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A coarse Jacquet–Zagier trace formula for GL(n), with applications

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We establish a coarse Jacquet–Zagier trace identity for GL(n) over a global field. We prove the absolute convergence in $\mathrm{Re}(s) > 1$, and obtain holomorphic continuation under almost all character twists. In particular, we compute all P-regular orbital integrals in the geometric side, and the contributions from constant terms and nondegenerate terms of Eisenstein series in the continuous spectrum, and further derive meromorphic continuations of them.

As an application, we give a conditional proof of the Dedekind conjecture, which asserts that given any finite extension E/F of number fields, the ratio $\zeta_E(s)/\zeta_F(s)$ of the Dedekind zeta functions is entire, which is only known for $[E:F] \leq 4$. Some nonvanishing results are also obtained.

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1. Introduction

1A. ***The Rankin–Selberg–Zagier integral*** $\int \mathrm{K}_0(x,x)E(x,s)\,dx$**.** Let $\boldsymbol{\Gamma}$ be an arithmetic subgroup of $\boldsymbol{G} = \mathrm{SL}_2(\mathbb{R})$ with finite covolume. A nice φ on $\boldsymbol{G}$ defines an integral operator $R(\varphi)$ on $L^2(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$ with kernel K^{φ}. With respect to the Casimir operator the space $L^2(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$ decomposes into the direct sum of $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$, the space of cusp forms, and its orthogonal complement $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})^{\perp}$. Correspondingly, $\mathrm{K}^{\varphi} = \mathrm{K}^{\varphi}_0 + \mathrm{K}^{\varphi}_{\mathrm{ER}}$, where K^{φ}_0 (resp. $\mathrm{K}^{\varphi}_{\mathrm{ER}}$) is the kernel of the restriction of $R(\varphi)$ to $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$ (resp. $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})^{\perp}$). The Selberg trace formula is the identity obtained by substituting $\mathrm{K}^{\varphi} - \mathrm{K}^{\varphi}_{\mathrm{ER}}$ for K^{φ}_0 and computing the trace

$$\mathrm{Tr}\, R(\varphi)\,|_{L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})} = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \mathrm{K}^{\varphi}_0(x,x)\,dx = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \big[\mathrm{K}^{\varphi}(x,x) - \mathrm{K}^{\varphi}_{\mathrm{ER}}(x,x)\big]\,dx. \tag{1-1}$$

Although the function $\mathrm{K}^{\varphi}_0(x,x)$ decays rapidly over $\boldsymbol{\Gamma}\backslash\boldsymbol{G}$, its counterparts $\mathrm{K}^{\varphi}(x,x)$ and $\mathrm{K}^{\varphi}_{\mathrm{ER}}(x,x)$ do not. Hence some truncation is typically required to carry out the right-hand side of the integration (1-1).

Zagier [38; 39] introduced the integral

$$I^{\varphi}_0(s) = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \mathrm{K}^{\varphi}_0(x,x)E(x,s)\,dx = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \big[\mathrm{K}^{\varphi}(x,x) - \mathrm{K}^{\varphi}_{\mathrm{ER}}(x,x)\big]E(x,s)\,dx, \tag{1-2}$$

where $E(x,s)$ is an Eisenstein series. He obtained a meromorphic continuation of the right-hand side of (1-2) and recovered the Selberg trace formula by computing the residue at $s = 1$. This approach is more convenient and computationally simpler than truncation-based proofs. Furthermore, $I^{\varphi}_0(s)$ gives more information than the Selberg trace formula, including the divisibility by $\zeta(s)$. Jacquet and Zagier [18] extended the formula to GL(2) over a number field F, providing a new proof of the holomorphy of adjoint L-functions and implying the Arthur–Selberg trace formula. However, the Jacquet–Zagier trace formula has only been developed for GL(2) so far.

Our aim is to generalize the Jacquet–Zagier trace formula to higher ranks, building on the equality of the geometric and spectral expansions of the meromorphic function

$$I_0^{\varphi}(s,\tau):=\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0^{\varphi}(x,x)E(x,s)\,dx, \tag{1-3}$$

where F is a global field, $G=\mathrm{GL}(n)$, and $E(x,s)$ is an Eisenstein series induced from a Hecke character τ. We anticipate a generalized Jacquet–Zagier trace formula for GL(n) of the form

$$I_0^{\varphi}(s,\tau)=I_{\mathrm{Geo}}^{\varphi}(s,\tau)-I_{\mathrm{ER}}^{\varphi}(s,\tau), \tag{1-4}$$

where $I_{\mathrm{Geo}}^{\varphi}(s,\tau)$ and $I_{\mathrm{ER}}^{\varphi}(s,\tau)$ capture (formally) the geometric and noncuspidal spectral contributions, respectively.

However, the convergence problems and complex analytic behaviors of $I_{\mathrm{Geo}}^{\varphi}(s,\tau)$ and $I_{\mathrm{ER}}^{\varphi}(s,\tau)$ are significant. In this study, we present several innovative techniques that address convergence, meromorphic continuation, and divisibility of L-series, leading to a coarse derivation of the Jacquet–Zagier trace formula for GL(n). By regularizing (1-4), we obtain an identity involving various L-functions, roughly of the form:

$$\sum_{\pi}L(s,\pi,\mathrm{Ad})\approx\sum_{[E:F]\le n}\frac{\zeta_E(s)}{\zeta_F(s)}+\sum\frac{\text{L-S }L\text{-functions}}{\zeta_F(s)}+\sum\frac{\text{R-S }L\text{-functions}}{\zeta_F(s)},$$

where L-S means Langlands–Shahidi and R-S refers to Rankin–Selberg for *nondiscrete* representations, which should be ‘obvious’ holomorphic multiples of $\zeta_F(s)$. This formula establishes a connection between the Dedekind conjecture on the entireness of quotients of the Dedekind zeta function $\zeta_E(s)/\zeta_F(s)$ (algebraic side) and the Selberg’s conjecture regarding the entireness of adjoint L-functions $L(s,\pi,\mathrm{Ad})$ (automorphic side). Further details can be found in Section 1B2. This provides a new example of using the general Langlands program to answer some basic questions about number fields.

1B. *Statement of the main results.*

1B1. *The Jacquet–Zagier trace formula for* GL(n). We present a generalization of the trace formula, which encompasses several independent results (Theorems C, D, E, and F). These results can be summarized informally as follows.

Theorem A. *Let notation be as in Section 2. Let* $\mathrm{Re}(s)>1$. *Let* φ *be a test function on* $G(\mathbb{A}_F)$. *Then* $I_0^{\varphi}(s;\tau)$ *admits a regularized geometric-spectral expansion*

$$I_0^{\varphi}(s;\tau)=I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)-I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)+I_{\mathrm{Sing}}^{\varphi}(s,\tau)-I_{\mathrm{Whi}}^{\varphi}(s,\tau), \tag{1-5}$$

where each integral on the right-hand side converges in the region $\mathrm{Re}(s)>1$, *and can be meromorphically continued to* $s\in\mathbb{C}$. *Moreover*:

- $I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)$ *can be expressed as a finite sum of Dedekind zeta functions associated with certain étale algebras of degree* $\le n$. (See Theorem C for details.)

- $I^{\varphi}_{P,\mathrm{Reg}}(s,\tau)$ *can be written as a finite sum of intertwining operators. It turns out to be a holomorphic multiple of*
$$\frac{\Lambda(s,\tau)\,\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\,\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\,\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})}.$$
(See Theorem D for details.)
- $I^{\varphi}_{\mathrm{Whi}}(s,\tau)$ *is an infinite sum over cuspidal data* χ *of Rankin–Selberg L-functions attached to* χ. (See Theorems E and F.)
- *If* $\tau^k \neq \mathbf{1}$ *for* $1 \le k \le n$, *then* (1-5) *has an analytic continuation to* $\mathbb{C}$, *with* $I^{\varphi}_{P,\mathrm{Reg}}(s,\tau)/\Lambda(s,\tau)$ *and* $I^{\varphi}_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ *being entire.*

A precise definition of the integrals on the right side of (1-5) can be found in Section 2C.

Remarks 1.2. (i) The expansion (1-5) generalizes Jacquet and Zagier's formula for GL(2) to GL(n). A restricted version was obtained by Flicker [8] under some choice of test functions φ so that only regular elliptic part of $I^{\varphi}_{\mathrm{Geo,Reg}}(s,\tau)$ shows up on the right-hand side of (1-5). New ideas of our proof are briefly summarized in Section 2B below.

(ii) $I^{\varphi}_{\mathrm{Sing}}(s,\tau)$ is defined geometrically, and it appears essentially when $n \geq 3$. In general, $I^{\varphi}_{\mathrm{Sing}}(s,\tau)$ should always be reduced to Jacquet–Zagier trace formula (1-5) in *smaller ranks*. For certain applications, one can easily eliminate it by choosing a suitable test function (see, for instance, Theorem B in Section 1B2). Also, a detailed analytic continuation of $I^{\varphi}_{\mathrm{Sing}}(s,\tau)/\Lambda(s,\tau)$ (for general test function φ) is given in [36] when $n \le 4$.

1B2. *Some applications.* The formula (1-5) conveys interesting information between L-functions defined analytically and algebraically. It gives an explicit relation among Rankin–Selberg L-functions, Langlands–Shahidi L-functions and Hecke L-functions associated to field extensions. In fact, we shall deduce from it that *holomorphy of certain adjoint L-functions* for GL(n) implies the *Dedekind conjecture* for degree n extensions:

Conjecture 1.3 (τ-twisted Dedekind conjecture). *Let E/F be an extension of global fields. Then $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is holomorphic when $s \neq 1$, where $\Lambda(\cdot,\cdot)$ denotes the complete Hecke L-function, and $N_{E/F}$ is the relative norm.*

When $\tau = \mathbf{1}$ is trivial, this conjecture is conventionally called the *Dedekind conjecture*, and is known to be true when E/F is Galois by the work of Aramata and Brauer (see Chapter 1 of [24]) or has a solvable Galois closure by the work of Uchida [32] and van der Waall [33]. The Dedekind conjecture is the principal case of Artin's holomorphy conjecture. The τ-twisted version of Conjecture 1.3 has been proved by Murty [26] when E/F is either Galois or has a solvable closure. However, the general case (or even the case of general degree-5 extensions) is not yet known.

When $n = 2$, [18] provides a connection between adjoint L-functions associated to cuspidal representations of GL(2)$/F$ and $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ when E/F is quadratic. It was noted in [19] that, at least for degree/rank n up to 5, the two families seem to be related on a nuts-and-bolts level in the theory of integral representations, in addition to the relationships suggested by [18].

Let $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ be the set of cuspidal representations of $G(\mathbb{A}_F)$ of central character ω^{-1}. Recall that the twisted adjoint L-function is defined by

$$\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)=\frac{\Lambda(s,\pi\times\widetilde{\pi}\otimes\tau)}{\Lambda(s,\tau)},\quad \pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}). \tag{1-6}$$

Let $\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})\subset\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ be a subset of cuspidal representations π such that π has a supercuspidal component. Following [18], Flicker [8] used a simple trace formula to conclude, modulo the key Lemma 4 in loc. cit., that Conjecture 1.3 implies holomorphy of $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ at $s\neq 1$, where $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$. However, this lemma is not correct, as Flicker himself pointed out in [9, p. 202]. Consequently, the asserted implication is not complete. Nevertheless, we have an implication in the *opposite* direction.

Theorem B. *Assume the twist adjoint L-functions $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ are holomorphic at $s\neq 1$ for all $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\mathbf{1})$. Then the τ-twisted Dedekind conjecture holds for all field extensions of E/F of degree n.*

Remarks 1.5. (i) Theorem B provides a new perspective in the study of the Dedekind conjecture, which is currently wide open when the degree is larger or equal to 5 — although, when $n=5$, there has been some progress towards the holomorphy of adjoint L-functions by integral representation (see [11]).

(ii) Suppose $\tau^k\neq\mathbf{1}$, $1\leq k\leq n$. With further investigation of $I_{\mathrm{Sing}}(s,\tau)$, we may conclude from Theorems A and B that $L(s,\pi,\mathrm{Ad}\otimes\tau)$ is holomorphic at $s\neq 1$ for all $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ if and only if the τ-twisted Dedekind conjecture holds for all field extensions of E/F of degree n.

In Section 8, we will see that the proof of Theorem B yields a nonvanishing result:

Corollary 1.6. *Let notation be as before. Let $n\geq 2$. Suppose there exists an extension E/F with degree $[E:F]=n$, and $\zeta_E(1/2)\neq 0$. Then there exists a $\pi=\pi(E)\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$, such that $L(1/2,\pi\times\widetilde{\pi})\neq 0$.*

Remark 1.7. In fact, Fröhlich [10] proved that there are infinitely many number fields F such that $\zeta_F(1/2)=0$. Since $L(s,\pi,\mathrm{Ad})$ is conjectured to be holomorphic, for *all* $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$, we have $L(1/2,\pi\times\widetilde{\pi})=0$ conjecturally.

2. Idea of proof and structure of the paper

This section outlines our strategy for establishing the coarse Jacquet–Zagier trace formula (1-5):

$$I_0^{\varphi}(s;\tau)=I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)-I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)+I_{\mathrm{Sing}}^{\varphi}(s,\tau)-I_{\mathrm{Whi}}^{\varphi}(s,\tau).$$

Definitions of these integrals will be provided in Sections 2A–2C. Additional details can be found in Section 3.

2A. ***Basic notation.***

2A1. *Number fields.* Let F be a number field with ring of adeles $\mathbb{A}_F$. Let Σ_F be the set of places of F. Denote by $\Sigma_{F,\mathrm{fin}}$ (resp. $\Sigma_{F,\infty}$) the set of nonarchimedean (resp. archimedean) places. For $v \in \Sigma_F$, we denote by F_v the corresponding local field and $\mathcal{O}_v$ its ring of integers with the maximal ideal $\mathfrak{p}_v$ and a uniformizer ϖ_v. We use $v \mid \infty$ to indicate an archimedean place v and write $v < \infty$ if v is nonarchimedean. Let $|\cdot|_v$ be the norm in F_v. Put $|\cdot|_\infty = \prod_{v\mid\infty} |\cdot|_v$ and $|\cdot|_{\mathrm{fin}} = \prod_{v<\infty} |\cdot|_v$. Let $|\cdot|_{\mathbb{A}_F} = |\cdot|_\infty \otimes |\cdot|_{\mathrm{fin}}$. We will simply write $|\cdot|$ for $|\cdot|_{\mathbb{A}_F}$ in calculation over $\mathbb{A}_F^\times$ or its quotient by $F^\times$.

2A2. *Some conventional notation.* For two meromorphic functions $h_1(s)$ and $h_2(s)$, we write $h_1(s) \propto h_2(s)$ if $h_1(s)/h_2(s)$ admits an analytic continuation to the whole complex plane. We will keep this $\propto$ notation throughout.

2A3. *Automorphic data.* Let $G = \mathrm{GL}(n)$. Let P be the standard parabolic subgroup of G of type $(n-1, 1)$. Let B be the Borel subgroup, consisting of upper triangular matrices. Let K be a fixed maximal compact subgroup of $G(\mathbb{A}_F)$. We will denote by $[H] := H(F)\backslash H(\mathbb{A}_F)$ for an algebraic group H over F.

Denote by Ξ_F the set of unitary characters on $F^\times\backslash\mathbb{A}_F^\times$ which are trivial on $\mathbb{R}_+^\times$. For any $\xi \in \Xi_F$, denote by $\Lambda(s, \xi)$ its complete Hecke L-function. For a topological space V, we denote by $\mathcal{S}(V)$ the space of Schwartz functions on V. Let $\Phi \in \mathcal{S}(\mathbb{A}_F^n)$. Let $\tau \in \Xi_F$ be fixed. Let $\eta = (0, \dots, 0, 1) \in F^n$. Set

$$f(x, \Phi, \tau; s) = \tau(\det x)\,|\det x|^s \int_{\mathbb{A}_F^\times} \Phi(\eta t x)\tau(t)^n |t|^{ns} d^\times t, \quad x \in G(\mathbb{A}_F), \tag{2-1}$$

which is a Tate integral for the complete L-function $\Lambda(ns, x.\Phi, \tau^n)$. It converges absolutely in $\mathrm{Re}(s) > 1/n$, and admits a meromorphic continuation elsewhere with a functional equation. Define the Eisenstein series

$$E_P(x, \Phi, \tau; s) = \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma x, \Phi, \tau; s), \tag{2-2}$$

which converges absolutely for $\mathrm{Re}(s) > 1$. Also, we define the integral

$$I_0^\varphi(s, \tau) = \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0^\varphi(x, x) E_P(x, \Phi, \tau; s)\, dx. \tag{2-3}$$

If there is no confusion in the context, we will write $f_\tau(x, s)$ or $f(x, s)$ instead of $f(x, \Phi, \tau; s)$ and omit the superscript φ in $I_*^\varphi(s; \tau)$ for simplicity, where $I_*^\varphi(s; \tau)$ is one of the functions in (1-5).

2A4. *The kernel function.* Denote by $\mathcal{H}(G(\mathbb{A}_F), \omega)$ the set of smooth functions $\varphi\colon G(\mathbb{A}_F) \to \mathbb{C}$, which is left and right K-finite for a compact subgroup K of $G(\mathbb{A}_F)$, transforms by a unitary character ω of $Z(\mathbb{A}_F)$, and has compact support modulo $Z(\mathbb{A}_F)$. Then $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$ defines an integral operator

$$R(\varphi) f(x) = \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(y) f(xy)\, dy \tag{2-4}$$

on the space $L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ of functions on $G(F)\backslash G(\mathbb{A}_F)$ which transform under $Z(\mathbb{A}_F)$ by ω^{-1} and are square integrable on $G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. This operator can be represented by the kernel function

$$\mathrm{K}^{\varphi}(x,y)=\sum_{\gamma\in Z(F)\backslash G(F)}\varphi(x^{-1}\gamma y).$$

We will omit the superscript φ and simply write $\mathrm{K}(x,y)$ for $\mathrm{K}^{\varphi}(x,y)$.

Recall that $L^2\big(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}\big)$ decomposes as a direct sum of the space $L_0^2\big(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}\big)$ of cusp forms and spaces $L_{\mathrm{Eis}}^2\big(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}\big)$ and $L_{\mathrm{Res}}^2\big(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}\big)$ defined using Eisenstein series and residues of Eisenstein series respectively. Then K splits up as $\mathrm{K}=\mathrm{K}_0+\mathrm{K}_{\mathrm{Eis}}+\mathrm{K}_{\mathrm{Res}}$. Explicitly,

$$\mathrm{K}_0(x,y)=\sum_{\pi}\mathrm{K}_{\pi}(x,y),\quad \text{where } \mathrm{K}_{\pi}(x,y)=\sum_{\phi\in\mathcal{B}_{\pi}}\pi(\varphi)\phi(x)\overline{\phi(y)}. \tag{2-5}$$

Here, π ranges over cuspidal automorphic representations, and $\mathcal{B}_{\pi}$ represents an orthonormal basis for the representation π.

2B. *Decomposition of the kernel function.* Starting with the spectral decomposition

$$\mathrm{K}_0(x,x)=\mathrm{K}(x,x)-(\mathrm{K}_{\mathrm{Eis}}(x,x)+\mathrm{K}_{\mathrm{Res}}(x,x)),$$

we will further decompose these kernel functions by algebraic and analytic/spectral expansions.

2B1. *The regular part.* Let $P_0=Z\backslash P$ is the mirabolic subgroup. Define

$$\mathfrak{S}:=\bigcup_{Q}\ \bigcup_{\gamma\in Z(F)\backslash Q(F)}\big\{p^{-1}\gamma p: p\in P_0(F)\big\}, \tag{2-6}$$

where Q's range through standard proper parabolic subgroups of G. Set

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x,x)=\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x), \tag{2-7}$$

We will show in Proposition 4.1 that the set $Z(F)\backslash G(F)-\mathfrak{S}$ consists of $P_0(F)$-conjugacy classes, resulting in regular $G(F)$-conjugacy classes. Notably, the stabilizers of elements in $Z(F)\backslash G(F)-\mathfrak{S}$ are direct sums of étale algebras over F with degrees $\leq n$. This distinction motivates the consideration of this set.

2B2. *The P-regular part.* Let N_P be the unipotent radical of P. Define

$$\mathrm{K}_{P,\mathrm{Reg}}(x,x)=\int_{[N_P]}\mathrm{K}_{\mathrm{Geo,Reg}}(ux,x)\,du, \tag{2-8}$$

where the subscript P indicates the constant term along $[N_P]=N_P(F)\backslash N_P(\mathbb{A}_F)$.

2B3. *Fourier expansion.* Let N be the unipotent radical of the standard Borel in G. Let θ being a fixed generic character on $[N]=N(F)\backslash N(\mathbb{A}_F)$.

We will show in Lemma 3.2 the Fourier expansion

$$\mathrm{K}_{\mathrm{Eis}}(x,x)+\mathrm{K}_{\mathrm{Res}}(x,x)=\int_{[N_P]}\mathrm{K}(ux,x)\,du+\sum_{k=2}^{n-1}\mathcal{F}_k\mathrm{K}(x,x)+\mathrm{K}_{\mathrm{Whi}}(x,x), \tag{2-9}$$

where $\mathcal{F}_k\mathrm{K}(x,x)$ represents the partial Fourier transforms of $\mathrm{K}(x,x)$ (see Proposition 3.1), and the generic part is given by

$$\mathrm{K}_{\mathrm{Whi}}(x,x)=\sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(u\delta x,x)\theta(u)\,du.$$

The integrand will be Whittaker functions, which explains the subscript 'Whi'.

Notably, the first two terms on the right side of (2-9) are expressed in terms of the entire kernel function $\mathrm{K}(x,x)$. Therefore, we can handle them by employing the geometric expansion of the kernel function.

2B4. *The singular part.* We define

$$\mathrm{K}_{\mathrm{Sing}}(x,x)=\sum_{\gamma\in\mathfrak{S}}\varphi(x^{-1}\gamma x)-\int_{[N_P]}\sum_{\gamma\in\mathfrak{S}}\varphi(x^{-1}u^{-1}\gamma x)\,du-\sum_{k=2}^{n-1}\mathcal{F}_k\mathrm{K}(x,x). \tag{2-10}$$

A precise definition will be given in Section 3B2. In [36], we provide explicit simplifications of this part in terms of φ for the cases when $n=3$ and $n=4$.

2B5. *Decomposition of* $\mathrm{K}_0(x,x)$. Combining (2-7), (2-8), (2-9) and (2-10) we then obtain an expansion of the cuspidal kernel function

$$\mathrm{K}_0(x,x)=\mathrm{K}_{\mathrm{Geo,Reg}}(x,x)-\mathrm{K}_{P,\mathrm{Reg}}(x,x)+\mathrm{K}_{\mathrm{Sing}}(x,x)-\mathrm{K}_{\mathrm{Whi}}(x,x). \tag{2-11}$$

This decomposition will be proved in Lemma 3.3.

2C. ***Decomposition of*** $\boldsymbol{I_0^{\varphi}(s,\tau)}$**.** Let $\mathrm{Re}(s)>1$. Unfold $E_P(x,\Phi,\tau;s)$ by substituting (2-2) into (2-3), and replace K_0 via (2-11) to obtain (formally)

$$I_0^{\varphi}(s;\tau)=I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)-I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)+I_{\mathrm{Sing}}^{\varphi}(s,\tau)-I_{\mathrm{Whi}}^{\varphi}(s,\tau), \tag{1-5}$$

Here

$$\begin{aligned}
I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)&:=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_{\mathrm{Geo,Reg}}(x,x)f(x,\Phi,\tau;s)\,dx,\\
I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)&:=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\mathrm{K}_{\mathrm{Geo,Reg}}(ux,x)f(x,\Phi,\tau;s)\,du\,dx,\\
I_{\mathrm{Whi}}^{\varphi}(s,\tau)&:=\int_{N(\mathbb{A}_F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{[N]}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(ux,vx)\theta(u)\overline{\theta}(v)f(x,\Phi,\tau;s)\,du\,dv\,dx,\\
I_{\mathrm{Sing}}^{\varphi}(s,\tau)&:=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_{\mathrm{Sing}}(x,x)f(x,\Phi,\tau;s)\,dx,
\end{aligned}$$

where $f(x,\Phi,\tau;s)$ is the section defined by (2-1). The proof of (1-5) will be derived in Section 3B4.

The main objective of this paper is to describe the analytic behavior of these integrals as meromorphic functions of s and establish their connections with various well-known L-functions.

For simplicity, we will omit the superscript φ in the integrals above.

2D. ***Structure of the paper.*** In Section 3, we use the Fourier expansion of functions on $P_0(F)\backslash G(\mathbb{A}_F)$ to obtain the decomposition (2-9). This manipulation allows us to shift the nongeneric part of the nondiscrete spectrum to the geometric side. Eventually we obtain the decomposition (1-5) in Section 3B.

In Section 4, we explore the algebraic structure of conjugacy classes in $Z(F)\backslash G(F)-\mathfrak{S}$ under the $P_0(F)$-adjoint action. Notably, the stabilizers of elements in $Z(F)\backslash G(F)-\mathfrak{S}$ correspond to direct sums of étale algebras over F. Consequently, we demonstrate that the function $I_{\mathrm{Geo,Reg}}(s,\tau)$ can be expressed as a sum of specific Artin L-series associated with these étale algebras. The detailed formulation of this result is provided in Theorem C in Section 4C.

In Section 5, we determine explicit representatives of $Z(F)\backslash G(F)-\mathfrak{S}$ as $P_0(F)$-conjugacy classes. Subsequently, we develop a geometric reduction technique to establish a connection between $I_{P,\mathrm{Reg}}(s,\tau)$ and certain intertwining operators. By employing Langlands' theory of intertwining operators, we establish the convergence and analytic properties of $I_{\mathrm{Geo,Reg}}(s,\tau)$. The results are summarized in Theorem D.

Next, our focus shifts to the function $I_{\mathrm{Whi}}(s,\tau)$, which pertains exclusively to the spectral side and is discussed in Sections 6–9.

- We encounter a challenge with Arthur's approach using modified truncation operators, as it is not suitable for our specific scenario due to the loss of $P(F)$-invariance when unfolding the Eisenstein series. Instead, we employ an alternative manipulation that reduces $I_{\mathrm{Whi}}(s,\tau)$ to a Mellin transform of the Kuznetsov trace formula, particularly Jacquet's *relative trace formula* for the pair (N,N) involving maximal unipotent radicals and generic characters. We establish that the relative trace formula can be bounded by a finite sum of gauges (Proposition 6.2). Consequently, we prove that $I_{\mathrm{Whi}}(s,\tau)$ is an absolutely convergent infinite sum of Mellin transforms, corresponding to certain Rankin–Selberg convolutions with *nondiscrete* automorphic representations when $\mathrm{Re}(s)$ exceeds a specific threshold. The precise results are presented in Theorem E in Section 6.
- In Section 7, we establish various properties of Rankin–Selberg periods associated with nondiscrete automorphic representations. These results play a crucial role in enhancing Theorem E and demonstrating the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ within the strip $0<\mathrm{Re}(s)<1$. For detailed information, refer to Theorem F. Notably, if $\tau^k\neq\mathbf{1}$ for all $1\leq k\leq n$, then $I_{\mathrm{Whi}}(s,\tau)$ is holomorphic in the region $\mathrm{Re}(s)>0$.
- In Section 8, we consolidate the results of Theorems C, D, E, and F to establish Equation (9-5). By employing specific test functions φ and addressing generalized Tate integrals, we subsequently prove Theorem B and Corollary 1.6 based on Equation (9-5).
- In Section 9, we address the special case when $\tau^k=\mathbf{1}$ for some $1\leq k\leq n$. In this scenario, the function $I_{\mathrm{Whi}}(s,\tau)$ exhibits singularities along the entire boundary $\mathrm{Re}(s)=1$. To overcome this, we aim to find a meromorphic continuation for $I_{\mathrm{Whi}}(s,\tau)$ that holds for *any* τ. This investigation is carried out in Section 9, where we establish Theorem G. By utilizing the Langlands–Shahidi method and analyzing residues, we obtain the meromorphic continuation for each individual term in

$I_{\mathrm{Whi}}(s,\tau)$. Theorem G provides the meromorphic continuation for $I_{\mathrm{Whi}}(s,\tau)$ in the general case of F being a function field. It also holds independent interest, as highlighted in [36].

2E. ***Some remarks on reading this paper.*** The integrals presented in (1-5) exhibit distinct characteristics, and each section is dedicated to analyzing one of these integrals. As a result, these sections can be considered somewhat independent from one another. Given the intricate nature of these integrals, we will introduce temporary notation specific to each section during the proof. While these ad hoc notational items may appear unfamiliar, readers are encouraged not to be overly concerned or confused, as their purpose is to facilitate the clarity and coherence of the respective section.

3. Fourier expansion and decomposition of $I_0(s,\tau)$

In this section, we use the Fourier expansion of the noncuspidal kernel function $\mathrm{K}_{\mathrm{ER}}(x,x)$ to decompose the cuspidal kernel function as follows:

$$\mathrm{K}_0(x,y)=\mathrm{K}_{\mathrm{Geo,Reg}}(x,y)-\mathrm{K}_{P,\mathrm{Reg}}(x,y)+\mathrm{K}_{\mathrm{Sing}}(x,y)-\mathrm{K}_{\mathrm{Whi}}(x,y). \tag{2-11}$$

Consequently, we obtain the decomposition

$$I_0(s,\tau)=I_{\mathrm{Geo,Reg}}(s,\tau)-I_{P,\mathrm{Reg}}(s,\tau)+I_{\mathrm{Sing}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau). \tag{1-5}$$

3A. ***Mirabolic Fourier expansions of automorphic functions.***

3A1. *Ad hoc notation.* To simplify the Fourier expansion discussion in this section, we introduce some notation that may deviate from standard conventions. Since it is specific to this section, readers need not be overly concerned with it.

Recall that B is the standard Borel of G (Section 2A3). Let N be the unipotent radical of B. For $1\le k\le n-1$, let B_{n-k} be the standard Borel subgroup (the subgroup consisting of nonsingular upper triangular matrices) of GL_{n-k}; let N_{n-k} be the unipotent radical of B_{n-k}. For any $i,\ j\in\mathbb{N}$, let $M_{i\times j}$ be the additive group scheme of $i\times j$ matrices. For $1\le k\le n-1$, define the unipotent radicals

$$N_k'=\left\{\begin{pmatrix} I_k & C\\ & D\end{pmatrix}: C\in M_{k\times(n-k)},\ D\in N_{n-k}\right\},$$

and the subgroup

$$N_k^*=\left\{\begin{pmatrix} I_{k-1} & C & \\ & 1 & \\ & & I_{n-k}\end{pmatrix}: C\in M_{(k-1)\times 1}(F)\right\}.$$

For $1\le k\le n-1$, set the generalized mirabolic subgroups

$$R_k=\left\{\begin{pmatrix} A & C\\ 0 & D\end{pmatrix}: A\in\mathrm{GL}_k,\ C\in M_{k\times(n-k)},\ D\in N_{n-k}\right\}.$$

In particular, $R_{n-1}=P_0$, which is the mirabolic subgroup of G.

Also we define $R_0=N_{(0,1,\dots,1)}:=N_{(1,1,\dots,1)}=N$ to be the unipotent radical of the standard Borel subgroup of GL_n.

3A2. *Mirabolic Fourier expansion.* Denote by Tr_F the trace map $\mathbb{A}_F \to \mathbb{A}_{\mathbb{Q}}$. Let $\psi_{\mathbb{Q}}$ be the additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ such that $\psi_{\mathbb{Q}}(t_\infty) = \exp(2\pi i t_\infty)$, for $t_\infty \in \mathbb{R} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$. Let $\psi_F = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_F$. For any $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in F^{n-1}$, define a character $\psi_\alpha : N(\mathbb{A}_F) \to \mathbb{C}$ by

$$\psi_\alpha(u) = \prod_{i=1}^{n-1} \psi_F\left(\alpha_i u_{i,i+1}\right), \quad \forall u = (u_{i,j})_{n\times n} \in N(\mathbb{A}_F). \tag{3-1}$$

Write $\psi_k = \psi_{(0,\dots,0,1,\dots,1)}$ (where the first $n-1-k$ components are 0 and the remaining k components are 1) and let $\theta = \psi_{(1,\dots,1)}$ be the standard generic character used to define Whittaker functions.

Proposition 3.1. *Let h be a continuous function on $P_0(F)\backslash G(\mathbb{A}_F)$. Then*

$$h(x) = \sum_{k=1}^{n} \sum_{\delta_k \in R_{k-1}(F)\backslash R_{n-1}(F)} \int_{[N_k^*]}\int_{[N_k']} h(uu'\delta_k x)\psi_{n-k}(u')\, du'\, du \tag{3-2}$$

if the right-hand side converges absolutely and locally uniformly. In particular, (3-2) *holds with $h(x) = \mathrm{K}^\varphi(x, y)$ for $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$ and for fixed $y \in G(\mathbb{A}_F)$.*

The proof follows the idea of Piatetski-Shapiro in [27]. See for example [35, §3.1] or [37, §2.4] (with certain mild adaptations).

3B. *Decomposition of $I_0(s, \tau)$.*

3B1. *Fourier expansion of* $\mathrm{K}_{\mathrm{ER}}(x, y)$. The objective of this section is to express $\mathrm{K}_{\mathrm{ER}}(x, y)$ in terms of its generic component and the complete kernel $\mathrm{K}(x, y)$.

Lemma 3.2. *Let notation be as in Section 3A. Then*

$$\mathrm{K}_{\mathrm{ER}}(x, y) = \mathrm{K}_{\mathrm{Whi}}(x, y) + \sum_{k=2}^{n} \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[N_k^*]}\int_{[N_k']} \mathrm{K}(uu'\delta_k x, y)\theta(u')\, du'\, du$$

where

$$\mathrm{K}_{\mathrm{Whi}}(x, y) = \sum_{\delta \in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\mathrm{Eis}}(u\delta x, \delta y)\theta(u)\, du.$$

Proof. $\mathrm{K}_{\mathrm{ER}}(x, y)$ is $P_0(F)$-invariant with respect to both variables. Given that $\mathrm{K}_0(x, y)$ decays rapidly, Proposition 3.1 applies to $h(x) = \mathrm{K}_{\mathrm{ER}}(x, y) = \mathrm{K}(x, y) - \mathrm{K}_0(x, y)$. Hence,

$$\mathrm{K}_{\mathrm{ER}}(x, y) = \sum_{k=1}^{n} \mathrm{K}_{\mathrm{ER}}^{(k)}(x, y), \tag{3-3}$$

where

$$\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y) = \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[N_k^*]}\int_{[N_k']} \mathrm{K}_{\mathrm{ER}}(uu'\delta_k x, y)\theta(u')\, du'\, du.$$

Recall that $\theta = \psi_{(1,1,\dots,1)}$ is the generic character of (3-1). By definition, $\theta(u') = \psi_{n-k}(u')$ for $y' \in [N_k']$.

When $k = 1$, N_k^* becomes trivial and $N_k' = N$. Consequently, we can replace K_{ER} with $\mathrm{K}_{\mathrm{Eis}}$ in the definition of $\mathrm{K}_{\mathrm{ER}}^{(1)}(x, y)$ since the residual spectrum is not generic [25, §V]. Define

$$\mathrm{K}_{\mathrm{Whi}}(x, y) := \mathrm{K}_{\mathrm{ER}}^{(1)}(x, y) = \sum_{\delta \in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\mathrm{Eis}}(u\delta x, \delta y)\theta(u)\, du. \tag{3-4}$$

Let $2 \le k \le n$. Let $V_k' = \mathrm{diag}(I_{k-1}, N_{n-k+1})$. Define V_k as the unipotent radical of the standard parabolic subgroup of type $(k-1, n-k+1)$. For any function ϕ on $G(\mathbb{A}_F)$, we have

$$\int_{[N_k^*]}\int_{[N_k']}\phi(uu'x)\theta(u')\,du'\,du = \int_{[V_k']}\int_{[V_k]}\phi(uu'x)\,du\,\theta(u')\,du', \quad x \in G(\mathbb{A}_F).$$

Since V_k is a nontrivial unipotent radical, then for all cusp forms ϕ on $G(F)\backslash G(\mathbb{A}_F)$,

$$\int_{[N_k^*]}\int_{[N_k']}\phi(uu'x)\theta(u')\,du'\,du = 0. \tag{3-5}$$

Combining (3-5) with the discrete spectral decomposition (2-5) one has

$$\int_{[N_k^*]}\int_{[N_k']}\mathrm{K}_0(uu'x, y)\theta(u')\,du'\,du = 0. \tag{3-6}$$

Since $\mathrm{K} = \mathrm{K}_{\mathrm{ER}} + \mathrm{K}_0$, this yields, for $2 \le k \le n$,

$$\iint \mathrm{K}_{\mathrm{ER}}(uu'x, y)\theta(u')\,du'\,du = \iint \mathrm{K}(uu'x, y)\theta(u')\,du'\,du,$$

where $u \in [N_k^*]$ and $u' \in [N_k']$. As a consequence, we have, for $2 \le k \le n$, that

$$\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y) = \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[N_k^*]}\int_{[N_k']}\mathrm{K}(uu'\delta_k x, y)\theta(u')\,du'\,du. \tag{3-7}$$

Now Lemma 3.2 follows from (3-3), (3-4), and (3-7). □

3B2. *The singular kernel* $\mathrm{K}_{\mathrm{Sing}}(x, y)$. Let $\mathrm{K}_{\mathrm{Geo,Reg}}(x, y)$ and $\mathrm{K}_{P,\mathrm{Reg}}(x, y)$ be defined by (2-7) and (2-8). Set

$$\mathrm{K}_{\mathrm{Geo,Sing}}(x, y) = \sum_{\gamma \in \mathfrak{S}} \varphi(x^{-1}\gamma y), \quad \mathrm{K}_{P,\mathrm{Sing}}(x, y) = \int_{[N_P]} \mathrm{K}_{\mathrm{Geo,Sing}}(ux, y)\,du. \tag{3-8}$$

Then $\mathrm{K}_{\mathrm{Geo,Sing}}(x, y) = \mathrm{K}(x, y) - \mathrm{K}_{\mathrm{Geo,Reg}}(x, y)$. Define

$$\mathrm{K}_{\mathrm{Sing}}(x, y) := \mathrm{K}_{\mathrm{Geo,Sing}}(x, y) - \mathrm{K}_{P,\mathrm{Sing}}(x, y) - \sum_{k=2}^{n-1} \mathrm{K}_{\mathrm{ER}}^{(k)}(x, y), \tag{2-10}$$

where $\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y)$ is defined by (3-7).

3B3. *Decomposition of* $\mathrm{K}_0(x, y)$.

Lemma 3.3. *Let notation be as before. Then*

$$\mathrm{K}_0(x, y) = \mathrm{K}_{\mathrm{Geo,Reg}}(x, y) - \mathrm{K}_{P,\mathrm{Reg}}(x, y) + \mathrm{K}_{\mathrm{Sing}}(x, y) - \mathrm{K}_{\mathrm{Whi}}(x, y). \tag{2-11}$$

Proof. By Lemma 3.2 and $\mathrm{K}_0(x, y) = \mathrm{K}(x, y) - \mathrm{K}_{\mathrm{ER}}(x, y)$ we obtain

$$\mathrm{K}_0(x, y) = \mathrm{K}(x, y) - \mathrm{K}_{\mathrm{Whi}}(x, y) - \sum_{k=2}^{n} \mathrm{K}_{\mathrm{ER}}^{(k)}(x, y), \tag{3-9}$$

where $\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y)$ is defined by (3-7). Note that $\mathrm{K}_{P,\mathrm{Sing}}(x, y) + \mathrm{K}_{P,\mathrm{Reg}}(x, y) = \mathrm{K}_{\mathrm{ER}}^{(n)}(x, y)$. Now (2-11) follows from (3-8), (2-10), and (3-9). □

3B4. *Decomposition of* $I_0(s,\tau)$. Unfolding the Eisenstein series into (2-3), we get

$$I_0(s,\tau)=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)f(x,\Phi,\tau;s)\,dx. \tag{3-10}$$

Therefore, substituting Lemma 3.3 into (3-10) we obtain

$$I_0(s,\tau)=I_{\mathrm{Geo,Reg}}(s,\tau)-I_{P,\mathrm{Reg}}(s,\tau)+I_{\mathrm{Sing}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau), \tag{1-5}$$

where the integrals on the right are defined in Section 2C.

In the next three sections, we will examine the analytic properties of the functions $I_{\mathrm{Geo,Reg}}(s,\tau)$, $I_{P,\mathrm{Reg}}(s,\tau)$, and $I_{\mathrm{Whi}}(s,\tau)$. The analytical behavior of $I_{\mathrm{Sing}}(s,\tau)$ has been studied in [36] for $n\leq 4$.

4. $I_{\mathrm{Geo,Reg}}(s,\tau)$ as Dedekind zeta functions

In this section we study $I_{\mathrm{Geo,Reg}}(s,\tau)$, which is defined by

$$I_{\mathrm{Geo,Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)}\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x)f(x,s)\,dx. \tag{4-1}$$

Decompose $G(F)$ into conjugacy classes. We will establish a relationship between a regular $G(F)$-conjugacy class $\mathcal{C}$ and a unique $P(F)$-conjugacy class $\mathcal{C}_0$. With this manipulation, we can transform the integral over the automorphic quotient $Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ into an integral over $Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, yielding

$$I_{\mathrm{Geo,Reg}}(s,\tau)=\sum_{\mathcal{C}\text{ regular}}I_{\mathcal{C}}(s,\tau)=\sum_{\mathcal{C}}\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma x)f(x,s)\,dx, \tag{4-2}$$

where γ is a suitable representative in $\mathcal{C}$. The inner integral factors through a Tate integral over the stabilizer $G_\gamma(\mathbb{A}_F)$, which is an étale algebra over F of degree n. We then compute this integral in terms of various Hecke L-functions.

4A. ***Structure of $G(F)$-conjugacy classes.*** Let B be the subgroup of upper triangular matrices of G. Let T (resp. N) be the Levi component (resp. unipotent radical) of B. For each $1\leq k\leq n-1$, let

$$w_k=\begin{pmatrix}I_{k-1}&&\\&S&\\&&I_{n-k-1}\end{pmatrix},\quad\text{where } S=\begin{pmatrix}&1\\1&\end{pmatrix}. \tag{4-3}$$

Let W_n be the group generated by elements w_i, $1\leq i\leq n-1$, identified with the Weyl group of G with respect to (B,T).

For each $1\leq k\leq n-1$, let Q_k be the standard maximal parabolic subgroup of G of type $(k,n-k)$. Note that $P=Q_{n-1}$. Set

$$Q_k(F)^{P(F)}=\{pqp^{-1}:p\in P(F),\ q\in Q_k(F)\},\ 1\leq k\leq n-1.$$

For $\gamma \in G(F)$, we denote by $G_\gamma(F)$ the centralizer of γ in $G(F)$. Recall that γ is regular if $\dim G_\gamma(F)$ is minimal (which also amounts to that the minimal polynomial of γ coincides with its characteristic polynomial).

Here are the two results in this section:

Proposition 4.1. *Let $\mathcal{C}$ be a regular $G(F)$-conjugacy classes in $G(F)$. There exists a $P(F)$-conjugacy class $\mathcal{C}_0$ such that*

$$\mathcal{C} = \mathcal{C}_0 \sqcup \bigcup_{k=1}^{n-1} \mathcal{C} \cap Q_k(F)^{P(F)}, \tag{4-4}$$

where $\sqcup$ denotes a disjoint union.

Proposition 4.2. *If $\mathcal{C}$ is an irregular $G(F)$-conjugacy class, then*

$$\mathcal{C} = \bigcup_{k=1}^{n-1} \mathcal{C} \cap Q_k(F)^{P(F)}.$$

4B. ***Decomposition of $G(F)$.*** The following is an analogue of Jordan canonical forms of matrices over a number field (see [34, Theorem 3.1], for instance).

Lemma 4.3. *Let V be a n-dimensional vector space over F, and $\mathscr{A} \in \operatorname{End}(V)$. Then there exist invariant subspaces $V_l \subseteq V$, $1 \leq l \leq r$, such that*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r, \tag{4-5}$$

and for each i, both the minimal polynomial and the characteristic polynomial of $\mathscr{A}_{V_l} = \mathscr{A}\,|_{V_l}$ are of the form $\wp(\lambda)^k$, where $k \in \mathbb{N}_{\geq 1}$ and $\wp(\lambda) \in F[\lambda]$ is an irreducible polynomial over F. For each l, there exists a basis of V_l under which $\mathscr{A}_{V_l}$ has the quasirational canonical form

$$\mathcal{J}\left(\wp(\lambda)^k\right) := \begin{pmatrix} C(\wp) & & & \\ \boldsymbol{N} & C(\wp) & & \\ & \ddots & \ddots & \\ & & \boldsymbol{N} & C(\wp) \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & & \end{pmatrix}, \tag{4-6}$$

where $C(\wp)$ is the companion matrix of $\wp(\lambda)$.

Let $\gamma \in G(F)$ be regular and denote by $f(\lambda) = \wp_1(\lambda)^{e_1} \cdots \wp_m(\lambda)^{e_m}$ its characteristic polynomial, where $e_i \geq 1$, $\wp_i$ is monic and irreducible over F, $1 \leq i \leq m$. Let $F[\gamma]$ be the subalgebra of $n \times n$ matrices generated by F and γ. We can identify $F[\gamma]$ with the polynomial algebra $F[\lambda]/(f(\lambda))$. Since γ is regular, it follows from the Frobenius dimension formula [14, Theorem 3.16] that the dimension of the centralizer $M_{n\times n,\gamma}(F)$ of γ in $M_{n\times n}(F)$ is n. As $F[\gamma]$ is contained in $M_{n\times n,\gamma}(F)$ and has dimension n, we conclude that $G_\gamma(F) = M_{n\times n,\gamma}(F) \cap G(F) = F[\gamma]^\times$, where $F[\gamma]^\times$ denotes the set of invertible elements in $F[\gamma]$.

Recall that γ is elliptic if $[G_\gamma]$ has finite volume, or equivalently, if the minimal polynomial of γ is irreducible over F. Hence, $F[\gamma]$ is a field if γ is elliptic.

Lemma 4.4. *Let $\gamma \in G(F)$ be regular elliptic. For any $(a_1, a_2, \dots, a_n) \in F^n$, there exists a unique element $x \in F[\gamma]$ such that the last row of x is exactly $(a_1, a_2, \dots, a_n)$.*

Proof. Since γ is regular, $G_\gamma(F) = F[\gamma]^\times$, and $\dim F[\gamma] = n$. Let $\eta = (0, \dots, 0, 1) \in F^n$. Consider the linear map

$$\tau : F[\gamma] \to F^n, \quad x \mapsto \tau(x) = \eta x.$$

Since γ is elliptic, $F[\gamma]$ is a field, so any nonzero element is invertible. Consequently, the map τ is injective, and hence surjective. Thus τ is an isomorphism of n-dimensional F-vector spaces. The lemma follows. □

Remark 4.5. Let $\gamma \in G(F)$ be regular elliptic. Then $G(F) = P_0(F)F[\gamma]^\times$, where $P_0(F) = Z(F)\backslash P(F)$ is the mirabolic subgroup of $G(F)$. In fact, since τ is a bijection, given $g \in G(F)$, there exists $h \in F[\gamma]^\times$ such that $\eta g = \eta h$, which implies that $gh^{-1} \in P_0(F)$, i.e., $g \in P_0(F)F[\gamma]^\times$.

Lemma 4.6. *Let $\gamma \in G(F)$ be regular, and assume that the characteristic polynomial of γ has only one irreducible factor. Let $\mathcal{J}$ be the quasirational canonical form of γ. Then for any $(a_1, a_2, \dots, a_n) \in F^n$, there exists a unique element $x \in F[\mathcal{J}]$ such that the last row of x is exactly $(a_1, a_2, \dots, a_n)$.*

Proof. Let $f(\lambda) = \wp(\lambda)^e$ be the characteristic polynomial of γ, where $\wp(\lambda) \in F[\lambda]$ is irreducible. Then $\deg \wp = ne^{-1}$, and the quasirational canonical form $\mathcal{J}$ of γ has the structure

$$\mathcal{J} = \begin{pmatrix} C & & & \\ N & C & & \\ & \ddots & \ddots & \\ & & N & C \end{pmatrix} \in \mathrm{GL}_n(F), \tag{4-7}$$

where $C = C(\wp)$ is the companion matrix of $\wp(\lambda)$, and N is defined in (4-6). Without loss of generality, and for simplicity of notation, we may assume $\gamma = \mathcal{J}$.

By definition, $F[\gamma] = F[\lambda]/(\wp(\lambda)^e)$. Consider the filtration

$$\wp(\lambda)^{i-1} F[\lambda]/(f(\lambda)) \supseteq \wp(\lambda)^i F[\lambda]/(f(\lambda)), \quad 1 \le i \le e-1.$$

With respect to the basis $\{\lambda^i \wp(\lambda)^j : 0 \le i \le d-1,\ 0 \le j \le e-1\}$ for $F[\gamma]$ over F, each element in $F[\gamma]$ has the following type

$$\mathcal{S}_\gamma = \left\{ A = \begin{pmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{e-1} & \dots & A_2 & A_1 & A_0 \end{pmatrix},\ A_i \in M_{d\times d}(F),\ 0 \le i \le e-1 \right\}. \tag{4-8}$$

Since γ is regular, we have $G_\gamma(F) = F[\gamma]^\times$. Therefore,

$$F[\gamma]^\times = \{A \in \mathcal{S}_\gamma \cap \mathrm{GL}_n(F) : A\gamma = \gamma A\}.$$

Now, consider the equation $A\gamma = \gamma A$ for $A \in \mathcal{S}_\gamma$. Using the expression $\gamma = \mathcal{J}$ from (4-7), this is equivalent to the following system of Lyapunov-like equations

$$\begin{cases} CA_0 = A_0C, \\ \boldsymbol{N}A_0 + CA_1 = A_1C + A_0\boldsymbol{N}, \\ \quad\vdots \\ \boldsymbol{N}A_{e-2} + CA_{e-1} = A_{e-1}C + A_{e-2}\boldsymbol{N}. \end{cases}$$

Since $A_0 \in F[C]^\times$ and C is regular elliptic, A_0 commuting with C implies that there exists some $h_0(\lambda) \in F[\lambda]$ such that $A_0 = h_0(C)$. We may assume that $d_0 = \deg h_0 \le d-1$. Let $\eta_d = (0, \dots, 0, 1) \in F^d$ and write $\eta_d C^i = (b_1^{(i)}, b_2^{(i)}, \dots, b_d^{(i)})$, $1 \le i \le d-1$, for the last row of C^i. Let

$$X_{(i)} = \begin{pmatrix} 0 & b_1^{(i)} & b_2^{(i)} & \cdots & b_{d-1}^{(i)} \\ 0 & 0 & b_1^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_2^{(i)} \\ \vdots & & \ddots & \ddots & b_1^{(i)} \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in \mathrm{GL}_d(F).$$

Then, for any $1 \le i \le d-1$, $X = X_{(i)}$ is a solution to the Lyapunov-like equation $\boldsymbol{N}C^i + CX = XC + C^i\boldsymbol{N}$.

Write $A_0 = h_0(C) = c'_{d_0}C^{d_0} + c'_{d_0-1}C^{d_0-1} + \cdots + c'_1C + c'_0I_d$, $c'_{d_0} \neq 0$. Define

$$A_{h_0}^{\mathrm{sp}} = c'_{d_0}X_{(d_0)} + c'_{d_0-1}X_{(d_0-1)} + \cdots + c'_1X_{(1)}.$$

Clearly $A_1 = A_{h_0}^{\mathrm{sp}}$ gives a special solution of the equation $\boldsymbol{N}A_0 + CA_1 = A_1C + A_0\boldsymbol{N}$ (superscript "sp" stands for "special"). Given $A_0 = h_0(C)$ as above, the set

$$\mathcal{U}_1 = \{A_{h_0}^{\mathrm{sp}} + h_1(C) : h_1 \in F[\lambda],\ \deg h_1 \le d-1\}$$

gives all solutions to the equation $\boldsymbol{N}A_0 + CA_1 = A_1C + A_0\boldsymbol{N}$. In fact, on the one hand, elements in $\mathcal{U}_1$ satisfy the equation; on the other hand, let A'_1 be any solution to the equation, then $A_{h_0}^{\mathrm{sp}} - A'_1$ commutes with C; thus it is a polynomial of C, namely, $A'_1 \in \mathcal{U}_1$.

Note that $\boldsymbol{N}A_{h_0}^{\mathrm{sp}} = A_{h_0}^{\mathrm{sp}}\boldsymbol{N} = 0$. Substituting $A_1 = A_{h_0}^{\mathrm{sp}} + h_1(C)$ into the equation $\boldsymbol{N}A_1 + CA_2 = A_2C + A_1\boldsymbol{N}$, we get $\boldsymbol{N}h_1(C) + CA_2 = A_2C + h_1(C)\boldsymbol{N}$. Write $h_1(\lambda) = c''_{d_1}\lambda^{d_1} + c''_{d_1-1}\lambda^{d_1-1} + \cdots + c''_1\lambda + c''_0$, and set

$$A_{h_1}^{\mathrm{sp}} = c''_{d_1}X_{(d_1)} + c''_{d_1-1}X_{(d_1-1)} + \cdots + c''_1X_{(1)}.$$

Then $\mathcal{U}_2 = \{A_{h_1}^{\mathrm{sp}} + h_2(C) : h_2 \in F[\lambda],\ \deg h_2 \le d-1\}$ gives all solutions to the equation $\boldsymbol{N}A_1 + CA_2 = A_2C + A_1\boldsymbol{N}$. we define $\mathcal{U}_i$, $1 \le i \le e-1$ similarly, and set $\mathcal{U}_0 = \{h_0(C) : h_0 \in F[\lambda],\ \deg h_0 \le d-1\}$. These $\mathcal{U}_i$'s describe the structure of $F[\gamma]^\times$.

Given $\mathfrak{a} = (a_1, a_2, \dots, a_n) \in F^n$, by Lemma 4.4 one can find uniquely an $A_0 \in F[C]$ such that $\eta_d A_0 = (a_{n-d+1}, a_{n-d+2}, \dots, a_n)$. Set

$$\mathfrak{a}_i = (a_{(i-1)d+1}, a_{(i-1)d+2}, \dots, a_{id}), \quad 1 \le i \le e-1.$$

Let $1 \le i_0 \le e-1$. Assume that for any $0 \le i < i_0$ one can find uniquely an element $A_i \in M_{d\times d}(F)$ such that the last row of A_i is exactly $\mathfrak{a}_{e-i}$, then let $h_{i_0}(C) \in F[C]^\times$ be the unique element whose last row is $\mathfrak{a}_{e-i_0}$, and take $A_{i_0} = A^{\mathrm{sp}}_{h_{i_0}} + h_{i_0}(C)$. Then $\eta_d A_{i_0} = \eta_d h_{i_0}(C) = \mathfrak{a}_{e-i_0}$. Such an A_{i_0} is unique. Let A'_{i_0} be another matrix satisfying that $\eta_d A'_{i_0} = \mathfrak{a}_{e-i_0}$. Since A'_{i_0} is a solution of $NA_{i_0-1} + CX = XC + A_{i_0-1}N$, $A_{i_0} - A'_{i_0}$ commutes with C. Thus $A_{i_0} - A'_{i_0} \in F[C]$. Note that the last row of $A_{i_0} - A'_{i_0}$ is $\mathbf{0}$, so by the uniqueness from Lemma 4.4, $A_{i_0} - A'_{i_0} = 0$. This shows the uniqueness of A_{i_0}.

Hence, Lemma 4.6 follows. □

In Remark 4.5, we showed the decomposition $G(F) = P_0(F)F[\gamma]^\times$ for regular elliptic $\gamma \in G(F)$, using Lemma 4.4. We now generalize this decomposition to regular $\gamma \in G(F)$ by utilizing Lemma 4.6 instead of Lemma 4.4.

Lemma 4.7. *Let $\gamma \in G(F)$ be regular. Then there exists a finite set $\Gamma_{\mathrm{reg}} = \{\gamma_i \in G(F) : 0 \le i \le m_0\}$ such that*

(1) *$G(F) = \bigcup_{0\le i\le m_0} P_0(F)\gamma_i F[\gamma]^\times$, where P_0 is the mirabolic subgroup of G, and*

(2) *there is at most one $\gamma_i \in \Gamma_{\mathrm{reg}}$ satisfying that $\gamma_i F[\gamma]^\times \gamma_i^{-1} \not\subseteq \bigcup_{k=1}^{n-1} Q_k(F)$.*

Proof. Denote by $f(\lambda) = \wp_1(\lambda)^{e_1} \cdots \wp_m(\lambda)^{e_m}$ the characteristic polynomial of $\gamma \in G(F)$, where $e_i \ge 1$ and the $\wp_i$'s are distinct monic and irreducible polynomials over F, $1 \le i \le m$. Let $d_i = \deg \wp_i$. Then $d_1e_1 + d_2e_2 + \cdots + d_me_m = \deg f = n$.

Since γ is regular, by Lemma 4.3, γ is $G(F)$-conjugate to a matrix of the form

$$\gamma^* = \begin{pmatrix} \mathcal{J}(\wp_1^{e_1}) & & & \\ & \mathcal{J}(\wp_2^{e_2}) & & \\ & & \ddots & \\ & & & \mathcal{J}(\wp_m^{e_m}) \end{pmatrix}, \tag{4-9}$$

where $\mathcal{J}(\wp_i^{e_i}) \in \mathrm{GL}_{d_ie_i}(F)$, $1 \le i \le m$. We may assume $\gamma = \gamma^*$. Write $k_i = d_ie_i$, $1 \le i \le m$. For $1 \le i \le m$, let $\tilde{\eta}_i : F^n \to F^{k_i}$ be defined by

$$\mathfrak{a} = (a_1, a_2, \ldots, a_n) \mapsto \tilde{\eta}_i(\mathfrak{a}) = (a_{k_1+\cdots+k_{i-1}+1}, \ldots, a_{k_1+\cdots+k_{i-1}+k_i}).$$

For simplicity we write $\eta^{(i)}(\mathfrak{a})$ for the last d_i components of $\tilde{\eta}_i(\mathfrak{a})$, i.e.,

$$\eta^{(i)}(\mathfrak{a}) = (a_{k_1+\cdots+k_{i-1}+(e_i-1)d_i+1}, a_{k_1+\cdots+k_{i-1}+(e_i-1)d_i+2}, \ldots, a_{k_1+\cdots+k_{i-1}+k_i}), \quad 1 \le i \le m.$$

We then split $G(F)$ into a union of sets following the conditions on the last row of its elements and show that each of the sets is of the form $P_0(F)\gamma_i F[\gamma]^\times$ for some specific γ_i.

- For $1 \le i, j \le n$, let $\eta_j = (0, 0, \ldots, 0, 1) \in F^j$, and $\boldsymbol{M}_{i,j}$ be the $i \times j$ matrix whose first $i-1$ rows are zero vectors, and the bottom row is η_j. Write $\eta = \eta_n$ as before. Let $\mathcal{S} = \{\delta \in G(F) : \eta^{(i)}(\eta\delta) \ne \mathbf{0},\ 1 \le i \le m\}$

and

$$\gamma_0 = \begin{pmatrix} I_{k_1} & & & \\ & \ddots & & \\ & & I_{k_{m-1}} & \\ \boldsymbol{M}_{k_m,k_1} & \dots & \boldsymbol{M}_{k_m,k_{m-1}} & I_{k_m} \end{pmatrix}. \tag{4-10}$$

Applying Lemma 4.6 to each $\tilde{\eta}_i(\mathfrak{a}) \in F^{k_i}$, we find for each $1 \le i \le m$, for any $\delta \in \mathcal{S}$, a unique $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$, such that $\eta_{k_i} x_i = \tilde{\eta}_i \delta$. When writing x_i in the form in (4-8), the definition of $\mathcal{S}$ implies that $A_0 \neq 0$; thus $A_0 \in F[C]^\times$. As a consequence, $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$.

Let $x = \operatorname{diag}(x_1, \dots, x_m)$. Then $\eta\gamma_0 x = \eta\delta$. So $\delta(\gamma_0 x)^{-1} \in P_0(F)$, i.e., $\delta \in P_0(F)\gamma_0 F[\gamma]^\times$. Moreover, $P_0(F) \cap \gamma_0 F[\gamma]^\times \gamma_0^{-1} = \{I_n\}$. To see this, look at the last row of $\gamma_0 x \gamma_0^{-1}$. A straightforward computation shows that

$$\tilde{\eta}_i(\gamma_0 x \gamma_0^{-1}) = \eta_{k_i} x_i - \eta_{k_i}, \quad 1 \le i \le m-1,$$

and $\tilde{\eta}_m(\gamma_0 x \gamma_0^{-1}) = \eta_{k_m}$. By the uniqueness part of Lemma 4.6, it follows that $x_i = I_{k_i}$, $1 \le i \le m$.

- For any $1 \le l \le m-1$ and $1 \le i_1 < \cdots < i_l \le m$, let

$$\mathcal{S}_{(i_1,\dots,i_l)} = \{\delta \in G(F) : \eta^{(j)}(\eta\delta) = \boldsymbol{0} \iff j \in \{i_1, \dots, i_l\}\}.$$

For any $1 \le i \le m$ and $1 \le s < t \le m$, define the Weyl elements $w_i^{(1)}$ and $w_{s,t}^{(2)}$ as

$$w_i^{(1)} = \begin{pmatrix} I_{k_i'} & & & & & \\ & 0 & & \cdots & I_{d_i} & \\ & & I_{d_i} & & & \\ & \vdots & & \ddots & \vdots & \\ & & & I_{d_i} & & \\ & I_{d_i} & & \cdots & 0 & \\ & & & & & I_{k_i''} \end{pmatrix},$$

where $k_i' = k_1 + \cdots k_{i-1}$, $k_i'' = k_{i+1} + \cdots + k_m$ for $1 \le i \le m$, and

$$w_{s,t}^{(2)} = \begin{pmatrix} I_{k_s'} & & & & & \\ & 0 & & \cdots & I_{k_t} & \\ & & I_{k_{s+1}} & & & \\ & \vdots & & \ddots & \vdots & \\ & & & I_{k_{t-1}} & & \\ & I_{k_s} & & \cdots & 0 & \\ & & & & & I_{k_t''} \end{pmatrix}.$$

We write simply $w_{s,t}^{(2)} = I_n$ if $s = t$.

Given $\delta \in \mathcal{S}_{(i_1,\dots,i_l)}$, let $\eta\delta = \mathfrak{a} = (a_1, a_2, \dots, a_n) \in F^n$. Then $\eta^{(j)}(\mathfrak{a}) = \boldsymbol{0}$ if and only if $j \in \{i_1, \dots, i_l\}$. We define $x = \operatorname{diag}(x_1, x_2, \dots, x_m) \in F[\gamma]^\times$ as follows.

- If $\tilde{\eta}_j(\mathfrak{a}) \neq \boldsymbol{0}$, let $e_j^0 \le e_j - 1$ be the maximal integer such that

$$(a_{k_1+\cdots+(e_j^0-1)d_j+1}, a_{k_1+\cdots+k_{j-1}+(e_j^0-1)d_j+2}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq \boldsymbol{0}.$$

By Lemma 4.6, for each such j, one can find an element x_j of the form in (4-8) such that the $e_j^0 d_j$-row of x_j is exactly

$$(a_{k_1+\cdots+k_{j-1}+1}, a_{k_1+\cdots+k_{j-1}+1}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq \mathbf{0}.$$

– If $\tilde{\eta}_j(\mathfrak{a}) = 0$, we take x_j to be an arbitrary element in $F[\mathcal{J}(\wp_j^{e_j})]^\times$. Since $\eta^{(i)}(\mathfrak{a}) \neq \mathbf{0}$ for any $i \notin \{i_1, \dots, i_l\}$, we can pick uniquely elements $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$ such that their last row is $\tilde{\eta}_i(\mathfrak{a})$, $i \in \{1, 2, \dots, m\} \setminus \{i_1, \dots, i_l\}$.

Apply the transposition $\prod_{1 \le j \le l}(j, i_j)$ to the ordered n-set $(1, 2, \dots, n)$ to get another ordered n-set $(i_1, \dots, i_l, i'_{l+1}, \dots, i'_m)$. Then $\eta^{(i'_m)}(\mathfrak{a}) \neq 0$. Hence, the last row of $x_{i'_m}$ is $\eta_{k_{i'_m}}(\mathfrak{a})$. Define

$$\gamma_{(i_1,\dots,i_l)} = \begin{pmatrix} I_{k_{i_1}} & & & & & \\ & \ddots & & & & \\ & & I_{k_{i_l}} & & & \\ & & & I_{k_{i'_{l+1}}} & & \\ & & & & \ddots & \\ \boldsymbol{M}^*_{k_{i'_m}, k_{i_1}} & \cdots & \boldsymbol{M}^*_{k_{i'_m}, k_{i_l}} & \boldsymbol{M}_{k_{i'_m}, k_{i_{l+1}}} & \cdots & I_{k_{i'_m}} \end{pmatrix} \cdot \prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)},$$

where $\boldsymbol{M}^*_{i,k_j}$ refers to the $i \times k_j$ matrix whose first $i-1$ rows are zero vectors and whose bottom row is

$$(0, \cdots 0, v_{d_i+1}, v_{d_i+2}, \dots, v_{k_i - d_i}, v_1, v_2, \dots, v_{d_i}).$$

Here v_l denotes the l-th element of $(0, \dots, 0, 1, 0, \dots, 0) \in F^{k_i}$, where the only nonzero entry occurs in the $e_i^0 d_i$-position.

From the definitions, $\eta\gamma_{(i_1,\dots,i_l)} x = \mathfrak{a} = \eta\delta$. Hence, $\delta \in P_0(F)\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times$.

Recall that $\gamma = \gamma^*$, as defined in (4-9). Given any $x \in F[\gamma]^\times$, one checks directly that

$$\prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)} x \left(\prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)} \right)^{-1} \in Q_{d_{i_1}}(F).$$

Therefore, $\gamma_{(i_1,\dots,i_l)} x \gamma^{-1}_{(i_1,\dots,i_l)} \in Q_{d_{i_1}}(F)$, i.e., $\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times \gamma^{-1}_{(i_1,\dots,i_l)} \subseteq Q_{d_{i_1}}(F)$.

• Now we consider $\mathcal{S}^{(m)} = \{\delta \in G(F) : \eta^{(j)}(\eta\delta) = \mathbf{0},\ 1 \le j \le m\}$. Let $\delta \in \mathcal{S}^{(m)}$. Denote $\eta\delta$ by $\mathfrak{a} = (a_1, a_2, \dots, a_n)$. For each $1 \le j \le m$ such that $\tilde{\eta}_j(\mathfrak{a}) \neq \mathbf{0}$, denote by $e_j^0 \le e_j - 1$ the maximal integer such that

$$(a_{k_1+\cdots+(e_j^0-1)d_j+1}, a_{k_1+\cdots+k_{j-1}+(e_j^0-1)d_j+2}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq 0.$$

Likewise, for each such j, one can find an element x_j of the form in (4-8) such that the $e_j^0 d_j$-row of x_j is exactly

$$(a_{k_1+\cdots+k_{j-1}+1}, a_{k_1+\cdots+k_{j-1}+1}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}).$$

For the remaining j's, we take arbitrary $x_j \in F[\mathcal{J}(\wp_j^{e_j})]^\times$. Let $x = \operatorname{diag}(x_1, \dots, x_m)$. Pick arbitrarily a j_0 such that $\tilde{\eta}_{j_0}(\mathfrak{a}) \neq 0$. Let $j_0' \neq j_0$ be another integer. Set

$$w_{j_0,e_{j_0}}^{(1)} = \begin{pmatrix} I_{k'_{j_0}} & & & \\ & 0 & I_{(e_j-e_{j_0})d_{j_0}} & \\ & I_{e_{j_0}d_{j_0}} & 0 & \\ & & & I_{k''_{j_0}} \end{pmatrix}.$$

Let

$$\gamma_m = \begin{pmatrix} I_{k_{j_0'}} & & & & \\ & \ddots & & & \\ & & I_{k_1} & & \\ & & & \ddots & \\ \boldsymbol{M}^*_{k_{j_0},k_{j_0'}} & \cdots & \boldsymbol{M}_{k_{j_0},k_{i_1}} & \cdots & I_{k_{j_0}} \end{pmatrix} \cdot w_{j_0'}^{(1)} w_{1,j_0'}^{(2)} w_{j_0,e_{j_0}}^{(1)} w_{j_0,m}^{(2)}.$$

Then $\eta\gamma_m x = \eta\delta$, implying that $\delta \in P_0(F)\gamma_m F[\gamma]^\times$. Moreover, for any $x' \in F[\gamma]^\times$, we have $\gamma_m x' \gamma_m^{-1} \in Q_{d_{j_0'}}(F)$.

It follows from the discussion above that

$$\begin{aligned} G(F) &= \mathcal{S} \sqcup \bigcup_{l=1}^{m-1} \bigcup_{1\le i_1<\dots<i_l\le m} \mathcal{S}_{(i_1,\dots,i_l)} \sqcup \mathcal{S}^{(m)} \\ &= P_0(F)\gamma_0 F[\gamma]^\times \cup \bigcup_{\substack{1\le l\le m-1 \\ 1\le i_1<\dots<i_l\le m}} P_0(F)\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times \cup P_0(F)\gamma_m F[\gamma]^\times, \end{aligned}$$

where $\gamma_m F[\gamma]^\times \gamma_m^{-1}$ and each $\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times \gamma_{(i_1,\dots,i_l)}^{-1}$ are contained in some standard maximal parabolic subgroup, and $P_0(F) \cap \gamma_0 F[\gamma]^\times \gamma_0^{-1} = \{I_n\}$. □

Now we prove the result on the structure of conjugacy classes:

Proof of Proposition 4.1. Recall that $\mathcal{C}$ is a regular $G(F)$-conjugacy classes in $G(F)$. Let $\gamma \in \mathcal{C}$ be of the form (4-9). By Lemma 4.7 we have

$$G(F) = \bigcup_{0\le i\le m_0} P_0(F)\gamma_i F[\gamma]^\times,$$

where P_0 is the mirabolic subgroup of G, and each γ_i is constructed explicitly in the proof of Lemma 4.7. Hence, for $\delta \in G(F)$, there exists $p \in P_0(F)$ and $i \in \{0, 1, \dots, m_0\}$ and $x \in F[\gamma]^\times$, such that $\delta = p\gamma_i x$. So one has

$$\delta\gamma\delta^{-1} = p\gamma_i x\gamma x^{-1}\gamma_i^{-1}p^{-1} = p\gamma_i\gamma\gamma_i^{-1}p^{-1}.$$

If $i \ge 1$, from the construction of γ_i, we have $\delta\gamma\delta^{-1} \in \mathcal{C} \cap Q_j(F)^{P(F)}$, for some standard maximal parabolic subgroup Q_j of type $(j, n-j)$, $1 \le j \le n-1$. And for $i = 0$, $\delta\gamma\delta^{-1} = p\gamma_0\gamma\gamma_0^{-1}p^{-1}$, where γ_0 is defined by (4-10). Let $\gamma' = \gamma_0\gamma\gamma_0^{-1}$. If $\gamma' \in \bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}$, we take $\mathcal{C}_0$ to be the empty set.

If $\gamma' \notin \bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}$, we set $\mathcal{C}_0 = \{p\gamma' p^{-1} : p \in P(F)\}$, which is disjoint from $\bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}$. This completes the proof. □

Recall the Bruhat decomposition

$$G(F) = P(F) \cup \bigcup_{j=1}^{n-1} P(F) w_{n-1} w_{n-2} \cdots w_j N_j(F), \tag{4-11}$$

where $N_j(F) := (w_j w_{j+1} \cdots w_n N(F) w_n w_{n-1} \cdots w_j \cap N(F)) \backslash N(F)$.

Proof of Proposition 4.2. Let $g \in \mathcal{C}$ be a representative. Set $m(\lambda)$ (resp. $f(\lambda)$) to be its minimal polynomial (resp. characteristic polynomial) over F. Consider their primary decompositions over F:

$$m(\lambda) = \prod_{i \in I} \wp_i(\lambda)^{e_i'} \quad \text{and} \quad f(\lambda) = \prod_{i \in I} \wp_i(\lambda)^{e_i},$$

where the $\wp_i(\lambda)$'s are distinct irreducible monic polynomials over F and I is a finite index set such that $e_i > 0$ for all $i \in I$. Write $d_i = \deg \wp_i(\lambda)$, $\forall i \in I$. We may assume that $d_1 \leq d_2 \leq \cdots \leq d_{\#I}$. Also, write $d_0 = 0$. Since the conjugacy class $\mathcal{C}$ is irregular, $m(\lambda)$ is a proper factor of $f(\lambda)$. Thus we have the following cases:

Case I: Suppose $\#I = 1$. Then $m(\lambda) = \wp(\lambda)^{e'}$, $f(\lambda) = \wp(\lambda)^{e}$, and $0 < e' < e = d_1^{-1} n$. Let C be the companion matrix of $m(\lambda)$. Then by Lemma 4.3, g is $G(F)$-conjugate to some element $\tilde{g} = \operatorname{diag}(g_1, \ldots, g_m)$ with

$$g_j = \mathcal{J}_j = \begin{pmatrix} C & & & \\ \boldsymbol{N} & C & & \\ & \ddots & \ddots & \\ & & \boldsymbol{N} & C \end{pmatrix}$$

being the quasirational canonical form, and $m > 1$. Let $r_j := \operatorname{rank} g_j$, $1 \leq j \leq m$. We may assume $r_1 \leq r_2 \leq \cdots \leq r_m$.

If $h \in P(F)$, then $h\tilde{g}h^{-1} \in hQ_{r_1}(F)h^{-1}$; if $h \in G(F) - P(F)$, it can be written as $h = pw_{n-1} \cdots w_k u_k$, where $p \in P(F)$ and u_k is of the form

$$\begin{pmatrix} I_{k-1} & & \\ & 1 & * \\ & & I_{n-k} \end{pmatrix} \in Q_k(F).$$

Note that if $k > r_1$, then $w_{n-1} \cdots w_k u_k \in \operatorname{diag}(\mathrm{GL}_{r_1}, \mathrm{GL}_{n-r_1})$, which implies that $h\tilde{g}h^{-1} \in Q_{r_1}(F)^{P(F)}$. Hence, we may assume $k \leq r_1$.

Since $r_1 \leq r_m$, there exists a Weyl element $w \in \mathrm{GL}(r_m)$ such that

$$w g_m w^{-1} = \begin{pmatrix} g_1 & B' \\ & A' \end{pmatrix} \in \mathrm{GL}(r_m, F), \tag{4-12}$$

for some matrices A' and B'. Let $w' = \operatorname{diag}(w, I_{n-r_m})$. Denote by $w'' = w_{r_1-1} w_{r_1-2} \cdots w_k$ if $k < r_1$, and set $w'' = I_n$ if $k = r_1$. Let $g_1'' = w'' g_1 w''^{-1}$. Then there exists a Weyl element $w_0 \in P_0(F)$ such that

$$w_0w_{n-1}w_{n-2}\cdots w_k\tilde{g}w_k\cdots w_{n-2}w_{n-1}w_0^{-1}=\begin{pmatrix} g_m &&&& \\ & g_2 &&& \\ && \ddots && \\ &&& g_{m-1} & \\ &&&& g_1'' \end{pmatrix}. \tag{4-13}$$

Let $\tilde{w}=w''w'w_0w_{n-1}w_{n-2}\cdots w_k$. By (4-12) and (4-13) one has

$$\tilde{w}\tilde{g}\tilde{w}^{-1}=\begin{pmatrix} g_1'' & B'' & \\ & A'' & \\ && g_1'' \end{pmatrix} \tag{4-14}$$

for some matrices A'' and B''. Let ${}^tN_P$ be the transpose of N_P, the unipotent radical of P. By definition,

$$w_{n-1}\cdots w_ku_kw_k\cdots w_{n-1}\in {}^tN_P(F).$$

Note also that $w''w'w_0\in P_0(F)$. So $w''w'w_0$ lies inside the Levi component of $P_0(F)$. Hence, $\tilde{w}u_k\tilde{w}^{-1}\in {}^tN_P(F)$. Write $\tilde{w}u_k\tilde{w}^{-1}=u_k''u_k'$, where

$$u_k''=\begin{pmatrix} I_{r_1} && \\ & I_{n-2r_1} & \\ & M_2 & I_{r_1} \end{pmatrix}\in {}^t N_P(F),\quad u_k'=\begin{pmatrix} I_{r_1} && \\ & I_{n-2r_1} & \\ M_1 && M_3 \end{pmatrix}\in {}^tN_P(F),$$

with M_1 and M_3 being $r_1\times r_1$ matrices and M_2 being an $r_1\times(n-2r_1)$ matrix. In addition, the first r_1-1 rows of M_1 and M_2 are all zeros.

Since g_1'' is regular, by Lemma 4.6 there exist $r_1\times r_1$ matrices $\gamma_1\in F[g_1'']$ and $\gamma_3\in F[g_1'']^\times$ such that the last row of γ_1 (resp. γ_3) coincides with the last row of M_1 (resp. M_3). Write $\gamma_3=\left(\begin{smallmatrix}\mathfrak{a} & \mathfrak{b}\\ \mathfrak{c} & 1\end{smallmatrix}\right)$ and set

$$p_1=\begin{pmatrix} I_{n-r_1} && \\ & (\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1} & -(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{b} \\ && 1 \end{pmatrix}\in P_0(F)\cap Q_{r_1}(F).$$

Write $\gamma_1=\left(\begin{smallmatrix}\mathfrak{a}'\\ \mathfrak{c}'\end{smallmatrix}\right)$, where $\mathfrak{a}'$ is an $(r_1-1)\times r_1$ matrix. Set

$$p_2=\begin{pmatrix} I_{r_1} &&& \\ & I_{n-2r_1} && \\ -(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{a}'+(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{b}\mathfrak{c}' && I_{r_1-1} & \\ &&& 1 \end{pmatrix}\in P_0(F).$$

Then $p_2^{-1}u_k''p_2=u_k''$. Let $p'=p_2p_1$. A straightforward calculation yields

$$u_k'=p'\begin{pmatrix} I_{r_1} && \\ 0 & I_{n-2r_1} & \\ \gamma_1 & 0 & I_{r_1} \end{pmatrix}\begin{pmatrix} I_{r_1} && \\ 0 & I_{n-2r_1} & \\ & 0 & \gamma_3 \end{pmatrix}. \tag{4-15}$$

Since $\gamma_1\in F[g_1'']$ and $\gamma_3\in F[g_1'']^\times$, then $\gamma_3^{-1}g_1''\gamma_3=g_1''$, and

$$\begin{pmatrix} I_{r_1} && \\ 0 & I_{n-2r_1} & \\ \gamma_1 & 0 & I_{r_1} \end{pmatrix}\begin{pmatrix} g_1'' & B'' & \\ & A'' & \\ && g_1'' \end{pmatrix}\begin{pmatrix} I_{r_1} && \\ 0 & I_{n-2r_1} & \\ \gamma_1 & 0 & I_{r_1} \end{pmatrix}^{-1}\in Q_{r_1}(F). \tag{4-16}$$

By (4-14), (4-15) and (4-16), and since $p_1\in Q_{r_1}(F)$, we have $p_2^{-1}u_k'\tilde{w}\tilde{g}\tilde{w}^{-1}u_k'p_2\in Q_{r_1}(F)$.

Since $p_2u_k''=u_k''p_2$ and $u_k''\in Q_{r_1}(F)$,

$$\tilde{w}u_k\tilde{g}u_k^{-1}\tilde{w}^{-1}=u_k''u_k'\tilde{w}\tilde{g}\tilde{w}^{-1}u_k'^{-1}u_k''^{-1}\in p_2Q_{r_1}(F)p_2^{-1}\subset Q_{r_1}(F)^{P(F)}. \tag{4-17}$$

Recall that $h=pw_{n-1}\cdots w_ku_k$, and $p'':=w''w'w_0\in P_0(F)$. Then it follows form (4-17) that

$$h\tilde{g}h^{-1}=pp''^{-1}\tilde{w}u_k\tilde{g}u_k^{-1}\tilde{w}^{-1}p''p^{-1}\in Q_{r_1}(F)^{P(F)}.$$

Case II: Here g is $G(F)$-conjugate to some matrix $\tilde{g}=\mathrm{diag}(\tilde{g}_1,\dots,\tilde{g}_m)$, where each $\tilde{g}_i$ is of the form $\mathrm{diag}(g_{i,1},\dots,g_{i,m_i})$, with

$$g_{i,j}=\begin{pmatrix} C_{i,j} & & & \\ N & C_{i,j} & & \\ & \ddots & \ddots & \\ & & N & C_{i,j}\end{pmatrix}$$

and $C_{i,j}$ regular elliptic; and $\tilde{g}_i$ has characteristic polynomial $\wp_i(\lambda)^{e_i}$. Since g is irregular, so is $\tilde{g}$. Hence there must be some $1\le i\le m$ such that $\tilde{g}_i$ is irregular. We may assume $\tilde{g}_1$ is irregular and $\mathrm{rank}\, g_{1,1}\le \mathrm{rank}\, g_{1,2}\le\cdots\le\mathrm{rank}\, g_{1,m_1}$. Then a similar argument as in Case I shows that $h\tilde{g}h^{-1}\in Q_{r_1}(F)^{P(F)}$, where $r_1=\mathrm{rank}\, g_{1,1}$.

Proposition 4.2 then follows. □

4C. ***Algebraic expansion: P-regular conjugacy classes.*** As in Section 4A, let Q_k be the standard parabolic subgroup of G of type $(k,n-k)$.

- In Proposition 4.1 we show that for any regular $G(F)$-conjugacy classes $\mathcal{C}$ in $G(F)$, there exists a $P(F)$-conjugacy class $\mathcal{C}_0$, uniquely determined by $\mathcal{C}$, such that

$$\mathcal{C}=\mathcal{C}_0\cup\bigcup_{k=1}^{n-1}\mathcal{C}\cap Q_k(F)^{P(F)}.$$

- When $\mathcal{C}$ is a nonregular $G(F)$-conjugacy class, by Proposition 4.2, we have $\mathcal{C}=\bigcup_{k=1}^{n-1}\mathcal{C}\cap Q_k(F)^{P(F)}$. Take $\mathcal{C}_0$ to be the empty set in this case.

Consider the notations introduced in Section 2. Let $\mathfrak{S}=\bigcup_{k=1}^{n-1}(Z(F)\backslash Q_k(F))^{P_0(F)}$. Following the approach in [18], we handle the integral $I_{\mathrm{Geo,Reg}}(s,\tau)$ of (4-1) through the decomposition

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x,y)=\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x)=\sum_{\mathcal{C}}\mathrm{K}_{\mathcal{C}_0}(x,y), \tag{4-18}$$

where $\mathcal{C}$ encompasses all conjugacy classes in $G(F)/Z(F)$ and

$$\mathrm{K}_{\mathcal{C}_0}(x,y)=\sum_{\gamma\in\mathcal{C}_0}\varphi(x^{-1}\gamma y)=\sum_{\gamma\in\mathcal{C}-\mathfrak{S}}\varphi(x^{-1}\gamma y).$$

It is important to note that while $\mathrm{K}_{\mathcal{C}_0}(x,x)$ is not $G(F)$-invariant, it is $P(F)$-invariant. Hence, it is meaningful to integrate them over $Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ with respect to $f(x,s)$. For each $\mathcal{C}$, let (at least formally)

$$I_{\mathcal{C}}(s,\tau):=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_{\mathcal{C}_0}(x,x)f(x,s)\,dx.$$

By definition, $I_{\mathcal{C}}(s,\tau)=0$ unless $\mathcal{C}$ is regular. Inserting (4-18) into (4-1) yields

$$I_{\mathrm{Geo,Reg}}(s,\tau)=\sum_{\mathcal{C}\text{ regular}}I_{\mathcal{C}}(s,\tau). \tag{4-2}$$

Let $\mathcal{C}$ be a conjugacy class in $G(F)$. Denote by $f(\lambda;\mathcal{C})$ the characteristic polynomial of $\mathcal{C}$. Factor it into irreducible ones with multiplicities as

$$f(\lambda;\mathcal{C})=\prod_{i=1}^{g}\wp_i(\lambda;\mathcal{C})^{e_i}, \tag{4-19}$$

where $\wp_i(\lambda;\mathcal{C})\in F[\lambda]$ is an irreducible polynomial of degree f_i. We may assume $f_1\geq\cdots\geq f_g$. Write $\boldsymbol{f}=(f_1,\dots,f_g)\in\mathbb{Z}_{\geq 1}^g$ and $\boldsymbol{e}=(e_1,\dots,e_g)\in\mathbb{Z}_{\geq 1}^g$. Then $\langle\boldsymbol{f},\boldsymbol{e}\rangle:=\sum f_ie_i=n$.

Definition 4.8. Let notation be as before. We say $\mathcal{C}$ is of type $(\boldsymbol{f},\boldsymbol{e};g)$. Let $\Gamma_{\boldsymbol{f},\boldsymbol{e};g}$ be the collection of regular $G(F)$-conjugacy classes of type $(\boldsymbol{f},\boldsymbol{e};g)$.

With the above definition, we have the decomposition

$$\bigsqcup_{\mathcal{C}\text{ regular}}\mathcal{C}=\bigsqcup_{\substack{\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}_{\geq 1}^g\\ \langle\boldsymbol{f},\boldsymbol{e}\rangle=n}}\Gamma_{\boldsymbol{f},\boldsymbol{e};g}. \tag{4-20}$$

Let $\mathcal{C}\in\Gamma_{\boldsymbol{f},\boldsymbol{e};g}$. Let $\gamma_{\mathcal{C}}\in\mathcal{C}$ be a fixed element. Let $\lambda_{\boldsymbol{f},\boldsymbol{e};g}$ be the inverse of the representative defined by (4-10), with a slight adjustment in notation: $f_i=d_i$, $k_i=f_ie_i$, and $g=m$. Specifically,

$$\lambda_{\boldsymbol{f},\boldsymbol{e};g}=\begin{pmatrix}I_{f_1}&&&&&&\\&\ddots&&&&&\\&&I_{f_1}&&&&\\&&&\ddots&&&\\&&&&I_{f_g}&&\\&&&&&\ddots&\\ \boldsymbol{M}_{f_g,f_1}&\cdots&\boldsymbol{M}_{f_g,f_1}&\cdots&\boldsymbol{M}_{f_g,f_g}&\cdots&I_{f_g}\end{pmatrix}^{-1}\in G(F), \tag{4-21}$$

where, for an integer m, $\boldsymbol{M}_{f_g,m}$ is the $f_g\times m$ matrix in which the first f_g-1 rows are zero vectors, and the bottom row is $\eta_m=(0,\dots,0,1)\in F^m$. In particular, $\lambda_{\boldsymbol{f},\boldsymbol{e};g}=I_n$ if $g=1$ and $e_1=1$, i.e., $\mathcal{C}$ is regular elliptic.

By Proposition 4.1, we have, when $\mathrm{Re}(s)>1$,

$$\begin{aligned}I_{\mathcal{C}}(s,\tau)&=\int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)}\sum_{p\in P_0(F)}\varphi(x^{-1}p^{-1}\lambda_{\boldsymbol{f},\boldsymbol{e};g}^{-1}\gamma_{\mathcal{C}}\lambda_{\boldsymbol{f},\boldsymbol{e};g}px)f(x,s)\,dx\\&=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_{\mathcal{C}}x)f(\lambda_{\boldsymbol{f},\boldsymbol{e};g}^{-1}x,s)\,dx,\end{aligned}$$

supposing the integrals converge absolutely. Combining this with (4-20) we then deduce (at least formally) that, when $\mathrm{Re}(s) > 1$,

$$\sum_{\mathcal{C}} I_{\mathcal{C}}(s,\tau) = \sum_{\substack{\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}^g_{\geq 1}\\ \langle \boldsymbol{f},\boldsymbol{e}\rangle = n}} \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\mathcal{C}\in\Gamma_{\boldsymbol{f},\boldsymbol{e};g}} \varphi(x^{-1}\gamma_{\mathcal{C}}x) f(\lambda^{-1}_{\boldsymbol{f},\boldsymbol{e};g}x,s)\,dx. \tag{4-22}$$

Moreover, (4-22) would be rigorous if the right-hand side converges absolutely, which is indeed the case. To verify, we will consider each type $(\boldsymbol{f},\boldsymbol{e};g)$ separately in the following subsections.

4C1. ***Type* $(n;1)$.** We treat the conjugacy classes of type $(\boldsymbol{f},\boldsymbol{e};g) = ((n),(1);1)$ first, i.e., $e = g = 1$, which are exactly regular elliptic conjugacy classes. Set

$$I_{\mathrm{r.e.}}(s,\tau) = I^{\varphi}_{\mathrm{r.e.}}(s,\tau) = \sum_{\mathcal{C}\text{ regular elliptic}} I_{\mathcal{C}}(s,\tau).$$

Proposition 4.9. *Let notation be as before. For every field extension E/F of degree n, there is an analytic function $Q_E(s)$ such that*

$$I_{\mathrm{r.e.}}(s,\tau) = \frac{1}{n}\sum_{[E:F]=n} Q_E(s)\Lambda(s,\tau\circ N_{E/F}), \tag{4-23}$$

where the summation is taken over only finitely many E's, depending only on the test function φ.

Proof. Let $\Gamma_{\mathrm{r.e.}}$ be the set of regular elliptic elements in $G(F)/Z(F)$. Then

$$I_{\mathrm{r.e.}}(s,\tau) = \int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in\Gamma_{\mathrm{r.e.}}} \varphi(x^{-1}\gamma x) f(x,s)\,dx.$$

Denote by {r.e.} a set of representatives for the regular elliptic conjugacy classes in $\Gamma_{\mathrm{r.e.}}$. For any $\gamma\in\{\mathrm{r.e.}\}$, the centralizer of γ in $G(F)/Z(F)$ is $F[\gamma]^\times$. So

$$\sum_{\gamma\in\Gamma_{\mathrm{r.e.}}} \varphi(x^{-1}\gamma x) = \sum_{\gamma\in\{\mathrm{r.e.}\}} \sum_{\delta\in F[\gamma]^\times Z(F)\backslash G(F)} \varphi(x^{-1}\delta^{-1}\gamma\delta x). \tag{4-24}$$

By Lemma 4.4 and the remark after it, one has $G(F) = P(F)F[\gamma]^\times$. Since $P(F)\cap F[\gamma]^\times = Z(F)$, every element $\delta\in Z(F)\backslash G(F)$ can be written unique as $\delta = p\nu$, where $p\in Z(F)\backslash P(F)$ and $\nu\in F[\gamma]^\times$. Hence the inner sum of (4-24) could be taken over $p\in Z(F)\backslash P(F)$. Therefore,

$$I_{\mathrm{r.e.}}(s,\tau) = \int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in\{\mathrm{r.e.}\}} \sum_{p\in Z(F)\backslash P(F)} \varphi(x^{-1}p^{-1}\gamma p x) f(x,s)\,dx. \tag{4-25}$$

Consider the field extension E/F of degree n. Fix an algebraic closure $\overline{F}$ of F, then E embeds into $\overline{F}$. A conjugacy class is said to belong to E if it consists of the conjugates of an element $\gamma\in E^\times/F^\times - 1$ under the usual identification. We need to consider two cases, according to whether E/F is Galois.

The idea is to replace the summation over $\gamma\in\{\mathrm{r.e.}\}$ by summation over extensions E/F of degree n; and inside, summation over elements of E.

<u>Case E/F Galois:</u> When γ varies over $E^\times/F^\times$ each conjugacy class belongs to E exactly n times.

Case E/F not Galois: When γ varies over $E^\times/F^\times$ each conjugacy class belongs to E once; but the sets of conjugacy classes belonging to the n embeddings of E in $\bar{F}$ are identical.

In either case, the integral in (4-25) can be rewritten as

$$I_{\text{r.e.}}(s,\tau)=\frac{1}{n}\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{[E:F]=n}\sum_{\gamma\in E^\times/F^\times-\{1\}}\varphi(x^{-1}\gamma x)f(x,s)\,dx, \tag{4-26}$$

where the summation on the right is taken over all extensions E/F of degree n. However, only finitely many E's (independent of x) in the inner sum contribute to (4-26). To verify this assertion, we generalize the argument in [18, p. 17]. Consider the function $\beta: Z(\mathbb{A}_F)\setminus G(\mathbb{A}_F)\to\mathbb{A}_F^{n-1}$ defined by

$$\beta(x)=\left(\frac{\sigma_1(x)^n}{\sigma_n(x)},\frac{\sigma_2(x)^n}{\sigma_n(x)^2},\dots,\frac{\sigma_{n-1}(x)^n}{\sigma_n(x)^{n-1}}\right),$$

where $\sigma_i(x)$ is the i-th symmetric polynomial in the eigenvalues of x. Since $\sigma_n(x)=\det x\neq 0$, $\forall x\in G(\mathbb{A}_F)$, β is continuous. Hence it maps $\operatorname{supp}\varphi$ to a compact set in $G(\mathbb{A}_F)$. On the other hand, β is invariant under conjugation. Consequently, the set $\{\beta(\gamma):\gamma\in G(F)/Z(F),\ \varphi(x^{-1}\gamma x)\neq 0 \text{ for some } x\in G(\mathbb{A}_F)\}$ is the intersection of a compact set with a discrete set, hence is finite.

- If $\beta(\gamma)\neq 0$, the number of distinct fields $F[\gamma]$ with a given value of $\beta(\gamma)$ is at most n, thus finite.
- If $\beta(\gamma)=0$, the map $\gamma\mapsto\det\gamma$ from

$$G(\mathbb{A}_F)/Z(\mathbb{A}_F)\to\mathbb{A}_F^\times/\mathbb{A}_F^{\times n}U_F$$

(with $\mathbb{A}_F^{\times n}=\{a^n:a\in\mathbb{A}_F^\times\}$ and U_F the maximal compact subgroup of $\mathbb{A}_F^\times$) is continuous; so the image of $\operatorname{supp}\varphi$ is also compact. Since $\mathbb{A}_F^{\times n}$ has finite index in $\mathbb{A}_F^{\times n}U_F\cap F$, there are only finitely many values for the image $\det\gamma \mod \mathbb{A}_F^{\times n}$ with $\gamma\in G(F)/Z(F)$ and $\varphi(x^{-1}\gamma x)\neq 0$ for some $x\in G(\mathbb{A}_F)$. When $\beta(\gamma)=0$, $F[\gamma]$ is determined (up to embedding) by $\det\gamma$, so we are done.

Thus we can interchange integrals in (4-26) to get

$$I_{\text{r.e.}}(s,\tau)=\frac{1}{n}\sum_{[E:F]=n}I_E(s,\tau), \tag{4-27}$$

where $I_E(s,\tau)$ is defined by

$$\int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\substack{\gamma\in E^\times/F^\times\\ \gamma\neq 1}}\varphi(x^{-1}\gamma x)\int_{G_\gamma(\mathbb{A}_F)}\Phi(\eta t x)\tau(\det tx)\,|\det tx|^s\,d^\times t\,dx. \tag{4-28}$$

Here $\eta=(0,\dots,0,1)\in\mathbb{A}_F^n$, and G_γ is the centralizer of γ in G. Since γ is regular elliptic, we have $G_\gamma(\mathbb{A}_F)\simeq\mathbb{A}_E^\times$.

As noted in [18, p. 17], the function

$$x\mapsto\sum_{\lambda\in E^\times/F^\times-\{1\}}\varphi(x^{-1}\lambda x)$$

has compact support on $G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. Note that for almost all finite places v, $x_v\in G(\mathcal{O}_v)$ and Φ_v

is the characteristic function of $\mathcal{O}_v^n$. Then as a function of ηt_v, $\Phi_v(\eta t_v x_v)$ is also the characteristic function of $\mathcal{O}_v^n$. So $\Phi(\eta t x)$ is a Schwartz–Bruhat function for fixed $x \in G(\mathbb{A}_F)$. If we identify $G_\gamma(\mathbb{A}_F)$ with $\mathbb{A}_E^\times$, $\det|_{E^\times}\colon E^\times \to F^\times$ with the norm map $N_{E/F}$, $t \mapsto |\det t|_{\mathbb{A}_F}$ with the idele norm in E, and $\mathcal{S}(\mathbb{A}_F^n)$ with $\mathcal{S}(\mathbb{A}_E)$, we see that the inner integral in (4-28) is just the Tate integral for $\Lambda(s, \tau \circ N_{E/F})$, converging absolutely for $\mathrm{Re}(s) > 1$. So there is some holomorphic function $Q_E(s)$ of s such that $I_E(s,\tau) = Q_E(s)\Lambda(s, \tau \circ N_{E/F})$, giving a meromorphic continuation of $I_E(s,\tau)$ to the entire s-plane. Since the sum over E in (4-27) is finite, $I_{\text{r.e.}}(s,\tau)$ is well-defined for $\mathrm{Re}(s) > 1$, and admits a meromorphic continuation to $s \in \mathbb{C}$ of the form (4-23). □

4C2. *Type* $(\boldsymbol{f}, \boldsymbol{e}; 1)$. One special aspect in the regular elliptic case is the compact support modulo $G_\gamma(\mathbb{A}_F)$ of the function

$$x \mapsto \sum_{\lambda \in E^\times/F^\times - \{1\}} \varphi(x^{-1}\lambda x). \tag{4-29}$$

However, for general types $(\boldsymbol{f}, \boldsymbol{e}; g)$, this function may not have compact support. A notable example is the absence of compact support for regular unipotent conjugacy classes; see [18, p. 18], for example. Therefore, a different approach is required compared to the regular elliptic case discussed in Section 4C1.

In this section, we focus on the case $g = 1$. The results obtained here, combined with the proof of Proposition 4.9, will play a pivotal role in addressing general types in the subsequent section.

Let $\mathcal{C}$ be of type $(\boldsymbol{f}, \boldsymbol{e}; 1)$. We may choose a representative $\gamma \in \mathcal{C}_0$ as in Proposition 4.1, in its quasirational canonical form:

$$\gamma = \begin{pmatrix} C & & & & \\ \boldsymbol{N} & C & & & \\ & \boldsymbol{N} & C & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{N} & C \end{pmatrix},$$

where

$$\boldsymbol{N} = \begin{pmatrix} & & 1 \\ & 0 & \\ & \iddots & \\ 0 & & \end{pmatrix},$$

$C \in \mathrm{GL}(f_1)$ and there are $e_1 = n/f_1$ such C's in the partitioned matrix above. The stabilizer of γ is studied in Lemma 4.6. In particular, if A is the stabilizer of γ, then A must be of the form

$$A = \begin{pmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{e_1-1} & \dots & A_2 & A_1 & A_0 \end{pmatrix}, \tag{4-30}$$

with $A_i \in M_{f_1 \times f_1}(F)$, $1 \le i < e_1$, and $A_0 \in F(C)$. Let $P_{\boldsymbol{f},\boldsymbol{e};1}$ be the transpose of standard parabolic subgroup of G of type $(f_1, f_1, \dots, f_1)$. Let $K_{\boldsymbol{f},\boldsymbol{e};1}$ be a compact subgroup such that $G(\mathbb{A}_F) = P_{\boldsymbol{f},\boldsymbol{e};1}(\mathbb{A}_F) K_{\boldsymbol{f},\boldsymbol{e};1}$.

We can decompose $G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ as follows. Write $x \in G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ as

$$\boldsymbol{B}\begin{pmatrix} I_{f_1} & & & & \\ & D_1 & & & \\ & & D_2 D_1 & & \\ & & & \ddots & \\ & & & & D_{e_1-1}\cdots D_1 \end{pmatrix}\begin{pmatrix} T_0 & & & & \\ & T_1 & & & \\ & & T_2 & & \\ & & & \ddots & \\ & & & & T_{e_1-1} \end{pmatrix} k, \tag{4-31}$$

where

$$\boldsymbol{B} = \begin{pmatrix} I_{f_1} & & & & \\ & I_{f_1} & & & \\ & B_{3,2} & I_{f_1} & & \\ & \vdots & \ddots & \ddots & \\ & B_{e_1,2} & \dots & B_{e_1,e_1-1} & I_{f_1} \end{pmatrix}; \tag{4-32}$$

each D_j, $1 \le j \le e_1 - 1$, is in the stabilizer $G_C(\mathbb{A}_F)$ of C; each T_j, $0 \le j < e_1$, is in $G_C(\mathbb{A}_F)\backslash \mathrm{GL}(f_1, \mathbb{A}_F)$; and $k \in K_{\boldsymbol{f},\boldsymbol{e};1}$. Write $\boldsymbol{D} = \mathrm{diag}(I_{f_1}, D_1, \dots, D_{e_1-1}\cdots D_1)$ and $\boldsymbol{T} = \mathrm{diag}(T_0, T_1, \dots, T_{e_1-1})$. Note that (4-31) follows from Iwasawa decomposition and the unipotent term $\boldsymbol{B}$ is of the form (4-32) because its first f_1 columns can be absorbed by left multiplication of some stabilizer $A \in G_\gamma(\mathbb{A}_F)$ of shape (4-30). Write the inverse $\boldsymbol{B}^{-1}$ in the matrix form

$$\begin{pmatrix} I_{f_1} & & & & \\ & I_{f_1} & & & \\ & B_{3,2} & I_{f_1} & & \\ & \vdots & \ddots & \ddots & \\ & B_{e_1,2} & \dots & B_{e_1,e_1-1} & I_{f_1} \end{pmatrix}^{-1} = \begin{pmatrix} I_{f_1} & & & & \\ & I_{f_1} & & & \\ & B'_{3,2} & I_{f_1} & & \\ & \vdots & \ddots & \ddots & \\ & B'_{e_1,2} & \dots & B'_{e_1,e_1-1} & I_{f_1} \end{pmatrix}.$$

For each $B'_{i,j}$, we write $\widetilde{B}'_{i,j} = B'_{i,j}C - CB'_{i,j}$. Let $\mathcal{B}$ be the group of such $\boldsymbol{B}$'s.

By definition, the contribution from conjugacy classes of type $(\boldsymbol{f}, \boldsymbol{e}; 1)$ is

$$I_{\boldsymbol{f},\boldsymbol{e};1}(s) = \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{C \in \Gamma_{\boldsymbol{f},\boldsymbol{e};1}} \varphi(x^{-1}\gamma_C x) f(\lambda^{-1}_{\boldsymbol{f},\boldsymbol{e};1} x, s)\, dx. \tag{4-33}$$

Remark 4.10. In the forthcoming calculation of $I_{\boldsymbol{f},\boldsymbol{e};1}(s)$, we will use the unconventional notation $\boldsymbol{N}$, $\mathcal{B}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{T}$ to simplify the description of integrands involving complex matrices of specific types.

Recall that for two meromorphic functions $h_1(s)$ and $h_2(s)$, we write $h_1(s) \propto h_2(s)$ if $h_1(s)/h_2(s)$ admits an analytic continuation to the whole complex plane.

Proposition 4.11. *Let notation be as before. Then $I_{\boldsymbol{f},\boldsymbol{e};1}(s)$ converges absolutely when* $\mathrm{Re}(s) > 1$ *and*

$$I_{\boldsymbol{f},\boldsymbol{e};1}(s) \propto \prod_{j=1}^{e} \biggl(\sum_{[E_j:F]=f} Q_{E_j}(s) \Lambda_{E_j}\bigl(js - j + 1, (\tau \circ N_{E/F})^j\bigr) \biggr),$$

where the sum over number fields E_j is finite and Q_{E_j} is an entire function of s.

Proof. Rewrite (4-33) in the original form

$$I_{\boldsymbol{f},\boldsymbol{e};1}(s)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\gamma}\varphi(x^{-1}\gamma x)f(x,s)\,dx,$$

where γ runs over regular elements of type $(\boldsymbol{f},\boldsymbol{e};1)$. Then similar to the discussion in Proposition 4.9, the sum over γ is finite, depending only on the support of φ. We then switch the sum to get $I_{\boldsymbol{f},\boldsymbol{e};1}(s)=\sum_{\gamma}I_{\gamma}(s,\tau)$, where

$$I_{\gamma}(s,\tau):=\int_{G_{\gamma}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma x)\int_{G_{\gamma}(\mathbb{A}_F)}\Phi[(0,\dots,0,1)tx]\tau(\det tx)\,|\det tx|^{s}\,dt\,dx.$$

Substituting the decomposition (4-31) into the above integral, we obtain

$$I_{\gamma}(s,\tau)=\int_{k}\int_{(G_C(\mathbb{A}_F))^{e_1-1}}\iint_{\mathcal{B}}\varphi(k^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}\boldsymbol{B}^{-1}\gamma\boldsymbol{B}\boldsymbol{D}\boldsymbol{T}k)\int_{G_{\gamma}(\mathbb{A}_F)}\cdots,$$

where $\boldsymbol{T}$ (resp. k) ranges through $(G_C(\mathbb{A}_F)\backslash\mathrm{GL}(f_1,\mathbb{A}_F))^{e_1}$ (resp. $K_{\boldsymbol{f},\boldsymbol{e};1}$), and $\mathcal{B}$ is the domain for $\boldsymbol{B}$.

According to the preceding discussion we have

$$I_{\gamma}(s,\tau)=\iint_{(G_C(\mathbb{A}_F))^{e_1-1}}\iint_{\mathcal{B}}\varphi(k^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}M\boldsymbol{D}\boldsymbol{T}k)\int_{G_{\gamma}(\mathbb{A}_F)}\cdots,$$

where $M=\boldsymbol{B}^{-1}\gamma\boldsymbol{B}$ is of the form

$$\begin{pmatrix} C &&&& \\ \boldsymbol{N} & C &&& \\ B'_{3,2}\boldsymbol{N} & \widetilde{B}'_{3,2}+\boldsymbol{N} & C && \\ \vdots & \vdots & \ddots & \ddots & \\ B'_{e_1,2}\boldsymbol{N} & \dots & \dots & \widetilde{B}'_{e_1,e_1-1}+\boldsymbol{N} & C \end{pmatrix},$$

where a typical entry of the above matrix is of the form

$$\widetilde{B}'_{l,k}+B'_{l,k+1}\boldsymbol{N}+\widetilde{B}'_{l,k+1}\mu_{l,k}(B_{i,j},\boldsymbol{N},C)+\lambda_{l,k}(B_{i,j},\boldsymbol{N},C)$$

with $\mu_{l,k}$ and $\lambda_{l,k}$ being polynomials in $B_{i,j}$, $\boldsymbol{N}$, and C, $i\le l-1$ and $j\ge k+2$.

A straightforward computation shows that if the first column of $B'_{i,j}$ is determined, then $\widetilde{B}'_{i,j}=B'_{i,j}C-CB'_{i,j}$ is completely determined by its last f_1-1 columns. Therefore, when $B'_{l,k}\boldsymbol{N}$ is fixed, then $\widetilde{B}'_{l,k}+B'_{l,k+1}\boldsymbol{N}$ is an $f_1\times f_1$ matrix, linearly determined by the last f_1-1 columns of $B'_{l,k}$ and the first column of $B'_{l,k+1}$, and this correspondence is one to one. Thus, the mapping $\boldsymbol{B}\mapsto M$ establishes a one-to-one smooth correspondence. The matrices M that contribute are confined to a compact region determined by the choice of φ. Consequently, after a change of variables, it follows that the integral with respect to $\boldsymbol{B}$ has compact support in $\mathcal{B}$. Therefore,

$$\begin{aligned}I_{\gamma}(s,\tau)=\int_{k}\int_{(G_C(\mathbb{A}_F))^{e_1-1}}\int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}}\int_{\mathcal{B}}\varphi(k^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}M\boldsymbol{D}\boldsymbol{T}k)\delta^{-1}_{P_{\boldsymbol{f},\boldsymbol{e};1}}(\boldsymbol{T})\\ \times\int_{G_{\gamma}(\mathbb{A}_F)}\Phi[(0,\dots,0,1)A\boldsymbol{B}\boldsymbol{D}\boldsymbol{T}k]\tau(\det A\boldsymbol{D}\boldsymbol{T})\,|\det A\boldsymbol{D}\boldsymbol{T}|^{s}dA\,d\boldsymbol{B}\,d\boldsymbol{D}\,d\boldsymbol{T}\,dk,\end{aligned}$$

where $\delta_{\boldsymbol{P}_{\boldsymbol{f},\boldsymbol{e};1}}(\boldsymbol{T})$ is the modular character associated with $P_{\boldsymbol{f},\boldsymbol{e};1}(\mathbb{A}_F)$. Changing variables via $\boldsymbol{B} \mapsto \boldsymbol{D}\boldsymbol{T}\boldsymbol{B}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}$ we then obtain

$$I_\gamma(s,\tau) = \int_k \int_{(G_C(\mathbb{A}_F))^{e_1-1}} \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}} \int_{\mathcal{B}} \varphi(k^{-1}\boldsymbol{B}^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}\gamma\boldsymbol{D}\boldsymbol{T}\boldsymbol{B}k) \times \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\dots,0,1)A\boldsymbol{D}\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{D}T)\,|\det A\boldsymbol{D}\boldsymbol{T}|^s\,dA\,d\boldsymbol{B}\,d\boldsymbol{D}\,d\boldsymbol{T}\,dk.$$

Then $\boldsymbol{D}^{-1}\gamma\boldsymbol{D}$ is equal to

$$\begin{pmatrix} C &&&& \\ D_1^{-1}\boldsymbol{N} & C &&& \\ 0 & D_1^{-1}D_2^{-1}\boldsymbol{N}D_1 & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \dots & (D_1^{-1}\cdots D_{e_1-2}^{-1}D_{e_1-1}^{-1}\boldsymbol{N}D_{e_1-2}\cdots D_1) & C \end{pmatrix}.$$

We can identify $D_j^{-1}\boldsymbol{N}$ with an element x_j in $E_j^\times/F^\times - 1$, $1 \le j \le e_1 - 1$, where E_j is a field extension of F with $[E_j : F] = f_1$. Conjugate of $D_j^{-1}\boldsymbol{N}$ under this identification becomes a Galois action on x_j. Therefore, we have

$$I_\gamma(s,\tau) = \frac{1}{f_1^{e_1-1}} \prod_{j=1}^{e_1-1} \sum_{\substack{x_j \in E_j^\times/F^\times - 1 \\ [E_j:F]=f_1}} \int_{\mathbb{A}_{E_j}^\times} \prod_{j=1}^{e_1-1} \tau^j(N_{E_j/F}(x_j)) N_{E_j/F}(x_j)^{js-j+1} \times \int_k \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}} \int_{\mathcal{B}} \varphi(k^{-1}\boldsymbol{B}^{-1}\boldsymbol{T}^{-1}\gamma^{\boldsymbol{D}}\boldsymbol{T}\boldsymbol{B}k)\,|\det A_0|^{1-e_1} \times \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\dots,0,1)A\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{T})\,|\det A\boldsymbol{T}|^s\,dA\,d\boldsymbol{B}\,d\boldsymbol{T}\,dki\,dx_1\cdots dx_{e_1-1},$$

where the sum over the x_j's is finite and

$$\gamma^{\boldsymbol{D}} = \begin{pmatrix} C &&&& \\ D_1^{-1}\boldsymbol{N} & C &&& \\ 0 & D_2^{-1}\boldsymbol{N} & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \dots & D_{e_1-1}^{-1}\boldsymbol{N} & C \end{pmatrix} = \begin{pmatrix} C &&&& \\ x_1 & C &&& \\ 0 & x_2 & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \dots & x_{e_1-1} & C \end{pmatrix}.$$

A similar analysis to that from the proof of Proposition 4.9 shows the integral relative to $\boldsymbol{T}$ actually is over a compact set, since φ has compact support. Hence, the function

$$(x_1,\dots,x_{e_1-1}) \mapsto \int_k \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}} \int_{\mathcal{B}} \varphi(k^{-1}\boldsymbol{B}^{-1}\boldsymbol{T}^{-1}\gamma^{\boldsymbol{D}}\boldsymbol{T}\boldsymbol{B}k)\,|\det A_0|^{1-e_1} \times \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\dots,0,1)A\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{T})\,|\det A\boldsymbol{T}|^s\,dA\,d\boldsymbol{B}\,d\boldsymbol{T}\,dk$$

is Schwartz; and the function

$$\int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\dots,0,1)A\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{T})\,|\det A\boldsymbol{T}|^s |\det A_0|^{1-e_1}\,dA$$

is the Tate integral for $\Lambda(e_1 s, (\tau \circ N_{E/F})^{e_1})$. Therefore, Proposition 4.11 follows. $\square$

4C3. *Orbital integrals of general type.* Set

$$I_{\boldsymbol{f},\boldsymbol{e};g}(s)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\mathcal{C}\in\Gamma_{\boldsymbol{f},\boldsymbol{e};g}}\varphi(x^{-1}\gamma_{\mathcal{C}}x)f(\lambda^{-1}_{\boldsymbol{f},\boldsymbol{e};g}x,s)\,dx,$$

where $g\geq 1$, $\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}^g_{\geq 1}$, $\langle\boldsymbol{f},\boldsymbol{e}\rangle=n$, and $\mathrm{Re}(s)>1$. We may write $\boldsymbol{f}=(f_1,\dots,f_g)$ with $f_1\geq\cdots\geq f_g$, and $\boldsymbol{e}=(e_1,\dots,e_g)$.

Let E be a finite extension of F, and let χ be an idele class character of $\mathbb{A}_E^\times$. For a positive integer j, define

$$\Lambda_E[j](s,\chi):=\Lambda_E(js-j+1,\chi^j),$$

where $\Lambda(s,\chi)$ denotes the complete Hecke L-function associated with χ.

Proposition 4.12. *Let notation be as before. Then $I_{\boldsymbol{f},\boldsymbol{e};g}(s)$ converges absolutely in* $\mathrm{Re}(s)>1$ *and*

$$I_{\boldsymbol{f},\boldsymbol{e};g}(s)\propto\prod_{i=1}^{g}\prod_{j=1}^{e_i}\Bigg(\sum_{[E_{i,j}:F]=f_i}Q_{E_{i,j}}(s)\Lambda_{E_{i,j}}[j](s,\tau\circ N_{E_i/F})\Bigg), \tag{4-34}$$

where for each i and j, the innermost summation is taken over only finitely many fields $E_{i,j}$, depending only on the test function φ; and each $Q_{E_{i,j}}(s)$ is an entire function.

Proof. Let $\mathcal{C}$ be a regular conjugacy class of type $(\boldsymbol{f},\boldsymbol{e};g)$. By Lemma 4.3 and the proof of Lemma 4.6, we can write $\mathcal{C}=\{\gamma\}$, with a typical element γ given by

$$\gamma=\begin{pmatrix}\alpha_1&&&&&&&&\\&\ddots&&&&&&&\\&&\alpha_m&&&&&&\\&&&\alpha_{m+1}&&&&&\\&&&\vdots&\ddots&&&&\\&&&*&\cdots&\alpha_{m+1}&&&\\&&&&&&\ddots&&\\&&&&&&&\alpha_g&&\\&&&&&&&\vdots&\ddots&\\&&&&&&&*&\cdots&\alpha_g\end{pmatrix}.$$

Using this expression and the definition of $\lambda^{-1}_{\boldsymbol{f},\boldsymbol{e};g}$ from (4-21), it becomes evident that Proposition 4.12 follows by combining the analysis from the proof of Proposition 4.9 with that of Proposition 4.11. ☐

Gathering (4-20) with Proposition 4.12 we then obtain:

Theorem C. *Let notation be as before. Let* $\mathrm{Re}(s)>1$. *Then*

$$I_{\mathrm{Geo,Reg}}(s,\tau)\propto\sum_{g=1}^{n}\sum_{\substack{\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}^g_{\geq 1}\\ \langle\boldsymbol{f},\boldsymbol{e}\rangle=n}}\prod_{i=1}^{g}\frac{1}{f_i^{e_i}}\sum_{[E_i:F]=f_i}Q_{E_i}(s)\prod_{j=1}^{e_i}\Lambda_{E_i}[j](s,\tau\circ N_{E_i/F}), \tag{4-35}$$

where, for each i, $Q_{E_i}(s)$ is an entire function and the innermost summation is taken over only finitely many fields E_i, depending implicitly only on the test function φ. The right side of (4-35) *gives a meromorphic continuation of $I_{\mathrm{Geo,Reg}}(s,\tau)$ to the whole s-plane, with poles possible only at $1,\frac{1}{2},\frac{2}{3},\dots,1-\frac{1}{n}$.*

5. $I_{P,\mathrm{Reg}}(s,\tau)$ as intertwining operators

We now proceed to handle the function $I_{P,\mathrm{Reg}}(s,\tau)$. Our approach involves explicit geometric computations, ultimately reducing it to a finite sum of intertwining operators whose analytical behavior is known through the work of Langlands. Recall that, by definition,

$$I_{P,\mathrm{Reg}}(s,\tau):=\int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}u^{-1}\gamma x)\,du\,f(x,s)\,dx.$$

To simplify $I_{P,\mathrm{Reg}}(s,\tau)$, we express $Z(F)\backslash G(F)-\mathfrak{S}$ as a disjoint union of $G(F)$-conjugacy classes $\mathcal{C}$ modulo $Z(F)$. Furthermore, we decompose each $\mathcal{C}$ into a disjoint union of $P(F)$-conjugacy classes. By explicitly determining representatives for these $P(F)$-conjugacy classes, we can perform a change of variables to transform the integral over $Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ into an integral over $Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. This allows us to perform local calculations and establish a relationship between these integrals and the constant terms of certain Eisenstein series. Consequently:

Theorem D. *Let notation be as before, then $I_{P,\mathrm{Reg}}(s,\tau)$ converges absolutely in* $\mathrm{Re}(s)>1$. *Moreover, $I_{P,\mathrm{Reg}}(s,\tau)$ admits a meromorphic continuation. Precisely, one has*

$$I_{P,\mathrm{Reg}}(s,\tau)\propto\frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})}. \tag{5-1}$$

5A. *$P(F)$-conjugacy classes.*

5A1. *Ad hoc notation.* We introduce some unconventional notation specific to this section, designed to simplify the calculations. Readers are advised not to be overly concerned with its details.

- Let $\mathcal{C}$ be a $G(F)$-conjugacy class. Recall that we define $\mathcal{C}_0$ in (4-4) if $\mathcal{C}$ is regular. Set $\mathfrak{R}=\bigsqcup_{@\mathcal{C}\text{ is regular}}\mathcal{C}_0$. Then $\mathfrak{R}$ is a disjoint union of $P(F)$-conjugacy classes in $G(F)$ by Proposition 4.1. Propositions 4.1 and 4.2 give a decomposition of $G(F)$ as $P(F)$-conjugacy classes

$$G(F)=\mathfrak{R}\sqcup\bigcup_{k=1}^{n-1}Q_k(F)^{P(F)}. \tag{5-2}$$

- For any $2\le k\le n$, let P_k be the standard maximal parabolic subgroup of GL_k of type $(k-1,1)$. In particular, $P=P_n$. Let N_P be the unipotent radical of P.
- Recall that B is the Borel of G. Let T (resp. N) be the Levi component (resp. unipotent radical) of B.
- Let $w_1,\dots,w_{n-1}$ be the Weyl elements of $G(F)$ defined by (4-3) in Section 4A.
- For any $1\le k\le n-1$, write Q_k (resp. Q_k^*) for the standard parabolic subgroup of G of type $(k,n-k)$ (resp. $(k,1,\dots,1)$).

To compute $I_{P,\mathrm{Reg}}(s,\tau)$, it is necessary to explicitly choose representatives for $\mathfrak{R}$. From the construction in the proof of Proposition 4.1, we have an explicit algebraic description of the representatives for each $\mathcal{C}_0$. However, this algebraic construction is not particularly convenient for analytic parametrization.

In this section, our goal is to find representatives of $\mathfrak{R}$ that are more suitable for analytic purposes.

5A2. *Explicit representatives of* $\mathfrak{R}$. To narrow down the possible candidates for these representatives, we begin with the following result.

Lemma 5.2. *Let notation be as before. Set* $\mathcal{R}=\{w_{n-1}w_{n-2}\cdots w_1 b : b\in B(F)\}$. *Denote by* $\mathcal{R}^{P(F)}$ *the union of* $P(F)$*-conjugacy classes of elements in* $\mathcal{R}$. *Then*

$$\mathfrak{R}=\mathcal{R}^{P(F)}. \tag{5-3}$$

Proof. Recall the Bruhat decomposition

$$G(F)=P(F)\sqcup P(F)w_{n-1}P(F).$$

Due to the disjointness of different Bruhat cells, the $P(F)$-conjugacy class of g_1 does not intersect with that of g_2, for any $g_1\in P(F)$ and $g_2\in P(F)w_{n-1}P(F)$. Since $P(F)$-conjugacy classes of $P(F)$ lie in $P(F)$, we can reject all representatives in $P(F)$ and conclude that $P(F)$-conjugacy classes in $\mathfrak{R}$ are represented by elements in $w_{n-1}P(F)$.

For any

$$g=w_{n-1}\begin{pmatrix}A_{n-1} & b\\ & d_n\end{pmatrix}\in w_{n-1}P(F)\cap\mathfrak{R},$$

by Bruhat decomposition, either $A_{n-1}\in P_{n-1}(F)$ or $A_{n-1}\in P_{n-1}(F)w_{n-2}P_{n-1}(F)$, where P_{n-1} is the standard maximal parabolic subgroup of $\mathrm{GL}_{n-1}(F)$ of type $(n-2,1)$. If $A_{n-1}\in P_{n-1}(F)$, then $g\in Q_{n-2}(F)\subset\bigcup_{1\le k\le n-1}Q_k(F)^{P(F)}$. Thus $g\notin\mathfrak{R}$. Therefore, $A_{n-1}\in P_{n-1}(F)w_{n-2}P_{n-1}(F)$. So we can write

$$g^{(0)}=g=w_{n-1}\begin{pmatrix}I_{n-2} & c & b_1\\ & 1 & b_2\\ & & 1\end{pmatrix}w_{n-2}\begin{pmatrix}A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n\end{pmatrix}\in w_{n-1}Q^*_{n-1}(F),$$

which is conjugate by $w_{n-2}\begin{pmatrix}A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n\end{pmatrix}\in P(F)$ to

$$\begin{aligned}g^{(1)}&=w_{n-2}\begin{pmatrix}A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n\end{pmatrix}w_{n-1}\begin{pmatrix}I_{n-2} & c & b_1\\ & 1 & b_2\\ & & 1\end{pmatrix}\\ &=w_{n-2}w_{n-1}\begin{pmatrix}A_{n-2} & & c_{n-2}\\ & d_n & \\ & & d_{n-1}\end{pmatrix}\begin{pmatrix}I_{n-2} & c & b_1\\ & 1 & b_2\\ & & 1\end{pmatrix}\in w_{n-2}w_{n-1}Q^*_{n-2}(F).\end{aligned}$$

Again, apply Bruhat decomposition to $\mathrm{GL}_{n-2}(F)$ to conclude that either $A_{n-2}\in P_{n-2}(F)$ or $A_{n-2}\in P_{n-2}(F)w_{n-3}P_{n-2}(F)$, where P_{n-2} is the standard maximal parabolic subgroup of $\mathrm{GL}_{n-2}(F)$ of type $(n-3,1)$. If $A_{n-2}\in P_{n-2}(F)$, then $g^{(1)}\in Q_{n-3}(F)\subset\bigcup_{1\le k\le n-1}Q_k(F)^{P(F)}$, i.e., $g^{(1)}\notin\mathfrak{R}$. Therefore, $A_{n-2}\in P_{n-2}(F)w_{n-3}P_{n-2}(F)$. Likewise, $g^{(1)}$ is thus $P(F)$-conjugate to some $g^{(2)}\in w_{n-3}w_{n-2}w_{n-1}Q^*_{n-3}(F)$. Continue this process inductively to see that g is $P(F)$-conjugate to some element $g^{(n-2)}\in w_1w_2\cdots w_{n-1}Q^*_1(F)$.

Therefore, $\mathfrak{R} \subseteq \{p^{-1}\gamma p : \gamma \in w_1 w_2 \cdots w_{n-1} Q_1^*(F),\ p \in P(F)\}$. So we have

$$\begin{aligned}\{g^{-1} : g \in \mathfrak{R}\} &\subseteq \{p^{-1}\gamma p : \gamma \in Q_1^*(F) w_{n-1} \cdots w_2 w_1,\ p \in P(F)\}\\ &= \big\{p^{-1}\gamma p : \gamma \in w_{n-1} w_{n-2} \cdots w_1 B(F),\ p \in P(F)\big\},\end{aligned}$$

since $Q_1^*(F) = B(F) \subseteq P(F)$. Note that $\mathfrak{R}$ is stable under inversion. Hence,

$$\mathfrak{R} = \{g^{-1} : g \in \mathfrak{R}\} \subseteq \{p^{-1}\gamma p : \gamma \in w_{n-1} w_{n-2} \cdots w_1 B(F), \quad p \in P(F)\} = \mathcal{R}^{P(F)}.$$

Based on the construction, we observe that

$$\mathcal{R}^{P(F)} \cap \bigcup_{1 \le k \le n-1} Q_k(F) = \varnothing,$$

since Bruhat cells are disjoint. This implies, according to (5-2), that $\mathcal{R}^{P(F)} \subseteq \mathfrak{R}$. Hence, $\mathfrak{R} = \mathcal{R}^{P(F)}$. □

Now we determine representatives of $\mathcal{R}^{P(F)}$.

Lemma 5.3. *Let notation be as before. Then* $\mathfrak{R} = \widetilde{\mathcal{R}}^{P(F)}$, *where*

$$\widetilde{\mathcal{R}} := \left\{ w_{n-1} w_{n-2} \cdots w_1 \begin{pmatrix} I_{n-1} & \mathfrak{b} \\ & t \end{pmatrix} : t \in F^\times,\ \begin{pmatrix} I_{n-1} & \mathfrak{b} \\ & 1 \end{pmatrix} \in N_P(F) \right\}.$$

Proof. Let $b = \mathfrak{t}\mathfrak{u} \in B(F)$, where $\mathfrak{t} \in T(F)$ and $\mathfrak{u} \in N(F)$. By examining the rows on both sides, we can find $c_{i,j} \in F$ and $\mathfrak{u}' \in N_P(F)$ that satisfy the equation

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & 1 & c_{1,2} & \cdots & c_{1,n-1} \\ & & \ddots & & \vdots \\ & & & 1 & c_{n-2,n-1} \\ & & & & 1 \end{pmatrix} \mathfrak{t}\mathfrak{u} = \mathfrak{t}\mathfrak{u}' \begin{pmatrix} 1 & c_{1,2} & \cdots & c_{1,n-1} & 0 \\ & 1 & \cdots & c_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}. \tag{5-4}$$

In fact, the values of $c_{1,j}$, $1 \le j \le n-1$, are determined by b. For $i \ge 2$, the values of $c_{i,j}$ are determined by $c_{i-1,j}$ and b, where $1 \le j \le n-1$. Thus, these $c_{i,j}$ values are uniquely determined by b. Consequently, $\mathfrak{u}'$ is also uniquely determined.

Let $p \in P(F)$ be the matrix multiplying $\mathfrak{t}\mathfrak{u}'$ on the right-hand side of (5-4). Then (5-4) becomes

$$p w_{n-1} w_{n-2} \cdots w_1 b p^{-1} = w_{n-1} w_{n-2} \cdots w_1 \mathfrak{t}\mathfrak{u}'.$$

Write $\mathfrak{t} = \operatorname{diag}(t_1, \dots, t_n)$, $\mathfrak{a} = \operatorname{diag}(a_1, \dots, a_n) \in T(F)$, with $a_i = t_1^{-1} \cdots t_i^{-1}$, $1 \le i \le n$, and

$$\mathfrak{u}' = \begin{pmatrix} I_{n-1} & \mathfrak{b} \\ & 1 \end{pmatrix},$$

with $\mathfrak{b} = {}^t(b_1, \dots, b_{n-1})$. Then

$$\mathfrak{a}^{-1} w_{n-1} w_{n-2} \cdots w_1 \mathfrak{t}\mathfrak{u}'\mathfrak{a} = w_{n-1} w_{n-2} \cdots w_1 \begin{pmatrix} I_{n-1} & \mathfrak{b}' \\ & t \end{pmatrix} \in \widetilde{\mathcal{R}},$$

where $t = \det \mathfrak{t}$ and $\mathfrak{b}' = {}^t(b_1', \dots, b_{n-1}')$, with $b_i' = t_1 \cdots t_i b_i$, $1 \le i \le n-1$.

Therefore, Lemma 5.3 follows from Lemma 5.2. □

We shall show that elements in $\widetilde{\mathcal{R}}$ are not $P(F)$-conjugate to each other.

Lemma 5.4. *Let notation be as before. Then $\widetilde{\mathcal{R}}$ forms a complete set of representatives for $\mathfrak{R}$. In particular,* $\mathfrak{R}=\widetilde{\mathcal{R}}^{P(F)}$.

Proof. Let $w_{n-1}w_{n-2}\cdots w_1 b$ and $w_{n-1}w_{n-2}\cdots w_1 b'$ be two elements in $\widetilde{\mathcal{R}}$. Assume that there exists some $p_n\in P_n(F)=P(F)$ such that

$$p_n w_{n-1}w_{n-2}\cdots w_1 b p_n^{-1}=w_{n-1}w_{n-2}\cdots w_1 b'. \tag{5-5}$$

Then $w_{n-1}p_nw_{n-1}=w_{n-2}\cdots w_1b'p_nb^{-1}w_1\cdots w_{n-2}\in P(F)=Q_{n-1}(F)$. Since $p_n\in P(F)$, it is necessarily of the form

$$p_n=\begin{pmatrix}A_{n-2}&\mathfrak{c}_{n-1}&\mathfrak{c}_n\\ &a_{n-1}&0\\ &&a_n\end{pmatrix}\in\begin{pmatrix}GL_{n-2}(F)&*&*\\ &F^\times&0\\ &&F^\times\end{pmatrix}\subset Q_{n-2}(F).$$

Hence, $w_{n-2}w_{n-1}p_nw_{n-1}w_{n-2}=w_{n-3}\cdots w_1b'p_nb^{-1}w_1\cdots w_{n-3}\in Q_{n-2}(F)$, which implies that A_{n-2} lies in the maximal parabolic subgroup of $\mathrm{GL}_{n-2}(F)$ of type $(n-3,1)$, and the last component of $\mathfrak{c}_n$ must vanish. Repeating this process $n-3$ more times, we simplify (5-5) to

$$\begin{pmatrix}a_n&0&0&\dots&0\\ &a_1&c_{1,2}&\cdots&c_{1,n-1}\\ &&\ddots&&\vdots\\ &&&a_{n-2}&c_{n-2,n-1}\\ &&&&a_{n-1}\end{pmatrix}b\begin{pmatrix}a_1&c_{1,2}&\cdots&c_{1,n-1}&0\\ &a_2&\cdots&c_{2,n-1}&0\\ &&\ddots&\vdots&\vdots\\ &&&a_{n-1}&0\\ &&&&a_n\end{pmatrix}^{-1}=b'. \tag{5-6}$$

Note that the unipotent radical of b and b' are in $N_P(F)$. By the analysis towards equation (5-4) we see that all the $c_{i,j}$'s in (5-6) must vanish.

Write

$$b=\begin{pmatrix}I_{n-1}&\mathfrak{b}\\ &t\end{pmatrix}\quad\text{and}\quad b'=\begin{pmatrix}I_{n-1}&\mathfrak{b}'\\ &t'\end{pmatrix}.$$

Then (5-6) becomes

$$\begin{pmatrix}a_n&&&\\ &a_1&&\\ &&\ddots&\\ &&&a_{n-1}\end{pmatrix}\begin{pmatrix}I_{n-1}&\mathfrak{b}\\ &t\end{pmatrix}\begin{pmatrix}a_1&&&\\ &a_2&&\\ &&\ddots&\\ &&&a_n\end{pmatrix}^{-1}=\begin{pmatrix}I_{n-1}&\mathfrak{b}'\\ &t'\end{pmatrix}.$$

By comparing the determinants on both sides of the equality, we deduce that $t'=t$. Similarly, comparing the Levi components yields $a_1=a_2=\cdots=a_n$, leading to $\mathfrak{b}'=\mathfrak{b}$. Therefore, any two elements in $\widetilde{\mathcal{R}}$, are either equal or not conjugate to each other by $P(F)$. □

Now we consider for our purpose the decomposition of $Z(F)\backslash G(F)$ into $P(F)$-conjugacy classes. By (5-2) one has the decomposition

$$Z(F)\backslash G(F)=(Z(F)\backslash\mathfrak{R})\sqcup\mathfrak{S}, \tag{5-7}$$

where $\mathfrak{S}$ was defined in (2-6) in Section 2B.

Corollary 5.5. *Let notation be as before. Set $(F^\times)^n=\{t^n : t\in F^\times\}$, and let*

$$\widetilde{\mathcal{R}}^*=\Big\{\mathfrak{u}w_1w_2\cdots w_{n-1}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}: t\in F^\times/(F^\times)^n,\ \mathfrak{u}\in N_P(F)\Big\}. \tag{5-8}$$

Then $\widetilde{\mathcal{R}}^*$ forms a family of representatives of $Z(F)\backslash\mathfrak{R}$.

Proof. By Lemma 5.4, the set

$$\Big\{w_{n-1}w_{n-2}\cdots w_1\mathfrak{u}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}: t\in F^\times/(F^\times)^n,\ \mathfrak{u}\in N_P(F)\Big\} \tag{5-9}$$

forms a family of representatives of $Z(F)\backslash\mathfrak{R}$. Then the inverse of elements in the set defined in (5-9) also form a family of representatives of $Z(F)\backslash\mathfrak{R}$. Note that these inverses are bijectively $P_0(F)$-conjugate to $\widetilde{\mathcal{R}}^*$, then the proof follows. □

5B. *Holomorphic continuation.* In this section we shall prove Theorem D.

Recall the definition in Section 2C:

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\mathrm{K}_{\mathrm{Geo,Reg}}(ux,x)\,du f(x,s)\,dx,$$

where $\mathrm{Re}(s)>1$. By (5-7) and Corollary 5.5,

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x,y)=\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x)=\sum_{\gamma\in\widetilde{\mathcal{R}}^*}\sum_{p\in P_0(F)}\varphi(x^{-1}p^{-1}\gamma py).$$

As a consequence, we have (at least formally) the decomposition

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\sum_{\gamma\in\widetilde{\mathcal{R}}^*}\varphi(x^{-1}u^{-1}\gamma x)\,du f(x,s)\,dx.$$

5B1. *Ad hoc notation.* Recall that P is the parabolic subgroup of type $(n-1,1)$. Let M_P be the Levi component of P. Let $N^P:=N\cap M_P$; then $N=N^PN_P$. Set $A(\mathbb{A}_F)=Z(\mathbb{A}_F)\backslash T(\mathbb{A}_F)$ and $\tilde{w}=w_1w_2\cdots w_{n-1}$.

5B2. *Coordinate transforms of unipotent radicals.*

Lemma 5.6. *Let $c_{i,j}\in\mathbb{A}_F$, $1\le i<j\le n-1$. Define $u=(u_{i,j})_{1\le i,j\le n-1}\in N^P(\mathbb{A}_F)$ via the expression*

$$\begin{pmatrix}1&0&0&\dots&0\\&1&c_{1,2}&\cdots&c_{1,n-1}\\&&\ddots&&\vdots\\&&&1&c_{n-2,n-1}\\&&&&1\end{pmatrix}^{-1}\begin{pmatrix}1&c_{1,2}&\cdots&c_{1,n-1}&0\\&1&\cdots&c_{2,n-1}&0\\&&\ddots&\vdots&\vdots\\&&&1&0\\&&&&1\end{pmatrix}=\begin{pmatrix}u&*\\&1\end{pmatrix}. \tag{5-10}$$

Then each $1\le i<j<n$, $u_{i,j}=c_{i,j}+P_{i,j}$, where $P_{i,j}$ is a polynomial in the variables $c_{i',j'}\neq c_{i,j}$, $1\le i'<j'<n$.

Proof. The argument holds trivially for 2×2 matrices. Suppose Lemma 5.6 holds for matrices of rank less than n. Write

$$\begin{pmatrix}1 & c_{1,2} & \cdots & c_{1,n-1}\\ & \ddots & & \vdots\\ & & 1 & c_{n-2,n-1}\\ & & & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & c'_{1,2} & \cdots & c'_{1,n-1}\\ & \ddots & & \vdots\\ & & 1 & c'_{n-2,n-1}\\ & & & 1\end{pmatrix}.$$

By multiplying block matrices, we can determine $c'_{i,j}$ using $c_{i',j'}$ with $i'\le i$ and $j'\le j$. Based on the induction assumption, the $u_{i,j}$ take the form $c_{i,j}+P_{i,j}$ for $1\le i<n-1$ and $1\le j<n$. We then focus on investigating the last column, specifically $u_{i,n-1}$ for $1\le i<n-1$. Note that (5-10) becomes

$$\begin{pmatrix}1 & 0 & 0 & \dots & 0\\ & 1 & c'_{1,2} & \cdots & c'_{1,n-1}\\ & & \ddots & & \vdots\\ & & & 1 & c'_{n-2,n-1}\\ & & & & 1\end{pmatrix}\begin{pmatrix}1 & c_{1,2} & \cdots & c_{1,n-1} & 0\\ & 1 & \cdots & c_{2,n-1} & 0\\ & & \ddots & \vdots & \vdots\\ & & & 1 & 0\\ & & & & 1\end{pmatrix}=\begin{pmatrix}u & *\\ & 1\end{pmatrix}.$$

As a consequence, we obtain

$$u_{i,n-1}=c_{i,n-1}+c_{i+1,n-1}c'_{i-1,i}+\cdots+c_{n-2,n-1}c'_{i-1,n-3}+c'_{i-1,n-2},$$

which yields that $u_{i,n-1}-c_{i,n-1}$ is a polynomial in the variables $c_{i',j'}\neq c_{i,n-1}$. Hence, Lemma 5.6 follows from induction. □

Consider the smooth transformation defined by (5-10):

$$N^P(\mathbb{A}_F)\to N^P(\mathbb{A}_F),\quad (c_{i,j})_{1\le i,j<n}\mapsto (u_{i,j})_{1\le i,j<n}.$$

By Lemma 5.6, its Jacobian matrix is identically trivial. This will simplify the calculation of $I_{P,\mathrm{Reg}}(s,\tau)$ in the next subsection.

Set $c=(c_{i,j})_{1\le i,j<n}\in N^P(\mathbb{A}_F)$. Since $\begin{pmatrix}I_{n-1} & \\ & t\end{pmatrix}$ commutes with c, the relation (5-10) amounts to

$$c^{-1}\tilde{w}\begin{pmatrix}I_{n-1} & \\ & t\end{pmatrix}c=\tilde{w}\begin{pmatrix}u & *\\ & t\end{pmatrix},\quad \tilde{w}=w_1w_2\cdots w_{n-1}. \tag{5-11}$$

5B3. *Manipulation of $I_{P,\mathrm{Reg}}(s,\tau)$: integral transformations.* We have

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\sum_t\varphi\Bigl(x^{-1}u_1^{-1}\tilde{w}\begin{pmatrix}I_{n-1} & \\ & t\end{pmatrix}x\Bigr)\,du_1\,f(x,s)\,dx,$$

where $t\in F^\times/(F^\times)^n$. By Iwasawa decomposition, we may write $x=cu_2\mathrm{t}k$, where $c\in N^P(\mathbb{A}_F)$, $u_2\in N_P(\mathbb{A}_F)$, $\mathrm{t}=\mathrm{diag}(t_1,t_2,\dots,t_{n-1},1)\in A(\mathbb{A}_F)$, $k\in K$. Then

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K\int_{A(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\mathcal{I}\cdot f(cu_2\mathrm{t}k,s)\,du_2\,du_1\,\delta^{-1}(\mathrm{t})\,d^\times\mathrm{t}\,dk,$$

where δ is the modular character, and

$$\mathcal{I}:=\int_{N^P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\varphi\Bigl(k^{-1}\mathrm{t}^{-1}u_1^{-1}c^{-1}\tilde{w}\begin{pmatrix}I_{n-1} & \\ & t\end{pmatrix}cu_2\mathrm{t}k\Bigr)\,dc.$$

We can change variables through Lemma 5.6 or equation (5-11) to derive

$$\mathcal{I}=\int_{N^P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\varphi(k^{-1}\mathfrak{t}^{-1}u_1^{-1}\tilde{w}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}uu_2\mathfrak{t}k)\,du.$$

Recall that $f(x,s)$ is defined by

$$f(x,s)=\tau(\det x)\,|\det x|_{\mathbb{A}_F}^s\int_{\mathbb{A}_F^\times}\Phi[(0,\dots,t)x]\tau^n(t)|t|^{ns}d^\times t, \tag{5-12}$$

which is a Tate integral for the complete L-function $\Lambda(ns,\tau^n)$. By definition, $f(cu_2\mathfrak{t}k,s)$ equals $\tau(\det\mathfrak{t})\,|\det\mathfrak{t}|^s f(k,s)$. Therefore,

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K f(k,s)\,dk\int_{A(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\frac{\mathcal{I}\cdot\tau(\det\mathfrak{t})\,|\det\mathfrak{t}|^s}{\delta(\mathfrak{t})}du_2\,du_1\,d^\times\mathfrak{t}.$$

Lemma 5.7. *Let notation be as before. Then*

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K f(k,s)\,dk\int_{N_P(\mathbb{A}_F)}du_1\int_{N^P(\mathbb{A}_F)}du\int_{N_P(\mathbb{A}_F)}du_2\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\Delta_{s,\tau}^{(1)}(\mathfrak{t})$$
$$\times\int_{\mathbb{A}_F^\times}\sum_{t\in F^\times/(F^\times)^n}\varphi\left(k^{-1}u_1\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1^n t\prod_{i=2}^{n-1}t_i\end{pmatrix}\tilde{w}u_2uk\right)d^\times t,$$

where $d^\times\mathfrak{t}=d^\times t_1\,d^\times t_2\cdots d^\times t_{n-1}$, *and for any* $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\dots,t_{n-1},1)\in A(\mathbb{A}_F)$,

$$\Delta_{s,\tau}^{(1)}(\mathfrak{t})=\tau(t_1)^{\frac{n(n-1)}{2}}|t_1|_{\mathbb{A}_F}^{\frac{n(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\tau(t_i)^{n-i}|t_i|_{\mathbb{A}_F}^{(n-i)(s+1)}.$$

Proof. By the change of variables $u_1\mapsto\mathfrak{t}u_1\mathfrak{t}^{-1}$ and $uu_2\mapsto\mathfrak{t}uu_2\mathfrak{t}^{-1}$ we obtain

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K f(k,s)\,dk\int_{A(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\mathcal{J}\cdot\tau(\det\mathfrak{t})\,|\det\mathfrak{t}|^{s+1}du_2\,du_1\,d^\times\mathfrak{t},$$

where

$$\mathcal{J}=\int_{N^P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\varphi(k^{-1}u_1^{-1}\mathfrak{t}^{-1}\tilde{w}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}\mathfrak{t}uu_2k)\,du.$$

Since

$$\mathfrak{t}^{-1}\tilde{w}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}\mathfrak{t}\tilde{w}^{-1}=\begin{pmatrix}tt_1^{-1}&&&&\\&t_1t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-2}t_{n-1}^{-1}&\\&&&&t_{n-1}\end{pmatrix},$$

Lemma 5.7 follows after a change of variables. ☐

5B4. *Manipulation of $I_{P,\mathrm{Reg}}(s,\tau)$: reducing to intertwining operators.* Recall the test function φ has the central character ω. Let Ξ be the set of idele class characters on $\mathbb{A}_F^\times$, which is trivial on $\mathbb{R}_+^\times$. Denote by $\Xi_{\omega,n}$ the subset $\{\chi\in\Xi:\chi^n=\omega\}\subset\Xi$. Also, let $\Xi_{\tau,2}^n=\{\xi\in\Xi:\xi^2=\tau\}$ if n is even, and let $\Xi_{\tau,2}^n=\{\mathbf{1}\}$, the singleton, if n is odd. Then both $\#\Xi_{\omega,n}<\infty$ and $\#\Xi^n_{\tau,2}<\infty$. For a Hecke character χ, we define

$$\Delta^{\mathrm{od}}_{s,\tau,\chi}(\mathfrak{t})=\bar{\chi}(t_1)\tau(t_1)^{\frac{n-1}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\tau(t_i^{\frac{n+1}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}$$

when n is odd. For even n, we define

$$\Delta^{\mathrm{en}}_{s,\tau,\chi,\xi}(\mathfrak{t})=\bar{\chi}(t_1)\xi(t_1)\tau(t_1)^{\frac{n-2}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\xi(t_i)\tau(t_i^{\frac{n}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

We can employ a change of variables, specifically $(\mathbb{A}_F^\times)^n\cdot F^\times/(F^\times)^n=F^\times\cdot(F^\times\backslash\mathbb{A}_F^\times)^n$, to perform Poisson summation, following a similar approach as in [18, §2.4, Lemma]. This allows us to derive the following results:

- When n is odd, the integral $I_{P,\mathrm{Reg}}(s,\tau)$ is equal to

$$\int_K f(k,s)\,dk\int_{N_P(\mathbb{A}_F)}du_1\int_{N^P(\mathbb{A}_F)}du\int_{N_P(\mathbb{A}_F)}du_2\int_{\mathbb{A}_F^\times}\sum_{\chi\in\Xi_{\omega,n}}\Delta^{\mathrm{od}}_{s,\tau,\chi}(\mathfrak{t})\,d^\times t_1$$
$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}u_1\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\tilde{w}u_2uk\right)d^\times t_2\cdots d^\times t_{n-1}.$$

- When n is even, the integral $I_{P,\mathrm{Reg}}(s,\tau)$ is equal to

$$\int_K f(k,s)\,dk\int_{N_P(\mathbb{A}_F)}du_1\int_{N^P(\mathbb{A}_F)}du\int_{N_P(\mathbb{A}_F)}du_2\sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi^n_{\tau,2}}\Delta^{\mathrm{en}}_{s,\tau,\chi,\xi}(\mathfrak{t})$$
$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}u_1\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\tilde{w}u_2uk\right)d^\times t_1\cdots d^\times t_{n-1}.$$

Let $T_*(\mathbb{A}_F^\times)=\{\mathrm{diag}(1,t_1,t_2,\ldots,t_{n-1})\in T(\mathbb{A}_F):t_i\in\mathbb{A}_F^\times,\ 1\le i\le n-1\}$. Set

$$\iota:A(\mathbb{A}_F^\times)\to T_*(\mathbb{A}_F^\times),\quad \mathfrak{t}\mapsto\mathfrak{t}^\iota=\mathrm{diag}(1,t_2^{-1},t_3^{-1},\ldots,t_{n-1}^{-1},t_1).$$

Let $\delta_n=-\frac{1}{2}(1+(-1)^n)$. Define $\mathfrak{F}_{\chi,\xi}(x;k,s)=\mathfrak{F}_{\chi,\xi}(x;k,s,\varphi,\Phi,\tau)$ by

$$\mathfrak{F}_{\chi,\xi}(x;k,s)=\int_{N_P(\mathbb{A}_F)}\int_{N^P(\mathbb{A}_F)}\int_{A(\mathbb{A}_F^\times)}\varphi(k^{-1}u_1\mathfrak{t}^\iota xuk)\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})\,d^\times\mathfrak{t}\,du\,du_1,$$

where $\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})$ is defined by

$$\bar{\chi}(t_1)\xi(t_1)^{-\delta_n}\tau(t_1)^{\frac{n-1-\delta_n}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

Since φ has compact support modulo the center, $\mathfrak{F}_{\chi,\xi}(x;k,s)$ is well defined for any χ,ξ in $\mathrm{Re}(s)>1$.

Let $b=\mathfrak{t}\mathfrak{u}'\mathfrak{u}\in B(\mathbb{A}_F)$, where $\mathfrak{u}'\in N_P(\mathbb{A}_F)$, $\mathfrak{u}\in N^P(\mathbb{A}_F)$, $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\ldots,t_n)\in T(\mathbb{A}_F)$. By the change of variables $u\mapsto\mathfrak{u}^{-1}u$, $u_1\mapsto u_1\mathfrak{t}^\iota\mathfrak{t}\mathfrak{u}'^{-1}(\mathfrak{t}^\iota\mathfrak{t})^{-1}$, we obtain

$$\mathfrak{F}_{\chi,\xi}(bx;k,s)=\prod_{i=1}^{n}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}\cdot\mathfrak{F}_{\chi,\xi}(x;k,s). \tag{5-13}$$

Since the modular character of $T(\mathbb{A}_F)$ is $\delta(\mathrm{t})=\prod_{i=1}^{n}t_i^{n+1-2i}$, one has

$$\mathfrak{F}_{\chi,\xi}(x;k,s)\in\mathrm{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\big(\chi\xi^{\delta_n}\tau^{\lambda_1}|\cdot|_{\mathbb{A}_F}^{\lambda_1 s},\dots,\chi\xi^{\delta_n}\tau^{\lambda_{n-1}}|\cdot|_{\mathbb{A}_F}^{\lambda_{n-1}s},\chi\xi^{\delta_n}\tau^{\lambda_n}|\cdot|_{\mathbb{A}_F}^{\lambda_n s}\big),$$

where $\lambda_i=\frac{1}{2}(n+1-\delta_n)-i$ for $1\le i\le n$. Define

$$G_{\chi,\xi}(x;s)=G_{\chi,\xi}(x;s,\varphi,\Phi,\tau)=\int_K f(k,s)\mathfrak{F}_{\chi,\xi}(x;k,s)\,dk.$$

Therefore, at least formally one can write $I_{P,\mathrm{Reg}}(s,\tau)$ as a finite sum:

$$I_{P,\mathrm{Reg}}(s,\tau)=\sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi_{\tau,2}^{n}}\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\tilde{w}n;s)\,dn,\quad \mathrm{Re}(s)>1. \tag{5-14}$$

5B5. *Proof of Theorem D.* Now we show the absolute convergence of (5-14) and deduce its analytic behavior.

Let $\mathfrak{F}_{1,1,+}(x;k,s)=\mathfrak{F}_{1,1}(x;k,s,|\varphi|,|\Phi|,1)$ and $G_{1,1,+}(x;s)=G_{1,1}(x;s,|\varphi|,|\Phi|,1)$. Then the above interchange in the order of integrals is justified by Fubini's theorem on integrals of nonnegative functions. One then has

$$I_{P,\mathrm{Reg}}^{+}(s,\tau)=\sum_{\chi\in\Xi_{1,n}}\sum_{\xi\in\Xi_{1,2}^{n}}\int_{N_P(\mathbb{A}_F)}G_{1,1,+}(\tilde{w}n;s)\,dn,$$

where the sums are finite. Then $\int_{N_P(\mathbb{A}_F)}G_{1,1,+}(\tilde{w}n;s)\,dn$ converges absolutely in $\mathrm{Re}(s)>1$ according to Langlands' theory on intertwining operators. Therefore, by the dominated convergence theorem, $\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\tilde{w}n;s)\,dn$ converges absolutely in $\mathrm{Re}(s)>1$. It is thus a well defined intertwining operator. By Langlands' theory (cf. [20] or [30]) on intertwining operators, we have

$$\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\tilde{w}n;s)\,dn\propto\frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})},$$

where the last factor $\Lambda(ns,\tau^n)$ on the numerator comes from the Tate integral $f(k,s)$ from (5-12).

So (5-14) is well defined. Consequently (5-1) holds, since the sums in (5-14) are finite.

6. Convergence of the spectral side

In this section we shall deal with the spectral side

$$I_{\mathrm{Whi}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\mathrm{K}_{\mathrm{Eis}}(n_1x,x)\theta(n_1)\,dn_1\,f(x,s)\,dx, \tag{6-1}$$

where $\mathrm{K}_{\mathrm{Eis}}(x,y)$ is the Eisenstein part of the kernel function relative to a general test function φ in $\mathcal{H}(G(\mathbb{A}_F),\omega)$. The main concern here is the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ when $\mathrm{Re}(s)$ is large.

Typically one needs certain suitable regularization or truncation for $\mathrm{K}_{\mathrm{Eis}}$, which is slowly increasing.

In the GL(2) case this can be handled by the techniques in [29] or [40]. Arthur [1; 2; 3; 4] developed a truncation approach to successfully regularize the trace formula on general reductive groups. Arthur's truncation operators and their variants (as in [21; 13]) provide a powerful toolkit to manipulate the convergence problem in the (relative) trace formula.

However, these truncation operators seem to be not quite suitable for the function (6-1). One barrier is that the domain is not the usual automorphic quotient $Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$ but the much larger region $Z(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)$; thus the kernel in (6-1) is not $G(F)$-invariant now, which makes the usual truncation operators not work well here. One can do the spectral expansion of $\mathrm{K}_{\mathrm{Eis}}$ and apply Arthur's truncation Λ^T to the second Eisenstein series and show it can be integrated over a Siegel domain. With further covering process by Weyl elements conjugation, one can show (6-1) converges absolutely with $\mathrm{K}_{\mathrm{Eis}}(x, y)$ replaced by $\Lambda_2^T \mathrm{K}_{\mathrm{Eis}}(x, y)$, where Λ_2^T means the operator Λ^T is applied to the y-variable. See Section 5.4 in [35] for details. Nevertheless, taking Fourier coefficients in the first variable makes the geometric truncation difficult to control, since it is just $N(F)$-invariant. So it is not clear how to compute the spectrally truncated function as a polynomial of the parameter T and ultimately show that this polynomial is indeed a constant. (Here the letter T is a conventional notation for the truncation parameter, while elsewhere we use T to denote the torus.) For an individual cuspidal datum, one may develop an allied truncation operator as in [13], but the problem is to show that the sum over *all* cuspidal data is convergent.

We will propose an alternative way to verify the convergence of (6-1), making essential use of the Fourier transform. Our strategy is to reduce (6-1) to a Mellin transform of the Kuznetsov relative trace formula, which is majorized by a gauge. Then one obtains convergence of (6-1) when $\mathrm{Re}(s)$ is large enough.

Inserting the spectral expansion (6-7) of $\mathrm{K}_{\mathrm{ER}}(x, y)$ into (6-1), $I_{\mathrm{Whi}}(s, \tau)$ becomes

$$\int \sum_{\chi} \sum_{Q} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \sum_{\phi_1} \sum_{\phi_2} \langle \mathcal{I}_Q(\lambda, \varphi)\phi_2, \phi_1 \rangle W_1(x; \lambda) \overline{W_2(x; \lambda)}\, d\lambda\, f(x, s)\, dx. \tag{6-2}$$

Here, x ranges through $Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, χ ranges through cuspidal data, Q represents proper standard parabolic subgroups, and W_j denotes the Whittaker functions. Additional details can be found in (6-10) below. The absolute convergence of (6-2) is summarized in Theorem E at the end of this section.

6A. *Reduction to the relative trace formula of Kuznetsov type.*

Lemma 6.1. *Let* $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$. *Let* $\mathrm{K} = \mathrm{K}^{\varphi}$, $\mathrm{K}_0 = \mathrm{K}_0^{\varphi}$ *and* $\mathrm{K}_{\mathrm{Eis}} = \mathrm{K}_{\mathrm{Eis}}^{\varphi}$ *be the corresponding kernel functions. Then*

$$\mathrm{K}_0(x, y) = \sum_{\delta \in N(F)\backslash P_0(F)} \int_{[N]} \big(\mathrm{K}(n\delta x, \delta x) - \mathrm{K}_{\mathrm{Eis}}(n\delta x, \delta x)\big)\, \overline{\theta(n)}\, dn. \tag{6-3}$$

Proof. By the spectral decomposition (2-5) of $\mathrm{K}_0(x, y)$ we see it is cuspidal as a function of x. Applying Proposition 3.1 to the first variable of $\mathrm{K}_0(x, y)$ and take $y = x$ we then obtain

$$\mathrm{K}_0(x,y)=\sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\mathrm{K}_0(n\delta x,x)\,\overline{\theta(n)}\,dn.$$

Then (6-3) follows from the spectral decomposition $\mathrm{K}_0(x,y)=\mathrm{K}(x,y)-\mathrm{K}_{\mathrm{ER}}(x,y)$ and the automorphy of these functions relative to the second variable. Here we also note that the residual spectrum does not contribute to (6-3). □

Let $\mathrm{Re}(s)>1$ in this section. We then plug Lemma 6.1 into

$$I_0(s,\tau)=\int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)E(x,s)\,dx$$

and unfold the Eisenstein series $E(x,s)$ to obtain

$$I_0(s,\tau)=I_{\mathrm{Kl}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau),$$

where

$$I_{\mathrm{Kl}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\int_{[N]}\mathrm{K}(nx,x)\,\overline{\theta(n)}\,dn\,f(x,s)\,dx. \tag{6-4}$$

To establish the well-definedness of $I_{\mathrm{Whi}}(s,\tau)$, it is sufficient to demonstrate the convergence of $I_{\mathrm{Kl}}(s,\tau)$, given the rapid decay of K_0. We aim to show that $I_{\mathrm{Kl}}(s,\tau)$ converges for all $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$ and for $\mathrm{Re}(s)$ sufficiently large. By utilizing Cauchy's inequality and the convolution decomposition of φ, we obtain the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$.

Through a change of variables, we obtain the expression

$$I_{\mathrm{Kl}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}J_{\mathrm{Kuz}}(\varphi,x)f(x,s)\,dx,$$

where

$$J_{\mathrm{Kuz}}(\varphi,x)=\int_{[N]}\int_{[N]}\mathrm{K}(n_1x,n_2x)\theta(n_1)\,\bar{\theta}(n_2)\,dn_1dn_2$$

represents a relative trace formula of Kuznetsov type. Consequently, $I_{\mathrm{Kl}}(s,\tau)$ can be considered as the Mellin transform of $J_{\mathrm{Kuz}}(\varphi,x)$ since $f(x,s)$ essentially corresponds to $|\det x|^s$. To establish the convergence of $I_{\mathrm{Kl}}(s,\tau)$, we will demonstrate that $J_{\mathrm{Kuz}}(\varphi,x)$ is dominated by a gauge when $\mathrm{Re}(s)$ is sufficiently large.

Recall that, for $x=\mathrm{diag}(x_1\cdots x_{n-1},\ldots,x_1x_2,x_1,1)\in A(\mathbb{A}_F)=Z(\mathbb{A}_F)\backslash T(\mathbb{A}_F)$, a gauge $\mathcal{G}$ is a positive function of the form

$$\mathcal{G}(x)=\xi(x_1,x_2,\ldots,x_{n-1})\cdot|x_1x_2\cdots x_{n-1}|^{-M},$$

where $M\geq 0$ and ξ is a Schwartz–Bruhat function on $(\mathbb{A}_F^\times)^{n-1}$.

Proposition 6.2. *Let notation be as above. Then as a function of $x\in A(\mathbb{A}_F)$, $J_{\mathrm{Kuz}}(\varphi,x)$ is majorized by a finite sum of gauges on $A(\mathbb{A}_F)$.*

Proof. By the definition of the kernel function $\mathrm{K}(x,y)$ we have

$$J_{\mathrm{Kuz}}(\varphi,x)=\int_{[N]}\int_{[N]}\sum_{\gamma\in Z(F)\backslash G(F)}\varphi(x^{-1}n_1^{-1}\gamma n_2x)\,\theta(n_1)\,\bar{\theta}(n_2)\,dn_1dn_2,$$

which converges absolutely since $\mathrm{K}(x, y)$ is continuous and $[N]$ is compact.

Consider the double coset $Z(\mathbb{A}_F)N(F)\backslash G(F)/N(F)$, whose element is of the form wa, where w is a Weyl element and $a \in Z(F)\backslash T(F)$. Let

$$H_{wa} := \{(n_1, n_2) \in N \times N : n_1^{-1} wan_2 a^{-1} w^{-1} \in Z\}$$

be the stabilizer relative to the representative wa. Then

$$J_{\mathrm{Kuz}}(\varphi, x) = \sum_{wa \in \Phi} \int_{H_{wa}(F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1} n_1^{-1} wan_2 x)\theta(n_1)\overline{\theta}(n_2)\, dn_1 dn_2,$$

where Φ is a set of complete representatives for $Z(\mathbb{A}_F)N(F)\backslash G(F)/N(F)$. Then

$$J_{\mathrm{Kuz}}(\varphi, x) = \sum_{wa \in \Phi} C_{wa} \int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1} n_1^{-1} wan_2 x)\theta(n_1)\overline{\theta}(n_2)\, dn_1 dn_2,$$

where

$$C_{wa} = \int_{[H_{wa}]} \theta(n_1')\overline{\theta}(n_2')\, dn_1'\, dn_2'.$$

Call $wa \in \Phi$ *relevant* if $C_{wa} \neq 0$, i.e., $\theta(n_1')\overline{\theta}(n_2')$ is trivial on $H_{wa}(\mathbb{A}_F)$. Denote by Φ^* the set of relevant elements in Φ. By [16, p. 272, Proposition 1] one can take the following realization: Φ^* consists of wa, where w is the long Weyl element inside a standard parabolic subgroup $Q \subseteq G$ of type $(k_1, \dots, k_r)$, and $a \in Z(F)\backslash \operatorname{diag}(T_{k_1}(F), \dots, T_{k_r}(F))$ (modulo some further relations), with T_{k_j} being the maximal split torus of $\mathrm{GL}(k_j)$. For example, when $Q = B$ the Borel, then $w = I_n$ and $a = I_n$ and $H_{wa} = N$. Therefore,

$$J_{\mathrm{Kuz}}(\varphi, x) = \sum_{wa \in \Phi^*} \operatorname{vol}([H_{wa}]) J_{\mathrm{Kuz}}(\varphi, x; wa),$$

where

$$J_{\mathrm{Kuz}}(\varphi, x; wa) = \int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1} n_1^{-1} wan_2 x)\theta(n_1)\overline{\theta}(n_2)\, dn_1 dn_2.$$

By the definition of Φ^*, each w corresponds to a unique (i.e., the minimal one) parabolic subgroup Q containing w. Suppose $w \neq I_n$. Then by Levi decomposition it suffices to consider the extreme case where $Q = G$ and w is the longest.

Recall that the test function φ is K-finite. Hence there is some compact subgroup $K_0 \subset G(\mathbb{A}_{F,\mathrm{fin}})$ such that φ is right K_0-invariant. Let $K_0 = \prod_{v<\infty} K_{0,v}$. Note that $J_{\mathrm{Kuz}}(\varphi, x; wa) = \prod_{v\leq\infty} J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$, where

$$J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) = \int_{H_{wa}(F_v)\backslash N(F_v)\times N(F_v)} \varphi_v(x_v^{-1} n_1^{-1} wan_2 x)\theta_v(n_1)\overline{\theta}_v(n_2)\, dn_1 dn_2.$$

Then for each finite place v, $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$ is right $K_{0,v}$-invariant. So there exists a compact subgroup $N_{0,v} \subseteq K_{0,v} \cap N(F_v)$, depending only on φ_v, such that

$$J_{\mathrm{Kuz},v}(\varphi_v, x_v u_v; wa) = J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) \quad \text{for all } x_v \in A(F_v) \text{ and } u_v \in N_{0,v}.$$

On the other hand, $J_{\mathrm{Kuz},v}(\varphi_v, x_v u_v; wa) = \theta(x_v u_v x_v^{-1}) J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ such that $\theta(x_v u_v x_v^{-1}) = 1$ if and only if $|\alpha_i(x_v)|_v \le C_v$, where α_i's are the simple roots of $G(F)$ relative to B. Note that for all but finitely many $v < \infty$, $K_{0,v} = G(\mathcal{O}_{F,v})$. Thus we can take the corresponding $C_v = 1$. Hence for any $x_v \in A(F_v)$, $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) \neq 0$ implies that $|\alpha_i(x_v)|_v \le C_v$, $1 \le i \le n-1$, and $C_v = 1$ for all but finitely many finite places v. Denote the compact set by

$$A_{\varphi,\mathrm{fin}} = \bigl\{a = (a_v) \in A(\mathbb{A}_{F,\mathrm{fin}}) : |\alpha_i(a_v)|_v \le C_v,\ 1 \le i \le n-1\bigr\}.$$

Then $\operatorname{supp} J_{\mathrm{Kuz}}(\varphi, x; wa) \subseteq A(\mathbb{A}_{F,\infty}) A_{\varphi,\mathrm{fin}}$.

For each place v, we fix a conventional local height function $\|\cdot\|_v$ on $G(F_v)$. Let $y = \bigotimes_v (y_{i,j,v}) \in G(\mathbb{A}_F)$. Then $\|y_v\|_v = 1$ for almost all v. The height function $\|y\| = \prod_v \|y_v\|_v$ is therefore well defined by a finite product. Also, since

$$\operatorname{supp} J_{\mathrm{Kuz}}(\varphi, x; wa) \subseteq A(\mathbb{A}_{F,\infty}) A_{\varphi,\mathrm{fin}}$$

and by the compactness of $\operatorname{supp} \varphi_v$, we have $\|w^{-1} x_v w x_v a\|_v \le C_v'$ for some constant C_v' depending only on φ_v, $v < \infty$, and $C_v' = 1$ for almost all v's.

Now we investigate the archimedean $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$, i.e., $v \mid \infty$. Note that φ_v is a compactly supported on $Z(F_v) \backslash G(F_v)$. Then $J_{\mathrm{Kuz}}(\varphi_v, x_v; wa) = 0$ unless $n_{1,v}^{-1} y_v w n_{2,v} \in \operatorname{supp} \varphi_v$, where $y_v = x_v^{-1} w a x_v w^{-1}$. Hence $\|n_{1,v}^{-1} y_v w n_{2,v} w^{-1}\|_v \le C_v$ for some constant C_v depending only on φ. A straightforward computation (or Lemma 5.1 of [15]) shows that $\|n_{1,v}\|_v + \|n_{2,v}\|_v + \|y_v\|_v \le C_v'$ for some constant C_v' depending only on φ. So $\varphi_v(n_{1,v} y_v w n_{2,v})$ has compact support relative to $n_{1,v}$ and $n_{2,v}$. Therefore, $J_{\mathrm{Kuz}}(\varphi_v, x_v; wa) = 0$ unless $n_{1,v}$, $n_{2,v}$ run through a compact set of $N(F_v)$ and $\|y\|_v$ is bounded.

For $x = \operatorname{diag}(x_1, \dots, x_{n-1}, 1) \in A(\mathbb{A}_F)$, similarly to (3-1), we define an additive character

$$\psi_x(u) = \prod_{i=1}^{n-1} \psi_F(x_i u_{i,i+1}) \qquad \text{for } u = (u_{i,j})_{n \times n} \in N(\mathbb{A}_F).$$

Then $J_{\mathrm{Kuz}}(\varphi, x; wa)$ is equal to

$$\delta_w(x)^2 \int_{H_{wa}(\mathbb{A}_F) \backslash N(\mathbb{A}_F) \times N(\mathbb{A}_F)} \varphi(n_1^{-1} x^{-1} w a x n_2) \psi_x(n_1) \overline{\psi}_x(n_2)\, dn_1\, dn_2,$$

where δ_w is the modular character of the parabolic subgroup associated to w.

Since n_1 and n_2 lie in a compact set determined by $\operatorname{supp} \varphi$, then for a fixed $y \in A(\mathbb{A}_F)$, $v \mid \infty$, the v-th component of

$$\int_{H_{wa}(\mathbb{A}_F) \backslash N(\mathbb{A}_F) \times N(\mathbb{A}_F)} \varphi(n_1^{-1} y w n_2) \psi_x(n_1) \overline{\psi}_x(n_2)\, dn_1 dn_2$$

is a Schwartz function of x, being the Fourier transform of a compactly supported smooth function. Hence, $J_{\mathrm{Kuz}}(\varphi, x; wa)$ is majorized by a nonnegative Schwartz–Bruhat function $\Xi(x^{-1} w a x w^{-1}, x)$ on $A(\mathbb{A}_F)^2$ with

$$x^{-1} w a x w^{-1} \in A^* := \bigl\{b \in A(F) :\ \|b\| \le \prod_v C_v'\bigr\}.$$

By properties of the height $\|\cdot\|$ (see [5, p. 70], for instance), one has

$$\#(w^{-1}x\cdot A^{*}\cdot wx^{-1})\leq C\cdot\big(|x_1\cdots x_{n-1}|^{M}+|x_1\cdots x_{n-1}|^{-M}\big),$$

for some constants C and M depending on $\operatorname{supp}\varphi$. Therefore,

$$\sum_{a\in A(F)}|J_{\mathrm{Kuz}}(\varphi,x;wa)|\leq\sum_{a\in A^{*}}\Xi(w^{-1}xwxa,x)=\sum_{a\in w^{-1}x\cdot A^{*}\cdot wx^{-1}}\Xi(a,x),$$

which is majorized by $|x_1\cdots x_{n-1}|^{-M}\cdot\xi(x_1,\dots,x_{n-1})$ for some $M\geq 0$ and Schwartz–Bruhat function ξ.

The remaining case is that of $w=I_n$, i.e., $Q=B$. In this case

$$J_{\mathrm{Kuz}}(\varphi,x;wa)=\delta_w(x)\int_{N(\mathbb{A}_F)}\varphi(an)\overline{\psi}_x(n)\,dn$$

is the Fourier transform of a Schwartz–Bruhat function. So it is majorized by a gauge. Then Proposition 6.2 follows. □

As a consequence of Proposition 6.2 and the Iwasawa decomposition, $I_{\mathrm{Kl}}(s,\tau)$ converges absolutely when $\mathrm{Re}(s)$ is large enough. Therefore, $I_{\mathrm{Whi}}(s,\tau)$ converges when $\mathrm{Re}(s)$ is large enough.

To show the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ and thus to obtain meromorphic continuation, we need to analyze properties of $\mathrm{K}_{\mathrm{Eis}}$ by its spectral expansion.

6B. ***Spectral decomposition of the kernel function.*** In this subsection, we provide a concise overview of the spectral theory concerning automorphic representations of reductive groups. Subsequently, we apply these results to the noncuspidal kernel function K_{ER}. *The notation introduced in this section will be regularly employed in later discussions.*

Denote by H a general reductive group and Q a standard parabolic subgroup of H. Let M_Q (resp. N_Q) be the Levi component (resp. unipotent radical) of Q. Let $H^1(\mathbb{A}_F)=\{g\in H(\mathbb{A}_F):|\lambda(g)|_{\mathbb{A}_F}=1,\ \forall\lambda\in X(H)_F\}$, where $X(H)_F$ is space set of F-rational characters of H. Let $\mathfrak{a}_H=\mathrm{Hom}_{\mathbb{Z}}(X(H)_F,\mathbb{R})$. Let $\mathfrak{a}_H^{*}=X(H)_F\otimes\mathbb{R}$. Set $\mathfrak{a}_Q=\mathfrak{a}_{M_Q}$ and $\mathfrak{a}_Q^{*}=\mathfrak{a}_{M_Q}^{*}$. Let P_0 be a fixed minimal parabolic subgroup of H over F. Write $\mathfrak{a}_0$ (resp. $\mathfrak{a}_0^{*}$) for $\mathfrak{a}_{P_0}$ (resp. $\mathfrak{a}_{Q_0}^{*}$). These notations concur with those used by Arthur (see, e.g., [5, pp. 20–31].

For a standard parabolic subgroup Q of $G=\mathrm{GL}(n)$ with type $(n_1,\dots,n_r)$, we define the surjective homomorphism $\log_Q$ from $M_Q(\mathbb{A}_F)$ to $\mathfrak{a}_Q$ as follows:

$$\log_Q(q)=\log_Q(m)=(n_1^{-1}\log|\det m_1|,\dots,n_r^{-1}\log|\det m_l|),\tag{6-5}$$

where $q\in Q(\mathbb{A}_F)$ and $m=\mathrm{diag}(m_1,\dots,m_r)$ represents the Levi component, with $m_i\in\mathrm{GL}(n_i)/\mathbb{A}_F$ for $1\leq i\leq r$.

By spectral theory (see, for example, [2, pp. 256 and 263], or [5, §12]), the decomposition of the Hilbert space $L^2\big(Z_H(\mathbb{A}_F)N_Q(\mathbb{A}_F)M_Q(F)\backslash H(\mathbb{A}_F)\big)$ into right $H(\mathbb{A}_F)$-invariant subspaces is determined by the spectral data $\chi=\{(M,\sigma)\}$, where M is the Levi component of $P_1\cap M_Q$ for some standard parabolic subgroup P_1 of H, and σ is an element of $\mathcal{A}_0\big(Z_H(\mathbb{A}_F)M^1(F)\backslash M^1(\mathbb{A}_F)\big)$, the set of cuspidal automorphic representations of $Z_H(\mathbb{A}_F)M^1(F)\backslash M^1(\mathbb{A}_F)$. Here M^1 is defined in a similar way to H^1.

The class (M,σ) derives from the equivalence relation $(M,\sigma)\sim(M',\sigma')$ if and only if M is conjugate to M' by a Weyl group element w, and $\sigma'=\sigma^w$ on $Z_H(\mathbb{A}_F)\backslash M^1(\mathbb{A}_F)$. Let $\mathfrak{X}$ be the set of equivalence classes $\chi=\{(M,\sigma)\}$ of these pairs, we thus have

$$L^2(P):=L^2\big(Z_H(\mathbb{A}_F)N_Q(\mathbb{A}_F)M_Q(F)\backslash H(\mathbb{A}_F)\big)=\bigoplus_{\chi\in\mathfrak{X}}L^2(P)_\chi, \tag{6-6}$$

where $L^2(P)_\chi$ consists of functions $\phi\in L^2\big(Z_H(\mathbb{A}_F)N_Q(\mathbb{A}_F)M_Q(F)\backslash H(\mathbb{A}_F)\big)$ such that, for each standard parabolic subgroup Q' of H with $Q'\subset Q$, and almost all $x\in H(\mathbb{A}_F)$, the projection of the function

$$m\mapsto x.\phi_{Q'}(m)=\int_{N_{Q'}(F)\backslash N_{Q'}(\mathbb{A}_F)}\phi(nmx)\,dn$$

onto the space of cusp forms in $L^2\big(Z_H(\mathbb{A}_F)M_{Q'}(F)\backslash M^1_{Q'}(\mathbb{A}_F)\big)$ transforms under $M^1_{Q'}(\mathbb{A}_F)$ as a sum of representations σ, in which $(M_{Q'},\sigma)\in\chi$. If there is no such pair in χ, $x.\phi_{Q'}$ will be orthogonal to $\mathcal{A}_0\big(Z_H(\mathbb{A}_F)M_{Q'}(F)\backslash M^1_{Q'}(\mathbb{A}_F)\big)$. Denote by $\mathcal{H}_Q$ the space of such ϕ's. Let $\mathcal{H}_{Q,\chi}$ be the subspace of $\mathcal{H}_Q$ such that for any $(M,\sigma)\notin\chi$, with $M=M_{Q_1}$ and $Q_1\subset Q$, we have

$$\int_{M(F)\backslash M(\mathbb{A}_F)^1}\int_{N_{Q_1}(F)\backslash N_{Q_1}(\mathbb{A}_F)}\psi_0(m)\phi(nmx)\,dn=0,$$

for any $\psi_0\in L^2_0\left(M(F)\backslash M(\mathbb{A}_F)^1\right)_\sigma$, and almost all x. This leads us to Langlands' result to decompose $\mathcal{H}_Q$ as an orthogonal direst sum $\mathcal{H}_Q=\bigoplus_{\chi\in\mathfrak{X}}\mathcal{H}_{Q,\chi}$. Let $\mathcal{B}_Q$ be an orthonormal basis of $\mathcal{H}_Q$, then we can choose $\mathcal{B}_Q=\bigcup_{\chi\in\mathfrak{X}}\mathcal{B}_{Q,\chi}$, where $\mathcal{B}_{Q,\chi}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{Q,\chi}$. We may assume that vectors in each $\mathcal{B}_{Q,\chi}$ are K-finite and are pure tensors.

By spectral theory, one can expand $\mathrm{K}_{\mathrm{Eis}}(x,y)$ as

$$\sum_{\chi\in\mathfrak{X}}\sum_{Q\in\mathcal{Q}}\frac{1}{k_Q!(2\pi)^{k_Q}}\int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{Q,\chi}}E(x,\mathcal{I}_Q(\lambda,\varphi)\phi,\lambda)\overline{E(y,\phi,\lambda)}\,d\lambda, \tag{6-7}$$

where $\mathcal{Q}$ is the set of standard parabolic subgroups which are not G; and for any such Q, k_Q is the number of blocks of the Levi part of Q. Also, (6-7) converges absolutely [2, Lemma 2, p.263].

Let $\phi_2\in\mathfrak{B}_{Q,\chi}$. Then $\mathcal{I}_Q(\lambda,\varphi)\phi_2$ can be expanded by a linear combination of vectors in $\mathfrak{B}_{Q,\chi}$. As a consequence,

$$E(x,\mathcal{I}_Q(\lambda,\varphi)\phi_2,\lambda)=\sum_{\phi_1\in\mathfrak{B}_{Q,\chi}}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle E(x,\phi_1,\lambda).$$

Since φ is K-finite, then $\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\equiv 0$ for all but finitely many $\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}$, depending only on the K-type of φ.

For $1\le j\le 2$, $\phi_j\in\mathfrak{B}_{Q,\chi}$, and $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, we define the Whittaker function

$$W_j(x,\lambda)=W(x,\phi_j,\lambda):=\int_{N(\mathbb{A}_F)}\phi_j(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn, \tag{6-8}$$

where w_0 is the long element in the Weyl group W_n, and $\log_Q$ is defined by (6-5).

6C. ***Spectral expansion of $I_{\mathrm{Whi}}(s,\tau)$.*** By definition, we have

$$I_{\mathrm{Whi}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)f(x,s)\,dx.$$

where

$$\widehat{\mathrm{K}}_{\mathrm{ER}}(x,y):=\int_{[N]}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(n_1x,n_2y)\theta(n_1)\bar{\theta}(n_2)\,dn_1dn_2 \tag{6-9}$$

Set $X_G=Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ and $c_Q=k_Q!(2\pi)^{k_Q}$. One can unfold the Eisenstein series (see [31, pp. 123–124]) to rewrite (at least formally) the function $I_{\mathrm{Whi}}(s,\tau)$ as

$$\int_{X_G}\sum_{\chi\in\mathfrak{X}}\sum_{Q\in\mathcal{Q}}\frac{1}{c_Q}\int\sum_{\phi_1,\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}\,d\lambda\ f(x,s)\,dx, \tag{6-10}$$

where $\phi_i\in\mathfrak{B}_{Q,\chi}$, $1\le i\le 2$, and $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$.

Theorem E. *Let notation be as before. Then there exists a constant $c_\varphi>0$ depending only on φ such that $I_{\mathrm{Whi}}(s,\tau)$ converges for* $\mathrm{Re}(s)>c_\varphi$. *Moreover, when* $\mathrm{Re}(s)>c_\varphi$, *$I_{\mathrm{Whi}}(s,\tau)$ is equal to*

$$\sum_{\chi}\sum_{Q\in\mathcal{Q}}\frac{1}{c_Q}\int\sum_{\phi_1,\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\int_{X_G}W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)\,dx\,d\lambda, \tag{6-11}$$

where χ runs over the proper cuspidal data, i.e., χ is not of the form $\{(G,\pi)\}$, $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, and $\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}$. Particularly, as a function of s, $I_{\mathrm{Whi}}(s,\tau)$ is analytic in the right half plane $\{z:\mathrm{Re}(z)>c_\varphi\}$.

Proof. For $x\in Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ we write it into the Iwasawa coordinates: $x=ak$, where $a\in A(\mathbb{A}_F)$ and $k\in K$. Then

$$f(x,s):=f(x,\Phi,\tau;s)=\tau(\det a)\,|\det a|^s\int_{\mathbb{A}_F^\times}\Phi(\eta tk)\tau(t)^n|t|^{ns}d^\times t.$$

Therefore, $|f(x,s)|=|\det a|^{\mathrm{Re}(s)}h(k,s)$, where

$$h(k,s):=\left|\int_{\mathbb{A}_F^\times}\Phi(\eta tk)\tau(t)^n|t|^{ns}d^\times t\right|$$

is a nonnegative continuous function of k and converges absolutely when $\mathrm{Re}(s)>1/n$. Let $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$. Then by Proposition 6.2,

$$\int_{X_G}J_{\mathrm{Kuz}}(\varphi,x)|f(x,s)|\,dx=\int_K\int_{A(\mathbb{A}_F)}J_{\mathrm{Kuz}}(\varphi,ak)\,|\det a|^{\mathrm{Re}(s)}\delta^{-1}(a)\,d^\times a h(k,s)\,dk$$

converges when $\mathrm{Re}(s)$ is large. By Lemma 6.1 we have

$$J(\varphi,s):=\int_{X_G}\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)|f(x,s)|\,dx=\int_{X_G}J_{\mathrm{Kuz}}(\varphi,x)\cdot|f(x,s)|\,dx-J_0(\varphi,s),$$

where $\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)$ is defined by (6-9), and

$$J_0(\varphi,s)=\int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)\sum_{\delta\in P(F)\backslash G(F)}|f(\delta x,s)|\,dx.$$

Since the series $\sum_{\delta\in P(F)\backslash G(F)}|f(\delta x,s)|$ is slowly increasing and $\mathrm{K}_0(x,x)$ is rapidly decaying on

$Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$, then $J_0(\varphi, s)$ converges absolutely. Hence $J(\varphi, s)$ converges and is well defined.

Consider test functions of the form $\varphi_0 * \varphi_0^*$, where $\varphi_0^*(x) = \overline{\varphi_0(x^{-1})}$. Substituting the spectral expansion (6-7) into $J(\varphi, s)$, which is convergent, gives

$$\int_{X_G} \sum_{\chi} \sum_{Q \in \mathcal{Q}} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \left| W(x; \mathcal{I}_Q(\lambda, \varphi_0)\phi, \lambda) \right|^2 |f(x,s)| \, d\lambda \, dx < \infty, \tag{6-12}$$

where χ ranges over the proper cuspidal data, and

$$W(x; \mathcal{I}_Q(\lambda, \varphi_0)\phi, \lambda) = \int_{N(\mathbb{A}_F)} \big(\mathcal{I}_Q(\lambda, \varphi_0)\phi\big)(w_0 n x) e^{(\lambda + \rho_Q)\log_Q(w_0 n x)} \overline{\theta(n)} \, dn,$$

with w_0 being the long element in the Weyl group W_n.

For an arbitrary test function $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$, by the factorization theorem of Dixmier and Malliavin, one can write φ as a finite linear combination of convolutions $\varphi_{j,1} * \varphi_{j,2}^*$ with functions $\varphi_{j,i} \in C_c^r(G(\mathbb{A}_F))$, whose archimedean components are differentiable of arbitrarily high order r, $1 \le i \le 2$, and $j \in J$ is a finite set. Using the triangle and Cauchy–Schwarz inequalities, along with (6-12), we derive

$$\begin{aligned}
&\sum_{\chi \in \mathfrak{X}} \sum_{Q \in \mathcal{Q}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \int_{X_G} \Big| \sum_{\phi_1, \phi_2 \in \mathfrak{B}_{Q,\chi}} \langle \mathcal{I}_Q(\lambda, \varphi)\phi_2, \phi_1 \rangle W_1(x;\lambda) \overline{W_2(x;\lambda)} f(x,s) \Big| \, dx \, d\lambda \\
&\qquad \le \sum_{j \in J} \prod_{i=1}^{2} \Big(\sum_{\chi \in \mathfrak{X}} \sum_{Q \in \mathcal{Q}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \int_{X_G} \sum_{\phi \in \mathfrak{B}_{Q,\chi}} |W_{j,i}(x;\lambda)|^2 \cdot \big| f(x,s) \big| \, dx \, d\lambda \Big)^{1/2} < \infty,
\end{aligned}$$

where $W_{j,i}(x;\lambda) = W(x; \mathcal{I}_P(\lambda, \varphi_{j,i})\phi, \lambda)$, for any $1 \le i \le 2$, and $j \in J$. This proves the first part of Theorem E and provides an expression for $I_{\mathrm{Whi}}(s, \tau)$ as

$$\sum_{\chi} \sum_{Q \in \mathcal{Q}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \int_{X_G} \sum_{\phi_1, \phi_2} \langle \mathcal{I}_Q(\lambda, \varphi)\phi_2, \phi_1 \rangle W_1(x;\lambda) \overline{W_2(x;\lambda)} f(x,s) \, dx \, d\lambda. \tag{6-13}$$

For $\mathrm{Re}(s)$ large, we have

$$\int_{X_G} \big| W_1(x;\lambda) \overline{W_2(x;\lambda)} f(x,s) \big| \, dx < \infty.$$

Recall that $\langle \mathcal{I}_Q(\lambda, \varphi)\phi_2, \phi_1 \rangle \equiv 0$ for all but finitely many $\phi_1, \phi_2 \in \mathfrak{B}_{Q,\chi}$. Thus,

$$\sum_{\phi_1, \phi_2 \in \mathfrak{B}_{Q,\chi}} \big| \langle \mathcal{I}_Q(\lambda, \varphi)\phi_2, \phi_1 \rangle \big| \int_{X_G} \big| W_1(x;\lambda) \overline{W_2(x;\lambda)} f(x,s) \big| \, dx < \infty. \tag{6-14}$$

Therefore, we obtain (6-11) from (6-13) and (6-14). □

Remark 6.4. If the base field F is a function field, it has no archimedean places. Then the support of $W_j(x;\lambda)\,|_{A(\mathbb{A}_F)}$ is contained in $A_{\varphi,\mathrm{fin}}$ for all $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, $1 \le j \le 2$, which means the support of $\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)$ is compact. Also, in the function field case the sum over the χ's is only finite. Therefore, Theorem E is clear.

7. Rankin–Selberg convolutions for generic representations

By Theorem E, we see that when $\mathrm{Re}(s)$ is large, $I_{\mathrm{Whi}}(s,\tau)$ is equal to

$$I_{\mathrm{Whi}}(s,\tau)=\sum_{\chi}\sum_{Q\in\mathcal{Q}}\frac{1}{c_Q}\int\sum_{\phi_1,\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\Psi_{Q,\chi}(s,W_1,W_2;\lambda)\,d\lambda, \tag{7-1}$$

where $\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}$, $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, and

$$\Psi_{Q,\chi}(s,W_1,W_2;\lambda)=\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)\,dx.$$

Here the Whittaker function $W_j(x,\lambda)$ has been defined by

$$W_j(x,\lambda)=\int_{N(\mathbb{A}_F)}\phi_j(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn,\ 1\le j\le 2. \tag{6-8}$$

Our objective is to establish the meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to $\mathbb{C}$ and demonstrate that the quotient $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ is holomorphic for $s\neq 0,1$. To achieve this, we initiate the process by calculating each $\Psi_{Q,\chi}(s,W_1,W_2;\lambda)$ associated with a standard parabolic subgroup Q and a cuspidal datum $\chi=(M_Q,\sigma)\in\mathfrak{X}$.

Here is the arrangement of this section:

- In Section 7A, we recall some notation from Section 6B–Section 6C and introduce new notation regarding induced representations.
- In Sections 7B and 7C, we extend the local and global investigation of the Rankin–Selberg convolution $\Psi_{Q,\chi}(s,W_1,W_2;\lambda)$, respectively. In particular, we explicitly compute it at the unramified places. This generalizes the work of [17] as we elucidate the dependence on the spectral parameter λ, which is crucial for the meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$.
- In Section 7C, we combine the analysis in the previous sections and utilize the analytic properties of the period integrals to achieve a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ beyond the region of absolute convergence.

7A. *Some notation.* Let Q be a standard parabolic subgroup of $G=\mathrm{GL}(n)$ of type $(n_1,\dots,n_r)$. Let $\chi=(M_Q,\sigma)\in\mathfrak{X}$ be a cuspidal datum, where σ is a unitary automorphic representation of M of central character ω. Let $\mathfrak{B}_{Q,\chi}$ be an orthonormal basis of the Hilbert space $\mathcal{H}_{Q,\chi}$ defined in Section 6B.

By definition, there exist r cuspidal representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, $1\le i\le r$, such that $\sigma\simeq\pi_1\boxplus\pi_2\boxplus\cdots\boxplus\pi_r$. Let $\pi=\mathrm{Ind}_{Q(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\pi_1,\pi_2\ \dots,\pi_r)$. For any $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, set

$$\pi_\lambda=\mathrm{Ind}_{Q(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\big(\pi_1\otimes|\cdot|_{\mathbb{A}_F}^{\lambda_1},\pi_2\otimes|\cdot|_{\mathbb{A}_F}^{\lambda_2},\dots,\pi_r\otimes|\cdot|_{\mathbb{A}_F}^{\lambda_r}\big). \tag{7-2}$$

Then π_λ is also a unitary automorphic representation of $G(\mathbb{A}_F)$.

For $\phi_1,\phi_2\in\mathcal{B}_{Q,\chi}$ and a point $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, recall that

$$W_j(x,\lambda)=W(x,\phi_j,\lambda):=\int_{N(\mathbb{A}_F)}\phi_j(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn, \tag{6-8}$$

where w_0 is the long Weyl element, and $\log_Q$ is defined by (6-5). Define

$$\Psi(s, W_1, W_2; \lambda, \Phi) = \int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x;\lambda)\overline{W_2(x;\lambda)} f(x,s)\, dx. \tag{7-3}$$

Since $W_1(x;\lambda)$ and $W_2(x;\lambda)$ are dominated by some gauge, and $f(x,s)$ increases slowly when $\mathrm{Re}(s) > 1$, then $\Psi(s, W_1, W_2; \lambda, \Phi)$ converges absolutely and normally when $\mathrm{Re}(s)$ is large. For each $v \in \Sigma_F$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, denote by $\pi_v = \mathrm{Ind}_{M_Q(F_v)}^{G(F_v)}(\pi_{1,v}, \pi_{2,v}, \dots, \pi_{r,v})$ and

$$\pi_{\lambda,v} = \mathrm{Ind}_{M_Q(F_v)}^{G(F_v)}\big(\pi_{1,v} \otimes |\cdot|_{F_v}^{\lambda_1}, \pi_{2,v} \otimes |\cdot|_{F_v}^{\lambda_2}, \dots, \pi_{r,v} \otimes |\cdot|_{F_v}^{\lambda_r}\big).$$

Then $\pi = \bigotimes_v' \pi_v$ and $\pi_\lambda = \bigotimes_v' \pi_{\lambda,v}$. Recall that $f(x,s) = \prod_v f_v(x_v, s)$, where

$$f_v(x_v, s) = \tau_v(\det x_v)\, |\det x_v|_{F_v}^s \int_{Z(F_v)} \Phi_v[(0, \dots, t_v)x_v]\tau_v^n(t_v)|t_v|_{F_v}^{ns}\, d^\times t_v,$$

if $\Phi = \bigotimes_v' \Phi_v$. We can rewrite $\Psi(s, W_1, W_2; \lambda, \Phi)$ as

$$\int_{N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x;\lambda)\overline{W_2(x;-\bar{\lambda})}\Phi(\eta x)\tau(\det x)\, |\det x|_{\mathbb{A}_F}^s\, dx, \tag{7-4}$$

where $\eta = (0, \dots, 0, 1) \in F^n$. Factor $W_j(x;\lambda)$ as $\prod_{v\in\Sigma_F} W_{j,v}(x_v;\lambda)$, where

$$W_{j,v}(x_v;\lambda) = \int_{N(F_v)} \phi_{j,v}(w_0 n x_v) e^{(\lambda+\rho_Q)\log_Q(w_0 n x_v)}\overline{\theta(n)}\, dn,\ 1 \le j \le 2,$$

with $\phi_{j,v}$ being a local vector in the space of $\pi_{j,v}$.

We may assume $\Phi = \bigotimes_v' \Phi_v$. Then

$$\Psi(s, W_{1,v}, W_{2,v}; \lambda, \Phi) = \prod_{v\in\Sigma_F} \Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v),$$

where each local factor $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is defined to be

$$\int_{N(F_v)\backslash G(F_v)} W_{1,v}(x_v;\lambda)\overline{W_{2,v}(x_v;-\bar{\lambda})}\Phi_v(\eta x_v)\tau(\det x_v)\, |\det x_v|_{F_v}^s\, dx_v. \tag{7-5}$$

Here $W_{j,v}(x_v;\lambda) = \int_{N(F_v)} \phi_{j,v}(w_0 n x) e^{(\lambda+\rho_Q)H_{Q,v}(w_0 n x)}\overline{\theta(n)}\, dn$, $1 \le j \le 2$.

7B. *Local theory for $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$*. In this section, we shall study each local integral representation $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ defined via (7-5).

Let $v \in \Sigma_F$ be a fixed nonarchimedean place, and let $\widetilde{\pi}_{\lambda,v}$ be the contragredient of $\pi_{\lambda,v}$. Let ϖ_v be a uniformizer of $\mathcal{O}_{F,v}$, the ring of integers of F_v. Let $q_v = N_{F_v/\mathbb{Q}_p}(\varpi_v)$, where p is the rational prime such that v is above p. Set

$$R_v(s, W_{1,v}, W_{2,v}; \lambda) := \frac{\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1. \tag{7-6}$$

Proposition 7.1 (nonarchimedean case). *Let notation be as before. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have*:

(a) *$R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \le i \le r\}$.*

(b) *The local functional equation*

$$\frac{\Psi_v(1-s,\widetilde{W}_{1,v},\widetilde{W}_{2,v};-\bar{\lambda},\widehat{\Phi}_v)}{L_v(1-s,\widetilde{\pi}_{-\bar{\lambda},v}\otimes\bar{\tau}_v\times\pi_{\lambda,v})}=\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)\cdot\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})}$$

holds, where $\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)$ *is a polynomial in* $\{q_v^s,q_v^{-s},q_v^{\lambda_i},q_v^{-\lambda_i}:1\le i\le r\}$.

Proof. We prove part (a); part (b) will follow form [17].

Let $T(F_v)$ be the maximal split torus of $G(F_v)$. For $m\in\mathbb{Z}$, let $T^{(m)}(F_v)=\{t\in T(F_v):|\det t|_{F_v}=q_v^{-m}\}$. Using Iwasawa decomposition and the fact that $W_{i,v}$ and Φ_v are right $G(\mathcal{O}_{F,v})$-finite, we can rewrite $\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ as

$$\sum_{j\in J}\int_{T(F_v)}W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}\Phi_{j,v}(\eta a_v)\tau_v(\det a_v)\delta^{-1}(a_v)\,|\det a_v|_{F_v}^s\,da_v,$$

where the sum is over a finite set J, $W_{i,v}^{(j)}(a_v;\lambda)$ is the Whittaker function associated to some smooth functions in $\mathcal{H}_{Q,\chi}$, $1\le i\le 2$, and $\Phi_{j,v}$ is some Schwartz–Bruhat function.

For $1\le i\le 2$ and $j\in J$, $W_{i,v}^{(j)}(x_v;\lambda)$ is right $G(\mathcal{O}_{F,v})$-finite. So there exists a compact subgroup $N_{0,v}\subseteq G(\mathcal{O}_{F,v})\cap N(F_v)$, depending only on φ, such that $W_{i,v}^{(j)}(t_vu_v;\lambda)=W_{i,v}^{(j)}(t_v;\lambda)$, for all $t_v\in T(F_v)$ and $u_v\in N_{0,v}$. On the other hand, $W_{i,v}^{(j)}(t_vu_v;\lambda)=\theta_{t_v}(u_v)W_{i,v}^{(j)}(t_v;\lambda)$, where $\theta_{t_v}(n_v)=\theta(t_vn_vt_v^{-1})$, for any $n_v\in N(F_v)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ (hence not on λ) such that $\theta_{t_v}(u_v)=1$ if and only if $|\alpha_i(t_v)|\le C_v$, where α_i's are the simple roots of $G(F)$. Thus each $W_{i,v}^{(j)}(x_v;\lambda)$ is compactly supported for a fixed $\lambda\in i\mathfrak{a}_Q/i\mathfrak{a}_G$. Therefore, for a fixed λ, $\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ is a formal Laurent series in q_v^{-s}. Indeed, one can chose some nonnegative integer M independent of λ (but depending on π and φ), such that

$$\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)=\sum_{m\ge -M}\Psi_v^{(m)}(W_{1,v},W_{2,v};\lambda,\Phi_v)\cdot q_v^{-ms},$$

where $\Psi_v^{(m)}(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ is defined by the integral

$$\sum_{j\in J}\int_{T^{(m)}(F_v)}W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}\Phi_{j,v}(\eta a_v)\tau_v(\det a_v)\delta^{-1}(a_v)\,da_v.$$

Applying the above analysis on supp $W_{i,v}(a_v;\lambda)$, we see similarly that

$$\operatorname{supp}W_{i,v}^{(j)}(a_v;\lambda)\subseteq\left\{t\in T^{(m)}(F_v):|\alpha_l(t)|_{F_v}\le C_v^{(j)},\ 1\le l\le n-1\right\}$$

for some constants $C_v^{(j)}$. Hence, for each $j\in J$, $m\ge -N$ and $a_v\in T^{(m)}(F_v)$, the function

$$a_v\mapsto W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}$$

is analytic and is a formal Laurent series in $\{q_v^{-\lambda_i}:1\le i\le r\}$ by (2.5.2) of [17], and the function

$$a_v\mapsto W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}\Phi_{j,v}(\eta a_v)\tau(\det a_v)\delta^{-1}(a_v)$$

is locally constant. Therefore, $\Psi_v^{(m)}(W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is an analytic function of λ and is a formal Laurent series in $\{q_v^{-\lambda_i} : 1 \leq i \leq r\}$.

Since $\pi_{\lambda,v}$ is of Whittaker type, we can use Theorem 2.7 of [17] to see that, for fixed $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) \cdot L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})^{-1}$ is a polynomial in $\{q_v^s, q_v^{-s}\}$ with coefficients functions of λ. Moreover, $L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})^{-1}$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. So we can write

$$L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{\lambda,v})^{-1} = \sum_{|l| \leq N} Q_l(\lambda) q_v^{-ls},$$

where N is a positive integer and $Q_l(\lambda)$ are polynomials in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. Then for $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) \cdot L(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})^{-1}$ is equal to the sum over $m \geq -N - M$ of $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) q_v^{-ms}$, where

$$R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = \sum_{\substack{i+j=m \\ |i| \leq N, j \geq -M}} Q_i(\lambda) \Psi_v^{(m)}(W_{1,v}, W_{2,v}; \lambda, \Phi_v).$$

Since the sum on the right is finite, $R_l(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is analytic in λ. Moreover, it is a formal Laurent series in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. Therefore, Proposition 7.1(a) follows from the next claim:

Claim 7.2. *There exists some $M_0 \in \mathbb{Z}$, independent of $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, such that $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ vanishes for all $m \geq M_0$ and for all $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$. For each $m \in \mathbb{Z}$, $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is a polynomial in $\{q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq r\}$.*

Proof of Claim 7.2. Let $l \in \mathbb{Z}$. We define

$$\Lambda_l := \{\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G : R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0 \text{ for all } m \geq l\}.$$

Each Λ_l is closed as the function $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is analytic (hence continuous) in λ. Since

$$R_v(s, \lambda) = \sum_m R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) q_v^{-ms} \in \mathbb{C}[q_v^s, q_v^{-s}]$$

for fixed $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, there exists some $M(\lambda)$ such that $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0$ as long as $m \geq M(\lambda)$. Therefore, $i\mathfrak{a}_Q/i\mathfrak{a}_G$ is covered by the union of all Λ_l.

Noting that $i\mathfrak{a}_Q/i\mathfrak{a}_G \simeq R^{r-1}$ is a Banach space, by Baire category theorem there exists some Λ_{l_0} having nonempty interior, $\mathrm{Int}(\Lambda_{l_0})$, say. Thus, for any $\lambda \in \mathrm{Int}(\Lambda_{l_0})$, $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0$ for any $m \geq l_0$. Since $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is analytic for any $l \in \mathbb{Z}$, $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0$ for all $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, proving the first part. For the remaining part, we consider the functional equation (see [17]):

$$\frac{\Psi_v(1-s, \widetilde{W}_{1,v}, \widetilde{W}_{2,v}; -\bar{\lambda}, \widehat{\Phi}_v)}{L_v(1-s, \widetilde{\pi}_{-\bar{\lambda},v} \otimes \bar{\tau}_v \times \pi_{\lambda,v})} = \varepsilon_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta_v) \cdot \frac{\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})},$$

where $\varepsilon_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta_v)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$.

We can interpret the functional as an identity between formal Laurent series in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. The left side is of the form $\sum_{m_1 \geq -M_1} q_v^{m_1\lambda_i}$, while the right side is of the form $\sum_{m_2 \geq -M_2} q_v^{-m_2\lambda_i}$. Since they are equal, they must both be polynomials in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. This proves Claim 7.2, and with it Proposition 7.1. □

One will see that Proposition 7.1 is insufficient for our continuation in next few sections. Hence we need to compute

$$R_v(s, W_{1,v}, W_{2,v}; \lambda) := \frac{\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1, \tag{7-6}$$

more explicitly. We will do principal series case below since this is the only case we need for the meromorphic continuation of Rankin–Selberg convolutions– it suffices to consider the partial Euler factors which corresponds to the principal series at all but finitely many places.

Lemma 7.3. *Let v be a nonarchimedean place of F. Let π_v be induced from $B(F_v)$ by characters $\chi_{v,1}, \chi_{v,2}, \dots, \chi_{v,n}$. Assume that π_v is right K_v-finite. Let $\alpha \in T(F_v)$ and let $W_v(\alpha, \lambda)$ be a Whittaker function associated to $\pi_{v,\lambda}$ and α. Then $W_v(\alpha, \lambda)$ is of the form $\mathcal{B}_v(\alpha, \lambda)\mathcal{L}_v(\lambda)$, where $\mathcal{B}_v(\alpha, \lambda)$ is a holomorphic function of λ, and*

$$\mathcal{L}_v(\lambda) = \prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{v,i}\overline{\chi}_{v,j})^{-1}.$$

Proof. Starting with $n = 2$, we may assume that $\chi_{1,2} = \chi_{v,1}\chi_{v,2}^{-1}$ is unramified. Otherwise, the local L-function $L(s, \chi_1\overline{\chi}_2)$ is trivial, and Lemma 7.3 follows from Part (a) of Proposition 7.1. In view of

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ u^{-1} & 1 \end{pmatrix} \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 1 \\ 1 & 0 \end{pmatrix}$$

and the K_v-finiteness condition, one has

$$W_v(\alpha, \lambda) = \sum_{j \in \boldsymbol{J}} \sum_{l=1}^{\infty} c_j \int_{\varpi_v^{-l}\mathcal{O}_v^\times} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)\, du + W_v^\circ(\alpha, \lambda),$$
$$W_v^\circ(\alpha, \lambda) = \sum_{j \in \boldsymbol{J}} c_j \int_{\mathcal{O}_v} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)\, du = \sum_{j \in \boldsymbol{J}} c_j \int_{\mathcal{O}_v} \theta(\alpha u)\, du,$$

where j runs over a finite set $\boldsymbol{J}$ and the c_j's are constants; both $\boldsymbol{J}$ and the c_j's depend only on the K_v-type of π_v. For $u \in F_v^\times$, write $u = u^\circ\varpi_v^l$, where $u^\circ \in \mathcal{O}_v^\times = \mathcal{O}_v^\times$, and $l \in \mathbb{Z}$. Write $\alpha = \alpha^\circ\varpi_v^k$, where $\alpha^\circ \in \mathcal{O}_v^\times$. Recall that by definition the one sees that the conductor of θ_v is precisely the inverse different of F_v, which is $\mathfrak{D}_{F_v}^{-1} = \{x_v \in F_v : \mathrm{tr}_{F_v/\mathbb{Q}_p}(x_v) \in \mathbb{Z}_p\}$, where p is the characteristic of residue field of $\mathcal{O}_v$. Since $\mathfrak{D}_{F_v}^{-1}$ is a $\mathbb{Z}_p$-module of F_v, it has the representation $\mathfrak{D}_{F_v}^{-1} = \varpi_v^{-d}\mathcal{O}_v$, where $d \in \mathbb{N}_{\geq 0}$. Hence

$$I = \int_{\mathcal{O}_v} \theta(\alpha u)\, du = \int_{\mathcal{O}_v} \theta(\alpha^\circ u\varphi_v^k)\, du = \int_{\mathcal{O}_v} \theta(u\varpi_v^k)\, du$$

is vanishing if $k \leq -d - 1$. Clearly $I = 1$ if $k \geq -d$. Note that

$$\int_{\varpi_v^{-l}\mathcal{O}_v^\times} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)\, du = \chi_{12}(\varpi_v)^l|\varpi_v|_v^{(1+\lambda_1-\lambda_2)l} \int_{\varpi_v^{-l}\mathcal{O}_v^\times} \theta(\alpha u)\, du$$

is vanishing if $l \geq k+d+2$. Let $q_v = |\varpi_v|_v^{-1}$. Then one sees that

$$W_v(\alpha,\lambda) = C + C\sum_{l=1}^{k+d}(1-q_v^{-1})\chi_{12}(\varpi_v)^l q_v^{-(\lambda_1-\lambda_2)l} + C\cdot W_{re}, \tag{7-7}$$

where C is a constant depending only on F and K_v-type of ϕ_v and

$$W_{re} = \chi_{12}(\varpi_v)^{k+d+1} q_v^{-(k+d+1)(1+\lambda_1-\lambda_2)} \int_{\varpi_v^{-k-d-1}\mathcal{O}_v^\times} \theta(u\varpi_v^k)\,du. \tag{7-8}$$

Since θ is nontrivial on $\varpi_v^{-d-1}\mathcal{O}_v$, then $\int_{\varpi_v^{-k-d-1}\mathcal{O}_v} \theta(u\varpi_v^k)\,du = 0$. Note that $\varpi_v^{-k-d-1}\mathcal{O}_v^\times = \varpi_v^{-k-d-1}\mathcal{O}_v \setminus \varpi_v^{-k-d}\mathcal{O}_v$. Then

$$\int_{\varpi_v^{-k-d-1}\mathcal{O}_v^\times} \theta(u\varpi_v^k)\,du = \int_{\varpi_v^{-k-d-1}\mathcal{O}_v} \theta(u\varpi_v^k)\,du - \int_{\varpi_v^{-k-d}\mathcal{O}_v} \theta(u\varpi_v^k)\,du = -q_v^{k+d}.$$

Then it follows form (7-7) and (7-8) that $W_v(\alpha,\lambda)$ is equal to C multiplying

$$L = 1 + \sum_{l=1}^{k+d}(1-q_v^{-1})\chi_{12}(\varpi_v)^l q_v^{-(\lambda_1-\lambda_2)l} - \chi_{12}(\varpi_v)^{k+d+1} q_v^{-(k+d+1)(\lambda_1-\lambda_2)-1}.$$

An elementary computation leads to the identity

$$L = (1-\chi_{12}(\varpi_v)q_v^{-(1+\lambda_1-\lambda_2)}) \cdot P(\chi_{12}(\varpi_v)q_v^{-(\lambda_1-\lambda_2)}), \tag{7-9}$$

where $P(z) = (1-z^{k+d+1})\cdot(1-z)^{-1} = 1+z+\cdots+z^{k+d} \in \mathbb{C}[z]$.

Therefore, one has $W_v(\alpha,\lambda) = CQ(\chi_{12}(\varpi_v)q_v^{-(\lambda_1-\lambda_2)})\cdot L_v(1+\lambda_1-\lambda_2,\chi_{12})$, where $Q(z) = P(z)$ if $k \geq -d$ and $Q(z) \equiv 0$ otherwise. Taking $\mathcal{B}_v(\alpha,\lambda)$ to be the function $CQ(\chi_{12}(\varpi_v)q_v^{-(\lambda_1-\lambda_2)})$, we then obtain Lemma 7.3 in $n=2$ case. The general case follows from the recursion property of Whittaker functions [37, §2.3] and induction, since the integral with respect to $\chi_{l,j}$ is exactly the same as above, $1 \leq l < j \leq n$. □

Proposition 7.4. *Let v be a nonarchimedean place of F. Let π_v be induced from $B(F_v)$ by characters $\chi_{1,v}, \chi_{2,v}, \ldots, \chi_{n,v}$. Assume that π_v is right K_v-finite. Then the function $R_v(s, W_{1,v}, W_{2,v};\lambda)$ is of the form $Q_v(s,\lambda)\mathcal{L}_v(\lambda)$, where the function $Q_v(s,\lambda)$ lies in $\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq n]$ and*

$$\mathcal{L}_v(\lambda) := \prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j, \chi_{i,v}\overline{\chi}_{j,v})^{-1}\cdot L_v(1-\lambda_i+\lambda_j, \overline{\chi}_{i,v}\chi_{j,v})^{-1}.$$

Proof. By Lemma 7.3 the function

$$W_v(x_v;\phi_{1,v},\lambda)\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j, \chi_{i,v}\overline{\chi}_{j,v})$$

is in $\mathbb{C}[q_v^{\pm\lambda_j} : 1 \leq j \leq n]$. Applying the expansions in [17] and changing orders of summations we see that

$$R_v(s, W_{1,v}, W_{2,v};\lambda)\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j, \chi_{i,v}\overline{\chi}_{j,v})L_v(1-\lambda_i+\lambda_j, \overline{\chi}_{i,v}\chi_{j,v})$$

lies in $\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq n]$. □

In conjunction with the Langlands–Shahidi method, we have:

Corollary 7.5. *Let $v \in \Sigma_{F,\mathrm{fin}}$ be a finite place such that π_v is unramified and $\Phi_v = \Phi_v^\circ$ is the characteristic function of $G(\mathcal{O}_{F,v})$. Assume that $\phi_{1,v} = \phi_{2,v} = \phi_v^\circ$ is the unique $G(\mathcal{O}_{F,v})$-fixed vector in the space of π_v such that $\phi_v^0(I_n) = 1$. Then $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is equal to*

$$\prod_{1\le i<r}\prod_{i<j\le r} L_v(1+\lambda_i-\lambda_j, \pi_{i,v}\times\widetilde{\pi}_{j,v})^{-1}\cdot L_v(1-\lambda_i+\lambda_j, \widetilde{\pi}_{i,v}\times\pi_{j,v})^{-1}. \tag{7-10}$$

In particular, $R_v(s, \lambda)$ is independent of s.

Proof. Fix $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$. Let $W_{j,v}^\circ$ be the $G(\mathcal{O}_{F,v})$-invariant vectors such that $W_{j,v}^\circ(I_n) = 1$, $1 \le j \le 2$. Then by the computation from [17], we know that $\Psi_v(s, W_{1,v}^\circ, W_{2,v}^\circ; \lambda, \Phi_v^\circ)/L_v(s, \pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$ is equal to (7-10), where Φ_v° is the characteristic function of $\mathcal{O}_{F,v}^n$. Then Corollary 7.5 follows from induction and unramified computations of nonconstant Fourier coefficients of Eisenstein series (see [31], Chapter 7). □

Corollary 7.5 will be used in Section 9 to investigate the meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ for general τ, e.g., Theorem G.

Now we move to the archimedean case. In the current state of affairs the local L-functions

$$L_\infty(s, \pi_\lambda\times\tau\times\widetilde{\pi}_{-\lambda}) = \prod_{v\mid\infty} L_v(s, \pi_{\lambda,v}\times\tau_v\times\widetilde{\pi}_{-\lambda,v})$$

are not defined intrinsically through the integrals as in the nonarchimedean case, but rather extrinsically through the Langlands correspondence and then related to the integrals. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2^{1-s}\pi^{-s}\Gamma(s)$. Then by Langlands classification, each archimedean L-function $L_v(s, \pi_{\lambda,v}\times\tau_v\times\pi_{-\lambda,v})$ is of the form

$$\prod_{i\in I}\Gamma_{\mathbb{R}}(s+\mu_i)\prod_{j\in J}\Gamma_{\mathbb{C}}(s+\mu_j'), \tag{7-11}$$

where I and J are finite set of integers, and $\mu_i, \mu_j' \in \mathbb{C}$.

Combining results from [15] and well known estimates on archimedean Satake parameters (see [23], for example) one concludes the following result.

Proposition 7.6 (archimedean case). *Let notation be as before. Let $v \in \Sigma_{F,\infty}$ be an archimedean place.*

(a) *$\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ converges absolutely and normally in the right half plane $\{s\in\mathbb{C} : \mathrm{Re}(s) > 1-2/(n^2+1)\}$, uniformly in $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$. Moreover, it is bounded at infinity in any strip of finite width.*

(b) *The function $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is holomorphic in s and λ. Hence, $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = R_v(s, W_{1,v}, W_{2,v}; \lambda)L_v(s, \pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$ admits a meromorphic continuation to the whole complex plane.*

(c) *We have the local functional equation*

$$\frac{\Psi_v(1-s,\widetilde{W}_{1,v},\widetilde{W}_{2,v};-\bar{\lambda},\widehat{\Phi}_v)}{L_v(1-s,\widetilde{\pi}_{-\bar{\lambda},v}\otimes\bar{\tau}_v\times\pi_{\lambda,v})}=\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)\cdot\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})},$$

where $\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)$ *is a holomorphic function.*

Remark 7.7. It follows from Lemma 5.4 in [15] that if both $\pi_{1,v}$ and $\pi_{2,v}$ are tempered, then the Rankin–Selberg convolution $\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ converges absolutely and normally in the right half-plane $\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$, uniformly in $\lambda\in i\mathfrak{a}_Q/i\mathfrak{a}_G$.

7C. ***Global theory for*** $\boldsymbol{\Psi(s,W_1,W_2;\lambda)}$**.** In this section, we shall compute the global integral representation $\Psi(s,W_1,W_2;\lambda,\Phi)$ defined via (7-4).

Let $\widetilde{\pi}_{\lambda,v}$ be the contragredient of $\pi_{\lambda,v}$. Let ϖ_v be a uniformizer of $\mathcal{O}_{F,v}$, the ring of integers of F_v. Let $q_v=N_{F_v/\mathbb{Q}_p}(\varpi_v)$, where p is the rational prime such that v is above p. Define

$$R(s,W_1,W_2;\lambda):=\prod_{v\in\Sigma_F}\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})},\quad \mathrm{Re}(s)>1. \tag{7-12}$$

Then $R(s,W_1,W_2;\lambda)$ is holomorphic for any $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$. Putting together the local computations in the last section, we get:

Proposition 7.8 (global case). *Let notation be as before. Let $s\in\mathbb{C}$ be such that* $\mathrm{Re}(s)>1$.*Then*

(a) *The integral* $\Psi(s,W_1,W_2;\lambda,\Phi)$ *converges absolutely in* $\mathrm{Re}(s)>1$, *and it is bounded at infinity in any strip of finite width.*

(b) *We have the global functional equation for* $\mathrm{Re}(s)>1$:

$$\Psi(1-s,\widetilde{W}_1,\widetilde{W}_2;-\lambda,\tau^{-1},\widehat{\Phi})=\Psi(s,W_1,W_2;\lambda,\tau,\Phi).$$

(c) *For any fixed* $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, $R(s,W_1,W_2;\lambda)$ *can be continued to an entire function.*

7D. ***A refinement of Theorem E.*** Let notation be as before. For $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$ and $\phi_2\in\mathfrak{B}_{Q,\chi}$, define

$$R_\varphi(s,\lambda;Q,\chi)=\sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}}\frac{\langle\mathcal{I}_Q(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})},\quad \mathrm{Re}(s)>1, \tag{7-13}$$

where

$$\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda}):=\prod_{v\in\Sigma_F}L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$$

is the complete L-function. According to Propositions 7.1 and 7.6 the function

$$\Psi(s,W_1,W_2;\lambda)\,\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})^{-1}$$

is entire.

Theorem F. *Let notation be as before. Let* $0 < \mathrm{Re}(s) < 1$ *or* $\mathrm{Re}(s) > 1$. *Then*

$$I_{\mathrm{Whi}}(s,\tau) = \sum_{\chi} \sum_{Q \in \mathcal{Q}} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} R_\varphi(s,\lambda;Q,\chi)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})\,d\lambda \tag{7-14}$$

converges absolutely and uniformly on every compact subsets. In particular, Theorem E holds with $c_\varphi = 1$.

Proof. Fix a proper parabolic subgroup $Q \in \mathcal{Q}$ of type $(n_1, n_2, \ldots, n_r)$. Let $\mathfrak{X}_Q$ be the subset of cuspidal data $\chi = \{(M,\sigma)\}$ such that $M = M_Q$. Set

$$J_Q(s) = \sum_{\chi \in \mathfrak{X}_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} R_\varphi(s,\lambda;Q,\chi)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})\,d\lambda.$$

Let $M_Q = \mathrm{diag}(M_1, M_2, \ldots, M_r)$, where $M_i \in \mathrm{GL}(n_i)$, $1 \le i \le r$. We may write $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$, where $\sigma_i \in \mathcal{A}_0(M_i(F)\backslash M_i(\mathbb{A}_F))$. By the K-finiteness of φ, each σ_i has a fixed finite type, so its arithmetic conductor is bounded uniformly (depending only on φ). Write φ as a finite sum of convolutions $\varphi_\alpha * \varphi_\beta$. So

$$R_\varphi(s,\lambda;Q,\chi) = \sum_\alpha \sum_\beta \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \frac{\Psi(s, W_\alpha, W_\beta; \lambda)}{\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})},$$

where $W_\alpha(x;\lambda) = W(x, \mathcal{I}_P(\lambda,\varphi_\alpha)\phi;\lambda)$ and $W_\beta(x;\lambda) = W(x, \mathcal{I}_P(\lambda,\varphi_\beta)\phi;\lambda)$. For $v \in \Sigma_F$, we define

$$R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi) := \frac{\Psi_v(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v})}{L_v(s,\pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1/2,$$

as the local component. By the calculations at the unramified places in Section 7, we have $R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi) = 1$ for almost all v, and $R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$ is holomorphic at the finitely many remaining places. As a consequence,

$$\frac{\Psi(s, W_\alpha, W_\beta; \lambda)}{\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})} = \prod_{v \in \Sigma_F} R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$$

is a finite product of holomorphic functions. Here we have identified $R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$ with its holomorphic continuation.

Write $i\mathfrak{a}_Q^*/i\mathfrak{a}_G^* \ni \lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 + \cdots + \lambda_n = 0$. Let $s = \mu + i\gamma$ with $0 < \mu < 1$ and $\gamma \in \mathbb{R}$. Set $s' = \mu' + i\gamma$, where $\mu' = 100 + |\beta| + c_\varphi$. Here c_φ is the constant defined in Theorem E. Let

$$V(s,\lambda) = s^n (s-1)^n \prod_{i,j} (s - \lambda_i + \lambda_j)^n (s - 1 - \lambda_i + \lambda_j)^n.$$

Then $V(s,\lambda)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is entire. As a consequence,

$$J_{\alpha,\beta}(s) := V(s,\lambda)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \prod_{v \in \Sigma_F} R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$$

is entire. By Proposition 7.8 we have $\lim_{\gamma \to \infty} J_{\alpha,\beta}(s) = 0$. So one can apply the maximum principle to get $|J_{\alpha,\beta}(s)| \le \max\{|J_{\alpha,\beta}(1-s')|, |J_{\alpha,\beta}(s')|\}$. By the functional equation and the estimate on the

ε-factor, whose size is a power of the norm of the arithmetic conductor, we obtain

$$|J_{\alpha,\beta}(1-s')| \ll |J_{\alpha,\beta}(s')| \cdot M^{s'} \prod_{p|\infty} \left| \frac{L_v(1-s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)}{L_v(s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)} \right|,$$

where M is an integer determined by the K-type of φ (i.e., the finite places v such that φ_v is *not* $G(\mathcal{O}_v)$-invariant) and the implied constant is absolute. Note that $\lambda_j \in i\mathbb{R}$ and $\mathrm{Re}(s) \neq 0, 1$. So $V(s, \lambda) \gg_s 1$, with the implied constant depending on s. By Stirling's formula concerning gamma functions we obtain

$$\frac{|V(s', \lambda)|}{|V(s, \lambda)|} \cdot M^{s'} \prod_{p|\infty} \left| \frac{L_v(1-s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)}{L_v(s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)} \right| \ll_{s,\varphi} 1$$

where the implied constant depends on s and φ. Then

$$|R_\varphi(s, \lambda; Q, \chi)\, \Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})| \ll \sum_\alpha \sum_\beta \left| \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \Psi(s', W_\alpha, W_\beta; \lambda) \right|,$$

where the implied constant depends only on φ. So the absolute convergence of $J_Q(s)$ follows from Theorem E. The above argument also works for $1 < \mu \leq c_\varphi$. So Theorem F follows. □

Corollary 7.10. *Let notation be as before. Assume τ is such that $\tau^k \neq \mathbf{1}$ for all $1 \leq k \leq n$. Then*

$$I_{\mathrm{Whi}}(s, \tau) = \sum_\chi \sum_{Q \in \mathcal{Q}} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} R_\varphi(s, \lambda; Q, \chi)\, \Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})\, d\lambda$$

admits a holomorphic continuation to the whole s-plane. The function $I_{\mathrm{Whi}}(s, \tau)/\Lambda(s, \tau)$ is also entire.

Proof. Since $\tau^k \neq \mathbf{1}$ for all $1 \leq k \leq n$, we have $\pi_\lambda \otimes \tau \not\simeq \pi_\lambda$ for all λ. Then $\Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is entire. Hence the arguments in the proof of Theorem F (with $V(s, \lambda) \equiv 1$) work here for all $\mathrm{Re}(s) > 0$. Then the first part of Corollary 7.10 follows from the functional equation Proposition 7.8.

Note that $\Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is a product of $\Lambda(s, \tau)^n$ and some other Rankin–Selberg L-functions on $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$, $n_1, n_2 < n$. Hence, $I_{\mathrm{Whi}}(s, \tau)/\Lambda(s, \tau)$ is entire if $\tau^k \neq \mathbf{1}$ for all $1 \leq k \leq n$. □

We say τ is *exceptional* if $\tau^k = \mathbf{1}$ for some $1 \leq k \leq n$. For nonexceptional τ, Corollary 7.10 gives a holomorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ to $\mathrm{Re}(s) > 0$. However, if τ is exceptional, the holomorphic functions defined by (7-14) in $0 < \mathrm{Re}(s) < 1$ and $\mathrm{Re}(s) > 1$ are *not* compatible, i.e., they do not give a natural continuation of $I_{\mathrm{Whi}}(s, \tau)$. For example, when τ is trivial, (7-14) diverges for *all* s with $\mathrm{Re}(s) = 1$. A meromorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ in these (finitely many) exceptional cases will be given in Section 9.

8. The Dedekind conjecture: proof of main results

8A. ***Proof of Theorem A.*** From Theorems C, D, E and F, we conclude the first part of Theorem A, obtaining (1-5), namely, for $\mathrm{Re}(s) > 1$,

$$I_0^\varphi(s, \tau) = I^\varphi_{\mathrm{Geo,Reg}}(s, \tau) + I^\varphi_{P,\mathrm{Reg}}(s, \tau) + I^\varphi_{\mathrm{Sing}}(s, \tau) + I_{\mathrm{Whi}}(s, \tau).$$

Moreover, $I_0^{\varphi}(s,\tau)$, $I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)$, and $I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)$ admit meromorphic continuation to the whole s-plane. Consequently, $I_{\mathrm{Sing}}^{\varphi}(s,\tau)$ can be continued to a meromorphic function on $\mathbb{C}$.

Assume τ is such that $\tau^k \neq \mathbf{1}$, $1 \leq k \leq n$. By Corollary 7.10 the function $I_{\mathrm{Whi}}(s,\tau)$ has a meromorphic continuation to $\mathrm{Re}(s) > 0$. By the functional equation of Eisenstein series, we conclude that $I_{\mathrm{Whi}}(s,\tau)$ has a meromorphic continuation to the whole s-plane. Then Theorem A follows.

8B. *Proof of Theorem B.* In this section, our objective is to construct a suitable test function φ in Theorem A to prove Theorem B, i.e., holomorphy of adjoint L-functions for GL(n) implies the Dedekind conjecture for degree n.

8B1. *Auxiliary results.* We establish several important auxiliary results: Lemmas 8.1–8.5, which address key points essential to our analysis:

- distribution of the fractional part $\{\alpha \log p\}$ of $\alpha \log p$ as primes p traverse arithmetic progressions;
- description of conjugacy classes supported in compact sets;
- construction of the local test function as the matrix coefficient of a suitable supercuspidal representation;
- nonvanishing of the local integrals of $I_{\mathrm{Geo,Reg}}(s,\tau)$.

These results play a crucial role in our subsequent analysis of the nonvanishing of the geometric side (see Section 8B2).

Lemma 8.1. *Let $\alpha \in \mathbb{R}_{>0}$. Let $n, m \in \mathbb{Z}_{\geq 1}$. There are infinitely many rational primes p such that $p \equiv 1 \pmod{m}$ and $\{\alpha \log p\} \leq n^{-1} p^{-1/6}$, where $\{\cdot\}$ is the fractional part function.*

Proof. If $\alpha = 0$, then Lemma 8.1 boils down to Dirichlet's theorem.

Suppose $\alpha \neq 0$ henceforth. Let $k > 100n + 100\alpha\,|\log 2\alpha| + 10m\alpha$. By [28, §VIII.14a, p. 290] there exists a prime $p \in (e^{k/\alpha}, e^{k/\alpha} + e^{4k/(5\alpha)})$ such that $p \equiv 1 \pmod{m}$. Write $p^{\alpha} = e^k + \beta$, where $0 \leq \beta \leq (e^{k/\alpha} + e^{4k/(5\alpha)})^{\alpha} - e^k$. Then

$$\alpha \log p - k = \log(1 + \beta e^{-k}) \leq (1 + e^{-k/(5\alpha)})^{\alpha} - 1 \leq 2\alpha e^{-k/(5\alpha)} \leq n^{-1} p^{-1/6}.$$

Hence, Lemma 8.1 follows. □

For $v \in \Sigma_{F,\mathrm{fin}}$, denote by $e_v(\cdot)$ the standard evaluation on F_v normalized as $e_v(\varpi_v) = 1$. For $x_v = (x_{i,j}) \in G(F_v)$, we set

$$e_{\min}(x_v) = \min_{1 \leq i,j \leq n} e_v(x_{i,j}).$$

Let Ω be a compact set in $G(F_v)$. Set

$$\Omega^{-1} := \{x^{-1} : x \in \Omega\},$$
$$e_{\min}(\Omega) := \min_{x_v \in \Omega \cup \Omega^{-1}} e_{\min}(x_v) \leq 0.$$

Lemma 8.2. *Let notation be as before. Let $v \in \Sigma_{F,\mathrm{fin}}$. Let Ω be a compact set in $G(F_v)$. Let*

$$\gamma_0 = \begin{pmatrix} 0 & & \cdots & -c_0 \\ 1 & 0 & \cdots & -c_1 \\ & \ddots & \ddots & \vdots \\ & & 1 & -c_{n-1} \end{pmatrix} \in G(F) \hookrightarrow G(F_v). \tag{8-1}$$

Let $x_v = tuk \in G(F_v)$, where $t = \mathrm{diag}(t_1, \dots, t_n) \in T(F_v)$, $k \in G(\mathcal{O}_v)$, and $u = (u_{i,j}) \in N(F_v)$ satisfies that, for $1 \le i < j \le n$, either $e_v(u_{i,j}) < 0$ or $u_{i,j} = 0$. Suppose that $\det x_v \in \mathcal{O}_v^\times$ *and $x_v^{-1}\gamma_0 x_v \in \Omega$. There exists a polynomial $p(n) > 0$ of n, with coefficients determined by $e_v(c_i)$, $0 \le i \le n-1$, such that*

$$\begin{cases} p(n)e_{\min}(\Omega) \le e_v(t_i) \le -p(n)e_{\min}(\Omega) & \text{if } 1 \le i \le n, \\ p(n)e_{\min}(\Omega) \le e_v(u_{i,j}) \le -1 & \text{if } u_{i,j} \ne 0 \text{ and } 1 \le i < j \le n. \end{cases} \tag{8-2}$$

Proof. Write $u^{-1} = (u'_{i,j})$. Then $e_v(u'_{i,j}) < 0$ if $u'_{i,j} \ne 0$, $1 \le i < j \le n$. By a straightforward calculation we have

$$kx_v^{-1}\gamma_0 x_v k^{-1} = \begin{pmatrix} u'_{1,2} & u_{1,2}u'_{1,2}+t_1^{-1}t_2u'_{1,2} & \cdots & * \\ t_1t_2^{-1} & t_1t_2^{-1}u_{1,2}+u'_{2,2} & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & t_{n-1}t_n^{-1} & * \end{pmatrix} \in k\Omega k^{-1}. \tag{8-3}$$

Upon analyzing the corresponding elements in (8-3) individually, we obtain

$$\begin{cases} e_v(t_i) - e_v(t_{i+1}) \ge e_{\min}(\Omega) & \text{if } 1 \le i \le n-1, \\ e_v(u_{i,j}) \ge p_1(n)e_{\min}(\Omega) & \text{if } 1 \le i, j \le n, \end{cases} \tag{8-4}$$

where $p_1(n) > 0$ is an explicit polynomial of n, whose coefficients depends only on $e_v(c_i)$, $0 \le i \le n-1$. Note that

$$\gamma_0^{-1} = \begin{pmatrix} -c_0^{-1}c_1 & -c_0^{-1}c_2 & \cdots & -c_0^{-1} \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \in G(F).$$

Taking the transpose inverse of (8-3) we then obtain $M_1M_2 \in k\Omega^{-1}k^{-1}$, where

$$M_1 := \begin{pmatrix} t_1 & & & \\ t_1u_{1,2} & t_2 & & \\ \vdots & \ddots & \ddots & \\ t_1u_{1,n-1} & \cdots & t_{n-1}u_{n-1,n} & t_n \end{pmatrix},$$

and the matrix M_2 is defined by

$$M_2 := \begin{pmatrix} -c_0^{-1}c_1 & -c_0^{-1}c_2 & \cdots & -c_0^{-1} \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} t_1^{-1} & & & \\ t_2^{-1}u'_{1,2} & t_2^{-1} & & \\ \vdots & \ddots & \ddots & \\ t_n^{-1}u'_{1,n-1} & \cdots & t_n^{-1}u'_{n-1,n} & t_n^{-1} \end{pmatrix}.$$

For $2 \le i \le n$, denote by m_i the $(i, i-1)$-th entry of $M_1 M_2 \in k\Omega^{-1}k^{-1}$. Then

$$m_i = t_1 u_{1,i-1}(-c_0^{-1} c_{i-1} t_{i-1}^{-1} + \cdots - c_0^{-1} t_n^{-1} u'_{i-2,i-1}) + \cdots + t_i t_{i-1}^{-1}.$$

Here we extend the notation by setting $u'_{i-2,i} = 1$ if $i = 2$.

In the expansion of $m_i - t_i t_{i-1}^{-1}$, if a term contains $t_l^{-1} t_j$, then $l \le j$. Consequently, according to (8-4), there exists a polynomial $p_2(n)$, whose coefficients depend solely on $e_v(c_i)$ with $0 \le i \le n-1$, such that

$$e_v(m_i - t_i t_{i-1}^{-1}) \ge p_2(n) e_{\min}(\Omega).$$

Note that $e_v(m_i) \ge e_{\min}(\Omega)$ and $e_{\min}(\Omega) \le 0$. So

$$e_v(t_i t_{i-1}^{-1}) \ge \min\{e_v(m_i), p_2(n) e_{\min}(\Omega)\} \ge (p_2(n) + 1) e_{\min}(\Omega). \tag{8-5}$$

In addition, it follows from $\det x_v \in \mathcal{O}_v^\times$ that $e_v(t_1) + \cdots + e_v(t_n) = 0$. Hence, combining (8-4) and (8-5) we obtain (8-2). □

Remark 8.3. By performing some elementary calculations (albeit tedious), it is possible to explicitly determine the polynomial $p(n)$. However, for our specific purpose, it suffices to note that $p(n)$ depends solely on the value of n and the characteristic polynomial of γ_0.

Lemma 8.4. *Let j be a fixed odd integer coprime to n. Let $\gamma_0 \in G(F)$. Let $v \in \Sigma_{F,\mathrm{fin}}$ be a finite place such that $q_v = \#(\mathcal{O}_v/\mathfrak{p}_v)$ is sufficiently large, where $\mathfrak{p}_v$ is the maximal ideal. Let $J^1 := (1 + \mathfrak{p}_v)(1 + \mathcal{B}^{\frac{j+1}{2}})$, where $\mathcal{B} = \{b = (b_{i,j}) \in G(\mathcal{O}_v) : b_{i,j} \in \varpi_v \mathcal{O}_v,\ 1 \le j \le i \le n\}$. Then there exists a supercuspidal representation σ of $G(F_v)$ with the following properties*:

- *σ has depth j and trivial central character.*
- *There exists a matrix coefficient m_σ of σ such that*

$$m_\sigma(x_v^{-1} \gamma_{0,v} x_v) = c(\gamma_{0,v}, \sigma) \mathbf{1}_\Omega(x_v^{-1} \gamma_{0,v} x_v), \tag{8-6}$$

  where $\gamma_{0,v}$ is the embedding of γ_0 into $G(F_v)$, and
  - *$c(\gamma_{0,v}, \sigma)$ is a nonzero number depending only on F_v, $\gamma_{0,v}$ and σ;*
  - *Ω is a compact set with $-e_{\min}(\Omega) \ll 1$. Here the implied constant depends only on j and n.*

Proof. A construction for such a σ has been presented in [7, §1]. In this context, we follow the reinterpretation outlined in [12, §3] to emphasize the specific properties mentioned in equation (8-6).

Following the notations in [12, §3]. Let β be an $n \times n$ matrix with $\min_{i \in \mathbb{Z}}\{\beta \in \mathcal{B}^i\} = -j$ such that $L = F_v[\beta]$ is a totally ramified field extension of degree n with the property that L^* normalizes $\mathfrak{A} = \{b = (b_{i,j}) \in G(\mathcal{O}_v) : b_{i,j} \in \varpi_v \mathcal{O}_v, 1 \le j < i \le n\}$. We may take β to be "minimal" in the sense of [7, (1.4.14)]. Let σ be the supercuspidal representation associated with β constructed in loc. cit. (or [12, §3]). Then σ has depth j and trivial central character.

Since q_v is sufficiently large, we can define the unique simple character θ on J^1 as $\theta(x) = \psi \circ \mathrm{Tr}(\beta(x-1))$, where ψ is a fixed unramified additive character of F_v. See [7, §3] or [12, Definition 3.6].

Then there is a unique vector $\xi \in \sigma$ such that $g^{-1}J^1 g$ acts on $\pi(g)\xi$ by θ^g for all $g \in G(F_v)$, where $\theta^g(x) := \theta(g^{-1}xg)$. We define a matrix coefficient by

$$m_\sigma(x) := \langle \sigma(x)\sigma(\beta^{-1})\xi, \xi\rangle \mathbf{1}_{x\in J^1\beta},$$

which is a slight variant of the matrix coefficient defined in [12, (3.23)]. Then by loc. cit. we have

$$m_\sigma(x) = \psi \circ \mathrm{Tr}(\beta(x\beta^{-1}-1))\langle \xi, \xi\rangle = \psi \circ \mathrm{Tr}(x-\beta)\langle \xi, \xi\rangle.$$

Taking $x = x_v^{-1}\gamma_{0,v}x_v$, we obtain

$$m_\sigma(x_v^{-1}\gamma_{0,v}x_v) = \psi \circ \mathrm{Tr}(\beta(x_v^{-1}\gamma_{0,v}x_v\beta^{-1}-1))\langle \xi, \xi\rangle = \psi \circ \mathrm{Tr}(\gamma_{0,v}-\beta)\langle \xi, \xi\rangle.$$

Then Lemma 8.4 follows with $c(\gamma_{0,v}, \sigma) = \psi \circ \mathrm{Tr}(\gamma_{0,v}-\beta) \neq 0$ and $\Omega = J^1\beta$. □

Lemma 8.5. *Let $s_0 = \sigma_0 + it_0$ with $\sigma_0 \geq 1/2$ and $t_0 \geq 0$. Let $\gamma_0 \in G(F)$ be regular elliptic. Then there exists a finite place $v \in \Sigma_F$, a real-valued Schwartz function Φ_v on F_v^n, and a compactly supported smooth function φ_v on $\mathrm{PGL}_n(F_v)$ such that $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0) \neq 0$, where $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0)$ is defined by*

$$\int_{G_{\gamma_0}(F_v)\backslash G(F_v)} \varphi_v(x_v^{-1}\gamma_{0,v}x_v) \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau_v(\det t x_v)\,|\det t x_v|_v^{s_0}\, dt\, dx_v.$$

Proof. Set $\alpha = (2\pi)^{-1}t_0$. Let $m \geq 1$ be the norm of the arithmetic conductor of τ. Fix a sufficiently large rational p as in Lemma 8.1. Then $\{\alpha \log p\} \leq n^{-1}p^{-1/6}$. Let v be a place above p. Then $\{\alpha \log q_v\} \leq q_v^{-1/(6n)}$, where q_v is the cardinality of the residue field of F_v.

We may assume γ_0 is given in its companion matrix form (8-1). Since p is sufficiently large, $e_v(c_i) = 0$, $0 \leq i \leq n-1$. Here c_i's are coefficients of the characteristic polynomial of γ_0. Let j be a *fixed* odd integer with $(j, n) = 1$. We will adopt the notation in Lemma 8.4. Take $\varphi_v = m_\sigma$ constructed therein. Then

$$\varphi_v(x_v^{-1}\gamma_{0,v}x_v) = c(\gamma_{0,v}, \sigma)\mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v), \quad c(\gamma_{0,v}, \sigma) \neq 0.$$

Take $\Phi_v = \mathbf{1}_{\mathcal{O}_v^n}$. The integral $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0)$ becomes

$$c(\gamma_{0,v}, \sigma) \int_{G_{\gamma_0}(F_v)\backslash G(F_v)} \mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v) \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det t x_v)\,|\det t x_v|_v^{s_0}\, dt\, dx_v.$$

For each $x_v \in G_{\gamma_0}(F_v)\backslash G(F_v)$, we may assume $\det x_v \in \mathcal{O}_v^\times$. For all x_v such that $x_v^{-1}\gamma_{0,v}x_v \in \Omega$ and $\det x_v \in \mathcal{O}_v^\times$, by Lemma 8.2, we have uniformly that

$$\mathbf{1}_{\varpi_v^{e_1}\mathcal{O}_v}(\eta t) \leq \Phi_v(\eta t x_v) \leq \mathbf{1}_{\varpi_v^{e_2}\mathcal{O}_v}(\eta t), \tag{8-7}$$

where $e_1 \leq e_2$ are two integers depending only on n and j.

Note that $p \equiv 1 \pmod{m}$, which yields $\tau_v(\det t x_v) = 1$. Hence,

$$\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det t x_v)\,|\det t x_v|_v^{s_0} dt = \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\,|\det t x_v|_v^{s_0} dt.$$

Write $|\det tx_v|_v = q_v^{-l}$. By (8-7) and Lemma 8.2 we have $l \geq e_3$, where e_3 is a constant determined only by n and j. Note that

$$\mathrm{Re}(|\det tx_v|_v^{s_0}) = q_v^{-\sigma_0 l}\cos(t_0 l \log q_v) = q_v^{-\sigma_0 l}\cos(2\pi l\{\alpha \log q_v\}).$$

Since $\{\alpha \log q_v\} \leq q_v^{-1/(6n)}$, we have $\cos(2\pi l\{\alpha \log q_v\}) \geq 1/2$ for $l \leq 100^{-1} q_v^{1/(6n)}$. So

$$\mathrm{Re}(|\det tx_v|_v^{s_0}) \geq q_v^{-\sigma_0 l}/2, \quad l \leq 100^{-1} q_v^{1/(6n)}.$$

Therefore, we obtain

$$\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\,\mathrm{Re}(|\det tx_v|_v^{s_0})\,dt \gg 1 + O(\mathcal{E}), \tag{8-8}$$

where the implied constants depend only on n and j, and the "tail" $\mathcal{E}$ is defined by

$$\mathcal{E} := \sum_{l > 100^{-1} q_v^{\delta}} q_v^{-l} \ll q_v^{-q_v^{\delta}}, \quad \delta = \frac{1}{6n}.$$

By taking q_v sufficiently large we see that the right-hand side of (8-8) is positive. Hence,

$$\mathrm{Re}\Big(\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det tx_v)\,|\det tx_v|_v^{s_0}\,dt\Big) \geq c_{n,j},$$

where $c_{n,j} > 0$ is a constant determined by n and j. So

$$\mathrm{Re}\Big(\int_{G_{\gamma_0}(F_v)\backslash G(F_v)} \mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v)\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det tx_v)\,|\det tx_v|_v^{s_0}\,dt\,dx_v\Big) > 0,$$

namely, $\mathrm{Re}(\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0)/c(\gamma_{0,v}, \sigma)) \neq 0$. Hence, $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0) \neq 0$. □

8B2. *Proof of Theorem B.* Fix a field extension E/F of degree n. Let $s_0 \in \mathbb{C} - \{1\}$ be such that $\mathrm{Re}(s) \geq 1/2$. Let $\gamma_0 \in G(F)$ be such that $F[\gamma_0]^\times = E$. Although such γ_0 are not unique, we fix one γ_0. Consider the continuous map

$$\sigma : G(F) \to F^n, \quad \gamma \mapsto (a_{n-1}(\gamma), \dots, a_1(\gamma), a_0(\gamma)),$$

where the $a_i(\gamma)$ are the coefficients of the characteristic polynomial f_γ of γ, i.e., $f_\gamma(t) = \det(tI_n - \gamma) = t^n + a_{n-1}(\gamma)t^{n-1} + \cdots + a_1(\gamma)t + a_0(\gamma)$. Then σ extends to a continuous function $G(\mathbb{A}_F) \to \mathbb{A}_F^n$.

As γ ranges over $G(F)$, the image $\sigma(\gamma)$ becomes discrete in $\mathbb{A}_F^n$. Consequently, there exists a small neighborhood U_0 centered at the identity $I_n \in G(\mathbb{A}_F)$ such that the set $\{x^{-1}\gamma_0 x : x \in G_{\gamma_0}(\mathbb{A}_F)U_0\}$ does not contain any other rational γ distinct from γ_0. By appropriately shrinking the neighborhood, we can ensure that $\tau(\det x) = 1$ and $|\det x| = 1$ for $x \in U_0$. Denote by

$$T(s, x) = \int_{\mathbb{A}_E^\times} \Phi(\eta t x)\tau(\det tx)\,|\det tx|^s\,d^\times t.$$

Then, by Tate's thesis, $T(s, x)$ is an integral representation for $\Lambda(s, \tau \circ N_{E/F})$. So $T(s, x) = Q(s, x)\Lambda(s, \tau \circ N_{E/F})$, where $Q(s, x)$ is a function holomorphic in s and smooth in x, depending on Φ, τ, and E. Moreover, by Tate's thesis, one can choose Φ such that $Q(s,x)\equiv 1$ when $x=1$. Fix that choice of $\Phi=\bigotimes_v \Phi_v$. By continuity there exists a small neighborhood U_1 of $I_n\in G(\mathbb{A}_F)$ such that

$$\mathrm{Re}(Q(s_0,x))>1/2,\quad x\in U_1. \tag{8-9}$$

Let $C:=\{x^{-1}\gamma_0 x : x\in U_0\cap U_1\}$ be a compact subset of $G(\mathbb{A}_F)$. Write $C=\bigotimes_v C_v$. For $v\in\Sigma_F$, define $\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)$ by

$$\int_{G_{\gamma_0}(F_v)\backslash G(F_v)}\varphi_v(x_v^{-1}\gamma_{0,v}x_v)\int_{G_{\gamma_0}(F_v)}\Phi_v(\eta t x_v)\tau_v(\det t x_v)\,|\det t x_v|_v^{s_0}\,dt\,dx_v.$$

This notation has been used in Lemma 8.5. We will construct φ_v and Φ_v as follows.

Let v_* be the finite place defined in Lemma 8.5. In that lemma, we construct functions φ_{v_*} and $\Phi_{v_*}=\mathbf{1}_{\mathcal{O}_{v_*}^n}$. It follows that

$$\mathcal{I}_{v_*}(s_0,\varphi_{v_*},\Phi_{v_*},\gamma_0)\neq 0. \tag{8-10}$$

Note that q_{v_*} is sufficiently large, the v_*-th component of Φ coincides with $\mathbf{1}_{\mathcal{O}_{v_*}^n}$. Therefore, the notation Φ_{v_*}, which refers to both the function constructed in Lemma 8.5 and the v_*-th component of Φ, is consistent and compatible.

For $v\in\Sigma_F-\{v_*\}$, we take $\varphi_v=\mathbf{1}_{C_v}$. Let $\varphi=\bigotimes_{v\in\Sigma_F}\varphi_v$. By construction, there exists a finite set S of places, such that $C_v=K_v$ and $\Phi_v=\mathbf{1}_{\mathcal{O}_v^n}$ for $v\notin S$. Then the local component $Q_v(s_0,x_v)\equiv 1$ for $x_v\in C_v$, $v\notin S$. As a consequence, we obtain, for $v\notin S$, that

$$\prod_{v\notin S}\frac{\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)}{L_v(s_0,\tau_v\circ N_{E_v/F_v})}=\prod_{v\notin S}\int_{G_{\gamma_0}(F_v)\backslash G(F_v)}\varphi_v(x_v^{-1}\gamma_{0,v}x_v)\,dx_v>0. \tag{8-11}$$

By (8-9) and (8-10), $\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)\neq 0$, for $v\in S$. Hence,

$$\prod_{v\in S}\frac{\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)}{L_v(s_0,\tau_v\circ N_{E_v/F_v})}\neq 0. \tag{8-12}$$

It the follows from (8-11) and (8-12) that

$$\int_{G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_0 x)Q(s_0,x)\,dx=\prod_{v\in\Sigma_F}\frac{\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)}{L_v(s_0,\tau_v\circ N_{E_v/F_v})}\neq 0. \tag{8-13}$$

Substituting this choice of φ into Theorem A we obtain

$$\frac{I_0^{\varphi}(s,\tau)}{\Lambda(s,\tau)}=\frac{\Lambda(s,\tau\circ N_{E/F})}{n\Lambda(s,\tau)}\int_{G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_0 x)Q(s_0,x)\,dx. \tag{8-14}$$

Assume that the twisted adjoint L-function $L(s,\pi,\mathrm{Ad}\otimes\tau)$ is holomorphic outside $s\in\{0,1\}$ for all $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\mathbf{1})$, which is the subset of cuspidal representations with a supercuspidal component. By spectral expansion (2-5), the function $I_0^{\varphi}(s,\tau)/\Lambda(s,\tau)$ is holomorphic at $s=s_0$. Therefore, it follows from (8-13) and (8-14) that the meromorphic function $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is holomorphic at $s=s_0$.

Since s_0 is arbitrary with $\mathrm{Re}(s_0) \geq 1/2$, $\Lambda(s, \tau \circ N_{E/F})/\Lambda(s, \tau)$ is holomorphic in $\mathrm{Re}(s) \geq 1/2$ and $s \neq 1$. Utilizing the functional equation, we thus conclude that $\Lambda(s, \tau \circ N_{E/F})/\Lambda(s, \tau)$ is holomorphic at all $s \in \mathbb{C} - \{0, 1\}$. So the τ-twisted Dedekind conjecture holds. Then Theorem B follows.

Remark 8.6. It is conjectured in [18; 19] that the reverse direction also holds, namely, the τ-twisted Dedekind conjecture for all field extensions E/F of degree n should imply holomorphy of the τ-twisted adjoint L-functions. This is proved in [36] for $n \leq 4$.

Proof of Corollary 1.6. Let E be a field extension of F of degree n, such that $\zeta_E(1/2) \neq 0$. By the proof of Theorem B, one can choose some test function φ such that

$$\int_{G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(x^{-1}\gamma_0 x) Q(1/2, x)\, dx \neq 0.$$

By (8-14) we have $I_0^{\varphi}(1/2, \tau) \neq 0$.

Corollary 1.6 then follows from the spectral expansion (2-5) of the cuspidal kernel function $\mathrm{K}_0(x, x)$. □

9. Meromorphic continuation: the exceptional case

In Sections 6 and 7, we established the holomorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ to the complex plane, except in the case where τ is an exceptional representation, meaning $\tau^k = \mathbf{1}$ for some $1 \leq k \leq n$. It is important to note that the number of exceptional τ is finite.

The main focus of this section is to derive a specific meromorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ beyond the region $\mathrm{Re}(s) > 1$, specifically for these exceptional representations. However, it is important to mention that for general n, we lack a symmetrical description of this continuation process. Thus, we will restrict our analysis to cases where $n \leq 4$ in this paper.

To illustrate with a simplified example, let's focus on the case when $n = 2$ and make certain oversimplifications. We can approximate the expression as follows:

$$I_{\mathrm{Whi}}(s, \tau) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{(s-1-\lambda)(s-1+\lambda)}\, d\lambda, \quad \mathrm{Re}(s) > 1.$$

Now we fix s such that $1 < \mathrm{Re}(s) < 1 + \epsilon/2$, for a small $\epsilon > 0$. By shifting the contour, as explained in [18], we have:

$$I_{\mathrm{Whi}}(s, \tau) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{(s-1-\lambda)(s-1+\lambda)}\, d\lambda - \frac{1}{2(s-1)}. \tag{9-1}$$

The right side of (9-1) defines a meromorphic function in the region $1 - \epsilon/2 < \mathrm{Re}(s) < 1 + \epsilon/2$ with a simple pole at $s = 1$. Therefore, we obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ to the region $\mathrm{Re}(s) > 1 - \epsilon/2$. By repeating this process, we can achieve a meromorphic continuation to the entire complex plane with an explicit description of the poles.

Just as the above prototype, the genuine situation admits the same idea of continuation, but with more delicate techniques required, since $I_{\mathrm{Whi}}(s, \tau)$ is typically infinitely many sums of such integrals.

Details will be provided in the following subsections. Moreover, we find all possible explicit poles of the continuation of each such integral as well, and show they cancel with each other except for $s=1/2$, where $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ has at most a simple pole if $\tau^2=1$.

9A. *Notation.* We fix an *exceptional* unitary character τ, e.g., $\tau=\mathbf{1}$. Let $\mathcal{D}_\tau$ be a standard (open) zero-free region of $L(s,\tau)$ (see, for example, [6]). Fix such a $\mathcal{D}_\tau$ once and for all. We thus can form a domain

$$\mathcal{R}(1/2;\tau)^- := \{s\in\mathbb{C} : 2s\in\mathcal{D}_\tau\} \supsetneq \{s\in\mathbb{C} : \mathrm{Re}(s)\geq 1/2\}. \tag{9-2}$$

In Section 9C, we will obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to the region $\mathcal{R}(1/2;\tau)^-$. In conjunction with the functional equation we then obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to the whole complex plane. The zero-free region plays a role as the strip $\mathrm{Re}(s)>1-\epsilon/2$ in (9-1).

Let Q be a standard parabolic subgroup of G of type $(n_1,n_2,\dots,n_r)$. Let $M=\mathrm{diag}(M_1,M_2,\dots,M_r)$ be the Levi of Q. Let $\mathfrak{X}_Q$ be the subset of cuspidal data $\chi=\{(M,\sigma)\}$. We may write $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_r)$, where $\sigma_i\in\mathcal{A}_0(M_i(F)\backslash M_i(\mathbb{A}_F))$. Let π be a representation induced from $\chi=\{(M,\sigma)\}$.

For any $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*\simeq(i\mathbb{R})^{r-1}$, satisfying that $\lambda_1+\lambda_2+\cdots+\lambda_r=0$, we let $\boldsymbol{\kappa}=(\kappa_1,\kappa_2,\dots,\kappa_r)\in\mathbb{C}^{r-1}$ be such that

$$\begin{aligned}\kappa_j&=\lambda_j-\lambda_{j+1},\ 1\leq j\leq r-1,\\ \kappa_r&=\lambda_1-\lambda_r=\kappa_1+\kappa_2+\cdots+\kappa_{r-1}.\end{aligned} \tag{9-3}$$

Then we have a bijection $i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*\leftrightarrow i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, $\lambda\mapsto\boldsymbol{\kappa}$ given by (9-3), which induces a change of coordinates with $d\lambda=m_Q\,d\boldsymbol{\kappa}$, where m_Q is an absolute constant (the determinant of the transform (9-3)). So that we can write $\lambda=\lambda(\boldsymbol{\kappa})$. Recall that

$$R_\varphi(s,\lambda;Q,\chi)=\sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}}\frac{\langle\mathcal{I}_Q(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})},\quad \mathrm{Re}(s)>1, \tag{7-13}$$

where $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$ is the complete L-function. Then we can write

$$R_\varphi(s,\lambda;Q,\chi)=R_\varphi(s,\boldsymbol{\kappa};Q,\chi),\quad \Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})=\Lambda(s,\pi_{\boldsymbol{\kappa}}\otimes\tau\times\widetilde{\pi}_{-\boldsymbol{\kappa}}).$$

Let v be a place. Recall from Section 7B the definition

$$R_v(s,W_{1,v},W_{2,v};\lambda):=\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})},\quad \mathrm{Re}(s)>1. \tag{7-6}$$

If π_v is unramified, Φ_v is the characteristic function of $G(\mathcal{O}_{F,v})$, and $W_{1,v}=W_{2,v}$ is the normalized spherical vector in the Whittaker model; then, by Corollary 7.5, $R_v(s,W_{1,v},W_{2,v};\boldsymbol{\kappa}):=R_v(s,W_{1,v},W_{2,v};\lambda)$ is equal to

$$\prod_{1\leq i<r}\prod_{i<j\leq r}L_v(1+\boldsymbol{\kappa}_{i,j},\sigma_{i,v}\times\tilde{\sigma}_{j,v})^{-1}\cdot L_v(1-\boldsymbol{\kappa}_{i,j},\tilde{\sigma}_{i,v}\times\sigma_{j,v})^{-1}, \tag{9-4}$$

where $\boldsymbol{\kappa}_{i,j}=\kappa_i+\cdots+\kappa_{j-1}$. Since φ is K-finite, then there exists a finite set $S_{\varphi,\tau,\Phi}$ of finite places such that for *any* π from a cuspidal datum $\chi\in\mathfrak{X}_Q$ in the spectral side, $R_v(s,W_{1,v},W_{2,v};\boldsymbol{\kappa})$ is equal to the formula in (9-4) if $v \notin S_{\varphi,\tau,\Phi}$. In particular, $R_v(s, W_{1,v}, W_{2,v}; \boldsymbol{\kappa})$ is independent of s for all but finitely many places v. Moreover, $R_\varphi(s, \boldsymbol{\kappa}; Q, \chi)L(\boldsymbol{\kappa}, \pi, \widetilde{\pi})$ is holomorphic as a function of s and $\boldsymbol{\kappa}$, where

$$L(\boldsymbol{\kappa}, \pi, \widetilde{\pi}) := \prod_{1\le i<r} \prod_{i<j\le r} \Lambda(1+\boldsymbol{\kappa}_{i,j}, \sigma_i \times \tilde{\sigma}_j) \cdot \Lambda(1-\boldsymbol{\kappa}_{i,j}, \tilde{\sigma}_i \times \sigma_j).$$

Let $1 \le m, m' \le n$ be two integers. Let $\sigma \in \mathcal{A}_0(\mathrm{GL}_m(F)\backslash \mathrm{GL}_m(\mathbb{A}_F))$ and $\sigma' \in \mathcal{A}_0(\mathrm{GL}_{m'}(F)\backslash \mathrm{GL}_{m'}(\mathbb{A}_F))$. Fix $\epsilon_0 > 0$. For any $c' > 0$, let $\mathcal{D}_{c'}(\sigma, \sigma')$ be

$$\left\{\kappa = \beta + i\gamma : \beta \ge 1 - c' \cdot \left[\frac{(C(\sigma)C(\sigma'))^{-2(m+m')}}{(|\gamma|+3)^{2mm'[F:\mathbb{Q}]}}\right]^{\frac{1}{2}+\frac{1}{2(m+m')}-\epsilon_0}\right\} \tag{9-5}$$

if $\sigma' \not\simeq \tilde{\sigma}$; and let $\mathcal{D}_{c'}(\sigma, \sigma')$ denote the region

$$\left\{\kappa = \beta + i\gamma : \beta \ge 1 - c' \cdot \left[\frac{(C(\sigma))^{-8m}}{(|\gamma|+3)^{2mm'[F:\mathbb{Q}]}}\right]^{-\frac{7}{8}+\frac{5}{8m}-\epsilon_0}\right\} \tag{9-6}$$

if $\sigma' \simeq \tilde{\sigma}$. According to [6] and the Appendix of [22], there exists a constant $c_{m,m'} > 0$ depending only on m and m', such that $L(s, \sigma \times \sigma') \ne 0$ in $s \in \mathcal{D}_{c_{m,m'}}(\sigma, \sigma')$.

We may assume that c (depending only on n) is small such that $1 \pm 2\boldsymbol{\kappa}_{i,j} \in \mathcal{D}_c(\sigma_i, \tilde{\sigma}_j)$, and the boundary of $\mathcal{D}_c(\sigma_i, \tilde{\sigma}_j)$ lies in the strip $1 - 1/(n+4) < \mathrm{Re}(\kappa_j) < 1$, $1 \le i \le j \le r$. Let $\mathcal{D}_\chi = \bigcap_{1\le i\le j\le r} \mathcal{D}_c(\sigma_i, \tilde{\sigma}_j)$. Then $L(\boldsymbol{\kappa}, \pi, \widetilde{\pi}) \ne 0$ if $\boldsymbol{\kappa} \in \mathcal{D}_\chi^{r-1} = \mathcal{D}_\chi \times \cdots \times \mathcal{D}_\chi$.

Let $\mathcal{C}_\chi$ be the boundary of $\mathcal{D}_\chi$. For $\epsilon \in \{0, 1\}$, we define

$$\mathcal{C}_\chi(\epsilon) := \begin{cases} \mathcal{C}_\chi & \text{if } \epsilon = 1, \\ i\mathbb{R}, & \text{if } \epsilon = 0. \end{cases}$$

Set $\mathcal{C}_\chi(\boldsymbol{\epsilon}) = \mathcal{C}_\chi(\epsilon_1) \times \cdots \times \mathcal{C}_\chi(\epsilon_{r-1})$, $\epsilon_l \in \{0, 1\}$, $1 \le l \le r-1$.

Let $\mathrm{Re}(s) > 1$. For any $\phi \in \mathfrak{B}_{Q,\chi}$ and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_{r-1}) \in \{0, 1\}^{r-1}$, let

$$J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\boldsymbol{\epsilon})) = \int_{\mathcal{C}_\chi(\boldsymbol{\epsilon})} R_\varphi(s, \boldsymbol{\kappa}; Q, \chi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}})\, d\boldsymbol{\kappa}. \tag{9-7}$$

which is well defined because $J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\boldsymbol{\epsilon})) = J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0}))$ (by the Cauchy integral formula), where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^{r-1}$. Therefore, according to Theorem E, for any s with $\mathrm{Re}(s) > 1$,

$$\sum_{P} \frac{1}{c_Q} \sum_{\chi\in\mathfrak{X}_Q} \sum_{\phi\in\mathfrak{B}_{Q,\chi}} \left| \int_{\mathcal{C}_\chi(\boldsymbol{\epsilon})} R_\varphi(s, \boldsymbol{\kappa}; Q, \chi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}})\, d\boldsymbol{\kappa} \right| < \infty.$$

For any $\beta \ge 1/2$, we set

$$\mathcal{R}(\beta; \chi) = \{s \in \beta - 1 + \mathcal{D}_\chi\} \cap \{s \in \beta + 1 - \mathcal{D}_\chi\}. \tag{9-8}$$

9B. ***Meromorphic continuation via the zero-free region.*** Let $s \in \mathcal{D}_\chi$ with $\mathrm{Re}(s) > 1$. Define

$$\mathcal{F}(\boldsymbol{\kappa}; s) = \mathcal{F}(\boldsymbol{\kappa}; s, Q, \chi) = R_\varphi(s, \boldsymbol{\kappa}; Q, \chi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}}) \tag{9-9}$$

if χ is fixed in the context. Then by Proposition 7.4 in Section 5B we see that $\mathcal{F}(\boldsymbol{\kappa}; s)$ is equal to a holomorphic function multiplying

$$\prod_{k=1}^{r} \Lambda(s, \sigma_k \otimes \tau \times \tilde{\sigma}_k) \prod_{j=1}^{r-1} \prod_{i=1}^{j} \frac{\Lambda(s + \kappa_{i,j}, \sigma_i \otimes \tau \times \tilde{\sigma}_{j+1}) \Lambda(s - \kappa_{i,j}, \sigma_{j+1} \otimes \tau \times \tilde{\sigma}_i)}{\Lambda(1 + \kappa_{i,j}, \sigma_i \times \tilde{\sigma}_{j+1}) \Lambda(1 - \kappa_{i,j}, \sigma_{j+1} \times \tilde{\sigma}_i)}.$$

Let $\mathcal{G}(\boldsymbol{\kappa}; s) = \mathcal{G}(\boldsymbol{\kappa}; s, Q, \chi)$ denote the above product. The denominators of $\mathcal{G}(\boldsymbol{\kappa}; s)$ are L-functions from Langlands–Shahidi method, and they will play an important role in the meromorphic continuation across the critical line $\mathrm{Re}(s) = 1$.

Let $\mathcal{C}$ denote the boundary $\mathcal{C}_\chi(1)$ and (0) represent the imaginary axis. Analyzing the potential poles of $\mathcal{G}(\boldsymbol{\kappa}; s)$, along with contour shifting, yields an expression for the integral $J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0})) = J_{Q,\chi}(s; \phi, \mathcal{C}) - \mathcal{J}_\chi(s)$, where $\mathcal{J}_\chi(s)$ is defined by

$$2\pi i \sum_{j=1}^{r-1} \sum_{i=1}^{j} \int_{(0)} \cdots \int_{(0)} d\kappa_{j-1} \cdots d\kappa_1 \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{i,j}=s-1} \mathcal{F}(\boldsymbol{\kappa}; s)\, d\kappa_{r-1} \cdots d\kappa_{j+1}.$$

Here $\mathrm{Res}_{\kappa_{i,j}=s-1} \mathcal{F}(\boldsymbol{\kappa}; s)$ vanishes identically unless $\sigma_i \otimes \tau \simeq \sigma_{j+1}$, in which case one must have $n_i = n_{j+1}$. To obtain a meromorphic continuation of $J_{Q,\chi}(s; \phi, \mathcal{C})$ to the critical strip $0 < \mathrm{Re}(s) < 1$, we start with the following initial step:

Proposition 9.1. *Let notation be as before. Let* $\chi \in \mathfrak{X}_Q$. *Let* $s \in \mathcal{R}(\beta; \chi)$ *and* $\mathrm{Re}(s) > 1$. *Then*

$$\sum_{\phi \in \mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0})) = \sum_{\phi \in \mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s; \phi, \mathcal{C}) - \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{J}(s; \phi, \mathcal{C}), \tag{9-10}$$

where $\mathcal{C} = \mathcal{C}_\chi(1)$, *and the summand* $\mathcal{J}(s; \phi, \mathcal{C})$ *is defined to be*

$$\sum_{m=1}^{r-1} \sum_{\substack{j_m, j_{m-1}, \ldots, j_1 \\ 1 \le j_m < \cdots < j_1 \le r-1}} \cdots \sum c_{j_1, \ldots, j_m} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_m}=s-1} \cdots \operatorname*{Res}_{\kappa_{j_1}=s-1} \mathcal{F}(\boldsymbol{\kappa}; s) \frac{d\kappa_{r-1} \cdots d\kappa_1}{d\kappa_{j_m} \cdots d\kappa_{j_1}},$$

where the expression $d\kappa_{r-1} \cdots d\kappa_1 / (d\kappa_{j_m} \cdots d\kappa_{j_1})$ *denotes omitting* $d\kappa_{j_m}, \ldots, d\kappa_{j_1}$ *from* $d\kappa_{r-1} \cdots d\kappa_1$, *and the coefficients* $c_{j_1, \ldots, j_m}$ *are explicit integers. Each term in* (9-10) *converges absolutely within* $\mathcal{R}(1; \chi) \setminus 1$, *where* $\mathcal{R}(1; \chi)$ *is defined in* (9-8). *Consequently,* (9-10) *provides a meromorphic continuation of the function*

$$\sum_{\phi \in \mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0}))$$

to $\mathcal{R}(1; \chi)$, *potentially having a pole at* $s = 1$.

Proof. For any $1 \le j \le r-1$ and $1 \le i \le j$, if $n_i = n_{j+1}$, we can make the following change of variables to simplify the integral of $\mathrm{Res}_{\kappa_{i,j}=s-1} \mathcal{F}(\boldsymbol{\kappa}; s)$:

$$\lambda_i' = \lambda_j, \quad \lambda_j' = \lambda_i, \quad \lambda_l' = \lambda_l \text{ for } l \neq i, j;$$
$$\sigma_i' = \sigma_j, \quad \sigma_j' = \sigma_i, \quad \sigma_l' = \sigma_l \text{ for } l \neq i, j.$$

Let $\kappa_l=\lambda_l-\lambda_{l+1}$, $\kappa_l'=\lambda_l'-\lambda_{l+1}'$, $1\leq l\leq r-1$; and $\kappa_{l,m}'=\kappa_l'+\cdots+\kappa_m'$, $1\leq l\leq m\leq r-1$. To describe the relation between $\{\kappa_l:1\leq l\leq r-1\}$ and $\{\kappa_l':1\leq l\leq r-1\}$, we argue as follows:

Case $i=j-1$: A direct computation shows that

$$\kappa_{i-1}=\kappa_{i-1,i}',\qquad \kappa_i=-\kappa_i',\qquad \kappa_{i+1}=\kappa_{i,i+1}',\qquad \kappa_l=\kappa_l',\ \ 1\leq l\leq r-1,\ l\neq i-1,i,i+1.$$

Hence, $\operatorname{Re}(\kappa_l)=0$, $1\leq l\leq i=j-1$ amounts to $\operatorname{Re}(\kappa_l')=0$, $1\leq l\leq i=j-1$. The determinant of transition matrix is $\det\{\partial\kappa_l/\partial\kappa_m'\}_{1\leq l,m\leq r-1}=-1$. Note that $\kappa_j'=\kappa_{i,j}=s-1$, leading to

$$\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}$$
$$=-\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}'\cdots d\kappa_1'\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_j'=s-1}\mathcal{F}(\boldsymbol{\kappa};s,Q,\chi')\,d\kappa_{r-1}'\cdots d\kappa_{j+1}',$$

where χ' is the cuspidal datum attached to the representations $(\sigma_1',\ldots,\sigma_r')$. Hence $\chi'=\chi$ as an equivalent class.

Case $i\leq j-2$: A direct computation leads to

$$\kappa_{i-1}=\kappa_{i-1,j-1}',\qquad \kappa_i=-\kappa_{i+1,j-1}',\qquad \kappa_{j-1}=-\kappa_{i,j-2}',\qquad \kappa_j=\kappa_{i,j}',$$
$$\kappa_l=\kappa_l',\ \ 1\leq l\leq r-1,\ l\neq i-1,i,j-1,j.$$

In addition, $\operatorname{Re}(\kappa_l)=0$, $1\leq l\leq i=j-1$ amounts to $\operatorname{Re}(\kappa_l')=0$, $1\leq l\leq i=j-1$. Again we have $\det\{\partial\kappa_l/\partial\kappa_m'\}_{l,m}=-1$, and $\kappa_j'=\kappa_{i,j}$. Therefore,

$$\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}$$
$$=-\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}'\cdots d\kappa_1'\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa};s,Q,\chi')\,d\kappa_{r-1}'\cdots d\kappa_{j+1}'.$$

If $n_i\neq n_{j+1}$, then $\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)=0$. In all, we have

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}$$
$$=-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}.$$

Therefore, $\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C}_\chi(\mathbf{0}))-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C})$ is equal to

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\sum_{j=1}^{r-1}c_j'\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1},$$

where the c_j's are some explicit constants, depending only on the type of Q. Consider

$$\int_{(0)}\cdots\int_{(0)}d\kappa_{j_1-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j_1+1},\quad 1\leq j_1\leq r-1.$$

By the Cauchy integral formula we can write this as the sum of

$$\int_{\mathcal{C}}\cdots\int_{\mathcal{C}} d\kappa_{j_1-1}\cdots d\kappa_1 \int_{\mathcal{C}}\cdots\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_1}=s-1} \mathcal{F}(\boldsymbol{\kappa};s)\, d\kappa_{r-1}\cdots d\kappa_{j_1+1}$$

and

$$\sum_{j_2=1}^{j_1-1}\sum_{i_2=1}^{j_2} c'_{i_2,j_2}\int_{(0)}\cdots\int_{(0)} d\kappa_{j_2-1}\cdots d\kappa_1 \int_{\mathcal{C}}\cdots\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{i_2,j_2}=s-1}\ \operatorname*{Res}_{\kappa_{j_1}=s-1} \mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_{j_2+1}}{d\kappa_{j_1}},$$

where the c'_{i_2,j_2} are some explicit integers depending only on the type of Q.

One can do a similar analysis to replace $\kappa_{i_2,j_2}=s-1$ with $\kappa_{j_2}=s-1$. By induction (or just continuing this process until $m=r-1$) we obtain (9-10). Recall that by the definition in (9-9), we have

$$\mathcal{F}(\boldsymbol{\kappa};s)=\sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\Psi(s,W_1,W_2;\lambda).$$

Then $\mathcal{F}(\boldsymbol{\kappa};s)$ is a Schwartz function of $\boldsymbol{\kappa}$. Hence all the above integrals converge absolutely, and the proof is completed. □

Let notation be as in Proposition 9.1. Denote by $\mathcal{I}_0(s;\chi)$ the summand of the first term of the right-hand side of (9-10), i.e.,

$$\mathcal{I}_{0,\chi}(s)=\sum_{\phi\in\mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s;\phi,\mathcal{C}_\chi),\quad s\in\mathcal{R}(1;\chi),\ \operatorname{Re}(s)>1.$$

Proposition 9.2. *Let notation be as before. Let* $s\in\mathcal{R}(1;\chi)$ *and* $\operatorname{Re}(s)>1$. *Then*

$$\mathcal{I}_{0,\chi}(s)=\sum_{\phi\in\mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s;\phi,\mathcal{C}_\chi)+\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\mathcal{J}_\chi^0(s),\tag{9-11}$$

where the summand $\mathcal{J}_\chi^0(s)$ *is defined by*

$$\sum_{m=1}^{r-1}\ \sum_{\substack{j_m,j_{m-1},\dots,j_1\\ 1\le j_m<\cdots<j_1\le r-1}}\!\!\cdots\sum \tilde{c}_{j_1,\dots,j_m}\int_{(0)}\cdots\cdots\int_{(0)} \operatorname*{Res}_{\kappa_{j_m}=1-s}\cdots\operatorname*{Res}_{\kappa_{j_1}=1-s}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}},$$

where the $\tilde{c}_{j_1,\dots,j_m}$ *are some explicit integers, depending only on* Q. *Moreover, the terms in* (9-11) *converges absolutely and normally inside any bounded strip.*

We omit the proof, which is similar to that of Proposition 9.1.

9C. ***Meromorphic continuation inside the critical strip.*** Let $s\in\mathcal{R}(1;\chi)$ and $1\le m\le r-1$. Let $j_1,j_2,\dots,j_m$ be m integers such that $1\le j_m<\cdots<j_1\le r-1$. Consider the summand in the second term of (9-10):

$$\mathcal{I}_{m,\chi}(s):=\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\cdots\cdots\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_m}=s-1}\cdots\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}}.$$

Then each $\mathcal{I}_{m,\chi}(s)$ is naturally meromorphic in $\mathcal{R}(1;\chi)$ with a possible at $s=1$.

Theorem G. *Let $n \leq 4$ and consider the notation as previously defined. Take $\chi \in \mathfrak{X}_Q$. Assume that the adjoint L-function $L(s, \sigma, \mathrm{Ad} \otimes \tau)$ is holomorphic within the strip $0 < \mathrm{Re}(s) < 1$ for all cuspidal representations $\sigma \in \mathcal{A}_0(\mathrm{GL}_k(\mathbb{A}_F))$, where $1 \leq k \leq n-1$. Then, for any $0 \leq m \leq r-1$, the function*

$$\sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{I}_{m,\chi}(s), \quad s \in \mathcal{R}(1; \chi),$$

admits a meromorphic continuation to the area $\mathcal{R}(1/2; \tau)^-$, with simple poles possible only at $s \in \left\{\frac{1}{2}, \frac{2}{3}, \dots, 1 - \frac{1}{n}, 1\right\}$, where $\mathcal{R}(1/2; \tau)^-$ is defined in (9-2). For any $3 \leq k \leq n$, if $L((k-1)/k, \tau) = 0$, then $s = (k-1)/k$ is not a pole.

Remark 9.4. If F is a function field, the number of cuspidal data χ's appearing in the spectral decomposition for a fixed test function is finite. As a consequence, Theorem G provides the continuation of the entire spectral side in the function field scenario.

Remark 9.5. As can be seen from the proof, when $n \leq 3$ we can continue the functions $\sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{I}_{m,\chi}(s)$ to $\mathrm{Re}(s) > 1/3$. When $n = 4$, we can only continue $\sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{I}_{m,\chi}(s)$ to $\mathcal{R}(1/2; \tau)^-$, an open region just containing the right half-plane $\mathrm{Re}(s) \geq 1/2$. This is because some of its components involve $\Lambda(2s, \tau^2)^{-1}$ as a factor. The key ingredient is that $\mathcal{R}(1/2; \tau)^-$ is uniform with respect to $\chi \in \mathfrak{X}_Q$.

Let notation be as before. To simplify our computations below, we shall write, for any $\beta \in \mathbb{R}$, that $\mathcal{R}(\beta) = \mathcal{R}(\beta; \chi)$, $\mathcal{R}(\beta)^- = \mathcal{R}(\beta; \chi) \cap \{s : \mathrm{Re}(s) < \beta\}$, and $\mathcal{R}(\beta)^+ = \mathcal{R}(\beta; \chi) \cap \{s : \mathrm{Re}(s) > \beta\}$. We also use $S_{(a,b)}$ to denote the strip $a < \mathrm{Re}(s) < b$, for any $a < b$.

9C1. *Proof of Theorem G when $n = 3$.*

Proof. Let $n = 3$. Then there are two possibilities: $r = 2$ or $r = 3$. If $r = 2$, then the parabolic subgroup Q is maximal, and any associated cuspidal datum is of the form $\chi \simeq (\sigma_1, \sigma_2)$, where σ_1 is a cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_F)$ and σ_2 is a Hecke character on $\mathbb{A}_F^\times$. In this case, $\mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to an entire function multiplying

$$\frac{\Lambda(s + \kappa_1, \sigma_1 \otimes \tau \times \tilde{\sigma}_2)\Lambda(s - \kappa_1, \sigma_2 \otimes \tau \times \tilde{\sigma}_1)}{\Lambda(1 + \kappa_1, \sigma_1 \times \tilde{\sigma}_2)\Lambda(1 - \kappa_1, \sigma_2 \times \tilde{\sigma}_1)} \cdot \prod_{k=1}^{2} \Lambda(s, \sigma_k \otimes \tau \times \tilde{\sigma}_k). \tag{9-12}$$

Since each completed L-functions in (9-12) is entire inside $S_{(0,1)}$, then $\mathcal{F}(\boldsymbol{\kappa}, s)$ is holomorphic (after continuation) when $0 < \mathrm{Re}(s) < 1$. On the other hand, $\mathcal{F}(\boldsymbol{\kappa}, s)$ vanishes when $\mathrm{Im}(\kappa_1) \to \infty$. Let $\mathrm{Re}(s) > 1$. By the Cauchy integral formula,

$$J_{Q,\chi}(s; \phi, \mathcal{C}(\mathbf{0})) = \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s)\, d\kappa_1 = \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \int_{\mathcal{C}} \mathcal{F}(\boldsymbol{\kappa}, s)\, d\kappa_1,$$

which gives holomorphic continuation to $\mathrm{Re}(s) > 1 - \epsilon_1$, for some $\epsilon_1 > 0$. Hence we obtain holomorphic continuation of $J_{Q,\chi}(s; \phi, \mathcal{C}(\mathbf{0}))$ to $\mathrm{Re}(s) > 0$.

Now we handle the more complicated case where $r = 3$. In this case, cuspidal data χ correspond to (χ_1, χ_2, χ_3), where χ_i's are unitary Hecke characters such that $\chi_1\chi_2\chi_3 = \omega$, the fixed central character.

Then $\mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to

$$\mathcal{H}(s,\boldsymbol{\kappa})\,\Lambda(s,\tau)^3\prod_{j=1}^{2}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\tau\chi_i\overline{\chi}_{j+1})\,\Lambda(s-\kappa_{i,j},\tau\chi_{j+1}\overline{\chi}_i)}{\Lambda(1+\kappa_{i,j},\chi_i\overline{\chi}_{j+1})\,\Lambda(1-\kappa_{i,j},\chi_{j+1}\overline{\chi}_i)},\tag{9-13}$$

where $\mathcal{H}(s,\boldsymbol{\kappa})$ is an entire function and $\Lambda(s,\chi')$ is the completed Hecke L-function associated to the unitary Hecke character χ' over F. Let $\sum_{\phi}$ denote the sum over $\phi\in\mathfrak{B}_{Q,\chi}$. Then, by Proposition 9.1,

$$\begin{aligned}J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\sum_{\phi}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\,d\kappa_1-c_{1,2}\sum_{\phi}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\ -c_1\sum_{\phi}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\sum_{\phi}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1,\end{aligned}$$

for some integers c_1, c_2 and $c_{1,2}$; and $s\in\mathcal{D}_\chi$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the right-hand side of the above equality. Then $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $s\in\mathcal{R}(1)$. In particular, we get a meromorphic continuation inside $\mathcal{R}(1)^-$ with a possible pole at $s=1$.

Recall that, for meromorphic functions $A(s)$ and $B(s)$, we write $A(s)\propto B(s)$ if there exists some holomorphic function $C(s)$ such that $A(s)=C(s)B(s)$. By (9-13),

$$\begin{aligned}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\propto\frac{\Lambda(s-\kappa_2,\chi_1\overline{\chi}_2\tau)\,\Lambda(2s-1+\kappa_2,\chi_2\overline{\chi}_1\tau^2)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(1+\kappa_2,\chi_2\overline{\chi}_1)\,\Lambda(2-s-\kappa_2,\chi_1\overline{\chi}_2\tau^{-1})\,\Lambda(2-s,\tau^{-1})},\\ \operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\propto\frac{\Lambda(s-\kappa_1,\chi_2\overline{\chi}_1\tau)\,\Lambda(2s-1+\kappa_1,\chi_1\overline{\chi}_2\tau^2)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_1\overline{\chi}_2)\,\Lambda(2-s-\kappa_1,\chi_2\overline{\chi}_1\tau^{-1})\,\Lambda(2-s,\tau^{-1})},\\ \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.\end{aligned}$$

Hence by the Cauchy integral formula we have, for $s\in\mathcal{R}(1)^-$,

$$\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2=\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-2\pi i\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s),\tag{9-14}$$

where the right-hand side is holomorphic inside $1/2<\mathrm{Re}(s)<1$, since

$$\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.\tag{9-15}$$

From (9-15) we see $\int_{\mathcal{C}}\mathrm{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2$ has a potential pole at $s=2/3$ when $\tau^3=1$. Likewise, we have the continuation for $\int_{\mathcal{C}}\mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$:

$$\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1=\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-2\pi i\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s),\tag{9-16}$$

where the right-hand side is holomorphic inside $1/2<\mathrm{Re}(s)<1$, since

$$\mathrm{Res}_{\kappa_1=2-2s}\,\mathrm{Res}_{\kappa_2=s-1}\,\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.\tag{9-17}$$

From (9-17) we see $\int_{\mathcal{C}} \mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$ has a potential pole at $s=2/3$ when $\tau^3=1$. Now we deal with the remaining term, $\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$. By Proposition 9.2, for $s\in\mathcal{R}(1)^-$, there are integers c_1, c_2 and $c_{1,2}$ such that

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1=\sum_{\phi}\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\sum_{\phi}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$$
$$-c'_1\sum_{\phi}\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c'_2\sum_{\phi}\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1.$$

Using (9-13), one can compute the partial residues of $\mathcal{F}(\boldsymbol{\kappa},s)$:

$$\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_1,\chi_1\overline{\chi}_2\tau)\Lambda(2s-1-\kappa_1,\chi_2\overline{\chi}_1\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_1,\chi_2\overline{\chi}_1)\Lambda(2-s+\kappa_1,\chi_1\overline{\chi}_2\tau^{-1})\Lambda(2-s,\tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_2,\chi_2\overline{\chi}_1\tau)\Lambda(2s-1-\kappa_2,\chi_1\overline{\chi}_2\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_2,\chi_1\overline{\chi}_2)\Lambda(2-s+\kappa_2,\chi_2\overline{\chi}_1\tau^{-1})\Lambda(2-s,\tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Combining these formulas with the analytic behavior of the function $\mathrm{Res}_{\kappa_1=s-1}\mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ we conclude that $\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ admits a meromorphic continuation to $1/2<\mathrm{Re}(s)<1$, with a possible pole at $s=2/3$ when $\tau^3=1$. Denote by $J_{Q,\chi}^{(1/2,1)}(s;\phi,\mathcal{C}(\mathbf{0}))$ this continuation. Now we continue our meromorphic continuation to some open set containing $\mathrm{Re}(s)\geq 1/2$. Let $s\in\mathcal{R}(1/2)^+$. Then one can plug (9-14) and (9-16) into formulas for $\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ and $\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ and shift contours to see that $J_{Q,\chi}^{(1/2,1)}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\begin{aligned}
&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c'_1\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\quad-c'_2\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_1\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\\
&\quad+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&=\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c'_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\quad-c'_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\\
&\quad+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&\quad+c'_1\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c'_2\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s),
\end{aligned}$$

where the right side of the equality has a natural meromorphic continuation to the domain $\mathcal{R}(1/2)$. Denote by $J_{Q,\chi}^{1/2}(s;\phi,\mathcal{C}(\mathbf{0}))$ the last expression. Note that

$$\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)},$$

$$\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)}.$$

Also, when $s\in\mathcal{R}(1/2)$, $2-2s$ lies in the zero-free region of $L(s,\tau^{-2})$, then $\Lambda(2-2s,\tau^{-2})\neq 0$. So the last two terms of $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $\mathcal{R}(1/2)$ with a possible simple pole at $s=1/2$ when $\tau^2=1$. Hence, we have a meromorphic continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ to the region $\mathcal{R}(1/2)$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

Now consider $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$, where $s\in\mathcal{R}(1/2)^-$. Using Cauchy's formula to determine the analytic behaviors of $\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ and $\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$, we obtain that $J^{(1/3,1/2)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\begin{aligned}&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c'_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\quad-c'_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_1\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\\
&\quad+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&\quad+c'_1\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c'_2\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c_1\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&\quad+c_2\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s).\end{aligned}$$

Denote by $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the last expression. We have

$$\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)},$$

$$\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)},$$

so $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ has a holomorphic continuation to $1/3<\operatorname{Re}(s)<1/2$. In all, we obtain the meromorphic continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ to $S_{(1/3,1)}\cup\mathcal{R}(1)$:

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\begin{cases}J^{1}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1),\\ J^{(1/2,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in S_{(1/2,1)},\\ J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1/2),\\ J^{(1/3,1/2)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{S}(1/3,1/2).\end{cases}$$

In particular, $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$ and $s=2/3$ when $\tau^2=1$ and $\tau^3=1$, respectively. □

9C2. *Proof of Theorem G when $n=4$.* The case $n=4$ presents additional complexities compared to $n=3$, although they share a common underlying idea. Consequently, the proof follows a similar approach. However, the main challenge as n increases lies in determining the partial residues of each continuation. There are approximately $O(n^2)$ such multiple residues, but a straightforward and systematic description of them is currently unavailable. Therefore, we provide a proof by explicitly addressing all possible cases. Further computations and continuations can be found in the appendix of [35].

Proof. Let $n=4$. There are three possibilities: $r=2$, $r=3$ or $r=4$.

Case $r=2$: Here the parabolic subgroup Q is of type $(2,2)$, and any associated cuspidal datum is of the form $\chi\simeq(\sigma_1,\sigma_2)$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}(2,\mathbb{A}_F)$. In this case, $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to an entire function multiplying

$$\frac{\Lambda(s+\kappa_1,\sigma_1\otimes\tau\times\tilde{\sigma}_2)\Lambda(s-\kappa_1,\sigma_2\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(1+\kappa_1,\sigma_1\times\tilde{\sigma}_2)\Lambda(1-\kappa_1,\sigma_2\times\tilde{\sigma}_1)}\cdot\prod_{k=1}^{2}\Lambda(s,\sigma_k\otimes\tau\times\tilde{\sigma}_k). \tag{9-18}$$

Let $s\in\mathcal{R}(1)^+$. Since $\mathcal{F}(\boldsymbol{\kappa},s)$ vanishes when $\mathrm{Im}(\kappa_1)\to\infty$, by the Cauchy integral formula, we have

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-2\pi i\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s). \tag{9-19}$$

The term $\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_1\simeq\sigma_2\otimes\tau$. Hence

$$\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(2s-1,\sigma_1\otimes\tau^2\times\tilde{\sigma}_1)\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(2-s,\sigma_1\otimes\tau^{-1}\times\tilde{\sigma}_1)}.$$

So $\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation in $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$. Now the right-hand side of (9-19) is meromorphic inside $\mathcal{R}(1)$, with a possible pole at $s=1$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ in $\mathcal{R}(1)$. By Cauchy's formula,

$$\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1=\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1+2\pi i\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s), \tag{9-20}$$

where $s\in\mathcal{R}(1)^-$. By (9-18), $\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$ is holomorphic inside $S_{(1/2,1)}$; also, $\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_2\simeq\sigma_1\otimes\tau$, in which case

$$\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(2s-1,\sigma_2\otimes\tau^2\times\tilde{\sigma}_2)\Lambda(s,\sigma_2\otimes\tau\times\tilde{\sigma}_2)}{\Lambda(2-s,\sigma_2\otimes\tau^{-1}\times\tilde{\sigma}_2)}.$$

So $\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation to $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$. Substituting this with (9-20) into (9-19) we conclude that $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ admits a meromorphic continuation to the domain $\mathcal{R}(1/2)\cup S_{(1/2,1)}$. Denote by $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ this continuation. Hence we have

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\begin{cases}J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1),\\ J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in S_{(0,1)}.\end{cases}$$

By assumption, $\Lambda(s,\sigma_2\otimes\tau\times\tilde{\sigma}_2)\Lambda(s,\tau)^{-1}$ is holomorphic in $S_{(0,1)}$, so from the expressions above we see that $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))\Lambda(s,\tau)^{-1}$ admits a meromorphic continuation in $s\in S_{(1/3,1)}$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

Case $r=3$: In this case, the parabolic subgroup Q is of type $(2,1,1)$, and any associated cuspidal datum is of the form $\chi\simeq(\sigma_1,\chi_2,\chi_3)$, where σ_1 is a cuspidal representations of $\mathrm{GL}(2,\mathbb{A}_F)$; and χ_2, χ_3 are unitary Hecke characters on $\mathbb{A}_F^\times$. Since $\Lambda(s,\sigma_1\otimes\tau\times\chi_i)$ is entire, $2\le i\le 3$, then $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to an entire function $\mathcal{H}(\boldsymbol{\kappa},s)$ multiplying

$$\frac{\Lambda(s+\kappa_2,\chi_2\overline{\chi}_3\tau)\,\Lambda(s-\kappa_2,\chi_3\overline{\chi}_2\tau)\,\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)\,\Lambda(s,\tau)^2}{\Lambda(1+\kappa_2,\chi_2\overline{\chi}_3\tau)\,\Lambda(1-\kappa_2,\chi_3\overline{\chi}_2\tau)}. \tag{9-21}$$

Let $s\in\mathcal{R}(1)^+$. Since $\mathcal{F}(\boldsymbol{\kappa},s)$ vanishes as $\mathrm{Im}(\kappa_1)\to\infty$, by the Cauchy integral formula, $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\,d\kappa_2-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1. \tag{9-22}$$

The term $\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\chi_1\simeq\chi_2\otimes\tau$. Hence

$$\mathrm{Res}_{\kappa_2=s-1}\,\mathcal{F}(\boldsymbol{\kappa},s)=\mathcal{H}(s,\kappa_1)\frac{\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)\,\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(2-s,\tau^{-1})},$$

where $\mathcal{H}(s,\kappa_1)$ is an holomorphic function. So $\mathrm{Res}_{\kappa_2=s-1}\,\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation in $S_{(0,1)}$, with possible simple poles at $s=1/2$. Now the right side of (9-22) is meromorphic inside $\mathcal{R}(1)$, with a possible pole at $s=1$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ in $\mathcal{R}(1)$. Apply Cauchy's formula again to get

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\,d\kappa_2=\int_{\mathcal{C}}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1+\int_{\mathcal{C}}\mathrm{Res}_{\kappa_2=1-s}\,\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1, \tag{9-23}$$

where $s\in\mathcal{R}(1)^-$. By (9-21), $\int_{\mathcal{C}}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ is holomorphic inside $S_{(1/3,1)}$; also, $\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_2\simeq\sigma_1\otimes\tau$, in which case one has

$$\mathrm{Res}_{\kappa_2=1-s}\,\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)\,\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(2-s,\tau^{-1})}.$$

So $\int_{\mathcal{C}}\mathrm{Res}_{\kappa_2=1-s}\,\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$ admits a meromorphic continuation to $S_{(1/3,1)}$, with possible simple poles at $s=1/2$. Substituting this and (9-23) into (9-19) we conclude that $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ admits a meromorphic continuation to the domain $S_{(1/3,1)}$. Denote by $J^{(1/3,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ this continuation. Hence invoking the above discussion we have

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\begin{cases}J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1);\\ J^{(1/3,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in S_{(1/3,1)}.\end{cases}$$

By assumption $\Lambda(s,\sigma_2\otimes\tau\times\tilde{\sigma}_2)\,\Lambda(s,\tau)^{-1}$ is holomorphic in $S_{(0,1)}$, then from the expressions above we see that $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))\,\Lambda(s,\tau)^{-1}$ admits a meromorphic continuation in $s\in S_{(1/3,1)}$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

Case $r=4$: Here the parabolic subgroup Q is of type $(1,1,1,1)$, and any associated cuspidal datum is of the form $\chi\simeq(\chi_1,\chi_2,\chi_3,\chi_4)$, where χ_i's are unitary Hecke characters on $\mathbb{A}_F^\times$ such that $\chi_1\chi_2\chi_3\chi_4=\omega$. Then there exists an entire function $\mathcal{H}(s,\boldsymbol{\kappa})$ such that $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to

$$\mathcal{H}(s,\boldsymbol{\kappa})\Lambda(s,\tau)^4\prod_{j=1}^{3}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\tau\chi_i\bar{\chi}_{j+1})\Lambda(s-\kappa_{i,j},\tau\chi_{j+1}\bar{\chi}_i)}{\Lambda(1+\kappa_{i,j},\chi_i\bar{\chi}_{j+1})\Lambda(1-\kappa_{i,j},\chi_{j+1}\bar{\chi}_i)},\tag{9-24}$$

where $\Lambda(s,\chi')$ is the completed Hecke L-function associated to the unitary Hecke character χ' over F. Then by Proposition 9.1, when $s\in\mathcal{R}(1)^+$, $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\begin{aligned}
&\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2d\kappa_1-c_1\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2\\
&\quad-c_2\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_1-c_3\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2d\kappa_1\\
&\quad-c_{1,2}\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3-c_{1,3}\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\quad-c_{2,3}\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_{1,2,3}\sum_\phi\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s),
\end{aligned}$$

where the coefficients c_1, c_2, c_3, $c_{1,2}$, $c_{1,3}$, $c_{2,3}$ and $c_{1,2,3}$ are some absolute integers; and the sum with respect to ϕ in taken over $\phi\in\mathfrak{B}_{Q,\chi}$.

Due to the finiteness of $\mathfrak{B}_{Q,\chi}$ and rapidly decay of $\mathcal{F}(\boldsymbol{\kappa},s)$ as a function of $\boldsymbol{\kappa}$, each term in the above expression converges absolutely and locally uniformly. Hence we only need to consider each summand in the above expression. Denote by $\chi_{ij}=\chi_i\bar{\chi}_j$, $1\le i,j\le 4$. By (9-24) we have

$$\begin{aligned}
\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\propto\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(s-\kappa_{12},\chi_{31}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1-\kappa_1,\chi_{21})\Lambda(1+\kappa_2,\chi_{23})\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})}\\
&\quad\times\frac{\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1+\kappa_{12},\chi_{13}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1+\kappa_{12},\chi_{13})\Lambda(2-s-\kappa_{12},\chi_{31}\tau^{-1})\Lambda(2-s,\tau^{-1})},\\
\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\propto\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s+\kappa_{13},\chi_{14}\tau)\Lambda(s-\kappa_{13},\chi_{41}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1+\kappa_3,\chi_{34})\Lambda(1+\kappa_{13},\chi_{14})\Lambda(2-s-\kappa_1,\chi_{21}\tau^{-1})}\\
&\quad\times\frac{\Lambda(2s-1+\kappa_3,\chi_{34}\tau^2)\Lambda(2s-1+\kappa_1,\chi_{12}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{13},\chi_{41})\Lambda(2-s-\kappa_3,\chi_{43}\tau^{-1})\Lambda(2-s,\tau^{-1})},\\
\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\propto\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(s-\kappa_{23},\chi_{42}\tau)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_3,\chi_{34})\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})}\\
&\quad\times\frac{\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1+\kappa_{23},\chi_{24}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1+\kappa_{23},\chi_{24})\Lambda(2-s-\kappa_{23},\chi_{42}\tau^{-1})\Lambda(2-s,\tau^{-1})}.
\end{aligned}$$

From these expressions we see that $\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function times

$$\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-2+\kappa_1,\chi_{12}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(3-2s-\kappa_1,\chi_{21}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Likewise, $\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is a holomorphic multiple of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(2-s,\tau^{-1})^{-2}$ times

$$\frac{\Lambda(1-\kappa_2,\chi_{31})\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(3s-2+\kappa_2,\chi_{23}\tau^3)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})\Lambda(3-2s-\kappa_2,\chi_{32}\tau^{-2})},$$

and $\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the function

$$\frac{\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(3s-2+\kappa_3,\chi_{34}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_3,\chi_{34})\Lambda(3-2s-\kappa_3,\chi_{43}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

One can continue the computation to see that

$$\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

The above shows that $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ admits a meromorphic continuation to $s\in\mathcal{S}(1)$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the continuation. Then clearly $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is holomorphic when $s\in\mathcal{R}(1)^-$.

Let $s\in\mathcal{R}(1)^-$. Let $L(s,\tau)$ be the finite part of Hecke L-function with respect to τ. Then by the Cauchy integral formula (see the Appendix of [35]), we have:

Claim 9.6. *The integrals*

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_j},\quad\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_j=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_j},$$

for $j\in\{1,2,3\}$, and the integrals

$$\int_{\mathcal{C}}\operatorname{Res}_{\kappa_i=s-1}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_i\,d\kappa_j},\quad\int_{(0)}\operatorname{Res}_{\kappa_i=1-s}\operatorname{Res}_{\kappa_j=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_i\,d\kappa_j},$$

for $i,j\in\{1,2,3\}$, $i\neq j$, admit a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, this continuation has at most simple poles at possibly $s=3/4$, $s=2/3$ and $s=1/2$. If $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.

By Proposition 9.2, for $s\in\mathcal{R}(1)^-$, there are absolute integers $c_1',c_2',c_3',c_{1,2}',c_{1,3}',c_{2,3}',c_{1,2,3}'$ such that

$$\begin{aligned}
&\sum_{\phi}\int_{\mathcal{C}}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2d\kappa_1\\
&\quad=\sum_{\phi}\int_{(0)}\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2d\kappa_1-c_1'\sum_{\phi}\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2\\
&\qquad-c_2'\sum_{\phi}\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_1-c_3'\sum_{\phi}\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2d\kappa_1\\
&\qquad-c_{1,2}'\sum_{\phi}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3-c_{1,3}'\sum_{\phi}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\qquad-c_{2,3}'\sum_{\phi}\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_{1,2,3}'\sum_{\phi}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s),
\end{aligned}$$

where the sums are taken over $\phi\in\mathfrak{B}_{Q,\chi}$.

Due to the finiteness of $\mathfrak{B}_{Q,\chi}$ and the rapid decay of $\mathcal{F}(\boldsymbol{\kappa}, s)$ as a function of $\boldsymbol{\kappa}$ (see [31], for example), each term in the above expression converges absolutely and locally normally. Hence we only need to consider each summand in that expression. According to (9-24), we have

$$\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \propto \frac{\Lambda(s+\kappa_1, \chi_{12}\tau)\Lambda(s-\kappa_1, \chi_{21}\tau)\Lambda(s+\kappa_2, \chi_{23}\tau)\Lambda(s+\kappa_{12}, \chi_{13}\tau)}{\Lambda(1+\kappa_1, \chi_{12})\Lambda(1-\kappa_1, \chi_{21})\Lambda(1-\kappa_2, \chi_{32})\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})} \times \frac{\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(2s-1-\kappa_{12}, \chi_{31}\tau^2)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^3}{\Lambda(1-\kappa_{12}, \chi_{31})\Lambda(2-s+\kappa_{12}, \chi_{13}\tau^{-1})\Lambda(2-s, \tau^{-1})},$$

$$\operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \propto \frac{\Lambda(s+\kappa_1, \chi_{12}\tau)\Lambda(s+\kappa_3, \chi_{34}\tau)\Lambda(s+\kappa_{13}, \chi_{14}\tau)\Lambda(s-\kappa_{13}, \chi_{41}\tau)}{\Lambda(1-\kappa_1, \chi_{21})\Lambda(1-\kappa_3, \chi_{43})\Lambda(1+\kappa_{13}, \chi_{14})\Lambda(2-s+\kappa_1, \chi_{12}\tau^{-1})} \times \frac{\Lambda(2s-1-\kappa_3, \chi_{43}\tau^2)\Lambda(2s-1-\kappa_1, \chi_{21}\tau^2)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^3}{\Lambda(1-\kappa_{13}, \chi_{41})\Lambda(2-s+\kappa_3, \chi_{34}\tau^{-1})\Lambda(2-s, \tau^{-1})},$$

$$\operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \propto \frac{\Lambda(s+\kappa_3, \chi_{34}\tau)\Lambda(s-\kappa_3, \chi_{43}\tau)\Lambda(s+\kappa_2, \chi_{23}\tau)\Lambda(s+\kappa_{23}, \chi_{24}\tau)}{\Lambda(1-\kappa_2, \chi_{32})\Lambda(1-\kappa_3, \chi_{43})\Lambda(1+\kappa_3, \chi_{34})\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})} \times \frac{\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(2s-1-\kappa_{23}, \chi_{42}\tau^2)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^3}{\Lambda(1-\kappa_{23}, \chi_{42})\Lambda(2-s+\kappa_{23}, \chi_{24}\tau^{-1})\Lambda(2-s, \tau^{-1})}.$$

From these expressions we see that $\operatorname*{Res}_{\kappa_2=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(s+\kappa_1, \chi_{12}\tau)\Lambda(3s-2-\kappa_1, \chi_{21}\tau^3)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^2}{\Lambda(1-\kappa_1, \chi_{21})\Lambda(3-2s+\kappa_1, \chi_{12}\tau^{-2})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}.$$

Likewise, $\operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ equals some holomorphic function multiplying the product of $\Lambda(2s-1, \tau^2)^2\Lambda(s, \tau)^2\Lambda(2-s, \tau^{-1})^{-2}$ and

$$\frac{\Lambda(1+\kappa_2, \chi_{13})\Lambda(s+\kappa_2, \chi_{23}\tau)\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(3s-2-\kappa_2, \chi_{32}\tau^3)}{\Lambda(1-\kappa_2, \chi_{32})\Lambda(s-\kappa_2, \chi_{32}\tau)\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})\Lambda(3-2s+\kappa_2, \chi_{23}\tau^{-2})},$$

and $\operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying the function

$$\frac{\Lambda(s+\kappa_3, \chi_{34}\tau)\Lambda(3s-2-\kappa_3, \chi_{43}\tau^3)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^2}{\Lambda(1-\kappa_3, \chi_{43})\Lambda(3-2s+\kappa_3, \chi_{34}\tau^{-2})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}.$$

Moreover, one can continue the computation to see that

$$\operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \propto \frac{\Lambda(4s-3, \tau^4)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)}{\Lambda(4-3s, \tau^{-3})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}.$$

Let $s \in \mathcal{R}(1)^-$. The desired meromorphic continuation of

$$\int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_j=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \frac{d\kappa_1\, d\kappa_2\, d\kappa_3}{d\kappa_j}, \quad j \in \{1, 2, 3\},$$

and

$$\int_{(0)} \operatorname{Res}_{\kappa_i=1-s} \operatorname{Res}_{\kappa_j=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \frac{d\kappa_1\, d\kappa_2\, d\kappa_3}{d\kappa_i\, d\kappa_j}, \qquad i, j \in \{1, 2, 3\},\ i \neq j,$$

follows from Claim 9.6. Then Theorem G follows. □

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On the cohomology of tautological bundles over Quot schemes of curves

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We consider tautological bundles and their exterior and symmetric powers on the Quot scheme over the projective line. We prove and conjecture several statements regarding the vanishing of their higher cohomology, and we describe their spaces of global sections via tautological constructions. To this end, we make use of the embedding of the Quot scheme as an explicit local complete intersection in the product of two Grassmannians, studied by Strømme. This allows us to construct resolutions with vanishing cohomology for the tautological bundles and their exterior and symmetric powers. We further illustrate our approach with a few additional cohomological calculations.

1. Introduction

1.1. ***Tautological vector bundles.*** We consider the Quot scheme $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$ parametrizing rank-r degree-n quotients of the trivial bundle of rank N over $\mathbb{P}^1$:

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1} \to Q \to 0, \quad \text{rank } Q = r, \quad \deg Q = n.$$

It is a smooth projective variety which comes equipped with the universal sequence

$$0 \to \mathcal{S} \to \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$$

over $\mathbb{P}^1 \times \mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$. We let p and π denote the projections to the two factors.

The Quot scheme over the projective line is an important testing ground for ideas in moduli theory. It has rich geometry and bears ties to homogeneous and quiver varieties while not being one of them. A beautiful systematic study of $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$ was carried out in [Str], where the Quot scheme was shown to be rational, and the Betti numbers, generators for the Chow ring, and the nef cone were calculated. As noted in [Str], $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$ is a compactification of the space of degree-d morphisms from $\mathbb{P}^1$ to the Grassmannian $\mathbf{G}(N, r)$ of r-dimensional quotients of $\mathbb{C}^N$. With this point of view, in the 1990s, the Quot scheme was used effectively to calculate the small quantum cohomology ring of the Grassmannian, leading eventually to a calculation for all flag varieties [Be; CF1; CF2; K; C]. Further progress included the description of the equivariant cohomology ring in [BCS], the calculation of the effective cone [J], and the birational study [I] which established $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$ as a Mori dream space, among others.

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In this paper, we take up the problem of calculating the cohomology of Schur functors of tautological bundles over $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$. While our results are primarily in the setting of zero quotient rank ($r=0$), the method is available for any rank. Conjectures for any r are formulated in Section 1.3 below.

To start, note that for any line bundle $L \to \mathbb{P}^1$, there is an induced tautological complex of rank $n + r(\deg L + 1)$ over $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, r, n)$, given by

$$L^{[n]} = R\pi_\star(p^\star L \otimes \mathcal{Q}). \tag{1.1.1}$$

When $r=0$, $L^{[n]}$ is a *vector bundle* of rank n. (Note that $R^1\pi_\star(p^\star L \otimes \mathcal{Q}) = 0$ for $r=0$ since the support of $\mathcal{Q}$ is finite in each fiber of π.) We let $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$ denote the Quot scheme in this case, and show the following results.

Theorem 1.1.2. (1) *For all line bundles $L \to \mathbb{P}^1$ with $\deg L \ge n \ge k$, we have*

$$H^0\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \textstyle\bigwedge^k L^{[n]}\big) \cong \textstyle\bigwedge^k H^0(L^{\oplus N})$$

and the higher cohomology vanishes:

$$H^i\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \textstyle\bigwedge^k L^{[n]}\big) = 0, \quad i > 0.$$

(2) *More generally, assume $\deg L \ge n \ge k$ and let $p_1, \dots, p_t$ be nonnegative integers, $0 \le t \le N-1$. We have*

$$\operatorname{Ext}^i\big(\textstyle\bigwedge^{p_1} L^{[n]} \otimes \cdots \otimes \bigwedge^{p_t} L^{[n]}, \bigwedge^k L^{[n]}\big) = \begin{cases} \bigwedge^{k-|p|} H^0(L^{\oplus N}) & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

for $|p| = p_1 + \cdots + p_t \le k$. If $|p| > k$, all the above Ext groups vanish.

Theorem 1.1.3. (1) *For all line bundles $L \to \mathbb{P}^1$ with $\deg L \ge n \ge k$, we have*

$$H^0\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \operatorname{Sym}^k L^{[n]}\right) \cong \operatorname{Sym}^k H^0(L^{\oplus N})$$

and the higher cohomology vanishes

$$H^i\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \operatorname{Sym}^k L^{[n]}\right) = 0, \quad i > 0.$$

(2) *More generally, assume $\deg L \ge n \ge k$ and let $p_1, \dots, p_t$ be nonnegative integers, $0 \le t \le N-1$. Then*

$$\operatorname{Ext}^i\left(\textstyle\bigwedge^{p_1} L^{[n]} \otimes \cdots \otimes \bigwedge^{p_t} L^{[n]}, \operatorname{Sym}^k L^{[n]}\right) = \begin{cases} \operatorname{Sym}^{k-|p|} H^0(L^{\oplus N}) & \text{if } i = 0 \text{ and all } p_j \in \{0,1\}, \\ 0 & \text{otherwise}, \end{cases}$$

for $|p| \le k$. If $|p| > k$, all the above Ext groups vanish.

We expect that the vanishing of higher cohomology of $\bigwedge^k L^{[n]}$ and $\operatorname{Sym}^k L^{[n]}$ in Theorems 1.1.2 and 1.1.3 holds whenever $\deg L \ge -1$. For arbitrary exterior powers, the above vanishings cannot be accessed by the classical theorems. In the special case of the determinant line bundle $\bigwedge^n L^{[n]}$, the bound in our theorems improves the bound $\deg L \ge Nn - N - n$ obtained by Kodaira vanishing. The latter can be applied using the description of the ample cone in [Str] and a standard calculation of the canonical bundle via Grothendieck–Riemann–Roch.

It is easy to see how the sections of the bundles $\bigwedge^k L^{[n]}$ and $\mathrm{Sym}^k L^{[n]}$ over $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$ arise geometrically. Indeed, from the universal quotient

$$\mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0$$

over $\mathbb{P}^1 \times \mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$, after tensoring by L and pushing forward, we immediately obtain a map

$$H^0(L^{\oplus N}) \otimes \mathcal{O}_{\mathsf{Quot}} \to L^{[n]}.$$

Taking exterior and symmetric powers, and taking cohomology, we obtain morphisms

$$\Phi_k : \bigwedge^k H^0(L^{\oplus N}) \to H^0\big(\bigwedge^k L^{[n]}\big), \quad \Psi_k : \mathrm{Sym}^k H^0(L^{\oplus N}) \to H^0\left(\mathrm{Sym}^k L^{[n]}\right). \tag{1.1.4}$$

Our proofs will show that Φ_k and Ψ_k are isomorphisms when $\deg L \geq n \geq k$. Thus all sections of $\bigwedge^k L^{[n]}$ and $\mathrm{Sym}^k L^{[n]}$ are obtained tautologically, while the higher cohomology vanishes.

We further illustrate the techniques developed here by showing that:

Theorem 1.1.5. *For all line bundles $L, M \to \mathbb{P}^1$ with $\deg M \geq n$ and $0 \leq \deg M - \deg L \leq 1$, for all $p_1, \dots, p_t \geq 0$, not all zero, and $1 \leq t \leq N-1$, we have*

$$H^i\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \big(\bigwedge^{p_1} L^{[n]}\big)^\vee \otimes (\bigwedge^{p_2} M^{[n]}\big)^\vee \otimes \cdots \otimes \big(\bigwedge^{p_t} M^{[n]}\big)^\vee\big) = 0, \quad i \geq 0.$$

Of course, the case $L = M$ is contained in Theorems 1.1.2(2) and 1.1.3(2) for $k = 0$.

Although we focus for notational simplicity on the case of quotients of the trivial bundle, the same arguments easily apply to the Quot scheme $\mathsf{Quot}_{\mathbb{P}^1}(E, n)$ of finite quotients of an arbitrary vector bundle $E \to \mathbb{P}^1$.

Let a denote the largest degree appearing in the splitting of E as a direct sum of line bundles, and set $\alpha = -\deg E + (N-1)a$. For Theorems 1.1.2 and 1.1.3, we replace all instances of $H^0(L^{\oplus N})$ by $H^0(E \otimes L)$. For instance, Theorem 1.1.2(1) takes the form

$$H^0\big(\mathsf{Quot}_{\mathbb{P}^1}(E, n), \bigwedge^k L^{[n]}\big) \cong \bigwedge^k H^0(E \otimes L), \quad H^i\big(\mathsf{Quot}_{\mathbb{P}^1}(E, n), \bigwedge^k L^{[n]}\big) = 0, \quad i > 0,$$

whenever

$$\deg L \geq n + \alpha, \quad n \geq k \geq 0.$$

Remark 2.1.2 below explains how this bound emerges. Similarly, the analogue of Theorem 1.1.3(1) reads

$$H^0\left(\mathsf{Quot}_{\mathbb{P}^1}(E, n), \mathrm{Sym}^k L^{[n]}\right) \cong \mathrm{Sym}^k H^0(E \otimes L), \quad H^i\left(\mathsf{Quot}_{\mathbb{P}^1}(E, n), \mathrm{Sym}^k L^{[n]}\right) = 0, \quad i > 0.$$

The more general Theorem 1.1.2(2) and Theorem 1.1.3(2) are also correct after the analogous modifications. Likewise, Theorem 1.1.5 remains true under the assumption $\deg M \geq n + \alpha$.

For a smooth projective curve C of arbitrary genus, the holomorphic Euler characteristics of $\bigwedge^k L^{[n]}$, $\mathrm{Sym}^k L^{[n]}$ and $\left(\bigwedge^p L^{[n]}\right)^\vee$ on $\mathsf{Quot}_C(\mathbb{C}^N, n)$ were calculated in [OS] by equivariant localization. The following expectation regarding individual cohomology groups was also formulated in [OS, Question 20]:

$$H^\bullet\left(\mathsf{Quot}_C(\mathbb{C}^N, n), \bigwedge^k L^{[n]}\right) \cong \bigwedge^k H^\bullet(L^{\oplus N}) \otimes \mathrm{Sym}^{n-k} H^\bullet(\mathcal{O}_C). \tag{1.1.6}$$

The exterior and symmetric powers on the right are understood in the graded sense.

In the case when $C = \mathbb{P}^1$, our theorems confirm this expectation for $\deg L \geq n$. We offer additional modest evidence for $k = 1$ and L of arbitrary degree:

Corollary 1.1.7. *For all line bundles $L \to \mathbb{P}^1$, we have*

$$H^i\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), L^{[n]}\right) = 0, \quad i \geq 2.$$

These results and formula (1.1.6) reflect a full parallelism to the cohomology of tautological bundles over the Hilbert scheme of points on a surface computed in [D; Sc1; Sc2; Kr1; A]. For instance, for all line bundles $L \to X$ over smooth projective surfaces, we have

$$H^\bullet\left(X^{[n]}, \bigwedge^k L^{[n]}\right) = \bigwedge^k H^\bullet(X, L) \otimes \mathrm{Sym}^{n-k} H^\bullet(X, \mathcal{O}_X).$$

The Bridgeland–King–Reid correspondence plays a central role in the proof. We refer the reader to the beautiful article [Kr1] for state-of-the-art calculations in the surface case.

1.2. ***Proofs.*** To establish Theorems 1.1.2, 1.1.3 and 1.1.5, the key idea is to use the twofold Grothendieck embedding of the Quot scheme into a product of Grassmannians,

$$\iota : \mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \hookrightarrow \mathbf{G}_1 \times \mathbf{G}_2,$$

so that the image of ι is the zero locus of a regular section σ of an explicit homogeneous vector bundle

$$\mathcal{E} \to \mathbf{G}_1 \times \mathbf{G}_2.$$

This embedding as an explicit local complete intersection is specific to the Quot scheme (of quotients of all ranks) over the projective line. It was considered and studied in detail by Strømme [Str], who used it to derive information about the Chow ring. Our paper widens the picture, and shows that the embedding is also well-suited to the study of the tautological bundles.

Using the Koszul resolution for σ,

$$\cdots \to \bigwedge^2 \mathcal{E}^\vee \to \bigwedge^1 \mathcal{E}^\vee \to \mathcal{O}_{\mathbf{G}_1} \times \mathbf{G}_2 \to \mathcal{O}_{\mathrm{Quot}} \to 0,$$

we obtain resolutions

$$\cdots \to \mathcal{R}_2 \to \mathcal{R}_1 \to \mathcal{R}_0 \to \iota_\star \mathcal{F} \to 0$$

for each one of the tautological bundles $\mathcal{F}$ appearing in Theorems 1.1.2, 1.1.3 and 1.1.5. The resolutions thus obtained are special. Remarkably, we show that the terms $\mathcal{R}_\ell$ have vanishing cohomology for all

$\ell \geq 1$, while $\mathcal{R}_0$ has no higher cohomology. This allows us to control the cohomology of the tautological bundles and establish our results.

The argument makes crucial use of the Borel–Weil–Bott theorem on the two Grassmannians $\mathbf{G}_1, \mathbf{G}_2$, along with several combinatorial arguments involving the Littlewood–Richardson rule. In intermediate stages, statements of independent interest are established generally over arbitrary Grassmannians; we refer the reader to Section 3 for details. It takes a delicate interplay between Borel–Weil–Bott and Littlewood–Richardson vanishings to show that all higher terms $\mathcal{R}_\ell$, $\ell \geq 1$, of the resolution have no cohomology at all.

While the above theorems concern genus 0, we also obtain the following corollary in arbitrary genus. Let y be a variable. Setting

$$\bigwedge_y V := \sum_k y^k \bigwedge^k V, \quad \mathrm{Sym}_y V := \sum_k y^k \,\mathrm{Sym}^k V,$$

the result below recovers Theorem 1, a special case of Theorem 2, and Theorem 4 in [OS].

Corollary 1.2.1. *Let $L, M_1, \ldots, M_t \to C$ be line bundles over a smooth projective curve, where $1 \leq t \leq N-1$. Then*

$$\sum_{n=0}^{\infty} q^n \chi\Big(\mathsf{Quot}_C(\mathbb{C}^N, n), \bigwedge_y L^{[n]}\Big) = (1-q)^{-\chi(\mathcal{O}_C)}(1+qy)^{N\chi(L)}, \tag{1.2.1a}$$

$$\sum_{n=0}^{\infty} q^n \chi\Big(\mathsf{Quot}_C(\mathbb{C}^N, n), \bigotimes_{i=1}^{t} \big(\bigwedge_{y_i} M_i^{[n]}\big)^\vee\Big) = (1-q)^{-\chi(\mathcal{O}_C)}. \tag{1.2.1b}$$

Furthermore, in genus 0, *we have*

$$\sum_{n\geq k} q^n y^k \chi\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \mathrm{Sym}^k L^{[n]}\right) = (1-q)^{-1}(1-qy)^{-N\chi(L)}. \tag{1.2.1c}$$

We will show how to derive Corollary 1.2.1 from Theorems 1.1.2, 1.1.3 and 1.1.5 in Section 4.1.

Formulas (1.2.1a), (1.2.1b) and (1.2.1c) were previously established in [OS] based on reduction to genus 0 using universality statements as in [EGL; OS; St], and equivariant torus localization in genus 0. The localization calculation is combinatorially involved and relies on several mysterious simplifications. In the present paper, Theorems 1.1.2, 1.1.3 and 1.1.5 reflect an efficient and more conceptual approach to the full cohomology, which cannot be accessed by localization.

In all genera, Corollary 1.2.1 also holds for an arbitrary vector bundle E instead of the trivial bundle $\mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1}$, with the only modification that all instances of $N\chi(L)$ are replaced by $\chi(E \otimes L)$.

1.3. ***Higher rank.*** A natural direction is to apply the techniques of this paper to study the cohomology of the Schur functors of tautological bundles over $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r)$, when $r > 0$. In this setting, even the numerical K-theoretic invariants of these Schur functors are largely unexplored.

Recalling the definition of the tautological complex $L^{[n]}$ from (1.1.1), we propose the following conjectures.

Conjecture 1.3.1. *Let $n=(N-r)a+b$ with $0\le b<N-r$. Then for all line bundles $L\to\mathbb{P}^1$, we have*

$$\chi\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n,r),\textstyle\bigwedge^k L^{[n]}\big)=\binom{N\chi(L)}{k}$$

for all $k\le n+r(a+1)$.

For symmetric powers, we state the following

Conjecture 1.3.2. *Let $n=(N-r)a+b$ with $0\le b<N-r$. Then for all line bundles $L\to\mathbb{P}^1$, we have*

$$\chi\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n,r),\mathrm{Sym}^k L^{[n]}\big)=\binom{N\chi(L)+k-1}{k}$$

for all $k\le n+r(a+1)$.

Finally, for the dualized exterior powers, we have

Conjecture 1.3.3. *Let $r>0$ and write $n=ar+b$ with $0\le b<r$. Let $1\le t\le N-r-1$ and $p_1,\dots,p_t$ be nonnegative integers with $0<p_1+\cdots+p_t\le n+(N-r)(a+1)$. Then for all line bundles $L_1,\dots,L_t\to\mathbb{P}^1$, we have*

$$\chi\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n,r),\big(\textstyle\bigwedge^{p_1}L_1^{[n]}\big)^\vee\otimes\cdots\otimes\big(\textstyle\bigwedge^{p_t}L_t^{[n]}\big)^\vee\big)=0.$$

When $r=0$, Conjectures 1.3.1 and 1.3.2 recover Theorems 1.1.2(1) and 1.1.3(1), while the case $r=N-1$ can be verified by hand since the Quot scheme is a projective space. For all three conjectures, we checked the answer by computer in several other cases. The bounds on k appear to be sharp.

It is natural to expect that the three conjectures can be lifted in obvious fashion to cohomology. This suggests that Theorems 1.1.2(1), 1.1.3(1) and 1.1.5 continue to hold for higher-rank quotients subject to the appropriate bounds on k and the condition that the degree of the L's be nonnegative. While Strømme's construction is valid for quotients of arbitrary rank, the Borel–Weil–Bott arguments become more involved and will be left for future study.

The answers predicted by the conjectures stabilize as n becomes large with respect to N,k,r. Equivalently, for each fixed k, the generating series

$$\sum_{n=0}^{\infty}q^n\chi\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n,r),\textstyle\bigwedge^k L^{[n]}\big),\quad \sum_{n=0}^{\infty}q^n\chi\big(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n,r),\mathrm{Sym}^k L^{[n]}\big)$$

are given by rational functions with a (simple) pole only at $q=1$. It is natural to wonder whether this statement is correct for all partitions and for the associated Schur functors of $L^{[n]}$.

1.4. ***Subsequent developments.*** After our preprint appeared on `arXiv`, the conjectural identity (1.1.6) was extended to cover more general Ext groups involving tensor products of wedge powers of tautological bundles; see [Kr2, Conjecture 1.1]. The extended conjecture is consistent with the Euler characteristic calculations in [OS]. Theorem 1.1.2 and Theorem 1.1.5 establish part of the conjecture. We also refer the reader to [Kr2, Theorems 1.2, 1.3, 1.5] for related results. Both equation (1.1.6) and the conjecture formulated in [Kr2] were very recently confirmed in [MN] using different methods.

1.5. ***Plan of the paper.*** We review Strømme's embedding, construct the resolutions of the tautological bundles, and establish Theorems 1.1.2, 1.1.3 and 1.1.5 in Section 2. This relies on the Borel–Weil–Bott analysis of the resolutions which is carried out in Section 3. Corollaries 1.2.1 and 1.1.7 are proved in Section 4.

2. Grassmannian embedding of the Quot scheme and resolutions

2.1. ***Strømme's embedding.*** We begin by describing Strømme's construction which exhibits the Quot scheme over $\mathbb{P}^1$ as the zero locus of a regular section of a vector bundle over the product of two Grassmannians [Str].

For each integer $m \geq n$, the embedding takes the form

$$\iota_m : \mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \hookrightarrow \mathbf{G}(V_{m-1}, n) \times \mathbf{G}(V_m, n),$$

where

$$\mathbf{G}_1 = \mathbf{G}(V_{m-1}, n), \quad \mathbf{G}_2 = \mathbf{G}(V_m, n)$$

are the Grassmannians of n-dimensional *quotients* of two vector spaces of dimensions Nm and $N(m+1)$ respectively. We identify

$$V_{m-1} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-1)^{\oplus N}), \quad V_m = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)^{\oplus N}).$$

Explicitly, ι_m is the product of two Grothendieck embeddings. When $m \geq n$, each short exact sequence in the Quot scheme

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1} \to Q \to 0,$$

yields two exact sequences of vector spaces

$$0 \to H^0(S(m-1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m-1)^{\oplus N}) \to H^0(Q(m-1)) \to 0,$$

$$0 \to H^0(S(m)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m)^{\oplus N}) \to H^0(Q(m)) \to 0.$$

Then ι_m is given by the assignment

$$\left[0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1} \to Q \to 0\right] \mapsto \left[V_{m-1} \to H^0(Q(m-1))\right] \times \left[V_m \to H^0(Q(m))\right].$$

We write $W = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. There is a natural morphism

$$V_{m-1} \to V_m \otimes W,$$

obtained from the natural section cutting out the diagonal $\Delta \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\Delta) = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1),$$

tensoring by $\mathcal{O}_{\mathbb{P}^1}(m-1)$ on the first factor, and taking cohomology.

To describe the equations cutting out $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$ in $\mathbf{G}_1 \times \mathbf{G}_2$, let

$$0 \to \mathcal{A}_1 \to V_{m-1} \otimes \mathcal{O}_{\mathbf{G}_1} \to \mathcal{B}_1 \to 0,$$

$$0 \to \mathcal{A}_2 \to V_m \otimes \mathcal{O}_{\mathbf{G}_2} \to \mathcal{B}_2 \to 0,$$

be the tautological sequences over the two Grassmannians $\mathbf{G}_1$ and $\mathbf{G}_2$. Let pr_1 and pr_2 be the two projections on $\mathbf{G}_1 \times \mathbf{G}_2$. The sheaf

$$\mathcal{E} = \mathrm{pr}_1^\star \mathcal{A}_1^\vee \otimes W \otimes \mathrm{pr}_2^\star \mathcal{B}_2 \to \mathbf{G}_1 \times \mathbf{G}_2 \tag{2.1.1}$$

admits a natural section σ induced by the composition

$$\sigma : \mathrm{pr}_1^\star \mathcal{A}_1 \to V_{m-1} \otimes \mathcal{O}_{\mathbf{G}_1 \times \mathbf{G}_2} \to V_m \otimes W \otimes \mathcal{O}_{\mathbf{G}_1 \times \mathbf{G}_2} \to \mathrm{pr}_2^\star \mathcal{B}_2 \otimes W.$$

Strømme shows that the section σ is regular and vanishes exactly along $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$; see [Str, §4].

Remark 2.1.2. For an arbitrary vector bundle $E \to \mathbb{P}^1$ and m sufficiently large, we similarly have an embedding

$$\iota : \mathsf{Quot}_{\mathbb{P}^1}(E, n) \to \mathbf{G}_1 \times \mathbf{G}_2, \quad \mathbf{G}_1 = \mathbf{G}(V_{m-1}, n), \quad \mathbf{G}_2 = \mathbf{G}(V_m, n),$$

where

$$V_{m-1} = H^0(E(m-1)), \quad V_m = H^0(E(m)).$$

To obtain a precise lower bound for m, we need to ensure the vanishing

$$H^1(S(m-1)) = H^1(S(m)) = 0,$$

for every exact sequence $0 \to S \to E \to Q \to 0$. Let

$$E = \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1}(a_i),$$

and set a to be the largest of the summand degrees a_i, $1 \le i \le N$. If

$$S = \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1}(s_i),$$

then we have $s_i \le a$, $1 \le i \le N$, since there is an injection

$$\mathcal{O}_{\mathbb{P}^1}(s_i) \to S \to E \to \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1}(a).$$

As $\sum_{i=1}^N s_i = \deg E - n$, we also have

$$s_i \ge \deg E - n - (N-1)a, \quad 1 \le i \le N.$$

Thus for

$$m \ge n + (N-1)a - \deg E,$$

we obtain $H^1(S(m-1)) = H^1(S(m)) = 0$, as wished.

With no additional conditions on m, Strømme's arguments show that $\mathrm{Quot}_{\mathbb{P}^1}(E, n)$ is also cut out by a regular section σ of the tautological bundle (2.1.1) over $\mathbf{G}_1 \times \mathbf{G}_2$. Consequently, the arguments below presented for the case of trivial E also carry over without change to arbitrary E.

2.2. *Resolutions.* As a result of the above discussion, the section σ induces a Koszul resolution

$$\cdots \to \bigwedge^2 \mathcal{E}^\vee \to \bigwedge^1 \mathcal{E}^\vee \to \mathcal{O}_{\mathbf{G}_1 \times \mathbf{G}_2} \to \mathcal{O}_{\mathsf{Quot}} \to 0. \tag{2.2.1}$$

Note that if $\deg L = m$, by the definition of the embedding ι_m we have

$$L^{[n]} = \iota_m^\star \operatorname{pr}_2^\star \mathcal{B}_2.$$

Hence, tensoring (2.2.1) with $\operatorname{pr}_2^\star \bigwedge^k \mathcal{B}_2$, we obtain the resolution

$$\cdots \to \bigwedge^2 \mathcal{E}^\vee \otimes \operatorname{pr}_2^\star \bigwedge^k \mathcal{B}_2 \to \bigwedge^1 \mathcal{E}^\vee \otimes \operatorname{pr}_2^\star \bigwedge^k \mathcal{B}_2 \to \operatorname{pr}_2^\star \bigwedge^k \mathcal{B}_2 \to (\iota_m)_\star \big(\bigwedge^k L^{[n]}\big) \to 0$$

over $\mathbf{G}_1 \times \mathbf{G}_2$. We set

$$\mathcal{V}_\ell = \bigwedge^\ell \mathcal{E}^\vee \otimes \operatorname{pr}_2^\star \bigwedge^k \mathcal{B}_2, \quad \ell \geq 0.$$

This yields the resolution

$$\cdots \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{V}_0 \to (\iota_m)_\star \big(\bigwedge^k L^{[n]}\big) \to 0, \tag{2.2.2}$$

corresponding to Theorem 1.1.2(1).

Theorem 1.1.2(2) is conceptually analogous, but the notation becomes slightly more involved. For this reason, it may be helpful to present the simpler case (1) first. To treat case (2), we consider the bundle

$$\mathcal{F} = \big(\bigwedge^{p_1} L^{[n]}\big)^\vee \otimes \cdots \otimes \big(\bigwedge^{p_t} L^{[n]}\big)^\vee \otimes \big(\bigwedge^k L^{[n]}\big). \tag{2.2.3a}$$

By similar reasoning, we are led to the resolution

$$\cdots \to \mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0 \to (\iota_m)_\star \mathcal{F} \to 0, \tag{2.2.3b}$$

where

$$\mathcal{X}_\ell = \bigwedge^\ell \mathcal{E}^\vee \otimes \big(\operatorname{pr}_2^\star \bigwedge^{p_1} \mathcal{B}_2^\vee \otimes \operatorname{pr}_2^\star \bigwedge^{p_2} \mathcal{B}_2^\vee \otimes \cdots \otimes \operatorname{pr}_2^\star \bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \operatorname{pr}_2^\star \bigwedge^k \mathcal{B}_2\big).$$

Proposition 2.2.4. *In the resolution* (2.2.2),

$$\cdots \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{V}_0 \to (\iota_m)_\star \big(\bigwedge^k L^{[n]}\big) \to 0,$$

the sheaves $\mathcal{V}_\ell$ have no cohomology for $\ell \geq 1$, while the sheaf $\mathcal{V}_0$ has no higher cohomology.

More generally, in the resolution (2.2.3b), *the sheaves $\mathcal{X}_\ell$ have no cohomology for $\ell \geq 1$, while the sheaf $\mathcal{X}_0$ has no higher cohomology.*

For the symmetric products, we similarly define

$$\mathcal{W}_\ell = \textstyle\bigwedge^\ell \mathcal{E}^\vee \otimes \mathrm{pr}_2^\star \mathrm{Sym}^k \mathcal{B}_2,$$

and have an analogous resolution

$$\cdots \to \mathcal{W}_2 \to \mathcal{W}_1 \to \mathcal{W}_0 \to (\iota_m)_\star \left(\mathrm{Sym}^k L^{[n]}\right) \to 0. \tag{2.2.5}$$

This corresponds to Theorem 1.1.3(1). For the more general Theorem 1.1.3(2), we let

$$\mathcal{G} = \left(\textstyle\bigwedge^{p_1} L^{[n]}\right)^\vee \otimes \cdots \otimes \left(\textstyle\bigwedge^{p_t} L^{[n]}\right)^\vee \otimes \mathrm{Sym}^k L^{[n]}. \tag{2.2.6a}$$

Setting

$$\mathcal{Y}_\ell = \textstyle\bigwedge^\ell \mathcal{E}^\vee \otimes \left(\mathrm{pr}_2^\star \textstyle\bigwedge^{p_1} \mathcal{B}_2^\vee \otimes \mathrm{pr}_2^\star \textstyle\bigwedge^{p_2} \mathcal{B}_2^\vee \otimes \cdots \otimes \mathrm{pr}_2^\star \textstyle\bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \mathrm{pr}_2^\star \mathrm{Sym}^k \mathcal{B}_2\right),$$

we obtain the resolution

$$\cdots \to \mathcal{Y}_2 \to \mathcal{Y}_1 \to \mathcal{Y}_0 \to (\iota_m)_\star \mathcal{G} \to 0. \tag{2.2.6b}$$

Proposition 2.2.7. *In the resolution* (2.2.5)

$$\cdots \to \mathcal{W}_2 \to \mathcal{W}_1 \to \mathcal{W}_0 \to (\iota_m)_\star \left(\mathrm{Sym}^k L^{[n]}\right) \to 0,$$

the sheaves $\mathcal{W}_\ell$ have no cohomology if $\ell \geq 1$ and $\deg L \geq n \geq k$, *while $\mathcal{W}_0$ has no higher cohomology.*

More generally, under the same assumptions, in the resolution (2.2.6b), *the sheaves $\mathcal{Y}_\ell$ have no cohomology for $\ell \geq 1$, while the sheaf $\mathcal{Y}_0$ has no higher cohomology.*

A further analysis is needed for Theorem 1.1.5. The case $L = M$ is already covered either by Theorem 1.1.2(2) or Theorem 1.1.3(2) for $k = 0$. Thus we may assume $\deg M = m \geq n$ and $\deg L = m - 1$. In this case, we have

$$L^{[n]} = \iota_m^\star \mathrm{pr}_1^\star \mathcal{B}_1, \quad M^{[n]} = \iota_m^\star \mathrm{pr}_2^\star \mathcal{B}_2.$$

Thus for the bundle

$$\mathcal{H} = \left(\textstyle\bigwedge^{p_1} L^{[n]}\right)^\vee \otimes \left(\textstyle\bigwedge^{p_2} M^{[n]}\right)^\vee \otimes \cdots \otimes \left(\textstyle\bigwedge^{p_t} M^{[n]}\right)^\vee$$

which appears in Theorem 1.1.5, we obtain a resolution

$$\cdots \to \mathcal{U}_2 \to \mathcal{U}_1 \to \mathcal{U}_0 \to (\iota_m)_\star \mathcal{H} \to 0, \tag{2.2.8}$$

where

$$\mathcal{U}_\ell = \textstyle\bigwedge^\ell \mathcal{E}^\vee \otimes \left(\mathrm{pr}_1^\star \textstyle\bigwedge^{p_1} \mathcal{B}_1^\vee \otimes \mathrm{pr}_2^\star \textstyle\bigwedge^{p_2} \mathcal{B}_2^\vee \otimes \cdots \otimes \mathrm{pr}_2^\star \textstyle\bigwedge^{p_t} \mathcal{B}_2^\vee\right).$$

Proposition 2.2.9. *For $p_1, \ldots, p_t$ not all zero, the cohomology of $\mathcal{U}_\ell$ vanishes for all $\ell \geq 0$.*

2.2.1. *The main theorems.* Before turning our attention to the proofs of the above propositions, we note that our main Theorems 1.1.2, 1.1.3 and 1.1.5 follow immediately from them.

For Theorem 1.1.2, we use Proposition 2.2.4. To establish case (1) of the theorem, we make use of the resolution (2.2.2). The associated spectral sequence shows that the higher cohomology of $\bigwedge^k L^{[n]}$ vanishes, while in degree zero we have

$$H^0\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \textstyle\bigwedge^k L^{[n]}\right) = H^0(\mathbf{G}_1 \times \mathbf{G}_2, \mathcal{V}_0).$$

Recalling that $\mathcal{V}_0 = \mathrm{pr}_2^\star \bigwedge^k \mathcal{B}_2$, we compute

$$H^0(\mathbf{G}_1 \times \mathbf{G}_2, \mathcal{V}_0) = H^0\big(\mathbf{G}_2, \textstyle\bigwedge^k \mathcal{B}_2\big) = \bigwedge^k V_m = \bigwedge^k H^0(\mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1}(m)) = \bigwedge^k H^0(L^{\oplus N}),$$

as needed.

For part (2) of Theorem 1.1.2, we note that

$$\mathrm{Ext}^i\big(\textstyle\bigwedge^{p_1} L^{[n]} \otimes \cdots \otimes \bigwedge^{p_t} L^{[n]}, \bigwedge^k L^{[n]}\big) = H^i(\mathcal{F}),$$

where the bundle $\mathcal{F}$ is defined in (2.2.3a). Using the resolution (2.2.3b) and Proposition 2.2.4, we see that the only contribution to the cohomology of $\mathcal{F}$ comes from the term

$$H^0(\mathbf{G}_1 \times \mathbf{G}_2, \mathcal{X}_0) = H^0\big(\mathcal{B}_2^\vee \otimes \cdots \otimes \textstyle\bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \bigwedge^k \mathcal{B}_2\big) = \bigwedge^{k-|p|} H^0(\mathbf{G}_2, \mathcal{B}_2) = \bigwedge^{k-|p|} H^0(L^{\oplus N})$$

for $|p| = p_1 + \cdots + p_t \leq k \leq n$. The second equality requires further explanation. Since we need additional notation, the argument will be presented later; see equation (2.3.11) in the proof of Proposition 2.2.4.

Turning to Theorem 1.1.3, for part (1), we make use of the resolution (2.2.5) and Proposition 2.2.7. This time, the initial term $\mathcal{W}_0 = \mathrm{pr}_2^\star \mathrm{Sym}^k \mathcal{B}_2$ has sections

$$H^0(\mathbf{G}_1 \times \mathbf{G}_2, \mathcal{W}_0) = H^0(\mathbf{G}_2, \mathrm{Sym}^k \mathcal{B}_2) = \mathrm{Sym}^k V_m = \mathrm{Sym}^k H^0(\mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1}(m)) = \mathrm{Sym}^k H^0(L^{\oplus N}).$$

For part (2), we note

$$\mathrm{Ext}^i\big(\textstyle\bigwedge^{p_1} L^{[n]} \otimes \cdots \otimes \bigwedge^{p_t} L^{[n]}, \mathrm{Sym}^k L^{[n]}\big) = H^i(\mathcal{G}),$$

where the sheaf $\mathcal{G}$ was defined in (2.2.6a). Using the resolution (2.2.6b) and Proposition 2.2.7, the only nontrivial contribution to the cohomology of $\mathcal{G}$ comes from the sheaf $\mathcal{Y}_0$ and it equals

$$H^0\big(\mathbf{G}_2, \textstyle\bigwedge^{p_1} \mathcal{B}_2^\vee \otimes \cdots \otimes \bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \mathrm{Sym}^k \mathcal{B}_2\big) = \mathrm{Sym}^{k-|p|} H^0(\mathbf{G}_2, \mathcal{B}_2) = \mathrm{Sym}^{k-|p|} H^0(L^{\oplus N}).$$

This requires that all $p_j \in \{0, 1\}$ and $|p| \leq k$. The cohomology vanishes altogether if this condition fails. The first equality will be explained after we develop more notation; see equation (2.3.13) in the proof of Proposition 2.2.7 below.

Finally, for Theorem 1.1.5, the argument uses Proposition 2.2.9. This time around, the cohomology vanishes for all terms of the resolution (2.2.8) and in all degrees. □

2.3. ***Analysis of the resolutions.*** We now turn to Propositions 2.2.4, 2.2.7, 2.2.9 and deduce them from the Grassmannian vanishing results of Section 3. We begin by making the terms of the resolutions more explicit.

2.3.1. *Partitions and Cauchy's formula.* We use standard terminology on partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$. We set

$$|\lambda| = \sum_{i=1}^{r} \lambda_i,$$

and we let $\lambda^\dagger$ be the transpose partition obtained by exchanging the rows and columns of the Young diagram of λ.

Throughout, we will always assume that our base scheme is defined over a field of characteristic 0.

For each partition λ, we let $\mathbf{S}_\lambda$ denote the associated Schur functor (for example, these are defined in [W, §2.1] where they are called $L_{\lambda^\dagger}$). For a partition λ and any vector bundle $V \to Y$ over a base Y, there is an associated vector bundle $\mathbf{S}_\lambda(V) \to Y$. The cases $\lambda = (1^k)$ and $\lambda = (k)$ correspond to the k-th exterior and k-th symmetric powers, respectively.

The vector bundles $\mathbf{S}_\lambda(V) \to Y$ are also defined when λ is not a partition but rather an arbitrary dominant weight $\lambda_1 \geq \cdots \geq \lambda_r$, where $r = \operatorname{rank}(V)$, and we now allow the entries to be negative. We have

$$\mathbf{S}_{-\lambda}(V) = \mathbf{S}_\lambda(V^\vee),$$

where $-\lambda$ denotes the sequence $-\lambda_r \geq \cdots \geq -\lambda_1$. In addition,

$$\mathbf{S}_\lambda(V) \otimes \det V = \mathbf{S}_{\lambda+(1^r)}(V).$$

If V, $W \to Y$ are two vector bundles, Cauchy's identity

$$\bigwedge^\ell (V \otimes W) = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(V) \otimes \mathbf{S}_\lambda(W)$$

holds, where the sum is over all partitions λ of size ℓ. We only need to consider those partitions λ with at most $\operatorname{rank}(W)$ rows and at most $\operatorname{rank}(V)$ columns since the term is 0 otherwise. Applying this formula to the bundle $\mathcal{E}$ whose section cuts out the Quot scheme, we obtain

$$\bigwedge^\ell \mathcal{E}^\vee = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee). \tag{2.3.1a}$$

Here, λ is a partition with at most $2n$ rows and the number of columns at most equal to

$$\operatorname{rank}(\mathcal{A}_1) = \dim V_{m-1} - n = \dim V_m - N - n \leq \dim V_m - n - 1. \tag{2.3.1b}$$

In the discussion below, the abbreviation $d = \dim V_m$ will often be used.

Using (2.3.1a), we immediately obtain the expressions

$$\mathcal{V}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \bigwedge^k \mathcal{B}_2\big), \tag{2.3.2a}$$

and

$$\mathcal{W}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \operatorname{Sym}^k \mathcal{B}_2\big) \tag{2.3.2b}$$

corresponding to Theorem 1.1.2(1) and Theorem 1.1.3(1). For the second halves of the two theorems, the expressions are slightly more complicated due to additional wedge powers:

$$\mathcal{X}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \bigwedge^{p_1} \mathcal{B}_2^\vee \otimes \cdots \otimes \bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \bigwedge^k \mathcal{B}_2\big). \tag{2.3.2c}$$

and

$$\mathcal{Y}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \textstyle\bigwedge^{p_1} \mathcal{B}_2^\vee \otimes \cdots \otimes \bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \operatorname{Sym}^k \mathcal{B}_2\big). \tag{2.3.2d}$$

Finally, we have

$$\mathcal{U}_\ell = \bigoplus_{|\lambda|=\ell} \big(\mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \otimes \textstyle\bigwedge^{p_1} \mathcal{B}_1^\vee\big) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \bigwedge^{p_2} \mathcal{B}_2^\vee \otimes \cdots \otimes \bigwedge^{p_t} \mathcal{B}_2^\vee\big). \tag{2.3.2e}$$

2.3.2. *The Borel–Weil–Bott theorem.* To compute the cohomology of the above bundles, we use the Borel–Weil–Bott theorem [B]. For integers $0 < n < d$, let $\mathbf{G} = \mathbf{G}(d, n)$ denote the Grassmannian of n-dimensional *quotients* of a d-dimensional vector space, and let $\mathcal{A}, \mathcal{B} \to \mathbf{G}$ denote the tautological subbundle and quotient. For a partition

$$\mu = (\mu_1, \dots, \mu_{d-n})$$

with $d - n$ rows, we form the string

$$\rho + (0, \mu) = (d-1, d-2, \dots, 1, 0) + (\underbrace{0, \dots, 0}_{n}, \mu_1, \dots, \mu_{d-n}). \tag{2.3.3}$$

Theorem 2.3.4 (Borel–Weil–Bott). *The bundle $\mathbf{S}_\mu(\mathcal{A})$ has at most one nonzero cohomology group. Furthermore, if the string $\rho + (0, \mu)$ contains repetitions, then all cohomology groups of $\mathbf{S}_\mu(\mathcal{A})$ vanish.*

This formulation can be found in [W, Corollary 4.1.9]. We are using a few translations. First, the Weyl functors K_γ defined there are isomorphic to the Schur functors used here; see [W, Proposition 2.1.18(c)]. Moreover, the dual of the tautological subbundle $\mathcal{A}$ is the quotient bundle on the dual Grassmannian, and on the latter space, $\mathbf{S}_\mu(\mathcal{A})$ corresponds to $\mathcal{V}(0, \mu)$ in the notation of [W].

We note from Theorem 2.3.4 that $\mathbf{S}_\mu(\mathcal{A})$ has no cohomology provided there exists j such that

$$j \le \mu_j \le n + j - 1. \tag{2.3.5a}$$

Let us record the "dual" rephrasing of condition (2.3.5a) which is also useful here. For a partition ν with n rows, the bundle $\mathbf{S}_\nu(\mathcal{B}^\vee)$ over the Grassmannian $\mathbf{G}(d, n)$ has no cohomology provided that there exists j such that

$$j \le \nu_j \le d - n + j - 1. \tag{2.3.5b}$$

In Proposition 2.2.9, we also consider the bundle

$$\mathbf{S}_\mu(\mathcal{A}) \otimes \textstyle\bigwedge^p \mathcal{B}^\vee = \mathbf{S}_{-\mu}\left(\mathcal{A}^\vee\right) \otimes \mathbf{S}_{(1^p)} \mathcal{B}^\vee$$

for $0 \le p \le n$. Again by the Borel–Weil–Bott theorem [W, Corollary 4.1.9], all cohomology vanishes provided that the string

$$(d-1, \dots, 0) + (-\mu_{d-n}, \dots, -\mu_1, \underbrace{1, \dots, 1}_{p}, \underbrace{0, \dots, 0}_{n-p}) \tag{2.3.6a}$$

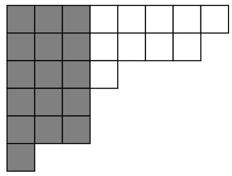

Figure 1. A partition of 2-index $i = 3$.

has repetitions. That happens when there exists j such that

$$j - 1 \leq \mu_j \leq n + j - 1 \text{ and } \mu_j \neq j + p - 1. \tag{2.3.6b}$$

Of course, the case $p = 0$ recovers (2.3.5a). In fact, for $p = 0$, the string (2.3.3) has repetitions if and only if the same holds for (2.3.6a).

2.3.3. *Indices of partitions.* The following definition is not standard but is crucial for our arguments.

Definition 2.3.7. Let n be a nonnegative integer. Let $\lambda \neq 0$ be a partition satisfying the condition that

($*$) for all j, the number of boxes in the j-th column of λ is either $< j$ or $\geq n + j$.

Let i denote the largest index j such that the j-th column has $\geq n + j$ boxes. We refer to i as the *n-index* of λ. If λ does not satisfy ($*$), we leave the n-index undefined.

It may help to visualize partitions λ of n-index i. There are i "long" columns with at least $n + i$ boxes, while the remaining columns are "short" containing at most i boxes. In Figure 1, the long columns are shown in gray, while the short columns are white. Thus, for a partition λ of n-index i, we have

$$\lambda_{i+1} = \cdots = \lambda_{i+n} = i. \tag{2.3.8}$$

The following variation is needed for Proposition 2.2.9 and is connected to condition (2.3.6b) above.

Definition 2.3.9. Let $0 \leq p \leq n$ be integers. Let $\lambda \neq 0$, $\lambda \neq (1^p)$ be a partition satisfying the condition that

($**$) for all j, the number of boxes in the j-th column of λ is either $< j - 1$ or $\geq n + j$ or equal to $j + p - 1$.

Let i denote the largest index j such that the j-th column has $\geq n + j$ boxes. We refer to i as the *(p, n)-index* of λ, when defined. The case $p = 0$ corresponds to the n-index defined above.

The partition $\lambda = (1^p)$ is not considered here. The reason is that λ satisfies ($**$), yet no column has at least $n + 1$ boxes, so the index is undefined.

For a partition λ of (p, n)-index i, two shapes are possible:

(a) The partition λ has i "long" columns with $\geq n + i$ boxes, and the remaining columns are "short", having $\leq i - 1$ boxes.

(b) The partition λ has i "long" columns with $\geq n + i$ boxes, the $(i + 1)$st column has $p + i$ boxes, and the remaining columns are "short", having $\leq i$ boxes.

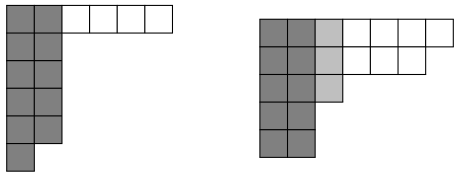

Figure 2. Partitions of (1, 3)-index $i = 2$.

The partitions in Figure 2 satisfy (a) and (b) respectively. For case (b), the long and short columns are shown in dark gray and white, while the middle $(i+1)$st column is lighter gray. In both cases

$$i \le \lambda_{i+1} \le i+1, \quad \ldots, \quad i \le \lambda_{i+n} \le i+1. \tag{2.3.10}$$

Proof of Proposition 2.2.4. For simplicity, we consider the case of the resolution $\mathcal{V}_\bullet$ first. Recall from (2.3.2a) that

$$\mathcal{V}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \textstyle\bigwedge^k \mathcal{B}_2\big).$$

For $\ell = 0$, we must have $\lambda = 0$, and $\mathcal{V}_0 = \mathrm{pr}^\star \bigwedge^k \mathcal{B}_2$ has no higher cohomology.

When $\ell \ge 1$, we have $\lambda \ne 0$. For a partition $\lambda \ne 0$ appearing in the above sum, we distinguish two mutually exclusive situations:

(†) there exists j such that $j \le \lambda_j^\dagger \le n + j - 1$, or

(∗) for all j, the number of boxes in the j-th column of λ is either $< j$ or $\ge n + j$.

Of course, condition (∗) already appeared in Definition 2.3.7. In case (†), we noted in (2.3.5a) that $\mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1)$ has no cohomology. In case (∗), Proposition 3.1.2 in Section 3 below shows that

$$\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \textstyle\bigwedge^k \mathcal{B}_2$$

has no cohomology either. Consequently, $\mathcal{V}_\ell$ has no cohomology when $\ell \ge 1$, establishing the first half of Proposition 2.2.4.

For the second half, recall from (2.3.2c) that

$$\mathcal{X}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \textstyle\bigwedge^{p_1} \mathcal{B}_2^\vee \otimes \cdots \otimes \bigwedge^{p_t} \mathcal{B}_2^\vee \otimes \bigwedge^k \mathcal{B}_2\big).$$

When $\ell \ne 0$, then either $\lambda \ne 0$ satisfies condition (†), which guarantees cohomology vanishing for the Schur bundle on the first factor, or else $\lambda \ne 0$ satisfies (∗). The latter situation yields vanishing on the second factor by Proposition 3.1.4(1) below. The hypothesis of the proposition is verified since λ was seen in (2.3.1b) to have at most

$$\dim V_m - N - n \le \dim V_m - (t+1) - n$$

columns and $t \le N - 1$.

In the proof of Theorem 1.1.2 in Section 2.2.1, we also claimed that the cohomology of $\mathcal{X}_0$ is given by

$$H^0\big(\mathbf{G}_2, \textstyle\bigwedge^{p_1}\mathcal{B}_2^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}_2^\vee \otimes \bigwedge^k \mathcal{B}_2\big) = H^0\big(\mathbf{G}_2, \bigwedge^{k-|p|}\mathcal{B}_2\big) = \bigwedge^{k-|p|} H^0(\mathbf{G}_2, \mathcal{B}_2). \tag{2.3.11}$$

Here we assume $|p| = p_1 + \cdots + p_t \le k \le n$, while the answer is understood to be 0 if this assumption fails. We can now justify the first equality using Pieri's rule combined with Borel–Weil–Bott. This is certainly well-known, but it appears easier to write the argument than to find a reference.

Consider an arbitrary Grassmannian $\mathbf{G}(d,n)$ with tautological quotient $\mathcal{B}$, and assume $t \le d-n$. The latter is true in our setting since $t \le N-1 \le \dim V_m - n$. We inspect the tensor product

$$\textstyle\bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \bigwedge^k \mathcal{B} = \bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \bigwedge^{n-k}\mathcal{B}^\vee \otimes \det \mathcal{B}. \tag{2.3.12}$$

By Pieri's rule, tensorization by $\bigwedge^p \mathcal{B}^\vee$ has the effect of adding p boxes, no two in the same row. Thus, if $\mathbf{S}_\nu(\mathcal{B}^\vee)$ is any summand of $\bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \bigwedge^{n-k}\mathcal{B}^\vee$, then we create at most $t+1$ columns. Hence

$$1 \le \nu_1 \le t+1 \le d-n+1.$$

On the other hand, tensorization by $\det \mathcal{B}$ subtracts 1 box from all the rows. Consequently, the summands $\mathbf{S}_\mu(\mathcal{B}^\vee)$ that appear in (2.3.12) satisfy

$$\mu_1 = \nu_1 - 1.$$

In general, we have $1 \le \mu_1 \le d-n$, so Borel–Weil–Bott shows that $\mathbf{S}_\mu(\mathcal{B}^\vee)$ has no cohomology; see condition (2.3.5b). There is one exception corresponding to $\mu_1 = 0$. In this case, we must have $\nu_1 = 1$, which forces $\nu = (1^{|p|+n-k})$ and then $-\mu = (1^{k-|p|})$. This yields the term $\mathbf{S}_\mu(\mathcal{B}^\vee) = \mathbf{S}_{-\mu}(\mathcal{B}) = \bigwedge^{k-|p|}\mathcal{B}$, and justifies (2.3.11). □

Proof of Proposition 2.2.7. We consider the simpler case of the resolution $\mathcal{W}_\bullet$ first. By (2.3.2b) we have

$$\mathcal{W}_\ell = \bigoplus_{|\lambda|=\ell} \mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1) \boxtimes \big(\mathbf{S}_\lambda(\mathcal{B}_2^\vee \oplus \mathcal{B}_2^\vee) \otimes \operatorname{Sym}^k \mathcal{B}_2\big).$$

The argument is similar to that of Equation (2.2.2). This time, to deal with case $(*)$ we invoke Proposition 3.1.3.

The analysis of the bundles $\mathcal{Y}_\ell$ for $\ell \ge 1$ in the general case uses Proposition 3.1.4(2) instead.

In Section 2.2.1, we also needed the cohomology of the bundle $\mathcal{Y}_0$. To this end, we show that on an arbitrary Grassmannian $\mathbf{G}(d,n)$, for $t \le d-n$ and positive integers $p_1, \dots, p_t > 0$, the bundle $\bigwedge^{p_1}\mathcal{B}^\vee \otimes \ldots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \operatorname{Sym}^k \mathcal{B}$ has no cohomology unless $p_1 = \cdots = p_t = 1$ and $k \ge t$, in which case there is only one nontrivial cohomology group

$$H^0\left(\textstyle\bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \operatorname{Sym}^k \mathcal{B}\right) = \operatorname{Sym}^{k-t} H^0(\mathcal{B}). \tag{2.3.13}$$

Indeed, let $\mathbf{S}_\mu(\mathcal{B}^\vee)$ be a summand of $\bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \operatorname{Sym}^k \mathcal{B}$. Dualizing, we have that $\mathbf{S}_{-\mu}(\mathcal{B}^\vee)$

is a summand of

$$\bigwedge^{p_1}\mathcal{B}\otimes\cdots\otimes\bigwedge^{p_t}\mathcal{B}\otimes\mathrm{Sym}^k\mathcal{B}^\vee=\bigwedge^{n-p_1}\mathcal{B}^\vee\otimes\cdots\otimes\bigwedge^{n-p_t}\mathcal{B}^\vee\otimes\mathrm{Sym}^k\mathcal{B}^\vee\otimes(\det\mathcal{B})^t.$$

Let $\mathbf{S}_\nu(\mathcal{B}^\vee)$ be a summand of $\bigwedge^{n-p_1}\mathcal{B}^\vee\otimes\cdots\otimes\bigwedge^{n-p_t}\mathcal{B}^\vee\otimes\mathrm{Sym}^k\mathcal{B}^\vee$. Applying the Pieri rules, we start with the partition (k) corresponding to $\mathrm{Sym}^k\,\mathcal{B}^\vee$, to which we add $n-p_1,\ldots,n-p_t$ boxes respectively, such that at each stage we do not add two boxes to the same row. Therefore

$$0\le\nu_n\le t$$

(unless $n=1$, which can be treated separately). If $\nu_n=t$, then $\nu_2=\cdots=\nu_{n-1}=t$. In this case we can say a bit more. Note that the last $n-1$ rows of ν each contain t boxes, and $n-p_1,\ldots,n-p_t$ are all $\le n-1$. For this to be possible, equality must hold, hence $p_1=\cdots=p_t=1$. Moreover, all boxes have to be added to the last $n-1$ rows, and so none can be added to the first row. Hence ν is the partition

$$\nu_1=k,\quad \nu_2=\cdots=\nu_n=t.$$

For this to make sense, we must have $\nu_1=k\ge t=\nu_2$.

Finally, each $\mathbf{S}_{-\mu}(\mathcal{B}^\vee)=\mathbf{S}_\nu(\mathcal{B}^\vee)\otimes(\det\mathcal{B})^t$ satisfies $\mu_{n+1-i}+\nu_i=t$. Since $0\le\nu_n\le t$, we also have $0\le\mu_1\le t$. Since $t\le d-n$, it follows by condition (2.3.5b) that we only have nontrivial cohomology for $\mathbf{S}_\mu(\mathcal{B}^\vee)$ when $\mu_1=0$. This case corresponds to $\nu_n=t$. We have seen above this means $\nu_1=k$ and $\nu_2=\cdots=\nu_n=t$. This yields $\mu=(0,\ldots,0,t-k)$ and $\mathbf{S}_\mu(\mathcal{B}^\vee)=\mathrm{Sym}^{k-t}\mathcal{B}$, which implies the claim. □

Proof of Proposition 2.2.9.. We need to inspect

$$\mathcal{U}_\ell=\bigoplus_{|\lambda|=\ell}\left(\mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1)\otimes\bigwedge^{p_1}\mathcal{B}_1^\vee\right)\boxtimes\left(\mathbf{S}_\lambda(\mathcal{B}_2^\vee\oplus\mathcal{B}_2^\vee)\otimes\bigwedge^{p_2}\mathcal{B}_2^\vee\otimes\cdots\otimes\bigwedge^{p_t}\mathcal{B}_2^\vee\right).$$

The case $\ell=0$ follows from either (2.3.11) or (2.3.13) with $k=0$.

Let $\ell\neq0$. When $\lambda^\dagger$ satisfies (2.3.6b) (for $p=p_1$), the first factor $\mathbf{S}_{\lambda^\dagger}(\mathcal{A}_1)\otimes\bigwedge^{p_1}\mathcal{B}_1^\vee$ has no cohomology. Otherwise, λ satisfies condition $(**)$ from Definition 2.3.9. In this case, we claim the second factor has no cohomology.

Indeed, when $\lambda=(1^{p_1})$, we have that $\mathbf{S}_\lambda=\bigwedge^{p_1}$ is an exterior power and

$$\bigwedge^{p_1}(\mathcal{B}_2^\vee\oplus\mathcal{B}_2^\vee)\cong\bigoplus_{j=0}^{p_1}\bigwedge^j\mathcal{B}_2^\vee\otimes\bigwedge^{p_1-j}\mathcal{B}_2^\vee.$$

As in the above analysis of (2.3.11), every summand $\mathbf{S}_\mu(\mathcal{B}_2^\vee)$ which appears in the second factor then satisfies $1\le\mu_1\le t+1$ by the Pieri rule. Since $t+1\le N\le\dim V_m-n=d-n$, we get vanishing by Borel–Weil–Bott and (2.3.5b).

When $\lambda\neq(1^{p_1})$, letting i denote the (p_1,n)-index of λ, the second factor

$$\mathbf{S}_\lambda(\mathcal{B}_2^\vee\oplus\mathcal{B}_2^\vee)\otimes\bigwedge^{p_2}\mathcal{B}_2^\vee\otimes\cdots\otimes\bigwedge^{p_t}\mathcal{B}_2^\vee$$

has no cohomology by Proposition 3.1.4(3). □

3. Cohomology vanishing on the Grassmannian

3.1. ***Overview.*** We establish the vanishing results which played a crucial role in the proofs of Propositions 2.2.4, 2.2.7 and 2.2.9.

We continue to write $\mathbf{G} = \mathbf{G}(d, n)$ for the Grassmannian of n-dimensional quotients of a d-dimensional vector space, and $\mathcal{B} \to \mathbf{G}$ for the tautological rank-n quotient.

Recall from Section 2.3.2 that $\mathbf{S}_\delta(\mathcal{B}^\vee)$ has no cohomology provided that there exists j such that

$$j \le \delta_j \le d - n + j - 1. \tag{3.1.1}$$

We will establish three results. The first corresponds to the resolution $\mathcal{V}_\bullet$, while the second pertains to the resolution $\mathcal{W}_\bullet$. Together, these already capture the main ideas. The third result covers the resolutions $\mathcal{X}_\bullet$, $\mathcal{Y}_\bullet$ and $\mathcal{U}_\bullet$. Although the notation in this case is more involved, the argument does not require new ideas. We present these results separately for clarity.

Proposition 3.1.2. *Let $\lambda \ne 0$ be a partition that fits in the $(2n) \times (d-n-1)$ rectangle and assume that λ has n-index i. For every summand $\mathbf{S}_\delta(\mathcal{B}^\vee) \subset \mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \bigwedge^\bullet(\mathcal{B})$, the partition δ satisfies condition* (3.1.1) *with $j = i$.*

In particular, all cohomology of $\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \bigwedge^\bullet(\mathcal{B})$ vanishes.

Proposition 3.1.3. *Let $\lambda \ne 0$ be a partition that fits in the $(2n) \times (d-n-1)$ rectangle and assume that λ has n-index i.*

1. *If $i < n$, then for every summand $\mathbf{S}_\delta(\mathcal{B}^\vee) \subset \mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \mathrm{Sym}^\bullet(\mathcal{B})$, the partition δ satisfies condition* (3.1.1) *with $j = i$.*

   In particular, all cohomology of $\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \mathrm{Sym}^\bullet(\mathcal{B})$ vanishes.

2. *If $i = n$ and $k \le n$, then for every summand $\mathbf{S}_\delta(\mathcal{B}^\vee) \subset \mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \mathrm{Sym}^k(\mathcal{B})$, the partition δ satisfies condition* (3.1.1) *with $j = i$.*

   In particular, all cohomology of $\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \mathrm{Sym}^k(\mathcal{B})$ vanishes.

Proposition 3.1.4. *Let $\lambda \ne 0$ be a partition that fits in the $(2n) \times (d-n-t-1)$ rectangle, for some $t \ge 0$.*

1. *Assume that λ has n-index i. Let $p_1, \dots, p_t$ be nonnegative integers and let $0 \le k \le n$. Then every summand*
$$\mathbf{S}_\delta(\mathcal{B}^\vee) \subset \mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \textstyle\bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \bigwedge^k \mathcal{B}$$
   satisfies condition (3.1.1) *with $j = i$.*

   In particular, all cohomology of $\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \bigwedge^k \mathcal{B}$ vanishes.

2. *Assume that λ has n-index i. Let $p_1, \dots, p_t$ be nonnegative integers and let $0 \le k \le n$. All summands of*
$$\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \textstyle\bigwedge^{p_1}\mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_t}\mathcal{B}^\vee \otimes \mathrm{Sym}^k(\mathcal{B})$$
   satisfy condition (3.1.1) *for $j = i$, and therefore this bundle has no cohomology.*

(3) *Assume* $\lambda \neq (1^p)$ *has* (p, n)*-index* i, *for some* $0 \le p \le n$. *Let* $p_1, \dots, p_{t-1}$ *be nonnegative integers. Then every summand*

$$\mathbf{S}_\delta(\mathcal{B}^\vee) \subset \mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \bigwedge^{p_1} \mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_{t-1}} \mathcal{B}^\vee$$

satisfies condition (3.1.1) *with* $j = i$.

In particular, all cohomology of $\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee) \otimes \bigwedge^{p_1} \mathcal{B}^\vee \otimes \cdots \otimes \bigwedge^{p_{t-1}} \mathcal{B}^\vee$ *vanishes.*

3.2. *Littlewood–Richardson coefficients.* For the proofs of the above propositions, we need a few preliminaries about the Littlewood–Richardson coefficients. The material below is well-known, but to establish the notation, we recall several definitions and basic facts, some of which can be found in [SS, §§2, 3].

For two partitions α and β of the same size $|\alpha| = |\beta|$, write $\alpha \ge \beta$ (α *dominates* β) if, for all m, we have

$$\alpha_1 + \cdots + \alpha_m \ge \beta_1 + \cdots + \beta_m.$$

Note that $\alpha \ge \beta$ if and only if $\alpha^\dagger \le \beta^\dagger$.

Given partitions α, β, γ, let $c^\gamma_{\alpha,\beta}$ denote the Littlewood–Richardson coefficient, which is the multiplicity of the Schur functor $\mathbf{S}_\gamma$ in the tensor product $\mathbf{S}_\alpha \otimes \mathbf{S}_\beta$.

The coefficient $c^\gamma_{\alpha,\beta}$ counts the number of Littlewood–Richardson tableaux. These are fillings of the skew tableau of shape γ/α with content β (i.e., for all i, the label i appears exactly β_i times) such that the following two properties hold:

- (*semistandard*) In each row, the entries are weakly increasing from left to right, and in each column, the entries are strictly increasing from top to bottom.
- (*lattice word property*) Let w be the word (called reading word) obtained by reading the entries in each row from right to left, starting with the top row and going down. For each i and m, let $w_i(m)$ be the number of times that i appears in the first m entries of w. Then for all m and i, we have $w_i(m) \ge w_{i+1}(m)$.

We collect a few facts about these coefficients in the next result.

Proposition 3.2.1. (1) *For any complex vector bundles* V, W, *we have*

$$\mathbf{S}_\gamma(V \oplus W) \cong \bigoplus_{\alpha,\beta} (\mathbf{S}_\alpha(V) \otimes \mathbf{S}_\beta(W))^{\oplus c^\gamma_{\alpha,\beta}},$$

where the sum is over all partitions α, β.

(2) *If* $c^\gamma_{\alpha,\beta} \neq 0$, *then* $|\gamma| = |\alpha| + |\beta|$.

(3) *If* $c^\gamma_{\alpha,\beta} \neq 0$, *then* γ *contains both* α *and* β, *i.e.*, $\gamma_i \ge \max(\alpha_i, \beta_i)$ *for all* i.

(4) *In a Littlewood–Richardson tableau of shape* γ/α *and type* β, *all occurrences of the number* i *must appear in rows* i *and later.*

As a consequence, if $c^{\gamma}_{\alpha,\beta} \neq 0$, *then* $\alpha+\beta$ *dominates* γ, *i.e., for all* m, *we have*

$$\sum_{i=1}^{m}(\alpha_i+\beta_i) \geq \sum_{i=1}^{m}\gamma_i.$$

(5) *If* $c^{\gamma}_{\alpha,\beta} \neq 0$, *then* γ *dominates* $\alpha \cup \beta$ (*this is the partition obtained from all of the rows of* α *and* β *placed one after the other according to their lengths*).

Proof. (1) See [SS, (4.5)] for a derivation.

(2) and (3) are clear from the interpretation in terms of tableaux.

(4) We prove this by induction on i. If $i=1$, there is nothing to show. Now suppose the statement is true for i. Suppose that there is a Littlewood–Richardson tableau in which $i+1$ appears in the first i rows. Let w be the reading word of this tableau. By the lattice word property, this instance of $i+1$ cannot appear in a row before the earliest (relative to w) instance of i, so it must appear in row i, and it must appear to the left of i in the tableau. However, this violates the semistandard condition.

(5) This is a consequence of (4) since $\alpha \cup \beta = (\alpha^{\dagger}+\beta^{\dagger})^{\dagger}$ and $c^{\gamma}_{\alpha,\beta} = c^{\gamma^{\dagger}}_{\alpha^{\dagger},\beta^{\dagger}}$. □

3.3. ***Lemmas.*** To carry out the proofs of Propositions 3.1.2, 3.1.3, and 3.1.4, we first establish a few supporting results.

First, by Proposition 3.2.1(1), we have

$$\mathbf{S}_{\lambda}(\mathcal{B}^{\vee} \oplus \mathcal{B}^{\vee}) \cong \bigoplus_{\alpha,\beta}(\mathbf{S}_{\alpha}(\mathcal{B}^{\vee}) \otimes \mathbf{S}_{\beta}(\mathcal{B}^{\vee}))^{\oplus c^{\lambda}_{\alpha,\beta}} \cong \bigoplus_{\alpha,\beta,\gamma} \mathbf{S}_{\gamma}(\mathcal{B}^{\vee})^{\oplus c^{\lambda}_{\alpha,\beta}c^{\gamma}_{\alpha,\beta}}. \tag{3.3.1}$$

Here, the number of rows of the partitions α, β, γ is less than or equal to n, while the number of rows in the partition λ is less than or equal to $2n$.

We assume first that λ is contained in the $(2n) \times (d-n-1)$ rectangle as needed in Proposition 3.1.2 and 3.1.3.

We reserve i *to be the* n*-index of* λ.

Pick a triple α, β, γ such that $\mathbf{S}_{\gamma}(\mathcal{B}^{\vee}) \neq 0$, and $c^{\lambda}_{\alpha,\beta}c^{\gamma}_{\alpha,\beta} \neq 0$. We will deduce a number of restrictions on the partitions α, β, γ.

Lemma 3.3.2. *We have* $\alpha_i \geq i$.

Proof. Suppose that $\alpha_i < i$. Since λ has n-index i, recall from (2.3.8) that $\lambda_{i+1} = \cdots = \lambda_{i+n} = i$ and thus $\lambda_i \geq i$. Then the i-th column of the skew shape λ/α has at least $n+1$ boxes (in rows i through $n+i$). But then any valid Littlewood–Richardson tableau of shape λ/α needs at least $n+1$ labels (because of the semistandard condition). This implies that $\beta_{n+1} > 0$, contradicting the fact that β has at most n rows. □

Lemma 3.3.3. *We have*

$$(\alpha_1+\beta_1)+\cdots+(\alpha_i+\beta_i) \leq i(d-n+i-1).$$

Proof. By Proposition 3.2.1(5), we know that $\lambda \geq \alpha \cup \beta$, so that

$$\lambda_1 + \cdots + \lambda_{2i} \geq (\alpha \cup \beta)_1 + \cdots + (\alpha \cup \beta)_{2i} \geq (\alpha_1 + \beta_1) + \cdots + (\alpha_i + \beta_i). \tag{3.3.3a}$$

Since $\lambda_j \leq d - n - 1$ for $j = 1, \ldots, i$ and $\lambda_j \leq i$ for $j = i + 1, \ldots, 2i$, the lemma follows. □

Lemma 3.3.4. *We have $i + 1 \leq \gamma_i \leq d - n + i - 1$.*

Proof. We know that $\alpha_i \geq i$ from Lemma 3.3.2. If in fact $\alpha_i \geq i + 1$, then we can use Proposition 3.2.1(3) to conclude that $\gamma_i \geq i + 1$.

Otherwise, we have $\alpha_i = i$. Suppose that $\gamma_i = i$. Then the i-th row of γ/α has no boxes. Since $c^{\lambda}_{\alpha,\beta} \neq 0$, there is a Littlewood–Richardson tableau of shape λ/α and type β. Since $\lambda_{i+n} = i$, the i-th column of λ/α has at least $i + n - \alpha^{\dagger}_i$ boxes, so that β has at least $i + n - \alpha^{\dagger}_i$ rows (by the semistandard condition).

Next, there is also a Littlewood–Richardson tableau of shape γ/α and type β. The integers in the interval $[i, i + n - \alpha^{\dagger}_i]$ cannot go in the first $i - 1$ rows of γ/α by Proposition 3.2.1(4), and cannot go in the i-th row since it is empty. Thus, these numbers must go in rows $i + 1$ or higher. Again, since $\gamma_i = i$, they are also constrained to the first i columns of γ/α as well. Now, in γ/α, the i-th column only has boxes in rows $\alpha^{\dagger}_i + 1, \ldots, n$, at most. Consequently, the labels $[i, i + n - \alpha^{\dagger}_i]$ can only be placed in rows $\alpha^{\dagger}_i + 1, \ldots, n$.

Suppose it is possible to do this. Consider the subdiagram D of γ/α consisting of boxes that are filled with entries $\geq i$. If we subtract $i - 1$ from every entry, we claim that the result is a valid Littlewood–Richardson tableau of shape D. Subtracting the same amount from each entry does not affect any of the semistandard inequalities. Furthermore, if w is the reading word of the Littlewood–Richardson tableau of γ/α of type β that we're considering, and w' is the reading word of its restriction to D, then in the notation of Section 3.2, for $j \geq i$ and any m, we have $w'_j(m) = w_j(m)$. In particular, then $w'_j(m) \geq w'_{j+1}(m)$ for all $j \geq i$ and hence the result of subtracting $i - 1$ from all entries of D has the lattice word property.

But then we have too many labels: in fact, at least $n - \alpha^{\dagger}_i + 1$ labels and only $n - \alpha^{\dagger}_i$ rows to put them into. We have a contradiction to Proposition 3.2.1(4) and hence $\gamma_i \geq i + 1$.

Finally, again by Proposition 3.2.1(4), we have $\alpha + \beta \geq \gamma$. Using Lemma 3.3.3, we obtain

$$i\gamma_i \leq \gamma_1 + \cdots + \gamma_i \leq i(d - n + i - 1), \tag{3.3.4a}$$

and hence $\gamma_i \leq d - n + i - 1$.[1] □

Lemma 3.3.5. (1) *If $i < n$, then $\gamma_{i+1} \geq i$.*

(2) *If $i = n$, then $\gamma_n \geq 2n$.*

Proof. Since $\lambda_{i+1} = i$, we have $\alpha_{i+1} \leq i$ by Proposition 3.2.1(3).

Let $c = i - \alpha_{i+1}$. Then λ/α contains the subrectangle occupying rows $i + 1, \ldots, i + n$ and columns $\alpha_{i+1} + 1, \ldots, i$, which has n rows and c columns. Since $c^{\lambda}_{\alpha,\beta} \neq 0$, we can fill λ/α with content β.

[1] We could relax the condition that λ is contained in the $(2n) \times (d - n - 1)$ rectangle here. It would suffice to know that $\lambda_1 + \cdots + \lambda_i \leq i(d - n - 1) + i - 1$.

By examining the $n \times c$ subrectangle and using the semistandard property, we obtain $\beta_n \geq c$. Let β' be the result of subtracting c from all parts of β. Then

$$\mathbf{S}_\beta(\mathcal{B}^\vee) = (\det \mathcal{B}^\vee)^{\otimes c} \otimes \mathbf{S}_{\beta'}(\mathcal{B}^\vee)$$

since $\operatorname{rank}(\mathcal{B}^\vee) = n$. Hence to compute $\mathbf{S}_\alpha(\mathcal{B}^\vee) \otimes \mathbf{S}_\beta(\mathcal{B}^\vee)$, we can first add c to all values of $(\alpha_1, \dots, \alpha_n)$ and then tensor with $\mathbf{S}_{\beta'}(\mathcal{B}^\vee)$.

In particular, if $i < n$, then $\gamma_{i+1} \geq \alpha_{i+1} + c = i$, again by Proposition 3.2.1(3). Otherwise, if $i = n$, since $\alpha_{n+1} = 0$ we find $c = n$. Using Lemma 3.3.2, we have $\alpha_n \geq n$, and thus $\gamma_n \geq \alpha_n + c \geq 2n$. □

3.4. ***Vanishing.*** We continue to use the notation from the previous section.

Proof of Proposition 3.1.2. Assume λ fits in the $(2n) \times (d-n-1)$ rectangle, and let $\mathbf{S}_\gamma(\mathcal{B}^\vee)$ be a summand of $\mathbf{S}_\lambda(\mathcal{B}^\vee \oplus \mathcal{B}^\vee)$. Consider the tensor product $\mathbf{S}_\gamma(\mathcal{B}^\vee) \otimes \bigwedge^k(\mathcal{B})$. First we use that $\bigwedge^k \mathcal{B} = \det(\mathcal{B}) \otimes \bigwedge^{n-k} \mathcal{B}^\vee$ and $\bigwedge^{n-k} \mathcal{B}^\vee = \mathbf{S}_{(1^{n-k})}(\mathcal{B}^\vee)$. The Pieri rule describes the outcome of tensoring with $\bigwedge^{n-k} \mathcal{B}^\vee$. The result is a sum over partitions where we add $n-k$ boxes, no two in the same row. Tensoring with $\det(\mathcal{B})$ is the same as subtracting 1 from all entries. Therefore, for any summand $\mathbf{S}_\delta(\mathcal{B}^\vee)$ of this tensor product, we have $\gamma_i - 1 \leq \delta_i \leq \gamma_i$. Hence we conclude from Lemma 3.3.4 that

$$i \leq \delta_i \leq d - n + i - 1,$$

completing the argument in this case. □

Proof of Proposition 3.1.3. The Pieri rule applied to symmetric powers tells us that $\mathbf{S}_\nu(\mathcal{B}^\vee)$ is a summand of $\mathbf{S}_\mu(\mathcal{B}^\vee) \otimes \operatorname{Sym}^k(\mathcal{B}^\vee)$ if and only if $|\nu| = |\mu| + k$ and the interlacing property $\nu_j \geq \mu_j \geq \nu_{j+1}$ holds for all j. In fact, it makes no difference if some entries of ν and μ are negative since we can make them nonnegative by twisting by powers of $\det(\mathcal{B}^\vee)$ and untwisting after.

If $\mathbf{S}_\delta(\mathcal{B}^\vee)$ is a summand of $\mathbf{S}_\gamma(\mathcal{B}^\vee) \otimes \operatorname{Sym}^k(\mathcal{B})$, we obtain by dualizing that $\mathbf{S}_{-\delta}(\mathcal{B}^\vee)$ is a summand of $\mathbf{S}_{-\gamma}(\mathcal{B}^\vee) \otimes \operatorname{Sym}^k(\mathcal{B}^\vee)$. Thus, $|\gamma| = |\delta| + k$ and the interlacing property gives

$$\gamma_{j+1} \leq \delta_j \leq \gamma_j.$$

(Thus, if $\mathbf{S}_\delta(\mathcal{B}^\vee)$ is a summand of $\mathbf{S}_\gamma(\mathcal{B}^\vee) \otimes \operatorname{Sym}^k(\mathcal{B})$, then $\mathbf{S}_\gamma(\mathcal{B}^\vee)$ is a summand of $\mathbf{S}_\delta(\mathcal{B}^\vee) \otimes \operatorname{Sym}^k(\mathcal{B}^\vee)$.)

Consider the n-index i of λ. If $i < n$, then Lemma 3.3.5 tells us that $\gamma_{i+1} \geq i$. The interlacing property then forces $\delta_i \geq i$. If $i = n$, then $\gamma_n \geq 2n$. Since γ is obtained from δ by adding k boxes, we have

$$\delta_n \geq \gamma_n - k \geq 2n - k \geq n$$

since we assume that $k \leq n$.

In any case, under either assumption, we have shown that $\delta_i \geq i$ and also $\delta_i \leq \gamma_i \leq d - n + i - 1$ by Lemma 3.3.4. This is what we set out to prove. □

Proof of Proposition 3.1.4. Assume now that λ is contained in the $(2n)\times(d-n-t-1)$ rectangle. For case (1), for a partition λ of n-index i, we have

$$\lambda_j \leq d-n-t-1 \text{ for } j\leq i, \quad \lambda_j \leq i \text{ for } i+1\leq j\leq 2i.$$

Thus, by (3.3.3a) and (3.3.4a), we have

$$i\gamma_i \leq \gamma_1+\cdots+\gamma_i \leq \lambda_1+\cdots+\lambda_{2i} \leq i(d-n-t-1)+i\cdot i \Longrightarrow \gamma_i \leq d-n-t-1+i.$$

We also have $\gamma_i \geq i+1$ by Lemma 3.3.4. We inspect the summands

$$\mathbf{S}_\delta(\mathcal{B}^\vee)\subset \mathbf{S}_\gamma(\mathcal{B}^\vee)\otimes \bigwedge^{p_1}\mathcal{B}^\vee\otimes \bigwedge^{p_2}\mathcal{B}^\vee\otimes\cdots\otimes\bigwedge^{p_t}\mathcal{B}^\vee\otimes\bigwedge^{k}\mathcal{B}.$$

We use $\bigwedge^k\mathcal{B}=\bigwedge^{n-k}\mathcal{B}^\vee\otimes\det\mathcal{B}$. By repeated application of the Pieri rule, and taking into account that tensorization by $\det\mathcal{B}$ subtracts one box from each entry, we conclude

$$\gamma_i-1\leq\delta_i\leq\gamma_i+t.$$

The conclusion follows since

$$i+1\leq\gamma_i\leq d-n-t-1+i\Longrightarrow i\leq\delta_i\leq d-n+i-1.$$

For (2), we saw in the proof of Proposition 3.1.3 that all summands $\mathbf{S}_\delta(\mathcal{B}^\vee)$ of $\mathbf{S}_\lambda(\mathcal{B}^\vee\oplus\mathcal{B}^\vee)\otimes\operatorname{Sym}^k\mathcal{B}$ satisfy

$$i\leq\delta_i\leq d-n-t-1+i,$$

where the modified upper bound is due to the different size of the rectangle that contains λ. By the Pieri rule, tensorization by $(\bigwedge^{\bullet}\mathcal{B}^\vee)^{\otimes t}$ can only increase the lengths of rows, and if so by at most t boxes. Thus all summands $\mathbf{S}_\nu(\mathcal{B}^\vee)$ of $\mathbf{S}_\lambda(\mathcal{B}^\vee\oplus\mathcal{B}^\vee)\otimes\bigwedge^{p_1}\mathcal{B}^\vee\otimes\cdots\otimes\bigwedge^{p_t}\mathcal{B}^\vee\otimes\operatorname{Sym}^k\mathcal{B}$ satisfy

$$i\leq\nu_i\leq d-n+i-1,$$

as claimed.

Finally, we prove (3). If $\mathbf{S}_\gamma(\mathcal{B}^\vee)$ is a summand of $\mathbf{S}_\lambda(\mathcal{B}^\vee\oplus\mathcal{B}^\vee)$, then by the same reasoning as in Lemma 3.3.2 we have $i\leq\alpha_i\leq\gamma_i$. By (2.3.10)

$$\lambda_1,\ldots,\lambda_i\leq d-n-t-1,\quad \lambda_{i+1},\ldots,\lambda_{2i}\leq i+1.$$

Using (3.3.3a) and (3.3.4a), we obtain

$$i\gamma_i\leq\gamma_1+\cdots+\gamma_i\leq\lambda_1+\cdots+\lambda_{2i}\leq i(d-n-t-1)+i(i+1)\Longrightarrow\gamma_i\leq d-n-t+i.$$

By repeated application of the Pieri rule, all summands $\mathbf{S}_\delta(\mathcal{B}^\vee)$ of $\mathbf{S}_\gamma(\mathcal{B}^\vee)\otimes\bigwedge^{p_1}\mathcal{B}^\vee\otimes\cdots\otimes\bigwedge^{p_{t-1}}\mathcal{B}^\vee$ satisfy $\gamma_i\leq\delta_i\leq\gamma_i+(t-1)$. Since $i\leq\gamma_i\leq d-n-t+i$, we have $i\leq\delta_i\leq d-n+i-1$. Therefore, condition (3.1.1) is satisfied for δ and $j=i$, and all cohomology vanishes. ☐

4. Corollaries

4.1. ***Corollary 1.2.1 and universality.*** We explain the universality arguments needed to derive Corollary 1.2.1 from the genus 0 computations in Theorems 1.1.2, 1.1.3, and 1.1.5.

Regarding equation (1.2.1a), we have the factorization

$$\sum_{n=0}^{\infty} q^n \chi\big(\mathsf{Quot}_C(\mathbb{C}^N, n), \textstyle\bigwedge_y L^{[n]}\big) = \mathsf{A}^{\chi(\mathcal{O}_C)} \cdot \mathsf{B}^{\chi(L)} \tag{4.1.1}$$

where $\mathsf{A}, \mathsf{B} \in 1 + q\,\mathbb{Q}[y][\![q]\!]$ are two universal power series whose coefficients may depend on N but not on the pair (C, L). This factorization is by now a standard fact; see for instance [EGL; OS; St] for various incarnations of this statement. To establish (1.2.1a), we show that

$$\mathsf{A} = (1-q)^{-1}, \quad \mathsf{B} = (1+qy)^N.$$

Specializing $C = \mathbb{P}^1$ and $\deg L = \ell \geq n$ in (4.1.1), and using Theorem 1.1.2(1) we obtain

$$[q^n]\mathsf{A} \cdot \mathsf{B}^{\ell+1} = \sum_{k=0}^{n} y^k \binom{N\chi(L)}{k},$$

where the brackets denote extracting the relevant coefficient in the q-expansion. By direct calculation, we also have

$$[q^n](1-q)^{-1} \cdot \big((1+qy)^N\big)^{\ell+1} = \sum_{k=0}^{n} y^k \binom{N\chi(L)}{k}.$$

It remains to explain that the coefficients

$$[q^n]\mathsf{A} \cdot \mathsf{B}^{\ell+1} \text{ for all } \ell \geq n$$

determine the series A, B at most uniquely. We argue inductively, each coefficient at a time. Explicitly, we write

$$\mathsf{A} = 1 + a_1 q + a_2 q^2 + \cdots, \quad \mathsf{B} = 1 + b_1 q + b_2 q^2 + \cdots.$$

Then

$$[q^n]\mathsf{A} \cdot \mathsf{B}^{\ell+1} = a_n + (\ell+1) b_n + \text{lower-order terms in } n.$$

The lower-order terms are determined by the induction hypothesis. The inductive step follows since the principal terms $a_n + (\ell+1)b_n$ for all $\ell \geq n$ determine a_n, b_n at most uniquely.

For (1.2.1b) the argument is similar, using the factorization

$$\sum_{n=0}^{\infty} q^n \chi\Big(\mathsf{Quot}_C(\mathbb{C}^N, n), \bigotimes_{i=1}^{t} \big(\textstyle\bigwedge_{y_i} M_i^{[n]}\big)^{\vee}\Big) = \mathsf{A}^{\chi(\mathcal{O}_C)} \cdot \mathsf{B}_1^{\chi(M_1)} \cdots \mathsf{B}_t^{\chi(M_t)}.$$

This time, we specialize $C = \mathbb{P}^1$, and

$$M_1 = M_2 = \cdots = M_t = M, \quad \deg M = m \geq n.$$

By Theorem 1.1.5, we have

$$[q^n]\mathsf{A}\cdot(\mathsf{B}_1\cdots\mathsf{B}_t)^{m+1}=1 \text{ for all } m\geq n.$$

Indeed, the only nonzero contribution appears from the free term $y_1=\cdots=y_t=0$ and yields the answer $\chi(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n),\mathcal{O})=1$ since $\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n)$ is rational. By the above reasoning, the series

$$\mathsf{A}=(1-q)^{-1},\quad \mathsf{B}_1\cdots\mathsf{B}_t=1$$

are uniquely determined. Next, we set

$$M_1=L,\quad M_2=\cdots=M_t=M,$$

where $\deg L=\deg M-1=m-1\geq n-1$. This time around, Theorem 1.1.5 implies

$$[q^n](\mathsf{A}\cdot\mathsf{B}_1^{-1})\cdot(\mathsf{B}_1\cdots\mathsf{B}_t)^{m+1}=1\Longrightarrow \mathsf{A}\cdot\mathsf{B}_1^{-1}=(1-q)^{-1},\quad \mathsf{B}_1\cdots\mathsf{B}_t=1.$$

Therefore

$$\mathsf{A}=(1-q)^{-1},\quad \mathsf{B}_1=\cdots=\mathsf{B}_t=1,$$

and (1.2.1b) follows.

Equation (1.2.1c) uses Theorem 1.1.3(1). Indeed, we have the factorization

$$\sum_{n=0}^{\infty}q^n\chi\left(\mathsf{Quot}_C(\mathbb{C}^N,n),\operatorname{Sym}_y L^{[n]}\right)=\mathsf{A}^{\chi(\mathcal{O}_C)}\cdot\mathsf{B}^{\chi(L)},$$

where $\mathsf{A},\mathsf{B}\in 1+q\mathbb{Q}[\![y]\!][\![q]\!]$. Fix $n\geq 1$. By Theorem 1.1.3, if $\deg L=\ell\geq n$, we have

$$[q^n]\,\mathsf{A}\cdot\mathsf{B}^{\ell+1}=\sum_{k\geq 0}y^k\chi\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n),\operatorname{Sym}^k L^{[n]}\right)=\sum_{k=0}^{n}y^k(-1)^k\binom{-N(\ell+1)}{k}\mod y^{n+1}.$$

Both sides of this identity are polynomials in $(\ell+1)$. (On the left hand side, these polynomials depend on the first q-coefficients $a_1,\ldots,a_n,b_1,\ldots,b_n$ of A,B considered modulo y^{n+1}.) Consequently, the same equality holds for all values of ℓ without restrictions. Hence, for all L, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}q^ny^k\chi\left(\mathsf{Quot}_{\mathbb{P}^1}(\mathbb{C}^N,n),\operatorname{Sym}^k L^{[n]}\right)&=\sum_{n=0}^{\infty}q^n\left(\sum_{k=0}^{n}y^k(-1)^k\binom{-N(\ell+1)}{k}\right)\\&=(1-q)^{-1}\cdot(1-qy)^{-N\chi(L)}\end{aligned}$$

as stated in (1.2.1c). □

4.2. ***Corollary 1.1.7.*** We analyze the cohomology groups of $L^{[n]}$ for all line bundles $L\to\mathbb{P}^1$ using a few simple considerations. The corollary can also be derived by combining the methods of [BGS, Corollary 9.3] when adapted to the case of the projective line, followed by a calculation on the symmetric product.

Let $p\in\mathbb{P}^1$ and write $\mathcal{Q}_p=\mathcal{Q}|_{p\times\mathsf{Quot}}$. The exact sequence

$$0\to L(-p)\to L\to L_p\to 0$$

yields an exact sequence over Quot:

$$0 \to L(-p)^{[n]} \to L^{[n]} \to \mathcal{Q}_p \to 0. \tag{4.2.1}$$

When $\deg L \geq n+1$, the bundles $L^{[n]}$ and $L(-p)^{[n]}$ carry no higher cohomology by Theorems 1.1.2(1) or 1.1.3(1) for $k = 1$. Taking cohomology in (4.2.1), we obtain

$$H^i(\mathcal{Q}_p) = 0, \quad i \geq 1. \tag{4.2.2}$$

We go back to (4.2.1) written for arbitrary L, not necessarily sufficiently positive. Considering cohomology again and using (4.2.2), we obtain

$$H^i(L^{[n]}) = H^i(L(-p)^{[n]}), \quad i \geq 2. \tag{4.2.3}$$

Since $H^i(L^{[n]}) = 0$ for $\deg L \geq n$ and $i \geq 2$, it follows from (4.2.3) that $H^i(L^{[n]}) = 0$ for all $i \geq 2$ and all L. □

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Admissible pairs and p-adic Hodge structures, I: Transcendence of the de Rham lattice

Sean Howe and Christian Klevdal

For an algebraically closed nonarchimedean extension $C/\mathbb{Q}_p$, we define a Tannakian category of *p-adic Hodge structures* over C that is a local, p-adic structural analog of the global, archimedean category of $\mathbb{Q}$-Hodge structures in complex geometry. In this setting the filtrations of classical Hodge theory must be enriched to lattices over a complete discrete valuation ring, Fontaine's integral de Rham period ring B_{dR}^+, and a pure p-adic Hodge structure is then a $\mathbb{Q}_p$-vector space equipped with a B_{dR}^+-lattice satisfying a natural condition analogous to the transversality of the complex Hodge filtration with its conjugate. We show p-adic Hodge structures are equivalent to a full subcategory of *basic* objects in the category of *admissible pairs*, a toy category of cohomological motives over C that is equivalent to the isogeny category of rigidified Breuil–Kisin–Fargues modules and closely related to Fontaine's p-adic Hodge theory over p-adic subfields. As an application, we characterize basic admissible pairs with complex multiplication in terms of the transcendence of p-adic periods. This generalizes an earlier result for one-dimensional formal groups and is an unconditional, local, p-adic analog of a global, archimedean characterization of CM motives over $\mathbb{C}$ conditional on the standard conjectures, the Hodge conjecture, and the Grothendieck period conjecture (known unconditionally for abelian varieties by work of Cohen, Shiga and Wolfart).

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1. Introduction

The de Rham comparison isomorphism equips the rational singular cohomology of a complex algebraic variety X with a Hodge filtration defined over $\mathbb{C}$. After choosing a basis for the singular cohomology, the modulus of this Hodge filtration is a complex point in a partial flag variety defined over $\mathbb{Q}$, and a fundamental question in algebraic geometry is to relate algebraic and arithmetic properties of these period moduli to algebraic and arithmetic properties of the equations defining X (see, e.g., [10]). In particular, one would like to understand which algebraic conditions on the flag variety correspond to algebraic conditions on the coefficients of the equations defining X, and vice versa. The goal of this paper and its sequel is to explore a local, p-adic analog of this question, where complex algebraic varieties are replaced with rigid analytic varieties over a complete algebraically closed extension of $\mathbb{Q}_p$ (eventually embedded in the more versatile geometric framework of diamonds). In this first part we focus purely on the theory over a point, which can be handled algebraically without knowledge of the modern foundations of p-adic geometry.

A fundamental example over $\mathbb{C}$ is Schneider's 1937 result [30] on the transcendence of the j-invariant. Recall that a complex elliptic curve E can be presented analytically as a quotient $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ for $\tau \in \mathbb{H}^{\pm} \subset \mathbb{P}^1(\mathbb{C})$ or algebraically as the solution set of a Weierstrass equation $y^2 = 4x^3 + ax + b$. The j-invariant

$$j = 1728\frac{a^3}{a^3 - 27b^2} = q^{-1} + 744 + 196884q + \cdots \quad (q = e^{2\pi i \tau})$$

is independent of choices and controls the field of definition of E as an algebraic variety. Schneider's theorem says that if j and τ are both algebraic over $\mathbb{Q}$ then E has complex multiplication (CM). Conversely, if E has CM then τ is quadratic imaginary and j is contained in an abelian extension of $\mathbb{Q}(\tau)$. In [19], one author proved an analog where elliptic curves are replaced with one-dimensional p-divisible formal groups, singular homology is replaced with the p-adic Tate module, and the Hodge filtration is replaced with the Hodge–Tate filtration: let $C/\mathbb{Q}_p$ be an algebraically closed nonarchimedean[1] extension, let $\mathcal{O}_C$ be the valuation ring in C with maximal ideal $\mathfrak{m}_C$ and residue field $\kappa = \mathcal{O}_C/\mathfrak{m}_C$, and let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ in κ. For W the p-typical Witt vectors, let $C_0 = W(\kappa)[1/p]$, which we identify with the maximal complete absolutely unramified subextension of C. Let $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p] \subseteq C_0$. For a subfield $K \subseteq C$, $\overline{K}$ is its algebraic closure in C.

For $G/\mathcal{O}_C$ a one-dimensional p-divisible formal group, we say G is $\overline{C}_0$-analytic if there is a finite extension K/C_0 in C and a choice of formal coordinate such that the power series defining the group law has coefficients in $\mathcal{O}_K$. We say the Hodge–Tate filtration $\mathrm{Lie}\, G(1) \subseteq T_pG \otimes C$ is $\overline{C}_0$-analytic if it is defined over $\overline{C}_0$, i.e., there is a basis of $\mathrm{Lie}\, G(1)$ consisting of vectors in $T_pG \otimes \overline{C}_0$. We say G has complex multiplication (CM) if it admits isogenies by a semisimple $\mathbb{Q}_p$-algebra of dimension equal to the height of G (i.e., the rank of the free $\mathbb{Z}_p$-module T_pG).

Theorem 1.0.1 [19]. *Let $G/\mathcal{O}_C$ be a one-dimensional p-divisible formal group. If G and the Hodge–Tate filtration are both $\overline{C}_0$-analytic, then G has CM. Conversely, if G has CM, then the Hodge–Tate filtration is defined over a finite extension $K/\mathbb{Q}_p$ of degree equal to the height of G, and G is defined over an abelian extension of K.*

Schneider's theorem is a global, archimedean transcendence result because the transcendence considered in both the defining equations of varieties and in cohomology is with respect to an archimedean extension of the global field $\mathbb{Q}$. By contrast, Theorem 1.0.1 is a local, p-adic transcendence result because it describes the transcendence of numbers in C over p-adic subfields. In both cases, the new result was the condition for CM, while the more refined information about fields of definition in the CM case was already known. In the following, we will focus on similar characterizations of CM in more general settings and mostly leave aside the refined information on fields of definition.

1.1. *Transcendence results.* Schneider's theorem was generalized by Cohen [11] and Shiga and Wolfart [34] to all abelian varieties using the Wüstholz analytic subgroup theorem. Conditional on strong conjectures, we further generalize to all motives over $\mathbb{C}$ (i.e., to the cohomology of all smooth projective complex varieties). We say an object M in a connected neutral Tannakian category has complex multiplication (CM) if its Tannakian structure group, i.e., the automorphism group of any fiber functor on the Tannakian subcategory $\langle M \rangle$ generated by M, is a torus.

Theorem A. *Assume the standard conjectures, the Hodge conjecture, and the Grothendieck period conjecture. Let M be an element of* $\mathrm{Mot}(\mathbb{C})$, *the Tannakian category of pure motives over $\mathbb{C}$ with*

[1] A nonarchimedean field is a field that is complete for a nonarchimedean absolute value.

$\mathbb{Q}$-coefficients [1]. *Then M has CM if and only if $M \in \mathrm{Mot}(\overline{\mathbb{Q}})$ and the Hodge filtration on the Betti realization of M is $\overline{\mathbb{Q}}$-algebraic.*

The main result of this paper, Theorem B, is an unconditional local, p-adic analog of Theorem A. It includes, as a special case, a generalization of Theorem 1.0.1 from one-dimensional formal groups to all isoclinic formal groups (see Example 1.1.3), analogous to the Cohen–Shiga–Wolfart generalization of Schneider's theorem from elliptic curves to all complex abelian varieties. Once the general formalism is established, the proof is almost identical to the proof of Theorem A (a motivated reader can find the proof of the latter in Section 6.1 before continuing the introduction). To state it, we first introduce some definitions.

The analog of $\mathrm{Mot}(\mathbb{C})$ will be the $\mathbb{Q}_p$-linear Tannakian category $\mathrm{AdmPair}^{\mathrm{basic}}(C)$ of *basic admissible pairs* over C. It is a full Tannakian subcategory of a category $\mathrm{AdmPair}(C)$ of admissible pairs over C, a toy category of cohomological motives over C which we give an overview of now. (For precise definitions see Section 5.) The objects of $\mathrm{AdmPair}(C)$ are isocrystals[2] equipped with some extra structure. To describe this structure, recall that Fontaine has defined a natural complete discrete valuation ring B_{dR}^+ with residue field C; as usual, we write $B_{\mathrm{dR}} = \mathrm{Frac}(B_{\mathrm{dR}}^+)$. Noncanonically, we have $B_{\mathrm{dR}}^+ \cong C[\![t]\!]$ and $B_{\mathrm{dR}} \cong C((t))$. The extra structure we consider is that of a B_{dR}^+-lattice after base change to B_{dR}. An object of $\mathrm{AdmPair}(C)$ is thus a pair $(W, \mathcal{L}_{\text{ét}})$, where W is an isocrystal and $\mathcal{L}_{\text{ét}} \subseteq W_{B_{\mathrm{dR}}} = W \otimes_{\breve{\mathbb{Q}}_p} B_{\mathrm{dR}}$ is a B_{dR}^+-lattice subject to an admissibility condition (see Definition 5.1.1). It is *basic* if the slope grading on the isocrystal induces a grading also on the lattice $\mathcal{L}_{\text{ét}}$, and we write $\mathrm{AdmPair}^{\mathrm{basic}}(C)$ for the full Tannakian subcategory of basic admissible pairs. Note that for an admissible pair $(W, \mathcal{L}_{\text{ét}})$, the lattice $\mathcal{L}_{\text{ét}}$ induces a trace filtration $F^\bullet W_C$ (called the *Hodge filtration* on W_C: $F^i W_C$ is the image of $W_{B_{\mathrm{dR}}^+} \cap t^i \mathcal{L}_{\text{ét}}$ under the specialization map $W_{B_{\mathrm{dR}}^+} \to W_C$).

Admissible pairs arise, for example, from the cohomology of smooth proper formal schemes over $\mathcal{O}_C$ (using A_{inf} or prismatic cohomology [5; 4], see Example 5.1.5). In this case, the isocrystal comes from the crystalline cohomology of the special fiber and the lattice comes from the comparison with étale cohomology and refines the Hodge filtration. There are equivalent presentations of this category, e.g., as the isogeny category of rigidified Breuil–Kisin–Fargues modules, but here we prefer this elementary perspective to emphasize the connection with classical definitions of period domains in p-adic Hodge theory.

There is a canonical inclusion $\overline{C}_0 \subset B_{\mathrm{dR}}^+$. We say an admissible pair is $\overline{C}_0$-analytic if the Hodge filtration on W_C is defined over $\overline{C}_0$ and if $\mathcal{L}_{\text{ét}}$ is obtained as the convolution of the Hodge filtration on $W_{\overline{C}_0}$ and the valuation filtration on B_{dR}, that is, if $\mathcal{L}_{\text{ét}} = \sum_{i \in \mathbb{Z}} F^{-i} B_{\mathrm{dR}} \cdot F^i W_{\overline{C}_0}$. The $\overline{C}_0$-analytic admissible pairs form a full Tannakian subcategory $\mathrm{AdmPair}(\overline{C}_0)$ of $\mathrm{AdmPair}(C)$. We similarly have a full Tannakian subcategory $\mathrm{AdmPair}(\overline{\breve{\mathbb{Q}}}_p)$ of $\overline{\breve{\mathbb{Q}}}_p$-analytic admissible pairs.

[2] An isocrystal for us is always a finite-dimensional $\breve{\mathbb{Q}}_p$-vector space equipped with a semilinear automorphism (semilinearity with respect to the Frobenius lift on $\breve{\mathbb{Q}}_p$). One could also setup the theory using E-isocrystals where $\breve{\mathbb{Q}}_p$ is replaced with $\breve{E}$ for $E/\mathbb{Q}_p$ a finite extension, but the additional generality gained is subsumed by consideration of objects in our category with endomorphisms by E, and in particular by allowing more general G-structure.

Remark 1.1.1. The rationality of an admissible pair is determined by the rationality of the étale lattice with respect to the $\breve{\mathbb{Q}}_p$-vector space underlying the isocrystal. In particular, for admissible pairs coming from cohomology of smooth proper formal schemes over $\mathcal{O}_C$, if the formal scheme is defined over $\mathcal{O}_K$ for a complete subfield $K \subseteq \overline{C}_0$ then the admissible pair is $\overline{C}_0$-analytic.

There is also a natural *linear* realization of $\mathrm{AdmPair}(C)$ given by a functor to the category of $\mathbb{Q}_p$-vector spaces equipped with a B_{dR}^+-lattice after base change to B_{dR}. In the cohomological setting, the vector space is $\mathbb{Q}_p$-étale cohomology, the lattice is a canonical deformation of de Rham cohomology, and the trace filtration is the Hodge–Tate filtration on C-étale cohomology. Moreover, in this case the data depends only on the rigid analytic generic fiber (it determines the underlying Breuil–Kisin–Fargues module, but not the rigidification), and in fact such cohomological pairs exist for any smooth proper rigid analytic variety over C.

For a pair $(V, \mathcal{L}_{\mathrm{dR}})$ consisting of a $\mathbb{Q}_p$-vector space V and a B_{dR}^+-lattice $\mathcal{L}_{\mathrm{dR}} \subseteq V_{B_{\mathrm{dR}}}$, we say $\mathcal{L}_{\mathrm{dR}}$ is $\overline{C}_0$-analytic (resp. $\overline{\mathbb{Q}}_p$-analytic) if the associated filtration on V_C is defined over $\overline{C}_0$ (resp. $\overline{\mathbb{Q}}_p$) and $\mathcal{L}_{\mathrm{dR}}$ is the convolution of this filtration on $V_{\overline{C}_0}$ (resp. $V_{\overline{\mathbb{Q}}_p}$) and the valuation filtration on B_{dR}. In the cohomological setting, the rationality of $\mathcal{L}_{\mathrm{dR}}$ against the underlying $\mathbb{Q}_p$-vector space V is divorced from the rationality of the variety itself — it is thus a natural analog of the rationality of the complex Hodge filtration against singular cohomology in complex geometry.

The following unconditional local, p-adic analog of Theorem A is our main result:

Theorem B. *Let M be an element of* $\mathrm{AdmPair}^{\mathrm{basic}}(C)$, *the Tannakian category of basic admissible pairs over C. If $M \in \mathrm{AdmPair}^{\mathrm{basic}}(\overline{C}_0)$ and the de Rham lattice on the étale realization of M is $\overline{C}_0$-analytic, then M has CM. Conversely, if M has CM, then $M \in \mathrm{AdmPair}^{\mathrm{basic}}(\breve{\overline{\mathbb{Q}}}_p)$ and the de Rham lattice is $\overline{\mathbb{Q}}_p$-analytic.*

A key motivational observation is that we can put a condition on a B_{dR}^+-latticed $\mathbb{Q}_p$-vector space that is completely analogous to the transversality condition of a filtration with its complex conjugate in the definition of a weight n pure Hodge structure. This condition allows us to cut out a category of *p-adic Hodge structures* which is equivalent to the category of *basic* admissible pairs. The relation between basic admissible pairs and p-adic Hodge structures is a natural *structural* analog of the relation between motives over $\mathbb{C}$ and complex Hodge structures, and this is why the basic hypothesis appears in Theorem B.

Example 1.1.2. If $E/\mathcal{O}_C$ is an elliptic curve with reduction E_κ, $\kappa = \mathcal{O}_C/\mathfrak{m}_C$, the admissible pair attached to the degree 1 cohomology of E is basic if and only if

(1) E_κ is supersingular, or

(2) E_κ is ordinary and, up to isogeny, E is the canonical lift of E_κ.

Example 1.1.3. The category of basic admissible pairs includes as a full subcategory the category of p-divisible groups up-to-isogeny over $\mathcal{O}_C$ equipped with a lifting of the slope decomposition from the special fiber (this is illustrated in Example 1.1.2 by passing from E to $E[p^\infty]$ with its slope decomposition). This includes, e.g., all isoclinic p-divisible formal groups (the height n one-dimensional case treated in

[19] is equivalent to isoclinic of slope $1/n$), and the field of rationality of the admissible pair is equivalent to the field of rationality of the p-divisible group up to isogeny. In these cases, the de Rham lattice is equivalent to the Hodge–Tate filtration, and thus Theorem B is indeed a generalization Theorem 1.0.1.

Remark 1.1.4. For $M \in \mathrm{AdmPair}(\overline{C}_0)$, the Hodge–Tate filtration is canonically split (e.g., by the theory of Hodge–Tate Galois representations). Thus there is a Hodge–Tate grading on the étale realization, not just a Hodge–Tate filtration. It is then a short step from results of Sen to see that if the Hodge–Tate grading is defined over $\overline{C}_0$, then M has CM — this is made explicit, e.g., in [33, Theorem 3], which computes the transcendence degree of the field of definition of the Hodge–Tate grading as the dimension of the motivic Galois group quotiented by its center.

Our result thus implies that, in the basic case, algebraicity of the Hodge–Tate *grading* is equivalent to algebraicity of the de Rham lattice. Our proof does not proceed directly through this equivalence, though we do make use of the associated Galois representation. We emphasize that, in cases where the Hodge–Tate filtration uniquely determines the de Rham lattice, it is a priori a much weaker condition to require that the Hodge–Tate filtration be defined over $\overline{C}_0$ than to require that the Hodge–Tate grading be defined over $\overline{C}_0$. In particular, this equivalence does not hold outside of the basic case: for example, when M is attached to an ordinary elliptic curve, the Hodge–Tate filtration is always algebraic but the Hodge–Tate grading is only algebraic if the Serre–Tate coordinate is a root of unity (this can be deduced from [33, Theorem 3] — the condition on q is equivalent to requiring that the associated admissible pair has CM, cf. also Example 1.1.2).

For our purposes — especially in the continuation article (which we will refer to as Part II) — it is crucial to have a result in terms of the Hodge–Tate filtration or de Rham lattice. Indeed, unlike, the Hodge–Tate grading, these exist in families, making it possible to formulate more general results about special subvarieties in the relative setting (this is closely related to the fact that, for a general admissible pair over C, there is a Hodge–Tate filtration but no canonical splitting).

Remark 1.1.5. It follows that our analogy between basic admissible pairs / p-adic Hodge structures and motives over $\mathbb{C}$ / Hodge structures works well for transcendence theory. However, as exhibited by Example 1.1.2, the admissible pairs in the cohomology of an algebraic variety are not always basic. even in the basic case, this analogy is *not* good for discussing algebraic cycles. One reason is that, to detect algebraic cycles via cohomology, one needs a global structure on the coefficients analogous to the $\mathbb{Q}$-structure on singular cohomology. In our setting, this is possible only on the crystalline side where the natural candidate is the Kottwitz global isocrystal of Scholze's [31] conjectural cohomology for varieties over $\overline{\mathbb{F}}_p$. Assuming this cohomology theory exists, the analog of the Hodge conjecture in this setting would following from a crystalline Tate conjecture for the global isocrystal combined with the variational p-adic Hodge conjecture of [15].

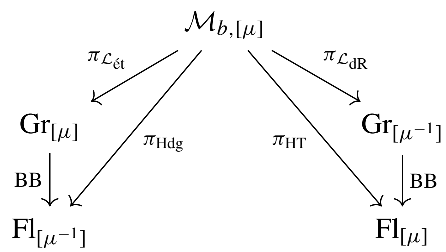
Remark 1.1.6. In Theorem 6.3.2 we give an extension of Theorem B that applies outside of the basic case. In [19] a more general statement was conjectured, but it seems likely that the version given here is actually optimal (see Section 6.3).

1.2. ***Structure groups and exactness of filtrations.*** The categories of admissible pairs and p-adic Hodge structures are interesting beyond Theorem B. Indeed, the moduli spaces of admissible pairs, which we will study in Part II, include the local Shimura varieties of Rapoport–Viehmann [27] and Scholze [32], and their nonminuscule generalizations due to Scholze [32]. Here the basic case is classically the most interesting, e.g., in cohomological constructions of Langlands correspondences.

In constructing these moduli spaces, just as in the theory of complex period domains and Shimura varieties, it is natural to consider admissible pairs with G-structure for $G/\mathbb{Q}_p$ a linear algebraic group. In the complex setting and in most past work in the p-adic setting, one takes G reductive. When one is interested in the complex Hodge theory of complex projective varieties, there is good reason for this: the polarization forces the Tannakian structure group to be reductive because it gives rise to a compact mod center real form. There is an analogous construction of an inner form in the p-adic case, but unfortunately the necessary compactness can only hold for groups of type A_n so the polarization does not help (see Remark 4.5.2 for discussion). In fact, the restrictions on a p-adic Hodge structure that would enforce reductivity via this argument are so severe that we could not in good conscience impose them in the development of the theory! Note that this plays a role even for the classical local Shimura varieties with G reductive, as there can be special subvarieties corresponding to nonreductive subgroups (see Part II).

Theorem B can be reformulated as a statement about period maps for admissible pairs with G-structure. Black-boxing all of the p-adic geometry, we now describe this formulation. This will segue us to a subtle point in the theory caused by the existence of nonreductive structure groups and will lead to our second main result.

Let $G/\mathbb{Q}_p$ be a connected linear algebraic group and let $\mathcal{G}$ be an admissible pair with G-structure. Attached to $\mathcal{G}$ there are two natural invariants — the first is a G-isocrystal, classified by an element in the Kottwitz set $B(G)$ of twisted conjugacy classes in $G(\breve{\mathbb{Q}}_p)$, and the second is an element of the double coset space $G(B_{\mathrm{dR}}^+)\backslash G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ measuring the position of the étale lattice relative to the isocrystal (or rather the de Rham lattice which it spans over B_{dR}^+). If this double coset contains $\mu(t)$ for any μ in a conjugacy class $[\mu]$ of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ and any choice of uniformizer t for B_{dR}^+, we say G is of type $[\mu]$. In particular, if we fix a $b \in G(\breve{\mathbb{Q}}_p)$ and conjugacy class $[\mu]$, there is then a natural infinite level moduli $\mathcal{M}_{b,[\mu]}$ of admissible pairs with G-structure of type $[\mu]$ equipped with a trivialization of the underlying G-isocrystal to b and a trivialization of the étale fiber functor. There are also moduli $\mathrm{Gr}_{[\mu^{\pm 1}]}$ parameterizing B_{dR}^+-lattices in relative position $[\mu^{\pm 1}]$, and Białynicki-Birula maps from these to the flag varieties $\mathrm{Fl}_{[\mu^{\mp 1}]}$ parameterizing filtrations of type $[\mu^{\mp 1}]$. Together, we obtain a diagram of period maps:



The period $\pi_{\mathcal{L}_{\text{ét}}}$ determines the rationality of a G-admissible pair, while $\pi_{\mathcal{L}_{\text{dR}}}$ is the de Rham lattice period. The map BB is the Białynicki-Birula map, which sends a lattice to its induced filtration (the Hodge filtration for the étale lattice, and the Hodge–Tate filtration for the de Rham lattice). If $[\mu]$ is minuscule, then BB is an isomorphism and it suffices to use only the filtration periods π_{Hdg} and π_{HT}. In this case the diagram is analogous to the following uniformization diagram for a Shimura variety $\text{Sh}_{G,K}$ attached to a Shimura datum (G, X) and level structure K,

$$\begin{array}{ccc} & X \times G(\mathbb{A}_f)/K & \\ \swarrow^{J} & & \searrow^{\pi} \\ \text{Sh}_{G,K}(\mathbb{C}) \cong \bigsqcup_{i \in I} \Gamma_i \backslash X^+ & & \text{Fl}_{[\mu^{-1}]}(\mathbb{C}) \end{array}$$

where $[\mu]$ is the class of Hodge cocharacters, π classifies the Hodge filtration, I is a finite index set, and the Γ_i are subgroups of $G(\mathbb{Q})$ determined by K.

Remark 1.2.1. At this point, one might reasonably be confused about the μ's and μ^{-1}'s. To preserve our sanity, we adopted the simple convention that a cocharacter μ of G defines a decreasing filtration on any representation of V of G such that $F^p V = \bigoplus_{i \geq p} V[i]$, for $V[i]$ the weight spaces where μ acts by z^i. Then, $\text{Fl}_{[\mu]}$ is the moduli of decreasing filtrations of this type. Thus, $\text{Fl}_{[\mu]}$ agrees on the nose with the flag variety denoted $\text{Fl}^{\text{std}}_{G,\mu^{-1}}$ in [9, p. 660] but, as an argument in favor of our choice, note that $\text{Fl}_{[\mu]}$ is also canonically identified with the flag variety denoted $\text{Fl}_{G,\mu}$ in loc cit. Indeed, $\text{Fl}_{G,\mu}$ is defined by attaching to μ an increasing filtration on any representation V of G such that $F_p V = \bigoplus_{i \leq p} V[-i]$. Under the canonical identification of increasing and decreasing filtrations by negating the indices, this agrees with our definition of the decreasing filtration attached to μ. The acrobatics here arise from the convention that the Hodge cocharacter attached to a complex Hodge structure acts by z^{-p} on the subspace $H^{p,q}$ of type (p, q). Recall though that there are several good reasons for this convention [13, p. 252]!

A point of $\mathcal{M}_{b,[\mu]}(C)$ is *special* if the Tannakian structure group of the associated G-admissible pair is a torus. Theorem B is then nearly equivalent to this:

Corollary C. *Suppose b is basic and $x \in \mathcal{M}_{b,[\mu]}(C)$. If $\pi_{\mathcal{L}_{\text{dR}}}(x) \in \text{Gr}_{[\mu^{-1}]}(\overline{C}_0)$ and $\pi_{\mathcal{L}_{\text{ét}}}(x) \in \text{Gr}_{[\mu]}(\overline{C}_0)$, then x is special. Conversely, if x is special, then $\pi_{\mathcal{L}_{\text{dR}}}(x) \in \text{Gr}_{[\mu^{-1}]}(\overline{\mathbb{Q}}_p)$ and $\pi_{\mathcal{L}_{\text{ét}}}(x) \in \text{Gr}_{[\mu]}(\breve{\overline{\mathbb{Q}}}_p)$.*

We note that BB is a surjection on C-points but not an isomorphism when μ is not minuscule; however, it induces a bijection on $\overline{C}_0$-points via the convolution construction used earlier. In Part I there will be no need to define these various spaces and maps — we will make the C-points together with their Galois action explicit in Section 5.6 without any p-adic geometry.

For G reductive, the spaces and maps appearing in this diagram were constructed in [32]. In this case, the double cosets $G(B^+_{\text{dR}})\backslash G(B_{\text{dR}})/G(B^+_{\text{dR}})$ are exhausted by the Cartan decomposition, so any G-admissible pair has a type and these diagrams for varying $[\mu]$ give a complete picture. When G is not reductive, the Cartan decomposition is no longer exhaustive, but we still say $\mathcal{G}$ is of type $[\mu]$ if it lies in the double coset attached to $[\mu]$ and that it is *good* if this holds for some $[\mu]$.

In Part II we will generalize the construction of moduli spaces to the nonreductive case. The main difficulty is tied to the other main result of Part I:

Theorem D. *The following are equivalent for a G-admissible pair $\mathcal{G}$:*

(1) *$\mathcal{G}$ is good (resp. has type $[\mu]$).*
(2) *The Hodge–Tate filtration for $\mathcal{G}$ is an exact functor from* Rep G *to filtered C-vector spaces (resp. of type $[\mu]$).*
(3) *The Hodge filtration for $\mathcal{G}$ is an exact functor from* Rep G *to filtered C-vector spaces (resp. of type $[\mu^{-1}]$).*

Remark 1.2.2. Lemma 3.25 of [2] is false.[3] Indeed, it is equivalent to the claim that the Hodge–Tate filtration is exact on the entire category of admissible pairs, which would contradict this theorem. That the Hodge–Tate filtration is not exact can be seen already in a simple motivating example (see Example 4.3.4).

1.3. ***Related work, Parts II and III.*** The category of $\mathrm{AdmPair}(\overline{C}_0)$ is equivalent to a Fontaine category of admissible $\overline{C}_0$-filtered φ-modules over C_0 (see Remark 5.5.2 for related discussion). The Tannakian structure groups in this context have been extensively studied, especially in regards to classification of their possible simple factors — see [33; 37]. With some modifications, most of Theorem B and its proof could be phrased entirely in this classical language. However, there are conceptual benefits in organizing things from C down, and this perspective will be indispensable for the relative theory in Parts II and III.

As noted at the start of the introduction, the category of admissible pairs over C is equivalent to the category of rigidified Breuil–Kisin–Fargues modules. Anschütz [2], building on observations in [5], has studied this category from a different perspective. In particular, he showed it is Tannakian and connected. Anschütz also classified the CM admissible pairs, recovering a description that is essentially equivalent to an earlier result of Serre [33, théorème 6] on Hodge–Tate representations — see Section 5.7 for further discussion. The category of admissible pairs as presented here is also implicit to varying degrees throughout the literature on local Shimura varieties (especially in [32]), but the category of p-adic Hodge structures and the emphasis on the structural analogy between Hodge structures/motives over $\mathbb{C}$ and p-adic Hodge structures/basic admissible pairs is a new contribution in our work.

As far as we are aware, Theorem 1.0.1 recalled above and Serre's [33, théorème 3] — computation of the transcendence degree of the extension generated by Hodge–Tate periods (see Remark 1.1.4) — are the only previous results on p-adic transcendence questions of the nature considered here. In Part II, we prove a bianalytic Ax–Lindemann theorem for basic local Shimura varieties and their nonminuscule generalizations. This implies, in particular, a bianalytic characterization of higher-dimensional special subvarieties analogous to the characterization of special points in Corollary C. It includes the zero-dimensional case, and thus subsumes Theorem B and Corollary C. However, many of the ingredients developed here in Part I will also be used in Part II. This organization separates out the purely algebraic aspects of the argument appearing in Part I from the geometric aspects in Part II, which require a more substantial

[3] Erratum at https://janschuetz.perso.math.cnrs.fr/cm_bkf_modules_erratum.pdf.

technical apparatus (the theory of locally spatial diamonds). We hope this separation will make the proofs and formalism more easily understood and motivated, and that the results will be more accessible to a reader whose interest is born from experience in complex transcendence.

The relative theory introduced in Part II to prove our Ax–Lindemann theorem will only use constant coefficients since the moduli spaces in question include trivializations of the underlying $\mathbb{Q}_p$-vector space or isocrystal. To formulate a more robust theory (e.g., so that the objects form a v-stack), it is better to allow arbitrary local systems (of $\mathbb{Q}_p$-vector spaces for p-adic Hodge structures or isocrystals for admissible pairs). This general definition has the downside of allowing for many objects that cannot possibly arise from geometry — in fact, this can be seen already over a p-adic field and is related to our restriction to algebraically closed residue field in the definition of good reduction here (see Remark 5.4.2). In Part III we complete our results by setting up a general formalism allowing variation of the coefficients and then cutting out a nice subcategory of variations of p-adic Hodge structure and formulating a potential good reduction conjecture for these.

1.4. ***Outline.*** In Section 2 we recall some results and definitions for Tannakian categories, isocrystals, the twistor line, and the Fargues–Fontaine curve. It can be ignored to begin with and returned to as needed; the main novelty is in some aspects of our parallel treatment of the twistor line and the Fargues–Fontaine curve.

In Section 3, we first recall some standard results on the relation between filtered and latticed vector spaces in a formulation that is convenient for our purposes. Afterwards, the majority of the section is dedicated to a proof of Theorem 3.2.6, which implies Theorem D above. The proof relies on a careful study of the behavior of types of lattices under extensions (Theorem 3.5.1) that will also be crucial to establishing good properties of moduli spaces for nonreductive groups in Part II.

In Section 4, we define our category of p-adic Hodge structures and establish its basic properties. We motivate the definition using an approach to real Hodge structures via the twistor line that originates in work of Simpson; we learned of the analogy between the twistor line and the Fargues–Fontaine curve from Laurent Fargues, so that the originality here can be attributed mainly to our naïveté in taking this analogy more literally than might initially seem a good idea! An important point for Part II is our interpretation of the Mumford–Tate group using Hodge–Tate lines in Section 4.5 (see also Section 5.3), analogous to the characterization of Mumford–Tate groups of complex Hodge structures using Hodge tensors.

In Section 5, we define the category of admissible pairs, explaining how it falls out naturally from the category of p-adic Hodge structures. We show that the category of p-adic Hodge structures is equivalent to the subcategory of basic admissible pairs, and then extend many of the results about p-adic Hodge structures to admissible pairs. That this category is connected Tannakian is due to Anschütz [2] (building on remarks in [5]) in the context of rigidified Breuil–Kisin–Fargues modules, and this and many of the other results can also be argued in essentially the same way as for p-adic Hodge structures. In Section 5.4 and Section 5.5 we study $\overline{C}_0$-analytic admissible pairs and their associated Galois representations — the two key points are that $\overline{C}_0$-analytic admissible pairs are automatically good (Theorem 5.4.4), and that the Galois representation associated to an $\overline{C}_0$-analytic admissible pair has open image in the motivic Galois group (Corollary 5.5.3). We describe the period maps of Corollary C in Section 5.6.

In Section 6 we prove Theorems A and B. The proofs are essentially the same, and that is more or less the point — since we are working with a toy category of motives that is further removed from algebraic cycles, the work of the earlier sections establishes all of the nice properties analogous to the standard conjectures, the Hodge conjecture, and the Grothendieck period conjecture that are needed to give an unconditional result. We encourage the reader who is primarily interested in Theorem B to turn now to Section 6 and read the short proof of Theorem A and the surrounding discussion, as this will give a firm footing for the path we take later. We conclude by giving an extension of Theorem B to the nonbasic case, Theorem 6.3.2, and discussing the precise relation of the present work with [19, Conjecture 4.1].

1.5. ***Notation.*** Throughout this paper $p > 0$ is a prime number. A nonarchimedean field is a field that is complete with respect to a nonarchimedean absolute value such that the residue field is of characteristic p. A p-adic field is a discretely valued nonarchimedean field of characteristic zero whose residue field is perfect, and is strict if the residue field is algebraically closed.

2. Preliminaries

2.1. *Tannakian categories.*

2.1.1. *Notation.* All categories considered are k-linear for some field k (which should be clear from the context), and all functors are assumed to be k-linear.

A Tannakian category $\mathcal{C}$ over a field k is a rigid abelian tensor category where the endomorphism ring of the unit object is k, and such that there exists an exact tensor functor $\omega : \mathcal{C} \to \mathrm{Vect}(k')$ (called a fiber functor) for k' an extension of k. Given a fiber functor ω as above, there is an affine group scheme $\mathrm{Aut}^{\otimes}\omega$ over k' whose R points are the tensor automorphisms of $\omega \otimes R$. Further, ω induces an equivalence of categories $\tilde{\omega} : \mathcal{C}_{k'} \xrightarrow{\sim} \mathrm{Rep}\,\mathrm{Aut}^{\otimes}(\omega)$. The Tannakian category $\mathcal{C}$ is neutral if there exists a fiber functor ω over k. A Tannakian subcategory $\mathcal{C}'$ of $\mathcal{C}$ is a strictly full subcategory that is closed under direct sum, tensor products, duals and subquotients. Given a collection of objects M of $\mathcal{C}$, we denote by $\langle M \rangle$ the strictly full Tannakian subcategory generated by M, i.e., the smallest strictly full subcategory closed under direct sum, tensor products, duals and subquotients.

For G an affine group scheme over k, we denote by $\mathrm{Rep}\,G$ the Tannakian category over k of algebraic representations of G on finite-dimensional k-vector spaces. It is neutralized by the standard fiber functor $\omega_{\mathrm{std}} = \omega_{\mathrm{std},G} : \mathrm{Rep}\,G \to \mathrm{Vect}(k)$, $(V, \rho) \mapsto V$. Any fiber functor $\omega : \mathrm{Rep}\,G \to \mathrm{Vect}(k)$ determines an étale G-torsor $\mathrm{Isom}^{\otimes}(\omega_{\mathrm{std}}, \omega)$ over $\mathrm{Spec}\,k$, and hence a cohomology class in $H^1(k, G)$. Two fiber functors ω, ω' are isomorphic if and only if they have the same cohomology class. If $f : G \to H$ is a morphism of affine group schemes over k, then pullback $f^* : \mathrm{Rep}\,H \to \mathrm{Rep}\,G$ is an exact tensor functor that commutes with the standard fiber functors, and Tannakian duality states that conversely any exact tensor functor $F : \mathrm{Rep}\,H \to \mathrm{Rep}\,G$ such that $\omega_{\mathrm{std},G} \circ F = \omega_{\mathrm{std},H}$ is f^* for a uniquely determined morphism $f : G \to H$.

For $\mathcal{C}$ a Tannakian category over k with fiber functor ω over k', we write[4] G–$\mathcal{C}$ for the category (groupoid) of objects in $\mathcal{C}$ with G-structure, i.e., the objects being exact tensor functors $Q : \mathrm{Rep}\,G \to \mathcal{C}$

[4] It would be more principled (but cumbersome) to include the choice of fiber functor ω in the notation; but the choice will be clear from the context. When considering vector bundles we often implicitly take it to be evaluation at a geometric point.

such that $\omega \circ Q$ is isomorphic to $\omega_{\mathrm{std}} \otimes k'$, and morphisms are natural transformations of tensor functors. If $f : G \to H$ is a morphisms of affine group schemes over k, then composition along $\operatorname{Rep} H \xrightarrow{f^*} \operatorname{Rep} G$ gives a functor $f_* : G\text{–}\mathcal{C} \to H\text{–}\mathcal{C}$.

Lemma 2.1.1. *Let G be a connected linear algebraic group over a characteristic 0 field k, let U be the unipotent radical of G, and let $G = MU$ be a Levi decomposition with inclusion map $s : M \hookrightarrow G$ and projection $\pi : G \to M$. Then for any semisimple Tannakian category $\mathcal{C}$ over k with fiber functor over k', and any object F of $G\text{–}\mathcal{C}$, $F \cong (s \circ \pi)_* F$. In particular, s_* is essentially surjective.*

Proof. Let $\mathcal{T}$ be the functor on k-algebras sending R to the set of isomorphisms $F \otimes R \xrightarrow{\sim} (s \circ \pi)_* F \otimes R$ that induce the canonical identification $\pi_* F \otimes R = \pi_*(s \circ \pi)_* F \otimes R$. By definition, this is a quasitorsor for the functor $\mathcal{U}$ sending R to the automorphisms of $F \otimes R$ that induce the identity on $\pi_* F$. We claim $\mathcal{T}$ is an étale torsor, and that $\mathcal{U}$ is a unipotent group scheme over k. Given this claim, we obtain the result since $H^1_{\text{ét}}(\operatorname{Spec} k, \mathcal{U})$ is trivial (because we have assumed k has characteristic zero, $\mathcal{U}$ has a filtration by normal subgroups with quotients $\mathbb{G}_a$ so that this is reduced to the usual vanishing of $H^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_a)$).

To establish the claim, we first note that, by replacing $\mathcal{C}$ with $\langle F(\operatorname{Rep} G)\rangle$, we may in particular assume $\mathcal{C}$ is finitely generated, and thus that k' is a finite extension of k. We write ω for the given fiber functor over k' on $\mathcal{C}$ and fix an identification of $\omega \circ F$ with the fiber functor $\omega_{\mathrm{std}} \otimes k'$ on $\operatorname{Rep} G$. Then, writing H for the algebraic group over k' of automorphisms of $\omega \otimes k'$, Tannakian duality gives that $\mathcal{C}_{k'} = \operatorname{Rep} H$ and that $F \otimes k'$ is induced by a homomorphism $f : H \to G_{k'}$, which is a closed immersion by [14, Proposition 2.21]. It follows that the automorphisms of $F \otimes k'$ are identified with the centralizer of f in $G_{k'}$, and thus $\mathcal{U}_{k'}$ is identified with the centralizer of f in $U_{k'}$, which is unipotent as a closed subgroup of a unipotent group. From the description of $\mathcal{C}$ by descent from k' as in, e.g., [14, p. 35], we find $\mathcal{U}$ can be defined via descent data from $\mathcal{U}_{k'}$; thus it is an algebraic group over k which is unipotent because $\mathcal{U}_{k'}$ is. Moreover, there is a point of $\mathcal{T}$ over k'; thus $\mathcal{T}$ is an étale torsor: since f is a closed immersion, $f^{-1}(U_{k'})$ is contained in the unipotent radical of H, which (since $\mathcal{C}$ is semisimple and thus H is reductive) is trivial. It follows from the main theorem of [25] that $f(H)$ is contained in a Levi subgroup M' of $G_{k'}$ which is conjugate to M by an element $u \in U(k')$. In particular, $\operatorname{Ad}(u) \circ f = s \circ \pi \circ f$, and the action of u gives the isomorphism $F \otimes k' \cong (s \circ \pi)_* F \otimes k'$. □

2.1.2. *Canonical G-structure.* Throughout the paper, we find ourselves frequently talking about the canonical $\operatorname{Aut}^{\otimes}(\omega)$-structure attached to a neutralized Tannakian category $(\mathcal{C}, \omega)$ over a field k, which we now clarify. Setting $G = \operatorname{Aut}^{\otimes}(\omega)$, we have the following commutative (on the nose) diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\tilde{\omega}\quad} & \operatorname{Rep} G \\ & {\scriptstyle \omega}\searrow \quad \swarrow{\scriptstyle \omega_{\mathrm{std}}} & \\ & \operatorname{Vect}(k) & \end{array}$$

where $\tilde{\omega}$ is an exact tensor functor and an equivalence of categories. By [29, Proposition I.4.4.2], we can choose a quasi-inverse $Q : \operatorname{Rep} G \to \mathcal{C}$ that is a tensor functor and such that $\tilde{\omega} \circ Q$ (resp. $Q \circ \tilde{\omega}$) is isomorphic to $\mathrm{Id}_{\mathrm{Rep}\,G}$ (resp. $\mathrm{Id}_{\mathcal{C}}$) as a tensor functor. Thus such a Q is an exact tensor functor (it is exact as an equivalence of abelian categories). If we fix such a Q and isomorphism $\tilde{\omega} \circ Q \cong \mathrm{Id}_{\mathrm{Rep}\,G}$, we obtain an isomorphism of tensor functors

$$\omega \circ Q = (\omega_{\mathrm{std}} \circ \tilde{\omega}) \circ Q = \omega_{\mathrm{std}} \circ (\tilde{\omega} \circ Q) \cong \omega_{\mathrm{std}} \circ \mathrm{Id}_{\mathrm{Rep}\,G} = \omega_{\mathrm{std}}.$$

The choice of a pair consisting of such a Q and an isomorphism $\tilde{\omega} \circ Q = \mathrm{Id}_{\mathrm{Rep}\,G}$ is unique up to unique isomorphism. By abuse of notation, we will sometimes refer to Q as the canonical G-structure and sometimes refer to Q with this identification as the canonical G-structure.

2.1.3. *Automorphisms of Tannakian categories.* Suppose G is an affine algebraic group over a perfect field k and let $Z(G)$ denote the center of G (an affine algebraic group over k). If $z \in Z(G)(k)$ then for each $V \in \mathrm{Rep}\,G$, $\rho(z) \in \mathrm{GL}(V)$ is an isomorphism of representations. Hence z determines a tensor automorphism of the identity functor on $\mathrm{Rep}\,G$, denoted by $z\,\cdot$. It will be useful to record the basic observation that $Z(G)$ accounts for all automorphisms of Tannakian categories.

Lemma 2.1.2. *Suppose $F : \mathrm{Rep}\,G \to \mathcal{C}$ is a fully faithful exact tensor functor with $\mathcal{C}$ a Tannakian category over k. Then for each tensor automorphism $\alpha : F \xrightarrow{\sim} F$, there is a unique $\beta \in Z(G)(k)$ such that $\alpha_V = F(\beta\cdot)$ for all $V \in \mathrm{Rep}\,G$.*

Proof. As F is fully faithful, for each $V \in \mathrm{Rep}\,G$, there is a unique $\beta_V \in \mathrm{Hom}_G(V, V)$ such that $F(\beta_V) = \alpha_V$ and $\beta_W \circ f = f \circ \beta_V$ whenever $f \in \mathrm{Hom}_G(V, W)$. Thus $\beta = \{\beta_V\}$ defines an element of $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{std}})(k) = G(k)$. Moreover, $\beta g = g\beta$ for all $g \in G(k)$ since $\beta_V \in \mathrm{Hom}_G(V, V)$ and hence $\beta \in Z(G)(k)$. □

2.2. *Isocrystals.* We introduce Kottwitz' categories of real and p-adic isocrystals. References are [22] for p-adic isocrystals and [31, Construction 9.3] and [24] for real isocrystals.

2.2.1. *Real isocrystals.* Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ denote the Hamiltonian quaternions and let $\mathbb{C} = \mathbb{R} + \mathbb{R}i \subseteq \mathbb{H}$. Let $W_{\mathbb{R}} = \mathbb{C}^{\times} \sqcup j\mathbb{C}^{\times}$, the Weil group of $\mathbb{R}$.

A real isocrystal is a semilinear representation of $W_{\mathbb{R}}$ on a finite-dimensional complex vector space whose restriction to $\mathbb{C}^{\times}$ is algebraic. Here the semilinearity is for the natural map $W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ sending j to the nontrivial element (this can be realized as the conjugation action on $\mathbb{C} \subseteq \mathbb{H}$).

We write $\mathrm{Kt}_{\mathbb{R}}$ for the Kottwitz category of real isocrystals.[5] The category $\mathrm{Kt}_{\mathbb{R}}$ is a semisimple Tannakian category over $\mathbb{R}$ (though it is not neutral). We can describe the simple objects explicitly: for each $\lambda \in \frac{1}{2}\mathbb{Z}$, $\lambda = a/b$ with $b > 0$ and $\gcd(a, b) = 1$, let

$$D_\lambda := \mathbb{C}[\pi]/(\pi^b - (-1)^a),$$

with $\mathbb{C}^{\times}$ acting by z^a and j acting as multiplication by π composed with complex conjugation on the coefficients. Then each simple object in $\mathrm{Kt}_{\mathbb{R}}$ is isomorphic to D_λ for a unique λ in $\frac{1}{2}\mathbb{Z}$.

[5] In [31, Construction 9.3], $\mathrm{Kt}_{\mathbb{R}}$ is described as the category of graded quaternionic vector spaces; this is easily seen to be equivalent to our definition by using the action of j to give the semilinear automorphism of loc. cit.

2.2.2. *p-adic isocrystals.* Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ and let $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p]$. We will frequently be given a complete algebraically closed nonarchimedean extension $C/\mathbb{Q}_p$, in which case we take $\overline{\mathbb{F}}_p$ to be the algebraic closure of $\mathbb{F}_p$ in $\mathcal{O}_C/\mathfrak{m}_C$ and identify $\breve{\mathbb{Q}}_p$ with the completion of the maximal unramified algebraic extension of $\mathbb{Q}_p$ in C.

A p-adic isocrystal is a finite-dimensional $\breve{\mathbb{Q}}_p$-vector space equipped with a semilinear automorphism $\varphi_W : W \xrightarrow{\sim} W$, i.e., an additive isomorphism such that $\varphi_W(\lambda w) = \varphi_{\breve{\mathbb{Q}}_p}(\lambda)\varphi_W(w)$ for $\lambda \in \breve{\mathbb{Q}}_p$, $w \in W$, where $\varphi_{\breve{\mathbb{Q}}_p}$ is the automorphism of $\breve{\mathbb{Q}}_p$ induced by the p-power Frobenius on $\overline{\mathbb{F}}_p$. We write $\mathrm{Kt}_{\mathbb{Q}_p}$ for the category of p-adic isocrystals. By the Dieudonné–Manin classification, the category $\mathrm{Kt}_{\mathbb{Q}_p}$ is a semisimple Tannakian category over $\mathbb{Q}_p$ (though it is not neutral) and we can describe the simple objects explicitly: for $\lambda \in \mathbb{Q}$, $\lambda = a/b$ with $b > 0$ and $\gcd(a, b) = 1$, let

$$D_\lambda := \breve{\mathbb{Q}}_p[\pi]/(\pi^b - p^a),$$

with Frobenius φ_{D_λ} corresponding to the semilinear map $\pi\varphi_{\breve{\mathbb{Q}}_p}$ (i.e., $\varphi_{\breve{\mathbb{Q}}_p}$ on the coefficients of a polynomial in π followed by multiplication by π). Each simple object in $\mathrm{Kt}_{\mathbb{Q}_p}$ is isomorphic to one of the form D_λ for $\lambda \in \mathbb{Q}$.

2.2.3. *Isocrystals with G-structure.* We recall results of [22] in this subsection. For convenience we write $\sigma = \varphi_{\breve{\mathbb{Q}}_p} : \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$ for the lift of the p-power Frobenius. Let G be a connected linear algebraic group over $\mathbb{Q}_p$. If $b : \mathrm{Rep}\, G \to \mathrm{Kt}_{\mathbb{Q}_p}$ is an exact tensor functor, then composition with $\omega_{\mathrm{Kt}} : \mathrm{Kt}_{\mathbb{Q}_p} \to \mathrm{Vect}(\breve{\mathbb{Q}}_p)$ gives a fiber functor valued in $\breve{\mathbb{Q}}_p$-vector spaces. The $G_{\breve{\mathbb{Q}}_p}$-torsor $\mathrm{Isom}^{\otimes}(\omega_{\mathrm{Kt}} \circ b, \omega_{\mathrm{std}} \otimes \breve{\mathbb{Q}}_p)$ yields an element of $H^1(\breve{\mathbb{Q}}_p, G)$ which, as G is connected and linear and $\breve{\mathbb{Q}}_p$ has cohomological dimension 1, is trivial by Steinberg's theorem. Thus we can (and do) fix an identification $\omega_{\mathrm{Kt}} \circ b = \omega_{\mathrm{std}} \otimes \breve{\mathbb{Q}}_p$. For each $V \in \mathrm{Rep}\, G$, the map $\varphi_{b(V)}\sigma^{-1} : V_{\breve{\mathbb{Q}}_p} \to V_{\breve{\mathbb{Q}}_p}$ is a $\breve{\mathbb{Q}}_p$-linear automorphism such that $f \circ \varphi_{b(V)}\sigma^{-1} = \varphi_{b(W)}\sigma^{-1} \circ f$ for any map $f : V \to W$ in $\mathrm{Rep}\, G$. In short, $\varphi_b\sigma^{-1} \in \mathrm{Aut}^{\otimes}(\omega_{\mathrm{std}})(\breve{\mathbb{Q}}_p) = G(\breve{\mathbb{Q}}_p)$. By a slight abuse of notation, we will let b denote both the element $\varphi_b\sigma^{-1} \in G(\breve{\mathbb{Q}}_p)$ (so $\varphi_b = b\sigma$) and the tensor functor; any $b \in G(\breve{\mathbb{Q}}_p)$ determines the corresponding tensor functor

$$b : \mathrm{Rep}\, G \to \mathrm{Kt}_{\mathbb{Q}_p}, \qquad (V, \rho) \mapsto V_b := (V_{\breve{\mathbb{Q}}_p}, \rho(b)\sigma).$$

For $b, b' \in G(\breve{\mathbb{Q}}_p)$, any isomorphism $b \cong b'$ of G-isocrystals is given by an element $g \in G(\breve{\mathbb{Q}}_p)$ such that $b' = gb\sigma(g)^{-1}$, in which case we say that b, b' are σ-conjugate. Thus, the groupoid G–$\mathrm{Kt}_{\mathbb{Q}_p}$ is equivalent to the groupoid with objects $b \in G(\breve{\mathbb{Q}}_p)$, and $\mathrm{Hom}(b, b') = \{g \in G(\breve{\mathbb{Q}}_p) : b' = gb\sigma(g)^{-1}\}$. If we write $B(G)$ for the quotient of $G(\breve{\mathbb{Q}}_p)$ by the relation of σ-conjugacy then $B(G)$ is the set of isomorphism classes of G-isocrystals. As $\mathrm{Kt}_{\mathbb{Q}_p}$ is semisimple, Lemma 2.1.1 shows that $B(G) = B(G/U)$ for $U \subseteq G$ the unipotent radical (more precisely, Lemma 2.1.1 gives injectivity of $B(G) \to B(G/U)$ while surjectivity is immediate since $G(\breve{\mathbb{Q}}_p)$ surjects onto $(G/U)(\breve{\mathbb{Q}}_p)$). Kottwitz [23] gives a complete description of $B(G)$ for G reductive.

For $b \in G(\breve{\mathbb{Q}}_p)$, the $\breve{\mathbb{Q}}_p$-linear extension of the G-isocrystal b is an exact tensor functor from $\mathrm{Rep}\, G$ to $\mathbb{Q}$-graded $\breve{\mathbb{Q}}_p$-vector spaces. Semisimplicity and the description of the simple objects D_λ implies it is induced (via Tannakian duality) by a morphism $\nu_b : \mathbb{D}_{\breve{\mathbb{Q}}_p} \to G_{\breve{\mathbb{Q}}_p}$ where $\mathbb{D} = \varprojlim_{[n]} \mathbb{G}_m$ is the pro-torus with character group $\mathbb{Q}$. In our notation, $\mathbb{D}_{\breve{\mathbb{Q}}_p}$ acts by z^λ on the D_λ isotypic component. The $G(\breve{\mathbb{Q}}_p)$-conjugacy class of ν_b is fixed by σ and an invariant of the σ-conjugacy class of b. This leads to the Newton map

$$\overline{\nu} : B(G) \to \mathcal{N}(G) := \left(\mathrm{Hom}(\mathbb{D}, G)/G^{\mathrm{ad}}\right)(\mathbb{Q}_p).$$

The latter notation means the $\mathbb{Q}_p$-points of the scheme of conjugacy classes of morphisms $\mathbb{D} \to G$, which can in turn be identified with $(X_*(T)^+ \otimes \mathbb{Q})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ once a Borel pair $T \subseteq B \subseteq G_{\overline{\mathbb{Q}}_p}$ is chosen.

We call a G-isocrystal b *basic* if the slope homomorphism ν_b is central, in which case ν_b is independent of the choice of b in $[b]$ and is a well-defined element of $\mathrm{Hom}(\mathbb{D}, G)$. We write $B(G)_{\mathrm{basic}}$ for the subset of basic isomorphism classes. Note that for $M = G/U$, the identification $B(G) = B(M)$ induces an inclusion $B(G)_{\mathrm{basic}} \subseteq B(M)_{\mathrm{basic}}$, but if the center of M acts nontrivially on U then the inclusion may be strict.

Example 2.2.1. If G is the group of upper triangular matrices in GL_2 then, taking $M \leq G$ to be the subgroup of diagonal matrices, $b = \mathrm{diag}(p, 1)$ is in $B(M)_{\mathrm{basic}}$ but is not in $B(G)_{\mathrm{basic}}$.

For b: $\mathrm{Rep}\, G \to \mathrm{Kt}_{\mathbb{Q}_p}$ a G-isocrystal, we write G_b for the functor on $\mathbb{Q}_p$-algebras

$$R \mapsto \mathrm{Aut}^{\otimes}(b \otimes_{\mathbb{Q}_p} R),$$

where the latter term is the group of tensor automorphisms of the R-linear extension of b. Identifying b with an element of $G(\breve{\mathbb{Q}}_p)$ as above identifies G_b with the functor of [28, Proposition 1.12] and hence G_b is represented by a connected linear algebraic group over $\mathbb{Q}_p$. There is a canonical closed immersion $G_{b,\breve{\mathbb{Q}}_p} \hookrightarrow G_{\breve{\mathbb{Q}}_p}$ that identifies $G_b(\mathbb{Q}_p)$ with the fixed points of $g \mapsto b\sigma(g)b^{-1}$ on $G(\breve{\mathbb{Q}}_p)$.

When b is basic, the canonical closed immersion $G_{b,\breve{\mathbb{Q}}_p} \to G_{\breve{\mathbb{Q}}_p}$ is an isomorphism: one fun way to see this is to note that in this case the adjoint action on $\mathrm{Lie}\, G_{\breve{\mathbb{Q}}_p}$ gives the trivial isocrystal, so it admits a $\mathbb{Q}_p$-basis such that elements in a small enough $\mathbb{Z}_p$-lattice exponentiate to elements of $G_b(\mathbb{Q}_p)$; thus we see that $\dim \mathrm{Lie}\, G_b \geq \dim \mathrm{Lie}\, G$ (equality of groups follows since G is, by assumption, a connected linear algebraic group and $G_{b,\breve{\mathbb{Q}}_p}$ is a closed subgroup with the same Lie algebra as $G_{\breve{\mathbb{Q}}_p}$). In a slightly more useful way, following [24, §5.2], we may σ-conjugate to assume b satisfies $(b\sigma)^n = z\sigma^n$ for some positive integer n and central element z. Then the image b_{ad} of b in the adjoint group $G^{\mathrm{ad}}(\breve{\mathbb{Q}}_p)$ is fixed by σ^n so $b_{\mathrm{ad}} \in G(\mathbb{Q}_{p^n})$ and G_b is the associated inner form obtained by this conjugation.

2.3. ***The twistor line.*** As in Section 2.2.1, let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ denote the Hamiltonian quaternions, let $\mathbb{C} = \mathbb{R} + \mathbb{R}i \subseteq \mathbb{H}$, and let $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$, the Weil group of $\mathbb{R}$.

We view $\mathbb{H} = \mathbb{C} + j\mathbb{C}$ as a complex vector space via right multiplication. We let $W_{\mathbb{R}}$ act on $\mathbb{H}$ by right multiplication and on $\mathbb{C}$ by the natural map $W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then, the natural action of $W_{\mathbb{R}}$ on $\mathrm{Cont}(\mathbb{H}, \mathbb{C})$, $(w \cdot f)(x) = wf(xw)$ preserves the space $\mathbb{C}[X, Y]$ of polynomial functions on $\mathbb{H}$ (where $X(a + jb) = a$ and $Y(a + jb) = b$). Concretely, we have $(z \cdot f)(X, Y) = f(Xz, Yz)$ for $z \in \mathbb{C}^\times$ and

$$(j \cdot f)(X, Y) = \overline{f(-\overline{Y}, \overline{X})} = \bar{f}(-Y, X),$$

where $\bar{f}$ denotes complex conjugation on the coefficients of the polynomial f.

Let $\mathbb{C}(k)$ denote the semilinear representation of $W_{\mathbb{R}}$ on $\mathbb{C}$ where j acts by complex conjugation and z acts by z^{2k} (this is the real isocrystal D_{2k} in the notation of Section 2.2.1). We define a scheme over $\mathbb{R}$:

$$\tilde{\mathbb{P}}^1 := \mathrm{Proj}\Big(\bigoplus_{k\in\mathbb{Z}} (\mathbb{C}[X, Y] \otimes_{\mathbb{C}} \mathbb{C}(-k))^{W_{\mathbb{R}}}\Big) = \mathrm{Proj}\Big(\bigoplus_{k\in\mathbb{Z}} \mathbb{C}[X, Y]_{2k}^{j=1}\Big).$$

The natural bundle $\mathcal{O}_{\tilde{\mathbb{P}}}(1)$ has an $\mathbb{R}$-basis of global sections, namely

$$S = iXY, \quad T = \tfrac{1}{2}(iX^2 - iY^2), \quad U = \tfrac{1}{2}(X^2 + Y^2),$$

and these define a closed embedding $\tilde{\mathbb{P}}^1 \hookrightarrow \mathbb{P}^2_{\mathbb{R}}$ identifying $\tilde{\mathbb{P}}^1$ with the vanishing set $S^2 + T^2 + U^2 = 0$. In particular, $\tilde{\mathbb{P}}^1$ is the Brauer–Severi variety for $\mathbb{H}$. Alternatively, we can see this using the inclusion of the graded ring into $\mathbb{C}[X, Y]$ to obtain a map $\mathbb{P}^1_{\mathbb{C}} \to \tilde{\mathbb{P}}^1$ which factors through an isomorphism $\mathbb{P}^1_{\mathbb{C}} = \tilde{\mathbb{P}}^1 \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$.

We write $\infty_{\mathbb{C}}$ for the point $\mathrm{Spec}\,\mathbb{C} \xrightarrow{[1:0]} \mathbb{P}^1_{\mathbb{C}} \to \tilde{\mathbb{P}}^1$. The action of

$$U(1) = \{a + bi \,|\, a^2 + b^2 = 1\} \subseteq \mathbb{C} \subseteq \mathbb{H}$$

on $\tilde{\mathbb{P}}^1$ induced by left multiplication on $\mathbb{H}$ is identified with the action on $\mathbb{P}^1_{\mathbb{C}}$ by $z \cdot [X : Y] = [zX : z^{-1}Y]$. This action fixes $\infty_{\mathbb{C}}$ and its conjugate $0_{\mathbb{C}}$. The points $\infty_{\mathbb{C}}$ and $0_{\mathbb{C}}$ underlie the same closed point in $\tilde{\mathbb{P}}^1$, cut out by $S = 0$, and this is the unique closed point fixed by $U(1)$.

Example 2.3.1. The map $\mathbb{P}^1_{\mathbb{C}} \to \tilde{\mathbb{P}}^1$ is finite étale thus induces an isomorphism between the formal neighborhood of $[1 : 0]$ in $\mathbb{P}^1_{\mathbb{C}}$ and the formal neighborhood of the closed point underlying $\infty_{\mathbb{C}}$ in $\tilde{\mathbb{P}}^1$. In particular, we may choose as uniformizing parameter $t = Y/X$ on which $U(1)$ acts by z^{-2} and identify the formal completion with $\mathbb{C}[\![t]\!]$. Note that the action of $U(1)$ by left multiplication on $\tilde{\mathbb{P}}^1$ extends to all of $\mathbb{H}^\times$, and thus in particular to $W_{\mathbb{R}}$; the action of j^2 in $W_{\mathbb{R}}$ is trivial, and j also fixes this closed point but acts by complex conjugation on the residue field. Note also that the subgroup $W_{\mathbb{R}}$ is the full stabilizer in $\mathbb{H}^\times$ of this closed point.

Remark 2.3.2. If we consider the full action by $\mathbb{H}^\times$ instead of just $U(1)$, then we obtain an identification of the closed points of $\tilde{\mathbb{P}}^1$ with $\mathbb{H}^\times/W_{\mathbb{R}}$ by taking the orbit of the closed point underlying $\infty_{\mathbb{C}}$.

We could also arrive at this construction by observing that the conjugation action of $\mathbb{H}$ on itself preserves $\mathbb{R}$ and the norm form; thus it preserves the orthogonal complement of $\mathbb{R}$, $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. It then preserves the norm-zero curve in $\mathbb{P}^2$, which is $\mathbb{P}(\mathbb{R}i + \mathbb{R}j + \mathbb{R}k)$. If we write the coefficients as $Si + Tj + Uk$ then this curve is $S^2 + T^2 + U^2 = 0$. The action of $\mathbb{H}^\times$ is transitive and the stabilizer of the closed point $S = 0$ is $W_{\mathbb{R}}$ so we again obtain this identification.

In fact, this can be upgraded to an identification of analytic spaces

$$\tilde{\mathbb{P}}^{1,\mathrm{an}} = \mathbb{H}^\times/W_{\mathbb{R}},$$

where $\mathbb{H}^\times \subseteq \mathbb{H} = \mathbb{C} + j\mathbb{C}$ gives the complex structure, $\tilde{\mathbb{P}}^{1,\mathrm{an}}$ is formed as the quotient of the locally ringed space $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ by the semilinear Galois action as in [20], and the action of $W_{\mathbb{R}}$ on the sheaf of functions is as described above for polynomials.

There is a classification theorem for vector bundles on $\tilde{\mathbb{P}}^1$ akin to Grothendieck's classification theorem for vector bundles on $\mathbb{P}^1_{\mathbb{C}}$. For comparison with the p-adic case, it is useful to realize this via a functor from the category of real isocrystals $\mathrm{Kt}_{\mathbb{R}}$ (see Section 2.2.1) to vector bundles on $\tilde{\mathbb{P}}^1$. Attached to any $W \in \mathrm{Kt}_{\mathbb{R}}$, we obtain a vector bundle $\mathcal{E}(W)$ on $\tilde{\mathbb{P}}^1$ as the sheaf associated to the graded module

$$\bigoplus_{k\in\mathbb{Z}} (\mathbb{C}[X, Y] \otimes_{\mathbb{C}} W(-k))^{W_{\mathbb{R}}}$$

where $W(-k) := W \otimes_{\mathbb{C}} \mathbb{C}(-k)$. We write $\mathcal{O}_{\tilde{\mathbb{P}}^1}(\lambda) := \mathcal{E}(D_{-\lambda})$.

Remark 2.3.3. The line bundles $\mathcal{O}_{\tilde{\mathbb{P}}^1}(k)$ arising from the Proj construction are canonically identified with the bundles $\mathcal{E}(D_{-k})$ for $k \in \mathbb{Z}$, so that the terminology will not cause confusion. By construction they are also identified with the restriction of $\mathcal{O}_{\mathbb{P}^2}(k)$ to $\tilde{\mathbb{P}}^1$ realized as the vanishing locus $S^2 + T^2 + U^2 = 0$. For $\lambda \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, $\mathcal{O}_{\tilde{\mathbb{P}}^1}(\lambda)$ can be identified with the pushforward of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda)$ to $\tilde{\mathbb{P}}^1$, whereas if $\lambda \in \mathbb{Z}$, the pushforward of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda)$ is identified with $\mathcal{O}_{\tilde{\mathbb{P}}^1}(\lambda)^{\oplus 2}$.

Theorem 2.3.4. *Every vector bundle on $\tilde{\mathbb{P}}^1$ is a direct sum of stable bundles. Any stable bundle has slope $\lambda \in \frac{1}{2}\mathbb{Z}$, and for any $\lambda \in \frac{1}{2}\mathbb{Z}$, $\mathcal{O}_{\tilde{\mathbb{P}}^1}(\lambda)$ is the unique up to isomorphism stable bundle of slope λ. The assignment $W \mapsto \mathcal{E}(W)$ is a faithful and essentially surjective exact tensor functor, and it restricts to an equivalence between the category of D_λ-isotypic real isocrystals and the category of semistable vector bundles on $\tilde{\mathbb{P}}^1$ of slope $-\lambda$.*

Remark 2.3.5. The functor $\mathcal{E}$ does not give an equivalence between $\mathrm{Kt}_{\mathbb{R}}$ and vector bundles on $\tilde{\mathbb{P}}^1$ because for vector bundles there are maps going up in slope (e.g., the maps from $\mathcal{O}_{\tilde{\mathbb{P}}^1}$ to $\mathcal{O}_{\tilde{\mathbb{P}}^1}(1)$ given by the global sections S, T, and U).

Remark 2.3.6. By the GAGA theorem of [20], vector bundles on $\tilde{\mathbb{P}}^1$ are equivalent to those on the real analytic variety $\tilde{\mathbb{P}}^{1,\mathrm{an}}$. If we use the uniformization of Remark 2.3.2, then the functor from $\mathrm{Kt}_{\mathbb{R}}$ can be realized by sending W to $\mathcal{O}_{\mathbb{H}^\times} \otimes_{\mathbb{C}} W$ and then using the diagonal action of $W_{\mathbb{R}}$ to descend it to $\tilde{\mathbb{P}}^{1,\mathrm{an}}$.

2.4. ***Period rings and the Fargues–Fontaine curve.*** Let C be an algebraically closed nonarchimedean extension of $\mathbb{Q}_p$. We now recall the construction of some of Fontaine's period rings and the Fargues–Fontaine curve attached to C. All material recalled here can be found in [18]. After defining some rings, our presentation of the material will mirror our presentation of the twistor line in Section 2.3.

First, recall that reduction mod p induces a bijection

$$\mathcal{O}_{C^\flat} := \mathcal{O}_C^\flat = \lim\bigl(\mathcal{O}_C \xleftarrow{x^p} \mathcal{O}_C \xleftarrow{x^p} \cdots\bigr) = \lim\bigl(\mathcal{O}_C/p \xleftarrow{x^p} \mathcal{O}_C/p \xleftarrow{x^p} \cdots\bigr).$$

The rightmost description equips $\mathcal{O}_{C^\flat}$ with a ring structure. We write $x \mapsto x^\sharp$ for the multiplicative map to $\mathcal{O}_C$ given by projection to the first component in the middle description. For $|\cdot|_C$ the absolute value on C, the assignment $|x| := |x^\sharp|_C$ is an absolute value on $\mathcal{O}_{C^\flat}$. Let $\varpi \in \mathcal{O}_{C^\flat}$ with $\varpi^\sharp = p$, i.e., $\varpi = (p, p^{1/p}, \ldots)$. Then $\mathrm{Frac}\,\mathcal{O}_{C^\flat} = \mathcal{O}_{C^\flat}[1/\varpi]$ and there is a natural identification

$$C^\flat := \lim\bigl(C \xleftarrow{x^p} C \xleftarrow{x^p} \cdots\bigr) = \mathcal{O}_{C^\flat}[1/\varpi].$$

The obvious extension of the absolute value makes $C^\flat$ into a nonarchimedean field in characteristic p with valuation ring $\mathcal{O}_{C^\flat}$. The surjection $\mathcal{O}_{C^\flat} \to \mathcal{O}_C/p$ induces, by the universal property of the p-typical Witt vectors functor W, a surjection

$$\theta : W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C.$$

We have $\theta([x]) = x^\sharp$, where $[x]$ denotes the multiplicative lift. The kernel of θ is principal (generated, e.g., by $[\varpi] - p$). Let A_{crys} be the p-complete divided power envelope of $\ker\theta$ and let $B^+_{\mathrm{crys}} = A_{\mathrm{crys}}[1/p]$. If we fix a compatible system (ζ_{p^n}) of p-power roots of unity in C, we can define $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_C^\flat$ and then the formal power series defining $\log([\epsilon]) = \log(1 + ([\epsilon] - 1))$ converges in A_{crys} to an element[6] t. We write B^+_{dR} for the completion of B^+_{crys} along $\ker\theta$; t is a generator for the kernel of the natural extension $\theta : B^+_{\mathrm{dR}} \to C$. We write $B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[1/t]$, $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[1/t]$. The Frobenius on $W(\mathcal{O}_{C^\flat})$ extends to an endomorphism φ on A_{crys}, B^+_{crys}, and B_{crys}. If $C = \overline{K}^\wedge$ for K a p-adic field, then $\mathrm{Gal}(\overline{K}/K)$ acts on everything in sight.

The inclusion $\overline{\mathbb{F}}_p \subseteq \mathcal{O}_C/\mathfrak{m}_C$ lifts canonically to an inclusion of $\overline{\mathbb{F}}_p$ into $\mathcal{O}_C/p$ and thus into $\mathcal{O}_{C^\flat}$ since $\overline{\mathbb{F}}_p$ is perfect. Thus B^+_{crys} is canonically a $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p]$-algebra compatibly with the Frobenius lifts. We write $\breve{\mathbb{Q}}_p(k)$ for $\breve{\mathbb{Q}}_p$ equipped with the semilinear automorphism $\varphi_{\breve{\mathbb{Q}}_p(k)} = p^{-k}\varphi_{\breve{\mathbb{Q}}_p}$ (the isocrystal D_{-k} in the notation of Section 2.2.2).

Definition 2.4.1. The (algebraic) Fargues–Fontaine curve of $C^\flat$ is defined by

$$\mathbb{F} = \mathbb{F}_{C^\flat} = \mathrm{Proj}\left(\bigoplus_{k\in\mathbb{Z}} \left(B^+_{\mathrm{crys}} \otimes_{\breve{\mathbb{Q}}_p} \breve{\mathbb{Q}}_p(k)\right)^{\varphi=1}\right) = \mathrm{Proj}\left(\bigoplus_{k\in\mathbb{Z}} (B^+_{\mathrm{crys}})^{\varphi=p^k}\right).$$

The point $\infty_C \in \mathbb{F}_{C^\flat}$ is induced by the surjection $\theta : B^+_{\mathrm{crys}} \to C$, and this induces an identification of B^+_{dR} with the completed local ring at ∞_C.

Remark 2.4.2. The analytic Fargues–Fontaine curve $\mathbb{F}^{\mathrm{an}} = \mathbb{F}^{\mathrm{an}}_{C^\flat}$ of $C^\flat$ is the adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ formed by taking the quotient of

$$Y := (\mathrm{Spa}(W(\mathcal{O}_{C^\flat}), W(\mathcal{O}_{C^\flat}))\backslash V([\varpi]p)$$

by the (properly discontinuous) action of the Frobenius φ.

There is a natural functor $D \mapsto \mathcal{E}(D)$ from $\mathrm{Kt}_{\mathbb{Q}_p}$ to vector bundles on $\mathbb{F}$ sending D to the sheaf attached to the graded module

$$\bigoplus_{k\in\mathbb{Z}} (B^+_{\mathrm{crys}} \otimes_{\breve{\mathbb{Q}}_p} D(k))^{\varphi=1} = \bigoplus_{k\in\mathbb{Z}} (B^+_{\mathrm{crys}} \otimes_{\breve{\mathbb{Q}}_p} D)^{\varphi=p^k}.$$

(The φ in the superscript is the diagonal action). For $\lambda \in \mathbb{Q}$ we write $\mathcal{O}_{\mathbb{F}}(\lambda) := \mathcal{E}(D_{-\lambda})$, where $D_{-\lambda}$ is as in Section 2.2.2.

[6]Commonly referred to as the p-adic $2\pi i$ because it is the period describing the comparison between de Rham and étale cohomology for $\mathbb{G}_m$.

Theorem 2.4.3. *Every vector bundle on $\mathbb{F}$ is a direct sum of stable bundles, and for any $\lambda \in \mathbb{Q}$, $\mathcal{O}_{\mathbb{F}}(\lambda)$ is the unique up to isomorphism stable bundle of slope λ. The assignment $W \mapsto \mathcal{E}(W)$ is a faithful and essentially surjective exact tensor functor, and it restricts to an equivalence between the category of $D_{-\lambda}$-isotypic isocrystals and the category of semistable vector bundles on $\mathbb{F}$ of slope λ.*

Example 2.4.4. $\mathcal{O}_{\mathbb{F}}(0) = \mathcal{E}(D_0) = \mathcal{O}_{\mathbb{F}}$. More generally, there is an obvious functor from $\mathbb{Q}_p$-vector spaces to isocrystals inducing an equivalence with the category of 0-isotypic isocrystals, and an inverse functor is given by taking φ_W-invariants. Composed with $\mathcal{E}$, we obtain an equivalence between $\mathbb{Q}_p$-vector spaces and vector bundles on $\mathbb{F}$ that are semistable of slope zero. This functor is naturally isomorphic to $V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{F}}$, and the inverse functor is given by global sections.

Remark 2.4.5. The functor $\mathcal{E}$ has a simple interpretation on $\mathbb{F}^{\text{an}}$. Indeed, the vector bundle $W \otimes_{\breve{\mathbb{Q}}_p} \mathcal{O}_Y$ on the cover Y of Remark 2.4.2 admits descent data to $\mathbb{F}^{\text{an}}$ via the diagonal action of Frobenius. Denote by $\mathcal{E}^{\text{an}}$ the resulting functor to vector bundles on $\mathbb{F}^{\text{an}}$, and let $\mathcal{O}_{\mathbb{F}^{\text{an}}}(\lambda) = \mathcal{E}^{\text{an}}(D_{-\lambda})$. Then for $k \in \mathbb{Z}$ one finds

$$H^0(\mathbb{F}^{\text{an}}, \mathcal{O}(k)) = (B^+_{\text{crys}})^{\varphi = p^k},$$

and using this one obtains a map of ringed spaces $\mathbb{F}^{\text{an}} \to \mathbb{F}$ that induces a GAGA equivalence of vector bundles identifying $\mathcal{E}^{\text{an}}$ with $\mathcal{E}$.

2.4.1. *G-bundles on the Fargues–Fontaine curve.* Let G be a connected linear algebraic group over $\mathbb{Q}_p$. A G-bundle on $\mathbb{F}$ is an exact tensor functor from $\operatorname{Rep} G$ to the category of vector bundles on $\mathbb{F}$. If b is a G-isocrystal, then the composition of b with the functor $\mathcal{E}$ yields a G-bundle $\mathcal{E}_b$ on $\mathbb{F}$. If G is reductive, then every G-bundle on $\mathbb{F}$ is isomorphic to $\mathcal{E}_b$ for a G-isocrystal b [17].

2.5. *Modifications of vector bundles.* Let X be a connected scheme that is locally the spectrum of a PID, $\infty \in |X|$ a closed point, $j : X \setminus \infty \hookrightarrow X$, and $\mathcal{E}$ a vector bundle. By assumption, the completed local ring $\hat{\mathcal{O}}_{X,\infty}$ is a DVR, and we denote by $\hat{\mathcal{E}}_\infty$ the sections of $\mathcal{E}$ over $\operatorname{Spec} \hat{\mathcal{O}}_{X,\infty}$, a finite rank free $\hat{\mathcal{O}}_{X,\infty}$-module. Now suppose $\mathcal{L} \subset \hat{\mathcal{E}}_\infty \otimes \operatorname{Frac}(\hat{\mathcal{O}}_{X,\infty})$ is a $\hat{\mathcal{O}}_{X,\infty}$-lattice. By a result of Beauville and Laszlo [3], there is a unique vector bundle $\mathcal{E}_{\mathcal{L}}$ with $(\mathcal{E}_{\mathcal{L}})|_U = \mathcal{E}|_U$ and $\widehat{(\mathcal{E}_{\mathcal{L}})}_\infty = \mathcal{L}$. Explicitly, $\mathcal{E}_{\mathcal{L}} \subset j_*\mathcal{E}|_U$ has sections

$$\mathcal{E}_{\mathcal{L}}(V) = \{s \in \mathcal{E}(V - \infty) : s|_{\operatorname{Frac}(\hat{\mathcal{O}}_{X,\infty})} \in \mathcal{L}\}.$$

Definition 2.5.1. The vector bundle $\mathcal{E}_{\mathcal{L}}$ is the modification of the vector bundle $\mathcal{E}$ by the lattice $\mathcal{L}$ at ∞.

3. Filtrations and lattices

Let L be an algebraically closed field of characteristic zero and let B^+ be a complete discretely valued L-algebra with algebraically closed residue field C. We write $\theta : B^+ \to C$ for the quotient, which equips C with the structure of an L-algebra such that θ is a map of L-algebras. Let B denote the fraction field of B^+, and let $F^\bullet B$ denote the valuation filtration on B. For convenience, we fix a generator t for the maximal ideal in B^+ (so, in particular, $F^i B = t^i B^+$).

Example 3.0.1. (1) We could take $B^+ = \mathbb{C}[[t]]$, $B = \mathbb{C}((t))$, $L = C = \mathbb{C}$.

(2) If C is an algebraically closed nonarchimedean extension of $\mathbb{Q}_p$, let $\kappa = \mathcal{O}_C/\mathfrak{m}_C$ be the residue field, let $C_0 = W(\kappa)[1/p]$ be the maximally absolutely unramified subextension, let $L = \overline{C}_0$, the algebraic closure of C_0 in C, and let $B^+ = B^+_{\mathrm{dR}}$, $B = B_{\mathrm{dR}}$. By construction (see Section 2.4), B^+_{dR} is a C_0-algebra compatibly with the canonical map $\theta : B^+_{\mathrm{dR}} \to C$, and Hensel's lemma extends this to a map of $\overline{C}_0$-algebras. Note that any p-adic field contained in C is automatically contained in $\overline{C}_0$. In this case it is convenient to take t to be the "p-adic $2\pi i$", whose definition depends on the choice of a compatible system of p-power roots of unity in C thus is well-defined only up to multiplication by an element of $\mathbb{Z}_p^\times$.

(3) By the Cohen structure theorem [36, Tag 0C0S], we can always choose a section $C \to B^+$ of L-algebras so that $B^+ = C[[t]]$ and $B = C((t))$, but such a section is noncanonical in general and may not respect other structure — in particular, if K is a p-adic field and $C = \overline{K}^\wedge$, then in the setting of (2) such a section cannot be chosen to be continuous for the natural topology on B^+_{dR} or to respect the natural actions of $\mathrm{Gal}(\overline{K}/K)$.

In this section, we study the relation between B^+-latticed L-vector spaces and C-filtered L-vector spaces. In the rest of the paper we will apply this study principally in case (2) above to understand the relation between the de Rham and étale lattices and the Hodge and Hodge–Tate filtrations in p-adic Hodge theory. We will also consider case (1) in our motivational description of real Hodge structures.

In Section 3.1 we recall some basic constructions relating C-filtered and B^+-latticed L-vector spaces. In the sections following, we shift perspectives slightly and explore a more delicate relation between bilatticed B-vector spaces and filtered C-vector spaces that plays an important role in describing the elementary invariants of p-adic Hodge structures and admissible pairs. The main result is Theorem 3.2.6, which will immediately imply Theorem D after the relevant definitions are in place.

3.1. *First relations.*

Definition 3.1.1. (1) For K any field, $\mathrm{Vect}^f(K)$ is the category of filtered K-vector spaces, i.e., pairs $(V, F^\bullet V)$ where V is a finite-dimensional K-vector space and $F^\bullet V$ is a separated and exhaustive decreasing filtration of V. A morphism

$$(V, F^\bullet V) \to (V', F^\bullet V')$$

is a morphism of K-vector spaces $f : V \to V'$ such that $f(F^k V) \subseteq F^k V'$ for all $k \in \mathbb{Z}$. It is strict if $f(F^k V) = f(V) \cap F^k V'$ for all $k \in \mathbb{Z}$.

(2) With L, B^+, B as above, $\mathrm{Vect}^{B^+\text{–latticed}}(L)$ is the category of B^+-latticed L-vector spaces consisting of pairs $(V, \mathcal{L})$ where V is a finite-dimensional L-vector space and $\mathcal{L} \subseteq V_B$ is a B^+-lattice, i.e., a B^+-submodule $\mathcal{L} \subset V_B$ such that $\mathcal{L} \otimes_{B^+} B \to V_B$ is an isomorphism. A morphism

$$(V, \mathcal{L}) \to (V', \mathcal{L}')$$

is a morphism of L-vector spaces $f : V \to V'$ such that $f(\mathcal{L}) \subseteq \mathcal{L}'$. A morphism is *strict* if $f(\mathcal{L}) = f(V)_B \cap \mathcal{L}'$.

These are rigid tensor categories with the usual tensor products and duals of vector spaces / modules / filtered vector spaces. In each category, a complex is exact if it is exact as a complex of vector spaces and each morphism is strict — in the filtered categories, this is equivalent to requiring that the complex obtained by applying the associated graded functor is exact. None of these exact categories is abelian: in all cases there are nonstrict morphisms, while strictness is precisely the condition that the natural quotient filtration/lattice on the coimage agree with the natural submodule filtration/lattice on the image.

Remark 3.1.2. In later sections we also consider variants of these categories where we fix some additional structure such as an underlying $\mathbb{Q}_p$-vector space or isocrystal.

There are natural functors between latticed and filtered vector spaces:

(1) We denote by $\mathrm{BB} : \mathrm{Vect}^{B^+\text{–latticed}}(L) \to \mathrm{Vect}^f(C)$ the Białynicki-Birula functor

$$(V, \mathcal{L}) \mapsto (V_C, \mathrm{trFil}^\bullet_{\mathcal{L}} V_C)$$

where $\mathrm{trFil}^k_{\mathcal{L}} V_C$ is the image of $(F^k B \cdot \mathcal{L}) \cap V_{B^+}$ in $V_C = V_{B^+} \otimes_{B^+} C$. We call $\mathrm{trFil}^\bullet_{\mathcal{L}} V_C$ the trace filtration of $\mathcal{L}$ on V_C.

(2) We denote by $\mathcal{L}_{\mathrm{can}} : \mathrm{Vect}^f(L) \to \mathrm{Vect}^{B^+\text{–latticed}}(L)$ the canonical lattice functor

$$(V, F^\bullet V_L) \mapsto (V, \mathcal{L}_{F^\bullet V_L})$$

where $\mathcal{L}_{F^\bullet V_L} := \sum_i F^{-i} B \cdot F^i V_L \subseteq V_B$.

The following lemma is elementary:

Lemma 3.1.3. *Both* BB *and* $\mathcal{L}_{\mathrm{can}}$ *are tensor functors*, $\mathcal{L}_{\mathrm{can}}$ *is fully faithful and exact*, *and* $\mathrm{BB} \circ \mathcal{L}_{\mathrm{can}}$ *is naturally identified with extension of scalars*

$$\mathrm{Vect}^f(L) \to \mathrm{Vect}^f(C), \quad (V, F^\bullet V_L) \mapsto (V_C, (F^\bullet V_L)_C).$$

Example 3.1.4. Unlike $\mathcal{L}_{\mathrm{can}}$, the functor BB is not exact: consider the exact sequence in $\mathrm{Vect}^{B^+\text{–latticed}}(L)$ (recall that t is a generator for $F^1 B$)

$$0 \to (L, B^+) \xrightarrow{\cdot e_1} \bigl(L^2, B^+ e_1 + B^+\bigl(e_2 + \tfrac{1}{t} e_1\bigr)\bigr) \to (L, B^+) \to 0.$$

Applying BB produces a sequence of filtered vector spaces that is exact as a sequence of vector spaces but such that the morphisms are not strict. Indeed, taking the zeroth graded part after applying BB gives the sequence

$$0 \to C \to 0 \to C \to 0.$$

In certain situations a filtration is uniquely determined by the corresponding lattice. We will recall a version of this with G-structure. Recall that we take the Tannakian viewpoint, that is, ω in the definition is an exact K-linear $\otimes$-functor.

Definition 3.1.5. Let K be a subfield of L and let G be a connected linear algebraic group over K.

(1) A *filtration* on a G-bundle $\omega : \mathrm{Rep}\, G \to \mathrm{Vect}(M)$ (for $M = K$ or C) is an exact tensor functor $F^\bullet\omega : \mathrm{Rep}\, G \to \mathrm{Vect}^f(M)$ and an identification $\omega \xrightarrow{\sim} \mathrm{Forget} \circ F^\bullet\omega$ where Forget is the forgetful functor $\mathrm{Vect}^f(M) \to \mathrm{Vect}(M)$.

(2) A *lattice* on a G-bundle $\omega : \mathrm{Rep}\, G \to \mathrm{Vect}(K)$ is an exact tensor functor $\omega_{\mathcal{L}} : \mathrm{Rep}\, G \to \mathrm{Vect}^{B^+\text{–latticed}}(K)$ and an identification $\omega \xrightarrow{\sim} \mathrm{Forget} \circ \omega_{\mathcal{L}}$ where Forget is the forgetful functor $\mathrm{Vect}^{B^+\text{–latticed}}(K) \to \mathrm{Vect}(K)$.

Finally, if ω is a G-bundle over K, and K' is an extension of K contained in C (resp. L) then a K'-filtration (resp. lattice) on ω is a filtration (resp. lattice) on $\omega \otimes K'$.

Remark 3.1.6. Our primary interest is when ω is the trivial G-bundle, i.e., $\omega = \omega_{\mathrm{std}}$ for the standard fiber functor on $\mathrm{Rep}\, G$.

Filtrations on the trivial G-bundle are parameterized by the points of the flag variety Fl_G. Indeed, any filtration on ω_{std} is split by [29, chapitre 4, théorème 2.4 of], thus isomorphic to the filtration attached to a cocharacter μ of G,

$$F^i_\mu(V, \rho) = \bigoplus_{j \geq i} V[j],$$

where $V[j]$ is weight j component, i.e., the isotypic component for the character $z \mapsto z^j$ of $\mathbb{G}_m$ under the action $\rho \circ \mu$. Two filtrations F_μ and $F_{\mu'}$ are isomorphic if and only if μ and μ' are conjugate, so that this conjugacy class $[\mu]$ is a natural invariant of a filtered G-bundle and $\mathrm{Fl}_{G,C} = \bigsqcup_{[\mu]} \mathrm{Fl}_{[\mu],C}$ where $[\mu]$ runs over all conjugacy classes of cocharacters of G_C and $\mathrm{Fl}_{[\mu]}$, which is naturally defined over the field of definition of $[\mu]$, parameterizes filtrations of the trivial G-bundle of type $[\mu]$ — the action of G on the point $F_\mu \in \mathrm{Fl}_{[\mu]}$ induces, after base change to the field of definition of μ, an isomorphism $\mathrm{Fl}_{[\mu]} = G/P_\mu$ where P_μ is the stabilizer of F_μ.

Lattices on the trivial G-bundle are parameterized by the coset

$$\mathrm{Gr}_G(C) := G(B)/G(B^+).$$

In the p-adic setting these will be the C-points of a B^+_{dR}-affine Grassmannian — for now, however, it suffices to treat this as notation. To see this classification, choose a trivialization $\mathrm{triv}_{\mathcal{L}} : \omega_{\mathrm{std}} \otimes B^+ \xrightarrow{\sim} \mathcal{L} \circ \omega_{\mathcal{L}}$ where $\mathcal{L}$ denotes the forgetful functor from $\mathrm{Vect}^{B^+\text{–latticed}}(K)$ to B^+-modules (this is possible since B^+ is a complete DVR with algebraically closed residue field C). The composition of $\mathrm{triv}_{\mathcal{L}} \otimes_{B^+} B$ with the extension of scalars to B of the fixed $\omega_{\mathrm{std}} \xrightarrow{\sim} \mathrm{Forget} \circ \omega_{\mathcal{L}}$ gives an element $g \in \mathrm{Aut}^{\otimes}(\omega_{\mathrm{std},B}) = G(B)$. Any other choice of trivialization of $\mathcal{L} \circ \omega_{\mathcal{L}}$ differs from triv by an element of $G(B^+)$, so the coset $gG(B^+) \in G(B)/G(B^+)$ only depends on the lattice $\omega_{\mathcal{L}}$. In the opposite direction, attached to a coset $gG(B^+)$, we consider the lattice on the trivial G-bundle

$$\omega_g : (V, \rho) \mapsto (V, \rho(g) \cdot V_{B^+})$$

and the above argument shows every lattice on the trivial G-bundle is isomorphic to one of this form. If G is reductive, the Cartan decomposition gives

$$\mathrm{Gr}_G(C) = \bigsqcup \mathrm{Gr}_{[\mu]}(C), \quad \mathrm{Gr}_{[\mu]}(C) := G(B^+) t^\mu G(B^+)/G(B^+) \subseteq G(B)/G(B^+),$$

where $[\mu]$ runs over all conjugacy classes of cocharacters of G_L (since L is algebraically closed, these are the same as the conjugacy classes of G_C), and we note that the open Schubert cell $\mathrm{Gr}_{[\mu]}(C)$ is independent of the choice of a representative μ and of the choice of uniformizer t. We caution that this decomposition is only disjoint at the level of points — once the affine Grassmannian is equipped with a geometric structure there are closure relations determined by the Bruhat order.

If G is not reductive, we still have $\bigsqcup_{[\mu]} \mathrm{Gr}_{[\mu]}(C) \subseteq \mathrm{Gr}_G(C)$, but this subset is no longer exhaustive. This will be studied further in Section 3.2.

In any case, it is an immediate computation that if $g = g^+ t^\mu G(B^+) \in \mathrm{Gr}_{[\mu]}$ for $g^+ \in G(B^+_{\mathrm{dR}})$ then $\mathrm{BB} \circ \omega_g = \overline{g^+} \cdot F_{\mu^{-1}}$ where $\bar{\bullet}$ denotes reduction $G(B^+) \to G(C)$. For each $[\mu]$ we thus have the following $G(L)$-equivariant commutative diagram:

$$\begin{array}{ccc} & \mathrm{Gr}_{[\mu]}(C) \hookrightarrow \mathrm{Gr}_G(C) & \\ {\scriptstyle \mathcal{L}_{\mathrm{can}}} \nearrow & & \searrow {\scriptstyle \mathrm{BB}} \\ \mathrm{Fl}_{[\mu^{-1}]}(L) & \hookrightarrow & \mathrm{Fl}_{[\mu^{-1}]}(C) \end{array}$$

Proposition 3.1.7 (criteria for the filtration to determine the lattice). (1) *If the weights of* $[\mu]$ *in the adjoint action on* $Lie G$ *are* ≤ 1, *then* $\mathrm{BB} : \mathrm{Gr}_{[\mu]}(C) \to \mathrm{Fl}_{[\mu^{-1}]}(C)$ *is a bijection* (*when G is reductive, this is equivalent to requiring the weights lie in* $\{-1, 0, 1\}$, *i.e., that* $[\mu]$ *is minuscule*).

(2) *Suppose* $\mathfrak{G}$ *is a group acting on* B^+ *preserving the maximal ideal, that* $K = C^{\mathfrak{G}} \subseteq L$ *for the induced action on* C, *and that* $\mathfrak{G}$ *acts on* $K \cdot t$ *by an infinite order character. If G and* $[\mu]$ *are defined over* K, *then* $\mathrm{Fl}_{[\mu^{-1}]}(K)$ *is identified with the* $\mathfrak{G}$*-invariants in* $\mathrm{Gr}_{[\mu]}(C)$ (*for the natural action of* $\mathfrak{G}$ *on* $\mathrm{Gr}_{G,[\mu]}(C) \subseteq G(B)/G(B^+)$ *induced by the action on* $G(B)$)*.*

Proof. We first treat (1). For general μ, the stabilizer of $t^\mu G(B^+)$ for the left multiplication action of $G(B^+)$ is $t^\mu G(B^+) t^{-\mu} \cap G(B^+)$. If the weights of μ in the adjoint representation are all ≤ 1, then this is exactly the preimage under reduction to $G(C)$ of the C-points of the connected subgroup whose Lie algebra is $F^0_{\mu^{-1}} \mathrm{Lie}\, G$ (i.e., the sum of the weight ≤ 0 eigenspaces for μ), and this connected subgroup is also the stabilizer of the filtration $F^\bullet_{\mu^{-1}} \omega_{\mathrm{std}}$. Since the map from $\mathrm{Gr}_{[\mu]}(C)$ to $\mathrm{Fl}_{[\mu^{-1}]}(C)$ is equivariant and the action of $G(B^+)$ is transitive on each, we conclude that it is a bijection.

For the second condition, it suffices to show that the canonical filtration induces an equivalence between $\mathrm{Vect}^f(K)$ and the category of B^+-latticed K-vector spaces $(V, \mathcal{L})$ such that the lattice is invariant under the semilinear action of $\mathfrak{G}$ on V_B. To see this, first note that a lattice invariant under $\mathfrak{G}$ must come from a $K[\![t]\!]$-lattice in $V_{K((t))}$: indeed, by multiplying by a power of t, we may assume $V_{B^+} \supseteq \mathcal{L} \supseteq t^n V_{B^+}$, then, arguing by induction on n, the $\mathfrak{G}$-invariant lattice $\mathcal{L}' = \mathcal{L} \cap t V_{B^+}$ has a basis in $t V_{K[\![t]\!]}$. If we write W_C for the image of $\mathcal{L}$ in V_C, it is invariant under $\mathfrak{G}$, thus of the form $W_C = W \otimes_K C$ for W a subspace of V. Since the image of $\mathcal{L}$ in V_C is contained in the image of $\frac{1}{t}\mathcal{L}'$, we may arrange our basis $e_1, \ldots, e_n$ for $\mathcal{L}'$ in $t V_{K[\![t]\!]}$ so that for $d = \dim W$, the image of $\frac{1}{t}e_1, \ldots, \frac{1}{t}e_d$ spans W. Then $\frac{1}{t}e_1, \ldots, \frac{1}{t}e_d, e_{d+1}, \ldots, e_n$ are a basis for $\mathcal{L}$ in $V_{K[\![t]\!]}$.

With this established, the equivalence is a standard result about the Rees construction for $K((t))$, using that the image of $\mathfrak{G}$ acting on t is Zariski dense in $\mathbb{G}_m$ by assumption. □

Example 3.1.8. If $G = \mathrm{GL}_n$, then a lattice on the trivial G-bundle is equivalent to a lattice $\mathcal{L}$ in B_{dR}^n and a filtration on the trivial G-bundle is equivalent to a filtration of C^n. In the first case, to be minuscule means that there is an $i \in \mathbb{Z}$ such that $F^{i+1}B \cdot \mathcal{L} \subset B^{+n} \subset F^i B \cdot \mathcal{L}$. In the second case, to be minuscule means that there an $i \in \mathbb{Z}$ such that $\mathrm{gr}^j C^n = 0$ for $j \neq i, i+1$.

Corollary 3.1.9. *A B^+-lattice on a vector space is trivial (i.e., $\mathcal{L}_{\mathrm{dR}} = V_{B_{\mathrm{dR}}^+}$) if and only if the induced filtration is trivial.*

Proof. This is a very special case of Proposition 3.1.7(1). □

Example 3.1.10. In the setting of Example 3.0.1(2), suppose $K \subseteq C$ is a p-adic field, $C = \overline{K}^\wedge$, and $\mathfrak{G} = \mathrm{Gal}(\overline{K}/K)$. Then Proposition 3.1.7(2) applies. It also applies if $K = C = \mathbb{C}$ and $\mathfrak{G} = U(1)(\mathbb{R})$ acting on t by $z \mapsto z^i$ for $i \neq 0$.

At this point it is natural to ask: for G not reductive, if we have a lattice $\omega_{\mathcal{L}}$ on the trivial G-bundle whose classifying point does not lie in $\mathrm{Gr}_{[\mu]}(C)$ for any $[\mu]$, then what can we say about $\mathrm{BB} \circ \omega_{\mathcal{L}}$? The main result to be developed in the remainder of this section, Theorem 3.2.6, implies that this is equivalent to inexactness of $\mathrm{BB} \circ \omega_{\mathcal{L}}$ (i.e., failure to be a filtered G-bundle)

Example 3.1.11. We can interpret Example 3.1.4 as a lattice on the trivial $\mathbb{G}_a$-bundle. It is classified by the point $\frac{1}{t} \cdot B^+ \in B/B^+ = \mathbb{G}_a(B)/\mathbb{G}_a(B^+)$. This lies outside of any Schubert cell — indeed, the only cocharacter of $\mathbb{G}_a$ is trivial, so only the trivial coset B^+ itself lies in a Schubert cell. The failure of the short exact sequence in Example 3.1.4 to remain exact after passing to filtered vector spaces illustrates the failure of $\mathrm{BB} \circ \omega_{\mathcal{L}}$ to be exact if $\omega_{\mathcal{L}}$ lies outside a Schubert cell.

3.2. *Bilatticed B-vector spaces and exactness of filtrations.* We now define a coarser category than $\mathrm{Vect}^{B^+\text{–latticed}}(L)$ through which BB factors and whose isomorphism classes will capture the invariant $[\mu]$ discussed above. Using this category, we will formulate and prove the general principle relating the Schubert cells and the failure of exactness of BB.

Definition 3.2.1. Let $\mathrm{Vect}^{\mathrm{bl}}(B)$ be the category of bilatticed B-vector spaces, i.e., finite-dimensional B-vector spaces equipped with a pair $\mathcal{L}_1, \mathcal{L}_2$ of B^+-lattices. A morphism $(V, \mathcal{L}_1, \mathcal{L}_2) \to (V', \mathcal{L}_1', \mathcal{L}_2')$ is a morphism of B-vector spaces $f : V \to V'$ such that $f(\mathcal{L}_i) \subseteq \mathcal{L}_i'$ for $i = 1, 2$. A morphism is *strict* if $f(\mathcal{L}_i) = f(V) \cap \mathcal{L}_i'$ for $i = 1, 2$. A complex of bilatticed B-vector spaces is exact if it is exact as a complex of B-vector spaces and each morphism is strict.

If V is a B-vector space and $\mathcal{L} \subset V$ is a B^+-lattice, we consider the filtration on V by the B^+-submodules $F^\bullet B \cdot \mathcal{L}$. Using the filtrations of this form, we obtain two L-linear tensor functors from $\mathrm{Vect}^{\mathrm{bl}}(B)$ to $\mathrm{Vect}^f(C)$:

$$\mathrm{BB}_1 : (V, \mathcal{L}_1, \mathcal{L}_2) \mapsto (\mathcal{L}_{1,C}, \mathrm{trFil}^\bullet_{\mathcal{L}_2} \mathcal{L}_{1,C}) \text{ and } \mathrm{BB}_2 : (V, \mathcal{L}_1, \mathcal{L}_2) \mapsto (\mathcal{L}_{2,C}, \mathrm{trFil}^\bullet_{\mathcal{L}_1} \mathcal{L}_{2,C})$$

where $\mathrm{trFil}^\bullet_{\mathcal{L}_i}\mathcal{L}_{j,C}$ is the trace filtration as in Section 3.1 formed by taking the image of $(F^\bullet B\cdot\mathcal{L}_i)\cap\mathcal{L}_j$ in $\mathcal{L}_{j,C}$.

Example 3.2.2. We define two functors from $\mathrm{Vect}^{B^+\text{–latticed}}(L)$ to $\mathrm{Vect}^{\mathrm{bl}}(B)$:

$$\mathrm{bl}_1 : (V,\mathcal{L}) \mapsto (V_B, V_{B^+}, \mathcal{L}), \quad \mathrm{bl}_2 : (V,\mathcal{L}) \mapsto (V_B, \mathcal{L}, V_{B^+}).$$

They differ by the involution of $\mathrm{Vect}^{\mathrm{bl}}(B)$ swapping the two lattices, and for each $i=1,2$ the following diagram commutes up to a natural isomorphism:

$$\begin{array}{ccc} \mathrm{Vect}^{B^+\text{–latticed}}(L) & \xrightarrow{\mathrm{bl}_i} & \mathrm{Vect}^{\mathrm{bl}}(B) \\ & \searrow^{\mathrm{BB}} & \downarrow \mathrm{BB}_i \\ & & \mathrm{Vect}^f(C) \end{array}$$

The functors bl_i, $i=1,2$ are exact, so any failure of exactness in BB is visible in the functors BB_i. The isomorphism class in $\mathrm{Vect}^{\mathrm{bl}}$ will give the invariant describing the Schubert cell for a B^+-lattice on the trivial G-bundle.

There is a simple relation between the associated gradeds for the BB_i:

Lemma 3.2.3. *For each $j\in\mathbb{Z}$, scalar multiplication induces an isomorphism of functors from* $\mathrm{Vect}^{\mathrm{bl}}(B)$ *to* $\mathrm{Vect}(C)$

$$\mathrm{gr}^j B\otimes(\mathrm{gr}^{-j}\circ\mathrm{BB}_1) \xrightarrow{\sim} \mathrm{gr}^j\circ\mathrm{BB}_2.$$

Proof. We have

$$\begin{aligned} \mathrm{gr}^{-i}_{\mathcal{L}_2}\mathcal{L}_{1,C} &= \big((t^{-i}\mathcal{L}_2)\cap\mathcal{L}_1\big)\,\big/\,\big((t^{-i+1}\mathcal{L}_2)\cap\mathcal{L}_1+(t^{-i}\mathcal{L}_2)\cap t\mathcal{L}_1\big),\\ \mathrm{gr}^{i}_{\mathcal{L}_1}\mathcal{L}_{2,C} &= \big(\mathcal{L}_2\cap(t^{i}\mathcal{L}_1)\big)\,\big/\,\big(t\mathcal{L}_2\cap(t^{i}\mathcal{L}_1)+\mathcal{L}_2\cap(t^{i+1}\mathcal{L}_1)\big), \end{aligned}$$

and multiplying by t^i is an isomorphism from the first to the second. □

Remark 3.2.4. In p-adic Hodge theory, this is the usual isomorphism from the Tate twists of the graded components for the Hodge filtration to the graded components for the Hodge–Tate filtration.

For G a connected linear algebraic group over L, in the following a filtered G-bundle will mean a filtered G-bundle over $\mathrm{Spec}\,C$, i.e., an exact rigid tensor functor $\omega_f : \mathrm{Rep}\,G\to\mathrm{Vect}^f(C)$. Recall from the previous subsection that the isomorphism classes are parameterized by the type of the filtration, a conjugacy class of cocharacters $[\mu]_{\omega_f}$ of G.

A bilatticed G-bundle (over B) is an exact L-linear tensor functor

$$\omega_{\mathrm{bl}} : \mathrm{Rep}\,G\to\mathrm{Vect}^{\mathrm{bl}}(C).$$

The induced functors $\mathcal{L}_i\circ\omega_{\mathrm{bl}}$ from $\mathrm{Rep}\,G$ to B^+-modules are G-bundles on $\mathrm{Spec}\,B^+$, and thus are trivial (because the residue field C is algebraically closed). Choosing trivializations for each, the isomorphism

$\mathcal{L}_2 \otimes B \xrightarrow{\sim} V \xrightarrow{\sim} \mathcal{L}_1 \otimes B$ induces an automorphism of the trivial G-bundle over $\operatorname{Spec} B$, thus an element of $G(B)$. The image in the double coset space

$$G(B^+)\backslash G(B)/G(B^+)$$

is independent of the choice of trivializations, and this assignment identifies the double cosets with the isomorphism classes of bilatticed G-bundle.

As above, if G is reductive, then the Cartan decomposition identifies this double coset space with the conjugacy classes of cocharacters of G (a conjugacy class is matched with the unique double coset containing $t^\mu = \mu(t)$ for any element μ in the conjugacy class). For a general connected linear algebraic G we can assign in this way distinct double cosets to each conjugacy class $[\mu]$, but not all double cosets are of this form.

Definition 3.2.5. A bilatticed G-bundle ω_{bl} is *good* if it lies in the double coset $G(B^+)t^\mu G(B^+)$ of a conjugacy class of cocharacters $[\mu]$. In this case we write $[\mu]_{\omega_{\mathrm{bl}}}$ for this conjugacy class and call it the *type* of the good bilatticed G-bundle.

We now explain the connection with exactness of filtrations: If G is reductive, then since $\operatorname{Rep} G$ is semisimple it is easy to verify that for any bilatticed G-bundle ω_{bl}, $\mathrm{BB}_i \circ \omega_{\mathrm{bl}}$ is exact, i.e., is a filtered G-bundle (this is because BB_i preserves split exact sequences). For G nonreductive this may not be the case, as can be seen by combining Example 3.2.2 and Example 3.1.11. In fact:

Theorem 3.2.6. *The following are equivalent for a bilatticed G-bundle ω_{bl}:*

(1) *ω_{bl} is good.*

(2) *$\mathrm{BB}_1 \circ \omega_{\mathrm{bl}}$ is a filtered G-bundle (i.e., is exact).*

(3) *$\mathrm{BB}_2 \circ \omega_{\mathrm{bl}}$ is a filtered G-bundle (i.e., is exact).*

In this case, if the type of ω_{bl} is $[\mu]$, then the type of $\mathrm{BB}_1 \circ \omega_{\mathrm{bl}}$ is $[\mu^{-1}]$ and the type of $\mathrm{BB}_2 \circ \omega_{\mathrm{bl}}$ is $[\mu]$.

To show this, we first observe that it is immediate from the definitions that if ω_{bl} is good then the $\mathrm{BB}_i \circ \omega_{\mathrm{bl}}$ are exact with the claimed types — indeed, to say that ω_{bl} is good of type $[\mu]$ is the same as saying that it is isomorphic to the functor

$$\operatorname{Rep} G \to \mathrm{Vect}^{\mathrm{bl}}(B),\ (\rho, V) \mapsto (V_B, V_{B^+}, \rho(\mu(t)) \cdot V_{B^+})$$

and for this bilatticed G-bundle the associated filtrations are exactly those given by μ^{-1} and μ. It is also immediate from Lemma 3.2.3 that (2) $\leftrightarrow$ (3) — indeed, a functor $\operatorname{Rep} G \to \mathrm{Vect}^f(C)$ is exact if and only if the associated graded is exact.

Thus what remains is to show that if $\mathrm{BB}_1 \circ \omega_{\mathrm{bl}}$ (or $\mathrm{BB}_2 \circ \omega_{\mathrm{bl}}$) is exact then ω_{bl} is good. This turns out to be intimately related to the behavior of types under extensions. Indeed, this is illustrated in Example 3.1.11: the associated sequence of filtered vector spaces fails to be exact, and the type of the extension is $(1, -1)$ while the type of the associated graded is $(0, 0)$. When a short exact sequence of bilatticed vector spaces does give rise to a short exact sequence of filtered vector spaces, then it is easy to see that the type stays the same (simply take the associated graded for the short exact sequence of filtered vector spaces). Our strategy for finishing the proof of Theorem 3.2.6 is thus to show that the good bilatticed G-bundles are exactly the ones such that, for any faithful representation V of G, the type of $\omega_{\mathrm{bl}}(V)$ agrees with the type of its associated graded (for the filtration by subobjects corresponding to the unipotent radical U of G).

Example 3.2.7. Consider the following sequence of filtered vector spaces $0 \to S \to V \to Q \to 0$, where the maps in the first row are inclusion and projection:

$$\begin{aligned} &F^k,\ k \le 0: && 0 \to Ce_1 \to Ce_1 + Ce_2 \to Ce_2 \to 0, \\ &F^1: && 0 \to 0 \to Ce_1 \xrightarrow{0} Ce_2 \to 0, \\ &F^k,\ k \ge 2: && 0 \to 0 \to 0 \to 0 \to 0. \end{aligned}$$

This is a short exact sequence of vector spaces and the maps respect the filtrations but are not strict exact. However, both V and $S \oplus Q$ have type $(1, 0)$. This particular example cannot arise from an exact sequence of bilatticed vector spaces because it is easy to check that for a strict surjection of bilatticed vector spaces, the induced map on filtered vector spaces is surjective on the smallest nonzero part of the filtration (whereas the map on F^1 in this example is the zero map!).

3.3. ***Dominance and extensions.*** In Example 3.1.11, as discussed above, the type of the extension does not change in an arbitrary way: the type $(1, -1)$ of the extension is not equal to the type $(0, 0)$ of the associated graded, but it does lie above it in the standard dominance partial ordering. We recall this partial order now, then prove that the type of an extension always rises over the type of the graded; we will need to bootstrap from this computation to identify when equality holds.

Recall that a conjugacy class of cocharacters $[\mu]$ of GL_n is equivalent to a multiset of n integers, where i appears with multiplicity equal to the multiplicity of the character $z \to z^i$ of any representative of $[\mu]$ acting on the standard representation of GL_n. We will write such a multiset as $[\mu] = (\mu_1, \ldots, \mu_n)$ where $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$. Then, if $[\mu] = (\mu_1, \ldots, \mu_n)$ and $[\nu] = (\nu_1, \ldots, \nu_n)$ we say $[\mu] \le [\nu]$ if for each k, $\sum_{i=1}^k \mu_i \le \sum_{i=1}^k \nu_i$. This is the dominance partial order.

For V a bilatticed vector space of type $[\mu] = (\mu_1, \ldots, \mu_n)$, $\mu_1 \ge \ldots \ge \mu_n$, we would like to give a characterization of this number $\sum_{i=1}^k \mu_i$ in terms of V. To that end, first note that it can be thought of as the largest integer appearing in the type of the k-th exterior power $\Lambda^k V$. On the other hand, if we define the *order*, $\mathrm{ord}(W)$, of a bilatticed vector space $W = (B, \mathcal{L}_1, \mathcal{L}_2)$ to be smallest i such that $t^i \mathcal{L}_1 \subseteq \mathcal{L}_2$, then it is trivial to see that if W has type $[\nu] = (\nu_1, \ldots, \nu_n)$ then $\mathrm{ord}(W) = \nu_1$. Thus $\sum_{i=1}^k \mu_i = \mathrm{ord}(\Lambda^k V)$, and this gives us a useful way to get a hold of these numbers. Using this interpretation, we can see that the type always rises in extensions: the proof is almost identical to that of [21, Lemma 1.2.3].

Proposition 3.3.1. *$V \in \mathrm{Vect}^{\mathrm{bl}}(B)$ be equipped with a filtration by strict subobjects $F^i V$, and let $\mathrm{gr}^\bullet(V) = \bigoplus_i F^i V / F^{i+1} V$, an object of $\mathrm{Vect}^{\mathrm{bl}}(B)$. If $\mathrm{gr}^\bullet V$ is of type $[\mu]$ and V is of type $[\nu]$ then $[\mu] \le [\nu]$.*

Proof. By induction on the length of the filtration by strict subobjects, it suffices to treat the case where the filtration has two steps, i.e., where we have a strict exact sequence in $\mathrm{Vect}^{\mathrm{bl}}(B)$:

$$0 \to S \to V \to Q \to 0.$$

We want to show the type of V lies above the type of $S \oplus Q$.

Thus we need to show $\mathrm{ord}(\Lambda^k V) \geq \mathrm{ord}(\Lambda^k(S \oplus Q))$ for each k. To that end, first note $\Lambda^k(S \oplus Q)) = \bigoplus_j \Lambda^j S \otimes \Lambda^{k-j} Q$; thus $\mathrm{ord}(\Lambda^k(S \oplus Q))) = \max_j \mathrm{ord}(\Lambda^j S \otimes \Lambda^{k-j} Q)$. On the other hand, $\Lambda^k V$ has a filtration by strict subojects such that the associated graded terms are $\Lambda^j S \otimes \Lambda^{k-j} Q$, so $\mathrm{ord}(\Lambda^k V) \geq \max_j \mathrm{ord}(\Lambda^j S \otimes \Lambda^{k-j} Q)$. □

Remark 3.3.2. We can also define the *order* of a filtered vector space $(V, F^\bullet V)$ to be the largest k such that $F^k V \neq 0$; a simple computation shows BB_2 preserves order.

3.4. *The action of unipotent elements on affine Schubert cells.*

To make the proofs explicit and free up the letter B for "Borel", for the remainder of Section 3 we fix an isomorphism $B^+ = C[[t]]$ and write $C[[t]]$ in place of B^+ and $C((t))$ in place of B. See Example 3.0.1(3).

We also fix some notation for this subsection only: let $G = \mathrm{GL}_n$, let $T \subseteq \mathrm{GL}_n$ be the diagonal maximal torus, and let $\mathcal{B}$ denote the set of Borel subgroups containing T. Let B^+ be the upper-triangular Borel with unipotent subgroup U^+, and let B^- be lower triangular Borel with unipotent subgroup U^-. We write $X_*(T)$ for the group of cocharacters of T. For each root α there is a coroot $\alpha^\vee \in X_*(T)$; if α is positive (by which we mean with respect to B^+) we call $\alpha^\vee$ a positive coroot. We call a cocharacter $\mu \in X_*(T)$ dominant if $\alpha(\mu) \geq 0$ for each positive root α, and let $X_*(T)^+$ be the set of such dominant cocharacters. (Concretely, these are the cocharacters $\mu(z) = \mathrm{diag}(z^{a_1}, \ldots, z^{a_n})$ for integers $a_1 \geq a_2 \geq \cdots \geq a_n$.)

For cocharacters $\mu, \mu' \in X_*(T)$, we write $\mu \leq \mu'$ if $\mu' - \mu$ is a nonnegative sum of positive coroots. On the dominant cocharacters this is equivalent to the dominance relation discussed in the previous subsection plus the condition that $\det \mu = \det \mu'$.

We write $K = \mathrm{GL}_n(C[[t]])$. Every conjugacy class $[\mu]$ of cocharacters of GL_n has a unique dominant representative μ, so the Cartan decomposition reads

$$\mathrm{GL}_n(C((t))) = \bigsqcup_{\mu \in X_*(T)^+} K t^\mu K.$$

For $B \in \mathcal{B}$ with unipotent subgroup U, the Iwasawa decomposition is

$$\mathrm{GL}_n(C((t))) = \bigsqcup_{\mu \in X_*(T)} U(C((t))) t^\mu K.$$

Essentially, we need to compare these two decompositions. The following lemma can be deduced from results of Bruhat and Tits [7] in a more general setting. We give a direct proof then explain in Remark 3.4.2 how it can be deduced from [7].

Lemma 3.4.1. *Let $B \in \mathcal{B}$ with unipotent subgroup U. Let $\mu \in X_*(T)^+$ be a dominant cocharacter, and let $u \in U(C((t)))$. Let $\nu \in X_*(T)^+$ be the dominant cocharacter such that $u \cdot t^\mu K \subseteq K t^\nu K$. Then*:

(1) $\mu \leq \nu$.

(2) *If $B = B^-$ then $\mu = \nu$ if and only if $u \cdot t^\mu K = t^\mu K$, or, equivalently,*

$$t^{-\mu} u t^\mu \in U(C[\![t]\!]) = U(C((t))) \cap K.$$

(3) *If $B = B^+$ then $\mu = \nu$ if and only if $u \in U(C[\![t]\!]) = K \cap U(C((t)))$.*

Proof. Part (1) is an immediate consequence of Proposition 3.3.1. Indeed, if we consider the bilatticed vector space $(C((t))^n, C[\![t]\!]^n, ut^\mu \cdot C[\![t]\!]^n)$, then it has a filtration by strict subojects coming from the standard filtration stabilized by B, and the associated graded is isomorphic $(C((t))^n, C[\![t]\!]^n, t^\mu \cdot C[\![t]\!]^n)$.

For the rest, we first note that (2) and (3) are equivalent: suppose (2) holds, and suppose $u \in U^+$ is such that $u \cdot t^\mu \in K t^\mu K$. Then, by taking transpose, $t^\mu u^t = (t^\mu u^t t^{-\mu}) t^\mu \in K t^\mu K$, and since $(t^\mu u^t t^{-\mu}) \in U^-$, (2) gives that $u^t = t^{-\mu}(t^\mu u^t t^{-\mu}) t^\mu \in U^-(C[\![t]\!])$, and thus $u \in U^+(C[\![t]\!])$. In the other direction, suppose (3) holds, and let $u \in U^-$ be such that $ut^\mu \in K t^\mu K$. Then, taking transpose, $t^\mu u^t = (t^\mu u^t t^{-\mu}) t^\mu \in K t^\mu K$ and by (3), $t^\mu u^t t^{-\mu} \in U^+(C[\![t]\!])$. Taking transpose again, we find $t^{-\mu} u^t t^\mu \in U^-(C[\![t]\!])$.

It thus suffices to prove (3). The if direction is clear, so, it remains only to show that if $u \notin U^+(C[\![t]\!])$ then $\nu > \mu$. To that end, let i be the index of the first row of u whose entries are not in $C[\![t]\!]$, and let $j > i$ be the first index such that $u_{i,j} \notin U^+(C[\![t]\!])$. Then, for $V = (C((t))^n, C[\![t]\!]^n, ut^\mu \cdot C[\![t]\!]^n)$, and $F_i V$ the standard increasing filtration of V attached to B^+, one finds that the type of $F_j V / F_{i-1} V$ is strictly larger than the type $(\mu_i, \dots, \mu_j)$ of its associated graded. Indeed, we use the opposite order

$$\overline{\mathrm{ord}}(V) = \max\{k : t^{-k} \mathcal{L}_2 \subseteq \mathcal{L}_1\},$$

which picks out the smallest integer in the type of V. The vector $u_{i,j} t^{\mu_j} e_i + \cdots + t^{\mu_j} e_j$ is in the second lattice of $F_j V / F_{i-1} V$, so if $t^{-k} \mathcal{L}_2(F_j V / F_{i-1} V) \subseteq \mathcal{L}_1(F_j V / F_{i-1} V)$ then $k \leq \mu_j + v(u_{i,j}) < \mu_j$ (where v denotes the additive t-adic valuation). Hence $\overline{\mathrm{ord}}(F_j V / F_{i-1} V) \leq \mu_j + v(u_{i,j}) < \mu_j$, which implies the type of $F_j V / F_{i-1} V$ is strictly greater than the type $(\mu_i, \dots, \mu_j)$ of its associated graded by Proposition 3.3.1. Applying Proposition 3.3.1 to the filtration $0 \subseteq F_{i-1} V \subseteq F_j V \subseteq V$ and the nonstrict inequalities that the type of $F_{i-1} V$ is greater than or equal to $(\mu_1, \dots, \mu_{i-1})$ and the type of $V / F_j V$ is greater than or equal to $(\mu_{j+1}, \dots, \mu_n)$, we conclude. □

Remark 3.4.2. Alternatively, Lemma 3.4.1(1) follows from [7, Corollary 4.3.17] and Lemma 3.4.1(2) follows from [7, Proposition 4.4.4-(ii)]; one can then obtain Lemma 3.4.1(3) by using the equivalence between (2) and (3) explained in the proof above. To make it possible to follow these citations, we explain how our notation compares to the notation in [7, Chapter 4]: For D the dominant chamber for the roots of a Borel B containing T, the group $\hat{\mathcal{B}}^0_D$ of loc. cit. is the group generated by the set of diagonal matrices with entries in $C((t))$ such that all of the entries have the same valuation and the $C((t))$-points of the unipotent subgroup of the *opposite* Borel to B. In particular, the fixed chamber $\boldsymbol{D}$ in loc. cit. corresponds in our setup to the dominant chamber for the roots of B^+, so that $\hat{\mathcal{B}}^0$, which by definition in loc. cit. is $\hat{\mathcal{B}}^0_{\boldsymbol{D}}$, contains our U^-.

We need refinements of parts (2)–(3) that apply to arbitrary $B \in \mathcal{B}$. We fix as above a dominant cocharacter $\mu \in X_*(T)^+$ and $B \in \mathcal{B}$ with unipotent radical U. The conjugation action of μ induces a decomposition $U = U_{>0} \cdot U_0 \cdot U_{<0}$ where $\mathrm{Lie}\, U_{>0}$ (resp. $\mathrm{Lie}\, U_0$, resp. $\mathrm{Lie}\, U_{<0}$) consists of all roots α in $\mathrm{Lie}\, U$ such that $\alpha(\mu) > 0$ (resp. $= 0$, resp. < 0). Note that since μ is dominant $U_{>0} \le U \cap U^+$ and $U_{<0} \le U \cap U^-$, while U_0, the subgroup of elements in U centralized by μ, can contain a mix of both positive and negative root subgroups. By [12, Theorem 3.3.11] this decomposition exists already for any closed subgroup $U' \subseteq U$ preserved by conjugation by t^μ, and is compatible with the decomposition of U, i.e.,

$$U' = U'_{>0} U'_0 U'_{<0} \text{ for } U'_{>0} = U' \cap U_{>0},\ U'_0 = U' \cap U_0, \text{ and } U'_{<0} = U' \cap U_{<0}. \tag{1}$$

Lemma 3.4.3. *With notation as above, let $u \in U'(C((t)))$ and let $u = u_{>0} u_0 u_{<0}$ denote its product decomposition. Then $u \cdot t^\mu \in K t^\mu K$ if and only if all of the following hold*:

(1) $u_{>0} \in U'_{>0} \cap K$.

(2) $u_0 = t^{-\mu} u_0 t^\mu \in U'_0 \cap K$.

(3) $t^{-\mu} u_{<0} t^\mu \in U'_{<0} \cap K$.

In this case, $u \cdot t^\mu K = u_{>0} \cdot t^\mu K = u_{>0} u_0 \cdot t^\mu K$.

Proof. It follows from the identities $u t^\mu = u_{>0} u_0 u_{<0} t^\mu = (u_{>0} u_0) t^\mu (t^{-\mu} u_{<0} t^\mu) = u_{>0} t^\mu (u_0 t^{-\mu} u_{<0} t^\mu)$ that if (1)–(3) are satisfied then $u \cdot t^\mu \in K t^\mu K$ and the last claim also holds. It remains to show that, if $u \cdot t^\mu \in K t^\mu K$ then (1)–(3) hold.

Using Equation (1), we may assume $U' = U$. We further refine $U_0 = U_{0,+} \cdot U_{0,-}$ where $U_{0,+} = U_0 \cap U^+$ and $U_{0,-} = U_0 \cap U^-$. We thus write $u = u_{>0} u_{0,+} u_{0,-} u_{<0}$. We note moreover that $U_{0,-}$ normalizes $U_{<0}$ and commutes with t^μ, so we can rewrite this as

$$u = u_{>0} u_{0,+} u_{0,-} u_{<0} = u_{>0} u_{0,+} u'_{<0} u_{0,-}, \quad \text{where } u'_{<0} = u_{0,-} u_{<0} u_{0,-}^{-1}.$$

We now choose a chain of adjacent Borel subgroups $B^+ = B_0, B_1, B_2, \dots, B_l = B$ such that, for each $1 \le i \le l$, there is a unique root α_i such that α_i is in $\mathrm{Lie}\, U^+$ and $-\alpha_i$ is in $\mathrm{Lie}\, U$ and to move from B_{i-1} to B_i we swap α_i for $-\alpha_i$. For each i, $-\alpha_i$ is a weight of $\mathrm{Lie}\, U_{<0}$ or $\mathrm{Lie}\, U_{0,-}$, and because $U_{0,-}$ normalizes $U_{<0}$, we can and do choose this chain so that $-\alpha_1, \dots, -\alpha_s$ lie in $U_{<0}$ and $-\alpha_{s+1}, \dots, -\alpha_l$ lie in $U_{0,-}$. Thus we may write

$$u = u_{>0} u_{0,+} u'_{<0} u_{0,-} = u_{>0} u_{0,+} u'_{-\alpha_1} \dots u'_{-\alpha_s} u_{-\alpha_{s+1}} \cdots u_{-\alpha_l},$$

so that $u'_{<0} = u'_{-\alpha_1} \cdots u'_{-\alpha_s}$ and $u_{0,-} = u_{-\alpha_{s+1}} \cdots u_{-\alpha_l}$ for unique elements $u'_{-\alpha_i}$, $i = 1, \dots, s$, and $u_{-\alpha_i}$, $i = s+1, \dots, l$, of the root subgroups $U_{-\alpha_i}(C((t)))$. Now suppose that $u_{0,-}$ is not in K. Then at least one factor $u_{-\alpha_i}$ must not be in K; let $s+1 \le i \le l$ be the index of the last root factor that is not in K. Since t^μ acts trivially by conjugation on U_0, we may pass the factors starting at i through t^μ to obtain

$$u \cdot t^{\mu} K = u_{>0} u_{0,+} u'_{<0} u_{-\alpha_{s+1}} \cdots u_{-\alpha_l} t^{\mu} K = u_{>0} u_{0,+} u'_{<0} u_{-\alpha_{s+1}} \cdots u_{-\alpha_{i-1}} t^{\mu} u_{-\alpha_i} K.$$

We may fix an identification of $U_{-\alpha_i}(C((t)))$ with $C((t))$ such that $u_{-\alpha_i} = t^{-m}$ for $m > 0$. From the following computation with 2×2 matrices, applied to the principal SL_2 containing U_{α_i} and $U_{-\alpha_i}$ (the latter as the lower triangular unipotents and the former as the upper triangular unipotents),

$$\begin{bmatrix} 1 & 0 \\ t^{-m} & 1 \end{bmatrix} = \begin{bmatrix} 1 & t^m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^m & 0 \\ 0 & t^{-m} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & t^m \end{bmatrix}.$$

we find

$$u \cdot t^{\mu} K = u_{>0} u_{0,+} u'_{<0} u_{-\alpha_{s+1}} \cdots u_{-\alpha_{i-1}} v_{\alpha_i} t^{\mu + m\alpha_i^{\vee}} K,$$

where v_{α_i} is in the root subgroup for α_i. The product s on the left of $t^{\mu+m\alpha_i^{\vee}}$ lies in the unipotent radical of B_{i-1}. If we conjugate by the Weyl element w sending $\mu + m\alpha_i^{\vee}$ to a dominant weight μ', this does not change the Schubert cell, so we get $wsw^{-1}t^{\mu'} \in Kt^{\mu}K$. By construction, μ' is strictly larger than μ in the lexicographic order. Since the element wsw^{-1} lies in the unipotent radical of $wB_{i-1}w^{-1}$, Lemma 3.4.1(1) implies that $wsw^{-1}t^{\mu'}$ lies in $Kt^{\nu}K$ for $\nu \geq \mu'$, but as lexicographic order refines dominance order, we see ν is strictly greater than μ in the lexicographic order, in contradiction to $\nu = \mu$.

Thus we have $u_{0,-} \in K$, and we may pass it through t^{μ} to obtain

$$u \cdot t^{\mu} K = u_{>0} u_{0,+} u'_{<0} \cdot t^{\mu} K.$$

Arguing similarly to the above with $u'_{<0} = u'_{-\alpha_1} \cdots u'_{-\alpha_s}$, we find that $t^{-\mu} u'_{<0} t^{\mu} \in K$ so we may pass it through t^{μ} to obtain

$$u \cdot t^{\mu} K = u_{>0} u_{0,+} \cdot t^{\mu} K.$$

Since $u_{>0} u_{0,+} \in B^{+}(C((t)))$, we conclude that $u_{>0} u_{0,+}$ is in K by Lemma 3.4.1(3). Thus, so is each factor. We have already established $u_{0,-} \in K$ and $t^{-\mu} u'_{<0} t^{\mu} \in K$; thus, since $u_{0,-}$ commutes with $t^{\pm\mu}$, we find also that

$$t^{-\mu} u_{<0} t^{\mu} = t^{-\mu} u_{0,-}^{-1} u'_{<0} u_{0,-} t^{\mu} = u_{0,-}^{-1} (t^{-\mu} u'_{<0} t^{\mu}) u_{0,-} \in K. \qquad \square$$

3.5. *Conclusion.* We now give the promised specialization result refining and generalizing Proposition 3.3.1.

Theorem 3.5.1. *Let G be a connected linear algebraic group over C with Levi decomposition $G = M \ltimes U$. Let $g \in G(C((t)))$ with $g = mu$ and let $[\mu]$ be the conjugacy class of cocharacters of M identifying the Schubert cell containing m (i.e., $m \in M(C[[t]])t^{\mu}M(C[[t]])$). For any representation $\rho : G \to \mathrm{GL}_n$, if $[\nu]$ is the conjugacy class of cocharacters indexing the Schubert cell containing $\rho(g)$, then $[\nu] \geq [\rho \circ \mu]$ (the conjugacy class of cocharacters indexing the Schubert cell containing $\rho(m)$). If ρ is faithful, then $[\nu] = [\rho \circ \mu]$ if and only if $g \in G(C[[t]])t^{\mu}G(C[[t]])$.*

Proof. It changes nothing to multiply on the left and right by an element of $M(C[[t]])$, so we may assume $m = t^{\mu}$. Then, replacing G with $\mathbb{G}_m \ltimes U$, we may assume G is solvable. Then its image under ρ lies in a Borel subgroup, and conjugating ρ suitably by an element of $\mathrm{GL}_n(C)$ we may assume that $\rho \circ \mu$ is dominant and this is a standard Borel $B \in \mathcal{B}$. Then $\rho(u)$ factors through the unipotent radical of B, and the first claim follows from Lemma 3.4.1. If the representation is faithful then U is a subgroup of the unipotent radical of B preserved under conjugation by t^μ, so if $[\mu] = [\nu]$ then Lemma 3.4.3 gives

$$ut^\mu = u_{>0}u_0u_{<0}t^\mu = u_{>0}u_0t^\mu t^{-\mu}u_{<0}t^\mu \in U(C[\![t]\!])t^\mu U(C[\![t]\!]). \qquad \square$$

Conclusion of proof of Theorem 3.2.6. Recall that it remains only to show (3) $\implies$ (1) in Theorem 3.2.6. To that end, let ω_{bl} be a bilatticed G-bundle, and suppose $\omega_f := \mathrm{BB}_2 \circ \omega_{\mathrm{bl}}$ is a filtered G-bundle. Pick a representative $g \in G(C((t)))$ for the double coset corresponding to ω_{bl}, and write $g = mu$ for a Levi decomposition $G = MU$. Let $[\mu]$ be the conjugacy class of cocharacters of M describing the Schubert cell of m. We want to show $g \in G(C[\![t]\!])t^\mu G(C[\![t]\!])$.

To that end, let $\rho : G \to \mathrm{GL}(V)$ be a faithful representation. By Theorem 3.5.1, it suffices to show $[\rho \circ \mu]$ is the type of $\rho_*\omega_{\mathrm{bl}}$, i.e., that the bilatticed vector space associated to $\rho(g)$ is of type $[\rho \circ \mu]$.

Let $\mathfrak{u} := \mathrm{Lie}\, U$. Then $\rho(G)$ is contained in the parabolic subgroup stabilizing the filtration of V by $\mathfrak{u}^i V$. The action on the associated graded is through M; thus the associated graded for this filtration is of type $[\rho \circ \mu]$. Since the associated filtration by subobjects in the category of filtered vector spaces after applying BB_2 is strict by assumption, we conclude the filtration on V is also of type $[\rho \circ \mu]$. Since this agrees with the type of $\rho_*\omega_{\mathrm{bl}}$, we are done. $\square$

4. Real and p-adic Hodge structures

In this section we will define the category of p-adic Hodge structures and establish some of its basic properties. By way of motivation, we begin in Section 4.1 by recalling a geometric perspective on the definition of a real Hodge structure via the theory of vector bundles on the twistor line (due to Simpson [35]) that leads to a definition of a category of extended real Hodge structures containing real Hodge structures as a full subcategory. This perspective will be mirrored in our definition of p-adic Hodge structures in Section 4.2, replacing the twistor line with the Fargues–Fontaine curve. With these definitions in place, in Section 4.3 and Section 4.4 we give a symmetric treatment of the basic structural properties of both extended real and p-adic Hodge structures — in particular, we discuss Mumford–Tate groups and some fundamental invariants. In Section 4.5 we explain how to compute the Mumford–Tate group of a p-adic Hodge structure using Hodge–Tate lines, analogous to the Hodge tensors in classical Hodge theory. It is important here to give a criterion that applies to nonreductive groups; we explain more about the lack of reductivity in this theory and the relation to polarizability in classical Hodge theory in Remark 4.5.2.

4.1. ***Definition of (extended) real Hodge structures.*** Traditionally, a real Hodge structure of weight k is defined to be a pair $(V, F^\bullet V_\mathbb{C})$ where V is a finite-dimensional real vector space and $F^\bullet$ is a decreasing filtration on $V_\mathbb{C}$ satisfying the following transversality condition with its complex conjugate

$$F^p V_\mathbb{C} \oplus \overline{F^{k-p+1} V_\mathbb{C}} = V_\mathbb{C}.$$

Our goal is to reinterpret this transversality condition using the theory of vector bundles on the twistor line $\tilde{\mathbb{P}}^1$, which can be constructed as the solution set of $U^2 + V^2 + W^2 = 0$ in $\mathbb{P}^2_\mathbb{R}$. Other constructions of

$\tilde{\mathbb{P}}^1$ and the theory of vector bundles on $\tilde{\mathbb{P}}^1$ are discussed in detail in Section 2.3. The key points are:

(1) $\tilde{\mathbb{P}}^1/\mathbb{R}$ is a smooth projective curve with a canonical point $\infty_{\mathbb{C}} \in \tilde{\mathbb{P}}^1(\mathbb{C})$.

(2) There is an action of the circle group $U(1) = a + bi$, $a^2 + b^2 = 1$ on $\tilde{\mathbb{P}}^1$ fixing a unique closed point, the one underlying $\infty_{\mathbb{C}}$, and we may identify the completed local ring at $\infty_{\mathbb{C}}$ with $\mathbb{C}[[t]]$, where the coordinate t is chosen so that $U(1)$ acts on t by $z \cdot t = z^{-2}t$.

(3) There is a slope formalism for vector bundles on $\tilde{\mathbb{P}}^1$. Any vector bundle is a direct sum of stable bundles, any stable bundle has slope $\lambda \in \frac{1}{2}\mathbb{Z}$, and for any such λ there is a unique up-to-isomorphism stable bundle $\mathcal{O}_{\tilde{\mathbb{P}}^1}(\lambda)$ of slope λ (these are matched with the real isocrystals of Section 2.2.1).

Now, suppose given a real vector space V and a filtration $F^\bullet V_{\mathbb{C}}$ on its complexification. By the canonical lattice construction of Section 3.1 and Proposition 3.1.7(2), there is a unique promotion of $F^\bullet V_{\mathbb{C}}$ to a $U(1)$-equivariant lattice in $V_{\mathbb{C}((t))}$, $\mathcal{L} := \sum_i t^{-i}\mathbb{C}[[t]] \cdot F^i V_{\mathbb{C}}$. Let $(V \otimes_{\mathbb{R}} \mathcal{O}_{\tilde{\mathbb{P}}^1})_{\mathcal{L}}$ be the modification of $(V \otimes_{\mathbb{R}} \mathcal{O}_{\tilde{\mathbb{P}}^1})$ by the lattice $\mathcal{L}$ at ∞ (see Definition 2.5.1). By a result of Simpson [35],

Lemma 4.1.1. $(V, F^\bullet V_{\mathbb{C}})$ *is a weight k Hodge structure if and only if the modified vector bundle* $(V \otimes_{\mathbb{R}} \mathcal{O}_{\tilde{\mathbb{P}}^1})_{\mathcal{L}}$ *is semistable of slope $k/2$.*

Proof. Here we just recall why the modification is semistable if it is a weight k Hodge structure. First, observe that, by the constructions in Section 2.3, it suffices to check that the pullback of the modification to $\mathbb{P}^1$ is a direct sum of $\mathcal{O}(k)$. One can commute the pullback with the modification, so this is equivalent to modifying $V_{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ at ∞ by the Hodge filtration and at $-\infty$ by its complex conjugate. If we write $V_{\mathbb{C}} = \bigoplus h^{p,q}$, then the filtrations both break up as a direct sum over this decomposition, where the Hodge filtration on $h^{p,q}$ is concentrated in degree p and its conjugate in degree q. If we fix a basis for $h^{p,q}$, this further decomposes into modifications of one-dimensional trivial bundles along filtrations with these concentrations. The resulting modification is a line bundle of degree $p + q = k$; thus $\mathcal{O}(k)$, and we conclude. □

Thus, the canonical filtration realizes the category of $\mathbb{C}$-filtered real vector spaces as a full rigid tensor subcategory of $\mathbb{C}[[t]]$-latticed real vector spaces, and the condition to be a weight k Hodge structure can be formulated entirely in terms of the lattice. This motivates the next definition.

Definition 4.1.2. The category of *extended real Hodge structures* of weight k is the full subcategory of $\mathbb{C}[[t]]$-latticed real vector spaces (Definition 3.1.1) consisting of $(V, \mathcal{L})$ such that $(V \otimes_{\mathbb{R}} \mathcal{O}_{\tilde{\mathbb{P}}^1})_{\mathcal{L}}$ is semistable of slope $k/2$. The category of extended real Hodge structures is the full subcategory of $\mathbb{Z}$-graded $\mathbb{C}[[t]]$-latticed real vector spaces consisting of those whose degree k component is a weight k extended real Hodge structure.

By Lemma 4.1.1 and Proposition 3.1.7(2), the canonical lattice embeds the category of real Hodge structures as a full rigid tensor subcategory of extended real Hodge structures — the ones such that the lattice is preserved by $U(1)$. The functor BB of Section 3.1 attaches to any extended real Hodge structure $(V, \mathcal{L})$ a Hodge filtration on $V_{\mathbb{C}}$, recovering the Hodge filtration on real Hodge structures. In general the lattice contains more information.

4.2. ***Definition and examples of p-adic Hodge structures.*** Let $C/\mathbb{Q}_p$ be an algebraically closed nonarchimedean extension. In [18], Fargues and Fontaine constructed from C a 1-dimensional scheme over $\operatorname{Spec}\mathbb{Q}_p$, $\mathbb{F} = \mathbb{F}_{C^\flat,\mathbb{Q}_p}$, the Fargues–Fontaine curve, that behaves in every way like a smooth proper curve over $\mathbb{Q}_p$ except that it is of infinite type. The geometry of $\mathbb{F}$ encodes the relation between Fontaine's period rings. The construction is recalled in more detail in Section 2.4; the key points for this section are:

(1) There is a canonical closed point ∞_C and identification of the residue field $\kappa(\infty_C)$ with C. The completed local ring of ∞_C is the complete discrete valuation ring B_{dR}^+ and the complement of ∞_C is the spectrum of the principal ideal domain $B_{\mathrm{crys}}^{\varphi=1}$.

(2) If $C = \overline{K}^\wedge$ for a p-adic field K, then there is an action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathbb{F} = \mathbb{F}_{\mathbb{Q}_p,C^\flat}$ and ∞_C is the unique closed point fixed by $\mathrm{Gal}(\overline{K}/K)$; the induced action on the residue field C is the standard one, and $t \in B_{\mathrm{dR}}^+$, the "p-adic $2\pi i$" which generates the maximal ideal in B_{dR}^+, transforms by the p-adic cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$.

(3) There is a slope formalism for vector bundles on $\mathbb{F}$, and any semistable vector bundle has slope $\lambda \in \mathbb{Q}$ and is a direct sum of copies of the unique up-to-isomorphism stable bundle $\mathcal{O}_{\mathbb{F}}(\lambda)$ of slope λ (these are matched with the p-adic isocrystals of Section 2.2.2).

Our definition of a p-adic Hodge structure will be analogous to the definition of extended real Hodge structures but using $\mathbb{F}$ instead of $\tilde{\mathbb{P}}^1$.

Definition 4.2.1. The category of *p-adic Hodge structures* over C of weight $\lambda \in \mathbb{Q}$ is the full subcategory of B_{dR}^+-latticed $\mathbb{Q}_p$-vector spaces (Definition 3.1.1) consisting of $(V, \mathcal{L})$ such that $(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{F}})_{\mathcal{L}}$ — the modification of $V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{F}}$ by the lattice $\mathcal{L}$ at ∞_C as in Definition 2.5.1 — is semistable of slope $-\lambda/2$. The category of p-adic Hodge structures over C is the full subcategory of $\mathbb{Q}$-graded B_{dR}^+-latticed $\mathbb{Q}_p$-vector spaces consisting of those whose component of degree λ is a weight λ p-adic Hodge structure over C.

Remark 4.2.2. The convention on weights is chosen so that the Tate structure $\mathbb{Q}_p(k)$ of Example 4.2.5 has weight $-2k$.

Remark 4.2.3. If $K \subseteq C$ is a p-adic field and $F^\bullet V_K$ is a filtration of V_K, then we can promote it to a lattice $\mathcal{L}$ on $V_{B_{\mathrm{dR}}}$ using the canonical lattice construction. Using this, one could define an analog of real Hodge structures instead of extended real Hodge structures, however, outside of the minuscule case where the filtration and lattice are equivalent, this construction does not arise in any natural way that we are aware of — for example, Theorem B implies that, for a p-adic Hodge structure coming from geometry over a p-adic field, the lattice will never come from such a filtration unless the p-adic Hodge structure has complex multiplication.

The category $\mathrm{HS}(C)$ of p-adic Hodge structures is a full subcategory of the category of $\mathbb{Q}$-graded B_{dR}^+-latticed $\mathbb{Q}_p$-vector spaces, and is stable under the natural tensor product and dual functors. The forgetful functor

$$\omega_{\text{ét}} : \mathrm{HS}(C) \to \mathrm{Vect}(\mathbb{Q}_p)$$

is a faithful exact tensor functor. The functor BB from Section 3.1 induces a Hodge–Tate filtration F^i_{HT} on $\omega_{\text{ét}} \otimes C$, with

$$F^i_{\mathrm{HT}}(V, \mathcal{L}) = \text{image in } V_C \text{ of } (V_{B^+_{\mathrm{dR}}} \cap (F^i B_{\mathrm{dR}}) \cdot \mathcal{L})$$

It is a tensor functor, but we caution that it is not exact, even when restricted to p-adic Hodge structures — see Example 4.3.4 below.

Example 4.2.4. If X/C is a smooth proper rigid analytic variety then, by [5, Theorem 13.1], for each degree i, we obtain a B^+_{dR}-latticed $\mathbb{Q}_p$-vector space

$$(H^i_{\text{ét}}(X, \mathbb{Q}_p), \mathcal{L}^i_{\mathrm{dR}})$$

where $\mathcal{L}^i_{\mathrm{dR}}$ is a canonical deformation of $H^i_{\mathrm{dR}}(X)$ to B^+_{dR} embedded as a B^+_{dR}-lattice inside $H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}$. The induced filtration on $H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes C$ is the Hodge–Tate filtration. If $\mathfrak{X}/\mathcal{O}_C$ is a smooth proper formal model, then

$$(H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes \mathcal{O}_{\mathbb{F}})_{\mathcal{L}^i_{\mathrm{dR}}}$$

is semistable if and only if the isocrystal determined by the i-th crystalline cohomology of the special fiber is $i/2$-isotypic. In particular, if it is a p-adic Hodge structure it is of weight i, but in many cases it is not a p-adic Hodge structure, for example if X is an elliptic curve with ordinary reduction.

Example 4.2.5. For $k \in \mathbb{Z}$, the Tate p-adic Hodge structure of weight $-2k$ is

$$\mathbb{Q}_p(k) := (\mathbb{Q}_p \cdot t^k, B^+_{\mathrm{dR}}) \cong (\mathbb{Q}_p, F^{-k} B^+_{\mathrm{dR}}).$$

The Hodge–Tate filtration on $\mathbb{Q}_p(k) \otimes C$ is concentrated in degree k (i.e., the only nonzero grade is in degree k). For $k \geq 0$, $\mathbb{Q}_p(k)$ is isomorphic to the dual of the p-adic Hodge structure $(H^{2k}_{\text{ét}}(\mathbb{P}^k, \mathbb{Q}_p), \mathcal{L}^{2k}_{\mathrm{dR}})$ of the previous example. We have natural identifications $\mathbb{Q}_p(a) \otimes \mathbb{Q}_p(b) = \mathbb{Q}_p(a+b)$ and $\mathbb{Q}_p(k)^\vee = \mathbb{Q}_p(-k)$.

It is clear that any 1-dimensional B^+_{dR}-latticed $\mathbb{Q}_p$-vector space is isomorphic to $\mathbb{Q}_p(k)$ for some k, so these give all of the one-dimensional p-adic Hodge structures.

4.3. ***Structural properties of extended real and p-adic Hodge structures.*** We work in one of two situations:

<u>real</u>: $K = \mathbb{R}$, $C = \mathbb{C}$, $X = \tilde{\mathbb{P}}^1$, $B^+ = \mathbb{C}[[t]]$, $B = \mathbb{C}((t))$, $\infty = \infty_{\mathbb{C}}$.

<u>p-adic</u>: $K = \mathbb{Q}_p$, $C/\mathbb{Q}_p$ is an algebraically closed nonarchimedean extension, $X = \mathbb{F}_{C^\flat}$, $B^+ = B^+_{\mathrm{dR}}$, $B = B_{\mathrm{dR}}$, $\infty = \infty_C$.

In the first case, by a Hodge structure we mean an extended real Hodge structure. In the second case, by a Hodge structure we mean a p-adic Hodge structure. We write $\mathrm{HS}(C)$ for the category of Hodge structures. In both cases, BB induces a functor from B^+-latticed K-vector spaces to C-filtered K-vector spaces. On extended real Hodge structures, this is the Hodge filtration. On p-adic Hodge structures, this is the Hodge–Tate filtration (we emphasize this distinction because the Hodge filtration will arise at a different point in the p-adic theory — see Section 5). We caution that in both cases BB is not exact on $\mathrm{HS}(C)$; see Example 4.3.4.

Lemma 4.3.1. *The functor $(V, \mathcal{L}) \mapsto (V \otimes \mathcal{O}_X)_{\mathcal{L}}$ from B^+-latticed K-vector spaces to vector bundles on X is exact.*

Proof. Let $f : (V, \mathcal{L}) \to (V', \mathcal{L}')$ be a morphism. We write

$$T = f(V), \quad \mathcal{M}_1 = f(\mathcal{L}) \subset T_B, \quad \mathcal{M}_2 = f(V)_B \cap \mathcal{L}' \subset T_B,$$

so that, by assumption, $\mathcal{M}_1 \subseteq \mathcal{M}_2$. The inclusion of lattices induces an exact sequence of coherent sheaves on X

$$0 \to (T \otimes \mathcal{O}_X)_{\mathcal{M}_1} \to (T \otimes \mathcal{O}_X)_{\mathcal{M}_2} \to \infty_*(\mathcal{M}_1/\mathcal{M}_2) \to 0.$$

and we deduce that a morphism is strict if and only if the image of the induced map of vector bundles is saturated. The exactness is then immediate. □

Lemma 4.3.2. *The category of Hodge structures is abelian.*

Proof. It suffices to show that the category of Hodge structures of a fixed weight λ is abelian, and the only nontrivial thing to show is that the morphisms are all strict, i.e., that the coimage is equal to the image.

We can see this equality through slope considerations: proceeding as in the previous lemma, let $f : (V, \mathcal{L}) \to (V', \mathcal{L}')$ be a morphism of Hodge structures of weight λ and write

$$T = f(V), \quad \mathcal{M}_1 = f(\mathcal{L}) \subset T_B, \quad \mathcal{M}_2 = f(V)_B \cap \mathcal{L}' \subset T_B.$$

We have $\mathcal{M}_1 \subset \mathcal{M}_2$ by assumption, and we want to deduce that $\mathcal{M}_1 = \mathcal{M}_2$.

We have an exact sequence of coherent sheaves on X:

$$0 \to (T \otimes \mathcal{O}_X)_{\mathcal{M}_1} \to (T \otimes \mathcal{O}_X)_{\mathcal{M}_2} \to \infty_*(\mathcal{M}_2/\mathcal{M}_1) \to 0.$$

Thus, the slope λ_2 of $(T \otimes \mathcal{O}_X)_{\mathcal{M}_2}$ is at least as large as the slope λ_1 of $(T \otimes \mathcal{O}_X)_{\mathcal{M}_1}$, with equality if and only if $\mathcal{M}_1 = \mathcal{M}_2$. Now, the map f realizes $(T \otimes \mathcal{O}_X)_{\mathcal{M}_1}$ as a quotient of $(V \otimes \mathcal{O}_X)_{\mathcal{L}}$, which is semistable of slope $\lambda/2$; thus $\lambda_1 \geq \lambda/2$. On the other hand, $(T \otimes \mathcal{O}_X)_{\mathcal{M}_2}$ is a subbundle of $(V' \otimes \mathcal{O}_X)_{\mathcal{L}'}$, which is also semistable of slope $\lambda/2$, so its slope is $\lambda_2 \leq \lambda/2$. We conclude, having shown that

$$\lambda/2 \leq \lambda_1 \leq \lambda_2 \leq \lambda/2.$$

□

Lemma 4.3.3. *Suppose that*

$$0 \to (V_1, \mathcal{L}_1) \to (V_2, \mathcal{L}_2) \to (V_3, \mathcal{L}_3) \to 0$$

is an exact sequence of B^+-latticed K-vector spaces. If any two of the terms are Hodge structures of the same weight λ, then so is the third.

Proof. By Lemma 4.3.1, we obtain an exact sequence of vector bundles

$$0 \to (V_1 \otimes \mathcal{O}_X)_{\mathcal{L}_1} \to (V_2 \otimes \mathcal{O}_X)_{\mathcal{L}_2} \to (V_3 \otimes \mathcal{O}_X)_{\mathcal{L}_3} \to 0.$$

The result follows as semistable bundles of a fixed slope satisfy the two out of three property in a short exact sequence. □

Example 4.3.4. The B^+-latticed K-vector space $\left(K^2,\, B^+e_1 + B^+\left(\frac{1}{t}e_1 + e_2\right)\right)$ is a Hodge structure. Indeed, this follows from Lemma 4.3.3 since it is an extension of the trivial Hodge structure by itself in the category of B^+-latticed K-vector spaces. Note that the exact sequence defining this extension does not remain exact after applying the filtration functor (compare Example 3.1.4).

Theorem 4.3.5. *The category of* $\mathrm{HS}(C)$ *of Hodge structures is a connected neutral Tannakian category over K with fiber functor* $\omega_{\acute{e}t} : (V, \mathcal{L}) \mapsto V$.

Proof. In Lemma 4.3.2 we have shown the category is abelian, and it is immediate from the definition of the tensor product and dual that it is neutral Tannakian over K (with the forgetful fiber functor to $\mathrm{Vect}(K)$). It remains to show it is connected: by [14, Corollary 2.22], it suffices to show that if V is an object such that the strictly full subcategory $\langle V\rangle_\oplus$ whose objects are those isomorphic to subquotients of $V^{\oplus k}$ for some k is stable under tensor product then V is trivial. Suppose V is such an object and write a and b for the minimum and maximum of the set $\{i \mid \mathrm{Gr}^i(V_C) \neq 0\}$. Then, for any object V' in $\langle V\rangle_\oplus$, $\mathrm{Gr}^i(V'_C) \neq 0$ implies $i \in [a, b]$.

Now, if $a = b = 0$, then V is trivial by Corollary 3.1.9 and we are done. Otherwise, either $a < 0$ or $b > 0$ — the arguments in the two cases are parallel, so we treat just the case $b > 0$. The filtration is a tensor functor, so $F^{2b}(V^{\otimes 2}) = (F^b V)^{\otimes 2}$ and $F^{2b+1}(V^{\otimes 2}) = 0$. We conclude that $\mathrm{Gr}^{2b}(V^{\otimes 2}) \neq 0$ and thus, by the considerations of the previous paragraph, $V^{\otimes 2} \notin \langle V\rangle_\oplus$ since $2b \notin [a, b]$. □

4.4. *Invariants of Hodge structures.* We continue with the notation of Section 4.3. Given a Hodge structure V over C, the *Mumford–Tate* group $\mathrm{MT}(V) = \mathrm{Aut}^\otimes(\omega_{\acute{e}t}|_{\langle V\rangle})$ is the Tannakian structure group of the Tannakian subcategory $\langle V\rangle$ generated by V. It is a closed subgroup of $\mathrm{GL}(V)$, and by Theorem 4.3.5 is connected.

Example 4.4.1. (1) For V the Hodge structure of Example 4.3.4, $\mathrm{MT}(V) = \mathbb{G}_a$.

(2) For $\mathbb{Q}_p(k)$ the Tate p-adic Hodge structure of Example 4.2.5, $\mathrm{MT}(\mathbb{Q}_p(k)) = \mathbb{G}_m$ if $k \neq 0$ and is trivial for $k = 0$.

We extend this definition to the category G–$\mathrm{HS}(C)$ of Hodge structures with G-structure (see Section 2.1.1) for G/K a connected linear algebraic group. An object of G–$\mathrm{HS}(C)$ is an exact tensor functor such that $\omega_{\acute{e}t} \circ \mathcal{G}$ is isomorphic to ω_{std}, and we define $\mathrm{MT}(\mathcal{G})$ to be the automorphism group of $\omega_{\acute{e}t}$ restricted to the Tannakian subcategory of $\mathrm{HS}(C)$ generated by the essential image of $\mathcal{G}$. As in Section 2.1.2, $\mathcal{G}$ has a canonical refinement to a Hodge structure with $\mathrm{MT}(\mathcal{G})$-structure, and a choice of identification $\mathrm{triv}_{\acute{e}t} : \omega_{\acute{e}t} \circ \mathcal{G} = \omega_{\mathrm{std}}$ identifies $\mathrm{MT}(\mathcal{G})$ with a closed subgroup of G.

A natural exact tensor functor ω_{Isoc} from $\mathrm{HS}(C)$ to the category of isocrystals Kt_K of Section 2.2.2

can be defined as follows. Recall the simple objects D_λ of Kt_K defined in Section 2.2, and in the real case let $D_\lambda = 0$ for $\lambda \notin \frac{1}{2}\mathbb{Z}$. Denote by $\mathcal{D}_\lambda$ the division algebra $\mathrm{End}(D_{-\lambda}) = \mathrm{End}(\mathcal{O}_X(\lambda))$. Then,

$$\omega_{\mathrm{Isoc}}\Big(\bigoplus_{w\in\mathbb{Q}} (V_w, \mathcal{L}_w)\Big) = \bigoplus_{w\in\mathbb{Q}} \mathrm{Hom}\big(\mathcal{O}(w/2), (V_w \otimes \mathcal{O}_X)_{\mathcal{L}_w}\big) \otimes_{\mathcal{D}_{w/2}} D_{-w/2}.$$

Thus, to any $\mathcal{G} \in G$–$\mathrm{HS}(C)$, we can associate the G-isocrystal $\omega_{\mathrm{Isoc}} \circ \mathcal{G}$. Isomorphism classes of G-isocrystals are classified by the Kottwitz set $B(G)$ (see Section 2.2.2) and we write $[b]_{\mathcal{G}}$ for the point classifying $\omega_{\mathrm{Isoc}} \circ \mathcal{G}$. By definition of $\mathrm{HS}(C)$, the slope homomorphism is central in $\mathrm{MT}(\mathcal{G})$ (up to multiplication by -2 it is equal to the weight homomorphism determining the $\mathbb{Q}$-grading). In particular, if $\mathcal{G} = \mathrm{MT}(\mathcal{G})$, $[b]_{\mathcal{G}}$ is basic. We will often assume this latter condition or equivalently that the weight morphism is central since it can always be arranged after a reduction of structure group.

There is also a natural exact tensor functor ω_{bl} from B^+-latticed vector spaces to $\mathrm{Vect}^{\mathrm{bl}}(B)$ (see Section 3) defined by

$$\omega_{\mathrm{bl}}((V, \mathcal{L})) \mapsto (V_B, \mathcal{L}, V \otimes B^+).$$

We say $\mathcal{G} \in G$–$\mathrm{HS}(C)$ is *good* if $\omega_{\mathrm{bl}} \circ \mathcal{G}$ is a good bilatticed G-bundle, in which case we write $[\mu]_{\mathcal{G}} := [\mu]_{\omega_{\mathrm{bl}}\circ\mathcal{G}}$ for the classifying conjugacy class of $\overline{K}$-cocharacters. By Theorem 3.2.6, $\mathcal{G}$ is good if and only if $\mathrm{BB} \circ \mathcal{G} : \mathrm{Rep}\, G \to \mathrm{Vect}^f(C)$ is a filtered G-bundle (i.e., is exact), and then $[\mu]_{\mathcal{G}}$ is also the type of this filtered G-bundle.

Remark 4.4.2. One also obtains a filtered G-bundle of type $[\mu^{-1}]_{\mathcal{G}}$ through BB_1, which is a filtration on $\omega_{\mathrm{Isoc}} \otimes C$. In the p-adic case, we will treat this perspective in Section 5.

Example 4.4.3. (1) If G is reductive, any $\mathcal{G} \in G$–$\mathrm{HS}(C)$ is good.

(2) Example 4.3.4 and Example 3.1.4 show that the $\mathbb{G}_a$-Hodge structure of Example 4.4.1 is not good.

When $G = \mathrm{MT}(\mathcal{G})$ or, more generally, $[b]_{\mathcal{G}}$ is basic, the invariant $[\mu]_{\mathcal{G}}$ determines $[b]_{\mathcal{G}}$: we explain this only in the p-adic case. Recall that Kottwitz ([23, §6]) has defined for any connected reductive group $G/\mathbb{Q}_p$ and conjugacy class $[\mu]$ of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ a subset $B(G, [\mu]) \subset B(G)$, and that any such subset contains a unique *basic* element (i.e., an element such that the slope homomorphism is central). We extend the definition of $B(G, [\mu])$ to any connected linear algebraic group as follows: write U for the unipotent radical of G. As explained in Section 2.2.3, since isocrystals are a semisimple category, $B(G) = B(G/U)$. The projection $G \to G/U$ also identifies the conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ with those of $(G/U)_{\overline{\mathbb{Q}}_p}$. Thus we may declare $B(G, [\mu]) = B(G/U, [\mu])$. For G nonreductive we continue to call an element of $B(G)$ basic if the slope morphism is central; thus, although $B(G/U, [\mu])$ has a unique basic element, this element may not be basic in G; thus $B(G, [\mu])$ either has one or zero basic elements.

The following should also be true in the real case, but we only prove it in the p-adic case (due to the reference to [8]):

Theorem 4.4.4. *In the p-adic case, if $\mathcal{G} \in G$–$\mathrm{HS}(C)$ is good and the weight morphism is central (for instance, if $G = \mathrm{MT}(\mathcal{G})$) then $[b]_{\mathcal{G}}$ is the unique basic element in $B(G, [\mu^{-1}]_{\mathcal{G}})$.*

Proof. Recall from above that the weight morphism being central is equivalent to $[b]_{\mathcal{G}}$ being basic. Then, for G reductive, this is [8, Proposition 3.5.3]: note that, because of our choice of ordering of the lattices when defining the associated bilatticed vector space, the relevant double-coset for applying Proposition 3.5.3 of [8] is $G(B_{\mathrm{dR}}^{+})\mu^{-1}(t)G(B_{\mathrm{dR}}^{+})$. In the definition of Schubert cells in [8, pp. 683–684], there is also an inverse appearing on the cocharacter, so this is exactly the double coset associated to the Schubert cell appearing in [8, Proposition 3.5.3].

Now, because $\mathcal{G}$ is good, the type of $\omega_{\mathrm{bl}} \circ \mathcal{G}$ is the same as the type of $\omega_{\mathrm{bl}} \circ \mathcal{G}^{\mathrm{ss}}$ where $\mathcal{G}^{\mathrm{ss}}$ denotes the composed functor $\operatorname{Rep} G/U \to \operatorname{Rep} G \xrightarrow{\mathcal{G}} \mathrm{HS}(C)$ — indeed, the map on isomorphism classes is induced by the quotient map on double cosets in Section 3. □

4.5. *Hodge–Tate lines.* In this section we assume we are in the p-adic case, and refer to the real case only for analogy. It would be possible to proceed symmetrically, but we wish to use the term Hodge–Tate lines and reserve the term Hodge lines for a related but distinct concept in Section 5.

A powerful tool in the study of Mumford–Tate groups in classical Hodge theory is through a characterization using Hodge tensors. We now give a similar characterization in the p-adic setting. In classical Hodge theory one typically works in the polarizable case, but here it is necessary to adjust slightly to allow for nonreductive structure groups: indeed, because our Mumford–Tate groups are not necessarily reductive, if $\mathcal{G} \in G\text{–}\mathrm{HS}(C)$ for G reductive, $\mathrm{MT}(\mathcal{G})$ may not be observable in G, i.e., it may not be realizable as the stabilizer of a vector in a representation. Thus we must consider all even integer weights instead of only weight zero tensors.

Suppose V is a p-adic Hodge structure. For $k \in \mathbb{Z}$, the space of weight $2k$ Hodge–Tate lines in V is

$$\mathrm{HT}^{2k}(V) := \left(\mathrm{Hom}_{\mathrm{HS}(C)}(\mathbb{Q}_p(-k), V_{2k}) \setminus \{0\}\right) / \mathbb{Q}_p^{\times}.$$

By evaluation, $\mathrm{HT}^{2k}(V)$ is identified with a projective subspace of $\mathbb{P}(V_{2k})$.

Theorem 4.5.1. *Suppose G is a connected linear algebraic group and $\mathcal{G} \in G\text{–}\mathrm{HS}(C)$ is equipped with a trivialization $\omega_{\acute{e}t} \circ \mathcal{G} = \omega_{\mathrm{std}}$. Then $\mathrm{MT}(\mathcal{G}) \leq G$ is the subgroup of G preserving every line $\ell \in \mathrm{HT}^{2k}(\mathcal{G}(V))$ for every $V \in \operatorname{Rep} G$ and $k \in \mathbb{Z}$.*

Proof. It is immediate that $\mathrm{MT}(\mathcal{G})$ stabilizes these lines, since they underlie one-dimensional sub-p-adic Hodge structures in $\mathcal{G}(V)$. On the other hand, $\mathrm{MT}(\mathcal{G})$ is a closed subgroup of G, so there is some line in some representation of G such that $\mathrm{MT}(\mathcal{G})$ is equal to the stabilizer of that line. In particular, this line corresponds to a 1-dimensional sub-p-adic Hodge structure, and by Example 4.2.5, it must be a Tate p-adic Hodge structure. □

Remark 4.5.2. Why do we not assume a condition that implies reductivity of Mumford–Tate groups? Recall that in classical Hodge theory, the Mumford–Tate groups of $\mathbb{Q}$-Hodge structures arising from algebraic geometry are always reductive. This follows from the existence of a polarization: indeed, if $(V, F^{\bullet}V_{\mathbb{C}})$ is a $\mathbb{Q}$-Hodge structure, then a polarization on V can be defined as a bilinear form $\langle\, , \rangle$ on V (alternating or symmetric depending on the parity of the weight) such that the map $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$

defining the Hodge structure (for $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ the Deligne torus) factors through the similitude group G of $\langle\, , \rangle$ and such that, for c denoting complex conjugation on $G^{\mathrm{ad}}(\mathbb{C})$, $\mathrm{Ad}h(i) \circ c$ is a Cartan involution for $G^{\mathrm{ad}}(\mathbb{C})$ (note $h(i)$ is the Weil operator). Thus the involution $\mathrm{Ad}h(i) \circ c$ defines a compact real form of $\mathrm{MT}(V)/Z(G) \cap \mathrm{MT}(V)$. We conclude that the unipotent radical of this real form and thus also of the Mumford–Tate group itself is trivial, and thus the Mumford–Tate group is reductive.

This can be reinterpreted from the perspective adopted at the beginning of this section: first, giving the inner product is equivalent to refining V to a G-Hodge structure $\mathcal{G}$ where $G \subseteq \mathrm{GL}_V$ is the similitude group. Now, the involution $\mathrm{Ad}h(i) \circ c$ defines an inner form that can be checked to be equivalent to the automorphism group of the induced real G-isocrystal.

This interpretation can be transported directly to the p-adic setting: Suppose $G/\mathbb{Q}_p$ is a connected reductive group and $\mathcal{G} \in G\text{–}\mathrm{HS}(C)$. The automorphism group $J(\mathbb{Q}_p)$ of the G-isocrystal $\omega_{\mathrm{Isoc}} \circ \mathcal{G}$ consists of the $\mathbb{Q}_p$-points of an inner form J of G (the inner form G_b for b a representative of $[b]_{\mathcal{G}}$). If $J/Z_J(\mathbb{Q}_p)$ is compact, then the $\mathbb{Q}_p$-points of the corresponding inner form of $\mathrm{MT}(\mathcal{G})/\mathrm{MT}(\mathcal{G}) \cap Z(G)$ are also compact (as a closed subgroup of a compact group); thus this inner form has no unipotent radical so neither does $\mathrm{MT}(\mathcal{G})$. Note any choice of element $b \in G(\breve{\mathbb{Q}}_p)$ representing $[b]_{\mathcal{G}}$ plays the role of the Weil operator $h(i)$.

If $G = \mathrm{GL}_n$, then $J(\mathbb{Q}_p)$ is compact mod center exactly when b corresponds to an isoclinic isocrystal of slope a/n for $(a, n) = 1$ — indeed, the endomorphisms of this isocrystal are the central simple algebra of Brauer class a/n, which is a division algebra exactly when a is coprime to n. Recalling that $[b]_{\mathcal{G}}$ is uniquely determined by $[\mu]_{\mathcal{G}}$, this occurs exactly when, writing $[\mu]_{\mathcal{G}} = (a_1, a_2, \dots, a_n)$, $\sum a_i$ is coprime to n. Indeed, in this case, the relation between $[\mu]$ and b is just that the Newton polygon b is the straight line starting at $(0, 0)$ and ending at the endpoint of the Hodge polygon.

Unfortunately, as explained in [26], up to the obvious manipulations, this captures all possibilities. Thus, there are no compact inner forms that arise in this way from similitude groups (outside of the exceptional overlap with groups of type A_1), so having a bilinear pairing, e.g., from a polarization, does not help at all!

5. Admissible pairs

In this section we work in the p-adic context (see 4.3), so $C/\mathbb{Q}_p$ is an algebraically closed nonarchimedean extension. Suppose $(V, \mathcal{L}_{\mathrm{dR}})$ is a p-adic Hodge structure and let $W = \omega_{\mathrm{Isoc}}(V)$. By construction, there is a canonical identification $W_{B^+_{\mathrm{dR}}} = \mathcal{L}_{\mathrm{dR}}$ and thus $V_{B_{\mathrm{dR}}} = W_{B_{\mathrm{dR}}}$. Letting $\mathcal{L}_{\text{ét}} := V_{B^+_{\mathrm{dR}}}$, we obtain a B^+_{dR}-latticed isocrystal $(W, \mathcal{L}_{\text{ét}})$, and we can recover V as

$$V = H^0(\mathbb{F}_{C^\flat}, \mathcal{E}(W)_{\mathcal{L}_{\text{ét}}}).$$

This motivates the study of a semilinear category of *admissible pairs* over C, to be defined precisely below, and the above construction will identify $\mathrm{HS}(C)$ with a natural subcategory of admissible pairs. In the introduction, we explained that admissible pairs were a natural toy category of cohomological motives because, in the cohomological setting, their rationality reflects the rationality of defining equations. However, the fact that we can attach an admissible pair to any p-adic Hodge structure means this local, p-adic setting is much better behaved than its global, archimedean analog — to any p-adic Hodge structure we can attach a motivic object, so we are not left to wonder which ones come from motives!

The rest of this section develops the essential properties of admissible pairs. Many of the proofs of structural results are the same or very similar to the proofs for extended real or p-adic Hodge structures given in Section 4, so we move more quickly through the basic structural material.

Remark 5.0.1. It would be possible to continue as in Section 4 by developing a symmetric theory for both p-adic and extended real Hodge structures, but this quickly loses touch with the aspects that are classically interesting in real Hodge theory (one would even arrive ultimately at a trivial "transcendence" result in this case; see Remark 6.2.2). However, although the symmetric treatment served us well in Section 4 as a tool for better understanding the arguments in the p-adic case, in this section it will be clearer to consider only the p-adic case and focus instead on the connections with classical notions in p-adic Hodge theory.

5.1. *Definitions, examples, and first properties.*

Definition 5.1.1. (1) A B_{dR}^{+}*-latticed isocrystal* is a pair $(W, \mathcal{L})$ where W is an isocrystal and $\mathcal{L} \subseteq W_{B_{\mathrm{dR}}}$ is a B_{dR}^{+}-lattice. A morphism $(W, \mathcal{L}_{\text{ét}}) \to (W', \mathcal{L}'_{\text{ét}})$ is a morphism of isocrystals $f : W \to W'$ such that $f(\mathcal{L}_{\text{ét}}) \subseteq \mathcal{L}'_{\text{ét}}$. It is strict if $f(\mathcal{L}_{\text{ét}}) = f(W)_{B_{\mathrm{dR}}} \cap \mathcal{L}'_{\text{ét}}$. A complex of B_{dR}^{+}-latticed isocrystals is exact if it is exact as a complex of isocrystals and each morphism is strict.

(2) An *admissible pair* is a B_{dR}^{+}-latticed isocrystal $(W, \mathcal{L}_{\text{ét}})$ such that $\mathcal{E}(W)_{\mathcal{L}_{\text{ét}}}$ — the modification of the vector bundle $\mathcal{E}(W)$ on $\mathbb{F}_{C^\flat}$ associated to W by the lattice $\mathcal{L}_{\text{ét}}$ at ∞ as in Definition 2.5.1 — is semistable of slope zero.

(3) An admissible pair $(W, \mathcal{L})$ is *basic* if the isotypic decomposition $W = \bigoplus_{\lambda \in \mathbb{Q}} W_\lambda$ induces a decomposition $\mathcal{L} = \bigoplus \mathcal{L}_\lambda$, $\mathcal{L}_\lambda := W_\lambda \otimes B_{\mathrm{dR}} \cap \mathcal{L}$.

We write $\mathrm{AdmPair}(C)$ for the category of admissible pairs and $\mathrm{AdmPair}^{\mathrm{basic}}(C)$ for the full subcategory of basic admissible pairs.

The category $\mathrm{AdmPair}(C)$ has realizations to isocrystals and $\mathbb{Q}_p$-vector spaces, which in the geometric case (Example 5.1.5) correspond to rational crystalline and p-adic étale cohomology. We begin by defining these realization functors and the additional structures on them, then show that $\mathrm{AdmPair}(C)$ is a neutral Tannakian category over $\mathbb{Q}_p$.

The isocrystalline realization

$$\omega_{\mathrm{Isoc}} : \mathrm{AdmPair}(C) \to \mathrm{Isoc}, \quad (W, \mathcal{L}) \mapsto W, \tag{2}$$

is an exact tensor functor. By abuse of notation, we also denote the composition with the forgetful functor $\mathrm{Isoc} \to \mathrm{Vect}(\breve{\mathbb{Q}}_p)$ by ω_{Isoc}. The étale realization is

$$\omega_{\text{ét}} : \mathrm{AdmPair}(C) \to \mathrm{Vect}(\mathbb{Q}_p), \quad (W, \mathcal{L}) \mapsto H^0(\mathbb{F}_{C^\flat}, \mathcal{E}(W)_{\mathcal{L}}), \tag{3}$$

where $\mathcal{E}(W)_{\mathcal{L}}$ is the modified vector bundle on the Fargues–Fontaine curve $\mathbb{F}_{C^\flat}$. This is also an exact tensor functor, moreover we will see in Theorem 5.1.6 that AdmPair(C) is a neutral Tannakian category over $\mathbb{Q}_p$ with fiber functor $\omega_{\text{ét}}$.

If $(W, \mathcal{L})$ is an admissible pair, the B_{dR}-linear extension of the restriction morphism

$$\omega_{\text{ét}}(W) = H^0(\mathbb{F}_{C^\flat}, \mathcal{E}(W)_{\mathcal{L}}) \to H^0(\operatorname{Spec} B_{\mathrm{dR}}, \mathcal{E}(W)_{\mathcal{L}}|_{\operatorname{Spec} B_{\mathrm{dR}}}) = W \otimes B_{\mathrm{dR}}$$

is an isomorphism $\omega_{\text{ét}}(W) \otimes B_{\mathrm{dR}} \cong \omega_{\mathrm{Isoc}}(W) \otimes B_{\mathrm{dR}}$. These isomorphisms are functorial and give a canonical de Rham comparison isomorphism

$$c_{\mathrm{dR}} : \omega_{\text{ét}} \otimes B_{\mathrm{dR}} \xrightarrow{\sim} \omega_{\mathrm{Isoc}} \otimes B_{\mathrm{dR}} \tag{4}$$

between B_{dR}-valued fiber functors on AdmPair(C).

The isocrystalline and étale realization functors on AdmPair(C) can be enriched to land in B^+_{dR}-latticed vector spaces by the étale and de Rham lattice functors, respectively given by

$$\omega_{\mathcal{L}_{\text{ét}}}(W, \mathcal{L}) = \mathcal{L}, \qquad \omega_{\mathcal{L}_{\mathrm{dR}}}(W, \mathcal{L}) = c_{\mathrm{dR}}^{-1}(W \otimes B^+_{\mathrm{dR}}).$$

Remark 5.1.2. We call $\mathcal{L}$ the étale lattice because it is the image of $\omega_{\text{ét}}(W, \mathcal{L}) \otimes B^+_{\mathrm{dR}}$ under c_{dR}. The term "de Rham lattice" matches the terminology in Section 4 via the functor ω_{HS} of Theorem 5.1.6; in the geometric setting (Example 5.1.5) the de Rham lattice is a canonical deformation of de Rham cohomology.

Using the Białynicki-Birula functor from B^+_{dR}-latticed vector spaces to C-filtered vector spaces, the étale lattice gives the Hodge filtration $F^\bullet_{\mathrm{Hdg}}$ on W_C and the de Rham lattice gives the Hodge–Tate filtration $F^\bullet_{\mathrm{HT}}$ on $\omega_{\text{ét}}(W, \mathcal{L})_C$. Specifically, for $(W, \mathcal{L}) \in$ AdmPair(C), the Hodge filtration is

$$F^i_{\mathrm{Hdg}} = \text{image in } W_C \text{ of } \left(W_{B^+_{\mathrm{dR}}} \cap (F^i B_{\mathrm{dR}}) \cdot \mathcal{L}\right)$$

and the Hodge–Tate filtration is

$$F^i_{\mathrm{HT}} = \text{image in } \omega_{\text{ét}}(W, \mathcal{L})_C \text{ of } \left(\omega_{\text{ét}}(W, \mathcal{L})_{B^+_{\mathrm{dR}}} \cap (F^i B_{\mathrm{dR}}) \cdot W_{B^+_{\mathrm{dR}}}\right).$$

While the Hodge and Hodge–Tate filtrations give tensor functors valued in $\mathrm{Vect}^f(C)$, we caution that they are not exact (by essentially the same computation as Example 4.3.4).

Lemma 5.1.3. *For an admissible pair $(W, \mathcal{L})$, the following are equivalent.*

(1) *$(W, \mathcal{L})$ is trivial, i.e., isomorphic to a direct sum of $(\breve{\mathbb{Q}}_p, B^+_{\mathrm{dR}})$.*

(2) *The Hodge filtration on W_C is trivial.*

(3) *The Hodge–Tate filtration on $\omega_{\text{ét}}(W, \mathcal{L})_C$ is trivial.*

Proof. The type of the Hodge–Tate filtration is the inverse of the type of the Hodge filtration, as follows from Lemma 3.2.3, so (2) holds if and only if (3) holds. It is immediate that (1) implies (2). To see (2) implies (1), note that if the Hodge filtration is trivial then the lattice is trivial by Corollary 3.1.9. Then $\mathcal{E}(W)_{\mathcal{L}} = \mathcal{E}(W)$, so admissibility implies the isocrystal W is also trivial. $\square$

Before continuing with the general structure of AdmPair(C), we give examples of admissible pairs.

Example 5.1.4. Recall from Section 2.2 that, for k in $\mathbb{Z}$, D_{-k} denotes the isocrystal $\breve{\mathbb{Q}}_p$ with Frobenius acting by p^{-k}. We define the Tate admissible pair

$$\breve{\mathbb{Q}}_p(k) := (D_{-k}, F^k B_{\mathrm{dR}}^+ \cdot D_{-k}).$$

For ω_{HS} as in Theorem 5.1.6 below, we have a canonical identification $\omega_{\mathrm{HS}}(\breve{\mathbb{Q}}_p(k)) = \mathbb{Q}_p(k)$, for $\mathbb{Q}_p(k)$ the Tate p-adic Hodge structure of Example 4.2.5. The Hodge filtration on $\breve{\mathbb{Q}}_p(k) \otimes C$ is concentrated in degree $-k$.

Since every one-dimensional isocrystal is isoclinic, every one-dimensional admissible pair is basic. It follows from Example 4.2.5 and Theorem 5.1.6 that, up to isomorphism, the Tate admissible pairs give all one-dimensional admissible pairs.

Example 5.1.5. If $\mathfrak{X}/\mathcal{O}_C$ is a smooth proper formal scheme with generic fiber X, a rigid analytic variety over C, the results of [5] provide cohomological admissible pairs $(H^i_{\mathrm{crys}}(\mathfrak{X}_\kappa/W(\kappa))[1/p], \mathcal{L}_{\text{ét}})$ where $\mathcal{L}_{\text{ét}} = H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}^+$. The associated B_{dR}^+-latticed $\mathbb{Q}_p$-vector space obtained by applying $(\omega_{\text{ét}}, \omega_{\mathcal{L}_{\mathrm{dR}}})$ is that of Example 4.2.4.

The obvious tensor and dual on B_{dR}^+-latticed isocrystals preserve admissible pairs, and we have:

Theorem 5.1.6. AdmPair(C) *is a connected neutral Tannakian category with fiber functor $\omega_{\acute{e}t}$. It is equivalent to the isogeny category of rigidified Breuil–Kisin–Fargues modules* $\mathrm{BKF}^\circ_{\mathrm{rig}}$ *of* [2]. *The functor $\omega^{\mathcal{L}}_{\acute{e}t} = (\omega_{\acute{e}t}, \omega_{\mathcal{L}_{\mathrm{dR}}})$ induces an equivalence*

$$\omega_{\mathrm{HS}} : \mathrm{AdmPair}^{\mathrm{basic}}(C) \xrightarrow{\sim} \mathrm{HS}(C).$$

The fiber functor $\omega_{\text{ét}}$ on HS(C) (with the Hodge–Tate filtration) defined in the previous section is canonically identified with $\omega_{\text{ét}}$ (with the Hodge–Tate filtration) on basic admissible pairs via ω_{HS}, so that the overlap of notation will cause no confusion.

Proof. By [2, Theorem 3.19], the category $\mathrm{BKF}^\circ_{\mathrm{rig}}$ is equivalent to the category of quadruples $(\mathcal{F}, \mathcal{F}', \beta, \alpha)$ where $\mathcal{F}$ and $\mathcal{F}'$ are vector bundles on $\mathbb{F}$ with $\mathcal{F}$ trivial (equivalently, semistable of slope zero), $\alpha : \mathcal{F}|_{\mathbb{F}\backslash\infty_C} \xrightarrow{\sim} \mathcal{F}'|_{\mathbb{F}\backslash\infty_C}$, and $\beta : \bigoplus \mathrm{gr}^\lambda \mathcal{F}' \xrightarrow{\sim} \mathcal{F}'$ (where the graded pieces are for the slope filtration). There is a natural functor from AdmPair(C) to this category

$$(W, \mathcal{L}_{\text{ét}}) \mapsto (\mathcal{E}(W)_{\mathcal{L}}, \mathcal{E}(W), \alpha_{\mathrm{can}}, \beta_{\mathrm{can}})$$

where α_{can} is the isomorphism given by the modification construction and β_{can} is the canonical isomorphism of $\mathcal{E}(W)$ with its slope graded. It is an exercise in the definitions to verify this is an equivalence.

The remaining properties claimed for AdmPair(C) then follow from the corresponding properties established in [2]. However, we can also justify them directly as in Section 4. Indeed, the proof that the category is abelian is almost identical to the proof of Lemma 4.3.2, using that the modifications in each graded piece are semistable of the same slope. That it is neutral Tannakian is then clear, and the connectedness follows as in the proof of Theorem 4.3.5 using Lemma 5.1.3.

It remains to see that $\omega_{\text{ét}}^{\mathcal{L}}$ induces an equivalence between basic admissible pairs and p-adic Hodge structures. The functor ω_{HS} is given by

$$(W, \mathcal{L}) = \bigoplus_{\lambda\in\mathbb{Q}} (W_\lambda, \mathcal{L}_\lambda) \mapsto \bigoplus_{w\in\mathbb{Q}} \omega_{\text{ét}}^{\mathcal{L}}(W_{-w/2}, \mathcal{L}_{-w/2})$$

and we have described the inverse at the beginning of this section. □

5.2. *G-structure and motivic Galois groups of admissible pairs.* For $G/\mathbb{Q}_p$ a connected linear algebraic group, recall (from Section 2.1) the category G–AdmPair(C) of admissible pairs with G-structure (or G-admissible pairs for short) has objects exact tensor functors $\mathcal{G} : \text{Rep}\, G \to \text{AdmPair}(C)$ such that $\omega_{\text{ét}} \circ \mathcal{G}$ is isomorphic to ω_{std}. A G-admissible pair is basic if it factors through $\text{AdmPair}^{\text{basic}}(C)$.

Definition 5.2.1. If W is an admissible pair, then the *motivic Galois group* of W, $\text{MG}(W)$, is the automorphism group of $\omega_{\text{ét}}|_{\langle W\rangle}$.

Any admissible pair W has a canonical $\text{MG}(W)$-structure. More generally, for any $\mathcal{G} \in G$–AdmPair(C), we define $\text{MG}(\mathcal{G})$ to be the automorphism group of $\omega_{\text{ét}}$ restricted to the Tannakian subcategory generated by the essential image of $\mathcal{G}$. Then, as in Section 2.1.2, $\mathcal{G}$ has a canonical refinement to a $\text{MG}(\mathcal{G})$-admissible pair, and a choice of isomorphism $\omega_{\text{ét}} \circ \mathcal{G} = \omega_{\text{std}}$ identifies $\text{MG}(\mathcal{G})$ with a closed subgroup of G. If $\mathcal{G}$ is basic, then $\text{MG}(\mathcal{G})$ is canonically identified with $\text{MT}(\omega_{\text{HS}} \circ \mathcal{G})$, the Mumford–Tate group of the associated G-p-adic Hodge structure, via Theorem 5.1.6. For any $\mathcal{G} \in G$–AdmPair(C), the isomorphism class of $\omega_{\text{Isoc}} \circ \mathcal{G}$ is classified by an element of $B(G)$, denoted $[b]_{\mathcal{G}}$.

Example 5.2.2. A G-admissible pair $\mathcal{G}$ factors through $\text{AdmPair}(C)^{\text{basic}}$ if and only if the slope morphism is central in the motivic Galois group; if $G = \text{MG}(\mathcal{G})$, this is equivalent to $[b]_{\mathcal{G}}$ being basic. A good example to keep in mind is an admissible pair arising as an extension of $\breve{\mathbb{Q}}_p$ by $\breve{\mathbb{Q}}_p(1)$. The trivial extension is basic, with motivic Galois group $\mathbb{G}_m$, and the others are nonbasic and have motivic Galois group $\mathbb{G}_m \ltimes \mathbb{G}_a$. If we view these extensions as arising from elliptic curves in a Serre–Tate disk, then only the lifts isogenous to the canonical lift (i.e., with Serre–Tate coordinate a root of unity) give rise to basic admissible pairs (see Example 1.1.2).

We write ω_{bl} for the functor to bilatticed B_{dR}-vector spaces

$$(W, \mathcal{L}_{\text{ét}}) \mapsto (W_{B_{\text{dR}}}, \omega_{\mathcal{L}_{\text{dR}}}(W, \mathcal{L}) \cong W_{B_{\text{dR}}^+}, \mathcal{L}_{\text{ét}})$$

It is an exact tensor functor, and extends the functor ω_{bl} on p-adic Hodge structures under the equivalence of Theorem 5.1.6. Note that the Hodge filtration on W_C is given by $\text{BB}_1 \circ \omega_{\text{bl}}$, while the Hodge–Tate filtration on $\omega_{\text{ét}}(W) \otimes_{\mathbb{Q}_p} C = \mathcal{L}_{\text{ét}} \otimes_{B_{\text{dR}}^+} C$ is given by $\text{BB}_2 \circ \omega_{\text{bl}}$.

Definition 5.2.3. If $\mathcal{G} \in G$–AdmPair(C), we say $\mathcal{G}$ is good if $\omega_{\text{bl}} \circ \mathcal{G}$ is a good bilatticed G-bundle as in Definition 3.2.5, and write $[\mu]_{\mathcal{G}}$ for the associated type.

By Theorem 3.2.6, $\mathcal{G}$ is good if and only if the Hodge filtration (resp. Hodge–Tate filtration) composed with $\mathcal{G}$, $\text{Rep}\, G \to \text{Vect}^f(C)$ is a filtered G-bundle (and so exact), and then $[\mu^{-1}]_{\mathcal{G}}$ (resp. $[\mu]_{\mathcal{G}}$) is the type of the Hodge (resp. Hodge–Tate) filtered G-bundle. If $\mathcal{G}$ is not basic then the invariant $[b]_{\mathcal{G}}$ is no longer uniquely determined by $[\mu]_{\mathcal{G}}$, but the proof of Theorem 4.4.4 extends immediately to show

Theorem 5.2.4. *If G is a connected linear algebraic group and $\mathcal{G} \in G$–*AdmPair(C) *is good, then $[b]_{\mathcal{G}}$ is contained in $B(G, [\mu^{-1}]_{\mathcal{G}})$.*

5.3. *Hodge lines.* Suppose W is an admissible pair. For $k \in \mathbb{Z}$, the space of weight $2k$ Hodge lines in W is

$$\mathrm{Hdg}^{2k}(W) := \big(\mathrm{Hom}_{\mathrm{AdmPair}(C)}(\breve{\mathbb{Q}}_p(-k), W) \setminus \{0\}\big) / \mathbb{Q}_p^\times.$$

By evaluation, $\mathrm{Hdg}^{2k}(W)$ is a projective subspace of $\mathbb{P}(W^{\varphi_W = p^k})$, where we note

$$W^{\varphi_W = p^k} = \mathrm{Hom}_{\mathrm{Isoc}}(D_k, W)$$

is a finite-dimensional $\mathbb{Q}_p$-vector space (recall D_k from Section 2.2 denotes the 1-dimensional isocrystal $\breve{\mathbb{Q}}_p$ with Frobenius acting by p^k).

We also may identify $\mathrm{Hdg}^{2k}(W)$ with a projective subspace of $\mathbb{P}(\omega_{\text{ét}}(W))$ by evaluation after application of $\omega_{\text{ét}}$. If W is basic with associated p-adic Hodge structure V, $\mathrm{Hdg}^{2k}(W) = \mathrm{HT}^{2k}(V)$ compatibly with this identification, and essentially the same proof as Theorem 4.5.1 yields an extension to all admissible pairs:

Theorem 5.3.1. *Suppose $G/\mathbb{Q}_p$ is a connected linear algebraic group and $\mathcal{G} \in G$–*AdmPair(C) *is equipped with a trivialization $\omega_{\acute{e}t} \circ \mathcal{G} = \omega_{\mathrm{std}}$. Then* $\mathrm{MG}(\mathcal{G}) \leq G$ *is the subgroup of G preserving every line $\ell \in \mathrm{Hdg}^{2k}(\mathcal{G}(V))$ for each $V \in \mathrm{Rep}\, G$, $k \in \mathbb{Z}$.*

Remark 5.3.2. Let $W \in \mathrm{AdmPair}(C)$ and let $V = \omega_{\text{ét}}(W)$. When we identify $\mathrm{Hdg}^{2k}(W)$ with a projective subspace of $\mathbb{P}(V)$, it lies inside the projectivization of

$$\mathrm{Hom}_{B^+_{\mathrm{dR}}\text{-latticed } \mathbb{Q}_p\text{-vector spaces}}(\mathbb{Q}_p(-k), (V, W \otimes B^+_{\mathrm{dR}})).$$

This is an equality when W is k-isotypic, but in general this set of homomorphisms can be larger, so that the Hodge lines cannot typically be determined only using the data of the B^+_{dR}-latticed $\mathbb{Q}_p$-vector space attached to a general admissible pair.

5.4. *$\overline{C}_0$-analyticity of an admissible pair and its de Rham lattice.* We now give precise definitions of the rationality properties of the lattices associated to an admissible pair, as needed for the main theorem. The rationality of the étale lattice determines the field of definition of an admissible pair, and we consider the subcategory consisting of those objects that can be defined over some strict p-adic subfield of C (recall a nonarchimedean field is strict p-adic if its value group is discrete and its residue field is algebraically closed). Note that any strict p-adic subfield is contained in $\overline{C}_0$ and contains $\breve{\mathbb{Q}}_p$.

Definition 5.4.1. Let $(W, \mathcal{L})$ be an admissible pair over C and let $V = \omega_{\text{ét}}(W, \mathcal{L})$.

(1) The de Rham lattice $\omega_{\mathcal{L}_{\mathrm{dR}}}(W, \mathcal{L})$ of $(W, \mathcal{L})$ is $\overline{C}_0$-*analytic* if there is a filtration $F^\bullet$ on $V_{\overline{C}_0}$ such that $\omega_{\mathcal{L}_{\mathrm{dR}}}(W, \mathcal{L}) = \mathcal{L}_{\mathrm{can}}(F^\bullet V_{\overline{C}_0}) = \sum_{i \in \mathbb{Z}} F^{-i} V_{\overline{C}_0} \otimes F^i B_{\mathrm{dR}}$. For $K \subseteq C$ a strict p-adic subfield, we say that the de Rham lattice is defined over K (or is K-analytic) if $F^\bullet$ is defined over K.

(2) $(W, \mathcal{L})$ is $\overline{C}_0$-*analytic* if there is a filtration $F^\bullet$ on $W_{\overline{C}_0}$ such that

$$\mathcal{L} = \mathcal{L}_{\mathrm{can}}(F^\bullet W_{\overline{C}_0}) = \sum_{i \in \mathbb{Z}} F^{-i} W_{\overline{C}_0} \otimes_{\overline{C}_0} F^i B_{\mathrm{dR}}.$$

For $K \subseteq C$ a strict p-adic subfield, a $\overline{C}_0$-analytic admissible pair $(W, \mathcal{L})$ has *good reduction over K* if this filtration is defined over K.

We write $\mathrm{AdmPair}(\overline{C}_0)$ for the full subcategory of $\mathrm{AdmPair}(C)$ consisting of $\overline{C}_0$-analytic admissible pairs, and an admissible pair with G-structure $\mathcal{G}$ is $\overline{C}_0$-analytic if it factors through $\mathrm{AdmPair}(\overline{C}_0)$. For $K \subseteq C$ a strict p-adic subfield, we write $\mathrm{AdmPair}^{\text{good-red}}(K)$ for the full subcategory of $\mathrm{AdmPair}(C)$ consisting of admissible pairs with good reduction over K, and an admissible pair with G-structure $\mathcal{G}$ has good reduction over K if it factors through $\mathrm{AdmPair}^{\text{good-red}}(K)$.

Remark 5.4.2. In Part III we will give a definition of admissible pairs over an arbitrary locally spatial diamond. For the diamond $\mathrm{Spd} K$, K a p-adic field, this will amount to an admissible pair $(W, \mathcal{L})$ over $C = \overline{K}^\wedge$ equipped with a semilinear action of $\mathrm{Gal}(\overline{K}/K)$ on W that preserves $\mathcal{L}$ for the induced semilinear action on $W_{B_{\mathrm{dR}}}$. An admissible pair with good reduction over K will then be one where the semilinear action is unramified; if the residue field of K is algebraically closed, then the semilinear action is trivial and we recover the notion above. We make the definition above only in the case of algebraically closed residue field to avoid introducing this semilinear action, which is irrelevant for our transcendence results.

Note that any $\overline{C}_0$-analytic admissible pair has good reduction over a sufficiently large strict p-adic subfield $K \subseteq C$. Thus, if $\mathcal{G}$ is a $\overline{C}_0$-analytic admissible pair with G-structure, and since $\mathrm{Rep}\, G$ has a tensor generator, one can always find a strict p-adic subfield $K \subseteq C$ such that $\mathcal{G}$ has good reduction over K.

On the category of admissible pairs with good reduction, there is a canonical splitting of the Hodge–Tate filtration. Below we write $\mathfrak{G}_K = \mathrm{Gal}(\overline{K}/K)$.

Lemma 5.4.3. *Let $(W, \mathcal{L}) \in \mathrm{AdmPair}(\overline{C}_0)$, and let $V = \omega_{ét}(W, \mathcal{L})$.*

(1) *If $\mathcal{L} = \mathcal{L}_{\mathrm{can}}(F^\bullet W_{\overline{C}_0})$ for a filtration $F^\bullet$ on $W_{\overline{C}_0}$, then*

$$F^\bullet W_{\overline{C}_0} = \varinjlim_{K \subset \overline{C}_0} \left(F^\bullet_{\mathrm{Hdg}} W_C\right)^{\mathfrak{G}_K},$$

where K ranges over discretely valued subfields of $\overline{C}_0$. In particular, $F^\bullet W_{\overline{C}_0}$ is uniquely determined by $(W, \mathcal{L})$ and recovered via the Hodge filtration.

(2) *For each i, the natural map $F^{-i} W_{\overline{C}_0} \otimes F^i B_{\mathrm{dR}} \subseteq \mathcal{L} \cong V \otimes B^+_{\mathrm{dR}}$ yields a commutative diagram*

$$\begin{array}{ccc} F^{-i} W_{\overline{C}_0} \otimes F^i B_{\mathrm{dR}} & \hookrightarrow & V \otimes B^+_{\mathrm{dR}} \\ \downarrow & & \downarrow \\ \mathrm{gr}^{-i} W_{\overline{C}_0} \otimes C(i) & \hookrightarrow & V \otimes C \end{array}$$

which induces an isomorphism

$$\mathrm{gr}^{-i} W_{\overline{C}_0} \otimes_{\overline{C}_0} C(i) \xrightarrow{\sim} \mathrm{gr}^i V_C.$$

(3) *The previous isomorphisms yield a functorial splitting of the Hodge–Tate filtration on* $\mathrm{AdmPair}(\overline{C}_0)$

$$(W, \mathcal{L}) \mapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^{-i} W_{\overline{C}_0} \otimes C(i) \xrightarrow{\sim} V_C.$$

Proof. Functoriality in (3) follows from (1) since the Hodge filtration is functorial. It is immediate from the definitions

$$\mathcal{L}_{\mathrm{can}}(F^\bullet) = \sum_{i\in\mathbb{Z}} F^{-i} W_{\overline{C}_0} \otimes F^i B_{\mathrm{dR}} \quad \text{and} \quad F^\bullet_{\mathrm{Hdg}} = (t^\bullet\mathcal{L} \cap W_{B^+_{\mathrm{dR}}})/(t^\bullet\mathcal{L} \cap t W_{B^+_{\mathrm{dR}}})$$

that $F^\bullet_{\mathrm{Hdg}} = F^\bullet \otimes C$ and (1) follows by taking the colimit over Galois invariants. (2) then follows from (1) and the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t^{i+1} W_{B^+_{\mathrm{dR}}} \cap t\mathcal{L} & \longrightarrow & t^{i+1} W_{B^+_{\mathrm{dR}}} \cap \mathcal{L} & \longrightarrow & F^{i+1}_{\mathrm{HT}} V_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t^{i} W_{B^+_{\mathrm{dR}}} \cap t\mathcal{L} & \longrightarrow & t^{i} W_{B^+_{\mathrm{dR}}} \cap \mathcal{L} & \longrightarrow & F^{i}_{\mathrm{HT}} V_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{-i+1}_{\mathrm{Hdg}} W_C \otimes C(i) & \longrightarrow & F^{-i}_{\mathrm{Hdg}} W_C \otimes C(i) & \longrightarrow & \mathrm{gr}^i V_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

□

As a consequence, we find $\overline{C}_0$-analytic admissible pairs avoid some of the perversities one encounters regarding types for general admissible pairs:

Theorem 5.4.4. *Any $\overline{C}_0$-analytic admissible pair with G-structure is good.*

Proof. The splitting in Lemma 5.4.3 is functorial, so the Hodge–Tate filtration on $\omega_{\text{ét}} \circ \mathcal{G}$ is exact and thus the result follows from Theorem D. □

Example 5.4.5. Building on Example 4.3.4, one easily finds that $\mathrm{Ext}^1_{\mathrm{AdmPair}(C)}(\mathrm{triv}, \mathrm{triv}) = B_{\mathrm{dR}}/B^+_{\mathrm{dR}}$ and that the only good extension (viewed as a $\mathbb{G}_a$-admissible pair) is the trivial one corresponding to the zero coset. By Theorem 5.4.4 we conclude that $\mathrm{Ext}^1_{\mathrm{AdmPair}(\overline{C}_0)}(\mathrm{triv}, \mathrm{triv}) = 0$. This is an incarnation of the fact that a self-extension of the trivial representation of the Galois group of a p-adic field is crystalline if and only if it is unramified (see Section 5.5 below).

Example 5.4.6 (Existence of G-admissible pairs with fixed invariants). Combining Theorem 5.4.4 and Theorem 5.2.4, we find a $\overline{C}_0$-analytic G-admissible pair is good with invariants $[\mu]$ and b such that $b \in B(G, [\mu^{-1}])$. We claim furthermore that given any $b \in B(G, [\mu^{-1}])$, there exists a $\overline{C}_0$-analytic G-admissible pair with invariants b and $[\mu]$ (so far it is not obvious that there is any G-admissible pair with these fixed invariants!). Indeed, choosing a Levi decomposition $G = MU$, we can always push-out from M to G, so it suffices to assume G is reductive. In the reductive case, this existence follows from [27, Proposition 3.1].

5.5. ***Galois representations for admissible pairs with good reduction.*** In this subsection we take K to be a strict p-adic field and assume $C=\overline{K}^\wedge$. Then $\omega_{\acute{e}t}|_{\mathrm{AdmPair}^{\text{good-red}}(K)}$ promotes naturally to a functor $\omega_{\mathfrak{G}_K}:\mathrm{AdmPair}^{\text{good-red}}(K)\to\mathrm{Rep}_{\mathbb{Q}_p}\mathfrak{G}_K$. Indeed, for $(W,\mathcal{L})$ in $\mathrm{AdmPair}^{\text{good-red}}(K)$ the Hodge filtration of W_C is defined over K and $\mathcal{L}=\sum_{i\in\mathbb{Z}}F^{-i}_{\mathrm{Hdg}}W_K\otimes_K F^iB_{\mathrm{dR}}$ is preserved by the $\mathfrak{G}_K$ action on $W_{B_{\mathrm{dR}}}=W\otimes B_{\mathrm{dR}}$ (acting on B_{dR}). Consequently there is an induced action ρ of $\mathfrak{G}_K$ on $\omega_{\acute{e}t}(W,\mathcal{L})=H^0(\mathbb{F},\mathcal{E}(W)_{\mathcal{L}})$, which is encoded by the following commutative diagram for each $\sigma\in\mathfrak{G}_K$:

$$\begin{array}{ccc} \omega_{\acute{e}t}(W,\mathcal{L})\otimes B_{\mathrm{dR}} & \xrightarrow{c_{\mathrm{dR}}} & W\otimes B_{\mathrm{dR}}=\omega_{\mathrm{Isoc}}(W,\mathcal{L})\otimes B_{\mathrm{dR}} \\ \Big\downarrow{\scriptstyle \rho(\sigma)\otimes\sigma} & & \Big\downarrow{\scriptstyle \mathrm{Id}\otimes\sigma} \\ \omega_{\acute{e}t}(W,\mathcal{L})\otimes B_{\mathrm{dR}} & \xrightarrow{c_{\mathrm{dR}}} & W\otimes B_{\mathrm{dR}}=\omega_{\mathrm{Isoc}}(W,\mathcal{L})\otimes B_{\mathrm{dR}} \end{array} \tag{5}$$

The following is essentially the crystalline comparison as reinterpreted by Fargues and Fontaine in this special case (see Remark 5.5.2 below).

Theorem 5.5.1. *The functor $\omega_{\mathfrak{G}_K}$ is fully faithful; its essential image is stable under subobjects and is contained in the subcategory of crystalline representations of $\mathfrak{G}_K$.*

Proof. Let $(W,\mathcal{L})\in\mathrm{AdmPair}^{\text{good-red}}(K)$. Let $V=\omega_{\acute{e}t}(W,\mathcal{L})$ and let ρ be the Galois representation on V. We first explain how to reconstruct $(W,\mathcal{L})$ from (V,ρ).

Because $V\otimes\mathcal{O}_{\mathbb{F}}|_{\mathbb{F}\setminus\infty_C}=\mathcal{E}(W)|_{\mathbb{F}\setminus\infty_C}$, we have by construction an isomorphism $V_{B_{\mathrm{crys}}}=W_{B_{\mathrm{crys}}}$. Taking $\mathfrak{G}_K$-invariants thus recovers W_{K_0} (showing that V is crystalline), compatibly with the natural Frobenius action on both sides (with $K_0=B_{\mathrm{crys}}^{\mathfrak{G}_K}$). Since K_0 has an algebraically closed residue field, the Dieudonné–Manin classification for Frobenius modules over K_0 shows $W=\sum_{\lambda=\frac{a}{b}}\breve{\mathbb{Q}}_p\cdot(W_{K_0})^{\varphi^b=p^a}$, so we recover W. Once we have W, we recover $\mathcal{L}$ as $\mathcal{L}=V_{B^+_{\mathrm{dR}}}\subseteq V_{B_{\mathrm{dR}}}=W_{B_{\mathrm{dR}}}$.

From this it is clear that $\omega_{\mathfrak{G}_K}$ is fully faithful — indeed, any endomorphism of W or V is determined by its action on $W_{B_{\mathrm{dR}}}=V_{B_{\mathrm{dR}}}$. If $S\subseteq V$ is a subspace preserved by the Galois action, then the exact sequence

$$0\to(S_{B_{\mathrm{crys}}})^{\mathfrak{G}_K}\to(V_{B_{\mathrm{crys}}})^{\mathfrak{G}_K}=W_{K_0}\to((V/S)_{B_{\mathrm{crys}}})^{\mathfrak{G}_K},$$

plus the relations $\dim_{\mathbb{Q}_p}V=\dim_{\breve{\mathbb{Q}}_p}W$, $\dim F_{K_0}(S_{B_{\mathrm{crys}}})^{\mathfrak{G}_K}\le\dim_{\mathbb{Q}_p}S$ and $\dim_{K_0}((V/S)_{B_{\mathrm{crys}}})^{\mathfrak{G}_K}\le\dim_{\mathbb{Q}_p}V/S$ show that $(S_{B_{\mathrm{crys}}})^{\mathfrak{G}_K}$ is a sub-φ-module of W_{K_0} with the same dimension as S. Passing to the associated isocrystal $W'\subset W$ as above, we find $W'_{B_{\mathrm{dR}}}=S_{B_{\mathrm{dR}}}$ and thus $(W',\mathcal{L}\cap W'_{B_{\mathrm{dR}}})=(W',S_{B^+_{\mathrm{dR}}})$ is a subobject corresponding to S. □

Remark 5.5.2. Of course, the essential image of this functor is the category of crystalline representations of $\mathfrak{G}_K$, and the construction in the proof explains also the relation to Fontaine's category of filtered φ-modules over K_0 when we take into account that the $\mathfrak{G}_K$-invariant lattices on $W_{B_{\mathrm{dR}}}$ are precisely the ones in the image of $\mathcal{L}_{\mathrm{can}}$. We do not discuss this further here as it is not necessary for our present purposes; in Part III we will explain this equivalence more generally in the context of Remark 5.4.2 where we allow also K with non-algebraically closed residue field.

If $\mathcal{G}$ is a G-admissible pair with good reduction over K, then we obtain a continuous representation $\rho : \mathfrak{G}_K \to G(\mathbb{Q}_p)$ after choosing a trivialization $\omega_{\text{ét}} \circ \mathcal{G} \cong \omega_{\text{std}}$. Theorem 5.5.1 implies that the induced map from the Tannakian structure group of $\omega_{\mathfrak{G}_K} \circ \mathcal{G}$ to $\mathrm{MG}(\mathcal{G})$ is an isomorphism. The former is the Zariski closure of the compact subgroup $\rho(\mathfrak{G}_K)$ in $G(\mathbb{Q}_p)$; thus the image of ρ is Zariski-dense in $\mathrm{MG}(\mathcal{G})$. In fact, we can do slightly better:

Corollary 5.5.3. *With notation as above, $\rho(\mathfrak{G}_K)$ is an open subgroup of $\mathrm{MG}(\mathcal{G})(\mathbb{Q}_p)$.*

Proof. Lemma 5.4.3 implies ρ is a Hodge–Tate representation. By [33, théorème 1], its image is open in its Zariski closure, which, by the above discussion, is $\mathrm{MG}(\mathcal{G})$. □

We record for later use the following remark regarding a Galois theoretic consequence of the rationality of the de Rham lattice.

Lemma 5.5.4. *If $\mathcal{G}$ is a G-admissible pair with good reduction over K, **and** the de Rham lattice $\omega_{\mathcal{L}_{\mathrm{dR}}} \circ \mathcal{G}$ on $\omega_{\acute{e}t} \circ \mathcal{G}$ is also defined over K, then the de Rham lattice $\omega_{\mathcal{L}_{\mathrm{dR}}} \circ \mathcal{G} \subset (\omega_{\acute{e}t} \otimes B_{\mathrm{dR}}) \circ \mathcal{G}$ is preserved by the B_{dR}-linear extension $\rho \otimes \mathrm{Id}_{B_{\mathrm{dR}}}$ of the action of $\mathfrak{G}_K$ on $\omega_{\acute{e}t}$.*

Proof. By definition there is an exact K-filtration F on $\omega_{\text{ét}} \circ \mathcal{G}$ such that $\omega_{\mathcal{L}_{\mathrm{dR}}} \circ \mathcal{G} = \mathcal{L}_{\mathrm{can}} \circ F$. Consequently $\mathcal{L}_{\mathrm{can}} \circ F = \sum_{i \in \mathbb{Z}} F^{-i}(\omega_{\text{ét}} \circ \mathcal{G}) \otimes_K F^i B_{\mathrm{dR}}$ is visibly preserved by $\mathrm{Id}_{\omega_{\text{ét}} \circ \mathcal{G}} \otimes \sigma^{-1}$ for $\sigma \in \mathfrak{G}_K$; while (5) shows it is preserved by $\rho(\sigma) \otimes \sigma$. Thus it is preserved by $\rho(\sigma) \otimes \mathrm{Id}_{B_{\mathrm{dR}}}$. □

5.6. *Periods.* We now describe some period constructions. We note that this kind of analysis goes back at least to Fargues [16] in the Lubin–Tate case. We write $\omega_{\mathrm{HT}}, \omega_{\mathrm{Hdg}}$ for the Hodge–Tate and Hodge–Tate filtrations on $\omega_{\text{ét}}, \omega_{\mathrm{Isoc}}$ respectively, and view them as tensor functors $\mathrm{AdmPair}(C) \to \mathrm{Vect}^f(C)$ (which are not exact, but are exact when restricted to $\mathrm{AdmPair}(\overline{C}_0)$).

Let G be a connected linear algebraic group over $\mathbb{Q}_p$ and fix an element $b \in G(\breve{\mathbb{Q}}_p)$. Let $\mathcal{G}_b$ be the G-isocrystal $(V, \rho) \mapsto (V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, \rho(b)\varphi_{\breve{\mathbb{Q}}_p})$. For $\mathcal{G}$ a G-admissible pair with underlying isocrystal of type $[b] \in B(G)$, fix trivializations

$$\mathrm{triv}_{\mathrm{Isoc}} : \omega_{\mathrm{Isoc}} \circ \mathcal{G} \cong \mathcal{G}_b \text{ and } \mathrm{triv}_{\text{ét}} : \omega_{\text{ét}} \circ \mathcal{G} \cong \omega_{\mathrm{std}}.$$

Note that $\mathrm{triv}_{\mathrm{Isoc}}$ also induces a isomorphism $\mathrm{Forget} \circ \omega_{\mathrm{Isoc}} \circ \mathcal{G} \cong \omega_{\mathrm{std}} \otimes \breve{\mathbb{Q}}_p$; we also call this $\mathrm{triv}_{\mathrm{Isoc}}$ and allow $\omega_{\mathrm{Isoc}} \circ \mathcal{G}$ to be valued in either isocrystals or $\breve{\mathbb{Q}}_p$-vector spaces depending on the context. Then

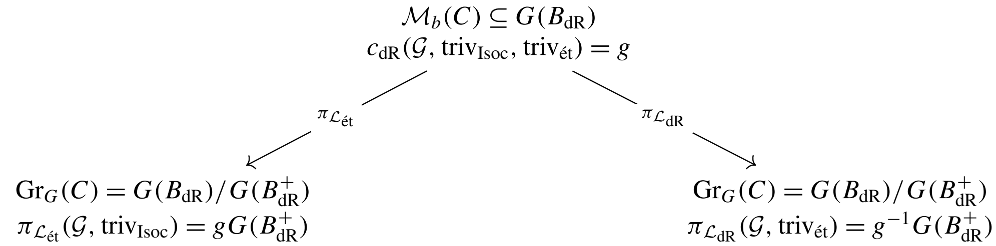
$$\omega_{\mathrm{std}} \otimes B_{\mathrm{dR}} =_{\mathrm{triv}_{\text{ét}}} \omega_{\text{ét}} \circ \mathcal{G} \otimes B_{\mathrm{dR}} \xrightarrow{c_{\mathrm{dR}}} \omega_{\mathrm{Isoc}} \circ \mathcal{G} \otimes B_{\mathrm{dR}} =_{\mathrm{triv}_{\mathrm{Isoc}}} \omega_{\mathrm{std}} \otimes B_{\mathrm{dR}}$$

is given by $c_{\mathrm{dR}}(\mathcal{G}, \mathrm{triv}_{\text{ét}}, \mathrm{triv}_{\mathrm{Isoc}}) = \mathrm{triv}_{\mathrm{Isoc}} \circ c_{\mathrm{dR}} \circ \mathrm{triv}_{\text{ét}}^{-1} \in G(B_{\mathrm{dR}})$. This period matrix, combined with knowledge of $[b]$, uniquely determines the triple $(\mathcal{G}, \mathrm{triv}_{\text{ét}}, \mathrm{triv}_{\mathrm{Isoc}})$ up to isomorphism of $\mathcal{G}$ matching the trivializations. In other words, if we write $\mathcal{M}_b(C)$ for the set of such isomorphism classes, then we obtain an injection $c_{\mathrm{dR}} : \mathcal{M}_b(C) \to G(B_{\mathrm{dR}})$. We also have (see Section 3 for the notation):

(1) The *de Rham lattice period* $\pi_{\mathcal{L}_{\mathrm{dR}}}(\mathcal{G}, \mathrm{triv}_{\text{ét}}) \in \mathrm{Gr}_G(C)$ classifying the B_{dR}^+-lattice $\omega_{\mathcal{L}_{\mathrm{dR}}} \circ \mathcal{G}$ on the trivial G-bundle $\omega_{\text{ét}} \circ \mathcal{G} =_{\mathrm{triv}_{\text{ét}}} \omega_{\mathrm{std}}$.

(2) The *étale lattice period* $\pi_{\mathcal{L}_{\text{ét}}}(\mathcal{G}, \text{triv}_{\text{Isoc}}) \in \text{Gr}_G(C)$ classifying the B_{dR}^+-lattice $\omega_{\mathcal{L}_{\text{ét}}} \circ \mathcal{G}$ on the trivial G-bundle $\omega_{\text{Isoc}} \circ \mathcal{G} =_{\text{triv}_{\text{Isoc}}} \omega_{\text{std}} \otimes \breve{\mathbb{Q}}_p$.

These are related by the following diagram:



Remark 5.6.1. If $C = \overline{K}^{\wedge}$ for a p-adic field K, then the period mappings $\pi_{\mathcal{L}_{\text{ét}}}$ and $\pi_{\mathcal{L}_{\text{dR}}}$ are $\mathfrak{G}_K = \text{Gal}(\overline{K}/K)$-equivariant for the natural actions. Indeed, if $g \in G(B_{\text{dR}})$, the lattice $gG(B_{\text{dR}}^+)$ on the trivial G-bundle sends a representation $V \in \text{Rep}\, G$ to the lattice $gV_{B_{\text{dR}}^+} \subset V_{B_{\text{dR}}}$. Then

$$\sigma \cdot gV_{B_{\text{dR}}^+} = \sigma(g)\sigma^{-1}(V_{B_{\text{dR}}^+}) = \sigma(g)V_{B_{\text{dR}}^+}$$

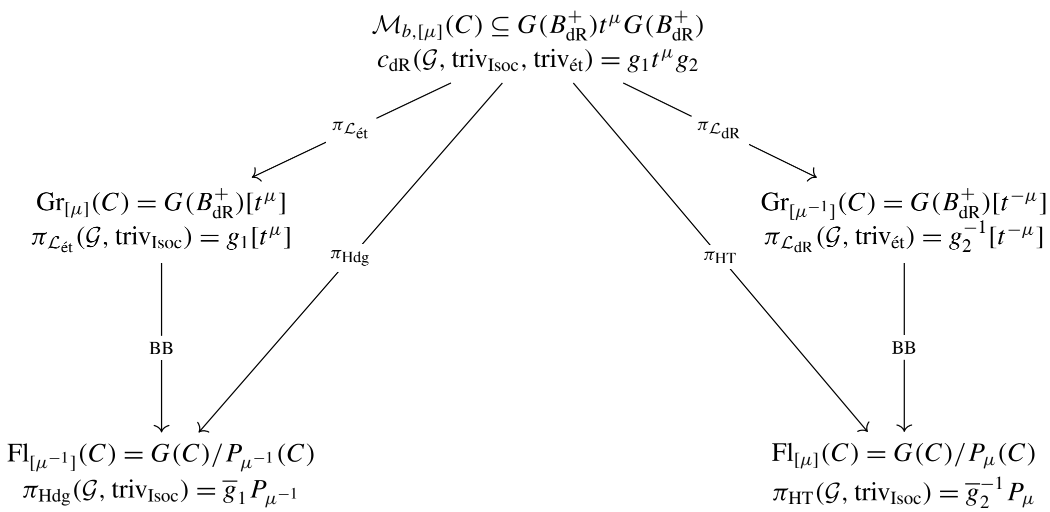
since σ preserves B_{dR}^+.

For $[\mu]$ such that $b \in B(G, [\mu])$, write $\mathcal{M}_{b,[\mu]}(C) \subseteq \mathcal{M}_b$ for the subset consisting of $\mathcal{G}$ that are good of type $[\mu]$. This is equivalent to requiring that $c_{\text{dR}}(\mathcal{G}, \text{triv}_{\text{Isoc}}, \text{triv}_{\text{ét}})$ lies in the cell $G(B_{\text{dR}}^+)t^\mu G(B_{\text{dR}}^+)$. In this case we also have:

(1) The Hodge–Tate filtration period $\pi_{\text{HT}}(\mathcal{G}, \text{triv}_{\text{ét}}) \in \text{Fl}_{[\mu]}(C)$ classifying the filtration ω_{HT} on the trivial G-bundle $\omega_{\text{ét}} \otimes C =_{\text{triv}_{\text{ét}}} \omega_{\text{std}} \otimes C$.

(2) The Hodge filtration period $\pi_{\text{Hdg}}(\mathcal{G}, \text{triv}_{\text{Isoc}}) \in \text{Fl}_{[\mu^{-1}]}(C)$ classifying the filtration ω_{Hdg} on the trivial G-bundle $\omega_{\text{Isoc}} \otimes C =_{\text{triv}_{\text{Isoc}}} \omega_{\text{std}} \otimes C$.

Writing $[t^\mu] := t^\mu G(B_{\text{dR}}^+)/G(B_{\text{dR}}^+) \in \text{Gr}_G(C)$, the diagram is then refined to



Note $\mathcal{G} \in G$–AdmPair(C) is $\overline{C}_0$-analytic if and only if, for one (equivalently, any) choice of $\mathrm{triv}_{\mathrm{Isoc}}$, $\pi_{\mathcal{L}_{\text{ét}}}([\mathcal{G}, \mathrm{triv}_{\mathrm{Isoc}}])$ is in the image of the canonical lattice map

$$\mathcal{L}_{\mathrm{can}} : \mathrm{Fl}_G(\overline{C}_0) \to \mathrm{Gr}_G(C).$$

In particular, to detect the $\overline{C}_0$-analytic points we may assume $C = \overline{C}_0^\wedge$ so that by Proposition 3.1.7(2) these are exactly the points of $\mathrm{Gr}_G(C)$ stabilized by a finite index subgroup $\mathfrak{I}$ of $\mathrm{Gal}(\overline{C}_0/C_0)$. If $K = \overline{C}_0^{\mathfrak{I}}$, the points stabilized by $\mathfrak{I}$ are exactly those G-admissible pairs factoring through $\mathrm{AdmPair}^{\text{good-red}}(K)$. For each $V \in \mathrm{Rep}\, G$, we have the crystalline Galois representation $\mathfrak{I} \to \mathrm{GL}(\omega_{\text{ét}} \circ \mathcal{G}(V))$ from Section 5.4. These are compatible for varying $V \in \mathrm{Rep}\, G$, and arise via Tannakian theory from a crystalline G-valued representation

$$\rho = \rho(\mathcal{G}, \mathrm{triv}_{\text{ét}}) : \mathfrak{I} \to G(\mathbb{Q}_p), \qquad \rho(\sigma) = \mathrm{triv}_{\text{ét}}^{-1} \circ \rho_{\text{ét}}(\sigma) \circ \mathrm{triv}_{\text{ét}}.$$

The next lemma shows how this can be recovered from the period of $\mathcal{G}$ (relative to the trivializations $\mathrm{triv}_{\text{ét}}, \mathrm{triv}_{\mathrm{Isoc}}$).

Lemma 5.6.2. *For $g = c_{\mathrm{dR}}(\mathcal{G}, \mathrm{triv}_{\mathrm{Isoc}}, \mathrm{triv}_{\text{ét}}) \in G(B_{\mathrm{dR}})$ as above, the associated Galois representation $\rho : \mathfrak{I} \to G(\mathbb{Q}_p)$ is recovered by the action on g by*

$$\sigma(g) = g\rho(\sigma),$$

Proof. To see this, we trace definitions noting that the action of σ on c_{dR} is

$$\sigma \cdot c_{\mathrm{dR}} = (\mathrm{Id}_{\omega_{\mathrm{Isoc}}} \otimes \sigma) \circ c_{\mathrm{dR}} \circ (\mathrm{Id}_{\omega_{\text{ét}}} \otimes \sigma^{-1}).$$

We substitute the defining property (5) for $\rho_{\text{ét}}$,

$$\mathrm{Id}_{\omega_{\mathrm{Isoc}}} \otimes \sigma = c_{\mathrm{dR}} \circ (\rho_{\text{ét}}(\sigma) \otimes \sigma) \circ c_{\mathrm{dR}}^{-1},$$

to get

$$\sigma \cdot c_{\mathrm{dR}} = c_{\mathrm{dR}} \circ \rho_{\text{ét}}(\sigma) \otimes \mathrm{Id}_{B_{\mathrm{dR}}}.$$

Putting all of this together, we see that

$$\begin{aligned}
\sigma(g) &= (\mathrm{Id}_{\omega_{\mathrm{std}}} \otimes \sigma) \circ g \circ (\mathrm{Id}_{\omega_{\mathrm{std}}} \otimes \sigma^{-1}),\\
&= (\mathrm{Id}_{\omega_{\mathrm{std}}} \otimes \sigma) \circ (\mathrm{triv}_{\mathrm{Isoc}}^{-1} \circ c_{\mathrm{dR}} \circ \mathrm{triv}_{\text{ét}}) \circ (\mathrm{Id}_{\omega_{\mathrm{std}}} \otimes \sigma^{-1}),\\
&= \mathrm{triv}_{\mathrm{Isoc}}^{-1} \circ (\mathrm{Id}_{\omega_{\mathrm{Isoc}}} \otimes \sigma) \circ c_{\mathrm{dR}} \circ (\mathrm{Id}_{\omega_{\text{ét}}} \otimes \sigma^{-1}) \circ \mathrm{triv}_{\text{ét}},\\
&= \mathrm{triv}_{\mathrm{Isoc}}^{-1} \circ (\sigma \cdot c_{\mathrm{dR}}) \circ \mathrm{triv}_{\text{ét}},\\
&= \mathrm{triv}_{\mathrm{Isoc}}^{-1} \circ (c_{\mathrm{dR}} \circ \rho_{\text{ét}}(\sigma) \otimes \mathrm{Id}_{B_{\mathrm{dR}}}) \circ \mathrm{triv}_{\text{ét}},\\
&= (\mathrm{triv}_{\mathrm{Isoc}}^{-1} \circ c_{\mathrm{dR}} \circ \mathrm{triv}_{\text{ét}}) \circ (\mathrm{triv}_{\text{ét}}^{-1} \circ \rho_{\text{ét}}(\sigma) \otimes \mathrm{Id}_{B_{\mathrm{dR}}} \circ \mathrm{triv}_{\text{ét}}),\\
&= g\rho(\sigma).
\end{aligned}$$

□

We obtain the following precise analog of the Grothendieck period conjecture:

Corollary 5.6.3. *If $\mathcal{G}$ is a $\overline{C}_0$-analytic G-admissible pair and $G = \mathrm{MG}(\mathcal{G})$, then $c_{\mathrm{dR}} \circ \mathcal{G}$ is a generic point of the torsor of isomorphisms between $\omega_{ét} \circ \mathcal{G}$ and $\omega_{\mathrm{Isoc}} \circ \mathcal{G}$.*

Proof. We need to show that the $\mathfrak{J}$-orbits of $c_{\mathrm{dR}} \circ \mathcal{G}$ are Zariski dense in $\mathrm{Isom}^{\otimes}(\omega_{\text{ét}} \circ \mathcal{G}, \omega_{\mathrm{Isoc}} \circ \mathcal{G})$, but this follows since the image of ρ is Zariski dense in G by Corollary 5.5.3. □

5.7. *Admissible pairs with complex multiplication.* An admissible pair M has CM if its motivic Galois group $\mathrm{MG}(M)$ is a torus. We now give the easy direction of Theorem B and Corollary C:

Proposition 5.7.1. *Suppose $\mathcal{G} \in G$–$\mathrm{AdmPair}(C)$ has complex multiplication. Then, for any choice of $\mathrm{triv}_{ét}$ and $\mathrm{triv}_{\mathrm{Isoc}}$, $\pi_{\mathcal{L}_{ét}}(\mathcal{G}, \mathrm{triv}_{\mathrm{Isoc}})$ is $\overline{\breve{\mathbb{Q}}}_p$-analytic and $\pi_{\mathcal{L}_{\mathrm{dR}}}(\mathcal{G}, \mathrm{triv}_{ét})$ is $\overline{\mathbb{Q}}_p$-analytic. In particular, if $M \in \mathrm{AdmPair}(C)$ has complex multiplication then it is $\overline{\breve{\mathbb{Q}}}_p$-analytic and its de Rham lattice is $\overline{\mathbb{Q}}_p$-analytic.*

Proof. By functoriality of periods, we may assume $G = T$ is a torus. Then every cocharacter is minuscule so each cell of the affine Grassmannian is equal to the flag variety and consists of a single point, and is thus defined over a finite extension of the base field (which is $\mathbb{Q}_p$ for the de Rham lattice and $\breve{\mathbb{Q}}_p$ for the étale lattice). □

For completeness, we recall now the classification of CM admissible pairs, a result due to Anschütz [2] in the language of Breuil–Kisin–Fargues modules and essentially equivalent to a result of Serre [33, théorèmes 5 et 6] on abelian Galois representations. This classification will not be used in any of our results.

We write $\mathrm{AdmPair}^{\mathrm{CM}}(C)$ for the Tannakian subcategory of CM admissible pairs. By Proposition 5.7.1, $\mathrm{AdmPair}^{\mathrm{CM}}(C) \subseteq \mathrm{AdmPair}(\overline{\breve{\mathbb{Q}}}_p)$. In fact, the motivic Galois group S of $\mathrm{AdmPair}^{\mathrm{CM}}(C)$ is the maximal abelian quotient of the motivic Galois group of $\mathrm{AdmPair}(\overline{\breve{\mathbb{Q}}}_p)$ — there can be no additive component because the only $\overline{C}_0$-analytic extension of the trivial admissible pair by itself is trivial by Example 5.4.5. Note also that, for T a torus, any T-admissible pair is basic, so $\mathrm{AdmPair}^{\mathrm{CM}}(C)$ is equivalent via Theorem 5.1.6 to the category $\mathrm{HS}^{\mathrm{CM}}(C)$ of p-adic Hodge structures with complex multiplication.

Consider the pro-torus over $\mathbb{Q}_p$, $S' = \lim_{\mathbb{Q}_p \subseteq E \subseteq C, [E:\mathbb{Q}_p]<\infty} \mathrm{Res}^E_{\mathbb{Q}_p} \mathbb{G}_m$, where the transition maps are the norm maps. Then the character group $X^*(S')$ is the space of locally constant functions on $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and there is a cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to S'_{\overline{\mathbb{Q}}_p}$ corresponding to

$$\mu^* : X^*(S') \to X^*(\mathbb{G}_m) = \mathbb{Z},\ f \mapsto f(\mathrm{Id}).$$

Then $\mu^{-1}(t)$ classifies an S'-admissible pair whose associated S'-p-adic Hodge structure is classified by $\mu(t)$ (the weight homomorphism corresponds to the map on character groups given by integrating $f/2$). Explicitly, if we consider the representation of S' determine by $S' \twoheadrightarrow \mathrm{Res}^E_{\mathbb{Q}_p} \mathbb{G}_m$ and the standard representation of the restriction of scalars given by E acting on itself by multiplication, then the associated admissible pair (resp. p-adic Hodge structure) is given by the covariant Dieudonné module of a Lubin–Tate formal group for $\mathcal{O}_E$ (resp. its Tate module). Arguing as in [2] or [33], one finds the induced map $S \to S'$ is an isomorphism.

6. Transcendence

In this section we prove Theorem B and Corollary C, and give a refinement that applies outside of the basic case. Before giving these proofs, in Section 6.1 we describe in more detail the analogous transcendence results over $\mathbb{C}$.

6.1. ***Conditional and unconditional results over*** $\mathbb{C}$**.** Let $A/\mathbb{C}$ be an abelian variety. The kernel of the exponential $\mathrm{Lie}\,A \to A(\mathbb{C})$ is naturally identified with $H_1(A(\mathbb{C}), \mathbb{Z})$. The Hodge filtration is the kernel of the induced map

$$H_1(A(\mathbb{C}), \mathbb{C}) = H_1(A(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} \to \mathrm{Lie}\,A,$$

which can canonically be identified with $\omega_{A^\vee}$, the invariant differentials on the dual abelian variety $A^\vee$. This weight -1 $\mathbb{Q}$-Hodge structure determines A up to isogeny, and we say A has complex multiplication if this weight -1 Hodge structure does — this is consistent with our earlier definitions, since $\mathbb{Q}$-Hodge structures form a Tannakian category, and is equivalent to the usual definition that $\mathrm{End}(A) \otimes \mathbb{Q}$ contains a semisimple commutative subalgebra of dimension $2 \dim A$. We say A is defined over $\overline{\mathbb{Q}}$ if the equations defining A can be chosen to have coefficients in $\overline{\mathbb{Q}}$, i.e., if there is an abelian variety over $\overline{\mathbb{Q}}$ whose base change to $\mathbb{C}$ is A.

Theorem 6.1.1 (Cohen [11]; Shiga and Wolfart [34]; generalizing Schneider [30]). *An abelian variety $A/\mathbb{C}$ has complex multiplication if and only if A is defined over $\overline{\mathbb{Q}}$ and its Hodge filtration is defined over $\overline{\mathbb{Q}}$ (i.e., the subspace $\omega_{A^\vee} \subset H_1(A(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$ admits a basis whose elements are $\overline{\mathbb{Q}}$-linear combinations of elements in $H_1(A(\mathbb{C}), \mathbb{Q})$).*

Abelian varieties up to isogeny are precisely the weight one motives over $\mathbb{C}$. The proof of this theorem makes use of the equivalence of this subcategory with the category of weight -1 $\mathbb{Q}$-Hodge structures and the Wüstholz analytic subgroup theorem, a linear transcendence result. Theorem A gives a conditional generalization of this result to all motives over $\mathbb{C}$, and the assumptions match up with these ingredients: we assume the standard conjectures so that there is a good Tannakian category of motives, we assume the Hodge conjecture so that Hodge structures give a fully faithful realization, and we assume the Grothendieck period conjecture as a nonlinear transcendence result to get us started.

We now explain the setup in detail, then prove Theorem A: Assume the standard conjectures, and let $\mathrm{Mot}(\mathbb{C})$ be the category of pure motives over $\mathbb{C}$ with $\mathbb{Q}$-coefficients. It is equipped with a Betti realization $\omega_B : \mathrm{Mot}(\mathbb{C}) \to \mathrm{Vect}(\mathbb{Q})$ and an algebraic de Rham realization $\omega_{\mathrm{dR}} : \mathrm{Mot}(\mathbb{C}) \to \mathrm{Vect}(\mathbb{C})$. The latter admits a Hodge filtration $F^\bullet\omega_{\mathrm{dR}} : \mathrm{Mot}(\mathbb{C}) \to \mathrm{Vect}^f(\mathbb{C})$, and there is a canonical comparison isomorphism $c : \omega_{\mathrm{dR}} \to \omega_B \otimes \mathbb{C}$. The category $\mathrm{Mot}(\overline{\mathbb{Q}})$ of motives over $\overline{\mathbb{Q}}$ is a full subcategory, and the restriction of $F^\bullet\omega_{\mathrm{dR}}$ to $\mathrm{Mot}(\overline{\mathbb{Q}})$ factors canonically through $\mathrm{Vect}^f(\overline{\mathbb{Q}})$. Both $\mathrm{Mot}(\mathbb{C})$ and $\mathrm{Mot}(\overline{\mathbb{Q}})$ are Tannakian categories over $\mathbb{Q}$, neutralized by ω_B. Moreover,

$$\mathrm{Isom}^\otimes\big(\omega_{\mathrm{dR}}|_{\mathrm{Mot}(\overline{\mathbb{Q}})}, \omega_B \otimes \overline{\mathbb{Q}}|_{\mathrm{Mot}(\overline{\mathbb{Q}})}\big)$$

is a scheme over $\overline{\mathbb{Q}}$ and the Grothendieck period conjecture says that

$$c \in \mathrm{Isom}^{\otimes}(\omega_{\mathrm{dR}}|_{\mathrm{Mot}(\overline{\mathbb{Q}})}, \omega_B \otimes \overline{\mathbb{Q}}|_{\mathrm{Mot}(\overline{\mathbb{Q}})})(\mathbb{C})$$

is a generic point, i.e., is $\overline{\mathbb{Q}}$-Zariski dense. There is also an exact tensor functor from $\mathrm{Mot}(\mathbb{C})$ to the category $\mathbb{Q}$–HS of $\mathbb{Q}$-Hodge structures sending $M \in \mathrm{Mot}(\mathbb{C})$ to $(\omega_B(M), c(M)(F^\bullet \omega_{\mathrm{dR}}(M)))$, and the Hodge conjecture implies this is fully faithful.

Proof of Theorem A. The algebraicity for CM-motives is well known, so let us assume both M and the Hodge filtration are defined over $\overline{\mathbb{Q}}$. Let G be the motivic Galois group of M, i.e., $G = \mathrm{Aut}^{\otimes}(\omega_B|_{\langle M \rangle})$. It is a connected linear algebraic group over $\mathbb{Q}$ (e.g., by the Hodge conjecture and the connectedness of Mumford–Tate groups), and we have a natural tensor equivalence $\mathrm{Rep}\, G \to \langle M \rangle$ and an identification of ω_{std} on $\mathrm{Rep}\, G$ with ω_B on $\langle M \rangle$. If we fix also a trivialization $\omega_{\mathrm{dR}}|_{\langle M \rangle} = \omega_{\mathrm{std}} \otimes \overline{\mathbb{Q}}$ then we obtain an identification

$$\mathrm{Isom}^{\otimes}(\omega_{\mathrm{dR}}|_{\langle M \rangle}, \omega_B \otimes \overline{\mathbb{Q}}|_{\langle M \rangle}) = \mathrm{Isom}^{\otimes}(\omega_{\mathrm{std}} \otimes \overline{\mathbb{Q}}, \omega_{\mathrm{std}} \otimes \overline{\mathbb{Q}}) = G_{\overline{\mathbb{Q}}}$$

and, by the Grothendieck period conjecture, the de Rham comparison isomorphism c is a generic point in $G_{\overline{\mathbb{Q}}}(\mathbb{C})$. Splitting the filtration of $\omega_{\mathrm{dR}}|_{\langle M \rangle}$ gives a cocharacter μ of $G_{\overline{\mathbb{Q}}}$ so that the Hodge filtration is F_μ. The classifying point for the induced fully faithful functor (by the Hodge conjecture) $\mathrm{Rep}\, G \to \mathbb{Q}$–HS is

$$c \cdot F_\mu \in \mathrm{Fl}_{[\mu]}(\mathbb{C})$$

Since the orbit map for F_μ, $G_{\overline{\mathbb{Q}}} \to \mathrm{Fl}_{[\mu]}$ is dominant, this is a generic point of $\mathrm{Fl}_{[\mu]}/\overline{\mathbb{Q}}$, so if the Hodge filtration is $\overline{\mathbb{Q}}$-algebraic (with respect to the Betti rational structure), we deduce $c \cdot F_\mu = \mathrm{Fl}_{[\mu]}$ and hence the stabilizer P_μ of F_μ is G. If this is the case, then each element of $G(\mathbb{Q})$ stabilizes the Hodge filtration thus induces an automorphism of the induced tensor functor from $\mathrm{Rep}\, G$ to $\mathbb{Q}$-Hodge structures. By Lemma 2.1.2 the only automorphisms are by $Z(G)(\mathbb{Q})$ so we conclude $G(\mathbb{Q}) \subseteq Z(G)$. Since the $\mathbb{Q}$-points of a connected linear algebraic group over $\mathbb{Q}$ are Zariski dense by [6, Corollary 18.3], we conclude $G = Z(G)$, i.e., G is abelian. It is thus a product of a torus and an additive group, but the additive part is trivial as there are no nontrivial extensions of the trivial Hodge structure by itself (this is already true at the level of filtrations)[7]. Thus, G is a torus. □

6.2. ***Proof of Theorem B and Corollary C.*** Given the setup of Section 5.6, Corollary C is an immediate consequence of Theorem B since for any connected linear algebraic group G, $\mathrm{Rep}\, G$ admits a single tensor generator. We now prove Theorem B. To that end, we fix an algebraically closed nonarchimedean extension $C/\mathbb{Q}_p$. Let $\kappa = \mathcal{O}_C/\mathfrak{m}_C$ be the residue field, and let $C_0 = W(\kappa)[1/p]$, so that any p-adic subfield of C is contained in the algebraic closure $\overline{C}_0$ of C_0 in C.

In Proposition 5.7.1 we saw that any CM admissible pair over C is $\breve{\overline{\mathbb{Q}}}_p$-analytic and has $\overline{\mathbb{Q}}_p$-analytic de Rham lattice. Suppose now that $M \in \mathrm{AdmPair}^{\mathrm{basic}}(\overline{C}_0)$, and that the de Rham lattice of M is also

[7] Of course we could also exclude the possibility of an additive factor by using that G is reductive because of the polarization on motives, but the given argument better mirrors the p-adic case.

$\overline{C}_0$-analytic. We may thus fix a finite extension K/C_0 such that M is an admissible pair with good reduction over K and such that the de Rham lattice is also defined over K. Let $\mathfrak{I} = \mathrm{Gal}(\overline{C}_0/K)$.

Let G be the motivic Galois group of M, and let

$$\mathcal{G} : \mathrm{Rep}\, G \to \mathrm{AdmPair}^{\text{good-red}}(K) \subseteq \mathrm{AdmPair}(C)$$

be the canonical G-structure for M, so that there is a canonical trivialization $\omega_{\text{ét}} \circ \mathcal{G} = \omega_{\text{std}}$. Fix also a trivialization $\omega_{\text{Isoc}} \otimes_{\breve{\mathbb{Q}}_p} K = \omega_{\text{std}}$ (viewed as a fiber functor to $\mathrm{Vect}(K)$). As in Section 5.6, c_{dR} corresponds (via the chosen trivializations) to an element $g \in G(B_{\text{dR}})$, and the de Rham lattice is classified by

$$g^{-1}G(B_{\text{dR}}^{+}) \in \mathrm{Gr}_G(C) = G(B_{\text{dR}})/G(B_{\text{dR}}^{+}).$$

As in Lemma 5.6.2, we find that $\mathfrak{I}$ acts on g by

$$\sigma(g) = g\rho(\sigma) \qquad \sigma \in \mathfrak{I},$$

where $\rho : \mathfrak{I} \to G(\mathbb{Q}_p)$ is the associated crystalline Galois representation. The de Rham lattice period mapping is $\mathfrak{I}$ equivariant (Remark 5.6.1) so the Galois action on the de Rham lattice period multiplies it on the left $\rho(\sigma)^{-1}$. By assumption, the de Rham lattice period is preserved by this action (Lemma 5.5.4), so we conclude that $\rho(\sigma) \in G(\mathbb{Q}_p)$ preserves the de Rham lattice. The induced functor from $\mathrm{Rep}\, G$ to p-adic Hodge structures is fully faithful and $\rho(\sigma)$ is an automorphism of this functor so Lemma 2.1.2 shows that $\rho(\sigma) \in Z(G)(\mathbb{Q}_p)$. However, the image of ρ is Zariski dense in G by Corollary 5.5.3, so we conclude $Z(G) = G$, so G is abelian. It is thus a product of a torus part and an additive part, but the additive part must be trivial because there is no nontrivial $\overline{C}_0$-analytic extension of the trivial admissible pair by itself (Example 5.4.5). Thus G is a torus.

Remark 6.2.1. The proof is similar to the proof for one-dimensional formal groups given in [19], but at the time we only understood weaker tools — indeed, in [19] we used Tate's full faithfulness of the p-adic Tate module over a p-adic field in place of the crystalline comparison here, and the Scholze–Weinstein classification in place of the equivalence between basic admissible pairs and p-adic Hodge structures here. Happily, we have skipped over the analog of the abelian varieties step (where progress has halted in the archimedean theory) and gone straight to an unconditional analog of Theorem A. Nonetheless, it would still be interesting to find another proof for isoclinic formal groups mirroring the use of the Wüstholz analytic subgroup theorem in Theorem 6.1.1.

Remark 6.2.2. The same methods would give an unconditional theorem for extended real Hodge structures, but the result is not very interesting: in this case, the condition for an extended real Hodge structure analogous to having an $\overline{C}_0$-analytic de Rham lattice is simply to be a real Hodge structure, but every real Hodge structure has complex multiplication (indeed, the motivic Galois group of the category of real Hodge structures is the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$). Thus in the archimedean case one must use an underlying global structure on the coefficients to have an interesting transcendence theory, whereas in the nonarchimedean case there is already a rich purely local transcendence theory.

6.3. ***Beyond the basic case.*** In [19, Conjecture 4.1], one of the authors made a conjecture based on the result for one-dimensional formal groups that does not really make sense as written (regretfully, we failed to noticed at the time that isogeny Breuil–Kisin–Fargues modules were not an abelian category!). The natural correction is to use rigidified Breuil–Kisin–Fargues modules. Doing so, the conjecture becomes that a $\overline{C}_0$-analytic admissible pair with $\overline{C}_0$-analytic Hodge–Tate filtration admits complex multiplication. Theorem B thus proves a weaker statement where we require the stronger condition that the de Rham lattice is $\overline{C}_0$-analytic and restrict ourselves to basic admissible pairs. Note that in the case of minuscule cocharacter, including all settings that apply to p-divisible groups, the Hodge–Tate filtration uniquely determines the de Rham lattice, so that the corrected conjecture is now proved in these basic minuscule cases. With a more complete understanding of the structures that allow us to prove Theorem B, it no longer seems reasonable to conjecture that $\overline{C}_0$-analyticity of the Hodge–Tate filtration alone would suffice beyond the minuscule case — it would be interesting to have an example!

With the correction described above, the conjecture does not hold outside the basic case. Indeed, already the admissible pair attached to an ordinary elliptic curve without complex multiplication gives an example where it fails. This may appear at odds with the full results of [19], which also treat one-dimensional p-divisible groups with an étale part, but note that in [19] we did not take into account the rigidification and characterized CM purely in terms of endomorphisms; without the rigidification the étale part of a p-divisible group can always be split off from the connected part over $\mathcal{O}_C$.

In the nonbasic counterexample given by an ordinary elliptic curve, note that the slope filtration still lifts, and each graded part for the slope filtration has complex multiplication. In fact, this holds in general, as we explain now.

The slope filtration on the category of isocrystals is the increasing $\mathbb{Q}$-filtration $F_\lambda(W) = \bigoplus_{\lambda' \leq \lambda} W_{\lambda'}$, where $W_{\lambda'}$ denotes the $D_{\lambda'}$-isotypic component of W. For $\mathcal{M} \subseteq \mathrm{AdmPair}(C)$ a Tannakian subcategory, we say the slope filtration lifts to $\mathcal{M}$ if $F_\lambda(W)$ underlies a subobject for any $(W, \mathcal{L}_{\text{ét}}) \in \mathcal{M}$. Note that it is equivalent to say that $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{Isoc}}|_{\mathcal{M}})$, where ω_{Isoc} is viewed as a fiber functor to $\mathrm{Vect}(\breve{\mathbb{Q}}_p)$, preserves the slope filtration.

Example 6.3.1. The slope filtration lifts to the subcategory of basic admissible pairs (because the slope grading does!).

Theorem 6.3.2. *Let $M \in \mathrm{AdmPair}(\overline{C}_0)$ and suppose the de Rham lattice of M is $\overline{C}_0$-analytic. Then the slope filtration lifts to $\langle M \rangle$, and the associated graded has complex multiplication.*

Proof. If the slope filtration lifts then the associated graded is basic, $\overline{C}_0$-analytic, and has $\overline{C}_0$-analytic de Rham lattice, so Theorem B implies it has complex multiplication. Thus it remains only to show that the slope filtration lifts.

For this, we may proceed as in the proof of Theorem B to obtain $\mathfrak{I}$ and ρ such that, for $\sigma \in \mathfrak{I}$, $\rho(\sigma)$ preserves the de Rham lattice. This implies that $\rho(\sigma)$ induces an automorphism of the vector bundle $\mathcal{E}(W)$ for any $(W, \mathcal{L}_{\text{ét}}) \in \langle M \rangle$. Such an automorphism preserves the slope filtration of $\mathcal{E}(W)$, since morphisms of semistable vector bundles on $\mathbb{F}_{C^\flat}$ only go up in slope. Thus, after we use c_{dR} to identify $G(B_{\mathrm{dR}})$ with

$\mathrm{Aut}^{\otimes}(\omega_{\mathrm{Isoc}})(B_{\mathrm{dR}})$, we find g preserves the slope filtration of $W_{B_{\mathrm{dR}}}$. But since the image of ρ is Zariski dense[8] in $G_{B_{\mathrm{dR}}}$ by Corollary 5.5.3, we conclude that $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{Isoc}}|_{\langle M\rangle})$, viewed as a fiber functor to $\breve{\mathbb{Q}}_p$, preserves the slope filtration. In other words, the slope filtration lifts to $\langle M\rangle$. □

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Local square mean in the hyperbolic circle problem

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Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a finite volume Fuchsian group. The hyperbolic circle problem is the estimation of the number of elements of the Γ-orbit of z in a hyperbolic circle around w of radius R, where z and w are given points of the upper half-plane and R is a large number. An estimate with error term $e^{2R/3}$ is known, and this has not been improved for any group. Petridis and Risager proved that in the special case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ taking $z = w$ and averaging over z locally the error term can be improved to $e^{(7/12+\epsilon)R}$. Here we show such an improvement for the local L^2-norm of the error term. Our estimate is $e^{(9/14+\epsilon)R}$, which is better than the pointwise bound $e^{2R/3}$ but weaker than the bound of Petridis and Risager for the local average.

1. Introduction

1.1. ***Statement of the main result.*** Let $\mathbb{H}$ be the upper half-plane. For $z, w \in \mathbb{H}$ let

$$u(z,w) = \frac{|z-w|^2}{4\,\mathrm{Im}\, z\,\mathrm{Im}\, w}; \tag{1-1}$$

this is closely related to the hyperbolic distance $\rho(z,w)$ of z and w, namely we have $1 + 2u = \cosh\rho$. The elements of the group $\mathrm{PSL}_2(\mathbb{R})$ act on $\mathbb{H}$ by linear fractional transformations, which are isometries of the hyperbolic plane. Let $d\mu_z = dx\,dy/y^2$; this measure is invariant with respect to the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For $z \in \mathbb{H}$ and $X > 2$ define

$$N(z,X) := \bigl|\{\gamma \in \Gamma : 4u(\gamma z, z) + 2 \le X\}\bigr|;$$

this is the number of points γz in the hyperbolic circle around z of radius $\cosh^{-1}(X/2)$, so the estimation of this quantity is called the hyperbolic circle problem. We know that

$$|N(z,X) - 3X| = O_z(X^{2/3}); \tag{1-2}$$

this is an unpublished theorem of Selberg, but it is proved also in [L-P]; see also [I, Theorem 12.1]. Let $\mathcal{F}$ be the closure of the standard fundamental domain of Γ, i.e.,

$$\mathcal{F} = \bigl\{z \in \mathbb{C} : \mathrm{Im}\, z > 0, -\tfrac{1}{2} \le \mathrm{Re}\, z \le \tfrac{1}{2},\ |z| \ge 1\bigr\}. \tag{1-3}$$

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The goal of this paper is to prove the following theorem.

Theorem 1.1. *Let* $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, *let* $\mathcal{F}$ *be as in* (1-3), *and let* $\Omega \subseteq \mathcal{F}$ *be a compact set. For any* $\epsilon > 0$, *we have*

$$\left(\int_\Omega (N(z,X) - 3X)^2 \, d\mu_z\right)^{1/2} = O_{\Omega,\epsilon}(X^{9/14+\epsilon}).$$

Remark 1.1. The significance of the theorem is that the estimate is better on average than the pointwise bound $X^{2/3}$.

Remark 1.2. Let f be a smooth nonnegative function that is compactly supported on $\mathcal{F}$ and let $\epsilon > 0$. It was proved in [P-R] that $\int_{\mathcal{F}} f(z)(N(z,X) - 3X)\, d\mu_z = O_{f,\epsilon}(X^{7/12+\epsilon})$.

Remark 1.3. The bound (1-2) remains valid if we take any finite-volume Fuchsian group (a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting discontinuously on $\mathbb{H}$ and having a fundamental domain of finite volume with respect to $d\mu_z$) in place of $\mathrm{PSL}_2(\mathbb{Z})$, provided the main term is defined including all small Laplace eigenvalues. The analogue of the theorem of [P-R] mentioned in Remark 1.2 was proved in [B1] for any finite volume Fuchsian group with exponent $\frac{5}{8}$ in place of $\frac{7}{12}$.

Remark 1.4. It would be interesting to extend Theorem 1.1 for any finite volume Fuchsian group in place of $\mathrm{PSL}_2(\mathbb{Z})$ with some exponent smaller than $\frac{2}{3}$, similarly as the theorem of [P-R] was extended in [B1]. Our present proof uses arithmetic tools, so it might be extended only for groups similar to $\mathrm{PSL}_2(\mathbb{Z})$.

Remark 1.5. Several other kinds of average results in the hyperbolic circle problem were proved in [C] and [C-R].

1.2. *Outline of the proof.* We take an integer $J \geq 2$, which will be fixed to be large enough in terms of ϵ. We also take a parameter d, which will tend to ∞ together with X, and we assume $X^{2/3} \leq d = X^{1-\delta}$ with some fixed $\delta > 0$. We take the sum

$$N_{d,J}(z,X) := \sum_{j=0}^{J} (-1)^j \binom{J}{j} \int_1^2 \eta_0(\tau) N(z, X - jd\tau)\, d\tau,$$

where η_0 is a given nonnegative smooth function on $(0,\infty)$ such that $\eta_0(\tau) = 0$ for $\tau \notin [1,2]$, and $\int_1^2 \eta_0(\tau)\, d\tau = 1$. Then the $j = 0$ term equals $N(z,X)$, but the terms $j \neq 0$ are smoothed versions of $N(z,X)$. It can be proved by spectral methods that for $z \in \Omega$ the $j \neq 0$ terms can be replaced by their main terms with an error term $O_\Omega(X/\sqrt{d})$. One gets from these spectral estimates that

$$N_{d,J}(z,X) = N(z,X) - 3X + O_\Omega(X/\sqrt{d}) \tag{1-4}$$

for $z \in \Omega$. If we take d larger than $X^{2/3}$, this error term will be smaller than $X^{2/3}$. One can also see easily that the contribution of the nonhyperbolic $\gamma \in \Gamma$ to $N_{d,J}(z,X)$ is $O_{\Omega,\epsilon}(X^{1/2+\epsilon})$. Therefore, for the proof of Theorem 1.1 it is enough to estimate

$$\int_{\mathcal{F}} (N_{d,J,\mathrm{hyp}}(z,X))^2 \, d\mu_z, \tag{1-5}$$

where $N_{d,J,\mathrm{hyp}}(z,X)$ is the contribution of the hyperbolic $\gamma\in\Gamma$ to $N_{d,J}(z,X)$. We will give an expression for (1-5) whose most essential part will be an expression of the form

$$\sum_{\substack{t_1,t_2\\ f^2\neq(t_1^2-4)(t_2^2-4)}} h(t_1^2-4,t_2^2-4,f)\sum_{j_1,j_2=0}^{J}(-1)^{j_1+j_2}\binom{J}{j_1}\binom{J}{j_2}F_{X,d}(t_1,t_2,f,j_1,j_2), \tag{1-6}$$

where $t_1,t_2>2$ and f run over the integers, the factor $F_{X,d}(t_1,t_2,f,j_1,j_2)$ is an analytic expression, and $h(t_1^2-4,t_2^2-4,f)$ has the following arithmetic meaning. If $d_1,d_2,t\in\mathbb{Z}$, then $h(d_1,d_2,t)$ denotes the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of pairs (Q_1,Q_2) of quadratic forms $Q_i(X,Y)=A_iX^2+B_iXY+C_iY^2$ with integer coefficients satisfying that the discriminant of Q_i is d_i, and the codiscriminant $B_1B_2-2A_1C_2-2A_2C_1$ of Q_1 and Q_2 is t.

Now, (1-6) can be estimated in the following way. For certain ranges of the parameters t_1,t_2 and f we will show that if these three parameters are fixed, then the summation over j_1,j_2 will be negligibly small. This will follow simply from the mean-value theorem of differential calculus, using that J is large enough. For those ranges of t_1,t_2 and f where this reasoning does not work, we estimate every term of the summation separately. In this way we get an upper bound for (1-6) of size $d^{5/2}X^{-1/2+\epsilon}$. Balancing it with the square of the error term in (1-4) we get the theorem choosing $d=X^{5/7}$.

We note that $h(d_1,d_2,t)$ was studied in the papers [H-W] and [M]. They gave explicit formulas for $h(d_1,d_2,t)$ but only under restrictive conditions for the parameters, so we cannot apply their results. Therefore we prove a general upper bound for $h(d_1,d_2,t)$ and apply it in the proof of Theorem 1.1. It would be interesting to investigate in the future whether it is possible to improve the estimate in Theorem 1.1 using an explicit formula instead of our upper bound.

1.3. ***Structure of the paper.*** In Section 2 we give a general formula for the inner product of two automorphic functions $\sum_{\gamma\in\Gamma_{t_i}} m_i(u(z,\gamma z))$, where the m_i are test functions, $t_1,t_2>2$ are integers, and Γ_{t_i} is the set of elements of $\mathrm{SL}_2(\mathbb{Z})$ with trace t_i. The class numbers $h(t_1^2-4,t_2^2-4,f)$ occur in that formula. In Section 3 we give an upper bound for $h(t_1^2-4,t_2^2-4,f)$, and in Section 4 we investigate the special functions appearing in the formula of Section 2 in the case when m_i are characteristic functions as in the circle problem. In Section 5 we begin the proof of Theorem 1.1 by giving the spectral estimate and bounding the contribution of nonhyperbolic elements. In Section 6 we complete the proof by estimating the square integral (1-5).

2. Inner product of automorphic functions and class numbers of pairs of quadratic forms

Our main goal in this section is to prove Lemma 2.2, which relates the inner product of two automorphic functions of a special kind to class numbers of pairs of quadratic forms. Before that we give the necessary definitions and prove an easy lemma to be used later.

2.1. ***Definitions and an upper bound.*** We start by taking a positive discriminant s and introducing the set $\mathcal{Q}_s$ of quadratic forms with discriminant s. Let s be a positive integer with $s\equiv 0,1$ modulo 4 and define

$$\mathcal{Q}_s := \{Q(X,Y) = AX^2 + BXY + CY^2 : A, B, C \in \mathbb{Z},\ B^2 - 4AC = s\}.$$

If $\tau = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and Q is a quadratic form, let us define the quadratic form Q^τ by $Q^\tau(X,Y) = Q(aX+bY, cX+dY)$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts in this way on $\mathcal{Q}_s$. When $s = t^2 - 4$, the set $\mathcal{Q}_s$ can be identified with elements in $\mathrm{SL}_2(\mathbb{Z})$ with trace t. Indeed, if $t > 2$ is an integer, let

$$\Gamma_t = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a + d = t\}.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on Γ_t by conjugation. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_t$, let $Q_\gamma(X,Y) = cX^2 + (d-a)XY - bY^2$. It is easy to see (see [B2], p. 119) that the map $\gamma \mapsto Q_\gamma$ is a one-to-one correspondence between Γ_t and $\mathcal{Q}_s$ with $s = t^2 - 4$, and also between the conjugacy classes of Γ_t over $\mathrm{SL}_2(\mathbb{Z})$ and the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{Q}_s$. More precisely: If $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma \in \Gamma_t$, then $Q_{\tau^{-1}\gamma\tau} = Q_\gamma^\tau$. The fixed points of γ on $\mathbb{R}$ are exactly the roots of the quadratic polynomial $Q_\gamma(X, 1)$.

For $d_1, d_2, t \in \mathbb{Z}$, let $\mathcal{Q}_{d_1,d_2,t}$ be the subset of $\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2}$ consisting of those pairs (Q_1, Q_2) of quadratic forms having codiscriminant t. In other words, writing

$$Q_1(X,Y) = A_1X^2 + B_1XY + C_1Y^2, \quad Q_2(X,Y) = A_2X^2 + B_2XY + C_2Y^2, \tag{2-1}$$

we require that the discriminant of Q_j be d_j $(j = 1, 2)$ and that

$$B_1B_2 - 2A_1C_2 - 2A_2C_1 = t. \tag{2-2}$$

It is easy to check that if $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $(Q_1, Q_2) \in \mathcal{Q}_{d_1,d_2,t}$, then $(Q_1^\tau, Q_2^\tau) \in \mathcal{Q}_{d_1,d_2,t}$. Hence $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_{d_1,d_2,t}$. Let us denote by $h(d_1, d_2, t)$ the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{Q}_{d_1,d_2,t}$.

If $t_1 > 2$, $t_2 > 2$ are integers, let $\mathcal{R}_{t_1,t_2}$ be the subset of $\mathcal{Q}_{t_1^2-4} \times \mathcal{Q}_{t_2^2-4}$ consisting of those pairs (Q_1, Q_2) of quadratic forms satisfying

$$Q_1 = \lambda Q_2 \quad \text{for } some \lambda \in \mathbb{Q}. \tag{2-3}$$

Note that $\mathcal{R}_{t_1,t_2}$ is empty unless $(t_1^2 - 4)/(t_2^2 - 4) \in \mathbb{Q}^2$. It is easy to check that if $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$, then $(Q_1^\tau, Q_2^\tau) \in \mathcal{R}_{t_1,t_2}$. Hence $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{R}_{t_1,t_2}$. Let $\mathcal{R}^*_{t_1,t_2}$ denote a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{R}_{t_1,t_2}$.

If $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$, we can define a nonnegative real number $n(Q_1, Q_2)$ in the following way. Using the bijection $\gamma \mapsto Q_\gamma$ defined earlier, let $\gamma_i \in \Gamma_{t_i}$ be such that $Q_{\gamma_i} = Q_i$ for $i = 1, 2$. The γ_i are uniquely determined. The fixed points on $\mathbb{R}$ of the hyperbolic transformations γ_1 and γ_2 are the same by (2-3), since they are the roots of the polynomial $Q_1(X, 1) = \lambda Q_2(X, 1)$. Denoting the centralizer of a hyperbolic element γ in $\mathrm{SL}_2(\mathbb{Z})$ by $C(\gamma)$ it is well-known and easily proved that

$$C(\gamma) = \{\tau \in \mathrm{SL}_2(\mathbb{Z}) : \tau z_1 = z_1,\ \tau z_2 = z_2\}, \tag{2-4}$$

where z_1 and z_2 are the fixed points of γ. Therefore $C(\gamma_1) = C(\gamma_2)$. The image of $C(\gamma_1)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is infinite cyclic, i.e., there is a $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$C(\gamma_1) = \{\pm\gamma_0^l \in \mathrm{SL}_2(\mathbb{Z}) : l \in \mathbb{Z}\}. \tag{2-5}$$

Let $N(\gamma)$ denote the norm of a hyperbolic transformation γ, see p. 19 of [I]. Let us define $n(Q_1, Q_2) := |\log N(\gamma_0)|$; this quantity is well-defined. It can be seen that if $\tau \in \mathrm{SL}_2(\mathbb{Z})$, then $n(Q_1^\tau, Q_2^\tau) = n(Q_1, Q_2)$. Finally, if $t_1 > 2$, $t_2 > 2$ are integers, let us define

$$E_{t_1,t_2} := \sum_{(Q_1,Q_2)\in\mathcal{R}^*_{t_1,t_2}} n(Q_1, Q_2). \tag{2-6}$$

The following lemma will be enough for handling E_{t_1,t_2} during the proof of Theorem 1.1.

Lemma 2.1. *If $2 < t_1 \le t_2$ are integers, then $E_{t_1,t_2} \ll_\epsilon t_2^{1+\epsilon}$ for every $\epsilon > 0$.*

Proof. If $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$ and $\gamma_i \in \Gamma_{t_i}$ is such that $Q_{\gamma_i} = Q_i$ for $i = 1, 2$, then for γ_0 satisfying (2-5) we clearly have $|\log N(\gamma_0)| \le |\log N(\gamma_2)| \ll \log t_2$. So it is enough to show that the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{R}_{t_1,t_2}$ is $\ll_\epsilon t_2^{1+\epsilon}$. If $Q_2 \in \mathcal{Q}_{t_2^2-4}$ is given, then there are at most two possibilities for Q_1 to have $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$, so it is enough to show that the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{Q}_{t_2^2-4}$ is $\ll_\epsilon t_2^{1+\epsilon}$. This follows from [Bu, Proposition 3.3 and formula (3.1)]. The lemma is proved. □

2.2. ***The formula for the inner product.*** If $t > 2$ is an integer and m is a compactly supported bounded function on $[0, \infty)$, then for $z, w \in \mathbb{H}$ write

$$m(z, w) = m(u(z, w)) \tag{2-7}$$

by an abuse of notation; see (1-1) for $u(z, w)$. For $z \in \mathbb{H}$ define

$$M_{t,m}(z) = \sum_{\gamma\in\Gamma_t} m(z, \gamma z). \tag{2-8}$$

The main result of this subsection, Lemma 2.2, expresses the inner product of two such functions M_{t_1,m_1}, M_{t_2,m_2} in terms of the quantities E_{t_1,t_2} and $h(t_1^2-4, t_2^2-4, f)$ defined above. We need also the following definitions to state the lemma.

Let $t_1, t_2 > 2$ be real numbers and let m_1, m_2 be compactly supported bounded functions on $[0, \infty)$. Let us write

$$\mathcal{J}(t_1, t_2, m_1, m_2) := \int_{-\pi/2}^{\pi/2} m_1\left(\frac{t_1^2-4}{4\cos^2\theta}\right) m_2\left(\frac{t_2^2-4}{4\cos^2\theta}\right) \frac{d\theta}{\cos^2\theta}, \tag{2-9}$$

and for every real F with $|F| \ne 1$ let us write

$$\mathcal{I}(t_1, t_2, F, m_1, m_2) := \iint \frac{m_1\big(\frac{1}{4}(t_1^2-4)(1+S^2)\big) m_2\big(\frac{1}{4}(t_2^2-4)(1+T^2)\big)}{\sqrt{S^2+T^2+2FTS+1-F^2}}\, dS\, dT, \tag{2-10}$$

where we integrate over the set

$$\{(S, T) \in \mathbb{R}^2 : S^2 + T^2 + 2FTS + 1 - F^2 > 0\}. \tag{2-11}$$

Remark 2.1. The integral (2-9) is absolutely convergent, because m_1 and m_2 are compactly supported and bounded. The absolute convergence of the integral in (2-10) is trivial in the case $|F|<1$, because then we always have $S^2+T^2+2FTS\geq 0$. In the case $|F|>1$ we use the linear substitution $u=S+\big(F+\sqrt{F^2-1}\big)T$, $v=S+\big(F-\sqrt{F^2-1}\big)T$. Since m_1 and m_2 are compactly supported, we have in (2-10) that $|S|$ and $|T|$ are bounded from above. But then we have also $|u|,|v|<C$ with some $C>0$. The condition in (2-11) reads as $uv\geq F^2-1$, therefore we have also $|u|,|v|>c$ with some $c>0$. Now, integrating over the set defined by the conditions $c<|u|,\ |v|<C$ and $uv\geq F^2-1$, we clearly have $\iint (1/\sqrt{uv+1-F^2})\,du\,dv<\infty$. Thus (2-10) is absolutely convergent also for $|F|>1$.

Lemma 2.2. *Let $t_1,t_2>2$ be integers and let m_1,m_2 be compactly supported bounded functions on* $[0,\infty)$. *Then, using equation* (2-8), *the integral*

$$\int_{\mathcal{F}} M_{t_1,m_1}(z)M_{t_2,m_2}(z)\,d\mu_z \tag{2-12}$$

equals the sum of

$$\mathcal{J}(t_1,t_2,m_1,m_2)E_{t_1,t_2} \tag{2-13}$$

and

$$\sum_{\substack{f\in\mathbb{Z}\\ f^2\neq(t_1^2-4)(t_2^2-4)}} h(t_1^2-4,t_2^2-4,f)\,\mathcal{I}\left(t_1,t_2,\frac{f}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}},m_1,m_2\right). \tag{2-14}$$

(*See equations* (2-1) *and* (2-2), *with the subsequent paragraph, for* $h(t_1^2-4,t_2^2-4,f)$, *equation* (2-6) *for* E_{t_1,t_2}, *and* (2-9), (2-10) *for the functions* $\mathcal{J}$ *and* $\mathcal{I}$.)

Remark 2.2. We will see later that the class numbers $h(t_1^2-4,t_2^2-4,f)$ are finite, see Lemma 3.1 and Remark 3.1. The sum (2-14) is actually finite, because for large enough $|f|$ the function $\mathcal{I}$ vanishes there. This can be seen easily from (2-10) and (2-11).

For the proof of Lemma 2.2 we need a few preliminary lemmas. To state the first one, we give some definitions using the notations of Lemma 2.2.

Write $G:=\Gamma_{t_1}\times\Gamma_{t_2}$, and let G_0 be the set of those elements $(\gamma_1,\gamma_2)\in G$ for which the set of fixed points on $\mathbb{R}$ of γ_1 and of γ_2 are the same. If (γ_1,γ_2), $(\gamma_1^*,\gamma_2^*)\in G$, we say that (γ_1,γ_2) and (γ_1^*,γ_2^*) are $\mathrm{SL}_2(\mathbb{Z})$-equivalent if there is an element $\tau\in\mathrm{SL}_2(\mathbb{Z})$ such that $\tau^{-1}\gamma_i\tau=\gamma_i^*$ for $i=1,2$. We denote by G_0^* a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of G_0, and by $(G\setminus G_0)^*$ a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $G\setminus G_0$.

Lemma 2.3. *Let $t_1,t_2>2$ be integers and let m_1,m_2 be compactly supported bounded functions on* $[0,\infty)$. *Recall equation* (2-8). *Then*

$$\int_{\mathcal{F}} M_{t_1,m_1}(z)M_{t_2,m_2}(z)\,d\mu_z \tag{2-15}$$

equals the sum of

$$\sum_{(\gamma_1,\gamma_2)\in G_0^*}\int_{C(\gamma_1)\backslash\mathbb{H}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z \quad \textit{and} \quad \sum_{(\gamma_1,\gamma_2)\in(G\setminus G_0)^*}\int_{\mathbb{H}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z.$$

Remark 2.3. To avoid confusion we emphasize that $G \setminus G_0$ denotes set difference, while $C(\gamma_1)\backslash\mathbb{H}$ denotes quotient on the left.

Proof. An element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\operatorname{tr}\gamma > 2$ determines a hyperbolic transformation of $\mathbb{H}$; see Section 1.5 of [I]. Hence γ has two different fixed points on $\mathbb{R}$. Assume that $\gamma_1 \in \Gamma_{t_1}$, $\gamma_2 \in \Gamma_{t_2}$, $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\tau^{-1}\gamma_1\tau = \gamma_1, \quad \tau^{-1}\gamma_2\tau = \gamma_2. \tag{2-16}$$

It is clear by (2-4) that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then (2-16) is true if and only if $\tau = \pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. If $(\gamma_1, \gamma_2) \in G_0$, then by (2-4) we see that $C(\gamma_1) = C(\gamma_2)$, and (2-16) is true if and only if $\tau \in C(\gamma_1)$.

From the definitions we see that (2-15) equals

$$\sum_{\gamma_1\in\Gamma_{t_1}} \sum_{\gamma_2\in\Gamma_{t_2}} \int_{\mathcal{F}} m_1(z, \gamma_1 z) m_2(z, \gamma_2 z)\, d\mu_z.$$

We partition G into $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes. Since for $\tau \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\int_{\mathcal{F}} m_1(z, \tau^{-1}\gamma_1\tau z) m_2(z, \tau^{-1}\gamma_2\tau z)\, d\mu_z = \int_{\tau\mathcal{F}} m_1(z, \gamma_1 z) m_2(z, \gamma_2 z)\, d\mu_z,$$

our considerations above give the lemma. □

Lemma 2.4. *Let* $\gamma_1 = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ *and* $\gamma_2 = \left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ *be hyperbolic elements of* $\mathrm{SL}_2(\mathbb{R})$, *and assume that the set of fixed points of* γ_1 *and the set of fixed points of* γ_2 *are disjoint. Let*

$$F := F(\gamma_1, \gamma_2) = \frac{(d-a)(D-A) + 2bC + 2Bc}{\sqrt{(d+a)^2-4}\sqrt{(D+A)^2-4}}. \tag{2-17}$$

Let us write $t_1 = a + d$, $t_2 = A + D$, *and assume* $t_1, t_2 > 2$. *Let* m_1, m_2 *be compactly supported bounded functions on* $[0, \infty)$ *and use* (2-7). *Then we have*

$$\int_{\mathbb{H}} m_1(z, \gamma_1 z) m_2(z, \gamma_2 z)\, d\mu_z = \mathcal{I}(t_1, t_2, F, m_1, m_2), \tag{2-18}$$

where the function $\mathcal{I}$ *is defined in* (2-10) *and* (2-11).

Proof. It is easy to check that $F(\gamma_1, \gamma_2) = F(\tau^{-1}\gamma_1\tau, \tau^{-1}\gamma_2\tau)$ for $\tau \in \mathrm{SL}_2(\mathbb{R})$. Since (2-18) also remains the same if we write $\tau^{-1}\gamma_1\tau$ and $\tau^{-1}\gamma_2\tau$ in place of γ_1 and γ_2, and since we can choose τ in such a way that $\tau^{-1}\gamma_1\tau$ is diagonal, for the proof of the lemma we may assume that γ_1 is diagonal.

So assume that $b = c = 0$. Then by the conditions we have $BC \neq 0$. It can be easily computed by the definitions that

$$u(z, \gamma_1 z) = \frac{(d-a)^2|z|^2}{4\operatorname{Im}^2 z}, \quad u(z, \gamma_2 z) = \frac{\left|Cz^2 + (D-A)z - B\right|^2}{4\operatorname{Im}^2 z}.$$

Hence, if $z = x + iy$, using $ad = 1$ and $AD - BC = 1$ we get after some computations that

$$u(z, \gamma_1 z) = \frac{(a+d)^2-4}{4} + \frac{(d-a)^2x^2}{4y^2}, \tag{2-19}$$

$$u(z, \gamma_2 z) = \frac{(A+D)^2-4}{4} + \frac{(Cx^2+(D-A)x-B+Cy^2)^2}{4y^2}. \tag{2-20}$$

Since $m(z,w)$ is defined through the function u, we will be able to compute the left-hand side of (2-18) using (2-19) and (2-20).

Let us use the substitution

$$q := \frac{x}{y}, \quad r := \frac{Cx^2 + (D-A)x - B + Cy^2}{y}. \tag{2-21}$$

Then the determinant of the Jacobi matrix $\frac{dq\,dr}{dx\,dy}$ is

$$\det\begin{pmatrix} 1/y & (2Cx+(D-A))/y \\ -x/y^2 & C-(Cx^2+(D-A)x-B)/y^2 \end{pmatrix} = \frac{B+Cx^2+Cy^2}{y^3}.$$

It is not hard to check that

$$\frac{B+Cx^2+Cy^2}{y} = r - (D-A)q + \frac{2B}{y} \tag{2-22}$$

and

$$\frac{B}{y^2} + \frac{r-(D-A)q}{y} - (C+Cq^2) = 0. \tag{2-23}$$

From (2-22) and (2-23) we get

$$\left(\frac{B+Cx^2+Cy^2}{y}\right)^2 = ((D-A)q-r)^2 + 4B(C+Cq^2).$$

Hence if we want to compute the left-hand side of (2-18) by the substitution (2-21), then on the one hand we see that for q and r we have the condition

$$((D-A)q-r)^2 + 4B(C+Cq^2) \geq 0. \tag{2-24}$$

On the other hand, in the case $BC > 0$ we see from the quadratic equation (2-23) that for every real q and r satisfying (2-24) there is exactly one $y > 0$ and real x satisfying (2-21). Similarly, in the case $BC < 0$ we see from the quadratic equation (2-23) that $((D-A)q-r)/B > 0$, i.e., combined with (2-24) we must have

$$\frac{(D-A)q-r}{B} \geq 2\sqrt{-\frac{C+Cq^2}{B}}. \tag{2-25}$$

If the left side of (2-25) is larger than the right side, then we have two positive solutions of (2-23) in $1/y$. If (2-25) holds with equality, then we have a double root.

Putting everything together we see that the left-hand side of (2-18) equals

$$\int_{-\infty}^{\infty} \int_{A_q} f(r,q)\,dr\,dq \tag{2-26}$$

for $BC > 0$, and the left-hand side of (2-18) equals

$$2\int_{-\infty}^{\infty} \int_{A_q^+} f(r,q)\,dr\,dq \tag{2-27}$$

for $BC < 0$, where

$$f(r,q) := \frac{m_1\big(\frac14((a+d)^2-4)+\frac14(d-a)^2q^2\big)m_2\big(\frac14((A+D)^2-4)+\frac14 r^2\big)}{\sqrt{((D-A)q-r)^2+4B(C+Cq^2)}},$$

$$A_q := \{r \in \mathbb{R} : ((D-A)q-r)^2+4B(C+Cq^2) \geq 0\},$$

and for $BC < 0$ we write

$$A_q^+ := \left\{ r \in \mathbb{R} : \frac{(D-A)q-r}{B} \geq 2\sqrt{-\frac{C+Cq^2}{B}} \right\}.$$

For $BC < 0$ we define also

$$A_q^- := \left\{ r \in \mathbb{R} : \frac{(D-A)q-r}{B} \leq -2\sqrt{-\frac{C+Cq^2}{B}} \right\}.$$

We see that for $BC < 0$ we have $-A_{-q}^+ = A_q^-$, so, since $f(r,q) = f(-r,-q)$, for $BC < 0$ we have that (2-27) equals

$$\int_{-\infty}^{\infty}\int_{A_q^+} f(r,q)\,dr\,dq + \int_{-\infty}^{\infty}\int_{A_q^-} f(r,q)\,dr\,dq.$$

Since A_q is the disjoint union of A_q^+ and A_q^- for $BC < 0$, we finally get that the left-hand side of (2-18) equals (2-26) also in the case $BC < 0$. Apply the substitution

$$S = q, \quad T = \frac{-r}{\sqrt{(D+A)^2-4}}$$

in (2-26). Recalling (2-17), $b = c = 0$ and $BC \neq 0$ we have that

$$(a+d)^2-4 = (a-d)^2, \quad (D-A)^2+4BC = (D+A)^2-4, \quad F^2 = \frac{(D-A)^2}{(D+A)^2-4} \neq 1.$$

Taking into account that $\mathcal{I}(t_1,t_2,F,m_1,m_2)$ is even in F, we get (2-18) for the case $b = c = 0$. But we have seen that then the lemma is completely proved. □

Lemma 2.5. *Let $\gamma_1 \in \Gamma_{t_1}$, $\gamma_2 \in \Gamma_{t_2}$, where $t_i > 2$ for $i = 1,2$. Assume that γ_1 and γ_2 have the same fixed points. Then*

$$\int_{C(\gamma_1)\backslash\mathbb{H}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z = \mathcal{J}(t_1,t_2,m_1,m_2)\left|\log N(\gamma_0)\right|, \tag{2-28}$$

where $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ is a generator of the centralizer $C(\gamma_1)$; see (2-5). (The function $\mathcal{J}$ is defined in (2-9).)

Proof. We may assume that $N(\gamma_0) > 1$. We can choose $\tau \in \mathrm{SL}_2(\mathbb{R})$ in such a way that $\tau^{-1}\gamma_i\tau z = \lambda_i z$ for every $z \in \mathbb{H}$ and $0 \leq i \leq 2$ with $\lambda_0 = N(\gamma_0)$ and $\lambda_i = N(\gamma_i)^{\epsilon_i}$ for $i = 1,2$, where $\epsilon_i \in \{-1,1\}$. The fundamental domain of the group $\tau^{-1}C(\gamma_1)\tau$ in $\mathbb{H}$ is the subset $\{1 \leq |z| < N(\gamma_0)\}$. Then the left-hand side of (2-28) equals

$$\int_{\{z\in\mathbb{H}:\,1\leq|z|<N(\gamma_0)\}} m_1(z,\tau^{-1}\gamma_1\tau z)m_2(z,\tau^{-1}\gamma_2\tau z)\,d\mu_z. \tag{2-29}$$

By the substitution $z = re^{i(\frac{\pi}{2}+\theta)}$ with $1 \le r < N(\gamma_0)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ we get using $d\mu_z = \dfrac{dr\, d\theta}{r\cos^2\theta}$ that (2-29) equals

$$\int_{-\pi/2}^{\pi/2}\int_1^{N(\gamma_0)} m_1\left(\frac{\lambda_1+\lambda_1^{-1}-2}{4\cos^2\theta}\right) m_2\left(\frac{\lambda_2+\lambda_2^{-1}-2}{4\cos^2\theta}\right)\frac{dr\, d\theta}{r\cos^2\theta}.$$

It is clear that $\lambda_1^{1/2}+\lambda_1^{-1/2}=t_1$, $\lambda_2^{1/2}+\lambda_2^{-1/2}=t_2$, since the trace is invariant under conjugation. The lemma is proved. □

Proof of Lemma 2.2. We use the one-to-one correspondence between Γ_{t_i} and $\mathcal{Q}_{t_i^2-4}$ for $i = 1, 2$.

Using the notation of Lemma 2.3, we see that if $(\gamma_1, \gamma_2) \in G$, then $(\gamma_1, \gamma_2) \in G_0$ holds if and only if $Q_{\gamma_1} = \lambda Q_{\gamma_2}$ with some $\lambda \in \mathbb{Q}$. Indeed, $Q_{\gamma_1} = \lambda Q_{\gamma_2}$ holds with some $\lambda \in \mathbb{Q}$ if and only if the polynomials $Q_{\gamma_1}(X, 1)$ and $Q_{\gamma_2}(X, 1)$ have the same roots, i.e., if and only if γ_1 and γ_2 have the same fixed points.

It is clear that $(\gamma_1, \gamma_2), (\gamma_1^*, \gamma_2^*) \in G$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent if and only if there is a $\tau \in \mathrm{SL}_2(\mathbb{Z})$ such that $(Q_{\gamma_1}^\tau, Q_{\gamma_2}^\tau) = (Q_{\gamma_1^*}, Q_{,\gamma_2^*})$. We note also that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then for the quantity $F(\gamma_1, \gamma_2)$ defined in (2-17) we have $F(\gamma_1, \gamma_2) = /(f)\sqrt{t_1^2-4}\sqrt{t_2^2-4}$, with

$$Q_{\gamma_1}(X, Y) = A_1X^2 + B_1XY + C_1Y^2, \quad Q_{\gamma_2}(X, Y) = A_2X^2 + B_2XY + C_2Y^2, \tag{2-30}$$

$$f = B_1B_2 - 2A_1C_2 - 2A_2C_1. \tag{2-31}$$

We show that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then $f^2 \ne (t_1^2-4)(t_2^2-4)$. Indeed, writing $d_i := t_i^2 - 4$ and using $d_i = B_i^2 - 4A_iC_i$ for $i = 1, 2$, we easily get from (2-30) and (2-31) that

$$d_2^2A_1^2 - 2fd_2A_1A_2 + d_1d_2A_2^2 = d_2(B_2A_1 - B_1A_2)^2,$$
$$d_2^2C_1^2 - 2fd_2C_1C_2 + d_1d_2C_2^2 = d_2(B_2C_1 - B_1C_2)^2.$$

Assume $f^2 = d_1d_2$. Then the left-hand sides above are squares, and since $d_2 = t_2^2-4$ cannot be a square, we get $B_2A_1 - B_1A_2 = 0$, $B_2C_1 - B_1C_2 = 0$. One has the identity

$$(A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1) = \frac{1}{4}\Big(f^2 - \prod_{i=1}^{2}(B_i^2 - 4A_iC_i)\Big), \tag{2-32}$$

with f defined in (2-31). We also get $A_1C_2 - A_2C_1 = 0$ from (2-32). It follows that the vectors (A_1, B_1, C_1) and (A_2, B_2, C_2) are linearly dependent, hence $(\gamma_1, \gamma_2) \in G_0$, which is a contradiction.

We note finally that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then γ_1 and γ_2 have no fixed points in common.

With these considerations, applying Lemmas 2.3, 2.4, 2.5 and the definitions we obtain the lemma. □

3. Estimates on the number of equivalence classes of quadratic forms

Recall the definitions of $\mathcal{Q}_{d_1,d_2,t}$ and $h(d_1, d_2, t)$ from (2-1), (2-2) and the subsequent paragraph. In this section we will give several upper bounds for $h(d_1, d_2, t)$ itself and for certain sums containing $h(d_1, d_2, t)$.

Let $\gcd(n_1, n_2, \ldots, n_r)$ be the greatest common divisor of the integers $n_1, n_2, \ldots, n_r$. The integer part of a real number x is denoted by $[x]$.

3.1. ***A general upper bound for $h(d_1, d_2, t)$.*** Our aim in this subsection is to prove Lemma 3.1. The upper bound we give for $h(d_1, d_2, t)$ will be smaller than $(1 + |d_1 d_2 t|)^\epsilon$ for any fixed $\epsilon > 0$ in many cases. This is not always true, but the exceptions are rare.

For any finite set of integers $n_1, n_2, \ldots, n_r$ we write

$$S(n_1, n_2, \ldots, n_r) = \max\{k \geq 1 : k^2 \mid \gcd(n_1, n_2, \ldots, n_r)\}. \tag{3-1}$$

Denote by $\tau(n)$ the number of divisors and by $\omega(n)$ the number of distinct prime divisors of a nonzero integer n.

Lemma 3.1. *Assume that $d_1, d_2, t \in \mathbb{Z}$ and d_i is not a square of an integer* $(i = 1, 2)$. *Assume also that $t^2 - d_1 d_2 \neq 0$. Then*

$$h(d_1, d_2, t) \leq C \cdot 2^{\omega(t^2 - d_1 d_2)} \tau(t^2 - d_1 d_2) S(d_1, d_2, t^2), \tag{3-2}$$

where $C > 0$ is an absolute constant.

Remark 3.1. Even the finiteness of $h(d_1, d_2, t)$ is not completely obvious. For a short proof of this fact using the theory of algebraic groups see Appendix I of [M].

For the case $d_i = t_i^2 - 4$ for $i = 1, 2$ with integers $t_i > 2$, which is our primary interest, one can give a trivial upper bound for $h(d_1, d_2, t)$ using Lemma 2.2. For simplicity let us consider the case when $t_1^2 - 4$, $t_2^2 - 4$ and t all lie in $[X, 2X]$ with a large real number X. Then choosing m_1 and also m_2 in Lemma 2.2 to be the characteristic function of the interval $[0, CX]$ with a suitable absolute constant C, one can show the trivial bound $h(t_1^2 - 4,\ t_2^2 - 4,\ t) \ll X$. Indeed, the coefficient of $h(t_1^2 - 4,\ t_2^2 - 4,\ t)$ is bounded from below by a positive constant, every term in (2-14) and (2-13) is nonnegative, and one can prove that (2-12) equals $O(X)$ in this case. The estimate (3-2) gives better than $h(t_1^2 - 4,\ t_2^2 - 4,\ t) \ll X$ even in the worst case, when $S(t_1^2 - 4,\ t_2^2 - 4,\ t^2)$ is as large as $\sqrt{X}$. But the S-function is often much smaller than $\sqrt{X}$, so the bound given in Lemma 3.1 is much stronger than the trivial bound.

To prepare the proof of Lemma 3.1 we need two preliminary lemmas. We introduce the notation

$$C_{d_1,d_2,t} := \{(x, y) \in \mathbb{R}^2 : d_2 x^2 + d_1 y^2 - 2txy = 1\}.$$

In the first lemma we prove general statements for any two different points of $C_{d_1,d_2,t}$. In the second one we show that if we have any element of $\mathcal{Q}_{d_1,d_2,t}$, then we can parametrize the rational points of $C_{d_1,d_2,t}$.

Lemma 3.2. *Let d_1, d_2, t be as in Lemma 3.1, and assume that $(x_i, y_i) \in C_{d_1,d_2,t}$ for $i = 1, 2$ and $(x_1, y_1) \neq (x_2, y_2)$. Then*

$$d_2(x_1 - x_2)^2 + d_1(y_1 - y_2)^2 - 2t(x_1 - x_2)(y_1 - y_2) \neq 0 \tag{3-3}$$

and

$$(d_1 y_1 - t x_1)(y_1 - y_2) + (d_2 x_1 - t y_1)(x_1 - x_2) \neq 0. \tag{3-4}$$

Proof. Let S_1 and S_2 be the quantities appearing in (3-3) and (3-4), respectively. One can check that

$$S_2 = \begin{pmatrix} y_1 & x_1 \end{pmatrix} \begin{pmatrix} d_1 & -t \\ -t & d_2 \end{pmatrix} \begin{pmatrix} y_1 - y_2 \\ x_1 - x_2 \end{pmatrix} \tag{3-5}$$

and

$$2S_2 + \sum_{i=1}^{2} (-1)^i (d_2 x_i^2 + d_1 y_i^2 - 2t x_i y_i) = S_1.$$

Since $(x_i, y_i) \in C_{d_1,d_2,t}$ for $i = 1, 2$, we have $S_1 = 2S_2$. Hence it is enough to show that $S_1 \neq 0$. Assume for a contradiction that $S_1 = 0$. Then the right-hand side of (3-5) is 0, but this is true also by exchanging the role of (x_1, y_1) and (x_2, y_2), so we get

$$\begin{pmatrix} y_1 & x_1 \\ y_2 & x_2 \end{pmatrix} \begin{pmatrix} d_1 & -t \\ -t & d_2 \end{pmatrix} \begin{pmatrix} y_1 - y_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $\left(\begin{smallmatrix} y_1 - y_2 \\ x_1 - x_2 \end{smallmatrix}\right)$ is nonzero and $\det\left(\begin{smallmatrix} d_1 & -t \\ -t & d_2 \end{smallmatrix}\right) \neq 0$ since $t^2 - d_1 d_2 \neq 0$, so we must have $\det\left(\begin{smallmatrix} y_1 & x_1 \\ y_2 & x_2 \end{smallmatrix}\right) = 0$. Hence $(x_2, y_2) = \lambda(x_1, y_1)$ with some constant $\lambda \neq 1$, so $S_1 = (1-\lambda)^2 \neq 0$ by our assumptions. This is a contradiction, and the lemma is proved. □

Lemma 3.3. *Let d_1, d_2, t be as in Lemma 3.1, and let $Q_i(X, Y) = A_i X^2 + B_i XY + C_i Y^2$ $(i = 1, 2)$ be such that $(Q_1, Q_2) \in \mathcal{Q}_{d_1,d_2,t}$. Assume that $A_1 B_2 - A_2 B_1 \neq 0$. Define*

$$R(X, Y) = (A_1 B_2 - A_2 B_1) X^2 + 2XY(A_1 C_2 - A_2 C_1) + Y^2 (B_1 C_2 - B_2 C_1).$$

Then for $x, y \in \mathbb{Q}$ the following two statements are equivalent.

(i) $(x, y) \in C_{d_1,d_2,t}$.

(ii) *There are $a, b \in \mathbb{Q}$ such that $R(a, b) \neq 0$ and writing $x_{a,b} := \dfrac{Q_1(a, b)}{R(a, b)}$, $y_{a,b} := \dfrac{Q_2(a, b)}{R(a, b)}$ we have $x = x_{a,b}$, $y = y_{a,b}$.*

Proof. By a straightforward computation using the definitions we get the identity

$$d_2 (Q_1(a, b))^2 + d_1 (Q_2(a, b))^2 - 2t Q_1(a, b) Q_2(a, b) = (R(a, b))^2. \tag{3-6}$$

Introduce the abbreviations

$$a_1 = \frac{A_1}{A_1 B_2 - A_2 B_1}, \quad a_2 = \frac{A_2}{A_1 B_2 - A_2 B_1}. \tag{3-7}$$

Note that writing $a = 1$, $b = 0$ in (3-6) we get $(a_1, a_2) \in C_{d_1,d_2,t}$. We first assume (ii). Then (i) follows at once from (3-6).

We now assume (i). If $(x, y) = (a_1, a_2)$, then we can take $a = 1$, $b = 0$. So let us assume that $(x, y) \neq (a_1, a_2)$. It is easy to see that if $a, b \in \mathbb{Q}$, then

$$Q_1(a, b) - a_1 R(a, b) = \frac{b(a\alpha + b\beta)}{(A_1 B_2 - A_2 B_1)}, \quad Q_2(a, b) - a_2 R(a, b) = \frac{b(\gamma a + \delta b)}{(A_1 B_2 - A_2 B_1)} \tag{3-8}$$

with

$$\alpha := B_1(A_1B_2 - A_2B_1) + 2A_1(A_2C_1 - A_1C_2) = tA_1 - d_1A_2, \tag{3-9}$$

$$\beta := C_1(A_1B_2 - A_2B_1) + A_1(C_1B_2 - C_2B_1), \tag{3-10}$$

$$\gamma := B_2(A_1B_2 - A_2B_1) + 2A_2(A_2C_1 - A_1C_2) = -tA_2 + d_2A_1, \tag{3-11}$$

$$\delta := C_2(A_1B_2 - A_2B_1) + A_2(C_1B_2 - C_2B_1). \tag{3-12}$$

Let us write $g := \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$. One can compute that

$$\det g = 2(A_2B_1 - A_1B_2)\big((A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1)\big).$$

The last factor, in larger parentheses, equals $\frac{1}{4}(t^2 - d_1d_2)$ by (2-32) and (2-31). Hence $t^2 - d_1d_2 \neq 0$ and $A_1B_2 - A_2B_1 \neq 0$ imply $\det g \neq 0$.

Let us take $a, b \in \mathbb{Q}$ in the following way:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}. \tag{3-13}$$

By (3-8) and (3-13) we then easily get

$$\begin{pmatrix} Q_1(a,b) - a_1R(a,b) \\ Q_2(a,b) - a_2R(a,b) \end{pmatrix} = \frac{b \det g}{A_1B_2 - A_2B_1} \begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}. \tag{3-14}$$

So this is true if $(x, y) \neq (a_1, a_2)$, and a, b are defined by (3-9)–(3-13).

Assume that $b = 0$. Then by (3-9), (3-11) and (3-13) we get

$$(d_1A_2 - tA_1)(y - a_2) + (-tA_2 + d_2A_1)(x - a_1) = 0.$$

By (3-7), $(a_1, a_2), (x, y) \in C_{d_1,d_2,t}$ and $(x, y) \neq (a_1, a_2)$. This contradicts (3-4). So we have $b \neq 0$.

Assume that $R(a, b) = 0$. Then (3-6) and (3-14) imply that

$$d_2(x - a_1)^2 + d_1(y - a_2)^2 - 2t(x - a_1)(y - a_2) = 0.$$

But this contradicts (3-3). So we have $R(a, b) \neq 0$.

Then (3-14) clearly implies

$$\begin{pmatrix} x_{a,b} - a_1 \\ y_{a,b} - a_2 \end{pmatrix} = \frac{b \det g}{R(a,b)(A_1B_2 - A_2B_1)} \begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}.$$

Hence $(x_{a,b}, y_{a,b}) \neq (a_1, a_2)$, since we assumed $(x, y) \neq (a_1, a_2)$. We would like to show that $(x, y) = (x_{a,b}, y_{a,b})$. If this is false, then (a_1, a_2), $(x_{a,b}, y_{a,b})$ and (x, y) are three distinct points lying on a line, all belonging to $C_{d_1,d_2,t}$. Hence the equation

$$d_2(a_1 + q(x - a_1))^2 + d_1(a_2 + q(y - a_2))^2 - 2t(a_1 + q(x - a_1))(a_2 + q(y - a_2)) = 1$$

has three different real solutions in q. The coefficient of q^2 is nonzero by (3-3), so this is a contradiction. The lemma is proved. □

For the next proof we need the following notation. If p is a prime and $n \neq 0$ is an integer, let us denote by $\nu_p(n)$ the largest nonnegative integer such that $p^{\nu_p(n)}$ divides n.

Proof of Lemma 3.1. If $\mathcal{Q}_{d_1,d_2,t}$ is empty, then $h(d_1, d_2, t) = 0$ and there is nothing to prove. So assume in the sequel that $\mathcal{Q}_{d_1,d_2,t} \neq \varnothing$. We divide the proof into four parts, which we formulate as claims.

Claim A. *Let $Q_i(X, Y) = A_i X^2 + B_i XY + C_i Y^2$ $(i = 1, 2)$ be quadratic forms such that $(Q_1, Q_2) \in \mathcal{Q}_{d_1,d_2,t}$. Then replacing (Q_1, Q_2) by an element in its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class we can achieve $B_2 A_1 - B_1 A_2 \neq 0$.*

Proof. We first show the weaker statement that replacing (Q_1, Q_2) by an element in its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class we may assume that $(B_1, B_2) \neq (0, 0)$. If $\tau = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$Q_i^\tau(X, Y) = Q_i(X + bY, Y) = A_i X^2 + (B_i + 2A_i b) XY + C_i^* Y^2 \tag{3-15}$$

for $i = 1, 2$ with some C_i^*. If $(B_1 + 2A_1 b, B_2 + 2A_2 b) = (0, 0)$ for every integer b, then $A_i = B_i = 0$ for $i = 1, 2$. But this is impossible, since this would imply $d_1 = d_2 = t = 0$, but this contradicts $t^2 - d_1 d_2 \neq 0$.

Hence we may assume that $(B_1, B_2) \neq (0, 0)$. Let $B_1 \neq 0$, say. Assume for a contradiction that $B_2 A_1 - B_1 A_2 = 0$ and $B_2 C_1 - B_1 C_2 = 0$. Then $(A_2, C_2) = \lambda(A_1, C_1)$ with $\lambda = B_2/B_1$, hence $C_2 A_1 - C_1 A_2 = 0$. So the matrix $\left(\begin{smallmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{smallmatrix}\right)$ has rank 1, hence its lines are linearly dependent. But this contradicts $t^2 - d_1 d_2 \neq 0$.

So we may assume that $B_2 A_1 - B_1 A_2 \neq 0$ or $B_2 C_1 - B_1 C_2 \neq 0$. But applying the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we can exchange the roles of A_i and C_i. Claim A follows. □

Claim B. $\mathcal{C}_{d_1,d_2,t} \neq \varnothing$.

Proof. By Claim A there is an element $(Q_1, Q_2) \in \mathcal{Q}_{d_1,d_2,t}$ whose coefficients satisfy $B_2 A_1 - B_1 A_2 \neq 0$. Taking $a = 1$, $b = 0$ in Lemma 3.3(ii) we see by that lemma that there are numbers $x, y \in \mathbb{Q}$ such that $(x, y) \in \mathcal{C}_{d_1,d_2,t}$. Claim B is proved. □

Claim C. *There exists a subset $\mathcal{A}$ of $\mathbb{Z}^2$ of size $2\tau(t^2 - d_1 d_2)$ such that every $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of $\mathcal{Q}_{d_1,d_2,t}$ contains an element (Q_1, Q_2) with coefficients $Q_i(X, Y) = A_i X^2 + B_i XY + C_i Y^2$ $(i = 1, 2)$ such that $(A_1, A_2) \in \mathcal{A}$.*

Proof. Fix $x, y \in \mathbb{Q}$ such that $(x, y) \in \mathcal{C}_{d_1,d_2,t}$, this is possible by Claim B. We also fix relatively prime integers m and n and a nonzero $s \in \mathbb{Q}$ such that $x = sm$, $y = sn$. Let us take an arbitrary $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of $\mathcal{Q}_{d_1,d_2,t}$. We know by Claim A that we can take an element (Q_1, Q_2) in this equivalence class such that we have $B_2 A_1 - B_1 A_2 \neq 0$ for their coefficients. Then it follows from Lemma 3.3 that there are $a, b \in \mathbb{Q}$ such that $Q_1(a, b) = qx$, $Q_2(a, b) = qy$ with some $q \in \mathbb{Q}$, $q \neq 0$. We may clearly assume here that $a, b \in \mathbb{Z}$ and $(a, b) = 1$. Taking $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with some suitable c and d we then see that replacing (Q_1, Q_2) by an element in its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class we may assume that for their coefficients we have $A_1 = rx$, $A_2 = ry$ with some $r \in \mathbb{Q}$, $r \neq 0$. Observe also that

$$\gcd(A_1, A_2) \mid (t^2 - d_1 d_2) \tag{3-16}$$

follows from the definition of d_1, d_2 and t. Then recalling $x = sm$, $y = sn$ we see that $A_1 = \delta m$, $A_2 = \delta n$ with some integer δ dividing $t^2 - d_1 d_2$. Claim C is proved. □

Claim D. *Let the integers A_1 and A_2 be given and assume that there are N inequivalent elements in $\mathcal{Q}_{d_1,d_2,t}$ of the form $Q_i(X,Y) = A_i X^2 + B_i XY + C_i Y^2$ having these fixed coefficients A_1 and A_2. Recall* (3-1). *Then we have, with an absolute implied constant,*

$$N = O\bigl(2^{\omega(t^2-d_1d_2)} S(d_1, d_2, t^2)\bigr). \tag{3-17}$$

Proof. We can assume $A_i \neq 0$ for $i = 1, 2$ by the assumption that d_i is not a square. Also, by the identity (3-6) with $a = 1$, $b = 0$ there are only two possibilities for $B_2 A_1 - B_1 A_2$. So we can fix D such that there are at least $N/2$ inequivalent forms having the fixed coefficients A_1, A_2 and having $B_2 A_1 - B_1 A_2 = D$. But we see from (3-15) that the residue of B_1 modulo $2A_1$ determines the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of (Q_1, Q_2) once A_1, A_2 and $B_2 A_1 - B_1 A_2$ is given. Therefore it is enough to estimate the possible values of B_1 modulo A_1 (for given A_1, A_2 and $D = B_2 A_1 - B_1 A_2$) by the right-hand side of (3-17).

We have

$$B_i^2 \equiv d_i \pmod{A_i} \tag{3-18}$$

for $i = 1, 2$. Let p be a prime, and let us denote $\nu_p(A_1)$ by α and $\nu_p(A_2)$ by β. We consider two cases.

<u>Case $\nu_p(d_1) < \alpha$:</u> We use (3-18) with $i = 1$. There is a solution only if $\nu_p(d_1) = 2k$ for some integer k, and then we must have $\nu_p(B_1) \geq k$ and $(B_1/p^k)^2 \equiv (d_1/p^{2k})$ modulo $p^{\alpha-2k}$. Since $\alpha - 2k > 0$, we get from this congruence that there are at most $2(1 + \nu_p(2))$ possibilities for B_1/p^k modulo $p^{\alpha-2k}$. Hence we finally get that there are at most $2(1 + \nu_p(2))p^{[\nu_p(d_1)/2]}$ possibilities for B_1 modulo p^α.

<u>Case $\nu_p(d_1) \geqslant \alpha$:</u> We use again (3-18) with $i = 1$, and we see that $\nu_p(B_1) \geq \alpha/2$. So there are at most $p^{[\alpha/2]}$ possibilities for B_1 modulo p^α.

Hence in both cases there are at most $2(1 + \nu_p(2))p^{\min([\frac{1}{2}\nu_p(d_1)],[\frac{1}{2}\alpha])}$ possibilities for B_1 modulo p^α. Similarly, there are at most $2(1 + \nu_p(2))p^{\min([\frac{1}{2}\nu_p(d_2)],[\frac{1}{2}\beta])}$ possibilities for B_2 modulo p^β. Since $D = B_2 A_1 - B_1 A_2$ is fixed, we see that if B_2 is given modulo p^β, then $D + B_1 A_2 = B_2 A_1$ is given modulo $p^{\alpha+\beta}$, hence B_1 is given modulo p^α. Taking into account (3-16) we finally get that for every prime p there are at most

$$2(1 + \nu_p(2)) \min\bigl(p^{[\nu_p(d_1)/2]}, p^{[\nu_p(d_2)/2]}, p^{[\nu_p(t^2-d_1d_2)/2]}\bigr)$$

possibilities for B_1 modulo $p^{\nu_p(A_1)}$. We apply this to every prime divisor p of A_1 that also divides $t^2 - d_1 d_2$. If $p \mid A_1$ but p does not divide $t^2 - d_1 d_2$, then by (3-16) p does not divide A_2, and so $D = B_2 A_1 - B_1 A_2$ implies that B_1 is determined modulo $p^{\nu_p(A_1)}$. So for the number of possible values of B_1 modulo A_1 we have the upper bound

$$C\Bigl(\prod_{p \mid t^2-d_1d_2} 2\Bigr) \prod_p \min\bigl(p^{[\nu_p(d_1)/2]}, p^{[\nu_p(d_2)/2]}, p^{[\nu_p(t^2-d_1d_2)/2]}\bigr) \tag{3-19}$$

with an absolute constant C. Formula (3-19) proves Claim D. □

Claims C and D imply Lemma 3.1 at once. □

3.2. ***Upper bounds for certain special averages of $h(d_1, d_2, t)$.*** When we apply Lemma 3.1, we will have numbers d_i of special form $d_i = t_i^2 - 4$, and we will have certain triple sums of $S(t_1^2-4, t_2^2-4, f^2)$, where t_1, t_2, f run over integers. We will use the trivial upper bounds $S(t_1^2-4, t_2^2-4, f^2) \le S(t_1^2-4, t_2^2-4)$ and $S(t^2-4, t^2-4, f^2) \le S(t^2-4, f^2)$, and we will use Lemmas 3.5, 3.6 and 3.7 below. We need a preliminary lemma.

Lemma 3.4. *Let $t_1, t_2 > 2$ be integers, $t_1 \ne t_2$, and let $E = \gcd(t_1^2-4, t_2^2-4)$.*

(i) *There is a divisor e of E such that*

$$e \ge c\sqrt{E}, \quad e \mid (t_1 - \delta t_2).$$

with an absolute constant $c > 0$ and with some $\delta \in \{-1, 1\}$.

(ii) $E \le |t_1 - t_2|(t_1 + t_2)$.

Proof. Part (ii) follows at once from the fact that E divides $t_1^2 - t_2^2$, so it remains to show part (i). Let $p \mid E$ be a prime. Then $p^{\nu_p(E)}$ divides $(t_1-t_2)(t_1+t_2)$, so writing $\alpha := \nu_p(t_1-t_2)$ and $\beta := \nu_p(t_1+t_2)$ we have $\nu_p(E) \le \alpha + \beta$. If $m = \min(\alpha, \beta)$, then $m \le \nu_p(2t_1)$, so $2m \le \nu_p(4t_1^2)$. But $0 < \nu_p(E) \le \nu_p(4t_1^2 - 16)$, so if $m > 0$, then we must have $p = 2$. If $p = 2$ and $m > 2$, then we have $\nu_2(4t_1^2) > 4$, and so $\nu_2(4t_1^2 - 16) = 4$, hence $\nu_2(E) \le 4$. It follows for every prime p that $p^{\nu_p(E)}$ divides either $16(t_1 - t_2)$ or $16(t_1 + t_2)$. Then there is a decomposition $E = e_1 e_2$ such that $\gcd(e_1, e_2) = 1$, and e_1 divides $16(t_1 - t_2)$, e_2 divides $16(t_1 + t_2)$. The lemma is proved. □

Lemma 3.5. *Let $3 \le a < b \le c \le 2a$ be integers. Recall the definition* (3-1). *For any $\epsilon > 0$ we have*

$$\sum_{t_1=a}^{c-1} \sum_{\substack{a \le t_2 \le c-1 \\ 0 < |t_2 - t_1| \le b-a}} S(t_1^2-4, t_2^2-4) \ll_\epsilon a^{1/2+\epsilon}(c-a)(b-a)^{1/2} \tag{3-20}$$

$$\sum_{t_1=a}^{c-1} \sum_{\substack{a \le t_2 \le c-1 \\ 0 < |t_2 - t_1|}} \frac{S(t_1^2-4, t_2^2-4)}{\sqrt{|t_1-t_2|}} \ll_\epsilon a^{1/2+\epsilon}(c-a). \tag{3-21}$$

Remark 3.2. With the very crude estimate $S(t_1^2-4, t_2^2-4) \ll a$ the trivial bound in (3-20) would be $(c-a)(b-a)a$, so in (3-20) we save roughly $\sqrt{(b-a)a}$.

Proof. Inequality (3-21) follows from (3-20) using a dyadic subdivision. To prove (3-20) we may assume $b = c$, since the general case follows by dividing the summation over t_1 into $O\left(\frac{c-a}{b-a}\right)$ subsums.

So let $b = c$. By Lemma 3.4(ii), we have $\gcd(t_1^2-4, t_2^2-4) \ll (b-a)b$. Then by Lemma 3.4(i) and because $S(t_1^2-4, t_2^2-4) \le \sqrt{\gcd(t_1^2-4, t_2^2-4)}$ the left-hand side of (3-20) is

$$\ll \sum_{t_1=a}^{b-1} \Bigg(\sum_{\substack{E \mid t_1^2-4 \\ E \ll (b-a)b}} \sqrt{E} \sum_{\substack{e \mid E \\ e \ge c\sqrt{E}}} \sum_{\delta \in \{-1,1\}} \sum_{\substack{a \le t_2 < b \\ e \mid t_2 - \delta t_1}} 1 \Bigg),$$

and for a given t_1 the quantity in parentheses is

$$\ll \sum_{\substack{E \mid t_1^2-4 \\ E \ll (b-a)b}} \sqrt{E} \sum_{\substack{e \mid E \\ e \geq c\sqrt{E}}} \left(1 + \frac{b-a}{e}\right) \ll b^{\epsilon}\left(\sqrt{(b-a)b} + b - a\right).$$

The lemma follows. □

Lemma 3.6. *Let $3 \leq a < b \leq 2a$ be integers. For every $\epsilon > 0$ we have*

$$\sum_{t=a}^{b-1} \max\{k \geq 1 : k^2 \mid t^2 - 4\} \ll_{\epsilon} a^{1+\epsilon}\sqrt{b-a}, \tag{3-22}$$

$$\sum_{t_1=a}^{b-1} \sum_{t_2=a}^{b-1} S(t_1^2 - 4, t_2^2 - 4) \ll_{\epsilon} a^{\epsilon}\left(a\sqrt{b-a} + a^{1/2}(b-a)^{3/2}\right). \tag{3-23}$$

Remark 3.3. Estimating every summand by $O(a)$ in (3-22) the trivial bound in (3-22) would be $(b-a)a$, so in (3-20) we save roughly $\sqrt{b-a}$.

Proof. Statement (3-23) follows at once from Lemma 3.5 and (3-22), so we deal only with (3-22). It is enough to show that for any integer $1 \leq K \leq 2a$ we have

$$K \sum_{t=a}^{b-1} \sum_{k=K}^{2K} \sum_{\substack{d>1 \\ t^2-4=dk^2}} \mu^2(d) \ll_{\epsilon} a^{\epsilon} a\sqrt{b-a}. \tag{3-24}$$

A trivial upper bound for the left-hand side of (3-24) is $K(b-a)$.

Let d be fixed and assume $t^2 - 4 = dk^2$. Then $\alpha := \frac{1}{2}(t + k\sqrt{d})$ is an algebraic integer, since it is a root of the equation $x^2 - tx + 1$. We also see that α is a unit in the ring R of algebraic integers of the real quadratic field $\mathbb{Q}(\sqrt{d})$. By the Dirichlet unit theorem, there is a unit $1 < \epsilon \in R$ such that every unit of R has the form $\pm\epsilon^l$ with integer l. One has $\epsilon = \frac{1}{2}(a + b\sqrt{d})$ with integers a, b, where $b \neq 0$. Then $\epsilon^{-1} = \delta \cdot \frac{1}{2}(a - b\sqrt{d})$ with $\delta \in \{-1, 1\}$, hence $\epsilon = b\sqrt{d} + \delta\epsilon^{-1}$, so $\epsilon > \sqrt{d} - 1 \geq \sqrt{2} - 1$. But $\alpha = \epsilon^l$ for some positive integer l and $\alpha \leq t \leq 2a$. So we have proved that for a fixed d there are at most $C \log a$ possibilities for the pair (t, k), with an absolute constant C. We have $d \ll a^2/K^2$ in (3-24); thus the left side of (3-24) is $\ll_{\epsilon} a^{2+\epsilon}/K$, and hence $\ll_{\epsilon} a^{\epsilon} \min(K(b-a), a^2/K)$, given the observation made after (3-24). This minimum here is clearly $\ll a\sqrt{b-a}$, and the lemma is proved. □

Lemma 3.7. *Let $t > 2$ be an integer and let $1 \leq A \ll t^2$. For any $\epsilon > 0$,*

$$\sum_{\substack{f \in \mathbb{Z} \\ t^2-4-A \leq |f| < t^2-4}} \frac{S(t^2 - 4, f^2)}{\sqrt{t^2 - 4 - |f|}} \ll_{\epsilon} t^{\epsilon}\sqrt{A}.$$

Remark 3.4. We save roughly t here with respect to the trivial bound.

Proof. The left-hand side is at most

$$\sum_{k^2 | t^2-4} k \sum_{\substack{g\in\mathbb{Z} \\ 0<\frac{t^2-4}{k}-|g|\le\frac{A}{k}}} \frac{1}{\sqrt{k}\sqrt{\frac{t^2-4}{k}-|g|}},$$

and the inner sum here is $\ll (1/\sqrt{k})\sqrt{A/k} = \sqrt{A}/k$. We used here that the inner sum is empty if $A < k$. The lemma is proved. □

4. Identities and estimates for special functions

In this section we consider the functions $\mathcal{I}(t_1, t_2, F, m_1, m_2)$ and $\mathcal{J}(t_1, t_2, m_1, m_2)$ defined in (2-10) and (2-9) for the special case when the functions m_i are characteristic functions of some intervals $[0, x_i]$ for $i = 1, 2$. This case will be important in our application. In the first subsection we prove an identity for this $\mathcal{I}$-function for every $1 \neq F > 0$; in the second and third subsections we use it to give estimates for the cases $F > 1$ and $F < 1$, respectively. In the last subsection we compute $\mathcal{J}(t_1, t_2, m_1, m_2)$ for the above-mentioned special case.

4.1. ***Computing $\mathcal{I}(t_1, t_2, F, m_1, m_2)$ when the m_i are characteristic functions.*** For $S_0, T_0, F > 0$, $F \neq 1$ define

$$Z(S_0, T_0, F) := \iint \frac{1}{\sqrt{S^2+T^2+2FTS+1-F^2}}\, dS\, dT, \tag{4-1}$$

where we integrate over the set

$$\{(S, T) \in \mathbb{R}^2 : |S| \le S_0, |T| \le T_0,\ S^2+T^2+2FTS+1-F^2 > 0\}.$$

By the reasoning of Remark 2.1 we see that (4-1) is absolutely convergent. One can see that (4-1) is divergent for $F = 1$, but we do not need that case. It is clear that the function $\mathcal{I}$ can be expressed in terms of the function Z in the case when the m_i are characteristic functions.

Lemma 4.1. *Let $S_0, T_0, F > 0$, $F \neq 1$. We have*

$$Z(S_0, T_0, F) = 2J(S_0, T_0, F) + 2J(T_0, S_0, F),$$

where we write

$$J(S_0, T_0, F) := \int_{\substack{|y|\le T_0/S_0 \\ (1+y^2+2Fy)S_0^2>F^2-1}} \frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}}{1+y^2+2Fy}\, dy \tag{4-2}$$

in the case $F > 1$, and

$$J(S_0, T_0, F) := \int_{|y|\le T_0/S_0} \frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}}{1+y^2+2Fy}\, dy - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sqrt{1-F^2}}{1+y^2+2Fy}\, dy$$

in the case $F < 1$.

Proof. It is clear by the substitution $(S,T)\mapsto(-S,-T)$ that the $S<0$ and $S>0$ parts of the integral (4-1) have the same value. For $S>0$ we make the substitution $y=T/S$, and we get

$$Z(S_0,T_0,F)=2\int_{-\infty}^{\infty}\int\frac{S}{\sqrt{S^2+y^2S^2+2FyS^2+1-F^2}}\,dS\,dy,$$

where the inner integral is taken over the set

$$\bigl\{S\in\mathbb{R}:0\le S\le S_0,\,|Sy|\le T_0,\ S^2+y^2S^2+2FyS^2+1-F^2>0\bigr\}. \tag{4-3}$$

If $F<1$, the last condition is always true. If $F>1$, then $1+y^2+2Fy>0$ should hold, otherwise (4-3) is empty. For a fixed y we integrate in S over the interval

$$\sqrt{\frac{F^2-1}{1+y^2+2Fy}}\le S\le\min(S_0,T_0/|y|)$$

in the case $F>1$, and we integrate over

$$0\le S\le\min(S_0,T_0/|y|)$$

in the case $F<1$. We consider separately the cases $|y|\le T_0/S_0$ and $|y|\ge T_0/S_0$. We can compute the S-integral in each case. Making the substitution $y\mapsto 1/y$ in the case $|y|\ge T_0/S_0$ we obtain the lemma. □

4.2. *The case F > 1.* In Lemma 4.3 we express the function $J(S_0,T_0,F)$ defined in Lemma 4.1 in terms of a simple function in the case $F>1$. Then we give estimates for this simple function in Lemma 4.4 and for higher derivatives of a variant of this function in Lemma 4.6.

Lemma 4.2. *Let $S_0,T_0>0$ and $F>1$. For the function $J(S_0,T_0,F)$ defined in* (4-2) *we have*

$$J(S_0,T_0,F)=\int_{H(S_0,T_0,F)}\frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}}{1+y^2+2Fy}\,dy, \tag{4-4}$$

where

$$H(S_0,T_0,F):=\begin{cases}[-T_0/S_0,T_0/S_0] & \text{if } T_0/S_0\le 1 \text{ and } 1<F\le A(S_0,T_0),\\ [-T_0/S_0,C(F,S_0)]\cup[D(F,S_0),T_0/S_0] & \\ & \text{if } T_0/S_0\ge 1 \text{ and } 1<F\le A(S_0,T_0),\\ [D(F,S_0),T_0/S_0] & \text{if } A(S_0,T_0)\le F\le B(S_0,T_0),\\ \qquad\varnothing & \text{if } F\ge B(S_0,T_0),\end{cases} \tag{4-5}$$

where we write

$$A(S_0,T_0):=\sqrt{(1+S_0^2)(1+T_0^2)}-S_0T_0,\quad B(S_0,T_0):=\sqrt{(1+S_0^2)(1+T_0^2)}+S_0T_0, \tag{4-6}$$

$$C(F,S_0):=-F-\sqrt{F^2-1}\sqrt{1+S_0^2}/S_0,\quad D(F,S_0):=-F+\sqrt{F^2-1}\sqrt{1+S_0^2}/S_0. \tag{4-7}$$

We mean every statement in such a way that if we write an interval $[a,b]$, this implicitly means that $a\le b$.

Proof. One can check that $(1+y^2+2Fy)S_0^2 > F^2-1$ holds if and only if $y < C(F,S_0)$ or $y > D(F,S_0)$. The following three claims can be checked by direct computation. For the proof of Claim 2 we use the obvious fact that $F - S_0T_0 > -\sqrt{(1+S_0^2)(1+T_0^2)}$.

Claim 1. *The sign of*

$$\frac{\sqrt{F^2-1}\sqrt{1+S_0^2}}{S_0} - \left|\frac{T_0}{S_0} - F\right| \tag{4-8}$$

equals the sign of $F - A(S_0,T_0)$.

Claim 2. *The sign of*

$$\frac{\sqrt{F^2-1}\sqrt{1+S_0^2}}{S_0} - \left(\frac{T_0}{S_0} + F\right) \tag{4-9}$$

equals the sign of $F - B(S_0,T_0)$.

Claim 3. *The sign of* $T_0/S_0 - 1$ *equals the sign of* $T_0/S_0 - A(S_0,T_0)$.

We get from Claim 1 that if $T_0/S_0 \le 1$ and $1 < F \le A(S_0,T_0)$, then $D(F,S_0) \le -T_0/S_0$, and this gives the first branch of (4-5).

If $\frac{T_0}{S_0} \ge 1$ and $1 < F \le A(S_0,T_0)$, we get from Claims 3 and 1 that

$$F + \frac{\sqrt{F^2-1}\sqrt{1+S_0^2}}{S_0} \le \frac{T_0}{S_0}, \tag{4-10}$$

and this implies the second branch.

If $A(S_0,T_0) \le F \le B(S_0,T_0)$, then $-T_0/S_0 \le D(F,S_0) \le T_0/S_0$ by Claims 1 and 2, and $C(F,S_0) \le -T_0/S_0$ by Claim 1. This proves the third branch.

If $F \ge B(S_0,T_0)$, then $D(F,S_0) \ge T_0/S_0$ by Claim 2, and $C(F,S_0) \le -T_0/S_0$ by Claim 1. This gives the last branch, and the lemma is proved. □

Lemma 4.3. *Use the notation of Lemma 4.2. Write* $\sigma = 1 + 1/S_0^2$, *and for* $0 < y < 1$ *let*

$$\Phi(y) = \Phi(S_0, y) := \int_0^y \frac{\sigma r^2}{(1-r^2)(r^2+\sigma-1)} dr. \tag{4-11}$$

Then for $1 < F \le B(S_0,T_0)$ *we have* $J(S_0,T_0,F) = S_0(\Phi(y_1) + \epsilon\Phi(y_2))$, *where*

$$y_1 = y_1(S_0,T_0,F) := \sqrt{1 - \frac{(1+S_0^2)(F^2-1)}{(T_0+S_0F)^2}}, \quad y_2 = y_2(S_0,T_0,F) := \sqrt{1 - \frac{(1+S_0^2)(F^2-1)}{(T_0-S_0F)^2}},$$

and $\epsilon = \epsilon(S_0,T_0,F)$ *is defined by*

$$\epsilon = \begin{cases} 1 & \text{if } T_0/S_0 > 1 \text{ and } 1 < F \le A(S_0,T_0), \\ -1 & \text{if } T_0/S_0 < 1 \text{ and } 1 < F \le A(S_0,T_0), \\ 0 & \text{if } A(S_0,T_0) < F \le B(S_0,T_0). \end{cases}$$

Assuming $1 < F \le B(S_0, T_0)$ *for* $i = 1$ *and* $1 < F \le A(S_0, T_0)$ *for* $i = 2$ *we have, for* $i = 1, 2$,

$$0 \le y_i \le 1 - c_1 \frac{F-1}{(1+S_0)^{c_2}(1+T_0)^{c_2}} \tag{4-12}$$

with some positive absolute constants c_1, c_2.

Proof. Note that $1 < F \le A(S_0, T_0)$ implies $T_0/S_0 \ne 1$ by Claim 3, so ϵ is well-defined. We get the statement $0 \le y_i \le 1$ by Claims 1 and 2 above. Then (4-12) follows by easy estimates using the conditions $S_0, T_0 > 0$ and $1 < F \le B(S_0, T_0)$.

To compute $J(S_0, T_0, F)$ we use (4-4). This is the same formula as (4-2), but the integration set is given there explicitly. Use the substitution

$$r = r(y) = \sqrt{1 - \frac{\sigma(F^2-1)}{(y+F)^2}} = \sqrt{\frac{1+y^2+2Fy+(F^2-1)(1-\sigma)}{(y+F)^2}}. \tag{4-13}$$

We have a positive number under the square root by (4-2). It is clear from the conditions and the definitions of $C(F, S_0)$ and $D(F, S_0)$ that the sign of $y + F$ is constant on each of the four intervals which are present in (4-5); hence r is well-defined and strictly monotone on each of those intervals. It is easy to check that

$$\frac{\sqrt{1+y^2+2Fy+(F^2-1)(1-\sigma)}}{1+y^2+2Fy} \left| \frac{dy}{dr} \right| = \frac{\sigma r^2}{\left(1-r^2\right)(r^2+\sigma-1)}. \tag{4-14}$$

We have $r(C(F, S_0)) = r(D(F, S_0)) = 0$, hence by Lemma 4.2 we get the present lemma. □

Lemma 4.4. *Let* $S_0 > 0$ *and* $0 < y < 1$. *For the function* $\Phi(S_0, y)$ *of Lemma 4.3 we have the estimates*

$$S_0 \Phi(S_0, y) \ll \begin{cases} S_0^3 y^3 & \text{if } S_0 \ge 1 \text{ and } 0 < y \le 1/(2S_0), \\ S_0 y & \text{if } S_0 \ge 1 \text{ and } 1/(2S_0) \le y \le \frac{1}{2}, \\ S_0 y^3 & \text{if } S_0 \le 1 \text{ and } y \le \frac{1}{2}. \end{cases} \tag{4-15}$$

Finally we have in every case

$$\Phi(S_0, y) \ll \log \frac{1}{1-y}. \tag{4-16}$$

The implied constants are absolute in formulas (4-15) *and* (4-16).

Proof. We have from the definitions that

$$S_0 \Phi(S_0, y) = S_0 \int_0^y \frac{S_0^2 r^2 + r^2}{(1-r^2)(S_0^2 r^2 + 1)} \, dr.$$

Every estimate follows easily. □

We recall Faà di Bruno's formula. If F and G are smooth functions and $H(x) = F(G(x))$, then for every $j \ge 1$ we have

$$H^{(j)}(x) = \sum_{l=1}^{j} \sum_{k=(k_1,\ldots,k_j)\in H_{j,l}} a_{j,l,k} F^{(l)}(G(x)) \prod_{i=1}^{j} (G^{(i)}(x))^{k_i}, \tag{4-17}$$

with some constants $a_{j,l,k}$, where

$$H_{j,l} = \Big\{(k_1, \dots, k_j) \in \mathbb{Z}^j : k_i \ge 0, \sum_{i=1}^{j} k_i = l, \sum_{i=1}^{j} i k_i = j\Big\}. \tag{4-18}$$

This can be seen by induction using the chain rule.

Lemma 4.5. *Let $S_0 > 0$ and $\sigma = 1 + 1/S_0^2$.*

(i) *Recall the definition of $\Phi(y)$ from* (4-11). *Write $\phi(t) = \Phi(1/t)$. For every $j \ge 1$ and $t > 1$ we have*

$$\phi^{(j)}(t) \ll_j \frac{1}{t(t-1)^j}$$

uniformly in S_0.

(ii) *For $Y > \sqrt{\sigma}$ let $G(Y) = Y/\sqrt{Y^2 - \sigma}$. Then for every $j \ge 1$ we have*

$$G^{(j)}(Y) \ll_j \begin{cases} \left(\frac{Y}{Y^2-\sigma}\right)^j \frac{Y}{\sqrt{Y^2-\sigma}} & \text{for } \sqrt{\sigma} < Y \le 2\sqrt{\sigma}, \\ \frac{\sigma}{Y^{j+2}} & \text{for } Y \ge 2\sqrt{\sigma}, \end{cases} \tag{4-19}$$

uniformly in S_0.

(iii) *For $Y > \sqrt{\sigma}$ let $H(Y) = \phi(G(Y))$. Then for every $j \ge 1$ and $Y > \sqrt{\sigma}$ we have*

$$H^{(j)}(Y) \ll_j \frac{\sqrt{Y^2-\sigma}}{Y} \left(\frac{Y}{Y^2-\sigma}\right)^j \tag{4-20}$$

uniformly in S_0.

Proof. By (4-11) and the substitution $r \mapsto 1/r$ we have

$$\phi(t) = \int_t^\infty \frac{\sigma}{(r^2-1)(1+r^2(\sigma-1))}\, dr.$$

The integrand here equals $\frac{1}{r^2-1} - \frac{1}{r^2+S_0^2}$. Considering separately the cases $t \ge 2$ and $1 < t \le 2$ we obtain (i) easily.

For (ii), note that if $j \ge 1$, the left-hand side of (4-19) is a linear combination of terms of the form $Y^l/\big(\sqrt{Y^2-\sigma}\big)^{j+l}$, where $0 \le l \le j+1$ and $j+l$ is odd. Here clearly $l = j+1$ gives the largest term, and we get the first branch of (4-19). The second branch follows easily from the Taylor expansion

$$G(Y) = \frac{1}{\sqrt{1-\sigma Y^{-2}}} = 1 + \frac{\sigma}{2Y^2} + \sum_{m=2}^{\infty} a_m \frac{\sigma^m}{Y^{2m}}, \tag{4-21}$$

where the a_m are absolute constants such that $\sum_{m=1}^{\infty} |a_m|\, r^m < \infty$ for every $0 < r < 1$.

For the proof of (iii) we use Faà di Bruno's formula (4-17), and we see that it is enough to estimate terms of the form

$$\phi^{(l)}(G(Y)) \prod_{i=1}^{j} (G^{(i)}(Y))^{k_i}, \tag{4-22}$$

where $1 \le l \le j$ and the k_i satisfy the conditions in (4-18).

If $Y \geq 2\sqrt{\sigma}$, then $1 < G(Y) \leq 2/\sqrt{3}$, and we get from (i) and (4-19) that (4-22) is

$$\ll_j \frac{1}{(G(Y)-1)^l} \prod_{i=1}^{j} \left(\frac{\sigma}{Y^{i+2}}\right)^{k_i} = \left(\frac{\sigma}{Y^2(G(Y)-1)}\right)^l Y^{-j},$$

where we used the conditions in (4-18). We see from (4-21) that $1 \ll \frac{\sigma}{Y^2(G(Y)-1)} \ll 1$, so we get (iii) for $Y \geq 2\sqrt{\sigma}$.

If $\sqrt{\sigma} < Y \leq 2\sqrt{\sigma}$, then $G(Y) \geq 2/\sqrt{3}$, and we get from (i) and (4-19) that (4-22) is

$$\ll_j \frac{1}{(G(Y))^{l+1}} \prod_{i=1}^{j} \left(\left(\frac{Y}{Y^2-\sigma}\right)^i \frac{Y}{\sqrt{Y^2-\sigma}}\right)^{k_i} = \frac{1}{G(Y)} \left(\frac{Y/G(Y)}{\sqrt{Y^2-\sigma}}\right)^l \left(\frac{Y}{Y^2-\sigma}\right)^j,$$

where we used the conditions in (4-18). By the definition of $G(Y)$ we get (iii) also for this case. The lemma is proved. □

Lemma 4.6. *Let $S_0 > 0$, $F > 1$, $t \geq 3$, $\tau \in \{-1, 1\}$ be given. If $x > t^2 - 4$, write*

$$T_0 = T_0(x) := \sqrt{\frac{x}{t^2-4} - 1} \quad \text{and} \quad R(x) := \frac{|\tau F S_0 + T_0(x)|}{S_0\sqrt{F^2-1}}. \tag{4-23}$$

Let the number σ and the function H be defined as in Lemma 4.5, and let us define $K(x) = H(R(x))$ for $x \in H_{S_0,F,t,\tau}$, where

$$H_{S_0,F,t,1} := \{x > t^2-4 : F < B(S_0, T_0(x))\} \quad \text{and} \quad H_{S_0,F,t,-1} := \{x > t^2-4 : F < A(S_0, T_0(x))\}$$

(see (4-6)). *Then K is well-defined. If $\tau = -1$, then $K(x)$ is a smooth function for $\{x \in H_{S_0,F,t,-1} : T_0(x) < S_0\}$ and also for $\{x \in H_{S_0,F,t,-1} : T_0(x) > S_0\}$. For every $j \geq 1$ and every x satisfying the above conditions we have*

$$K^{(j)}(x) \ll_j (x - t^2 + 4)^{-j} \max\left(1, \left(\frac{T_0|T_0 - FS_0|}{\sqrt{(S_0^2+1)(T_0^2+1)}(A(S_0,T_0)-F)}\right)^j\right) \quad \text{for } \tau = -1,$$

$$K^{(j)}(x) \ll_j (x - t^2 + 4)^{-j} \max\left(1, \left(\frac{T_0(T_0 + FS_0)}{\sqrt{(S_0^2+1)(T_0^2+1)}\,(B(S_0,T_0)-F)}\right)^j\right) \quad \text{for } \tau = 1.$$

Proof. From Claims 1 and 2 we have $R(x) > \sqrt{\sigma}$; hence $K(x)$ is well-defined. Also, for $\tau = -1$ we have $|-FS_0 + T_0| = FS_0 - T_0$ in the case $S_0 > T_0$, and $|-FS_0 + T_0| = T_0 - FS_0$ in the case $S_0 < T_0$. This follows from the conditions, using Claim 3. We cannot have $T_0 = S_0$ if $\tau = -1$, because $T_0 = S_0$ implies $A(S_0, T_0) = 1$, so $1 < F < A(S_0, T_0)$ is impossible. Hence if $\tau = -1$, then $R(x)$ is indeed a smooth function for $T_0 < S_0$, and also for $T_0 > S_0$, so we can speak about the derivatives of K.

We see from Faà di Bruno's formula (4-17) that it is enough to estimate terms of the form

$$H^{(l)}(R(x)) \prod_{i=1}^{j} (R^{(i)}(x))^{k_i}, \tag{4-24}$$

where $1 \leq l \leq j$ and the k_i satisfy the conditions in (4-18). It is clear from the definitions that

$$R^{(i)}(x) \ll_i \frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{F^2-1}}. \quad \text{for } i \geq 1.$$

Hence, using also Lemma 4.5(iii) we get that (4-24) is

$$\ll_j \frac{\sqrt{(R(x))^2-\sigma}}{R(x)} \left(\frac{R(x)}{(R(x))^2-\sigma}\right)^l \prod_{i=1}^{j} \left(\frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{F^2-1}}\right)^{k_i}.$$

By the conditions in (4-18) and the definition of T_0 in (4-23), this equals

$$\frac{\sqrt{(R(x))^2-\sigma}}{R(x)} \left(\frac{R(x)T_0}{\left((R(x))^2-\sigma\right) S_0\sqrt{F^2-1}}\right)^l (x-t^2+4)^{-j}.$$

Note that $\frac{\sqrt{(R(x))^2-\sigma}}{R(x)} \leq 1$. From the definition of R in (4-23) is easy to compute, using also $\sigma = 1+1/S_0^2$, that

$$S_0\sqrt{F^2-1}\,\frac{(R(x))^2-\sigma}{R(x)} = \frac{(S_0^2+1)(T_0^2+1)-(F-\tau S_0T_0)^2}{|\tau FS_0+T_0|}.$$

The lemma is proved. □

4.3. *The case $F < 1$.* In Lemma 4.7 we give a new expression for the function $J(S_0, T_0, F)$ defined in Lemma 4.1 in the case $F < 1$, and we also give upper bounds for the new expression. In Lemma 4.8 we give another new expression for $J(S_0, T_0, F)$, expressing it in terms of a simple function. Then in Lemma 4.10 we give estimates for higher derivatives of a variant of this simple function.

Lemma 4.7. *Let* $0 < S_0$, $0 < T_0 < T_0^*$, $0 < F < 1$. *We have*

$$J(S_0,T_0,F)+J(T_0,S_0,F) = K(S_0,T_0,F)+K(T_0,S_0,F), \tag{4-25}$$

where

$$K(S_0,T_0,F) := \int_{|y|\leq T_0/S_0} \frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}-\sqrt{1-F^2}}{1+y^2+2Fy}\,dy.$$

We also have

$$K(S_0,T_0,F) \ll \frac{S_0T_0}{\sqrt{1-F^2}}, \quad K(S_0,T_0^*,F)-K(S_0,T_0,F) \ll \frac{S_0(T_0^*-T_0)}{\sqrt{1-F^2}}. \tag{4-26}$$

Proof. The first statement follows from Lemma 4.1 and

$$\int_{|y|\leq T_0/S_0} \frac{dy}{1+y^2+2Fy} + \int_{|y|\leq S_0/T_0} \frac{dy}{1+y^2+2Fy} = \int_{-\infty}^{\infty} \frac{dy}{1+y^2+2Fy},$$

which follows by the substitution $y \to 1/y$. We have

$$\frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}-\sqrt{1-F^2}}{1+y^2+2Fy} = \frac{S_0^2}{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}+\sqrt{1-F^2}} \leq \frac{S_0^2}{\sqrt{1-F^2}}.$$

The lemma follows. □

Lemma 4.8. *Let $S_0, T_0 > 0$ and $F < 1$. Write $\sigma = 1 + 1/S_0^2$, and for $-1 < t < 1$ let*

$$V(t) = V(S_0, t) := \int_0^t \frac{\sigma}{(1-r^2)(1+(\sigma-1)r^2)}\,dr. \tag{4-27}$$

Then

$$J(S_0, T_0, F) = S_0(V(s_1) - V(s_2)) - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sqrt{1-F^2}}{1+y^2+2Fy}\,dy,$$

where

$$s_1 = s_1(S_0, T_0, F) := \frac{S_0F + T_0}{\sqrt{(S_0F+T_0)^2 + (1+S_0^2)(1-F^2)}},$$

$$s_2 = s_2(S_0, T_0, F) := \frac{S_0F - T_0}{\sqrt{(S_0F-T_0)^2 + (1+S_0^2)(1-F^2)}}.$$

Proof. We use the substitution

$$r = r(y) = \frac{y+F}{\sqrt{(y+F)^2 + \sigma(1-F^2)}}$$

in the first integral in the definition of $J(S_0, T_0, F)$. It is easy to check that

$$\frac{\sqrt{1+y^2+2Fy+(1-F^2)(\sigma-1)}}{1+y^2+2Fy}\,dy = \frac{\sigma}{(1-r^2)(1+(\sigma-1)r^2)}\,dr,$$

and the lemma follows. □

Lemma 4.9. *Let $S_0 > 0$, $\sigma = 1 + 1/S_0^2$, and let V be as in* (4-27).

(i) *For every $j \geq 1$ and $-1 < t < 1$ we have*

$$V^{(j)}(t) \ll_j (1-|t|)\left(\frac{1}{|t|+S_0} + \frac{1}{1-|t|}\right)^{j+1}$$

uniformly in S_0.

(ii) *For $-\infty < Y < \infty$ let $g(Y) = Y/\sqrt{Y^2+\sigma}$. Then for every $j \geq 1$ we have*

$$g^{(j)}(Y) \ll_j \left(\frac{1}{\sqrt{\sigma}}\right)^j \quad \textit{for } |Y| \leq 2\sqrt{\sigma}, \tag{4-28}$$

$$g^{(j)}(Y) \ll_j \frac{\sigma}{|Y|^{j+2}} \quad \textit{for } |Y| \geq 2\sqrt{\sigma}, \tag{4-29}$$

uniformly in S_0.

(iii) *For $-\infty < Y < \infty$ let $h(Y) = V(g(Y))$. Then for every $j \geq 1$ and $Y > 0$ we have*

$$h^{(j)}(Y) \ll_j \sqrt{\sigma}(1+|Y|)^{-j}$$

uniformly in S_0.

Proof. For the proof of (i) note that

$$\frac{\sigma}{(1-r^2)(1+(\sigma-1)r^2)}=\frac{1}{1-r^2}+\frac{1}{S_0^2+r^2}=\frac{1}{1-r^2}+\frac{1/2S_0}{S_0+ir}+\frac{1/2S_0}{S_0-ir}.$$

Considering first the case $|t|\geq\frac{1}{2}$, and then if $|t|\leq\frac{1}{2}$, then considering separately $|t|\leq S_0$ and $|t|\geq S_0$ we get (i). In (ii) we can assume $Y\geq 0$, and then the proof is completely similar to that of Lemma 4.5(ii).

For the proof of (iii) we use Faà di Bruno's formula (4-17), and we see that it is enough to estimate terms of the form

$$V^{(l)}(g(Y))\prod_{i=1}^{j}(g^{(i)}(Y))^{k_i}, \tag{4-30}$$

where $1\leq l\leq j$ and the k_i satisfy the conditions in (4-18).

If $|Y|\geq 2\sqrt{\sigma}$, then $|g(Y)|\geq 2/\sqrt{5}$, and we get from (i) and (4-29) that (4-30) is

$$\ll_j \frac{1}{(1-|g(Y)|)^l}\prod_{i=1}^{j}\left(\frac{\sigma}{|Y|^{i+2}}\right)^{k_i}=\left(\frac{\sigma}{|Y|^2\,(1-|g(Y)|)}\right)^l|Y|^{-j},$$

where we used the conditions in (4-18). It is easy to see that $1\ll(\sigma/|Y|^2)/(1-|g(Y)|)\ll 1$, so taking into account $\sigma\geq 1$ we get (iii) for $|Y|\geq 2\sqrt{\sigma}$. If $|Y|\leq 2\sqrt{\sigma}$, then $|g(Y)|\leq 2/\sqrt{5}$, and we get from (i) and (4-28) that (4-30) is

$$\ll_j\left(\frac{1}{|g(Y)|+S_0}+1\right)^{l+1}\prod_{i=1}^{j}\left(\left(\frac{1}{\sqrt{\sigma}}\right)^i\right)^{k_i}=\left(\frac{1}{|g(Y)|+S_0}+1\right)^{l+1}\left(\frac{1}{\sqrt{\sigma}}\right)^j,$$

where we used the conditions in (4-18). If $S_0\gg 1$, then $1\ll\sigma\ll 1$, and we get (iii). If $S_0\ll 1$, then $1/\sqrt{\sigma}\ll S_0\ll 1/\sqrt{\sigma}$, $|Y|/\sqrt{\sigma}\ll|g(Y)|\ll|Y|/\sqrt{\sigma}$, and

$$\left(\frac{1}{|g(Y)|+S_0}+1\right)^{l+1}\left(\frac{1}{\sqrt{\sigma}}\right)^j\ll\left(\frac{1}{|g(Y)|+S_0}\right)\left(\frac{1}{|g(Y)|\sqrt{\sigma}+S_0\sqrt{\sigma}}\right)^j.$$

The lemma follows. □

Lemma 4.10. *Let $S_0>0$, $F<1$, $t\geq 3$, $\tau\in\{-1,1\}$ be given. If $x>t^2-4$, write*

$$T_0=T_0(x):=\sqrt{\frac{x}{t^2-4}-1}\quad\textit{and}\quad r(x):=\frac{FS_0+\tau T_0}{S_0\sqrt{1-F^2}}. \tag{4-31}$$

Let the number σ and the function h be defined as in Lemma 4.9, and let us define $k(x)=h(r(x))$ for every x satisfying $x>t^2-4$. Then for every $j\geq 1$ and every $x>t^2-4$ we have

$$k^{(j)}(x)\ll_j\sqrt{\sigma}(x-t^2+4)^{-j}\max\left(1,\left(\frac{T_0}{S_0\sqrt{1-F^2}+|\tau FS_0+T_0|}\right)^j\right). \tag{4-32}$$

Proof. We see from Faà di Bruno's formula (4-17) that it is enough to estimate terms of the form

$$h^{(l)}(r(x))\prod_{i=1}^{j}(r^{(i)}(x))^{k_i}, \tag{4-33}$$

where $1 \le l \le j$ and the k_i satisfy the conditions in (4-18). It is clear from the definitions that for $i \ge 1$ we have

$$r^{(i)}(x) \ll_i \frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{1-F^2}}.$$

Hence, using also Lemma 4.9(iii), we get that (4-33) is

$$\ll_j \sqrt{\sigma}\,(1+|r(x)|)^{-l}\prod_{i=1}^{j}\left(\frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{1-F^2}}\right)^{k_i}.$$

Using the conditions in (4-18) and the relations (4-31) we obtain the lemma. □

4.4. ***Computing $\mathcal{J}(t_1, t_2, m_1, m_2)$ when the m_i are characteristic functions.*** For $x > 0$ introduce the notation

$$k_x(y) = 1 \text{ for } 0 \le y \le x, \quad k_x(y) = 0 \text{ for } y > x. \tag{4-34}$$

Lemma 4.11. *Let $t_i > 2$ and $x_i > 0$ for $i = 1, 2$. Then $\mathcal{J}\left(t_1, t_2, k_{x_1/4}, k_{x_2/4}\right)$ is nonzero only if $x_i > t_i^2-4$ for $i = 1, 2$. Assuming that this is true, we have*

$$\mathcal{J}\left(t_1, t_2, k_{x_1/4}, k_{x_2/4}\right) = 2\frac{\sqrt{1-m}}{\sqrt{m}},$$

where $m := \max\left(\frac{t_1^2-4}{x_1}, \frac{t_2^2-4}{x_2}\right)$.

Proof. The statement is trivial for $m \ge 1$, so let us assume $m < 1$. Then by definition we have

$$\mathcal{J}(t_1, t_2, k_{x_1/4}, k_{x_2/4}) = \int_{-\arccos\sqrt{m}}^{\arccos\sqrt{m}} \frac{d\theta}{\cos^2\theta} = 2\frac{\sin\left(\arccos\sqrt{m}\right)}{\cos\left(\arccos\sqrt{m}\right)}.$$

The lemma is proved. □

5. First steps of the proof of Theorem 1.1

Let η_0 be a nonnegative smooth function on $(0, \infty)$ vanishing outside of $\tau \in [1, 2]$ and such that

$$\int_1^2 \eta_0(\tau)\,d\tau = 1. \tag{5-1}$$

Recall the definition of $k_x(y)$ from (4-34). For $x > 0$, $D > 0$, define

$$k_{x,D}(y) := \frac{1}{D}\int_D^{2D} \eta_0\left(\frac{\tau}{D}\right) k_x(y+\tau)\,d\tau \tag{5-2}$$

for $y \ge 0$. We will use also the notations of Theorem 1.1.

5.1. ***A spectral estimate.*** Our aim is to prove Lemma 5.2, whose result will show that for a smoothed version of the hyperbolic circle problem one can give a good estimate by spectral methods. We first need some notation.

The hyperbolic Laplace operator is denoted by $\Delta := y^2\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)$. Let $\{u_j(z) : j \geq 0\}$ be a complete orthonormal system of Maass forms for $\mathrm{PSL}_2(\mathbb{Z})$ (the function $u_0(z)$ being constant), and let $\Delta u_j = \left(-\frac{1}{4} - t_j^2\right)u_j$, where $t_0 = \frac{i}{2}$ and t_j is real for $j > 0$.

If m is a compactly supported bounded function on $[0, \infty)$, let (see [I, (1.62)])

$$g_m(a) = 2q_m\left(\frac{e^a + e^{-a} - 2}{4}\right), \quad \text{where } q_m(v) = \int_0^\infty \frac{m(v+\tau)}{\sqrt{\tau}}\,d\tau, \tag{5-3}$$

and for any complex r let

$$h_m(r) = \int_{-\infty}^{\infty} g_m(a)e^{ira}\,da. \tag{5-4}$$

For simplicity introduce the abbreviations $h_x = h_{k_x}$ and $h_{x,D} = h_{k_{x,D}}$.

Lemma 5.1. *Assume* $1 < D < x/10$. *For every integer* $j \geq 0$ *and all* $r \geq 1$ *we have*

$$h_{x,D}(r) \ll_j \frac{x^{1/2}}{r^{3/2}} \min\left(1, \frac{x}{Dr}\right)^j + \frac{x^{1/2}}{r^{5/2}}. \tag{5-5}$$

We also have for every real r

$$h_{x,D}(r) \ll x^{1/2}\log x. \tag{5-6}$$

Furthermore, *we have*

$$h_{x,D}\left(\frac{i}{2}\right) = 4\pi x - 4\pi D\int_1^2 \eta_0(\tau)\tau\,d\tau. \tag{5-7}$$

Proof. It is easy to see from (5-3), (5-4) and (5-2) that

$$h_{x,D}(r) = \frac{1}{D}\int_D^{2D} \eta_0\left(\frac{\tau}{D}\right) h_{x-\tau}(r)\,d\tau \tag{5-8}$$

for every complex r. Now we apply Lemma 2.4 of [C] for the function $h_{x-\tau}(r)$ choosing $R = R(\tau)$ in that lemma in such a way that

$$\tfrac{1}{2}\cosh R(\tau) - \tfrac{1}{2} = x - \tau. \tag{5-9}$$

Applying part (d) of that lemma we see that $h_{x-\tau}\left(\frac{i}{2}\right) = 4\pi(x-\tau)$, and taking into account (5-1) we get (5-7). The estimate (5-6) follows from a trivial estimation of (2.6) of [C]. Finally, for the proof of (5-5) we apply part (a) of Lemma 2.4 of [C]. Applying it in (5-8) we get for $r \geq 1$ that

$$h_{x,D}(r) = \frac{2\sqrt{2\pi}}{r^{3/2}D}\int_D^{2D} \eta_0\left(\frac{\tau}{D}\right)\sqrt{\sinh R(\tau)}\cos\left(rR(\tau) - \tfrac{3}{4}\pi\right)d\tau + O\left(\frac{x^{1/2}}{r^{5/2}}\right).$$

Through the substitution $R = R(\tau)$ and use of (5-9) this equals

$$\frac{\sqrt{2\pi}}{r^{3/2}D}\int_{R_1}^{R_2} \eta_0\left(\frac{1 + 2x - \cosh R}{2D}\right)(\sinh R)^{3/2}\cos\left(rR - \tfrac{3}{4}\pi\right)dR + O\left(\frac{x^{1/2}}{r^{5/2}}\right),$$

where $\cosh R_1 = 1 + 2x - 4D$ and $\cosh R_2 = 1 + 2x - 2D$. Repeated partial integration gives (5-5). The lemma is proved. □

Lemma 5.2. *Assume* $1 < D < x/10$ *and* $z \in \Omega$. *Then*

$$\sum_{\gamma \in \Gamma} k_{x,D}(u(\gamma z, z)) = 12x - 12D \int_1^2 \eta_0(\tau)\tau\, d\tau + O_\Omega\left(\frac{x}{\sqrt{D}} + x^{1/2}\log x\right). \tag{5-10}$$

Proof. It is clear by (5-3) and (5-4) that the function $h_{x,D}(r)$ satisfies [I, condition (1.63)], i.e., it is even, it is holomorphic in the strip $|\operatorname{Im} r| \le \frac{1}{2} + \epsilon$ and $h_{k_{x,D}}(r) = O((1+|r|)^{-2-\epsilon})$ in this strip for some $\epsilon > 0$. Then we get from Theorem 7.4 of [I], using again the abbreviation $h_{x,D} = h_{k_{x,D}}$, that the left-hand side of (5-10) equals

$$\sum_{j=0}^{\infty} h_{x,D}(t_j)|u_j(z)|^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_{x,D}(r)\left|E\left(z, \tfrac{1}{2} + ir\right)\right|^2 dr$$

for any $z \in \mathbb{H}$, where $E(z,s)$ is the Eisenstein series for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$; see [I, Chapter 3]. For the fundamental domain $\mathcal{F}$ defined in (1-3) we have $|\mathcal{F}| = \pi/3$ by [I, (6.33) and (3.26)]; hence $|u_0(z)|^2$ equals $3/\pi$ for every z. By Lemma 5.1 above and by [I, Proposition 7.2], we get the lemma. □

5.2. *Nonhyperbolic elements.* We give an easy estimate for the contribution of the nonhyperbolic elements in the hyperbolic circle problems.

Lemma 5.3. *Let* $z \in \Omega$ *and* $X > 2$. *Then for every* $\epsilon > 0$ *we have*

$$\left|\left\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) : |\operatorname{tr}\gamma| \le 2,\ 4u(\gamma z, z) \le X - 2\right\}\right| \ll_{\Omega,\epsilon} X^{1/2+\epsilon}. \tag{5-11}$$

Proof. Write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By [I, (1.9), (1.11)] we have

$$4u(\gamma z, z) = \frac{|cz^2 + (d-a)z - b|^2}{\operatorname{Im}^2 z}.$$

It is easy to compute that if $z = x + iy$, then

$$\operatorname{Im}\left(cz^2 + (d-a)z - b\right) = 2cxy + (d-a)y \quad \text{and} \quad \operatorname{Re}\left(cz^2 + (d-a)z - b\right) = c(x^2 - y^2) + (d-a)x - b.$$

Hence if $z \in \Omega$, then the second inequality in (5-11) gives

$$2cx + d - a \ll_\Omega \sqrt{X}, \tag{5-12}$$

$$c(x^2 + y^2) + b \ll_\Omega \sqrt{X}. \tag{5-13}$$

By the first inequality in (5-11) we get from (5-12) that

$$d = -cx + O_\Omega(\sqrt{X}), \quad a = cx + O_\Omega(\sqrt{X}),$$

and from these relations and (5-13) we get

$$1 = ad - bc = -c^2x^2 + c^2(x^2 + y^2) + O_\Omega\left(\sqrt{X}\,(\sqrt{X} + |c|)\right).$$

This implies $c = O_\Omega(\sqrt{X})$, and so (5-12) gives $d - a \ll_\Omega \sqrt{X}$. Then there are $O_\Omega(\sqrt{X})$ possibilities for the pair (a, d). If a and d are given with $ad \neq 1$, then $bc = ad - 1$ implies that there are $O_\epsilon(X^\epsilon)$ possibilities for the pair (b, c). Finally, if $ad = 1$, then $bc = 0$, and so (5-13) implies that there are $\ll_\Omega \sqrt{X}$ possibilities for the pair (b, c). The lemma is proved. □

5.3. ***Reduction to the estimation of a square integral on the fundamental domain.*** Let us take an integer $J \geq 2$; it will be fixed but we will choose it sufficiently large. Let d be a parameter that will be chosen optimally later, at the moment we assume that $d \geq 100$ and $100Jd \leq X$.

Let us define

$$N_{d,J}(z, X) := \sum_{j=0}^{J} (-1)^j \binom{J}{j} \int_1^2 \eta_0(\tau) N(z, X - jd\tau)\, d\tau. \tag{5-14}$$

Then using (5-1) we see that $N_{d,J}(z, X)$ equals

$$N(z, X) + \sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{\gamma\in\Gamma} \int_1^2 \eta_0(\tau) k_{\frac{X-2}{4}}\big(u(\gamma z, z) + \tfrac{1}{4} jd\tau\big)\, d\tau,$$

which equals

$$N(z, X) + \sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{\gamma\in\Gamma} k_{\frac{X-2}{4}, \frac{jd}{4}}(u(\gamma z, z))$$

by (5-2). Applying Lemma 5.2 this equals

$$N(z, X) + O_{\Omega,J}\left(\frac{X}{\sqrt{d}} + X^{1/2}\log X\right) + \sum_{j=1}^{J} (-1)^j \binom{J}{j} \Big(3X - 3jd \int_1^2 \eta_0(\tau)\tau\, d\tau\Big)$$

for $z \in \Omega$. Now, it follows from the binomial theorem, taking into account that $j\binom{J}{j} = J\binom{J-1}{j-1}$ for $1 \leq j \leq J$, that

$$\sum_{j=1}^{J} (-1)^j \binom{J}{j} = -1 \quad \text{and} \quad \sum_{j=1}^{J} (-1)^j j \binom{J}{j} = 0.$$

Hence we have proved that for $z \in \Omega$

$$N_{d,J}(z, X) = N(z, X) - 3X + O_{\Omega,J}\left(\frac{X}{\sqrt{d}} + X^{1/2}\log X\right). \tag{5-15}$$

Recalling the notation $M_{t,m}$ from (2-8) we get from Lemma 5.3 that

$$N(z, X) = O_{\Omega,\epsilon}(X^{1/2+\epsilon}) + \sum_{t>2} M_{t,k_{\frac{X-2}{4}}}(z)$$

for $z \in \Omega$, $X > 2$ and for any $\epsilon > 0$. Hence by (5-14) we see that

$$N_{d,J}(z, X) = O_{\Omega,\epsilon,J}(X^{1/2+\epsilon}) + \int_1^2 \eta_0(\tau) \Big(\sum_{t>2} \sum_{j=0}^{J} (-1)^j \binom{J}{j} M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)\, d\tau$$

for $z \in \Omega$, $\epsilon > 0$. By Cauchy–Schwarz we have that

$$\Big(\int_1^2 \eta_0(\tau)\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)\,d\tau\Big)^2$$

is

$$\ll \int_1^2\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)^2\,d\tau.$$

Hence, using also (5-15) we finally get that

$$\int_\Omega (N(z,X)-3X)^2\,d\mu_z \tag{5-16}$$

is

$$O_{\Omega,\epsilon,J}\left(\frac{X^2}{d}+X^{1+\epsilon}+\int_1^2\int_{\mathcal{F}}\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)^2\,d\mu_z\,d\tau\right) \tag{5-17}$$

if $\epsilon > 0$, $d \geq 100$ and $100Jd \leq X$.

We will show in Section 6 that if $\epsilon > 0$ is given and the integer $J \geq 2$ is fixed to be large enough in terms of ϵ, then we have

$$\int_{\mathcal{F}}\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)^2\,d\mu_z \ll_\epsilon X^\epsilon\frac{d^{5/2}}{\sqrt{X}} \tag{5-18}$$

uniformly for $1 \leq \tau \leq 2$ and

$$X^{2/3} \leq d \leq X^{99/100}. \tag{5-19}$$

Assume that (5-18) is true. Then we see from (5-16) and (5-17) that (5-16) equals

$$O_{\Omega,\epsilon}\left(\frac{X^2}{d}+X^\epsilon\frac{d^{5/2}}{\sqrt{X}}\right) \tag{5-20}$$

for any d satisfying (5-19). Note that we choose J in terms of ϵ, so we do not have to denote the dependence on J in (5-20). Choosing $d = X^{5/7}$ we obtain Theorem 1.1. So it is enough to show the estimate (5-18).

6. Conclusion

The goal of this section is to prove the estimate (5-18).

6.1. ***Application of Lemma 2.2 and basic observations.*** It is easy to see that if $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and the trace of γ is $t > 2$, then we have $u(\gamma z, z) \geq \frac{1}{4}(t^2-4)$ for every $z \in \mathbb{H}$. Therefore, the contribution of the terms $t > \sqrt{X+2}$ to the sum (5-18) is 0. We can take integers $1 \leq I \ll \log X$ and

$$3 = a_1 < a_2 < \cdots < a_I < 1+\sqrt{X+2} \leq a_{I+1} < 2+\sqrt{X+2} \tag{6-1}$$

such that

$$a_{i+1} \leq \tfrac{3}{2}a_i, \quad 3+\sqrt{X+2}-a_i \leq 2\big(3+\sqrt{X+2}-a_{i+1}\big) \tag{6-2}$$

for $1 \le i \le I$. By the Cauchy–Schwarz inequality we get for every $1 \le \tau \le 2$ that

$$\int_{\mathcal{F}} \Big(\sum_{t>2} \sum_{j=0}^{J} (-1)^j \binom{J}{j} M_{t,k_{\frac{X-jd\tau-2}{4}}}(z) \Big)^2 d\,\mu_z \ll \log X \sum_{i=1}^{I} U_i \tag{6-3}$$

with

$$U_i = U_i(\tau) := \int_{\mathcal{F}} \Big(\sum_{t=a_i}^{a_{i+1}-1} \sum_{j=0}^{J} (-1)^j \binom{J}{j} M_{t,k_{\frac{X-jd\tau-2}{4}}}(z) \Big)^2 d\mu_z. \tag{6-4}$$

By Lemma 2.2 we have for every $1 \le i \le I$ that U_i equals the sum of

$$\sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} E_{t_1,t_2} S_{t_1,t_2} \tag{6-5}$$

with the two complementary pieces

$$\sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \sum_{\substack{f \in \mathbb{Z} \\ f^2 < (t_1^2-4)(t_2^2-4)}} h(t_1^2-4, t_2^2-4, f) R_{t_1,t_2,f}, \tag{6-6}$$

and

$$\sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \sum_{\substack{f \in \mathbb{Z} \\ f^2 > (t_1^2-4)(t_2^2-4)}} h(t_1^2-4, t_2^2-4, f) R_{t_1,t_2,f}, \tag{6-7}$$

with the abbreviations

$$a_{j_1,j_2} := (-1)^{j_1+j_2} \binom{J}{j_1} \binom{J}{j_2}, \tag{6-8}$$

$$S_{t_1,t_2} := \sum_{j_1=0}^{J} \sum_{j_2=0}^{J} a_{j_1,j_2} \mathcal{J}\Big(t_1, t_2, k_{\frac{X-j_1d\tau-2}{4}}, k_{\frac{X-j_2d\tau-2}{4}} \Big) \tag{6-9}$$

$$R_{t_1,t_2,f} := \sum_{j_1=0}^{J} \sum_{j_2=0}^{J} a_{j_1,j_2} \mathcal{I}\Big(t_1, t_2, \frac{f}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}}, k_{\frac{X-j_1d\tau-2}{4}}, k_{\frac{X-j_2d\tau-2}{4}} \Big). \tag{6-10}$$

By (2-10) and Lemma 4.11 we see that the functions $\mathcal{I}$ and $\mathcal{J}$ from (6-9) and (6-10) can be nonzero only in the case

$$t_1^2 - 4 \le X - j_1 d\tau - 2, \quad t_2^2 - 4 \le X - j_2 d\tau - 2. \tag{6-11}$$

If (6-11) is true, then by (2-10) and (4-1) we have

$$\mathcal{I}\Big(t_1, t_2, \frac{f}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}}, k_{\frac{X-j_1d\tau-2}{4}}, k_{\frac{X-j_2d\tau-2}{4}} \Big) = Z(S_0, T_0, F), \tag{6-12}$$

and by Lemma 4.11 we have

$$\mathcal{J}\Big(t_1, t_2, k_{\frac{X-j_1d\tau-2}{4}}, k_{\frac{X-j_2d\tau-2}{4}} \Big) = 2\min(S_0, T_0), \tag{6-13}$$

where we use the abbreviations

$$S_0 = S_0(j_1, t_1) = \sqrt{\frac{X-j_1d\tau-2}{t_1^2-4} - 1}, \quad T_0 = T_0(j_2, t_2) = \sqrt{\frac{X-j_2d\tau-2}{t_2^2-4} - 1}, \tag{6-14}$$

and

$$F = F(t_1, t_2, f) = \left| \frac{f}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}} \right|. \tag{6-15}$$

We now consider the sum (6-7). Assume that (6-11) holds. Then by Lemma 4.1 and the last line of (4-5) we see that (6-12) can be nonzero only if

$$|f| < B(S_0, T_0)\sqrt{t_1^2-4}\sqrt{t_2^2-4}. \tag{6-16}$$

If (6-11) is true and (6-16) holds for some f in (6-7), then we have

$$\frac{|f|}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}} - 1 \geq \frac{1}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}\left(|f| + \sqrt{t_1^2-4}\sqrt{t_2^2-4}\right)} \gg X^{-c} \tag{6-17}$$

with some absolute constant c. If (6-11) holds, then we determine (6-12) by Lemmas 4.1 and 4.3. In some cases we will apply the upper bounds of Lemma 4.4. By (6-17) and (4-12) we see that when we apply these lemmas for the estimation of (6-12) we always have $\log \frac{1}{1-y} \ll \log X$. So assuming (6-11) we get that for any f in (6-7) the value of (6-12) is

$$\ll \left(\sqrt{\frac{X - j_1 d\tau - 2}{t_1^2-4} - 1} + \sqrt{\frac{X - j_2 d\tau - 2}{t_2^2-4} - 1} \right) \log X. \tag{6-18}$$

We note finally that if $a_i \leq t_1, t_2 < a_{i+1}$ for some i, then

$$(t_1t_2 - 5)^2 + \tfrac{1}{6}t_1t_2 < (t_1^2-4)(t_2^2-4) \leq (t_1t_2-4)^2, \tag{6-19}$$

since by the assumption $a_{i+1} \leq \frac{3}{2}a_i$ made in (6-2) we have $\frac{2}{3} \leq t_2/t_1 \leq \frac{3}{2}$, and we also have $t_1t_2 \geq 9$. So there is an absolute constant $c_0 > 0$ such that if $a_i \leq t_1, t_2 < a_{i+1}$, then

$$(t_1t_2 - 5) + c_0 \leq \sqrt{(t_1^2-4)(t_2^2-4)} \leq t_1t_2 - 4. \tag{6-20}$$

6.2. ***The case of very large a_i.*** Assume that we have

$$\sqrt{X+2} - a_i = O\left(\frac{d}{\sqrt{X}} X^{\delta} \right) \tag{6-21}$$

for some $\delta > 0$ chosen small enough in terms of ϵ. Consider first (6-7). Since it is easy to see that $B(S,T) - 1 \leq S^2 + T^2$ for any $S, T \geq 0$, the number of integers f in (6-7) satisfying (6-16) is

$$\ll 1 + \sqrt{t_1^2-4}\sqrt{t_2^2-4}\left(\left(\frac{X - j_1 d\tau - 2}{t_1^2-4} - 1 \right) + \left(\frac{X - j_2 d\tau - 2}{t_2^2-4} - 1 \right) \right) \ll dX^{\delta},$$

where in the last step we used (6-21), (6-1) and (6-2). On the other hand, we get by (6-11), (6-12), (6-18), (6-21) and (6-2) that (6-10) is always $\ll_\delta (\sqrt{d}/\sqrt{X})X^{\delta}$ for every such f. Hence for i satisfying (6-21) we have, applying also Lemma 3.1, (3-23) and (6-21) that (6-7) is

$$\ll_\delta X^{3\delta} \frac{d^{3/2}}{\sqrt{X}} \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} S(t_1^2-4, t_2^2-4) \ll_\delta X^{5\delta} d^{3/2} \left(\left(\frac{d}{\sqrt{X}} \right)^{1/2} + \frac{(d/\sqrt{X})^{3/2}}{X^{1/4}} \right),$$

where we used (6-1), (6-2). By (6-3) we see that its contribution is acceptable in (5-18).

We now consider (6-6). We get by (6-11), (6-12), Lemma 4.1, (4-25), the first relation in (4-26) and (6-21) that (6-10) is $\ll_\delta X^\delta d/\sqrt{(t_1^2-4)\left(t_2^2-4\right)-f^2}$. Hence for i satisfying (6-21) we have, applying also Lemma 3.1, that (6-6) is

$$\ll_\delta X^{2\delta} d \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \sum_{\substack{f\in\mathbb{Z}\\ f^2<(t_1^2-4)(t_2^2-4)}} \frac{S(t_1^2-4,\, t_2^2-4,\, f^2)}{\sqrt{(t_1^2-4)(t_2^2-4)-f^2}}. \tag{6-22}$$

In the $t_1 \neq t_2$ part we use $S(t_1^2-4, t_2^2-4, f^2) \leq S(t_1^2-4, t_2^2-4)$, and we easily see that the sum over f in (6-22) is $\ll S(t_1^2-4, t_2^2-4)$. Then applying (3-20) with $b=c$ and (6-21) we see that the $t_1 \neq t_2$ part of (6-22) is

$$\ll_\delta X^{3\delta} d\sqrt{a_i}\left(\frac{d}{\sqrt{X}}X^\delta\right)^{3/2} \ll_\delta X^{5\delta}\frac{d^{5/2}}{\sqrt{X}},$$

which is acceptable in (5-18). For the $t_1=t_2$ part of (6-22) we use Lemma 3.7 and we get that the sum over f in (6-22) is $\ll_\delta X^\delta$, and so the $t_1=t_2$ part of (6-22) is $\ll_\delta X^{3\delta} d\big(\frac{d}{\sqrt{X}}X^\delta\big)$, which is smaller than our estimate for the $t_1 \neq t_2$ part. So we have proved that assuming (6-21) the contribution of (6-6) is also acceptable in (5-18).

Assuming (6-21) in (6-5) we clearly have $\min(S_0, T_0) \ll_\delta X^\delta\sqrt{d}/\sqrt{X}$; hence using also (6-9), (6-13) and Lemma 2.1 we get that (6-5) is

$$\ll_\delta \frac{X^{2\delta}\sqrt{d}}{\sqrt{X}}\left(\frac{d}{\sqrt{X}}X^\delta\right)^2\sqrt{X} = \frac{X^{4\delta}d^{5/2}}{X}.$$

Hence in the case of (6-21) we get that the contribution of U_i in (5-18) is acceptable. We may assume from now on that

$$\sqrt{X+2}-a_i > X^{\delta_0}\frac{d}{\sqrt{X}} \tag{6-23}$$

with some $\delta_0>0$, which is fixed in terms of ϵ.

6.3. ***Easy consequences of*** **(6-23).** Observe that (6-23) implies the following relations, for all real numbers $0 \leq j_1, j_2, j \leq J$ and integers $a_i \leq t_1, t_2, t \leq a_{i+1}$:

$$\frac{X^{1/4}\sqrt{\sqrt{X}-a_i}}{a_i} \ll S_0(j,t_1), T_0(j,t_2) \ll \frac{X^{1/4}\sqrt{\sqrt{X}-a_i}}{a_i}, \tag{6-24}$$

$$\frac{X}{a_i^2} \ll 1+S_0^2(j,t_1), 1+T_0^2(j,t_2) \ll \frac{X}{a_i^2}, \tag{6-25}$$

$$T_0(j_1,t)-T_0(j_2,t) = O\left(\frac{d}{X^{1/4}a_i\sqrt{\sqrt{X}-a_i}}\right) = S_0(j_1,t)-S_0(j_2,t), \tag{6-26}$$

$$S_0(j_1,t_1)-T_0(j_2,t_2) = \frac{X(t_2^2-t_1^2)\left((t_1^2-4)(t_2^2-4)\right)^{-1}}{S_0(j_1,t_1)+T_0(j_2,t_2)} + O\left(\frac{dX^{-1/4}}{\sqrt{\sqrt{X}-a_i}a_i}\right) \tag{6-27}$$

We have in general that $\frac{d}{dT}\big(\sqrt{1+S^2}\sqrt{1+T^2}+ST\big)=\frac{\sqrt{1+S^2}T}{\sqrt{1+T^2}}+S$ and

$$\frac{d}{dT}\big(\sqrt{1+S^2}\sqrt{1+T^2}-ST\big)=\frac{\sqrt{1+S^2}\,T}{\sqrt{1+T^2}}-S=\frac{(T-S)(T+S)}{\sqrt{1+T^2}\big(\sqrt{1+S^2}T+\sqrt{1+T^2}S\big)};$$

hence we get from (6-24)–(6-27) and the mean-value theorem that

$$B\big(S_0(j_1,t_1),T_0(j,t_2)\big)-B\big(S_0(j_1,t_1),T_0(0,t_2)\big)\ll d/a_i^2 \tag{6-28}$$

and

$$A\big(S_0(j_1,t_1),T_0(j,t_2)\big)-A\big(S_0(j_1,t_1),T_0(0,t_2)\big)\ll\left(\frac{|t_2-t_1|}{a_i}+\frac{d}{X}\right)\frac{d}{X-a_i^2} \tag{6-29}$$

for all real numbers $0\le j_1, j\le J$ and integers $a_i\le t_1,t_2\le a_{i+1}$.

We will also need later the easily proved general identity

$$\frac{T_0}{S_0}-A(S_0,T_0)=\frac{(1+S_0^2)(T_0^2-S_0^2)}{S_0T_0(1+S_0^2)+S_0^2\sqrt{(1+S_0^2)(1+T_0^2)}}. \tag{6-30}$$

This implies by (6-24) and (6-25) that assuming (6-23) we have

$$\left|\frac{T_0}{S_0}-1\right|\ll\left|\frac{T_0}{S_0}-A\left(S_0,T_0\right)\right|\ll\left|\frac{T_0}{S_0}-1\right| \tag{6-31}$$

for every choice $S_0=S_0(j_1,t_1)$, $T_0=T_0\left(j_2,t_2\right)$ with any real numbers $0\le j_1,j_2\le J$ and integers $a_i\le t_1,t_2\le a_{i+1}$. We also see from (6-30) that the signs of $T_0/S_0-A(S_0,T_0)$ and T_0/S_0-1 are the same. Therefore we get from (6-31) that assuming (6-23) we have

$$\left|\frac{T_0(j_2,t_2)}{S_0(j_1,t_1)}-1\right|\ll\left|\frac{T_0(j_2,t_2)}{S_0(j_1,t_1)}-F\right|\ll\left|\frac{T_0(j_2,t_2)}{S_0(j_1,t_1)}-1\right| \tag{6-32}$$

for any real numbers $0\le j_1,j_2\le J$ and any $1<F\le A\big(S_0(j_1,t_1),T_0(j_2,t_2)\big)$.

Assuming (6-23) we see from (6-2) that (6-11) is always true. Then it follows by (6-9) and (6-13) that (6-5) equals

$$2\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{t_2=a_i}^{a_{i+1}-1}E_{t_1,t_2}\sum_{j_1=0}^{J}\sum_{j_2=0}^{J}a_{j_1,j_2}\min\big(S_0(j_1,t_1),\,T_0(j_2,t_2)\big). \tag{6-33}$$

We also see that (6-12) always holds. Then by Lemma 4.1 and (6-10) we get for any f that

$$R_{t_1,t_2,f}=2\sum_{j_1=0}^{J}\sum_{j_2=0}^{J}a_{j_1,j_2}\big(J(S_0,T_0,F)+J(T_0,S_0,F)\big). \tag{6-34}$$

By the change of variables $j_1\mapsto j_2$, $t_1\mapsto t_2$ we see that substituting (6-34) into (6-7) and (6-6) the contributions of $J(S_0,T_0,F)$ and $J(T_0,S_0,F)$ in (6-7) are the same.

6.4. *Estimating* (6-5). Assume besides (6-23) that we have

$$|t_2-t_1|>\frac{da_i}{X}X^{\delta} \tag{6-35}$$

for some $\delta > 0$ which is fixed in terms of ϵ. Then we see from (6-27) and (6-24) that the sign of $S_0(j_1,t_1)-T_0(j_2,t_2)$ is the same for every pair $0 \le j_1, j_2 \le J$. But then that part of (6-33) where (6-35) holds is 0, since $\sum_{j_1=0}^{J} a_{j_1,j_2} = 0$ for every j_2, and $\sum_{j_2=0}^{J} a_{j_1,j_2} = 0$ for every j_1 by (6-8). So we may assume in (6-33) that

$$|t_2 - t_1| \ll \frac{da_i}{X} X^{\delta} \tag{6-36}$$

for some $\delta > 0$ which is chosen small enough in terms of ϵ. Then we see using (6-24) and Lemma 2.1 that (6-5) is $\ll_\delta X^{2\delta} X^{1/4} \sqrt{\sqrt{X} - a_i}\, a_i (1 + da_i/X) \ll_\delta X^{2\delta} \sqrt{X} d$, which is acceptable in (5-18).

6.5. *Estimating* (6-6). Assume besides (6-23) that in (6-6) we have (6-36) and

$$1 - \left(\frac{d}{X - a_i^2} \right)^2 X^{\delta} < F < 1 \tag{6-37}$$

for some $\delta > 0$ which is chosen small enough in terms of ϵ. By (6-34) and (4-25) we have

$$R_{t_1,t_2,f} = 2 \sum_{j_1=0}^{J} \sum_{j_2=0}^{J} a_{j_1,j_2} \big(K(S_0, T_0, F) + K(T_0, S_0, F) \big) \tag{6-38}$$

in the case $F < 1$, and by the substitutions $j_1 \mapsto j_2$, $t_1 \mapsto t_2$ we see that the contributions of $K(S_0, T_0, F)$ and $K(T_0, S_0, F)$ in (6-6) are the same. Hence it is enough to consider the contribution of $K(S_0, T_0, F)$. Applying the second relation in (4-26) we see for fixed t_1, t_2, f and j_1 that

$$\sum_{j_2=0}^{J} (-1)^{j_2} \binom{J}{j_2} K\left(S_0, T_0(j_2, t_2), F\right) \ll \max_{0 \le j_2 < J} \frac{|S_0| \left| T_0(j_2+1, t_2) - T_0(j_2, t_2) \right|}{\sqrt{1 - F^2}}.$$

The parameters are written here only in the case of T_0, since only this variable depends on j_2. By (6-24) and (6-26) this is $\ll d/(a_i^2 \sqrt{1-F^2})$. Hence using (6-8), (6-38) and Lemma 3.1 we get that that part of (6-6) where (6-36) and (6-37) hold is

$$\ll_\delta \frac{dX^{\delta}}{a_i^2} \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{\substack{a_i \le t_2 < a_{i+1} \\ |t_2 - t_1| \ll da_i X^{\delta-1}}} \sum_{\substack{f \in \mathbb{Z} \\ 0 < 1-F < \left(\frac{d}{X-a_i^2}\right)^2 X^{\delta}}} \frac{S(t_1^2-4, t_2^2-4, f^2)}{\sqrt{1-F^2}}. \tag{6-39}$$

From (6-20) we see that the innermost sum is

$$\ll a_i \sum_f \frac{S(t_1^2-4, t_2^2-4, f^2)}{\sqrt{(t_1t_2-4) - |f|}}, \tag{6-40}$$

where the sum ranges over all f such that

$$t_1t_2 - 5 - \sqrt{(t_1^2-4)(t_2^2-4)} \left(\frac{d}{X - a_i^2} \right)^2 X^{\delta} \le |f| \le t_1t_2 - 5.$$

In the case $t_1 \neq t_2$ we use $S(t_1^2-4, t_2^2-4, f^2) \le S(t_1^2-4, t_2^2-4)$, and we see that (6-40) is

$$\ll_\delta X^\delta a_i\left(1+\frac{da_i}{X-a_i^2}\right) S(t_1^2-4, t_2^2-4).$$

So, applying also (3-20) we get that the $t_1 \neq t_2$ part of (6-39) is

$$\ll_\delta X^{3\delta}\left(\frac{d}{a_i}+\frac{d^2}{X-a_i^2}\right)\left(\min\bigl(a_i, \sqrt{X}-a_i\bigr)\left(a_i\frac{da_i}{X}\right)^{1/2}\right) \ll_\delta X^{3\delta}\frac{d^{5/2}}{\sqrt{X}},$$

which is acceptable in (5-18). In the case $t_1 = t_2$ we estimate (6-40) by Lemma 3.7 and we get that (6-40) is

$$\ll_\delta X^\delta a_i\left(1+(t_1^2-4)\left(\frac{d}{X-a_i^2}\right)^2\right)^{1/2}.$$

So the $t_1 = t_2$ part of (6-39) is

$$\ll_\delta X^{2\delta}\frac{d}{a_i}\min\bigl(a_i, \sqrt{X}-a_i\bigr)\left(1+a_i\frac{d}{X-a_i^2}\right) \ll_\delta X^{2\delta}\frac{d^2}{\sqrt{X}}.$$

Hence that part of (6-6) where (6-36) and (6-37) hold is acceptable in (5-18).

So it is enough to consider that part of (6-6) where at least one of the conditions (6-36) and (6-37) is false. We prove that this part is negligible. We use (6-34), and we recall that the contributions of $J(S_0, T_0, F)$ and $J(T_0, S_0, F)$ in (6-6) are the same. By Lemma 4.8 and $\sum_{j_2=0}^{J} a_{j_1,j_2} = 0$ we see that it is enough to show that for fixed t_1, t_2, f, j_1 the sum

$$\sum_{j_2=0}^{J}(-1)^{j_2}\binom{J}{j_2} V\left(s_i\left(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f)\right)\right) \tag{6-41}$$

is negligibly small for $i = 1, 2$. Observe that by the notation of Lemma 4.10, using $t = t_2$, $S_0 = S_0(j_1, t_1)$, $F = F(t_1, t_2, f)$ and $\tau = 1$ for $i = 1$, $\tau = -1$ for $i = 2$ we have

$$V\bigl(s_i(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f)\bigr) = k(X - j_2 d\tau - 2).$$

By Taylor's formula with remainder (see Theorem 7.6 of [A], for example), the value of (6-41) is $\ll d^J \max_{X-2-Jd\tau \le x \le X-2}\left|k^{(J)}(x)\right|$. Then by Lemma 4.10 and (6-24) we see that (6-41) is

$$\ll \sqrt{1+\frac{1}{S_0(j_1,t_1)^2}}\, d^J (X-a_i^2)^{-J} \max\left(1, \left(\sqrt{1-F^2}+\left|\tau F+\frac{T_0(j,t_2)}{S_0(j_1,t_1)}\right|\right)^{-J}\right), \tag{6-42}$$

with some real number $0 \le j \le J$. We see that if (6-37) is false, then this is negligibly small, since J is fixed to be large enough. So we can assume that (6-37) is true but (6-36) is false. We show that (6-42) is negligibly small. If

$$\left|\tau F+\frac{T_0(j,t_2)}{S_0(j_1,t_1))}\right| \gg X^\delta \frac{d}{X-a_i^2}, \tag{6-43}$$

then this is true. So we may assume that (6-43) is false. But then using also (6-37) and the triangle inequality, taking into account (6-23) we get

$$\left|\tau+\frac{T_0(j,t_2)}{S_0(j_1,t_1)}\right| \ll X^{\delta}\frac{d}{X-a_i^2}. \tag{6-44}$$

This is impossible for $\tau = 1$ for small δ by (6-23), so we may assume $\tau = -1$. Hence, using (6-27) and (6-44) with $\tau = -1$, applying also (6-24) we get $|t_2 - t_1| \ll X^{\delta} da_i/X$. But this is a contradiction, since we assumed that (6-36) is false. So that part of (6-6) where at least one of the conditions (6-36) and (6-37) is false is also negligibly small. Consequently (6-6) is acceptable in (5-18).

6.6. ***A new expression for* (6-7).** Recall that substituting (6-34) into (6-7) the contributions of $J(S_0, T_0, F)$ and $J(T_0, S_0, F)$ in (6-7) are the same. Hence, applying also Lemma 4.3 we get that (6-7) equals

$$4\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{t_2=a_i}^{a_{i+1}-1}\left(A_{t_1,t_2}+B_{t_1,t_2}-C_{t_1,t_2}\right) \tag{6-45}$$

with

$$A_{t_1,t_2} := \sum_{\substack{0\le j_1,j_2\le J\\ f\in\mathbb{Z}\\ 1<F\le B(S_0,T_0)}} h(t_1^2-4,t_2^2-4,f)a_{j_1,j_2}S_0\Phi(y_1(S_0,T_0,F)), \tag{6-46}$$

$$B_{t_1,t_2} := \sum_{\substack{0\le j_1,j_2\le J\\ f\in\mathbb{Z},\,T_0\ge S_0\\ 1<F\le A(S_0,T_0)}} h(t_1^2-4,t_2^2-4,f)a_{j_1,j_2}S_0\Phi\left(y_2(S_0,T_0,F)\right), \tag{6-47}$$

$$C_{t_1,t_2} := \sum_{\substack{0\le j_1,j_2\le J\\ f\in\mathbb{Z},\,T_0\le S_0\\ 1<F\le A(S_0,T_0)}} h(t_1^2-4,t_2^2-4,f)a_{j_1,j_2}S_0\Phi\left(y_2(S_0,T_0,F)\right). \tag{6-48}$$

6.7. ***The contribution of B_{t_1,t_2} and C_{t_1,t_2}.*** Assume besides (6-23) that

$$1<|F|\le A(S_0,T_0) \tag{6-49}$$

and (6-36) holds with some $\delta > 0$ which is chosen small enough in terms of ϵ. Applying (6-17), (4-12) and (4-16) we get that the terms $\Phi\left(y_2(S_0, T_0, F)\right)$ in (6-47), (6-48) are always $O(\log X)$. Using

$$A(S_0,T_0)-1=\frac{(S_0-T_0)^2}{\sqrt{(1+S_0^2)(1+T_0^2)}+S_0T_0+1}\le\frac{(S_0-T_0)^2}{\sqrt{(1+S_0^2)(1+T_0^2)}},$$

(6-27), (6-25), (6-24) and (6-36) we see that the number of integers f satisfying (6-49) and (6-15) is $\ll 1 + X^{2\delta}(d^2a_i^2/X)/(X - a_i^2)$. So we get, applying also Lemma 3.1 that the contribution to (6-45) of the terms B_{t_1,t_2}, C_{t_1,t_2} satisfying (6-49) and (6-36) is

$$\ll_\delta X^{3\delta}\left(\frac{\sqrt{X-a_i^2}}{a_i}+\frac{d^2a_i/X}{\sqrt{X-a_i^2}}\right)\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2\le a_{i+1}-1\\ |t_2-t_1|\ll a_i\frac{d}{X}X^\delta}} S(t_1^2-4,t_2^2-4).$$

By (3-20) and (3-22) this is

$$\ll_\delta X^{4\delta}\left(\frac{\sqrt{X-a_i^2}}{a_i}+\frac{d^2a_i/X}{\sqrt{X-a_i^2}}\right)\left(a_i\sqrt{\sqrt{X}-a_i}+(\sqrt{X}-a_i)a_i\left(\frac{d}{X}\right)^{1/2}\right),$$

which in turn is $\ll_\delta X^{4\delta}d^{5/2}/\sqrt{X}$ by (6-1), (6-2) and (5-19). Hence we have proved that the contribution to (6-45) of the terms B_{t_1,t_2}, C_{t_1,t_2} satisfying (6-49) and (6-36) is acceptable in (5-18).

Now consider the contribution of those terms B_{t_1,t_2}, C_{t_1,t_2} to (6-45) for which the inequalities

$$1<F\le A\big(S_0(j_1,t_1),T_0(0,t_2)\big)-\frac{d\,|t_2-t_1|}{a_i(X-a_i^2)}X^\delta \tag{6-50}$$

and (6-35) hold for some $\delta>0$ which is fixed in terms of ϵ. We want to prove that this contribution is negligibly small. We first show that for fixed t_1, t_2, j_1 and f the conditions in the summations in (6-47) and (6-48) are independent of $0\le j_2\le J$. It is enough to see that we have $F\le A\big(S_0(j_1,t_1),T_0(j,t_2)\big)$ for every $0\le j\le J$, and the sign of $S_0(j_1,t_1)-T_0(j,t_2)$ is the same for every $0\le j\le J$. These statements follow easily from (6-29), (6-27) and (6-24). Hence for fixed t_1, t_2, j_1 and f satisfying $a_i\le t_1,t_2<a_{i+1}$, $0\le j_1\le J$ and the conditions (6-50), (6-35) we have that either each $0\le j_2\le J$ satisfies the conditions of the summations in (6-47), or each $0\le j_2\le J$ satisfies the conditions of the summations in (6-48). Consequently, recalling (6-8) we see that it is enough to show that for every fixed t_1, t_2, j_1 and f satisfying the conditions just mentioned the sum

$$\sum_{j_2=0}^{J}(-1)^{j_2}\binom{J}{j_2}\Phi\big(y_2(S_0(j_1,t_1),T_0(j_2,t_2),F(t_1,t_2,f))\big) \tag{6-51}$$

is negligibly small. In the notation of Lemma 4.6, using $\tau=-1$ and $t=t_2$, $S_0=S_0(j_1,t_1)$, $F=F(t_1,t_2,f)$ there, we have

$$\Phi\big(y_2(S_0(j_1,t_1),T_0(j_2,t_2),F(t_1,t_2,f))\big)=K(X-j_2d\tau-2).$$

Theorem 7.6 of [A] shows that (6-51) is $\ll d^J\max_{X-2-Jd\tau\le x\le X-2}\big|K^{(J)}(x)\big|$. By Lemma 4.6, (6-32), (6-27), (6-24), (6-23), (6-1), (6-2) this is

$$\ll d^J(X-a_i^2)^{-J}\max\left(1,\left(\frac{|t_2-t_1|}{a_i\,(A(S_0,T_0)-F)}\right)^J\right).$$

From (6-50), (6-29) and (6-23) we see that this is negligibly small, since J is fixed to be large enough in terms of ϵ. Hence we have proved that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-50) and (6-35) hold is negligibly small.

Consider the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35) holds and we have

$$A\left(S_0(j_1,t_1),T_0(0,t_2)\right)-\frac{d\,|t_2-t_1|}{a_i(X-a_i^2)}X^{\delta}<F\le A\left(S_0(j_1,t_1),T_0(j_2,t_2)\right) \tag{6-52}$$

for some $\delta>0$ which is small enough in terms of ϵ. Using also (6-29) this shows that the number of possible values of the integers f satisfying (6-52) is

$$\ll 1+\frac{d\,|t_2-t_1|\,a_i}{X-a_i^2}X^{\delta}. \tag{6-53}$$

It is easy to compute that

$$y_2(S_0,T_0,F)=\sqrt{\frac{(B(S_0,T_0)+F)(A(S_0,T_0)-F)}{(T_0-S_0F)^2}},$$

so using (6-24), (6-25), (6-27), (6-32) and (6-52) we get

$$y_2(S_0,T_0,F)\ll_\delta X^{\delta}\sqrt{\frac{da_i}{X\,|t_2-t_1|}},\quad S_0y_2(S_0,T_0,F)\ll_\delta X^{\delta}\sqrt{\frac{d(\sqrt{X}-a_i)}{a_i\sqrt{X}\,|t_2-t_1|}}. \tag{6-54}$$

Now, if $a_i\ge\sqrt{X}/2$, then we have $S_0\ll 1$ by (6-24), and so by Lemma 4.4, (4-12) and (6-17) we get

$$S_0\Phi\left(y_2(S_0,T_0,F)\right)\ll_\delta S_0y_2^3(S_0,T_0,F)X^{\delta}\ll_\delta X^{4\delta}\sqrt{\frac{d^3(\sqrt{X}-a_i)}{X^2\,|t_2-t_1|^3}}. \tag{6-55}$$

If $a_i\le\sqrt{X}/2$, then we have $S_0\gg 1$ by (6-24), and by the second relation in (6-54), Lemma 4.4, (4-12), (6-17) we see that if $|t_2-t_1|\ll d/a_i$, then

$$S_0\Phi\left(y_2(S_0,T_0,F)\right)\ll_\delta X^{\delta}S_0y_2(S_0,T_0,F)\ll_\delta X^{2\delta}\sqrt{\frac{d}{a_i\,|t_2-t_1|}}, \tag{6-56}$$

while if $|t_2-t_1|\gg d/a_i$, then

$$S_0\Phi\left(y_2(S_0,T_0,F)\right)\ll_\delta X^{\delta}S_0^3y_2^3(S_0,T_0,F)\ll_\delta X^{4\delta}\sqrt{\frac{d^3}{a_i^3\,|t_2-t_1|^3}}. \tag{6-57}$$

If $a_i\ge\sqrt{X}/2$, then by (6-35) and (5-19) we see that the second term is larger than the first one in (6-53). Then by (6-53) and (6-55) we see that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35) and (6-52) hold is

$$\ll_\delta X^{6\delta}\frac{d^{5/2}}{X^{3/4}\sqrt{X-a_i^2}}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\ |t_2-t_1|\ge da_iX^{\delta-1}}}\frac{S(t_1^2-4,t_2^2-4)}{\sqrt{|t_2-t_1|}} \tag{6-58}$$

in the case $a_i\ge\frac{1}{2}\sqrt{X}$. By (3-21) we have that the sum over t_1,t_2 is $\ll_\delta X^{\delta}(\sqrt{X}-a_i)X^{1/4}$; hence (6-58) is acceptable in (5-18).

If $a_i \le \frac{1}{2}\sqrt{X}$, then by (6-53) and (6-56) the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35), (6-52) and $|t_2-t_1| \ll d/a_i$ hold is $\ll_\delta$ than the sum of

$$X^{4\delta}\frac{d^2}{X}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\ 0<|t_2-t_1|\ll d/a_i}} S(t_1^2-4,t_2^2-4) \tag{6-59}$$

and

$$X^{4\delta}\sqrt{\frac{d}{a_i}}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\ 0<|t_2-t_1|}} \frac{S(t_1^2-4,t_2^2-4)}{\sqrt{|t_2-t_1|}}. \tag{6-60}$$

By (3-20) we get that (6-59) is $\ll_\delta X^{5\delta-1}d^2a_i^{3/2}(d/a_i)^{1/2} \ll_\delta X^{5\delta}d^{5/2}/\sqrt{X}$. We estimate (6-60) by (3-21); the result is an upper bound $X^{5\delta}(d/a_i)^{1/2}a_i^{3/2} \ll X^{5\delta}\sqrt{d}\sqrt{X}$, which is smaller than $d^{5/2}/\sqrt{X}$ by (5-19).

If $a_i \le \frac{1}{2}\sqrt{X}$ and $|t_2-t_1| \gg d/a_i$, then by (5-19) the second term in (6-53) is larger than the first one. Hence by (6-53) and (6-57) we see in the case $a_i \le \frac{1}{2}\sqrt{X}$ that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35), (6-52) and $|t_2-t_1| \gg d/a_i$ hold is

$$\ll_\delta X^{6\delta}\frac{d^{5/2}}{a_i^{1/2}X}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\ 0<|t_2-t_1|}} \frac{S(t_1^2-4,t_2^2-4)}{\sqrt{|t_2-t_1|}} \ll_\delta X^{7\delta}\frac{d^{5/2}a_i}{X},$$

where in the last step we used (3-21). This is again acceptable in (5-18).

We examined every case, so we proved that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35) and (6-52) hold is acceptable in (5-18). Using also the previous estimates we see that the whole contributions of B_{t_1,t_2} and C_{t_1,t_2} in (6-45) is acceptable in (5-18).

6.8. ***The contribution of A_{t_1,t_2}.*** Now consider that part of the contribution of A_{t_1,t_2} in (6-45) where

$$1 < F \le B(S_0(j_1,t_1),T_0(0,t_2)) - \frac{d}{a_i^2}X^\delta \tag{6-61}$$

for some $\delta > 0$ which is fixed in terms of ϵ. We show that for fixed t_1, t_2, j_1 and f the condition in the summation in (6-46) is independent of $0 \le j_2 \le J$. It is enough to see that we have $F \le B\big(S_0(j_1,t_1),T_0(j,t_2)\big)$ for every $0 \le j \le J$, and this follows by (6-28). Hence for fixed t_1, t_2, j_1 and f satisfying $a_i \le t_1, t_2 < a_{i+1}$, $0 \le j_1 \le J$ and (6-61) each $0 \le j_2 \le J$ satisfies the conditions of the summation in (6-46). Recalling (6-8) we then see that if we can show that for every fixed t_1, t_2, j_1 and f satisfying the above-mentioned conditions the sum

$$\sum_{j_2=0}^{J}(-1)^{j_2}\binom{J}{j_2}\Phi\big(y_1(S_0(j_1,t_1),\,T_0(j_2,t_2),\,F(t_1,t_2,f))\big) \tag{6-62}$$

is negligibly small, then we will get that that part of the contribution of A_{t_1,t_2} in (6-45) where (6-61) is true is negligibly small. Observe that by the notations of Lemma 4.6, using $\tau = 1$ and $t = t_2$, $S_0 = S_0(j_1,t_1)$,

$F = F(t_1, t_2, f)$ there we have

$$\Phi\bigl(y_1(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f))\bigr) = K(X - j_2 d\tau - 2). \tag{6-63}$$

Theorem 7.6 of [A] gives that (6-62) is $\ll d^J \max_{X-Jd\tau-2\le x\le X-2} |K^{(J)}(x)|$. By Lemma 4.6 and (6-24) this is

$$\ll d^J \max\biggl((X-a_i^2)^{-J}, \biggl(\frac{1}{a_i^2(B(S_0,T_0)-F)}\biggr)^J\biggr).$$

By (6-61), (6-28) and (6-23) this is negligibly small, since J is fixed to be large enough in terms of ϵ. Hence that part of the contribution of A_{t_1,t_2} in (6-45) where (6-61) holds is negligibly small.

Now consider that part of the contribution of A_{t_1,t_2} in (6-45) where

$$B\bigl(S_0(j_1,t_1), T_0(0,t_2)\bigr) - \frac{d}{a_i^2} X^\delta < F \le B\bigl(S_0(j_1,t_1), T_0(j_2,t_2)\bigr) \tag{6-64}$$

for some $\delta > 0$ chosen small enough in terms of ϵ. It is easy to compute that

$$y_1(S_0,T_0,F) = \sqrt{\frac{(B(S_0,T_0)-F)(A(S_0,T_0)+F)}{(T_0+S_0F)^2}}. \tag{6-65}$$

By (6-25), (6-64) and (5-19) we see that $X/a_i^2 \ll B(S_0,T_0) \ll X/a_i^2$ and $B(S_0,T_0) - F = o(X/a_i^2)$. So

$$S_0 y_1(S_0,T_0,F) \ll \frac{\sqrt{B(S_0,T_0)-F}}{\sqrt{B(S_0,T_0)}} = o(1). \tag{6-66}$$

Then applying Lemma 4.4 (note that the middle case of (4-15) cannot hold by (6-66)), and also using (4-12) and (6-17), we get in every case that

$$S_0\Phi\bigl(S_0, y_1(S_0,T_0,F)\bigr) \ll_\delta X^\delta (S_0+S_0^3) y_1^3(S_0,T_0,F) \ll_\delta \frac{X^{1+\delta}\,(B(S_0,T_0)-F)^{3/2}}{a_i^2 S_0^2\,(B(S_0,T_0))^{3/2}};$$

the second inequality follows from (6-25) and (6-66). By (6-24), (6-25) and (6-64) this gives

$$S_0\Phi\bigl(S_0, y_1(S_0,T_0,F)\bigr) \ll_\delta X^{3\delta}\frac{d^{3/2}}{X(\sqrt{X}-a_i)}. \tag{6-67}$$

The number of possible values of the integers f satisfying (6-64) is $\ll_\delta X^\delta d$. Therefore, using also Lemma 3.1, we get that that part of the contribution of A_{t_1,t_2} in (6-45) where (6-64) holds is

$$\ll_\delta X^{5\delta}\frac{d^{5/2}}{X(\sqrt{X}-a_i)}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{t_2=a_i}^{a_{i+1}-1} S(t_1^2-4, t_2^2-4) \ll_\delta \frac{X^{6\delta}d^{5/2}}{X\bigl(\sqrt{X}-a_i\bigr)}(\sqrt{X}-a_i)a_i, \tag{6-68}$$

where in the last step we used (3-23), noting that $a(b-a)$ is an upper bound there for both terms. This estimate is again acceptable in (5-18). So we proved that the contribution of A_{t_1,t_2} in (6-45) is acceptable in (5-18), hence the whole sum (6-45) is acceptable. The proof of (5-18) is now complete, so Theorem 1.1 is also proved.

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