{der}}$ is perfect and (ii) the $\mathbb{F}_p[H'_{\mathbb{F}}]$-cosocle $\bar{\mathfrak{g}}^{\mathrm{der}}$ of $\mathfrak{g}^{\mathrm{der}}$ is irreducible and $H_{\mathbb{F}}$ acts trivially on $\mathrm{Ker}(\mathfrak{g}^{\mathrm{der}} \to \bar{\mathfrak{g}}^{\mathrm{der}})$.
- **(csc)** The cosocle of $\mathfrak{g}^{\mathrm{der}}$ does not contain the trivial $H'_{\mathbb{F}}$-module $\mathbb{F}_p$.
- **(van)** The cohomology $H^1(H'_{\mathbb{F}}, \mathfrak{g})$ vanishes.
- **(sch)** The mod p Schur multiplier group $H^2(H'_{\mathbb{F}}, \mathbb{F}_p)$ vanishes.
- **(n-s)** The extension $1 \to \mathfrak{g}^{\mathrm{der}} \to H'_{W_2(\mathbb{F})} \to H'_{\mathbb{F}} \to 1$ is nonsplit.

Remark 2.6. It is straightforward to see that the following conditions are equivalent: (i) condition **(csc)**, (ii) $\mathrm{Hom}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}}, \mathbb{F}_p) = 0$, (iii) $H_0(H'_{\mathbb{F}}, \mathfrak{g}^{\mathrm{der}}) = 0$.

Remark 2.7. We have the implications

(l-ge)(ii) $\Longrightarrow$ **(csc)**, **(l-ge)**(i)+(ii) $\Longrightarrow$ **(l-cl)**, **(l-ge)**(ii)+(iii) $\Longrightarrow$ **(l-un)**.

[3]For any $\mathbb{F}[G]$-module V there is a canonical homomorphism $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[G]}(V)$.

In Section 3 we discuss in great detail the case when $\mathscr{G}$ is connected, $\mathscr{G}^{\mathrm{der}}$ is absolutely simple, $H_{\mathbb{F}} = \mathscr{G}^{\mathrm{der}}(\mathbb{F})$ and $H'_{\mathbb{F}} \supset [H_{\mathbb{F}}, H_{\mathbb{F}}]$. Because of the Lie-part of Corollary 3.5 we like to think that **(l-ge)** describes the 'generic' behavior of $\mathfrak{g}^{\mathrm{der}}$. As was just observed, **(l-ge)** implies **(l-un)** and **(l-cl)**. The latter conditions will be a crucial input about the Lie algebra $\mathfrak{g}^{\mathrm{der}}$ in our main theorems on certain **un**iversal deformation rings and on **cl**osed subgroups of $\mathscr{G}(R)$, for $R \in \widehat{Ar}_{W(\mathbb{F})}$, respectively.

We shall now state the main technical results of this work and use them to derive Theorem 1.1. For this let R be in $\widehat{Ar}_{W(\mathbb{F})}$ and consider the canonical reduction

$$\bar{\rho}_R : H_R \to \mathscr{G}^{\mathrm{der}}(\mathbb{F}) \subset \mathscr{G}(\mathbb{F})$$

as a $\mathscr{G}(\mathbb{F})$-valued representation of H_R. A $\mathscr{G}$-deformation of $\bar{\rho}_R$ to a ring A (in $\widehat{Ar}_{W(\mathbb{F})}$) is a $\mathrm{Ker}(\mathscr{G}(A) \to \mathscr{G}(\mathbb{F}))$-conjugacy class of continuous homomorphisms $\rho_A : H_R \to \mathscr{G}(A)$ such that $\rho_A \equiv \bar{\rho}_R \pmod{\mathfrak{m}_A}$.

Lemma 2.8. *If* **(ct)** *holds, then the functor*

$$D_{\bar{\rho}_R} : \widehat{Ar}_{W(\mathbb{F})} \to \mathbf{Sets},\ A \mapsto \{\mathscr{G}\text{-valued deformations of } \bar{\rho}_R \text{ to } A\}$$

is pro-representable within $\widehat{Ar}_{W(\mathbb{F})}$.

Proof. If H_R satisfies the finiteness condition formulated by Mazur, and in particular if R is finite discrete, the result follows from [Til96, Theorem 3.3] (an adaption of [Maz89, Proposition 1]). The argument in the general case bears some similarities to the proof of Theorem 2.3 in [dSL97], from which the above result follows in the case $\mathscr{G} = \mathrm{GL}_n$.

In the general case, we let $(I_j)_{j\in J}$ be the set of proper open ideals of R, so that all I_j are contained in $\mathfrak{m}_R$ and the quotients $R_j := R/I_j$ are finite discrete local Artin rings with residue field $\mathbb{F}$. We have $R = \lim_j R_j$ and $H_R = \lim_j H_{R_j}$ as filtered inverse limits in $\widehat{Ar}_{W(\mathbb{F})}$ and in profinite groups, respectively, and where the R_j and H_{R_j} are finite with the discrete topology. Then by [Til96, Theorem 3.3], we have for each j a universal pair $(R_j^u, [\rho_j^u])$ for $D_{\bar{\rho}_{R_j}}$, where $\rho_j^u : H_{R_j} \to \mathscr{G}(R_j^u)$ is a representative of the universal deformation class.

Let j, k be in J such that I_k is contained in I_j, and denote by $h_{k,j} : H_{R_k} \to H_{R_j}$ the map induced from $R/I_k \to R/I_j$. Now observe that by the universality of ρ_k^u we have a unique homomorphism $\pi_{k,j} : R_k \to R_j$ in $\widehat{Ar}_{W(\mathbb{F})}$ such that $\mathscr{G}(\pi_{k,j}) \circ \rho_k^u = \rho_j^u \circ h_{k,j}$. From the universality of the $(R_j^u)_{j\in J}$ one moreover easily deduces that the maps $\pi_{j,k}$ form an inverse system in $\widehat{Ar}_{W(\mathbb{F})}$. Define R^u as the filtered inverse lim R_j^u under these maps and ρ^u as the map $H_R \to \mathscr{G}(R^u)$ that is the filtered inverse limit of the $\rho_j^u : H_{R_j} \to \mathscr{G}(H_{R_j})$. One verifies that $(R^u, [\rho^u])$ is a universal object for $D_{\bar{\rho}_R}$. □

Remark 2.9. This proof, with obvious modifications, works for any residual continuous representation $\Pi \to \mathscr{G}(\mathbb{F})$ with Π any profinite group.

Definition 2.10. We denote the universal ring representing $D_{\bar{\rho}_R}$ by $R_{\bar{\rho}_R}$ and a representative of the universal deformation by $\rho_{\bar{\rho}_R} : H_R \to \mathscr{G}(R_{\bar{\rho}_R})$.

The first main technical result of this article, which we shall prove in Section 5, is the following:

Theorem 5.2. *Suppose* $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ *satisfies conditions* **(ct)**, **(van)**, **(n-s)**, **(l-un)** *and one of* **(csc)** *or* $\mathscr{G} = \mathscr{G}^{\mathrm{der}}$. *Then the canonical inclusion* $\iota : H_R \to \mathscr{G}(R)$ *represents the universal deformation of* $D_{\bar{\rho}_R}$, *and in particular* $R_{\bar{\rho}_R} = R$.

In Remark 5.6, we shall explain the relation between Theorem 5.2 and the results of Dorobisz, Eardly and Manoharmayum.

For the second main result, let H denote any closed subgroup of $\mathscr{G}(R)$ such that the image of H in $\mathscr{G}(\mathbb{F})$ is equal to $H'_{\mathbb{F}}$. In Lemma 4.2 we shall prove that there exists a unique closed subgroup $H^c \subset H$ that contains the closure $\overline{[H, H]}$ of the commutator subgroup and for which $H^c/\overline{[H, H]} \to H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$ is an isomorphism. Note that by definition H^c is a closed subgroup of H that surjects onto $H_{\mathbb{F}}$. We shall also show that $H \mapsto H^c$ commutes with passing from R to a quotient.

Definition 2.11. The group H is called $H_{\mathbb{F}}$*-perfect* if $H = H^c$.

Definition 2.12. For any H as above, define $H^{(0)} := H$ and, inductively, for any $i \geq 1$ define $H^{(i)} := (H^{(i-1)})^c$. The $H^{(i)}$ define a descending sequence of closed subgroups of H.

The $H_{\mathbb{F}}$*-perfection* of H is defined as $H^{(\infty)} := \bigcap_i H^{(i)}$.

Note that for any artinian quotient of R the procedure defining $H^{(\infty)}$ stops after finitely many steps. From R being a filtered inverse limit of such rings and from Lemma 4.2, it follows that $H^{(\infty)}$ is $H_{\mathbb{F}}$-perfect. We call a closed subgroup H of H_R *residually full* if under the reduction map H surjects onto $H_{\mathbb{F}}$.

In Section 6 we treat the second main technical result of this article:

Theorem 6.1. *Let* $H \subset H_R$ *be a closed subgroup that is residually full. Suppose that* **(ct)**, **(n-s)** *and* **(van)** *hold, and that either* **(l-ge)** *holds or that* **(l-cl)** *and* **(sch)** *hold. Then there exists a closed* $W(\mathbb{F})$*-subalgebra* A *of* R *such that* $H^{(\infty)}$ *is conjugate to* $H_A \subset \mathscr{G}(R)$.

In Remark 6.8 we shall explain its relation to a result by Manoharmayum.

Proof of Theorem 1.1. In Corollary 3.5 we show that under the hypotheses of Theorem 1.1 conditions **(pf)**, **(ct)**, **(l-ge)**, **(sch)**, **(n-s)** and **(van)** hold. By Remark 2.7, also condition **(l-un)** holds. Hence Theorem 1.1(a) and (c) are immediate from Theorem 5.2 and Theorem 6.1. Using the universal deformation property of $\mathrm{id} : \mathscr{G}(R) \to \mathscr{G}(R)$, the proof of (b) follows from (a), (c) and Lemma 4.3. □

We end this section with some remarks on our standard hypotheses from Assumption 2.1 in relation to speculations about compatible systems of Galois representations attached to pure motives with coefficients; cf. the foundational reference [Ser94] and also [Hui25].

Let G be an affine group scheme over a number field F such that the identity component G^o of G is reductive. Denote by P_F the set of finite places of F, by F_λ the completion of F at $\lambda \in P_F$, by $\mathscr{O}_\lambda$ its ring of integers and by $\mathbb{F}_\lambda$ its residue field. Let k be a number field with absolute Galois group Γ_k; we write Frob_v for a Frobenius automorphism of the finite place $v \in P_k$. We write ℓ_v and ℓ_λ for the rational prime below v and λ, respectively, we set $S_\lambda := \{v \in P_k \mid \ell_v = \ell_\lambda\}$. Then an F-rational G-compatible system $\rho_\bullet$ of Γ_k consists of a finite set S of places of P_k, for each $\lambda \in P_F$ a continuous representation

$\rho_\lambda : \Gamma_k \to G(F_\lambda)$ that is unramified outside $S \cup S_\lambda$, and for each $v \in P_k \setminus S$ a semisimple $G(\overline{\mathbb{Q}})$-conjugacy class t_v in $G(\overline{\mathbb{Q}})$ such that for all $\lambda \in P_F$ and $v \in P_k \setminus (S \cup S_\lambda)$, the semisimplification of $\rho_\lambda(\mathrm{Frob}_v)$ is conjugate to t_v in $G(\overline{F}_\lambda)$.

For any smooth projective variety X over k and fixed $i \geq 0$, the i-th étale ℓ-adic cohomologies over all primes ℓ form a $\mathbb{Q}$-rational GL_n-compatible system with $n = \dim H^i_{\mathrm{et}}(X, \mathbb{Q}_\ell)$. If in addition one has a projector in the sense of Grothendieck motives for the i-th étale cohomologies of X, say defined over a number field F, then multiplying with the projector defines an F-rational compatible system. Similarly, it is expected that to a G'-valued cuspidal automorphic representation with Hecke field F' and Langlands dual G of G' one can attach a an F'-rational compatible system over a finite extension F of F'.

Given a G-compatible system, a first expectation is that there exists a reductive subgroup H of G such that, up to conjugation for each $\lambda \in P_F$, the group $\rho_\lambda(\Gamma_k)$ is Zariski dense in $H(F_\lambda)$. So let us assume from now on that G is chosen so that $\rho_\lambda(\Gamma_k)$ is Zariski dense in $G(F_\lambda)$ for all λ. The group G should then be the motivic Galois group of the F-rational compatible system. There is a finite Galois extension k' of k such that $\mathrm{Gal}(k'/k)$ is isomorphic to $G(F_\lambda)/G^o(F_\lambda)$, and a result of Serre on compatible systems says that k' is independent of λ.

By forming the quotient modulo the center of G^o, let us next assume that G^o is semisimple of adjoint type. The system remains compatible. Because G^o is of adjoint type, by [Pin98] the compatible system $(\rho_\lambda : \Gamma_{k'} \to G^o(F_\lambda))$ thus obtained should arise from an F_{tr}-rational G'-compatible system where $F_{\mathrm{tr}} \subset F$ is the field of traces of its adjoint representation and $G = G' \times_{F_{\mathrm{tr}}} F$. We now make two hypotheses for the remaining discussion:

(a) The group G^o is absolutely simple.

(b) The G-compatible system, and not only its restriction to $\Gamma_{k'}$, is F_{tr}-rational.

Condition (a) is an intrinsic condition on the motive giving rise to $(\rho_\lambda)_\lambda$. We assume it in order to fit our context. Concerning (b), observe that the result of Pink guarantees that for each $\lambda \in P_F$ and $\lambda' \in P_{F_{\mathrm{tr}}}$ below λ, there is connected reductive group $H_{\lambda'}$ defined over $(F_{\mathrm{tr}})_{\lambda'}$ whose base change to F_λ is $G^o \times_F F_\lambda$. Pink's result does not guarantee that the groups come from a global group H defined over F_{tr}. This should be expected and this is one of our requirements in (b). The other requirement is that there is an extension G' of a finite group by H defined over F_{tr}, such that $G = G' \times_{F_{\mathrm{tr}}} F$. We want to alert the reader at this point, that G^o is simply connected and of adjoint type, and hence inner-twist like phenomena do not occur. We think that it is an interesting question to ask if (b) can always be expected, or if there are natural sufficient conditions for it to hold. Jointly with A. Conti, the first author plans to explore this further in some particular cases.

The group G has an integral model $\mathscr{G}$ over an open nonempty subscheme U of $\operatorname{Spec} \mathscr{O}_F$. For places λ in $|U| \subset P_F$ the group $\mathscr{G}^o(\mathscr{O}_\lambda)$ is maximal hyperspecial. An expectation that one has in this context, cf. [Lar95], is that for almost all $\lambda \in |U|$ the condition (hyp_λ) holds: the subgroup $\rho_\lambda(\Gamma_{k'})$ of $G^o(F_\lambda)$ itself is maximal hyperspecial, and hence conjugate to $\mathscr{G}^o(\mathscr{O}_\lambda)$. The following result is now a direct consequence of the above discussion and Corollary 3.5.

Proposition 2.13. *Let $\rho_\bullet$ be an F-rational G-compatible system of representations of Γ_k with G^o absolutely simple and of adjoint type. Suppose condition* (hyp$_\lambda$) *holds for all but finitely many $\lambda \in P_F$.*[4] *Set $H_\lambda := \bar{\rho}_\lambda(\Gamma_k)$ and $H'_\lambda := H_\lambda \cap \mathscr{G}^o(\mathbb{F}_\lambda)$. Then for all but finitely many $\lambda \in P_F$ the tuple $(\mathscr{G}_{W(\mathbb{F}_\lambda)}, \mathbb{F}_\lambda, H_\lambda, H'_\lambda)$ satisfies Assumption 2.1 and conditions* **(pf)**, **(ct)**, **(l-ge)**, **(van)**, **(sch)**, **(n-s)**.

3. Discussion of basic hypotheses for Chevalley groups

In this section, until the end of Section 3.6, we fix the following set-up.

Conditions 3.1. (a) $\mathscr{G}$ is a connected absolutely simple linear algebraic group over $\mathbb{F}$,

(b) $H'_{\mathbb{F}}$ is the image of $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ under $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \to \mathscr{G}(\mathbb{F})$, and

(c) $H_{\mathbb{F}}$ is any subgroup of $\mathscr{G}(\mathbb{F})$ that contains $H'_{\mathbb{F}}$.

We shall investigate the validity of Assumption 2.1 and the conditions formulated in Conditions 2.5 for the tuple $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$. We rely on well-known results from the literature. To get them into the precise shape needed, we often need to prove auxiliary results. We shall investigate our basic hypotheses separately, each in its own subsection. In Section 3.7 we shall consider a slight variation of this basic setup. It will be useful to have the following lists that will be used to describe some exceptional behavior:

$$\mathscr{E}_{\mathbf{(pf)}} := \{\mathrm{SL}_2(\mathbb{F}_2),\ \mathrm{SL}_2(\mathbb{F}_3),\ \mathrm{SU}_3(\mathbb{F}_2),\ \mathrm{Sp}_4(\mathbb{F}_2),\ G_2(\mathbb{F}_2)\} \tag{3-1}$$

$$\mathscr{E}_{\mathbf{(sch)}} := \left\{ \begin{matrix} (A_1, \mathbb{F}_?)_{?\in\{4,9\}}, (A_2, \mathbb{F}_?)_{?\in\{2,4\}}, (A_3, \mathbb{F}_2), (B_2, \mathbb{F}_2), (B_3, \mathbb{F}_?)_{?\in\{2,3\}}, (C_3, \mathbb{F}_2), \\ (D_4, \mathbb{F}_2), (F_4, \mathbb{F}_2), (G_2, \mathbb{F}_?)_{?\in\{3,4\}}, ({}^2A_3, \mathbb{F}_?)_{?\in\{2,3\}}, ({}^2A_5, \mathbb{F}_2), ({}^2E_6, \mathbb{F}_2) \end{matrix} \right\} \tag{3-2}$$

$$\mathscr{E}_{\mathbf{(n\text{-}s)}} := \left\{ \begin{matrix} (\mathrm{SL}_2, \mathbb{F}_?)_{?\in\{2,3\}}, (\mathrm{PGL}_2, \mathbb{F}_?)_{?\in\{2,3,4\}}, (\mathrm{SL}_3, \mathbb{F}_2), (\mathrm{PGL}_3, \mathbb{F}_2), \\ (\mathrm{SU}_3, \mathbb{F}_2), (\mathrm{PGU}_3, \mathbb{F}_2), (\mathrm{PGU}_4, \mathbb{F}_2), (\mathrm{SO}_6, \mathbb{F}_2) \end{matrix} \right\} \tag{3-3}$$

The following result combines the results from Sections 3.1 to 3.6.

Theorem 3.2. *Suppose that $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Conditions 3.1*

(a) *If $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is not in $\mathscr{E}_{\mathbf{(pf)}}$, then* **(pf)** *holds and $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Assumption 2.1.*

(b) *If* (*type*, $\mathbb{F}$) $\notin \mathscr{E}_{\mathbf{(sch)}}$, *then* **(sch)** *holds.*

(c) *Suppose that $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$. Then $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies*:

  (i) **(l-ge)** *if $\mathscr{G}$ is of type A_n and $p \nmid n+1$, or if $\mathscr{G}$ is of type B_n, C_n, D_n, E_7 or F_4 and $p \neq 2$, or if $\mathscr{G}$ is of type E_6 or G_2 and $p \neq 3$, or if $\mathscr{G}$ is of type E_8.*

  (ii) **(l-cl)** *if $\mathscr{G}$ is Lie-simply connected of type A_n, $n \geq 2$, D_n, E_n, or $\mathscr{G}$ is of type A_1, B_n, C_n, F_4 and $p \neq 2$, or of type G_2 and $p \neq 3$.*[5]

  (iii) **(l-un)** *unless $\mathscr{G}$ is Lie-intermediate of type A_n with $p | n+1$ or D_n with n odd and $p = 2$, or $\mathscr{G}$ is of type B_2 or F_4 and $p = 2$, or of type G_2 and $p = 3$.*

(d) *If $\mathscr{G}$ is Lie-simply connected*, *then* **(csc)** *holds.*

[4]In particular, $\rho_\lambda(\Gamma_k)$ is open in $G(F_\lambda)$ and Zariski-dense in $G \times_F F_\lambda$ for all these λ.

[5]For types A_n, D_n, E_6, E_7, this gives restrictions only if $p|n+1$, $p=2$, $p=3$, $p=2$, respectively.

type	A_n	B_n	C_n	D_n, (n odd)	D_n, (n even)	E_6	E_7	E_8	F_4	G_2
$\mathscr{Z}$	μ_{n+1}	μ_2	μ_2	μ_4	$\mu_2 \times \mu_2$	μ_3	μ_2	μ_1	μ_1	μ_1

Table 1. Centers of absolutely simple simply connected $\mathscr{G}$.

(e) **(ct)** *holds* $\iff Z(\mathfrak{g}) = 0 \iff \mathscr{G}$ *is of Lie-adjoint type.*

(f) *Condition* **(n-s)** *holds if and only if* $(\mathscr{G}, \mathbb{F})$ *is not* $\mathscr{E}_{\textbf{(n-s)}}$.

(g) *Condition* **(van)** *holds if* $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{\textbf{(pf)}}$, $(type, \mathbb{F}) \notin \mathscr{E}_{\textbf{(sch)}} \cup \{(A_1, \mathbb{F}_5)\}$, **(ct)** *holds, and if further one of the following holds*:

  (i) *If type* $= C_n$, *then* $|\mathbb{F}| \notin \{2, 3, 4, 5, 9\}$.

  (ii) *If* $\mathscr{G}$ *is nonsplit (and hence of type* A, D *or* E_6), *then* $|\mathbb{F}| \geq 4$.

Remark 3.3. Conditions **(ct)** and **(l-cl)** can hold simultaneously only if **(l-ge)**(ii) holds; and in the present situation the latter implies **(l-ge)**.

Notation 3.4. For the conjunction of $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{\textbf{(pf)}}$, (type $\mathscr{G}$, $\mathbb{F}) \notin \mathscr{E}_{\textbf{(sch)}}$ and $(\mathscr{G}, \mathbb{F}) \notin \mathscr{E}_{\textbf{(n-s)}}$, we shall simply write $(\mathscr{G}, \mathbb{F}) \notin \mathscr{E}$, and say that $(\mathscr{G}, \mathbb{F})$ is *not exceptional*.

We state an immediate consequence of Theorem 3.2.

Corollary 3.5. *Suppose that* $p \geq 5$, *that* $p \nmid n+1$ *if* $\mathscr{G}$ *is of type* A_n *and that* $\mathbb{F} \neq \mathbb{F}_5$ *if* $\mathscr{G}$ *is of type* A_1 *or* C_n. *Then* **(pf)**, **(sch)**, **(l-ge)**, **(ct)**, **(n-s)** *and* **(van)** *hold.*

We fix the following notation throughout this section for an absolutely simple group $\mathscr{G}$ over $\mathbb{F}$. By $\phi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}$ and $\phi^{\mathrm{ad}} : \mathscr{G} \to \mathscr{G}^{\mathrm{ad}}$ we denote the central isogenies from the simply connected cover of $\mathscr{G}$ and to its adjoint group, respectively. We define $\phi := \phi^{\mathrm{ad}} \circ \phi^{\mathrm{sc}}$ and write $\mathscr{Z} := \operatorname{Ker} \phi$ for the center of $\mathscr{G}^{\mathrm{sc}}$. We set $\mathscr{Z}' := \operatorname{Ker} \phi^{\mathrm{sc}}$ and $\mathscr{Z}'' := \operatorname{Ker} \phi^{\mathrm{ad}}$, so that there is a short exact sequence $1 \to \mathscr{Z}' \to \mathscr{Z} \to \mathscr{Z}'' \to 1$. Cf. A.13.

We remind the reader of the well-known structure of $\mathscr{Z}$ given in Table 1; see [MT11, Table 9.2 and Table 24.2]. Here μ_n is the finite flat group scheme that is the kernel of $\mathscr{G}_m \to \mathscr{G}_m, \alpha \mapsto \alpha^n$. In particular $\mathscr{Z}$ is étale over $\mathbb{F}$ and $\operatorname{Lie} \mathscr{Z} = 0$ if and only if p does not divide the order of $\mathscr{Z}$. We also recall the following fact for the convenience of the reader, since it is used repeatedly.

Fact 3.6 [Mil17, 10.14]. Let $u : \mathscr{G} \to \mathscr{G}'$ be a morphism of affine algebraic groups over a field. Then $\operatorname{Lie}(\ker u) = \ker(\operatorname{Lie} u)$ by [Mil17, 10.14].

The homomorphisms on Lie algebras induced from the above morphisms of algebraic groups are denoted $\mathrm{d}\phi^{\mathrm{sc}} : \mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g}$, $\mathrm{d}\phi^{\mathrm{ad}} : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}$ and $\mathrm{d}\phi = \mathrm{d}\phi^{\mathrm{ad}} \circ \mathrm{d}\phi^{\mathrm{sc}}$. We set $\mathfrak{z} := \operatorname{Ker} \mathrm{d}\phi = \operatorname{Lie}(\mathscr{Z})$ and $\mathfrak{z}^* := \operatorname{Coker} \mathrm{d}\phi$, so that $\dim \mathfrak{z} = \dim \mathfrak{z}^*$; see [Pin98, Proposition 1.11(a)]. We also define $\mathfrak{z}' := \operatorname{Ker} \mathrm{d}\phi^{\mathrm{sc}}$ and $\mathfrak{z}'' := \operatorname{Ker} \mathrm{d}\phi^{\mathrm{ad}}$, so that $\mathfrak{z}' = \operatorname{Lie} \mathscr{Z}'$ and $\mathfrak{z}'' = \operatorname{Lie} \mathscr{Z}''$. Note that [Pin98] usually requires that $\mathscr{G}$ be adjoint, so that $\mathfrak{z}' = \mathfrak{z}$. The action of $\mathscr{G}^{\mathrm{sc}}$ (resp. $\mathscr{G}$, $\mathscr{G}^{\mathrm{ad}}$) on $\mathfrak{g}^{\mathrm{sc}}$ (resp. $\mathfrak{g}$, $\mathfrak{g}^{\mathrm{ad}}$) is via the adjoint action, and hence it factors via the canonical map to $\mathscr{G}^{\mathrm{ad}}$. In particular, all Lie algebras and homomorphisms between them that we just defined are modules under any of the algebraic groups $\mathscr{G}^{\mathrm{sc}}$, $\mathscr{G}$ and $\mathscr{G}^{\mathrm{ad}}$, and thus also modules for $H_{\mathbb{F}}$. For any Lie algebra $\mathfrak{h}$ over $\mathbb{F}$, we write $\mathfrak{h}_{\overline{\mathbb{F}}}$ for $\mathfrak{h} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ where $\overline{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$. We note that if $\mathfrak{h}$ is any of the Lie algebras above, then $\mathfrak{h}_{\overline{\mathbb{F}}}$ carries a representation of $\mathscr{G}^{\mathrm{ad}}(\overline{\mathbb{F}})$. Adapting the notation of [Hog82-I; Hog82-II] to our needs, we call $\mathscr{G}$ *Lie-simply connected* if $\mathrm{d}\phi^{\mathrm{sc}}$ is an isomorphism, *Lie-adjoint* if $\mathrm{d}\phi^{\mathrm{ad}}$ is an isomorphism, and *Lie-intermediate* otherwise.

Following [Pin98], in our notation (!), we let $\bar{\mathfrak{g}}$ be the image of $\mathfrak{g}^{\mathrm{sc}}$ under $\mathrm{d}\phi$, and we write $\mathrm{d}\phi : \mathfrak{g}^{\mathrm{sc}} \to \bar{\mathfrak{g}}$ and $\mathrm{incl} : \bar{\mathfrak{g}} \to \mathfrak{g}^{\mathrm{ad}}$ for the induced homomorphisms, of Lie algebras and of $\mathscr{G}^{\mathrm{ad}}$-representations. By [Pin98, Proposition 1.10] there is a Lie algebra and $\mathscr{G}^{\mathrm{ad}}$-module $\hat{\mathfrak{g}}$ such that one has a pushout as well as a pullback diagram of Lie algebras and $\mathscr{G}^{\mathrm{ad}}$-representations

$$\begin{tikzcd} \hat{\mathfrak{g}} \arrow[r, two heads] & \mathfrak{g}^{\mathrm{ad}} \\ \mathfrak{g}^{\mathrm{sc}} \arrow[u, hook] \arrow[r, two heads, "\mathrm{d}\phi"] & \bar{\mathfrak{g}}. \arrow[u, hook, "\mathrm{incl}"'] \end{tikzcd} \tag{3-4}$$

We will see in Section 3.7 that in fact $\hat{\mathfrak{g}}$ occurs as the Lie algebra of a certain reductive group constructed from $\mathscr{G}^{\mathrm{ad}}$ (or $\mathscr{G}^{\mathrm{sc}}$). The usefulness of $\hat{\mathfrak{g}}$ can also be seen in Sections 3.4 and 3.5.

3.1. *Condition* (pf). Note again that until the end of Section 3.6 we assume Conditions 3.1.

Theorem 3.7 (Tits; see [MT11, Theorem 24.17]). *If $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is not in the list $\mathscr{E}_{\mathbf{(pf)}}$ from* (3-1), *then the group $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is perfect, and its quotient $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})/Z(\mathscr{G}^{\mathrm{sc}}(\mathbb{F}))$ is simple.*

Corollary 3.8. *Write Z for $\mathscr{Z}'(\mathbb{F})$.*

(a) *The map ϕ^{sc} induces short exact sequences $1 \to Z \to \mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \to H'_{\mathbb{F}} \to 1$ and $1 \to H'_{\mathbb{F}} \to \mathscr{G}(\mathbb{F}) \to Z \to 1$.*

(b) *The group Z is finite abelian of order prime to p.*

(c) *Suppose that $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is not in $\mathscr{E}_{\mathbf{(pf)}}$, then $H'_{\mathbb{F}}$ is perfect and $H'_{\mathbb{F}} = [\mathscr{G}(\mathbb{F}), \mathscr{G}(\mathbb{F})]$.*

Proof. The argument seems to be well-known, but we could not find a complete reference, so we give some details: The map ϕ^{sc} induces the short exact sequence $1 \to \mathscr{Z}' \to \mathscr{G}^{\mathrm{sc}} \to \mathscr{G} \to 1$ of group schemes. Applying (noncommutative) flat cohomology yields the 5-term left exact sequence of pointed sets

$$0 \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{G}^{\mathrm{sc}}) \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{G}) \to H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \to H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{G}^{\mathrm{sc}});$$

in fact because $\mathscr{Z}'$ is central, by [Gir71, Proposition 3.4.3] the first three nontrivial arrows are group homomorphisms. Moreover for smooth group schemes flat and étale cohomology in degrees 0 and 1 coincide; for both see [Mil80, III.4.5, III.4.7, III.4.8]. This yields

$$0 \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \to \mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \xrightarrow{\phi^{\mathrm{sc}}} \mathscr{G}(\mathbb{F}) \to H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \to H^1_{\mathrm{et}}(\mathbb{F}, \mathscr{G}^{\mathrm{sc}}). \tag{3-5}$$

By a result of Lang, we have $H^1_{\mathrm{et}}(\mathbb{F}, \mathscr{G}^{\mathrm{sc}}) = 1$; see [PR94, Theorem 6.1]. The sequence also implies that $H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \cong \mathrm{Ker}(\phi^{\mathrm{sc}}) = Z$. Note next that the group scheme $\mathscr{Z}'$ is a product of group schemes μ_n, and for the latter ones flat cohomology in degrees 0 and 1 can be computed via $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ by the same arguments that were used above, now using Hilbert 90, from the 4-term exact sequence of groups

$$1 \longrightarrow H^0_{\mathrm{fl}}(\mathbb{F}, \mu_n) \longrightarrow \mathbb{F}^\times \xrightarrow{\alpha \mapsto \alpha^n} \mathbb{F}^\times \longrightarrow H^1_{\mathrm{fl}}(\mathbb{F}, \mu_n) \longrightarrow 1.$$

Because $\mathbb{F}^\times$ is finite cyclic of order prime to p, it follows that $H^0_{\mathrm{fl}}(\mathbb{F}, \mu_n) \cong H^1_{\mathrm{fl}}(\mathbb{F}, \mu_n)$ is finite cyclic of order prime to p. Hence $H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \cong H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \cong Z$ and Z is finite abelian of order prime to p, which proves (b).

Moreover, the exact sequence (3-5) reduces to

$$1 \to Z \to \mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \xrightarrow{\varphi^{\mathrm{sc}}} \mathscr{G}(\mathbb{F}) \to Z \to 1.$$

(a) follows from this exact sequence because by Conditions 3.1, we have that $H'_{\mathbb{F}}$ is the image of $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ by φ^{sc}

To prove (c), note first that by Theorem 3.7 the group $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is perfect, and hence so is $\mathrm{Im}(\phi^{\mathrm{sc}}) = H'_{\mathbb{F}}$. It remains to show that $H'_{\mathbb{F}}$ is equal to $[\mathscr{G}(\mathbb{F}), \mathscr{G}(\mathbb{F})]$. Since the cokernel of ϕ^{sc} is abelian, the group $H'_{\mathbb{F}}$ contains $[\mathscr{G}(\mathbb{F}), \mathscr{G}(\mathbb{F})]$. But $H'_{\mathbb{F}}$ cannot be larger, since as a perfect group it has no nontrivial abelian quotients. □

Remark 3.9. The group Z is cyclic unless $\mathscr{G}$ is of type D_n with n even, $p > 2$ and ϕ^{ad} is an isomorphism in which case Z is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Corollary 3.10. *Suppose that $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is not in $\mathscr{E}_{(\mathbf{pf})}$. Then the pair $(H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Assumption 2.1.*

Proof. By Corollary 3.8 the group $H'_{\mathbb{F}}$ is perfect. Therefore it suffices to find a subgroup $M_{\mathbb{F}}$ of $H_{\mathbb{F}}$ that is of order prime to p and such that $M_{\mathbb{F}}$ surjects onto $H_{\mathbb{F}}/H'_{\mathbb{F}}$. Since $\mathscr{G}(\mathbb{F})/H'_{\mathbb{F}}$ is finite abelian, it suffices to construct $M_{\mathbb{F}}$ in the case where $H_{\mathbb{F}} = \mathscr{G}(\mathbb{F})$: if $M_{\mathbb{F}} H'_{\mathbb{F}} = \mathscr{G}(\mathbb{F})$, then the product of the kernel of $M_{\mathbb{F}} \to \mathscr{G}(\mathbb{F})/H_{\mathbb{F}}$ with $H'_{\mathbb{F}}$ will be equal to $H_{\mathbb{F}}$ for any $H_{\mathbb{F}}$ with $H'_{\mathbb{F}} \subset H_{\mathbb{F}} \subset \mathscr{G}(\mathbb{F})$.

Let $\mathscr{T} \subset \mathscr{G}$ be a maximal torus. Then by A.14 its inverse image $\mathscr{T}^{\mathrm{sc}}$ is a maximal torus in $\mathscr{G}^{\mathrm{sc}}$ and one has a short exact sequence $1 \to \mathscr{Z}' \to \mathscr{T}^{\mathrm{sc}} \to \mathscr{T} \to 1$. Arguing as in the proof of Corollary 3.8, we apply flat cohomology to obtain the exact sequence

$$0 \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{T}^{\mathrm{sc}}) \to H^0_{\mathrm{fl}}(\mathbb{F}, \mathscr{T}) \to H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \to H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{T}^{\mathrm{sc}}),$$

and again for the smooth group schemes $\mathscr{T}$ and $\mathscr{T}^{\mathrm{sc}}$ flat and étale cohomology coincides. This time by Lang's theorem, cf. [PR94, Theorem 6.1], the term $H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{T}^{\mathrm{sc}})$ vanishes, and following the same steps as in the proof of Corollary 3.8 we arrive at the 4-term exact sequence

$$1 \longrightarrow \mathscr{Z}'(\mathbb{F}) \longrightarrow \mathscr{T}^{\mathrm{sc}}(\mathbb{F}) \xrightarrow{\phi^{\mathrm{sc}}} \mathscr{T}(\mathbb{F}) \longrightarrow H^1_{\mathrm{fl}}(\mathbb{F}, \mathscr{Z}') \longrightarrow 1.$$

The argument in the proof of Corollary 3.8 showes that the first and last terms have the same (finite) cardinality, and it follows that $\mathscr{T}(\mathbb{F})/\phi^{\mathrm{sc}}(\mathscr{T}^{\mathrm{sc}}(\mathbb{F}))$ has the same cardinality as $\mathscr{Z}'(\mathbb{F})$, and by Corollary 3.8, as $\mathscr{G}(\mathbb{F})/H'_{\mathbb{F}}$. Because $\mathscr{T}^{\mathrm{sc}}$ is the fiber product of $\mathscr{T}$ with $\mathscr{G}^{\mathrm{sc}}$ over $\mathscr{G}$, the resulting square of $\mathbb{F}$-points and elementary group theory yield a natural inclusion

$$\mathscr{T}(\mathbb{F})/\phi^{\mathrm{sc}}(\mathscr{T}^{\mathrm{sc}}(\mathbb{F})) \hookrightarrow \mathscr{G}(\mathbb{F})/\phi^{\mathrm{sc}}(\mathscr{G}^{\mathrm{sc}}(\mathbb{F})) \overset{3.8(a)}{=} \mathscr{G}(\mathbb{F})/H'_{\mathbb{F}}.$$

By our consideration on cardinalities, it must be an isomorphism. Therefore we can take $M_{\mathbb{F}} := \mathscr{T}(\mathbb{F})$ which is clearly of order prime to p. □

Remark 3.11. The groups in the list $\mathscr{E}_{\mathbf{(pf)}}$ are discussed in [MT11, Remark 24.18]. Analyzing them in more detail, using [CCN+85, I.3.5], one can verify that Assumption 2.1 is satisfied for the pairs $(H'_{\mathbb{F}}, H'_{\mathbb{F}})$ (any type) and the pair $(G_2(\mathbb{F}_3), H_{\mathbb{F}})$, but not for any other pair $(H_{\mathbb{F}}, H'_{\mathbb{F}})$ with $\mathscr{G}(\mathbb{F}) \in \mathscr{E}_{\mathbf{(pf)}}$ exceptional. We leave the details to the reader.

For later use, we also note the following immediate consequence of Corollary 3.8.

Corollary 3.12. *For $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$ one has $H^1(\mathscr{G}(\mathbb{F}), \mathbb{F}_p) = \mathrm{Hom}(\mathscr{G}, \mathbb{F}_p) = 0$.*

3.2. *Condition* (sch).

Theorem 3.13 [Ste81, Theorem 1.1]. *Let $\mathscr{G}$ be as in Conditions 3.1. Then the mod p Schur multiplier $H^2(\mathscr{G}^{\mathrm{sc}}(\mathbb{F}), \mathbb{F}_p)$ vanishes, unless (type, $\mathbb{F}$) is in the list $\mathscr{E}_{\mathbf{(sch)}}$ from* (3-2).

Remark 3.14. In fact, the result stated in [Ste81, Theorem 1.1] is slightly different. It asserts that the Schur multiplier $H_2(\mathscr{G}(\mathbb{F}), \mathbb{Z})$ vanishes whenever (type, $\mathbb{F}$) $\notin \mathscr{E}_{\mathbf{(sch)}}$, and that for (type, $\mathbb{F}$) $\in \mathscr{E}_{\mathbf{(sch)}}$ the group $H_2(\mathscr{G}(\mathbb{F}), \mathbb{Z})$ is finite and of order a power of p. The relation to Theorem 3.13 is given by the universal coefficient theorem. It gives the short exact sequence

$$0 \to \mathrm{Ext}^1(H_1(\mathscr{G}(\mathbb{F}), \mathbb{Z}), A) \to H^2(\mathscr{G}(\mathbb{F}), A) \to \mathrm{Hom}(H_2(\mathscr{G}(\mathbb{F}), \mathbb{Z}), A) \to 0.$$

Because $\mathscr{G}(\mathbb{F})$ is finite and $\mathbb{Z}$ is torsion free, the left hand term is zero, and $A = \mathbb{Z}/(p)$ gives the above theorem. Another consequence is that all finite central extensions of $\mathscr{G}(\mathbb{F})$ of order prime to p are trivial.

Remark 3.15. (a) In the statement of [Ste81, Theorem 1.1] there is a typo: the group $A_2(3)$ should be $A_3(2)$ (see (5), §2.3 of [Ste81]). Moreover, to obtain the above list from Theorem 1.1 in [Ste81], one should take into account that $B_2(2)$ is isomorphic to the symmetric group S_6, which has nontrivial mod p Schur multiplier (cf. [Ste81, (6) in §3.3]), as well as the exceptional isomorphism $B_3(2) \simeq C_3(2)$.

(b) One can find the same list of exceptions in [Gri80, Table 1]. The groups considered in [Gri80] are simple groups of Lie type of the form $H'_{\mathbb{F}} = [\mathscr{G}(\mathbb{F}), \mathscr{G}(\mathbb{F})]$, where $\mathscr{G}$ is of adjoint type.

Corollary 3.16. *Suppose that (type, $\mathbb{F}$) $\notin \mathscr{E}_{\mathbf{(sch)}}$. Then* **(sch)** *holds.*

Proof. In the simply connected case this result is Theorem 3.13, taking into account that $H'_{\mathbb{F}} = [\mathscr{G}(\mathbb{F}), \mathscr{G}(\mathbb{F})] = \mathscr{G}(\mathbb{F})$. In the general case, one applies Corollary 3.8(a), (b) to control kernel and cokernel of $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \to \mathscr{G}(\mathbb{F})$, and then concludes using the Hochschild–Serre spectral sequence from group cohomology. □

3.3. ***Condition* (ct).** We begin with the following result of Hogeweij concerning the center:

Theorem 3.17 [Hog82-I; Hog82-II]. *One has*

$$\operatorname{Ker}(\mathrm{d}\phi^{\mathrm{ad}} : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}) \overset{\text{Fact 3.6}}{=} \operatorname{Lie} Z(\mathscr{G}) = Z(\mathfrak{g}).$$

Because $Z(\mathscr{G})$ is finite, $Z(\mathfrak{g})$ is nonzero if and only if p divides the order of $Z(\mathscr{G})$, i.e., if and only if $\mathscr{G}$ is Lie-adjoint.

More concretely, from Table 1 one has $\mathfrak{z} = \operatorname{Ker}(\mathrm{d}\phi : \mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g}^{\mathrm{ad}}) \neq 0$ if and only if one of the following conditions is satisfied:

(a) $\mathfrak{g}$ *is of type A_n and $p|(n+1)$.*

(b) $\mathfrak{g}$ *is of type B_n, C_n or D_n (n odd) or E_7 and $p = 2$.*

(c) $\mathfrak{g}$ *is of type E_6, and $p = 3$.*

(d) $\mathfrak{g}$ *is of type D_n (n even), and $p = 2$.*

In cases (a)–(c) *one has* $\dim \mathfrak{z} = 1$, *in case* (d) *one has* $\dim \mathfrak{z} = 2$.

For general $\mathfrak{g}$ *one has $Z(\mathfrak{g}) \neq 0$ if and only if $\mathfrak{z} \neq 0$ and either* $\mathrm{d}\phi^{\mathrm{sc}} : \mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g}$ *is bijective in cases* (a)–(c), *or* $\operatorname{Ker}(\mathrm{d}\phi^{\mathrm{sc}} : \mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g})$ *has dimension at most* 1 *in case* (d).

Finally, $Z(\mathfrak{g})$ is an $\mathbb{F}[H_{\mathbb{F}}]$-submodule of $\mathfrak{g}$ on which $H_{\mathbb{F}}$ acts trivially.

Proof. It is a basic fact that $\operatorname{Lie} Z(\mathscr{G})$ is a Lie subalgebra of $Z(\mathfrak{g})$; see [Mil17, Proposition 10.33]. To prove the displayed formula, note that $\dim_{\mathbb{F}} \operatorname{Lie} \mu_m = 1$ if $p|m$, and $\dim_{\mathbb{F}} \operatorname{Lie} \mu_m = 0$, otherwise. Hence one can read off from Table 1 the dimension of $Z(\mathscr{G}^{\mathrm{sc}})$.

Next note that $\phi^{\mathrm{sc}} \mapsto \operatorname{Ker} \phi^{\mathrm{sc}}$ defines a bijection between the (central) isogenies $\phi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}$ and the subgroups of $Z(\mathscr{G}^{\mathrm{sc}})$. Moreover $Z(\mathscr{G})$ is equal to $Z(\mathscr{G}^{\mathrm{sc}})$ modulo $\operatorname{Ker} \phi^{\mathrm{sc}}$. This allows one to read off $\dim \operatorname{Lie} Z(\mathscr{G})$ from Table 1 for all $\mathscr{G}$ (as a function of p and ϕ^{sc}). One now compares this dimension with that of $Z(\mathfrak{g})$ given in [Hog82-I, Table 1] and observes equality in all cases, so that the stated formula holds. The remaining assertions on $Z(\mathfrak{g})$ follow immediately from [Hog82-I, Table 1].

The assertion on the $\mathbb{F}[H_{\mathbb{F}}]$-module structure is straightforward from $Z(\mathfrak{g}) = \operatorname{Lie} Z(\mathscr{G})$: $Z(\mathscr{G}) \subset \mathscr{G}$ is a stable subgroup under the adjoint action, and the latter is trivial on $Z(\mathscr{G})$. Passing to Lie algebras, $Z(\mathfrak{g}) \subset \mathfrak{g}$ is an $H_{\mathbb{F}}$-stable subalgebra on which $H_{\mathbb{F}}$ acts trivially. □

Proposition 3.18. *($\mathscr{G}$, $\mathbb{F}$, $H_{\mathbb{F}}$, $H'_{\mathbb{F}}$) satisfies* **(ct)** *if and only if $Z(\mathfrak{g}) = 0$.*

Proof. The centers of the various $\mathscr{G}^{\mathrm{sc}}$ are described in Table 1. For a general $\mathscr{G}$, the central isogeny $\phi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}$, introduced above Table 1, gives a short exact sequence $1 \to \operatorname{Ker}(\phi^{\mathrm{sc}}) \to Z(\mathscr{G}^{\mathrm{sc}}) \to Z(\mathscr{G}) \to 1$. From the explicit description in Table 1, it follows that $Z(\mathscr{G})$ is smooth (over $W(\mathbb{F})$) if and only if p does not divide the order of $Z(\mathscr{G})$, and the latter is equivalent to $\operatorname{Lie} Z(\mathscr{G})$ being trivial. By Theorem 3.17 this is equivalent to $Z(\mathfrak{g})$ being zero.

It remains to show that $H^0(H'_{\mathbb{F}}, \mathfrak{g}) = 0$ provided that $Z(\mathfrak{g}) = 0$. This can be read off from [Pin98, Proposition 1.11]. Note that there one considers $\mathfrak{g} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$; but this does not affect the vanishing of $H^0(H'_{\mathbb{F}}, \mathfrak{g})$. Also, $\mathfrak{z}$ in [Pin98] is defined as $\operatorname{Ker}(\mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g}^{\mathrm{ad}})$; but we have identified this with $Z(\mathfrak{g}^{\mathrm{sc}})$ in Theorem 3.17.

Pink's tables, building on [His84; Hog82-I; Hog82-II], display all possible decomposition series of $\mathfrak{g}$ as $H'_{\mathbb{F}}$-modules, and he explains that precisely the factors contributing to the center $Z(\mathfrak{g})$ contain nontrivial $H'_{\mathbb{F}}$-invariant elements; the other factors (denoted $\bar{\mathfrak{g}}$, $\bar{\mathfrak{g}}_s$, $\bar{\mathfrak{g}}_l$) do not. $\square$

Remark 3.19. If $Z(\mathfrak{g})$ is nonzero, we shall obtain in Section 3.7 an "ambient group" of $\mathscr{G}$ for which **(ct)** holds; see in particular Theorem 3.52(a).

3.4. *Condition* (lie). This subsection is devoted to collecting results on the structure of the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$. Recall the maps ϕ, ϕ^{sc} and ϕ^{ad} in $\mathscr{G}^{\mathrm{sc}} \to \mathscr{G} \to \mathscr{G}^{\mathrm{ad}}$, and their induced maps $\mathrm{d}\phi$, $\mathrm{d}\phi^{\mathrm{sc}}$ and $\mathrm{d}\phi^{\mathrm{ad}}$ on Lie algebras from the paragraphs following Corollary 3.5. Throughout this subsection, we write G for the finite Chevalley group given by the image of $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ in $\mathscr{G}^{\mathrm{ad}}(\mathbb{F})$ under the map ϕ. We are going to consider the structure of the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$, both as a Lie algebra and as an $\mathbb{F}_p[G]$-module. Note right away that the action of $H'_{\mathbb{F}}$ on $\mathfrak{g}$ factors via G as it is trivial on the center of $H'_{\mathbb{F}}$, and this explains why assertions for G typically imply the same assertions for $H'_{\mathbb{F}}$.

Our main references will be [Hog82-I; Hog82-II; His84; Pin98]. Since the last two references work over an algebraically closed field, we begin with two lemmas to descend from $\overline{\mathbb{F}}$ to $\mathbb{F}$. For an $\mathbb{F}$-vector space V, we write $V_{\overline{\mathbb{F}}}$ for the base change $V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$.

Lemma 3.20. *Let $\mathfrak{h}$ be a Lie algebra over $\mathbb{F}$.*

(a) *If $\mathfrak{a} \subset \mathfrak{h}$ is an ideal, then $\mathfrak{a}_{\overline{\mathbb{F}}}$ is an ideal of $\mathfrak{h}_{\overline{\mathbb{F}}}$ and the assignment $\mathfrak{a} \mapsto \mathfrak{a}_{\overline{\mathbb{F}}}$ satisfies $\dim_{\mathbb{F}} \mathfrak{a} = \dim_{\overline{\mathbb{F}}} \mathfrak{a}_{\overline{\mathbb{F}}}$ and is injective. In particular, if $\mathfrak{h}_{\overline{\mathbb{F}}}$ is simple, then so is $\mathfrak{h}$; and if $\mathfrak{h}$ splits as $\mathfrak{a} \oplus \mathfrak{b}$ for two ideals $\mathfrak{a}$, $\mathfrak{b}$, then $\mathfrak{h}_{\overline{\mathbb{F}}}$ splits as $\mathfrak{a}_{\overline{\mathbb{F}}} \oplus \mathfrak{b}_{\overline{\mathbb{F}}}$.*

(b) *Under $\mathfrak{h} \mapsto \mathfrak{h}_{\overline{\mathbb{F}}}$, we have $[\mathfrak{h}_{\overline{\mathbb{F}}}, \mathfrak{h}_{\overline{\mathbb{F}}}] = [\mathfrak{h}, \mathfrak{h}]_{\overline{\mathbb{F}}}$ and $Z(\mathfrak{h})_{\overline{\mathbb{F}}} = Z(\mathfrak{h}_{\overline{\mathbb{F}}})$. In particular $\mathfrak{h}$ is perfect if and only if $\mathfrak{h}_{\overline{\mathbb{F}}}$ is perfect.*

Proof. Let $(X_i)_{i\in I}$ be a basis of $\mathfrak{h}$ such that $(X_i)_{i\in J}$ for a subset $J \subset I$ is a basis for $\mathfrak{a}$.

Because $\mathfrak{a}$ is an ideal, all brackets $[X_i, X_j]$, for $i \in I$ and $j \in J$, lie in the $\mathbb{F}$-span of $(X_i)_{i\in J}$. Since $\mathfrak{a}_{\overline{\mathbb{F}}}$ is also spanned by $(X_i)_{i\in J}$, this in turn implies that $\mathfrak{a}_{\overline{\mathbb{F}}}$ is an ideal of $\mathfrak{h}_{\overline{\mathbb{F}}}$. The injectivity of $\mathfrak{a} \mapsto \mathfrak{a}_{\overline{\mathbb{F}}}$ follows from the injectivity for the corresponding map on $\mathbb{F}$-vector spaces; the remaining parts are immediate and this completes the proof of (a).

For (b) note that the $\mathbb{F}$-span of the brackets $[X_i, X_{i'}]$, $i, i' \in I$, is $[\mathfrak{h}, \mathfrak{h}]$; their $\overline{\mathbb{F}}$-span is $[\mathfrak{h}_{\overline{\mathbb{F}}}, \mathfrak{h}_{\overline{\mathbb{F}}}]$. This implies the first assertion, and also the last. To see the remaining assertion, observe that $Z(\mathfrak{h}) = \{Y \in \mathfrak{h} \mid \forall i \in I : [Y, X_i] = 0\}$. This is a linear system of equations. Hence an $\mathbb{F}$-basis of $Z(\mathfrak{h})$ is also an $\overline{\mathbb{F}}$-basis of $Z(\mathfrak{h}_{\overline{\mathbb{F}}})$, and thus also (b) is proved. $\square$

Lemma 3.21. *Let V be an $\mathbb{F}[G]$-module and let N be an $\mathbb{F}$-vector subspace. Suppose that $N_{\overline{\mathbb{F}}} \subset V_{\overline{\mathbb{F}}}$ is invariant under G. Then N is an $\mathbb{F}[G]$-submodule of V.*

Proof. Let $(v_i)_{i\in I}$ be a basis of V over $\mathbb{F}$ such that there exists $J \subset I$ such that $(v_i)_{i\in J}$ is a basis of N. Let $j \in J$. The G-invariance of $N_{\overline{\mathbb{F}}}$ implies that $gv_j = \sum_{j'\in J} \lambda_{jj'} v_j$ for some $\lambda_{jj'} \in \overline{\mathbb{F}}$. The fact that V carries a G action implies that $gv_j = \sum_{i'\in I} \mu_{ji'} v_{i'}$ for suitable $\mu_{ji'} \in \mathbb{F}$. The basis property of $(v_i)_{i\in I}$

yields $\mu_{jj'} = \lambda_{jj'}$ for $j' \in J$ and $\mu_{ji'} = 0$ for $i' \in I \setminus J$. Hence $Gv_j \in N$ for all $j \in J$, and this implies the lemma. □

In many cases, the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$ is simple, both as a Lie algebra and as an $\mathbb{F}[G]$-module. From [His84, Hauptsatz und Korollare], [Hog82-I, Corollary 2.7] and Lemma 3.20, the following is immediate:

Proposition 3.22. *The Lie algebra $\mathfrak{g}$ is simple if and only if none of the following holds*:

(a) *$\mathfrak{g}$ is of type A_n and p divides $n+1$.*

(b) *$\mathfrak{g}$ is of type B_n, C_n, D_n, E_7 or F_4 and $p = 2$.*

(c) *$\mathfrak{g}$ is of type E_6 or G_2 and $p = 3$.*

If $\mathfrak{g}$ is simple as a Lie algebra, it is simple as an $\mathbb{F}[G]$-module.

The following result is a complete classification of when $\mathfrak{g}$ is perfect. It follows directly from [Hog82-I; Hog82-II] using Lemma 3.20.

Proposition 3.23. *Suppose that $p \neq 2$ if $\mathscr{G}$ is of type A_1, B_2 or C_n. Then $\mathfrak{g}$ is perfect if and only if $\mathscr{G}$ is Lie-simply connected. Moreover, the map $\mathrm{d}\phi^{\mathrm{sc}}$ can fail to be an isomorphism in the following cases only*:

(a) *$\mathfrak{g}$ is of type A_n and p divides $n+1$.*

(b) *$\mathfrak{g}$ is of type B_n, $n \geq 3$, D_n, $n \geq 4$, or E_7 and $p = 2$.*

(c) *$\mathfrak{g}$ is of type E_6 and $p = 3$.*

The next result is again essentially due to Hiss and Hogeweij with some additions by Pink. There is also much overlap with [Vasiu, Theorem 3.10]. Recall the exact sequences $0 \to \mathfrak{z} \to \mathfrak{g}^{\mathrm{sc}} \to \bar{\mathfrak{g}} \to 0$ and $0 \to \bar{\mathfrak{g}} \to \mathfrak{g}^{\mathrm{ad}} \to \mathfrak{z}^* \to 0$ of $\mathscr{G}^{\mathrm{ad}}$-representations in which $\mathfrak{z} \cong \mathfrak{z}^*$ are the trivial representation; see [Pin98, Proposition 1.11(a)].

Theorem 3.24. *Suppose that $p \neq 2$ if $\mathscr{G}$ is of type A_1, B_n, C_n or F_4 and that $p \neq 3$ if $\mathscr{G}$ is of type G_2. Let H be any group with $G \subset H \subset \mathscr{G}^{\mathrm{ad}}(\mathbb{F})$.*

(a) *As a Lie algebra and as an $\mathbb{F}[H]$-module, $\bar{\mathfrak{g}}$ is absolutely simple and nontrivial.*

Assume from now on that p divides the order of $Z(\mathscr{G}^{\mathrm{sc}})$.

(b) *The socle of $\mathfrak{g}^{\mathrm{sc}}$ as a Lie algebra and as an $\mathbb{F}[H]$-module is $\mathfrak{z}$.*

(c) *The cosocle of $\mathfrak{g}^{\mathrm{ad}}$ as a Lie algebra and as an $\mathbb{F}[H]$-module is $\mathfrak{z}^*$.*

(d) *$\mathscr{G}$ is Lie-simply connected if and only if $\mathfrak{z}' = 0$; it is Lie-adjoint if and only if $\mathfrak{z}'' = 0$.*

(e) *If $\mathfrak{z}'$ and $\mathfrak{z}''$ are nonzero, then $\dim \mathfrak{z}' = \dim \mathfrak{z}'' = 1$ and either of the following holds*:

  (i) *$\mathscr{G}$ is Lie-intermediate of type A_n with $p^2 | n+1$ or of type D_n with n odd and $p = 2$, and then $\mathfrak{g} \cong \bar{\mathfrak{g}} \oplus \mathfrak{z}''$ as Lie algebras and as $\mathbb{F}[H]$-modules.*

(ii) *$\mathscr{G}$ is Lie-intermediate of type D_n with n even and $p = 2$, and then $\mathfrak{g}$ possesses a unique composition series*

$$0 \subsetneq \mathfrak{z}'' \subsetneq [\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$$

as a Lie algebra and as an $\mathbb{F}[H]$-module, with $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{z}'$ and $[\mathfrak{g}, \mathfrak{g}]/\mathfrak{z}'' \cong \bar{\mathfrak{g}}$.

Proof. It clearly suffices to prove the result for $H = G$. The assertions on Lie algebras are from Table 1 of [Hog82-I] — strictly speaking that reference deals with the case of $\mathfrak{g}_{\overline{\mathbb{F}}}$; but by Lemma 3.20 this suffices to deduce the above results over $\mathbb{F}$ from the corresponding ones over $\overline{\mathbb{F}}$. The assertion on the $\mathbb{F}[G]$-module structure is stated, over $\overline{\mathbb{F}}$, in a similar way in [Pin98, Proposition 1.11]. It completes previous work from [His84; Hog82-I; Hog82-II].

(a) Consider [Hog82-I, Table 1]. Since $\bar{\mathfrak{g}}_{\overline{\mathbb{F}}}$ only depends on the type of $\mathscr{G}$, we may assume that $\mathscr{G}$ is simply connected, i.e., that $\mathfrak{g}_{\overline{\mathbb{F}}}$ of universal type in the sense of op.cit. In all cases that we allow, all nontrivial ideals are contained in $Z(\mathfrak{g}_{\overline{\mathbb{F}}})$. Thus $\bar{\mathfrak{g}}_{\overline{\mathbb{F}}} \simeq \mathfrak{g}_{\overline{\mathbb{F}}}/Z(\mathfrak{g}_{\overline{\mathbb{F}}})$ is simple as a Lie algebra. That $\bar{\mathfrak{g}}_{\overline{\mathbb{F}}} = \mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}}/Z(\mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}})$ is a simple $\overline{\mathbb{F}}[G]$-module is [His84, Hauptsatz]. Because $\dim_{\mathbb{F}} \bar{\mathfrak{g}} > 1$, the action cannot be trivial.

(b) Note first that under the hypothesis for (b)–(e) we have $\mathfrak{z} \neq 0$ by Theorem 3.17. By [Hog82-I, Table 1], the only nontrivial ideals of $\mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}}$ are contained in the centre $\mathfrak{z}_{\overline{\mathbb{F}}} = Z(\mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}})$. By Lemma 3.20, the nontrivial ideals of $\mathfrak{g}^{\mathrm{sc}}$ are thus contained in $\mathfrak{z}$. By (a), the socle of $\mathfrak{g}^{\mathrm{sc}}$ as a Lie algebra is either $\mathfrak{z}$ or $\mathfrak{g}^{\mathrm{sc}}$. If it was $\mathfrak{g}^{\mathrm{sc}}$, then there should be an ideal $\mathfrak{a}$ such that $\mathfrak{z} \oplus \mathfrak{a} = \mathfrak{g}^{\mathrm{sc}}$, and after enlarging coefficients we would have $\mathfrak{z}_{\overline{\mathbb{F}}} \oplus \mathfrak{a}_{\overline{\mathbb{F}}} = \mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}}$. But [Hog82-I, Table 1] displays no such ideal.

We now consider $\mathfrak{g}^{\mathrm{sc}}$ as an $\mathbb{F}[H]$-module. From Theorem 3.17 we see that $0 \neq \mathfrak{z} \subset \mathfrak{g}$ is an $\mathbb{F}[H]$-submodule on which H acts trivially. Hence it is part of the $\mathbb{F}[H]$-socle of $\mathfrak{g}^{\mathrm{sc}}$. If the socle was strictly larger, then by (a) it would be all of $\mathfrak{g}$, and by semisimplicity of the socle there would be a subrepresentation $\mathfrak{h} \subset \mathfrak{g}$ that would map isomorphically to $\bar{\mathfrak{g}}$. But then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$. This contradicts [His84, Hauptsatz], which asserts also that $\mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}}$ is indecomposable as an $\overline{\mathbb{F}}[G]$-module.

(c) The proof is dual to that of (b). From the ideals displayed in [Hog82-I, Table 1], it follows that $\mathfrak{z}^*$ is an abelian Lie algebra, and by [Pin98, Proposition 1.11(a)] we have that the action of H on $\mathfrak{z}^*$ is trivial. Moreover from [Hog82-I, Table 1] it follows $\bar{\mathfrak{g}} = [\mathfrak{g}^{\mathrm{ad}}, \mathfrak{g}^{\mathrm{ad}}]$ is absolutely simple and contained in all nontrivial Lie ideals of $\mathfrak{g}^{\mathrm{ad}}$ and from [Pin98, Proposition 1.11(b)] which is based on [His84] and our hypothesis on the types, it follows that $\bar{\mathfrak{g}}$ is absolutely irreducible and contained in all nontrivial subrepresentations of $\mathfrak{g}^{\mathrm{ad}}$. Hence arguing as in (b) one deduces (c).

(d) Follows from $\dim_{\mathbb{F}} \mathfrak{g}^{\mathrm{sc}} = \dim_{\mathbb{F}} \mathfrak{g} = \dim_{\mathbb{F}} \mathfrak{g}^{\mathrm{ad}}$ and the definition of $\mathfrak{z}'$ and $\mathfrak{z}''$ as the kernels of $\mathrm{d}\phi^{\mathrm{sc}}$ and of $\mathrm{d}\phi^{\mathrm{ad}}$, respectively.

(e) If both $\mathfrak{z}'$ and $\mathfrak{z}''$ are nonzero, then neither $\mathrm{d}\phi^{\mathrm{sc}}$ nor $\mathrm{d}\phi^{\mathrm{ad}}$ is an isomorphism, and so $\mathscr{G}$ is Lie-intermediate. Looking at [Hog82-I, Table 1], this can only happen if $\mathscr{G}$ is of type A_n, with $p^2 | (n+1)$, or $p = 2$ and types D_{2n} or D_{2n+1}, and in either case $\dim \mathfrak{z}'' = 1$. Because $\phi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}$ is a universal central isogeny, we have the exact sequence $1 \to \operatorname{Ker} \phi^{\mathrm{sc}} \to Z(\mathscr{G}^{\mathrm{sc}}) \to Z(\mathscr{G}) \to 1$. Applying the functor

$\mathrm{Lie}(\,\cdot\,)$ yields the left exact sequence

$$0 \to \mathfrak{z}' \to \mathfrak{z} \xrightarrow{\mathrm{d}\phi^{\mathrm{sc}}} \mathfrak{z}''. \tag{3-6}$$

In case A_n, we have $\dim_{\mathbb{F}} \mathfrak{z} = 1 = \dim_{\mathbb{F}} \mathfrak{z}''$. From the above sequence and $\mathfrak{z}' \neq 0$ we deduce that $\mathfrak{z}' = \mathfrak{z}$, and so $\dim_{\mathbb{F}} \mathfrak{z}' = 1$ and $\mathrm{d}\phi^{\mathrm{sc}}(\mathfrak{z}) = 0$. Next we establish (i) for A_n. (The case D_n with n odd is analogous). According to [Hog82-I, Table 1], the nontrivial ideals of $\mathfrak{g}_{\overline{\mathbb{F}}}$ are $\mathfrak{z}''_{\overline{\mathbb{F}}} = Z(\mathfrak{g}_{\overline{\mathbb{F}}})$ of dimension 1 and $[\mathfrak{g}_{\overline{\mathbb{F}}}, \mathfrak{g}_{\overline{\mathbb{F}}}]$ of codimension 1 of $\mathfrak{g}_{\overline{\mathbb{F}}}$. It follows from Lemma 3.20 that the nontrivial ideals of $\mathfrak{g}$ are $\mathfrak{z}''$ of dimension 1 and $[\mathfrak{g}, \mathfrak{g}]$ of codimension 1. Now the image of $\mathrm{d}\phi^{\mathrm{sc}}$, which is isomorphic to $\bar{\mathfrak{g}}$ and hence simple, is also a Lie subalgebra of $\mathfrak{g}$ of codimension 1. Since $\dim_{\mathbb{F}} \mathfrak{g} - 1 = \dim_{\mathbb{F}} \bar{\mathfrak{g}} > 1 = \dim_{\mathbb{F}} \mathfrak{z}''$, we must have $\bar{\mathfrak{g}} \cong [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}'' \cap [\mathfrak{g}, \mathfrak{g}] = 0$, and now (i) follows for A_n and the Lie algebra structure.

In both cases A_n and D_n with n odd, we know that $\mathfrak{z}''$ is a trival $\mathbb{F}[G]$-module of $\mathbb{F}$-dimension 1. Moreover, the image of $\mathrm{d}\phi^{\mathrm{sc}}$ is an $\mathbb{F}[G]$-submodule of $\mathfrak{g}$. We have already seen that $\mathrm{Im}\,\mathrm{d}\phi^{\mathrm{sc}} = [\mathfrak{g}, \mathfrak{g}] \cong \bar{\mathfrak{g}}$ and $\mathfrak{z}''$ have trivial intersection, and thus we have $\mathfrak{g} = \mathfrak{z}'' \oplus [\mathfrak{g}, \mathfrak{g}]$ as $\mathbb{F}[H]$-modules.

Suppose now that $p = 2$ and that the type is D_n with n even. Then $Z(\mathscr{G}^{\mathrm{sc}}) \cong \mu_2 \times \mu_2$, and because $\mathscr{G}$ is neither of simply connected nor of adjoint type, we must have $\mathrm{Ker}\,\phi^{\mathrm{sc}} \cong \mu_2 \cong Z(\mathscr{G})$. It follows that (3-6) is also exact on the right, and moreover that $\dim_{\mathbb{F}} \mathfrak{z}' = \dim_{\mathbb{F}} \mathfrak{z}'' = 1$. From [Hog82-I, Table 1] and using Lemma 3.20, we find that $\mathfrak{z}''$ and $[\mathfrak{g}, \mathfrak{g}]$ are the only nontrivial ideals of $\mathfrak{g}$. Moreover, $\mathfrak{z}''$ is one dimensional while $[\mathfrak{g}, \mathfrak{g}]$ has codimension 1 in $\mathfrak{g}$. At the same time $\mathrm{d}\phi^{\mathrm{sc}}(\mathfrak{g}^{\mathrm{sc}})$ is an ideal of codimension 1, and as a Lie algebra it is perfect, because this holds for $\mathfrak{g}^{\mathrm{sc}}$ by Proposition 3.23. Since $\dim_{\mathbb{F}} \mathfrak{g} > 2$, we deduce $\mathrm{d}\phi^{\mathrm{sc}}(\mathfrak{g}^{\mathrm{sc}}) = [\mathfrak{g}, \mathfrak{g}]$, and we have $\bar{\mathfrak{g}} = \mathfrak{g}^{\mathrm{sc}}/\mathfrak{z} \cong [\mathfrak{g}, \mathfrak{g}]/\mathfrak{z}''$. Thus as a Lie algebra we have the composition series described in (ii), and since $\mathfrak{g}$ has no other ideals, it is the unique composition series.

It remains to understand the $\mathbb{F}[H]$-module structure of $\mathfrak{g}$. By (b) we have that $\mathfrak{z} \subset \mathfrak{g}^{\mathrm{sc}}$ is an $\mathbb{F}[H]$-submodule on which H acts trivially. Hence $\mathfrak{z}'$ has the same property. Because $\mathrm{d}\phi^{\mathrm{sc}}$ is H-equivariant, it follows that $\mathfrak{z}'' = \mathrm{d}\phi^{\mathrm{sc}}(\mathfrak{z})$ and $[\mathfrak{g}, \mathfrak{g}] = \mathrm{d}\phi^{\mathrm{sc}}(\mathfrak{g}^{\mathrm{sc}})$ are $\mathbb{F}[H]$-submodules of $\mathfrak{g}$. Moreover H acts trivially on $\mathfrak{z}''$ and by (a) the quotient $[\mathfrak{g}, \mathfrak{g}]/\mathfrak{z}'' \cong \bar{\mathfrak{g}}$ is absolutely irreducible and nontrivial. Since $\mathfrak{z}''$ lies in $[\mathfrak{g}, \mathfrak{g}]$ and is the kernel of $\mathrm{d}\phi^{\mathrm{ad}}$, we have an injection $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \hookrightarrow \mathfrak{g}^{\mathrm{ad}}/\bar{\mathfrak{g}} = \mathfrak{z}^*$ as $\mathbb{F}[H]$-modules. This shows that H acts trivially on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, and for dimension reasons the latter is isomorphic to $\mathfrak{z}'$ as an $\mathbb{F}[H]$-module.

It remains to prove the uniqueness of the $\mathbb{F}[H]$ composition series in (ii). Let $\mathfrak{s}$ be the socle and $\mathfrak{c}$ the cosocle of $\mathfrak{g}$. We need to show that the canonical inclusion $\mathfrak{z}'' \hookrightarrow \mathfrak{s}$ and surjection $\mathfrak{c} \twoheadrightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ are isomorphisms. By [Pin98, Proposition 1.11-(b)], we know that $\bar{\mathfrak{g}}_{\overline{\mathbb{F}}}$ is the cosocle of $\mathfrak{g}^{\mathrm{sc}}_{\overline{\mathbb{F}}}$ as $\overline{\mathbb{F}}[G]$-module, and hence also as $\overline{\mathbb{F}}[H]$-module. Passing to the quotient by $\mathfrak{z}'_{\overline{\mathbb{F}}}$ and applying Lemma 3.20, we find that $\mathfrak{z}''$ is the socle of $[\mathfrak{g}, \mathfrak{g}]$, and it follows that $\mathfrak{s} \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{z}''$. Quotienting by $\mathfrak{z}''$, we have $\mathfrak{s}/\mathfrak{z}'' \oplus \bar{\mathfrak{g}} \subset \mathfrak{g}^{\mathrm{ad}}$ as $\mathbb{F}[H]$-modules. However again by [Pin98, Proposition 1.11-(b)], the socle of $\mathfrak{g}^{\mathrm{ad}}_{\overline{\mathbb{F}}}$ as an $\mathbb{F}[G]$-module is $\bar{\mathfrak{g}}_{\overline{\mathbb{F}}}$, and from Lemma 3.20, we deduce $\mathfrak{s}/\mathfrak{z}'' = 0$, which shows the first isomorphism. The argument for $\mathfrak{c}$ is dual but analogous. □

Remark 3.25. For $p = 2$ and $\mathscr{G}^{\mathrm{sc}}$ of type A_1, we have $[\mathfrak{g}^{\mathrm{sc}}, \mathfrak{g}^{\mathrm{sc}}] = Z(\mathfrak{g}^{\mathrm{sc}})$, from [Hog82-I, Table 1]. Thus $\bar{\mathfrak{g}}$ is abelian as a Lie algebra. However, if $q \geq 4$, it is indecomposable as an $\overline{\mathbb{F}}[G]$-module; see [His84, Hauptsatz].

Remark 3.26. For those $\mathscr{G}$ not of type A_1 that are not included in Theorem 3.24, the representation $\bar{\mathfrak{g}}$ possesses a nontrivial composition series, as an H-module and as a Lie algebra. Much of this is related to some nonstandard isogenies between types B_n and C_n or from F_4 to F_4 if $p = 2$ and from type G_2 to G_2 if $p = 3$; see [Pin98, Section 1].

Remark 3.27. Assertions (b) and (c) of Theorem 3.24 are still true in the cases, $p = 2$, type C_n with n even, and type B_n with $n \geq 4$ even, as can be directly verified from [His84; Hog82-I; Hog82-II]. Also, in all cases, i.e., also those excluded in Theorem 3.24, the maximal $\mathbb{F}_p[G]$-submodule of $\mathfrak{g}^{\mathrm{sc}}$ on which G acts trivially is $\mathfrak{z}$, by the Hauptsatz of [His84].

Next, we study the canonical map $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\bar{\mathfrak{g}})$. For this we need the following result:

Lemma 3.28. *Let Γ be a finite group, let $L \supset K$ be any extensions of fields, and let V be a $K[\Gamma]$-module.*

(a) *If $L \otimes_K V$ is completely reducible as an $L[\Gamma]$-module, then V is completely reducible as a $K[\Gamma]$-module.*

(b) *Suppose that V is completely reducible as a $K[\Gamma]$-module, that $\mathrm{End}_{K[\Gamma]}(V)$ contains a finite field extension E of K and that $\dim_L \mathrm{End}_{L[\Gamma]}(L \otimes_K V) = \dim_K E$. Then we have $\mathrm{End}_{K[\Gamma]}(V) = E$ and V is irreducible as a $K[\Gamma]$-module.*

Proof. Lacking a reference for the certainly well-known results of the lemma, we give a proof: We deduce (a) from the following result of M. Deuring and E. Noether: If W and W' are finite dimensional $K[\Gamma]$-modules such that $L \otimes_K W \cong L \otimes_K W'$ as $L[\Gamma]$-modules, then $W \cong W'$; see [CR62, Theorem (29.11)].

If now $V_0 = 0 \subsetneq V_1 \subsetneq \cdots V_{i-1} \subsetneq V_i = V$ is a composition series of V with simple quotients, then by our hypothesis on $L \otimes_K V$ we have $L \otimes_K V \cong L \otimes_K (\oplus_{j=1}^{i} V_i / V_{i-1})$. It follows from the Noether–Deuring theorem that V is completely reducible.

For part (b) note first that $\mathrm{End}_{K[\Gamma]}(V) \otimes_K L$ injects into $\mathrm{End}_{L[\Gamma]}(L \otimes_K V)$, because for any finite-dimensional K-vector space V one has $\mathrm{End}_K(V) \otimes_K L \cong \mathrm{End}_L(L \otimes_K V)$. We deduce that $E \otimes_K L$ injects L-linearly into $\mathrm{End}_{L[\Gamma]}(L \otimes_K V)$, and using our dimension hypothesis we see that $E \otimes_K L \to \mathrm{End}_{L[\Gamma]}(L \otimes_K V)$ is an isomorphism, and hence the inclusion $E \hookrightarrow \mathrm{End}_{K[\Gamma]}(V)$ must be an isomorphism as well. Because V is semisimple and E is a field, we also deduce that V is a simple $K[\Gamma]$-module. $\square$

Proposition 3.29. *Suppose that $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{(\mathbf{pf})}$. Denote by $\mathfrak{g}^{\text{h.w.}}$ the $\mathbb{F}[G]$-subquotient of $\mathfrak{g}$ of highest weight. Then $\mathfrak{g}^{\text{h.w.}}$ is irreducible as an $\mathbb{F}_p[H'_{\mathbb{F}}]$-module and the canonical map $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\text{h.w.}})$ is an isomorphism.*

The proposition expresses the fact that $\mathbb{F}$ is the smallest coefficient field over which the absolutely irreducible $\mathbb{F}[G]$-module $\mathfrak{g}^{\text{h.w.}}$ can be defined. Lacking a reference, we give a proof.

Proof. In [Pin98, Proposition 1.11] a composition series of $\mathfrak{g}_{\overline{\mathbb{F}}}$ as an $\overline{\mathbb{F}}[G]$-module is given. According to [Hog82-I, Table 1] there is a filtration of $\mathbb{F}$-Lie subalgebras of $\mathfrak{g}$ whose scalar extension to $\overline{\mathbb{F}}$ is the decomposition series from [Pin98], and hence by Lemma 3.21, the filtration deduced from [Hog82-I;

Hog82-II] is also one of $\mathbb{F}[G]$-modules. It follows that the highest weight subquotient of $\mathfrak{g}_{\overline{\mathbb{F}}}$ is defined over $\mathbb{F}$, i.e., that $\mathfrak{g}^{\text{h.w.}}$ is defined and absolutely irreducible. Hence by Lemma 3.28(a), the $\mathbb{F}_p[G]$-module $\mathfrak{g}^{\text{h.w.}}$ is completely reducible.

We consider $\mathfrak{g}^{\text{h.w.}} \otimes_{\mathbb{F}_p} \mathbb{F}$ as a $\mathbb{F}[H'_{\mathbb{F}}]$-module. Using the decomposition

$$\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\simeq} \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)} \mathbb{F}, \quad \alpha \otimes \beta \mapsto (\alpha\sigma(\beta))_{\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)},$$

we find $\mathfrak{g}^{\text{h.w.}} \otimes_{\mathbb{F}_p} \mathbb{F} \cong \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)} \mathfrak{g}^{\text{h.w.}} \otimes^{\sigma}_{\mathbb{F}} \mathbb{F}$, where each tensor product uses a different Galois automorphism. In the next paragraph we shall use the Steinberg tensor theorem and Steinberg's theory of irreducible representations of finite Chevalley groups to deduce that the $\mathfrak{g}^{\text{h.w.}} \otimes^{\sigma}_{\mathbb{F}} \mathbb{F}$ are pairwise non-isomorphic.

Let $\mathfrak{M}$ denote the set of irreducible restricted (algebraic) representations of $\mathscr{G}^{\text{sc}}$; their number is p^{ℓ} where ℓ denotes the rank of $\mathscr{G}^{\text{sc}}$; see [Ste63, p. 36]. Let $n := [\mathbb{F} : \mathbb{F}_p]$. Note that for $n = 1$ there is obviously nothing to show. By [Ste63, Theorem 7.4 and 9.3] every irreducible representation V of $\mathscr{G}^{\text{sc}}(\mathbb{F})$ can be written as $\bigotimes_{i=0}^{n-1} V_i^{(i)}$ for a unique choice $(V_0, \dots, V_{n-1}) \in \mathfrak{M}^n$, and where the superscript ${}^{(i)}$ denotes the i-th Frobenius twist of V_i; here the V_i are supposed to be considered as representations of $\mathscr{G}^{\text{sc}}(\mathbb{F})$ and therefore one has $W^{(n)} = W$ for $W \in \mathfrak{M}$. Choose the V_i for $\mathfrak{g}^{\text{h.w.}}$, considered as a representation of $\mathscr{G}^{\text{sc}}(\mathbb{F})$ via the surjection $\mathscr{G}^{\text{sc}}(\mathbb{F}) \to H'_{\mathbb{F}}$. Let $\mathrm{Fr} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ be the geometric Frobenius automorphism $\alpha \mapsto \alpha^{1/p}$. It is elementary to see that $\mathfrak{g}^{\text{h.w.}} \otimes^{\mathrm{Fr}^i}_{\mathbb{F}} \mathbb{F} = \mathfrak{g}^{\text{h.w.},(i)}$. It follows that $\mathfrak{g}^{\text{h.w.}} \otimes^{\mathrm{Fr}^j}_{\mathbb{F}} \mathbb{F} \cong \bigotimes_{i=0}^{n-1} V_i^{(i+j)}$. Denote by $\underline{w} := (w_0, \dots, w_{n-1})$ the tuple of p-restricted weights such that $\sum_{i=0}^{n-1} w_i p^i$ is the highest weight of $\mathfrak{g}^{\text{h.w.}}$. The p-restricted weights w_i are linear combinations of the fundamental weights with coefficients in $\{0, \dots, p-1\}$. Then it remains to show that no cyclic permutation of $\underline{w}$ except for the identity, fixes the list $\underline{w}$.

By [Bou68, Planches], the highest weight of $\mathfrak{g}^{\text{h.w.}}$ in terms of a basis of fundamental weights $\varpi_1, \dots, \varpi_\ell$, depending on the type, is given in the following table:

type	A_ℓ	B_2	$B_\ell,\ \ell\geq 3$	$C_\ell,\ \ell\geq 3$	$D_\ell,\ \ell\geq 4$	E_6	E_7	E_8	F_4	G_2
h.w.	$\varpi_1 + \varpi_\ell$	$2\varpi_2$	ϖ_2	$2\varpi_1$	ϖ_2	ϖ_2	ϖ_1	ϖ_8	ϖ_1	ϖ_2

(3-7)

As a $\mathbb{Z}$-linear combination in the basis $(\varpi_i)_{i=1,\dots,\ell}$, only the coefficients 0, 1 and 2 occur, and 2 only occurs for types B_ℓ and C_ℓ and $p = 2$. Hence apart from this exception the highest weight of $\mathfrak{g}^{\text{h.w.}}$ itself is p-restricted, and thus the $\mathfrak{g}^{\text{h.w.},(i)}$, $i = 0, \dots, n-1$ are pairwise non-isomorphic. If we are in the case B_ℓ or C_ℓ and $p = 2$, then $\underline{w} = (0, \varpi_2, 0, \dots, 0)$ or $\underline{w} = (0, \varpi_1, 0, \dots, 0)$, respectively, and again no nontrivial cyclic permutation fixes $\underline{w}$, and so also in this case the $\mathfrak{g}^{\text{h.w.},(i)}$, $i = 0, \dots, n-1$ are pairwise non-isomorphic.

We now have seen that $\mathfrak{g}^{\text{h.w.}} \otimes_{\mathbb{F}_p} \mathbb{F} \cong \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)} \mathfrak{g}^{\text{h.w.}} \otimes^{\sigma}_{\mathbb{F}} \mathbb{F}$ is a decomposition into pairwise non-isomorphic absolutely irreducible modules. This implies

$$\mathrm{End}_{\mathbb{F}[H'_{\mathbb{F}}]}(\mathbb{F} \otimes_{\mathbb{F}_p} \mathfrak{g}^{\text{h.w.}}) \cong \mathbb{F} \times \cdots \times \mathbb{F}$$

with n factors on the right, so that $\dim_{\mathbb{F}} \operatorname{End}_{\mathbb{F}[H'_{\mathbb{F}}]}(\mathbb{F}\otimes_{\mathbb{F}_p}\mathfrak{g}^{\text{h.w.}}) = n = [\mathbb{F}:\mathbb{F}_p]$. The proposition now follows from Lemma 3.28(b). $\square$

Corollary 3.30. *Suppose that $p \neq 2$ if $\mathscr{G}$ is of type B_n, C_n or F_4 and that $p \neq 3$ if $\mathscr{G}$ is of type G_2, and suppose that $\mathscr{G}^{\text{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$, so that $\mathscr{G}^{\text{sc}}(\mathbb{F})$ is perfect. Then $\bar{\mathfrak{g}}$ is irreducible as an $\mathbb{F}_p[H'_{\mathbb{F}}]$-module and the canonical map $\mathbb{F} \to \operatorname{End}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\bar{\mathfrak{g}})$ is an isomorphism.*

Proof. By our hypotheses and Theorem 3.24, the module $\bar{\mathfrak{g}}$ is absolutely irreducible, as a module over $\mathbb{F}[H'_{\mathbb{F}}]$, and it is not difficult to see that $\bar{\mathfrak{g}} \cong \mathfrak{g}^{\text{h.w.}}$ in all cases of the corollary. Hence the result follows from Proposition 3.29. $\square$

From what we proved in Section 3.4 so far, the following result is now immediate.

Corollary 3.31. *If $\mathscr{G}^{\text{sc}}(\mathbb{F})$ is not in $\mathscr{E}_{\mathbf{(pf)}}$, then the triple $(\mathscr{G}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies* **(l-ge)** *if $\mathscr{G}$ is of type A_n and $p \nmid n+1$, or if $\mathscr{G}$ is of type B_n, C_n, D_n, E_7 or F_4 and $p \neq 2$, or if $\mathscr{G}$ is of type E_6 or G_2 and $p \neq 3$, or if $\mathscr{G}$ is of type E_8.*

For **(l-un)** we also need the following result, where for an $\mathbb{F}[G]$-module V we denote by $\operatorname{soc}(V)$ and $\operatorname{csoc}(V)$ the socle and cosocle of V, respectively:

Lemma 3.32. *Let W be an $\mathbb{F}[G]$-module that is irreducible over $\mathbb{F}_p[G]$, carries a nontrivial G-action and for which the natural ring map $\mathbb{F} \to \operatorname{End}_{\mathbb{F}_p[G]}(W)$ is an isomorphism. Suppose that V is an $\mathbb{F}[G]$-module that satisfies one of the following conditions*:

(a) *The module V sits in a short exact sequence $0 \to \operatorname{soc}(V) \to V \to \operatorname{csoc}(V) \to 0$ such that $\{\operatorname{soc}(V), \operatorname{csoc}(V)\} = \{\mathbb{F}^r, W\}$ for some $r \geq 1$.*

(b) *The module V possesses a 3-step filtration $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V$ such that $V_1 = \operatorname{soc}(V) \cong \mathbb{F}$, $V/V_2 = \operatorname{csoc}(V) \cong \mathbb{F}$ and $V_2/V_1 \cong W$.*

(c) *The module V sits in a short exact sequence $0 \to \operatorname{soc}(V) \to V \to \operatorname{csoc}(V) \to 0$ such that $\{\operatorname{soc}(V), \operatorname{csoc}(V)\} = \{\mathbb{F} \oplus W, V'\}$ for some irreducible $\mathbb{F}[G]$-module V' such that $\mathbb{F}, V', W$ are pairwise non-isomorphic.*

(d) *The module V possesses a unique filtration $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq V_4 = V$ with irreducible $\mathbb{F}[G]$-subquotients $Q_i = V_i/V_{i-1}$, $i = 1, \dots, 4$, and moreover one of the following holds*:

   (i) *$Q_1 \cong Q_3 \cong \mathbb{F}$ and $\{Q_2, Q_4\} = \{W, V'\}$ for some irreducible nontrivial $\mathbb{F}[G]$-module V' such that $\mathbb{F}, V', W$ are pairwise non-isomorphic.*

   (ii) *$Q_2 \cong Q_4 \cong \mathbb{F}$ and $\{Q_1, Q_3\} = \{W, V'\}$ for some irreducible nontrivial $\mathbb{F}[G]$-module V' such that $\mathbb{F}, V', W$ are pairwise non-isomorphic.*

Then the natural map $\mathbb{F} \to \operatorname{End}_{\mathbb{F}_p[G]}(V)$ is an isomorphism.

Proof. The proofs are all fairly standard. They use properties of the socle and cosocle of modules over group rings. For instance if $V \to V'$ is an $\mathbb{F}[G]$-module homomorphism, then it induces maps $\operatorname{soc}(V) \to \operatorname{soc}(V')$ and $\operatorname{csoc}(V) \to \operatorname{csoc}(V')$. Case (b) can use case (a) and case (d) can benefit from (a) and sometimes (b). As a sample proof we give one half of the proof in case (c):

Suppose that $\mathrm{soc}(V) = \mathbb{F} \oplus W$ and $\mathrm{csoc}(V) = V'$, and let $\phi : V \to V$ be in $\mathrm{End}_{\mathbb{F}_p[G]}(V)$. Since ϕ preserves the socle, it induces a map $\phi|_{\mathbb{F}\oplus W}$ in $\mathrm{End}_{\mathbb{F}_p[G]}(\mathbb{F} \oplus W)$. The latter ring is isomorphic to $\mathbb{F} \times \mathbb{F}$ by our hypothesis on $\mathrm{End}_{\mathbb{F}_p[G]}(W)$ and the fact that W is irreducible and non-isomorphic to $\mathbb{F}_p$. Replacing ϕ by $\phi' := \phi - \lambda\, \mathrm{id}_V$, we may assume that ϕ' restricts to the zero map on W. We must prove that $\phi' = 0$.

The map ϕ' induces a homomorphism $\bar{\phi}' : V/W \to V$. Suppose first that $\bar{\phi}'$ is injective. Then V/W is a direct summand of V, so that $V \cong W \oplus V/W$, since W and V' are irreducible over $\mathbb{F}_p[G]$ and of different dimensions and since they are not isomorphic to $\mathbb{F}_p$. But the $\mathrm{csoc}(V)$ surjects onto $V' \oplus W$, which contradicts our hypothesis. Hence $\mathrm{Ker}\, \bar{\phi}' \neq 0$. We note that V/W is a nontrivial extension of V' by $\mathbb{F}$, since otherwise $\mathrm{csoc}(V)$ would surject onto $\mathbb{F} \oplus V'$ which is not allowed. Thus $\mathrm{Ker}\, \bar{\phi}' \in \{\mathbb{F}, V/W\}$, and we need to rule out the first case.

Suppose $\mathbb{F} = \mathrm{Ker}\, \bar{\phi}'$. Then ϕ vanishes on $\mathbb{F} \oplus W$ and induces an injective homomorphism $V' \cong V/(\mathbb{F} \oplus W) \hookrightarrow V$. But then $V' \subset \mathrm{soc}(V)$, which contradicts the hypotheses of (c). This completes the proof of (c). ☐

Corollary 3.33. *Suppose that $\mathscr{G}^{\mathrm{sc}}(\mathbb{F})$ is not in $\mathscr{E}_{\mathbf{(pf)}}$ and one of the following conditions holds*:

(a) $\mathfrak{g}$ *is irreducible*, *i.e.*, **(l-ge)**(ii) *holds.*

(b) *$\mathscr{G}$ is Lie-simply connected or Lie-adjoint and of type A_n with $p|n+1$ and $(\mathbb{F}, n) \neq (\mathbb{F}_2, 2)$ or $\mathscr{G}$ is of type D_n or of type E_7 with $p = 2$, or $\mathscr{G}$ is of type E_6 with $p = 3$.*

(c) *$\mathscr{G}$ is Lie-intermediate of type D_n with $p = 2$ and n even.*

(d) *$\mathscr{G}$ is of type B_n or C_n, $n \geq 3$, and $p = 2$.*

Then the natural map $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g})$ is an isomorphism.

Proof. If **(l-ge)**(ii) holds, then $\mathfrak{g} = \mathfrak{g}^{\mathrm{h.w.}}$ and we can directly apply Proposition 3.29. In all other cases we apply Lemma 3.32. The case (b) here reduces to (a) of Lemma 3.32, the case (c) here reduces to (b), the case (d) here reduces to (c) if n is odd and to (d) if n is even. To see that one can apply Lemma 3.32, note first that by [Pin98, Proposition 1.11] we have filtrations of $\mathfrak{g}_{\overline{\mathbb{F}}}$ as an $\overline{\mathbb{F}}[G]$-module with the properties described in Lemma 3.32(a)–(d), and that by [Hog82-I, Table 1] there is a filtration of $\mathbb{F}$-Lie subalgebras of $\mathfrak{g}$ whose scalar extension to $\overline{\mathbb{F}}$ is that of [Pin98, Proposition 1.11]. Hence by Lemma 3.21, the filtration descends to one of $\mathbb{F}[G]$-modules of $\mathfrak{g}$, as does the uniqueness property stated in [Pin98, Proposition 1.11]. This establishes the required properties on socles, cosocles and filtrations needed in Lemma 3.32(a)–(d). That $\mathbb{F}$, V', W in Lemma 3.32 are pairwise non-isomorphic is most easily deduced from [His84, diagrams in Hauptsatz], where it is observed that they have pairwise distinct dimensions. ☐

Remark 3.34. In the following situations not covered by Corollary 3.33, the canonical map $\mathbb{F} \to \mathrm{End}_{\mathbb{F}_p[G]}(\mathfrak{g})$ is not an isomorphism:

(a) $\mathscr{G}$ is of type B_2 or F_4 and $p = 2$ or of type G_2 and $p = 3$; here $\mathrm{End}_{\mathbb{F}_p[G]}(\mathfrak{g}) \cong \mathbb{F}[\varepsilon]$. These are the cases where the Ree and Suzuki groups occur.

(b) $\mathscr{G}$ is of Lie-intermediate type A_n with $p|n+1$ or D_n with $p = 2$ and n odd; here $\mathrm{End}_{\mathbb{F}_p[G]}(\mathfrak{g}) \cong \mathbb{F} \times \mathbb{F}$.

Lemma 3.35. *Condition* **(l-un)**(i) *holds unless* $\mathscr{G}$ *is of type* B_2.

Proof. For any $\mathfrak{g}$, the commutator Lie subalgebras $[\mathfrak{g}, \mathfrak{g}]$ and the quotients $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ are given in [Hog82-I, Table 1]. We have that the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is of the form $\mathbb{F}^r$ with $r \in \{0, 1, 2\}$ except for the following cases: $p = 2$ and $\mathscr{G}$ is of type A_1, types C_l for $l > 2$ with $\mathscr{G}$ Lie-universal, and type B_2. If we are not in these exceptional cases then $\mathfrak{g}^{\text{h.w.}}$ is a JH-factor of $[\mathfrak{g}, \mathfrak{g}]$, and so **(l-un)**(i) holds. For type A_1 and $p = 2$ we have $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}$ and the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is nontrivial irreducible $\mathbb{F}[G]$-module. For C_l with $l > 2$ and $\mathscr{G}$ Lie-universal, the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is irreducible but not isomorphic to $\mathfrak{g}^{\text{h.w.}}$ by [His84, Hauptsatz]. $\square$

Combining the last two results, we obtain:

Corollary 3.36. *If* $\mathscr{G}^{\text{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$, *then* **(l-un)** *holds unless* $\mathscr{G}$ *is Lie-intermediate of type* A_n *with* $p|n+1$ *or* D_n *with n odd and* $p = 2$, *or* $\mathscr{G}$ *is of type* B_2 *or* F_4 *and* $p = 2$, *or of type* G_2 *and* $p = 3$.

Lastly we turn to **(l-cl)**.

Lemma 3.37. *If* $\mathscr{G}^{\text{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$, *then* **(l-cl)** *holds if* $\mathscr{G}$ *is Lie-simply connected of type* A_n, $n \geq 2$, D_n, E_n, *or* $\mathscr{G}$ *is of type* A_1, B_n, C_n, F_4 *and* $p \neq 2$, *or of type* G_2 *and* $p \neq 3$.

Proof. Condition **(l-cl)**(i), namely that $\mathfrak{g}$ is perfect, is completely described in Proposition 3.23. It requires $\mathscr{G}$ to be Lie-simply connected and that the type of $\mathscr{G}$ is not A_1, B_2 or C_n if $p = 2$. Condition **(l-cl)**(ii), that $\operatorname{soc}(\mathfrak{g}) \cong \mathbb{F}^r$ for $r \geq 0$ and $\operatorname{csoc}(\mathfrak{g}) = \bar{\mathfrak{g}}$ (and hence $\operatorname{csoc}(\mathfrak{g}) = \mathfrak{g}^{\text{h.w.}}$), can be read off from [Pin98, Proposition 11.1]. It requires us to further exclude B_n, F_4 of $p = 2$ and G_2 if $p = 3$. Note also Theorem 3.17 concerning the $H_{\mathbb{F}}$-action. $\square$

Let us also state the complete result concerning **(csc)**.

Proposition 3.38. *If* $\mathscr{G}$ *is Lie-simply connected*, *then* **(csc)** *holds.*

Proof. It follows from [His84, Hauptsatz] that in the case where $\mathscr{G}$ is Lie-simply connected the cosocle of $\mathfrak{g}_{\overline{\mathbb{F}}}$ contains no copy of the trivial $H'_{\mathbb{F}}$-module $\overline{\mathbb{F}}$. Hence the cosocle of $\mathfrak{g}$ cannot contain a copy of $\mathbb{F}$; and this implies **(csc)**. $\square$

3.5. ***Condition*** **(van).** This subsection collects some known results on the vanishing of $H^1(H'_{\mathbb{F}}, \mathfrak{g})$. We begin with the following lemma that, when combined with results we shall recall later, suggests that it is most natural to expect the vanishing of $H^1(H'_{\mathbb{F}}, \hat{\mathfrak{g}})$ in (almost) all cases (similar to [TZ70, Theorem 9] for $\mathscr{G} = \mathrm{GL}_n$), with $\hat{\mathfrak{g}}$ as in diagram (3-4). The lemma will be used later in Section 3.7; it is also useful to determine $H^1(H'_{\mathbb{F}}, \mathfrak{g})$ in some cases.

Lemma 3.39. *Suppose that* $(\mathscr{G}, \mathbb{F})$ *lies neither in* $\mathscr{E}_{\mathbf{(pf)}}$ *nor in* $\mathscr{E}_{\mathbf{(sch)}}$. *Then the following are equivalent*:

(a) $H^1(H'_{\mathbb{F}}, \hat{\mathfrak{g}}) = 0$.

(b) $\dim_{\mathbb{F}} H^1(H'_{\mathbb{F}}, \mathfrak{g}) = \dim_{\mathbb{F}} Z(\mathfrak{g})$.

(c) $\dim_{\mathbb{F}} H^1(H'_{\mathbb{F}}, \bar{\mathfrak{g}}) = \dim_{\mathbb{F}} \mathfrak{z}$.

Remark 3.40. The proof shows that if $d\phi^{\text{sc}} : \mathfrak{g}^{\text{sc}} \to \mathfrak{g}$ is an isomorphism, then (a)$\Longleftrightarrow$(b) only requires that $(\mathscr{G}, \mathbb{F})$ is not in $\mathscr{E}_{\mathbf{(pf)}}$; the same holds when $d\phi^{\text{ad}} : \mathfrak{g} \to \mathfrak{g}^{\text{ad}}$ is an isomorphism for (b)$\Longleftrightarrow$(c).

Proof. We first assume that $d\phi^{sc} : \mathfrak{g}^{sc} \to \mathfrak{g}$ is an isomorphism. To prove (a)$\Longleftrightarrow$(b) consider the short exact sequence $0 \to \mathfrak{g}^{sc} \to \hat{\mathfrak{g}} \to \mathfrak{z}^* \to 0$ and the resulting long exact sequence

$$0 \to H^0(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \to H^0(H'_{\mathbb{F}}, \hat{\mathfrak{g}}) \to \mathfrak{z}^* \to H^1(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \to H^1(H'_{\mathbb{F}}, \hat{\mathfrak{g}}) \to H^1(H'_{\mathbb{F}}, \mathfrak{z}^*) \to \cdots$$

of group cohomology. Note that by Theorem 3.24 and Remark 3.27, the module $\mathfrak{z}$ is the maximal submodule of $\mathfrak{g}^{sc}$ on which $H'_{\mathbb{F}}$ acts trivially. Now by [Pin98, Proposition 1.11] the socle of $\mathfrak{g}^{ad}$ contains no submodule on which $H'_{\mathbb{F}}$ acts trivially, and hence diagram (3-4) implies that $\mathfrak{z}$ is also the maximal submodule of $\hat{\mathfrak{g}}$ on which $H'_{\mathbb{F}}$ acts trivially. Hence $H^0(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \to H^0(H'_{\mathbb{F}}, \hat{\mathfrak{g}})$ is an isomorphism in the above sequence. Also we have $H^1(H'_{\mathbb{F}}, \mathfrak{z}^*) \cong H^1(H'_{\mathbb{F}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathfrak{z}^*$, because $\mathfrak{z}^*$ is trivial as a $H'_{\mathbb{F}}$-module. Therefore $H^1(H'_{\mathbb{F}}, \mathfrak{z}^*) = 0$ by Corollary 3.12, so that $0 \to \mathfrak{z}^* \to H^1(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \to H^1(H'_{\mathbb{F}}, \hat{\mathfrak{g}}) \to 0$ is a short exact sequence. This implies (a)$\Longleftrightarrow$(b).

To prove (b)$\Longleftrightarrow$(c) consider the short exact sequence $0 \to \mathfrak{z} \to \mathfrak{g}^{sc} \to \bar{\mathfrak{g}} \to 0$ and the resulting long exact sequence of group cohomology

$$0 \to \mathfrak{z} \to H^0(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \to H^0(H'_{\mathbb{F}}, \bar{\mathfrak{g}}) \to H^1(H'_{\mathbb{F}}, \mathfrak{z}) \to H^1(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \to H^1(H'_{\mathbb{F}}, \bar{\mathfrak{g}}) \to H^2(H'_{\mathbb{F}}, \mathfrak{z}) \to \cdots.$$

We deduce $H^1(H'_{\mathbb{F}}, \mathfrak{z}) = 0$ as in the previous paragraph (there for $\mathfrak{z}^*$ instead of $\mathfrak{z}$). Moreover we obtain $H^2(H'_{\mathbb{F}}, \mathfrak{z}) = 0$ from Corollary 3.16. Since $\mathfrak{z} = Z(\mathfrak{g}^{sc})$ we showed (b)$\Longleftrightarrow$(c).

From now on we no longer assume that $d\phi^{sc} : \mathfrak{g}^{sc} \to \mathfrak{g}$ is an isomorphism. Instead, we consider the short exact sequences

$$0 \to \operatorname{Ker} d\phi^{sc} \to \mathfrak{g}^{sc} \to \operatorname{Im} d\phi^{sc} \to 0 \tag{3-8}$$

and

$$0 \to \operatorname{Im} d\phi^{sc} \to \mathfrak{g} \to \operatorname{Coker} d\phi^{sc} \to 0. \tag{3-9}$$

We shall prove that (b) for $\mathfrak{g}^{sc}$ and (b) for $\mathfrak{g}$ are equivalent: Using (3-8) and $\operatorname{Ker} d\phi^{sc} \subset \mathfrak{z}$, and arguing as in (b)$\Longleftrightarrow$(c), we deduce the isomorphism

$$H^1(H'_{\mathbb{F}}, \mathfrak{g}^{sc}) \xrightarrow{\simeq} H^1(H'_{\mathbb{F}}, \operatorname{Im} d\phi^{sc}).$$

Note now that by Theorem 3.24 and Remark 3.27, the module $Z(\operatorname{Im} d\phi^{sc})$ is the maximal submodule of $\operatorname{Im} d\phi^{sc}$ on which $H'_{\mathbb{F}}$ acts trivially, and it is also the maximal submodule of $\mathfrak{g}$ on which $H'_{\mathbb{F}}$ acts trivially. Therefore we can now argue as in (a)$\Longleftrightarrow$(b) in the first paragraph using the sequence (3-9), to deduce the short exact sequence

$$0 \to \operatorname{Coker} d\phi^{sc} \to H^1(H'_{\mathbb{F}}, \operatorname{Im} d\phi^{sc}) \to H^1(H'_{\mathbb{F}}, \mathfrak{g}) \to 0.$$

Since $\dim_{\mathbb{F}} \mathfrak{g}^{sc} = \dim_{\mathbb{F}} \mathfrak{g}$, we have that $\dim_{\mathbb{F}} \operatorname{Coker} d\phi^{sc} = \dim_{\mathbb{F}} \ker d\phi^{sc}$. Thus, (b) for $\mathfrak{g}^{sc}$ is equivalent to $\dim_{\mathbb{F}} \ker d\phi^{sc} + \dim_{\mathbb{F}} H^1(H'_{\mathbb{F}}, \mathfrak{g}) = \dim_{\mathbb{F}} \mathfrak{z}$. We may now combine the above with the short exact sequence

$$0 \to \ker d\phi^{sc} \to \mathfrak{z} \to Z(\operatorname{Im} d\phi^{sc}) \to 0,$$

to deduce that (b) for $\mathfrak{g}^{sc}$ and (b) for $\mathfrak{g}$ are equivalent. ∎

The following result is essentially due to Cline, Jones, Parshall, Scott for the A-D-E types and due to Völklein for the others, as we shall indicate in its proof.

Theorem 3.41. *Suppose that* $(\mathscr{G}, \mathbb{F})$ *satisfies the following conditions*

(i) $(\mathit{type}, \mathbb{F})$ *is neither* $(A_1, \mathbb{F}_2)$ *nor* $(A_1, \mathbb{F}_5)$.

(ii) *If type* $= C_n$, *then* $\#\mathbb{F} \notin \{4, 9\}$.

(iii) *If* $\mathscr{G}$ *is nonsplit* (*and hence of types* A, D *or* E_6), *then* $|\mathbb{F}| \geq 4$.

Then

$$\dim_{\mathbb{F}} H^1(H'_{\mathbb{F}}, \mathfrak{g}) = \dim_{\mathbb{F}} Z(\mathfrak{g}).$$

Proof. Suppose first that $\mathscr{G}$ is split. If the type is C_n and $\#\mathbb{F} \in \{2, 3, 5\}$, then the result is proved in [Put12, Theorem G]. If one excludes the two cases in (i) and assumes for type C_n that $\#\mathbb{F}$ is not in $\{2, 3, 4, 5, 9\}$, then the assertion is covered by [Völ89b, Theorem and Remarks (a)]. Concerning [Völ89b, Remarks (a)], note that the assertion is not quite immediate from [Völ89a, Corollary 2]. Some additional work is required since the module V defined and used in [Völ89a] is not equal to $\bar{\mathfrak{g}}$ for $\mathscr{G}$ of type B_n, C_n, F_4 and G_2, cf. [His84, diagrams in Hauptsatz]. Hence in addition to [Völ89a, Corollary 2] one has to prove the vanishing of $H^1(H'_{\mathbb{F}}, W)$ for certain subfactors W of $\mathfrak{g}$ directly. This is not difficult.

Suppose finally, that $\mathscr{G}$ is nonsplit over $\mathbb{F}$. Since we assume $|\mathbb{F}| \geq 4$ it is shown in [CPS77, Table 3], in all cases claimed, that $H^1(\mathscr{G}^{\mathrm{sc}}(\mathbb{F}), \bar{\mathfrak{g}}) = \dim_{\mathbb{F}} \mathfrak{z}$. It follows from Lemma 3.39 that $H^1(\mathscr{G}^{\mathrm{sc}}(\mathbb{F}), \hat{\mathfrak{g}}) = 0$, since all exceptions in that lemma occur for $|\mathbb{F}| < 4$ in the nonsplit case. By inflation-restriction and Corollary 3.8(a) we deduce $H^1(H'_{\mathbb{F}}, \hat{\mathfrak{g}}) = 0$, and then again from Lemma 3.39, the assertion. □

Remark 3.42. Concerning the excluded A_1 cases, a direct computation shows that

$$\begin{aligned} &H^1(\mathrm{SL}_2(\mathbb{F}_2), \mathfrak{sl}_2) \cong \mathbb{F}_2 \cong Z(\mathfrak{sl}_2), && H^1(\mathrm{SL}_2(\mathbb{F}_5), \mathfrak{sl}_2) \cong \mathbb{F}_5 \ncong Z(\mathfrak{sl}_2) = 0, \\ &H^1(\mathrm{PGL}_2(\mathbb{F}_2), \mathfrak{pgl}_2) \cong \mathbb{F}_2 \ncong Z(\mathfrak{pgl}_2) = 0, && H^1(\mathrm{PGL}_2(\mathbb{F}_5), \mathfrak{pgl}_2) = 0 = Z(\mathfrak{pgl}_2). \end{aligned}$$

The results on the Lie algebra centers are straightforward. For the cohomology computations over $\mathbb{F}_2$ one can simply use the Hochschild–Serre spectral sequence; over $\mathbb{F}_5$, the result for SL_2 can be found in [CPS75, Table 4.5] and that for PGL_2 can be easily derived via Hochschild–Serre from [Fla92, Lemma 1.2] which states $H^1(\mathrm{GL}_2(\mathbb{F}_5), \mathfrak{gl}_2) = 0$.

Combining Lemma 3.39 and Theorem 3.41, we obtain:

Corollary 3.43. *Condition* **(van)** *holds if* $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$, $(\mathit{type}, \mathbb{F}) \notin \mathscr{E}_{\mathbf{(sch)}} \cup \{(A_1, \mathbb{F}_5)\}$, **(ct)** *holds*, *and if one of the following holds*:

(ii) *If type* $= C_n$, *then* $\#\mathbb{F} \notin \{4, 9\}$.

(iii) *If* $\mathscr{G}$ *is nonsplit* (*and hence of types* A, D *or* E_6), *then* $|\mathbb{F}| \geq 4$.

Remark 3.44. The authors only recently became aware of the work [Put12, Theorem G], which proves Theorem 3.41 in case (ii) if $\mathbb{F}$ is a prime field of order at most 5. We plan to see if similar methods could give a complete result in the C_n case, and perhaps also the A and D cases excluded in (iii). The hope would be that there is only a finite number of cases, i.e., of pairs $(type, \mathbb{F})$, in which $H^1(H'_{\mathbb{F}}, \hat{\mathfrak{g}})$ does not vanish, and that one might perhaps be able to completely classify these cases. In light of Lemma 3.39 and Remark 3.42 this would seem the best result one could hope for.

3.6. ***Condition* (n-s).** In this subsection we recall two results from [Vasiu] on **(n-s)**. Let $\mathscr{G}$ be an absolutely simple semisimple group scheme defined over $W(\mathbb{F})$, and consider the exact sequence of $\mathbb{F}_p[\mathscr{G}(\mathbb{F})]$-modules

$$1 \to \mathfrak{g} \to \mathscr{G}(W_2(\mathbb{F})) \xrightarrow{\pi_{\mathscr{G}}} \mathscr{G}(\mathbb{F}) \to 1, \tag{3-10}$$

where $W_2(\mathbb{F}) = W(\mathbb{F})/p^2W(\mathbb{F})$.

The most comprehensive study of the splitness of such a sequence (and also for general semisimple $\mathscr{G}$) is to our knowledge carried out in [Vasiu].

Theorem 3.45 (Vasiu). *Suppose $\mathscr{G}$ over $W(\mathbb{F})$ is absolutely simple. Then* (3-10) *is nonsplit if* $|\mathbb{F}| \geq 5$, *or in the following cases*:

(a) $\mathbb{F} = \mathbb{F}_2$, *$\mathscr{G}$ is of adjoint type and the type is not in* $\{A_1, A_2, {}^2A_2, {}^2A_3\}$, *or $\mathscr{G}$ is simply connected and the type is not in* $\{A_1, A_2, {}^2A_2\}$.

(b) $\mathbb{F} = \mathbb{F}_3$, *$\mathscr{G}$ is of adjoint type or simply connected and the type is not* A_1.

(c) $\mathbb{F} = \mathbb{F}_4$, *$\mathscr{G}$ is of adjoint type and the type is not A_1 or $\mathscr{G}$ is simply connected.*

Moreover for the exceptions listed in (*a*)–(*c*) *it is known that* (3-10) *is split.*

In addition, (3-10) *is split if $\mathscr{G}$ is of type ${}^2A_3$ but neither simply connected nor of adjoint type.*

Proof. The case $|\mathbb{F}| \geq 5$ follows from [Vasiu, Proposition 4.4.1]. Cases (a)–(c) and the assertion thereafter follow from [Vasiu, Theorem 4.5]; note that there is an omission in his result: the reduction $\mathrm{SL}_2(\mathbb{Z}_2) \to \mathrm{SL}_2(\mathbb{F}_2)$ has a splitting; see [Dor16, Proposition 20]. The last line, can be deduced from the second to last paragraph of the proof of [Vasiu, Theorem 4.5]: As described there, one has a degree 2 cover $\mathrm{SO}_6^- \to \mathrm{PGU}_4$ (note $D_3 = A_3$ as Dynkin diagrams), which can be found in [CCN+85, p. 26] or in [MT11, Table 22.1]. Vasiu deduces from [CCN+85, p. 26] a splitting of $\mathscr{G}(\mathbb{Z}_2) \to \mathscr{G}(\mathbb{F}_2)$ for $\mathscr{G}$ of type SO_6^-. □

To have a more complete result, we clarify the relation between the sequence (3-10) being split for $\mathscr{G}$ and for its universal cover $\phi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}$. The first important point to notice is that the proof of Corollary 3.8(a) holds over the base ring $W_2(\mathbb{F})$ instead of $\mathbb{F}$ with next to no changes. Recalling $\mathscr{Z}' = \operatorname{Ker} \phi^{\mathrm{sc}}$, $Z' := \mathscr{Z}'(\mathbb{F})$, we obtain short exact sequences

$$1 \to \mathfrak{z}' \times Z' \to \mathscr{G}^{\mathrm{sc}}(W_2(\mathbb{F})) \to E_2 \to 1 \quad \text{and} \quad 1 \to E_2 \to \mathscr{G}(W_2(\mathbb{F})) \to \mathfrak{z}' \times Z' \to 1,$$

in which E_2 is defined as the image of $\mathscr{G}^{\mathrm{sc}}(W_2(\mathbb{F})) \to \mathscr{G}(W_2(\mathbb{F}))$ under ϕ^{sc}.

Let us now consider the following commutative diagram of abstract groups with exact rows, in which the group E_3 is defined as the pullback from row 4, and the π_j are the maps they label:

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{g}^{\mathrm{sc}} \arrow[r] \arrow[d, equal] & E_1 := \mathscr{G}^{\mathrm{sc}}(W_2(\mathbb{F})) \arrow[r, "\pi_{\mathscr{G}^{\mathrm{sc}}}"] \arrow[d, "\mathrm{mod}\,\mathfrak{z}'\times Z'"] & \mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \arrow[r] \arrow[d, "\mathrm{mod}\,Z'"] & 1 \\
1 \arrow[r] & \mathfrak{g}_2 := \mathfrak{g}^{\mathrm{sc}}/\mathfrak{z}' \arrow[r] \arrow[d, hook] & E_2 \arrow[r, "\pi_2"] \arrow[d, hook] & H'_{\mathbb{F}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathfrak{g}_3 := \mathfrak{g} \arrow[r] \arrow[d, equal] & E_3 \arrow[r, "\pi_3"] \arrow[d, hook] & H'_{\mathbb{F}} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathfrak{g} \arrow[r] & E_4 := \mathscr{G}(W_2(\mathbb{F})) \arrow[r, "\pi_4:=\pi_{\mathscr{G}}"] & \mathscr{G}(\mathbb{F}) \arrow[r] & 1.
\end{tikzcd}$$

Note that each of the extensions in rows 1 to 4 defines a corresponding class γ_i in a second group cohomology: $\gamma_1 \in H^2(\mathscr{G}^{\mathrm{sc}}(\mathbb{F}), \mathfrak{g}^{\mathrm{sc}})$, $\gamma_i \in H^2(H'_{\mathbb{F}}, \mathfrak{g}_i)$ for $i = 2, 3$ and $\gamma_4 \in H^2(\mathscr{G}(\mathbb{F}), \mathfrak{g})$. In each row i the extension is split if and only if $\gamma_i = 0$. One has the following result:

Lemma 3.46. *If* **(n-s)** *holds for $\mathscr{G}$ it holds for $\mathscr{G}^{\mathrm{sc}}$. If conversely* **(n-s)** *holds for $\mathscr{G}^{\mathrm{sc}}$ and if $\mathscr{G}^{\mathrm{sc}}$ is not in $\mathscr{E}_{\mathbf{(pf)}}$ or in $\mathscr{E}_{\mathbf{(sch)}}$, then* **(n-s)** *holds for $\mathscr{G}$.*

Proof. It is clear that if row 1 splits, then so does row 2, by passing to quotients. Also it is trivial that if row 2 splits, then so does row 3. Suppose that row 3 is split. Let $s : \mathscr{G}(\mathbb{F}) \to E_4$ be a set-theoretic splitting that defines the class γ_4. Then its restriction to $H'_{\mathbb{F}}$ defines a set-theoretic splitting $H'_{\mathbb{F}} \to E_3$, i.e., a representative of γ_3. By hypothesis, $\gamma_3 = 0$. Now the Hochschild–Serre spectral sequence for $1 \to H_{\mathbb{F}} \to \mathscr{G}(\mathbb{F}) \to Z' \to 1$ degenerates with $\mathfrak{g}$ coefficients, because Z' is of order prime to p. Hence restriction is an isomorphism $H^2(\mathscr{G}(\mathbb{F}), \mathfrak{g}) \to H^2(H'_{\mathbb{F}}, \mathfrak{g})^{Z'}$, and it follows that $\gamma_4 = 0$.

We now go in the opposite direction. It is clear that a splitting of row 4 restricts to a splitting of row 3. Suppose now that row 3 is split via some $s : H'_{\mathbb{F}} \to E_3$. Composing this with the surjection $E_3 \to E_3/E_2$ induces a homomorphism $H'_{\mathbb{F}} \to E_3/E_2 \cong \mathfrak{z}'$. Note that the target is an elementary abelian p-group. As $\mathscr{G}^{\mathrm{sc}}$ is not in $\mathscr{E}_{\mathbf{(pf)}}$, the map $H'_{\mathbb{F}} \to E_3/E_2$ is zero by Corollary 3.12. Hence s takes values in E_2.

Suppose finally that row 2 is split, via some homomorphism $s : H'_{\mathbb{F}} \to E_2$. The splitting in row 2 gives a commuting diagram

$$\begin{tikzcd}
 & \mathfrak{z}' \times Z' \arrow[d] \\
\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \arrow[r, dashed] \arrow[dr, "s\circ(\mathrm{mod}\,\mathfrak{z}'\times Z')"'] & E_1 \arrow[d] \\
 & E_2.
\end{tikzcd}$$

We first wish to lift to the dotted homomorphism. The obstruction lies in $H^2(\mathscr{G}^{\mathrm{sc}}(\mathbb{F}), \mathfrak{z}' \times Z')$. Here Z' is finite abelian of order prime to p and $\mathfrak{z}'$ is a p-torsion finite abelian group. As we assume that $\mathscr{G}^{\mathrm{sc}}$ lies not in $\mathscr{E}_{\mathbf{(sch)}}$, it follows from Remark 3.14 that the obstruction vanishes. Let $s' : \mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \to E_1$ be a homomorphism lifting s. Consider the map $\psi : \mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \to \mathfrak{z}' \times Z'$, $g \mapsto s'(g)g^{-1}$. Since it takes values in the center $\mathfrak{z} \times Z'$ of $\mathscr{G}^{\mathrm{sc}}(W_2(\mathbb{F}))$, it is easily seen to be a homomorphism. For the same reason

$\psi^{-1} \circ s' : \mathcal{G}^{\mathrm{sc}}(\mathbb{F}) \to E_1$ is a homomorphism. It clearly lifts s and hence row 1 is split. This completes the proof. □

The following result is immediate from Theorem 3.45 and Lemma 3.46.

Corollary 3.47. *Suppose $\mathcal{G}$ over $W(\mathbb{F})$ is absolutely simple. Then* **(n-s)** *holds if and only if $(\mathcal{G}, \mathbb{F})$ is not in the list $\mathcal{E}_{\textbf{(n-s)}}$ given in* (3-3).

3.7. *z-Lie-balanced groups.* In this subsection we introduce a particular kind of reductive group over $W(\mathbb{F})$ that is frequently encountered in applications and for which the analog Theorem 3.52 of Theorem 3.2 has fewer exceptions, for instances for **(l-cl)** or **(l-un)**.

Let us begin by a construction inspired from [Art02, p. 474] which uses ideas from z-extensions introduced by Langlands and Kottwitz. Let $\mathcal{H}$ be a connected reductive group over $W(\mathbb{F})$ whose special fiber $\mathcal{H}_{\mathbb{F}}$ is absolutely simple and such that the natural map $\mathfrak{h} = \mathrm{Lie}\,\mathcal{H}_{\mathbb{F}} \to \mathfrak{h}^{\mathrm{ad}}$ is an isomorphism. The kernel of the central isogeny $\mathcal{H}^{\mathrm{sc}} \to \mathcal{H}$ is described in Table 1. Let $\pi : \mathcal{H}' \to \mathcal{H}$ be a central isogeny such that $\mathcal{H}^{\mathrm{sc}} \to \mathcal{H}'$ is étale. This implies that the natural map $\mathfrak{h}^{\mathrm{sc}} \to \mathfrak{h}'$ is an isomorphism and that $Z(\mathcal{H}')^o$ lies in $\mathrm{Ker}\,\pi$. Choose a homomorphism $\mathrm{Ker}\,\pi \hookrightarrow \mathcal{T}$ over $W(\mathbb{F})$ into a torus whose induced map $Z(\mathfrak{h}') \to \mathfrak{t} = \mathrm{Lie}(\mathcal{T}_{\mathbb{F}})$ on Lie algebras is an isomorphism; so $\mathcal{T}$ has rank 0, 1 or 2. Define $\mathcal{G}' := (\mathcal{H}' \times \mathcal{T}) / \mathrm{Ker}\,\pi$ with $\mathrm{Ker}\,\pi$ embedded diagonally. Then one has a short exact sequence of reductive groups

$$1 \to \mathcal{T} \to \mathcal{G}' \to \mathcal{H} \to 1 \tag{3-11}$$

with the following properties:

(a) The map $\mathcal{G}' \to \mathcal{H}$ is a central extension.
(b) The natural map $\mathcal{H}' \to \mathcal{G}'^{,\mathrm{der}}$ is an isomorphism and $\mathcal{H}'$ is Lie-simply connected.
(c) The image of $\mathcal{T}$ in $\mathcal{G}'$ is the identity component of the center of $\mathcal{G}'$.
(d) The Lie algebra $\mathfrak{g}' = \mathrm{Lie}\,\mathcal{G}'$ is isomorphic to $\hat{\mathfrak{h}}$.
(e) The induced sequence of Lie algebras $0 \to \mathfrak{t} \to \mathfrak{g}' \to \mathfrak{h} \to 0$ is exact.

Definition 3.48. We call a connected reductive group $\mathcal{G}'$ over $W(\mathbb{F})$

(a) *Lie-balanced* if it arises as a $\mathcal{G}'$ via the above construction.
(b) *z-Lie-balanced* if there exists a central extension $\mathcal{G}' \to \mathcal{G}''$ to a Lie-balanced $\mathcal{G}''$ with smooth kernel of multiplicative type such that the induced morphism $\mathcal{G}'^{,\mathrm{der}} \to \mathcal{G}''^{,\mathrm{der}}$ is a central isogeny.

Example 3.49. The groups GL_n and GSp_n are z-Lie-balanced over $W(\mathbb{F})$ for any finite field $\mathbb{F}$. They are Lie-balanced if p divides n, for GL_n, or if $p = 2$, for GSp_n.

If $\mathcal{G}$ is almost simple over $W(\mathbb{F})$ and if $\mathfrak{g}$ is simple, cf. Proposition 3.22, then $\mathcal{G}$ is Lie-balanced.

Lemma 3.50. *Let $\mathcal{G}'$ be z-Lie-balanced and let $\pi' : \mathcal{G}' \to \mathcal{G}''$ be a central extension* (Definition 3.48(b)).

(a) *The identity component $Z(\mathcal{G}')^o$ of $Z(\mathcal{G}')$ is a torus, and hence $Z(\mathcal{G}')$ is smooth.*

(b) *Let* $\mathfrak{t} := \mathrm{Lie}\,\mathrm{Ker}\,\pi'$ *and* $\hat{\mathfrak{h}} := \mathrm{Lie}\,\mathscr{G}''$. *Then* $0 \to \mathfrak{t} \to \mathrm{Lie}\,\mathscr{G}' \to \hat{\mathfrak{h}} \to 0$ *is a short exact sequence as Lie algebras and as* $H_{\mathbb{F}}$*-modules and* $\mathfrak{t}$ *is an abelian Lie subalgebra which carries a trivial* $H_{\mathbb{F}}$*-action.*

Proof. Because π' is central and surjective with smooth kernel of multiplicative type, we have a short exact sequence of centers $1 \to \mathrm{Ker}\,\pi' \to Z(\mathscr{G}') \to Z(\mathscr{G}'') \to 1$ with $\mathrm{Ker}\,\pi'$ smooth of multiplicative type. By the hypothesis on $\mathscr{G}''$ the group $Z^o(\mathscr{G}'')$ is a torus. Therefore $Z(\mathscr{G}')$ is smooth and of multiplicative type. This implies (a).

To see (b) note that because π' is central the claimed properties on $\mathfrak{t}$ are clear. Also the displayed sequence is clearly left exact. The right exactness follows from smoothness and dimension considerations: One has $\dim_{\mathrm{Krull}} \mathscr{G}' = \dim_{\mathrm{Krull}} \mathscr{G}'' + \dim_{\mathrm{Krull}} \mathrm{Ker}\,\pi'$. Passing to tangent spaces at the identity and using the smoothness, we obtain $\dim_{\mathbb{F}} \mathrm{Lie}\,\mathscr{G}' = \dim_{\mathbb{F}} \mathrm{Lie}\,\mathscr{G}'' + \dim_{\mathbb{F}} \mathfrak{t}$, and using $\mathrm{Lie}\,\mathscr{G}'' = \hat{\mathfrak{h}}$, we are done. $\square$

For the remainder of this subsection, the following set of conditions will be important:

Conditions 3.51. The algebraic group $\mathscr{G}'$ is z-Lie-balanced over $W(\mathbb{F})$, $H'_{\mathbb{F}} := [\mathscr{G}'(\mathbb{F}), \mathscr{G}'(\mathbb{F})]$, and $H_{\mathbb{F}}$ is a subgroup of $\mathscr{G}'(\mathbb{F})$ that contains $H'_{\mathbb{F}}$.

Theorem 3.52. *Suppose that* $(\mathscr{G}', \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ *satisfies Conditions 3.51. Let* $\mathscr{G} := \mathscr{G}'^{,\mathrm{der}}$. *Then the tuple satisfies Assumption 2.1, and the following hold*:

(a) **(pf)**, **(ct)**, **(csc)** *and* **(n-s)** *hold unless* (*type*, $\mathbb{F}$) *for the group* $\mathscr{G}$ *is in the list*

$$(A_1, \mathbb{F}_?)_{? \in \{2,3\}}, \quad ({}^2A_2, \mathbb{F}_2), \quad ({}^2A_3, \mathbb{F}_2), \quad (B_2, \mathbb{F}_2), \quad (G_2, \mathbb{F}_2).$$

(b) **(van)** *holds unless* (*type*, $\mathbb{F}$) *for* $\mathscr{G}$ *is in the list*

$$(A_1, \mathbb{F}_?)_{? \in \{2,3,5\}}, \quad ({}^2A_2, \mathbb{F}_2), \quad (C_n, \mathbb{F}_?)_{n \geq 2, ? \in \{2,3,4,5,9\}}, \quad (G_2, \mathbb{F}_2)$$

or $\mathscr{G}^{\mathrm{der}}$ *is nonsplit, and hence of types* A, D *or* E_6, *and* $|\mathbb{F}| < 4$.

(c) *Assuming* $\mathscr{G}^{\mathrm{sc}}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$, *concerning* **(l-?)** *the following hold*:

(i) **(l-ge)**, *if* $\mathscr{G}$ *is of type* A_n *and* $p \nmid n+1$, *or of type* B_n, C_n, D_n, E_7 *or* F_4 *and* $p \neq 2$, *or if* $\mathscr{G}$ *is of type* E_6 *or* G_2 *and* $p \neq 3$, *or if* $\mathscr{G}$ *is of type* E_8.

(ii) **(l-un)**, *unless* $\mathscr{G}$ *is of type* B_2 *or* F_4 *and* $p = 2$, *or of type* G_2 *and* $p = 3$.

(iii) **(l-cl)**, *if* $\mathscr{G}$ *is of type* A_n, $n \geq 2$, D_n *or* E_n, *or* $\mathscr{G}$ *is of type* A_1, B_n, C_n, F_4 *and* $p \neq 2$, *or of type* G_2 *and* $p \neq 3$.

Proof. Part (a) of Assumption 2.1 is clear, since $\mathscr{G}'$ is connected reductive. To see Assumption 2.1(b) note that $H_{\mathbb{F}}$ is generated by $H'_{\mathbb{F}} = \mathscr{G}(\mathbb{F})$ together with $Z(\mathscr{G}')(\mathbb{F})$ and that the latter group is of order prime to p.

(a) Conditions **(pf)**, **(csc)** and **(n-s)** only depend on $H'_{\mathbb{F}}$ and $\mathscr{G}$. Hence we can directly apply Corollary 3.8(c), Proposition 3.38 and Corollary 3.47, noting that here the map $\mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g}$ is an isomorphism. The list in (a) is the union of the relevant sublists of the exceptional lists $\mathscr{E}_{\mathbf{(pf)}}$ and $\mathscr{E}_{\mathbf{(n\text{-}s)}}$ that occur in the two corollaries. It remains to see that **(ct)** can only fail for groups in that list. That $Z(\mathscr{G}')$ is smooth follows from

Lemma 3.50(a). To see that $Z(\operatorname{Lie}\mathscr{G}') \to H^0(H_{\mathbb{F}}, \mathfrak{g}')$ is an isomorphism, we consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Lie}(\operatorname{Ker}\pi) & \longrightarrow & \operatorname{Lie} Z(\mathscr{G}') & \longrightarrow & \operatorname{Lie} Z(\mathscr{G}'') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(H_{\mathbb{F}}, \mathfrak{t}) & \longrightarrow & H^0(H_{\mathbb{F}}, \mathfrak{g}') & \longrightarrow & H^0(H_{\mathbb{F}}, \mathfrak{g}''), \end{array}$$

induced from Lemma 3.50(b), where $\pi : \mathscr{G}' \to \mathscr{G}''$ is a surjection as in Definition 3.48. The top row is exact by the proof of Lemma 3.50. The arrow on the left is an isomorphism, again by Lemma 3.50(b), and so it suffices to show that the arrow on the right is an isomorphism, i.e., we may assume that $\mathscr{G}' = \mathscr{G}''$. But then $\operatorname{Lie}\mathscr{G}'' = \hat{\mathfrak{h}}$ for $\mathfrak{h} = \operatorname{Lie}\mathscr{G}^{\mathrm{der}}$, and now the isomorphism property follows from Theorem 3.24(b) and the last line of Remark 3.27.

(b) Using the sequence in Lemma 3.50(b) together with Corollary 3.12, we may assume that $\mathscr{G}' = \mathscr{G}''$ for the proof; the exceptions ruled out in Corollary 3.12 are part of the list in (b). From the definition of Lie-balanced, we know that $\mathfrak{g}^{\mathrm{sc}} \to \mathfrak{g}$ is an isomorphism. By Remark 3.40, for Lemma 3.39(a)$\Longleftrightarrow$(b) we only need $\mathscr{G}(\mathbb{F}) \notin \mathscr{E}_{\mathbf{(pf)}}$. For Lemma 3.39(b) to hold, we need to avoid the pairs $(\mathscr{G}, \mathbb{F})$ listed as exceptions in Theorem 3.41. The combined list of exceptions is that given in (b).

(c) The conditions **(l-?)** only depend on $\mathscr{G}$, which by construction is Lie-simply connected. Therefore (c) follows from Corollary 3.31, Corollary 3.36 and Lemma 3.37. ☐

Remark 3.53. It is possible to construct a sequence as in (3-11) also for cases where $\mathscr{H}'$ is Lie-intermediate (and not only for Lie-simply connected cases, as done here). However in this situation the analog of Theorem 3.52(b) on **(van)** does not hold. Since this condition is crucial for our later applications, we did not pursue this here.

Corollary 3.54. *The tuple* $(\mathrm{GL}_n, \mathbb{F}, H_{\mathbb{F}}, \mathrm{SL}_n(\mathbb{F}))$ *for* $H_{\mathbb{F}}$ *a subgroup of* $\mathrm{GL}_n(\mathbb{F})$ *that contains* $\mathrm{SL}_n(\mathbb{F})$ *satisfies Assumption 2.1, and the following hold*:

(a) **(pf)**, **(ct)**, **(van)**, **(csc)** *and* **(n-s)** *hold unless* $(n, \mathbb{F})$ *is in the list* $(2, \mathbb{F}_?)_{? \in \{2,3,5\}}$.

(b) **(l-un)** *holds unconditionally*, **(l-cl)** *holds if and only if* $(n, \operatorname{Char}\mathbb{F}) \neq (2, 2)$.

4. Preparations

Throughout this section we require that Assumption 2.1 holds for $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$.

Our first result clarifies the definition of H_R from (2-1) based on Convention 2.4.

Lemma 4.1. *Let R be in $\widehat{Ar}_{W(\mathbb{F})}$.*

(a) *There exist subgroups $M_R^o \subset \mathscr{G}^o(R)$ and $M_R \subset \mathscr{G}(R)$ that under reduction modulo $\mathfrak{m}_R$ map isomorphically to $M_{\mathbb{F}}^o$ and $M_{\mathbb{F}}$, respectively.*

(b) *The possible M_R^o from (a) form a conjugacy class under* $\operatorname{Ker}\big(\mathscr{G}^o(R) \to \mathscr{G}^o(\mathbb{F})\big)$, *and so do the M_R*;

(c) *the groups M_R^o and M_R normalize H'_R.*

(d) *The group* $H_R^o := H_R' M_R^o \subset \mathscr{G}^o(R)$ *is independent of the choice of* M_R^o.

(e) *If* $\mathscr{G}/\mathscr{G}^o$ *acts trivially on* $\mathscr{G}^o/\mathscr{G}^{\mathrm{der}}$, *then also* $H_R = H_R' M_R \subset \mathscr{G}(R)$ *is independent of any choices.*

Proof. Parts (a) and (b) follow from the profinite version of the Schur–Zassenhaus theorem given in [Bos91, Proposition 2.1]: Since $M_{\mathbb{F}}^o$ has order prime to p, the cohomology groups $H^i(M_{\mathbb{F}}^o, \mathfrak{g})$, $i \geq 1$, vanish. Now the kernel of $\mathscr{G}^o(R) \to \mathscr{G}^o(\mathbb{F})$ is a pro-p group. Therefore an inductive argument using $H^2(M_{\mathbb{F}}^o, \mathfrak{g}) = 0$ shows the existence of a lift as in (a); its uniqueness up to conjugation, claimed in (b), follows from $H^1(M_{\mathbb{F}}^o, \mathfrak{g}) = 0$, again by induction. The same argument works for M_R as well. Since $\mathscr{G}^{\mathrm{der}}(R)$ is characteristic in $\mathscr{G}(R)$, see Proposition A.18, it is normalized by M_R^o and M_R. Because $M_{\mathbb{F}}^o$ and $M_{\mathbb{F}}$ normalize $H_{\mathbb{F}}'$, part (c) follows.

For (d) note that by construction $H_R^o \cap \mathscr{G}^{\mathrm{der}}(R) = H_R'$. Hence it suffices to show that

$$H_R^o/H_R' \cong \mathscr{G}^{\mathrm{der}}(R) H_R^o/\mathscr{G}^{\mathrm{der}}(R) = \mathscr{G}^{\mathrm{der}}(R) M_R^o/\mathscr{G}^{\mathrm{der}}(R) \subset \mathscr{G}^o(R)/\mathscr{G}^{\mathrm{der}}(R)$$

is independent of any choices. However $\mathscr{G}^o(R)/\mathscr{G}^{\mathrm{der}}(R)$ is abelian and the conjugation action of M_R^o induced from (b) is thus trivial. This shows (d). The proof of (e) is analogous, using the hypotheses of (e) and again assertion (b). □

The following lemma introduces in our setting an important substitute H^c for the commutator subgroup $[H, H]$ for a closed subgroup H of H_R that surjects onto $H_{\mathbb{F}}$ under reduction modulo $\mathfrak{m}_R$, and that takes the residual image $H_{\mathbb{F}}$ into account.

Lemma 4.2. *Let* R *be in* $\widehat{Ar}_{W(\mathbb{F})}$ *and let* $H \subset \mathscr{G}(R)$ *be a closed subgroup that surjects onto* $H_{\mathbb{F}}$. *Then there exists a unique closed subgroup* $H^c \subset H$ *that contains the closure* $\overline{[H, H]}$ *of the commutator subgroup and for which* $H^c/\overline{[H, H]} \to H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$ *is an isomorphism.*

Moreover if I *is an ideal of* R *and if* $\bar{H}$ *denotes the image of* H *in* $H_{R/I}$, *which is a closed subgroup, then under reduction modulo* I *the group* H^c *maps onto* $\bar{H}^c$. *In particular, the reduction of* H^c *modulo* $\mathfrak{m}_R$ *is* $H_{\mathbb{F}}$.

Proof. The group $H^{\mathrm{ab}} := H/\overline{[H, H]}$ surjects onto $H_{\mathbb{F}}^{\mathrm{ab}} := H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$ and the kernel is a pro-p group. Since $H_{\mathbb{F}}^{\mathrm{ab}}$ is of order prime to p, there exists a lift $H^{\mathrm{ab},c}$ of $H_{\mathbb{F}}^{\mathrm{ab}}$ to H^{ab} that is unique up to conjugation. Because H^{ab} is abelian, in fact $H^{\mathrm{ab},c}$ is unique. Clearly H^c must be the inverse image of $H^{\mathrm{ab},c}$ under the canonical surjection $H \to H^{\mathrm{ab}}$.

It remains to prove the second assertion. It is immediate that the reduction of $[H, H]$ modulo I equals $[\bar{H}, \bar{H}]$. Consider the reduction maps

$$H^{\mathrm{ab}} \longrightarrow \bar{H}^{\mathrm{ab}} := \bar{H}/\overline{[\bar{H}, \bar{H}]} \longrightarrow H_{\mathbb{F}}^{\mathrm{ab}}.$$

The above construction defines subgroups $H^{\mathrm{ab},c} \subset H^{\mathrm{ab}}$ and $\bar{H}^{\mathrm{ab},c} \subset \bar{H}^{\mathrm{ab}}$. Since the image of $H^{\mathrm{ab},c}$ in $\bar{H}^{\mathrm{ab}}$ satisfies the properties required for $\bar{H}^{\mathrm{ab},c}$, the uniqueness of $\bar{H}^{\mathrm{ab},c}$ shows that $H^{\mathrm{ab},c}$ maps isomorphically to $\bar{H}^{\mathrm{ab},c}$ under reduction. This implies (b). □

Lemma 4.3. *Let R be in $\widehat{Ar}_{W(\mathbb{F})}$ and let $H \subset H_R$ be a closed subgroup that surjects onto $H_{\mathbb{F}}$. If either* **(l-ge)**(*ii*) *holds, or if $H = H_R$ and* **(csc)** *holds, then $H^c = H$.*

Proof. By an inverse limit argument and Lemma 4.2, it suffices to prove the lemma for Artinian R. Here, by an inductive argument, and again by Lemma 4.2, it will suffice to prove the following: Let $I \subset R$ be an ideal such that $I \cong \mathbb{F}$ as an R-algebra, let $\bar{H}$ be the image of H in $H_{R/I}$. Then $\bar{H}^c = \bar{H}$ implies that $H^c = H$. To see this consider the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook, "\iota"] & \bar{H} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathfrak{g}^{\mathrm{der}} \otimes I \cong \mathfrak{g}^{\mathrm{der}} \arrow[r] & H_R \arrow[r] & H_{R/I} \arrow[r] & 0,
\end{tikzcd}$$

where N is the kernel of $H \to \bar{H}$. If N is the trivial group, then $H \to \bar{H}$ is an isomorphism, and the result is clear. Suppose now that N is nontrivial. Then either by **(l-ge)**(ii), or by hypotheses **(csc)** if $H = H_R$, we must have $N = \mathfrak{g}^{\mathrm{der}}$. In either case, $H_0(H'_{\mathbb{F}}, \mathfrak{g}^{\mathrm{der}}) = 0$, i.e., $\mathfrak{g}^{\mathrm{der}}$ is the $\mathbb{F}_p$-span of $\{gX - X \mid X \in \mathfrak{g}^{\mathrm{der}}, g \in H'_{\mathbb{F}}\}$. This implies $N = [N, H] \subset [H, H]$. Thus H^c contains N. Since $\bar{H}^c = \bar{H}$ by hypothesis, we deduce $H^c = H$. $\square$

Remark 4.4. Suppose that as an $\mathbb{F}_p[H'_{\mathbb{F}}]$-module no Jordan–Hölder factor of $\mathfrak{g}^{\mathrm{der}}$ is trivial, i.e., that $\mathrm{Hom}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(N, \mathbb{F}_p) = 0$ for all submodules N of $\mathfrak{g}^{\mathrm{der}}$. Then the above argument shows that $H = H^c$ for any closed subgroup $H \subset H_R$.

If on the other hand there exists an $\mathbb{F}_p[H'_{\mathbb{F}}]$-submodule N with a nonzero $\mathbb{F}_p[H'_{\mathbb{F}}]$-homomorphism $N \to \mathbb{F}_p$, then N is a normal subgroup of $H_{\mathbb{F}[\varepsilon]}$, and $H := NH_{\mathbb{F}}$ is a subgroup of $H_{\mathbb{F}[\varepsilon]}$ with $H^c \subsetneq H$.

Lemma 4.5. *Consider a diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook, "\iota"] & H_{\mathbb{F}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{g}^{\mathrm{der}} \arrow[r] & H_{\mathbb{F}[\varepsilon]} \arrow[r] & H_{\mathbb{F}} \arrow[r] \arrow[l, bend right, "\sigma"'] & 0
\end{tikzcd} \tag{4-1}$$

where ι is an inclusion and σ the canonical splitting. Then the following hold:

(a) *Suppose $N = 0$. If* **(van)** *holds, then ι is conjugate to σ via an element of* $\mathfrak{g}^{\mathrm{der}}$.

(b) *Suppose $N \neq 0$. Suppose either* (1) *that* **(l-ge)**(*ii*) *holds, or* (2) *that* **(l-cl)**(*ii*) *and* **(sch)** *hold, and that $H = H^c$. Then $H = H_{\mathbb{F}[\varepsilon]}$.*

Proof. If $N = 0$, then ι is a splitting of the bottom sequence and by **(van)** all such splittings are conjugate by an element in $\mathfrak{g}^{\mathrm{der}}$. Thus it remains to prove (b), and we assume $N \neq 0$. Under condition **(l-ge)**(ii), we deduce $N = \mathfrak{g}^{\mathrm{der}}$, and we are done.

Suppose now in (b) that (2) holds. Because of **(l-cl)**(ii), we have to rule out that N lies in $\mathrm{Ker}(\mathfrak{g}^{\mathrm{der}} \to \bar{\mathfrak{g}}^{\mathrm{der}})$. Assume on the contrary that N lies in this kernel, so that $N \cong \mathbb{F}_p^r$ for some $r \geq 1$. As $H_{\mathbb{F}}/H'_{\mathbb{F}}$ is of order prime to p, the Hochschild–Serre spectral sequence yields $H^2(H_{\mathbb{F}}, N) \cong H^2(H'_{\mathbb{F}}, N)^{H_{\mathbb{F}}/H'_{\mathbb{F}}}$, and the latter is zero by **(sch)**. Hence H is a semidirect product $N \rtimes H_{\mathbb{F}}$. By hypothesis, $H_{\mathbb{F}}$ acts trivially on N, and thus $H = N \times H_{\mathbb{F}}$. One computes $[H, H] \subset \sigma(H_{\mathbb{F}}) \subsetneq H$, and this contradicts $H = H^c$. Thus we must have $H = H_{\mathbb{F}[\varepsilon]}$. ☐

Lemma 4.6. *Consider the diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook, "\iota"] & H_{\mathbb{F}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{g}^{\mathrm{der}} \arrow[r] & H_{W_2(\mathbb{F})} \arrow[r] & H_{\mathbb{F}} \arrow[r] & 0 \end{tikzcd}$$

where ι is the inclusion. Assume **(n-s)** *and either* **(l-ge)**(*ii*) *or* **(l-cl)**(*ii*) *and* **(sch)**. *Then $H = H_{W_2(\mathbb{F})}$.*

Proof. Because of **(n-s)** the subgroup N must be nontrivial. If in addition **(l-ge)**(ii) holds, then $N = \mathfrak{g}^{\mathrm{der}}$ because $\mathfrak{g}^{\mathrm{der}}$ is irreducible as an $\mathbb{F}_p[H'_{\mathbb{F}}]$-module, and thus ι is an isomorphism.

Suppose now that **(l-cl)**(ii), **(n-s)** and **(sch)** hold. We immediately deduce from **(sch)** and **(n-s)** that N cannot be a submodule of $\mathrm{Ker}(\mathfrak{g}^{\mathrm{der}} \to \bar{\mathfrak{g}}^{\mathrm{der}})$, and then from **(l-cl)**(ii) that N surjects onto the cosocle of $\mathfrak{g}^{\mathrm{der}}$. But as a submodule of $\mathfrak{g}^{\mathrm{der}}$ that surjects onto the cocoscle of $\mathfrak{g}^{\mathrm{der}}$, we must have $N = \mathfrak{g}^{\mathrm{der}}$. This completes the proof. ☐

Example 4.7. If $\mathfrak{g}^{\mathrm{der}}$ possesses a nontrivial $\mathbb{F}_p[H'_{\mathbb{F}}]$-homomorphism to $\mathbb{F}_p$, then one cannot expect the conclusion of Lemma 4.5(b) or of Lemma 4.6 to hold. For a concrete example, let $\mathscr{G} = \mathrm{PGL}_p$ and $H_{\mathbb{F}} = \mathrm{PGL}_p(\mathbb{F})$ with $p = \mathrm{Char}\,\mathbb{F}$ and $\#\mathbb{F} \notin \{2, 3, 4, 5, 9\}$, so that **(sch)** and **(n-s)** hold. One has a surjective homomorphism $\mathrm{pdet} : \mathrm{PGL}_p(R) \to R^\times/R^{\times p}$ induced from det. Let $R \in \{W_2(\mathbb{F}), \mathbb{F}[\varepsilon]\}$ and define $H \subset \mathrm{PGL}_p(\mathbb{F}[\varepsilon])$ as the kernel of $\mathrm{pdet} : \mathrm{PGL}_p(R) \to R^\times/R^{\times p} \cong (\mathbb{F}, +)$. Then H surjects onto $H_{\mathbb{F}}$. But H is properly contained in $\mathrm{PGL}_p(R)$, it is a proper extension of $H_{\mathbb{F}}$, and it satisfies $H = H_{\mathbb{F}}[H, H]$.

For the following result, let $\mathscr{G}^{[i]}(R)$ denote the kernel of $\mathscr{G}^{\mathrm{der}}(R) \to \mathscr{G}^{\mathrm{der}}(R/\mathfrak{m}_R^i)$ for $i \geq 1$ and $R \in \widehat{Ar}_{W(\mathbb{F})}$, and recall from [Pin98, Section 6] that one has natural exponential maps giving isomorphisms $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathbb{F}} \mathfrak{m}_R^i/\mathfrak{m}_R^{i+1} \xrightarrow{\simeq} \mathscr{G}^{[i]}(R)/\mathscr{G}^{[i+1]}(R)$ for any $i \geq 1$.

Lemma 4.8 (cf. [MW86, Appendix, Proposition 2]). *Let R be an Artinian ring in $\widehat{Ar}_{W(\mathbb{F})}$. Assume that $\mathfrak{m}_R^{n+1} = 0$ and $\mathfrak{m}_R^n \neq 0$ for some $n \geq 2$. Consider a diagram*

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook, "\iota"] & H_{R/\mathfrak{m}_R^n} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{g}^{\mathrm{der}} \otimes_{\mathbb{F}} \mathfrak{m}_R^n \arrow[r] & H_R \arrow[r] & H_{R/\mathfrak{m}_R^n} \arrow[r] & 0 \end{tikzcd}$$

with exact rows and ι the inclusion.

(a) *The intersection of N with the commutator subgroup $[\mathscr{G}^{[n-1]}(R) \cap H, \mathscr{G}^{[1]}(R) \cap H]$ contains $[\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}] \otimes_{\mathbb{F}} \mathfrak{m}_R^n$.*

(b) *If* **(l-cl)**(*i*) *holds, then $H = H_R$.*

Proof. Clearly the multiplication map $\mathfrak{m}_R^{n-1}\otimes\mathfrak{m}_R\to\mathfrak{m}_R^n$ factors via

$$\mathfrak{m}_R^{n-1}/\mathfrak{m}_R^n\otimes\mathfrak{m}_R/\mathfrak{m}_R^2\to\mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}.$$

It follows that the commutator map

$$\mathscr{G}^{[n-1]}(R)\times\mathscr{G}^{[1]}(R)\to\mathscr{G}^{[n]}(R),\ (g,h)\mapsto ghg^{-1}h^{-1}$$

factors via

$$\mathscr{G}^{[n-1]}(R)/\mathscr{G}^{[n]}(R)\times\mathscr{G}^{[1]}(R)/\mathscr{G}^{[2]}(R)\cong\left(\mathfrak{g}^{\mathrm{der}}\otimes_{\mathbb{F}}\mathfrak{m}_R^{n-1}/\mathfrak{m}_R^n\right)\times\left(\mathfrak{g}^{\mathrm{der}}\otimes_{\mathbb{F}}\mathfrak{m}_R^1/\mathfrak{m}_R^2\right)\to\mathscr{G}^{[n]}(R).$$

For $X,Y\in\mathfrak{g}^{\mathrm{der}}$ and $f\in\mathfrak{m}_R^{n-1}$, $g\in\mathfrak{m}_R$, the last map is given by

$$(X\otimes f,Y\otimes g)\mapsto[X,Y]\otimes fg\in\mathfrak{g}^{\mathrm{der}}\otimes_{\mathbb{F}}\mathfrak{m}_R^n\cong\mathscr{G}^{[n]}(R).$$

Now $H\cap\mathscr{G}^{[i]}/H\cap\mathscr{G}^{[i+1]}$ is naturally isomorphic to $\mathscr{G}^{[i]}/\mathscr{G}^{[i+1]}$ for $i=1,\dots,n-1$, since H surjects onto $H_{R/\mathfrak{m}_R^n}$. Hence forming (products of) commutators in H, we see that the commutator subgroup $[\mathscr{G}^{[n-1]}(R)\cap H,\mathscr{G}^{[1]}(R)\cap H]$ must contain $[\mathfrak{g}^{\mathrm{der}},\mathfrak{g}^{\mathrm{der}}]\otimes_{\mathbb{F}}\mathfrak{m}_R^n\subset\mathfrak{g}^{\mathrm{der}}\otimes_{\mathbb{F}}\mathfrak{m}_R^n\cong\mathscr{G}^{[n]}(R)$. This proves (a). Part (b) is obvious since **(l-cl)**(i) asserts that $[\mathfrak{g}^{\mathrm{der}},\mathfrak{g}^{\mathrm{der}}]=\mathfrak{g}^{\mathrm{der}}$. □

For any $R\in\widehat{Ar}_{W(\mathbb{F})}$ denote by π_R the canonical reduction $H_R\to H_{\mathbb{F}}$, and for any ideal I of R by $\pi_{R,I}$ the canonical reduction $H_R\to H_{R/I}$.

Corollary 4.9. *Let R in $\widehat{Ar}_{W(\mathbb{F})}$ be Artinian and let $H\subset H_R$ be a subgroup such that $\pi_R(H)=H_{\mathbb{F}}$. Suppose that* **(l-cl)**(*i*) *holds. Then $H=H_R$ assuming one of the following*:

(a) *one has $\pi_{R,\mathfrak{m}_R^2}(H)=H_{R/\mathfrak{m}_R^2}$*;

(b) *one has $\pi_{R,(p,\mathfrak{m}_R^2)}(H)=H_{R/(p,\mathfrak{m}_R^2)}$ and condition* **(n-s)** *and either* **(l-ge)** *or* **(l-cl)**(*ii*) *and* **(sch)** *hold.*

Proof. Let us first show that (b) implies the condition in (a). For this, for simplicity of notation, we may assume that $\mathfrak{m}_R^2=0$, and we need to show that then $H=H_R$. If $p=0$ in R then there is nothing to show. Otherwise $R\cong W_2(\mathbb{F})[x_1,\dots,x_n]/(p,x_1,\dots,x_n)^2$ for some n, and there are surjective ring homomorphisms onto $R/(p,\mathfrak{m}_R^2)$ and onto $W_2(\mathbb{F})$. The kernels of the induced group homomorphisms $\pi_{R,(p)}:H_R\to H_{R/(p,\mathfrak{m}_R^2)}$ and $\pi_{R,(x_1,\dots,x_n)}:H_R\to H_{W_2(\mathbb{F})}$ intersect trivially. Now the restriction of $\pi_{R,(p)}$ to H is surjective by hypothesis and that of $\pi_{R,(x_1,\dots,x_n)}$ to H by Lemma 4.6, using the hypotheses in (b). This implies $H=H_R$, as had to be shown.

To deduce $H=H_R$ from (a) one proceeds by induction over rings R such that $\mathfrak{m}_R^n=0$, starting at $n=2$, and one applies Lemma 4.8(b) in the induction step. □

Corollary 4.10. *Let R be an Artinian ring in $\widehat{Ar}_{W(\mathbb{F})}$. Let $\phi:H_R\to H_{\mathbb{F}[\varepsilon]}$ be any group homomorphism such that $\pi_{\mathbb{F}[\varepsilon]}\circ\phi=\pi_R$. Suppose that* **(l-un)** *holds. Then ϕ factors via $\pi_{R,\mathfrak{m}_R^2}:H_R\to H_{R/\mathfrak{m}_R^2}$.*

If in addition condition **(n-s)** *holds, then ϕ factors via $\pi_{R,(p,\mathfrak{m}_R^2)}:H_R\to H_{R/(p,\mathfrak{m}_R^2)}$.*

Proof. For the first part we induct on $n\geq 2$ and rings R such that $\mathfrak{m}_R^n=0$. The case $n=2$ is clear by hypothesis. In the induction step we assume that $\mathfrak{m}_R^{n+1}=0$. We claim that ϕ factors via

$\pi_{R,\mathfrak{m}_R^n} : H_R \to H_{R/\mathfrak{m}_R^n}$, and by induction this implies the result. We have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{G}^{[1]}(R) \arrow[r] \arrow[d] & H_R \arrow[r] \arrow[d, "\phi"] & H_{\mathbb{F}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{g}^{\mathrm{der}} \arrow[r] & H_{\mathbb{F}[\varepsilon]} \arrow[r] & H_{\mathbb{F}} \arrow[r] & 0. \end{tikzcd}$$

Because $\mathfrak{g}^{\mathrm{der}}$ is abelian as a group under $+$, the group $[\mathscr{G}^{[1]}(R), \mathscr{G}^{[1]}(R)]$ lies in the kernel of ϕ. Hence by Lemma 4.8(a), the restriction of ϕ to $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathbb{F}} \mathfrak{m}_R^n \cong \mathscr{G}^{[n]}(R)$ contains $[\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}] \otimes_{\mathbb{F}} \mathfrak{m}_R^n$ in its kernel. By **(l-un)**(ii), any $\mathbb{F}_p[H'_{\mathbb{F}}]$-module homomorphism $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathbb{F}} \mathfrak{m}_R^n \to \mathfrak{g}^{\mathrm{der}}$ is either surjective or trivial. If we apply this to ϕ and observe that **(l-un)**(i) holds, we see that ϕ restricted to $\mathscr{G}^{[n]}(R)$ is zero, proving the claim.

For the second part, we may assume $\mathfrak{m}_R^2 = 0$ and in addition that $p \notin \mathfrak{m}_R^2$, since otherwise there is nothing to show. The embedding $W_2(\mathbb{F}) \to R$ gives the subgroup $H_{W_2(\mathbb{F})} \subset H_R$. Now assume that $\phi(\operatorname{Ker} \pi_{W_2(\mathbb{F})}) \neq 0$, else we are done. Then ϕ restricted to $H_{W_2(\mathbb{F})}$ defines a nonzero homomorphism from $\mathfrak{g}^{\mathrm{der}} \cong \operatorname{Ker} \pi_{W_2(\mathbb{F})}$ to $\mathfrak{g}^{\mathrm{der}} \cong \operatorname{Ker} \pi_{\mathbb{F}[\varepsilon]}$. By **(l-un)**(ii), it follows that any such homomorphism is multiplication by a scalar in $\mathbb{F}$, and since the map is nonzero the scalar must be nontrivial. This implies that the restriction of ϕ to $H_{W_2(\mathbb{F})}$ defines an isomorphism $H_{W_2(\mathbb{F})} \to H_{\mathbb{F}[\varepsilon]}$. But this is absurd since the extension $H_{\mathbb{F}[\varepsilon]} \to H_{\mathbb{F}}$ is split while $H_{W_2(\mathbb{F})} \to H_{\mathbb{F}}$ is nonsplit by **(n-s)**. □

Remark 4.11. If there is a splitting of $H_{W_2(\mathbb{F})} \to H_{\mathbb{F}}$ and if **(van)** holds, then all split extensions of $H_{\mathbb{F}}$ by $\mathfrak{g}^{\mathrm{der}}$ are equivalent, and in particular $H_{W_2(\mathbb{F})} \cong H_{\mathbb{F}[\varepsilon]}$.

5. Rings as universal deformation rings

Throughout this section, we assume that $\mathscr{G}$ is connected and that $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Assumption 2.1. In particular, this means that $H_{\mathbb{F}} = M_{\mathbb{F}} H'_{\mathbb{F}}$ with $M_{\mathbb{F}} \subset \mathscr{G}(\mathbb{F})$ of order prime to p. From now on we fix some $R \in \widehat{Ar}_{W(\mathbb{F})}$. Then $H_R = H_R^o = M_R H'_R$ from Lemma 4.1 is independent of any choices.

Recall $\bar{\rho}_R$, $D_{\bar{\rho}_R}$ and $R_{\bar{\rho}_R}$ from Definition 2.10 and the paragraphs preceding it. Concerning the image of $\rho_A \in D_{\bar{\rho}_R}(A)$ for $A \in \widehat{Ar}_{W(\mathbb{F})}$, let us first observe the following, which is trivial if $\mathscr{G} = \mathscr{G}^{\mathrm{der}}$.

Lemma 5.1. *For an $H_{\mathbb{F}}$-perfect closed subgroup H of H_R, an $A \in \widehat{Ar}_{W(\mathbb{F})}$ and $[\rho_A] \in D_{\bar{\rho}_R}(A)$ with representative ρ_A, one has $\rho_A(H) \subset H_A$. If moreover* **(csc)** *holds, then $\rho_A(H_R) \subset H_A$.*

Proof. Since $\mathscr{G} = \mathscr{G}^o$, the composition of ρ_A with the canonical surjection $\pi : \mathscr{G}(A) \to \mathscr{G}(A)/\mathscr{G}^{\mathrm{der}}(A)$ has abelian image. Therefore $H \to \pi \circ \rho_A(H)$ factors via $H/\overline{[H, H]}$, which by the construction of H^c in Lemma 4.2 and our hypothesis $H = H^c$ is isomorphic to $H_{\mathbb{F}}/[H_{\mathbb{F}}, H_{\mathbb{F}}]$. By Assumption 2.1, the latter group is of order prime to p. Since $H'_{\mathbb{F}}$ maps to $\mathscr{G}^{\mathrm{der}}(\mathbb{F})$ under $\bar{\rho}$, the restriction of $\pi \circ \rho_A$ to H will factor via $M_{\mathbb{F}}/M_{\mathbb{F}} \cap \mathscr{G}^{\mathrm{der}}(\mathbb{F})$. As in the proof of Lemma 4.1(d) we find that $\pi \circ \rho_A(H)$ lies in the unique lift of $M_{\mathbb{F}}/M_{\mathbb{F}} \cap \mathscr{G}^{\mathrm{der}}(\mathbb{F})$ to $\mathscr{G}(R)/\mathscr{G}^{\mathrm{der}}(R)$. Since H maps to $H_{\mathbb{F}}$ under $\bar{\rho}_R$, we deduce $\rho_A(H) \subset H_A$. For the last claim note that under **(csc)** we have $H_R^c = H_R$ by Lemma 4.3. □

We have the following generalization of [EM16; Dor16] from GL_n to arbitrary $\mathscr{G}$, under the same basic hypotheses; cf. Remark 5.6.

Theorem 5.2. *Suppose $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies conditions* **(ct)**, **(van)**, **(n-s)**, **(l-un)** *and one of* **(csc)** *or $\mathscr{G} = \mathscr{G}^{\mathrm{der}}$. Then the canonical inclusion $\iota : H_R \to \mathscr{G}(R)$ represents the universal deformation of $D_{\bar\rho_R}$, and in particular $R_{\bar\rho_R} = R$.*

Recall that **(l-un)** and **(csc)** are implied by **(l-ge)**(ii) and (iii).

Remark 5.3. For tuples $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ as in Theorem 3.2, with $\mathscr{G}$ absolutely simple, condition **(van)** and **(ct)** can only hold under **(l-ge)** and **(lie-ad)**. For tuples $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ as in Theorem 3.52, where $\mathscr{G}$ is connected reductive but $\mathscr{G}^{\mathrm{der}}$ is absolutely simple, conditions **(van)** and **(ct)** are compatible with **(lie-sc)**.

From Theorem 5.2 and Theorem 3.2 we deduce:

Corollary 5.4. *If $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Conditions 3.1, then $\iota : H_R \to \mathscr{G}(R)$ represents the universal deformation of $D_{\bar\rho_R}$ under the conjunction of the following conditions*:

(a) *$(\mathscr{G}, \mathbb{F})$ is not exceptional in the sense of Notation 3.4, and $(\text{type}\ \mathscr{G}, \mathbb{F}) \notin \{(A_1, \mathbb{F}_5)\}$.*

(b) *$\mathscr{G}$ is of Lie-adjoint type.*

(c) *If type $\mathscr{G} = C_n$, then $|\mathbb{F}| \notin \{2, 3, 4, 5, 9\}$,*

(d) *If $\mathscr{G}$ is nonsplit (and hence of types A, D or E_6), then $|\mathbb{F}| \geq 4$.*

(e) *If $\mathscr{G}$ is of type B_2 or F_4 then $p \neq 2$, if $\mathscr{G}$ is of type G_2, then $p \neq 3$.*

From Theorem 5.2 and Theorem 3.52 we deduce:

Corollary 5.5. *If $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Conditions 3.51, then $\iota : H_R \to \mathscr{G}(R)$ represents the universal deformation of $D_{\bar\rho_R}$ under the conjunction of the following conditions*:

(a) *$(\text{type}\ \mathscr{G}, \mathbb{F}) \notin \{(A_1, \mathbb{F}_?)_{? \in \{2,3,5\}}, (C_n, \mathbb{F}_?)_{n \geq 2, ? \in \{2,3,4,5,9\}}\}$.*

(b) *If $\mathscr{G}^{\mathrm{der}}$ is nonsplit, then $|\mathbb{F}| \geq 4$.*

(c) *If $\mathscr{G}$ is of type B_2 or F_4 then $p \neq 2$, if $\mathscr{G}$ is of type G_2, then $p \neq 3$.*

Remark 5.6. For $\mathscr{G} = \mathrm{GL}_n$ and $H_{\mathbb{F}} = H'_{\mathbb{F}} = \mathrm{SL}_n(\mathbb{F})$, Corollary 5.5 shows that Theorem 5.2 completely recovers [Dor16, Theorem 1] and [EM16, Main Theorem].

By an inverse limit argument as in the proof of Lemma 2.8, it will suffice to give the proof of Theorem 5.2 when R is noetherian. So for the remainder of this section, we assume $R \in \widehat{Ar}_{W(\mathbb{F})}$ noetherian.

The proof of Theorem 5.2 consists then of the following main steps. First we compute the dimension of the mod p tangent space of $R_{\bar\rho_R}$. From this deformation theory implies Theorem 5.2 in an elementary way in the case where R is formally smooth over $W(\mathbb{F})$. Finally we reduce the case of general R to that of formally smooth R.

Lemma 5.7. *Suppose $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies conditions* **(ct)**, **(van)**, **(n-s)**, **(l-un)** *and one of* **(csc)** *or $\mathscr{G} = \mathscr{G}^{\mathrm{der}}$. Then the mod p tangent spaces of R and $R_{\bar\rho_R}$ have the same dimension.*

Proof. Define $d := \dim_{\mathbb{F}} \mathfrak{m}_R/(p, \mathfrak{m}_R^2)$, and denote by d' the dimension of the mod p tangent space of $R_{\bar\rho_R}$.

From Corollary 4.10 and also Lemma 5.1, which requires **(n-s)** and **(l-un)** and **(csc)** if $\mathscr{G} \neq \mathscr{G}^{\mathrm{der}}$, we deduce that any deformation $[\rho_{\mathbb{F}[\varepsilon]}]$ in $D_{\bar\rho_R}(\mathbb{F}[\varepsilon])$ factors via $H_{R/(p,\mathfrak{m}_R^2)}$, and hence that $d' = \dim_{\mathbb{F}} H^1(H_{R/(p,\mathfrak{m}_R^2)}, \mathfrak{g})$. Now the inflation restriction sequence of group cohomology yields

$$0 \to H^1(H_{\mathbb{F}}, \mathfrak{g}) \to H^1(H_{R/(p,\mathfrak{m}_R^2)}, \mathfrak{g}) \to H^1(\mathscr{G}^{\mathrm{der},[1]}(R/(p, \mathfrak{m}_R^2)), \mathfrak{g})^{H_{\mathbb{F}}} \tag{5-1}$$

Because $M_{\mathbb{F}}$ is of order prime to p, and again by inflation restriction, the left term $H^1(H_{\mathbb{F}}, \mathfrak{g})$ is isomorphic to $H^1(H'_{\mathbb{F}}, \mathfrak{g})^{M_{\mathbb{F}}/(M_{\mathbb{F}} \cap H'_{\mathbb{F}})}$, and thus zero by **(van)**. Moreover the abelian group $\mathscr{G}^{\mathrm{der},[1]}(R/(p, \mathfrak{m}_R^2))$, which as an $H_{\mathbb{F}}$-module is isomorphic to $\mathfrak{g}^{\mathrm{der}} \otimes \mathfrak{m}_R/(p, \mathfrak{m}_R^2)$, acts trivially on $\mathfrak{g}$, and so the right hand side can be identified with

$$\mathrm{Hom}_{\mathbb{F}_p[H_{\mathbb{F}}]}((\mathfrak{g}^{\mathrm{der}})^d, \mathfrak{g}) = \mathrm{Hom}_{\mathbb{F}_p[H_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}}, \mathfrak{g})^d.$$

If $\mathscr{G} = \mathscr{G}^{\mathrm{der}}$, then $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}}$. Otherwise **(csc)** yields $\mathrm{Hom}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}}, \mathbb{F}_p) = 0$, so that $\mathrm{Hom}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}) = \mathrm{Hom}_{\mathbb{F}_p[H'_{\mathbb{F}}]}(\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}})$. We conclude $d' \leq d$ from **(l-un)**(ii).

For the converse inequality, consider the diagram

$$\begin{tikzcd} H_R \arrow[rr, "\rho_{\bar\rho_R}"] \arrow[dr, "\iota"'] & & \mathscr{G}(R_{\bar\rho_R}) \arrow[dl, "\alpha"] \\ & \mathscr{G}(R) & \end{tikzcd} \tag{5-2}$$

provided by the universality of $R_{\bar\rho_R}$, that commutes up to conjugation by $\mathrm{Ker}(\mathscr{G}(R) \to \mathscr{G}(\mathbb{F}))$, and for some unique homomorphism $\alpha : R_{\bar\rho_R} \to R$ in $\widehat{Ar}_{W(\mathbb{F})}$. Recall that R is the ring used to define H_R. Denote by α_2 the homomorphism $R_{\bar\rho_R}/(p, \mathfrak{m}_{R_{\bar\rho_R}}^2) \longrightarrow R/(p, \mathfrak{m}_R^2)$ induced from α. Then $H_{R/(p,\mathfrak{m}_R^2)}$ is the image of H_R in $\mathscr{G}(R/(p, \mathfrak{m}_R^2))$, and thus it must lie in the image of the homomorphism

$$\mathscr{G}(R_{\bar\rho_R}/(p, \mathfrak{m}_{R_{\bar\rho_R}}^2)) \longrightarrow \mathscr{G}(R/(p, \mathfrak{m}_R^2))$$

induced from α_2. Since $\pi_{R,(p,\mathfrak{m}_R^2)} : H_R \to H_{R/(p,\mathfrak{m}_R^2)}$ is surjective, it follows that α_2 must be surjective. This yields $d' \geq d$ by comparing cardinalities. □

Corollary 5.8. *Theorem 5.2 holds for $R = W(\mathbb{F})[\![x_1, \dots, x_d]\!]$ for any integer $d \geq 0$.*

Proof. Consider diagram (5-2). It implies that the homomorphism

$$\alpha : R_{\bar\rho_R} \longrightarrow R = W(\mathbb{F})[\![x_1, \dots, x_d]\!]$$

is surjective, as it is surjective on mod p tangent spaces. At the same time, we know from Lemma 5.7 that d is the dimension of the mod p tangent space of $R_{\bar\rho_R}$. Hence by the Cohen structure theorem and Nakayama's lemma we have a surjective homomorphism

$$\beta : R = W(\mathbb{F})[\![x_1, \dots, x_d]\!] \longrightarrow R_{\bar\rho_R}.$$

For dimension reasons it follows that the composite $\alpha \circ \beta$ must have trivial kernel, so that β is an isomorphism. But then the same argument shows that α is an isomorphism, proving Corollary 5.8. □

Proof of Theorem 5.2. Choose a surjective ring homomorphism

$$\pi : S := W(\mathbb{F})[[x_1, \dots, x_d]] \longrightarrow R$$

in $\widehat{Ar}_{W(\mathbb{F})}$, denote the induced map $\mathscr{G}(S) \longrightarrow \mathscr{G}(R)$ by $\mathscr{G}(\pi)$. Let now $A \in \widehat{Ar}_{W(\mathbb{F})}$ and let ρ_A represent a $\mathscr{G}$-valued deformation of $\bar\rho_R$ to A. Then $\rho_A \circ \mathscr{G}(\pi)$ is a $\mathscr{G}$-valued deformation of $\bar\rho_S := \bar\rho_R \circ \mathscr{G}(\pi) : H_S \to \mathscr{G}(\mathbb{F})$ with H_S from formula (2-1). We consider the diagram

$$\begin{tikzcd} H_S \arrow[rr, "\mathscr{G}(\pi)"] \arrow[dr, "\mathscr{G}(\alpha)"'] & & H_R \arrow[dl, "\rho_A"] \\ & \mathscr{G}(A) & \end{tikzcd} \tag{5-3}$$

where $\mathscr{G}(\alpha) : H_S \longrightarrow \mathscr{G}(A)$ is induced from a unique homomorphism $\alpha : S \to A$ in $\widehat{Ar}_{W(\mathbb{F})}$ using the universality of the inclusion $H_S \to \mathscr{G}(S)$ for $\mathscr{G}$-deformations of $\bar\rho_S$ established in Corollary 5.8. Note that a priori, the diagram commutes only up to strict equivalence, i.e., up to conjugation by an element g of $\mathscr{G}(A)$ that surjects to the identity in $\mathscr{G}(\mathbb{F})$. However by replacing the group homomorphism ρ_A by its conjugate by g, we can assume from now on that Equation (5-3) commutes.

A priori, ρ_A is only a group homomorphism. The main point we need to establish is that it is induced from a unique ring homomorphism $R \to A$ in $\widehat{Ar}_{W(\mathbb{F})}$. We first show that $\ker\pi \subset \ker\alpha$. For this consider $s \in S \setminus \operatorname{Ker}\alpha$. We need to show that $\pi(s) \neq 0$. Let ξ be a root of $\mathscr{G}$ with root group $\mathscr{U}_\xi \subset \mathscr{G}$ and let $x_\xi : \mathbb{G}_a \to \mathscr{U}_\xi$ be an isomorphism, all defined over $W(\mathbb{F})$. Then $\mathscr{G}(\alpha)(x_\xi(s)) = x_\xi(\alpha(s))$ is nonzero. This implies that $\mathscr{G}(\pi)(x_\xi(s)) = x_\xi(\pi(s))$ is nonzero.

By the previous paragraph we have $\ker\pi \subset \ker\alpha$. This gives a factorization $\alpha = \beta\pi$ for a unique ring homomorphism $\beta : R \to A$, by the homomorphism theorem for rings. Hence $\mathscr{G}(\alpha) = \mathscr{G}(\beta) \circ \mathscr{G}(\pi)$. Because $\mathscr{G}(\pi) : H_S \to H_R$ is a surjective homomorphism of groups, by the homomorphism theorem for groups, we have $\rho_A = \mathscr{G}(\beta)$, and this can only hold for a unique ring homomorphism β in $\widehat{Ar}_{W(\mathbb{F})}$. Thus we have directly established the universal property of R, i.e., that $R \cong R_{\bar\rho_R}$. □

Remark 5.9. Suppose that **(l-un)** and **(csc)** hold on the structure of $\mathfrak{g}$ as a Lie algebra and as an $H'_{\mathbb{F}}$-module. Suppose also that $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies either Conditions 3.1 or Conditions 3.51, and that $H_{\mathbb{F}} = H'_{\mathbb{F}}$. If then the assertion of Corollary 5.8 holds for $d = 0$, then one can deduce from (5-1), running backward the argument of Lemma 5.7, that $H^1(H_{\mathbb{F}}', \mathfrak{g}) = 0$. This means that the direct proofs of Theorem 5.2 for $\mathscr{G} = \mathrm{GL}_n$ given in [Dor16] and [EM16] essentially also reprove, without making this explicit, the vanishing of $H^1(\mathscr{G}^{\mathrm{der}}(\mathbb{F}), \mathfrak{g})$ for $\mathscr{G} = \mathrm{GL}_n$ — except for some small values of $\#\mathbb{F}$ and n.

6. Closed subgroups of $H_R \subset \mathscr{G}(R)$

Throughout this section, we assume that $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Assumption 2.1. By R we denote a ring in $\widehat{Ar}_{W(\mathbb{F})}$ and by $H \subset H_R$ a closed subgroup that under π_R surjects onto $H_{\mathbb{F}}$. We advise the reader to recall the $H_{\mathbb{F}}$-perfection of H defined in Definition 2.12 and the discussion around it. Note in addition that if **(l-ge)**(ii) holds, then $H = H^{(\infty)}$ by Lemma 4.3.

The following result was motivated by [Man15] and [MW86]. It sheds some light on the structure of certain closed subgroups of $\mathscr{G}(R)$ for $R \in \widehat{Ar}_{W(\mathbb{F})}$, which might be useful for studying images of Galois representations attached to automorphic forms.

Theorem 6.1. *Let $H \subset H_R$ be a closed subgroup that surjects onto $H_{\mathbb{F}}$. Suppose that* **(ct)**, **(n-s)** *and* **(van)** *hold, and that either* **(l-ge)** *holds or that* **(l-cl)** *and* **(sch)** *hold. Then there exists a closed $W(\mathbb{F})$-subalgebra A of R such that $H^{(\infty)}$ is conjugate to $H_A \subset \mathscr{G}(R)$.*

Observe that if R is noetherian, then A need not share this property. Its tangent space could be infinite dimensional. However if H surjects onto $H_{\mathbb{F}}$ and is open in H_R, so that H contains $\mathscr{G}^{[i]}$ for some $i \geq 1$, then following the proof of Lemma 4.3, H^c still contains $\mathscr{G}^{[i]}$, and hence so does $H^{(\infty)}$. From this one easily deduces that in this case A is noetherian if R is so, and hence that $A \in \widehat{Ar}_{W(\mathbb{F})}$.

The main idea for the proof of Theorem 6.1 is to let deformation theory determine the sought for ring A. For this we may, and from now on will, assume that H is $H_{\mathbb{F}}$-perfect. Since the inclusion $\iota : H \subset \mathscr{G}(R)$ factors via the universal pair, we have a diagram

$$\begin{tikzcd} H \arrow[rr, "\rho_{\bar\rho_H}"] \arrow[dr, "\iota"'] & & \mathscr{G}(R_{\bar\rho_H}) \arrow[dl, "\mathscr{G}(\alpha)"] \\ & \mathscr{G}(R) & \end{tikzcd} \tag{6-1}$$

that commutes up to strict equivalence, for a unique ring homomorphism $\alpha : R_{\bar\rho_H} \to R$. After conjugation of ι by an element of $\mathscr{G}(R)$ that reduces to the identity in $\mathscr{G}(\mathbb{F})$, we may assume the diagram commutes. This means that we replace H by a conjugate; this may necessitate a new choice for $M_{W(\mathbb{F})}$ inside $\mathscr{G}(W(\mathbb{F}))$ (and thus of M_R and H_R). Note that $M_R \subset H$, since H surjects onto $H_{\mathbb{F}} \subset M_{\mathbb{F}}$.

So from now on, we assume that H lies in $\mathscr{G}(A)$ for $A := \alpha(R_{\bar\rho_H}) \subset R$, so that $R_{\bar\rho_H} \to A$ is surjective.[6] We let M_A be the image of $M_R \subset H$ under $\mathscr{G}(\alpha) \circ \rho_{\bar\rho_H}$, i.e., it is simply equal to M_R under ι. Then $H_A = H_R \cap \mathscr{G}(A) = M_A H'_R \cap \mathscr{G}(A)$ and H_A contains $\iota(H)$.

If we compose ι with the canonical map $\mathscr{G}(A) \longrightarrow \mathscr{G}(A/(p, \mathfrak{m}_A^2))$, we obtain a homomorphism $H \to H_{A/(p,\mathfrak{m}_A^2)}$ such that $R_{\bar\rho_H} \to A/(p, \mathfrak{m}_A^2)$ is again surjective.

Lemma 6.2. *The homomorphism $H \to H_{A/(p,\mathfrak{m}_A^2)}$ is surjective.*

Proof. Let $A \to \mathbb{F}[\varepsilon]$ be any surjection in $\widehat{Ar}_{W(\mathbb{F})}$. Since A is a quotient of $R_{\bar\rho_H}$, the induced deformation $\big[\, \rho_{\mathbb{F}[\varepsilon]} : H \to \mathscr{G}(\mathbb{F}[\varepsilon]) \,\big]$ is nontrivial. Note also that the image $\bar H$ of H in $H_{\mathbb{F}[\varepsilon]}$ satisfies $\bar H^c = \bar H$ because of Lemma 4.2(b). Since we assume that **(van)** holds and that either **(l-ge)**(ii) holds or that **(l-cl)** and **(sch)** hold, we deduce from Lemma 4.5 that $\rho_{\mathbb{F}[\varepsilon]}$ is surjective. Since $A \to \mathbb{F}[\varepsilon]$ was arbitrary, the lemma is proved. □

[6] The non-noetherian context may be avoided below by replacing A by any of its Artinian quotients.

Proof of Theorem 6.1. Because of the previous lemma, we are in a position to apply Corollary 4.9(b). Its hypotheses are satisfied, since we assume that **(n-s)** holds and either **(l-ge)** or **(l-cl)** and **(sch)** hold. This immediately yields $H = H_A$. □

From Theorem 6.1 and Theorem 3.2 we deduce:

Corollary 6.3. *Suppose that $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Conditions 3.1. Then under the conjunction of the following conditions, any closed subgroup $H \subset H_R$ that is residually full is a conjugate of $H_A \subset \mathscr{G}(R)$ for a closed $W(\mathbb{F})$-subalgebra $A \in \widehat{Ar}_{W(\mathbb{F})}$ of R:*

(a) *$(\mathscr{G}, \mathbb{F})$ is not exceptional in the sense of Notation 3.4, and $(\text{type}\, \mathscr{G}, \mathbb{F}) \notin \{(A_1, \mathbb{F}_5)\}$.*

(b) *$\mathscr{G}$ is of type A_n and $p \nmid n+1$, or $\mathscr{G}$ is of type B_n, C_n, D_n, E_7 or F_4 and $p \neq 2$, or $\mathscr{G}$ is of type E_6 or G_2 and $p \neq 3$, or $\mathscr{G}$ is of type E_8; i.e.,* **(l-ge)** *holds.*

(c) *If type $\mathscr{G} = C_n$, then $|\mathbb{F}| \notin \{3, 5, 9\}$.*

(d) *If $\mathscr{G}$ is nonsplit (and hence of types A, D or E_6), then $|\mathbb{F}| \geq 4$.*

From Theorem 6.1 and Theorem 3.52 we deduce:

Corollary 6.4. *Suppose that $(\mathscr{G}, \mathbb{F}, H_{\mathbb{F}}, H'_{\mathbb{F}})$ satisfies Conditions 3.51. Then under the conjunction of the following conditions, for any residually full closed subgroup H of H_R, there exists a closed $W(\mathbb{F})$-subalgebra $A \in \widehat{Ar}_{W(\mathbb{F})}$ of R such that $H^{(\infty)}$ is conjugate to $H_A \subset \mathscr{G}(R)$.*

(a) *$(\text{type}\, \mathscr{G}, \mathbb{F}) \notin \{(A_1, \mathbb{F}_?)_{? \in \{3,5\}}, (C_n, \mathbb{F}_?)_{n \geq 2, ? \in \{3,5,9\}}\}$.*

(b) *$\mathscr{G}$ is of type A_n, $n \geq 2$, D_n or E_n, or $\mathscr{G}$ is of type A_1, B_n, C_n, F_4 and $p \neq 2$, or $\mathscr{G}$ is of type G_2 and $p \neq 3$.*

(c) *If $\mathscr{G}^{\mathrm{der}}$ is nonsplit, then $|\mathbb{F}| \geq 4$.*

Remark 6.5. If **(l-ge)**(ii) or if **(l-cl)**(ii) and **(sch)** are not satisfied, then Example 4.7 shows that the conclusion of Theorem 6.1 need not hold.

The proof of Theorem 6.1 is built via Corollary 4.9(b) on $[\mathfrak{g}^{\mathrm{der}}, \mathfrak{g}^{\mathrm{der}}] = \mathfrak{g}^{\mathrm{der}}$. This condition is satisfied in the setup of Corollary 3.54. However in the particular case $\mathscr{G}' = \mathrm{GL}_2$ and $\mathrm{Char}\, \mathbb{F} = 2$ the Lie group $\mathfrak{g}^{\mathrm{der}}$ is not perfect — for all other pairs $(n, \mathrm{Char}\, \mathbb{F})$ it is. The following example shows that the conclusion of Theorem 6.1 does not hold for $(n, \mathrm{Char}\, \mathbb{F}) = (2, 2)$:

Example 6.6. Let $(\mathscr{G}, \mathrm{Char}\, \mathbb{F}) = (\mathrm{GL}_2, 2)$, and let $R = \mathbb{F}[x_1, \dots, x_d]/(x_1, \dots, x_d)^3$ be in $\widehat{Ar}_{W(\mathbb{F})}$. Let $\mathfrak{m}' \subset \mathfrak{m}_R$ be the $\mathbb{F}$-linear span of $\{x_1, \dots, x_d, x_1^2, \dots, x_d^2\}$. Define $H \subset \mathrm{SL}_2(R)$ as the subgroup generated by $H_1 \cup \dots \cup H_4$ for

$$H_1 = \mathrm{SL}_2(\mathbb{F}),\ H_2 = \{\left(\begin{smallmatrix} 1+a+a^2 & 0 \\ 0 & 1-a \end{smallmatrix}\right) \mid a \in \mathfrak{m}_R\},\ H_3 = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathfrak{m}'\},\ H_4 = \{\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \mid c \in \mathfrak{m}'\}.$$

Then H is a subgroup of $\mathrm{SL}_2(R)$ which surjects onto $\mathrm{SL}_2(R/\mathfrak{m}_R^2)$. One can verify that any element in H can be written in a unique way as a product $\gamma_1 \cdot \dots \cdot \gamma_4$ with $\gamma_i \in H_i$. Then the order of $\mathrm{SL}_2(R)$ divided by the order of H is $\#\mathbb{F}^{2 \dim_{\mathbb{F}} \mathfrak{m}_R/\mathfrak{m}'} = \#\mathbb{F}^{d(d-1)}$. Hence H is a proper subgroup of $\mathrm{SL}_2(R)$ unless $d = 1$. In particular, for $d > 1$ and $|\mathbb{F}| > 4$, we see that Theorem 6.1 requires condition **(l-cl)**.

The above counterexample is optimal in the following sense.

Proposition 6.7. *Let* $(\mathscr{G}, \operatorname{Char}\mathbb{F}) = (\mathrm{GL}_2, 2)$ *and suppose that* $\mathbb{F} \neq \mathbb{F}_2$. *Let* R *in* $\widehat{Ar}_{W(\mathbb{F})}$ *satisfy*

$$\dim_{\mathbb{F}} \mathfrak{m}_R/\mathfrak{m}_R^2 = 1.$$

Then $H = \mathrm{SL}_2(R)$ *for any closed subgroup* $H \subset \mathrm{SL}_2(R)$ *that surjects onto* $\mathrm{SL}_2(R/(p, \mathfrak{m}_R^2))$.

Proof. If R is a quotient of $\mathbb{F}[[X]]$, the result follows from [DP12, Theorem 3.6]; this is a rather technical proof. If R is a quotient of $W(\mathbb{F})$, the result is a special case of [Man15, Main Theorem].[7] However the second case also has a simple natural direct proof; see [Vasiu], for example. Our conditions imply that **(n-s)**, **(sch)** and **(l-cl)**(ii) hold. Then Lemma 4.6 implies that $H_R = H_{W_2(\mathbb{F})}$ for $R = W_2(\mathbb{F})$. Denote by Γ_n the kernel of $\mathrm{SL}_2(W_{n+1}(\mathbb{F})) \to \mathrm{SL}_2(W_n(\mathbb{F}))$, for $n \geq 1$, with the map being the canonical reduction. Then the p-power map $X \mapsto X^p$ induces an isomorphism $\Gamma_n \to \Gamma_{n+1}$. From this one deduces easily that for any quotient $W_n(\mathbb{F})$ of R the group H_R surjects onto $H_{W_n(\mathbb{F})}$; the p-power map replaces the use commutators in the proof of Corollary 4.9(b). □

Remark 6.8. The content of [Man15, Main Theorem] is the following result. Let R be in $\widehat{Ar}_{W(\mathbb{F})}$. Let H be a closed subgroup of $\mathrm{GL}_n(R)$ whose image in $\mathrm{GL}_n(\mathbb{F})$ contains $\mathrm{SL}_n(\mathbb{F})$. Suppose that $|\mathbb{F}| \geq 4$ and $\mathbb{F} \neq \mathbb{F}_4$ if $n = 3$ and $\mathbb{F} \neq \mathbb{F}_5$ if $n = 2$. Denote by W_R the image of structure map $W(\mathbb{F}) \to R$. Then H contains a $\mathrm{GL}_n(R)$-conjugate of $\mathrm{SL}_n(W_R)$.

Our Theorem 6.1 generalizes [Man15, Main Theorem] in all cases, except for $(n, p) = (2, 2)$. Example 6.6 shows that our theorem cannot be expected to hold for $(n, p) = (2, 2)$.

It is possible to reduce the statement of [Man15, Main Theorem] to the methods treated here, by reducing it to Proposition 6.7 in the following way: Let R be in $\widehat{Ar}_{W(\mathbb{F})}$. Choose a descending filtration by ideals I_m of R with $I_1 = \mathfrak{m}_R$ such that $I_m/I_{m+1} \cong \mathbb{F}$ for $m \geq 1$ and so that $\bigcap_m W_R + I_m = W_R$. Then $W_R + I_m/pW_R + I_{m+1} \cong \mathbb{F}[\varepsilon]$ for all $m \geq 1$. And by induction on n and *up to conjugation*, one can find a descending sequence of closed subgroups $H_m \subset H$ such that $H_\infty := \bigcap_m H_m \subset \mathrm{SL}_n(W_R)$ and H_∞ surjects onto $\mathrm{SL}_n(\mathbb{F})$ under reduction. Now Proposition 6.7 implies [Man15, Main Theorem].

Appendix: Primer on affine group schemes over a base

In this appendix, we gather definitions and results, frequently used in this article, on various types of affine group schemes over arbitrary base schemes. For further details we refer to [Con14]. We assume familiarity with the theory of affine algebraic groups over a field as in [Bor91; Mil17; Spr98].

Throughout this appendix we fix an arbitrary base scheme S and an affine group scheme $\mathscr{G}$ over S. We write π for the structure morphism $\mathscr{G} \to S$. In the main body of this work, S will typically be the spectrum of a complete discrete valuation ring with finite residue field.

[7] There appears to be a small error in [Man15, Theorem 3.5]; for $\mathrm{SL}_2(\mathbb{F}_4)$ the mod 2 Schur multiplier is nontrivial.

A.1. As the map $\pi : \mathscr{G} \to S$ is assumed to be affine, it is separated and quasicompact; see [GW10, Proposition-Definition 12.1]. If π is furthermore smooth, then it is also flat and locally of finite presentation, and hence of finite presentation; see [GW10, Definition 10.34].

A.2 (see [DG70, II.4, in part 1.1, 1.2, 1.4, 4.8] or [SGA 3_{I}, II.3. and II.4].). The *Lie algebra* $\mathrm{Lie}(\mathscr{G}/S)$ of $\mathscr{G}$ over S is the sheaf of $\mathscr{O}_S$-modules which on affine S-schemes $\operatorname{Spec} R$ takes the value

$$\mathrm{Lie}(\mathscr{G}/S)(\operatorname{Spec} R) = \mathrm{Ker}(\mathscr{G}(R[\varepsilon]) \to \mathscr{G}(R)),$$

where $R[\varepsilon]$ is the ring of dual numbers over R. This defines a functor from affine S-schemes of finite type to coherent $\mathscr{O}_S$-modules. Moreover for any affine S-scheme $\operatorname{Spec} R$ there is an obvious action of the abstract group $\mathscr{G}(R)$ on the abelian subgroup $\mathrm{Lie}(\mathscr{G}/S)(\operatorname{Spec} R)$ of $\mathscr{G}(R[\varepsilon])$, and this induces the adjoint action $\mathrm{Ad} : \mathscr{G} \to \mathrm{Aut}_S(\mathrm{Lie}(\mathscr{G}/S))$. Passing to Lie algebras defines a morphism

$$\mathrm{Lie}(\mathscr{G}/S) \to \mathrm{End}(\mathrm{Lie}(\mathscr{G}/S)),\ X \mapsto [X, \cdot].$$

In fact the latter defines a Lie bracket on $\mathrm{Lie}(\mathscr{G}/S)$ by $(X, Y) \mapsto [X, Y]$. One can also identify $\mathrm{Lie}(\mathscr{G}/S)$ with the $\mathscr{O}_S$-module of invariant derivations in the relative tangent sheaf $\mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_{\mathscr{G}}, \mathscr{O}_{\mathscr{G}})$, and then the Lie bracket is identified with the commutator bracket in $\mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_{\mathscr{G}}, \mathscr{O}_{\mathscr{G}})$. If $S = \operatorname{Spec} k$, we write $\mathrm{Lie}(\mathscr{G})$ for $\mathrm{Lie}(\mathscr{G}/S)$.

If $\mathrm{Lie}(\mathscr{G}/S)$ is locally free of finite rank, the formation of $\mathrm{Lie}(\mathscr{G}/S)$ commutes with any base change; otherwise one has to require that $S \to S'$ is flat. If $\mathscr{G}$ is smooth over S, then $\mathrm{Lie}(\mathscr{G}/S)$ is a locally free $\mathscr{O}_S$-module of rank the relative dimension of $\mathscr{G}$ over S.

A finitely presented closed subgroup scheme $\mathscr{H} \subset \mathscr{G}$ over S is called *normal*, if for all $S' \in \mathbf{Sch}_S$ the subgroup $\mathscr{H}(S') \subset \mathscr{G}(S')$ is normal.

Proposition A.3 (identity component, [Gro66, 15.6.5] and [Con14, bottom p. 81]). *Suppose $\mathscr{G}$ is smooth over S. Then there exists a unique open subgroup scheme $\mathscr{G}^o \subset \mathscr{G}$ such that $(\mathscr{G}^o)_s$ is the identity component of $\mathscr{G}_s$ for all $s \in S$. The subgroup scheme $\mathscr{G}^o$ is normal in $\mathscr{G}$. The formation $\mathscr{G} \mapsto \mathscr{G}^o$ commutes with any base change.*

Definition A.4 (identity component). If $\mathscr{G}$ is smooth over S, it is called *connected* over S, and π is called *connected*, if $\mathscr{G} = \mathscr{G}^o$.

For a finitely generated $\mathbb{Z}$-module $(M, +, 0_M)$, let $\mathbb{Z}[M]$ be the Hopf algebra with multiplication defined by $m_1 \otimes m_2 \mapsto m_1 + m_2$, comultiplication by $m \mapsto m \otimes m$, counit $1_{\mathbb{Z}} \mapsto 0_M$, coinverse $m \mapsto -m$, and let $D(M)$ be the corresponding affine group scheme over $\mathbb{Z}$; it is flat and of finite type.

Definition A.5 (multiplicative type and tori, [Con14, Appendix B]).

(a) The group scheme $\mathscr{G}$ is called of *of multiplicative type* over S, and π is of *multiplicative type*, if there is an fppf covering $\{S_i\}$ of S such that for all i there are finitely generated abelian groups M_i and isomorphisms $\mathscr{G} \times_S S_i \cong D(M_i) \times_{\operatorname{Spec} \mathbb{Z}} S_i$.

(b) The group scheme $\mathscr{G}$ is called a *torus* if it is of multiplicative type and if there is a covering as in (a) with all M_i free over $\mathbb{Z}$.[8]

(c) The group scheme $\mathscr{G}$ is called a *split torus* if $\mathscr{G} \cong D(M) \times_{\operatorname{Spec}\mathbb{Z}} S$ with M a free finitely generated $\mathbb{Z}$-module.

Proposition 14.51(6) of [GW10] shows that group schemes of multiplicative type are affine.

Definition A.6 (reductivity and semisimplicity). The group scheme $\mathscr{G}$[9] is called *reductive* or *semisimple* over S, and π is called *reductive* or *semisimple*, if π is smooth and if for all geometric points $\bar{s}$ of S the fiber $\mathscr{G}^o_{\bar{s}}$ is reductive or semisimple, respectively.

The definition of reductivity in SGA3 is more restrictive than Definition A.6(b), as noted in [Con14, § 3.1]. It also requires $\mathscr{G}$ to be connected. The more general definition above is justified by the following result:

Proposition A.7 [Con14, Proposition 3.1.3]. *Suppose π is smooth. Then $\mathscr{G}$ is reductive if and only if $\mathscr{G}^o$ is reductive. In this case $\mathscr{G}^o$ is clopen in $\mathscr{G}$, the quotient $\mathscr{G}/\mathscr{G}^o$ exists and it is étale over S and of finite presentation.*

Since being smooth and being affine are preserved under any base change, and since geometric fibers of any base change are geometric fibers of a given scheme we also have:

Proposition A.8. *If π is reductive or semisimple, then the respective property is preserved under any base change.*

Let $\mathbf{Sch}_S$ be the category of schemes over S and **Gps** that of abstract groups.

Proposition A.9 (center, [Con14, Remark 2.2.5 and Theorem 3.3.4]). *Suppose π is smooth and has connected geometric fibers, i.e., for all geometric point $\bar{s}$ of S the fiber $\mathscr{G}_{\bar{s}}$ is connected. Then the functor*

$$\mathbf{Sch}_S \to \mathbf{Gps},\ S' \mapsto \{g \in \mathscr{G}(S') \mid \forall g' \in \mathscr{G}(S') : gg' = g'g\}$$

is representable by a finitely presented closed subgroup scheme $Z_{\mathscr{G}}$ of $\mathscr{G}$ over S.

If in addition π is connected reductive, then $Z_{\mathscr{G}}$ is of multiplicative type, affine and flat over S, and the formation of $Z_{\mathscr{G}}$ commutes with any base change.

Definition A.10 (center). The group scheme $Z_{\mathscr{G}}$ is called the *center* of $\mathscr{G}$.

Even if π is reductive, the scheme $Z_{\mathscr{G}}$ need not be smooth over S, as is witnessed for instance by $\mathscr{G} = \mathrm{SL}_p$ and $S = \operatorname{Spec}\mathbb{F}_p$, in which case $Z_{\mathscr{G}}$ is the finite flat group scheme μ_p which is not smooth over $\mathbb{F}_p$.

Definition A.11 (tori and splitness). Let π be reductive and let S and $\mathscr{G}$ be connected.[10]

[8] Equivalently: $\mathscr{G}$ is a torus if and only if π is smooth, connected and of multiplicative type.

[9] Recall that we assume that π is affine.

[10] For S not connected, the definition is more complicated, but we do not need it.

(a) A *maximal torus in* $\mathscr{G}$ is a closed S-subgroup $\mathscr{T} \subset \mathscr{G}$ such that for any geometric point $\bar{s}$ of S the fiber $\mathscr{T}_{\bar{s}}$ is a maximal torus in the reductive group $\mathscr{G}_{\bar{s}}$.

(b) One calls $\mathscr{G}$ *split reductive over* S if it contains a maximal torus $\mathscr{T}$ that is split over S *and for each root* $\alpha \in \mathrm{Hom}(\mathscr{T}, \mathbb{G}_{m,S})$ *the root space* $\mathrm{Lie}(\mathscr{G}/S)_\alpha$ *is free over* $\mathscr{O}_S$ *of rank* 1.

Definition A.12. Suppose $\mathscr{G}$ and $\mathscr{G}'$ are connected semisimple over S.

(a) A homomorphism $\phi : \mathscr{G} \to \mathscr{G}'$ is called an *isogeny* if f is finite, flat and surjective, and it is called a *central isogeny* if in addition $\mathrm{Ker}\,\phi$ lies in the center of $\mathscr{G}$.

(b) $\mathscr{G}$ is called of *adjoint type* if $Z_{\mathscr{G}} = 1$.

(c) $\mathscr{G}$ is called *simply connected* if all its geometric fibers are simply connected.

Since for $\mathscr{G}$ reductive the formation of $Z_{\mathscr{G}}$ commutes with any base change, a connected semisimple group $\mathscr{G}$ is of adjoint type if and only if all its geometric fibers are of adjoint type.

Theorem A.13 [Con14, Exercise 6.5.2, Proposition 3.3.5]. *Assume π is semisimple.*

(a) *There exists a semisimple, simply connected group scheme $\pi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to S$ and a central isogeny $\phi^{\mathrm{sc}} : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}$ over S, and the pair $(\mathscr{G}^{\mathrm{sc}}, \phi^{\mathrm{sc}})$ is unique up to unique isomorphism.*

(b) *The maps $\phi^{\mathrm{ad}} : \mathscr{G} \to \mathscr{G}/Z(\mathscr{G})$ and $\phi : \mathscr{G}^{\mathrm{sc}} \to \mathscr{G}^{\mathrm{sc}}/Z(\mathscr{G}^{\mathrm{sc}})$ are central isogenies, the S-group schemes $\mathscr{G}/Z(\mathscr{G})$ and $\mathscr{G}^{\mathrm{sc}}/Z(\mathscr{G}^{\mathrm{sc}})$ are isomorphic and semisimple of adjoint type, and under any isomorphism, the map ϕ factors uniquely via ϕ^{ad}.*

(c) *The map ϕ^{sc} induces a short exact sequence of finite flat S-group schemes*

$$1 \to \mathrm{Ker}\,\phi^{\mathrm{sc}} \xrightarrow{\phi^{\mathrm{sc}}} Z(\mathscr{G}^{\mathrm{sc}}) \to Z(\mathscr{G}) \to 1.$$

Proposition A.14 [Con14, Proposition 3.3.5]. *Let $\phi : \mathscr{G} \to \mathscr{G}'$ be a central isogeny of connected reductive groups. Then $\mathscr{T}' \mapsto \phi^{-1}\mathscr{T}$ defines a bijection between maximal tori of $\mathscr{G}'$ and maximal tori of $\mathscr{G}$.*

Given an S-scheme X carrying a $\mathscr{G}$-action, an important construction is that of a quotient $X/\mathscr{G}$. The course taken in SGA3 is as follows: embed the category $\mathbf{Sch}_S$ via Yoneda into the category of functors $\mathbf{Sch}_S^{\mathrm{opp}} \to \mathbf{Sets}$ by sending X to $h_X : T \mapsto \mathrm{Hom}_S(T, X)$. Equip the latter category with the fppf-topology; we write $\mathrm{Sh}(\mathbf{Sch}_S)_{\mathrm{fppf}}$. It turns out that that any h_X lie in $\mathrm{Sh}(\mathbf{Sch}_S)_{\mathrm{fppf}}$. One calls $F \in \mathrm{Sh}(\mathbf{Sch}_S)_{\mathrm{fppf}}$ representable if it is isomorphic to h_X for some $X \in \mathbf{Sch}_S$. Now given X, $\mathscr{G}$ as above, consider the presheaf $h_X/h_{\mathscr{G}} : T \mapsto h_X(T)/h_{\mathscr{G}}(T)$ and let $(h_X/h_{\mathscr{G}})^{\mathrm{sh}}$ be the associated sheaf in $\mathrm{Sh}(\mathbf{Sch}_S)_{\mathrm{fppf}}$.

Definition A.15. One calls $Y \in \mathbf{Sch}_S$ an *fppf-quotient* of X by $\mathscr{G}$ if $h_Y \cong (h_X/h_{\mathscr{G}})^{\mathrm{sh}}$.

In general Y need not exist. If it exists, an important result of Raynaud gives a comparison with (universal) geometric quotients: if $\mathscr{G}$ is smooth and affine, if X is locally of finite type and if the action is strictly free, i.e., $\mathscr{G} \times_S X \to X \times_S X$, $(g, x) \mapsto (gx, x)$ is an immersion, then geometric and fppf-quotients agree; see [EvdGM14, Chapter 4].

A special case is when X itself is an S-group scheme $\mathscr{H}$ and $\mathscr{G}$ is a closed normal subgroup of $\mathscr{H}$. Then $h_{\mathscr{H}}/h_{\mathscr{G}}$ carries a group law, a unit section and an inversion. This passes to the fppf-sheafification. Hence if $\mathscr{H}/\mathscr{G}$ exists as an fppf-quotient, it is automatically an S-group scheme.

Proposition A.16 (derived subgroup, [Con14, Theorem 5.3.1]). *Let π be connected and reductive.*

(a) *The fppf-sheafification of the* ***commutator subfunctor*** $S' \mapsto [\mathscr{G}(S'), \mathscr{G}(S')]$ *on Sch_S is representable by a semisimple closed normal S-subgroup $\mathscr{G}^{\mathrm{der}} \subset \mathscr{G}$.*

(b) *The fppf-quotient $\mathscr{G}/\mathscr{G}^{\mathrm{der}}$ is representable by a torus.*

(c) *The quotient map $\mathscr{G} \to \mathscr{G}/\mathscr{G}^{\mathrm{der}}$ is initial among all homomorphisms from $\mathscr{G}$ to an abelian sheaf, and the formation of $\mathscr{G}^{\mathrm{der}}$ commutes with any base change on S.*

Hence if π is connected and semisimple, then $\mathscr{G} = \mathscr{G}^{\mathrm{der}}$, since a nontrivial torus quotient would violate the required semisimplicity of all fibers of $\mathscr{G}$.

Definition A.17. Suppose that π is reductive. Then we define $\mathscr{G}^{\mathrm{der}}$ as $(\mathscr{G}^o)^{\mathrm{der}}$.

If π is reductive, then the S-subgroup scheme $\mathscr{G}^{\mathrm{der}}$ is closed and normal in $\mathscr{G}$: By Proposition A.3, the group scheme $\mathscr{G}^o$ is closed and normal in $\mathscr{G}$. To conclude observe that $\mathscr{G}^{\mathrm{der}}$ is closed in $\mathscr{G}^o$ and that normality is a property of the underlying functor of points, so that one can use that for an abstract group G and a normal subgroup N of G the commutator subgroup $[N, N]$ is normal in G.

Proposition A.18. *If π is reductive and $\mathscr{G}/\mathscr{G}^o$ is finite étale, then the following hold:*

(a) *The quotient $\mathscr{G}/\mathscr{G}^{\mathrm{der}}$ is fppf-representable by a smooth affine S-group scheme.*

(b) *The S-group scheme $\mathscr{G}/\mathscr{G}^{\mathrm{der}}$ is an extension of the finite étale S-group scheme $\mathscr{G}/\mathscr{G}^o$ by the torus $\mathscr{G}^o/\mathscr{G}^{\mathrm{der}}$.*

Proof. Consider the fppf-sheaf $\overline{\mathscr{G}} := (h_{\mathscr{G}}/h_{\mathscr{G}^{\mathrm{der}}})^{\mathrm{sh}}$ on $\mathbf{Sch}_S$. To prove (a) we need to show that it is representable by a smooth affine group scheme. Let $S' \to S$ be an fppf-cover over which $\pi_0 := \mathscr{G}/\mathscr{G}^o$ becomes a finite constant group scheme and each component has an S'-point. Then $\mathscr{G}_{S'}$ is a disjoint union $\bigsqcup_{g\in\pi_0} \hat{g}\mathscr{G}^o_{S'}$ where each $\hat{g}$ is a representative in $\mathscr{G}(S')$ of $g \in \pi_0$. Now $\mathscr{G}^o_{S'}/\mathscr{G}^{\mathrm{der}}_{S'}$ exists as a smooth affine fppf-quotient by Proposition A.16. Hence so does $\mathscr{G}_{S'}/\mathscr{G}^{\mathrm{der}}_{S'} = \bigsqcup_{g\in\pi_0} \hat{g}\mathscr{G}^o_{S'}/\mathscr{G}^{\mathrm{der}}_{S'}$. It follows that $\overline{\mathscr{G}}$ after restriction to $\mathbf{Sch}_{S'}$ is representable by a smooth affine S'-scheme. By descent, see the proof of [Sti09, Theorem 14, § 5.4], it follows that $\overline{\mathscr{G}}$ is representable by a scheme over S, and this proves (a).

For the proof of (b) observe that it suffices to show that the natural diagram

$$1 \to \mathscr{G}^o/\mathscr{G}^{\mathrm{der}} \to \mathscr{G}/\mathscr{G}^{\mathrm{der}} \to \mathscr{G}/\mathscr{G}^o \to 1$$

of fppf-presheaves is a short exact sequence of groups under any evaluation at $T \in \mathbf{Sch}_S$. This remains true under fppf-sheafification, and hence also for the representing schemes that exist by (a), Proposition A.16 and the hypotheses. This proves (b). ☐

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Projectivity and effective global generation of determinantal line bundles on quiver moduli

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We give a moduli-theoretic treatment of the existence and properties of moduli spaces of semistable quiver representations, avoiding methods from geometric invariant theory. Using the existence criteria of Alper, Halpern-Leistner and Heinloth, we show that for many stability functions, the stack of semistable representations admits an adequate moduli space, and prove that this moduli space is proper over the moduli space of semisimple representations. We construct a natural determinantal line bundle that descends to a semiample line bundle on the moduli space and provide new effective bounds for global generation. For an acyclic quiver, we show that this line bundle is ample, thus giving a modern proof of the fact that the moduli space is projective.

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1. Introduction

Many important natural problems in linear algebra such as Jordan normal forms, the classification of tuples of matrices up to simultaneous conjugation, and the classification of tuples of subspaces in a fixed vector space can all be encoded as moduli problems for representations of a quiver, where properties of the classification problem are described by the geometry of the moduli space. Moreover, quiver moduli spaces closely interact with other important moduli spaces and play a fundamental role in representation theory.

The theory of quivers and their representations goes back to Gabriel, who showed that the quivers for which the moduli problem is *discrete* are precisely the Dynkin quivers [22]. After pioneering work of Kac [31], the study of *continuous* moduli problems began with King's construction [32] of moduli spaces of quiver representations using geometric invariant theory (GIT), in which the GIT notion of stability obtains a reinterpretation as stability of representations with respect to a stability function. For the trivial stability function, all quiver representations are semistable and the moduli space is just an affine GIT quotient which classifies semisimple quiver representations. For a nontrivial stability function, the moduli space of semistable representations is projective over the affine GIT quotient. The ring of invariants turns out to be generated by taking traces along oriented cycles [36], so in particular when the quiver is acyclic, the semisimple moduli space is a point and moduli spaces of semistable quiver representations are projective varieties. An excellent survey of the geometry of these moduli spaces is given in [42]; for further details on their GIT construction, see also [9].

In this paper, we instead take an approach that combines Alper's theory of adequate moduli spaces with determinantal line bundle techniques to provide a construction of projective moduli spaces of semistable representations of an acyclic quiver that avoids the methods of GIT. Our method follows the blueprint set out in recent papers on the projectivity of moduli spaces of stable curves [10; 33], semistable vector bundles on curves [5], and semistable vector bundles on stacky curves [12; 13]. Let us recall the basic steps in these approaches:

(1) Interpret the moduli problem as an algebraic stack $\mathcal{M}$ of finite type.

(2) Prove that $\mathcal{M}$ admits an adequate moduli space M, which is a proper algebraic space.

(3) Find a line bundle on $\mathcal{M}$ which descends to an ample line bundle on M.

In the case of stable curves [10], the second step uses the Keel–Mori theorem for proper Deligne–Mumford stacks, whereas the analogue for the moduli space of semistable vector bundles on a curve [5] relies on the recent existence result on adequate moduli spaces of algebraic stacks [6].

In the case of stable curves, the construction of the ample line bundle in the third step above is due to Kollár [33], who considers the determinant of the direct image of a relative pluricanonical bundle on the universal family over the moduli stack. The case of vector bundles on a curve [5], which similarly centers around proving ampleness of a certain determinantal line bundle constructed from the universal vector bundle, follows arguments by Esteves and Popa [19, Section 5; 20, Section 3] improving upon the original GIT-free approach of Faltings [21].

There are several good reasons for pursuing such an approach: (i) it provides an intrinsic moduli-theoretic proof, (ii) it illustrates the theory of adequate moduli spaces, and (iii) it can yield insight into how to construct projective moduli spaces in situations where GIT cannot be applied. It should be noted however that this approach does not automatically yield projectivity, and we need to rely on the properties of the specific moduli problem.

Main results. Working over a noetherian base scheme S, we will moduli-theoretically define the stack $\mathcal{M}_{d,S}$ parameterizing families of representations of a quiver Q of dimension vector d. To a stability function θ we associated the open substack $\mathcal{M}^{\theta\text{-ss}}_{d,S} \subseteq \mathcal{M}_{d,S}$ of θ-semistable representations and construct a natural determinantal line bundle $\mathcal{L}_\theta$ on it.

We then give moduli-theoretic proofs that these stacks are Θ-reductive and S-complete and deduce that both stacks admit adequate moduli spaces by applying the existence result of Alper, Halpern-Leistner and Heinloth [6]. A reader who is used to working in characteristic 0 can read the word “adequate” as “good”, since the two notions only differ in positive characteristic.

The following is a summary of the main results of the paper.

Theorem A. *Let Q be an acyclic quiver and let S be a noetherian scheme.*

(i) *The stack $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ of θ-semistable quiver representations over S admits an adequate moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ which is proper over S.*

(ii) *The line bundle $\mathcal{L}_\theta$ on $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ descends to a relatively ample line bundle L_θ on $\mathrm{M}^{\theta\text{-ss}}_{d,S}$. In particular, $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is projective over S.*

The last part of (ii) is of course well-known (when working over a field usually), but the novelty here is that we illustrate how it can be obtained using the modern methods of algebraic stacks and adequate moduli spaces.

Our methods yield partial results also when Q has oriented cycles. In this case we require that the stability function θ is of the form $-\langle \_, \beta\rangle$ for a dimension vector β with $\langle d, \beta\rangle = 0$, where $\langle \_, \_\rangle$ denotes the Euler pairing; this condition holds for every stability function when Q is acyclic by Lemma 2.3.5. Under this assumption we prove the analogues of (i) and (ii):

(i′) *The stack $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ of θ-semistable quiver representations admits a separated adequate moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$, and the semisimplification map $\mathrm{M}^{\theta\text{-ss}}_{d,S} \to \mathrm{M}_{d,S}$ on adequate moduli spaces is proper.*

(ii′) *When $S = \operatorname{Spec} k$ is the spectrum of a field, the line bundle $\mathcal{L}_\theta$ on $\mathcal{M}^{\theta\text{-ss}}_{d,k}$ descends to a semiample line bundle L_θ on $\mathrm{M}^{\theta\text{-ss}}_{d,k}$.*

In this setting, there are two limitations to our approach. First, our methods are unable to handle the true analogue of (ii) in the non-acyclic case – that $\mathrm{M}_{d,k}$ is affine and the semisimplification map $\mathrm{M}_{d,k}^{\theta\text{-ss}} \to \mathrm{M}_{d,k}$ is projective; this statement usually follows from the methods of GIT. Second, we can only develop (ii′) over a field.

It should moreover be possible to remove the condition on θ in the non-acyclic case. It is currently used to produce sections of determinantal line bundles, which we use to check local reductivity of the stack of semistable representations in order to apply the existence criteria [6, Theorem 5.4] in arbitrary characteristic. If S has characteristic 0, we can use good moduli spaces instead of adequate moduli spaces, and the required local reductivity follows from [4; 6]. We also use determinantal sections in the proof of semiampleness.

We prove (i) and (i′) in Corollary 5.5.7 and Proposition 5.6.1. The main projectivity result (ii) is given in Theorem 6.2.1. We prove (ii′) in Proposition 6.1.1.

Our proof that the semisimplification map $\mathrm{M}_{d,S}^{\theta\text{-ss}} \to \mathrm{M}_{d,S}$ is proper follows an adaptation of Langton's semistable reduction argument for semistable coherent sheaves [35], see Proposition 5.6.2. It says that if a representation of Q over a discrete valuation ring R has semistable generic fiber, then there exists a subrepresentation which agrees at the generic point such that its special fiber is semistable. For an acyclic quiver, we moreover argue moduli-theoretically that $\mathrm{M}_{d,S}^{\theta\text{-ss}}$ is proper over S (Corollary 5.6.4) and that the adequate moduli space $\mathrm{M}_{d,S}$ of all representations is isomorphic to S (Proposition 5.6.5).

The proof of projectivity in Theorem A(ii) (when Q is acyclic) is obtained by bootstrapping from $\operatorname{Spec} k$ to $\operatorname{Spec} \mathbb{Z}$ and finally to an arbitrary base S. The main idea over a field k is to show that the line bundle $\mathcal{L}_\theta$ is semiample, and that the induced map $\mathrm{M}_{d,k}^{\theta\text{-ss}} \to \mathbb{P}_k^n$ is finite, and thus the proper algebraic space $\mathrm{M}_{d,k}^{\theta\text{-ss}}$ is in fact a projective variety. Instead of appealing to methods from GIT, we give a new moduli-theoretic proof of global generation of a power of $\mathcal{L}_\theta$ inspired by the approach of Esteves and Popa [19; 20] for moduli of vector bundles on curves using dimension-counting techniques.

Let us outline how we produce sections of $\mathcal{L}_\theta$. Since Q is acyclic, we can write $\theta = -\langle \_, \beta \rangle$ for a dimension vector β with $\langle d, \beta \rangle = 0$. For $m > 0$ and a representation N of dimension vector $m\beta$, we define a 2-term complex $\mathcal{E}_N^\bullet$ on $\mathcal{M}_{d,S}$ whose associated determinant line bundle is $\mathcal{L}_\theta^{\otimes m}$ and, since $\langle d, \beta \rangle = 0$, comes with a section σ_N which is nonzero at a representation $M \in \mathcal{M}_{d,S}$ if and only if $\operatorname{Hom}(M, N) = \operatorname{Ext}(M, N) = 0$.

The sections constructed in this way are often called *determinantal semi-invariants* in the representation theory literature, and were first studied by Schofield [43]. A key result is that determinantal semi-invariants span the global sections of powers of determinantal line bundles; GIT-based proofs are due to Derksen and Weyman [14], Domokos and Zubkov [17], and Schofield and Van den Bergh [45].

Interestingly, this approach enables us to produce new effective bounds for global generation, by keeping track of the estimates in the dimension counting. This is analogous to the effective bounds from [19; 20] for moduli of vector bundles on curves. Moreover, the bound is independent of the orientation of the quiver. We show the following result combining Proposition 4.2.1 and Proposition 6.1.1.

Theorem B. *Let k be a field and let Q be a quiver. Let λ be the negative of the minimal eigenvalue of the Tits form. If m is a positive integer greater than $\lambda\|d\|^2$, then $\mathcal{L}_\theta^{\otimes m}$ is globally generated on $\mathcal{M}_{d,k}^{\theta\text{-ss}}$.*

In Remark 6.1.2 we comment further on the context for this result.

Similarities and differences with vector bundles on curves. There is an important parallel between moduli of semistable quiver representations and moduli of semistable vector bundles on a smooth projective curve: both parameterize objects in a category of global dimension 1 and have smooth moduli stacks. Some aspects of this dictionary, including similarities between their constructions via GIT and symplectic reduction (over $k = \mathbb{C}$), are described in [28].

However, we also see several instances where this dictionary breaks down. One example, mentioned above, is the preservation of stability under natural dualities: Serre duality preserves stability and semistability of vector bundles on curves, but the Auslander–Reiten translations are only a partial duality, and, although they preserve semistability, they only preserve stability under some additional assumptions (see Lemma 4.3.4).

As a second example, the theory of elementary Hecke modifications for vector bundles on curves does not have an immediate analogue for quiver representations (lacking a notion of torsion sheaves), meaning that the proof in [5] cannot be directly translated to quiver representations. We can nevertheless find a close enough analogue of Hecke modifications (as in the proof of Proposition 4.5.2), so that we can stay close to the global structure of the proof for vector bundles.

A third difference is that for a curve of genus $g \geq 2$, the moduli space of stable vector bundles of any rank and degree is non-empty with dimension determined by the Euler pairing, the analogue of which fails for quiver representations — the stable locus may well be empty, in which case the dimension of the semistable locus is difficult to control. This results in us adopting an alternative approach to the dimension counts for quiver representations in Section 4.

Structure of the paper. In Section 2 we recall quiver representations, their stability properties, and the Auslander–Reiten translations. In Section 3 we construct the stack of quiver representations from the ground up and prove its algebraicity moduli-theoretically, as well as explain how to produce determinantal line bundles and their sections. Section 4 is the technical heart of the paper, where we prove the key vanishing results required for later sections. In Section 5 we construct adequate moduli spaces for stacks of representations by verifying the conditions for the existence result of [6]. Here we also discuss semistable reduction. The main results, namely the projectivity of the adequate moduli space for Q acyclic as well as the effective bounds on global generation, are finally established in Section 6. In the Appendix we explain how one can obtain the same results using Halpern-Leistner's theory of stability for stacks; this gives an alternative modern proof, albeit one that still relies on methods of GIT and only works in characteristic 0.

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2. Background on quiver representations

We recall here some terminology from the theory of quiver representations that will be used throughout this paper. We refer to [15] for more details.

2.1. ***Quiver representations.*** A *quiver* $Q = (Q_0, Q_1, s, t)$ is a finite directed graph with vertex set Q_0, arrow set Q_1, and maps $s, t : Q_1 \to Q_0$ that assign to each arrow its source and target. A *path* in a quiver is a sequence of composable arrows; that is, the target of the previous arrow is the source of the next arrow. We formally include a path of length 0 at every vertex $i \in Q_0$. A vertex $i \in Q_0$ is a *source* (respectively a *sink*) of Q if there are no arrows whose target (respectively source) equals i. We will assume that Q is connected throughout, and in Section 3 we will see why this is a harmless assumption to make.

An *oriented cycle* in a quiver is a path of positive length starting and ending at the same vertex; a special case is a *loop*, which is an arrow whose source and target are equal. A quiver is *acyclic* if it has no oriented cycles. Note that if Q is acyclic, there is an *admissible ordering* of Q_0, meaning that $i < j$ whenever there is an arrow $i \to j$; in particular, an acyclic quiver has both a source and a sink.

Given a field k, a *k-representation* of Q is a tuple $M = ((M_i)_{i \in Q_0}, (M_a)_{a \in Q_1})$ consisting of finite-dimensional k-vector spaces M_i for each vertex $i \in Q_0$ and k-linear maps $M_a : M_{s(a)} \to M_{t(a)}$ for each arrow $a \in Q_1$. The *dimension vector* $\underline{\dim}(M) \in \mathbb{N}^{Q_0}$ of M is the tuple $(\dim(M_i))_{i \in Q_0}$. A *morphism* of k-representations $\phi : M \to N$ is a tuple of k-linear maps $(\phi_i : M_i \to N_i)_{i \in Q_0}$ such that $\phi_{t(a)} \circ M_a = N_a \circ \phi_{s(a)}$ for every arrow $a \in Q_1$. The representations of Q over k form an abelian category $\operatorname{rep}_k Q$. If $k \subset k'$ is a field extension, there is a base change functor $(\_) \otimes_k k' : \operatorname{rep}_k Q \to \operatorname{rep}_{k'} Q$ that preserves dimension vectors.

A representation $M \neq 0$ is called *simple* if it has exactly two subrepresentations 0 and M, and *semisimple* if it is the direct sum of simple representations. Any representation has a finite filtration by simple representations, making $\operatorname{rep}_k Q$ a category of *finite length*. For each vertex $i \in Q_0$, there is a simple representation $S(i)$ with $S(i)_i = k$ and $S(i)_j = 0$ for $j \neq i$. If Q is acyclic, these are the only simple representations.

A representation $M \neq 0$ is called *indecomposable* if it cannot be written as the direct sum of two nonzero subrepresentations. By the Krull–Remak–Schmidt theorem, every representation can be written as a direct sum of indecomposable subrepresentations in an essentially unique way; we will use this fact without further mention.

The *Euler pairing* or *Euler form* of Q is $\langle \_, \_ \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$, where

$$\langle \alpha, \beta \rangle := \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{s(a)} \beta_{t(a)}.$$

For representations M and N of Q, we write $\langle M, N\rangle := \langle \underline{\dim}(M), \underline{\dim}(N)\rangle$. Given an ordering $Q_0 = \{1, 2, \ldots, n\}$ of the vertices of Q, one has an isomorphism $\mathbb{Z}^{Q_0} \cong \mathbb{Z}^n$. Hence the Euler pairing is represented by a matrix $A \in \mathrm{Mat}_n(\mathbb{Z})$, called the *Euler matrix*, which satisfies

$$\langle \alpha, \beta\rangle = \alpha^{\mathrm{T}} A \beta \qquad \text{for all } \alpha, \beta \in \mathbb{Z}^n.$$

If Q is acyclic, then for an admissible ordering of the vertices, the Euler matrix is upper unitriangular and hence invertible over $\mathbb{Z}$. In particular, when Q is acyclic, the Euler pairing is perfect.

2.2. ***Modules over the path algebra.*** Let k be a field. The *path algebra* of Q is the k-algebra kQ with basis given by all paths in Q, including a path ϵ_i of length 0 at each vertex i, and multiplication given by the concatenation of paths; see for example [15, Section 1.5]. The category $\mathrm{rep}_k\, Q$ of k-representations of Q is equivalent to the category kQ-mod of finite-dimensional left modules over kQ. The path algebra is finite-dimensional if and only if Q is acyclic. If Q is not acyclic we will also need to consider the category $\mathrm{Rep}_k\, Q$ of not necessarily finite-dimensional representations, which is equivalent to the category kQ-Mod of all left modules over the path algebra.

For each $i \in Q_0$, there are projective and injective representations $P(i) = kQ\epsilon_i$ and $I(i) = (\epsilon_i kQ)^*$; thus for each $j \in Q_0$

- $P(i)_j$ is the k-vector space with basis the set of paths from i to j,
- $I(i)_j$ is the k-vector space dual to the one whose basis is the set of paths from j to i.

The representation $P(i)$ is finite-dimensional if and only if there is no path from i to any vertex in an oriented cycle. Similarly, $I(i)$ is finite-dimensional if and only if there is no path from any vertex in an oriented cycle to i.

We will focus on the case of projective modules, as injective modules are the dual notion; see for example [15, Section 2.2] and Section 2.4 below. The projective representations $P(i)$ are indecomposable, and in the case when Q is acyclic, these are the only indecomposable projective representations of Q up to isomorphism. For any representation M of Q, we have a canonical isomorphism

$$\mathrm{Hom}(P(i), M) \cong M_i. \tag{1}$$

The category $\mathrm{rep}_k\, Q$ is *hereditary*, meaning that any subrepresentation of a projective representation is again projective and dually any quotient of an injective representation is injective. In particular, $\mathrm{rep}_k\, Q$ has homological dimension at most one, so we will write $\mathrm{Ext}(M, N)$ for $\mathrm{Ext}^1(M, N)$. In fact, any representation M has a *canonical projective resolution*

$$0 \to \bigoplus_{a \in Q_1} M_{s(a)} \otimes P(t(a)) \to \bigoplus_{i \in Q_0} M_i \otimes P(i) \to M \to 0 \tag{2}$$

and an analogous canonical injective resolution, working in $\mathrm{Rep}_k\, Q$ in case Q is not acyclic. Applying the functor Hom(_, N) to (2) and using (1) gives the useful exact sequence

$$0 \to \operatorname{Hom}(M, N) \to \bigoplus_{i \in Q_0} \operatorname{Hom}_k(M_i, N_i) \to \bigoplus_{a \in Q_1} \operatorname{Hom}_k(M_{s(a)}, N_{t(a)}) \to \operatorname{Ext}(M, N) \to 0, \tag{3}$$

where the middle morphism is given by $(f_i)_{i \in Q_0} \mapsto (f_{t(a)} \circ M_a - N_a \circ f_{s(a)})_{a \in Q_1}$. From this we deduce that the Euler pairing computes the Euler characteristic:

$$\langle M, N \rangle = \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}(M, N).$$

In particular, $\langle P(i), M \rangle = \dim(M_i)$ since $\operatorname{Ext}(P(i), M) = 0$ as $P(i)$ is projective.

2.3. ***Stability of representations.*** Following [32; 40] we introduce a standard notion of stability for a representation of Q. A *stability function* on Q is a linear map $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$. By convention we will write $\theta(M)$ instead of $\theta(\underline{\dim}\, M)$ for a representation M.

Given a stability function θ, a k-representation M of Q is called

- *θ-semistable* if $\theta(M) = 0$ and $\theta(M') \leq 0$ for every subrepresentation $M' \subseteq M$;
- *θ-stable* if it is θ-semistable and it has exactly two subrepresentations $M' \subseteq M$ with $\theta(M') = 0$, namely $M' = 0$ and $M' = M$ with $M \neq 0$;
- *geometrically θ-stable* if $M \otimes_k k'$ is stable for every field extension k'/k;
- *θ-polystable* if it is a direct sum of θ-stable representations;
- *geometrically θ-polystable* if $M \otimes_k k'$ is polystable for every field extension k'/k.

Remark 2.3.1. There is no need for a notion of geometric θ-semistability since a representation M is θ-semistable if and only if $M \otimes_k k'$ is for some field extension $k \subset k'$ [29, Proposition 2.4]. A representation M is geometrically θ-stable if either one of $M \otimes_k k^s$ or $M \otimes_k \bar{k}$ is θ-stable, where k^s and $\bar{k}$ denote a separable and an algebraic closure of k respectively; see [29, Corollary 2.12]. In particular, over an algebraically closed field, θ-stability and geometric θ-stability coincide. Over a perfect field, θ-polystability and geometric θ-polystability coincide, although a field extension may give rise to more stable factors [41, Lemma 4.2].

Example 2.3.2. This example shows that polystability may not be preserved under field extension when the base field is not perfect. Consider the Jordan quiver Q:

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

Isomorphism classes of representations of Q are given by square matrices up to conjugation. We consider the trivial stability function $\theta = 0$ and give an example of a 2-dimensional representation M of Q which is stable over $\mathbb{F}_2(t)$ but not polystable over $\mathbb{F}_2(\sqrt{t})$. The representation M is given over $\mathbb{F}_2(t)$ by the matrix

$$\begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix},$$

whose only eigenvalue is $1+\sqrt{t}$, so M is simple and in particular stable over $\mathbb{F}_2(t)$. Over $\mathbb{F}_2(\sqrt{t})$, this matrix is similar to

$$\begin{bmatrix}1 & 1+\sqrt{t}\\ 1 & \sqrt{t}\end{bmatrix}\begin{bmatrix}1 & t\\ 1 & 1\end{bmatrix}\begin{bmatrix}\sqrt{t} & 1+\sqrt{t}\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1+\sqrt{t} & 1\\ 0 & 1+\sqrt{t}\end{bmatrix}$$

and since it is a 2×2 Jordan block, it gives a semistable representation that no longer splits into a direct sum of simple representations.

Every θ-semistable representation M has a *Jordan–Hölder filtration*

$$0=M^0\subsetneq M^1\subsetneq M^2\subsetneq\cdots\subsetneq M^{r-1}\subsetneq M^r=M$$

with the property that the quotients $M^\ell/M^{\ell-1}$ are θ-stable for $\ell=1,\ldots,r$. The filtration is not unique, but the isomorphism type of the *associated graded* representation

$$\operatorname{gr} M:=\bigoplus_{\ell=1}^{r} M^\ell/M^{\ell-1}$$

is independent of the filtration.

To define a second type of filtration, we introduce a slope function μ_θ which associates to a dimension vector $d\in\mathbb{N}^{Q_0}\setminus\{0\}$ the rational number

$$\mu_\theta(d):=\frac{\theta(d)}{\sum_{i\in Q_0} d_i}.$$

We say that a representation M is μ_θ-semistable if $\mu_\theta(M')\leq\mu_\theta(M)$ for every subrepresentation $M'\subseteq M$. Observe that when $\theta(M)=0$, stability with respect to μ_θ and θ coincide. Moreover, an arbitrary representation M has a *Harder–Narasimhan filtration*

$$0=M^0\subsetneq M^1\subsetneq M^2\subsetneq\cdots\subsetneq M^{r-1}\subsetneq M^r=M$$

such that M^i/M^{i-1} is μ_θ-semistable for every $i=1,\ldots,r$ and

$$\mu_\theta(M^1/M^0)>\mu_\theta(M^2/M^1)>\ldots>\mu_\theta(M^r/M^{r-1}).$$

See for example [42, Section 4]. This filtration is unique and the representation M^1 is called the *maximally destabilizing* subrepresentation. More generally, one can define different slope functions by giving positive weights to the dimensions in the denominator; these give the same notion of semistability for dimension vectors d for which $\theta(d)=0$ but possibly different Harder–Narasimhan filtrations [27, Section 5.2].

We record some elementary properties of semistable representations, see [25, Lemma 2.6 and Proposition 2.7] for a proof.

Proposition 2.3.3. *Let M and N be μ_θ-semistable k-representations of the same slope.*

(i) *If $f: M\to N$ is any morphism, then* $\ker(f)$, $\operatorname{im}(f)$ *and* $\operatorname{coker}(f)$ *are μ_θ-semistable.*

(ii) *If M and N are θ-stable, then any nonzero morphism $M\to N$ is an isomorphism. In particular, if M is geometrically θ-stable, then the canonical map $k\to$* $\operatorname{End}(M)$ *is an isomorphism.*

Given $\alpha, \beta \in \mathbb{Z}^{Q_0}$ we define stability functions $\theta_\alpha, \eta_\beta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ by

$$\theta_\alpha(d) := \langle \alpha, d \rangle \qquad \text{and} \qquad \eta_\beta(d) := -\langle d, \beta \rangle. \tag{4}$$

When Q is an acyclic quiver, the matrix of the Euler form is upper-unitriangular, hence invertible, and therefore defines an isomorphism $\mathbb{Z}^{Q_0} \cong (\mathbb{Z}^{Q_0})^\vee$. This proves the following.

Lemma 2.3.4. *Suppose Q is an acyclic quiver and let $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be a stability function.*

(a) *There is a unique $\alpha \in \mathbb{Z}^{Q_0}$ such that $\theta = \theta_\alpha = \langle \alpha, \_ \rangle$, given by $\alpha_i = \theta(I(i))$.*

(b) *There is a unique $\beta \in \mathbb{Z}^{Q_0}$ such that $\theta = \eta_\beta = -\langle \_, \beta \rangle$, given by $\beta_i = -\theta(P(i))$.*

The next lemma shows how stability functions for acyclic quivers are in fact given by dimension vectors.

Lemma 2.3.5. *Suppose Q is an acyclic quiver and let $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be a stability function for which there exists a semistable representation M such that* $\operatorname{supp} M = Q_0$.

(a) *Let $\alpha \in \mathbb{Z}^{Q_0}$ be the unique vector such that $\theta = \theta_\alpha$. Then $\alpha \in \mathbb{N}^{Q_0}$.*

(b) *Let $\beta \in \mathbb{Z}^{Q_0}$ be the unique vector such that $\theta = \eta_\beta$. Then $\beta \in \mathbb{N}^{Q_0}$.*

Proof. We show that the entries of β are non-negative. The proof for α is analogous. As $\beta_i = -\theta(P(i))$, we need to show that $\theta(P(i)) \leq 0$ for every i. We choose an admissible ordering of the vertices of Q and write $Q_0 = \{1, \dots, n\}$. We show the claim by descending induction on $i \in \{1, \dots, n\}$. For $i = n$, the vertex i is a sink and therefore $P(i)$ is simple. Choose a nonzero vector $x \in M_i$ and let $f \in \operatorname{Hom}(P(i), M) \cong M_i$ be the corresponding morphism. As $f \neq 0$ and $P(i)$ is simple, f is injective. Since M is θ-semistable, we see that $\theta(P(i)) \leq 0$.

Now let $i < n$. Again, choose any nonzero element $x \in M_i$ and consider the corresponding morphism $f : P(i) \to M$. The kernel K of f is again projective and a proper subrepresentation of $P(i)$, and thus $K \cong \bigoplus_{j>i} P(j)^{\oplus m_j}$ for some multiplicities $m_j \geq 0$. By the inductive assumption, we know that $\theta(P(j)) \leq 0$ for all $j > i$, which implies that $\theta(K) \leq 0$. Now $f : P(i) \to M$ gives rise to an injective homomorphism

$$P(i)/K \hookrightarrow M$$

which by θ-semistability of M implies that

$$0 \geq \theta(P(i)/K) = \theta(P(i)) - \theta(K),$$

and so $\theta(P(i)) \leq \theta(K) \leq 0$. $\square$

Without the acyclicity assumption Lemma 2.3.5 can fail.

Example 2.3.6. Consider the quiver

$$Q:\ \begin{tikzcd} 1 \arrow[r, bend left, shift left] \arrow[r, bend left, shift right] & 2 \arrow[l, bend left, shift left] \arrow[l, bend left, shift right] \end{tikzcd}\ .$$

The stability function θ given by the inner product with $(-3, 3)$ equals θ_α for $\alpha = (-1, 1)$, and η_β for

$\beta = (1, -1)$. Note that since the Euler matrix $\left(\begin{smallmatrix} 1 & -2 \\ -2 & 1 \end{smallmatrix}\right)$ is invertible over the rationals, both α and β are uniquely determined, but neither of them is a dimension vector. Consider the dimension vector $d = (1, 1)$, so that $\theta(d) = 0$. There exists a θ-semistable representation with dimension vector d: it suffices that one of the maps from vertex 2 to vertex 1 is non-zero. In this case there is only one non-trivial subrepresentation, which necessarily is of dimension $(1, 0)$ and hence does not destabilize the representation.

2.4. ***Auslander–Reiten translations.*** The final standard construction for quivers and their representations that we need to recall is that of Auslander–Reiten translations. These are endofunctors of the category of quiver representations, and their interaction with θ-stability will be discussed in Section 4.3.

The *opposite quiver* of a quiver $Q = (Q_0, Q_1, s, t)$ is the quiver $Q^{\mathrm{op}} := (Q_0, Q_1, t, s)$, where all arrows have been reversed. The path algebra of Q^{op} is canonically isomorphic to the opposite algebra of kQ. Taking the dual k-vector space gives a contravariant functor D from the category kQ-mod to kQ^{op}-mod, or equivalently between the category of representations of Q and Q^{op}. The duality functor D is an antiequivalence of categories and exchanges injective and projective modules.

Let Q be an acyclic quiver. Consider a representation M as a left module over the path algebra kQ and kQ as a bimodule over itself. The *Auslander–Reiten translates* of M are the left kQ-modules

$$\tau M := \mathrm{D}\,\mathrm{Ext}(M, kQ) \quad \text{and} \quad \tau^{-} M := \mathrm{Ext}(\mathrm{D}\, M, kQ),$$

and these constructions provide two endofunctors of the category $kQ\text{-mod} \cong \mathrm{rep}_k\, Q$, called the *Auslander–Reiten translations*. It follows from the construction that

$$\tau P = \tau^{-} I = 0$$

whenever P is projective and I is injective. The following proposition records the key properties we will need later; for the proof, see [14, Section 6.4].

Proposition 2.4.1. *Let M and N be representations of Q. The Auslander–Reiten translations τ and τ^{-} satisfy the following properties*:

(i) **(*partial inverse property*)** *We have $\tau^{-}\tau M \cong M$ and $\tau\tau^{-} N \cong N$, provided that M has no projective summands and N has no injective summands.*

(ii) **(*Auslander–Reiten duality*)** *We have isomorphisms of k-vector space valued functors*

$$\mathrm{Hom}(\_, \tau M) \cong \mathrm{Ext}(M, \_)^{\vee} \quad \textit{and} \quad \mathrm{Hom}(\tau^{-} N, \_) \cong \mathrm{Ext}(\_, N)^{\vee}.$$

In particular τ^{-} is the left adjoint to τ. If M has no projective summands and N has no injective summands, we have additional isomorphisms

$$\mathrm{Ext}(\_, \tau M) \cong \mathrm{Hom}(M, \_)^{\vee} \quad \textit{and} \quad \mathrm{Ext}(\tau^{-} N, \_) \cong \mathrm{Hom}(\_, N)^{\vee},$$

and in particular

$$\langle \_, \tau M \rangle = -\langle M, \_ \rangle \quad \textit{and} \quad \langle \tau^{-} N, \_ \rangle = -\langle \_, N \rangle.$$

3. Moduli stacks of representations and determinantal line bundles

In this section, we introduce moduli stacks parameterizing representations of a quiver with fixed dimension vector and study their first properties. Throughout d will denote a dimension vector for the fixed quiver Q.

3.1. ***The moduli stack of all representations.*** Let S be a fixed base scheme. For an algebraic stack T over S, a *family $\mathcal{F}$ of quiver representations* of dimension vector d consists of locally free sheaves $\mathcal{F}_i$ of rank d_i over T for each $i \in Q_0$ and homomorphisms $\mathcal{F}_a : \mathcal{F}_{s(a)} \to \mathcal{F}_{t(a)}$ of $\mathcal{O}_T$-modules for each $a \in Q_1$. If $f : T' \to T$ is a morphism of stacks over S, we obtain a family of representations $\mathcal{F}_{T'} := f^*\mathcal{F}$ on T' by pullback along f. If T is an S-scheme and $x \in T$ is a point, then pulling back along the inclusion of the residue field $\operatorname{Spec} \kappa(x) \hookrightarrow T$ gives a $\kappa(x)$-representation which we denote by $\mathcal{F}|_x$.

Definition 3.1.1. The moduli stack $\mathcal{M}_{d,S}$ of representations of Q of dimension vector d is the category fibered in groupoids over the big étale site of the category of S-schemes whose objects are pairs $(T, \mathcal{F})$, where T is an S-scheme and $\mathcal{F}$ is a family of representations of Q of dimension vector d over T. A morphism $(T', \mathcal{F}') \to (T, \mathcal{F})$ is the data of a morphism $f : T' \to T$ of S-schemes together with morphisms $\phi_i : \mathcal{F}_i \to f_*\mathcal{F}'_i$ of $\mathcal{O}_T$-modules such that the squares

$$\begin{tikzcd}
\mathcal{F}_{s(a)} \arrow[r, "\phi_{s(a)}"] \arrow[d, "f_*\mathcal{F}'_a"'] & f_*\mathcal{F}'_{s(a)} \arrow[d, "\mathcal{F}_a"] \\
\mathcal{F}_{t(a)} \arrow[r, "\phi_{t(a)}"'] & f_*\mathcal{F}'_{t(a)}
\end{tikzcd}$$

commute for every $a \in Q_1$ and whose adjoints are isomorphisms $f^*\mathcal{F}_i \to \mathcal{F}'_i$.

We will frequently omit one of the subscripts in $\mathcal{M}_{d,S}$ when either the base scheme or the dimension vector is clear from the context. In Section 4 and Section 6, we will take our base scheme to be $S = \operatorname{Spec} A$ for a ring A, in which case we may denote $\mathcal{M}_{d,S}$ by $\mathcal{M}_{d,A}$ or simply $\mathcal{M}_A$. Similarly, when we below define the substacks $\mathcal{M}^{\theta\text{-ss}}_{d,S}$, the moduli spaces $\mathrm{M}_{d,S}$ and $\mathrm{M}^{\theta\text{-ss}}_{d,S}$, and the representation space $\mathrm{R}_{d,S}$, we apply that same conventions for the subscripts.

We will make the harmless assumption that Q is connected: if $Q = Q' \sqcup Q''$ with corresponding decomposition $d = (d', d'')$ with $d' \in \mathbb{N}^{Q'_0}$, $d'' \in \mathbb{N}^{Q''_0}$, then $\mathcal{M}_{d,S}$ is isomorphic to the product of moduli stacks of representations of Q' and Q'' of dimension vectors d' and d'', respectively, and similarly for all other constructions.

The stack $\mathcal{M}_{d,S}$ commutes with base change, meaning that if $S' \to S$ is a morphism of schemes, then we have a cartesian diagram

$$\begin{tikzcd}
\mathcal{M}_{d,S'} \arrow[r] \arrow[d] & \mathcal{M}_{d,S} \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

where in the top horizontal map, a family of representations $\mathcal{F}$ on $T \to S'$ is viewed as a family on $T \to S$

via the composition $T \to S' \to S$. In particular, the stack $\mathcal{M}_{d,\mathbb{Z}}$ over the final object $\operatorname{Spec}\mathbb{Z}$ is universal in the sense that for any scheme S, the stack $\mathcal{M}_{d,S}$ is obtained by base change from $\mathcal{M}_{d,\mathbb{Z}}$ by the structure morphism $S \to \operatorname{Spec}\mathbb{Z}$.

There is a universal family of representations $\mathcal{F}^{\mathrm{univ}}$ of dimension vector d on the stack $\mathcal{M}_{d,S}$. If T is an S-scheme and $\mathcal{F}$ is a family of representations of dimension vector d on T, there exists a unique morphism $f : T \to \mathcal{M}_{d,S}$ such that $\mathcal{F} \cong f^*\mathcal{F}^{\mathrm{univ}}$.

Proposition 3.1.2. *The diagonal of the stack $\mathcal{M}_{d,S}$ is represented by affine S-schemes.*

Proof. We follow a similar argument in [47, Tag 08K9]. Let $\mathcal{F}$ and $\mathcal{G}$ be two representations over an S-scheme T and consider the 2-cartesian diagram

$$\begin{tikzcd} \mathbb{I}\mathrm{som}(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d] & T \arrow[d, "(\mathcal{F},\mathcal{G})"] \\ \mathcal{M}_{d,S} \arrow[r, "\Delta"] & \mathcal{M}_{d,S} \times \mathcal{M}_{d,S} \end{tikzcd}$$

The fiber product $\mathbb{I}\mathrm{som}(\mathcal{F},\mathcal{G})$ is the functor associating to a T-scheme U the set of isomorphisms $\mathcal{F}_U \cong \mathcal{G}_U$ of representations of Q over U. This functor is represented by an affine T-scheme constructed as follows. First, for each $i \in Q_0$, the functor of maps $\mathcal{F}_i \to \mathcal{G}_i$ of $\mathcal{O}_T$-modules is represented by the affine T-scheme

$$\mathbb{V}_i := \underline{\operatorname{Spec}}_{\mathcal{O}_T} \operatorname{Sym}^\bullet \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}_i, \mathcal{G}_i)^\vee.$$

Next, the functor parameterizing homomorphisms of representations $\mathcal{F} \to \mathcal{G}$ is represented by the closed subscheme

$$\mathbb{H}\mathrm{om}(\mathcal{F},\mathcal{G}) := \underline{\operatorname{Spec}}_{\mathcal{O}_T} \operatorname{Sym}^\bullet \mathcal{K} \subseteq \prod_{i\in Q_0} \mathbb{V}_i,$$

where $\mathcal{K} := \operatorname{coker}\phi^\vee$ is the cokernel of the morphism dual to the map

$$\phi : \bigoplus_{i\in Q_0} \mathcal{H}om(\mathcal{F}_i,\mathcal{G}_i) \to \bigoplus_{a\in Q_1} \mathcal{H}om(\mathcal{F}_{s(a)},\mathcal{G}_{t(a)})$$
$$(\alpha_i)_{i\in Q_0} \mapsto \left(\mathcal{G}_a \circ \alpha_{s(a)} - \alpha_{t(a)} \circ \mathcal{F}_a\right)_{a\in Q_1}$$

Finally, $\mathbb{I}\mathrm{som}(\mathcal{F},\mathcal{G})$ is represented by the base change of the morphism

$$\mathbb{H}\mathrm{om}(\mathcal{F},\mathcal{G}) \times_T \mathbb{H}\mathrm{om}(\mathcal{G},\mathcal{F}) \to \mathbb{H}\mathrm{om}(\mathcal{F},\mathcal{F}) \times_T \mathbb{H}\mathrm{om}(\mathcal{G},\mathcal{G}),\ (\phi,\psi) \mapsto (\psi\circ\phi, \phi\circ\psi)$$

along the section $\sigma : T \to \mathbb{H}\mathrm{om}(\mathcal{F},\mathcal{F}) \times_T \mathbb{H}\mathrm{om}(\mathcal{G},\mathcal{G})$ given by $(\mathrm{id}_{\mathcal{F}}, \mathrm{id}_{\mathcal{G}})$. As a section of a separated morphism, σ is a closed embedding, and so $\mathbb{I}\mathrm{som}(\mathcal{F},\mathcal{G})$ is a closed subscheme of an affine T-scheme, hence itself affine over T. $\square$

We will next construct a smooth atlas for $\mathcal{M}_{d,S}$. For each vertex $i \in Q_0$, let $V_i^d := \mathcal{O}_S^{\oplus d_i}$ denote the standard free $\mathcal{O}_S$-module of rank d_i. For each arrow $a \in Q_1$, we set

$$\mathbb{A}_{a,S} = \mathbb{A}_a := \underline{\operatorname{Spec}}_{\mathcal{O}_S} \operatorname{Sym}^\bullet_{\mathcal{O}_S} \left(\mathcal{H}om(V^d_{s(a)}, V^d_{t(a)})^\vee\right) \cong \mathbb{A}_S^{d_{s(a)}d_{t(a)}}.$$

Definition 3.1.3. The space of representations of Q of dimension vector d is the affine S-space

$$\mathrm{R}_{d,S} = \mathrm{R}_d := \prod_{a \in Q_1} \mathbb{A}_a.$$

Each affine space $\mathbb{A}_a$ parameterizes a universal linear map $\varphi_a : \mathcal{O}_{\mathbb{A}_a}^{\oplus s(a)} \to \mathcal{O}_{\mathbb{A}_a}^{\oplus t(a)}$. Letting $\pi_a : \mathrm{R}_d \to \mathbb{A}_a$ denote the projection, we obtain a canonical family of representations $\mathcal{F}^{\mathrm{can}}$ on R_d as follows. For each vertex $i \in Q_0$, we set $\mathcal{F}_i^{\mathrm{can}} = \mathcal{O}_{\mathrm{R}_d}^{\oplus d_i}$ and for each arrow $a \in Q_1$ we set

$$\mathcal{F}_a^{\mathrm{can}} = \pi_a^* \varphi_a : \mathcal{O}_{\mathrm{R}_d}^{\oplus d_{s(a)}} \to \mathcal{O}_{\mathrm{R}_d}^{\oplus d_{t(a)}}.$$

This induces a morphism $\varphi : \mathrm{R}_d \to \mathcal{M}_d$. We can view R_d as representing the functor on S-schemes that to an S-scheme T associates the set of representations $\mathcal{F}$ over T with $\mathcal{F}_i = \mathcal{O}_T^{\oplus d_i}$ for each $i \in Q_0$.

The S-group scheme

$$\mathrm{G}_{d,S} = \mathrm{G}_d := \prod_{i \in Q_0} \mathrm{GL}_{d_i} \tag{5}$$

acts on R_d as follows. The T-points of G_d are tuples $g = (g_i)_{i \in Q_0}$ where g_i is an automorphism of $\mathcal{O}_T^{\oplus d_i}$. If $T \to \mathrm{R}_d$ corresponds to a representation $\mathcal{F} = (\mathcal{F}_a : \mathcal{O}_T^{\oplus d_{s(a)}} \to \mathcal{O}_T^{\oplus d_{t(a)}})_{a \in A}$, then the action of g sends $\mathcal{F}$ to the representation $\mathcal{F}' = (\mathcal{F}'_a = g_{t(a)} \circ \mathcal{F}_a \circ g_{s(a)}^{-1})_{a \in A}$.

Proposition 3.1.4. *The morphism $\varphi : \mathrm{R}_d \to \mathcal{M}_d$ is schematic, smooth, and surjective, and induces an isomorphism of stacks*

$$[\mathrm{R}_d / G_d] \cong \mathcal{M}_d. \tag{6}$$

In particular, the stack $\mathcal{M}_d$ is smooth and of finite type over S.

Proof. By Proposition 3.1.2, the diagonal of $\mathcal{M}_d$ is affine, in particular representable by schemes, and this implies that φ is schematic.

Let T be an S-scheme and $T \to \mathcal{M}_d$ a morphism corresponding to a representation $\mathcal{G}$ over T. The fiber product $\mathcal{T} := \mathrm{R}_d \times_{\mathcal{M}_d} T$ is isomorphic to the functor that sends a morphism $g : U \to T$ of S-schemes to the set of isomorphisms $\phi = (\phi_i : \mathcal{O}_U^{\oplus d_i} \xrightarrow{\sim} g^* \mathcal{G}_i)_{i \in Q_0}$ of $\mathcal{O}_U$-modules. The group $\mathrm{G}_d(U)$ acts on $\mathcal{T}$ by sending ϕ to $g \cdot \phi = (\phi_i \circ g_i^{-1})_{i \in Q_0}$ for $g = (g_i)_{i \in Q_0} \in \mathrm{G}_d(U)$, making $\mathcal{T}$ into a G_d-torsor over T. This implies that ϕ is smooth and surjective. In particular, $\mathcal{M}_d$ is smooth and of finite type over S since R_d is.

The induced morphism $[\mathrm{R}_d / \mathrm{G}_d] \to \mathcal{M}_d$ is given as follows. For an S-scheme T, a map $T \to [\mathrm{R}_d / \mathrm{G}_d]$ corresponds to a G_d-torsor $\mathcal{T}$ over T and a G_d-equivariant morphism $\pi : \mathcal{T} \to \mathrm{R}_d$. This induces a representation $\mathcal{G}'$ on $\mathcal{T}$ given by $\mathcal{G}'_i = \mathcal{O}_{\mathcal{T}}^{\oplus d_i}$ and $\mathcal{G}'_a = \pi^* \mathcal{F}_a^{\mathrm{can}}$. From the equivariance, we deduce descent data for the sheaves $\mathcal{G}'_i$ and the maps $\mathcal{G}'_a$ on the fppf cover $\mathcal{T} \to T$, giving a representation $\mathcal{G}$ over T which determines a map $T \to \mathcal{M}_d$.

A quasi-inverse is given as follows. To an S-scheme T and a map $T \to \mathcal{M}_d$ corresponding to a representation $\mathcal{G}$, we assign the G_d-torsor $\mathcal{T} = \prod_{i \in Q_0} \mathbb{I}\mathrm{som}(\mathcal{O}_T^{\oplus d_i}, \mathcal{G}_i)$ over T. Each vector bundle $\mathcal{G}_i$ trivializes after pulling back to $\mathcal{T}$, so the pullback of $\mathcal{G}$ provides a G_d-equivariant morphism $\mathcal{T} \to \mathrm{R}_d$. □

We observe that since R_d is separated and G_d is affine over S, the above proposition gives a second proof that $\mathcal{M}_d$ has affine diagonal. We gave a direct argument in Proposition 3.1.2 to emphasize the moduli-theoretic point of view.

3.2. ***The moduli stack of semistable representations.*** For a stability function θ, a family $\mathcal{F}$ of representations over a scheme T is said to be a family of θ-semistable, respectively geometrically θ-stable, representations if for each point $x \in T$, the restriction $\mathcal{F}|_x$ is a θ-semistable, respectively geometrically θ-stable, over the residue field $\kappa(x)$. By Remark 2.3.1, this is equivalent to requiring that given a field k and a morphism $\operatorname{Spec} k \to T$ with image x, the k-representation $\mathcal{F}|_x \otimes_{\kappa(x)} k$ is θ-semistable, respectively geometrically θ-stable.

Definition 3.2.1. The substack $\mathcal{M}^{\theta\text{-ss}}_{d,S}$, respectively $\mathcal{M}^{\theta\text{-s}}_{d,S}$, of $\mathcal{M}_{d,S}$ is the full subcategory whose objects are those $(T, \mathcal{F}) \in \mathcal{M}_{d,S}$ for which $\mathcal{F}$ is a family of θ-semistable, respectively geometrically θ-stable, representations.

Since θ-semistability and geometric θ-stability are pointwise conditions on families of representations, it is clear that both moduli stacks are preserved under base change along morphisms $S' \to S$. If $\theta = 0$ is the trivial stability function, then $\mathcal{M}^{\theta\text{-ss}}_d = \mathcal{M}_d$ since every representation is θ-semistable. We will next see that $\mathcal{M}^{\theta\text{-ss}}_{d,S} \subseteq \mathcal{M}_{d,S}$ and $\mathcal{M}^{\theta\text{-s}}_{d,S} \subseteq \mathcal{M}_{d,S}$ are open substacks, or equivalently that in a family $\mathcal{F}$ of representations over T, the locus of points $x \in T$ such that $\mathcal{F}|_x$ is θ-semistable (respectively geometrically θ-stable) is open.

Toward this end, we first construct a relative version of a quiver Grassmannian as follows. For a family $\mathcal{F}$ of d-dimensional representations over T and a dimension vector $d' < d$, consider the functor $(\mathrm{Sch}/T)^{\mathrm{op}} \to (\mathrm{Sets})$ that sends $f : U \to T$ to the set of families of d'-dimensional subrepresentations $\mathcal{F}' \subseteq f^*\mathcal{F}$ with locally free quotient. We claim that this functor is representable by a projective T-scheme $\mathrm{Gr}(d', \mathcal{F}/T)$. Indeed, it is represented by a closed subscheme of the T-fiber product $\prod_{i \in Q_0} \mathrm{Gr}_i$ of the Grassmannian bundles $\mathrm{Gr}_i := \mathrm{Gr}(d'_i, \mathcal{F}_i) \to T$ parameterizing rank d'_i locally free subsheaves of $\mathcal{F}_i$ with locally free quotients. More precisely,

$$\mathrm{Gr} := \mathrm{Gr}(d', \mathcal{F}/T) \subset \prod_{i \in Q_0} \mathrm{Gr}_i$$

is the scheme-theoretic intersection over all $a \in Q_1$ of the scheme-theoretic vanishing loci of the compositions

$$p^*_{s(a)} \mathcal{S}_{s(a)} \xrightarrow{p^*_{s(a)} \iota_{s(a)}} p^* \mathcal{F}_{s(a)} \xrightarrow{p^* \mathcal{F}_a} p^* \mathcal{F}_{t(a)} \xrightarrow{p^*_{t(a)} \eta_{t(a)}} p^*_{t(a)} \mathcal{Q}_{t(a)},$$

where $0 \to \mathcal{S}_i \xrightarrow{\iota_i} \pi_i^* \mathcal{F} \xrightarrow{\eta_i} \mathcal{Q}_i \to 0$ denotes the universal exact sequence on Gr_i and $p_i : \mathrm{Gr} \to \mathrm{Gr}_i$ and $p : \mathrm{Gr} \to T$ are the natural projections.

Lemma 3.2.2. *In a family of quiver representations $\mathcal{F}$ of dimension vector d over a scheme T, the set of θ-semistable, respectively geometrically θ-stable, representations is open.*

Proof. For each nonzero dimension vector $d' < d$, let $\pi_{d'} : \mathrm{Gr}(d', \mathcal{F}/T) \to T$ denote the relative quiver Grassmannian of d'-dimensional subrepresentations of $\mathcal{F}$. Since $\mathrm{Gr}(d', \mathcal{F}/T)$ is proper over T, the image

$T_{d'} \subseteq T$ of $\pi_{d'}$ is closed. Consider the two subsets

$$Z_1 = \bigcup_{\substack{0<d'<d:\\ \theta(d')\geq\theta(d)}} T_{d'} \quad \text{and} \quad Z_2 = \bigcup_{\substack{0<d'<d:\\ \theta(d')>\theta(d)}} T_{d'}.$$

As there are only finitely many dimension vectors d' with $d' < d$, the subsets Z_1 and Z_2 are closed in T, and consequently their complements $U_1 = T \setminus Z_1$ and $U_2 = T \setminus Z_2$ are open. We claim that U_1 and U_2 are the loci of geometrically θ-stable and θ-semistable representations parameterized by $\mathcal{F}$ respectively. Let $x \in T$ be a point. If $x \in Z_1$, then $x = \pi_{d'}(y)$ for some $d' < d$ with $\theta(d') \geq \theta(d)$ and some $y \in \mathrm{Gr}(d', \mathcal{F}/T)$, meaning that the representation $\mathcal{F}|_x \otimes_{\kappa(x)} \kappa(y)$ has a subrepresentation of dimension vector d'. Thus, $\mathcal{F}|_x$ is not geometrically θ-stable.

Conversely, if $\mathcal{F}|_x$ is not geometrically θ-stable, then for some field extension $k/\kappa(x)$, the representation $\mathcal{F}|_x \otimes_{\kappa(x)} k$ has a subrepresentation of dimension vector d' with $\theta(d') \geq \theta(d)$, which gives a map $\operatorname{Spec} k \to \mathrm{Gr}(d', \mathcal{F}/T)$ whose composite with $\pi_{d'}$ has image x; thus $x \in Z_1$.

The argument for Z_2 is exactly the same, except by Remark 2.3.1 the question of whether $\mathcal{F}|_x$ is θ-semistable is insensitive to extending the residue field $\kappa(x)$. □

Remark 3.2.3. The following example shows that θ-stability is *not* an open condition. Consider the Jordan quiver from Example 2.3.2 and the family of 2-dimensional representations over $\mathbb{R}$ parameterized by $\mathbb{A}^1_{\mathbb{R}}$ with coordinate t given by the matrix $M_t := \left(\begin{smallmatrix} t & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For the trivial stability function $\theta = 0$, all representations are semistable and $M_t \otimes_{\mathbb{R}} \mathbb{C}$ is polystable unless $t = \pm 2$, but over $\mathbb{R}$ a point M_t is stable if and only if M_t has no real eigenvalues i.e. $|t| < 2$; this set is not even constructible, let alone Zariski-open.

The next result directly follows from Lemma 3.2.2.

Corollary 3.2.4. *For a dimension vector d and a stability function θ, the moduli stacks*

$$\mathcal{M}^{\theta\text{-s}}_{d,S} \subseteq \mathcal{M}^{\theta\text{-ss}}_{d,S} \subseteq \mathcal{M}_{d,S} \tag{7}$$

are open substacks of $\mathcal{M}_{d,S}$. In particular, both $\mathcal{M}^{\theta\text{-s}}_{d,S}$ and $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ are smooth and of finite type and have affine diagonal over S.

We can express these moduli stacks as quotient stacks: the open subschemes

$$\mathrm{R}^{\theta\text{-s}}_{d,S} \subseteq \mathrm{R}^{\theta\text{-ss}}_{d,S} \subseteq \mathrm{R}_{d,S},$$

parameterizing geometrically θ-stable and θ-semistable representations are subschemes invariant under the action of G_d, and we obtain identifications

$$\mathcal{M}^{\theta\text{-s}}_{d,S} \cong [\mathrm{R}^{\theta\text{-s}}_{d,S}/\mathrm{G}_d], \quad \mathcal{M}^{\theta\text{-ss}}_{d,S} \cong [\mathrm{R}^{\theta\text{-ss}}_{d,S}/\mathrm{G}_d]$$

for the open substacks in Corollary 3.2.4, similar to (6) in Proposition 3.1.4. This perspective, i.e., the stacky perspective on GIT quotients, is the key to studying cohomology vanishing using Teleman quantization in [7].

Remark 3.2.5. One interesting way in which the moduli theory of quiver representations differs from that of vector bundles on a curve of genus $g \geq 2$ is that the analogous moduli stacks for stable vector bundles are always non-empty. For any integers $r \geq 1$ and d there always exists a *stable* vector bundle of rank r and degree d [38, Lemma 4.3], making the analogue of (7) an inclusion of dense open substacks. However, the following standard example shows that whether or not $\mathcal{M}_{d,S}^{\theta\text{-s}}$ or $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ is empty may depend on θ.

Example 3.2.6. Consider a representation of the A_2-quiver $\bullet \xrightarrow{a} \bullet$ over a field k of dimension vector $d_n = (n, n)$ for a positive integer n. We can reduce the study of stability functions θ such that $\theta(d_n) = 0$ to three cases:

$\underline{\theta = (0, 0)}$: all representations of dimension vector d_n are semistable, but none of them can be geometrically stable, as the unique subrepresentation of dimension vector $(0, n)$ has slope 0.

$\underline{\theta = (-1, 1)}$: no representation of dimension vector d_n can be geometrically stable and even semistable, as the unique subrepresentation of dimension vector $(0, n)$ is a destabilizing subrepresentation.

$\underline{\theta = (1, -1)}$: a representation $M_a : k^n \to k^n$ is semistable if and only if M_a is injective, while it is geometrically stable if and only if $n = 1$ and M_a is injective.

For an acyclic quiver, a general criterion for the existence of a stable representation is given in [1].

3.3. *Determinantal line bundles.* In this section, we explain how to construct line bundles $\mathcal{L}_\theta$ on the stack $\mathcal{M}_d$ depending on a stability function $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$. For an S-scheme T and a family $\mathcal{F}$ of representations of Q over T, we define a line bundle on T

$$\mathcal{L}_{\theta,\mathcal{F}} := \bigotimes_{i \in Q_0} \det(\mathcal{F}_i)^{\otimes -\theta_i},$$

where $\theta_i = \theta(e_i)$ for $i \in Q_0$, with e_i being the basis vector corresponding to the simple $S(i)$ as defined in Section 2.1. We list some basic properties of this construction.

Proposition 3.3.1. (i) *The assignment $\theta \mapsto \mathcal{L}_{\theta,\mathcal{F}}$ defines a homomorphism* $\mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z}) \to \mathrm{Pic}(T)$.

(ii) *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of families of representations on T, then there is a canonical isomorphism $\mathcal{L}_{\theta,\mathcal{F}'} \otimes_{\mathcal{O}_T} \mathcal{L}_{\theta,\mathcal{F}''} \cong \mathcal{L}_{\theta,\mathcal{F}}$.*

(iii) *If $f : T' \to T$ is a morphism of S-schemes, then there is a canonical isomorphism $f^*\mathcal{L}_{\theta,\mathcal{F}} \cong \mathcal{L}_{\theta,f^*\mathcal{F}}$.*

Proof. Part (i) is clear from the definition. For part (ii), see for example [47, Tag 0FJB], and for part (iii), see [47, Tag 0FJY]. □

For every scheme T and morphism $T \to \mathcal{M}_d$ (corresponding to a family $\mathcal{F}$ over T as above), we have constructed a line bundle $\mathcal{L}_{\theta,\mathcal{F}}$ on T and as these line bundles are compatible with morphisms $T' \to T$ as in (iii) above, we obtain a line bundle $\mathcal{L}_\theta$ on $\mathcal{M}_d$. Equivalently, we can define $\mathcal{L}_\theta$ by applying the above construction to the universal representation $\mathcal{F}^{\mathrm{univ}}$ over $\mathcal{M}_d$. Moreover, this construction $\theta \mapsto \mathcal{L}_\theta$ defines a group homomorphism $\mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z}) \to \mathrm{Pic}(\mathcal{M}_d)$, which is analogous to the one constructed in [30, Definition 8.1.1].

Proposition 3.3.2. *The map* $\mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z}) \to \mathrm{Pic}(\mathcal{M}_{d,\mathbb{Z}})$ *given by* $\theta \mapsto \mathcal{L}_\theta$ *is an isomorphism.*

Proof. By Proposition 3.1.4, we have $\mathcal{M}_d \cong [\mathrm{R}_d/\mathrm{G}_d]$, so the Picard group of $\mathcal{M}_d$ is the group $\mathrm{Pic}^{\mathrm{G}_d}(\mathrm{R}_d)$ of G_d-equivariant line bundles on R_d (see Definition 3.1.3 for the definition of R_d and the group G_d). Since R_d is an affine space, its Picard group is trivial and a G_d-equivariant structure on $\mathcal{O}_{\mathrm{R}_d}$ is equivalent to a character of G_d. Since the character group of GL_n is $\mathbb{Z}$ (generated by the determinant), we see G_d has character group $\mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z})$. Hence the above map corresponds to the isomorphisms

$$\mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z}) \cong X^*(\mathrm{G}_d) \cong \mathrm{Pic}^{\mathrm{G}_d}(\mathrm{R}_d) \cong \mathrm{Pic}(\mathcal{M}_d),$$

where $X^*(\mathrm{G}_d)$ denotes the character group of G_d. □

Suppose now that we are given two families of representations $\mathcal{F}$ and $\mathcal{G}$ on an S-scheme T. We can construct a 2-term complex of locally free sheaves concentrated in degrees -1 and 0 by setting

$$\mathcal{E}^\bullet_{\mathcal{F},\mathcal{G}} : \bigoplus_{i\in Q_0} \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}_i, \mathcal{G}_i) \xrightarrow{d_{\mathcal{F},\mathcal{G}}} \bigoplus_{a\in Q_1} \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}_{s(a)}, \mathcal{G}_{t(a)}), \tag{8}$$

where the differential of degree one is given by

$$d_{\mathcal{F},\mathcal{G}} : (\phi_i)_{i\in Q_0} \mapsto (\phi_{t(a)} \circ \mathcal{F}_a - \mathcal{G}_a \circ \phi_{s(a)})_{a\in Q_1}.$$

As a first application of (8), we see that the dimensions

$$\dim_{\kappa(x)} \mathrm{Hom}(\mathcal{F}|_x, \mathcal{G}|_x) \quad \text{and} \quad \dim_{\kappa(x)} \mathrm{Ext}(\mathcal{F}|_x, \mathcal{G}|_x)$$

are upper semicontinuous functions on T — by comparing with (3), we see that the latter vector space is nothing but the fiber of the quasi-coherent sheaf $\mathrm{coker}(d_{\mathcal{F},\mathcal{G}})$, while the former is dual to the fiber of $\mathrm{coker}(d^\vee_{\mathcal{F},\mathcal{G}})$, both of which are finitely presented.

Our second application of (8) is a moduli-theoretic construction of determinantal semi-invariants mentioned in the introduction. If the two families $\mathcal{F}$ and $\mathcal{G}$ satisfy the condition $\langle \underline{\dim}\,\mathcal{F}, \underline{\dim}\,\mathcal{G}\rangle = 0$, then

$$\mathrm{rk}(\mathcal{E}^{-1}_{\mathcal{F},\mathcal{G}}) = \sum_{i\in Q_0} \mathrm{rk}(\mathcal{F}_i)\,\mathrm{rk}(\mathcal{G}_i) = \sum_{a\in Q_1} \mathrm{rk}(\mathcal{F}_{s(a)})\,\mathrm{rk}(\mathcal{G}_{t(a)}) = \mathrm{rk}(\mathcal{E}^0_{\mathcal{F},\mathcal{G}}),$$

and thus we obtain a global section of the determinant of $\mathcal{E}^\bullet_{\mathcal{F},\mathcal{G}}$:

$$\sigma_{\mathcal{F},\mathcal{G}} = \det(d_{\mathcal{F},\mathcal{G}}) : \mathcal{O}_T \to \det(\mathcal{E}^\bullet_{\mathcal{F},\mathcal{G}}) = \det(\mathcal{E}^{-1}_{\mathcal{F},\mathcal{G}})^\vee \otimes \det(\mathcal{E}^0_{\mathcal{F},\mathcal{G}}).$$

Suppose that the stability function $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is of the form $\eta_\beta = -\langle \_, \beta\rangle$ for a dimension vector $\beta \in \mathbb{N}^{Q_0}$; that is, we have

$$\theta_i = -\langle \underline{\dim}\, S(i), \beta\rangle = -\beta_i + \sum_{a\in Q_1 : s(a)=i} \beta_{t(a)}.$$

By applying the construction of the 2-term complex to the universal family $\mathcal{F} := \mathcal{F}^{\mathrm{univ}}$ on $\mathcal{M}_d$ and a family $\mathcal{G}$ of representations on $\mathcal{M}_d$ of dimension vector $m\beta$ for $m \in \mathbb{N}$ such that $\mathcal{G}_i$ is *free* for each $i \in Q_0$, we get an associated determinantal line bundle that is isomorphic to $\mathcal{L}_\theta^{\otimes m}$, as we have

$$\mathcal{H}om_{\mathcal{O}_{\mathcal{M}_d}}(\mathcal{F}_i, \mathcal{G}_i) \cong \mathcal{F}_i^{\oplus - m\beta_i}, \quad \mathcal{H}om_{\mathcal{O}_{\mathcal{M}_d}}(\mathcal{F}_{s(a)}, \mathcal{G}_{t(a)}) \cong \mathcal{F}_{s(a)}^{\oplus - m\beta_{t(a)}},$$

and so

$$\det(\mathcal{E}^\bullet_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}}) := \det(\mathcal{E}^{-1}_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}})^\vee \otimes \det(\mathcal{E}^0_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}}) \cong \bigotimes_{i \in Q_0} \det(\mathcal{F}_i)^{\otimes - m\theta_i} = \mathcal{L}_\theta^{\otimes m}.$$

If moreover $\theta(d) = -\langle d, \beta\rangle = 0$ then the ranks of the two terms in the complex $\mathcal{E}^\bullet_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}}$ are equal, so the determinant of the differential $d_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}}$ gives a section

$$\sigma_{\mathcal{G}} = \sigma_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}} := \det(d_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}}) : \mathcal{O}_{\mathcal{M}_d} \to \mathcal{L}_\theta^{\otimes m}.$$

The most important case is when $\mathcal{G} = V \otimes \mathcal{O}_{\mathcal{M}_d}$ is a "constant family", that is, the pullback of a family V on S, in which case we denote the section by

$$\sigma_V : \mathcal{O}_{\mathcal{M}_d} \to \mathcal{L}_\theta^{\otimes m}.$$

The following result, corresponding to [43, Theorem 1.1], describes the vanishing locus of σ_V.

Proposition 3.3.3. *Let d be a dimension vector and θ a stability function with $\theta(d) = 0$. Let k be a field and $x : \operatorname{Spec} k \to \mathcal{M}_d^{\theta\text{-ss}}$ a morphism corresponding to a representation M over k, and let $y \in S$ denote its image in S.*

(a) *If $\theta = \theta_\alpha$ for a dimension vector α and V is a family of representation of dimension vector $m\alpha$ on S, then the section σ_V of $\mathcal{L}_\theta^{\otimes m}$ is nonzero at x if and only if*

$$\mathrm{Hom}_k(V|_y \otimes_{\kappa(y)} k, M) = 0, \quad \textit{or equivalently } \mathrm{Ext}(V|_y \otimes_{\kappa(y)} k, M) = 0.$$

(b) *If $\theta = \eta_\beta$ for a dimension vector β and V is a family of representations of dimension vector $m\beta$ on S, then the section σ_V of $\mathcal{L}_\theta^{\otimes m}$ is nonzero at x if and only if*

$$\mathrm{Hom}(M, V|_y \otimes_{\kappa(y)} k) = 0, \quad \textit{or equivalently } \mathrm{Ext}(M, V|_y \otimes_{\kappa(y)} k) = 0.$$

Proof. To prove (b), we note that the fiber of the complex $\mathcal{E}^\bullet_{\mathcal{F}^{\mathrm{univ}},\mathcal{G}}$ at x is identified with the complex

$$\bigoplus_{i \in Q_0} \mathrm{Hom}_k(M_i, V_i|_y \otimes_{\kappa(y)} k) \longrightarrow \bigoplus_{a \in Q_1} \mathrm{Hom}_k(M_{s(a)}, V_{t(a)}|_y \otimes_{\kappa(y)} k) ,$$

defined in (3) whose kernel and cokernel are the k-vector spaces $\mathrm{Hom}(M, V)$ and $\mathrm{Ext}(M, V)$ respectively, and these vanish if and only if the differential of the complex is invertible, if and only if its determinant is nonzero.

For part (a) we instead consider the complex $\mathcal{E}^\bullet_{\mathcal{G},\mathcal{F}^{\mathrm{univ}}}$ and proceed similarly. $\square$

4. Vanishing results

In this section we take our base scheme to be $S = \operatorname{Spec} k$ where $k = \overline{k}$ is an algebraically closed field. The results obtained here will be used in later sections to construct adequate moduli spaces for stacks of semistable representations and to show that they are projective when the quiver is acyclic.

The results in this section are inspired by the proof of global generation of determinantal line bundles on the moduli of vector bundles on a curve using dimension-counting techniques in [20]. We will obtain parallel results in the case of moduli of quiver representations. The starting point is Proposition 3.3.3, which describes the non-vanishing locus of determinantal semi-invariants. In Section 4.1, for a given semistable representation M over k, we will find another representation N satisfying $\operatorname{Hom}(M, N) = \operatorname{Ext}(M, N) = 0$ by showing that in the appropriate representation space, the locus of those N for which $\operatorname{Hom}(M, N) \neq 0$ has positive codimension; this will enable us to show a power of the determinantal line bundle on the moduli space of semistable representations is globally generated in Section 6. For acyclic quivers, we study the preservation of (semi)stability under the Auslander–Reiten translations in Section 4.3 and use this to establish generic vanishing of Ext groups in Section 4.4, in order to ultimately prove a key result in Section 4.5 required to later show ampleness of the determinantal line bundle.

Throughout this section, we will formulate many results for stability functions of the form (a) $\theta = \theta_\alpha$ and (b) $\theta = \eta_\beta$, where α and β are dimension vectors. This condition is automatically satisfied if Q is acyclic and there exists a semistable representation supported on Q_0; see Lemma 2.3.5. We only give the proof in one of these two cases, as the other is proved analogously. Since we will ultimately be concerned with case (b) $\theta = \eta_\beta$ in Section 6, we mostly apply these results in this case and thus give the proofs in this case.

4.1. ***Characterizing semistable representations.*** We begin with the key dimension estimates. Consider $d', d'' \in \mathbb{N}^{Q_0}$ and write $d = d' + d''$. Let M and N be representations of dimension vector d' and d'' respectively. Define the following subsets of R_d:

$$B_{d'} = B_{d',d} = \{V \in \mathrm{R}_d \mid \text{there exists } V' \subset V \text{ with } \underline{\dim}\, V' = d'\}, \tag{9}$$

$$K_M = K_{M,d} = \{V \in \mathrm{R}_d \mid \text{there exists an injection } M \hookrightarrow V\}, \tag{10}$$

$$Q_N = Q_{N,d} = \{V \in \mathrm{R}_d \mid \text{there exists a surjection } V \twoheadrightarrow N\}. \tag{11}$$

Lemma 4.1.1. *We have the following estimates*:

(i) $\operatorname{codim}_{\mathrm{R}_d} B_{d'} \geq -\langle d', d''\rangle$.

(ii) $\operatorname{codim}_{\mathrm{R}_d} K_M \geq 1 - \langle d', d\rangle$.

(iii) $\operatorname{codim}_{\mathrm{R}_d} Q_N \geq 1 - \langle d, d''\rangle$.

Proof. View R_d as parameterizing representations V where $V_i = k^{\oplus d_i}$ for each i. Let $V'_i \subseteq V_i$ be the subspace spanned by the d'_i first standard basis vectors and define the subset of all representations for which V' is a subrepresentation:

$$S := \{V \in \mathrm{R}_d \mid V_a(V'_{s(a)}) \subseteq V'_{t(a)} \text{ for all } a \in Q_1\}.$$

A representation $V \in \mathrm{R}_d$ has a subrepresentation of dimension vector d' if and only if it lies in the G_d-saturation of S, that is,

$$B_{d'} = \mathrm{G}_d \cdot S.$$

Consider the parabolic subgroup $P \subseteq \mathrm{G}_d$ given by

$$P := \{g \in \mathrm{G}_d \mid g_i(V_i') \subseteq V_i' \text{ for all } i \in Q_0\}.$$

The subgroup P acts on S, which implies that the action map $\mathrm{G}_d \times S \to \mathrm{R}_d$ factors as

$$\begin{tikzcd} \mathrm{G}_d \times S \arrow[d] \arrow[dr] & \\ \mathrm{G}_d \times^P S \arrow[r, dashed, "\exists!"] & \mathrm{R}_d \end{tikzcd}$$

where $\mathrm{G}_d \times^P S$ is the associated fiber bundle [46, Proposition 4]. Thus, from the surjection $\mathrm{G}_d \times^P S \to B_{d'}$ we obtain the bound

$$\dim B_{d'} \leq \dim \left(\mathrm{G}_d \times^P S\right) = \dim \mathrm{G}_d + \dim S - \dim P$$

and hence

$$\operatorname{codim}_{\mathrm{R}_d} B_{d'} \geq \dim \mathrm{R}_d - \dim S - (\dim \mathrm{G}_d - \dim P) = \operatorname{codim}_{\mathrm{R}_d} S - \operatorname{codim}_{\mathrm{G}_d} P.$$

Now S is a linear subspace of R_d of codimension $\sum_{a \in Q_1} d'_{s(a)} d''_{t(a)}$ and P is a subgroup of G_d of codimension $\sum_{i \in Q_0} d'_i d''_i$, so we have

$$\operatorname{codim}_{\mathrm{R}_d} B_{d'} \geq \sum_{a \in Q_1} d'_{s(a)} d''_{t(a)} - \sum_{i \in Q_0} d'_i d''_i = -\langle d', d'' \rangle,$$

proving (i).

We have projection maps $p : S \to \mathrm{R}_{d'}$ and $q : S \to \mathrm{R}_{d''}$ taking a representation V to the subrepresentation V' and the quotient $V'' = V/V'$ respectively. Identifying M and N with points in $\mathrm{R}_{d'}$ and $\mathrm{R}_{d''}$ and letting $O_M \subset \mathrm{R}_{d'}$ and $O_N \subset \mathrm{R}_{d''}$ denote their orbits under the actions of $G_{d'}$ and $G_{d''}$ respectively, we see that

$$K_M = \mathrm{G}_d \cdot p^{-1}(O_M), \quad Q_N = \mathrm{G}_d \cdot q^{-1}(O_N).$$

To prove (ii) and (iii), we note the group P acts on $\mathrm{R}_{d'}$ and $\mathrm{R}_{d''}$ via its natural projections to $\mathrm{G}_{d'}$ and $\mathrm{G}_{d''}$, and that the projections p and q are P-equivariant under these actions. Thus, P acts on the preimages $p^{-1}(O_M)$ and $q^{-1}(O_N)$, so we have surjections

$$\mathrm{G}_d \times^P p^{-1}(O_M) \to K_M, \quad \mathrm{G}_d \times^P q^{-1}(O_N) \to Q_N,$$

from which we obtain the bounds

$$\operatorname{codim}_{\mathrm{R}_d} K_M \geq \dim \mathrm{R}_d - \dim p^{-1}(O_M) - \operatorname{codim}_{\mathrm{G}_d} P$$

and

$$\operatorname{codim}_{\mathrm{R}_d} Q_N \geq \dim \mathrm{R}_d - \dim q^{-1}(O_N) - \operatorname{codim}_{\mathrm{G}_d} P.$$

Now the map p is a projection along a linear subspace of dimension $\sum_{a\in Q_1} d''_{s(a)}d_{t(a)}$ so

$$\dim p^{-1}(O_M) = \dim O_M + \sum_{a\in Q_1} d''_{s(a)}d_{t(a)}.$$

On the other hand,

$$\dim O_M = \dim \mathrm{G}_{d'} - \dim \mathrm{Stab}_{\mathrm{G}_{d'}}(M) \leq \dim \mathrm{G}_{d'} - 1 = \sum_{i\in Q_0} (d'_i)^2 - 1,$$

as the stabilizer of any representation contains the multiplicative group $\mathbb{G}_{\mathrm{m}}$. Thus,

$$\begin{aligned}\mathrm{codim}_{\mathrm{R}_d} K_M &\geq \dim \mathrm{R}_d - \dim \mathrm{G}_{d'} + 1 - \mathrm{codim}_{\mathrm{G}_d} P\\ &= \sum_{a\in Q_1} d_{s(a)}d_{t(a)} - \sum_{i\in Q_0}(d'_i)^2 + 1 - \sum_{a\in Q_1} d''_{s(a)}d_{t(a)} - \sum_{i\in Q_0} d'_i d''_i\\ &= 1 + \sum_{a\in Q_1} d'_{s(a)}d_{t(a)} - \sum_{i\in Q_0} d'_i d_i\\ &= 1 - \langle d', d\rangle,\end{aligned}$$

proving (ii).

Similarly, since q is a projection along a linear subspace of dimension $\sum_{a\in Q_1} d_{s(a)}d'_{t(a)}$ and we have $\dim O_N \leq \dim \mathrm{G}_{d''} - 1$, the corresponding calculation yields (iii). □

As in [44] we say that a property of representations holds for a *general* representation of dimension vector d if there exists a nonempty open substack $\mathcal{U}$ of $\mathcal{M}_d$ such that the property holds for all M in $\mathcal{U}$. This is equivalent to giving a nonempty G_d-invariant open subscheme of R_d. Note that since $\mathcal{M}_d$ is irreducible, any nonempty open substack is dense, and in particular any finitely many general properties will hold simultaneously on a dense open substack.

Let $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be a stability function. We will use Lemma 4.1.1 to relate various vanishing results with θ-semistability for representations.

Lemma 4.1.2. *Let M be a θ-semistable representation and let $\epsilon \in \mathbb{N}^{Q_0}$.*

(a) *Suppose $\theta = \theta_\alpha$ for a dimension vector α. If m is sufficiently large, then for a general representation V of dimension vector $m\alpha + \epsilon$, every nonzero map $f : V \to M$ satisfies $\theta(\mathrm{im}\, f) = 0$. In fact, it suffices to take*

$$m > \frac{\langle \gamma, \gamma\rangle - \langle \epsilon, \gamma\rangle}{\langle \alpha, \gamma\rangle} \tag{12}$$

for the finitely many dimension vectors $0 < \gamma < \underline{\dim}\, M$ such that $\langle \alpha, \gamma\rangle < 0$.

(b) *Suppose $\theta = \eta_\beta$ for a dimension vector β. If m is sufficiently large, then for a general representation V of dimension vector $m\beta + \epsilon$, every nonzero map $f : M \to V$ satisfies $\theta(\mathrm{im}\, f) = 0$. In fact, it suffices to take*

$$m > \frac{\langle \gamma, \gamma\rangle - \langle \gamma, \epsilon\rangle}{\langle \gamma, \beta\rangle} \tag{13}$$

for the finitely many dimension vectors $0 < \gamma < \underline{\dim}\, M$ such that $\langle \gamma, \beta\rangle < 0$.

Proof. We only prove (b), as (a) is dual. Since M is θ-semistable, if $f : M \to V$ is any nonzero map, then $\theta(\operatorname{im} f) \geq 0$, so it suffices to rule out the case $\theta(\operatorname{im} f) > 0$.

Let $0 < \gamma < \underline{\dim}\, M$ be the dimension vector of a quotient representation of M such that $\theta(\gamma) = -\langle \gamma, \beta \rangle > 0$. The subset

$$B := \{V \in \mathrm{R}_{m\beta+\epsilon} \mid \text{there exists } f \in \operatorname{Hom}(M, V) \text{ such that } \underline{\dim} \operatorname{im} f = \gamma\}$$

is contained in the set $B_{\gamma,m\beta+\epsilon}$ defined in (9) so from Lemma 4.1.1 we deduce that

$$\operatorname{codim}_{\mathrm{R}_{m\beta+\epsilon}} B \geq \operatorname{codim}_{\mathrm{R}_{m\beta+\epsilon}} B_{\gamma,m\beta+\epsilon} \geq -\langle \gamma, m\beta + \epsilon - \gamma \rangle = -m\langle \gamma, \beta \rangle - \langle \gamma, \epsilon \rangle + \langle \gamma, \gamma \rangle.$$

If m satisfies the inequality in (13), we see that $\operatorname{codim}_{\mathrm{R}_{m\beta+\epsilon}} B > 0$, so for a general representation V of dimension vector $m\beta + \epsilon$, there are no maps $f : M \to V$ with $\underline{\dim} \operatorname{im} f = \gamma$. □

Using the above lemma, we obtain Hom-vanishing conditions for stable and semistable representations with respect to θ_α and η_β.

Corollary 4.1.3. *Let M be a θ-stable representation of dimension d and let $\epsilon \in \mathbb{N}^{Q_0}$.*

(a) *Suppose $\theta = \theta_\alpha$ for some $\alpha \in \mathbb{N}^{Q_0}$ and assume that $\langle \epsilon, d \rangle \leq 0$. If m satisfies* (12), *then* $\operatorname{Hom}(V, M) = 0$ *for a general representation V of dimension vector $m\alpha + \epsilon$.*

(b) *Suppose $\theta = \eta_\beta$ for some $\beta \in \mathbb{N}^{Q_0}$ and assume that $\langle d, \epsilon \rangle \leq 0$. If m satisfies* (13), *then* $\operatorname{Hom}(M, V) = 0$ *for a general representation V of dimension vector $m\beta + \epsilon$.*

Proof. We prove (b). By Lemma 4.1.2 we have $\theta(\operatorname{im} f) = 0$ for a general representation V of dimension vector $m\beta + \epsilon$ and any nonzero map $f : M \to V$, so since M is θ-stable, any such nonzero map is injective. However, by Lemma 4.1.1(ii), the locus $K_M \subseteq \mathrm{R}_{m\beta+\epsilon}$ defined in (10) of representations V for which there exists an injection $M \hookrightarrow V$ has codimension

$$\operatorname{codim}_{\mathrm{R}_{m\beta+\epsilon}} K_M \geq 1 - \langle d, m\beta + \epsilon \rangle = 1 - \langle d, \epsilon \rangle \geq 1$$

since by assumption $\langle d, \beta \rangle = 0$ and $\langle d, \epsilon \rangle \leq 0$. Thus, a general representation V does not admit such an injection.

The proof of (a) is dual and uses Q_M as defined in (11) in place of K_M. □

Corollary 4.1.4. *Let M be a θ-semistable representation and let $\epsilon \in \mathbb{N}^{Q_0}$.*

(a) *Suppose that $\theta = \theta_\alpha$ for a dimension vector α and assume that $\langle \epsilon, \gamma \rangle \leq 0$ for the dimension vectors γ of all θ-stable subquotients of M. If m satisfies* (12), *then for a general representation V of dimension vector $m\alpha + \epsilon$, we have* $\operatorname{Hom}(V, M) = 0$.

(b) *Suppose that $\theta = \eta_\beta$ for a dimension vector β and assume that $\langle \gamma, \epsilon \rangle \leq 0$ for the dimension vectors γ of all θ-stable subquotients of M. If m satisfies* (13), *then for a general representation V of dimension vector $m\beta + \epsilon$, we have* $\operatorname{Hom}(M, V) = 0$.

Proof. We show the (b), as (a) is analogous. Let $M^1, \ldots, M^r$ denote the θ-stable subquotients of a Jordan–Hölder filtration of M. By assumption $\langle \underline{\dim}\, M^\ell, \epsilon \rangle \leq 0$, so by Corollary 4.1.3, a general representation V

of dimension vector $m\beta + \epsilon$ satisfies

$$\mathrm{Hom}(M^{\ell}, V) = 0$$

for each ℓ. By breaking up the Jordan–Hölder filtration of M into short exact sequences, we inductively deduce that $\mathrm{Hom}(M, V)$ vanishes for a general V of dimension vector $m\beta + \epsilon$. □

Corollary 4.1.4 can be used to derive a characterization of semistability in terms of vanishing of Hom and Ext. Note that for representations M and N such that $\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = 0$, we have $\mathrm{Hom}(M, N) = 0$ if and only if $\mathrm{Ext}(M, N) = 0$.

Proposition 4.1.5. *Let θ be a stability function and let M be a representation with $\theta(M) = 0$.*

(a) *If $\theta = \theta_\alpha$ for a dimension vector α, then M is θ-semistable if and only if there exists $m > 0$ and a representation V of dimension vector $m\alpha$ such that* $\mathrm{Hom}(V, M) = 0$.

(b) *If $\theta = \eta_\beta$ for a dimension vector β, then M is θ-semistable if and only if there exists $m > 0$ and a representation V of dimension vector $m\beta$ such that* $\mathrm{Hom}(M, V) = 0$.

Proof. We will prove (b), as the proof of (a) is dual. The forward implication of (b) is a special case of Corollary 4.1.4 with $\epsilon = 0$. For the other direction, let $M' \subseteq M$ be a subrepresentation of dimension vector d'. After applying $\mathrm{Hom}(\_, V)$ to the short exact sequence

$$0 \to M' \to M \to M/M' \to 0,$$

we get $\mathrm{Ext}(M', V) = 0$, and therefore

$$\begin{aligned} m\eta_\beta(\underline{\dim}\, M') &= -\langle \underline{\dim}\, M', m\beta\rangle = -\dim \mathrm{Hom}(M', V) + \dim \mathrm{Ext}(M', V) \\ &= -\dim \mathrm{Hom}(M', V) \le 0. \end{aligned}$$

This shows that M is η_β-semistable. □

Remark 4.1.6. Proposition 4.1.5 appears as [45, Corollary 1.1] in characteristic 0, using possibly infinite-dimensional representations when Q is not acyclic, and with a proof using GIT methods. Our result stays completely within the realm of finite-dimensional representations and holds in all characteristics. In arbitrary characteristic the forward implication is proved in [11, Corollary 2].

In addition, Proposition 4.1.5 is the quiver analogue of Faltings's characterization of semistability for vector bundles, and more generally Higgs bundles, on a curve [21, Theorem I.2].

4.2. ***Effective bounds for vanishing of* Hom.** We use the inequalities of Lemma 4.1.2 to derive an upper bound for the vanishing of Hom and Ext that only depends on the underlying undirected graph of Q. This will turn into an upper bound for global generation of determinantal line bundles in Section 6.

Recall from Section 2.1 that we denote the Euler matrix of Q by A for a choice of an ordering of the vertices. Let $B = \frac{1}{2}\left(A + A^{\mathrm{T}}\right)$ be the symmetrization of A. These matrices A and B define the same quadratic form, called the *Tits form*, that for any vector $x \in \mathbb{Z}^{Q_0}$ associates its self-pairing:

$$\langle x, x\rangle = x^{\mathrm{T}} A x = x^{\mathrm{T}} B x.$$

Notice that the Tits form is independent of the orientation of the quiver. In addition to the Euler pairing, we will also use the standard inner product on $\mathbb{Z}^{Q_0}$ and write the induced norm of x as $\|x\|$.

Proposition 4.2.1. *Let $d \in \mathbb{N}^{Q_0}$ be a dimension vector and let θ be a stability function such that $\theta(d) = 0$. Denote*

$$\lambda := -\min\{\mu \mid \mu \text{ eigenvalue of } B\}$$

and let m be a positive integer greater than $\lambda\|d\|^2$.

(a) *If $\theta = \theta_\alpha = \langle \alpha, \_\rangle$ for a dimension vector α, then for every θ-semistable representation M of dimension vector d, a general representation V of dimension vector $m\alpha$ satisfies*

$$\operatorname{Hom}(V, M) = \operatorname{Ext}(V, M) = 0.$$

(b) *If $\theta = \eta_\beta = -\langle \_, \beta\rangle$ for a dimension vector β, then for every θ-semistable representation M of dimension vector d, a general representation V of dimension vector $m\beta$ satisfies*

$$\operatorname{Hom}(M, V) = \operatorname{Ext}(M, V) = 0.$$

Proof. We will only prove (b), as the argument for (a) is identical and leads to the same bound. Given a θ-semistable representation M of dimension vector d and a positive integer m, Corollary 4.1.4 with $\epsilon = 0$ implies that as soon as

$$m > f(\gamma) := \frac{\langle \gamma, \gamma\rangle}{\langle \gamma, \beta\rangle}$$

for the finitely many dimension vectors $0 < \gamma < d$ for which $\langle \gamma, \beta\rangle < 0$, we have $\operatorname{Hom}(M, V) = \operatorname{Ext}(M, V) = 0$ for a general representation V of dimension vector $m\beta$.

Clearly it is enough to consider only those γ for which $\langle \gamma, \gamma\rangle < 0$, since otherwise $f(\gamma) \le 0$. If Q is either a Dynkin or extended Dynkin quiver, there are no such γ, and we may take any $m \ge 1$. Hence we will assume that Q is not of these types, or equivalently $\lambda > 0$. We now prove the claim by showing that $f(\gamma) \le \lambda||d||^2$ for all dimension vectors $0 < \gamma < d$ for which $\langle \gamma, \beta\rangle < 0$ and $\langle \gamma, \gamma\rangle < 0$.

Since the denominator of $f(\gamma)$ is assumed to be negative, in order to obtain an upper bound for $f(\gamma)$, we need to minimize the numerator. Notice that since B is symmetric, the minimal value of the Tits form on the unit sphere is

$$-\lambda = \min\left\{\langle \rho, \rho\rangle \mid \rho \in \mathbb{R}^{Q_0}, \|\rho\| = 1\right\}.$$

For $\gamma < d$ that satisfies $\langle \gamma, \beta\rangle < 0$ and $\langle \gamma, \gamma\rangle \le 0$, write $\gamma = \|\gamma\|\gamma_0$ where $\|\gamma_0\| = 1$. We now have the following inequalities between nonnegative numbers

$$\|\gamma\| < \|d\|, \qquad \frac{1}{|\langle \gamma, \beta\rangle|} \le 1, \qquad |\langle \gamma_0, \gamma_0\rangle| \le \lambda,$$

where the third one follows from the assumption $\langle \gamma, \gamma\rangle < 0$. Combining these inequalities gives the estimate

$$f(\gamma) = \frac{\langle \gamma, \gamma\rangle}{\langle \gamma, \beta\rangle} \le \frac{\|\gamma\|^2 \cdot |\langle \gamma_0, \gamma_0\rangle|}{|\langle \gamma, \beta\rangle|} \le \frac{\|d\|^2\lambda}{|\langle \gamma, \beta\rangle|} \le \lambda\|d\|^2. \qquad \square$$

Example 4.2.2. The n-Kronecker quiver

$$Q: \ 1 \overset{\textstyle\longrightarrow}{\underset{\textstyle\longrightarrow}{\vdots}} 2$$

has Tits matrix $\begin{pmatrix} 1 & -\frac{n}{2} \\ -\frac{n}{2} & 1 \end{pmatrix}$. It follows that its eigenvalues are $1 \pm \frac{n}{2}$ and so $\lambda = \frac{n}{2} - 1$, which is positive when $n = 1$ and zero when $n = 2$.

Remark 4.2.3. In Example 4.2.2, the case where $n = 1, 2$ is a Dynkin, resp. extended Dynkin quiver. As observed in the proof of Proposition 4.2.1 any $m \geq 1$ will work. This is consistent with the effective basepoint-freeness results following from Proposition 4.2.1 discussed in Section 6.1, because by [16, Theorem 3.1] the moduli spaces are all affine or projective spaces.

In certain cases, the effective bounds described in Proposition 4.2.1 are not optimal. For instance let Q be the $(n+1)$-Kronecker quiver, $d = (1, 1)$, and $\theta = \eta_\beta$ with $\beta = (n, 1)$. Proposition 4.2.1 shows that $\mathrm{Hom}(M, V) = 0$ for M a θ-semistable representation of Q of dimension d and V a general representation of dimension $m\beta$ for $m \geq 2(n-1)$. A direct computation shows that it is in fact enough to let m be any positive integer.

4.3. *Auslander–Reiten translations and semistability.* In this section we investigate how stability behaves under Auslander–Reiten translations, and therefore assume throughout that Q is acyclic. For this reason, whenever we write $\theta = \theta_\beta$ or $\theta = \eta_\beta$, we implicitly assume that β is a dimension vector.

Only Lemma 4.3.1 from this section is used as an ingredient to our main theorem. The other results are included to give a complete picture of the interaction between stability and the translation functors, which does not seem to appear in the literature.

Let $\tau, \tau^- : \mathrm{rep}_k Q \to \mathrm{rep}_k Q$ denote the Auslander–Reiten translations defined in Section 2.4. Let θ be a stability function for Q; recall from Lemma 2.3.4 that since Q is acyclic, the stability function θ can be identified with θ_α (resp. η_β) for a unique dimension vector α (resp. β).

Lemma 4.3.1. *Let M be a θ-semistable representation of Q.*

(a) *If $\theta = \theta_\alpha$, then $\tau^- M$ is η_α-semistable.*

(b) *If $\theta = \eta_\beta$, then τM is θ_β-semistable.*

Proof. We only show the second assertion, as the first follows similarly. By Proposition 4.1.5(b) there exists $m > 0$ and a representation V of dimension vector $m\beta$ such that $\mathrm{Hom}(M, V) = 0$. Assume first that M has no projective summands. By Auslander–Reiten duality we have

$$\mathrm{Ext}(V, \tau M) \cong \mathrm{Hom}(M, V)^\vee = 0.$$

Moreover, from Proposition 2.4.1(ii) we have

$$\dim \mathrm{Hom}(V, \tau M) - \dim \mathrm{Ext}(V, \tau M) = \langle V, \tau M \rangle = -\langle M, V \rangle = -m \langle \underline{\dim}\, M, \beta \rangle = 0$$

since M is η_β-semistable. Hence also $\mathrm{Hom}(V, \tau M) = 0$, and so τM is θ_β-semistable by Proposition 4.1.5(a).

In the general case, we can decompose M into indecomposable summands and this decomposition is unique up to isomorphism. Thus, we can write $M = U \oplus P$, where P is projective and U has no projective summands. Since M is η_β-semistable, both summands U and P are also η_β-semistable. As U has no projective summands, we conclude that τU is θ_β-semistable. Moreover,

$$\tau M = \tau U \oplus \tau P = \tau U,$$

and so τM is θ_β-semistable as well. $\square$

Lemma 4.3.2. *Let M be a θ-stable representation.*

(a) *Suppose $\theta = \theta_\alpha$. If* $\operatorname{supp} M \not\subset \operatorname{supp} \alpha$, *then* $\operatorname{supp} M \setminus \operatorname{supp} \alpha = \{j\}$ *and $M \cong I_{Q'}(j)$, where $I_{Q'}(j)$ is the indecomposable injective of the full subquiver Q' supported on* $\operatorname{supp} \alpha \cup \{j\}$, *viewed as a representation of Q.*

(b) *Suppose $\theta = \eta_\beta$. If* $\operatorname{supp} M \not\subset \operatorname{supp} \beta$, *then* $\operatorname{supp} M \setminus \operatorname{supp} \beta = \{j\}$ *and $M \cong P_{Q'}(j)$, where $P_{Q'}(j)$ is the indecomposable projective of the full subquiver Q' supported on* $\operatorname{supp} \beta \cup \{j\}$, *viewed as a representation of Q.*

Proof. We again only show (b). Assume that $\operatorname{supp} M \not\subset \operatorname{supp} \beta$. Let $Q'' \subset Q' \subseteq Q$ denote the full subquivers on $\operatorname{supp} M \setminus \operatorname{supp} \beta$ and $\operatorname{supp} M \cup \operatorname{supp} \beta$ respectively. We first observe that if V and W are representations supported on Q', then it follows from (3) that

$$\operatorname{Hom}_Q(V, W) = \operatorname{Hom}_{Q'}(V, W) \quad \text{and} \quad \operatorname{Ext}_Q(V, W) = \operatorname{Ext}_{Q'}(V, W),$$

so we may drop the subscripts. The subquiver Q'' is also acyclic so it has a sink j, and by assumption $\beta_j = 0$ and $\dim M_j > 0$. Consider the projective representation $P_{Q'}(j)$ of Q'. For any representation V supported on Q', we have

$$\langle P_{Q'}(j), V\rangle = \dim \operatorname{Hom}(P_{Q'}(j), V) = \dim V_j$$

because $\operatorname{Ext}(P_{Q'}(j), V) = 0$ and $\operatorname{Hom}(P_{Q'}(j), V) \cong V_j$. This implies that

$$\eta_\beta(P_{Q'}(j)) = -\langle \underline{\dim}\, P_{Q'}(j), \beta\rangle = -\beta_j = 0$$

whereas $\operatorname{Hom}(P_{Q'}(j), M) \cong M_j \neq 0$. Let $f \in \operatorname{Hom}(P_{Q'}(j), M)$ be a nonzero homomorphism and consider its kernel P. We have $P \subsetneq P_{Q'}(j)$, and since the category $\operatorname{rep}_k Q'$ is hereditary, P is again a projective representation of Q'.

Suppose that $P \neq 0$. Any indecomposable direct summand of P is of the form $P_{Q'}(i)$ for some vertex $i \in Q'_0$ for which there exists a path $j \to i$ in Q' because it must embed into $P_{Q'}(j)$, and this path cannot have length 0 because $P_j = 0$. As j is a sink of Q'', we must have $i \in \operatorname{supp} \beta$. This shows that

$$\eta_\beta(P_{Q'}(i)) = -\beta_i < 0.$$

We conclude that $\eta_\beta(P) < 0$ and therefore $\eta_\beta(P_{Q'}(j)/P) > 0$. However, $P_{Q'}(j)/P$ embeds into M via f, which contradicts the fact that M is stable. Thus, we have $P = 0$ and the map $f : P_{Q'}(j) \to M$ is injective. Since the image of f is a nonzero subrepresentation of M with $\eta_\beta(\operatorname{im} f) = \eta_\beta(P_{Q'}(j)) = 0$, we conclude that f must also be surjective and thus $M \cong P_{Q'}(j)$.

If j' is another sink of Q'', then the same argument shows that also $M \cong P_{Q'}(j')$, which implies that $j = j'$ and so $Q'' = \{j\}$ as claimed. □

Lemma 4.3.3. *Let M be a θ-stable representation of dimension vector d and let $\epsilon \in \mathbb{N}^{Q_0}$.*

(a) *If $\theta = \theta_\alpha$ and $\langle \epsilon, d \rangle > 0$, then $m\alpha + \epsilon \geq d$ for $m \gg 0$.*

(b) *If $\theta = \eta_\beta$ and $\langle d, \epsilon \rangle > 0$, then $m\beta + \epsilon \geq d$ for $m \gg 0$.*

Proof. As above, we just prove the second claim. The result is clear if $\operatorname{supp} d \subseteq \operatorname{supp} \beta$. By Lemma 4.3.2, the only other case is that $\operatorname{supp} M \setminus \operatorname{supp} \beta = \{j\}$ and $M \cong P'(j)$, the indecomposable projective of the full subquiver on $\operatorname{supp} \beta \cup \{j\}$. In this case we have $d_j = (\underline{\dim}\, P'(j))_j = 1$, while $\epsilon_j \geq \langle d, \epsilon \rangle > 0$, so in particular $\epsilon_j \geq d_j$. □

Lemma 4.3.4. *Let M be a θ-stable representation.*

(a) *If $\theta = \theta_\alpha$, then either $\tau^- M$ is η_α-stable, or M is isomorphic to an injective representation I of the full subquiver Q' supported on* $\operatorname{supp} M \cup \operatorname{supp} \alpha$, *viewed as a representation of Q.*

(b) *If $\theta = \eta_\beta$, then either τM is θ_β-stable, or M is isomorphic to a projective representation P of the full subquiver Q' supported on* $\operatorname{supp} M \cup \operatorname{supp} \beta$, *viewed as a representation of Q.*

Proof. We again only show (b). Suppose that M is not of the form P as in the statement. We know by Lemma 4.3.1(b) that τM is θ_β-semistable and want to conclude that it is θ_β-stable. Viewing M as a representation of the full subquiver Q' supported on $\operatorname{supp} M \cup \operatorname{supp} \beta$, our assumption means that M is not projective. Therefore, by Lemma 4.3.2(b), we have $\operatorname{supp} \beta = Q'_0$. Let $\tau M \twoheadrightarrow U$ be a surjection such that $\theta_\beta(U) = 0$ and write $U = \bigoplus_\ell U^\ell$ as a direct sum of indecomposables. Each U^ℓ is a quotient of the semistable representation τM, so $\theta_\beta(U^\ell) \geq 0$, and since these quantities sum to 0, we have $\theta_\beta(U^\ell) = 0$ for each ℓ. Thus, we may assume that U is itself indecomposable.

The quotient U cannot be injective, because if $U \cong I(i)$ for some $i \in Q'$, then

$$0 = \theta_\beta(U) = \langle \beta, \underline{\dim}\, I(i) \rangle = \beta_i,$$

whereas $\beta_i > 0$ since $i \in \operatorname{supp} \beta$. Thus, $U \cong \tau\tau^- U$ by Proposition 2.4.1(i). By Proposition 2.4.1(ii), the functor τ^- is a left adjoint, thus right exact, so $\tau^- U$ is a quotient of M, as $\tau^-\tau M$ is in any case a quotient of M. Using Proposition 2.4.1(ii) we obtain

$$\eta_\beta(\tau^- U) = -\langle \underline{\dim}\, \tau^- U, \beta \rangle = \langle \beta, \underline{\dim}\, U \rangle = \theta_\beta(U) = 0.$$

As M is η_β-stable, we have either $\tau^- U = 0$ or $\tau^- U = M$, and since $U \cong \tau\tau^- U$, we conclude that either $U = 0$ or $U = \tau M$ which proves the claim. □

4.4. ***Generic vanishing of Ext.*** We continue to assume that Q is acyclic.

Lemma 4.4.1. *Let M be a θ-semistable representation and let $\epsilon \in \mathbb{N}^{Q_0}$.*

(a) *If $\theta = \theta_\alpha$ and $\epsilon = \underline{\dim}\, P$ for some projective representation P, then for all sufficiently large integers m, a general representation V of dimension vector $m\alpha + \epsilon$ satisfies* $\operatorname{Ext}(V, M) = 0$. *In fact, it is enough that*

$$m > \frac{\langle \gamma, \gamma \rangle}{\langle \gamma, \alpha \rangle} \quad \text{for all dimension vectors} \quad \gamma < \underline{\dim}\, \tau^{-} M.$$

(b) *If $\theta = \eta_\beta$ and $\epsilon = \underline{\dim}\, I$ for some injective representation I, then for all sufficiently large integers m, a general representation V of dimension vector $m\beta + \epsilon$ satisfies* $\operatorname{Ext}(M, V) = 0$. *In fact, it is enough that*

$$m > \frac{\langle \gamma, \gamma \rangle}{\langle \beta, \gamma \rangle} \quad \text{for all dimension vectors} \quad \gamma < \underline{\dim}\, \tau M.$$

Proof. We prove (b). By Lemma 4.3.1, the representation τM is θ_β-semistable, so Corollary 4.1.4 (a) with $\epsilon = 0$ implies that for m satisfying the inequality, there exists a representation V' of dimension vector $m\beta$ such that $\operatorname{Hom}(V', \tau M) = 0$. By Auslander–Reiten duality, this implies $\operatorname{Ext}(M, V') \cong \operatorname{Hom}(V', \tau M)^\vee = 0$.

Now the representation $V = V' \oplus I$ has dimension vector $m\beta + \epsilon$ and satisfies

$$\operatorname{Ext}(M, V) = \operatorname{Ext}(M, V') \oplus \operatorname{Ext}(M, I) = 0$$

since I is injective. Thus, by upper semicontinuity this must hold for a general representation of dimension vector $m\beta + \epsilon$. □

4.5. ***Separating stable representations.*** In this section, we assume that Q is acyclic and $\theta = \eta_\beta$ for concreteness, and leave formulating the dual statements for $\theta = \theta_\alpha$ to the reader. Our aim is to prove the following result.

Theorem 4.5.1. *Let $M^0, M^1, \ldots, M^r$ be non-isomorphic η_β-stable representations. For all sufficiently large integers m, there exists a representation N of dimension vector $m\beta$ such that*

$$\operatorname{Hom}(M^0, N) \neq 0, \quad \textit{and} \quad \operatorname{Hom}(M^\ell, N) = 0 \quad \text{for } \ell = 1, \ldots, r. \tag{14}$$

This will be used in Theorem 6.2.1 below by considering two polystable representations M and M' such that $M^1, \ldots, M^r$ are the non-isomorphic stable summands of M' while M^0 appears as a stable summand of M. In view of Proposition 3.3.3, we will see that σ_N separates the polystable representations M and M'; this will ultimately enable us to prove ampleness of the determinantal line bundle on the moduli space of semistable representations in Section 6.

Our argument is inspired by the proof of a similar statement for moduli of vector bundles on a curve, due to Esteves and Popa [19, Section 5; 20, Section 3].

We break up the proof of Theorem 4.5.1 into several steps.

Proposition 4.5.2. *For $M^0, M^1, \ldots, M^r$ as in Theorem 4.5.1 and for all sufficiently large integers m, there exists a representation N such that*

(i) $\operatorname{Hom}(M^0, N) \neq 0$;

(ii) $\operatorname{Hom}(M^\ell, N) = 0 \quad \text{for } \ell = 1, \ldots, r$;

(iii) $\underline{\dim}\, N = m\beta + \epsilon$, *where* ϵ *is the dimension vector of an injective representation and* $\operatorname{supp} \epsilon$ *is disjoint from* $\operatorname{supp} M^0$.

Before proving Proposition 4.5.2 we first use it to establish Theorem 4.5.1.

Proof of Theorem 4.5.1. Let N be as in Proposition 4.5.2. If $\epsilon = 0$, we are already done, so assume $\epsilon > 0$. It suffices to find a subrepresentation $N' \subset N$ that satisfies properties (i)-(iii) with $\epsilon' < \epsilon$, since repeating the construction results in a sequence of subrepresentations with the same properties, and the sequence must terminate at a subrepresentation with $\epsilon = 0$.

By assumption we have $\epsilon = \underline{\dim}\, I$ for some nonzero injective representation I. Since $\langle V, I\rangle = \dim \operatorname{Hom}(V, I) \geq 0$ for any representation V, and since β is a dimension vector since Q is acyclic, we have

$$\dim \operatorname{Hom}(N, I) = \langle N, I\rangle = \langle m\beta + \epsilon, \epsilon\rangle \geq \langle \epsilon, \epsilon\rangle = \dim \operatorname{Hom}(I, I) \geq 1,$$

so there is a nonzero map $f : N \to I$. Let $N' \subset N$ and I' denote the kernel and cokernel of f respectively and set $\epsilon' = \underline{\dim}\, I'$. Note that $\epsilon' < \epsilon$. We claim that N' satisfies properties (i)-(iii).

To verify (i) and (ii) for N', we apply $\operatorname{Hom}(M^\ell, \_)$ to the exact sequence

$$0 \to N' \to N \xrightarrow{f} I$$

to get

$$0 \to \operatorname{Hom}(M^\ell, N') \to \operatorname{Hom}(M^\ell, N) \to \operatorname{Hom}(M^\ell, I).$$

Since $\operatorname{Hom}(M^\ell, N) = 0$ for $\ell \geq 1$, we also have $\operatorname{Hom}(M^\ell, N') = 0$, giving (ii). For (i), we have $\operatorname{Hom}(M^0, I) = 0$ since $\operatorname{supp} I \cap \operatorname{supp} M^0 = \varnothing$, and so

$$\operatorname{Hom}(M^0, N') \cong \operatorname{Hom}(M^0, N) \neq 0.$$

For (iii), from the exact sequence $0 \to N' \to N \to I \to I' \to 0$ we obtain

$$\underline{\dim}\, N' = \underline{\dim}\, N - \underline{\dim}\, I + \underline{\dim}\, I' = m\beta + \epsilon - \epsilon + \epsilon' = m\beta + \epsilon'.$$

Clearly $\operatorname{supp} \epsilon' \cap \operatorname{supp} M^0 = \varnothing$ since $\operatorname{supp} \epsilon' \subseteq \operatorname{supp} \epsilon$, and moreover, as I' is a quotient of I, it is also injective since $\operatorname{rep}_k Q$ is hereditary. Thus, we have proven (iii) for N'. □

The rest of the section is devoted to proving Proposition 4.5.2. Fix an admissible ordering of Q_0 as in Section 2.1, let $i_0 \in \operatorname{supp} M^0$ be the minimal vertex in the support of M^0, and set $\epsilon_0 := \underline{\dim}\, I(i_0)$. Note that for any dimension vector ξ we have

$$\langle \xi, \epsilon_0\rangle = \xi_{i_0}$$

and that $\operatorname{supp} \epsilon_0 \cap \operatorname{supp} M^0 = \{i_0\}$ by the choice of i_0.

Lemma 4.5.3. *Let $M^0, M^1, \dots, M^r$ be as in Theorem 4.5.1 and let $m_\ell = \dim(M^\ell)_{i_0}$, where i_0 is the minimal vertex in* $\operatorname{supp} M^0$. *For all sufficiently large integers m, there exists a representation V of dimension vector $m\beta + \epsilon_0$ such that*

- $\operatorname{Ext}(M^\ell, V) = 0$ *for each $\ell = 0, \dots, r$, and so*

$$\dim \operatorname{Hom}(M^\ell, V) = \langle \underline{\dim}\, M^\ell, m\beta + \epsilon_0\rangle = \langle \underline{\dim}\, M^\ell, \epsilon_0\rangle = m_\ell;$$

- *For $\ell = 0, \dots, r$, every nonzero map $M^\ell \to V$ is injective*;
- *For $\ell = 1, \dots, r$, every nonzero map $f : M^0 \oplus M^\ell \to V$ satisfies*

$$\langle \underline{\dim} \ker f, \beta\rangle = 0.$$

Proof. Since each M^ℓ and each $M^0 \oplus M^\ell$ is η_β-semistable and ϵ_0 is the dimension vector of an injective representation, we obtain the claim by applying Lemma 4.4.1(b) to each M^ℓ for $\ell = 0, \dots, r$, and by applying Lemma 4.1.2(b) to each M^ℓ for $\ell = 0, \dots, r$ as well as each $M^0 \oplus M^\ell$ for $\ell = 1, \dots, r$. □

Denote the cokernel of ϕ by W, so that we have an exact sequence

$$0 \longrightarrow M^0 \xrightarrow{\ \phi\ } V \longrightarrow W \longrightarrow 0.$$

Lemma 4.5.4. *If $\ell \geq 1$ and $\psi : M^\ell \to V$ is a nonzero map, then the induced map $\overline{\psi} : M^\ell \to W$ is injective.*

Proof. By Lemma 4.5.3, the map ψ is injective and the kernel K of the induced map

$$(\phi, \psi) : M^0 \oplus M^\ell \longrightarrow V$$

satisfies $\langle \underline{\dim}\, K, \beta\rangle = 0$. Since M^0 and M^ℓ are η_β-stable and non-isomorphic, the only nonzero subrepresentations of $M^0 \oplus M^\ell$ with this property are M^0, M^ℓ, and $M^0 \oplus M^\ell$. Since both ϕ and ψ are injective, we must have $K = 0$. The successive inclusions

$$M^0 \hookrightarrow M^0 \oplus M^\ell \xrightarrow{(\phi,\psi)} V$$

induce a short exact sequence

$$0 \longrightarrow M^\ell \xrightarrow{\ \overline{\psi}\ } W \longrightarrow V/(M^0 \oplus M^\ell) \longrightarrow 0,$$

so we see that the induced map $\overline{\psi} : M^\ell \xrightarrow{\psi} V \to W$ is injective. □

Lemma 4.5.5. *There exists a hyperplane $H \subset V_{i_0}$ such that $\phi_{i_0}(M^0_{i_0}) \subseteq H$ but $\psi_{i_0}(M^\ell_{i_0}) \nsubseteq H$ for every $\ell \geq 1$ and every nonzero $\psi : M^\ell \to V$.*

Proof. As before, let $m_\ell := \dim(M^\ell)_{i_0}$. Consider the Grassmannian $\operatorname{Gr}(m_\ell, W_{i_0})$ of m_ℓ-dimensional subspaces of the vector space W_{i_0}. By Lemma 4.5.4, we obtain a morphism

$$\mathbb{P}(\operatorname{Hom}(M^\ell, V)) \longrightarrow \operatorname{Gr}(m_\ell, W_{i_0})$$

that sends a map $\psi : M^\ell \to V$ to the subspace $\mathrm{im}(\overline{\psi})_{i_0} \subseteq W_{i_0}$. The image $X_\ell \subseteq \mathrm{Gr}(m_\ell, W_{i_0})$ of this morphism has dimension at most $\dim \mathbb{P}(\mathrm{Hom}(M^\ell, V)) = m_\ell - 1$.

For a hyperplane $P \subseteq W_{i_0}$, denote by $Y_{\ell,P} \subseteq \mathrm{Gr}(m_\ell, W_{i_0})$ the Schubert variety of subspaces contained in P. Since the codimension of $Y_{\ell,P}$ is m_ℓ, by the Bertini–Kleiman theorem there exists a hyperplane P such that $Y_{\ell,P} \cap X_\ell = \varnothing$ for each $\ell = 1, \ldots, r$. The preimage $H \subseteq V_{i_0}$ of this P satisfies the conditions in the lemma. □

Proof of Proposition 4.5.2. Let $H \subset V_{i_0}$ be a hyperplane as in Lemma 4.5.5 and let $p : V_{i_0} \to k$ be a linear map with kernel H. Recall that restricting to V_{i_0} gives an isomorphism $\mathrm{Hom}_Q(V, I(i_0)) \xrightarrow{\sim} \mathrm{Hom}_k(V_{i_0}, k)$. Let π be the unique map $V \to I(i_0)$ corresponding to p under this isomorphism. We claim that the representation $N = \ker \pi$ satisfies conditions (i)-(ii) in the statement.

Note that a morphism $f : M^\ell \to V$ factors through N if and only if the composition $\pi \circ f : M^\ell \to I(i_0)$ is zero, if and only if the composition $p \circ f_{i_0} : M^\ell_{i_0} \to I(i_0)_{i_0}$ is zero, if and only if $f_{i_0}(M^\ell_{i_0}) \subset H$. Thus, it follows from the choice of H that $\phi : M^0 \to V$ factors through N but no nonzero map $M^\ell \to V$ does for $\ell = 1, \ldots, r$, which proves (i) and (ii).

For (iii), we note that the representation $I = \mathrm{coker}\, \pi$ is injective, as it is a quotient of $I(i_0)$ and the category of representations is hereditary. Setting $\epsilon = \underline{\dim}\, I$, the exact sequence

$$0 \to N \to V \xrightarrow{\pi} I(i_0) \to I \to 0$$

implies that

$$\underline{\dim}\, N = \underline{\dim}\, V - \underline{\dim}\, I(i_0) + \underline{\dim}\, I = m\beta + \epsilon_0 - \epsilon_0 + \epsilon = m\beta + \epsilon.$$

Finally, as $\mathrm{supp}\, I(i_0) \cap \mathrm{supp}\, M^0 = \{i_0\}$ and $(\mathrm{im}\, \pi)_{i_0} = (I(i_0))_{i_0}$, we must have $\mathrm{supp}\, \epsilon \cap \mathrm{supp}\, M^0 = \varnothing$. □

5. Moduli spaces of quiver representations

In this section we show that under certain assumptions (see Remark 5.5.8), stacks of semistable quiver representations admit adequate moduli spaces by verifying the existence criteria of [6].

Throughout this section we let S denote a noetherian scheme. The locally noetherian hypothesis is required for the notions of Θ-reductivity and S-completeness to be well-defined and for the existence criteria to be applicable as in Section 5.2, while we add a quasi-compactness condition to ensure that points specialize to closed points, so that the local reductivity in Section 5.4 is better behaved.

5.1. ***Good and adequate moduli spaces.*** We begin by recalling the definition of good and adequate moduli spaces due to Alper [2; 3].

Definition 5.1.1. Let $\mathcal{X}$ be a quasi-separated algebraic stack over S. An *adequate moduli space* is a quasi-compact quasi-separated morphism $f : \mathcal{X} \to X$ to an algebraic space X over S such that

(i) f is *adequately affine*, meaning that for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras and every section s of $f_*\mathcal{B}$ over a smooth morphism $\mathrm{Spec}\, A \to X$, some power of s lifts to a section of $f_*\mathcal{A}$ over $\mathrm{Spec}\, A$, and

(ii) the natural morphism $\mathcal{O}_X \to f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

The map f is a *good moduli space* if instead of (i), we have the stronger hypothesis

(i′) f is *cohomologically affine*, meaning that the functor $f_* : \mathrm{Qcoh}\,\mathcal{X} \to \mathrm{Qcoh}\,X$ is exact.

The notion of a good moduli space is inspired by good quotients in GIT.

Example 5.1.2. [2, Section 13] If G is a linearly reductive algebraic group over a field k acting on an affine k-scheme $\mathrm{Spec}(A)$, then a good moduli space of the quotient stack is given by the affine GIT quotient:

$$[\mathrm{Spec}(A)/G] \to \mathrm{Spec}(A) /\!\!/ G := \mathrm{Spec}(A^G).$$

More generally, if G acts on a quasi-projective k-scheme X with a fixed ample G-linearization and $X^{\mathrm{ss}} \to X /\!\!/ G$ denotes the corresponding GIT quotient, then

$$[X^{\mathrm{ss}}/G] \to X /\!\!/ G$$

is a good moduli space.

In this example, it is vital that G is linearly reductive, since $\pi : \mathrm{B}G \to \mathrm{Spec}(k)$ is cohomologically affine if and only if G is linearly reductive, that is, the functor π_*, which corresponds to taking G-invariants, is exact [2, Proposition 12.2]. In characteristic 0, the notions of linearly reductive and reductive coincide; however, in positive characteristic, many reductive groups, GL_n among them, are not linearly reductive.

In order to bridge this gap, Alper introduced the broader notion of adequate moduli spaces in [3] which also covers many interesting cases in positive characteristic. In particular a flat, separated, finitely presented group scheme G over S is reductive if and only if the morphism $\mathrm{B}G \to S$ is adequately affine. Consequently, Alper's notion of adequate moduli space enables a generalization of GIT to stacks for all reductive groups in arbitrary characteristic.

Example 5.1.3. [3, Section 9] If G is a smooth affine reductive group over a field k acting on an affine k-scheme $\mathrm{Spec}(A)$, then

$$[\mathrm{Spec}(A)/G] \to \mathrm{Spec}(A) /\!\!/ G := \mathrm{Spec}(A^G)$$

is an adequate moduli space. More generally, for a reductive k-group G acting on a quasi-projective k-scheme X with an ample G-linearization,

$$[X^{\mathrm{ss}}/G] \to X /\!\!/ G$$

is an adequate moduli space.

Since the stacks $\mathcal{M}_d$ and $\mathcal{M}_d^{\theta\text{-ss}}$ can both be described as quotient stacks via such a GIT setup (see Proposition 3.1.4), one can conclude that these moduli stacks have adequate moduli spaces given by the GIT quotient. We will instead take a modern approach which avoids GIT and establish the existence of adequate moduli spaces by applying the criteria of [6].

We summarize some of the properties of good and adequate moduli spaces that will be relevant for us. Recall that S is assumed to be noetherian, hence quasi-separated, which is a standing assumption in the works we build on.

Theorem 5.1.4. *Let $\mathcal{X}$ be an algebraic stack of finite type over S, and let $f : \mathcal{X} \to X$ be an adequate moduli space.*

(i) [3, Theorems 5.3.1 and 7.2.1] *The map f is surjective, universally closed, and initial for maps to schemes and to separated algebraic spaces over S.*
[2, Theorem 6.6] *If f is a good moduli space, it is initial for maps to algebraic spaces.*
In particular, adequate moduli spaces which are schemes and good moduli spaces are unique up to a unique isomorphism.

(ii) [3, Theorem 6.3.3] *The algebraic space X is of finite type over S.*

(iii) [3, Theorem 5.3.1] *For an algebraically closed $\mathcal{O}_S$-field k, the map $f \times_S k$ identifies two k-points $x, y : \operatorname{Spec} k \to \mathcal{X} \times_S k$ if and only if the closures of $\{x\}$ and $\{y\}$ in $|\mathcal{X} \times_S k|$ intersect.*

(iv) [3, Proposition 5.2.9(1)] *If $X' \to X$ is a flat morphism of algebraic spaces, then $\mathcal{X} \times_X X' \to X'$ is an adequate moduli space. In particular, if $S' \to S$ is a flat morphism of schemes, then $\mathcal{X} \times_S S' \to X \times_S S'$ is an adequate moduli space.*
[2, Proposition 4.7(i)] *If $\mathcal{X} \to X$ is a good moduli space, then for any morphism $X' \to X$ the base change $\mathcal{X} \times_X X' \to X'$ is a good moduli space.*

(v) [3, Proposition 5.2.9(3)] *More generally, if $X' \to X$ is any morphism of algebraic spaces, then the morphism $\mathcal{X} \times_X X' \to X'$ is a universal homeomorphism. In particular, if $S' \to S$ is any morphism of schemes, then the closed points of $\mathcal{X} \times_S S'$ are in natural bijection with those of $X \times_S S'$.*

Condition (ii) in Definition 5.1.1 allows us to prove the following.

Lemma 5.1.5. *Let $\mathcal{X}$ be an algebraic stack over a scheme S and let $f : \mathcal{X} \to X$ be an adequate moduli space.*

(i) *The pullback f^* is fully faithful for vector bundles on X.*

*In particular, given a vector bundle $\mathcal{F}$ on $\mathcal{X}$, there is up to isomorphism at most one vector bundle F on X such that $\mathcal{F} \cong f^*F$. If such an F exists, we say that $\mathcal{F}$ **descends** to X.*

(ii) *Let $X' \to X$ be an fpqc cover, let $\mathcal{X}' \to X'$ be the base change of $\mathcal{X}$ along X' with the second projection morphism $u : \mathcal{X}' \to \mathcal{X}$, and let $\mathcal{F}$ be a vector bundle on $\mathcal{X}$. If $u^*\mathcal{F}$ descends to X', then $\mathcal{F}$ descends to X.*

(iii) *If a line bundle $\mathcal{L}$ on $\mathcal{X}$ is generated by finitely many global sections, then $\mathcal{L}$ descends to X.*

Proof. (i) Given vector bundles F and G on X, using the projection formula and the condition $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, we have

$$\operatorname{Hom}_{\mathcal{X}}(f^*F, f^*G) = \operatorname{Hom}_X(F, f_*f^*G) = \operatorname{Hom}_X(F, f_*\mathcal{O}_{\mathcal{X}} \otimes G) = \operatorname{Hom}_X(F, G).$$

(ii) The vector bundle $\mathcal{F}' = u^*\mathcal{F}$ comes equipped with a canonical descent datum; that is, there is a canonical isomorphism $\sigma : \mathrm{pr}_1^*\mathcal{F}' \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{F}'$, where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{X}'$ are the projections, and σ satisfies the cocycle condition on $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$. By assumption, $\mathcal{F}'$ descends to a vector bundle F' on X', and by Theorem 5.1.4(iv), X' is an adequate moduli space of $\mathcal{X}'$, so it follows from (i) that σ descends to an isomorphism $\tau : \mathrm{pr}_1^*F' \cong \mathrm{pr}_2^*F'$, where $\mathrm{pr}_1, \mathrm{pr}_2 : X' \times_X X' \to X'$ are the projections. Moreover $X' \times_X X' \times_X X'$ is an adequate moduli space for $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$, so the isomorphism τ satisfies the cocycle condition on $X' \times_X X' \times_X X'$. In other words, we obtain a descent datum for F', so there exists a unique vector bundle F on X whose restriction to X' is F', and since the descent datum for F pulls back to that of $\mathcal{F}$, it follows that $f^*F \cong \mathcal{F}$.

(iii) Let $s_0, \ldots, s_n \in \Gamma(\mathcal{X}, \mathcal{L})$ be global sections that generate $\mathcal{L}$ and denote by $\phi : \mathcal{X} \to \mathbb{P}^n$ the morphism induced by these sections. Since the adequate moduli space $f : \mathcal{X} \to X$ is initial for maps to separated algebraic spaces, there is a map $\psi : X \to \mathbb{P}^n$ such that $\phi = \psi \circ f$. This implies that the line bundle $L := \psi^*\mathcal{O}_{\mathbb{P}^n}(1)$ satisfies $f^*L = \mathcal{L}$, proving the claim. □

If $f : \mathcal{X} \to X$ is a good moduli space, then a more general result holds: the pullback f^* is fully faithful for all quasi-coherent sheaves, and in fact quasi-coherent sheaves satisfy descent along f [37, Lemma 2.12]. We do not know if the analogous result holds for adequate moduli spaces.

5.2. ***Existence criteria for moduli spaces.*** The existence criteria for good and adequate moduli spaces of [6] are expressed in terms of two valuative criteria for algebraic stacks, known as Θ-reductivity and S-completeness, that we now recall. Both are certain codimension-2 filling conditions and, in order to specify them, we introduce two quotient stacks associated to a given discrete valuation ring.

For a DVR R with uniformizer $\pi \in R$, fraction field K, and residue field κ, we define

$$\Theta_R = [\mathrm{Spec}(R[t])/\mathbb{G}_\mathrm{m}] \quad \text{and} \quad \overline{\mathrm{ST}}_R = \left[\mathrm{Spec}\left(\frac{R[s,t]}{st-\pi}\right)/\mathbb{G}_\mathrm{m}\right],$$

where $\mathbb{G}_\mathrm{m}$ acts with weight 1 on s and weight -1 on t. We denote by $0 \in \Theta_R$ and $0 \in \overline{\mathrm{ST}}_R$ the unique closed point in each stack.

Definition 5.2.1. An algebraic stack $\mathcal{X}$ of finite type over S is Θ-*reductive*, respectively S-*complete*, if for any discrete valuation ring R and any commutative diagram of solid arrows on the left, respectively on the right,

$$\begin{tikzcd} \Theta_R \setminus \{0\} \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed, "\exists!"] & S \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathrm{ST}}_R \setminus \{0\} \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"] & S \end{tikzcd} \tag{15}$$

there exists a unique dashed arrow making the diagram commute.

Remark 5.2.2. A stack $\mathcal{X}$ satisfies *Hartogs's principle* if for any regular local ring A of dimension 2 with closed point $0 \in \mathrm{Spec}\, A$ and any map $\mathrm{Spec}\, A \setminus \{0\} \to \mathcal{X}$, there exists a unique extension $\mathrm{Spec}\, A \to \mathcal{X}$

[6, Remark 3.51]. In particular, it follows from descent that $\mathcal{X}$ satisfying Hartogs's principle is both Θ-reductive and S-complete. Moreover, if $\mathcal{X}$ has affine diagonal, any such extension is unique if it exists.

We can now state the existence criteria of Alper, Halpern-Leistner and Heinloth (in a slightly more restrictive version).

Theorem 5.2.3 [6, Theorem 5.4]. *Let $\mathcal{X}$ be an algebraic stack of finite type, with affine stabilizers and separated diagonal, over a noetherian scheme S.*

(i) *If S is a scheme of characteristic* 0, *then $\mathcal{X}$ admits a separated good moduli space over S if and only if $\mathcal{X}$ is Θ-reductive and S-complete.*

(ii) *If $\mathcal{X}$ is locally reductive (see Definition 5.2.4 below), then $\mathcal{X}$ admits a separated adequate moduli space if and only if $\mathcal{X}$ is Θ-reductive and S-complete.*

When $\mathcal{X}$ has affine diagonal — as it will have in our case — the condition that it has affine stabilizers and separated diagonal is automatic.

Definition 5.2.4. A quasi-separated algebraic stack $\mathcal{X}$ with affine stabilizers is *locally reductive* if every point of $\mathcal{X}$ specializes to a closed point and for every closed point $x \in \mathcal{X}$, there exists a pointed étale morphism $([\operatorname{Spec}(A)/\operatorname{GL}_n], w) \to (X, x)$ inducing an isomorphism of stabilizers at w.

If $\mathcal{X}$ is S-complete, the automorphism group of any closed point is reductive [6, Proposition 3.47]. In characteristic 0, this implies that $\mathcal{X}$ has étale local presentations by quotient stacks of the form $[\operatorname{Spec}(A)/G]$ with G linearly reductive [4]. In positive characteristic, one instead has to assume the existence of such local presentations. In Section 5.5, we show that the stacks $\mathcal{M}_{d,S}$ and $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ are Θ-reductive and S-complete, which implies that they have good moduli spaces when S is a scheme of characteristic 0. However, to obtain adequate moduli spaces in positive characteristic, we show in Section 5.4 that $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ is locally reductive under the additional assumption $\theta = \theta_\beta$ or $\theta = \eta_\beta$ for a dimension vector β.

5.3. *Points of moduli spaces of quiver representations.* In this section, we describe closed points of the moduli stacks $\mathcal{M}_{d,S}$ and $\mathcal{M}_{d,S}^{\theta\text{-ss}}$.

Lemma 5.3.1. *Let k be a field and let M be a θ-semistable representation of dimension vector d. The point $[\operatorname{gr} M] \in |\mathcal{M}_{d,k}^{\theta\text{-ss}}|$ corresponding to the associated graded object* $\operatorname{gr} M$ *of the Jordan–Hölder filtration lies in the closure of $[M]$. Consequently, a closed point of $|\mathcal{M}_{d,k}^{\theta\text{-ss}}|$ corresponds to a geometrically polystable representation.*

Proof. Given a nonsplit short exact sequence $0 \to N' \to N \to N'' \to 0$ of representations where $\underline{\dim}\, N = d$, the line $\mathbb{A}^1$ in $\operatorname{Ext}(N'', N')$ spanned by this class parameterizes a family of representations $\mathcal{N}$ such that $\mathcal{N}_t \cong N$ for $t \neq 0$ and $\mathcal{N}_0 \cong N' \oplus N''$. Considering the induced map $\mathbb{A}^1 \to \mathcal{M}_d$, we see that the point $[N' \oplus N''] \in |\mathcal{M}_d|$ is in the closure of the point $[N]$.

If now

$$0 = M^0 \subset M^1 \subset M^2 \subset \cdots \subset M^{r-1} \subset M^r = M$$

is a Jordan–Hölder filtration of M, then by inductively applying the above argument, we see that the closure of $[M]$ contains the point corresponding to

$$\left(\bigoplus_{\ell=1}^{i} M^{\ell}/M^{\ell-1}\right) \oplus M/M^{i}$$

for each $i = 1, \ldots, r$, hence in particular the point $[\operatorname{gr} M]$.

Now if M corresponds to a closed point in $|\mathcal{M}_{d,k}^{\theta\text{-ss}}|$, then it defines the same point as its base change M' to an algebraically closed field. By the above, $[\operatorname{gr} M']$ lies in the closure of $[M']$ which only contains one point, hence $M' = \operatorname{gr} M'$, and so M is geometrically polystable. □

Proposition 5.3.2. *Let $\pi : \mathcal{M}_{d,S}^{\theta\text{-ss}} \to S$ denote the structure morphism and let $p \in |\mathcal{M}_{d,S}^{\theta\text{-ss}}|$ be a point. If $\pi(p) \in S$ is closed and p is represented by a geometrically polystable representation M defined over a finite extension L of the residue field $\kappa(\pi(p))$, then p is closed. If S is a Jacobson scheme, the converse holds.*

Proof. Suppose that $\pi(p)$ is closed and that p is represented by a geometrically polystable representation M defined over a finite extension L of $\kappa(\pi(p))$. Since π is continuous, the fiber $\pi^{-1}(\pi(p))$ is closed, so we may assume $S = \operatorname{Spec} k$ where $k = \kappa(\pi(p))$. Suppose for a contradiction that p is not closed. The closed substack with underlying set $\overline{\{p\}}$ with its reduced substack structure is of finite type over k, hence contains a closed point p' represented by a map $\operatorname{Spec} L' \to \mathcal{M}_{d,k}^{\theta\text{-ss}}$ corresponding to a representation M'. If M' is not geometrically polystable, then there is a finite extension L'' such that $M' \otimes_{L'} L''$ is not polystable, and by Lemma 5.3.1, the point $[\operatorname{gr}(M' \otimes_{L'} L'')]$ is in the closure of p', contrary to the fact that p' is closed. Thus, M' is geometrically polystable.

Let now K be a compositum of L and L' over k. On the one hand, by assumption $p \neq p'$, so $M \otimes K$ and $M' \otimes K$ are not isomorphic. On the other hand, for any stable summand E of $M \otimes K$, by upper semicontinuity we have

$$\dim \operatorname{Hom}(E, M \otimes K) \leq \dim \operatorname{Hom}(E, M' \otimes K).$$

Since both $M \otimes K$ and $M' \otimes K$ are polystable of the same dimension vector and the above dimensions give the multiplicities of the stable summand E, we conclude they must be isomorphic, which gives a contradiction. Thus, p is closed.

Conversely, suppose that S is Jacobson and $p \in |\mathcal{M}_{d}^{\theta\text{-ss}}|$ is closed. Since π is of finite type, the image point $\pi(p) \in S$ is closed by [47, Tag 01TB]. On the other hand since p is in particular closed in the fiber $\pi^{-1}(\pi(p)) = \mathcal{M}_{d,\kappa(\pi(p))}^{\theta\text{-ss}}$, by the Nullstellensatz p is represented by a map $\operatorname{Spec} L \to \mathcal{M}_{d}^{\theta\text{-ss}}$ corresponding to a representation M over L, where L is a finite extension of $\kappa(\pi(p))$. If M is not geometrically polystable, then there is a finite extension L' of L such that the point $[\operatorname{gr}(M \otimes L')]$ is distinct from p but lies in the closure of p by Lemma 5.3.1. Thus, M must be geometrically polystable. □

If S is not Jacobson, the image of a closed point $p \in |\mathcal{M}_{d,S}^{\theta\text{-ss}}|$ in S may not be closed.

Example 5.3.3. Let R be a discrete valuation ring with uniformizer π and fraction field K, and consider the representation $K \to K$ of the Jordan quiver given by multiplication by π^{-1}. It corresponds to a closed point in $\mathcal{M}_{1,\operatorname{Spec} R}$ whose image in $\operatorname{Spec} R$ is not closed.

Corollary 5.3.4. *Over an algebraically closed field k, the closed points of $\mathcal{M}_d$ are in bijection with isomorphism classes of semisimple k-representations.*

Proof. We have $\mathcal{M}_d = \mathcal{M}_d^{\theta\text{-ss}}$ for the stability function $\theta = 0$, with respect to which a representation is polystable if and only if it is semisimple. □

Remark 5.3.5. As an application of the classification of closed points of $\mathcal{M}_d^{\theta\text{-ss}}$, one can show that the line bundle $\mathcal{L}_\theta$ on $\mathcal{M}_d^{\theta\text{-ss}}$ descends to the moduli space $\mathrm{M}_d^{\theta\text{-ss}}$ (whose existence we will obtain in Corollary 5.5.7 and Remark 5.5.8), provided we work over a noetherian base of characteristic 0, so that we can apply [2, Theorem 10.3]. Indeed it suffices to show the stabilizer of any closed geometric point $x : \operatorname{Spec} k \to \mathcal{M}_d^{\theta\text{-ss}}$ acts trivially on $x^*\mathcal{L}_\theta$ and since such closed points correspond to θ-polystable representations, whose automorphism groups are products of general linear groups, one can directly check the action on $x^*\mathcal{L}_\theta$ is trivial. We do not give the details, as we will prove in greater generality that $\mathcal{L}_\theta$ descends in Section 6.

5.4. *Local reductivity.* We first give a criterion to ensure that points on a stack specialize to closed points.

Lemma 5.4.1. *If $\mathcal{X}$ is a quasi-compact algebraic stack with quasi-compact diagonal, then every point in the topological space $|\mathcal{X}|$ specializes to a closed point.*

Proof. By [47, Tag 0DQN], $|\mathcal{X}|$ is a spectral topological space, and by [26, Theorem 6], a spectral topological space underlies some affine scheme, meaning that there is a homeomorphism $|\mathcal{X}| \cong |\operatorname{Spec} A|$ for some commutative ring A. But on an affine scheme, every point specializes to a closed point. □

We remark that Lemma 5.4.1 is not true for arbitrary quasi-compact stacks, not even algebraic spaces. For example, if k is a field of characteristic 0 and X is the quotient of $\mathbb{A}_k^1$ by the free action of $\mathbb{Z}$ given by the dual action $n \cdot x \mapsto x + n$ on $k[x]$, then X has infinitely many points but the trivial topology.

Now we will start applying Lemma 5.4.1 by proving that the moduli stack of all representations is locally reductive.

Lemma 5.4.2. *The stack $\mathcal{M}_{d,S}$ is locally reductive.*

Proof. Since the stack $\mathcal{M}_{d,S}$ is of finite type over the noetherian base scheme S, it is quasi-compact, and by Proposition 3.1.2 the diagonal of $\mathcal{M}_{d,S}$ is affine, hence quasi-compact, so it follows from Lemma 5.4.1 that every point of $|\mathcal{X}|$ specializes to a closed point.

Thus, it remains to find étale local quotient presentations. By covering S by affine open subschemes, we may assume that S itself is affine. Since $\mathcal{M}_{d,S}$ is now a global quotient stack of the affine scheme $\mathrm{R}_{d,S}$ by the reductive group scheme $\mathrm{G}_{d,S}$, by choosing an embedding $\mathrm{G}_{d,S} \hookrightarrow \mathrm{GL}_{N,S}$, we have

$$\mathcal{M}_{d,S} \cong [\mathrm{R}_{d,S}/\mathrm{G}_{d,S}] \cong [X/\mathrm{GL}_{N,S}],$$

where X is the quotient of the affine scheme $\mathrm{R}_{d,S} \times_S \mathrm{GL}_{N,S}$ by the free diagonal action of $\mathrm{G}_{d,S}$. Since X is an affine scheme, this presentation shows that $\mathcal{M}_{d,S}$ is locally reductive. □

We next show that $\mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ is locally reductive using the determinantal sections constructed in Section 4.1. We comment on the (non-)necessity of the condition on the stability function θ in Remark 5.5.8. First we make an observation that will be used in the proof of Proposition 5.4.4 and Theorem 6.3.1.

Remark 5.4.3. (i) Let k be a field, $B \twoheadrightarrow k$ a surjection of rings, and let V be a representation of dimension vector d over k. We can extend V to a representation $\mathcal{V}$ over $\operatorname{Spec} B$ as follows. Since V_i is free for each vertex i, we can take $\mathcal{V}_i = B^{\oplus d_i}$, and since each map $V_a : V_{s(a)} \to V_{t(a)}$ is represented by a matrix with entries in k, we can lift these matrices to B to obtain $\mathcal{V}_a$.

(ii) If $x \in \operatorname{Spec} A$ is a point in an affine scheme and k is a finite separable extension of the residue field $\kappa(x)$, we can find an étale morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ whose fiber over x is $\operatorname{Spec} k$. For instance, we can take $A = A_a[T]/(f)$, where $f \in A[T]$ is a monic polynomial whose image in $\kappa(x)[T]$ is the minimal polynomial of a primitive element of k over $\kappa(x)$ and $a \in A$ is chosen suitably.

Proposition 5.4.4. *If $\theta = \theta_\beta$ or $\theta = \eta_\beta$ for a dimension vector β, then the stack $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ is locally reductive.*

Proof. We consider the case $\theta = \eta_\beta$. By Corollary 3.2.4, $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ is quasi-compact with quasi-compact diagonal, so by Lemma 5.4.1 every point specializes to a closed point. We are again left with finding étale local quotient presentations as in Definition 5.2.4. The question is Zariski local on S, so we may assume $S = \operatorname{Spec} C$ is affine. We begin by reducing to the case $S = \operatorname{Spec} \mathbb{Z}$. Namely, suppose $[\operatorname{Spec} A/\operatorname{GL}_N] \to \mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ is étale and consider the diagram

$$\begin{tikzcd}
{[\operatorname{Spec} A/\operatorname{GL}_N] \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} C} \arrow[r] \arrow[d] & {[\operatorname{Spec} A/\operatorname{GL}_N]} \arrow[d] \\
\mathcal{M}^{\theta\text{-ss}}_{d,C} \arrow[r] \arrow[d] & \mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \arrow[d] \\
\operatorname{Spec} C \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

The top-left vertical arrow is étale, and moreover

$$[\operatorname{Spec} A/\operatorname{GL}_N] \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} C \cong [\operatorname{Spec}(A \otimes_{\mathbb{Z}} C)/\operatorname{GL}_N]$$

is of the required form, so base changing an étale cover of $\mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ by quotient presentations yields the desired cover of $\mathcal{M}^{\theta\text{-ss}}_{d,C}$. Thus, we may assume $S = \operatorname{Spec}\mathbb{Z}$.

Since $\operatorname{Spec}\mathbb{Z}$ is Jacobson, by Proposition 5.3.2 any closed point of $\mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ is represented by a map $x : \operatorname{Spec} k \to \mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ corresponding to a θ-semistable representation M defined over a finite field k. Using Proposition 4.1.5, we can find a representation $\overline{V}$ over $\bar{k}$ with $\underline{\dim}\, \overline{V} = m\beta$ for some $m > 0$ such that

$$\operatorname{Hom}(M \otimes \bar{k}, \overline{V}) = \operatorname{Ext}(M \otimes \bar{k}, \overline{V}) = 0.$$

The representation $\overline{V}$ is defined over some finite extension k' of k, meaning that $\overline{V} \cong V' \otimes_{k'} \bar{k}$ for some representation V' over k'. Note that

$$\operatorname{Hom}(M \otimes k', V') \otimes \bar{k} = \operatorname{Hom}(M \otimes \bar{k}, \overline{V}) = 0,$$

hence $\operatorname{Hom}(M \otimes k', V') = 0$. We now replace k by k' and M by $M \otimes_k k'$, as the two define the same point of $\mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$. Since k is separable over its prime field $\mathbb{F}_p$, using Remark 5.4.3 we can find an étale map $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{Z}$ whose fiber over $\operatorname{Spec} \mathbb{F}_p$ is $\operatorname{Spec} k$ and an extension $\mathcal{V}$ of V to $\operatorname{Spec} B$.

Now consider the cartesian diagram

$$\begin{tikzcd} \mathcal{M}^{\theta\text{-ss}}_{d,B} \arrow[r, "f"] \arrow[d] & \mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

The bottom morphism is étale, hence so is the top morphism f. Moreover, f induces an isomorphism on automorphism groups because it arises as the base change of a morphism of schemes [47, Tag 0DUB]. Observe that the map x can be factored through f since we chose B that admits a quotient isomorphic to k. Now the representation $\mathcal{V}$ defines a section σ of the line bundle $\mathcal{L}_\theta^{\otimes m}$ on the larger stack $\mathcal{M}_{d,B}$ and by Proposition 3.3.3 this section is nonzero at x. Let $\mathcal{U} \subset \mathcal{M}_{d,B}$ be the nonvanishing locus of σ. Using Proposition 4.1.5(b) we see that in fact $\mathcal{U} \subseteq \mathcal{M}^{\theta\text{-ss}}_{d,B}$.

We claim that $\mathcal{U}$ is of the desired form $[\operatorname{Spec} A/\operatorname{GL}_N]$. To see this, recall that $\mathcal{M}_{d,B} \cong [\mathrm{R}_{d,B}/\mathrm{G}_{d,B}]$ and let $\varphi : \mathrm{R}_{d,B} \to \mathcal{M}_{d,B}$ denote the quotient map. The preimage $U = \varphi^{-1}(\mathcal{U}) \subset \mathrm{R}_{d,B}$ is $\mathrm{G}_{d,B}$-invariant, and moreover U is the nonvanishing locus of the section $\varphi^*\sigma$, hence affine since $\varphi^*\mathcal{L}_\theta \cong \mathcal{O}_{\mathrm{R}_{d,B}}$. Thus, we see that $\mathcal{U} \cong [U/\mathrm{G}_{d,B}]$. Finally, since we can embed $\mathrm{G}_{d,B}$ into $\mathrm{GL}_{N,B}$ as a closed subgroup for a suitable N, we can write $[U/\mathrm{G}_{d,B}] \cong [\operatorname{Spec} A/\operatorname{GL}_{N,B}]$ as in Lemma 5.4.2. In conclusion, we have exhibited an étale neighborhood $[\operatorname{Spec} A/\operatorname{GL}_N] \to \mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ of the point x. □

5.5. *Θ-reductivity and S-completeness for quiver representations.* In this section we give a direct moduli-theoretic proof that the stacks $\mathcal{M}_{d,S}$ and $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ are Θ-reductive and S-complete. First, we show that the stack $\mathcal{M}_{d,S}$ satisfies Hartogs's principle. For this, we use the following version of Hartogs's lemma.

Lemma 5.5.1. *Let A be a regular local ring of dimension* 2 *with closed point* $0 \in \operatorname{Spec} A$.

(i) *Any locally free sheaf $\mathcal{F}$ of finite rank on* $\operatorname{Spec} A \setminus \{0\}$ *is free.*

(ii) *If $\mathcal{F}$ is a free sheaf of finite rank on* $\operatorname{Spec} A$, *then any automorphism of $\mathcal{F}$ over* $\operatorname{Spec} A \setminus \{0\}$ *extends uniquely to* $\operatorname{Spec} A$.

Proof. For (i), note that since $\operatorname{Spec} A$ is noetherian, the coherent sheaf $\mathcal{F}$ on $\operatorname{Spec} A \setminus \{0\}$ admits a coherent extension $\mathcal{G}$ to all of $\operatorname{Spec} A$. The double dual $\mathcal{G}^{\vee\vee}$ is reflexive, hence free since A is regular local of dimension 2. Since $\mathcal{G}^{\vee\vee}$ agrees with $\mathcal{G}$ wherever $\mathcal{G}$ is locally free, we see that $\mathcal{G}^{\vee\vee}$ is also an extension of $\mathcal{F}$, and so $\mathcal{F}$ is free itself.

For (ii), let $n = \operatorname{rk} \mathcal{F}$. After choosing an isomorphism $\mathcal{F} \cong \mathcal{O}^{\oplus n}$, an automorphism ϕ of $\mathcal{F}|_{\operatorname{Spec} A\setminus\{0\}}$ corresponds to an invertible $n \times n$ matrix consisting of functions on $\operatorname{Spec} A \setminus \{0\}$. By the usual Hartogs's lemma, these functions extend uniquely to functions on all of $\operatorname{Spec} A$. The determinant of the resulting matrix is nonzero at every codimension 1 point of $\operatorname{Spec} A$, hence is a unit of A, so the corresponding endomorphism of $\mathcal{F}$ is invertible. $\square$

Proposition 5.5.2. *For any regular local ring A of dimension 2 with closed point $0 \in \operatorname{Spec} A$, any morphism $\operatorname{Spec} A \setminus \{0\} \to \mathcal{M}_{d,S}$ extends uniquely to $\operatorname{Spec} A \to \mathcal{M}_{d,S}$. In particular, $\mathcal{M}_{d,S}$ is both Θ-reductive and S-complete by Remark 5.2.2.*

Proof. The morphism $\operatorname{Spec} A \setminus \{0\} \to \mathcal{M}_d$ corresponds to a family

$$\mathcal{F} = ((\mathcal{F}_i)_{i \in Q_0}, (\mathcal{F}_a : \mathcal{F}_{s(a)} \to \mathcal{F}_{t(a)})_{a \in Q_1})$$

over $\operatorname{Spec} A \setminus \{0\}$. By Lemma 5.5.1, each vector bundle $\mathcal{F}_i$ on $\operatorname{Spec} A \setminus \{0\}$ extends uniquely to $\widetilde{\mathcal{F}}_i$ on $\operatorname{Spec} A$ and each map $\mathcal{F}_a$ extends uniquely to $\widetilde{\mathcal{F}}_a : \widetilde{\mathcal{F}}_{s(a)} \to \widetilde{\mathcal{F}}_{t(a)}$ on $\operatorname{Spec} A$. This family $\widetilde{\mathcal{F}}$ thus defines the unique morphism $\operatorname{Spec} A \to \mathcal{M}_d$ extending the given one. $\square$

We are now in a position to apply Theorem 5.2.3 to $\mathcal{M}_{d,S}$, which we know is locally reductive by Lemma 5.4.2, and has affine diagonal by Proposition 3.1.2.

Corollary 5.5.3. *The stack $\mathcal{M}_{d,S}$ admits a separated adequate moduli space $\mathrm{M}_{d,S}$.*

We next turn our attention to the stack $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ of θ-semistable representations. As the following example shows, the stack $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ does not in general satisfy Hartogs's principle.

Example 5.5.4. Consider the 2-Kronecker quiver

$$Q: \; 1 \rightrightarrows 2$$

and the stability function $\theta(n_1, n_2) = n_1 - n_2$. Let k be a field and $\mathcal{F}$ be the family of representations of Q of dimension vector $d = (1, 1)$ over $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$, where the arrows are multiplication by x and multiplication by y. The representation $\mathcal{F}_t$ is stable when $t \in \mathbb{A}^2 \setminus \{0\}$ but unstable when $t = 0$, as it is destabilized by the subrepresentation $k \rightrightarrows 0$. In other words, the family $\mathcal{F}$ gives a map $\mathbb{A}^2 \setminus \{0\} \to \mathcal{M}_d^{\theta\text{-s}}$ whose unique extension to a map $\mathbb{A}^2 \to \mathcal{M}_d$ does not factor through $\mathcal{M}_d^{\theta\text{-ss}}$.

In order to establish Θ-reductivity and S-completeness, we need a moduli-theoretic interpretation of families of representations over Θ_R and $\overline{\mathrm{ST}}_R$. Throughout we will use R to denote a DVR with fraction field K and residue field $\kappa = R/\pi$, where π is a uniformizer of R.

Proposition 5.5.5. *The stack $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ is Θ-reductive.*

Proof. Since $\Theta_R \setminus \{0\}$ is the union of $\operatorname{Spec}(R)$ and Θ_K over $\operatorname{Spec}(K)$, a morphism $\Theta_R \setminus \{0\} \to \mathcal{M}_d^{\theta\text{-ss}}$ is given by a family $\mathcal{F}$ of θ-semistable representations over $\operatorname{Spec}(R)$ with a filtration

$$0 = \mathcal{F}_K^0 \subset \mathcal{F}_K^1 \subset \cdots \subset \mathcal{F}_K^r = \mathcal{F}_K$$

of the generic fiber $\mathcal{F}_K$ such that the successive quotients $\mathcal{F}_K^\ell / \mathcal{F}_K^{\ell-1}$ are θ-semistable.

Since $\mathcal{M}_d$ is Θ-reductive by Proposition 5.5.2, the filtration $\mathcal{F}_K^\bullet$ extends uniquely to a filtration $\mathcal{F}^\bullet$ of $\mathcal{F}$. Thus it suffices to verify that the associated graded of this filtration over the special fiber is

θ-semistable, or equivalently that $\mathcal{F}_\kappa^\ell/\mathcal{F}_\kappa^{\ell-1}$ are all θ-semistable. Since θ is constant in flat families, we have $\theta(\mathcal{F}_\kappa^\ell) = \theta(\mathcal{F}_K^\ell) = 0$, and as $\mathcal{F}_\kappa^\ell$ is a subrepresentation of the θ-semistable representation $\mathcal{F}_\kappa$ of the same slope, it is also θ-semistable and consequently all the $\mathcal{F}_\kappa^\ell/\mathcal{F}_\kappa^{\ell-1}$ are θ-semistable. □

Proposition 5.5.6. *The stack $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ is S-complete.*

Proof. Let $\overline{\mathrm{ST}}_R \setminus \{0\} \to \mathcal{M}_d^{\theta\text{-ss}}$ be a morphism. By Proposition 5.5.2 the morphism extends uniquely to $\overline{\mathrm{ST}}_R \to \mathcal{M}_d$, so we must show that the image of $0 \in \overline{\mathrm{ST}}_R$ is contained in $\mathcal{M}_d^{\theta\text{-ss}}$.

The morphism $\overline{\mathrm{ST}}_R \to \mathcal{M}_d$ is equivalent to a diagram of representations

$$\cdots \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{F}^{\ell-1} \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{F}^{\ell} \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{F}^{\ell+1} \underset{t}{\overset{s}{\rightleftarrows}} \cdots$$

of Q over Spec R, such that

- each map s and t is injective,
- $s \circ t = t \circ s = \pi$,
- s is an isomorphism for $\ell \gg 0$ and t is an isomorphism for $\ell \ll 0$, and
- the induced maps $s : \mathcal{F}^{\ell-1}/t\mathcal{F}^\ell \to \mathcal{F}^\ell/t\mathcal{F}^{\ell+1}$ and $t : \mathcal{F}^{\ell+1}/s\mathcal{F}^\ell \to \mathcal{F}^\ell/s\mathcal{F}^{\ell-1}$ are injective.

The restriction to $\overline{\mathrm{ST}}_R \setminus \{0\} \cong \operatorname{Spec} R \coprod_{\operatorname{Spec} K} \operatorname{Spec} R$ corresponds to the two θ-semistable representations

$$\mathcal{E} := \operatorname{colim}(\mathcal{F}^{\ell-1} \xleftarrow{t} \mathcal{F}^\ell) \quad \text{and} \quad \mathcal{F} := \operatorname{colim}(\mathcal{F}^\ell \xrightarrow{s} \mathcal{F}^{\ell+1})$$

such that the restrictions $\mathcal{E}_K$ and $\mathcal{F}_K$ to Spec K are isomorphic. Over the closed subsets $\Theta_\kappa \xhookrightarrow{s=0} \overline{\mathrm{ST}}_R$ and $\Theta_\kappa \xhookrightarrow{t=0} \overline{\mathrm{ST}}_R$ we obtain filtrations

$$\cdots \xhookrightarrow{t} \mathcal{F}^{\ell+1}/s\mathcal{F}^\ell \xhookrightarrow{t} \mathcal{F}^\ell/s\mathcal{F}^{\ell-1} \xhookrightarrow{t} \cdots \hookrightarrow \mathcal{E}_\kappa \quad \text{and} \quad \cdots \xhookrightarrow{s} \mathcal{F}^{\ell-1}/t\mathcal{F}^\ell \xhookrightarrow{s} \mathcal{F}^\ell/t\mathcal{F}^{\ell+1} \xhookrightarrow{s} \cdots \hookrightarrow \mathcal{F}_\kappa,$$

respectively. The image of $0 \in \overline{\mathrm{ST}}_R$ corresponds to the common associated graded

$$\bigoplus_{\ell \in \mathbb{Z}} \frac{\mathcal{F}^\ell/t\mathcal{F}^{\ell+1}}{s(\mathcal{F}^{\ell-1}/t\mathcal{F}^\ell)} \cong \bigoplus_{\ell \in \mathbb{Z}} \frac{\mathcal{F}^\ell/s\mathcal{F}^{\ell-1}}{t(\mathcal{F}^{\ell+1}/s\mathcal{F}^\ell)} \cong \bigoplus_{\ell \in \mathbb{Z}} \frac{\mathcal{F}^\ell}{s\mathcal{F}^{\ell-1} + t\mathcal{F}^{\ell+1}},$$

which we must show is θ-semistable.

By assumption we have for all ℓ

$$0 = \theta(\mathcal{E}_\kappa) \geq \theta(\mathcal{F}^\ell/s\mathcal{F}^{\ell-1}) = \theta(\mathcal{F}^\ell) - \theta(\mathcal{F}^{\ell-1}),$$

since $\mathcal{F}^{\ell-1} \cong s\mathcal{F}^{\ell-1} \subseteq \mathcal{F}^\ell$. Similarly,

$$0 = \theta(\mathcal{F}_\kappa) \geq \theta(\mathcal{F}^\ell/t\mathcal{F}^{\ell+1}) = \theta(\mathcal{F}^\ell) - \theta(\mathcal{F}^{\ell+1}).$$

Thus, we must have $\theta(\mathcal{F}^\ell) = \theta(\mathcal{F}^{\ell+1})$ for all ℓ. Moreover, as the value of θ is constant in flat families and $\mathcal{F}_K^\ell \cong \mathcal{F}_K$, we must have $\theta(\mathcal{F}^\ell) = 0$ for all ℓ. Thus, the quotients $\mathcal{F}^\ell/t\mathcal{F}^{\ell+1}$ are all semistable with $\theta(\mathcal{F}^\ell/t\mathcal{F}^{\ell+1}) = 0$, and the same is true for the quotients $(\mathcal{F}^\ell/t\mathcal{F}^{\ell+1})/s(\mathcal{F}^{\ell-1}/t\mathcal{F}^\ell)$. □

Corollary 5.5.7. *If $\theta = \theta_\beta$ or $\theta = \eta_\beta$ for a dimension vector β, then the stack $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ admits an adequate moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$, separated over S.*

Proof. We have verified that $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ is locally reductive in Proposition 5.4.4, Θ-reductive in Proposition 5.5.5, and S-complete in Proposition 5.5.6. It moreover has affine diagonal, and thus affine stabilizers and separated diagonal. Therefore, it admits an adequate moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$, separated over S, by [6, Theorem 5.4] (see Theorem 5.2.3). □

Recall that, if Q is acyclic, then any stability function θ for which there exists a semistable representation supported on Q_0 can be written in this form by Lemma 2.3.5.

Remark 5.5.8. It follows from the GIT construction that $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ is locally reductive for an arbitrary stability function θ. We are currently unable to remove the hypothesis $\theta = \theta_\beta$ or $\theta = \eta_\beta$ using our methods.

When the base scheme S has characteristic 0, S-completeness implies local reductivity [6, Proposition 3.47, Theorem 2.2], and it follows that the stack $\mathcal{M}^{\theta\text{-ss}}_{d,S}$ admits a separated *good* moduli space for any choice of θ.

Corollary 5.5.7 also follows from GIT, as explained in [8, Theorem 1.5]. We have given a purely moduli-theoretic argument by appealing to the existence result for adequate moduli spaces.

5.6. ***Langton's semistable extension theorem for quiver representations.*** In this section we will show that when Q is an acyclic quiver, the adequate moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is proper over S, where S is a noetherian scheme. This is a particular instance of Proposition 5.6.1 describing properness of maps between adequate moduli spaces.

Proposition 5.6.1. *Let Q be a (not necessarily acyclic) quiver and let θ be a stability function. The morphism*

$$\mathrm{M}^{\theta\text{-ss}}_{d,S} \to \mathrm{M}_{d,S}$$

on adequate moduli spaces is proper whenever the adequate moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ exists (see Corollary 5.5.7).

The proof of Proposition 5.6.1 relies on the next result, which is an analogue of the main result of [35].

Proposition 5.6.2. *Let R be a DVR with uniformizer π, fraction field K and residue field κ. Let M be a representation over R such that the generic fiber $M \otimes_R K$ is θ-semistable. There exists a subrepresentation $M' \subseteq M$ such that $M' \otimes_R K$ and $M \otimes_R K$ are isomorphic, and $M' \otimes_R \kappa$ is θ-semistable.*

Proof. If $\overline{M} := M \otimes_R \kappa$ is θ-semistable, then there is nothing to prove. If this is not the case, let $\overline{F}$ be the maximal destabilizing subrepresentation of $\overline{M}$. This defines a subrepresentation $M^{(1)} \subset M$ of Q in the following way. For every $i \in Q_0$, let $(f_i^1, \dots, f_i^{s_i}, e_i^{s_i+1}, \dots, e_i^{d_i})$ be a basis of $\overline{M}_i$ extending a basis $f_i^1, \dots, f_i^{s_i}$ of $\overline{F}$. We can further lift these to bases of each M_i, which we denote by $(\tilde{f}_i^1, \dots, \tilde{f}_i^{s_i}, \tilde{e}_i^{s_i+1}, \dots, \tilde{e}_i^{d_i})$. For every $i \in Q_0$, we define $M_i^{(1)}$ as the subset of M_i spanned by $(\tilde{f}_i^1, \dots, \tilde{f}_i^{s_i}, \pi\tilde{e}_i^{s_i+1}, \dots, \pi\tilde{e}_i^{d_i})$. For every $a : i \to j$, the restriction of M_a to $M_i^{(1)}$ lands in $M_j^{(1)}$, thus this defines a representation $M^{(1)}$ of Q. If $\overline{M^{(1)}}$ is θ-semistable, then we are done. Otherwise, let $\overline{F}_1$ be the maximal destabilizing subrepresentation of $\overline{M^{(1)}}$ which, following the above procedure, defines a subrepresentation $M^{(2)} \subset M^{(1)}$. We can apply the arguments of [35, Section 5] to show that this procedure will terminate, i.e., there is an n such that $\overline{M^{(n)}}$ is θ-semistable. $\square$

Proof of Proposition 5.6.1. The map $\mathrm{M}_{d,S}^{\theta\text{-ss}} \to \mathrm{M}_{d,S}$ is separated and of finite type over S since both $\mathrm{M}_{d,S}^{\theta\text{-ss}}$ and $\mathrm{M}_{d,S}$ are, so it suffices to show that it is universally closed. Moreover, since the adequate moduli space map $\mathcal{M}_{d,S}^{\theta\text{-ss}} \to \mathrm{M}_{d,S}^{\theta\text{-ss}}$ is surjective, it is enough to show that the map $\mathcal{M}_{d,S}^{\theta\text{-ss}} \to \mathrm{M}_{d,S}$ is universally closed. This is local on the base scheme S, so since the formation of the adequate moduli space commutes with base change along open embeddings, we may assume that $S = \operatorname{Spec} B$ for a noetherian ring B.

To show that $\mathcal{M}_{d,S}^{\theta\text{-ss}} \to \mathrm{M}_{d,S}$ is universally closed, we verify the valuative criterion for universal closedness [47, Tag 0H2C]: if for any DVR R with fraction field K and the square of solid arrows in the diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r, dashed] \arrow[d, dashed] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{M}_{d,S}^{\theta\text{-ss}} \arrow[d] \\
\operatorname{Spec} R' \arrow[r, dashed] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathrm{M}_{d,S}
\end{tikzcd}$$

commutes, there exists a field extension K' of K, a DVR $R' \subset K'$ dominating R, and a dashed diagonal arrow $\operatorname{Spec} R' \to \mathcal{M}_{d,S}^{\theta\text{-ss}}$ making the whole diagram commute, then the rightmost morphism is universally closed.

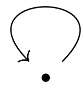
Since $S = \operatorname{Spec} B$ is affine, we have by Proposition 3.1.4 that $\mathcal{M}_{d,S} = [\operatorname{Spec} A / \mathrm{GL}_N]$, where A is a polynomial ring over B. Moreover, since B is noetherian, the map $\pi : \mathcal{M}_{d,S} \to \mathrm{M}_{d,S}$ is of finite type by Theorem 5.1.4(ii). Thus, we may apply [6, Theorem A.8] to find a finite extension K' of K, a DVR $R' \supseteq R$ dominating R, and a morphism $\psi : \operatorname{Spec} R' \to \mathcal{M}_{d,S}$ such that the diagram of solid arrows

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[dd] & \operatorname{Spec} K \arrow[r] \arrow[dd] & \mathcal{M}_{d,S}^{\theta\text{-ss}} \arrow[d, "\iota"] \\
 & & \mathcal{M}_{d,S} \arrow[d, "\pi"] \\
\operatorname{Spec} R' \arrow[r] \arrow[uurr, dashed, "\psi'"] \arrow[urr, "\psi"] & \operatorname{Spec} R \arrow[r] & \mathrm{M}_{d,S}
\end{tikzcd}$$

commutes.

The map $\psi : \operatorname{Spec} R' \to \mathcal{M}_{d,S}$ corresponds to a family of representations over R' with θ-semistable generic fiber. By Proposition 5.6.2, there exists another morphism $\psi' : \operatorname{Spec} R' \to \mathcal{M}_{d,S}^{\theta\text{-ss}}$ such that the restrictions of ψ and ψ' to $\operatorname{Spec} K'$ agree. Since R' is a DVR and $\mathrm{M}_{d,S}$ is separated, this implies that the two morphisms $\pi \circ \psi$ and $\pi \circ \iota \circ \psi'$ are equal [47, Tag 03KU]. Thus, the top and bottom rows together with the arrow ψ' in the above diagram are what we set out to construct. $\square$

If Q is not acyclic, then $\mathrm{M}_{d,k}$ is not proper over $\operatorname{Spec}(k)$. Consider for instance the Jordan quiver



In this case all representations are θ-semistable, since the only stability function is the zero function. For an explicit example (using the notation of Proposition 5.6.2) that illustrates how the valuative criterion for properness fails it suffices to consider the one-dimensional representation $M_a : K \to K$, $1 \mapsto \pi^{-1}$ over K. Then M has no lift to any representation over R. In fact, the space of d-dimensional representations of Q where $d = (1)$ is represented by the affine line, because the group $\mathrm{G}_{d,k} \cong \mathbb{G}_{\mathrm{m}}$ acts trivially on $\mathrm{R}_{d,k} = \mathbb{A}^1$.

Proposition 5.6.3. *If Q is acyclic, the stack $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ is universally closed over S.*

Proof. We check the valuative criterion [47, Tag 0H2C], which translates to: for a discrete valuation ring R with uniformizer π and fraction field K, if M is a semistable representation over K, there exists a semistable representation N over R and an isomorphism $\phi : M \xrightarrow{\sim} N|_K$. By Proposition 5.6.2, it suffices to find such a family N without requiring that $N \otimes_R (R/\pi)$ is semistable.

Choose an admissible ordering of $Q_0 = \{1, \dots, n\}$ and a K-basis of M_i for each $i \in Q_0$. In these bases, the maps $M_a : M_{s(a)} \to M_{t(a)}$ are given by matrices A_a over K. For each i, let N_i be a free R-module with the same basis. We define integers m_i for $i \in Q_0$ by setting $m_1 = 0$ and inductively choosing m_i in such a way that for each arrow a with $t(a) = i$, the matrix $N_a = \pi^{m_i - m_{s(a)}} A_a$ has entries in R. Now we can set $N = (\{N_i\}_{i \in Q_0}, \{N_a\}_{a \in Q_1})$, and $\phi : M \xrightarrow{\sim} N|_K$ is given by taking ϕ_i to be multiplication by π^{m_i}. □

Corollary 5.6.4. *If Q is acyclic, the adequate moduli space $\mathrm{M}_{d,S}^{\theta\text{-ss}}$ is proper over S.*

Proof. The map $\mathrm{M}_{d,S}^{\theta\text{-ss}} \to S$ is separated by Corollary 5.5.7, so it suffices to show that it is universally closed, and since the map $\mathcal{M}_{d,S}^{\theta\text{-ss}} \to \mathrm{M}_{d,S}^{\theta\text{-ss}}$ is surjective, this is equivalent to $\mathcal{M}_{d,S}^{\theta\text{-ss}} \to S$ being universally closed, which is Proposition 5.6.3. □

The following gives a complete description of the moduli space $\mathrm{M}_{d,S}$ when Q is acyclic.

Proposition 5.6.5. *If Q is acyclic, the structure morphism $\mathrm{M}_{d,S} \to S$ is an isomorphism.*

Proof. Consider first the case $S = \operatorname{Spec} k$ for a field k. We claim that the stack $\mathcal{M}_{d,k}$ has a unique closed point. Indeed, by Proposition 5.3.2, any closed point is represented by a semisimple representation M defined over a finite extension L of k. Since Q is acyclic, the only such semisimple representation is $\bigoplus_{i \in Q_0} S(i)^{\oplus d_i}$ which is already defined over k. Thus, by Theorem 5.1.4(iii), the adequate moduli space M_d has a unique closed point which is defined over k, and since M_d is of finite type over k, this point must be the only one. Finally, as the stack $\mathcal{M}_d$ is reduced, so is M_d, and thus we conclude that $\mathrm{M}_d \cong \operatorname{Spec} k$.

Let now S be a noetherian scheme. For any point $x \in S$, the base change map

$$\mathrm{M}_{d,\kappa(x)} \to \mathrm{M}_{d,S} \times_S \operatorname{Spec} \kappa(x)$$

is bijective by Theorem 5.1.4(v), and by the above $\mathrm{M}_{d,\kappa(x)}$ is isomorphic to $\operatorname{Spec} \kappa(x)$. Since $\mathrm{M}_{d,S} \to S$ is also proper by Corollary 5.6.4, it is finite by Zariski's main theorem [47, Tag 0A4X]. Thus, we have

$M_{d,S} \cong \underline{\mathrm{Spec}}_{\mathcal{O}_S} \mathcal{A}$ for a sheaf $\mathcal{A}$ of finite $\mathcal{O}_S$-algebras. Moreover, since for any $x \in S$, the induced map $\mathcal{O}_S|_x \to \mathcal{A}|_x$ is an isomorphism, the structure map $\mathcal{O}_S \to \mathcal{A}$ is surjective, so $\mathrm{M}_{d,S} \to S$ is a closed embedding. On the other hand, the composition $\mathcal{M}_{d,S} \to \mathrm{M}_{d,S} \to S$ is scheme-theoretically surjective, hence so is $\mathrm{M}_{d,S} \to S$, so it must be an isomorphism. □

A more general result regarding the structure of $\mathrm{M}_{d,S}$ is proved using GIT-methods by Donkin in [18, Theorem and Remark], generalizing the result for fields in characteristic 0 due to Le Bruyn and Procesi [36, Theorem 1].

6. Projectivity of the adequate moduli space

The aim of this section is to prove that the moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is projective over S when the quiver Q is acyclic. Recall that we defined a line bundle $\mathcal{L}_\theta$ on $\mathcal{M}^{\theta\text{-ss}}_d$ in Section 3.3. We begin by showing that $\mathcal{L}_\theta$ is semiample even if Q is not acyclic, which will imply Theorem B from the introduction. After this, we show with increasing generality that $\mathcal{L}_\theta$ is relatively ample over S.

6.1. *Global generation over a field.* Let Q be a quiver, d a dimension vector, and θ a stability function such that $\theta(d) = 0$.

Proposition 6.1.1. *Suppose k is a field and the stability function θ is of the form $\theta = \theta_\beta$ or $\theta = \eta_\beta$ for a dimension vector β. The line bundle $\mathcal{L}_\theta$ on $\mathcal{M}^{\theta\text{-ss}}_{d,k}$ is semiample and descends to a line bundle L_θ on the moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,k}$. In fact, if $m \in \mathbb{N}$ satisfies the inequality* (13), *then $\mathcal{L}_\theta^{\otimes m}$ is generated by finitely many global sections, and if k is algebraically closed, these sections can be taken to be of the form σ_V for a representation V of dimension vector $m\beta$.*

Proof. We give the proof when $\theta = \eta_\beta$, the other case follows similarly. Assume first that k is algebraically closed. Let $p \in \mathcal{M}^{\theta\text{-ss}}_{d,k}$ be a closed point corresponding to a θ-semistable representation M. By Corollary 4.1.4(b), a general representation V of dimension vector $m\beta$ satisfies $\mathrm{Hom}(M, V) = 0$, and by Proposition 3.3.3, the associated section σ_V of $\mathcal{L}_\theta^{\otimes m}$ is nonzero at the point p. Since $\mathcal{M}^{\theta\text{-ss}}_{d,k}$ is quasi-compact, the non-vanishing loci of finitely many such sections cover $\mathcal{M}^{\theta\text{-ss}}_{d,k}$.

Let now k be an arbitrary field. By the above, the pullback of $\mathcal{L}_\theta^{\otimes m}$ to $\mathcal{M}^{\theta\text{-ss}}_{d,\bar{k}}$ is generated by finitely many global sections when m satisfies (13), so the same holds for $\mathcal{L}_\theta^{\otimes m}$ by for example [48, Exercise 18.2.I].

To prove that $\mathcal{L}_\theta$ descends to $\mathrm{M}^{\theta\text{-ss}}_{d,k}$, we let $m > 0$ be an integer satisfying (13) so that both $\mathcal{L}_\theta^{\otimes m}$ and $\mathcal{L}_\theta^{\otimes m+1}$ are generated by finitely many globally sections. It follows from Lemma 5.1.5(iii) that $\mathcal{L}_\theta^{\otimes m}$ and $\mathcal{L}_\theta^{\otimes m+1}$ descend to line bundles L_m and L_{m+1} on $\mathrm{M}^{\theta\text{-ss}}_{d,k}$. Now $L_\theta := L_{m+1} \otimes L_m^\vee$ pulls back to $\mathcal{L}_\theta^{\otimes m+1} \otimes (\mathcal{L}_\theta^{\otimes m})^\vee = \mathcal{L}_\theta$, so we see that $\mathcal{L}_\theta$ itself descends. □

Proof of Theorem B. This now follows from Proposition 4.2.1 and Proposition 6.1.1, with the bound in Theorem B being derived just from the Euler matrix for Q and the dimension vector d, and not the more implicit bound in (13). □

Remark 6.1.2. Effective basepoint-freeness results as in Theorem B are of interest in general, and for moduli spaces in particular. For moduli of vector bundles on (smooth projective) curves there has been significant progress; for an overview, see [39, Section 7.2]. The moduli space $\mathrm{M}_C(r, \mathcal{L})$ of semistable vector bundles of rank r and fixed determinant $\mathcal{L}$ on a curve C of genus g has Picard rank 1 and its Picard group is generated by a determinantal line bundle. The best known bound on the basepoint-freeness of the linear system associated to the kth multiple of the generator is quadratic in the rank r (but independent of the genus g). These bounds are similar to the one in Theorem B, which are also quadratic in the entries of the dimension vector.

Conjecturally [39, Section 7.5] the true bound for basepoint-freeness on moduli of vector bundles is *linear* in the rank, and thus of the same order as the *square root* of the dimension of the moduli space.

For moduli of quiver representations one also expects room for improvement. Consider the following two ways in which $\mathbb{P}^n$ can be realized as a moduli space of quiver representations. First, using dimension vector $(1, 1)$ for the $(n+1)$-Kronecker quiver as in Example 4.2.2 and stability function $(1, -1)$ we obtain a bound linear in n, yet the Picard group is generated by the very ample line bundle $\mathcal{O}(1)$, hence the bound should be constant. On the other hand, following [25, page 218] we can also realize it as the moduli space for the 2-Kronecker quiver using dimension vector (n, n). Again using Example 4.2.2 we see that $\lambda = 0$, and thus the effective basepoint-freeness bound says that the generator is globally generated. See also Remark 4.2.3 for the case of general Dynkin and extended Dynkin quivers.

In general, Fujita's conjecture predicts a bound linear in the dimension of the moduli space. We obtain a bound that is quadratic in the entries of the dimension vector, and thus of the same order as the dimension of the moduli space which also grows quadratically in the entries of the dimension vector. This can be compared to Kollár's general effective basepoint-freeness result [34, Theorem 1.1], which is very far from the predicted bound.

6.2. ***Projectivity over a field.*** From now on, we assume that Q is acyclic, in which case Lemma 2.3.5(b) implies that $\theta = \eta_\beta$ for a unique dimension vector $\beta \in \mathbb{N}^{Q_0}$. We now prove Theorem A(ii) over a field.

Theorem 6.2.1. *Let k be a field and assume that Q is acyclic. The line bundle $\mathcal{L}_\theta$ descends to an ample line bundle L_θ on the moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,k}$. In particular, the moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ of θ-semistable representations with dimension vector d is a projective variety.*

Proof. Suppose first that k is algebraically closed. By Proposition 6.1.1, the line bundle $\mathcal{L}_\theta$ is semiample and descends to a line bundle L_θ. To show that L_θ is ample, it suffice to show that for m satisfying (13), the map $\phi : \mathrm{M}^{\theta\text{-ss}}_d \to \mathbb{P}^n$ induced by the complete linear series of $L_\theta^{\otimes m}$ is finite. For convenience, we denote $\mathcal{L} = \mathcal{L}_\theta^{\otimes m}$ and $L = L_\theta^{\otimes m}$. We first claim that ϕ has finite fibers. If not, there exists a smooth, proper, connected curve C and a nonconstant map $\gamma : C \to \mathrm{M}^{\theta\text{-ss}}_{d,k}$ such that the composition $\phi \circ \gamma : C \to \mathbb{P}^n$ is constant. This means that the line bundle $\gamma^* L$ has degree 0 on C, so any section of any power of $\gamma^* L$ is constant. We will show that this is impossible.

By Theorem 5.1.4(iii) and Proposition 5.3.2, the k-points of $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ correspond to θ-polystable representations under the adequate moduli space map $f : \mathcal{M}^{\theta\text{-ss}}_{d,k} \to \mathrm{M}^{\theta\text{-ss}}_{d,k}$. Given a polystable representation, there are only finitely many polystable representations of the same dimension vector with the same isomorphism classes of stable summands. Thus, since the image of C in $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ contains infinitely many k-points, it in particular contains two points p and p' corresponding to polystable representations M and M' such that one of the stable summands of M does not appear in M'. By Theorem 4.5.1, there exists $m' > 0$ and a representation V of dimension vector $mm'\beta$ such that

$$\operatorname{Hom}(M, V) \neq 0 \quad \text{and} \quad \operatorname{Hom}(M', V) = 0.$$

The representation V induces a section σ_V of $\mathcal{L}^{\otimes m'}$, and by Proposition 3.3.3 we have $\sigma_V(M) = 0$ but $\sigma_V(M') \neq 0$. There is a section $t \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m'))$ such that $\sigma_V = f^*\phi^*t$, and the section $s = \phi^*t \in \Gamma(\mathrm{M}^{\theta\text{-ss}}_{d,k}, L^{\otimes m'})$ has the property that $s(p) = 0$ but $s(p') \neq 0$. Hence, $\gamma^*(s)$ is a nonconstant section of $\gamma^*L^{\otimes m'}$, which gives a contradiction. Thus, ϕ has finite fibers.

Since $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ is proper by Corollary 5.6.4, the map ϕ is proper, hence finite by Zariski's main theorem [47, Tag 0A4X]. Thus, $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ is projective and L_θ is ample. This concludes the case of k algebraically closed.

Now let k be an arbitrary field and let $\bar{k}$ be an algebraic closure. By the case of an algebraically closed field, $\mathcal{L}_{\theta,\bar{k}}$ descends to an ample line bundle $L_{\theta,\bar{k}}$ on $\mathrm{M}^{\theta\text{-ss}}_{d,\bar{k}}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{M}^{\theta\text{-ss}}_{d,\bar{k}} \arrow[r] \arrow[d] & \mathcal{M}^{\theta\text{-ss}}_{d,k} \arrow[d] \\
\mathrm{M}^{\theta\text{-ss}}_{d,\bar{k}} \arrow[r] \arrow[d] & \mathrm{M}^{\theta\text{-ss}}_{d,k} \arrow[d] \\
\operatorname{Spec} \bar{k} \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

By Theorem 5.1.4(iv), the base change morphism $\mathrm{M}^{\theta\text{-ss}}_{d,\bar{k}} \to \mathrm{M}^{\theta\text{-ss}}_{d,k} \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is an isomorphism, so it follows from Lemma 5.1.5(ii) that there exists a line bundle L_θ on $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ whose pullback to $\mathrm{M}^{\theta\text{-ss}}_{d,\bar{k}}$ is $\bar{L}_\theta$.

Finally, we claim that L_θ is ample. Since $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ is proper over k, by [47, Tag 0D38], it suffices to show that for any coherent sheaf F on $\mathrm{M}^{\theta\text{-ss}}_{d,k}$, there exists n_0 such that

$$\mathrm{H}^i(\mathrm{M}^{\theta\text{-ss}}_{d,k}, F \otimes L_\theta^{\otimes n}) = 0$$

for all $i > 0$ and all $n \geq n_0$. By flat base change, we have

$$\mathrm{H}^i(\mathrm{M}^{\theta\text{-ss}}_{d,k}, F \otimes L_\theta^{\otimes n}) \otimes_k \bar{k} \cong \mathrm{H}^i(\mathrm{M}^{\theta\text{-ss}}_{d,\bar{k}}, \bar{F} \otimes \bar{L}_\theta^{\otimes n}),$$

where $\bar{F}$ denotes the pullback of F to $\mathrm{M}^{\theta\text{-ss}}_{d,\bar{k}}$. By the first part of the proof, $\bar{L}_\theta$ is ample, and so such an n_0 exists. □

6.3. ***Projectivity over a general base.*** We are now ready to prove Theorem A(ii), namely that $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is projective over an arbitrary noetherian base scheme S. Here we use the notion of projectivity from [47, Tag 01W8], and not the stronger notion of H-projectivity.

Theorem 6.3.1. *Suppose Q is acyclic and S is a noetherian scheme. The line bundle $\mathcal{L}_\theta$ descends to an S-ample line bundle L_θ on the moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$. In particular, $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is projective over S.*

Proof. Recall that $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is proper over S by Corollary 5.6.4. We begin by reducing to the case when the base scheme is $\operatorname{Spec}\mathbb{Z}$. Consider the commuting diagram

$$\begin{tikzcd}
\mathcal{M}^{\theta\text{-ss}}_{d,S} \arrow[rr, "\iota_{\mathcal{M}}"] \arrow[d, "f_S"] \arrow[dr, "f_{\mathbb{Z}}|_S"] & & \mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \arrow[d, "f_{\mathbb{Z}}"] \\
\mathrm{M}^{\theta\text{-ss}}_{d,S} \arrow[r, "g"] \arrow[dr, "\pi_S"'] & \mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times S \arrow[r, "\iota_{\mathrm{M}}"] \arrow[d, "\pi_{\mathbb{Z}}|_S"] & \mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \arrow[d, "\pi_{\mathbb{Z}}"] \\
& S \arrow[r, "\iota"] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd} \tag{16}$$

Suppose we know that $\mathcal{L}_{\theta,\mathbb{Z}}$ descends to a $\mathbb{Z}$-ample line bundle $L_{\theta,\mathbb{Z}}$ on $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$. This means that there exists a closed embedding $j : \mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \hookrightarrow \mathbb{P}^n_{\mathbb{Z}}$ such that $j^*\mathcal{O}(1) = L_\theta^{\otimes m}$ for some $m > 0$, and in particular $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ is a scheme. By base change, we obtain a closed embedding $j_S : \mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times S \hookrightarrow \mathbb{P}^n_S$ such that $j_S^*\mathcal{O}(1) = \iota_M^* L^{\otimes m}_{\theta,\mathbb{Z}}$. Moreover, since

$$f_S^* g^* \iota_M^* L_{\theta,\mathbb{Z}} = \iota_{\mathcal{M}} f_{\mathbb{Z}}^* L_{\theta,\mathbb{Z}} = \iota_{\mathcal{M}} \mathcal{L}_{\theta,\mathbb{Z}} = \mathcal{L}_{\theta,S},$$

we see that $\mathcal{L}_{\theta,S}$ descends to the line bundle $L_{\theta,S} = g^*\iota_M^* L_{\theta,\mathbb{Z}}$ and that $L^{\otimes m}_{\theta,S} = g^* j_S^* \mathcal{O}(1)$.

Now by Theorem 5.1.4(v), the map g has finite fibers, and it is proper since $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is. Thus, g is finite by [47, Tag 0A4X]. This implies firstly that $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is affine over the scheme $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times S$, hence itself a scheme, and secondly by [47, Tag 0B5V] that $L_{\theta,S}$ is ample.

We now proceed to prove the theorem over $\operatorname{Spec}\mathbb{Z}$. First of all, we show that $\mathcal{L}_{\theta,\mathbb{Z}}$ descends to the moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$. As in the proof of Proposition 6.1.1, it suffices to show that $\mathcal{L}^{\otimes m}_{\theta,\mathbb{Z}}$ descends for all sufficiently large integers m, and by combining Lemma 5.1.5(ii) and Lemma 5.1.5(iii), it is enough to show that for all such $m > 0$, there exists an étale cover of $\operatorname{Spec}\mathbb{Z}$ by affine schemes $\operatorname{Spec} A \to \operatorname{Spec}\mathbb{Z}$ such that $\mathcal{L}^{\otimes m}_{\theta,A}$ is globally generated.

By Proposition 6.1.1, for all sufficiently large integers $m > 0$ and for all primes p the line bundle $\mathcal{L}^{\otimes m}_{\theta,\overline{\mathbb{F}}_p}$ is generated by determinantal sections $\sigma_0, \dots, \sigma_n$ corresponding to representations $V_0, \dots, V_n$ of dimension vector $m\beta$ over $\overline{\mathbb{F}}_p$. These representations are defined over a finite extension k of $\mathbb{F}_p$, and using Remark 5.4.3 we can find an étale neighborhood $\operatorname{Spec} B \to \operatorname{Spec}\mathbb{Z}$ of $\operatorname{Spec} k$ and extensions $\mathcal{V}_i$ of each V_i to B. The representations $\mathcal{V}_i$ define global sections $\widetilde{\sigma}_i$ of $\mathcal{L}^{\otimes m}_{\theta,B}$ over $\mathcal{M}^{\theta\text{-ss}}_{d,B}$ which pull back to σ_i in $\mathcal{M}^{\theta\text{-ss}}_{d,\overline{\mathbb{F}}_p}$. Thus, the locus $\mathcal{U}$ on $\mathcal{M}^{\theta\text{-ss}}_{d,B}$ over which the $\widetilde{\sigma}_i$ generate $\mathcal{L}^{\otimes m}_{\theta,B}$ contains $\mathcal{M}^{\theta\text{-ss}}_{d,k}$.

Since the structure morphism $\mathcal{M}^{\theta\text{-ss}}_{d,B} \to \operatorname{Spec} B$ is closed by Proposition 5.6.3, the image of the complement of $\mathcal{U}$ is closed in $\operatorname{Spec} B$ and does not contain $\operatorname{Spec} k$, so replacing $\operatorname{Spec} B$ by an affine open neighborhood of $\operatorname{Spec} k$, we may assume that the sections $\widetilde{\sigma}_i$ generate $\mathcal{L}^{\otimes m}_{d,B}$. Choosing such an étale neighborhood $\operatorname{Spec} B$ for each prime p provides us with the required étale cover of $\operatorname{Spec}\mathbb{Z}$.

Let $L_{\mathbb{Z}}$ denote the line bundle on $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ whose pullback to $\mathcal{M}^{\theta\text{-ss}}_{d,\mathbb{Z}}$ is $\mathcal{L}_{\mathbb{Z}} = \mathcal{L}_{\theta,\mathbb{Z}}$, and similarly define $L_{\overline{\mathbb{F}}_p}$ on $\mathrm{M}^{\theta\text{-ss}}_{d,\overline{\mathbb{F}}_p}$. We know that $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is proper, so by [47, Tag 0D3A] it suffices to show that the restriction of $L_{\mathbb{Z}}$ to $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{F}_p$ is ample, and to do this, it suffices to show that the pullback of $L_{\mathbb{Z}}$ to $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ is ample.

Now consider the diagram (16) with $S = \operatorname{Spec} \overline{\mathbb{F}}_p$. As above, the base change morphism g is finite, so it follows that if $g^*\iota_{\mathrm{M}}^* L_{\mathbb{Z}}$ is ample on $\mathrm{M}^{\theta\text{-ss}}_{d,\overline{\mathbb{F}}_p}$, then $\iota_{\mathrm{M}}^* L_{\mathbb{Z}}$ is ample on $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$. Now, on $\mathcal{M}^{\theta\text{-ss}}_{d,\overline{\mathbb{F}}_p}$ we have isomorphisms of line bundles

$$f^*_{\overline{\mathbb{F}}_p} L_{\overline{\mathbb{F}}_p} = \mathcal{L}_{\overline{\mathbb{F}}_p} = \iota^*_{\mathcal{M}} \mathcal{L}_{\mathbb{Z}} = \iota^*_{\mathcal{M}} f^*_{\mathbb{Z}} L_{\mathbb{Z}} = f^*_{\overline{\mathbb{F}}_p} g^* \iota^*_{\mathrm{M}} L_{\mathbb{Z}},$$

which by Lemma 5.1.5(i) implies that $L_{\overline{\mathbb{F}}_p} = g^*\iota^*_{\mathrm{M}} L_{\mathbb{Z}}$. However, we know from Theorem 6.2.1 that $L_{\overline{\mathbb{F}}_p}$ is ample on $\mathrm{M}^{\theta\text{-ss}}_{d,\overline{\mathbb{F}}_p}$, so $\iota^*_{\mathrm{M}} L$ is ample on $\mathrm{M}^{\theta\text{-ss}}_{d,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$. □

Appendix: Projectivity using Theta-stability

In this appendix we present a short argument for projectivity that uses Halpern-Leistner's theory of stability for stacks. Using this theory gives a shorter argument than King's or the GIT-free approach outlined in the body of this paper, albeit it relies on a theorem ([23, Theorem 5.6.1 (2)]) that we treat as a black box, and only holds in characteristic 0. The theorem says that if a stack $\mathcal{M}$ admits a good moduli space, then its semistable locus does too, and the latter good moduli space is projective over the former. In particular, if the good moduli space of $\mathcal{M}$ is a point, then the good moduli space of the semistable locus is a projective variety.

Let k be a field, which we assume to be of characteristic 0, so that in the context of Theorem A.2, we get a good (and not merely adequate) moduli space. As in Section 5.5, we define the stack Θ_k as $[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m}}]$. We will denote the closed point of Θ_k by 0, and let 1 denote the point corresponding to the open orbit $\mathbb{A}^1 \setminus \{0\}$. As explained in [6, Corollary 7.13], a morphism $f : \Theta_k \to \mathcal{M}$ to a moduli stack of objects in an abelian category corresponds to a filtration, and the closed point corresponds to the associated graded of the filtration. As in [23, Section 3.2], we will say that a filtration is *non-degenerate* if the induced morphism $\mathbb{G}_{\mathrm{m}} \to \operatorname{Aut}(f(0))$ of sheaves of groups has finite kernel.

For a k-scheme T we write $\Theta_T = \Theta_k \times T$. Following [24, Tag 00F3] and [6, Corollary 7.13], for an algebraic stack $\mathcal{X}$, we define $\operatorname{Filt} \mathcal{X}$, called the *stack of $\mathbb{Z}$-weighted filtrations*, to be the stack corresponding to the pseudofunctor

$$\operatorname{Filt} \mathcal{X} : T \to \operatorname{Maps}(\Theta_T, \mathcal{X}).$$

From now on, let $\mathcal{X}$ be an algebraic stack locally of finite type and with affine automorphism groups over a noetherian base scheme S.

If $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$, then we can define a locally constant *weight function* $\mathrm{wt}_{\mathcal{L}} : |\operatorname{Filt} \mathcal{X}| \to \mathbb{Z}$:

$$\mathrm{wt}_{\mathcal{L}} : (f : \Theta_k \to \mathcal{X}) \mapsto \mathrm{wt}_{\mathbb{G}_{\mathrm{m}}}(\mathcal{L}_{|f(0)}),$$

where the $\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{L}_{|f(0)}$ is induced by $\mathbb{G}_{\mathrm{m}} = \operatorname{Aut}_{\Theta_k}(0) \to \operatorname{Aut}_{\mathcal{X}}(f(0))$.

Having a weight function allows one to define a notion of semistability for points of stacks; see [23, Section 4.1] or [6, Section 7.3]. We say that a point $p \in |\mathcal{X}|$ is

(i) *$\mathcal{L}$-unstable* (or *Θ-unstable* as in [6, Section 7.3]) if there is a nondegenerate filtration $f \in \operatorname{Filt} \mathcal{X}$ with $f(1) = p$ and $\operatorname{wt}_{\mathcal{L}}(f) < 0$;

(ii) *$\mathcal{L}$-semistable* if it is not $\mathcal{L}$-unstable.

Let Q be a quiver, possibly with oriented cycles. We fix a dimension vector d and a stability function θ such that $\theta(d) = 0$. Let $\mathcal{M}_{d,S}$ be the moduli stack of representations of Q of dimension vector d over a noetherian scheme S (Definition 3.1.1). Let $\mathcal{F}^{\mathrm{univ}}$ be the universal family on $\mathcal{M}_{d,S}$. Recall that we have the following line bundle on $\mathcal{M}_d$ (Section 3.3):

$$\mathcal{L}_\theta = \bigotimes_{i \in Q_0} (\det \mathcal{F}_i^{\mathrm{univ}})^{\otimes -\theta_i}.$$

This line bundle induces a weight function and hence a notion of semistability on $\mathcal{M}_{d,S}$ which we will show coincides with King's notion of θ-stability (Section 2.3).

Lemma A.1. *Fix a dimension vector d and a stability function θ such that $\theta(d) = 0$.*

(i) *The weight function for a filtration $f : \Theta_k \to \mathcal{M}_{d,S}$ has the following formula:*

$$\operatorname{wt}_{\mathcal{L}_\theta}(f) = -\sum_{n \in \mathbb{Z}} n \cdot \theta\left(\operatorname{gr}_n M\right) = -\sum_{n \in \mathbb{Z}} \theta\left(F_n M\right),$$

where f corresponds to a representation M with filtration $\ldots \subset F_{n+1}M \subset F_n M \subset \ldots$ and $\operatorname{gr}_n M := F_n M / F_{n+1} M$ so that $f(0) = \operatorname{gr} M$.

(ii) *A representation M is θ-semistable if and only if it is $\mathcal{L}_\theta$-semistable.*

Proof. To prove the first part, the weight calculation is as follows:

$$\begin{aligned} \operatorname{wt}_{\mathcal{L}_\theta}(f) &= \operatorname{wt}_{\mathbb{G}_{\mathrm{m}}}\Big(\bigotimes_{i \in Q_0} (\det \mathcal{F}_i^{\mathrm{univ}})^{\otimes -\theta_i}\Big)\Big|_{f(0)} = \operatorname{wt}_{\mathbb{G}_{\mathrm{m}}}\Big(\bigotimes_{i \in Q_0} (\det(\operatorname{gr} M)_i)^{\otimes -\theta_i}\Big) \\ &= -\sum_{i \in Q_0} \theta_i \cdot \operatorname{wt}_{\mathbb{G}_{\mathrm{m}}} \det((\operatorname{gr} M)_i). \end{aligned}$$

The $\mathbb{G}_{\mathrm{m}}$-weight corresponds to the grading weight, so $\operatorname{wt}_{\mathbb{G}_{\mathrm{m}}}(\det \operatorname{gr}_n M_i) = n \dim M_i$, and therefore:

$$\operatorname{wt}_{\mathcal{L}_\theta}(f) = -\sum_{i \in Q_0} \theta_i \cdot \sum_{n \in \mathbb{Z}} n \dim \operatorname{gr}_n M_i = -\sum_{n \in \mathbb{Z}} n \cdot \theta\left(\operatorname{gr}_n M\right).$$

The second equality follows from the fact that θ is additive in short exact sequences, so $\theta\left(\underline{\dim} \operatorname{gr}_n M\right) = \theta\left(\underline{\dim} F_n M\right) - \theta\left(\underline{\dim} F_{n+1} M\right)$.

Both equalities are used to prove the equivalence in the second part. Suppose first that M is θ-unstable, so that there is a subrepresentation $M' \subset M$ such that $\theta(\underline{\dim} M') > 0$. We can view it as a two-step filtration $f = (M' \subset M)$ with $f(1) = M$ whose associated graded factors are $\operatorname{gr}_0 = M/M'$, $\operatorname{gr}_1 = M'$ while all other components are zero. Then by (i):

$$\operatorname{wt}_{\mathcal{L}_\theta}(f) = -\theta(\operatorname{gr}_1 M) = -\theta(M') < 0.$$

This shows that M is $\mathcal{L}_\theta$-unstable.

Conversely, assume that M is $\mathcal{L}_\theta$-unstable and let f be the destabilizing filtration with $\mathrm{wt}_{\mathcal{L}_\theta}(f) < 0$. By (i), this is equivalent to

$$\sum_{n\in\mathbb{Z}} \theta(F_n M) > 0,$$

so there exists at least one value of n for which $\theta(F_n M) > 0$. Since $\theta(M) = \theta(d) = 0$ by assumption, we see that $F_n M \subset M$ is a proper subrepresentation that destabilizes M. □

Theorem A.2. *Let Q be a quiver, d a dimension vector and θ a stability function with $\theta(d) = 0$. Fix a noetherian scheme S defined over $\mathbb{Q}$. Then $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is an algebraic space which is projective over $\mathrm{M}_{d,S}$. In particular, if $S = \operatorname{Spec} k$ for a field k of characteristic 0 and Q is acyclic, then $\mathrm{M}^{\theta\text{-ss}}_{d,k}$ is a projective variety.*

Proof. By Corollary 5.5.3 and since S is of characteristic 0, the stack $\mathcal{M}_{d,S}$ admits a good moduli space $\mathrm{M}_{d,S}$. By Lemma A.1(ii), the substack $\mathcal{M}^{\theta\text{-ss}}_{d,S} \subset \mathcal{M}_{d,S}$ coincides with the substack of $\mathcal{L}_\theta$-semistable objects, so we can apply [23, Theorem 5.6.1(2)] to $\pi = \mathrm{id}_{\mathcal{M}_d} : \mathcal{M}_d \to \mathcal{M}_d$ and conclude that this substack admits a good moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$, yielding the commutative diagram

$$\begin{tikzcd} \mathcal{M}^{\theta\text{-ss}}_{d,S} \arrow[r, hook] \arrow[d] & \mathcal{M}_{d,S} \arrow[d] \\ \mathrm{M}^{\theta\text{-ss}}_{d,S} \arrow[r] & \mathrm{M}_{d,S} \end{tikzcd}$$

where vertical arrows are good moduli spaces and the bottom morphism $\mathrm{M}^{\theta\text{-ss}}_{d,S} \to \mathrm{M}_{d,S}$ is projective. □

Using the description $\mathcal{M}_{d,S} \cong [\mathrm{R}_{d,S}/\mathrm{G}_{d,S}]$ from Proposition 3.1.4 and the fact that the good moduli space of such a global quotient stack is given by the ring of invariants, then we can also get a more streamlined proof for quivers with cycles that the good moduli space $\mathrm{M}^{\theta\text{-ss}}_{d,S}$ is projective-over-affine. It would be interesting to find a proof that $\mathrm{M}_{d,S}$ is affine without methods from GIT.

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Higher modularity of elliptic curves over function fields

Adam Logan and Jared Weinstein with an interlude by Masato Kuwata

We investigate a notion of "higher modularity" for elliptic curves over function fields. Given such an elliptic curve E and an integer $r \geq 1$, we say that E is r-modular when there is an algebraic correspondence between a stack of r-legged shtukas, and the r-fold product of E considered as an elliptic surface. The (known) case $r = 1$ is analogous to the notion of modularity for elliptic curves over $\mathbb{Q}$. Our main theorem is that if $E/\mathbb{F}_q(t)$ is a nonisotrivial elliptic curve with tame fibers whose conductor has degree 4, then E is 2-modular. Ultimately, the proof uses properties of K3 surfaces. Along the way we prove a result of independent interest: A K3 surface admits a finite morphism to a Kummer surface attached to a product of elliptic curves if and only if its Picard lattice is rationally isometric to the Picard lattice of such a Kummer surface.

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1. Introduction

Let K be a function field over a finite field $\mathbb{F}_q$, and let E/K be an elliptic curve. In this article we investigate a notion of "higher modularity" for E/K.

1.1. *Analytic modularity, geometric modularity.* Let E be an elliptic curve over the rational numbers $\mathbb{Q}$, with conductor N. We review what it means for $E/\mathbb{Q}$ to be *modular*. Following [Ulm04], we might distinguish between two notions of modularity for $E/\mathbb{Q}$:

(1) **(*analytic modularity*)** There exists a cuspidal holomorphic form f of weight 2 and level $\Gamma_0(N)$ such that $L(f,\chi,s) = L(E,\chi,s)$ for all Dirichlet characters χ.

(2) **(*geometric/motivic modularity*)** There exists a nonconstant morphism $X_0(N) \to E$ defined over $\mathbb{Q}$, where $X_0(N)$ is the modular curve of level $\Gamma_0(N)$.

The two notions are equivalent. If E is geometrically modular, the required cusp form f may be found by pulling back a nonzero holomorphic differential form on E through $X_0(N) \to E$. Conversely, if E is analytically modular, there exists an elliptic curve quotient $\operatorname{Jac} X_0(N) \to E'$ corresponding to its cusp form f. The equality of L-functions implies that the representations of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the rational ℓ-adic Tate modules of E and E' are isomorphic. A theorem of Faltings [Fal83] implies that E and E' are isogenous, from which we deduce that E is geometrically modular.

In any case, the Shimura–Taniyama conjecture is a theorem [BCDT01]; every elliptic curve $E/\mathbb{Q}$ is analytically modular, hence geometrically modular. An important consequence is that the Heegner point on $X_0(N)$ for an imaginary quadratic field F satisfying the Heegner hypothesis relative to N gives rise to a Heegner point $P_F \in E(F)$. The celebrated theorem of Gross and Zagier [GZ86] relates the height of P_F to the derivative $L'(E,F,1)$. The theorem of Kolyvagin [Kol90] shows that if P_F is not torsion, then it generates a finite-index subgroup of $E(F)$, and furthermore one gets finiteness of the Shafarevich–Tate group.

Now suppose instead E is an elliptic curve over a function field K with field of scalars $\mathbb{F}_q$. We assume that E is nonisotrivial, meaning that $j(E) \notin \mathbb{F}_q$. Then $L(E,\chi,s)$ is entire for every Hecke character χ. (In fact $L(E,\chi,s)$ is a polynomial in q^{-s}.) The analogue of analytic modularity for E is the condition that there exists a cuspidal automorphic representation π of GL_2 over K, such that $L(\pi,\chi,s) = L(E,\chi,s)$ for all χ. The analytic modularity of E follows from a theorem of Deligne together with Weil's converse theorem (see [Ulm04] for a discussion).

The analogue of geometric modularity for E is the subject of this paper. The interest here hinges on the correct analogue for the modular curve. Let $X/\mathbb{F}_q$ be the nonsingular projective curve with function field K. Let N be the conductor of E, with support $\Sigma \subset |X|$. As a natural starting point, we may consider the Drinfeld modular curve $\mathrm{DrMod}(\Gamma_0(N_f);\infty)$. Here $\infty \in |X|$ is a place of split multiplicative reduction for E; thus $N = N_f + (\infty)$ for a divisor N_f which is prime to ∞. If A is the ring of functions on X regular away from ∞, then $\mathrm{DrMod}(\Gamma_0(N_f);\infty)$ parametrizes Drinfeld A-modules of rank 2 endowed with $\Gamma_0(N_f)$-structure. There is a morphism $\mathrm{DrMod}(\Gamma_0(N_f);\infty) \to X \smallsetminus \Sigma$ (sending a Drinfeld module to its "characteristic"), whose fibers are smooth curves. Let $\mathrm{DrMod}(\Gamma_0(N_f);\infty)_K$ be the generic fiber of this morphism, and let $\mathrm{DrMod}(\Gamma_0(N_f);\infty)_{\overline{K}}$ be its base change to an algebraic closure $\overline{K}/K$.

Drinfeld's computation [Dri74] of the cohomology of Drinfeld modular curves shows that $H^1(E_{\overline{K}})$ appears as a summand of $H^1(\mathrm{DrMod}(\Gamma_0(N_f);\infty)_{\overline{K}})$. (We mean étale cohomology with $\mathbb{Q}_\ell$-coefficients, considered as a representation of the Galois group of K.) We may once again apply Faltings' theorem to conclude that there is a nonconstant morphism $\mathrm{DrMod}(\Gamma_0(N_f);\infty)_K \to E$. If K'/K is a separable quadratic extension satisfying the Heegner hypothesis relative to E, there is a Heegner point $P_{K'} \in E(K')$, whose height is related to the derivative $L'(E/K',1)$; see [YZ19, Remark 1.5]. If $L'(E/K',1) \neq 0$, then $P_{K'}$ is not a torsion point. But also we have that the rank of $E(K')$ is ≤ 1, since the "easy inequality" of the Birch and Swinnerton-Dyer conjecture is a theorem for function fields. We conclude immediately that the rank is exactly 1, and so the conjecture of Birch and Swinnerton-Dyer holds for E'/K.

1.2. *Stacks of shtukas, and the definition of higher modularity.* The notion of geometric modularity in the function field setting may be generalized far beyond the Drinfeld modular curve. Namely, for $r = 1, 2, \dots$ we have the moduli stack of r-legged shtukas of rank 2, which lies over the r-fold product X^r over $\mathbb{F}_q$. These were introduced by Drinfeld [Dri80] in the case $r = 2$ to establish the Langlands correspondence for GL_2. For general r, these are examples of the spaces used by V. Lafforgue [Laf18] to prove the automorphic-to-Galois direction of the Langlands correspondence for general reductive groups.

The spaces of shtukas relevant to us are relative to the group $G = \mathrm{PGL}_2$. To define them, we need an effective divisor N of X, with support Σ. We also need a subset $\Sigma_\infty \subset \Sigma$, such that each $v \in \Sigma_\infty$ appears with multiplicity 1 in N. Finally, we need an integer $r \geq 1$ satisfying the parity condition $\#\Sigma_\infty \equiv r \pmod 2$. Following [YZ19] we let

$$\lambda_r : \mathrm{Sht}^r_G(\Gamma_0(N);\Sigma_\infty) \to (X \smallsetminus \Sigma)^r$$

denote the moduli stack of G-shtukas with $\Gamma_0(N)$-level structure, with r "moving legs" in $X \smallsetminus \Sigma$, and "fixed legs" at the places in Σ_∞ (these are analogous to the archimedean places in the number field setting). These legs are "minuscule"; i.e., they are the type considered by Drinfeld in his original definition. We save the precise definition of $\mathrm{Sht}^r_G(\Gamma_0(N);\Sigma_\infty)$ for Section 2. It is a Deligne–Mumford stack, generically smooth of relative dimension r over $(X \smallsetminus \Sigma)^r$. When r is even and $\Sigma_\infty = \varnothing$ we simply write $\mathrm{Sht}^r_G(\Gamma_0(N))$.

For example, when $r = 0$, $\mathrm{Sht}^0_G(\Gamma_0(N))$ is the discrete set $G(K)\backslash G(\mathbb{A}_K)/\Gamma_0(N)$. When $r = 1$ and $\Sigma_\infty = \{\infty\}$ is a singleton, one of the connected components of $\mathrm{Sht}^1_G(\Gamma_0(N); \{\infty\})$ is isomorphic over $X \smallsetminus \Sigma$ to a quotient of $\mathrm{DrMod}(\Gamma_0(N_f); \infty)$; this is the shtuka correspondence [Mum78].

We can now start to describe our notion of "higher modularity" for our elliptic curve E/K. Let $U = X \smallsetminus \Sigma$, and let $\mathcal{E} \to U$ be the family of elliptic curves with generic fiber E. The idea goes like this: Consider the family $h^1(\mathcal{E})$ of motives over U. Let $h^1(\mathcal{E})^{\boxtimes r}$ be its r-th external tensor power, meaning the family of motives over U^r whose fiber over $(s_1, \dots, s_r)$ is $h^1(\mathcal{E}_{s_1}) \otimes \cdots \otimes h^1(\mathcal{E}_{s_r})$. For E to be r-modular should mean that $h^1(\mathcal{E})^{\boxtimes r}$ is a quotient of the (compactly supported) cohomology of a stack of r-legged shtukas.

The precise condition involves the existence of a degree 0 algebraic correspondence between $\mathrm{Sht}^r := \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)$ and $\mathcal{E}^r$. By a degree 0 algebraic correspondence, we mean a $\mathbb{Q}$-linear combination of irreducible closed substacks

$$Z \subset \mathrm{Sht}^r \times_{U^r} \mathcal{E}^r$$

such that $Z \to U^r$ is proper with r-dimensional fibers.

Let η (resp., η_r) be the generic point of X (resp., X^r), and let $\overline{\eta}_r \to \eta_r$ be an algebraic closure lying over $\overline{\eta}^r \to \eta^r$. Let ℓ be a prime not dividing q. The ℓ-adic cycle class of $Z_{\overline{\eta}_r}$ gives rise to a map

$$p_Z : H^r_c(\mathrm{Sht}^r_{\overline{\eta}_r}, \mathbb{Q}_\ell) \to \bigotimes_{i=1}^r H^1(\mathcal{E}_{\overline{\eta}}, \mathbb{Q}_\ell) \tag{1.2.1}$$

(Explanation: the ℓ-adic cycle class of $Z_{\overline{\eta}_r}$ belongs to $H^{2r}_c(\mathrm{Sht}^r_{\overline{\eta}_r} \times (\mathcal{E}_{\overline{\eta}})^r, \mathbb{Q}_\ell)(r)$, which maps via the Künneth isomorphism to $H^r_c(\mathrm{Sht}^r_{\overline{\eta}_r}, \mathbb{Q}_\ell)(r) \otimes \bigotimes_{i=1}^r H^1(\mathcal{E}_{\overline{\eta}}, \mathbb{Q}_\ell)$. Poincaré duality identifies the latter with the space of maps as in (1.2.1).) For $Z = \sum_j a_j Z_j$ a formal $\mathbb{Q}$-linear combination of correspondences Z_j as above, we define $p_Z = \sum_j a_j p_{Z_j}$.

Definition 1.2.1. Let $K/\mathbb{F}_q$ be a function field, and let E/K be a nonisotrivial elliptic curve whose conductor N has support Σ. Let $U = X \smallsetminus \Sigma$, and let $\mathcal{E} \to U$ be the family of elliptic curves with generic fiber E. We say that E/K is *r-modular* if for some $\Sigma_\infty \subset \Sigma$ there exists a degree 0 algebraic correspondence Z between $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)$ and $\mathcal{E}^r$ such that the map p_Z of (1.2.1) is surjective.

Remark 1.2.2. When r is odd, we must assume that E/K has at least one place of multiplicative reduction in order to have an interesting notion of r-modularity.

Conjecture 1.2.3. *A nonisotrivial elliptic curve E/K is r-modular for all $r \geq 1$.*

The main goal of this article is to give the first examples of 2-modular and 3-modular elliptic curves. Our results apply to what is arguably the simplest class of elliptic fibrations, namely the *extremal rational elliptic fibrations* $\mathcal{E} \to X$. Here "rational" means that $\mathcal{E}$ is birational to $\mathbb{P}^2$ (and consequently $X \cong \mathbb{P}^1$), and "extremal" means that $\mathcal{E} \to X$ has Mordell–Weil rank 0. (The generic rational elliptic fibration has Mordell–Weil rank 8.) For such a fibration we necessarily have $\deg N = 4$ and $L(E, s) = 1$ identically. Conversely, if an elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ has degree 4 conductor, then it is extremal rational.

The extremal rational elliptic fibrations $\mathcal{E} \to X$ over a field of characteristic 0 are classified in [MP86]. The number of singular geometric fibers of a extremal rational elliptic fibration is either two, three, or four, with two occurring only if $\mathcal{E} \to X$ is isotrivial or if it has wild fibers (which happens only in characteristics 2 and 3). A nonisotrivial extremal rational elliptic fibration always has at least one multiplicative fiber, and so there is always a nontrivial notion of higher modularity for $\mathcal{E}$.

Theorem 1.2.4. *Let $\mathcal{E} \to X$ be a nonisotrivial tame extremal rational elliptic fibration with generic fiber E. Then E is 2-modular.*

In the next two subsections we will (a) explain why one should believe Conjecture 1.2.3, and (b) sketch the proof of Theorem 1.2.4.

1.3. ***Relation to the Tate conjecture.*** Here we motivate the definition of higher modularity. Using analytic modularity (which is a theorem) combined with prior results on the cohomology of stacks of shtukas, we will reduce Conjecture 1.2.3 to a sufficiently strong version of the Tate conjecture. Let $\eta = \operatorname{Spec} K$ be the generic point of X, and let $\bar{\eta} = \operatorname{Spec} \overline{K}$ be a geometric generic point.

We use the normalization of the global Langlands correspondence found in [Dri80], adapted to the group $G = \mathrm{PGL}_2$. This is a bijection between isomorphism classes:

- cuspidal automorphic representations π of G with coefficients in $\overline{\mathbb{Q}}_\ell$, and
- irreducible representations $\sigma : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2\, \overline{\mathbb{Q}}_\ell$, such that $\det \sigma = \overline{\mathbb{Q}}_\ell(-1)$.

Suppose that π and σ correspond. Let N be the conductor of π, in the sense that $\dim \pi^{\Gamma_0(N)} = 1$, and let Σ be the support of N. Let $U = X \smallsetminus \Sigma$. Then σ factors through a representation of $\pi_1(U, \bar{\eta})$ of conductor N. The correspondence is characterized by the property that for all $v \in U$ with residue field $\mathbb{F}_{q_v}$ and Frobenius Frob_v, we have $L(s - 1/2, \pi_v) = \det(1 - \sigma(\mathrm{Frob}_v) q_v^{-s})^{-1}$.

Remark 1.3.1. We have chosen this normalization of the global Langlands correspondence so that the H^1 of a nonisotrivial elliptic curve over K corresponds to a cuspidal automorphic representation of G. Under the "usual" normalization of the correspondence, the Langlands parameter of an automorphic representation of G is a homomorphism $\mathrm{Gal}(\overline{K}/K) \to \mathrm{SL}_2\, \overline{\mathbb{Q}}_\ell$. This is largely an aesthetic choice; working with G rather than GL_2 saves us certain notational headaches having to do with the center of GL_2.

We now reference some facts about the relation between the cohomology of stacks of shtukas and the global Langlands correspondence. Let $r \geq 1$, write η^r for the r-fold product of η over $\mathbb{F}_q$, and write η_r for the generic point of X^r. Let $\bar{\eta}_r \to \eta_r$ be an algebraic closure lying over $\bar{\eta}^r \to \eta^r$. Then there is a homomorphism

$$\pi_1(U^r, \bar{\eta}_r) \to \pi_1(U, \bar{\eta})^r \tag{1.3.1}$$

The cohomology $H_c^r(\mathrm{Sht}_G^r(\Gamma_0(N); \Sigma_\infty)_{\bar{\eta}_r}, \mathbb{Q}_\ell)$ admits commuting actions of $\pi_1(U^r, \bar{\eta}_r)$ and the Hecke algebra $T_0(N)$ for G with level $\Gamma_0(N)$. The following proposition is along the lines of the Kottwitz conjecture for Shimura varieties. It appears later as Proposition 2.3.1. We warn that it is conditional on the extension of the main results of Xue in [Xue20a] and [Xue20b] to the situation $\Sigma_\infty \neq \varnothing$.

Proposition 1.3.2. *Assume that* $\#\Sigma_\infty \leq 1$. *Let* $\sigma : \pi_1(U, \bar{\eta}) \to \mathrm{GL}_2\,\overline{\mathbb{Q}}_\ell$ *be an irreducible representation of conductor* N *with* $\det \sigma = \overline{\mathbb{Q}}_\ell(-1)$. *Then as representations of* $\pi_1(U^r, \bar{\eta}_r)$, *the external tensor power* $\sigma^{\boxtimes r}$ *appears as a subspace of* $H^r_c(\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)_{\bar{\eta}_r}, \overline{\mathbb{Q}}_\ell)$. *(Here* $\pi_1(U^r, \bar{\eta}_r)$ *acts on* $\sigma^{\boxtimes r}$ *by means of the homomorphism* (1.3.1).*)*

Now suppose E/K is a nonisotrivial elliptic curve of conductor N. Let $\mathcal{E} \to U$ be the family of elliptic curves with generic fiber E. Let σ be the 2-dimensional representation of $\pi_1(U, \bar{\eta})$ on $H^1(\mathcal{E}_{\bar{\eta}}, \mathbb{Q}_\ell)$. Then σ is irreducible, and (owing to the Weil pairing on E) we have $\det \sigma = \mathbb{Q}_\ell(-1)$. By Proposition 1.3.2, there exists a $\pi_1(U^r, \bar{\eta}_r)$-equivariant embedding $\sigma^{\boxtimes r} \hookrightarrow H^r_c(\mathrm{Sht}^r_{\bar{\eta}_r}, \mathbb{Q}_\ell)$, where $\mathrm{Sht}^r = \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)$. Therefore we have embeddings

$$(\sigma \otimes \sigma(1))^{\boxtimes r} \hookrightarrow H^r_c(\mathrm{Sht}^r_{\bar{\eta}_r}, \mathbb{Q}_\ell) \otimes H^1(\mathcal{E}_{\bar{\eta}}, \mathbb{Q}_\ell)^{\boxtimes r}(r) \hookrightarrow H^{2r}_c(\mathrm{Sht}^r_{\bar{\eta}_r} \times \mathcal{E}^r_{\bar{\eta}_r}, \mathbb{Q}_\ell(r))$$

The representation $(\sigma \otimes \sigma(1))^{\boxtimes r}$ contains a $\pi_1(U^r, \bar{\eta}_r)$-invariant vector (again due to the Weil pairing), and therefore so does $H^{2r}_c(\mathrm{Sht}^r_{\bar{\eta}_r} \times \mathcal{E}^r_{\bar{\eta}_r}, \mathbb{Q}_\ell(r))$. If the stack of shtukas were a projective variety, the Tate conjecture would predict that w is the class of an algebraic correspondence Z. This is why we have claimed that a sufficiently strong version of the Tate conjecture predicts the algebraic correspondence Z appearing in Definition 1.2.1.

1.4. ***Strategy of proof of 2-modularity.*** Let N be an effective divisor of $X = \mathbb{P}^1_{\mathbb{F}_q}$ of degree 4 with support Σ. As before, let $U = X \smallsetminus \Sigma$. Let $\Sigma_\infty \subset \Sigma$ be a set of places appearing with multiplicity 1 in N, such that $\#\Sigma_\infty$ is even. The stack of shtukas in this situation can be described by simple equations, at least up to birational equivalence. More precisely, each fiber of

$$\mathrm{Sht}^2_G(\Gamma_0(N); \Sigma_\infty) \to U^2$$

is birational to an elliptic surface (Theorem 2.5.1). For q large these elliptic surfaces are of general type, which makes it hard to imagine finding the desired algebraic correspondence directly. Instead, we found an unexpected relation between shtuka spaces with different numbers of legs, which we have called the "coincidence map". The coincidence map is a rational map

$$c : \mathrm{Sht}^2_G(\Gamma_0(N); \Sigma_\infty) \dashrightarrow \mathrm{Sht}^1_G(\Gamma_0(N); \Sigma'_\infty).$$

between stacks of shtukas with 2 and 1 legs, respectively; its domain of definition meets every fiber over U^2. Here $\Sigma'_\infty \subset \Sigma_\infty$ is another set of places appearing with multiplicity 1 in N, such that $\#\Sigma'_\infty$ is odd. The coincidence map fits into a cartesian diagram of varieties with rational maps:

$$\begin{tikzcd} \mathrm{Sht}^2_G(\Gamma_0(N); \Sigma_\infty) \arrow[r, dashed, "c\times\lambda_2"] \arrow[d, dashed] & \mathrm{Sht}^1_G(\Gamma_0(N); \Sigma'_\infty) \times U^2 \arrow[d, "\lambda_1\times\mathrm{id}"] \\ \mathrm{Coinc}^3_G(\Gamma_0(N); \Sigma''_\infty) \arrow[r, "\lambda_3"'] & U^3 \end{tikzcd} \tag{1.4.1}$$

Here, $\mathrm{Coinc}^3_G(\Gamma_0(N); \Sigma''_\infty)$ is a moduli space of what we have called "3-legged coincidences": these are certain modifications of vector bundles on the projective line, but Frobenius is not involved (and indeed the space of coincidences can be defined in any characteristic). The set of archimedean places Σ'' is the symmetric difference between Σ and Σ'. The leg map λ_3 in (1.4.1) is a double cover of U^3, branched over a divisor of degree $(2, 2, 2)$.

Now suppose E/K is a nonisotrivial elliptic curve of conductor N, corresponding to a family of elliptic curves $\mathcal{E} \to U$. Let ∞ be a place of multiplicity 1 in N. We apply the fact that E is 1-modular, which is to say there is a finite morphism

$$\mathrm{Sht}^1_G(\Gamma_0(N), \{\infty\}) \to \mathcal{E} \tag{1.4.2}$$

commuting with the maps to U. Combining (1.4.1) with (1.4.2), we obtain a dominant rational map:

$$\mathrm{Sht}^2_G(\Gamma_0(N)) \dashrightarrow \mathcal{Z}^2(\mathcal{E}) \tag{1.4.3}$$

where $\mathcal{Z}^2(\mathcal{E})$ is defined as the cartesian product

$$\begin{tikzcd} \mathcal{Z}^2(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E} \times U^2 \arrow[d] \\ \mathrm{Coinc}^3_G(\Gamma_0(N); \{\infty\}) \arrow[r] & U^3 \end{tikzcd} \tag{1.4.4}$$

We have a map $\mathcal{Z}^2(\mathcal{E}) \to U^2$ obtained by composing the upper row of (1.4.4) with the projection; with respect to this, the morphism (1.4.3) commutes with the maps to U^2. By replacing $\mathcal{E} \to U$ with the complete elliptic fibration over $\mathbb{P}^1$ in (1.4.4), it is possible to redefine $\mathcal{Z}^2(\mathcal{E})$ so that the morphism $\mathcal{Z}^2(\mathcal{E}) \to U^2$ is proper and smooth; we do this. Then each fiber of $\mathcal{Z}^2(\mathcal{E}) \to U^2$ is the base change of a rational elliptic fibration by a double cover of $\mathbb{P}^1$ ramified at 2 points. Generically, such a base change is a K3 surface of Picard rank 18. Our plan of attack now shifts to the study of K3 surfaces.

Recall that the Kummer surface $\mathrm{Km}(A)$ of an abelian surface A is the K3 surface obtained by resolving singularities on the quotient $A/[-1]$. If A is the product of generic nonisogenous elliptic curves, then $\mathrm{Km}(A)$ has Picard rank 18.

We prove the following theorem:

Theorem 1.4.1. *Let S be a* K3 *surface over an algebraically closed field. Assume there is an isometry* $\mathrm{Pic}\, S \otimes \mathbb{Q} \cong \mathrm{Pic}\, K \otimes \mathbb{Q}$, *where K is a Kummer surface of the form* $\mathrm{Km}(E_1 \times E_2)$, *where E_1, E_2 are nonisogenous elliptic curves. Then there exists a finite morphism from S to a Kummer surface of this form.*

Remark 1.4.2. Theorem 1.4.1 is in the spirit of a result of Buskin [Bus19, Theorem 1.1] which states that if S and S' are two complex K3 surfaces, then a Hodge isometry $H^2(S, \mathbb{Q}) \cong H^2(S', \mathbb{Q})$ is necessarily induced from an algebraic correspondence between S and S'.

Recall our family of K3 surfaces $\mathcal{Z}^2(\mathcal{E}) \to U^2$. The idea now is to apply Theorem 1.4.1 (suitably modified to work in families) to conclude the existence of a family of abelian surfaces $\mathcal{A} \to U^2$, which factors as a product of elliptic curves étale-locally on U^2, together with an isogeny $\mathcal{Z}^2(\mathcal{E}) \to \mathrm{Km}(\mathcal{A})$.

The next phase of the operation is to find an isogeny between $\mathrm{Km}(\mathcal{A})$ and $\mathrm{Km}(\mathcal{E}^2)$ over U^2, where $\mathcal{E} \to U$ is the given family of elliptic curves. This implies that $\mathcal{E}$ is 2-modular: there is then a dominant rational map $\mathrm{Sht}^2 \dashrightarrow \mathrm{Km}(\mathcal{E}^2)$ over U^2, which lifts over $\mathcal{E}^2 \dashrightarrow \mathrm{Km}(\mathcal{E}^2)$ to produce (after taking Zariski closures) the desired correspondence between Sht^2 and $\mathcal{E}^2$.

We can leverage prior knowledge of the cohomology of Sht^2 to force an isogeny between $\mathrm{Km}(\mathcal{A})$ and $\mathrm{Km}((\mathcal{E}')^2)$, where $\mathcal{E}' \to U$ is some family of elliptic curves of conductor N, a posteriori 2-modular (Theorem 4.2.2). In the case that $\mathcal{E}$ is semistable (meaning N is multiplicity-free), this is enough to force an isogeny between $\mathcal{E}$ and $\mathcal{E}'$, and so $\mathcal{E}$ is 2-modular as well. In the other cases, explicit calculations were necessary to find the isogeny between $\mathcal{Z}^2(\mathcal{E})$ and $\mathrm{Km}(\mathcal{E}^2)$.

The phenomenon of the coincidence map applies to spaces of shtukas with arbitrarily many legs, and opens up an avenue of attack to prove r-modularity for any r.

Theorem 1.4.3. *Assume q is odd. Let $E/\mathbb{F}_q(t)$ be the Legendre elliptic curve, with Weierstrass equation $y^2 = x(x-1)(x-t)$. Then E is 3-modular.*

This E has conductor $N = (0) + (1) + 2(\infty)$. Theorem 1.4.3 was proved by finding a birational map between two families of Calabi–Yau threefolds $\mathcal{Z}^3(\mathcal{E})$ and $\mathrm{Km}(\mathcal{E}^3)$ over U^3, where $U = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. Here $\mathcal{Z}^3(\mathcal{E})$ is defined as a fiber product analogous to (1.4.4), and $\mathrm{Km}(\mathcal{E}^3)$ is a generalized Kummer variety.

1.5. *Application: Heegner–Drinfeld cycles on $\mathcal{E}^r$.* This article was inspired by the theorem of Yun and Zhang [YZ19] relating *Heegner–Drinfeld cycles* on stacks of shtukas, to the Taylor expansion of L-functions of automorphic forms. We explain here the contact between higher modularity and the conjecture of Birch and Swinnerton-Dyer for elliptic curves over function fields.

As before, we let X be a curve over $\mathbb{F}_q$ with function field K. Let $f : \mathcal{E} \to X$ be an elliptic fibration with generic fiber E/K. Let ℓ be a prime not dividing q. Recall [Tat95] that the inequality $\mathrm{ord}_{s=1} L(E/K, s) \geq \mathrm{rank}\, E(K)$ is known unconditionally, and that the following are equivalent:

(1) The conjecture of Birch and Swinnerton-Dyer (BSD) holds for E/K; i.e., $\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{rank}\, E(K)$.

(2) The Tate conjecture holds for the surface $\mathcal{E}/\mathbb{F}_q$. By this, we mean that $H^2(\mathcal{E}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))^{\mathrm{Frob}_q=1}$ is spanned by classes of divisors.

(3) The Tate–Shafarevich group $Ш(E/K)$ is finite.

We briefly review the connection between these statements. Let $\mathrm{NS}(\mathcal{E})$ be the Néron–Severi group of $\mathcal{E}$ (divisors modulo algebraic equivalence). Define a decreasing two-step filtration $\mathrm{Fil}^i \mathrm{NS}(\mathcal{E})$, with $\mathrm{Fil}^1 \mathrm{NS}(\mathcal{E})$ the subgroup of divisors whose intersection pairing with a fiber of f is zero, and with $\mathrm{Fil}^2 \mathrm{NS}(\mathcal{E})$ the subgroup generated by irreducible components of fibers. Then $\mathrm{NS}(\mathcal{E})/\mathrm{Fil}^1 \mathrm{NS}(\mathcal{E}) \cong \mathbb{Z}$ and $\mathrm{Fil}^1 \mathrm{NS}(\mathcal{E})/\mathrm{Fil}^2 \mathrm{NS}(\mathcal{E}) \cong E(K)$, the Mordell–Weil group. The identity section splits off the first filtrant: $\mathrm{NS}(\mathcal{E}) \cong \mathrm{Fil}^1 \mathrm{NS}(\mathcal{E}) \oplus \mathbb{Z}$.

Meanwhile, the Leray spectral sequence for the composition $\mathcal{E} \to X \to \mathrm{Spec}\, \mathbb{F}_q$ degenerates on the second page. As a result there is a decreasing two-step filtration $\mathrm{Fil}^i H^2(\mathcal{E}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ with graded pieces $H^2(X_{\overline{\mathbb{F}}_q}, f_*\mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell$, $H^1(X_{\overline{\mathbb{F}}_q}, R^1 f_*\mathbb{Q}_\ell(1))$, $H^0\big(X_{\overline{\mathbb{F}}_q}, R^2 f_*\mathbb{Q}_\ell(1)\big)$. This filtration is preserved by the action of Frobenius Frob_q. Once again, the identity section splits off the first filtrant: $H^2(\mathcal{E}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) \cong \mathrm{Fil}^1 H^2(\mathcal{E}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell$.

The cycle class map $\mathrm{NS}(\mathcal{E}) \otimes \mathbb{Q}_\ell \to H^2(\mathcal{E}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ respects the filtrations on either side, and induces an isomorphism on the first and third graded pieces. On the second graded pieces, we get an injective map

$$E(K) \otimes \mathbb{Q}_\ell \to V := H^1(X_{\overline{\mathbb{F}}_q}, R^1 f_*\mathbb{Q}_\ell(1))$$

which lands in the Frobenius-fixed part $V^{\mathrm{Frob}_q=1}$. The Tate conjecture for $\mathcal{E}/\mathbb{F}_q$ reduces to the statement that $E(K) \otimes \mathbb{Q}_\ell \to V^{\mathrm{Frob}_q=1}$ is an isomorphism. Note that $V^{\mathrm{Frob}_q=1}$ is the Selmer group for E/K (with $\mathbb{Q}_\ell$-coefficients), and the statement that $E(K) \otimes \mathbb{Q}_\ell \to V^{\mathrm{Frob}_q=1}$ is an isomorphism is equivalent to the statement that the ℓ-primary part of $Ш(E/K)$ is finite.

The vector space V is related to the L-function via

$$L(E/K, s) = \det(1 - q^{1-s}\,\mathrm{Frob}_q \mid V).$$

Thus $\mathrm{ord}_{s=1} L(E/K, s)$ is the dimension of the generalized 1-eigenspace for Frob_q acting on V, which a priori may be larger than $V^{\mathrm{Frob}_q=1}$. Thus we have inequalities:

$$\mathrm{rank}\, E(K) \le \dim V^{\mathrm{Frob}_q=1} \le \mathrm{ord}_{s=1} L(E/K, s)$$

If BSD holds for E/K, then the leftmost and rightmost quantities are equal, and consequently the Tate conjecture holds for $\mathcal{E}$, and as a bonus we also find that the generalized 1-eigenspace for Frob_q on V coincides with the eigenspace. Conversely, if the Tate conjecture holds for $\mathcal{E}$, Tate shows that the generalized 1-eigenspace for Frob_q on V coincides with the eigenspace, and therefore BSD holds for E/K. Note that the statement that the generalized 1-eigenspace for Frob_q on V coincides with the eigenspace is a consequence of the general conjecture that Frob_q acts semisimply on all $H^i(\mathcal{E}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$; this statement is sometimes packaged along with the Tate conjecture.

Let N be the conductor of E, and let Σ be the support of N. Assume that N is multiplicity-free. Let $\Sigma_\infty \subset \Sigma$ be a subset. For each $r \ge 0$ with the same parity as $\#\Sigma_\infty$, we have a space of r-legged shtukas $\lambda_r : \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty) \to X^r$. (The map is really defined over all of X^r, without removing Σ. In the following discussion we are working with the iterated shtukas as defined in [YZ19]. In particular $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)$ is a smooth Deligne–Mumford stack.) Abbreviate this as Sht^r.

Let K'/K be a quadratic extension corresponding to a (possibly branched) double cover $X' \to X$. We assume that X' is geometrically connected over $\mathbb{F}_q$, and that the branch points of $X' \to X$ are disjoint from Σ. Let us write $\Sigma = \Sigma_f \cup \Sigma_\infty$. Assume the Heegner hypothesis: all places in Σ_f are split in X', and all places in Σ_∞ are inert in X'. Let $\mathcal{E}' = \mathcal{E} \times_X X'$. Then (under an assumption on the expected behavior of the cohomology of the stacks of shtukas; see [YZ19, equation 1.9]), Yun and Zhang construct a *Heegner–Drinfeld class*

$$HD_{E,r} \in ((V')^{\otimes r})^{\mathrm{Frob}_q=1}$$

where V' is the analogue of V for $E' \to X'$.

The following theorem is claimed by Yun and Zhang (see [YZ19, Section 1.3.1]):

Theorem 1.5.1. *Let $r_0 \geq 0$ be the smallest integer r such that $HD_{E,r} \neq 0$. Then the Selmer rank $\dim(V')^{\mathrm{Frob}_q=1}$ is r_0, and HD_{E,r_0} spans the line $\bigwedge^{r_0}(V')^{\mathrm{Frob}_q=1}$.*

On the other hand, the main result [YZ19, Corollary 1.4] applied to this situation reads

$$(HD_{E,r}, HD_{E,r}) \doteq L^{(r)}(E/K', 1), \tag{1.5.1}$$

where $(\ ,\ ) : V \times V \to \mathbb{Q}_\ell$ is the natural symmetric pairing on V, and $\doteq$ means equality up to an explicit nonzero constant. If one knew definiteness of the restriction of the pairing to the Heegner–Drinfeld classes, one could conclude that the r_0 of Theorem 1.5.1 is the analytic rank of E/K', and therefore that Selmer rank = analytic rank. (This is true unconditionally in the case of analytic rank 3, using the Hodge index theorem for surfaces.) The upshot is that Heegner–Drinfeld classes can be used to construct a basis for the determinant of the Selmer group of E/K', but (unless the analytic rank is 1) one cannot use them to construct elements of the Mordell–Weil group.

Under the assumption that E is r-modular, the Heegner–Drinfeld classes actually arise from algebraic cycles on $(\mathcal{E}')^r$, in the following sense. Define the base change

$$(\mathrm{Sht}^r)' = \mathrm{Sht}^r \times_{X^r} X'.$$

The *Heegner–Drinfeld cycle* is a codimension r cycle with proper support:

$$\mathcal{HD}_r \in Z^r_c((\mathrm{Sht}^r)', \mathbb{Q}).$$

It is obtained as the locus of rank 2 X-shtukas which arise from rank 1 X'-shtukas. (To define $\mathcal{HD}_r$ one needs to make some auxiliary choices; see [YZ19, 1.1.3]. In particular one has to choose a place of X' above each $v \in \Sigma_f$.)

Let Z be an algebraic correspondence between Sht^r and $\mathcal{E}^r$ as in Definition 1.2.1 (extended over all of X^r). Such a correspondence induces a map on algebraic cycles:

$$p_Z : Z^r_c((\mathrm{Sht}^r)', \mathbb{Q}) \to Z^r((\mathcal{E}')^r, \mathbb{Q}).$$

Define

$$\mathcal{HD}_{E,r} := p_Z(\mathcal{HD}_r) \in Z^r((\mathcal{E}')^r, \mathbb{Q}).$$

Its cohomology class is an element

$$\mathrm{cl}(\mathcal{HD}_{E,r}) \in H^{2r}((\mathcal{E}')^r_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(r))$$

which is Frob_q-invariant.

From the Künneth formula and the splitting of $H^2(\mathcal{E}'_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ noted above, we have a surjective map

$$H^{2r}((\mathcal{E}')^r_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(r)) \to (V')^{\otimes r} \tag{1.5.2}$$

We expect that the image of $\mathrm{cl}(\mathcal{HD}_{E,r})$ in $(V')^{\otimes r}$ is $HD_{E,r}$. The moral of our story is this: suppose r is the Selmer rank of E/K'. Under the assumption of r-modularity, one still doesn't know that the Selmer group $(V')^{\mathrm{Frob}_q=1}$ is spanned by classes of algebraic cycles in $\mathcal{E}'$, but one does know that the determinant of the Selmer group is spanned by the class of an algebraic cycle in $(\mathcal{E}')^r$.

2. The stacks of shtukas for PGL_2

In this section we review the construction of the stack of r-legged shtukas $\mathrm{Sht}^r_G(\Gamma_0(N);\Sigma_\infty)$ for the group $G=\mathrm{PGL}_2$. We have tried to keep our notation consistent with [YZ19].

From Section 2.4 on we specialize to the case that the base curve X is $\mathbb{P}^1_{\mathbb{F}_q}$. In that case the shtuka space $\mathrm{Sht}^r(\Gamma;\Sigma_\infty)$ is amenable to concrete calculations.

2.1. ***Vector bundles of rank 2, fractional twists, Atkin–Lehner automorphisms, and passage to $G=\mathrm{PGL}_2$.*** Let F be a field, and let X/F be a smooth projective curve with fraction field K. For each closed point $x\in|X|$, let K_x be the completion of K at x. We have the stack Bun_2, which assigns to an F-scheme S the groupoid of rank 2 vector bundles $\mathcal{F}$ on $X\times S$. If $D\in\operatorname{Div}X$, we have the twist $\mathcal{F}(D):=\mathcal{F}\otimes\mathcal{O}_X(D)$. If $N\subset X$ is an effective divisor and $\mathcal{F}$ is a rank 2 vector bundle on $X\times S$, then a $\Gamma_0(N)$-structure on $\mathcal{F}$ is a rank 1 subbundle $\mathcal{L}_N\subset\mathcal{F}|_{N\times S}$.

In the special case that N is multiplicity-free with support Σ, a $\Gamma_0(N)$-structure may alternatively be described as a system of *fractional twists* $\mathcal{F}\big(\tfrac12 P\big)$ for each $P\in\Sigma$, which lies in between $\mathcal{F}$ and $\mathcal{F}(P)$:

$$\mathcal{F}\subset\mathcal{F}\big(\tfrac12 P\big)\subset\mathcal{F}(P)$$

The quotients $\mathcal{F}\big(\tfrac12 P\big)/\mathcal{F}$ and $\mathcal{F}(P)/\mathcal{F}\big(\tfrac12 P\big)$ are required to be rank 1 vector bundles on $P\times S$. Given the data of the $\mathcal{F}\big(\tfrac12 P\big)$, one can define for each $D\le N$ the twist $\mathcal{F}\big(\tfrac12 D\big)$, defined as the subbundle of $\mathcal{F}(D)$ generated by $\mathcal{F}$ and the $\mathcal{F}\big(\tfrac12 P\big)$ for each $P\in\operatorname{supp}D$. The line bundle $\mathcal{L}_N$ can be recovered as the kernel of $\mathcal{F}|_{N\times S}\to\mathcal{F}\big(\tfrac12 N\big)|_{N\times S}$.

We can go further than this and define $\mathcal{F}(D)$ for any D belonging to $\tfrac12\mathbb{Z}\Sigma$, in such a way that $\mathcal{F}(D+D')=\mathcal{F}(D)(D')$ for all $D,D'\in\tfrac12\mathbb{Z}N$. Then whenever $D\le D'$ for $D,D'\in\tfrac12\mathbb{Z}N$, we have an inclusion $\mathcal{F}(D)\subset\mathcal{F}(D')$ with cokernel supported on $D'-D$.

Keeping the hypothesis that N is multiplicity-free, we may write objects of $\mathrm{Bun}_2(\Gamma_0(N))$ as pairs $\mathcal{F}^\dagger=(\mathcal{F},\{\mathcal{F}(D)\})$, where $\{\mathcal{F}(D)\}$ is a system of fractional twists for $D\in\tfrac12\mathbb{Z}\Sigma$. For an object $\mathcal{F}^\dagger$ of $\mathrm{Bun}_2(\Gamma_0(N))$ and $D'\in\tfrac12\mathbb{Z}\Sigma$ we can form the fractional twist $\mathcal{F}^\dagger(D')$, simply by replacing each $\mathcal{F}(D)$ with $\mathcal{F}(D+D')$. The resulting automorphism of $\mathrm{Bun}_2(\Gamma_0(N))$ is called the *Atkin–Lehner automorphism* $\mathrm{AL}(D')$. These satisfy $\mathrm{AL}(D)\circ\mathrm{AL}(D')=\mathrm{AL}(D+D')$.[1]

A rank 2 vector bundle $\mathcal{F}$ on a curve is semistable if $2\deg\mathcal{L}\le\deg\mathcal{F}$ for all rank 1 subbundles $\mathcal{L}\subset\mathcal{F}$. The index of instability of a vector bundle is

$$\operatorname{inst}(\mathcal{F})=\max\nolimits_{\mathcal{L}}(2\deg\mathcal{L}-\deg\mathcal{F})$$

where $\mathcal{L}$ runs through rank 1 subbundles of $\mathcal{F}$. Note that $\operatorname{inst}(\mathcal{F})$ is invariant under (integral) twists by line bundles. This index allows us to define a stratification on $\mathrm{Bun}_2(\Gamma_0(N))$. For $i\ge 0$, let

$$\mathrm{Bun}_2(\Gamma_0(N))^{\le i}\subset\mathrm{Bun}_2(\Gamma_0(N))$$

[1] To be pedantic: a certain diagram of stacks 2-commutes.

denote the open substack determined by the condition that $\operatorname{inst}\mathcal{F} \leq i$ on all geometric points. Similarly define $\mathrm{Bun}_2(\Gamma_0(N))^{d,\leq i}$ as the degree d component. Then each $\mathrm{Bun}_2(\Gamma_0(N))^{d,\leq i}$ is an Artin stack of finite type.

As in [YZ17], our main focus is on shtukas for the group

$$G = \mathrm{PGL}_2\,.$$

Let Bun_G be the stack which assigns to an F-scheme S the groupoid of G-torsors $\mathcal{F}$ on $X \times S$. Then

$$\mathrm{Bun}_G \cong \mathrm{Bun}_2 / \mathrm{Pic}_X,$$

where the Picard stack Pic_X acts on Bun_2 by twisting. We may also define the stack $\mathrm{Bun}_G(\Gamma_0(N))$, classifying G-torsors $\mathcal{F}$ over $X \times S$ together with a reduction of $\mathcal{F}|_{N\times S}$ to the Borel subgroup of G.

The degree of an object of Bun_G is valued in $\mathbb{Z}/2\mathbb{Z}$, and we write $\mathrm{Bun}_G(\Gamma_0(N))^d$ $(d = 0, 1)$ for the appropriate component. Similarly as above we have $\mathrm{Bun}_G(\Gamma_0(N))^{d,\leq i}$, an Artin stack of finite type. The Atkin–Lehner automorphisms described above descend to $\mathrm{Bun}_G(\Gamma_0(N))$: for any subset $\Sigma_0 \subset \Sigma$ supported on places appearing with multiplicity 1 in N, we have the Atkin–Lehner automorphism $\mathrm{AL}\big(\tfrac{1}{2}\Sigma_0\big)$, which is in fact an involution. For two such subsets $\Sigma_0, \Sigma_0' \subset \Sigma$, we have the relation $\mathrm{AL}\big(\tfrac{1}{2}\Sigma_0\big) \circ \mathrm{AL}\big(\tfrac{1}{2}\Sigma_0'\big) = \mathrm{AL}\big(\tfrac{1}{2}\Sigma_0''\big)$, where $\Sigma_0'' \subset \Sigma$ is the symmetric difference of Σ_0 and Σ_0'.

2.2. ***Stacks of G-shtukas.*** Now let $\mathbb{F}_q$ be a finite field, and let $X/\mathbb{F}_q$ be a curve. The definition of shtukas we use coincides with that appearing in Varshavsky [Var04] for general reductive groups G, except that our shtukas feature a set of "archimedean places" Σ_∞, as in [YZ19].

Definition 2.2.1. Let S be an $\mathbb{F}_q$-scheme. Let G be GL_2 or PGL_2. Suppose given the following data:

- An effective divisor $N \subset X$, with support Σ.
- A subset $\Sigma_\infty \subset \Sigma$ of degree 1 places[2] appearing in N with multiplicity 1.
- An integer $r \geq 0$.
- Morphisms $x_i : S \to X \smallsetminus \Sigma$ for $i = 1, \dots, r$. Let $\gamma_i \subset X \times_{\mathbb{F}_q} S$ be the graph of x_i; this is a divisor of $X \times_{\mathbb{F}_q} S$.

A *G-shtuka with legs $x_1, \dots, x_r$, archimedean places Σ_∞, and $\Gamma_0(N)$-structure* is an isomorphism of G-torsors with $\Gamma_0(N)$-structure:

$$f : \mathcal{F}^\dagger|_{(X\times_{\mathbb{F}_q} S)\smallsetminus \bigcup_{i=1}^r \gamma_i} \xrightarrow{\sim} \mathrm{Frob}_S^* \,\mathcal{F}^\dagger\big(\tfrac{1}{2}\Sigma_\infty\big)|_{(X\times_{\mathbb{F}_q} S)\smallsetminus \bigcup_{i=1}^r \gamma_i}\,.$$

Here the "Frob_S" is shorthand for the endomorphism $\mathrm{id}_X \times \mathrm{Frob}_q$ on $X \times S$, where $\mathrm{Frob}_q : S \to S$ is the q-th power Frobenius morphism.

One can measure the failure of f to be an isomorphism along $\gamma_1, \dots, \gamma_r$ by means of an algebraic representation W of the r-fold product $\hat{G}^r$, where $\hat{G}$ is the Langlands dual group. We write

[2]The assumption that all places of Σ_∞ are degree 1 is just for notational convenience.

$\mathrm{Sht}_G^W(\Gamma_0(N);\Sigma_\infty)$ for the moduli stack of G-shtukas with $\Gamma_0(N)$-structure and archimedean places Σ_∞ which are bounded by W.

We explain here in explicit terms the condition that a shtuka as above is "bounded by W", in the case that $G=\mathrm{GL}_2$ (so that $\hat{G}=\mathrm{GL}_2$ as well) and W is the representation

$$W_\mu=\mathrm{std}^{\mu_1}\boxtimes\cdots\boxtimes\mathrm{std}^{\mu_r}$$

of $\hat{G}^r$, where std is the standard representation of GL_2 and $\mu=(\mu_1,\dots,\mu_r)\in\{1,-1\}^r$ is a vector of signs (with std^{-1} meaning the dual of std). Consider the following data:

- Vector bundles $\mathcal{G}_0,\mathcal{G}_1,\cdots,\mathcal{G}_r$ on $X\times_{\mathbb{F}_q}S$.
- For each $i=1,\dots,r$, an isomorphism $f_i:\mathcal{G}_{i-1}|_{(X\times_{\mathbb{F}_q}S)\setminus\gamma_i}\to\mathcal{G}_i|_{(X\times_{\mathbb{F}_q}S)\setminus\gamma_i}$. If $\mu_i=1$, then f_i is required to extend to a morphism of vector bundles $\mathcal{G}_{i-1}\to\mathcal{G}_i$ whose cokernel is length 1 supported on γ_i. If $\mu_i=-1$, then f_i^{-1} is required to have this property,

An isomorphism between $\mathcal{G}_0$ and $\mathcal{G}_r$ away from $\bigcup_i^r\gamma_i$ is bounded by W_μ if it factors into $f_r\circ\cdots\circ f_1$, where $f_1,\dots,f_r$ are as above. Comparing degrees of vector bundles, we find that such an isomorphism can exist only if $\sum_i\mu_i=\deg\mathcal{G}_r-\deg\mathcal{G}_0$.

From now on we fix

$$G=\mathrm{PGL}_2,$$

so that $\hat{G}=\mathrm{SL}_2$. Note that std is self-dual when considered as a representation of $\hat{G}$. Let us simply write

$$\mathrm{Sht}_G^r(\Gamma_0(N);\Sigma_\infty)$$

for the moduli stack of G-shtukas with $\Gamma_0(N)$-structure and archimedean places Σ_∞ bounded by

$$W_r:=\mathrm{std}^{\boxtimes r}.$$

We have an isomorphism

$$\mathrm{Sht}_G^r(\Gamma_0(N);\Sigma_\infty)\cong\mathrm{Sht}_{\mathrm{GL}_2}^{W_\mu}(\Gamma_0(N);\Sigma_\infty)/\operatorname{Pic}_X(\mathbb{F}_q),$$

where μ is any vector of signs satisfying $\sum_i\mu_i=\#\Sigma_\infty$. Hence a necessary condition for $\mathrm{Sht}_G^r(\Gamma_0(N);\Sigma_\infty)$ to be nonempty is the congruence

$$r\equiv\#\Sigma_\infty\pmod 2.$$

The stack $\mathrm{Sht}_G^r(\Gamma_0(N);\Sigma_\infty)$ is a Deligne–Mumford stack, and the morphism

$$\lambda^r:\mathrm{Sht}_G^r(\Gamma_0(N);\Sigma_\infty)\to(X\smallsetminus\Sigma)^r$$

sending a shtuka to its legs $x_1,\dots,x_r$ is generically smooth of relative dimension r.

Remark 2.2.2. Keeping the intermediate $\mathcal{G}_i$ as above, one obtains the notion of an iterated shtuka. The moduli space of iterated GL_2-shtukas bounded by $\mu=(1,-1)$ was originally considered by Drinfeld [Dri80]. If we had worked with iterated shtukas, the morphism λ^r would be smooth over $(X\smallsetminus\Sigma)^r$.

The spaces of iterated and noniterated shtukas coincide over the locus in X^r where the legs are pairwise distinct.

Example 2.2.3 (Relation to Drinfeld modular curves). In the special case that $r = 1$ and $\Sigma_\infty = \{\infty\}$ is a singleton, the stack $\mathrm{Sht}^1_G(\Gamma_0(N); \infty)$ is related to a Drinfeld modular curve. Namely, decompose N as $N = N_f + (\infty)$, and let $\mathrm{DrMod}(\Gamma_0(N_f); \infty)$ classify Drinfeld A-modules, where A is the ring of functions on X regular away from ∞, equipped with a $\Gamma_0(N_f)$-structure. The precise relationship is that there is an isomorphism

$$\mathrm{Sht}^1_G(\Gamma_0(N); \infty) \cong \mathrm{DrMod}(\Gamma_0(N_f); \infty) / \operatorname{Pic}^\circ_X(\mathbb{F}_q).$$

For $r > 1$ the stack $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)$ is generally not of finite type. To produce finite-type stacks, we need to pass to truncations. There is a map

$$\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty) \to \mathrm{Bun}_G(\Gamma_0(N))$$

sending a shtuka as in Definition 2.2.1 to its G-torsor $\mathcal{F}$. For each $i \geq 0$, let $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{\leq i}$ be the pullback under this map of $\mathrm{Bun}_G(\Gamma_0(N))^{\leq i}$. Then $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{\leq i}$ is of finite type.

2.3. ***Cohomology of stacks of shtukas.*** Here we prove Proposition 1.3.2, which states that irreducible Langlands parameters for G appear as subspaces of the cohomology of stacks of shtukas. We will need the main results of [Xue20a] and [Xue20b], which we now review.

Recall our conventions on generic points: η (resp., η_r) is the generic point of X (resp., X^r), and $\bar{\eta}_r \to \eta_r$ is an algebraic closure lying over $\bar{\eta}^r \to \eta^r$. Let Σ be the support of N, and let $U = X \smallsetminus \Sigma$.

The truncated stack of shtukas $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{\leq i}$ is of finite type over U^r, so it makes sense to define the cohomology with compact supports

$$H^r_c(\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{\leq i}_{\bar{\eta}_r}, \overline{\mathbb{Q}}_\ell),$$

a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space admitting an action of $\pi_1(U^r, \bar{\eta}_r)$. Define

$$H^r_c(\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)_{\bar{\eta}_r}, \overline{\mathbb{Q}}_\ell) := \varinjlim_{i \geq 0} H^r_c(\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{\leq i}_{\bar{\eta}_r}, \overline{\mathbb{Q}}_\ell),$$

and abbreviate

$$H^r_c := H^r_c(\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)_{\bar{\eta}_r}, \overline{\mathbb{Q}}_\ell).$$

Then H^r_c admits an action of the Hecke algebra $T_0(N)$ which commutes with the action of $\pi_1(U^r, \bar{\eta}_r)$. Let $(H^r_c)^{\mathrm{cusp}} \subset H^r_c$ be the cuspidal part. Assuming that $\Sigma_\infty = \varnothing$, Theorem 5.0.1 of [Xue20a] shows that $(H^r_c)^{\mathrm{cusp}}$ is finite-dimensional, and then Drinfeld's lemma extends the action of $\pi_1(U^r, \bar{\eta}_r)$ on $(H^r_c)^{\mathrm{cusp}}$ to an action of $\pi_1(U, \bar{\eta})^r$.

Proposition 2.3.1. *Assume that* $\#\Sigma_\infty \leq 1$. *When r is odd, assume that Theorem* 5.0.1 *of* [Xue20a] *extends to the case of* (r, Σ_∞). *Then as a representation of* $T_0(N) \times \pi_1(U, \bar{\eta})^r$, *there is a decomposition*

$$(H_c^r)^{\mathrm{cusp}} = \bigoplus_\pi \pi^{\Gamma_0(N)} \otimes \sigma_\pi^{\boxtimes r},$$

where π *runs over cuspidal representations of* G *containing a* $\Gamma_0(N)$*-fixed vector, and* σ_π *is the* 2*-dimensional representation of* $\pi_1(U, \bar{\eta})$ *corresponding to* π *under the global Langlands correspondence.*

Remark 2.3.2. When $r = 2$ and $\Sigma_\infty = \varnothing$, 2.3.1 is a special case of a theorem of Drinfeld [Dri80].

Proof. The formalism of excursion operators developed in [Laf18] furnishes a decomposition of $T_0(N) \times \pi_1(U, \bar{\eta})^r$-modules $(H_c^r)^{\mathrm{cusp}} = \bigoplus_\sigma (H_c^r)_\sigma^{\mathrm{cusp}}$ indexed by isomorphism classes of Langlands parameters $\sigma : \pi_1(U, \bar{\eta}) \to \mathrm{GL}_2\,\overline{\mathbb{Q}}_\ell$.

Let σ be an irreducible Langlands parameter for G. In the case that r is even and $\Sigma_\infty = \varnothing$, corollaire 2.1 to [LZ18, Proposition 1.2] identifies $(H_c^r)_\sigma^{\mathrm{cusp}}$ with $\pi^{\Gamma_0(N)} \otimes \sigma^{\boxtimes r}$, where π corresponds to σ under the Langlands correspondence. (We remark here that our $(H_c^r)^{\mathrm{cusp}}$ is the space denoted $H_{I,W}^{\mathrm{cusp}}$ in [LZ18], where I is a set of cardinality r and $W = \mathrm{std}^{\boxtimes I}$ is the external tensor product of I copies of the standard representation std of $\hat{G} = \mathrm{SL}_2$.) The proposition follows.

In the case that r is odd and $\Sigma_\infty = \{\infty\}$ is a singleton, the same technique as in [LZ18] applies (conditionally on the extension of Xue's theorem to this case) to give a finite-dimensional $T_0(N)$-module A_σ such that for all odd r, there is an isomorphism of $T_0(N) \times \pi_1(U, \bar{\eta})^r$-modules:

$$(H_c^r)_\sigma^{\mathrm{cusp}} \cong A_\sigma \otimes \sigma^{\boxtimes r}.$$

Setting $r = 1$, we have that $(H_c^1)_\sigma^{\mathrm{cusp}}$ is the cohomology of a Drinfeld modular curve, namely the curve which parametrizes Drinfeld modules over the ring $H^0(X \setminus \{\infty\}, \mathcal{O}_X)$. The behavior of the cohomology of Drinfeld modular curves was known to Drinfeld: $A_\sigma = \pi^{\Gamma_0(N)}$. We proceed as above. □

We can now complete the proof of Proposition 1.3.2. Suppose σ is an irreducible Langlands parameter of conductor N. Proposition 2.3.1 shows that $\sigma^{\boxtimes r}$ appears as a summand of $(H_c^r)^{\mathrm{cusp}}$. Therefore it is a subspace of H_c^r.

2.4. ***Vector bundles with level structures on*** $\mathbb{P}^1$**.** At this point we specialize our study of shtuka spaces to the case that the base curve X is the projective line. We make heavy use of the fact that the degree d component Bun_2^d admits a dense open substack containing only one isomorphism class, namely $\mathcal{O}(d/2)^{\oplus 2}$ or $\mathcal{O}((d-1)/2) \oplus \mathcal{O}((d+1)/2)$ as d is even or odd, respectively; these are exactly the objects with instability index ≤ 1. As a result, it becomes possible to give explicit equations for dense open substacks of the moduli stacks of shtukas.

The following lemmas, concerning $\mathrm{Bun}_G(\Gamma_0(N))$ for divisors N of degree 3 and 4, exemplify the simplifications that will occur.

Lemma 2.4.1. *Let F be a field, and let $X = \mathbb{P}^1_F$. Let N be an effective divisor of degree* 3. *There is a dense substack* $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$ *of* $\mathrm{Bun}_G(\Gamma_0(N))$, *such that each connected component* $\mathrm{Bun}_G(\Gamma_0(N))^{d,\omega}$ $(d = 0, 1)$ *is isomorphic to a single point* $\mathrm{Spec}\, F$.

Proof. We given an explicit description of $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$ in the case that $N = P_1 + P_2 + P_3$ where each P_i is F-rational, the other cases being similar. It is the locus of $\mathcal{F}^{\dagger}$ where the instability of each fractional twist $\mathcal{F}\big(\frac{1}{2}D\big)$ is ≤ 1, where D runs over divisors supported on N_1, N_2, N_3. One sees that each of its components $\mathrm{Bun}_G(\Gamma_0(N))^{d,\omega}$ is a single point. In degree 0, for instance, let $\mathcal{F}^{\dagger} \in \mathrm{Bun}^0_G(\Gamma_0(N))(F)$ be the object with $\mathcal{F}$ the trivial G-torsor. The level structure, represented by a triple of pairwise distinct elements of $\mathbb{P}^1(F)$, determines an object of $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$ if and only if the three points in the triple are pairwise distinct. (Translated into vector bundles, this is the observation that elementary modifications of $\mathcal{O}^{\oplus 2}$ at points P_1, P_2 along the lines $L_1, L_2 \subset F^2$ produce $\mathcal{O}(1) \oplus \mathcal{O}(1)$ generically, but will produce $\mathcal{O}(2) \oplus \mathcal{O}$ if $L_1 = L_2$.) As G acts simply transitively on such triples, we find an isomorphism $\mathrm{Bun}_G(\Gamma_0(N))^{0,\omega} \cong \mathrm{Spec}\, F$. □

Lemma 2.4.2. *Let F be a field, and let $X = \mathbb{P}^1_F$.*

(1) *Let $N = P_1 + P_2 + P_3 + P_4$ be a multiplicity-free divisor of degree* 4. *There exists a dense open substack* $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$, *stable under the Atkin–Lehner operators, each of whose connected components is isomorphic to $\mathbb{P}^1_F \smallsetminus N$. With respect to these isomorphisms, the Atkin–Lehner involution* $\mathrm{AL}\big(\frac{1}{2}(P_1 + P_2)\big)$ *is the unique involution which exchanges P_1 with P_2 and P_3 with P_4.*

(2) *Let $N = P_1 + P_2 + 2P_3$, with P_1, P_2, P_3 pairwise distinct. There exists a dense open substack* $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$, *stable under the Atkin–Lehner operators* (*supported only on P_1, P_2*), *each of whose connected components is isomorphic to $\mathbb{A}^1_F$.*

Proof. (1) Let $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$ be the locus of $\mathcal{F}^{\dagger}$ where (once again) all Atkin–Lehner twists $\mathcal{F}\big(\frac{1}{2}\Sigma\big)$ have instability index ≤ 1. This time, the $\Gamma_0(N)$-level structures $\mathcal{F}^{\dagger}$ on the trivial G-torsor correspond to quadruples of points of $\mathbb{P}^1$. One computes that $\mathcal{F}^{\dagger}$ lies in $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$ if and only if the quadruple is pairwise distinct and is not in the G-orbit of (P_1, P_2, P_3, P_4). The set of G-orbits of such quadruples is $\mathbb{P}^1_F \smallsetminus N$. (The claim about Atkin–Lehner involutions is not used in the sequel and we omit its proof.)

(2) Let $\mathrm{Bun}_2(\Gamma_0(N))^{\omega}$ be the preimage of the substack described in Lemma 2.4.1. The parameter space for $\Gamma_0(2P_3)$-structures lying over a given $\Gamma_0(P_3)$-structure is described by the fibers of $\mathrm{Res}^{2P_3}_{P_3} \mathbb{P}^1_F \to \mathbb{P}^1_F$, which are all $\mathbb{A}^1_F$. □

2.5. *Explicit presentations of shtuka spaces with $\Gamma_0(N)$-structure.* The following theorem and its proof give an explicit presentation of the space of shtukas with $\Gamma_0(N)$-structure, where N is a divisor of degree 3 or 4. It is a special case of the main results of [FF22]. Define $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{\omega}$ to be the dense open substack where the underlying G-bundle of the shtuka lies in the substack $\mathrm{Bun}_G(\Gamma_0(N))^{\omega}$ described by Lemmas 2.4.1 and 2.4.2.

Theorem 2.5.1. *Let $X = \mathbb{P}^1_{\mathbb{F}_q}$, let $r \geq 1$ be an integer, and let $\Sigma_\infty \subset |X|$ be a finite set of degree* 1 *points satisfying* $\#\Sigma_\infty \equiv r \pmod 2$. *Let $\eta_r = \mathrm{Spec}\, \mathbb{F}_q(t_1, \dots, t_r)$ be the generic point of X^r.*

(1) *Let N be an effective divisor of degree* 3, *such that each point of Σ_∞ has multiplicity* 1 *in N. The stack* $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^\omega_{\eta_r}$ *is a quasiprojective variety*, *each of whose connected components is rational*; *more precisely*, *there is a birational morphism from each component to* $(\mathbb{P}^1_{\eta_r})^r$.

(2) *Let N be a multiplicity-free effective divisor of degree* 4 *whose support contains Σ_∞. The stack* $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^\omega_{\eta_r}$ *is a quasiprojective variety. Each of its connected components is birational to a hypersurface in* $(\mathbb{P}^1_{\eta_r})^{r+1}$ *of degree* $(2, 2, \dots, 2, q+1)$.

(3) *Let* $N = P_1 + P_2 + 2P_3$ *be an effective divisor of degree* 4, *such that* P_1, P_2, P_3 *are pairwise distinct and* $\Sigma_\infty \subset \{P_1, P_2\}$. *The stack* $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^\omega_{\eta_r}$ *is a quasiprojective variety. Each of its connected components is birational to a hypersurface in* $(\mathbb{P}^1_{\eta_r})^{r+1}$ *of degree* $(2, 2, \dots, 2, q)$.

Remark 2.5.2. There are no cusp forms on PGL_2 of conductor N if $\deg N = 3$, whereas if $\deg N = 4$ and N is multiplicity-free, then the dimension of the space of cusp forms of conductor N is exactly q. (See [DF13, §7] for these results and related formulas.) So it is not surprising that nonrational varieties appear when passing from degree 3 to degree 4.

Remark 2.5.3. It is possible to extend Theorem 2.5.1 to include the cases $N = 2P_1 + 2P_2$, $N = P_1 + 3P_2$, and $N = 4P$. We refer the reader to [FF22] for details.

Proof. (1) By Lemma 2.4.1, $\mathrm{Bun}_G(\Gamma_0(N))^{d,\omega}$ is a point, corresponding to an object $\mathcal{F}^\dagger$ of $\mathrm{Bun}_G(\Gamma_0(N))(\mathbb{F}_q)$. Thus for a test scheme S, the shtukas in $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{d,\omega}(S)$ classify isomorphisms of G-torsors with $\Gamma_0(N)$-structure:

$$f : \mathcal{F}^\dagger|_{(X\times_{\mathbb{F}_q} S)\setminus \cup_i \gamma_i} \xrightarrow{\sim} \mathcal{F}^\dagger\big(\tfrac{1}{2}\Sigma_\infty\big)|_{(X\times_{\mathbb{F}_q} S)\setminus \cup_i \gamma_i}$$

which are bounded by W_r.

With Frobenius out of the picture, all that remains is linear algebra. Let $\mathcal{F}_1^\dagger$ and $\mathcal{F}_2^\dagger$ be any two lifts of $\mathcal{F}$ and $\mathcal{F}\big(\frac{1}{2}\Sigma_\infty\big)$ to $\mathrm{Bun}_2(\mathbb{F}_q)$, such that $\deg \mathcal{F}_2 = \deg \mathcal{F}_1 + r$. (This is possible due to the parity condition on r.) Note that the dimension of $\mathrm{Hom}(\mathcal{F}_1^\dagger, \mathcal{F}_2^\dagger)$ (meaning, morphisms of vector bundles, not necessarily injective, which preserve level structures) is $2r + 1$. The stack of shtukas $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{d,\omega}$ is isomorphic to the quotient by $\mathbb{G}_m$ of the moduli stack of morphisms of vector bundles with $\Gamma_0(N)$-structure $\mathcal{F}_1^\dagger \to \mathcal{F}_2^\dagger$ which are bounded by $\mathrm{std}^{\boxtimes r}$ at the legs. This is a quasiprojective variety; indeed it is the locally closed subvariety of

$$(\mathbb{P}^1)^r \times \mathbb{P}\,\mathrm{Hom}(\mathcal{F}_1^\dagger, \mathcal{F}_2^\dagger) \cong (\mathbb{P}^1)^r \times \mathbb{P}^{2r},$$

consisting of data $(x_1, \dots, x_r, f)$, where the divisor of $\det f$ is $\gamma_1 + \dots + \gamma_r$.

For each $i = 1, \dots, r$, the local behavior of f at γ_i gives an S-point of the affine Grassmannian Gr_G: this is the period morphism at the i-th leg. If the legs are pairwise disjoint, all the period morphisms land in the minuscule Schubert cell $\mathrm{Gr}_G^{\mathrm{std}} = \mathbb{P}^1$. We claim that over the generic point η_r, the product of the period morphisms

$$\Pi : \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{d,\omega}_{\eta_r} \to (\mathbb{P}^1_{\eta_r})^r$$

is a birational map. Indeed, let $x_1, \dots, x_r$ be generic legs, and consider the fiber of Π over $(u_1, \dots, u_r) \in (\mathbb{P}^1)^r$. This is the subspace of $f \in \mathbb{P}\operatorname{Hom}(\mathcal{F}_1^\dagger, \mathcal{F}_2^\dagger) \cong \mathbb{P}^{2r}$ such that f is generically an isomorphism, and such that $f(x_i)$ has rank 1, with kernel u_i. This is $2r$ linear conditions; for generic $(u_1, \dots, u_r)$ there is a unique solution.

A computation involving minors shows that the rational map $\Pi^{-1} : (\mathbb{P}^1_{\eta_r})^r \to \mathbb{P}^{2r}_{\eta_r}$ has degree $(2, \dots, 2)$.

(2) Now suppose N is multiplicity-free of degree 4. Let $N = N_0 + P$ for a point P disjoint from N_0. The quasiprojective variety $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{d,\omega}_{\eta_r}$ can be modeled on a subvariety of $\mathbb{P}^{2r} \times \mathbb{P}^1$, where the $\mathbb{P}^{2r}$ is as in part (1) and the $\mathbb{P}^1$ records the additional level structure at P. Namely, it is the space of pairs (f, v), where the divisor of $\det f$ is $\gamma_1 + \cdots + \gamma_r$, and f preserves the level structure at P in the sense that $f(P)(v) = \mathrm{Frob}_q(v)$. Part (1) shows that this is birational via Π to the subvariety Z of $(\mathbb{P}^1)^r \times \mathbb{P}^1$ consisting of pairs (u, v), where $f = \Pi^{-1}(u)$ satisfies $f(P)(v) = \mathrm{Frob}_q(v)$. We saw that Π^{-1} has degree $(2, \dots, 2)$, which is to say that $f(t)$ has coordinates which are degree $(2, \dots, 2)$ in $u = (u_1, \dots, u_r)$. Finally we observe that the equation $f(P)(v) = \mathrm{Frob}_q(v)$ has degree $q + 1$ in v.

(3) Finally, suppose $N = P_1 + P_2 + 2P_3 = N_0 + P_3$. The calculation is similar to (2), except that the additional level structure at P_3 has parameter space $\mathbb{A}^1$, this being the fiber of $\mathrm{Res}^{2P_3}_{P_3} \mathbb{P}^1 \to \mathbb{P}^1$. The operator $f(P)$ is an automorphism of this $\mathbb{A}^1$; i.e., an affine transformation. Thus $f(P)(v) = \mathrm{Frob}_q(v)$ has degree q in v. $\square$

Remark 2.5.4. Consider the case $r = 2$. Let N be a divisor of degree 4 as in part (2) or (3) of Theorem 2.5.1. Let $\eta_2 \cong \operatorname{Spec} \mathbb{F}_q(t_1, t_2)$ be the generic point of X^2. The theorem shows that each connected component of $\mathrm{Sht}^2_G(\Gamma_0(N); \Sigma_\infty)^\omega_{\eta_2}$ is birational to a hypersurface in $(\mathbb{P}^1_{\eta_2})^3$ of degree $(2, 2, q)$ or $(2, 2, q + 1)$. Since $(2, 2)$-curves in $(\mathbb{P}^1)^2$ are of genus 1, there is a desingularization, call it Z, which is a genus 1 fibration over $\mathbb{P}^1_{\eta_2}$. In fact $Z \to \mathbb{P}^1_{\eta_2}$ admits a section, so that Z is an elliptic surface. The nature of the singular fibers of $Z \to \mathbb{P}^1_{\eta_2}$ was studied in [FF22].

Specializing even further to the case $q = 2$ and $N = (0) + (1) + 2(\infty)$, one obtains a hypersurface Z defined over $\mathbb{F}_2(t_1, t_2)$ of degree $(2, 2, 2)$, which is a K3 elliptic surface. This project originated in 2019, when the second author sent the equation for Z to Noam Elkies, who (a) recognized Z as the universal K3 elliptic surface with a 6-torsion section, and (b) computed a finite morphism from Z to the Kummer surface $\mathrm{Km}(\mathcal{E}^2_{\eta_2})$, where $\mathcal{E} \to \mathbb{P}^1_{\mathbb{F}_2}$ is an elliptic fibration of conductor N. (More precisely, $\mathcal{E} \to \mathbb{P}^1_{\mathbb{F}_2}$ is the elliptic fibration with $\mathrm{I}_2, \mathrm{I}_6, \mathrm{IV}$ fibers at $0, 1, \infty$, with Mordell–Weil group $\mathbb{Z}/6\mathbb{Z}$.) This $\mathcal{E}$ was the first example of a elliptic fibration we knew to be 2-modular.

2.6. ***The coincidence map.*** Theorem 2.5.1 allows us to derive equations for shtuka spaces of level $\Gamma_0(N)$, where N is a degree 4 effective divisor of $\mathbb{P}^1_{\mathbb{F}_q}$ supported on at least one place ∞ of multiplicity 1. In an early phase of this line of research, we dealt directly with those equations in an attempt to prove 2-modularity of some elliptic fibrations of conductor N. To our surprise, a change of variables turned the defining equation for $\mathrm{Sht}^2_G(\Gamma_0(N))^{d,\omega}_{\eta_2}$ into one in which we could recognize an equation for the Drinfeld modular curve $\mathrm{Sht}^1_G(\Gamma_0(N);\infty)$. Eventually we found a generalization of this phenomenon, in the form of the coincidence map described in the introduction.

Let $r \geq 2$, and let $\Sigma_\infty, \Sigma'_\infty \subset \operatorname{Supp} N$ be sets of degree 1 places with multiplicity 1 in N, such that $\#\Sigma_\infty \equiv r \pmod 2$ and $\#\Sigma'_\infty \equiv 1 \pmod 2$. (Under our hypotheses on N, such sets exist for any r.) Let Σ''_∞ be the symmetric difference of these sets, so that $\#\Sigma''_\infty \equiv r+1 \pmod 2$. Our goal is to give a factorization of the variety of r-legged shtukas into a cartesian square:

$$\begin{tikzcd} \mathrm{Sht}^r_G(\Gamma_0(N);\Sigma_\infty) \arrow[r, dashed, "c\times\lambda_r"] \arrow[d, dashed, "\beta"'] & \mathrm{Sht}^1_G(\Gamma_0(N);\Sigma'_\infty)\times U^r \arrow[d, "\lambda_1\times\mathrm{id}"] \\ \mathrm{Coinc}^{r+1}_G(\Gamma_0(N);\Sigma''_\infty) \arrow[r, "\lambda_{r+1}"'] & U^{r+1} \end{tikzcd} \tag{2.6.1}$$

We have called the variety $\mathrm{Coinc}^{r+1}(\Gamma_0(N);\Sigma''_\infty)$ the moduli space of coincidences, and c the coincidence map. The surprising thing (to us) about this state of affairs was that the family of varieties $\mathrm{Coinc}^{r+1}(\Gamma_0(N);\Sigma''_\infty) \to U^{r+1}$ was symmetric not just in the first r copies of U as we would expect, but also in the final copy.

First we define the variety of coincidences. The "coincidences" classified by these spaces are modifications of vector bundles, but Frobenius is not involved and they may be defined in any characteristic. Therefore let F be any field, and let N be an effective divisor on $\mathbb{P}^1$ of degree 4. Let $\Sigma_\infty \subset \operatorname{Supp} N$ be a set of places appearing with multiplicity 1 in N, such that $\#\Sigma_\infty \equiv r \pmod 2$.

Let $\mathcal{H}^\dagger \in \mathrm{Bun}_G(\Gamma_0(N))(F)$ be the following "reducible object" of instability index 2. This will be the image of an object of $\mathrm{Bun}_2(\Gamma_0(N))(F)$ represented by the data $\mathcal{H} = \mathcal{O}(2)\oplus\mathcal{O}$ together with the $\Gamma_0(N)$-level structure which is just $\mathcal{O}_N$ embedded into $\mathcal{H}|_N$ along the second factor. Note that $\operatorname{Aut}\mathcal{H}^\dagger$ (considered as a group scheme) is $\mathbb{T}$, the diagonal torus in G. Thus $\mathrm{Bun}_G(\Gamma_0(N))^0$ contains a locally closed substack isomorphic to $B\mathbb{T}$, whose S-points classify those $\mathcal{H}^\dagger_0 \in \mathrm{Bun}_G(\Gamma_0(N))$ which are locally isomorphic to $\mathcal{H}^\dagger$.

We need $\mathcal{H}^\dagger$ for the following property:

Lemma 2.6.1. *Consider the stack $\mathcal{U}$ whose S-points classify tuples $(\mathcal{H}^\dagger_0, \mathcal{F}^\dagger, \tau, h)$, where $\mathcal{H}^\dagger_0$ is an object of $\mathrm{Bun}_G(\Gamma_0(N))^0(S)$ which is locally isomorphic to $\mathcal{H}^\dagger$, where $\mathcal{F}^\dagger \in \mathrm{Bun}_G(\Gamma_0(N))^{1,\omega}(S)$, where $\tau \in \mathbb{P}^1(S)$, and where*

$$h : \mathcal{H}^\dagger_0|_{\mathbb{P}^1\smallsetminus\gamma_\tau} \cong \mathcal{F}^\dagger|_{\mathbb{P}^1\smallsetminus\gamma_\tau}$$

is an isomorphism bounded by std *on τ. Then the projection $\mathcal{U} \mapsto \mathrm{Bun}_G(\Gamma_0(N))^{1,\omega}$ is an isomorphism. That is, $\mathcal{F}^\dagger$ determines the data $(\tau, \mathcal{H}^\dagger, h)$.*

Proof. A restatement of the lemma is that, given $\mathcal{F}^\dagger$, the space classifying isomorphisms h between $\mathcal{H}^\dagger$ and $\mathcal{F}^\dagger$ away from an unspecified leg and bounded by std there is a $\mathbb{T}$-torsor. The space of such h can be modeled in an ambient space $\mathbb{P}\operatorname{Hom}(\mathcal{O}(2)\oplus\mathcal{O}, \mathcal{O}(2)\oplus\mathcal{O}(1)) \cong \mathbb{P}^5$. The fact that h preserves

$\Gamma_0(N)$-level structures (recall that $\deg N = 4$) means that it belongs to a $\mathbb{P}^1$, but then two of the points on the $\mathbb{P}^1$ correspond to h with $\det h = 0$. The complement of these points is a $\mathbb{T}$-torsor. $\square$

Definition 2.6.2 (The stack of r-legged coincidences). A *G-coincidence* with legs $x_i : S \to \mathbb{P}^1$ (for $i = 1, \dots, r$), archimedean places Σ_∞, and $\Gamma_0(N)$-structure is an isomorphism of G-torsors with $\Gamma_0(N)$-structure:

$$f : \mathcal{H}_1^\dagger|_{\mathbb{P}^1_S \smallsetminus \bigcup_i \gamma_i} \xrightarrow{\sim} \mathcal{H}_2^\dagger|_{\mathbb{P}^1_S \smallsetminus \bigcup_i \gamma_i}$$

such that locally $\mathcal{H}_1^\dagger \cong \mathcal{H}^\dagger$ and $\mathcal{H}_2^\dagger \cong \mathcal{H}^\dagger\left(\frac{1}{2}\Sigma_\infty\right)$, and such that f is bounded by $\mathrm{std}^{\boxtimes r}$ at the legs. Let $\mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty)$ be the stack classifying such G-coincidences, and let

$$\mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty) \to (\mathbb{P}^1)^r$$

be the morphism sending a G-coincidence to its legs.

In other words, $\mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty)$ classifies such isomorphisms involving $\mathcal{H}^\dagger$ and $\mathcal{H}^\dagger\left(\frac{1}{2}\Sigma_\infty\right)$, modulo the action of $\mathbb{T} \times \mathbb{T}$.

Now we can construct the coincidence map c. Return to the setting in the overview, so that the base field is $\mathbb{F}_q$ and we have $\Sigma_\infty, \Sigma'_\infty, \Sigma''_\infty \subset \Sigma$, whose parities are $r, 1, r+1$, respectively. Consider the stack $\mathcal{V}$ whose S-points classify the following data:

- $\mathcal{F}^\dagger$ is an object of $\mathrm{Bun}_G(\Gamma_0(N))^{0,\omega}(S)$.
- $\mathcal{H}_1^\dagger, \mathcal{H}_2^\dagger$ are objects of $\mathrm{Bun}_G(\Gamma_0(N))^0(S)$ locally isomorphic to $\mathcal{H}^\dagger$.
- $f : \mathcal{F}^\dagger|_{X_S \smallsetminus \bigcup_{i=1}^r \gamma_i} \xrightarrow{\sim} \mathrm{Frob}^* \mathcal{F}^\dagger\left(\frac{1}{2}\Sigma_\infty\right)|_{X_S \smallsetminus \bigcup_{i=1}^r \gamma_i}$ is a shtuka bounded by $\mathrm{std}^{\boxtimes r}$ at the legs $x_1, \dots, x_r \in \mathbb{P}^1(S)$.
- $f' : \mathcal{F}^\dagger|_{X_S \smallsetminus \gamma_{r+1}} \xrightarrow{\sim} \mathrm{Frob}^* \mathcal{F}^\dagger\left(\frac{1}{2}\Sigma'_\infty\right)|_{X_S \smallsetminus \gamma_{r+1}}$ is a shtuka bounded by std at the leg $x_{r+1} \in \mathbb{P}^1(S)$.
- For $i = 1, 2$, $h_i : \mathcal{H}_i^\dagger|_{X_S \smallsetminus \gamma_\tau} \xrightarrow{\sim} \mathcal{F}^\dagger\left(\frac{1}{2}\Sigma'_\infty\right)|_{X_S \smallsetminus \gamma_\tau}$ is an isomorphism bounded by std at $\tau \in \mathbb{P}^1(S)$.

These are subject to a condition $(*)$ that the composition

$$g := h_2\left(\tfrac{1}{2}\Sigma''_\infty\right)^{-1} \circ f'\left(\tfrac{1}{2}\Sigma_\infty\right)^{-1} \circ f\left(\tfrac{1}{2}\Sigma'_\infty\right) \circ h_1, \tag{2.6.2}$$

which is a priori an isomorphism from $\mathcal{H}_1^\dagger$ to $\mathcal{H}_2\left(\frac{1}{2}\Sigma''_\infty\right)$ away from $x_1, \dots, x_{r+1}, \tau$ and bounded by $\mathrm{std}^{\boxtimes r+1} \boxtimes \mathrm{std}^{\otimes 2}$, extends over τ (more precisely, is bounded by the subrepresentation $\mathrm{std}^{\boxtimes r+1} \boxtimes \mathrm{triv}$).

Now, by Lemma 2.6.1, $\mathcal{F}^\dagger$ completely determines the data of τ and the h_i, so we see that $\mathcal{V}$ is a closed substack of $\mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{0,\omega} \times \mathrm{Sht}^1_G(\Gamma_0(N); \Sigma'_\infty)^{0,\omega}$. We claim that $\mathcal{V}$ is birational to the graph of a dominant rational map $c : \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^0 \dashrightarrow \mathrm{Sht}^1_G(\Gamma_0(N); \Sigma'_\infty)^0$.

Proof of claim: Given an r-legged shtuka f involving $\mathcal{F}^\dagger$, the space of shtukas f' with one indeterminate leg lives in an ambient space $\mathbb{P}\,\mathrm{Hom}(\mathcal{O}^2, \mathcal{O}(1) \oplus \mathcal{O}) \cong \mathbb{P}^5$, subject to 4 level conditions plus the additional linear condition $(*)$, which determines f' uniquely, at least for generic f.

We have defined a rational map

$$c : \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^0 \dashrightarrow \mathrm{Sht}^1_G(\Gamma_0(N); \Sigma'_\infty)^0$$

sending f to f'. There is also a morphism $\mathcal{V} \to \mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma''_\infty)$ whose output is the composition g above, inducing a rational map

$$\beta : \mathrm{Sht}^r_G(\Gamma_0(N); \Sigma_\infty)^{0,\omega} \dashrightarrow \mathrm{Coinc}^{r+1}_G(\Gamma_0(N); \Sigma''_\infty).$$

We have defined a commutative diagram of rational maps (2.6.1) at least on the degree 0 component. To see that it is cartesian, suppose g and f' are given. By Lemma 2.6.1 we get an h_1 and an h_2, from which it is possible to solve for f in (2.6.2). Finally, we may extend c and β to the degree 1 component using an Atkin–Lehner involution.

2.7. ***Explicit equations for the spaces of G-coincidences.*** Let F be a field, let $N \subset \mathbb{P}^1_F$ be an effective divisor of degree 4 with support Σ, let $U = \mathbb{P}^1_F \smallsetminus \Sigma$, and let $\Sigma_\infty \subset \Sigma$ be a subset consisting of points appearing in N with multiplicity 1. Then we have defined the space of G-coincidences $\mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty) \to U^r$ in Definition 2.6.2. The fiber of this morphism over the generic point $\eta = \mathrm{Spec}\, F(t_1, \dots, t_r)$ is, as we will see, a quasiprojective variety over η of dimension $r-3$. We record here some explicit equations for $\mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty)_{\eta_r}$. The case

$$N = (0) + (1) + 2(\infty)$$

being especially simple, we assume this. Fix Σ_∞ in the following way: $\varnothing$ if r is even, and $\{1\}$ if r is odd. Let $\mathcal{H}^\dagger_0$ be the object of $\mathcal{O}(2) \oplus \mathcal{O}$ of $\mathrm{Bun}_2(\Gamma_0(N))$, where the $\Gamma_0(N)$-level structure is $\mathcal{O}_N$ embedded in the second factor. Let $\mathcal{H}^\dagger_1$ be a twist of $\mathcal{H}^\dagger_0\big(\frac{1}{2}\Sigma_\infty\big)$ which has degree $r+2$: thus $\mathcal{H}_1$ is isomorphic to $\mathcal{O}((r+3)/2) \oplus \mathcal{O}((r+1)/2)$ if r is odd and $\mathcal{O}(r/2+2) \oplus \mathcal{O}(r/2)$ if r is even. We can model $\mathrm{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty)$ on the quotient by $\mathbb{T} \times \mathbb{T}$ of the subvariety $A \subset \mathbb{P}\,\mathrm{Hom}(\mathcal{H}^\dagger_0, \mathcal{H}^\dagger_1) \cong \mathbb{P}^{2r+3}$ describing those f for which $\det f$ is nonzero. Explicitly, A classifies matrices with polynomial entries

$$f(T) = \begin{pmatrix} a(T) & b(T) \\ c(T) & d(T) \end{pmatrix}$$

taken up to nonzero scalar, where the entries a, b, c, d have degrees bounded by $r/2$, $r/2+2$, $r/2-2$, $r/2$ (for r even) or $(r-1)/2$, $(r+3)/2$, $(r-3)/2$, $(r+1)/2$ (for r odd), such that $f(T)$ preserves the $\Gamma_0(N)$-level structure, and such that $\det f(T) = \prod_{i=1}^r (T - t_i)$ (again up to nonzero scalar). The preservation of level structure means:

- $b(0) = b(1) = 0$, and $\deg b(t) \le r/2$, if r is even,
- $b(0) = d(1) = 0$, and $\deg b(t) \le (r-1)/2$, if r is odd.

The action of $\mathbb{T} \times \mathbb{T}$ is by left and right translation on $f(T)$. This is a quasiprojective variety over η_r of dimension $r-3$.

Example 2.7.1. [The case $r = 3$.] In this case, $\mathrm{Coinc}^3_G(\Gamma_0(N); \Sigma_\infty)_{\eta_3} \to \eta_3$ is a separable double cover of $\eta_3 \cong \mathrm{Spec}\, F(t_1, t_2, t_3)$. For our values of N and Σ_∞, we have that $\mathrm{Coinc}^3_\circ(\Gamma_0(N); \{1\})_{\eta_3}$ is isomorphic to the 0-dimensional subvariety of $\mathbb{A}^3$ in variables a, b, d over $\eta_3 = \operatorname{Spec} F(t_1, t_2, t_3)$, which classifies matrices

$$f(T) = \begin{pmatrix} T-a & bT \\ 1 & (T-1)(T-d) \end{pmatrix}$$

which are singular at t_1, t_2, t_3. This translates into the equations

$$(t_i - 1)(t_i - a)(t_i - d) - bt_i = 0, \quad i = 1, 2, 3.$$

We can eliminate b to obtain two equations for a, d, revealing that these are the roots of the irreducible polynomial

$$s^2 - (t_1 + t_2 + t_3 - 1)s + t_1 t_2 t_3. \tag{2.7.1}$$

Then (2.7.1) is the equation for the double cover $\operatorname{Coinc}^3_G(\Gamma_0(N); \Sigma_\infty)_{\eta_3} \to \eta_3$ and its discriminant is $(t_1 + t_2 + t_3 - 1)^2 - 4t_1 t_2 t_3$. Considered as a quadratic polynomial in t_3, this discriminant is not a square for any specializations of t_1 and t_2 other than at 0, 1. (Indeed, when considered this way, the discriminant of the discriminant is $16t_1 t_2(t_1 - 1)(t_2 - 1)$, which is nonzero away from such values.) Therefore the double cover can be spread out into a family of double covers $\mathbb{P}^1_s \times U^2 \to \mathbb{P}^1_{t_3} \times U^2$ defined by the equation (2.7.1), which is nonsplit over every point of U^2.

Example 2.7.2. [The case $r = 4$.] $\operatorname{Coinc}^4_G(\Gamma_0(N))_{\eta_4}$ is isomorphic to the 1-dimensional subvariety of $\mathbb{A}^5_{\eta_4}$ in variables a_0, a_1, b, d_0, d_1 over the base $\eta_4 \cong F(t_1, t_2, t_3, t_4)$, which classifies matrices

$$f(T) = \begin{pmatrix} T^2+a_1T+a_0 & bT(T-1) \\ 1 & T^2+d_1T+d_0 \end{pmatrix}$$

which are singular at t_1, t_2, t_3, t_4. The closure of $\operatorname{Coinc}^4_G(\Gamma_0(N))_{\eta_4}$ in $\mathbb{A}^5_{\eta_4}$ contains 6 obvious rational points P_{ij}, one for each partition of $\{1, 2, 3, 4\}$ into i, j and i', j', defined by

$$f_{ij}(T) = \begin{pmatrix} (T-t_i)(T-t_j) & 0 \\ 1 & (T-t_{i'})(T-t_{j'}) \end{pmatrix}$$

Each of these points is nonsingular. There is also an obvious involution of $\operatorname{Coinc}^4_G(\Gamma_0(N))_{\eta_4}$ which exchanges (a_0, a_1) with (d_0, d_1).

The completion of this curve, call it $\mathcal{C}$, has genus 1. (One way to see this is that the projective closure in $\mathbb{P}^5$ has a unique singular point, and projection from this point to $\mathbb{P}^4$ gives a nonsingular curve of genus 1. Another way to see this is to observe that projection onto $\mathbb{A}^2$ via the final coordinates d_0, d_1 is an isomorphism of $\operatorname{Coinc}^4_G(\Gamma_0(N))_{\eta_4}$ onto an affine cubic whose projective closure is nonsingular.) A Weierstrass equation for $\mathcal{C}$ is

$$y^2 + e_1 xy = x^3 + (-e_2 + e_3 - 2e_4)x^2 + (1 - e_1 + e_2 - e_3 + e_4)e_4 x, \tag{2.7.2}$$

where $e_1, \dots, e_4$ are the elementary symmetric polynomials in $t_1, \dots, t_4$. The points P_{ij} generate the Mordell–Weil group of $\mathcal{C}$, which is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$. The points P_{ij} have coordinates

$$P_{ij}: \ (t_i(t_i-1)t_j(t_j-1), t_i(t_i-1)t_j(t_j-1)(-t_it_j + t_{i'}t_{j'} - t_{i'} - t_{j'}))$$

in the model (2.7.2); we remark that P_{12}, P_{13}, P_{14} are independent. The involution noted above is translation by the 2-torsion point T, which has coordinates $(0,0)$ in (2.7.2).

It turns out that $\mathcal{C}$ is the universal elliptic curve with these data, in the following sense.

Proposition 2.7.3. *Let $\mathcal{M}'_{1,4}$ be the moduli space of curves over k of genus 1 together with 4 marked points and also an element of order 2 in Pic^0. There is an isomorphism of $\eta = \operatorname{Spec} k(t_1, t_2, t_3, t_4)$ with the generic point of $\mathcal{M}'_{1,4}$, such that the pullback of the universal object is $(\mathcal{C}; O, P_{12}, P_{13}, P_{14}, T)$.*

Proof. We assume $\operatorname{char} k \neq 2$ for convenience. Let V be the group $\{(\pm 1, \pm 1)\}$ under multiplication, and let V act on $\overline{\mathcal{M}}'_{1,4}$ by $(\sigma_1, \sigma_2)(E; P_1, P_2, P_3, P_4; t) = (E; P_1, P_2, \sigma_1(P_3), \sigma_2(P_4); t)$, where negation is with respect to the origin P_1. We claim that $\overline{\mathcal{M}}'_{1,4}/V$ is birational to $\mathbb{A}^4_k$. Indeed, an elliptic curve with a point of order 2 has a model $y^2 = x(x^2 + rx + s)$ which is unique up to replacing (r, s) with $(\lambda^2 r, \lambda^4 s)$. Since negation is an automorphism, P_2 is only defined up to sign. Our birational map $\overline{\mathcal{M}}'_{1,4}/V \dashrightarrow \mathbb{A}^4_k$ sends $(E; P_1, P_2, P_3, P_4; t)$ to $(s/r^2, x(P_2), x(P_3), x(P_4)) \in \mathbb{A}^4_k$.

On the other hand, the data $(\mathcal{C}, O, P_{12}, P_{13}, P_{14}, T)$ defines a map $\eta \to \overline{\mathcal{M}}'_{1,4}$. One calculates [LW] that the composition $\eta \to \overline{\mathcal{M}}'_{1,4} \to \overline{\mathcal{M}}'_{1,4}/V \dashrightarrow \mathbb{A}^4_k$ is dominant of degree 4. We conclude that $\eta \to \overline{\mathcal{M}}'_{1,4}$ is an isomorphism from η onto the generic point of $\overline{\mathcal{M}}'_{1,4}$. $\square$

We can only speculate on the geometry of $\operatorname{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty)_{\eta_r}$ for general r. Let n_1, n_2 be the bounds on the degree of $c(T), d(T)$ in the description we gave above: thus $n_1 = r/2-2, (r-3)/2$ and $n_2 = r/2, (r+1)/2$ as r is even or odd, respectively. If r is odd, we have an additional condition that $d(1) = 0$, so let $n'_2 = r/2$ if r is even and $(r-1)/2$ if r is odd. In both cases we have $n_1 + n_2 = r-2$. Projection onto the pair $(c(T), d(T))$ or $(c(T), d(T)/(T-1))$, depending on the parity of r, gives a birational equivalence between $\operatorname{Coinc}^r_G(\Gamma_0(N); \Sigma_\infty)_{\eta_r}$ and a hypersurface C in $\mathbb{P}^{n_1} \times \mathbb{P}^{n'_2}$ of degree (n_1+1, n'_2+1). If such a hypersurface has canonical singularities, it has a resolution which is a (nonsingular) Calabi–Yau variety. For instance, this holds in the case $r = 5$, in which the variety C is birational to an elliptic K3 surface. We guess that it is true in general.

3. Isogenies between K3 surfaces

3.1. ***Motivation: 2-modularity of extremal rational elliptic fibrations.*** Let $E/\mathbb{F}_q(t)$ be a nonisotrivial elliptic curve with degree 4 conductor N, with corresponding rational elliptic fibration $\mathcal{E} \to \mathbb{P}^1$. Let Σ be the support of N, and let $U = \mathbb{P}^1 \setminus \Sigma$. In the introduction we sketched a plan to prove that E is 2-modular. In the first phase of the plan, we found a dominant rational map

$$\operatorname{Sht}^2_G(\Gamma_0(N)) \dashrightarrow \mathcal{Z}^2(\mathcal{E}),$$

where $\mathcal{Z}^2(\mathcal{E})$ is defined as a certain cartesian product:

$$\begin{tikzcd} \mathcal{Z}^2(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E} \times_{\mathbb{F}_q} U^2 \arrow[d] \\ \mathbb{P}^1 \times U^2 \arrow[r, "2:1"'] & \mathbb{P}^1 \times U^2 \end{tikzcd} \tag{3.1.1}$$

The lower horizontal arrow is a certain family of nonsplit double covers of $\mathbb{P}^1$ parametrized by U, derived from the moduli space of 3-legged coincidences. Now, the base change of a rational elliptic surface $\mathcal{A} \to \mathbb{P}^1$ by a nonsplit double cover $\mathbb{P}^1 \to \mathbb{P}^1$ is a K3 surface, unless the cover is ramified at a point where the elliptic surface has additive reduction. Thus $\mathcal{Z}^2(\mathcal{E}) \to U^2$ is a family of K3 surfaces. As explained in the introduction, in order to prove that E is 2-modular, it suffices to find a finite morphism $\mathcal{Z}^2(\mathcal{E}) \to \mathrm{Km}(\mathcal{E}^2)$ commuting with the maps to U^2. This is what we accomplish in this section and the next:

Theorem 3.1.1. *There exists a finite morphism $\mathcal{Z}^2(\mathcal{E}) \to \mathrm{Km}(\mathcal{E}^2)$ commuting with the maps to U^2.*

Remark 3.1.2. Theorem 3.1.1 can be stated in any characteristic, since the definition of the space of coincidences is independent of characteristic. In the cases where $\mathcal{E} \to \mathbb{P}^1$ is unstable our formulas confirm Theorem 3.1.1 in any characteristic.

Generally, it is an interesting problem to give sufficient conditions for two K3 surfaces to be isogenous; see [Muk87; Bus19; Yan22]. A necessary condition is some relation between the Picard lattices of the two surfaces. Naturally one wants to know if this is sufficient:

Question 3.1.3. Let S be a K3 surface over an algebraically closed field. Let Λ be a lattice such that $\mathrm{Pic}\, S \otimes \mathbb{Q}$ and $\Lambda \otimes \mathbb{Q}$ are isometric. Does there exist a K3 surface S' in correspondence with S such that $\mathrm{Pic}\, S' \cong \Lambda$?

For many values of Λ, operations on elliptic fibrations can be used to construct the correspondence. The main theorem of this section answers a special case of Question 3.1.3.

Theorem 3.1.4. *Let S be a* K3 *surface over an algebraically closed field. Suppose there is an isometry* $\mathrm{Pic}\, S \otimes \mathbb{Q} \cong \mathrm{Pic}\, K \otimes \mathbb{Q}$, *where $K = \mathrm{Km}(E_1 \times E_2)$ for two nonisogenous elliptic curves E_1 and E_2. Then there is a morphism of finite degree from S to a Kummer surface of this form.*

3.2. ***Elliptic fibrations on* K3 *surfaces.*** We begin with a few remarks on elliptic fibrations and especially elliptic K3 surfaces. Some standard references are [SS10] for elliptic fibrations and [Huy16] for K3 surfaces. Unless stated otherwise, by "surface" we mean a smooth projective surface over an algebraically closed field.

3.2.1. *Generalities on elliptic surfaces.* Let S be a surface. A *genus* 1 *fibration* on S is a morphism $S \to X$ to a curve whose geometric generic fiber is a smooth integral curve of genus 1. If $S \to X$ admits a section, we call it an *elliptic fibration*. In any case it always has a *multisection*, meaning a curve $C \subset S$ such that $C \to X$ is finite; the degree of the multisection is the degree of $C \to X$. An *elliptic surface* is a surface admitting an elliptic fibration, such that no fiber contains an exceptional curve (meaning a smooth rational curve of self-intersection -1). In this paper all elliptic surfaces are assumed to be nonisotrivial; in other words, they are not products of two curves.

For a surface S, the *Néron–Severi group* $\mathrm{NS}(S)$ is the group of divisors modulo algebraic equivalence. It is a finitely generated abelian group endowed with an intersection pairing, which we write as (v, w). This pairing is nondegenerate on the torsion-free part of $\mathrm{NS}(S)$. We write $\rho(S)$ for the rank of the torsion-free part of $\mathrm{NS}(S)$. If $S \to X$ is an elliptic fibration with general fiber F and section O, the *trivial lattice* $T \subset \mathrm{NS}(S)$ is generated by O and all irreducible components of fibers of $S \to X$. Suppose the reducible fibers occur at $R \subset X$. For $v \in R$, let $x_0, \dots, x_{m_v-1} \in \mathrm{NS}(S)$ be the irreducible components of the fiber S_v, where x_0 is the unique such component which meets O. As all fibers are equivalent to F, the trivial lattice T is the orthogonal direct sum of $\langle O, F\rangle$ and $\langle x_1, \dots, x_{m_v-1}\rangle$.

The isomorphism classes of singular fibers of an elliptic surface have a well-known classification by Kodaira symbol. For each such fiber, the corresponding summand of T is a root lattice of type A_n, D_n ($n \geq 4$), E_6, E_7, or E_8. Here our convention is that our root lattices are *negative definite*. We will make use of the standard facts about each Kodaira symbol without comment, e.g., the corresponding Dynkin diagram, multiplicity of components, discriminant, etc.

There are two useful formulas relating the geometry of an elliptic surface S to the singular fibers of an elliptic fibration on S. One is the Shioda–Tate formula, and the other is a formula for the Euler number of S.

The Mordell–Weil group of an elliptic fibration $S \to X$ is the group of sections $S(X)$. Equivalently it is the set of rational points of the generic fiber of $S \to X$, which is an elliptic curve. There is an isomorphism

$$NS(S)/T \xrightarrow{\sim} S(X) \tag{3.2.1}$$

sending a divisor to its generic fiber. This yields the Shioda–Tate formula [SS10, Proposition 6.6]:

$$\rho(S) = 2 + \sum_{v \in R} (m_v - 1) + \operatorname{rank} S(X)/(\text{torsion}) \tag{3.2.2}$$

The smooth locus in each singular fiber S_v of an elliptic fibration has a group structure, with neutral component either $\mathbb{G}_m$ (type I_n) or $\mathbb{G}_a$ (all other types). We call S_v a multiplicative or additive fiber, respectively. For an additive fiber v, let δ_v be the index of wild ramification, defined by the formula

$$\delta_v = v(\Delta) - 1 - m_v$$

where Δ is the discriminant of a Weierstrass equation for S over the local ring $\mathcal{O}_{X,v}$. Then $\delta_v \neq 0$ only if the ground field has characteristic 2 or 3. If $\delta_v = 0$ for all v, we say that $S \to X$ is tame.

For a projective variety Y, write $e(Y)$ for the Euler number (= topological Euler characteristic = alternating sum of Betti numbers of Y). For a singular fiber S_v of an elliptic fibration $S \to X$, we have

$$e(S_v) = \begin{cases} m_v, & \text{if } S_v \text{ is multiplicative} \\ m_v + 1, & \text{if } S_v \text{ is additive.} \end{cases} \tag{3.2.3}$$

We have the relations

$$\chi(S, \mathcal{O}_S) = \tfrac{1}{12} e(S), \tag{3.2.4}$$

where χ is the Euler characteristic, and

$$e(S) = \sum_v (e(S_v) + \delta_v), \tag{3.2.5}$$

where the sum is over the singular fibers of $S \to X$.

3.2.2. K3 *elliptic surfaces.* A smooth projective surface S is K3 if it has trivial canonical bundle and $H^1(S, \mathcal{O}_S) = 0$.

Let S be a K3 surface. Then on S, the notions of linear, algebraic, and numerical equivalence of divisors agree, and $\operatorname{Pic} S \cong \operatorname{NS}(S)$ is torsion-free. We refer to this group as the Picard lattice. It has signature $(1, \rho(S) - 1)$. By Riemann–Roch, an irreducible curve $C \subset S$ of arithmetic genus g has $(C, C) = 2g - 2$. This implies that $\operatorname{Pic} S$ is an even lattice.

In the case that $k = \mathbb{C}$ is the field of complex numbers, then $\operatorname{Pic} S$ is a primitive sublattice of $H^2(S, \mathbb{Z})$. For its part, $H^2(S, \mathbb{Z})$ is isomorphic to the K3 *lattice* $E_8^2 \oplus U^3$, where U is the hyperbolic plane. By the Lefschetz principle, this fact about $\operatorname{Pic} S$ extends to any field of characteristic 0. In positive characteristic, we can use the following lemma.

Proposition 3.2.1 [Huy16, Proposition 5.8, Chapter 9]. *Let S be a* K3 *surface of finite height over a perfect field k of characteristic p. Then S can be lifted to $W(k)$ in such a way that the specialization map from the Néron–Severi group of the general to the special fiber is an isomorphism.*

We are interested in those K3 surfaces which admit genus 1 fibrations. If S is K3 and $S \to X$ is a genus 1 fibration, then necessarily $X \cong \mathbb{P}^1$. An interesting fact about elliptic K3 surfaces S is, if their Picard rank is not too small, they typically admit several genus 1 fibrations which are distinct modulo the action of $\operatorname{Aut} S$. It will be important for us to recognize the genus 1 fibrations on S purely by examining the lattice $\operatorname{Pic} S$.

Proposition 3.2.2 [Pvv71, §3, Theorem 1]. *Assume that the characteristic of k is not* 2 *or* 3. *Let S be a* K3 *surface of finite height.*

(1) *Suppose F is a primitive class in* $\operatorname{Pic} S$ *with* $(F, F) = 0$. *Then there exists a genus* 1 *fibration* $S \to \mathbb{P}^1$ *whose fiber class is* $\mathcal{O}(\operatorname{Pic} S)$*-equivalent to F.*

(2) *Continuing, let d be the greatest common divisor of (F, D) for all $D \in \operatorname{Pic} S$. Then $S \to \mathbb{P}^1$ admits a multisection of degree d.*

(3) *Continuing further, suppose $d = 1$, so that $S \to \mathbb{P}^1$ is an elliptic fibration with fiber F and section O. Then $\langle F, O\rangle$ is the hyperbolic plane. The reducible fibers of $S \to \mathbb{P}^1$ correspond exactly to the root lattice summands of the root sublattice of $\langle F, O\rangle^\perp$. (The* ***root sublattice*** *is the sublattice spanned by vectors v with $(v, v) = -2$.)*

Proof. (1) One can apply standard results on $\operatorname{Pic} S$ [Huy16, Corollary 8.2.9] to find a sequence of smooth rational curves $C_1, \dots, C_r$ on S such that $F' = \pm\rho_{C_r} \circ \cdots \circ \rho_{C_1}(F)$ is nef, where $\rho_C(v) = v - (2(v, C)/(C, C))C$ is the reflection in the hyperplane orthogonal to a Picard class $C \in \operatorname{Pic} S$ satisfying $C^2 = -2$. Since ρ_C preserves the intersection pairing, we have $(F', F') = 0$. By [Huy16, Proposition 2.3.10], the linear system $|F'|$ has no base points, and the associated morphism to projective space factors through a genus 1 fibration $S \to \mathbb{P}^1$, with fiber equivalent to F.

(2) It is clear that $F' \equiv F \pmod d$ in $\operatorname{Pic} S$, so that there exists $D \in \operatorname{Pic} S$ with $(F', D) = d$. After replacing D with $D + nF'$ for $n \gg 0$, we may assume that D is effective. Then D is a multisection of the fibration of degree d.

(3) Let $\mathcal{C} \subset \langle F, O\rangle^\perp$ be the set of C that satisfy $(C, C) = -2$. By Riemann–Roch either C or $-C$ is effective, so let $\mathcal{C}^+$ be the set of effective elements. We claim that every element of $\mathcal{C}^+$ is a sum of classes of curves contained in fibers.

Indeed, write $C \in \mathcal{C}^+ = \sum a_i[C_i]$, where the a_i are nonnegative integers and the C_i are classes of irreducible curves. Since $[F]$ moves with empty base locus, it has nonnegative intersection with all curves, and positive intersection with all curves not contained in any member of the linear series $|F|$. In particular O is not among the C_i, because $[F] \cdot [O] = 1$. Therefore $O \cdot C_i \geq 0$ for all i, and so F is not one of the C_i either. On the other hand it is clear that if C_i is a curve in a fiber that does not meet O, then $[C_i] \in \mathcal{C}$.

Let R be the root sublattice of $\langle F, O\rangle^\perp$. From the above we see that R is freely generated by the classes of curves in fibers that do not meet O. For each reducible fiber we obtain a root lattice as in [SS10, §11.13], and distinct fibers give orthogonal sublattices of R. $\square$

Corollary 3.2.3. *Let S be a* K3 *surface of finite height. If the Picard rank $\rho(S)$ is at least* 5, *then S admits a genus* 1 *fibration.*

Proof. The quadratic form of $\operatorname{Pic} S$ is indefinite and represents zero over $\mathbb{Q}_p$ for all primes p. Therefore by Hasse–Minkowski it represents zero rationally; i.e., there exists $F \in \operatorname{Pic} S$ with $(F, F) = 0$. By Proposition 3.2.2, the surface S admits a genus 1 fibration. $\square$

Remark 3.2.4. If the characteristic of k is 2 or 3, it is possible that a nef divisor with self-intersection 0 does not actually define a genus 1 fibration in the sense we have defined it; it can happen that the generic fiber has a cusp. See [Huy16, proof of Proposition 2.3.10].

In the situation of Proposition 3.2.2, we have a genus 1 fibration on S admitting a multisection D of degree d. We may pass to the Jacobian of this fibration [Huy16, Chapter 11, §4]: this is another K3 surface S' admitting a Jacobian fibration. There is a finite morphism $S \to S'$ of degree d^2, sending a section P to $dP - D$. The following lemma identifies the Picard lattice of S'.

Lemma 3.2.5 [Keu00, Lemma 2.1]. *Let S be a* K3 *surface admitting a genus* 1 *fibration of multisection index d with fiber $F \in$* Pic S, *and let S' be the Jacobian of that fibration. Then* Pic S' *is isomorphic to the overlattice* (Pic S)$[F/d]$ *of* Pic S.

It is natural to ask about a sort of converse to Lemma 3.2.5. Suppose S is a K3 surface with Picard lattice L, and suppose $L' \supset L$ is an overlattice with L'/L cyclic of degree d. Does there exist a genus 1 fibration on S whose Jacobian has Picard lattice L'? This is true if the rank of L is at least 13, as we will see in Proposition 3.2.8.

Lemma 3.2.6. *Let R be a principal ideal domain, let M be a free R-module of finite rank, and let Q be a quadratic form on M. Let $N \subset M$ be a submodule such that the discriminant of $Q|_N$ is a unit in R. Then $M = N \oplus N^\perp$.*

Proof. First note that $N \cap N^\perp = 0$, as otherwise $Q|_N$ would have discriminant 0.

Let F be the fraction field of R, and let $N^\vee \subset M \otimes_R F$ be the dual lattice, i.e., the R-submodule of $M \otimes_R F$ consisting of those m for which $(m, N) \subset R$. Then $\operatorname{Hom}(N, R) \cong N^\vee$. By hypothesis, we have $N = N^\vee$. Let $m \in M$. Then we can find $n \in N$ such that $(m, x) = (n, x)$ for all $x \in N$. Then $m - n \in N^\perp$, and we conclude $N + N^\perp = M$. □

In the context of Lemma 3.2.6, we say that Q *primitively represents* $a \in R$ if there exists a primitive element $x \in M$ with $Q(x) = a$.

Lemma 3.2.7. *Let p be prime and let Q be a quadratic form over $\mathbb{Z}_p$ of rank $n \geq 3$ and unit discriminant. If $p = 2$, assume further that Q is even. Then Q represents all elements of $\mathbb{Z}_p$ primitively.*

Proof. Suppose p is odd. Then Q may be diagonalized [CS99, p. 369, Theorem 2]: $Q = \sum_{i=1}^n a_i x_i^2$. Since Q has unit discriminant, each $a_i \in \mathbb{Z}_p^\times$. A standard counting argument shows that $a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2$ primitively represents all elements of $\mathbb{F}_p$. By Hensel's lemma, Q primitively represents all of $\mathbb{Z}_p$.

If $p = 2$, then (loc. cit.) the form Q is isomorphic to the direct sum of a diagonal form $\sum_{i=1}^m a_i x_i^2$ with forms of the shape $2^r Q_{a,b,c}$, where $r \geq 1$ and $Q_{a,b,c}(x, y) = ax^2 + bxy + cy^2$ is a form of unit discriminant with $a, c \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_2^\times$. Note that $\operatorname{disc} 2^r Q_{a,b,c}$ has 2-adic valuation $2r$. Since Q has unit discriminant, all $a_i \in \mathbb{Z}_2^\times$, and all the r are 0. But then since Q is even, $m = 0$.

Thus Q is the direct sum of at least two forms $Q_{a,b,c}$. For each of these, we consider two cases. If $2|ac$ then the discriminant $b^2 - 4ac$ is congruent to 1 mod 8. It is therefore the square of a unit, and so Q is a product of two linear factors $(rx + sy), (r'x + s'y)$ that generate the space of linear forms in two variables over $\mathbb{Z}_2$. So Q represents every element of $\mathbb{Z}_2$ primitively in this case. Alternatively, if $a, c \in \mathbb{Z}_2^\times$, then $Q_{a,b,c}$ defines a nonsingular conic over $\mathbb{Z}_2$, so by Hensel's lemma $Q_{a,b,c}$ primitively represents all of $\mathbb{Z}_2^\times$. Since every element of $\mathbb{Z}_2$ is either a unit or the sum of two units, it follows that Q primitively represents all of $\mathbb{Z}_2$ if there are two factors of this type. □

Proposition 3.2.8. *Let S be a* K3 *surface with Picard number at least* 13. *Assume that the ground field has characteristic* 0, *or else that S has finite height. Let p be prime, and let D be a divisor of S which is not divisible by p in $L = \operatorname{Pic} S$. Assume that $p|(D, x)$ for all $x \in L$, and also that $p^2|(D, D)$. Equivalently, $L' = L[D/p] \subset \operatorname{Pic} S \otimes \mathbb{Q}$ is an overlattice of L of index p. If $p = 2$, assume that L' is even.* (*Since L is even, this condition is equivalent to* $8|(D, D)$.)

Then there exists a divisor class $D' \in D + pL$ such that the p-part of $\gcd_{x \in L}(D', x)$ *is p, and a genus* 1 *fibration $S \to \mathbb{P}^1$ with fiber D'. Hence, if $S' \to \mathbb{P}^1$ is the p-Jacobian J^p* (see [Huy16, Remark 11.4.1]) *of the fibration, then* $\operatorname{Pic} S' \cong L'$.

Proof. Using Proposition 3.2.1 we assume that the ground field is $\mathbb{C}$.

Let $M \subset \operatorname{Pic} S \otimes \mathbb{Z}_p$ be a $\mathbb{Z}_p$-sublattice of maximal rank among those which have unit discriminant. Write Q for the quadratic form on $\operatorname{Pic} S \otimes \mathbb{Z}_p$. By Lemma 3.2.6, we have $\operatorname{Pic} S \otimes \mathbb{Z}_p = M \oplus N$, with $N = M^{\perp}$. Then $Q|_N$ is divisible by p. Indeed, if $x \in N$ satisfies $Q(x) \in \mathbb{Z}_p^{\times}$, then $M \oplus \langle x \rangle$ would be a larger submodule with unit discriminant. In fact we claim that the intersection form on N is divisible by p, i.e., that $N \subset pN^{\vee}$. This follows automatically if p is odd. If $p = 2$, we also need to exclude the possibility of $x, x' \in N$ with $(x, x') \in \mathbb{Z}_2^{\times}$. But since we already know that $(x, x), (x', x') \in 2\mathbb{Z}_2$, we have $(x, x)(x', x') - (x, x')^2 \in \mathbb{Z}_2^{\times}$, and again we have constructed a larger submodule $M \oplus \langle x, x' \rangle$ with unit discriminant.

Recall that the discriminant group of a lattice Λ (over whatever PID base) is $D(\Lambda) = \Lambda^{\vee}/\Lambda$. Then $D(L) \otimes \mathbb{Z}_p \cong D(L \otimes \mathbb{Z}_p)$. On the other hand $L \otimes \mathbb{Z}_p = M \oplus N$ as above, and we have $D(M) = 0$, whereas

$$D(N) \otimes \mathbb{F}_p = N^{\vee}/(N, pN^{\vee}) = N^{\vee}/pN^{\vee} = N^{\vee} \otimes \mathbb{F}_p$$

since $N \subset pN^{\vee}$ as we observed above. We conclude from this that $\dim_{\mathbb{F}_p} D(L) \otimes \mathbb{F}_p = \operatorname{rank} N$.

Now we use that fact that $L = \operatorname{Pic} S = \operatorname{NS}(S)$ embeds primitively into the K3 lattice $H = H^2(X, \mathbb{Z})$. (This is because for any complex projective surface S, we have $\operatorname{NS}(S) \cong H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$.) Crucially, H is unimodular. Let $L^{\perp}$ be the orthogonal complement of L in H. The intersection pairing on H induces a pairing $D(L) \otimes H/(L \oplus L^{\perp}) \to \mathbb{Q}/\mathbb{Z}$.

This pairing puts $D(H)$ and $H/(L \oplus L^{\perp})$ into Pontrjagin duality. Proof: if $h \in H$ satisfies $(L^{\vee}, h) \subset \mathbb{Z}$, write $h = \ell + \ell^{\perp}$, where $\ell \in L \otimes \mathbb{Q}$, $\ell^{\perp} \in L^{\perp} \otimes \mathbb{Q}$. Then $(L^{\vee}, \ell) \subset \mathbb{Z}$ implies $\ell \in L^{\vee\vee} = L$ and therefore $\ell^{\perp} \in H \cap (L^{\perp} \otimes \mathbb{Q}) = L^{\perp}$ and $h \in L \oplus L^{\perp}$. Conversely if $\ell^{\vee} \in L^{\vee}$ satisfies $(\ell^{\vee}, H) \in \mathbb{Z}$, then $\ell^{\vee} \in H^{\vee} \cap L^{\vee} = H \cap L^{\vee} \subset H \cap (L \otimes \mathbb{Q}) = L$ because L is embedded primitively in H.

In particular $\dim_{\mathbb{F}_p} D(L) \otimes \mathbb{F}_p = \dim(H/(L \oplus L^{\perp})) \otimes \mathbb{F}_p \le \operatorname{rank} L^{\perp} = 22 - \operatorname{rank} L$, as one can see by extending a $\mathbb{Z}$-basis of L to H. Thus $\operatorname{rank} N \le 22 - \operatorname{rank} L$.

We have assumed $\rho(S) = \operatorname{rank} L \ge 13$, and so

$$\operatorname{rank} M = \operatorname{rank} L - \operatorname{rank} N \ge 2 \operatorname{rank} L - 22 \ge 4.$$

Our divisor D was assumed to satisfy $p|(D, x)$ for all $x \in L$. If we decompose D as $D = m + n$ with $m \in M$, $n \in N = M^{\perp}$, then $p|(m, x)$ for all $x \in M$, which is to say, $m \in pM^{\vee}$. But since M is unimodular, we find that $m \in pM$. We find that $n \equiv D \pmod{L \otimes p\mathbb{Z}_p}$. We have also assumed that $L[D/p]$ is a lattice, which implies that $p^2|(D, D)$; if $p = 2$ we have assumed $8|(D, D)$. The same statements are true when D is replaced with n. By Lemma 3.2.7, we can find a vector $x \in M$ such that $(x, x) = -(n, n)/p^2$. Then $D_0 = n + px \in L \otimes \mathbb{Z}_p$ satisfies $D_0 \equiv D \pmod{p}$ and $(D_0, D_0) = 0$.

By weak approximation on quadric hypersurfaces, there exists $D' \in L \otimes \mathbb{Q}$ satisfying $(D', D') = 0$ which is p-adically close to D_0. After clearing denominators, we may assume $D' \in L$. Now we may apply Proposition 3.2.2 to obtain the required genus 1 fibration. □

Corollary 3.2.9. *Let S be a finite height* K3 *surface of Picard number* ≥ 13, *and let p be an odd prime. Let $d = \operatorname{disc}\operatorname{Pic} S$, and let p^v be the largest power of p dividing d. Suppose we have an isometry* $\operatorname{Pic} S \otimes \mathbb{Q}_p \cong L \otimes \mathbb{Q}_p$, *where L is a unimodular $\mathbb{Z}_p$-lattice. Then there is a map of finite degree from S to a* K3 *surface S' for which* $\operatorname{disc}\operatorname{Pic} S' = d/p^v$.

Proof. The hypothesis on $\operatorname{Pic} S$ implies that v is even, as the discriminant of a quadratic form over a field is well-defined up to a square. It also implies that the Hasse invariant of $\operatorname{Pic} S \otimes \mathbb{Q}_p$ is trivial, as this is the case for $L \otimes \mathbb{Q}_p$. Indeed, we may diagonalize the quadratic form on L as $\sum_{i=1}^n a_i x_i^2$, with $a_i \in \mathbb{Z}_p^\times$, and then the Hasse invariant is $\prod_{i<j}(a_i, a_j)$ (Hilbert symbol). Each factor in the product is trivial (note that p is odd).

We proceed by induction on v, the case $v = 0$ being obvious. First suppose that $D(\operatorname{Pic} S)$ contains an element of order p^2, represented by $D^\vee \in (\operatorname{Pic} S)^\vee$. Then $D = p^2 D^\vee \in \operatorname{Pic} S$ satisfies the hypotheses of Proposition 3.2.8, in which case there is a degree p^2 map $S \to S'$ with $\operatorname{disc}\operatorname{Pic} S' = d/p^2$.

Therefore assume that $D(\operatorname{Pic} S) \otimes \mathbb{Z}_p$ is p-torsion, in which case its $\mathbb{F}_p$-dimension is v. The quadratic form on $\operatorname{Pic} S \otimes \mathbb{Z}_p$ is equivalent to the diagonal form

$$pa_1x_1^2 + \cdots + pa_vx_v^2 + a_{v+1}x_{v+1}^2 + \cdots + a_nx_n^2, \tag{3.2.6}$$

with each $a_i \in \mathbb{Z}_p^\times$. If $v > 2$, there exist $x_1, x_2, x_3 \in \mathbb{Z}_p$ such that $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 \equiv 0 \pmod p$ since every nonsingular conic over $\mathbb{F}_p$ has a rational point. Let $D \in \operatorname{Pic} S$ be p-adically close to $(x_1, x_2, x_3, 0, \dots) \in \operatorname{Pic} S \otimes \mathbb{Z}_p$. Then D satisfies the hypotheses of Proposition 3.2.8, and again we can remove a factor of p^2 from $\operatorname{disc}\operatorname{Pic} S$.

We are now reduced to the case $v = 2$. Standard formulas for the Hilbert symbol reveal that $(pa_1, a_i) = (pa_2, a_i)$ for all $i \geq 3$, so that the Hasse invariant of (3.2.6) is $(pa_1, pa_2) = (-a_1a_2/p)$ (Legendre symbol). But by the observation in the first paragraph, the Hasse invariant is trivial; i.e., $-a_1a_2$ is a square modulo p. Thus $a_1x_1^2 + a_2x_2^2$ represents 0 modulo p, and we proceed as in the previous paragraph. □

3.3. ***Kummer surfaces* $\mathbf{Km(E_1 \times E_2)}$ *of Picard rank 18.*** Here we recall the basic constructions and properties of the Kummer surface $\operatorname{Km}(A)$ attached to an abelian surface A. For the moment let us suppose we are in characteristic not 2. Denoting by ι the involution $x \mapsto -x$ on A, the quotient A/ι has rational double point singularities at each of the 16 fixed points of ι (namely, the 2-torsion points of A). Let $\operatorname{Km}(A)$ be the minimal resolution of A/ι. Then $\operatorname{Km}(A)$ is a K3 surface.

The Picard lattice $\operatorname{Pic}\operatorname{Km}(A) = \operatorname{NS}\operatorname{Km}(A)$ contains both $\operatorname{NS}(A)$ and the classes of the 16 exceptional divisors. In fact these generate $\operatorname{Pic}\operatorname{Km}(A)$ and we have:

$$\rho(\operatorname{Km}(A)) = 16 + \rho(A) \tag{3.3.1}$$

We will focus on the case that $A = E_1 \times E_2$ is a product of nonisogenous elliptic curves E_1, E_2. In this case, (3.3.1) gives $\rho(\mathrm{Km}(A)) = 18$. There is an elliptic fibration $\mathrm{Km}(E_1 \times E_2) \to \mathbb{P}^1$ given by projecting onto $E_2/\iota \cong \mathbb{P}^1$; the geometric fibers are all isomorphic to E_1. In fact this gives an alternate construction of $\mathrm{Km}(E_1 \times E_2)$: it is the quadratic twist of the constant elliptic fibration $E_1 \times \mathbb{P}^1 \to \mathbb{P}^1$ by the double cover $E_2 \to \mathbb{P}^1$. (Of course, we could have reversed the roles of E_1 and E_2 in this construction.)

The bad fibers of $\mathrm{Km}(E_1 \times E_2) \to \mathbb{P}^1$ occur at the four branch points of $E_2 \to \mathbb{P}^1$, and they are all of type $\mathrm{I}_0^* = \tilde{D}_4$. Therefore the trivial lattice of the fibration is $U \oplus D_4^{\oplus 4}$. We note here that $\operatorname{disc} D_4 = 4$. On the other hand the Mordell–Weil group of this fibration has order 4 (coming from the 2-torsion in E_1). We conclude from (3.2.1) that $\operatorname{Pic} \mathrm{Km}(E_1 \times E_2)$ has discriminant -16. The lattice $\operatorname{Pic} \mathrm{Km}(E_1 \times E_2)$ does not depend on the elliptic curves E_1 or E_2, so long as they are nonisogenous. In the complex setting, the transcendental lattice of $\mathrm{Km}(E_1 \times E_2)$ is $T_{E_1 \times E_2}(2) \cong (H^1(E_1, \mathbb{Z}) \otimes H^1(E_2, \mathbb{Z}))(2) \cong U^{\oplus 2}(2)$.

We are interested in the question of whether a given K3 surface S of rank 18 is isogenous to a Kummer surface of the form $\mathrm{Km}(E_1 \times E_2)$, and if so, how to find the elliptic curves E_1 and E_2. The following result relies on a deep theorem of Mukai.

Proposition 3.3.1. *Let S be a complex* K3 *surface such that* $\operatorname{Pic} S \otimes \mathbb{Q}$ *is isometric to* $\operatorname{Pic} \mathrm{Km}(E_1 \times E_2) \otimes \mathbb{Q}$ *for some nonisogenous elliptic curves E_1, E_2. Then there exists an isogeny between S and a Kummer surface of that form.* (*An isogeny between* K3 *surfaces S and S' is an algebraic cycle on $S \times S'$ inducing an isometry $H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$.*)

Sketch of proof. Let T_S be the transcendental lattice of S. Then T_S has signature $(2, 2)$, and is the complement of $\operatorname{Pic} \mathrm{Km}(E_1 \times E_2)$ in $H^2(S, \mathbb{Z})$. A calculation involving Hasse–Minkowski invariants shows that $T_S \otimes \mathbb{Q}$ is isometric to $U^{\oplus 2} \otimes \mathbb{Q}$. Via this isometry, the Hodge structure on T_S now determines a Hodge structure on $U^{\oplus 2}$, which is to say, a morphism of real groups $h : \mathbb{S} \to O(2, 2)$, where $\mathbb{S}$ is the Deligne torus. Now observe that there is an isomorphism $(g_1, g_2) \mapsto g_1 \otimes g_2$ from $(\mathrm{Sp}(2) \times \mathrm{Sp}(2))/\{\pm 1\}$ onto the neutral component of $O(2, 2)$. Therefore h factors through a morphism $\mathbb{S} \to (\mathrm{Sp}(2) \times \mathrm{Sp}(2))/\{\pm 1\}$. For $i = 1, 2$, let h_i be the projection of this morphism onto the i-th copy of $\mathrm{Sp}(2)/\{\pm 1\} \cong \mathrm{PGL}_2$; then h_i determines an elliptic curve E_i. Thus we have an isometry of rational Hodge structures:

$$T_S \otimes \mathbb{Q} \cong H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q}) \cong T_{\mathrm{Km}(E_1 \times E_2)} \otimes \mathbb{Q}$$

We now invoke a theorem of Mukai [Muk87, Corollary 1.10]: an isometry between rational Hodge structures of two K3 surfaces of rank ≥ 11 is always induced from an isogeny. □

In this section we prove an effective version of Proposition 3.3.1, whereby the elliptic curves E_1 and E_2 can in theory be computed from S, and where the isogeny is simply a finite rational map (which can also be computed). The idea is to leverage operations on elliptic fibrations. We build up the result in stages.

Proposition 3.3.2. *Let S be a* K3 *surface whose Picard lattice is isometric to* $\mathrm{Km}(E_1 \times E_2)$ *for some nonisogenous elliptic curves E_1, E_2. Then S is isomorphic to a Kummer surface of that form.*

Proof. Proposition 3.2.2 produces an elliptic fibration $S \to \mathbb{P}^1$ with I_0^* fibers at four points of $\mathbb{P}^1$. Let $E_2 \to \mathbb{P}^1$ be the elliptic curve branched at exactly these four points. Then the quadratic twist of $S \to \mathbb{P}^1$

by $E_2 \to \mathbb{P}^1$ has I_0 reduction at these points; i.e., it has good reduction everywhere and therefore must be a constant elliptic curve $E_1 \times \mathbb{P}^1$. Thus S is a quadratic twist of $E_1 \times \mathbb{P}^1$ by $E_2 \to \mathbb{P}^1$, and this is exactly $\mathrm{Km}(E_1 \times E_2)$. □

Remark 3.3.3. Recall that a *Shioda–Inose structure* [Mor84, Definition 6.1] on a complex K3 surface S is a rational map $S \to S'$ of degree 2 such that S' is a Kummer surface and $T_{S'} \cong 2T_S$, meaning that the Gram matrix of $T_{S'}$ is twice that of T_S. Morrison proves [Mor84, Theorem 6.3] that S has a Shioda–Inose structure if and only if $\operatorname{Pic} S \otimes \mathbb{Q}$ is isometric to $\operatorname{Pic} K \otimes \mathbb{Q}$ for some Kummer surface $K = \mathrm{Km}(A)$, if and only if T_S embeds primitively in U^3.

Proposition 3.3.4. *Assume that* $\operatorname{char} k \neq 2$. *Let S be a* K3 *surface whose Picard lattice has rank* 18 *and discriminant* -1. *Then there is a finite map from S to a* K3 *surface of the form* $\mathrm{Km}(E_1 \times E_2)$, *where* E_1, E_2 *are nonisogenous elliptic curves.*

Proof. The transcendental lattice of S is an even unimodular lattice of rank 4 and signature $(2,2)$, so it is isometric to U^2. By Remark 3.3.3, there is a Shioda–Inose structure on S. The codomain of this isogeny has transcendental lattice $U^2(2)$, which as we have seen is that of the Picard lattice of $\mathrm{Km}(E_1 \times E_2)$ for nonisogenous E_1, E_2. The result now follows from Proposition 3.3.2. □

We need a further lattice-theoretic result.

Proposition 3.3.5. *Every even lattice L of discriminant* -4^i *and signature* $(1,17)$ *is contained in a unimodular lattice*, *necessarily* $\mathrm{II}_{1,17} = U \oplus E_8^{\oplus 2}$.

Proof. We will show that if $i \geq 1$ then L is contained in a lattice of discriminant -4^{i-1}, and then the result follows by induction. The idea is that if $x \in D(L) = L^\vee / L$ is an element of order 2 satisfying $(x,x) \in 2\mathbb{Z}$ (i.e., x is isotropic in $D(L)[2]$), then $L' = L + \langle x \rangle$ is again an even lattice, and $\operatorname{disc} L' = \frac{1}{4} \operatorname{disc} L$.

We need the fact that (for general lattices L) the $\mathbb{F}_2$-dimension of $D(L)[2]$ has the same parity as the rank of L, which in our case is even. (Proof: Use the classification of lattices over $\mathbb{Z}_2$ to reduce to the case that the quadratic form is either x^2 or $2^r(ax^2 + bxy + cy^2)$ with unit discriminant where $a, c \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_2^\times$.) As a result the following four cases are exhaustive.

- Case 1: $D(L)$ contains an element a of order 2^i, where $i > 2$. We have $(a, 2^i a) \in \mathbb{Z}$, and therefore $(a,a) \in 2^{-i}\mathbb{Z}$. Now let $x = 2^{i-1}a$; then x has order 2 in $D(L)$ and $(x,x) = 2^{2i-2}(a,a) \in 2^{i-2}\mathbb{Z} \subset 2\mathbb{Z}$ as desired.
- Case 2: $D(L)$ contains two independent elements a, b of order 4. Let these be a, b. We then have $4(a,a), 4(b,b), 4(a,b) \in \mathbb{Z}$. If $4(a,a)$ and $4(b,b)$ are both odd, then $(2a+2b, 2a+2b) = 4(a,a) + 8(a,b) + 4(b,b)$ is even. Thus at least one of $2a, 2a+2b, 2b$ is isotropic in $D(L)[2]$.
- Case 3: $D(L)[2]$ contains $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$. Let S be the subset of $D(L)[2]$ consisting of x with $(x,x) \in \mathbb{Z}$; it is easy to see this is a subgroup. If $x, y \in D(L)[2]$ do not lie in S, each of $(x,x), (y,y) \in \mathbb{Z} + \frac{1}{2}$, and then $(x+y, x+y) = (x,x) + 2(x,y) + (y,y) \in \mathbb{Z}$. This shows that S has index at most 2 in $D(L)[2]$, so that S contains $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$. Let $s_1, s_2, s_3 \in S$ be independent elements. If any s_i satisfies

$(s_i, s_i) \in 2\mathbb{Z}$, it is isotropic. If $(s_i, s_j) \in \mathbb{Z}$ for any $i \neq j$, then $(s_i + s_j, s_i + s_j) \in 2\mathbb{Z}$. If not, then $(s_i, s_i) \in 1 + 2\mathbb{Z}$ for all i and $(s_i, s_j) \in 1/2 + \mathbb{Z}$ for all $i \neq j$, so $(s_1 + s_2 + s_3, s_1 + s_2 + s_3) \in 2\mathbb{Z}$.

- Case 4: $D(L) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let a_1, a_2, a_3 be the nonzero elements of $D(L)$. Assume that none of these are isotropic. By [Mor84, Theorem 2.8] we can embed L into the K3 lattice $U^3 + E_8^2$, and the complement is a lattice of signature $(2, 2)$ and discriminant 4. By [Mor84, Theorem 2.8] again we can embed this into U^3, with complement of signature $(1, 1)$ and discriminant -4. This lattice has the same discriminant form as L, by applying [Mor84, Lemma 2.4] twice. On the other hand, there are only two such lattices L_1, L_2, with Gram matrices $M_1 = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. (Proof: let x, y be a basis. If $(x, x)(y, y) < 0$ then we must have $(x, x) = 2, (y, y) = -2, (x, y) = 0$, or the same with x, y switched, and that is the first case. If not, we can take $0 \le (x, x) \le (y, y)$ and $(x, y) \ge 0$, and either we are in one of the cases above or we can decrease $(x, x) + (y, y)$ by replacing y by $y \pm x$). So L, being determined by its invariants [Mor84, Theorem 2.2], is the direct sum $M_i \oplus E_8^{\oplus 2}$ for $i = 1$ or 2. In both cases there is a vector in L that can be divided by 2, namely $(x, 0)$. □

We have reached the main theorem of this section.

Theorem 3.3.6. *Let Λ be the Picard lattice of (any) Kummer surface of the form* $\mathrm{Km}(E_1 \times E_2)$, *where E_1, E_2 are nonisogenous elliptic curves. Let S be a* K3 *surface. Assume there is an isometry* $\operatorname{Pic} S \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q}$. *Then there exists a finite morphism $S \to \mathrm{Km}(E_1 \times E_2)$ for nonisogenous elliptic curves E_1, E_2.*

Proof. Since Λ has discriminant -16, the hypothesis implies that $|\operatorname{disc} \operatorname{Pic} S|$ is a square. We apply Corollary 3.2.9 to all odd primes dividing $\operatorname{disc} \operatorname{Pic} S$ to obtain a finite morphism from $S \to S'$, where S' is a K3 surface with $\operatorname{disc} \operatorname{Pic} S' = -4^n$. By Proposition 3.3.5, there is an embedding of $L' = \operatorname{Pic} S'$ into L'', where $L'' = \mathrm{II}_{1,17}$ is the even unimodular lattice of signature $(1, 17)$. We can factor this embedding as $L' = L_0 \subset L_1 \subset \cdots \subset L_m = L''$, with $L_i/L_{i+1} \cong \mathbb{Z}/2\mathbb{Z}$ for each i. Successive applications of 3.2.8 give a finite morphism $S' \to S''$, where $\operatorname{Pic} S'' \cong L''$. Finally, by Proposition 3.3.4 there is a finite morphism from S'' to a Kummer surface of the form $\mathrm{Km}(E_1 \times E_2)$. □

We conclude this section with some remarks on potential extensions of Theorem 3.3.6 to the case of more general Kummer surfaces. Recall that if A is an abelian surface, the Picard rank of the Kummer surface $\mathrm{Km}(A)$ equals $16 + \operatorname{rank} \mathrm{NS}(A)$. In characteristic 0, the rank of $\mathrm{Km}(A)$ takes one of the values 17, 18, 19, 20. The cases of rank 19, 20 are easy to deal with.

Remark 3.3.7. If $\operatorname{rank} \operatorname{Pic} S > 18$, then a Shioda–Inose structure on S always exists [Mor84, Theorem 6.3, Corollary 6.4], so there is a map of degree 2 from S to a Kummer surface. Thus the analogue of Theorem 3.3.6 is true for such S.

We now turn to the case of rank 18. We pose the question of whether a K3 surface S of rank 18 should admit a finite morphism to a Kummer surface $\mathrm{Km}(A)$. Here, A would have to be an abelian surface admitting endomorphisms by an order $\mathcal{O}$ in a real (and possibly split) quadratic extension of $\mathbb{Q}$. Then $\operatorname{disc} \operatorname{Pic} \mathrm{Km}(A) = -16 \operatorname{disc} \mathcal{O}$. We have already treated the split case $\mathcal{O} = \mathbb{Z} \times \mathbb{Z}$, which corresponds to the case that A is a product of elliptic curves. Our methods can also treat the case of $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$.

Proposition 3.3.8. *Let S be a* K3 *surface. Let Λ be the Picard lattice of* (*any*) *Kummer surface of the form* $\mathrm{Km}(A)$, *where A is an abelian surface with* $\mathrm{End}\, A = \mathbb{Z}[\sqrt{2}]$. *Assume there is an isometry* $\mathrm{Pic}\, S \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q}$. *Then there exists a finite morphism $S \to \mathrm{Km}(A)$ for an abelian surface A of this sort.*

Sketch of proof. As in Corollary 3.2.9, we may remove all odd primes from the discriminant of $\mathrm{Pic}\, S$. Therefore assume that $\mathrm{disc}\,\mathrm{Pic}\, S = -2^{2i+1}$ for some $i \geq 0$. An even lattice of even rank may not have discriminant ± 2, so in fact we may assume $i \geq 1$.

We claim that $\mathrm{Pic}\, S$ is contained in a lattice of discriminant -8. We argue as in the proof of Proposition 3.3.5: referring to that proof, if $i \geq 2$ then one of the first three cases always holds and we can always find an overlattice of $\mathrm{Pic}\, S$ of index 2. Therefore let us assume that $i = 1$, so that $\mathrm{disc}\,\mathrm{Pic}\, S = -8$. Using the results of [CS99, Chapter 15], we can confirm that there is only one isomorphism class of even lattices with discriminant -8 and signature $(1, 17)$, namely $D_9 \oplus E_7 \oplus U$. We can now assume that $\mathrm{Pic}\, S$ is isomorphic to this lattice.

Invoking a computation with Hasse–Minkowski invariants, this forces the transcendental lattice of S to be isomorphic to $U \oplus \langle -2 \rangle \oplus \langle 4 \rangle$. (This is also a consequence of [EK14, Proposition 7].) Thus by [Mor84, Corollary 6.2] there is a Shioda–Inose structure $S \to S'$, where $S' = \mathrm{Km}(A)$ is the Kummer surface of an A satisfying $T_A \cong U \oplus \langle -2 \rangle \oplus \langle 4 \rangle$. The Néron–Severi lattice $\mathrm{NS}(A)$ of A, being the complement of T_A in $H^2(A, \mathbb{Z}) \cong U^{\oplus 3}$, is isomorphic to $\langle 2 \rangle \oplus \langle -4 \rangle$. The vector of length 2 in $\mathrm{NS}(A)$ is a principal polarization on A, and $\mathrm{End}\, A = \mathrm{NS}(A)$ is the quadratic order of discriminant -8, namely $\mathbb{Z}[\sqrt{-2}]$. □

Remark 3.3.9. It is easy to embed $D_9 \oplus E_7 \oplus U$ into the K3 lattice $E_8^{\oplus 2} \oplus U^{\oplus 3}$ with complement either $U \oplus \langle -2 \rangle \oplus \langle 4 \rangle$ or $U \oplus \langle 2 \rangle \oplus \langle -4 \rangle$. However, these need not be distinguished. Indeed, let L be the lattice $\langle 2 \rangle \oplus \langle -4 \rangle$ and let x, y be the given basis. Then $L \cong -L$, as one sees by changing to the basis $(x + y, 2x + y)$. (This is a manifestation of the fact that $\mathbb{Z}[\sqrt{2}]$ has a unit of norm -1.)

Finally we turn to the case of Picard rank 17.

Proposition 3.3.10. *Let S be a* K3 *surface. Let Λ be the Picard lattice of* (*any*) *Kummer surface of the form* $\mathrm{Km}(A)$, *where A is an abelian surface with* $\mathrm{End}\, A = \mathbb{Z}$. *Assume there is an isometry n* $\mathrm{Pic}\, S \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q}$, *where $n \in \{1, 2\}$. Then there exists a finite morphism $S \to \mathrm{Km}(A)$ for an abelian surface A of this sort.*

Sketch of proof. Again we apply the techniques of Corollary 3.2.9 and 3.3.5 to assume that $\mathrm{disc}\,\mathrm{Pic}\, S$ is either -2 or -4. In these two cases we must have $n = 1$ or 2 respectively. In both cases, the even lattice $\mathrm{Pic}\, S$ is uniquely determined by its discriminant and signature.

In the case of discriminant -2, we have $\mathrm{Pic}\, S \cong E_8 \oplus E_7 \oplus U$. This lattice is rationally isometric to Λ. The orthogonal complement of $\mathrm{Pic}\, S$ in $H^2(S, \mathbb{Z})$ is isomorphic to $A_1 \oplus U^{\oplus 2}$. The existence of a Shioda–Inose structure now follows from [Mor84, Corollary 6.4 (iii)].

In the case of discriminant -4, $\mathrm{Pic}\, S \cong L \oplus U$, where L is a lattice containing its root sublattice L_0 with index 2, and $L_0 \cong A_3 \oplus D_{12}$. Therefore by Proposition 3.2.2, there exists an elliptic fibration on S with 2-torsion section. Let $S \to S'$ be the quotient by the 2-torsion section. The induced map $T_{S'} \to T_S$ is not generally a rational isometry, but rather $T_{S'}$ is rationally isometric to $2T_S$: see [BSV17, §2.4].

Since the rank of T_S is odd, the discriminants of T_S and $T_{S'}$ differ by twice the square of a rational number. Since Pic S is the complement of T_S in a unimodular lattice, and similarly for Pic S', we find that disc Pic $S' = -2n^2$ for some integer $n \geq 1$. Also, we now have a rational isometry Pic $S' \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q}$. After once again removing squares from the discriminant we are in the case of the previous paragraph. □

4. Verification of 2-modularity for extremal rational elliptic fibrations

The goal of this section is to complete the proof of Theorem 1.2.4: every nonisotrivial tame extremal rational elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ over a finite field is 2-modular. In Section 4.1 we present an overview and classification of such fibrations, and show that when $\mathcal{E} \to \mathbb{P}^1$ is base changed along a double cover of $\mathbb{P}^1$, a K3 surface arises which is isogenous to a Kummer. In Section 4.2 we consider the family of K3 surfaces $\mathcal{Z}^2(\mathcal{E}) \to U^2$ that arose in the introduction, and prove the existence of an isogeny from $\mathcal{Z}^2(\mathcal{E})$ onto Km$(\mathcal{E}' \times \mathcal{E}')$, where $\mathcal{E}' \to U$ is a 2-modular elliptic fibration of the same conductor as $\mathcal{E}$. If $\mathcal{E}$ is semistable, this is enough to show that $\mathcal{E}$ and $\mathcal{E}'$ are isogenous, so that $\mathcal{E}$ is 2-modular as well. The remaining sections feature case-by-case calculations which show that $\mathcal{E}'$ and $\mathcal{E}$ are isogenous in the unstable cases as well. For those calculations, the theory of Shioda–Inose structures is indispensable.

4.1. ***Extremal rational elliptic fibrations and associated*** **K3** ***surfaces.*** A rational elliptic surface $\mathcal{E}$ over an algebraically closed field F is isomorphic to $\mathbb{P}^2$ blown up at 9 points (possibly infinitely near), so that $\rho(\mathcal{E}) = 10$ and $e(\mathcal{E}) = 12$. If in addition $\mathcal{E} \to \mathbb{P}^1$ is extremal, and if we also assume that the singular fibers are tame, then by the Shioda–Tate formula (3.2.2) we must have $\sum_v (m_v - 1) = 8$, where m_v is the number of irreducible components in the fiber $\mathcal{E}_v$. On the other hand by the Euler number formula (3.2.3) we have $\sum_v e(\mathcal{E}_v) = 12$, where $e(E_v)$ is m_v or $m_v + 1$ as v is multiplicative or additive. Therefore the singular fibers of $\mathcal{E} \to \mathbb{P}^1$ fall into one of the following three possibilities:

(1) four multiplicative fibers;

(2) two multiplicative fibers and one additive fiber;

(3) two additive fibers.

The case of two additive fibers can only occur if $\mathcal{E} \to \mathbb{P}^1$ is isotrivial (i.e., has constant j-invariant). We discard this case, and refer to the (1) as the semistable case and (2) as the unstable case.

Remark 4.1.1. There do exist nonconstant extremal rational elliptic fibrations with two singular fibers, for example the curve with Weierstrass equation

$$y^2 + txy = x^3 - t^5$$

in characteristic 2 has j-invariant t and singular fibers exactly at $t = 0, \infty$. The fiber at 0 is wild.

We present here the classification of semistable extremal rational elliptic fibrations $\mathcal{E} \to \mathbb{P}^1$. Each has exactly four fibers of multiplicative type. This classification is due to Beauville ([Bea82], which also gives Weierstrass equations and the connection to universal elliptic curves with level structure). See also [Ito02].

Proposition 4.1.2. *Let k be an algebraically closed field, and let $\mathcal{E} \to \mathbb{P}^1_k$ be a nonisotrivial semistable extremal rational elliptic fibration. Then the fibration $\mathcal{E} \to \mathbb{P}^1_k$ is determined up to isomorphism by its configuration of singular fibers. Up to an automorphism of $\mathbb{P}^1_k$, those configurations are as follows, grouped by isogeny class*:

singular fibers	locations	Mordell–Weil group	notes
$\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3$	$(1, \omega, \omega', \infty)^\dagger$	$(\mathbb{Z}/3\mathbb{Z})^2$	char $k \neq 3$
$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_9$		$\mathbb{Z}/3\mathbb{Z}$	
$\mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_4, \mathrm{I}_4$	$(-1, 1, 0, \infty)$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	char $k \neq 2$
$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_8$		$\mathbb{Z}/4\mathbb{Z}$	
$\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_6$	$\left(4, -\frac{1}{2}, 0, \infty\right)$	$\mathbb{Z}/6\mathbb{Z}$	char $k \neq 2, 3$
$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_5, \mathrm{I}_5$	$(\phi, \phi', 1, \infty)^\ddagger$	$\mathbb{Z}/5\mathbb{Z}$	char $k \neq 5$

† ω, ω' are roots of $x^2 + x + 1$. ‡ ϕ, ϕ' are roots of $x^2 - x - 1$.

The following proposition classifies the tame extremal rational elliptic fibrations in the unstable case. If such a fibration is nonisotrivial, then it has exactly three singular fibers, two additive and one multiplicative. As $\operatorname{Aut} \mathbb{P}^1$ acts triply transitively, it is no longer necessary to keep track of the locations of the singular fibers. We derive the following table from [MP86] and [Ito02] (in characteristics 2 and 3).

Proposition 4.1.3. *Let k be an algebraically closed field, and let $\mathcal{E} \to \mathbb{P}^1_k$ be an unstable nonisotrivial tame extremal rational elliptic fibration. Then the fibration is determined up to isomorphism by its configuration of singular fibers. The possible configurations are as follows, grouped by isogeny class*:

singular fibers	Mordell–Weil group	notes
$\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2$	$(\mathbb{Z}/2\mathbb{Z})^2$	char $k \neq 2$
$\mathrm{I}_4^*, \mathrm{I}_1, \mathrm{I}_1$	$\mathbb{Z}/2\mathbb{Z}$	
$\mathrm{I}_1^*, \mathrm{I}_1, \mathrm{I}_4$	$\mathbb{Z}/4\mathbb{Z}$	
$\mathrm{II}^*, \mathrm{I}_1, \mathrm{I}_1$	0	char $k \neq 2, 3$
$\mathrm{III}^*, \mathrm{I}_1, \mathrm{I}_2$	$\mathbb{Z}/2\mathbb{Z}$	char $k \neq 2$
$\mathrm{III}, \mathrm{I}_3, \mathrm{I}_6$	$\mathbb{Z}/6\mathbb{Z}$	char $k = 3$
$\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3$	0	char $k \neq 2, 3$
$\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6$	$\mathbb{Z}/6\mathbb{Z}$	char $k = 2$
$\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3$	$\mathbb{Z}/3\mathbb{Z}$	
$\mathrm{II}, \mathrm{I}_5, \mathrm{I}_5$	$\mathbb{Z}/5\mathbb{Z}$	char $k = 5$

Remark 4.1.4. In characteristic 2, the $\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6$ fibration is the specialization of the $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_6$ fibration, and in characteristic 5, the $\mathrm{II}, \mathrm{I}_5, \mathrm{I}_5$ fibration is the specialization of the $\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_5, \mathrm{I}_5$ fibration.

Fix a nonisotrivial extremal rational elliptic fibration $\mathcal{E} \to \mathbb{P}^1_F$. Let $C \to \mathbb{P}^1_F$ be a separable double cover, with C a rational curve; thus $C \to \mathbb{P}^1_F$ is ramified at two points. Assume these points are disjoint from the singular locus of $\mathcal{E} \to \mathbb{P}^1_F$. Consider the base change

$$S = C \times_{\mathbb{P}^1} \mathcal{E}.$$

Then S is a K3 surface. Indeed, S is a double cover of the rational surface $\mathcal{E}$ branched along the sextic described by the union of two cubics (namely, the fibers of $\mathcal{E}$ over the branch points of $C \to \mathbb{P}^1$).

Under our hypothesis that $\mathcal{E} \to \mathbb{P}^1$ is rational and extremal, we must have $\rho(\mathcal{E}) = 10 = 2 + 8$, where the 2 is from the identity O and fiber F, and the 8 is from irreducible components of fibers which do not cross O. Considering the elliptic fibration $S \to C$, the contribution to $\rho(S)$ from irreducible components of fibers is $2 \cdot 8 = 16$, and so $\rho(S) \geq 2 + 16 = 18$.

We now have a rational map of moduli spaces:

$$(\text{double covers of } \mathbb{P}^1 \text{ branched at 2 points}) \;\to\; (\text{K3 surfaces of Picard rank} \geq 18).$$

Both spaces are 2-dimensional, so we expect the generic double cover $C \to \mathbb{P}^1$ to produce a K3 surface S of rank 18, with Picard lattice not depending on C.

Proposition 4.1.5. *Assume that* Pic S *has rank* 18. *There exists a rational isometry*

$$\operatorname{Pic} S \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q},$$

where Λ is the Picard lattice of the Kummer surface associated to the product of two nonisogenous elliptic curves.

Proof. We have $\operatorname{Pic} \mathcal{E} \cong E_8 \oplus U$. Let $L \subset \operatorname{Pic} \mathcal{E}$ be the sublattice generated by components of reducible fibers not meeting the identity section, so that $L \oplus U$ embeds into $E_8 \oplus U$ with finite cokernel. We therefore have embeddings of lattices: $L^{\oplus 2} \oplus U \subset E_8 \oplus L \oplus U \subset E_8^{\oplus 2} \oplus U$.

Now Pic S contains a finite-index sublattice isomorphic to $L^{\oplus 2} \oplus U$, generated by reducible fibers and the identity section. By the observation above, $\operatorname{Pic} S \otimes \mathbb{Q} \cong (E_8^{\oplus 2} \oplus U) \otimes \mathbb{Q}$. A calculation involving Hasse–Minkowski invariants shows that the latter is isomorphic to $\Lambda \otimes \mathbb{Q}$. □

4.2. ***On the construction of 2-modular elliptic fibrations.*** With this background, we now turn to the question of 2-modularity for a tame extremal rational elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ of conductor N over a finite field. We have a family of K3 surfaces $\mathcal{Z}^2(\mathcal{E}) \to U^2$, obtained via base changing $\mathcal{E} \to \mathbb{P}^1$ by a family of double covers $\mathbb{P}^1 \to \mathbb{P}^1$ parametrized by U^2. Here is what we know so far about $\mathcal{Z}^2(\mathcal{E})$:

(1) There exists an isogeny between families of K3 surfaces $\mathcal{Z}^2(\mathcal{E}) \to \operatorname{Km}(\mathcal{A})$ over U^2. Here $\mathcal{A} \to U^2$ is a family of abelian varieties which, possibly after passing to a double cover of U^2, splits as a product of nonisogenous elliptic curves with transcendental j-invariant.

(2) There exists a dominant rational map $\operatorname{Sht}_G(\Gamma_0(N)) \dashrightarrow \mathcal{Z}^2(\mathcal{E})$ over U^2; see (1.4.3).

Remark 4.2.1. The first statement is obtained by applying 1.4.1 to $\mathcal{Z}^2(\mathcal{E})$. Strictly speaking, that theorem only applies over algebraically closed fields, but it can be made to work over the base U^2, with the proviso that the splitting of $\mathcal{A}$ into a product of elliptic curves may only happen over an étale double cover of U^2. The argument runs this way: consider the ring scheme $\operatorname{End}\mathcal{A}$ over U^2 which classifies endomorphisms of $\mathcal{A}$. Since $\mathcal{A}_{\bar{\eta}_2}$ splits as a product of nonisogenous, transcendental elliptic curves, the $\bar{\eta}_2$-points of $\operatorname{End}\mathcal{A}$ are $\mathbb{Z}\times\mathbb{Z}$. On the other hand, an appeal to the rigidity lemma shows that the union of those components of $\operatorname{End}\mathcal{A}$ which dominate U^2 together form an étale group scheme; thus one can think of this union as a representation of $\pi_1(U^2,\bar{\eta}_2)$ on the ring $\mathbb{Z}\times\mathbb{Z}$. There are two possibilities. If the representation is trivial, in which case $\mathcal{A}$ is the product of two families of elliptic curves. If the representation is nontrivial, $\pi_1(U^2,\bar{\eta}_2)$ permutes the factors of the $\mathbb{Z}\times\mathbb{Z}$; in this case $\mathcal{A}$ is the restriction of scalars of a family of elliptic curves over an étale double cover of U^2.

The main goal of the subsection is to prove the following theorem.

Theorem 4.2.2. *Let $\mathcal{A}\to U^2$ be a family of abelian surfaces. Assume*:

(1) *Étale-locally on U^2, the abelian surface $\mathcal{A}$ is isomorphic to a product of nonisogenous elliptic curves, each of which has transcendental j-invariant.*

(2) *There exists a dominant rational map from $\mathrm{Sht}^2_G(\Gamma_0(N))$ to $\mathrm{Km}(\mathcal{A})$ lying over U^2.*

Then there exists a nonisotrivial family of elliptic curves $\mathcal{E}'\to U$ of conductor bounded by N, and an isogeny $\mathrm{Km}(\mathcal{A})\to\mathrm{Km}(\mathcal{E}'\times_{\mathbb{F}_q}\mathcal{E}')$ over U^2. This $\mathcal{E}'$ is 2-modular.

The main players in the proof are the étale fundamental groups and their representations. Let us notate our étale fundamental groups as $\pi_1(U)$, $\pi_1(U^2)$, etc., the base point being understood to be $\bar{\eta}$ or $\bar{\eta}_2$. The Künneth theorem fails in characteristic p, in the sense that the natural map $\pi_1(U^2)\to\pi_1(U)^2$ is not an isomorphism. In fact it is neither injective nor surjective; see the discussion of Drinfeld's Lemma in [SW20, §1.1]. Write $\bar{U}$ (resp., $\bar{U}^2$) for the base change of U (resp., $\bar{U}^2$) from $\mathbb{F}_q$ to $\bar{\mathbb{F}}_q$. The various fundamental groups fit into a diagram with exact rows and columns (here Δ = diagonal map):

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathcal{K} \arrow[r, "="] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \pi_1(\bar{U}^2) \arrow[r] \arrow[d] & \pi_1(U^2) \arrow[r] \arrow[d] & \mathrm{Gal}_{\mathbb{F}_q} \arrow[r] \arrow[d, "\Delta"] & 1 \\
1 \arrow[r] & \pi_1(\bar{U})^2 \arrow[r] \arrow[d] & \pi_1(U)^2 \arrow[r] \arrow[d] & \mathrm{Gal}^2_{\mathbb{F}_q} \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & \mathrm{Gal}_{\mathbb{F}_q} \arrow[r, "="'] \arrow[d] & \mathrm{Gal}_{\mathbb{F}_q} \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

Now consider our family of abelian surfaces $f : \mathcal{A} \to U^2$. Let $H^2(\mathcal{A}, \mathbb{Q}_\ell)$ denote the fiber of $R^2 f_* \mathbb{Q}_\ell$ at the geometric generic point $\overline{\eta}_2$: this is a representation of $\pi_1(U^2)$. Within this there is the 4-dimensional transcendental subspace $H^2_{\text{trans}}(\mathcal{A}, \mathbb{Q}_\ell)$, meaning the subspace orthogonal to all algebraic classes.

Lemma 4.2.3. *The Tate twist $H^2_{trans}(\mathcal{A}, \overline{\mathbb{Q}}_\ell)(1)$ contains no nonzero vector fixed by an open subgroup of $\pi_1(U^2)$.*

Proof. Suppose otherwise, and that the nonzero vector is fixed by the fundamental group of a connected finite cover of U^2 with fraction field L. After a possible further extension of L, the base change $\mathcal{A}_L$ is isomorphic to $E_1 \times_L E_2$ for elliptic curves E_1, E_2 over L. Then $H^2_{\text{trans}}(\mathcal{A}_L, \mathbb{Q}_\ell)(1)$ is the $\mathbb{Q}_\ell$-linear dual of the tensor product $V_\ell(E_1) \otimes V_\ell(E_2)(-1)$, where V_ℓ means the rational Tate module. This is in turn isomorphic via the Weil pairing to $\text{Hom}(V_\ell(E_1), V_\ell(E_2))$.

The situation now is that $\text{Hom}_{\text{Gal}_L}(V_\ell(E_1), V_\ell(E_2)) \otimes \overline{\mathbb{Q}}_\ell$ is nonzero. We now appeal to the isogeny theorem over fields finitely generated over $\mathbb{F}_q$ [Zar14, Theorem 1.4] to conclude that E_1 and E_2 are isogenous over L, contrary to our hypothesis about $\mathcal{A}$. $\square$

We have assumed the existence of a dominant rational map from $\text{Sht}^2_G(\Gamma_0(N))$ to $\text{Km}(\mathcal{A})$. This induces a map on cohomology

$$H^2_c(\text{Sht}^2_G(\Gamma_0(N))_{\overline{\eta}_2}, \mathbb{Q}_\ell) \to H^2_{\text{trans}}(\mathcal{A}, \mathbb{Q}_\ell) \tag{4.2.1}$$

which is equivariant for the action of $\pi_1(U^2)$ on either side. The map in (4.2.1) is surjective: this reduces to a general fact about dominant rational maps between quasiprojective varieties; see [Kle68, Proposition 1.2.4].

Lemma 4.2.4. *The map in* (4.2.1) *is nonzero when restricted to the cuspidal subspace.*

Proof. The cuspidal subspace is the kernel of the "constant term map"

$$H^2_c(\text{Sht}^2_G(\Gamma_0(N))_{\overline{\eta}_2}, \mathbb{Q}_\ell)(1) \to H^0_c(\text{Sht}^2_M(\Gamma_0(N))_{\overline{\eta}_2}, \mathbb{Q}_\ell)$$

towards the compactly supported cohomology of a space of shtukas relative to $M \subset G$, where $M \cong \mathbb{G}_m$ is the Levi subgroup. (For the construction of the constant term map, see [Xue20a]. The Tate twist (1) appears to compensate for the fact that we have used constant coefficients rather than intersection cohomology on the G-shtuka side.) The cohomology of M-shtukas is known by class field theory: after base extension to $\overline{\mathbb{Q}}_\ell$, the image of the constant term map decomposes as a $\pi_1(U^2)$-module as direct sum of characters $\chi \boxtimes \chi^{-1}$, where χ runs over finite-order characters of $\pi_1(U)$ of conductor bounded by N. Note that all such characters become trivial when restricted to an open subgroup. For the map (4.2.1) to be trivial on the cuspidal subspace, it would mean that such a direct sum surjects onto $H^2_{\text{trans}}(\mathcal{A}, \overline{\mathbb{Q}}_\ell)(1)$, which violates Lemma 4.2.3. $\square$

Proposition 1.3.2 states that the action of $\pi_1(U^2)$ on the cuspidal subspace of $H^2_c(\text{Sht}^2_G(\Gamma_0(N)), \overline{\mathbb{Q}}_\ell)$ extends along the homomorphism $\pi_1(U^2) \to \pi(U)^2$, and decomposes into a direct sum of representations of $\pi(U)^2$ of the form $\sigma^{\boxtimes 2}$, where σ runs over irreducible representations of $\pi_1(U)$ with determinant

$\overline{\mathbb{Q}}_\ell(-1)$ of conductor bounded by N. Since each $\sigma^{\boxtimes 2}$ is irreducible of dimension 4, we find that there is a particular σ for which

$$H^2_{\mathrm{trans}}(\mathcal{A}, \overline{\mathbb{Q}}_\ell) \cong \sigma^{\boxtimes 2}|_{\pi_1(U^2)} \tag{4.2.2}$$

as representations of $\pi_1(U^2)$.

Lemma 4.2.5. *The family of abelian surfaces $\mathcal{A} \to U^2$ is isomorphic to $\mathcal{E}_1 \times_{U^2} \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are families of elliptic curves over U^2.*

Proof. Recall we have assumed that $\mathcal{A}$ splits into such a product over an étale cover. Assume this splitting doesn't occur globally over U^2. Then $\mathcal{A}$ is the restriction of scalars of a family of elliptic curves $\mathcal{E} \to V$ along a connected double cover $V \to U^2$. Consequently $H^2_{\mathrm{trans}}(\mathcal{A})$ is isomorphic to the *tensor-induced* representation of $\rho := H^1(\mathcal{E}_{\bar{\eta}_2}, \overline{\mathbb{Q}}_\ell)$ from $\pi_1(V)$ to $\pi(U^2)$. This can be modeled as the extension of $\rho^{\otimes 2}$ to $\pi_1(U^2)$ determined by the rule $s(v_1 \otimes v_2) = v_2 \otimes s^2 v_1$, where s is any representative for the nontrivial coset of $\pi_1(V)$ in $\pi_1(U^2)$. Such an element s always acts nontrivially on $\rho^{\otimes 2}$: to see this, choose v_1 to be an eigenvector of s^2, and choose v_2 to be linearly independent from v_1.

Recall that $\mathcal{K}$ is the kernel of $\pi_1(U^2) \to \pi_1(U)^2$. The isomorphism (4.2.2) shows that $\mathcal{K}$ acts trivially on $H^2_{\mathrm{trans}}(\mathcal{A}, \mathbb{Q}_\ell)$. By the above observation about the nontrivial action of s, it follows that $\mathcal{K} \subset \pi_1(V)$. Equivalently, there exists a finite connected cover $W \to U$ such that V is intermediate to $W^2 \to U^2$.

Let us now base change the entire story from U to W. We have a family of elliptic curves $\mathcal{E} \to W^2$, such that if $\rho = H^1(\mathcal{E}_{\bar{\eta}_2}, \overline{\mathbb{Q}}_\ell)$, then $\rho^{\otimes 2}$ extends along the homomorphism $\pi_1(W^2) \to \pi_1(W)^2$ to a representation of the form $\sigma^{\boxtimes 2}$.

Let $\mathcal{K}_W$ be the kernel of $\pi_1(W^2) \to \pi_1(W)^2$. An element of $\mathcal{K}_W$ acts trivially on $\rho^{\otimes 2}$, which implies that it must act as a scalar ± 1 on ρ. Therefore $\mathcal{K}_W$ acts trivially on the projective representation $P\rho$.

We find that the restriction of $P\rho$ to $\pi_1(\overline{W}^2)$ factors through a projective representation

$$P\rho : \pi_1(\overline{W})^2 \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell),$$

which is tantamount to two homomorphisms from $\pi_1(\overline{W})$ whose images commute with one another. By considering Zariski closures, there are two possibilities, each of which leads to a contradiction:

- $P\rho$ is trivial when restricted to one of the factors of $\pi_1(\overline{W})^2$. But then the same would be true of $P(\sigma^{\boxtimes 2})$, which is false.
- The image of each copy of $\pi_1(\overline{W})$ under $P\rho$ lies in the same torus in PGL_2. This would imply that the image of $\pi_1(W^2)$ in $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ lies in the normalizer of a torus. By the isogeny theorem, this is only possible if $\operatorname{End} \mathcal{E}$ is larger than $\mathbb{Z}$. This contradicts our hypothesis on $\mathcal{A}$: its elliptic curve factors have transcendental j-invariant. □

By Lemma 4.2.5, our abelian surface $\mathcal{A}$ is isomorphic to a product $\mathcal{E}_1 \times_{U^2} \mathcal{E}_2$, where each $\mathcal{E}_i \to U^2$ is a family of elliptic curves. Let ρ_i be the representation of $\pi(U^2)$ on $H^1(\mathcal{E}_{i,\bar{\eta}_2}, \overline{\mathbb{Q}}_\ell)$. The isomorphism in (4.2.2) becomes

$$\rho_1 \otimes \rho_2 \cong \sigma^{\boxtimes 2}|_{\pi_1(U^2)}. \tag{4.2.3}$$

Let $\mathrm{pr}_1, \mathrm{pr}_2$ refer to the projection maps $U^2 \to U$.

Lemma 4.2.6. *After possibly relabeling, the projective representation $P\rho_i$ is isomorphic to $pr_i^* P\sigma$, meaning the pullback of $P\sigma$ along the map $\pi_1(U^2) \to \pi(U)$ induced by pr_i.*

Proof. Since $\mathcal{K}$ acts trivially on $\rho_1 \otimes \rho_2$, it must act by scalars on each ρ_i, in fact by ± 1 due to the Weil pairing on each ρ_i. Therefore the projective representations $P\rho_i$ are trivial on $\mathcal{K}$, and so $P\rho_i|_{\pi_1(\overline{U}^2)}$ factors through a projective representation of $\pi_1(\overline{U})^2$. As such, $P\rho_i$ has to be trivial on one of the factors of $\pi_1(\overline{U})^2$. (The argument is similar to that given in the proof of Lemma 4.2.5.)

Considering (4.2.3), the only possibility (after possibly relabeling) is that $P\rho_i|_{\pi_1(\overline{U}^2)} \cong \mathrm{pr}_i^* P\sigma|_{\pi_1(\overline{U})}$. □

Lemma 4.2.7. *There are families of elliptic curves $\mathcal{E}'_1, \mathcal{E}'_2 \to U$ and characters $\chi_1, \chi_2 : \pi_1(U^2) \to \{\pm 1\}$ such that $\mathcal{E}_i \cong pr_i^* \mathcal{E}'_i \otimes \chi_i$ for $i = 1, 2$.*

Proof. We run the argument for $\mathcal{E}_1$ only. For the proof it will be convenient to distinguish the two copies of U: $U^2 = U_1 \times U_2$. We want to show that $\mathcal{E}_1$ is isotrivial relative to U_2.

Let $\bar{\eta}_1 \to U_1$ be a geometric generic point. The base change of $\mathcal{E}_1$ to $\bar{\eta}_1 \times U_2$ induces an action of the monodromy group $\pi_1(\bar{\eta}_1 \times U_2)$ on ρ_1; this action factors through the map $\pi_1(\bar{\eta}_1 \times U_2) \to \pi_1(U_1 \times U_2)$. Lemma 4.2.6 shows that this monodromy acts as a scalar, since the composition

$$\pi_1(\bar{\eta}_1 \times U_2) \to \pi_1(U_1 \times U_2) \to \pi_1(U_1)$$

is the identity. This is enough to show that $\mathcal{E}_1$ is isotrivial relative to U_2: its j-invariant lies in the coordinate ring of $\bar{\eta}_1$. But the j-invariant already lies in the coordinate ring of $U_1 \times U_2$, so it must lie in the coordinate ring of U_1.

Let $\mathcal{E}'_1 \to U_1$ be a family of elliptic curves with the same j-invariant; there exists a (possibly trivial) character $\chi_i : \pi_1(U^2) \to \{\pm 1\}$ such that $\mathcal{E}_1$ is isomorphic to the twist $\mathcal{E}'_1 \otimes \chi_i$. □

The following lemma completes the proof of Theorem 4.2.2:

Lemma 4.2.8. *There is a family of elliptic curves $\mathcal{E}' \to U$ of conductor bounded by N and an isogeny* $\mathrm{Km}(\mathcal{E}_1 \times_{U^2} \mathcal{E}_2) \to \mathrm{Km}(\mathcal{E}' \times_{\mathbb{F}_q} \mathcal{E}')$.

Proof. Let ρ'_i be the representation of $\pi_1(U_i)$ on $H^1(\mathcal{E}'_i, \overline{\mathbb{Q}}_\ell)$. According to Lemma 4.2.7, $\rho_i \cong \mathrm{pr}_i^* \rho'_i \otimes \chi_i$ as a representation of $\pi_1(U^2)$. The isomorphism (4.2.3) now reads

$$(\rho'_1 \boxtimes \rho'_2) \otimes \chi_1\chi_2 \cong \sigma^{\boxtimes 2}|_{\pi_1(U^2)}$$

Since $\mathcal{K}$ acts trivially on the external tensor products $\rho_1 \boxtimes \rho_2$ and $\sigma^{\boxtimes 2}$, it lies in the kernel of $\chi_1\chi_2$. Therefore $\chi_1\chi_2$ factors through the image of $\pi_1(U^2) \to \pi_1(U)^2$. It is possible to extend $\chi_1\chi_2$ to a character $\chi'_1 \boxtimes \chi'_2$ of $\pi_1(U)^2$ valued in $\{\pm 1\}$ such that

$$(\rho'_1 \boxtimes \rho'_2) \otimes (\chi'_1 \boxtimes \chi'_2) \cong \sigma \boxtimes \sigma,$$

in which case $\rho_1' \otimes \chi_1' \cong \rho_2' \otimes \chi_2' \cong \sigma$. By the isogeny theorem, $\mathcal{E}_1' \otimes \chi_1'$ and $\mathcal{E}_2' \otimes \chi_2'$ are isogenous to the same family of elliptic curves $\mathcal{E}' \to U$, which has conductor bounded by N (since σ does).

Since $\chi_1' \boxtimes \chi_2'$ extends $\chi_1\chi_2$, the products $\chi_1 \operatorname{pr}_1^* \chi_1'$ and $\chi_2 \operatorname{pr}_2^* \chi_2'$ represent the same character on $\pi_1(U^2)$; call this common character χ. Now note that the formation of the Kummer surface of a product of elliptic curves is insensitive to twisting both curves by a common quadratic character:

$$\operatorname{Km}(\mathcal{E}_1 \times_{U^2} \mathcal{E}_2) \cong \operatorname{Km}((\mathcal{E}_1 \otimes \chi) \times_{U^2} (E_2 \otimes \chi)) \cong \operatorname{Km}((\mathcal{E}_1' \otimes \chi_1') \times_{U^2} (\mathcal{E}_2' \otimes \chi_2')).$$

The latter is isogenous to $\operatorname{Km}(\mathcal{E}' \times_{\mathbb{F}_q} \mathcal{E}')$. □

Return now to the situation of Theorem 1.2.4. Let $\mathcal{E} \to \mathbb{P}^1$ be a tame extremal rational elliptic fibration over a finite field. Then on the one hand $\mathcal{Z}^2(\mathcal{E}) \to U^2$ is a family of K3 surfaces admitting a dominant rational map from $\operatorname{Sht}_G^2(\Gamma_0(N))$, and on the other, there is a finite map $\mathcal{Z}^2(\mathcal{E}) \to \operatorname{Km}(\mathcal{A})$ for a family of abelian surfaces $\mathcal{A} \to U^2$ which splits étale-locally as a product of nonisogenous elliptic curves with transcendental j-invariant. Theorem 4.2.2 applies, and we find a 2-modular elliptic fibration $\mathcal{E}' \to U$ of conductor bounded by N. Since N has degree 4 and $\mathcal{E}'$ is nontrivial, the conductor of $\mathcal{E}'$ must be exactly N.

We claim that $\mathcal{E}$ and $\mathcal{E}'$ are isogenous over U, which would imply that $\mathcal{E}$ is 2-modular as well. In the semistable cases, there is only one isogeny class of conductor N for p large enough, so the isogeny is automatic for those primes. For the remaining primes, we argue this way: The objects $\mathcal{E}$ and $\mathcal{Z}^2(\mathcal{E})$ can be defined in characteristic 0; we have found an isogeny between $\mathcal{Z}^2(\mathcal{E})$ and $\operatorname{Km}(\mathcal{E}' \times \mathcal{E}')$, where $\mathcal{E}'$ is an elliptic fibration in characteristic 0. After replacing $\mathcal{E}'$ with an isogenous fibration, we have shown that $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic modulo almost all primes; this shows immediately that they are isomorphic at all primes of good reduction.

In the unstable cases, there are multiple elliptic fibrations with the same conductor, and we could not find a theoretical argument for why $\mathcal{E}$ and $\mathcal{E}'$ should be isogenous. In all those cases however we were able to find an explicit isogeny.

The remaining material in this section consists of calculations performed on each isogeny class of tame extremal rational elliptic fibrations, indicating how to find the required isogeny $\mathcal{Z}(\mathcal{E}^2) \to \operatorname{Km}(\mathcal{E} \times \mathcal{E})$.

4.3. ***The*** $\mathbf{I_2^* I_2 I_2}$ ***(Legendre) fibration, with Mordell–Weil group*** $(\mathbb{Z}/2\mathbb{Z})^2$**.** The simplest case of Theorem 3.1.1 concerns the Legendre fibration $\mathcal{E} \to \mathbb{P}_t^1$, with equation

$$y^2 = x(x-1)(x-t) \tag{4.3.1}$$

over a field k of characteristic $\neq 2$. Then $\mathcal{E} \to \mathbb{P}_t^1$ has singular fibers $\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2$ at $t = \infty, 0, 1$, with split multiplicative reduction at $t = 1$. The conductor is $N = (0) + (1) + 2(\infty)$, and the Mordell–Weil group is $(\mathbb{Z}/2\mathbb{Z})^2$. Let $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. We write t_1, t_2 for the coordinates on U^2.

Let $\Sigma_\infty = \{1\}$ (this choice is unimportant); we computed in Example 2.7.1 that the space of coincidences $\operatorname{Coinc}^3(\Gamma_0(N); \Sigma_\infty) \cong \mathbb{P}_s^1 \times U^2$ is a double cover of $\mathbb{P}_t^1 \times U^2$, with equation

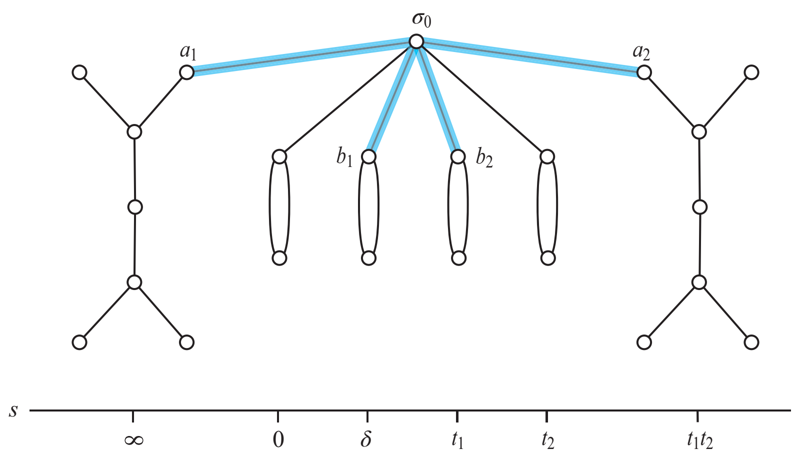
$$t = \frac{s(s - t_1 - t_2 + 1)}{s - t_1 t_2} \tag{4.3.2}$$

(obtained by solving for $t = t_3$ in (2.7.1)).

The elliptic fibration $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1_s \times U^2$ may be defined by substituting (4.3.2) into (4.3.1). For each pair $(t_1, t_2) \in U^2$, it has singular fibers of type $\mathrm{I}_2^*, \mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_2$ at $s = \infty, t_1t_2, 0, t_1, t_2$, and $\delta := t_1 + t_2 - 1$, respectively. Generically, the Mordell–Weil group is again $(\mathbb{Z}/2\mathbb{Z})^2$. From this we conclude that the Picard lattice of the generic fiber of $\mathcal{Z}^2(\mathcal{E}) \to U^2$ has rank 18 and determinant -16, which agrees with the corresponding data for the lattice of $\mathrm{Km}(A)$, where A is the product of two nonisogenous elliptic curves. In fact the two lattices are isomorphic, suggesting that we can find an isomorphism between $\mathcal{Z}^2(\mathcal{E})$ and such a Kummer surface. This is in fact the case: there is an *isomorphism* (not just a finite rational map) of K3 surfaces over U^2:

$$\mathcal{Z}^2(\mathcal{E}) \xrightarrow{\sim} \mathrm{Km}(\mathcal{E}^2)$$

To find the isomorphism, we should look for an elliptic fibration on $\mathcal{Z}^2(\mathcal{E})$ with four I_0^* fibers. One I_0^* configuration can be found within the union of the identity section σ_0, together with two of the I_2^*s and two of the I_2s, shown here as $2\sigma_0 + a_1 + a_2 + b_1 + b_2$:



By Proposition 3.2.2(3), there exists an elliptic fibration $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1$ containing this I_0^* as a fiber. We write it down explicitly: Introduce a function $w : \mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1$ by the substitutions (all are equivalent)

$$x = -\frac{(s-\delta)(s-t_1 w)}{(s-t_1t_2)(w-1)}, \quad x-1 = -\frac{(s-t_1)(s-t_2-(t_1-1)w)}{(s-t_1t_2)(w-1)}, \quad x-t = -\frac{(s-\delta)(s-t_1)w}{(s-t_1t_2)(w-1)}.$$

Making this substitution in $y^2 = x(x-1)(x-t)$ yields, after absorbing square factors into y,

$$y^2 = -w(w-1)(s-t_1t_2)(s-t_1w)(s-t_2-(t_1-1)w),$$

and then $s = -(w-t_2)u + t_1 w$ brings this into the form

$$y^2 = w(w-1)(w-t_2)u(u-1)(u-t_1),$$

which is the Kummer surface $\mathrm{Km}(\mathcal{E}^2)$.

We used software to find rather ungainly proofs of Theorem 3.1.1 for some other elliptic fibrations. But then Masato Kuwata recognized that in all those cases $\mathcal{Z}^2(\mathcal{E})$ is related to the *Inose surface* of $\mathcal{E}^2$. He graciously explained to us the connection in the following interlude.

4.4. ***Interlude by Masato Kuwata: Kummer surfaces and Inose surfaces.*** Throughout this section, the base field k has characteristic different from 2.

Let E_1 and E_2 be elliptic curves defined over k. Let ι be the inversion map $(P, Q) \mapsto (-P, -Q)$ on $E_1 \times E_2$, and let $\bar{S} = E_1 \times E_2/\langle\iota\rangle$ be the quotient by ι. The surface $\bar{S}$ has sixteen double points corresponding to the 2-torsion points of $E_1 \times E_2$. The Kummer surface associated with the product $E_1 \times E_2$, denoted by $\mathrm{Km}(E_1 \times E_2)$, is defined as the smooth surface obtained by blowing up these double points.

There are two obvious elliptic fibrations on $\mathrm{Km}(E_1 \times E_2)$ corresponding to the projections $E_1 \times E_2 \to E_i$ $(i = 1, 2)$:

$$\begin{tikzcd}[column sep=small] & \bar{S} = E_1 \times E_2/\langle\iota\rangle \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E_1/\{\pm 1\} \simeq \mathbb{P}^1 & & \mathbb{P}^1 \simeq E_2/\{\pm 1\} \end{tikzcd}$$

Oguiso showed in [Ogu89] that there are eleven different types of elliptic fibrations on $\mathrm{Km}(E_1 \times E_2)$ if k is algebraically closed. Generally, most of these fibrations (excepting the two above) will not be defined over k.

"Inose's pencil" [KS08] is a genus 1 fibration on $\mathrm{Km}(E_1 \times E_2)$ which is always defined over k. To define it, choose Weierstrass equations of E_1 and E_2:

$$E_1 : y^2 = x^3 + a_2x^2 + a_4x + a_6, \quad E_2 : y^2 = x^3 + a_2'x^2 + a_4'x + a_6'.$$

Then an affine model of $\mathrm{Km}(E_1 \times E_2)$ is given by

$$(z^3 + a_2z^2 + a_4z + a_6)t^2 = x^3 + a_2'x^2 + a_4'x + a_6'. \tag{4.4.1}$$

This equation can be viewed as a cubic curve in x, z over the function field $k(t)$. The map $\mathrm{Km}(E_1 \times E_2) \to \mathbb{P}^1$ given by $(x, z, t) \mapsto t$ defines a genus 1 fibration. This is Inose's pencil. It does not admit a section unless E_1 or E_2 has a k-rational point of order 2.

Let $J \to \mathbb{P}^1$ be the Jacobian of Inose's pencil (4.4.1). Using the formula in [ARVT05], we obtain the Weierstrass equation for J as follows:

$$\begin{aligned} Y^2 = X^3 + 4a_2a_2'X^2 + 16(a_2^2a_4' - 3a_4a_4' + a_4a_2'^2)X& \\ + \big(\Delta_{E_1}t^2 - (c_6a_6' - 32a_2a_4a_2'a_4' + 864a_6a_6' + c_6'a_6) + \Delta_{E_2}t^{-2}\big),& \end{aligned} \tag{4.4.2}$$

where

$$\begin{aligned} \Delta_{E_1} &= -16(4a_2^3a_6 - a_2^2a_4^2 - 18a_2a_4a_6 + 4a_4^3 + 27a_6^2), \\ \Delta_{E_2} &= -16(4a_2'^3a_6' - a_2'^2a_4'^2 - 18a_2'a_4'a_6' + 4a_4'^3 + 27a_6'^2), \\ c_6 &= -32(2a_2^3 - 9a_2a_4 + 27a_6), \quad c_6' = -32(2a_2'^3 - 9a_2'a_4' + 27a_6'). \end{aligned}$$

An alternate way to obtain (4.4.2) is to substitute $t = (t')^3$ into (4.4.1), and then use the rational point $(1 : (t')^2 : 0)$ to convert (4.4.1) into Weierstrass form (see [KK17, §2.1]).

Following [KK17], the *Inose surface* $\mathrm{Ino}(E_1 \times E_2)$ may be defined as the quotient of (4.4.2) by the involution $t \mapsto -t$. It has equation

$$\begin{aligned}\mathrm{Ino}(E_1 \times E_2) : Y^2 = X^3 &+ 4a_2a_2'X^2 + 16(a_2^2a_4' - 3a_4a_4' + a_4{a_2'}^2)X \\ &+ \bigl(\Delta_{E_1}T - (c_6a_6' + 32a_2a_4a_2'a_4' - 864a_6a_6' + c_6'a_6) + \Delta_{E_2}T^{-1}\bigr), \end{aligned} \tag{4.4.3}$$

where $T = t^2$ is the parameter. It has two II^* fibers, at $T = 0$ and ∞; all other singular fibers are irreducible. It is known that, if E_1 and E_2 are not isogenous to each other, the transcendental lattice of the Inose surface is isomorphic to $U \oplus U$, where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the hyperbolic plane, whereas the transcendental lattice of the Kummer surface is isomorphic to $U(2) \oplus U(2)$.

There is an involution on $\mathrm{Ino}(E_1 \times E_2)$ given by $i : (X, Y, T) \mapsto (X, -Y, \Delta_{E_2}/\Delta_{E_1}T)$. The quotient by this involution has equation

$$\begin{aligned}\mathrm{Ino}(E_1 \times E_2)/\langle i \rangle : Y^2 = X^3 &+ 4a_2a_2'(S^2 - 4\Delta_{E_1}\Delta_{E_2})X^2 \\ &+ 16(a_2^2a_4' - 3a_4a_4' + a_4{a_2'}^2)(S^2 - 4\Delta_{E_1}\Delta_{E_2})^2X \\ &- (16S + c_6a_6' - 32a_2a_4a_2'a_4' + 864a_6a_6' + c_6'a_6)(S^2 - 4\Delta_{E_1}\Delta_{E_2})^3. \end{aligned} \tag{4.4.4}$$

(Note the sign $-Y$ in the definition of the involution. Without it, the resulting quotient would be a rational surface.) In fact $\mathrm{Ino}(E_1 \times E_2)/\langle i \rangle$ is isomorphic to $\mathrm{Km}(E_1 \times E_2)$, and the quotient map is nothing but the rational map π_2 Shioda and Inose [SI77] used to construct the so-called Shioda–Inose structure:

$$\begin{tikzcd} \mathrm{Ino}(E_1 \times E_2) \arrow[dr, dashed, "\pi_2"'] & & E_1 \times E_2 \arrow[dl, dashed, "\pi_1"] \\ & \mathrm{Km}(E_1 \times E_2) & \end{tikzcd}$$

Here π_1 and π_2 are dominant rational maps of degree 2. The isomorphism between the quotient $\mathrm{Ino}(E_1 \times E_2)/\langle i \rangle$ and $\mathrm{Km}(E_1 \times E_2)$ is defined only over an extension of k containing all the 2-torsion points of both E_1 and E_2.

In general, there is always a k-rational morphism $\mathrm{Km}(E_1 \times E_2) \to \mathrm{Ino}(E_1 \times E_2)$ of degree 8, using the degree 2 multisection of Inose's pencil to get a degree 4 morphism $\mathrm{Km}(E_1 \times E_2) \to J$, followed by the degree 2 morphism $J \to \mathrm{Ino}(E_1 \times E_2)$.

4.5. ***The remaining unstable fibrations.*** We consider one from each isogeny class.

4.5.1. *The* $\mathrm{II}^*\mathrm{I}_1\mathrm{I}_1$ *fibration, with Mordell–Weil group* 0. Assume that $\operatorname{char} k \neq 2, 3$. Consider the elliptic fibration $\mathcal{E} \to \mathbb{P}^1_k$ defined by the Weierstrass equation

$$y^2 = x^3 - 3x - 2(2t - 1). \tag{4.5.1}$$

The singular fibers of $\mathcal{E} \to \mathbb{P}^1_k$ are of type $\mathrm{I}_1, \mathrm{I}_1, \mathrm{II}^*$ at $0, 1, \infty$, respectively.

Proposition 4.5.1. *There is an isomorphism $\mathcal{Z}^2(\mathcal{E}) \cong \mathrm{Ino}(\mathcal{E}^2)$ of varieties over U^2.*

Proof. For convenience we work over the generic fiber $\eta_2 = \operatorname{Spec} k(t_1, t_2)$ of U^2. We have $\mathcal{E}^2_{\eta_2} = E_1 \times E_2$, where for $i = 1, 2$, $E_i/k(t_1, t_2)$ is the elliptic curve

$$E_i : y^2 = x^3 - 3x - 2(2t_i - 1).$$

The Inose surface $\mathrm{Ino}(E_1 \times E_2)$ has equation

$$\mathrm{Ino}(E_1 \times E_2) : Y^2 = X^3 - 3X + 2\left(2t_1(t_1 - 1)T - (2t_1 - 1)(2t_2 - 1) + \frac{2t_2(t_2 - 1)}{T}\right),$$

with II^* fibers at $T = 0, \infty$. On the other hand the surface $\mathcal{Z}^2(\mathcal{E})$ is obtained by substituting

$$t = \frac{s(s - t_1 - t_2 + 1)}{s - t_1 t_2} \tag{4.5.2}$$

into (4.5.1), giving

$$Y^2 = X^3 - 3(s - t_1t_2)^4 X - 2\big(2s^2 - (2t_1 + 2t_2 - 1)s + t_1t_2\big)(s - t_1t_2)^5. \tag{4.5.3}$$

Like $\mathrm{Ino}(E_1 \times E_2)$, the fibration $\mathcal{Z}^2(\mathcal{E})$ also has two II^* fibers, located at $s = \infty, t_1t_2$. It is now easy to see that the linear transformation

$$s = \frac{t_2(Tt_1 - t_2 + 1)}{T}$$

transforms (4.5.3) to $\mathrm{Ino}(E_1 \times E_2)$. ☐

4.5.2. *The* $\mathrm{III}^*\mathrm{I}_2\mathrm{I}_1$ *fibration, with Mordell–Weil group* $\mathbb{Z}/2\mathbb{Z}$. Assume that $\operatorname{char} k \neq 2$. Let $\mathcal{E} \to \mathbb{P}^1_k$ be the elliptic fibration with Weierstrass equation

$$y^2 = x(x^2 - 2x + t). \tag{4.5.4}$$

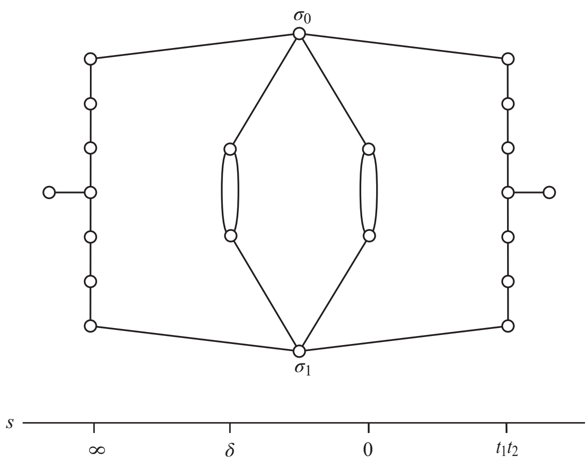
The singular fibers of $\mathcal{E} \to \mathbb{P}^1_k$ are of type $\mathrm{I}_2, \mathrm{I}_1, \mathrm{III}^*$ at $t = 0, 1, \infty$, respectively. The Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z}$.

Proposition 4.5.2. *There is a finite morphism $\mathcal{Z}^2(\mathcal{E}) \to \mathrm{Ino}(\mathcal{E}^2)$ of degree 4 of varieties over U^2.*

Proof. The surface $\mathcal{Z}^2(\mathcal{E})$ obtained by substituting (4.3.2) into (4.5.4) is an elliptic fibration with singular fibers of type $\mathrm{III}^*, \mathrm{III}^*, \mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1$ at $s = \infty, t_1t_2, 0, \delta = t_1 + t_2 - 1, t_1, t_2$, respectively, with Mordell–Weil group again $\mathbb{Z}/2\mathbb{Z}$. Let σ_0 and σ_1 be the identity and 2-torsion section, respectively. We display at the top left of the next page the III^* and I_2 fibers along with σ_0 and σ_1. This configuration contains two II^* fibers, highlighted in the diagram on the right.

By Proposition 3.2.2, there exists an elliptic fibration $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1$ with two II^* fibers. To compute it explicitly, we use the so-called "2-neighbor step" developed by Noam Elkies. For comprehensible accounts, see [Kum14; Sen17; Uts12]. Since the divisor of the function x equals $2(\sigma_1) - 2(\sigma_0)$, the pole of the function

$$w = \frac{x}{s - \delta}$$



coincides with the divisor colored in blue. Thus, w defines a genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with these II* fibers at $w=0$ and $w=\infty$, with equation

$$y^2=ws^4-3t_1t_2ws^3+(3t_1^2t_2^2w-2w^2)s^2-(t_1^3t_2^3w-4t_1t_2w^2-w^3)s-2t_1^2t_2^2w^2-(t_1+t_2-1)w^3. \quad (4.5.5)$$

This fibration has no section, but it does have a multisection D of degree 2. Indeed, any of the uncolored vertices in the figure represents a curve which meets each fiber with multiplicity 2. Therefore there is a degree 4 morphism from $\mathcal{Z}^2(\mathcal{E})$ onto the Jacobian J of the fibration. Standard formulas supply the Weierstrass equation for J in terms of the coefficients of the quartic in (4.5.5). After a further substitution $w=-t_2^2(t_2-1)T$, this Weierstrass equation becomes

$$Y^2=X^3+4X^2+(4t_1+4t_2-3t_1t_2)X+\left(-t_1^2(t_1-1)T+2t_1t_2-t_2^2(t_2-1)T^{-1}\right), \quad (4.5.6)$$

which we recognize as the equation for $\mathrm{Ino}(E_1\times E_2)$. □

In this case, each E_i has a 2-torsion point $(0,0)$, so the Inose pencil (4.4.1) on $\mathrm{Km}(E_1\times E_2)$ admits a section, and there is a double cover $\mathrm{Km}(E_1\times E_2)\to\mathrm{Ino}(E_1\times E_2)$.

4.5.3. *The* $\mathrm{IV}^*\mathrm{I}_3\mathrm{I}_1$ *fibration, with Mordell–Weil group* $\mathbb{Z}/3\mathbb{Z}$. Assume that $\operatorname{char} k\neq 2,3$. Let $\mathcal{E}\to\mathbb{P}^1_k$ be the elliptic fibration with Weierstrass equation

$$y^2=x^3+9x^2+24tx+16t^2. \quad (4.5.7)$$

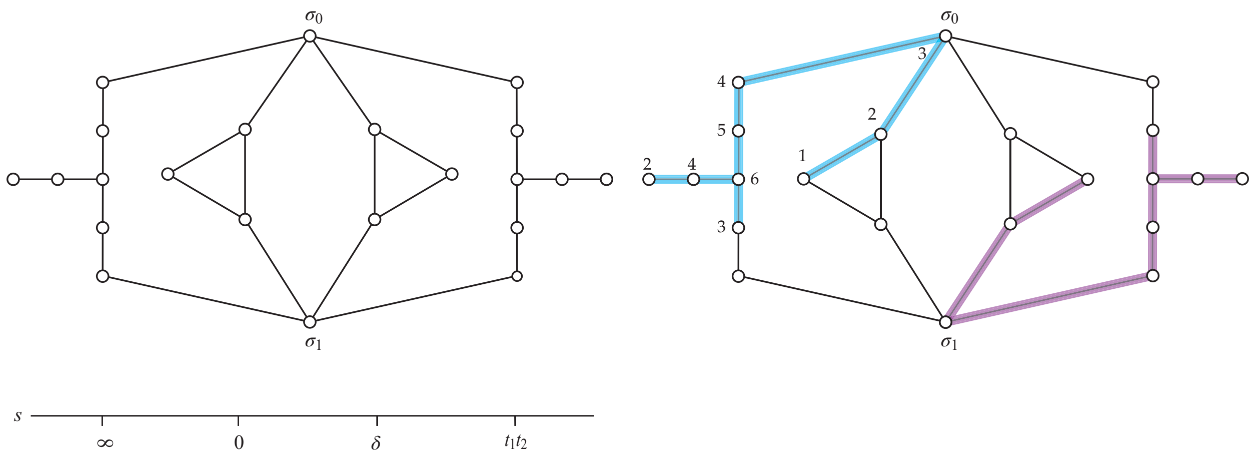
The singular fibers of $\mathcal{E}\to\mathbb{P}^1_k$ are of type $\mathrm{I}_3,\mathrm{I}_1,\mathrm{IV}^*$ at $t=0,1,\infty$, respectively. The Mordell–Weil group is $\mathbb{Z}/3\mathbb{Z}$, generated by $(0,4t)$.

Proposition 4.5.3. *There is a morphism* $\mathcal{Z}^2(\mathcal{E})\to\mathrm{Ino}(\mathcal{E}^2)$ *of degree* 9 *of varieties over* U^2.

Proof. The surface $\mathcal{Z}^2(\mathcal{E})$ is obtained by substituting $t=s(s-t_1-t_2+1)/(s-t_1t_2)$ into (4.5.7), giving

$$y^2=x^3+9(s-t_1t_2)^2x^2+24s(s-\delta)(s-t_1t_2)^3x+16s^2(s-\delta)^2(s-t_1t_2)^4. \quad (4.5.8)$$

It has singular fibers of type $\mathrm{IV}^*, \mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_1, \mathrm{I}_1$ at $s = \infty$, t_1t_2, 0, $\delta = t_1+t_2-1$, t_1, t_2, respectively, with Mordell–Weil group again $\mathbb{Z}/3\mathbb{Z}$. Let σ_0 be the identity section and let σ_1 be one of the 3-torsion sections. We display on the left the IV^* and I_3 fibers along with σ_0 and σ_1. This configuration contains two II^* fibers, highlighted on the right.



Therefore there exists a genus 1 fibration $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1 \times U^2$ with two II^* fibers. To perform a 3-neighbor step, we first find the tangent line at each point of the 3-torsion section $\sigma_1 = \big(0, 4s(s-t_1-t_2+1)(s-t_1t_2)^2\big)$:

$$y = 3(s-t_1t_2)x + 4s(s-t_1-t_2+1)(s-t_1t_2)^2.$$

Using this, we find an elliptic parameter

$$w = -\frac{t_2^3(t_2-1)}{8} \cdot \frac{y - 3(s-t_1t_2)x - 4s(s-t_1-t_2+1)(s-t_1t_2)^2}{s^2(s-t_1t_2)^4}, \tag{4.5.9}$$

which leads to a cubic curve in x' and s with parameter w:

$$t_2^6(t_2-1)^2x'^3 + 12t_2^3(t_2-1)w(s-t_1t_2)x' + 8t_2^3(t_2-1)w(s-t_1-t_2+1) - 8w^2s(s-t_1t_2)^2, \tag{4.5.10}$$

where $x' = 2s(s-t_1t_2)^2x$.

This fibration lacks a section, but it does have a multisection of degree 3. Indeed, any of the uncolored vertices in the figure represents a curve which meets each fiber with multiplicity 3. Therefore there exists a degree 9 map from $\mathcal{Z}^2(\mathcal{E})$ to the Jacobian J of the fibration. The Weierstrass equation for J is

$$Y^2 = X^3 - 3(8t_2-9)(8t_1-9)X + 2(32t_1^3(t_1-1)w - (8t_2^2-36t_2+27)(8t_1^2-36t_1+27) + 32t_2^3(t_2-1)w^{-1}),$$

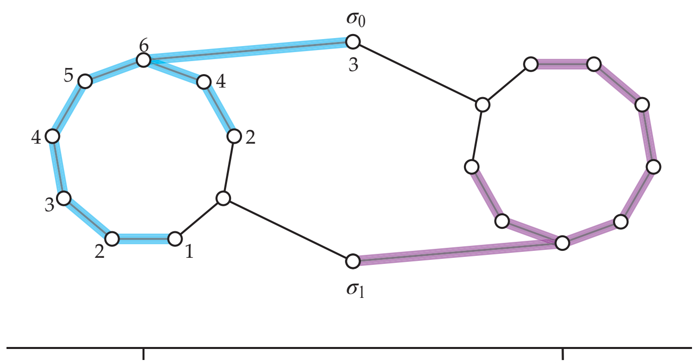
which coincides with the twist of $\mathrm{Ino}(\mathcal{E}^2)$ by -3. □

4.6. ***The semistable fibrations.*** The verification of Theorem 3.1.1 for the semistable extremal rational elliptic fibrations is more difficult, since now one must keep track of the locations of the singular fibers of $\mathcal{E} \to \mathbb{P}^1$. Each time, we found a genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with two II^* fibers, which means we can apply the techniques of Section 4.4, and conclude that there is a product of elliptic curves $\mathcal{E}_1 \times \mathcal{E}_2 \to U^2$ and a finite morphism $\mathcal{Z}^2(\mathcal{E}) \to \mathrm{Km}(\mathcal{E}_1 \times \mathcal{E}_2)$ commuting with the maps to U^2. We demonstrate this fact with the figures that follow.

However, we were not able to directly compute $\mathcal{E}_1 \times \mathcal{E}_2$ in all cases. The trouble is that in two of the cases, the genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with two II^* fibers has a multisection of degree 5 (resp., 6). There is currently no explicit formula for the Weierstrass equation of the Jacobian of a genus 1 curve with such a multisection. (Compare with [ARVT05], which has formulas for the Jacobian of a plane cubic, and with [AKM+01], which shows how to proceed for the intersection of two quadrics in $\mathbb{P}^3$.) Fisher [Fis18] gives a method for finding the equation in all degrees, and it has been implemented in Magma for degree 5. However, the implementation did not conclude within a reasonable time in the example that arose. For degree 6, even this resource is not available.

Remark 4.6.1. In all four cases, there is an elliptic parameter for the II^* fibration whose divisor is of the form $d(\sigma_1 - \sigma_0) + F$, where d is the torsion order of the semistable fibration and $\sigma_1 - \sigma_0$ generates the torsion group, while F is a fibral divisor for this fibration. We do not have a unified explanation for this fact. If we were also to consider the curves E with torsion subgroup $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, isogenous to the first two considered below, we would not find fibrations with two II^* fibers on $\mathcal{Z}^2(\mathcal{E})$. The reason for this is that the discriminant of the Picard lattice is t^2, where t is the torsion order, and so the multisection degree would have to be t. However, the discriminant group has no elements of order t, and therefore no such fibration exists by Lemma 3.2.5.

4.6.1. *The* $\mathrm{I}_9\mathrm{I}_1\mathrm{I}_1\mathrm{I}_1$ *fibration, with Mordell–Weil group* $\mathbb{Z}/3\mathbb{Z}$. The reducible fibers of $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1 \times U^2$ are of type $\mathrm{I}_9, \mathrm{I}_9$, and these together with the trivial section and a section of order 3 contain two II^* configurations as shown below:

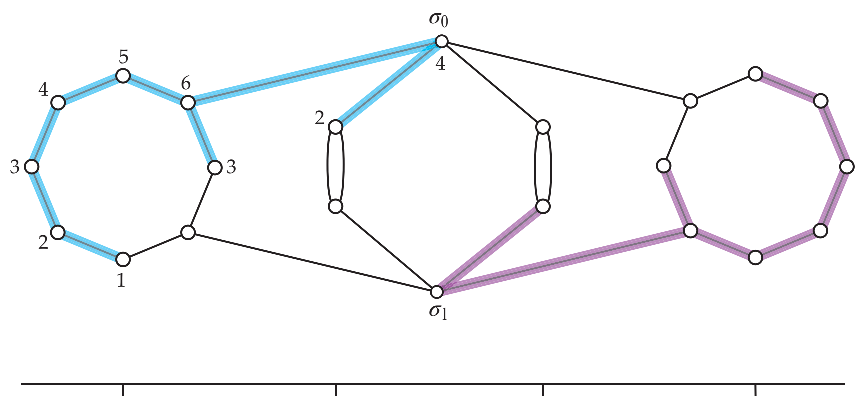


The genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with these two II^* fibers has a degree 3 multisection, represented by any of the uncolored vertices (or the other torsion section which is not shown). Therefore $\mathcal{Z}^2(\mathcal{E})$ admits a degree 9 map to the Jacobian of the fibration, which is an Inose surface.

In this case, we were able to make all of the equivalences explicit. The first step is to write down the elliptic parameter for the fibration with the two II^* fibers by exhibiting a function whose divisor is described by the figure. Computationally, we approached this by first writing down a projective model for $\mathcal{Z}^2(\mathcal{E})$ in $\mathbb{P}^6$ with five A_1 singularities and two A_4 singularities such that each of the two I_9 fibers consists of three curves of degree 1, two A_1 points, and one A_4 point. The other A_1 singularity is the zero section of the fibration, while the 3-torsion sections are curves of degree 2 in this model. (It may be surprising that such a nice model exists for a K3 surface of such large Picard rank and small discriminant.) In this model it is straightforward to exhibit, not only the elliptic parameter t, but also three functions $f_0, f_1, f_2 = 1$ whose restrictions to a smooth fiber generate the Riemann–Roch space of $\mathcal{O}(D)$, where D is the multisection of degree 3.

Once this was done, we wanted to find the image of the map $(f_0 : f_1 : f_2), (t : 1)$ from $\mathcal{Z}^2(\mathcal{E})$ to $\mathbb{P}^2 \times \mathbb{P}^1$, since the general fiber of the image of the map to $\mathbb{P}^1$ would be the cubic whose Jacobian has the two II^* fibers. This proved to be computationally difficult and we resorted to interpolation; however, the final result is rigorous, because we could verify the equations of the map and its codomain in Magma once we had found them. At that point it was a routine matter to use the formulas of Section 4.4, in particular (4.4.3), to verify that up to a change of coordinates and twist the Inose surface is isomorphic to $\mathrm{Km}(E_1 \times E_2)$, where the E_i are obtained by substituting t_i for t in the equation defining an elliptic curve over $\mathbb{Q}(t)$ with four bad fibers of type I_3 and no others. (This is isogenous to the curve with one I_9 and three I_1 fibers.) See [LW] for details.

4.6.2. *The* $\mathrm{I}_8\mathrm{I}_2\mathrm{I}_1\mathrm{I}_1$ *fibration, with Mordell–Weil group* $\mathbb{Z}/4\mathbb{Z}$. The reducible fibers of $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1 \times U^2$ are of type $\mathrm{I}_8, \mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_2$, and these together with the trivial section and a section of order 4 contain two II^* configurations as shown below:



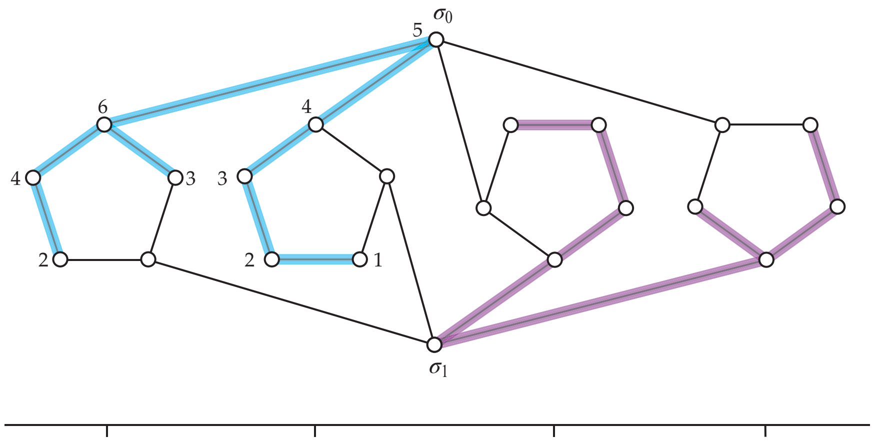
The genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with these two II^* fibers has a degree 4 multisection, represented by any of the uncolored vertices (or the other two torsion sections which are not shown). Therefore $\mathcal{Z}^2(\mathcal{E})$ admits a degree 16 map to the Jacobian of the fibration, which is an Inose surface.

4.6.3. *The* $\mathrm{I}_5\mathrm{I}_5\mathrm{I}_1\mathrm{I}_1$ *fibration, with Mordell–Weil group* $\mathbb{Z}/5\mathbb{Z}$. Assume that $\operatorname{char} k \neq 2$. Let $\mathcal{E} \to \mathbb{P}^1_k$ be the elliptic fibration with Weierstrass equation

$$y^2 = x^3 + (t^2+1)x^2 - 4t(t^2+t-1)x + 4t^2(t^2+1). \tag{4.6.1}$$

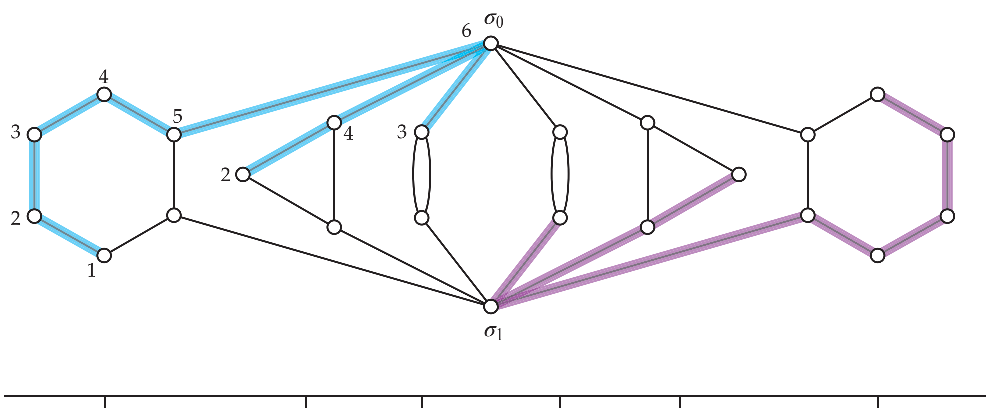
The reducible fibers of $\mathcal{E} \to \mathbb{P}^1_k$ are of type $\mathrm{I}_5, \mathrm{I}_5$ at $t = 0, \infty$. The Mordell–Weil group is $\mathbb{Z}/5\mathbb{Z}$, generated by $\sigma_1 = (2t, 4t)$.

The reducible fibers of $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1 \times U^2$ are of type $\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_5$, and these together with the 5-torsion section σ_1 and the 0-section σ_0 contain two II^* configurations as shown below:



The genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with these two II^* fibers has a degree 5 multisection, represented by any of the uncolored vertices (or the other four torsion sections which are not shown). Therefore $\mathcal{Z}^2(\mathcal{E})$ admits a degree 25 map to the Jacobian of the fibration, which is an Inose surface.

4.6.4. *The $\mathrm{I}_6\mathrm{I}_3\mathrm{I}_2\mathrm{I}_1$ fibration, with Mordell–Weil group $\mathbb{Z}/6\mathbb{Z}$.* The reducible fibers of $\mathcal{Z}^2(\mathcal{E}) \to \mathbb{P}^1 \times U^2$ are of type $\mathrm{I}_6, \mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_2$, and these together with the trivial section and a section of order 6 contain two II^* configurations as shown below:



The genus 1 fibration on $\mathcal{Z}^2(\mathcal{E})$ with these two II^* fibers has a degree 6 multisection, represented by any of the uncolored vertices (or the other four torsion sections which are not shown). Therefore $\mathcal{Z}^2(\mathcal{E})$ admits a degree 36 map to the Jacobian of the fibration, which is an Inose surface.

5. Calabi–Yau threefolds and 3-modularity

We discuss here the problem of proving that an elliptic fibration is 3-modular, at least for the unstable extremal rational elliptic surfaces of Proposition 4.1.3. Following the technique of proof for the case of 2-modularity, we consider a coincidence variety

$$\mathrm{Coinc}^4_G(\Gamma_0(N);\Sigma_\infty) \to (\mathbb{P}^1_F)^4$$

for an arbitrary algebraically closed field F. As we saw in Example 2.7.2, the generic fiber of this morphism is an affine open in an elliptic curve. Let $\eta_3 \cong \operatorname{Spec} F(t_1,t_2,t_3)$ be the generic point of $(\mathbb{P}^1_F)^3$, and let $\mathrm{Coinc}^4_G(\Gamma_0(N);\Sigma_\infty)_{\eta_3}$ be the fiber over η_3 in the projection onto the first three coordinates. Then projection onto the remaining coordinate $t=t_4$ defines an elliptic fibration with affine part $\mathrm{Coinc}^4_G(\Gamma_0(N);\Sigma_\infty)_{\eta_3}$, which we shall call $\mathcal{C}$.

The Weierstrass equation for $\mathcal{C}$ is

$$y^2+e_1xy = x^3+(-e_2+e_3-2e_4)x^2+(1-e_1+e_2-e_3+e_4)e_4x,$$

where $e_1,\dots,e_4$ are the elementary symmetric polynomials in t_1,t_2,t_3,t_4, and we take $t=t_4$ as the parameter on the base. As polynomials in t, the coefficients $a_i(t)$ of the fibration satisfy $\deg a_i \le i$, so that $\mathcal{C}$ is (at least over an algebraic closure of η_3) a rational elliptic surface. We compute that $\mathcal{C}\to\mathbb{P}^1_{\eta_3}$ has one singular fiber of type I_4 at $t=\infty$, two of type I_2 at $t=0,1$, and four of type I_1 at other points. Its Mordell–Weil group is $\mathbb{Z}^{\oplus 3}\oplus\mathbb{Z}/2\mathbb{Z}$.

Remark 5.0.1. The fibration $\mathcal{C}\to\mathbb{P}^1_{\eta_3}$ is the universal fibration of type no. 21 in the Oguiso–Shioda tables [OS91] classifying all rational elliptic fibrations.

Now let $\mathcal{E}\to\mathbb{P}^1_F$ be a rational elliptic fibration with multiplicative fibers at $0,1$ and additive fiber at ∞. Define a projective variety $\mathcal{Z}^3(\mathcal{E})$ over η_3 as the fiber product:

$$\begin{array}{ccc} \mathcal{Z}^3(\mathcal{E}) & \longrightarrow & \mathcal{E}\times\eta_3 \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathbb{P}^1_{\eta_3} \end{array} \tag{5.0.1}$$

The technique of proof of Theorem 1.2.4 reduces the 3-modularity of such elliptic fibrations to the following conjecture.

Conjecture 5.0.2. *There exists an algebraic correspondence between $\mathcal{Z}^3(\mathcal{E})$ and $\mathcal{E}^3_{\eta_3}$, such that the induced map $H^3(\mathcal{Z}^3(\mathcal{E}))\to H^3(\mathcal{E}^3_{\eta_3})$ is surjective over the transcendental part of $H^3(\mathcal{E}^3_{\eta_3})$. Here the H^3 can refer to ℓ-adic cohomology as a Galois representation, or (if $F=\mathbb{C}$) Betti cohomology as a family of Hodge structures.*

We were able to prove Conjecture 5.0.2 in the case that $\mathcal{E}$ is the Legendre fibration, by way of a study of Calabi–Yau threefolds.

5.1. ***Fiber products of two elliptic fibrations.*** The fiber product of two rational elliptic fibrations $S_1, S_2 \to \mathbb{P}^1$ over a common base was studied in [Sch88], as a means of constructing interesting Calabi–Yau threefolds. To see why such a threefold might result, let us model S_i as an equation of bidegree $(3, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Then the fiber product $S = S_1 \times_{\mathbb{P}^1} S_2$ is defined by equations of tridegree $(3, 0, 1)$, $(0, 3, 1)$ in $\mathbb{P} = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$, and so, by the adjunction formula, its canonical divisor is $K_S = (K_{\mathbb{P}})|_S \otimes \mathcal{O}(3, 3, 2) = \mathcal{O}_S$.

The fiber product S generally has singular points; to construct (nonsingular) Calabi–Yau threefolds, we must determine whether the canonical divisor is trivial on a desingularization of S. Recall that a *crepant resolution* $f : \tilde{S} \to S$ is a desingularization which does not affect the canonical class: $f^* K_S = K_{\tilde{S}}$. Thus if our fiber product $S = S_1 \times_{\mathbb{P}^1} S_2$ admits a crepant resolution, then its desingularization is a Calabi–Yau threefold.

It is not hard to see that the only possible singularities of S occur over those points $v \in \mathbb{P}^1$ where both S_v and S'_v are singular. It is shown in [Sch88] that as long as S_v and S'_v are multiplicative, then all singular points of S are ordinary double points, and consequently S admits a crepant resolution.

However, in the fiber product defining $\mathcal{Z}^3(\mathcal{E})$, the fibers of the two fibrations $\mathcal{C}, \mathcal{E} \to \mathbb{P}^1$ at ∞ are of multiplicative and additive type, respectively, and so we are not in the situation considered in [Sch88]. Nonetheless, the fiber product still has a crepant resolution:

Proposition 5.1.1. *Let $S_1, S_2 \to X$ be two elliptic fibrations. Suppose there is no point of X at which the fibers of S_1 and S_2 are both additive. Then $S = S_1 \times_X S_2$ admits a projective crepant resolution.*

Sketch of proof. Consider a singular point $P = (P_1, P_2)$ of S. Then P_i is a singular point of a fiber $(S_i)_v$ of S_i for $i = 1, 2$. In light of [Sch88] it suffices to assume that $(S_1)_v$ is additive and $(S_2)_v$ is multiplicative. We will only discuss the case that $(S_1)_v$ has an I_n^* fiber (indeed this is the only case we will use). Then P_1 falls into one of the following three cases: it belongs to a single nonreduced component of multiplicity 2, it is the intersection of such a component with a reduced component, or else it is the intersection of two nonreduced components.

The local equations of P for these three types are

$$x_1^2 - x_3 x_4 = 0, \quad x_1^2 x_2 - x_3 x_4 = 0, \quad x_1^2 x_2^2 - x_3 x_4 = 0,$$

as hypersurfaces in $\mathbb{A}^4$. (In the first case, for instance, let t be a local coordinate for X at v. Then the local equations for P_1, P_2 are $x_1^2 = t$ and $x_3 x_4 = t$, respectively, and so the fiber product has equation $x_1^2 = x_3 x_4$.) We analyze each case in turn.

First consider $x_1^2 - x_3 x_4 = 0$ as a hypersurface in $\mathbb{A}^4$. This is a product of $\mathbb{A}^1$ with a surface with an A_1 singularity. This hypersurface has canonical singularities in the sense of [Rei80]; we confirm that it has a crepant resolution.

Let $Q \subset \mathbb{P}^4$ be the projective closure of the hypersurface. Then Q is the cone over the cone over a smooth conic. Let $\pi : \tilde{Q} \subset \mathbb{P}^4 \times \mathbb{P}^2 \to Q$ be the blowup of Q along its singular locus S, which is a line in $\mathbb{P}^4$. A simple calculation shows that $\tilde{Q}$ is projective and smooth, and that the exceptional divisor E is isomorphic to $(\mathbb{P}^1)^2$. We claim also that $\pi : \tilde{Q} \to Q$ is a crepant resolution of Q.

Let π_1, π_2 be the two projections $(\mathbb{P}^1)^2 \to \mathbb{P}^1$. Identifying E with $(\mathbb{P}^1)^2$ and S with $\mathbb{P}^1$, we identify π_1 with $\pi|_E$. Let p be a point of the singular locus of Q and let $F = \pi^{-1}(p)$. Let G be a fiber of π_2, viewed as a curve on E. The canonical divisor K_Q of the quadric hypersurface $Q \subset \mathbb{P}^4$ is $K_Q = \mathcal{O}(-3)$, so we'd like to know that the canonical divisor $K_{\tilde{Q}}$ of $\tilde{Q} \subset \mathbb{P}^4 \times \mathbb{P}^2$ is $\pi^* K_Q = \mathcal{O}(-3, 0)$. We have $h^2(\tilde{Q}) = 2$, which forces $K_{\tilde{Q}} = \pi^* K_Q + cE$. To determine c, we use the adjunction formula. We have $(K_{\tilde{Q}} + E) \cdot E = K_E = -2F - 2G$. Clearly $\mathcal{O}(1,0)$ misses F (a general hyperplane doesn't pass through a given point) and hits G once (in the fiber above the point of intersection in $\mathbb{P}^4$), so the intersection is F. On the other hand, if we take a section S of $\mathcal{O}(1,0)$ containing the exceptional divisor (i.e., the strict transform of a hyperplane containing the singular locus of the quadric), then we can compute that $F \sim S \cdot E = (R + E) \cdot E = 2G + E^2$, so $E^2 = F - 2G$. It follows that $K_{\tilde{Q}} \cdot E = -3F$, and so $c = 0$ as desired.

We now consider the second case where the local equation is $x_1^2 x_2 - x_3 x_4 = 0$. The singular subscheme of the affine scheme defined by this equation has two components: a line $x_1 = x_3 = x_4 = 0$, and an embedded component $x_1^2 = x_2 = x_3 = x_4 = 0$. If we blow up the first of these (after projectivizing, to make the calculation easier) we obtain a variety whose singular subscheme misses the locus $x_1 = x_2 = x_3 = x_4 = 0$. In other words, the single blowup has resolved the singularity. In codimension 1 this is the same as the previous example, and it follows that this is likewise a crepant resolution.

Finally, in the third case, the local equation is $x_1^2 x_2^2 - x_3 x_4 = 0$. There are three components of the singular subscheme: two of the form $x_i = x_3 = x_4 = 0$ for $i = 1, 2$, and the embedded component $x_1^2 = x_2^2 = x_3 = x_4$. Blowing up the first component takes care of the embedded component and leaves a line of A_1 singularities, which we have already seen to have a crepant resolution. Note also that in this case we have converted two rational curves into divisors, which should increase h^2 and h^4 by 2 each, whereas in the previous examples the contribution was only 1. Further, since the resolution method is an actual blowup, rather than a small resolution, projectivity is automatic. $\square$

Remark 5.1.2. Similar considerations show a stronger result: The fiber product $S_1 \times_X S_2$ admits a crepant resolution if and only if, when two additive fibers come together at the same point of X, the fiber types belong to the following list:

$$(\mathrm{I}_n^*, \mathrm{II}),\ (\mathrm{I}_0^*, \mathrm{III}),\ (\mathrm{IV}^*, \mathrm{II}),\ (\mathrm{IV}, \mathrm{II}),\ (\mathrm{IV}, \mathrm{III}),\ (\mathrm{III}, \mathrm{III}),\ (\mathrm{III}, \mathrm{II}),\ (\mathrm{II}, \mathrm{II}).$$

Most of the other cases can be excluded by the observation that if the fibers both contain a nonreduced component, then there cannot be a crepant resolution. The local equation is $x_2^i - x_4^j = 0$ for $i, j > 1$, and this is singular in dimension 2, i.e., along a divisor.

Corollary 5.1.3. *The fiber product $\mathcal{Z}^3(\mathcal{E})$ is birational to a nonsingular projective Calabi–Yau threefold.*

5.2. *The Kummer threefold.* On the other hand, there is a separate Calabi–Yau threefold that we can associate to three elliptic curves E_1, E_2, E_3; this is a generalization of the Kummer surface associated to the product of two elliptic curves.

Definition 5.2.1. Let $Q = (E_1 \times E_2 \times E_3)/V$, where nontrivial elements of the group $V = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ act by negating two of the factors E_1, E_2, E_3 at a time.

Proposition 5.2.2. *The threefold Q admits a crepant resolution $\tilde{Q} \to Q$, where $\tilde{Q}$ is a Calabi–Yau threefold.*

We refer to any such resolution as a *Kummer threefold* and denote it by $\mathrm{Km}(E_1 \times E_2 \times E_3)$.

Proof. (Assuming $\operatorname{char} k \neq 2$.) First we consider how to embed Q into a toric variety. We can view each E_i as a hyperelliptic curve $y_i^2 = f_i(x_i, z_i)$ in weighted projective space $\mathbb{P}(1, 2, 1)$, where f is homogenous of degree 4. With respect to these coordinates, negation on E_i is $(x_i : y_i : z_i) \mapsto (x_i : -y_i : -z_i)$.

The product $E_1 \times E_2 \times E_3$ lives in $\mathbb{P}(1, 2, 1)^3$. The group V acts on $\mathbb{P}(1, 2, 1)^3$ in the evident matter. The quotient $T = \mathbb{P}(1, 2, 1)^3/V$ has coordinates $y = y_1 y_2 y_3, x_1, z_1, x_2, z_2, x_3, z_3$; it is the toric variety obtained by taking the quotient of $\mathbb{A}^7$ by $\mathbb{G}_m^3$ acting by the weights $(2, 1, 1, 0, 0, 0, 0)$, $(2, 0, 0, 1, 1, 0, 0)$, and $(2, 0, 0, 0, 0, 1, 1)$. Finally, Q is the hypersurface in T with equation

$$y^2 = \prod_{i=1}^{3} f_i(x_i, z_i).$$

Identify $\operatorname{Pic} T$ with $\mathbb{Z}^3$ by these three weight vectors. The canonical divisor of a toric variety is the negative of the sum of the toric divisors [Ful93, Proposition, section 4.3], so $K_T = \mathcal{O}(-4, -4, -4)$. Since the defining equation for Q has degree 4 with respect to each copy of $\mathbb{G}_m$, we have by the adjunction formula $K_Q = (K_T \otimes \mathcal{O}(Q))|_Q = 0$.

It remains to show that Q has a crepant resolution. The subschemes of $E_1 \times E_2 \times E_3$ where a nontrivial element of V has a fixed point are precisely those where two or three of the coordinates are points of order 1 or 2. Let V act on $\mathbb{A}^3$ by negating any two of the coordinates. This is a linear action by a subgroup of SL_3, so by a theorem of Roan, Ito, and Markushevich [Roa96, Theorem 1] there is a crepant resolution. The action of V on the tangent spaces of fixed points of $E_1 \times E_2 \times E_3$ is the same as for $\mathbb{A}^3$, so the result carries over to our case. □

Remark 5.2.3. The crepant resolution of Q is obtained by blowing up 48 curves. These are the images of those curves in $E_1 \times E_2 \times E_3$ of the form

$$\{P_1\} \times \{P_2\} \times E_3, \ \{P_1\} \times E_2 \times \{P_3\}, \ E_1 \times \{P_2\} \times \{P_3\},$$

where P_1, P_2, P_3 are 2-torsion points. The order of blowing up is significant: at each of the 64 points (P_1, P_2, P_3), there are 6 possible orders to choose from, and exchanging two adjacent ones amounts to a flop. However, for our purposes the choice makes no difference.

Remark 5.2.4. This construction generalizes to give a Calabi–Yau manifold $\mathrm{Km}(E_1 \times \cdots \times E_d)$ for d elliptic curves $E_1, \ldots, E_d$. This is the desingularization of the quotient $(E_1 \times \cdots E_d)/V$, where now $V \subset (\mathbb{Z}/2\mathbb{Z})^{\oplus d}$ is the subgroup where the sum of the coordinates is 0.

Proposition 5.2.5. *The Hodge diamond of* $\mathrm{Km}(E_1 \times E_2 \times E_3)$ *is*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 51 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 51 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Proof. We assume $k = \mathbb{C}$ to give a simple proof in terms of differential forms. Let $H^{1,0}$ and $H^{0,1}$ of E_i be spanned by $z_i, \bar{z}_i$ respectively. Then $H^1(E_1 \times E_2 \times E_3)$ is spanned by all the $z_i, \bar{z}_i$ (abusively using the same notation for forms on the E_i and their pullbacks to the product) and $H^n(E_1 \times E_2 \times E_3)$ is identified with $\bigwedge^n H^1$, as usual for an abelian variety. We can write $H^n = \bigoplus_{j=0}^n H^{j,n-j}$, where $H^{j,n-j}$ is spanned by the products of j holomorphic and $n-j$ antiholomorphic forms.

The negation map on E_i negates $z_i, \bar{z}_i$ while fixing the others. Thus V acts on H^1, H^2, H^3 with the following fixed subspaces:

(1) the fixed subspace on H^1 is trivial;

(2) the fixed subspace on H^2 is spanned by the $z_i \wedge \bar{z}_i$;

(3) the fixed subspace on H^3 is spanned by products of one form with each subscript.

In order to obtain the Hodge diamond of $\mathrm{Km}(E_1 \times E_2 \times E_3)$, we must consider the effect of the blowup. Each blowup of a rational curve replaces a subvariety with $h^{0,0} = h^{1,1} = 1$ by one with $h^{0,0} = h^{2,2} = 1$ and $h^{1,1} = 2$, so it increases $h^{1,1}$ and $h^{2,2}$ by 1. Since there are 48 such curves, we obtain the values claimed in the statement of the proposition. □

5.3. ***A birational map between a Kummer threefold and a fiber product of elliptic fibrations.*** Assume that $\operatorname{char} k \neq 2$. Let $\mathcal{E} \to \mathbb{P}^1_k$ be the Legendre fibration, with Weierstrass equation

$$y^2 = x(x-1)(x-t).$$

Let $\eta = \operatorname{Spec} k(t_1, t_2, t_3)$ be the generic point of $(\mathbb{P}^1_k)^3$, and as usual write $\mathcal{E}^3_\eta = E_1 \times E_2 \times E_3$. Thus E_i is the elliptic curve over η with Weierstrass equation $y^2 = x(x-1)(x-t_i)$. We are interested in two Calabi–Yau threefolds over η. On the one hand, we have the Kummer threefold $\mathrm{Km}(E_1 \times E_2 \times E_3)$, which is birational to the subvariety of the toric variety $T = \mathbb{A}^7/\mathbb{G}^3_m$ with equation

$$y^2 = \prod_{i=1}^{3} x_i z_i (x_i - t_i z_i) \tag{5.3.1}$$

On the other hand, we have the fiber product $\mathcal{Z}^3(\mathcal{E}) = \mathcal{C} \times_{\mathbb{P}^1_\eta} \mathcal{E}_\eta$, which is birational to a Calabi–Yau variety by Corollary 5.1.3.

Theorem 5.3.1. *The varieties $\mathcal{Z}^3(\mathcal{E})$ and $\mathrm{Km}(E_1 \times E_2 \times E_3)$ are birational over η. Hence Conjecture 5.0.2 is true for $\mathcal{E}$, and (if $k = \mathbb{F}_q$) then $\mathcal{E}$ is 3-modular.*

Proof. We found a birational equivalence between $\mathcal{Z}^3(\mathcal{E})$ and $\mathrm{Km}(E_1 \times E_2 \times E_3)$ by exhibiting K3 surface fibrations on each threefold, such that the generic fibers of the fibrations are isomorphic. The discovery of the birational equivalence was extraordinarily serendipitous: we reached for a few of the easiest possible K3 fibrations to construct on either side, and happened upon two pairs that matched. For the birational equivalence in terms of coordinates, we refer to the code [LW]. We only describe here the method we used to find it.

The K3 surface fibration on $\mathrm{Km}(E_1 \times E_2 \times E_3)$ is easy enough to describe. Recall that $\mathrm{Km}(E_1 \times E_2 \times E_3)$ is the desingularization of a subvariety Q of a toric variety T, with equation (5.3.1). Consider the rational map $T \dashrightarrow \mathbb{P}^1$ defined by the ratio $(y : x_1 x_2 x_3 z_1 z_2 z_3)$. We claim that its restriction to $Q \dashrightarrow \mathbb{P}^1$ has generic fiber a K3 surface. Indeed, the fiber over $u \in \mathbb{P}^1$ is the subvariety of $(\mathbb{P}^1)^2$ with equation

$$u^2 \prod_{i=1}^{3} x_i z_i = \prod_{i=1}^{3} (x_i - z_i)(x_i - t_i z_i), \tag{5.3.2}$$

which one checks is nonsingular for generic u. A nonsingular surface in $(\mathbb{P}^1)^3$ of degree $(2, 2, 2)$ is a K3 surface. This is our K3 fibration $\mathrm{Km}(E_1 \times E_2 \times E_3) \dashrightarrow \mathbb{P}^1$.

Now consider $\mathcal{Z}^3(\mathcal{E})$, which is the fiber product of two rational elliptic fibrations over a common base. This is naturally a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Let S be the set of prime components of the singular subscheme of $\mathcal{Z}^3(\mathcal{E})$ of dimension 1, and consider the linear system of $(1, 1, 1)$-forms vanishing on S. Our computations showed that this linear system gives a birational map from $\mathcal{Z}^3(\mathcal{E})$ to a quintic threefold $Z \subset \mathbb{P}^4$.

The quintic Z is singular along 8 lines and some isolated points. By considering spans of pairs of these lines, we found 9 planes $H \subset Z$. If such a plane has equations $\ell_0 = \ell_1 = 0$ for linear forms ℓ_0, ℓ_1 on $\mathbb{P}^4$, then the quintic defining Z can be written $\ell_0 f_0 + \ell_1 f_1$ for forms f_0, f_1 of degree 4. Then $[\ell_0 : \ell_1]$ defines a rational map $Z \dashrightarrow \mathbb{P}^1$, whose generic fiber is a quartic in $\mathbb{P}^3$; i.e., a K3 surface.

Experimentally, we counted points of fibers of these $Z \dashrightarrow \mathbb{P}^1$ over a finite field (choosing random values for t_1, t_2, t_3), until we found one which matched the point counts from fibers of $\mathrm{Km}(E_1 \times E_2 \times E_3) \dashrightarrow \mathbb{P}^1$. It was then possible to change the coordinate u on the $\mathbb{P}^1$ so that the point count matched fiber by fiber. This suggested that the generic fibers of the two fibrations, which are K3 surfaces over $k(t_1, t_2, t_3, u)$, were isomorphic, and indeed they were. We verified this by finding elliptic fibrations with the same configuration of singular fibers, and then directly observing that those elliptic fibrations are isomorphic. □

Remark 5.3.2. In contrast to the theory of elliptic fibrations on a K3 surface, which is well developed and draws on the very rich theory of lattice genera, K3 surface fibrations on Calabi–Yau varieties are not well understood. Such fibrations are described by certain extremal rays of the ample cone. However, there is no general theory that allows one to construct a set of orbit representatives for such extremal rays with respect to the action of the automorphism group. In any case, this would not be sufficient, because we might need to consider not only K3 fibrations on one particular Calabi–Yau model but on all equivalent ones. Though there are finitely many Calabi–Yau varieties birational to a given one up to isomorphism, it is not clear how to determine them all in practice or approximately how many there should be for a variety such as Q. It is known (the relevant facts are nicely summarized in [Fry01, Introduction]) that the ample cones of different Calabi–Yau models exhaust the movable cone, but one expects that for a Calabi–Yau variety of Picard number 51 such as Q the geometry of this partition would be exceedingly complicated. For example, we can blow up the 48 curves of singularities in any order (though blowups in two disjoint curves commute). The group of automorphisms acting on this set of Calabi–Yau models is small, and presumably there are many other ways to obtain models as well.

These problems have discouraged us from attempting to extend the result of Theorem 5.3.1 to other examples such as the elliptic surface with singular fibers $\mathrm{III}^*, \mathrm{I}_2, \mathrm{I}_1$. We have verified numerically that the $\mathbb{F}_p$-rational point counts match mod p and therefore expect a correspondence. If the analogy with surfaces holds, we might hope that there is a K3 surface fibration on each side such that the corresponding fibers are isogenous (just as in our proof of 2-modularity for this family we find genus 1 fibrations on both K3 surfaces whose fibers have the same number of points and that are therefore related by an isogeny). However, at our present level of understanding there is no possibility of finding it except by a lucky stab into a potentially enormous space of fibrations. The other families with one additive and two multiplicative fibers are more daunting still, since we would not expect any single fibration to be sufficient; we would need a third isogenous Calabi–Yau threefold to mediate between the two sides.

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