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A coarse Jacquet–Zagier trace formula for GL(n), with applications

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We establish a coarse Jacquet–Zagier trace identity for GL(n) over a global field. We prove the absolute convergence in $\mathrm{Re}(s) > 1$, and obtain holomorphic continuation under almost all character twists. In particular, we compute all P-regular orbital integrals in the geometric side, and the contributions from constant terms and nondegenerate terms of Eisenstein series in the continuous spectrum, and further derive meromorphic continuations of them.

As an application, we give a conditional proof of the Dedekind conjecture, which asserts that given any finite extension E/F of number fields, the ratio $\zeta_E(s)/\zeta_F(s)$ of the Dedekind zeta functions is entire, which is only known for $[E:F] \leq 4$. Some nonvanishing results are also obtained.

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1. Introduction

1A. ***The Rankin–Selberg–Zagier integral $\int \mathrm{K}_0(x,x)E(x,s)\,dx$.*** Let $\boldsymbol{\Gamma}$ be an arithmetic subgroup of $\boldsymbol{G} = \mathrm{SL}_2(\mathbb{R})$ with finite covolume. A nice φ on $\boldsymbol{G}$ defines an integral operator $R(\varphi)$ on $L^2(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$ with kernel K^φ. With respect to the Casimir operator the space $L^2(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$ decomposes into the direct sum of $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$, the space of cusp forms, and its orthogonal complement $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})^\perp$. Correspondingly, $\mathrm{K}^\varphi = \mathrm{K}^\varphi_0 + \mathrm{K}^\varphi_{\mathrm{ER}}$, where K^φ_0 (resp. $\mathrm{K}^\varphi_{\mathrm{ER}}$) is the kernel of the restriction of $R(\varphi)$ to $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})$ (resp. $L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})^\perp$). The Selberg trace formula is the identity obtained by substituting $\mathrm{K}^\varphi - \mathrm{K}^\varphi_{\mathrm{ER}}$ for K^φ_0 and computing the trace

$$\mathrm{Tr}\, R(\varphi)\,|_{L^2_0(\boldsymbol{\Gamma}\backslash\boldsymbol{G})} = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \mathrm{K}^\varphi_0(x,x)\,dx = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \big[\mathrm{K}^\varphi(x,x) - \mathrm{K}^\varphi_{\mathrm{ER}}(x,x)\big]\,dx. \tag{1-1}$$

Although the function $\mathrm{K}^\varphi_0(x,x)$ decays rapidly over $\boldsymbol{\Gamma}\backslash\boldsymbol{G}$, its counterparts $\mathrm{K}^\varphi(x,x)$ and $\mathrm{K}^\varphi_{\mathrm{ER}}(x,x)$ do not. Hence some truncation is typically required to carry out the right-hand side of the integration (1-1).

Zagier [38; 39] introduced the integral

$$I^\varphi_0(s) = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \mathrm{K}^\varphi_0(x,x)E(x,s)\,dx = \int_{\boldsymbol{\Gamma}\backslash\boldsymbol{G}} \big[\mathrm{K}^\varphi(x,x) - \mathrm{K}^\varphi_{\mathrm{ER}}(x,x)\big]E(x,s)\,dx, \tag{1-2}$$

where $E(x,s)$ is an Eisenstein series. He obtained a meromorphic continuation of the right-hand side of (1-2) and recovered the Selberg trace formula by computing the residue at $s = 1$. This approach is more convenient and computationally simpler than truncation-based proofs. Furthermore, $I^\varphi_0(s)$ gives more information than the Selberg trace formula, including the divisibility by $\zeta(s)$. Jacquet and Zagier [18] extended the formula to GL(2) over a number field F, providing a new proof of the holomorphy of adjoint L-functions and implying the Arthur–Selberg trace formula. However, the Jacquet–Zagier trace formula has only been developed for GL(2) so far.

Our aim is to generalize the Jacquet–Zagier trace formula to higher ranks, building on the equality of the geometric and spectral expansions of the meromorphic function

$$I_0^{\varphi}(s,\tau) := \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0^{\varphi}(x,x)E(x,s)\,dx, \tag{1-3}$$

where F is a global field, $G = \mathrm{GL}(n)$, and $E(x,s)$ is an Eisenstein series induced from a Hecke character τ. We anticipate a generalized Jacquet–Zagier trace formula for GL(n) of the form

$$I_0^{\varphi}(s,\tau) = I_{\mathrm{Geo}}^{\varphi}(s,\tau) - I_{\mathrm{ER}}^{\varphi}(s,\tau), \tag{1-4}$$

where $I_{\mathrm{Geo}}^{\varphi}(s,\tau)$ and $I_{\mathrm{ER}}^{\varphi}(s,\tau)$ capture (formally) the geometric and noncuspidal spectral contributions, respectively.

However, the convergence problems and complex analytic behaviors of $I_{\mathrm{Geo}}^{\varphi}(s,\tau)$ and $I_{\mathrm{ER}}^{\varphi}(s,\tau)$ are significant. In this study, we present several innovative techniques that address convergence, meromorphic continuation, and divisibility of L-series, leading to a coarse derivation of the Jacquet–Zagier trace formula for GL(n). By regularizing (1-4), we obtain an identity involving various L-functions, roughly of the form:

$$\sum_{\pi} L(s,\pi,\mathrm{Ad}) \approx \sum_{[E:F]\le n} \frac{\zeta_E(s)}{\zeta_F(s)} + \sum \frac{\text{L-S } L\text{-functions}}{\zeta_F(s)} + \sum \frac{\text{R-S } L\text{-functions}}{\zeta_F(s)},$$

where L-S means Langlands–Shahidi and R-S refers to Rankin–Selberg for *nondiscrete* representations, which should be ‘obvious’ holomorphic multiples of $\zeta_F(s)$. This formula establishes a connection between the Dedekind conjecture on the entireness of quotients of the Dedekind zeta function $\zeta_E(s)/\zeta_F(s)$ (algebraic side) and the Selberg’s conjecture regarding the entireness of adjoint L-functions $L(s,\pi,\mathrm{Ad})$ (automorphic side). Further details can be found in Section 1B2. This provides a new example of using the general Langlands program to answer some basic questions about number fields.

1B. *Statement of the main results.*

1B1. *The Jacquet–Zagier trace formula for* GL(n). We present a generalization of the trace formula, which encompasses several independent results (Theorems C, D, E, and F). These results can be summarized informally as follows.

Theorem A. *Let notation be as in Section 2. Let* $\mathrm{Re}(s) > 1$. *Let φ be a test function on $G(\mathbb{A}_F)$. Then $I_0^{\varphi}(s;\tau)$ admits a regularized geometric-spectral expansion*

$$I_0^{\varphi}(s;\tau) = I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau) - I_{P,\mathrm{Reg}}^{\varphi}(s,\tau) + I_{\mathrm{Sing}}^{\varphi}(s,\tau) - I_{\mathrm{Whi}}^{\varphi}(s,\tau), \tag{1-5}$$

where each integral on the right-hand side converges in the region $\mathrm{Re}(s) > 1$, *and can be meromorphically continued to $s \in \mathbb{C}$. Moreover*:

- $I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)$ *can be expressed as a finite sum of Dedekind zeta functions associated with certain étale algebras of degree $\le n$.* (See Theorem C for details.)

- $I^{\varphi}_{P,\mathrm{Reg}}(s,\tau)$ *can be written as a finite sum of intertwining operators. It turns out to be a holomorphic multiple of*
$$\frac{\Lambda(s,\tau)\,\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\,\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\,\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})}.$$
(See Theorem D for details.)
- $I^{\varphi}_{\mathrm{Whi}}(s,\tau)$ *is an infinite sum over cuspidal data* χ *of Rankin–Selberg* L*-functions attached to* χ. (See Theorems E and F.)
- *If* $\tau^k\neq\mathbf{1}$ *for* $1\leq k\leq n$, *then* (1-5) *has an analytic continuation to* $\mathbb{C}$, *with* $I^{\varphi}_{P,\mathrm{Reg}}(s,\tau)/\Lambda(s,\tau)$ *and* $I^{\varphi}_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ *being entire.*

A precise definition of the integrals on the right side of (1-5) can be found in Section 2C.

Remarks 1.2. (i) The expansion (1-5) generalizes Jacquet and Zagier's formula for GL(2) to GL(n). A restricted version was obtained by Flicker [8] under some choice of test functions φ so that only regular elliptic part of $I^{\varphi}_{\mathrm{Geo,Reg}}(s,\tau)$ shows up on the right-hand side of (1-5). New ideas of our proof are briefly summarized in Section 2B below.

(ii) $I^{\varphi}_{\mathrm{Sing}}(s,\tau)$ is defined geometrically, and it appears essentially when $n\geq 3$. In general, $I^{\varphi}_{\mathrm{Sing}}(s,\tau)$ should always be reduced to Jacquet–Zagier trace formula (1-5) in *smaller ranks*. For certain applications, one can easily eliminate it by choosing a suitable test function (see, for instance, Theorem B in Section 1B2). Also, a detailed analytic continuation of $I^{\varphi}_{\mathrm{Sing}}(s,\tau)/\Lambda(s,\tau)$ (for general test function φ) is given in [36] when $n\leq 4$.

1B2. *Some applications.* The formula (1-5) conveys interesting information between L-functions defined analytically and algebraically. It gives an explicit relation among Rankin–Selberg L-functions, Langlands–Shahidi L-functions and Hecke L-functions associated to field extensions. In fact, we shall deduce from it that *holomorphy of certain adjoint* L*-functions* for GL(n) implies the *Dedekind conjecture* for degree n extensions:

Conjecture 1.3 (τ-twisted Dedekind conjecture). *Let* E/F *be an extension of global fields. Then* $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ *is holomorphic when* $s\neq 1$, *where* $\Lambda(\cdot,\cdot)$ *denotes the complete Hecke* L*-function*, *and* $N_{E/F}$ *is the relative norm.*

When $\tau=\mathbf{1}$ is trivial, this conjecture is conventionally called the *Dedekind conjecture*, and is known to be true when E/F is Galois by the work of Aramata and Brauer (see Chapter 1 of [24]) or has a solvable Galois closure by the work of Uchida [32] and van der Waall [33]. The Dedekind conjecture is the principal case of Artin's holomorphy conjecture. The τ-twisted version of Conjecture 1.3 has been proved by Murty [26] when E/F is either Galois or has a solvable closure. However, the general case (or even the case of general degree-5 extensions) is not yet known.

When $n=2$, [18] provides a connection between adjoint L-functions associated to cuspidal representations of GL(2)$/F$ and $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ when E/F is quadratic. It was noted in [19] that, at least for degree/rank n up to 5, the two families seem to be related on a nuts-and-bolts level in the theory of integral representations, in addition to the relationships suggested by [18].

Let $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ be the set of cuspidal representations of $G(\mathbb{A}_F)$ of central character ω^{-1}. Recall that the twisted adjoint L-function is defined by

$$\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)=\frac{\Lambda(s,\pi\times\widetilde{\pi}\otimes\tau)}{\Lambda(s,\tau)},\quad \pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}). \tag{1-6}$$

Let $\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})\subset\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ be a subset of cuspidal representations π such that π has a supercuspidal component. Following [18], Flicker [8] used a simple trace formula to conclude, modulo the key Lemma 4 in loc. cit., that Conjecture 1.3 implies holomorphy of $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ at $s\neq 1$, where $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$. However, this lemma is not correct, as Flicker himself pointed out in [9, p. 202]. Consequently, the asserted implication is not complete. Nevertheless, we have an implication in the *opposite* direction.

Theorem B. *Assume the twist adjoint L-functions $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ are holomorphic at $s\neq 1$ for all $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\mathbf{1})$. Then the τ-twisted Dedekind conjecture holds for all field extensions of E/F of degree n.*

Remarks 1.5. (i) Theorem B provides a new perspective in the study of the Dedekind conjecture, which is currently wide open when the degree is larger or equal to 5 — although, when $n=5$, there has been some progress towards the holomorphy of adjoint L-functions by integral representation (see [11]).

(ii) Suppose $\tau^k\neq\mathbf{1}$, $1\leq k\leq n$. With further investigation of $I_{\mathrm{Sing}}(s,\tau)$, we may conclude from Theorems A and B that $L(s,\pi,\mathrm{Ad}\otimes\tau)$ is holomorphic at $s\neq 1$ for all $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ if and only if the τ-twisted Dedekind conjecture holds for all field extensions of E/F of degree n.

In Section 8, we will see that the proof of Theorem B yields a nonvanishing result:

Corollary 1.6. *Let notation be as before. Let $n\geq 2$. Suppose there exists an extension E/F with degree $[E:F]=n$, and $\zeta_E(1/2)\neq 0$. Then there exists a $\pi=\pi(E)\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$, such that $L(1/2,\pi\times\widetilde{\pi})\neq 0$.*

Remark 1.7. In fact, Fröhlich [10] proved that there are infinitely many number fields F such that $\zeta_F(1/2)=0$. Since $L(s,\pi,\mathrm{Ad})$ is conjectured to be holomorphic, for *all* $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$, we have $L(1/2,\pi\times\widetilde{\pi})=0$ conjecturally.

2. Idea of proof and structure of the paper

This section outlines our strategy for establishing the coarse Jacquet–Zagier trace formula (1-5):

$$I_0^{\varphi}(s;\tau)=I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)-I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)+I_{\mathrm{Sing}}^{\varphi}(s,\tau)-I_{\mathrm{Whi}}^{\varphi}(s,\tau).$$

Definitions of these integrals will be provided in Sections 2A–2C. Additional details can be found in Section 3.

2A. ***Basic notation.***

2A1. *Number fields.* Let F be a number field with ring of adeles $\mathbb{A}_F$. Let Σ_F be the set of places of F. Denote by $\Sigma_{F,\mathrm{fin}}$ (resp. $\Sigma_{F,\infty}$) the set of nonarchimedean (resp. archimedean) places. For $v \in \Sigma_F$, we denote by F_v the corresponding local field and $\mathcal{O}_v$ its ring of integers with the maximal ideal $\mathfrak{p}_v$ and a uniformizer ϖ_v. We use $v \mid \infty$ to indicate an archimedean place v and write $v < \infty$ if v is nonarchimedean. Let $|\cdot|_v$ be the norm in F_v. Put $|\cdot|_\infty = \prod_{v\mid\infty} |\cdot|_v$ and $|\cdot|_{\mathrm{fin}} = \prod_{v<\infty} |\cdot|_v$. Let $|\cdot|_{\mathbb{A}_F} = |\cdot|_\infty \otimes |\cdot|_{\mathrm{fin}}$. We will simply write $|\cdot|$ for $|\cdot|_{\mathbb{A}_F}$ in calculation over $\mathbb{A}_F^\times$ or its quotient by $F^\times$.

2A2. *Some conventional notation.* For two meromorphic functions $h_1(s)$ and $h_2(s)$, we write $h_1(s) \propto h_2(s)$ if $h_1(s)/h_2(s)$ admits an analytic continuation to the whole complex plane. We will keep this $\propto$ notation throughout.

2A3. *Automorphic data.* Let $G = \mathrm{GL}(n)$. Let P be the standard parabolic subgroup of G of type $(n-1, 1)$. Let B be the Borel subgroup, consisting of upper triangular matrices. Let K be a fixed maximal compact subgroup of $G(\mathbb{A}_F)$. We will denote by $[H] := H(F)\backslash H(\mathbb{A}_F)$ for an algebraic group H over F.

Denote by Ξ_F the set of unitary characters on $F^\times\backslash\mathbb{A}_F^\times$ which are trivial on $\mathbb{R}_+^\times$. For any $\xi \in \Xi_F$, denote by $\Lambda(s, \xi)$ its complete Hecke L-function. For a topological space V, we denote by $\mathcal{S}(V)$ the space of Schwartz functions on V. Let $\Phi \in \mathcal{S}(\mathbb{A}_F^n)$. Let $\tau \in \Xi_F$ be fixed. Let $\eta = (0, \dots, 0, 1) \in F^n$. Set

$$f(x, \Phi, \tau; s) = \tau(\det x)\, |\det x|^s \int_{\mathbb{A}_F^\times} \Phi(\eta t x)\tau(t)^n |t|^{ns} d^\times t, \quad x \in G(\mathbb{A}_F), \tag{2-1}$$

which is a Tate integral for the complete L-function $\Lambda(ns, x.\Phi, \tau^n)$. It converges absolutely in $\mathrm{Re}(s) > 1/n$, and admits a meromorphic continuation elsewhere with a functional equation. Define the Eisenstein series

$$E_P(x, \Phi, \tau; s) = \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma x, \Phi, \tau; s), \tag{2-2}$$

which converges absolutely for $\mathrm{Re}(s) > 1$. Also, we define the integral

$$I_0^\varphi(s, \tau) = \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0^\varphi(x, x) E_P(x, \Phi, \tau; s)\, dx. \tag{2-3}$$

If there is no confusion in the context, we will write $f_\tau(x, s)$ or $f(x, s)$ instead of $f(x, \Phi, \tau; s)$ and omit the superscript φ in $I_*^\varphi(s; \tau)$ for simplicity, where $I_*^\varphi(s; \tau)$ is one of the functions in (1-5).

2A4. *The kernel function.* Denote by $\mathcal{H}(G(\mathbb{A}_F), \omega)$ the set of smooth functions $\varphi\colon G(\mathbb{A}_F) \to \mathbb{C}$, which is left and right K-finite for a compact subgroup K of $G(\mathbb{A}_F)$, transforms by a unitary character ω of $Z(\mathbb{A}_F)$, and has compact support modulo $Z(\mathbb{A}_F)$. Then $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$ defines an integral operator

$$R(\varphi) f(x) = \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(y) f(xy)\, dy \tag{2-4}$$

on the space $L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ of functions on $G(F)\backslash G(\mathbb{A}_F)$ which transform under $Z(\mathbb{A}_F)$ by ω^{-1} and are square integrable on $G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. This operator can be represented by the kernel function

$$\mathrm{K}^{\varphi}(x, y) = \sum_{\gamma \in Z(F)\backslash G(F)} \varphi(x^{-1}\gamma y).$$

We will omit the superscript φ and simply write $\mathrm{K}(x, y)$ for $\mathrm{K}^{\varphi}(x, y)$.

Recall that $L^2\big(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\big)$ decomposes as a direct sum of the space $L^2_0\big(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\big)$ of cusp forms and spaces $L^2_{\mathrm{Eis}}\big(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\big)$ and $L^2_{\mathrm{Res}}\big(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\big)$ defined using Eisenstein series and residues of Eisenstein series respectively. Then K splits up as $\mathrm{K} = \mathrm{K}_0 + \mathrm{K}_{\mathrm{Eis}} + \mathrm{K}_{\mathrm{Res}}$. Explicitly,

$$\mathrm{K}_0(x, y) = \sum_{\pi} \mathrm{K}_{\pi}(x, y), \quad \text{where } \mathrm{K}_{\pi}(x, y) = \sum_{\phi \in \mathcal{B}_{\pi}} \pi(\varphi)\phi(x)\overline{\phi(y)}. \tag{2-5}$$

Here, π ranges over cuspidal automorphic representations, and $\mathcal{B}_{\pi}$ represents an orthonormal basis for the representation π.

2B. *Decomposition of the kernel function.* Starting with the spectral decomposition

$$\mathrm{K}_0(x, x) = \mathrm{K}(x, x) - (\mathrm{K}_{\mathrm{Eis}}(x, x) + \mathrm{K}_{\mathrm{Res}}(x, x)),$$

we will further decompose these kernel functions by algebraic and analytic/spectral expansions.

2B1. *The regular part.* Let $P_0 = Z\backslash P$ is the mirabolic subgroup. Define

$$\mathfrak{S} := \bigcup_{Q} \bigcup_{\gamma \in Z(F)\backslash Q(F)} \{p^{-1}\gamma p : p \in P_0(F)\}, \tag{2-6}$$

where Q's range through standard proper parabolic subgroups of G. Set

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x, x) = \sum_{\gamma \in Z(F)\backslash G(F) - \mathfrak{S}} \varphi(x^{-1}\gamma x), \tag{2-7}$$

We will show in Proposition 4.1 that the set $Z(F)\backslash G(F) - \mathfrak{S}$ consists of $P_0(F)$-conjugacy classes, resulting in regular $G(F)$-conjugacy classes. Notably, the stabilizers of elements in $Z(F)\backslash G(F) - \mathfrak{S}$ are direct sums of étale algebras over F with degrees $\leq n$. This distinction motivates the consideration of this set.

2B2. *The P-regular part.* Let N_P be the unipotent radical of P. Define

$$\mathrm{K}_{P,\mathrm{Reg}}(x, x) = \int_{[N_P]} \mathrm{K}_{\mathrm{Geo,Reg}}(ux, x)\, du, \tag{2-8}$$

where the subscript P indicates the constant term along $[N_P] = N_P(F)\backslash N_P(\mathbb{A}_F)$.

2B3. *Fourier expansion.* Let N be the unipotent radical of the standard Borel in G. Let θ being a fixed generic character on $[N] = N(F)\backslash N(\mathbb{A}_F)$.

We will show in Lemma 3.2 the Fourier expansion

$$\mathrm{K}_{\mathrm{Eis}}(x, x) + \mathrm{K}_{\mathrm{Res}}(x, x) = \int_{[N_P]} \mathrm{K}(ux, x)\, du + \sum_{k=2}^{n-1} \mathcal{F}_k \mathrm{K}(x, x) + \mathrm{K}_{\mathrm{Whi}}(x, x), \tag{2-9}$$

where $\mathcal{F}_k\mathrm{K}(x,x)$ represents the partial Fourier transforms of $\mathrm{K}(x,x)$ (see Proposition 3.1), and the generic part is given by

$$\mathrm{K}_{\mathrm{Whi}}(x,x)=\sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(u\delta x,x)\theta(u)\,du.$$

The integrand will be Whittaker functions, which explains the subscript 'Whi'.

Notably, the first two terms on the right side of (2-9) are expressed in terms of the entire kernel function $\mathrm{K}(x,x)$. Therefore, we can handle them by employing the geometric expansion of the kernel function.

2B4. *The singular part.* We define

$$\mathrm{K}_{\mathrm{Sing}}(x,x)=\sum_{\gamma\in\mathfrak{S}}\varphi(x^{-1}\gamma x)-\int_{[N_P]}\sum_{\gamma\in\mathfrak{S}}\varphi(x^{-1}u^{-1}\gamma x)\,du-\sum_{k=2}^{n-1}\mathcal{F}_k\mathrm{K}(x,x). \tag{2-10}$$

A precise definition will be given in Section 3B2. In [36], we provide explicit simplifications of this part in terms of φ for the cases when $n=3$ and $n=4$.

2B5. *Decomposition of* $\mathrm{K}_0(x,x)$. Combining (2-7), (2-8), (2-9) and (2-10) we then obtain an expansion of the cuspidal kernel function

$$\mathrm{K}_0(x,x)=\mathrm{K}_{\mathrm{Geo,Reg}}(x,x)-\mathrm{K}_{P,\mathrm{Reg}}(x,x)+\mathrm{K}_{\mathrm{Sing}}(x,x)-\mathrm{K}_{\mathrm{Whi}}(x,x). \tag{2-11}$$

This decomposition will be proved in Lemma 3.3.

2C. ***Decomposition of*** $\boldsymbol{I_0^{\varphi}(s,\tau)}$**.** Let $\mathrm{Re}(s)>1$. Unfold $E_P(x,\Phi,\tau;s)$ by substituting (2-2) into (2-3), and replace K_0 via (2-11) to obtain (formally)

$$I_0^{\varphi}(s;\tau)=I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)-I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)+I_{\mathrm{Sing}}^{\varphi}(s,\tau)-I_{\mathrm{Whi}}^{\varphi}(s,\tau), \tag{1-5}$$

Here

$$\begin{aligned}
I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)&:=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_{\mathrm{Geo,Reg}}(x,x)f(x,\Phi,\tau;s)\,dx,\\
I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)&:=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\mathrm{K}_{\mathrm{Geo,Reg}}(ux,x)f(x,\Phi,\tau;s)\,du\,dx,\\
I_{\mathrm{Whi}}^{\varphi}(s,\tau)&:=\int_{N(\mathbb{A}_F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{[N]}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(ux,vx)\theta(u)\overline{\theta}(v)f(x,\Phi,\tau;s)\,du\,dv\,dx,\\
I_{\mathrm{Sing}}^{\varphi}(s,\tau)&:=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_{\mathrm{Sing}}(x,x)f(x,\Phi,\tau;s)\,dx,
\end{aligned}$$

where $f(x,\Phi,\tau;s)$ is the section defined by (2-1). The proof of (1-5) will be derived in Section 3B4.

The main objective of this paper is to describe the analytic behavior of these integrals as meromorphic functions of s and establish their connections with various well-known L-functions.

For simplicity, we will omit the superscript φ in the integrals above.

2D. ***Structure of the paper.*** In Section 3, we use the Fourier expansion of functions on $P_0(F)\backslash G(\mathbb{A}_F)$ to obtain the decomposition (2-9). This manipulation allows us to shift the nongeneric part of the nondiscrete spectrum to the geometric side. Eventually we obtain the decomposition (1-5) in Section 3B.

In Section 4, we explore the algebraic structure of conjugacy classes in $Z(F)\backslash G(F)-\mathfrak{S}$ under the $P_0(F)$-adjoint action. Notably, the stabilizers of elements in $Z(F)\backslash G(F)-\mathfrak{S}$ correspond to direct sums of étale algebras over F. Consequently, we demonstrate that the function $I_{\mathrm{Geo,Reg}}(s,\tau)$ can be expressed as a sum of specific Artin L-series associated with these étale algebras. The detailed formulation of this result is provided in Theorem C in Section 4C.

In Section 5, we determine explicit representatives of $Z(F)\backslash G(F)-\mathfrak{S}$ as $P_0(F)$-conjugacy classes. Subsequently, we develop a geometric reduction technique to establish a connection between $I_{P,\mathrm{Reg}}(s,\tau)$ and certain intertwining operators. By employing Langlands' theory of intertwining operators, we establish the convergence and analytic properties of $I_{\mathrm{Geo,Reg}}(s,\tau)$. The results are summarized in Theorem D.

Next, our focus shifts to the function $I_{\mathrm{Whi}}(s,\tau)$, which pertains exclusively to the spectral side and is discussed in Sections 6–9.

- We encounter a challenge with Arthur's approach using modified truncation operators, as it is not suitable for our specific scenario due to the loss of $P(F)$-invariance when unfolding the Eisenstein series. Instead, we employ an alternative manipulation that reduces $I_{\mathrm{Whi}}(s,\tau)$ to a Mellin transform of the Kuznetsov trace formula, particularly Jacquet's *relative trace formula* for the pair (N,N) involving maximal unipotent radicals and generic characters. We establish that the relative trace formula can be bounded by a finite sum of gauges (Proposition 6.2). Consequently, we prove that $I_{\mathrm{Whi}}(s,\tau)$ is an absolutely convergent infinite sum of Mellin transforms, corresponding to certain Rankin–Selberg convolutions with *nondiscrete* automorphic representations when $\mathrm{Re}(s)$ exceeds a specific threshold. The precise results are presented in Theorem E in Section 6.
- In Section 7, we establish various properties of Rankin–Selberg periods associated with nondiscrete automorphic representations. These results play a crucial role in enhancing Theorem E and demonstrating the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ within the strip $0<\mathrm{Re}(s)<1$. For detailed information, refer to Theorem F. Notably, if $\tau^k\neq\mathbf{1}$ for all $1\leq k\leq n$, then $I_{\mathrm{Whi}}(s,\tau)$ is holomorphic in the region $\mathrm{Re}(s)>0$.
- In Section 8, we consolidate the results of Theorems C, D, E, and F to establish Equation (9-5). By employing specific test functions φ and addressing generalized Tate integrals, we subsequently prove Theorem B and Corollary 1.6 based on Equation (9-5).
- In Section 9, we address the special case when $\tau^k=\mathbf{1}$ for some $1\leq k\leq n$. In this scenario, the function $I_{\mathrm{Whi}}(s,\tau)$ exhibits singularities along the entire boundary $\mathrm{Re}(s)=1$. To overcome this, we aim to find a meromorphic continuation for $I_{\mathrm{Whi}}(s,\tau)$ that holds for *any* τ. This investigation is carried out in Section 9, where we establish Theorem G. By utilizing the Langlands–Shahidi method and analyzing residues, we obtain the meromorphic continuation for each individual term in

$I_{\mathrm{Whi}}(s,\tau)$. Theorem G provides the meromorphic continuation for $I_{\mathrm{Whi}}(s,\tau)$ in the general case of F being a function field. It also holds independent interest, as highlighted in [36].

2E. ***Some remarks on reading this paper.*** The integrals presented in (1-5) exhibit distinct characteristics, and each section is dedicated to analyzing one of these integrals. As a result, these sections can be considered somewhat independent from one another. Given the intricate nature of these integrals, we will introduce temporary notation specific to each section during the proof. While these ad hoc notational items may appear unfamiliar, readers are encouraged not to be overly concerned or confused, as their purpose is to facilitate the clarity and coherence of the respective section.

3. Fourier expansion and decomposition of $I_0(s,\tau)$

In this section, we use the Fourier expansion of the noncuspidal kernel function $\mathrm{K}_{\mathrm{ER}}(x,x)$ to decompose the cuspidal kernel function as follows:

$$\mathrm{K}_0(x,y)=\mathrm{K}_{\mathrm{Geo,Reg}}(x,y)-\mathrm{K}_{P,\mathrm{Reg}}(x,y)+\mathrm{K}_{\mathrm{Sing}}(x,y)-\mathrm{K}_{\mathrm{Whi}}(x,y). \tag{2-11}$$

Consequently, we obtain the decomposition

$$I_0(s,\tau)=I_{\mathrm{Geo,Reg}}(s,\tau)-I_{P,\mathrm{Reg}}(s,\tau)+I_{\mathrm{Sing}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau). \tag{1-5}$$

3A. ***Mirabolic Fourier expansions of automorphic functions.***

3A1. *Ad hoc notation.* To simplify the Fourier expansion discussion in this section, we introduce some notation that may deviate from standard conventions. Since it is specific to this section, readers need not be overly concerned with it.

Recall that B is the standard Borel of G (Section 2A3). Let N be the unipotent radical of B. For $1\leq k\leq n-1$, let B_{n-k} be the standard Borel subgroup (the subgroup consisting of nonsingular upper triangular matrices) of GL_{n-k}; let N_{n-k} be the unipotent radical of B_{n-k}. For any $i,\ j\in\mathbb{N}$, let $M_{i\times j}$ be the additive group scheme of $i\times j$ matrices. For $1\leq k\leq n-1$, define the unipotent radicals

$$N_k'=\left\{\begin{pmatrix} I_k & C\\ & D\end{pmatrix}: C\in M_{k\times(n-k)},\ D\in N_{n-k}\right\},$$

and the subgroup

$$N_k^*=\left\{\begin{pmatrix} I_{k-1} & C & \\ & 1 & \\ & & I_{n-k}\end{pmatrix}: C\in M_{(k-1)\times 1}(F)\right\}.$$

For $1\leq k\leq n-1$, set the generalized mirabolic subgroups

$$R_k=\left\{\begin{pmatrix} A & C\\ 0 & D\end{pmatrix}: A\in \mathrm{GL}_k,\ C\in M_{k\times(n-k)},\ D\in N_{n-k}\right\}.$$

In particular, $R_{n-1}=P_0$, which is the mirabolic subgroup of G.

Also we define $R_0=N_{(0,1,\dots,1)}:=N_{(1,1,\dots,1)}=N$ to be the unipotent radical of the standard Borel subgroup of GL_n.

3A2. *Mirabolic Fourier expansion.* Denote by Tr_F the trace map $\mathbb{A}_F \to \mathbb{A}_{\mathbb{Q}}$. Let $\psi_{\mathbb{Q}}$ be the additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ such that $\psi_{\mathbb{Q}}(t_\infty) = \exp(2\pi i t_\infty)$, for $t_\infty \in \mathbb{R} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$. Let $\psi_F = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_F$. For any $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in F^{n-1}$, define a character $\psi_\alpha : N(\mathbb{A}_F) \to \mathbb{C}$ by

$$\psi_\alpha(u) = \prod_{i=1}^{n-1} \psi_F\left(\alpha_i u_{i,i+1}\right), \quad \forall u = (u_{i,j})_{n\times n} \in N(\mathbb{A}_F). \tag{3-1}$$

Write $\psi_k = \psi_{(0,\dots,0,1,\dots,1)}$ (where the first $n-1-k$ components are 0 and the remaining k components are 1) and let $\theta = \psi_{(1,\dots,1)}$ be the standard generic character used to define Whittaker functions.

Proposition 3.1. *Let h be a continuous function on $P_0(F)\backslash G(\mathbb{A}_F)$. Then*

$$h(x) = \sum_{k=1}^{n} \sum_{\delta_k \in R_{k-1}(F)\backslash R_{n-1}(F)} \int_{[N_k^*]} \int_{[N_k']} h(uu'\delta_k x)\psi_{n-k}(u')\, du'\, du \tag{3-2}$$

if the right-hand side converges absolutely and locally uniformly. In particular, (3-2) *holds with $h(x) = \mathrm{K}^{\varphi}(x, y)$ for $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$ and for fixed $y \in G(\mathbb{A}_F)$.*

The proof follows the idea of Piatetski-Shapiro in [27]. See for example [35, §3.1] or [37, §2.4] (with certain mild adaptations).

3B. *Decomposition of $I_0(s, \tau)$.*

3B1. *Fourier expansion of* $\mathrm{K}_{\mathrm{ER}}(x, y)$. The objective of this section is to express $\mathrm{K}_{\mathrm{ER}}(x, y)$ in terms of its generic component and the complete kernel $\mathrm{K}(x, y)$.

Lemma 3.2. *Let notation be as in Section 3A. Then*

$$\mathrm{K}_{\mathrm{ER}}(x, y) = \mathrm{K}_{\mathrm{Whi}}(x, y) + \sum_{k=2}^{n} \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[N_k^*]} \int_{[N_k']} \mathrm{K}(uu'\delta_k x, y)\theta(u')\, du'\, du$$

where

$$\mathrm{K}_{\mathrm{Whi}}(x, y) = \sum_{\delta \in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\mathrm{Eis}}(u\delta x, \delta y)\theta(u)\, du.$$

Proof. $\mathrm{K}_{\mathrm{ER}}(x, y)$ is $P_0(F)$-invariant with respect to both variables. Given that $\mathrm{K}_0(x, y)$ decays rapidly, Proposition 3.1 applies to $h(x) = \mathrm{K}_{\mathrm{ER}}(x, y) = \mathrm{K}(x, y) - \mathrm{K}_0(x, y)$. Hence,

$$\mathrm{K}_{\mathrm{ER}}(x, y) = \sum_{k=1}^{n} \mathrm{K}_{\mathrm{ER}}^{(k)}(x, y), \tag{3-3}$$

where

$$\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y) = \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[N_k^*]} \int_{[N_k']} \mathrm{K}_{\mathrm{ER}}(uu'\delta_k x, y)\theta(u')\, du'\, du.$$

Recall that $\theta = \psi_{(1,1,\dots,1)}$ is the generic character of (3-1). By definition, $\theta(u') = \psi_{n-k}(u')$ for $y' \in [N_k']$.

When $k = 1$, N_k^* becomes trivial and $N_k' = N$. Consequently, we can replace K_{ER} with $\mathrm{K}_{\mathrm{Eis}}$ in the definition of $\mathrm{K}_{\mathrm{ER}}^{(1)}(x, y)$ since the residual spectrum is not generic [25, §V]. Define

$$\mathrm{K}_{\mathrm{Whi}}(x, y) := \mathrm{K}_{\mathrm{ER}}^{(1)}(x, y) = \sum_{\delta \in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\mathrm{Eis}}(u\delta x, \delta y)\theta(u)\, du. \tag{3-4}$$

Let $2 \leq k \leq n$. Let $V_k' = \mathrm{diag}(I_{k-1}, N_{n-k+1})$. Define V_k as the unipotent radical of the standard parabolic subgroup of type $(k-1, n-k+1)$. For any function ϕ on $G(\mathbb{A}_F)$, we have

$$\int_{[N_k^*]}\int_{[N_k']}\phi(uu'x)\theta(u')\,du'\,du = \int_{[V_k']}\int_{[V_k]}\phi(uu'x)\,du\,\theta(u')\,du', \quad x \in G(\mathbb{A}_F).$$

Since V_k is a nontrivial unipotent radical, then for all cusp forms ϕ on $G(F)\backslash G(\mathbb{A}_F)$,

$$\int_{[N_k^*]}\int_{[N_k']}\phi(uu'x)\theta(u')\,du'\,du = 0. \tag{3-5}$$

Combining (3-5) with the discrete spectral decomposition (2-5) one has

$$\int_{[N_k^*]}\int_{[N_k']}\mathrm{K}_0(uu'x, y)\theta(u')\,du'\,du = 0. \tag{3-6}$$

Since $\mathrm{K} = \mathrm{K}_{\mathrm{ER}} + \mathrm{K}_0$, this yields, for $2 \leq k \leq n$,

$$\iint \mathrm{K}_{\mathrm{ER}}(uu'x, y)\theta(u')\,du'\,du = \iint \mathrm{K}(uu'x, y)\theta(u')\,du'\,du,$$

where $u \in [N_k^*]$ and $u' \in [N_k']$. As a consequence, we have, for $2 \leq k \leq n$, that

$$\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y) = \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[N_k^*]}\int_{[N_k']}\mathrm{K}(uu'\delta_k x, y)\theta(u')\,du'\,du. \tag{3-7}$$

Now Lemma 3.2 follows from (3-3), (3-4), and (3-7). ☐

3B2. *The singular kernel* $\mathrm{K}_{\mathrm{Sing}}(x, y)$. Let $\mathrm{K}_{\mathrm{Geo,Reg}}(x, y)$ and $\mathrm{K}_{\boldsymbol{P},\mathrm{Reg}}(x, y)$ be defined by (2-7) and (2-8). Set

$$\mathrm{K}_{\mathrm{Geo,Sing}}(x, y) = \sum_{\gamma \in \mathfrak{S}} \varphi(x^{-1}\gamma y), \quad \mathrm{K}_{\boldsymbol{P},\mathrm{Sing}}(x, y) = \int_{[N_P]}\mathrm{K}_{\mathrm{Geo,Sing}}(ux, y)\,du. \tag{3-8}$$

Then $\mathrm{K}_{\mathrm{Geo,Sing}}(x, y) = \mathrm{K}(x, y) - \mathrm{K}_{\mathrm{Geo,Reg}}(x, y)$. Define

$$\mathrm{K}_{\mathrm{Sing}}(x, y) := \mathrm{K}_{\mathrm{Geo,Sing}}(x, y) - \mathrm{K}_{\boldsymbol{P},\mathrm{Sing}}(x, y) - \sum_{k=2}^{n-1}\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y), \tag{2-10}$$

where $\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y)$ is defined by (3-7).

3B3. *Decomposition of* $\mathrm{K}_0(x, y)$.

Lemma 3.3. *Let notation be as before. Then*

$$\mathrm{K}_0(x, y) = \mathrm{K}_{\mathrm{Geo,Reg}}(x, y) - \mathrm{K}_{\boldsymbol{P},\mathrm{Reg}}(x, y) + \mathrm{K}_{\mathrm{Sing}}(x, y) - \mathrm{K}_{\mathrm{Whi}}(x, y). \tag{2-11}$$

Proof. By Lemma 3.2 and $\mathrm{K}_0(x, y) = \mathrm{K}(x, y) - \mathrm{K}_{\mathrm{ER}}(x, y)$ we obtain

$$\mathrm{K}_0(x, y) = \mathrm{K}(x, y) - \mathrm{K}_{\mathrm{Whi}}(x, y) - \sum_{k=2}^{n}\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y), \tag{3-9}$$

where $\mathrm{K}_{\mathrm{ER}}^{(k)}(x, y)$ is defined by (3-7). Note that $\mathrm{K}_{\boldsymbol{P},\mathrm{Sing}}(x, y) + \mathrm{K}_{\boldsymbol{P},\mathrm{Reg}}(x, y) = \mathrm{K}_{\mathrm{ER}}^{(n)}(x, y)$. Now (2-11) follows from (3-8), (2-10), and (3-9). ☐

3B4. *Decomposition of* $I_0(s,\tau)$. Unfolding the Eisenstein series into (2-3), we get

$$I_0(s,\tau)=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)f(x,\Phi,\tau;s)\,dx. \tag{3-10}$$

Therefore, substituting Lemma 3.3 into (3-10) we obtain

$$I_0(s,\tau)=I_{\mathrm{Geo,Reg}}(s,\tau)-I_{P,\mathrm{Reg}}(s,\tau)+I_{\mathrm{Sing}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau), \tag{1-5}$$

where the integrals on the right are defined in Section 2C.

In the next three sections, we will examine the analytic properties of the functions $I_{\mathrm{Geo,Reg}}(s,\tau)$, $I_{P,\mathrm{Reg}}(s,\tau)$, and $I_{\mathrm{Whi}}(s,\tau)$. The analytical behavior of $I_{\mathrm{Sing}}(s,\tau)$ has been studied in [36] for $n\leq 4$.

4. $I_{\mathrm{Geo,Reg}}(s,\tau)$ as Dedekind zeta functions

In this section we study $I_{\mathrm{Geo,Reg}}(s,\tau)$, which is defined by

$$I_{\mathrm{Geo,Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)}\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x)f(x,s)\,dx. \tag{4-1}$$

Decompose $G(F)$ into conjugacy classes. We will establish a relationship between a regular $G(F)$-conjugacy class $\mathcal{C}$ and a unique $P(F)$-conjugacy class $\mathcal{C}_0$. With this manipulation, we can transform the integral over the automorphic quotient $Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ into an integral over $Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, yielding

$$I_{\mathrm{Geo,Reg}}(s,\tau)=\sum_{\mathcal{C}\text{ regular}}I_{\mathcal{C}}(s,\tau)=\sum_{\mathcal{C}}\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma x)f(x,s)\,dx, \tag{4-2}$$

where γ is a suitable representative in $\mathcal{C}$. The inner integral factors through a Tate integral over the stabilizer $G_\gamma(\mathbb{A}_F)$, which is an étale algebra over F of degree n. We then compute this integral in terms of various Hecke L-functions.

4A. ***Structure of $G(F)$-conjugacy classes.*** Let B be the subgroup of upper triangular matrices of G. Let T (resp. N) be the Levi component (resp. unipotent radical) of B. For each $1\leq k\leq n-1$, let

$$w_k=\begin{pmatrix}I_{k-1}&&\\&S&\\&&I_{n-k-1}\end{pmatrix},\quad\text{where } S=\begin{pmatrix}&1\\1&\end{pmatrix}. \tag{4-3}$$

Let W_n be the group generated by elements w_i, $1\leq i\leq n-1$, identified with the Weyl group of G with respect to (B,T).

For each $1\leq k\leq n-1$, let Q_k be the standard maximal parabolic subgroup of G of type $(k,n-k)$. Note that $P=Q_{n-1}$. Set

$$Q_k(F)^{P(F)}=\{pqp^{-1}:p\in P(F),\ q\in Q_k(F)\},\ 1\leq k\leq n-1.$$

For $\gamma \in G(F)$, we denote by $G_\gamma(F)$ the centralizer of γ in $G(F)$. Recall that γ is regular if $\dim G_\gamma(F)$ is minimal (which also amounts to that the minimal polynomial of γ coincides with its characteristic polynomial).

Here are the two results in this section:

Proposition 4.1. *Let $\mathcal{C}$ be a regular $G(F)$-conjugacy classes in $G(F)$. There exists a $P(F)$-conjugacy class $\mathcal{C}_0$ such that*

$$\mathcal{C} = \mathcal{C}_0 \sqcup \bigcup_{k=1}^{n-1} \mathcal{C} \cap Q_k(F)^{P(F)}, \tag{4-4}$$

where $\sqcup$ denotes a disjoint union.

Proposition 4.2. *If $\mathcal{C}$ is an irregular $G(F)$-conjugacy class, then*

$$\mathcal{C} = \bigcup_{k=1}^{n-1} \mathcal{C} \cap Q_k(F)^{P(F)}.$$

4B. *Decomposition of $G(F)$.* The following is an analogue of Jordan canonical forms of matrices over a number field (see [34, Theorem 3.1], for instance).

Lemma 4.3. *Let V be a n-dimensional vector space over F, and $\mathscr{A} \in \operatorname{End}(V)$. Then there exist invariant subspaces $V_l \subseteq V$, $1 \le l \le r$, such that*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r, \tag{4-5}$$

and for each i, both the minimal polynomial and the characteristic polynomial of $\mathscr{A}_{V_l} = \mathscr{A}\,|_{V_l}$ are of the form $\wp(\lambda)^k$, where $k \in \mathbb{N}_{\ge 1}$ and $\wp(\lambda) \in F[\lambda]$ is an irreducible polynomial over F. For each l, there exists a basis of V_l under which $\mathscr{A}_{V_l}$ has the quasirational canonical form

$$\mathcal{J}\left(\wp(\lambda)^k\right) := \begin{pmatrix} C(\wp) & & & \\ \boldsymbol{N} & C(\wp) & & \\ & \ddots & \ddots & \\ & & \boldsymbol{N} & C(\wp) \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & & \end{pmatrix}, \tag{4-6}$$

where $C(\wp)$ is the companion matrix of $\wp(\lambda)$.

Let $\gamma \in G(F)$ be regular and denote by $f(\lambda) = \wp_1(\lambda)^{e_1} \cdots \wp_m(\lambda)^{e_m}$ its characteristic polynomial, where $e_i \ge 1$, $\wp_i$ is monic and irreducible over F, $1 \le i \le m$. Let $F[\gamma]$ be the subalgebra of $n \times n$ matrices generated by F and γ. We can identify $F[\gamma]$ with the polynomial algebra $F[\lambda]/(f(\lambda))$. Since γ is regular, it follows from the Frobenius dimension formula [14, Theorem 3.16] that the dimension of the centralizer $M_{n\times n,\gamma}(F)$ of γ in $M_{n\times n}(F)$ is n. As $F[\gamma]$ is contained in $M_{n\times n,\gamma}(F)$ and has dimension n, we conclude that $G_\gamma(F) = M_{n\times n,\gamma}(F) \cap G(F) = F[\gamma]^\times$, where $F[\gamma]^\times$ denotes the set of invertible elements in $F[\gamma]$.

Recall that γ is elliptic if $[G_\gamma]$ has finite volume, or equivalently, if the minimal polynomial of γ is irreducible over F. Hence, $F[\gamma]$ is a field if γ is elliptic.

Lemma 4.4. *Let $\gamma \in G(F)$ be regular elliptic. For any $(a_1, a_2, \dots, a_n) \in F^n$, there exists a unique element $x \in F[\gamma]$ such that the last row of x is exactly $(a_1, a_2, \dots, a_n)$.*

Proof. Since γ is regular, $G_\gamma(F) = F[\gamma]^\times$, and $\dim F[\gamma] = n$. Let $\eta = (0, \dots, 0, 1) \in F^n$. Consider the linear map

$$\tau : F[\gamma] \to F^n, \quad x \mapsto \tau(x) = \eta x.$$

Since γ is elliptic, $F[\gamma]$ is a field, so any nonzero element is invertible. Consequently, the map τ is injective, and hence surjective. Thus τ is an isomorphism of n-dimensional F-vector spaces. The lemma follows. □

Remark 4.5. Let $\gamma \in G(F)$ be regular elliptic. Then $G(F) = P_0(F)F[\gamma]^\times$, where $P_0(F) = Z(F)\backslash P(F)$ is the mirabolic subgroup of $G(F)$. In fact, since τ is a bijection, given $g \in G(F)$, there exists $h \in F[\gamma]^\times$ such that $\eta g = \eta h$, which implies that $gh^{-1} \in P_0(F)$, i.e., $g \in P_0(F)F[\gamma]^\times$.

Lemma 4.6. *Let $\gamma \in G(F)$ be regular, and assume that the characteristic polynomial of γ has only one irreducible factor. Let $\mathcal{J}$ be the quasirational canonical form of γ. Then for any $(a_1, a_2, \dots, a_n) \in F^n$, there exists a unique element $x \in F[\mathcal{J}]$ such that the last row of x is exactly $(a_1, a_2, \dots, a_n)$.*

Proof. Let $f(\lambda) = \wp(\lambda)^e$ be the characteristic polynomial of γ, where $\wp(\lambda) \in F[\lambda]$ is irreducible. Then $\deg \wp = ne^{-1}$, and the quasirational canonical form $\mathcal{J}$ of γ has the structure

$$\mathcal{J} = \begin{pmatrix} C & & & \\ N & C & & \\ & \ddots & \ddots & \\ & & N & C \end{pmatrix} \in \mathrm{GL}_n(F), \tag{4-7}$$

where $C = C(\wp)$ is the companion matrix of $\wp(\lambda)$, and N is defined in (4-6). Without loss of generality, and for simplicity of notation, we may assume $\gamma = \mathcal{J}$.

By definition, $F[\gamma] = F[\lambda]/\left(\wp(\lambda)^e\right)$. Consider the filtration

$$\wp(\lambda)^{i-1} F[\lambda]/(f(\lambda)) \supseteq \wp(\lambda)^i F[\lambda]/(f(\lambda)), \quad 1 \le i \le e-1.$$

With respect to the basis $\left\{\lambda^i \wp(\lambda)^j : 0 \le i \le d-1,\ 0 \le j \le e-1\right\}$ for $F[\gamma]$ over F, each element in $F[\gamma]$ has the following type

$$\mathcal{S}_\gamma = \left\{ A = \begin{pmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{e-1} & \dots & A_2 & A_1 & A_0 \end{pmatrix},\ A_i \in M_{d\times d}(F),\ 0 \le i \le e-1 \right\}. \tag{4-8}$$

Since γ is regular, we have $G_\gamma(F) = F[\gamma]^\times$. Therefore,

$$F[\gamma]^\times = \left\{A \in \mathcal{S}_\gamma \cap \mathrm{GL}_n(F) : A\gamma = \gamma A\right\}.$$

Now, consider the equation $A\gamma = \gamma A$ for $A \in \mathcal{S}_\gamma$. Using the expression $\gamma = \mathcal{J}$ from (4-7), this is equivalent to the following system of Lyapunov-like equations

$$\begin{cases} CA_0 = A_0 C, \\ \boldsymbol{N}A_0 + CA_1 = A_1 C + A_0 \boldsymbol{N}, \\ \quad\vdots \\ \boldsymbol{N}A_{e-2} + CA_{e-1} = A_{e-1}C + A_{e-2}\boldsymbol{N}. \end{cases}$$

Since $A_0 \in F[C]^\times$ and C is regular elliptic, A_0 commuting with C implies that there exists some $h_0(\lambda) \in F[\lambda]$ such that $A_0 = h_0(C)$. We may assume that $d_0 = \deg h_0 \le d-1$. Let $\eta_d = (0, \ldots, 0, 1) \in F^d$ and write $\eta_d C^i = (b_1^{(i)}, b_2^{(i)}, \ldots, b_d^{(i)})$, $1 \le i \le d-1$, for the last row of C^i. Let

$$X_{(i)} = \begin{pmatrix} 0 & b_1^{(i)} & b_2^{(i)} & \cdots & b_{d-1}^{(i)} \\ 0 & 0 & b_1^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_2^{(i)} \\ \vdots & & \ddots & \ddots & b_1^{(i)} \\ 0 & \ldots & \ldots & 0 & 0 \end{pmatrix} \in \mathrm{GL}_d(F).$$

Then, for any $1 \le i \le d-1$, $X = X_{(i)}$ is a solution to the Lyapunov-like equation $\boldsymbol{N}C^i + CX = XC + C^i\boldsymbol{N}$.

Write $A_0 = h_0(C) = c'_{d_0} C^{d_0} + c'_{d_0-1} C^{d_0-1} + \cdots + c'_1 C + c'_0 I_d$, $c'_{d_0} \neq 0$. Define

$$A_{h_0}^{\mathrm{sp}} = c'_{d_0} X_{(d_0)} + c'_{d_0-1} X_{(d_0-1)} + \cdots + c'_1 X_{(1)}.$$

Clearly $A_1 = A_{h_0}^{\mathrm{sp}}$ gives a special solution of the equation $\boldsymbol{N}A_0 + CA_1 = A_1 C + A_0 \boldsymbol{N}$ (superscript "sp" stands for "special"). Given $A_0 = h_0(C)$ as above, the set

$$\mathcal{U}_1 = \{A_{h_0}^{\mathrm{sp}} + h_1(C) : h_1 \in F[\lambda],\ \deg h_1 \le d-1\}$$

gives all solutions to the equation $\boldsymbol{N}A_0 + CA_1 = A_1 C + A_0 \boldsymbol{N}$. In fact, on the one hand, elements in $\mathcal{U}_1$ satisfy the equation; on the other hand, let A'_1 be any solution to the equation, then $A_{h_0}^{\mathrm{sp}} - A'_1$ commutes with C; thus it is a polynomial of C, namely, $A'_1 \in \mathcal{U}_1$.

Note that $\boldsymbol{N}A_{h_0}^{\mathrm{sp}} = A_{h_0}^{\mathrm{sp}}\boldsymbol{N} = 0$. Substituting $A_1 = A_{h_0}^{\mathrm{sp}} + h_1(C)$ into the equation $\boldsymbol{N}A_1 + CA_2 = A_2 C + A_1 \boldsymbol{N}$, we get $\boldsymbol{N}h_1(C) + CA_2 = A_2 C + h_1(C)\boldsymbol{N}$. Write $h_1(\lambda) = c''_{d_1}\lambda^{d_1} + c''_{d_1-1}\lambda^{d_1-1} + \cdots + c''_1\lambda + c''_0$, and set

$$A_{h_1}^{\mathrm{sp}} = c''_{d_1} X_{(d_1)} + c''_{d_1-1} X_{(d_1-1)} + \cdots + c''_1 X_{(1)}.$$

Then $\mathcal{U}_2 = \{A_{h_1}^{\mathrm{sp}} + h_2(C) : h_2 \in F[\lambda],\ \deg h_2 \le d-1\}$ gives all solutions to the equation $\boldsymbol{N}A_1 + CA_2 = A_2 C + A_1 \boldsymbol{N}$. we define $\mathcal{U}_i$, $1 \le i \le e-1$ similarly, and set $\mathcal{U}_0 = \{h_0(C) : h_0 \in F[\lambda],\ \deg h_0 \le d-1\}$. These $\mathcal{U}_i$'s describe the structure of $F[\gamma]^\times$.

Given $\mathfrak{a} = (a_1, a_2, \ldots, a_n) \in F^n$, by Lemma 4.4 one can find uniquely an $A_0 \in F[C]$ such that $\eta_d A_0 = (a_{n-d+1}, a_{n-d+2}, \ldots, a_n)$. Set

$$\mathfrak{a}_i = (a_{(i-1)d+1}, a_{(i-1)d+2}, \ldots, a_{id}), \quad 1 \le i \le e-1.$$

Let $1 \le i_0 \le e-1$. Assume that for any $0 \le i < i_0$ one can find uniquely an element $A_i \in M_{d\times d}(F)$ such that the last row of A_i is exactly $\mathfrak{a}_{e-i}$, then let $h_{i_0}(C) \in F[C]^\times$ be the unique element whose last row is $\mathfrak{a}_{e-i_0}$, and take $A_{i_0} = A^{\mathrm{sp}}_{h_{i_0}} + h_{i_0}(C)$. Then $\eta_d A_{i_0} = \eta_d h_{i_0}(C) = \mathfrak{a}_{e-i_0}$. Such an A_{i_0} is unique. Let A'_{i_0} be another matrix satisfying that $\eta_d A'_{i_0} = \mathfrak{a}_{e-i_0}$. Since A'_{i_0} is a solution of $NA_{i_0-1} + CX = XC + A_{i_0-1}N$, $A_{i_0} - A'_{i_0}$ commutes with C. Thus $A_{i_0} - A'_{i_0} \in F[C]$. Note that the last row of $A_{i_0} - A'_{i_0}$ is $\mathbf{0}$, so by the uniqueness from Lemma 4.4, $A_{i_0} - A'_{i_0} = 0$. This shows the uniqueness of A_{i_0}.

Hence, Lemma 4.6 follows. □

In Remark 4.5, we showed the decomposition $G(F) = P_0(F)F[\gamma]^\times$ for regular elliptic $\gamma \in G(F)$, using Lemma 4.4. We now generalize this decomposition to regular $\gamma \in G(F)$ by utilizing Lemma 4.6 instead of Lemma 4.4.

Lemma 4.7. *Let $\gamma \in G(F)$ be regular. Then there exists a finite set $\Gamma_{\mathrm{reg}} = \{\gamma_i \in G(F) : 0 \le i \le m_0\}$ such that*

(1) $G(F) = \bigcup_{0\le i\le m_0} P_0(F)\gamma_i F[\gamma]^\times$, *where P_0 is the mirabolic subgroup of G, and*

(2) *there is at most one $\gamma_i \in \Gamma_{\mathrm{reg}}$ satisfying that $\gamma_i F[\gamma]^\times \gamma_i^{-1} \not\subseteq \bigcup_{k=1}^{n-1} Q_k(F)$.*

Proof. Denote by $f(\lambda) = \wp_1(\lambda)^{e_1} \cdots \wp_m(\lambda)^{e_m}$ the characteristic polynomial of $\gamma \in G(F)$, where $e_i \ge 1$ and the $\wp_i$'s are distinct monic and irreducible polynomials over F, $1 \le i \le m$. Let $d_i = \deg \wp_i$. Then $d_1e_1 + d_2e_2 + \cdots + d_me_m = \deg f = n$.

Since γ is regular, by Lemma 4.3, γ is $G(F)$-conjugate to a matrix of the form

$$\gamma^* = \begin{pmatrix} \mathcal{J}(\wp_1^{e_1}) & & & \\ & \mathcal{J}(\wp_2^{e_2}) & & \\ & & \ddots & \\ & & & \mathcal{J}(\wp_m^{e_m}) \end{pmatrix}, \tag{4-9}$$

where $\mathcal{J}(\wp_i^{e_i}) \in \mathrm{GL}_{d_ie_i}(F)$, $1 \le i \le m$. We may assume $\gamma = \gamma^*$. Write $k_i = d_ie_i$, $1 \le i \le m$. For $1 \le i \le m$, let $\tilde{\eta}_i : F^n \to F^{k_i}$ be defined by

$$\mathfrak{a} = (a_1, a_2, \ldots, a_n) \mapsto \tilde{\eta}_i(\mathfrak{a}) = (a_{k_1+\cdots+k_{i-1}+1}, \ldots, a_{k_1+\cdots+k_{i-1}+k_i}).$$

For simplicity we write $\eta^{(i)}(\mathfrak{a})$ for the last d_i components of $\tilde{\eta}_i(\mathfrak{a})$, i.e.,

$$\eta^{(i)}(\mathfrak{a}) = (a_{k_1+\cdots+k_{i-1}+(e_i-1)d_i+1}, a_{k_1+\cdots+k_{i-1}+(e_i-1)d_i+2}, \ldots, a_{k_1+\cdots+k_{i-1}+k_i}), \quad 1 \le i \le m.$$

We then split $G(F)$ into a union of sets following the conditions on the last row of its elements and show that each of the sets is of the form $P_0(F)\gamma_i F[\gamma]^\times$ for some specific γ_i.

• For $1 \le i, j \le n$, let $\eta_j = (0, 0, \ldots, 0, 1) \in F^j$, and $\boldsymbol{M}_{i,j}$ be the $i \times j$ matrix whose first $i-1$ rows are zero vectors, and the bottom row is η_j. Write $\eta = \eta_n$ as before. Let $\mathcal{S} = \{\delta \in G(F) : \eta^{(i)}(\eta\delta) \ne \mathbf{0},\ 1 \le i \le m\}$

and

$$\gamma_0 = \begin{pmatrix} I_{k_1} & & & \\ & \ddots & & \\ & & I_{k_{m-1}} & \\ \boldsymbol{M}_{k_m,k_1} & \dots & \boldsymbol{M}_{k_m,k_{m-1}} & I_{k_m} \end{pmatrix}. \tag{4-10}$$

Applying Lemma 4.6 to each $\tilde{\eta}_i(\mathfrak{a}) \in F^{k_i}$, we find for each $1 \le i \le m$, for any $\delta \in \mathcal{S}$, a unique $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$, such that $\eta_{k_i} x_i = \tilde{\eta}_i \delta$. When writing x_i in the form in (4-8), the definition of $\mathcal{S}$ implies that $A_0 \neq 0$; thus $A_0 \in F[C]^\times$. As a consequence, $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$.

Let $x = \operatorname{diag}(x_1, \dots, x_m)$. Then $\eta\gamma_0 x = \eta\delta$. So $\delta(\gamma_0 x)^{-1} \in P_0(F)$, i.e., $\delta \in P_0(F)\gamma_0 F[\gamma]^\times$. Moreover, $P_0(F) \cap \gamma_0 F[\gamma]^\times \gamma_0^{-1} = \{I_n\}$. To see this, look at the last row of $\gamma_0 x \gamma_0^{-1}$. A straightforward computation shows that

$$\tilde{\eta}_i(\gamma_0 x \gamma_0^{-1}) = \eta_{k_i} x_i - \eta_{k_i}, \quad 1 \le i \le m-1,$$

and $\tilde{\eta}_m(\gamma_0 x \gamma_0^{-1}) = \eta_{k_m}$. By the uniqueness part of Lemma 4.6, it follows that $x_i = I_{k_i}$, $1 \le i \le m$.

- For any $1 \le l \le m-1$ and $1 \le i_1 < \cdots < i_l \le m$, let

$$\mathcal{S}_{(i_1,\dots,i_l)} = \{\delta \in G(F) : \eta^{(j)}(\eta\delta) = \mathbf{0} \iff j \in \{i_1, \dots, i_l\}\}.$$

For any $1 \le i \le m$ and $1 \le s < t \le m$, define the Weyl elements $w_i^{(1)}$ and $w_{s,t}^{(2)}$ as

$$w_i^{(1)} = \begin{pmatrix} I_{k_i'} & & & & & \\ & 0 & & \cdots & I_{d_i} & \\ & & I_{d_i} & & & \\ & \vdots & & \ddots & \vdots & \\ & & & I_{d_i} & & \\ & I_{d_i} & & \cdots & 0 & \\ & & & & & I_{k_i''} \end{pmatrix},$$

where $k_i' = k_1 + \cdots k_{i-1}$, $k_i'' = k_{i+1} + \cdots + k_m$ for $1 \le i \le m$, and

$$w_{s,t}^{(2)} = \begin{pmatrix} I_{k_s'} & & & & & \\ & 0 & & \cdots & I_{k_t} & \\ & & I_{k_{s+1}} & & & \\ & \vdots & & \ddots & \vdots & \\ & & & I_{k_{t-1}} & & \\ & I_{k_s} & & \cdots & 0 & \\ & & & & & I_{k_t''} \end{pmatrix}.$$

We write simply $w_{s,t}^{(2)} = I_n$ if $s = t$.

Given $\delta \in \mathcal{S}_{(i_1,\dots,i_l)}$, let $\eta\delta = \mathfrak{a} = (a_1, a_2, \dots, a_n) \in F^n$. Then $\eta^{(j)}(\mathfrak{a}) = \mathbf{0}$ if and only if $j \in \{i_1, \dots, i_l\}$. We define $x = \operatorname{diag}(x_1, x_2, \dots, x_m) \in F[\gamma]^\times$ as follows.

- If $\tilde{\eta}_j(\mathfrak{a}) \neq \mathbf{0}$, let $e_j^0 \le e_j - 1$ be the maximal integer such that

$$(a_{k_1+\cdots+(e_j^0-1)d_j+1}, a_{k_1+\cdots+k_{j-1}+(e_j^0-1)d_j+2}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq \mathbf{0}.$$

By Lemma 4.6, for each such j, one can find an element x_j of the form in (4-8) such that the $e_j^0 d_j$-row of x_j is exactly

$$(a_{k_1+\cdots+k_{j-1}+1}, a_{k_1+\cdots+k_{j-1}+1}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq \mathbf{0}.$$

– If $\tilde{\eta}_j(\mathfrak{a}) = 0$, we take x_j to be an arbitrary element in $F[\mathcal{J}(\wp_j^{e_j})]^\times$. Since $\eta^{(i)}(\mathfrak{a}) \neq \mathbf{0}$ for any $i \notin \{i_1, \dots, i_l\}$, we can pick uniquely elements $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$ such that their last row is $\tilde{\eta}_i(\mathfrak{a})$, $i \in \{1, 2, \dots, m\} \setminus \{i_1, \dots, i_l\}$.

Apply the transposition $\prod_{1 \le j \le l}(j, i_j)$ to the ordered n-set $(1, 2, \dots, n)$ to get another ordered n-set $(i_1, \dots, i_l, i'_{l+1}, \dots, i'_m)$. Then $\eta^{(i'_m)}(\mathfrak{a}) \neq 0$. Hence, the last row of $x_{i'_m}$ is $\eta_{k_{i'_m}}(\mathfrak{a})$. Define

$$\gamma_{(i_1,\dots,i_l)} = \begin{pmatrix} I_{k_{i_1}} & & & & & \\ & \ddots & & & & \\ & & I_{k_{i_l}} & & & \\ & & & I_{k_{i'_{l+1}}} & & \\ & & & & \ddots & \\ \boldsymbol{M}^*_{k_{i'_m}, k_{i_1}} & \cdots & \boldsymbol{M}^*_{k_{i'_m}, k_{i_l}} & \boldsymbol{M}_{k_{i'_m}, k_{i_{l+1}}} & \cdots & I_{k_{i'_m}} \end{pmatrix} \cdot \prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)},$$

where $\boldsymbol{M}^*_{i,k_j}$ refers to the $i \times k_j$ matrix whose first $i-1$ rows are zero vectors and whose bottom row is

$$(0, \cdots 0, v_{d_i+1}, v_{d_i+2}, \dots, v_{k_i-d_i}, v_1, v_2, \dots, v_{d_i}).$$

Here v_l denotes the l-th element of $(0, \dots, 0, 1, 0, \dots, 0) \in F^{k_i}$, where the only nonzero entry occurs in the $e_i^0 d_i$-position.

From the definitions, $\eta\gamma_{(i_1,\dots,i_l)}x = \mathfrak{a} = \eta\delta$. Hence, $\delta \in P_0(F)\gamma_{(i_1,\dots,i_l)}F[\gamma]^\times$.

Recall that $\gamma = \gamma^*$, as defined in (4-9). Given any $x \in F[\gamma]^\times$, one checks directly that

$$\prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)} x \left(\prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)} \right)^{-1} \in Q_{d_{i_1}}(F).$$

Therefore, $\gamma_{(i_1,\dots,i_l)} x \gamma^{-1}_{(i_1,\dots,i_l)} \in Q_{d_{i_1}}(F)$, i.e., $\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times \gamma^{-1}_{(i_1,\dots,i_l)} \subseteq Q_{d_{i_1}}(F)$.

• Now we consider $\mathcal{S}^{(m)} = \{\delta \in G(F) : \eta^{(j)}(\eta\delta) = \mathbf{0},\ 1 \le j \le m\}$. Let $\delta \in \mathcal{S}^{(m)}$. Denote $\eta\delta$ by $\mathfrak{a} = (a_1, a_2, \dots, a_n)$. For each $1 \le j \le m$ such that $\tilde{\eta}_j(\mathfrak{a}) \neq \mathbf{0}$, denote by $e_j^0 \le e_j - 1$ the maximal integer such that

$$(a_{k_1+\cdots+(e_j^0-1)d_j+1}, a_{k_1+\cdots+k_{j-1}+(e_j^0-1)d_j+2}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq 0.$$

Likewise, for each such j, one can find an element x_j of the form in (4-8) such that the $e_j^0 d_j$-row of x_j is exactly

$$(a_{k_1+\cdots+k_{j-1}+1}, a_{k_1+\cdots+k_{j-1}+1}, \dots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}).$$

For the remaining j's, we take arbitrary $x_j \in F[\mathcal{J}(\wp_j^{e_j})]^\times$. Let $x = \operatorname{diag}(x_1, \dots, x_m)$. Pick arbitrarily a j_0 such that $\tilde{\eta}_{j_0}(\mathfrak{a}) \neq 0$. Let $j_0' \neq j_0$ be another integer. Set

$$w_{j_0,e_{j_0}}^{(1)} = \begin{pmatrix} I_{k'_{j_0}} & & & \\ & 0 & I_{(e_j-e_{j_0})d_{j_0}} & \\ & I_{e_{j_0}d_{j_0}} & 0 & \\ & & & I_{k''_{j_0}} \end{pmatrix}.$$

Let

$$\gamma_m = \begin{pmatrix} I_{k_{j_0'}} & & & & \\ & \ddots & & & \\ & & I_{k_1} & & \\ & & & \ddots & \\ M^*_{k_{j_0},k_{j_0'}} & \cdots & M_{k_{j_0},k_{i_1}} & \cdots & I_{k_{j_0}} \end{pmatrix} \cdot w_{j_0'}^{(1)} w_{1,j_0'}^{(2)} w_{j_0,e_{j_0}}^{(1)} w_{j_0,m}^{(2)}.$$

Then $\eta\gamma_m x = \eta\delta$, implying that $\delta \in P_0(F)\gamma_m F[\gamma]^\times$. Moreover, for any $x' \in F[\gamma]^\times$, we have $\gamma_m x' \gamma_m^{-1} \in Q_{d_{j_0'}}(F)$.

It follows from the discussion above that

$$\begin{aligned} G(F) &= \mathcal{S} \sqcup \bigcup_{l=1}^{m-1} \bigcup_{1\le i_1<\cdots<i_l\le m} \mathcal{S}_{(i_1,\dots,i_l)} \sqcup \mathcal{S}^{(m)} \\ &= P_0(F)\gamma_0 F[\gamma]^\times \cup \bigcup_{\substack{1\le l\le m-1 \\ 1\le i_1<\cdots<i_l\le m}} P_0(F)\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times \cup P_0(F)\gamma_m F[\gamma]^\times, \end{aligned}$$

where $\gamma_m F[\gamma]^\times \gamma_m^{-1}$ and each $\gamma_{(i_1,\dots,i_l)} F[\gamma]^\times \gamma_{(i_1,\dots,i_l)}^{-1}$ are contained in some standard maximal parabolic subgroup, and $P_0(F) \cap \gamma_0 F[\gamma]^\times \gamma_0^{-1} = \{I_n\}$. □

Now we prove the result on the structure of conjugacy classes:

Proof of Proposition 4.1. Recall that $\mathcal{C}$ is a regular $G(F)$-conjugacy classes in $G(F)$. Let $\gamma \in \mathcal{C}$ be of the form (4-9). By Lemma 4.7 we have

$$G(F) = \bigcup_{0\le i\le m_0} P_0(F)\gamma_i F[\gamma]^\times,$$

where P_0 is the mirabolic subgroup of G, and each γ_i is constructed explicitly in the proof of Lemma 4.7. Hence, for $\delta \in G(F)$, there exists $p \in P_0(F)$ and $i \in \{0, 1, \dots, m_0\}$ and $x \in F[\gamma]^\times$, such that $\delta = p\gamma_i x$. So one has

$$\delta\gamma\delta^{-1} = p\gamma_i x\gamma x^{-1}\gamma_i^{-1} p^{-1} = p\gamma_i\gamma\gamma_i^{-1} p^{-1}.$$

If $i \ge 1$, from the construction of γ_i, we have $\delta\gamma\delta^{-1} \in \mathcal{C} \cap Q_j(F)^{P(F)}$, for some standard maximal parabolic subgroup Q_j of type $(j, n-j)$, $1 \le j \le n-1$. And for $i = 0$, $\delta\gamma\delta^{-1} = p\gamma_0\gamma\gamma_0^{-1} p^{-1}$, where γ_0 is defined by (4-10). Let $\gamma' = \gamma_0\gamma\gamma_0^{-1}$. If $\gamma' \in \bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}$, we take $\mathcal{C}_0$ to be the empty set.

If $\gamma' \notin \bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}$, we set $\mathcal{C}_0 = \{p\gamma' p^{-1} : p \in P(F)\}$, which is disjoint from $\bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}$. This completes the proof. □

Recall the Bruhat decomposition

$$G(F) = P(F) \cup \bigcup_{j=1}^{n-1} P(F) w_{n-1} w_{n-2} \cdots w_j N_j(F), \tag{4-11}$$

where $N_j(F) := (w_j w_{j+1} \cdots w_n N(F) w_n w_{n-1} \cdots w_j \cap N(F)) \backslash N(F)$.

Proof of Proposition 4.2. Let $g \in \mathcal{C}$ be a representative. Set $m(\lambda)$ (resp. $f(\lambda)$) to be its minimal polynomial (resp. characteristic polynomial) over F. Consider their primary decompositions over F:

$$m(\lambda) = \prod_{i \in I} \wp_i(\lambda)^{e_i'} \quad \text{and} \quad f(\lambda) = \prod_{i \in I} \wp_i(\lambda)^{e_i},$$

where the $\wp_i(\lambda)$'s are distinct irreducible monic polynomials over F and I is a finite index set such that $e_i > 0$ for all $i \in I$. Write $d_i = \deg \wp_i(\lambda)$, $\forall i \in I$. We may assume that $d_1 \leq d_2 \leq \cdots \leq d_{\#I}$. Also, write $d_0 = 0$. Since the conjugacy class $\mathcal{C}$ is irregular, $m(\lambda)$ is a proper factor of $f(\lambda)$. Thus we have the following cases:

<u>Case I:</u> Suppose $\#I = 1$. Then $m(\lambda) = \wp(\lambda)^{e'}$, $f(\lambda) = \wp(\lambda)^{e}$, and $0 < e' < e = d_1^{-1} n$. Let C be the companion matrix of $m(\lambda)$. Then by Lemma 4.3, g is $G(F)$-conjugate to some element $\tilde{g} = \operatorname{diag}(g_1, \ldots, g_m)$ with

$$g_j = \mathcal{J}_j = \begin{pmatrix} C & & & \\ \boldsymbol{N} & C & & \\ & \ddots & \ddots & \\ & & \boldsymbol{N} & C \end{pmatrix}$$

being the quasirational canonical form, and $m > 1$. Let $r_j := \operatorname{rank} g_j$, $1 \leq j \leq m$. We may assume $r_1 \leq r_2 \leq \cdots \leq r_m$.

If $h \in P(F)$, then $h\tilde{g}h^{-1} \in hQ_{r_1}(F)h^{-1}$; if $h \in G(F) - P(F)$, it can be written as $h = pw_{n-1} \cdots w_k u_k$, where $p \in P(F)$ and u_k is of the form

$$\begin{pmatrix} I_{k-1} & & \\ & 1 & * \\ & & I_{n-k} \end{pmatrix} \in Q_k(F).$$

Note that if $k > r_1$, then $w_{n-1} \cdots w_k u_k \in \operatorname{diag}(\mathrm{GL}_{r_1}, \mathrm{GL}_{n-r_1})$, which implies that $h\tilde{g}h^{-1} \in Q_{r_1}(F)^{P(F)}$. Hence, we may assume $k \leq r_1$.

Since $r_1 \leq r_m$, there exists a Weyl element $w \in \mathrm{GL}(r_m)$ such that

$$w g_m w^{-1} = \begin{pmatrix} g_1 & B' \\ & A' \end{pmatrix} \in \mathrm{GL}(r_m, F), \tag{4-12}$$

for some matrices A' and B'. Let $w' = \operatorname{diag}(w, I_{n-r_m})$. Denote by $w'' = w_{r_1-1} w_{r_1-2} \cdots w_k$ if $k < r_1$, and set $w'' = I_n$ if $k = r_1$. Let $g_1'' = w'' g_1 w''^{-1}$. Then there exists a Weyl element $w_0 \in P_0(F)$ such that

$$w_0w_{n-1}w_{n-2}\cdots w_k\tilde{g}w_k\cdots w_{n-2}w_{n-1}w_0^{-1}=\begin{pmatrix}g_m&&&&\\&g_2&&&\\&&\ddots&&\\&&&g_{m-1}&\\&&&&g_1''\end{pmatrix}. \tag{4-13}$$

Let $\tilde{w}=w''w'w_0w_{n-1}w_{n-2}\cdots w_k$. By (4-12) and (4-13) one has

$$\tilde{w}\tilde{g}\tilde{w}^{-1}=\begin{pmatrix}g_1''&B''&\\&A''&\\&&g_1''\end{pmatrix} \tag{4-14}$$

for some matrices A'' and B''. Let ${}^tN_P$ be the transpose of N_P, the unipotent radical of P. By definition,

$$w_{n-1}\cdots w_ku_kw_k\cdots w_{n-1}\in {}^tN_P(F).$$

Note also that $w''w'w_0\in P_0(F)$. So $w''w'w_0$ lies inside the Levi component of $P_0(F)$. Hence, $\tilde{w}u_k\tilde{w}^{-1}\in {}^tN_P(F)$. Write $\tilde{w}u_k\tilde{w}^{-1}=u_k''u_k'$, where

$$u_k''=\begin{pmatrix}I_{r_1}&&\\&I_{n-2r_1}&\\&M_2&I_{r_1}\end{pmatrix}\in^t N_P(F),\quad u_k'=\begin{pmatrix}I_{r_1}&&\\&I_{n-2r_1}&\\M_1&&M_3\end{pmatrix}\in {}^tN_P(F),$$

with M_1 and M_3 being $r_1\times r_1$ matrices and M_2 being an $r_1\times(n-2r_1)$ matrix. In addition, the first r_1-1 rows of M_1 and M_2 are all zeros.

Since g_1'' is regular, by Lemma 4.6 there exist $r_1\times r_1$ matrices $\gamma_1\in F[g_1'']$ and $\gamma_3\in F[g_1'']^\times$ such that the last row of γ_1 (resp. γ_3) coincides with the last row of M_1 (resp. M_3). Write $\gamma_3=\left(\begin{smallmatrix}\mathfrak{a}&\mathfrak{b}\\\mathfrak{c}&1\end{smallmatrix}\right)$ and set

$$p_1=\begin{pmatrix}I_{n-r_1}&&\\&(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}&-(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{b}\\&&1\end{pmatrix}\in P_0(F)\cap Q_{r_1}(F).$$

Write $\gamma_1=\left(\begin{smallmatrix}\mathfrak{a}'\\\mathfrak{c}'\end{smallmatrix}\right)$, where $\mathfrak{a}'$ is an $(r_1-1)\times r_1$ matrix. Set

$$p_2=\begin{pmatrix}I_{r_1}&&&\\&I_{n-2r_1}&&\\-(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{a}'+(\mathfrak{a}-\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{b}\mathfrak{c}'&&I_{r_1-1}&\\&&&1\end{pmatrix}\in P_0(F).$$

Then $p_2^{-1}u_k''p_2=u_k''$. Let $p'=p_2p_1$. A straightforward calculation yields

$$u_k'=p'\begin{pmatrix}I_{r_1}&&\\0&I_{n-2r_1}&\\\gamma_1&0&I_{r_1}\end{pmatrix}\begin{pmatrix}I_{r_1}&&\\0&I_{n-2r_1}&\\&0&\gamma_3\end{pmatrix}. \tag{4-15}$$

Since $\gamma_1\in F[g_1'']$ and $\gamma_3\in F[g_1'']^\times$, then $\gamma_3^{-1}g_1''\gamma_3=g_1''$, and

$$\begin{pmatrix}I_{r_1}&&\\0&I_{n-2r_1}&\\\gamma_1&0&I_{r_1}\end{pmatrix}\begin{pmatrix}g_1''&B''&\\&A''&\\&&g_1''\end{pmatrix}\begin{pmatrix}I_{r_1}&&\\0&I_{n-2r_1}&\\\gamma_1&0&I_{r_1}\end{pmatrix}^{-1}\in Q_{r_1}(F). \tag{4-16}$$

By (4-14), (4-15) and (4-16), and since $p_1\in Q_{r_1}(F)$, we have $p_2^{-1}u_k'\tilde{w}\tilde{g}\tilde{w}^{-1}u_k'p_2\in Q_{r_1}(F)$.

Since $p_2u_k''=u_k''p_2$ and $u_k''\in Q_{r_1}(F)$,

$$\tilde{w}u_k\tilde{g}u_k^{-1}\tilde{w}^{-1}=u_k''u_k'\tilde{w}\tilde{g}\tilde{w}^{-1}u_k'^{-1}u_k''^{-1}\in p_2Q_{r_1}(F)p_2^{-1}\subset Q_{r_1}(F)^{P(F)}. \tag{4-17}$$

Recall that $h=pw_{n-1}\cdots w_ku_k$, and $p'':=w''w'w_0\in P_0(F)$. Then it follows form (4-17) that

$$h\tilde{g}h^{-1}=pp''^{-1}\tilde{w}u_k\tilde{g}u_k^{-1}\tilde{w}^{-1}p''p^{-1}\in Q_{r_1}(F)^{P(F)}.$$

Case II: Here g is $G(F)$-conjugate to some matrix $\tilde{g}=\operatorname{diag}(\tilde{g}_1,\dots,\tilde{g}_m)$, where each $\tilde{g}_i$ is of the form $\operatorname{diag}(g_{i,1},\dots,g_{i,m_i})$, with

$$g_{i,j}=\begin{pmatrix}C_{i,j}&&&\\ N&C_{i,j}&&\\ &\ddots&\ddots&\\ &&N&C_{i,j}\end{pmatrix}$$

and $C_{i,j}$ regular elliptic; and $\tilde{g}_i$ has characteristic polynomial $\wp_i(\lambda)^{e_i}$. Since g is irregular, so is $\tilde{g}$. Hence there must be some $1\le i\le m$ such that $\tilde{g}_i$ is irregular. We may assume $\tilde{g}_1$ is irregular and $\operatorname{rank} g_{1,1}\le\operatorname{rank} g_{1,2}\le\cdots\le\operatorname{rank} g_{1,m_1}$. Then a similar argument as in Case I shows that $h\tilde{g}h^{-1}\in Q_{r_1}(F)^{P(F)}$, where $r_1=\operatorname{rank} g_{1,1}$.

Proposition 4.2 then follows. □

4C. ***Algebraic expansion: P-regular conjugacy classes.*** As in Section 4A, let Q_k be the standard parabolic subgroup of G of type $(k,n-k)$.

- In Proposition 4.1 we show that for any regular $G(F)$-conjugacy classes $\mathcal{C}$ in $G(F)$, there exists a $P(F)$-conjugacy class $\mathcal{C}_0$, uniquely determined by $\mathcal{C}$, such that

$$\mathcal{C}=\mathcal{C}_0\cup\bigcup_{k=1}^{n-1}\mathcal{C}\cap Q_k(F)^{P(F)}.$$

- When $\mathcal{C}$ is a nonregular $G(F)$-conjugacy class, by Proposition 4.2, we have $\mathcal{C}=\bigcup_{k=1}^{n-1}\mathcal{C}\cap Q_k(F)^{P(F)}$. Take $\mathcal{C}_0$ to be the empty set in this case.

Consider the notations introduced in Section 2. Let $\mathfrak{S}=\bigcup_{k=1}^{n-1}(Z(F)\backslash Q_k(F))^{P_0(F)}$. Following the approach in [18], we handle the integral $I_{\mathrm{Geo,Reg}}(s,\tau)$ of (4-1) through the decomposition

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x,y)=\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x)=\sum_{\mathcal{C}}\mathrm{K}_{\mathcal{C}_0}(x,y), \tag{4-18}$$

where $\mathcal{C}$ encompasses all conjugacy classes in $G(F)/Z(F)$ and

$$\mathrm{K}_{\mathcal{C}_0}(x,y)=\sum_{\gamma\in\mathcal{C}_0}\varphi(x^{-1}\gamma y)=\sum_{\gamma\in\mathcal{C}-\mathfrak{S}}\varphi(x^{-1}\gamma y).$$

It is important to note that while $\mathrm{K}_{\mathcal{C}_0}(x,x)$ is not $G(F)$-invariant, it is $P(F)$-invariant. Hence, it is meaningful to integrate them over $Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ with respect to $f(x,s)$. For each $\mathcal{C}$, let (at least formally)

$$I_{\mathcal{C}}(s,\tau):=\int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_{\mathcal{C}_0}(x,x)f(x,s)\,dx.$$

By definition, $I_{\mathcal{C}}(s,\tau)=0$ unless $\mathcal{C}$ is regular. Inserting (4-18) into (4-1) yields

$$I_{\mathrm{Geo,Reg}}(s,\tau)=\sum_{\mathcal{C}\text{ regular}}I_{\mathcal{C}}(s,\tau). \tag{4-2}$$

Let $\mathcal{C}$ be a conjugacy class in $G(F)$. Denote by $f(\lambda;\mathcal{C})$ the characteristic polynomial of $\mathcal{C}$. Factor it into irreducible ones with multiplicities as

$$f(\lambda;\mathcal{C})=\prod_{i=1}^{g}\wp_i(\lambda;\mathcal{C})^{e_i}, \tag{4-19}$$

where $\wp_i(\lambda;\mathcal{C})\in F[\lambda]$ is an irreducible polynomial of degree f_i. We may assume $f_1\geq\cdots\geq f_g$. Write $\boldsymbol{f}=(f_1,\dots,f_g)\in\mathbb{Z}_{\geq 1}^g$ and $\boldsymbol{e}=(e_1,\dots,e_g)\in\mathbb{Z}_{\geq 1}^g$. Then $\langle\boldsymbol{f},\boldsymbol{e}\rangle:=\sum f_ie_i=n$.

Definition 4.8. Let notation be as before. We say $\mathcal{C}$ is of type $(\boldsymbol{f},\boldsymbol{e};g)$. Let $\Gamma_{\boldsymbol{f},\boldsymbol{e};g}$ be the collection of regular $G(F)$-conjugacy classes of type $(\boldsymbol{f},\boldsymbol{e};g)$.

With the above definition, we have the decomposition

$$\bigsqcup_{\mathcal{C}\text{ regular}}\mathcal{C}=\bigsqcup_{\substack{\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}_{\geq 1}^g\\ \langle\boldsymbol{f},\boldsymbol{e}\rangle=n}}\Gamma_{\boldsymbol{f},\boldsymbol{e};g}. \tag{4-20}$$

Let $\mathcal{C}\in\Gamma_{\boldsymbol{f},\boldsymbol{e};g}$. Let $\gamma_{\mathcal{C}}\in\mathcal{C}$ be a fixed element. Let $\lambda_{\boldsymbol{f},\boldsymbol{e};g}$ be the inverse of the representative defined by (4-10), with a slight adjustment in notation: $f_i=d_i$, $k_i=f_ie_i$, and $g=m$. Specifically,

$$\lambda_{\boldsymbol{f},\boldsymbol{e};g}=\begin{pmatrix}I_{f_1}&&&&&&&\\&\ddots&&&&&&\\&&I_{f_1}&&&&&\\&&&\ddots&&&&\\&&&&I_{f_g}&&&\\&&&&&\ddots&&\\\boldsymbol{M}_{f_g,f_1}&\cdots&\boldsymbol{M}_{f_g,f_1}&\cdots&\boldsymbol{M}_{f_g,f_g}&\cdots&I_{f_g}\end{pmatrix}^{-1}\in G(F), \tag{4-21}$$

where, for an integer m, $\boldsymbol{M}_{f_g,m}$ is the $f_g\times m$ matrix in which the first f_g-1 rows are zero vectors, and the bottom row is $\eta_m=(0,\dots,0,1)\in F^m$. In particular, $\lambda_{\boldsymbol{f},\boldsymbol{e};g}=I_n$ if $g=1$ and $e_1=1$, i.e., $\mathcal{C}$ is regular elliptic.

By Proposition 4.1, we have, when $\mathrm{Re}(s)>1$,

$$\begin{aligned}I_{\mathcal{C}}(s,\tau)&=\int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)}\sum_{p\in P_0(F)}\varphi(x^{-1}p^{-1}\lambda_{\boldsymbol{f},\boldsymbol{e};g}^{-1}\gamma_{\mathcal{C}}\lambda_{\boldsymbol{f},\boldsymbol{e};g}px)f(x,s)\,dx\\&=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_{\mathcal{C}}x)f(\lambda_{\boldsymbol{f},\boldsymbol{e};g}^{-1}x,s)\,dx,\end{aligned}$$

supposing the integrals converge absolutely. Combining this with (4-20) we then deduce (at least formally) that, when $\mathrm{Re}(s) > 1$,

$$\sum_{\mathcal{C}} I_{\mathcal{C}}(s,\tau) = \sum_{\substack{\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}^g_{\geq 1}\\ \langle \boldsymbol{f},\boldsymbol{e}\rangle = n}} \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\mathcal{C}\in\Gamma_{\boldsymbol{f},\boldsymbol{e};g}} \varphi(x^{-1}\gamma_{\mathcal{C}}x) f(\lambda^{-1}_{\boldsymbol{f},\boldsymbol{e};g}x,s)\,dx. \tag{4-22}$$

Moreover, (4-22) would be rigorous if the right-hand side converges absolutely, which is indeed the case. To verify, we will consider each type $(\boldsymbol{f},\boldsymbol{e};g)$ separately in the following subsections.

4C1. *Type* $(n;1)$. We treat the conjugacy classes of type $(\boldsymbol{f},\boldsymbol{e};g) = ((n),(1);1)$ first, i.e., $e = g = 1$, which are exactly regular elliptic conjugacy classes. Set

$$I_{\text{r.e.}}(s,\tau) = I^{\varphi}_{\text{r.e.}}(s,\tau) = \sum_{\mathcal{C}\text{ regular elliptic}} I_{\mathcal{C}}(s,\tau).$$

Proposition 4.9. *Let notation be as before. For every field extension E/F of degree n, there is an analytic function $Q_E(s)$ such that*

$$I_{\text{r.e.}}(s,\tau) = \frac{1}{n}\sum_{[E:F]=n} Q_E(s)\Lambda(s,\tau\circ N_{E/F}), \tag{4-23}$$

where the summation is taken over only finitely many E's, depending only on the test function φ.

Proof. Let $\Gamma_{\text{r.e.}}$ be the set of regular elliptic elements in $G(F)/Z(F)$. Then

$$I_{\text{r.e.}}(s,\tau) = \int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in\Gamma_{\text{r.e.}}} \varphi(x^{-1}\gamma x) f(x,s)\,dx.$$

Denote by $\{\text{r.e.}\}$ a set of representatives for the regular elliptic conjugacy classes in $\Gamma_{\text{r.e.}}$. For any $\gamma\in\{\text{r.e.}\}$, the centralizer of γ in $G(F)/Z(F)$ is $F[\gamma]^\times$. So

$$\sum_{\gamma\in\Gamma_{\text{r.e.}}} \varphi(x^{-1}\gamma x) = \sum_{\gamma\in\{\text{r.e.}\}} \sum_{\delta\in F[\gamma]^\times Z(F)\backslash G(F)} \varphi(x^{-1}\delta^{-1}\gamma\delta x). \tag{4-24}$$

By Lemma 4.4 and the remark after it, one has $G(F) = P(F)F[\gamma]^\times$. Since $P(F)\cap F[\gamma]^\times = Z(F)$, every element $\delta\in Z(F)\backslash G(F)$ can be written unique as $\delta = p\nu$, where $p\in Z(F)\backslash P(F)$ and $\nu\in F[\gamma]^\times$. Hence the inner sum of (4-24) could be taken over $p\in Z(F)\backslash P(F)$. Therefore,

$$I_{\text{r.e.}}(s,\tau) = \int_{P(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in\{\text{r.e.}\}} \sum_{p\in Z(F)\backslash P(F)} \varphi(x^{-1}p^{-1}\gamma p x) f(x,s)\,dx. \tag{4-25}$$

Consider the field extension E/F of degree n. Fix an algebraic closure $\overline{F}$ of F, then E embeds into $\overline{F}$. A conjugacy class is said to belong to E if it consists of the conjugates of an element $\gamma\in E^\times/F^\times - 1$ under the usual identification. We need to consider two cases, according to whether E/F is Galois.

The idea is to replace the summation over $\gamma\in\{\text{r.e.}\}$ by summation over extensions E/F of degree n; and inside, summation over elements of E.

<u>Case E/F Galois:</u> When γ varies over $E^\times/F^\times$ each conjugacy class belongs to E exactly n times.

Case E/F not Galois: When γ varies over $E^\times/F^\times$ each conjugacy class belongs to E once; but the sets of conjugacy classes belonging to the n embeddings of E in $\bar{F}$ are identical.

In either case, the integral in (4-25) can be rewritten as

$$I_{\text{r.e.}}(s,\tau)=\frac{1}{n}\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{[E:F]=n}\sum_{\gamma\in E^\times/F^\times-\{1\}}\varphi(x^{-1}\gamma x)f(x,s)\,dx, \tag{4-26}$$

where the summation on the right is taken over all extensions E/F of degree n. However, only finitely many E's (independent of x) in the inner sum contribute to (4-26). To verify this assertion, we generalize the argument in [18, p. 17]. Consider the function $\beta : Z(\mathbb{A}_F)\setminus G(\mathbb{A}_F)\to\mathbb{A}_F^{n-1}$ defined by

$$\beta(x)=\left(\frac{\sigma_1(x)^n}{\sigma_n(x)},\frac{\sigma_2(x)^n}{\sigma_n(x)^2},\dots,\frac{\sigma_{n-1}(x)^n}{\sigma_n(x)^{n-1}}\right),$$

where $\sigma_i(x)$ is the i-th symmetric polynomial in the eigenvalues of x. Since $\sigma_n(x)=\det x\neq 0$, $\forall x\in G(\mathbb{A}_F)$, β is continuous. Hence it maps $\operatorname{supp}\varphi$ to a compact set in $G(\mathbb{A}_F)$. On the other hand, β is invariant under conjugation. Consequently, the set $\{\beta(\gamma):\gamma\in G(F)/Z(F),\ \varphi(x^{-1}\gamma x)\neq 0 \text{ for some } x\in G(\mathbb{A}_F)\}$ is the intersection of a compact set with a discrete set, hence is finite.

- If $\beta(\gamma)\neq 0$, the number of distinct fields $F[\gamma]$ with a given value of $\beta(\gamma)$ is at most n, thus finite.
- If $\beta(\gamma)=0$, the map $\gamma\mapsto\det\gamma$ from

$$G(\mathbb{A}_F)/Z(\mathbb{A}_F)\to\mathbb{A}_F^\times/\mathbb{A}_F^{\times^n}U_F$$

(with $\mathbb{A}_F^{\times^n}=\{a^n : a\in\mathbb{A}_F^\times\}$ and U_F the maximal compact subgroup of $\mathbb{A}_F^\times$) is continuous; so the image of $\operatorname{supp}\varphi$ is also compact. Since $\mathbb{A}_F^{\times^n}$ has finite index in $\mathbb{A}_F^{\times^n}U_F\cap F$, there are only finitely many values for the image $\det\gamma \mod \mathbb{A}_F^{\times^n}$ with $\gamma\in G(F)/Z(F)$ and $\varphi(x^{-1}\gamma x)\neq 0$ for some $x\in G(\mathbb{A}_F)$. When $\beta(\gamma)=0$, $F[\gamma]$ is determined (up to embedding) by $\det\gamma$, so we are done.

Thus we can interchange integrals in (4-26) to get

$$I_{\text{r.e.}}(s,\tau)=\frac{1}{n}\sum_{[E:F]=n}I_E(s,\tau), \tag{4-27}$$

where $I_E(s,\tau)$ is defined by

$$\int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\substack{\gamma\in E^\times/F^\times\\ \gamma\neq 1}}\varphi(x^{-1}\gamma x)\int_{G_\gamma(\mathbb{A}_F)}\Phi(\eta t x)\tau(\det tx)\,|\det tx|^s\,d^\times t\,dx. \tag{4-28}$$

Here $\eta=(0,\dots,0,1)\in\mathbb{A}_F^n$, and G_γ is the centralizer of γ in G. Since γ is regular elliptic, we have $G_\gamma(\mathbb{A}_F)\simeq\mathbb{A}_E^\times$.

As noted in [18, p. 17], the function

$$x\mapsto\sum_{\lambda\in E^\times/F^\times-\{1\}}\varphi(x^{-1}\lambda x)$$

has compact support on $G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. Note that for almost all finite places v, $x_v\in G(\mathcal{O}_v)$ and Φ_v

is the characteristic function of $\mathcal{O}_v^n$. Then as a function of ηt_v, $\Phi_v(\eta t_v x_v)$ is also the characteristic function of $\mathcal{O}_v^n$. So $\Phi(\eta t x)$ is a Schwartz–Bruhat function for fixed $x \in G(\mathbb{A}_F)$. If we identify $G_\gamma(\mathbb{A}_F)$ with $\mathbb{A}_E^\times$, $\det|_{E^\times}\colon E^\times \to F^\times$ with the norm map $N_{E/F}$, $t \mapsto |\det t|_{\mathbb{A}_F}$ with the idele norm in E, and $\mathcal{S}(\mathbb{A}_F^n)$ with $\mathcal{S}(\mathbb{A}_E)$, we see that the inner integral in (4-28) is just the Tate integral for $\Lambda(s, \tau \circ N_{E/F})$, converging absolutely for $\mathrm{Re}(s) > 1$. So there is some holomorphic function $Q_E(s)$ of s such that $I_E(s,\tau) = Q_E(s)\Lambda(s, \tau \circ N_{E/F})$, giving a meromorphic continuation of $I_E(s,\tau)$ to the entire s-plane. Since the sum over E in (4-27) is finite, $I_{\mathrm{r.e.}}(s,\tau)$ is well-defined for $\mathrm{Re}(s) > 1$, and admits a meromorphic continuation to $s \in \mathbb{C}$ of the form (4-23). □

4C2. ***Type*** **$(\boldsymbol{f}, \boldsymbol{e}; 1)$.** One special aspect in the regular elliptic case is the compact support modulo $G_\gamma(\mathbb{A}_F)$ of the function

$$x \mapsto \sum_{\lambda \in E^\times/F^\times - \{1\}} \varphi(x^{-1}\lambda x). \tag{4-29}$$

However, for general types $(\boldsymbol{f}, \boldsymbol{e}; g)$, this function may not have compact support. A notable example is the absence of compact support for regular unipotent conjugacy classes; see [18, p. 18], for example. Therefore, a different approach is required compared to the regular elliptic case discussed in Section 4C1.

In this section, we focus on the case $g = 1$. The results obtained here, combined with the proof of Proposition 4.9, will play a pivotal role in addressing general types in the subsequent section.

Let $\mathcal{C}$ be of type $(\boldsymbol{f}, \boldsymbol{e}; 1)$. We may choose a representative $\gamma \in \mathcal{C}_0$ as in Proposition 4.1, in its quasirational canonical form:

$$\gamma = \begin{pmatrix} C & & & & \\ \boldsymbol{N} & C & & & \\ & \boldsymbol{N} & C & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{N} & C \end{pmatrix},$$

where

$$\boldsymbol{N} = \begin{pmatrix} & & 1 \\ & 0 & \\ \iddots & & \\ 0 & & \end{pmatrix},$$

$C \in \mathrm{GL}(f_1)$ and there are $e_1 = n/f_1$ such C's in the partitioned matrix above. The stabilizer of γ is studied in Lemma 4.6. In particular, if A is the stabilizer of γ, then A must be of the form

$$A = \begin{pmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{e_1-1} & \dots & A_2 & A_1 & A_0 \end{pmatrix}, \tag{4-30}$$

with $A_i \in M_{f_1 \times f_1}(F)$, $1 \le i < e_1$, and $A_0 \in F(C)$. Let $P_{\boldsymbol{f},\boldsymbol{e};1}$ be the transpose of standard parabolic subgroup of G of type $(f_1, f_1, \dots, f_1)$. Let $K_{\boldsymbol{f},\boldsymbol{e};1}$ be a compact subgroup such that $G(\mathbb{A}_F) = P_{\boldsymbol{f},\boldsymbol{e};1}(\mathbb{A}_F) K_{\boldsymbol{f},\boldsymbol{e};1}$.

We can decompose $G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ as follows. Write $x \in G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ as

$$\boldsymbol{B}\begin{pmatrix} I_{f_1} &&&& \\ & D_1 &&& \\ && D_2 D_1 && \\ &&& \ddots & \\ &&&& D_{e_1-1}\cdots D_1 \end{pmatrix}\begin{pmatrix} T_0 &&&& \\ & T_1 &&& \\ && T_2 && \\ &&& \ddots & \\ &&&& T_{e_1-1} \end{pmatrix} k, \tag{4-31}$$

where

$$\boldsymbol{B} = \begin{pmatrix} I_{f_1} &&&& \\ & I_{f_1} &&& \\ & B_{3,2} & I_{f_1} && \\ & \vdots & \ddots & \ddots & \\ & B_{e_1,2} & \dots & B_{e_1,e_1-1} & I_{f_1} \end{pmatrix}; \tag{4-32}$$

each D_j, $1 \le j \le e_1 - 1$, is in the stabilizer $G_C(\mathbb{A}_F)$ of C; each T_j, $0 \le j < e_1$, is in $G_C(\mathbb{A}_F)\backslash \mathrm{GL}(f_1, \mathbb{A}_F)$; and $k \in K_{\boldsymbol{f},\boldsymbol{e};1}$. Write $\boldsymbol{D} = \mathrm{diag}(I_{f_1}, D_1, \dots, D_{e_1-1}\cdots D_1)$ and $\boldsymbol{T} = \mathrm{diag}(T_0, T_1, \dots, T_{e_1-1})$. Note that (4-31) follows from Iwasawa decomposition and the unipotent term $\boldsymbol{B}$ is of the form (4-32) because its first f_1 columns can be absorbed by left multiplication of some stabilizer $A \in G_\gamma(\mathbb{A}_F)$ of shape (4-30). Write the inverse $\boldsymbol{B}^{-1}$ in the matrix form

$$\begin{pmatrix} I_{f_1} &&&& \\ & I_{f_1} &&& \\ & B_{3,2} & I_{f_1} && \\ & \vdots & \ddots & \ddots & \\ & B_{e_1,2} & \dots & B_{e_1,e_1-1} & I_{f_1} \end{pmatrix}^{-1} = \begin{pmatrix} I_{f_1} &&&& \\ & I_{f_1} &&& \\ & B'_{3,2} & I_{f_1} && \\ & \vdots & \ddots & \ddots & \\ & B'_{e_1,2} & \dots & B'_{e_1,e_1-1} & I_{f_1} \end{pmatrix}.$$

For each $B'_{i,j}$, we write $\widetilde{B}'_{i,j} = B'_{i,j}C - CB'_{i,j}$. Let $\mathcal{B}$ be the group of such $\boldsymbol{B}$'s.

By definition, the contribution from conjugacy classes of type $(\boldsymbol{f}, \boldsymbol{e}; 1)$ is

$$I_{\boldsymbol{f},\boldsymbol{e};1}(s) = \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{C \in \Gamma_{\boldsymbol{f},\boldsymbol{e};1}} \varphi(x^{-1}\gamma_C x) f(\lambda^{-1}_{\boldsymbol{f},\boldsymbol{e};1} x, s)\, dx. \tag{4-33}$$

Remark 4.10. In the forthcoming calculation of $I_{\boldsymbol{f},\boldsymbol{e};1}(s)$, we will use the unconventional notation $\boldsymbol{N}$, $\mathcal{B}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{T}$ to simplify the description of integrands involving complex matrices of specific types.

Recall that for two meromorphic functions $h_1(s)$ and $h_2(s)$, we write $h_1(s) \propto h_2(s)$ if $h_1(s)/h_2(s)$ admits an analytic continuation to the whole complex plane.

Proposition 4.11. *Let notation be as before. Then $I_{\boldsymbol{f},\boldsymbol{e};1}(s)$ converges absolutely when* $\mathrm{Re}(s) > 1$ *and*

$$I_{\boldsymbol{f},\boldsymbol{e};1}(s) \propto \prod_{j=1}^{e}\Bigg(\sum_{[E_j:F]=f} Q_{E_j}(s)\Lambda_{E_j}\big(js - j + 1, (\tau \circ N_{E/F})^j\big)\Bigg),$$

where the sum over number fields E_j is finite and Q_{E_j} is an entire function of s.

Proof. Rewrite (4-33) in the original form

$$I_{\boldsymbol{f},\boldsymbol{e};1}(s)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\gamma}\varphi(x^{-1}\gamma x)f(x,s)\,dx,$$

where γ runs over regular elements of type $(\boldsymbol{f},\boldsymbol{e};1)$. Then similar to the discussion in Proposition 4.9, the sum over γ is finite, depending only on the support of φ. We then switch the sum to get $I_{\boldsymbol{f},\boldsymbol{e};1}(s)=\sum_{\gamma}I_{\gamma}(s,\tau)$, where

$$I_{\gamma}(s,\tau):=\int_{G_{\gamma}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma x)\int_{G_{\gamma}(\mathbb{A}_F)}\Phi[(0,\dots,0,1)tx]\tau(\det tx)\,|\det tx|^{s}\,dt\,dx.$$

Substituting the decomposition (4-31) into the above integral, we obtain

$$I_{\gamma}(s,\tau)=\int_{k}\int_{(G_C(\mathbb{A}_F))^{e_1-1}}\iint_{\mathcal{B}}\varphi(k^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}\boldsymbol{B}^{-1}\gamma\boldsymbol{B}\boldsymbol{D}\boldsymbol{T}k)\int_{G_{\gamma}(\mathbb{A}_F)}\cdots,$$

where $\boldsymbol{T}$ (resp. k) ranges through $(G_C(\mathbb{A}_F)\backslash \mathrm{GL}(f_1,\mathbb{A}_F))^{e_1}$ (resp. $K_{\boldsymbol{f},\boldsymbol{e};1}$), and $\mathcal{B}$ is the domain for $\boldsymbol{B}$.

According to the preceding discussion we have

$$I_{\gamma}(s,\tau)=\iint_{(G_C(\mathbb{A}_F))^{e_1-1}}\iint_{\mathcal{B}}\varphi(k^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}M\boldsymbol{D}\boldsymbol{T}k)\int_{G_{\gamma}(\mathbb{A}_F)}\cdots,$$

where $M=\boldsymbol{B}^{-1}\gamma\boldsymbol{B}$ is of the form

$$\begin{pmatrix} C &&&&\\ \boldsymbol{N} & C &&&\\ B'_{3,2}\boldsymbol{N} & \widetilde{B}'_{3,2}+\boldsymbol{N} & C &&\\ \vdots & \vdots & \ddots & \ddots &\\ B'_{e_1,2}\boldsymbol{N} & \dots & \dots & \widetilde{B}'_{e_1,e_1-1}+\boldsymbol{N} & C\end{pmatrix},$$

where a typical entry of the above matrix is of the form

$$\widetilde{B}'_{l,k}+B'_{l,k+1}\boldsymbol{N}+\widetilde{B}'_{l,k+1}\mu_{l,k}(B_{i,j},\boldsymbol{N},C)+\lambda_{l,k}(B_{i,j},\boldsymbol{N},C)$$

with $\mu_{l,k}$ and $\lambda_{l,k}$ being polynomials in $B_{i,j}$, $\boldsymbol{N}$, and C, $i\le l-1$ and $j\ge k+2$.

A straightforward computation shows that if the first column of $B'_{i,j}$ is determined, then $\widetilde{B}'_{i,j}=B'_{i,j}C-CB'_{i,j}$ is completely determined by its last f_1-1 columns. Therefore, when $B'_{l,k}\boldsymbol{N}$ is fixed, then $\widetilde{B}'_{l,k}+B'_{l,k+1}\boldsymbol{N}$ is an $f_1\times f_1$ matrix, linearly determined by the last f_1-1 columns of $B'_{l,k}$ and the first column of $B'_{l,k+1}$, and this correspondence is one to one. Thus, the mapping $\boldsymbol{B}\mapsto M$ establishes a one-to-one smooth correspondence. The matrices M that contribute are confined to a compact region determined by the choice of φ. Consequently, after a change of variables, it follows that the integral with respect to $\boldsymbol{B}$ has compact support in $\mathcal{B}$. Therefore,

$$I_{\gamma}(s,\tau)=\int_{k}\int_{(G_C(\mathbb{A}_F))^{e_1-1}}\int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}}\int_{\mathcal{B}}\varphi(k^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}M\boldsymbol{D}\boldsymbol{T}k)\delta^{-1}_{P_{\boldsymbol{f},\boldsymbol{e};1}}(\boldsymbol{T})$$
$$\times\int_{G_{\gamma}(\mathbb{A}_F)}\Phi[(0,\dots,0,1)A\boldsymbol{B}\boldsymbol{D}\boldsymbol{T}k]\tau(\det A\boldsymbol{D}\boldsymbol{T})\,|\det A\boldsymbol{D}\boldsymbol{T}|^{s}dA\,d\boldsymbol{B}\,d\boldsymbol{D}\,d\boldsymbol{T}\,dk,$$

where $\delta_{P_{f,e;1}}(\boldsymbol{T})$ is the modular character associated with $P_{f,e;1}(\mathbb{A}_F)$. Changing variables via $\boldsymbol{B} \mapsto \boldsymbol{D}\boldsymbol{T}\boldsymbol{B}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}$ we then obtain

$$I_\gamma(s,\tau) = \int_k \int_{(G_C(\mathbb{A}_F))^{e_1-1}} \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}} \int_{\mathcal{B}} \varphi(k^{-1}\boldsymbol{B}^{-1}\boldsymbol{T}^{-1}\boldsymbol{D}^{-1}\gamma\boldsymbol{D}\boldsymbol{T}\boldsymbol{B}k) \\ \times \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\ldots,0,1)A\boldsymbol{D}\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{D}T)\,|\det A\boldsymbol{D}\boldsymbol{T}|^s\,dA\,d\boldsymbol{B}\,d\boldsymbol{D}\,d\boldsymbol{T}\,dk.$$

Then $\boldsymbol{D}^{-1}\gamma\boldsymbol{D}$ is equal to

$$\begin{pmatrix} C &&&& \\ D_1^{-1}\boldsymbol{N} & C &&& \\ 0 & D_1^{-1}D_2^{-1}\boldsymbol{N}D_1 & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \ldots & (D_1^{-1}\cdots D_{e_1-2}^{-1}D_{e_1-1}^{-1}\boldsymbol{N}D_{e_1-2}\cdots D_1) & C \end{pmatrix}.$$

We can identify $D_j^{-1}\boldsymbol{N}$ with an element x_j in $E_j^\times/F^\times - 1$, $1 \le j \le e_1 - 1$, where E_j is a field extension of F with $[E_j : F] = f_1$. Conjugate of $D_j^{-1}\boldsymbol{N}$ under this identification becomes a Galois action on x_j. Therefore, we have

$$I_\gamma(s,\tau) = \frac{1}{f_1^{e_1-1}} \prod_{j=1}^{e_1-1} \sum_{\substack{x_j \in E_j^\times/F^\times - 1 \\ [E_j:F]=f_1}} \int_{\mathbb{A}_{E_j}^\times} \prod_{j=1}^{e_1-1} \tau^j(N_{E_j/F}(x_j))N_{E_j/F}(x_j)^{js-j+1} \\ \times \int_k \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}} \int_{\mathcal{B}} \varphi(k^{-1}\boldsymbol{B}^{-1}\boldsymbol{T}^{-1}\gamma^{\boldsymbol{D}}\boldsymbol{T}\boldsymbol{B}k)\,|\det A_0|^{1-e_1} \\ \times \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\ldots,0,1)A\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{T})\,|\det A\boldsymbol{T}|^s\,dA\,d\boldsymbol{B}\,d\boldsymbol{T}\,dki\,dx_1\cdots dx_{e_1-1},$$

where the sum over the x_j's is finite and

$$\gamma^{\boldsymbol{D}} = \begin{pmatrix} C &&&& \\ D_1^{-1}\boldsymbol{N} & C &&& \\ 0 & D_2^{-1}\boldsymbol{N} & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \ldots & D_{e_1-1}^{-1}\boldsymbol{N} & C \end{pmatrix} = \begin{pmatrix} C &&&& \\ x_1 & C &&& \\ 0 & x_2 & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \ldots & x_{e_1-1} & C \end{pmatrix}.$$

A similar analysis to that from the proof of Proposition 4.9 shows the integral relative to $\boldsymbol{T}$ actually is over a compact set, since φ has compact support. Hence, the function

$$(x_1,\ldots,x_{e_1-1}) \mapsto \int_k \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^{e_1}} \int_{\mathcal{B}} \varphi(k^{-1}\boldsymbol{B}^{-1}\boldsymbol{T}^{-1}\gamma^{\boldsymbol{D}}\boldsymbol{T}\boldsymbol{B}k)\,|\det A_0|^{1-e_1} \\ \times \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\ldots,0,1)A\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{T})\,|\det A\boldsymbol{T}|^s\,dA\,d\boldsymbol{B}\,d\boldsymbol{T}\,dk$$

is Schwartz; and the function

$$\int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\ldots,0,1)A\boldsymbol{T}\boldsymbol{B}k]\tau(\det A\boldsymbol{T})\,|\det A\boldsymbol{T}|^s|\det A_0|^{1-e_1}\,dA$$

is the Tate integral for $\Lambda(e_1 s, (\tau \circ N_{E/F})^{e_1})$. Therefore, Proposition 4.11 follows. □

4C3. *Orbital integrals of general type.* Set

$$I_{\boldsymbol{f},\boldsymbol{e};g}(s)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\mathcal{C}\in\Gamma_{\boldsymbol{f},\boldsymbol{e};g}}\varphi(x^{-1}\gamma_{\mathcal{C}}x)f(\lambda_{\boldsymbol{f},\boldsymbol{e};g}^{-1}x,s)\,dx,$$

where $g\geq 1$, $\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}_{\geq 1}^{g}$, $\langle\boldsymbol{f},\boldsymbol{e}\rangle=n$, and $\mathrm{Re}(s)>1$. We may write $\boldsymbol{f}=(f_1,\dots,f_g)$ with $f_1\geq\cdots\geq f_g$, and $\boldsymbol{e}=(e_1,\dots,e_g)$.

Let E be a finite extension of F, and let χ be an idele class character of $\mathbb{A}_E^{\times}$. For a positive integer j, define

$$\Lambda_E[j](s,\chi):=\Lambda_E(js-j+1,\chi^j),$$

where $\Lambda(s,\chi)$ denotes the complete Hecke L-function associated with χ.

Proposition 4.12. *Let notation be as before. Then $I_{\boldsymbol{f},\boldsymbol{e};g}(s)$ converges absolutely in* $\mathrm{Re}(s)>1$ *and*

$$I_{\boldsymbol{f},\boldsymbol{e};g}(s)\propto\prod_{i=1}^{g}\prod_{j=1}^{e_i}\Biggl(\sum_{[E_{i,j}:F]=f_i}Q_{E_{i,j}}(s)\Lambda_{E_{i,j}}[j](s,\tau\circ N_{E_i/F})\Biggr),\tag{4-34}$$

where for each i and j, the innermost summation is taken over only finitely many fields $E_{i,j}$, depending only on the test function φ; and each $Q_{E_{i,j}}(s)$ is an entire function.

Proof. Let $\mathcal{C}$ be a regular conjugacy class of type $(\boldsymbol{f},\boldsymbol{e};g)$. By Lemma 4.3 and the proof of Lemma 4.6, we can write $\mathcal{C}=\{\gamma\}$, with a typical element γ given by

$$\gamma=\begin{pmatrix}\alpha_1&&&&&&&&\\&\ddots&&&&&&&\\&&\alpha_m&&&&&&\\&&&\alpha_{m+1}&&&&&\\&&&\vdots&\ddots&&&&\\&&&*&\cdots&\alpha_{m+1}&&&\\&&&&&&\ddots&&\\&&&&&&&\alpha_g&&\\&&&&&&&\vdots&\ddots&\\&&&&&&&*&\cdots&\alpha_g\end{pmatrix}.$$

Using this expression and the definition of $\lambda_{\boldsymbol{f},\boldsymbol{e};g}^{-1}$ from (4-21), it becomes evident that Proposition 4.12 follows by combining the analysis from the proof of Proposition 4.9 with that of Proposition 4.11. $\square$

Gathering (4-20) with Proposition 4.12 we then obtain:

Theorem C. *Let notation be as before. Let* $\mathrm{Re}(s)>1$. *Then*

$$I_{\mathrm{Geo,Reg}}(s,\tau)\propto\sum_{g=1}^{n}\sum_{\substack{\boldsymbol{f},\boldsymbol{e}\in\mathbb{Z}_{\geq 1}^{g}\\ \langle\boldsymbol{f},\boldsymbol{e}\rangle=n}}\prod_{i=1}^{g}\frac{1}{f_i^{e_i}}\sum_{[E_i:F]=f_i}Q_{E_i}(s)\prod_{j=1}^{e_i}\Lambda_{E_i}[j](s,\tau\circ N_{E_i/F}),\tag{4-35}$$

where, for each i, $Q_{E_i}(s)$ is an entire function and the innermost summation is taken over only finitely many fields E_i, depending implicitly only on the test function φ. The right side of (4-35) *gives a meromorphic continuation of $I_{\mathrm{Geo,Reg}}(s,\tau)$ to the whole s-plane, with poles possible only at $1,\frac{1}{2},\frac{2}{3},\dots,1-\frac{1}{n}$.*

5. $I_{P,\mathrm{Reg}}(s,\tau)$ as intertwining operators

We now proceed to handle the function $I_{P,\mathrm{Reg}}(s,\tau)$. Our approach involves explicit geometric computations, ultimately reducing it to a finite sum of intertwining operators whose analytical behavior is known through the work of Langlands. Recall that, by definition,

$$I_{P,\mathrm{Reg}}(s,\tau) := \int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)} \int_{[N_P]} \sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}} \varphi(x^{-1}u^{-1}\gamma x)\, du\, f(x,s)\, dx.$$

To simplify $I_{P,\mathrm{Reg}}(s,\tau)$, we express $Z(F)\backslash G(F)-\mathfrak{S}$ as a disjoint union of $G(F)$-conjugacy classes $\mathcal{C}$ modulo $Z(F)$. Furthermore, we decompose each $\mathcal{C}$ into a disjoint union of $P(F)$-conjugacy classes. By explicitly determining representatives for these $P(F)$-conjugacy classes, we can perform a change of variables to transform the integral over $Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ into an integral over $Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. This allows us to perform local calculations and establish a relationship between these integrals and the constant terms of certain Eisenstein series. Consequently:

Theorem D. *Let notation be as before, then $I_{P,\mathrm{Reg}}(s,\tau)$ converges absolutely in* $\mathrm{Re}(s)>1$. *Moreover, $I_{P,\mathrm{Reg}}(s,\tau)$ admits a meromorphic continuation. Precisely, one has*

$$I_{P,\mathrm{Reg}}(s,\tau) \propto \frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})}. \tag{5-1}$$

5A. *$P(F)$-conjugacy classes.*

5A1. *Ad hoc notation.* We introduce some unconventional notation specific to this section, designed to simplify the calculations. Readers are advised not to be overly concerned with its details.

- Let $\mathcal{C}$ be a $G(F)$-conjugacy class. Recall that we define $\mathcal{C}_0$ in (4-4) if $\mathcal{C}$ is regular. Set $\mathfrak{R} = \bigsqcup_{@\mathcal{C}\text{ is regular}} \mathcal{C}_0$. Then $\mathfrak{R}$ is a disjoint union of $P(F)$-conjugacy classes in $G(F)$ by Proposition 4.1. Propositions 4.1 and 4.2 give a decomposition of $G(F)$ as $P(F)$-conjugacy classes

$$G(F) = \mathfrak{R} \sqcup \bigcup_{k=1}^{n-1} Q_k(F)^{P(F)}. \tag{5-2}$$

- For any $2\le k\le n$, let P_k be the standard maximal parabolic subgroup of GL_k of type $(k-1,1)$. In particular, $P=P_n$. Let N_P be the unipotent radical of P.
- Recall that B is the Borel of G. Let T (resp. N) be the Levi component (resp. unipotent radical) of B.
- Let $w_1,\dots,w_{n-1}$ be the Weyl elements of $G(F)$ defined by (4-3) in Section 4A.
- For any $1\le k\le n-1$, write Q_k (resp. Q_k^*) for the standard parabolic subgroup of G of type $(k,n-k)$ (resp. $(k,1,\dots,1)$).

To compute $I_{P,\mathrm{Reg}}(s,\tau)$, it is necessary to explicitly choose representatives for $\mathfrak{R}$. From the construction in the proof of Proposition 4.1, we have an explicit algebraic description of the representatives for each $\mathcal{C}_0$. However, this algebraic construction is not particularly convenient for analytic parametrization.

In this section, our goal is to find representatives of $\mathfrak{R}$ that are more suitable for analytic purposes.

5A2. *Explicit representatives of* $\mathfrak{R}$. To narrow down the possible candidates for these representatives, we begin with the following result.

Lemma 5.2. *Let notation be as before. Set* $\mathcal{R}=\{w_{n-1}w_{n-2}\cdots w_1 b : b\in B(F)\}$. *Denote by* $\mathcal{R}^{P(F)}$ *the union of* $P(F)$*-conjugacy classes of elements in* $\mathcal{R}$. *Then*

$$\mathfrak{R}=\mathcal{R}^{P(F)}. \tag{5-3}$$

Proof. Recall the Bruhat decomposition

$$G(F)=P(F)\sqcup P(F)w_{n-1}P(F).$$

Due to the disjointness of different Bruhat cells, the $P(F)$-conjugacy class of g_1 does not intersect with that of g_2, for any $g_1\in P(F)$ and $g_2\in P(F)w_{n-1}P(F)$. Since $P(F)$-conjugacy classes of $P(F)$ lie in $P(F)$, we can reject all representatives in $P(F)$ and conclude that $P(F)$-conjugacy classes in $\mathfrak{R}$ are represented by elements in $w_{n-1}P(F)$.

For any

$$g=w_{n-1}\begin{pmatrix}A_{n-1} & b\\ & d_n\end{pmatrix}\in w_{n-1}P(F)\cap\mathfrak{R},$$

by Bruhat decomposition, either $A_{n-1}\in P_{n-1}(F)$ or $A_{n-1}\in P_{n-1}(F)w_{n-2}P_{n-1}(F)$, where P_{n-1} is the standard maximal parabolic subgroup of $\mathrm{GL}_{n-1}(F)$ of type $(n-2,1)$. If $A_{n-1}\in P_{n-1}(F)$, then $g\in Q_{n-2}(F)\subset\bigcup_{1\le k\le n-1}Q_k(F)^{P(F)}$. Thus $g\notin\mathfrak{R}$. Therefore, $A_{n-1}\in P_{n-1}(F)w_{n-2}P_{n-1}(F)$. So we can write

$$g^{(0)}=g=w_{n-1}\begin{pmatrix}I_{n-2} & c & b_1\\ & 1 & b_2\\ & & 1\end{pmatrix}w_{n-2}\begin{pmatrix}A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n\end{pmatrix}\in w_{n-1}Q^*_{n-1}(F),$$

which is conjugate by $w_{n-2}\begin{pmatrix}A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n\end{pmatrix}\in P(F)$ to

$$\begin{aligned}g^{(1)}&=w_{n-2}\begin{pmatrix}A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n\end{pmatrix}w_{n-1}\begin{pmatrix}I_{n-2} & c & b_1\\ & 1 & b_2\\ & & 1\end{pmatrix}\\ &=w_{n-2}w_{n-1}\begin{pmatrix}A_{n-2} & & c_{n-2}\\ & d_n & \\ & & d_{n-1}\end{pmatrix}\begin{pmatrix}I_{n-2} & c & b_1\\ & 1 & b_2\\ & & 1\end{pmatrix}\in w_{n-2}w_{n-1}Q^*_{n-2}(F).\end{aligned}$$

Again, apply Bruhat decomposition to $\mathrm{GL}_{n-2}(F)$ to conclude that either $A_{n-2}\in P_{n-2}(F)$ or $A_{n-2}\in P_{n-2}(F)w_{n-3}P_{n-2}(F)$, where P_{n-2} is the standard maximal parabolic subgroup of $\mathrm{GL}_{n-2}(F)$ of type $(n-3,1)$. If $A_{n-2}\in P_{n-2}(F)$, then $g^{(1)}\in Q_{n-3}(F)\subset\bigcup_{1\le k\le n-1}Q_k(F)^{P(F)}$, i.e., $g^{(1)}\notin\mathfrak{R}$. Therefore, $A_{n-2}\in P_{n-2}(F)w_{n-3}P_{n-2}(F)$. Likewise, $g^{(1)}$ is thus $P(F)$-conjugate to some $g^{(2)}\in w_{n-3}w_{n-2}w_{n-1}Q^*_{n-3}(F)$. Continue this process inductively to see that g is $P(F)$-conjugate to some element $g^{(n-2)}\in w_1w_2\cdots w_{n-1}Q^*_1(F)$.

Therefore, $\mathfrak{R} \subseteq \{p^{-1}\gamma p : \gamma \in w_1 w_2 \cdots w_{n-1} Q_1^*(F),\ p \in P(F)\}$. So we have

$$\begin{aligned}\{g^{-1} : g \in \mathfrak{R}\} &\subseteq \{p^{-1}\gamma p : \gamma \in Q_1^*(F) w_{n-1} \cdots w_2 w_1,\ p \in P(F)\}\\ &= \big\{p^{-1}\gamma p : \gamma \in w_{n-1} w_{n-2} \cdots w_1 B(F),\ p \in P(F)\big\},\end{aligned}$$

since $Q_1^*(F) = B(F) \subseteq P(F)$. Note that $\mathfrak{R}$ is stable under inversion. Hence,

$$\mathfrak{R} = \{g^{-1} : g \in \mathfrak{R}\} \subseteq \{p^{-1}\gamma p : \gamma \in w_{n-1} w_{n-2} \cdots w_1 B(F), \quad p \in P(F)\} = \mathcal{R}^{P(F)}.$$

Based on the construction, we observe that

$$\mathcal{R}^{P(F)} \cap \bigcup_{1 \leq k \leq n-1} Q_k(F) = \varnothing,$$

since Bruhat cells are disjoint. This implies, according to (5-2), that $\mathcal{R}^{P(F)} \subseteq \mathfrak{R}$. Hence, $\mathfrak{R} = \mathcal{R}^{P(F)}$. □

Now we determine representatives of $\mathcal{R}^{P(F)}$.

Lemma 5.3. *Let notation be as before. Then* $\mathfrak{R} = \widetilde{\mathcal{R}}^{P(F)}$, *where*

$$\widetilde{\mathcal{R}} := \left\{ w_{n-1} w_{n-2} \cdots w_1 \begin{pmatrix} I_{n-1} & \mathfrak{b} \\ & t \end{pmatrix} : t \in F^\times,\ \begin{pmatrix} I_{n-1} & \mathfrak{b} \\ & 1 \end{pmatrix} \in N_P(F) \right\}.$$

Proof. Let $b = \mathfrak{t}\mathfrak{u} \in B(F)$, where $\mathfrak{t} \in T(F)$ and $\mathfrak{u} \in N(F)$. By examining the rows on both sides, we can find $c_{i,j} \in F$ and $\mathfrak{u}' \in N_P(F)$ that satisfy the equation

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & 1 & c_{1,2} & \cdots & c_{1,n-1} \\ & & \ddots & & \vdots \\ & & & 1 & c_{n-2,n-1} \\ & & & & 1 \end{pmatrix} \mathfrak{t}\mathfrak{u} = \mathfrak{t}\mathfrak{u}' \begin{pmatrix} 1 & c_{1,2} & \cdots & c_{1,n-1} & 0 \\ & 1 & \cdots & c_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}. \tag{5-4}$$

In fact, the values of $c_{1,j}$, $1 \leq j \leq n-1$, are determined by b. For $i \geq 2$, the values of $c_{i,j}$ are determined by $c_{i-1,j}$ and b, where $1 \leq j \leq n-1$. Thus, these $c_{i,j}$ values are uniquely determined by b. Consequently, $\mathfrak{u}'$ is also uniquely determined.

Let $p \in P(F)$ be the matrix multiplying $\mathfrak{t}\mathfrak{u}'$ on the right-hand side of (5-4). Then (5-4) becomes

$$p w_{n-1} w_{n-2} \cdots w_1 b p^{-1} = w_{n-1} w_{n-2} \cdots w_1 \mathfrak{t}\mathfrak{u}'.$$

Write $\mathfrak{t} = \operatorname{diag}(t_1, \dots, t_n)$, $\mathfrak{a} = \operatorname{diag}(a_1, \dots, a_n) \in T(F)$, with $a_i = t_1^{-1} \cdots t_i^{-1}$, $1 \leq i \leq n$, and

$$\mathfrak{u}' = \begin{pmatrix} I_{n-1} & \mathfrak{b} \\ & 1 \end{pmatrix},$$

with $\mathfrak{b} = {}^t(b_1, \dots, b_{n-1})$. Then

$$\mathfrak{a}^{-1} w_{n-1} w_{n-2} \cdots w_1 \mathfrak{t}\mathfrak{u}'\mathfrak{a} = w_{n-1} w_{n-2} \cdots w_1 \begin{pmatrix} I_{n-1} & \mathfrak{b}' \\ & t \end{pmatrix} \in \widetilde{\mathcal{R}},$$

where $t = \det \mathfrak{t}$ and $\mathfrak{b}' = {}^t(b_1', \dots, b_{n-1}')$, with $b_i' = t_1 \cdots t_i b_i$, $1 \leq i \leq n-1$.

Therefore, Lemma 5.3 follows from Lemma 5.2. □

We shall show that elements in $\widetilde{\mathcal{R}}$ are not $P(F)$-conjugate to each other.

Lemma 5.4. *Let notation be as before. Then $\widetilde{\mathcal{R}}$ forms a complete set of representatives for $\mathfrak{R}$. In particular, $\mathfrak{R}=\widetilde{\mathcal{R}}^{P(F)}$.*

Proof. Let $w_{n-1}w_{n-2}\cdots w_1 b$ and $w_{n-1}w_{n-2}\cdots w_1 b'$ be two elements in $\widetilde{\mathcal{R}}$. Assume that there exists some $p_n\in P_n(F)=P(F)$ such that

$$p_n w_{n-1}w_{n-2}\cdots w_1 b p_n^{-1}=w_{n-1}w_{n-2}\cdots w_1 b'. \tag{5-5}$$

Then $w_{n-1}p_n w_{n-1}=w_{n-2}\cdots w_1 b' p_n b^{-1}w_1\cdots w_{n-2}\in P(F)=Q_{n-1}(F)$. Since $p_n\in P(F)$, it is necessarily of the form

$$p_n=\begin{pmatrix}A_{n-2} & \mathfrak{c}_{n-1} & \mathfrak{c}_n\\ & a_{n-1} & 0\\ & & a_n\end{pmatrix}\in\begin{pmatrix}GL_{n-2}(F) & * & *\\ & F^\times & 0\\ & & F^\times\end{pmatrix}\subset Q_{n-2}(F).$$

Hence, $w_{n-2}w_{n-1}p_n w_{n-1}w_{n-2}=w_{n-3}\cdots w_1 b' p_n b^{-1} w_1\cdots w_{n-3}\in Q_{n-2}(F)$, which implies that A_{n-2} lies in the maximal parabolic subgroup of $\mathrm{GL}_{n-2}(F)$ of type $(n-3,1)$, and the last component of $\mathfrak{c}_n$ must vanish. Repeating this process $n-3$ more times, we simplify (5-5) to

$$\begin{pmatrix}a_n & 0 & 0 & \dots & 0\\ & a_1 & c_{1,2} & \cdots & c_{1,n-1}\\ & & \ddots & & \vdots\\ & & & a_{n-2} & c_{n-2,n-1}\\ & & & & a_{n-1}\end{pmatrix} b \begin{pmatrix}a_1 & c_{1,2} & \cdots & c_{1,n-1} & 0\\ & a_2 & \cdots & c_{2,n-1} & 0\\ & & \ddots & \vdots & \vdots\\ & & & a_{n-1} & 0\\ & & & & a_n\end{pmatrix}^{-1}=b'. \tag{5-6}$$

Note that the unipotent radical of b and b' are in $N_P(F)$. By the analysis towards equation (5-4) we see that all the $c_{i,j}$'s in (5-6) must vanish.

Write

$$b=\begin{pmatrix}I_{n-1} & \mathfrak{b}\\ & t\end{pmatrix}\quad\text{and}\quad b'=\begin{pmatrix}I_{n-1} & \mathfrak{b}'\\ & t'\end{pmatrix}.$$

Then (5-6) becomes

$$\begin{pmatrix}a_n & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_{n-1}\end{pmatrix}\begin{pmatrix}I_{n-1} & \mathfrak{b}\\ & t\end{pmatrix}\begin{pmatrix}a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n\end{pmatrix}^{-1}=\begin{pmatrix}I_{n-1} & \mathfrak{b}'\\ & t'\end{pmatrix}.$$

By comparing the determinants on both sides of the equality, we deduce that $t'=t$. Similarly, comparing the Levi components yields $a_1=a_2=\cdots=a_n$, leading to $\mathfrak{b}'=\mathfrak{b}$. Therefore, any two elements in $\widetilde{\mathcal{R}}$, are either equal or not conjugate to each other by $P(F)$. □

Now we consider for our purpose the decomposition of $Z(F)\backslash G(F)$ into $P(F)$-conjugacy classes. By (5-2) one has the decomposition

$$Z(F)\backslash G(F)=(Z(F)\backslash\mathfrak{R})\sqcup\mathfrak{S}, \tag{5-7}$$

where $\mathfrak{S}$ was defined in (2-6) in Section 2B.

Corollary 5.5. *Let notation be as before. Set $(F^\times)^n=\{t^n : t\in F^\times\}$, and let*

$$\widetilde{\mathcal{R}}^*=\Big\{\mathfrak{u}w_1w_2\cdots w_{n-1}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}: t\in F^\times/(F^\times)^n,\ \mathfrak{u}\in N_P(F)\Big\}. \tag{5-8}$$

Then $\widetilde{\mathcal{R}}^*$ forms a family of representatives of $Z(F)\backslash\mathfrak{R}$.

Proof. By Lemma 5.4, the set

$$\Big\{w_{n-1}w_{n-2}\cdots w_1\mathfrak{u}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}: t\in F^\times/(F^\times)^n,\ \mathfrak{u}\in N_P(F)\Big\} \tag{5-9}$$

forms a family of representatives of $Z(F)\backslash\mathfrak{R}$. Then the inverse of elements in the set defined in (5-9) also form a family of representatives of $Z(F)\backslash\mathfrak{R}$. Note that these inverses are bijectively $P_0(F)$-conjugate to $\widetilde{\mathcal{R}}^*$, then the proof follows. ☐

5B. *Holomorphic continuation.* In this section we shall prove Theorem D.

Recall the definition in Section 2C:

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\mathrm{K}_{\mathrm{Geo,Reg}}(ux,x)\,du f(x,s)\,dx,$$

where $\mathrm{Re}(s)>1$. By (5-7) and Corollary 5.5,

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x,y)=\sum_{\gamma\in Z(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}\gamma x)=\sum_{\gamma\in\widetilde{\mathcal{R}}^*}\sum_{p\in P_0(F)}\varphi(x^{-1}p^{-1}\gamma p y).$$

As a consequence, we have (at least formally) the decomposition

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\sum_{\gamma\in\widetilde{\mathcal{R}}^*}\varphi(x^{-1}u^{-1}\gamma x)\,du f(x,s)\,dx.$$

5B1. *Ad hoc notation.* Recall that P is the parabolic subgroup of type $(n-1,1)$. Let M_P be the Levi component of P. Let $N^P:=N\cap M_P$; then $N=N^PN_P$. Set $A(\mathbb{A}_F)=Z(\mathbb{A}_F)\backslash T(\mathbb{A}_F)$ and $\tilde{w}=w_1w_2\cdots w_{n-1}$.

5B2. *Coordinate transforms of unipotent radicals.*

Lemma 5.6. *Let $c_{i,j}\in\mathbb{A}_F$, $1\le i<j\le n-1$. Define $u=(u_{i,j})_{1\le i,j\le n-1}\in N^P(\mathbb{A}_F)$ via the expression*

$$\begin{pmatrix}1&0&0&\dots&0\\&1&c_{1,2}&\cdots&c_{1,n-1}\\&&\ddots&&\vdots\\&&&1&c_{n-2,n-1}\\&&&&1\end{pmatrix}^{-1}\begin{pmatrix}1&c_{1,2}&\cdots&c_{1,n-1}&0\\&1&\cdots&c_{2,n-1}&0\\&&\ddots&\vdots&\vdots\\&&&1&0\\&&&&1\end{pmatrix}=\begin{pmatrix}u&*\\&1\end{pmatrix}. \tag{5-10}$$

Then each $1\le i<j<n$, $u_{i,j}=c_{i,j}+P_{i,j}$, where $P_{i,j}$ is a polynomial in the variables $c_{i',j'}\ne c_{i,j}$, $1\le i'<j'<n$.

Proof. The argument holds trivially for 2×2 matrices. Suppose Lemma 5.6 holds for matrices of rank less than n. Write

$$\begin{pmatrix} 1 & c_{1,2} & \cdots & c_{1,n-1} \\ & \ddots & & \vdots \\ & & 1 & c_{n-2,n-1} \\ & & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c'_{1,2} & \cdots & c'_{1,n-1} \\ & \ddots & & \vdots \\ & & 1 & c'_{n-2,n-1} \\ & & & 1 \end{pmatrix}.$$

By multiplying block matrices, we can determine $c'_{i,j}$ using $c_{i',j'}$ with $i'\le i$ and $j'\le j$. Based on the induction assumption, the $u_{i,j}$ take the form $c_{i,j}+P_{i,j}$ for $1\le i<n-1$ and $1\le j<n$. We then focus on investigating the last column, specifically $u_{i,n-1}$ for $1\le i<n-1$. Note that (5-10) becomes

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & 1 & c'_{1,2} & \cdots & c'_{1,n-1} \\ & & \ddots & & \vdots \\ & & & 1 & c'_{n-2,n-1} \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & c_{1,2} & \cdots & c_{1,n-1} & 0 \\ & 1 & \cdots & c_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} u & * \\ & 1 \end{pmatrix}.$$

As a consequence, we obtain

$$u_{i,n-1} = c_{i,n-1} + c_{i+1,n-1}c'_{i-1,i} + \cdots + c_{n-2,n-1}c'_{i-1,n-3} + c'_{i-1,n-2},$$

which yields that $u_{i,n-1}-c_{i,n-1}$ is a polynomial in the variables $c_{i',j'}\neq c_{i,n-1}$. Hence, Lemma 5.6 follows from induction. □

Consider the smooth transformation defined by (5-10):

$$N^P(\mathbb{A}_F)\to N^P(\mathbb{A}_F),\quad (c_{i,j})_{1\le i,j<n}\mapsto (u_{i,j})_{1\le i,j<n}.$$

By Lemma 5.6, its Jacobian matrix is identically trivial. This will simplify the calculation of $I_{P,\mathrm{Reg}}(s,\tau)$ in the next subsection.

Set $c=(c_{i,j})_{1\le i,j<n}\in N^P(\mathbb{A}_F)$. Since $\begin{pmatrix} I_{n-1} & \\ & t\end{pmatrix}$ commutes with c, the relation (5-10) amounts to

$$c^{-1}\tilde{w}\begin{pmatrix} I_{n-1} & \\ & t\end{pmatrix}c = \tilde{w}\begin{pmatrix} u & * \\ & t\end{pmatrix},\quad \tilde{w}=w_1w_2\cdots w_{n-1}. \tag{5-11}$$

5B3. *Manipulation of $I_{P,\mathrm{Reg}}(s,\tau)$: integral transformations.* We have

$$I_{P,\mathrm{Reg}}(s,\tau) = \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\sum_t \varphi\Big(x^{-1}u_1^{-1}\tilde{w}\begin{pmatrix} I_{n-1} & \\ & t\end{pmatrix}x\Big)\,du_1\,f(x,s)\,dx,$$

where $t\in F^\times/(F^\times)^n$. By Iwasawa decomposition, we may write $x=cu_2\mathrm{t}k$, where $c\in N^P(\mathbb{A}_F)$, $u_2\in N_P(\mathbb{A}_F)$, $\mathrm{t}=\mathrm{diag}(t_1,t_2,\dots,t_{n-1},1)\in A(\mathbb{A}_F)$, $k\in K$. Then

$$I_{P,\mathrm{Reg}}(s,\tau) = \int_K\int_{A(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)} \mathcal{I}\cdot f(cu_2\mathrm{t}k,s)\,du_2\,du_1\,\delta^{-1}(\mathrm{t})\,d^\times\mathrm{t}\,dk,$$

where δ is the modular character, and

$$\mathcal{I} := \int_{N^P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\varphi\Big(k^{-1}\mathrm{t}^{-1}u_1^{-1}c^{-1}\tilde{w}\begin{pmatrix} I_{n-1} & \\ & t\end{pmatrix}cu_2\mathrm{t}k\Big)\,dc.$$

We can change variables through Lemma 5.6 or equation (5-11) to derive

$$\mathcal{I}=\int_{N^P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\varphi(k^{-1}\mathfrak{t}^{-1}u_1^{-1}\tilde{w}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}uu_2\mathfrak{t}k)\,du.$$

Recall that $f(x,s)$ is defined by

$$f(x,s)=\tau(\det x)\,|\det x|_{\mathbb{A}_F}^s\int_{\mathbb{A}_F^\times}\Phi[(0,\dots,t)x]\tau^n(t)|t|^{ns}d^\times t, \tag{5-12}$$

which is a Tate integral for the complete L-function $\Lambda(ns,\tau^n)$. By definition, $f(cu_2\mathfrak{t}k,s)$ equals $\tau(\det\mathfrak{t})\,|\det\mathfrak{t}|^s f(k,s)$. Therefore,

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K f(k,s)\,dk\int_{A(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\frac{\mathcal{I}\cdot\tau(\det\mathfrak{t})\,|\det\mathfrak{t}|^s}{\delta(\mathfrak{t})}du_2\,du_1\,d^\times\mathfrak{t}.$$

Lemma 5.7. *Let notation be as before. Then*

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K f(k,s)\,dk\int_{N_P(\mathbb{A}_F)}du_1\int_{N^P(\mathbb{A}_F)}du\int_{N_P(\mathbb{A}_F)}du_2\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\Delta_{s,\tau}^{(1)}(\mathfrak{t})$$
$$\times\int_{\mathbb{A}_F^\times}\sum_{t\in F^\times/(F^\times)^n}\varphi\left(k^{-1}u_1\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1^n t\prod_{i=2}^{n-1}t_i\end{pmatrix}\tilde{w}u_2uk\right)d^\times t,$$

where $d^\times\mathfrak{t}=d^\times t_1\,d^\times t_2\cdots d^\times t_{n-1}$, *and for any* $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\dots,t_{n-1},1)\in A(\mathbb{A}_F)$,

$$\Delta_{s,\tau}^{(1)}(\mathfrak{t})=\tau(t_1)^{\frac{n(n-1)}{2}}|t_1|_{\mathbb{A}_F}^{\frac{n(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\tau(t_i)^{n-i}|t_i|_{\mathbb{A}_F}^{(n-i)(s+1)}.$$

Proof. By the change of variables $u_1\mapsto\mathfrak{t}u_1\mathfrak{t}^{-1}$ and $uu_2\mapsto\mathfrak{t}uu_2\mathfrak{t}^{-1}$ we obtain

$$I_{P,\mathrm{Reg}}(s,\tau)=\int_K f(k,s)\,dk\int_{A(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\mathcal{J}\cdot\tau(\det\mathfrak{t})\,|\det\mathfrak{t}|^{s+1}du_2\,du_1\,d^\times\mathfrak{t},$$

where

$$\mathcal{J}=\int_{N^P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\varphi(k^{-1}u_1^{-1}\mathfrak{t}^{-1}\tilde{w}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}\mathfrak{t}uu_2k)\,du.$$

Since

$$\mathfrak{t}^{-1}\tilde{w}\begin{pmatrix}I_{n-1}&\\&t\end{pmatrix}\mathfrak{t}\tilde{w}^{-1}=\begin{pmatrix}tt_1^{-1}&&&&\\&t_1t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-2}t_{n-1}^{-1}&\\&&&&t_{n-1}\end{pmatrix},$$

Lemma 5.7 follows after a change of variables. □

5B4. *Manipulation of $I_{P,\mathrm{Reg}}(s,\tau)$: reducing to intertwining operators.* Recall the test function φ has the central character ω. Let Ξ be the set of idele class characters on $\mathbb{A}_F^\times$, which is trivial on $\mathbb{R}_+^\times$. Denote by $\Xi_{\omega,n}$ the subset $\{\chi\in\Xi:\chi^n=\omega\}\subset\Xi$. Also, let $\Xi_{\tau,2}^n=\{\xi\in\Xi:\xi^2=\tau\}$ if n is even, and let $\Xi_{\tau,2}^n=\{\mathbf{1}\}$, the singleton, if n is odd. Then both $\#\Xi_{\omega,n}<\infty$ and $\#\Xi^n_{\tau,2}<\infty$. For a Hecke character χ, we define

$$\Delta^{\mathrm{od}}_{s,\tau,\chi}(\mathfrak{t})=\bar{\chi}(t_1)\tau(t_1)^{\frac{n-1}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\tau(t_i^{\frac{n+1}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}$$

when n is odd. For even n, we define

$$\Delta^{\mathrm{en}}_{s,\tau,\chi,\xi}(\mathfrak{t})=\bar{\chi}(t_1)\xi(t_1)\tau(t_1)^{\frac{n-2}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\xi(t_i)\tau(t_i^{\frac{n}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

We can employ a change of variables, specifically $(\mathbb{A}_F^\times)^n\cdot F^\times/(F^\times)^n=F^\times\cdot(F^\times\backslash\mathbb{A}_F^\times)^n$, to perform Poisson summation, following a similar approach as in [18, §2.4, Lemma]. This allows us to derive the following results:

- When n is odd, the integral $I_{P,\mathrm{Reg}}(s,\tau)$ is equal to

$$\int_K f(k,s)\,dk\int_{N_P(\mathbb{A}_F)}du_1\int_{N^P(\mathbb{A}_F)}du\int_{N_P(\mathbb{A}_F)}du_2\int_{\mathbb{A}_F^\times}\sum_{\chi\in\Xi_{\omega,n}}\Delta^{\mathrm{od}}_{s,\tau,\chi}(\mathfrak{t})\,d^\times t_1$$
$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}u_1\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\tilde{w}u_2uk\right)d^\times t_2\cdots d^\times t_{n-1}.$$

- When n is even, the integral $I_{P,\mathrm{Reg}}(s,\tau)$ is equal to

$$\int_K f(k,s)\,dk\int_{N_P(\mathbb{A}_F)}du_1\int_{N^P(\mathbb{A}_F)}du\int_{N_P(\mathbb{A}_F)}du_2\sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi^n_{\tau,2}}\Delta^{\mathrm{en}}_{s,\tau,\chi,\xi}(\mathfrak{t})$$
$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}u_1\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\tilde{w}u_2uk\right)d^\times t_1\cdots d^\times t_{n-1}.$$

Let $T_*(\mathbb{A}_F^\times)=\{\mathrm{diag}(1,t_1,t_2,\dots,t_{n-1})\in T(\mathbb{A}_F):t_i\in\mathbb{A}_F^\times,\ 1\le i\le n-1\}$. Set

$$\iota:A(\mathbb{A}_F^\times)\to T_*(\mathbb{A}_F^\times),\quad \mathfrak{t}\mapsto\mathfrak{t}^\iota=\mathrm{diag}(1,t_2^{-1},t_3^{-1},\dots,t_{n-1}^{-1},t_1).$$

Let $\delta_n=-\frac{1}{2}(1+(-1)^n)$. Define $\mathfrak{F}_{\chi,\xi}(x;k,s)=\mathfrak{F}_{\chi,\xi}(x;k,s,\varphi,\Phi,\tau)$ by

$$\mathfrak{F}_{\chi,\xi}(x;k,s)=\int_{N_P(\mathbb{A}_F)}\int_{N^P(\mathbb{A}_F)}\int_{A(\mathbb{A}_F^\times)}\varphi(k^{-1}u_1\mathfrak{t}^\iota xuk)\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})\,d^\times\mathfrak{t}\,du\,du_1,$$

where $\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})$ is defined by

$$\bar{\chi}(t_1)\xi(t_1)^{-\delta_n}\tau(t_1)^{\frac{n-1-\delta_n}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

Since φ has compact support modulo the center, $\mathfrak{F}_{\chi,\xi}(x;k,s)$ is well defined for any χ,ξ in $\mathrm{Re}(s)>1$.

Let $b=\mathfrak{t}\mathfrak{u}'\mathfrak{u}\in B(\mathbb{A}_F)$, where $\mathfrak{u}'\in N_P(\mathbb{A}_F)$, $\mathfrak{u}\in N^P(\mathbb{A}_F)$, $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\dots,t_n)\in T(\mathbb{A}_F)$. By the change of variables $u\mapsto\mathfrak{u}^{-1}u$, $u_1\mapsto u_1\mathfrak{t}^\iota\mathfrak{t}\mathfrak{u}'^{-1}(\mathfrak{t}^\iota\mathfrak{t})^{-1}$, we obtain

$$\mathfrak{F}_{\chi,\xi}(bx;k,s)=\prod_{i=1}^{n}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}\cdot\mathfrak{F}_{\chi,\xi}(x;k,s). \tag{5-13}$$

Since the modular character of $T(\mathbb{A}_F)$ is $\delta(\mathrm{t})=\prod_{i=1}^{n}t_i^{n+1-2i}$, one has

$$\mathfrak{F}_{\chi,\xi}(x;k,s)\in\mathrm{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\big(\chi\xi^{\delta_n}\tau^{\lambda_1}|\cdot|_{\mathbb{A}_F}^{\lambda_1 s},\dots,\chi\xi^{\delta_n}\tau^{\lambda_{n-1}}|\cdot|_{\mathbb{A}_F}^{\lambda_{n-1}s},\chi\xi^{\delta_n}\tau^{\lambda_n}|\cdot|_{\mathbb{A}_F}^{\lambda_n s}\big),$$

where $\lambda_i=\frac{1}{2}(n+1-\delta_n)-i$ for $1\le i\le n$. Define

$$G_{\chi,\xi}(x;s)=G_{\chi,\xi}(x;s,\varphi,\Phi,\tau)=\int_K f(k,s)\mathfrak{F}_{\chi,\xi}(x;k,s)\,dk.$$

Therefore, at least formally one can write $I_{P,\mathrm{Reg}}(s,\tau)$ as a finite sum:

$$I_{P,\mathrm{Reg}}(s,\tau)=\sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi_{\tau,2}^{n}}\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\tilde{w}n;s)\,dn,\quad \mathrm{Re}(s)>1. \tag{5-14}$$

5B5. *Proof of Theorem D.* Now we show the absolute convergence of (5-14) and deduce its analytic behavior.

Let $\mathfrak{F}_{1,1,+}(x;k,s)=\mathfrak{F}_{1,1}(x;k,s,|\varphi|,|\Phi|,1)$ and $G_{1,1,+}(x;s)=G_{1,1}(x;s,|\varphi|,|\Phi|,1)$. Then the above interchange in the order of integrals is justified by Fubini's theorem on integrals of nonnegative functions. One then has

$$I_{P,\mathrm{Reg}}^{+}(s,\tau)=\sum_{\chi\in\Xi_{1,n}}\sum_{\xi\in\Xi_{1,2}^{n}}\int_{N_P(\mathbb{A}_F)}G_{1,1,+}(\tilde{w}n;s)\,dn,$$

where the sums are finite. Then $\int_{N_P(\mathbb{A}_F)}G_{1,1,+}(\tilde{w}n;s)\,dn$ converges absolutely in $\mathrm{Re}(s)>1$ according to Langlands' theory on intertwining operators. Therefore, by the dominated convergence theorem, $\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\tilde{w}n;s)\,dn$ converges absolutely in $\mathrm{Re}(s)>1$. It is thus a well defined intertwining operator. By Langlands' theory (cf. [20] or [30]) on intertwining operators, we have

$$\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\tilde{w}n;s)\,dn\propto\frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})},$$

where the last factor $\Lambda(ns,\tau^n)$ on the numerator comes from the Tate integral $f(k,s)$ from (5-12).

So (5-14) is well defined. Consequently (5-1) holds, since the sums in (5-14) are finite.

6. Convergence of the spectral side

In this section we shall deal with the spectral side

$$I_{\mathrm{Whi}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\mathrm{K}_{\mathrm{Eis}}(n_1x,x)\theta(n_1)\,dn_1\,f(x,s)\,dx, \tag{6-1}$$

where $\mathrm{K}_{\mathrm{Eis}}(x,y)$ is the Eisenstein part of the kernel function relative to a general test function φ in $\mathcal{H}(G(\mathbb{A}_F),\omega)$. The main concern here is the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ when $\mathrm{Re}(s)$ is large.

Typically one needs certain suitable regularization or truncation for $\mathrm{K}_{\mathrm{Eis}}$, which is slowly increasing.

In the GL(2) case this can be handled by the techniques in [29] or [40]. Arthur [1; 2; 3; 4] developed a truncation approach to successfully regularize the trace formula on general reductive groups. Arthur's truncation operators and their variants (as in [21; 13]) provide a powerful toolkit to manipulate the convergence problem in the (relative) trace formula.

However, these truncation operators seem to be not quite suitable for the function (6-1). One barrier is that the domain is not the usual automorphic quotient $Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$ but the much larger region $Z(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)$; thus the kernel in (6-1) is not $G(F)$-invariant now, which makes the usual truncation operators not work well here. One can do the spectral expansion of $\mathrm{K}_{\mathrm{Eis}}$ and apply Arthur's truncation Λ^T to the second Eisenstein series and show it can be integrated over a Siegel domain. With further covering process by Weyl elements conjugation, one can show (6-1) converges absolutely with $\mathrm{K}_{\mathrm{Eis}}(x, y)$ replaced by $\Lambda_2^T\mathrm{K}_{\mathrm{Eis}}(x, y)$, where Λ_2^T means the operator Λ^T is applied to the y-variable. See Section 5.4 in [35] for details. Nevertheless, taking Fourier coefficients in the first variable makes the geometric truncation difficult to control, since it is just $N(F)$-invariant. So it is not clear how to compute the spectrally truncated function as a polynomial of the parameter T and ultimately show that this polynomial is indeed a constant. (Here the letter T is a conventional notation for the truncation parameter, while elsewhere we use T to denote the torus.) For an individual cuspidal datum, one may develop an allied truncation operator as in [13], but the problem is to show that the sum over *all* cuspidal data is convergent.

We will propose an alternative way to verify the convergence of (6-1), making essential use of the Fourier transform. Our strategy is to reduce (6-1) to a Mellin transform of the Kuznetsov relative trace formula, which is majorized by a gauge. Then one obtains convergence of (6-1) when $\mathrm{Re}(s)$ is large enough.

Inserting the spectral expansion (6-7) of $\mathrm{K}_{\mathrm{ER}}(x, y)$ into (6-1), $I_{\mathrm{Whi}}(s, \tau)$ becomes

$$\int \sum_{\chi}\sum_{Q}\frac{1}{c_Q}\int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*}\sum_{\phi_1}\sum_{\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}\,d\lambda\, f(x,s)\,dx. \tag{6-2}$$

Here, x ranges through $Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, χ ranges through cuspidal data, Q represents proper standard parabolic subgroups, and W_j denotes the Whittaker functions. Additional details can be found in (6-10) below. The absolute convergence of (6-2) is summarized in Theorem E at the end of this section.

6A. *Reduction to the relative trace formula of Kuznetsov type.*

Lemma 6.1. *Let* $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$. *Let* $\mathrm{K} = \mathrm{K}^{\varphi}$, $\mathrm{K}_0 = \mathrm{K}_0^{\varphi}$ *and* $\mathrm{K}_{\mathrm{Eis}} = \mathrm{K}_{\mathrm{Eis}}^{\varphi}$ *be the corresponding kernel functions. Then*

$$\mathrm{K}_0(x, y) = \sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\big(\mathrm{K}(n\delta x, \delta x) - \mathrm{K}_{\mathrm{Eis}}(n\delta x, \delta x)\big)\,\overline{\theta(n)}\,dn. \tag{6-3}$$

Proof. By the spectral decomposition (2-5) of $\mathrm{K}_0(x, y)$ we see it is cuspidal as a function of x. Applying Proposition 3.1 to the first variable of $\mathrm{K}_0(x, y)$ and take $y = x$ we then obtain

$$\mathrm{K}_0(x,y)=\sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\mathrm{K}_0(n\delta x,x)\,\overline{\theta(n)}\,dn.$$

Then (6-3) follows from the spectral decomposition $\mathrm{K}_0(x,y)=\mathrm{K}(x,y)-\mathrm{K}_{\mathrm{ER}}(x,y)$ and the automorphy of these functions relative to the second variable. Here we also note that the residual spectrum does not contribute to (6-3). □

Let $\mathrm{Re}(s)>1$ in this section. We then plug Lemma 6.1 into

$$I_0(s,\tau)=\int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)E(x,s)\,dx$$

and unfold the Eisenstein series $E(x,s)$ to obtain

$$I_0(s,\tau)=I_{\mathrm{Kl}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau),$$

where

$$I_{\mathrm{Kl}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\int_{[N]}\mathrm{K}(nx,x)\,\overline{\theta(n)}\,dn\,f(x,s)\,dx. \tag{6-4}$$

To establish the well-definedness of $I_{\mathrm{Whi}}(s,\tau)$, it is sufficient to demonstrate the convergence of $I_{\mathrm{Kl}}(s,\tau)$, given the rapid decay of K_0. We aim to show that $I_{\mathrm{Kl}}(s,\tau)$ converges for all $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$ and for $\mathrm{Re}(s)$ sufficiently large. By utilizing Cauchy's inequality and the convolution decomposition of φ, we obtain the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$.

Through a change of variables, we obtain the expression

$$I_{\mathrm{Kl}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}J_{\mathrm{Kuz}}(\varphi,x)f(x,s)\,dx,$$

where

$$J_{\mathrm{Kuz}}(\varphi,x)=\int_{[N]}\int_{[N]}\mathrm{K}(n_1x,n_2x)\theta(n_1)\,\bar{\theta}(n_2)\,dn_1dn_2$$

represents a relative trace formula of Kuznetsov type. Consequently, $I_{\mathrm{Kl}}(s,\tau)$ can be considered as the Mellin transform of $J_{\mathrm{Kuz}}(\varphi,x)$ since $f(x,s)$ essentially corresponds to $|\det x|^s$. To establish the convergence of $I_{\mathrm{Kl}}(s,\tau)$, we will demonstrate that $J_{\mathrm{Kuz}}(\varphi,x)$ is dominated by a gauge when $\mathrm{Re}(s)$ is sufficiently large.

Recall that, for $x=\mathrm{diag}(x_1\cdots x_{n-1},\dots,x_1x_2,x_1,1)\in A(\mathbb{A}_F)=Z(\mathbb{A}_F)\backslash T(\mathbb{A}_F)$, a gauge $\mathcal{G}$ is a positive function of the form

$$\mathcal{G}(x)=\xi(x_1,x_2,\dots,x_{n-1})\cdot|x_1x_2\cdots x_{n-1}|^{-M},$$

where $M\geq 0$ and ξ is a Schwartz–Bruhat function on $(\mathbb{A}_F^\times)^{n-1}$.

Proposition 6.2. *Let notation be as above. Then as a function of $x\in A(\mathbb{A}_F)$, $J_{\mathrm{Kuz}}(\varphi,x)$ is majorized by a finite sum of gauges on $A(\mathbb{A}_F)$.*

Proof. By the definition of the kernel function $\mathrm{K}(x,y)$ we have

$$J_{\mathrm{Kuz}}(\varphi,x)=\int_{[N]}\int_{[N]}\sum_{\gamma\in Z(F)\backslash G(F)}\varphi(x^{-1}n_1^{-1}\gamma n_2x)\,\theta(n_1)\,\bar{\theta}(n_2)\,dn_1dn_2,$$

which converges absolutely since $\mathrm{K}(x, y)$ is continuous and $[N]$ is compact.

Consider the double coset $Z(\mathbb{A}_F)N(F)\backslash G(F)/N(F)$, whose element is of the form wa, where w is a Weyl element and $a \in Z(F)\backslash T(F)$. Let

$$H_{wa} := \left\{(n_1, n_2) \in N \times N : n_1^{-1} wan_2 a^{-1} w^{-1} \in Z\right\}$$

be the stabilizer relative to the representative wa. Then

$$J_{\mathrm{Kuz}}(\varphi, x) = \sum_{wa \in \Phi} \int_{H_{wa}(F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1} n_1^{-1} wan_2 x)\theta(n_1)\overline{\theta}(n_2)\, dn_1 dn_2,$$

where Φ is a set of complete representatives for $Z(\mathbb{A}_F)N(F)\backslash G(F)/N(F)$. Then

$$J_{\mathrm{Kuz}}(\varphi, x) = \sum_{wa \in \Phi} C_{wa} \int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1} n_1^{-1} wan_2 x)\theta(n_1)\overline{\theta}(n_2)\, dn_1 dn_2,$$

where

$$C_{wa} = \int_{[H_{wa}]} \theta(n_1')\overline{\theta}(n_2')\, dn_1'\, dn_2'.$$

Call $wa \in \Phi$ *relevant* if $C_{wa} \neq 0$, i.e., $\theta(n_1')\overline{\theta}(n_2')$ is trivial on $H_{wa}(\mathbb{A}_F)$. Denote by Φ^* the set of relevant elements in Φ. By [16, p. 272, Proposition 1] one can take the following realization: Φ^* consists of wa, where w is the long Weyl element inside a standard parabolic subgroup $Q \subseteq G$ of type $(k_1, \dots, k_r)$, and $a \in Z(F)\backslash \operatorname{diag}(T_{k_1}(F), \dots, T_{k_r}(F))$ (modulo some further relations), with T_{k_j} being the maximal split torus of $\mathrm{GL}(k_j)$. For example, when $Q = B$ the Borel, then $w = I_n$ and $a = I_n$ and $H_{wa} = N$. Therefore,

$$J_{\mathrm{Kuz}}(\varphi, x) = \sum_{wa \in \Phi^*} \mathrm{vol}([H_{wa}]) J_{\mathrm{Kuz}}(\varphi, x; wa),$$

where

$$J_{\mathrm{Kuz}}(\varphi, x; wa) = \int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1} n_1^{-1} wan_2 x)\theta(n_1)\overline{\theta}(n_2)\, dn_1 dn_2.$$

By the definition of Φ^*, each w corresponds to a unique (i.e., the minimal one) parabolic subgroup Q containing w. Suppose $w \neq I_n$. Then by Levi decomposition it suffices to consider the extreme case where $Q = G$ and w is the longest.

Recall that the test function φ is K-finite. Hence there is some compact subgroup $K_0 \subset G(\mathbb{A}_{F,\mathrm{fin}})$ such that φ is right K_0-invariant. Let $K_0 = \prod_{v<\infty} K_{0,v}$. Note that $J_{\mathrm{Kuz}}(\varphi, x; wa) = \prod_{v \le \infty} J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$, where

$$J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) = \int_{H_{wa}(F_v)\backslash N(F_v)\times N(F_v)} \varphi_v(x_v^{-1} n_1^{-1} wan_2 x)\theta_v(n_1)\overline{\theta}_v(n_2)\, dn_1 dn_2.$$

Then for each finite place v, $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$ is right $K_{0,v}$-invariant. So there exists a compact subgroup $N_{0,v} \subseteq K_{0,v} \cap N(F_v)$, depending only on φ_v, such that

$$J_{\mathrm{Kuz},v}(\varphi_v, x_v u_v; wa) = J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) \quad \text{for all } x_v \in A(F_v) \text{ and } u_v \in N_{0,v}.$$

On the other hand, $J_{\mathrm{Kuz},v}(\varphi_v, x_v u_v; wa) = \theta(x_v u_v x_v^{-1}) J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ such that $\theta(x_v u_v x_v^{-1}) = 1$ if and only if $|\alpha_i(x_v)|_v \le C_v$, where α_i's are the simple roots of $G(F)$ relative to B. Note that for all but finitely many $v < \infty$, $K_{0,v} = G(\mathcal{O}_{F,v})$. Thus we can take the corresponding $C_v = 1$. Hence for any $x_v \in A(F_v)$, $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) \ne 0$ implies that $|\alpha_i(x_v)|_v \le C_v$, $1 \le i \le n-1$, and $C_v = 1$ for all but finitely many finite places v. Denote the compact set by

$$A_{\varphi,\mathrm{fin}} = \{a = (a_v) \in A(\mathbb{A}_{F,\mathrm{fin}}) : |\alpha_i(a_v)|_v \le C_v,\ 1 \le i \le n-1\}.$$

Then $\operatorname{supp} J_{\mathrm{Kuz}}(\varphi, x; wa) \subseteq A(\mathbb{A}_{F,\infty}) A_{\varphi,\mathrm{fin}}$.

For each place v, we fix a conventional local height function $\|\cdot\|_v$ on $G(F_v)$. Let $y = \bigotimes_v (y_{i,j,v}) \in G(\mathbb{A}_F)$. Then $\|y_v\|_v = 1$ for almost all v. The height function $\|y\| = \prod_v \|y_v\|_v$ is therefore well defined by a finite product. Also, since

$$\operatorname{supp} J_{\mathrm{Kuz}}(\varphi, x; wa) \subseteq A(\mathbb{A}_{F,\infty}) A_{\varphi,\mathrm{fin}}$$

and by the compactness of $\operatorname{supp} \varphi_v$, we have $\|w^{-1} x_v w x_v a\|_v \le C_v'$ for some constant C_v' depending only on φ_v, $v < \infty$, and $C_v' = 1$ for almost all v's.

Now we investigate the archimedean $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$, i.e., $v \mid \infty$. Note that φ_v is a compactly supported on $Z(F_v)\backslash G(F_v)$. Then $J_{\mathrm{Kuz}}(\varphi_v, x_v; wa) = 0$ unless $n_{1,v}^{-1} y_v w n_{2,v} \in \operatorname{supp} \varphi_v$, where $y_v = x_v^{-1} w a x_v w^{-1}$. Hence $\|n_{1,v}^{-1} y_v w n_{2,v} w^{-1}\|_v \le C_v$ for some constant C_v depending only on φ. A straightforward computation (or Lemma 5.1 of [15]) shows that $\|n_{1,v}\|_v + \|n_{2,v}\|_v + \|y_v\|_v \le C_v'$ for some constant C_v' depending only on φ. So $\varphi_v(n_{1,v} y_v w n_{2,v})$ has compact support relative to $n_{1,v}$ and $n_{2,v}$. Therefore, $J_{\mathrm{Kuz}}(\varphi_v, x_v; wa) = 0$ unless $n_{1,v}$, $n_{2,v}$ run through a compact set of $N(F_v)$ and $\|y\|_v$ is bounded.

For $x = \operatorname{diag}(x_1, \dots, x_{n-1}, 1) \in A(\mathbb{A}_F)$, similarly to (3-1), we define an additive character

$$\psi_x(u) = \prod_{i=1}^{n-1} \psi_F(x_i u_{i,i+1}) \qquad \text{for } u = (u_{i,j})_{n\times n} \in N(\mathbb{A}_F).$$

Then $J_{\mathrm{Kuz}}(\varphi, x; wa)$ is equal to

$$\delta_w(x)^2 \int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(n_1^{-1} x^{-1} w a x n_2) \psi_x(n_1) \overline{\psi}_x(n_2)\, dn_1\, dn_2,$$

where δ_w is the modular character of the parabolic subgroup associated to w.

Since n_1 and n_2 lie in a compact set determined by $\operatorname{supp} \varphi$, then for a fixed $y \in A(\mathbb{A}_F)$, $v \mid \infty$, the v-th component of

$$\int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(n_1^{-1} y w n_2) \psi_x(n_1) \overline{\psi}_x(n_2)\, dn_1 dn_2$$

is a Schwartz function of x, being the Fourier transform of a compactly supported smooth function. Hence, $J_{\mathrm{Kuz}}(\varphi, x; wa)$ is majorized by a nonnegative Schwartz–Bruhat function $\Xi(x^{-1} w a x w^{-1}, x)$ on $A(\mathbb{A}_F)^2$ with

$$x^{-1} w a x w^{-1} \in A^* := \{b \in A(F) :\ \|b\| \le \prod_v C_v'\}.$$

By properties of the height $\|\cdot\|$ (see [5, p. 70], for instance), one has

$$\#(w^{-1}x\cdot A^*\cdot wx^{-1})\leq C\cdot\big(|x_1\cdots x_{n-1}|^M+|x_1\cdots x_{n-1}|^{-M}\big),$$

for some constants C and M depending on $\operatorname{supp}\varphi$. Therefore,

$$\sum_{a\in A(F)}|J_{\mathrm{Kuz}}(\varphi,x;wa)|\leq\sum_{a\in A^*}\Xi(w^{-1}xwxa,x)=\sum_{a\in w^{-1}x\cdot A^*\cdot wx^{-1}}\Xi(a,x),$$

which is majorized by $|x_1\cdots x_{n-1}|^{-M}\cdot\xi(x_1,\dots,x_{n-1})$ for some $M\geq 0$ and Schwartz–Bruhat function ξ.

The remaining case is that of $w=I_n$, i.e., $Q=B$. In this case

$$J_{\mathrm{Kuz}}(\varphi,x;wa)=\delta_w(x)\int_{N(\mathbb{A}_F)}\varphi(an)\overline{\psi}_x(n)\,dn$$

is the Fourier transform of a Schwartz–Bruhat function. So it is majorized by a gauge. Then Proposition 6.2 follows. □

As a consequence of Proposition 6.2 and the Iwasawa decomposition, $I_{\mathrm{Kl}}(s,\tau)$ converges absolutely when $\mathrm{Re}(s)$ is large enough. Therefore, $I_{\mathrm{Whi}}(s,\tau)$ converges when $\mathrm{Re}(s)$ is large enough.

To show the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ and thus to obtain meromorphic continuation, we need to analyze properties of $\mathrm{K}_{\mathrm{Eis}}$ by its spectral expansion.

6B. ***Spectral decomposition of the kernel function.*** In this subsection, we provide a concise overview of the spectral theory concerning automorphic representations of reductive groups. Subsequently, we apply these results to the noncuspidal kernel function K_{ER}. *The notation introduced in this section will be regularly employed in later discussions.*

Denote by H a general reductive group and Q a standard parabolic subgroup of H. Let M_Q (resp. N_Q) be the Levi component (resp. unipotent radical) of Q. Let $H^1(\mathbb{A}_F)=\{g\in H(\mathbb{A}_F):|\lambda(g)|_{\mathbb{A}_F}=1,\ \forall\lambda\in X(H)_F\}$, where $X(H)_F$ is space set of F-rational characters of H. Let $\mathfrak{a}_H=\mathrm{Hom}_{\mathbb{Z}}(X(H)_F,\mathbb{R})$. Let $\mathfrak{a}_H^*=X(H)_F\otimes\mathbb{R}$. Set $\mathfrak{a}_Q=\mathfrak{a}_{M_Q}$ and $\mathfrak{a}_Q^*=\mathfrak{a}_{M_Q}^*$. Let P_0 be a fixed minimal parabolic subgroup of H over F. Write $\mathfrak{a}_0$ (resp. $\mathfrak{a}_0^*$) for $\mathfrak{a}_{P_0}$ (resp. $\mathfrak{a}_{Q_0}^*$). These notations concur with those used by Arthur (see, e.g., [5, pp. 20–31].

For a standard parabolic subgroup Q of $G=\mathrm{GL}(n)$ with type $(n_1,\dots,n_r)$, we define the surjective homomorphism $\log_Q$ from $M_Q(\mathbb{A}_F)$ to $\mathfrak{a}_Q$ as follows:

$$\log_Q(q)=\log_Q(m)=(n_1^{-1}\log|\det m_1|,\dots,n_r^{-1}\log|\det m_l|),\tag{6-5}$$

where $q\in Q(\mathbb{A}_F)$ and $m=\mathrm{diag}(m_1,\dots,m_r)$ represents the Levi component, with $m_i\in\mathrm{GL}(n_i)/\mathbb{A}_F$ for $1\leq i\leq r$.

By spectral theory (see, for example, [2, pp. 256 and 263], or [5, §12]), the decomposition of the Hilbert space $L^2\big(Z_H(\mathbb{A}_F)N_Q(\mathbb{A}_F)M_Q(F)\backslash H(\mathbb{A}_F)\big)$ into right $H(\mathbb{A}_F)$-invariant subspaces is determined by the spectral data $\chi=\{(M,\sigma)\}$, where M is the Levi component of $P_1\cap M_Q$ for some standard parabolic subgroup P_1 of H, and σ is an element of $\mathcal{A}_0\big(Z_H(\mathbb{A}_F)M^1(F)\backslash M^1(\mathbb{A}_F)\big)$, the set of cuspidal automorphic representations of $Z_H(\mathbb{A}_F)M^1(F)\backslash M^1(\mathbb{A}_F)$. Here M^1 is defined in a similar way to H^1.

The class (M,σ) derives from the equivalence relation $(M,\sigma)\sim(M',\sigma')$ if and only if M is conjugate to M' by a Weyl group element w, and $\sigma'=\sigma^w$ on $Z_H(\mathbb{A}_F)\backslash M^1(\mathbb{A}_F)$. Let $\mathfrak{X}$ be the set of equivalence classes $\chi=\{(M,\sigma)\}$ of these pairs, we thus have

$$L^2(P):=L^2\big(Z_H(\mathbb{A}_F)N_Q(\mathbb{A}_F)M_Q(F)\backslash H(\mathbb{A}_F)\big)=\bigoplus_{\chi\in\mathfrak{X}}L^2(P)_\chi, \tag{6-6}$$

where $L^2(P)_\chi$ consists of functions $\phi\in L^2\big(Z_H(\mathbb{A}_F)N_Q(\mathbb{A}_F)M_Q(F)\backslash H(\mathbb{A}_F)\big)$ such that, for each standard parabolic subgroup Q' of H with $Q'\subset Q$, and almost all $x\in H(\mathbb{A}_F)$, the projection of the function

$$m\mapsto x.\phi_{Q'}(m)=\int_{N_{Q'}(F)\backslash N_{Q'}(\mathbb{A}_F)}\phi(nmx)\,dn$$

onto the space of cusp forms in $L^2\big(Z_H(\mathbb{A}_F)M_{Q'}(F)\backslash M^1_{Q'}(\mathbb{A}_F)\big)$ transforms under $M^1_{Q'}(\mathbb{A}_F)$ as a sum of representations σ, in which $(M_{Q'},\sigma)\in\chi$. If there is no such pair in χ, $x.\phi_{Q'}$ will be orthogonal to $\mathcal{A}_0\big(Z_H(\mathbb{A}_F)M_{Q'}(F)\backslash M^1_{Q'}(\mathbb{A}_F)\big)$. Denote by $\mathcal{H}_Q$ the space of such ϕ's. Let $\mathcal{H}_{Q,\chi}$ be the subspace of $\mathcal{H}_Q$ such that for any $(M,\sigma)\notin\chi$, with $M=M_{Q_1}$ and $Q_1\subset Q$, we have

$$\int_{M(F)\backslash M(\mathbb{A}_F)^1}\int_{N_{Q_1}(F)\backslash N_{Q_1}(\mathbb{A}_F)}\psi_0(m)\phi(nmx)\,dn=0,$$

for any $\psi_0\in L^2_0\left(M(F)\backslash M(\mathbb{A}_F)^1\right)_\sigma$, and almost all x. This leads us to Langlands' result to decompose $\mathcal{H}_Q$ as an orthogonal direst sum $\mathcal{H}_Q=\bigoplus_{\chi\in\mathfrak{X}}\mathcal{H}_{Q,\chi}$. Let $\mathcal{B}_Q$ be an orthonormal basis of $\mathcal{H}_Q$, then we can choose $\mathcal{B}_Q=\bigcup_{\chi\in\mathfrak{X}}\mathcal{B}_{Q,\chi}$, where $\mathcal{B}_{Q,\chi}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{Q,\chi}$. We may assume that vectors in each $\mathcal{B}_{Q,\chi}$ are K-finite and are pure tensors.

By spectral theory, one can expand $\mathrm{K}_{\mathrm{Eis}}(x,y)$ as

$$\sum_{\chi\in\mathfrak{X}}\sum_{Q\in\mathcal{Q}}\frac{1}{k_Q!(2\pi)^{k_Q}}\int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{Q,\chi}}E(x,\mathcal{I}_Q(\lambda,\varphi)\phi,\lambda)\overline{E(y,\phi,\lambda)}\,d\lambda, \tag{6-7}$$

where $\mathcal{Q}$ is the set of standard parabolic subgroups which are not G; and for any such Q, k_Q is the number of blocks of the Levi part of Q. Also, (6-7) converges absolutely [2, Lemma 2, p.263].

Let $\phi_2\in\mathfrak{B}_{Q,\chi}$. Then $\mathcal{I}_Q(\lambda,\varphi)\phi_2$ can be expanded by a linear combination of vectors in $\mathfrak{B}_{Q,\chi}$. As a consequence,

$$E(x,\mathcal{I}_Q(\lambda,\varphi)\phi_2,\lambda)=\sum_{\phi_1\in\mathfrak{B}_{Q,\chi}}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle E(x,\phi_1,\lambda).$$

Since φ is K-finite, then $\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\equiv 0$ for all but finitely many $\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}$, depending only on the K-type of φ.

For $1\le j\le 2$, $\phi_j\in\mathfrak{B}_{Q,\chi}$, and $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, we define the Whittaker function

$$W_j(x,\lambda)=W(x,\phi_j,\lambda):=\int_{N(\mathbb{A}_F)}\phi_j(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn, \tag{6-8}$$

where w_0 is the long element in the Weyl group W_n, and $\log_Q$ is defined by (6-5).

6C. ***Spectral expansion of $I_{\mathrm{Whi}}(s,\tau)$.*** By definition, we have

$$I_{\mathrm{Whi}}(s,\tau)=\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)f(x,s)\,dx.$$

where

$$\widehat{\mathrm{K}}_{\mathrm{ER}}(x,y):=\int_{[N]}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(n_1x,n_2y)\theta(n_1)\bar{\theta}(n_2)\,dn_1dn_2 \tag{6-9}$$

Set $X_G=Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ and $c_Q=k_Q!(2\pi)^{k_Q}$. One can unfold the Eisenstein series (see [31, pp. 123–124]) to rewrite (at least formally) the function $I_{\mathrm{Whi}}(s,\tau)$ as

$$\int_{X_G}\sum_{\chi\in\mathfrak{X}}\sum_{Q\in\mathcal{Q}}\frac{1}{c_Q}\int\sum_{\phi_1,\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}\,d\lambda\; f(x,s)\,dx, \tag{6-10}$$

where $\phi_i\in\mathfrak{B}_{Q,\chi}$, $1\le i\le 2$, and $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$.

Theorem E. *Let notation be as before. Then there exists a constant $c_\varphi>0$ depending only on φ such that $I_{\mathrm{Whi}}(s,\tau)$ converges for* $\mathrm{Re}(s)>c_\varphi$. *Moreover, when* $\mathrm{Re}(s)>c_\varphi$, *$I_{\mathrm{Whi}}(s,\tau)$ is equal to*

$$\sum_{\chi}\sum_{Q\in\mathcal{Q}}\frac{1}{c_Q}\int\sum_{\phi_1,\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\int_{X_G}W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)\,dx\,d\lambda, \tag{6-11}$$

where χ runs over the proper cuspidal data, i.e., χ is not of the form $\{(G,\pi)\}$, $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, and $\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}$. Particularly, as a function of s, $I_{\mathrm{Whi}}(s,\tau)$ is analytic in the right half plane $\{z:\mathrm{Re}(z)>c_\varphi\}$.

Proof. For $x\in Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ we write it into the Iwasawa coordinates: $x=ak$, where $a\in A(\mathbb{A}_F)$ and $k\in K$. Then

$$f(x,s):=f(x,\Phi,\tau;s)=\tau(\det a)\,|\det a|^s\int_{\mathbb{A}_F^\times}\Phi(\eta tk)\tau(t)^n|t|^{ns}d^\times t.$$

Therefore, $|f(x,s)|=|\det a|^{\mathrm{Re}(s)}h(k,s)$, where

$$h(k,s):=\left|\int_{\mathbb{A}_F^\times}\Phi(\eta tk)\tau(t)^n|t|^{ns}d^\times t\right|$$

is a nonnegative continuous function of k and converges absolutely when $\mathrm{Re}(s)>1/n$. Let $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$. Then by Proposition 6.2,

$$\int_{X_G}J_{\mathrm{Kuz}}(\varphi,x)|f(x,s)|\,dx=\int_K\int_{A(\mathbb{A}_F)}J_{\mathrm{Kuz}}(\varphi,ak)\,|\det a|^{\mathrm{Re}(s)}\delta^{-1}(a)\,d^\times a h(k,s)\,dk$$

converges when $\mathrm{Re}(s)$ is large. By Lemma 6.1 we have

$$J(\varphi,s):=\int_{X_G}\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)|f(x,s)|\,dx=\int_{X_G}J_{\mathrm{Kuz}}(\varphi,x)\cdot|f(x,s)|\,dx-J_0(\varphi,s),$$

where $\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)$ is defined by (6-9), and

$$J_0(\varphi,s)=\int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)\sum_{\delta\in P(F)\backslash G(F)}|f(\delta x,s)|\,dx.$$

Since the series $\sum_{\delta\in P(F)\backslash G(F)}|f(\delta x,s)|$ is slowly increasing and $\mathrm{K}_0(x,x)$ is rapidly decaying on

$Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$, then $J_0(\varphi, s)$ converges absolutely. Hence $J(\varphi, s)$ converges and is well defined.

Consider test functions of the form $\varphi_0 * \varphi_0^*$, where $\varphi_0^*(x) = \overline{\varphi_0(x^{-1})}$. Substituting the spectral expansion (6-7) into $J(\varphi, s)$, which is convergent, gives

$$\int_{X_G} \sum_{\chi} \sum_{Q\in\mathcal{Q}} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \sum_{\phi\in\mathfrak{B}_{Q,\chi}} \left|W(x;\mathcal{I}_Q(\lambda,\varphi_0)\phi,\lambda)\right|^2 |f(x,s)|\,d\lambda\,dx < \infty, \tag{6-12}$$

where χ ranges over the proper cuspidal data, and

$$W(x;\mathcal{I}_Q(\lambda,\varphi_0)\phi,\lambda) = \int_{N(\mathbb{A}_F)} \big(\mathcal{I}_Q(\lambda,\varphi_0)\phi\big)\,(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn,$$

with w_0 being the long element in the Weyl group W_n.

For an arbitrary test function $\varphi \in \mathcal{H}(G(\mathbb{A}_F),\omega)$, by the factorization theorem of Dixmier and Malliavin, one can write φ as a finite linear combination of convolutions $\varphi_{j,1} * \varphi_{j,2}^*$ with functions $\varphi_{j,i} \in C_c^r(G(\mathbb{A}_F))$, whose archimedean components are differentiable of arbitrarily high order r, $1 \le i \le 2$, and $j \in J$ is a finite set. Using the triangle and Cauchy–Schwarz inequalities, along with (6-12), we derive

$$\begin{aligned}
&\sum_{\chi\in\mathfrak{X}} \sum_{Q\in\mathcal{Q}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \int_{X_G} \Big| \sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}} \langle \mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)} f(x,s)\Big|\,dx\,d\lambda \\
&\qquad \le \sum_{j\in J} \prod_{i=1}^{2} \Big(\sum_{\chi\in\mathfrak{X}} \sum_{Q\in\mathcal{Q}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \int_{X_G} \sum_{\phi\in\mathfrak{B}_{Q,\chi}} |W_{j,i}(x;\lambda)|^2 \cdot \big|f(x,s)\big|\,dx\,d\lambda\Big)^{1/2} < \infty,
\end{aligned}$$

where $W_{j,i}(x;\lambda) = W(x;\mathcal{I}_P(\lambda,\varphi_{j,i})\phi,\lambda)$, for any $1 \le i \le 2$, and $j \in J$. This proves the first part of Theorem E and provides an expression for $I_{\mathrm{Whi}}(s,\tau)$ as

$$\sum_{\chi} \sum_{Q\in\mathcal{Q}} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} \int_{X_G} \sum_{\phi_1,\phi_2} \langle \mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)} f(x,s)\,dx\,d\lambda. \tag{6-13}$$

For $\mathrm{Re}(s)$ large, we have

$$\int_{X_G} \big|W_1(x;\lambda)\overline{W_2(x;\lambda)} f(x,s)\big|\,dx < \infty.$$

Recall that $\langle \mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle \equiv 0$ for all but finitely many $\phi_1,\phi_2 \in \mathfrak{B}_{Q,\chi}$. Thus,

$$\sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}} \big|\langle \mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\big| \int_{X_G} \big|W_1(x;\lambda)\overline{W_2(x;\lambda)} f(x,s)\big|\,dx < \infty. \tag{6-14}$$

Therefore, we obtain (6-11) from (6-13) and (6-14). □

Remark 6.4. If the base field F is a function field, it has no archimedean places. Then the support of $W_j(x;\lambda)\,|_{A(\mathbb{A}_F)}$ is contained in $A_{\varphi,\mathrm{fin}}$ for all $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, $1 \le j \le 2$, which means the support of $\widehat{\mathrm{K}}_{\mathrm{ER}}(x,x)$ is compact. Also, in the function field case the sum over the χ's is only finite. Therefore, Theorem E is clear.

7. Rankin–Selberg convolutions for generic representations

By Theorem E, we see that when $\mathrm{Re}(s)$ is large, $I_{\mathrm{Whi}}(s,\tau)$ is equal to

$$I_{\mathrm{Whi}}(s,\tau)=\sum_{\chi}\sum_{Q\in\mathcal{Q}}\frac{1}{c_Q}\int\sum_{\phi_1,\phi_2}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_2,\phi_1\rangle\Psi_{Q,\chi}(s,W_1,W_2;\lambda)\,d\lambda, \tag{7-1}$$

where $\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}$, $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, and

$$\Psi_{Q,\chi}(s,W_1,W_2;\lambda)=\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)\,dx.$$

Here the Whittaker function $W_j(x,\lambda)$ has been defined by

$$W_j(x,\lambda)=\int_{N(\mathbb{A}_F)}\phi_j(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn,\ 1\le j\le 2. \tag{6-8}$$

Our objective is to establish the meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to $\mathbb{C}$ and demonstrate that the quotient $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ is holomorphic for $s\neq 0,1$. To achieve this, we initiate the process by calculating each $\Psi_{Q,\chi}(s,W_1,W_2;\lambda)$ associated with a standard parabolic subgroup Q and a cuspidal datum $\chi=(M_Q,\sigma)\in\mathfrak{X}$.

Here is the arrangement of this section:

- In Section 7A, we recall some notation from Section 6B–Section 6C and introduce new notation regarding induced representations.
- In Sections 7B and 7C, we extend the local and global investigation of the Rankin–Selberg convolution $\Psi_{Q,\chi}(s,W_1,W_2;\lambda)$, respectively. In particular, we explicitly compute it at the unramified places. This generalizes the work of [17] as we elucidate the dependence on the spectral parameter λ, which is crucial for the meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$.
- In Section 7C, we combine the analysis in the previous sections and utilize the analytic properties of the period integrals to achieve a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ beyond the region of absolute convergence.

7A. ***Some notation.*** Let Q be a standard parabolic subgroup of $G=\mathrm{GL}(n)$ of type $(n_1,\dots,n_r)$. Let $\chi=(M_Q,\sigma)\in\mathfrak{X}$ be a cuspidal datum, where σ is a unitary automorphic representation of M of central character ω. Let $\mathfrak{B}_{Q,\chi}$ be an orthonormal basis of the Hilbert space $\mathcal{H}_{Q,\chi}$ defined in Section 6B.

By definition, there exist r cuspidal representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, $1\le i\le r$, such that $\sigma\simeq\pi_1\boxplus\pi_2\boxplus\cdots\boxplus\pi_r$. Let $\pi=\mathrm{Ind}_{Q(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\pi_1,\pi_2\ \dots,\pi_r)$. For any $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, set

$$\pi_\lambda=\mathrm{Ind}_{Q(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\big(\pi_1\otimes|\cdot|_{\mathbb{A}_F}^{\lambda_1},\pi_2\otimes|\cdot|_{\mathbb{A}_F}^{\lambda_2},\dots,\pi_r\otimes|\cdot|_{\mathbb{A}_F}^{\lambda_r}\big). \tag{7-2}$$

Then π_λ is also a unitary automorphic representation of $G(\mathbb{A}_F)$.

For $\phi_1,\phi_2\in\mathcal{B}_{Q,\chi}$ and a point $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, recall that

$$W_j(x,\lambda)=W(x,\phi_j,\lambda):=\int_{N(\mathbb{A}_F)}\phi_j(w_0nx)e^{(\lambda+\rho_Q)\log_Q(w_0nx)}\overline{\theta(n)}\,dn, \tag{6-8}$$

where w_0 is the long Weyl element, and $\log_Q$ is defined by (6-5). Define

$$\Psi(s, W_1, W_2; \lambda, \Phi) = \int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x;\lambda)\overline{W_2(x;\lambda)} f(x,s)\, dx. \tag{7-3}$$

Since $W_1(x;\lambda)$ and $W_2(x;\lambda)$ are dominated by some gauge, and $f(x,s)$ increases slowly when $\mathrm{Re}(s) > 1$, then $\Psi(s, W_1, W_2; \lambda, \Phi)$ converges absolutely and normally when $\mathrm{Re}(s)$ is large. For each $v \in \Sigma_F$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, denote by $\pi_v = \mathrm{Ind}_{M_Q(F_v)}^{G(F_v)}(\pi_{1,v}, \pi_{2,v}, \dots, \pi_{r,v})$ and

$$\pi_{\lambda,v} = \mathrm{Ind}_{M_Q(F_v)}^{G(F_v)}\big(\pi_{1,v} \otimes |\cdot|_{F_v}^{\lambda_1}, \pi_{2,v} \otimes |\cdot|_{F_v}^{\lambda_2}, \dots, \pi_{r,v} \otimes |\cdot|_{F_v}^{\lambda_r}\big).$$

Then $\pi = \bigotimes_v' \pi_v$ and $\pi_\lambda = \bigotimes_v' \pi_{\lambda,v}$. Recall that $f(x,s) = \prod_v f_v(x_v, s)$, where

$$f_v(x_v, s) = \tau_v(\det x_v)\, |\det x_v|_{F_v}^s \int_{Z(F_v)} \Phi_v[(0, \dots, t_v)x_v]\tau_v^n(t_v)|t_v|_{F_v}^{ns}\, d^\times t_v,$$

if $\Phi = \bigotimes_v' \Phi_v$. We can rewrite $\Psi(s, W_1, W_2; \lambda, \Phi)$ as

$$\int_{N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x;\lambda)\overline{W_2(x;-\bar{\lambda})}\Phi(\eta x)\tau(\det x)\, |\det x|_{\mathbb{A}_F}^s\, dx, \tag{7-4}$$

where $\eta = (0, \dots, 0, 1) \in F^n$. Factor $W_j(x;\lambda)$ as $\prod_{v \in \Sigma_F} W_{j,v}(x_v;\lambda)$, where

$$W_{j,v}(x_v;\lambda) = \int_{N(F_v)} \phi_{j,v}(w_0 n x_v) e^{(\lambda+\rho_Q)\log_Q(w_0 n x_v)}\overline{\theta(n)}\, dn, \ 1 \le j \le 2,$$

with $\phi_{j,v}$ being a local vector in the space of $\pi_{j,v}$.

We may assume $\Phi = \bigotimes_v' \Phi_v$. Then

$$\Psi(s, W_{1,v}, W_{2,v}; \lambda, \Phi) = \prod_{v \in \Sigma_F} \Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v),$$

where each local factor $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is defined to be

$$\int_{N(F_v)\backslash G(F_v)} W_{1,v}(x_v;\lambda)\overline{W_{2,v}(x_v;-\bar{\lambda})}\Phi_v(\eta x_v)\tau(\det x_v)\, |\det x_v|_{F_v}^s\, dx_v. \tag{7-5}$$

Here $W_{j,v}(x_v;\lambda) = \int_{N(F_v)} \phi_{j,v}(w_0 n x) e^{(\lambda+\rho_Q)H_{Q,v}(w_0 n x)}\overline{\theta(n)}\, dn$, $1 \le j \le 2$.

7B. ***Local theory for $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$.*** In this section, we shall study each local integral representation $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ defined via (7-5).

Let $v \in \Sigma_F$ be a fixed nonarchimedean place, and let $\widetilde{\pi}_{\lambda,v}$ be the contragredient of $\pi_{\lambda,v}$. Let ϖ_v be a uniformizer of $\mathcal{O}_{F,v}$, the ring of integers of F_v. Let $q_v = N_{F_v/\mathbb{Q}_p}(\varpi_v)$, where p is the rational prime such that v is above p. Set

$$R_v(s, W_{1,v}, W_{2,v}; \lambda) := \frac{\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1. \tag{7-6}$$

Proposition 7.1 (nonarchimedean case). *Let notation be as before. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have*:

(a) *$R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \le i \le r\}$.*

(b) *The local functional equation*

$$\frac{\Psi_v(1-s,\widetilde{W}_{1,v},\widetilde{W}_{2,v};-\bar{\lambda},\widehat{\Phi}_v)}{L_v(1-s,\widetilde{\pi}_{-\bar{\lambda},v}\otimes\bar{\tau}_v\times\pi_{\lambda,v})}=\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)\cdot\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})}$$

holds, where $\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)$ *is a polynomial in* $\{q_v^s,q_v^{-s},q_v^{\lambda_i},q_v^{-\lambda_i}:1\le i\le r\}$.

Proof. We prove part (a); part (b) will follow form [17].

Let $T(F_v)$ be the maximal split torus of $G(F_v)$. For $m\in\mathbb{Z}$, let $T^{(m)}(F_v)=\{t\in T(F_v):|\det t|_{F_v}=q_v^{-m}\}$. Using Iwasawa decomposition and the fact that $W_{i,v}$ and Φ_v are right $G(\mathcal{O}_{F,v})$-finite, we can rewrite $\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ as

$$\sum_{j\in J}\int_{T(F_v)}W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}\Phi_{j,v}(\eta a_v)\tau_v(\det a_v)\delta^{-1}(a_v)\,|\det a_v|_{F_v}^s\,da_v,$$

where the sum is over a finite set J, $W_{i,v}^{(j)}(a_v;\lambda)$ is the Whittaker function associated to some smooth functions in $\mathcal{H}_{Q,\chi}$, $1\le i\le 2$, and $\Phi_{j,v}$ is some Schwartz–Bruhat function.

For $1\le i\le 2$ and $j\in J$, $W_{i,v}^{(j)}(x_v;\lambda)$ is right $G(\mathcal{O}_{F,v})$-finite. So there exists a compact subgroup $N_{0,v}\subseteq G(\mathcal{O}_{F,v})\cap N(F_v)$, depending only on φ, such that $W_{i,v}^{(j)}(t_vu_v;\lambda)=W_{i,v}^{(j)}(t_v;\lambda)$, for all $t_v\in T(F_v)$ and $u_v\in N_{0,v}$. On the other hand, $W_{i,v}^{(j)}(t_vu_v;\lambda)=\theta_{t_v}(u_v)W_{i,v}^{(j)}(t_v;\lambda)$, where $\theta_{t_v}(n_v)=\theta(t_vn_vt_v^{-1})$, for any $n_v\in N(F_v)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ (hence not on λ) such that $\theta_{t_v}(u_v)=1$ if and only if $|\alpha_i(t_v)|\le C_v$, where α_i's are the simple roots of $G(F)$. Thus each $W_{i,v}^{(j)}(x_v;\lambda)$ is compactly supported for a fixed $\lambda\in i\mathfrak{a}_Q/i\mathfrak{a}_G$. Therefore, for a fixed λ, $\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ is a formal Laurent series in q_v^{-s}. Indeed, one can chose some nonnegative integer M independent of λ (but depending on π and φ), such that

$$\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)=\sum_{m\ge -M}\Psi_v^{(m)}(W_{1,v},W_{2,v};\lambda,\Phi_v)\cdot q_v^{-ms},$$

where $\Psi_v^{(m)}(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ is defined by the integral

$$\sum_{j\in J}\int_{T^{(m)}(F_v)}W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}\Phi_{j,v}(\eta a_v)\tau_v(\det a_v)\delta^{-1}(a_v)\,da_v.$$

Applying the above analysis on supp $W_{i,v}(a_v;\lambda)$, we see similarly that

$$\operatorname{supp}W_{i,v}^{(j)}(a_v;\lambda)\subseteq\left\{t\in T^{(m)}(F_v):|\alpha_l(t)|_{F_v}\le C_v^{(j)},\ 1\le l\le n-1\right\}$$

for some constants $C_v^{(j)}$. Hence, for each $j\in J$, $m\ge -N$ and $a_v\in T^{(m)}(F_v)$, the function

$$a_v\mapsto W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}$$

is analytic and is a formal Laurent series in $\{q_v^{-\lambda_i}:1\le i\le r\}$ by (2.5.2) of [17], and the function

$$a_v\mapsto W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}(a_v;-\bar{\lambda})}\Phi_{j,v}(\eta a_v)\tau(\det a_v)\delta^{-1}(a_v)$$

is locally constant. Therefore, $\Psi_v^{(m)}(W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is an analytic function of λ and is a formal Laurent series in $\{q_v^{-\lambda_i} : 1 \le i \le r\}$.

Since $\pi_{\lambda,v}$ is of Whittaker type, we can use Theorem 2.7 of [17] to see that, for fixed $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) \cdot L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})^{-1}$ is a polynomial in $\{q_v^s, q_v^{-s}\}$ with coefficients functions of λ. Moreover, $L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})^{-1}$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \le i \le r\}$. So we can write

$$L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{\lambda,v})^{-1} = \sum_{|l| \le N} Q_l(\lambda) q_v^{-ls},$$

where N is a positive integer and $Q_l(\lambda)$ are polynomials in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \le i \le r\}$. Then for $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) \cdot L(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})^{-1}$ is equal to the sum over $m \ge -N-M$ of $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) q_v^{-ms}$, where

$$R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = \sum_{\substack{i+j=m \\ |i| \le N, j \ge -M}} Q_i(\lambda) \Psi_v^{(m)}(W_{1,v}, W_{2,v}; \lambda, \Phi_v).$$

Since the sum on the right is finite, $R_l(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is analytic in λ. Moreover, it is a formal Laurent series in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \le i \le r\}$. Therefore, Proposition 7.1(a) follows from the next claim:

Claim 7.2. *There exists some $M_0 \in \mathbb{Z}$, independent of $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, such that $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ vanishes for all $m \ge M_0$ and for all $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$. For each $m \in \mathbb{Z}$, $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is a polynomial in $\{q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \le i \le r\}$.*

Proof of Claim 7.2. Let $l \in \mathbb{Z}$. We define

$$\Lambda_l := \{\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G : R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0 \text{ for all } m \ge l\}.$$

Each Λ_l is closed as the function $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is analytic (hence continuous) in λ. Since

$$R_v(s, \lambda) = \sum_m R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) q_v^{-ms} \in \mathbb{C}[q_v^s, q_v^{-s}]$$

for fixed $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, there exists some $M(\lambda)$ such that $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0$ as long as $m \ge M(\lambda)$. Therefore, $i\mathfrak{a}_Q/i\mathfrak{a}_G$ is covered by the union of all Λ_l.

Noting that $i\mathfrak{a}_Q/i\mathfrak{a}_G \simeq R^{r-1}$ is a Banach space, by Baire category theorem there exists some Λ_{l_0} having nonempty interior, $\mathrm{Int}(\Lambda_{l_0})$, say. Thus, for any $\lambda \in \mathrm{Int}(\Lambda_{l_0})$, $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0$ for any $m \ge l_0$. Since $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ is analytic for any $l \in \mathbb{Z}$, $R_m(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = 0$ for all $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$, proving the first part. For the remaining part, we consider the functional equation (see [17]):

$$\frac{\Psi_v(1-s, \widetilde{W}_{1,v}, \widetilde{W}_{2,v}; -\bar{\lambda}, \widehat{\Phi}_v)}{L_v(1-s, \widetilde{\pi}_{-\bar{\lambda},v} \otimes \bar{\tau}_v \times \pi_{\lambda,v})} = \varepsilon_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta_v) \cdot \frac{\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})},$$

where $\varepsilon_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta_v)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \le i \le r\}$.

We can interpret the functional as an identity between formal Laurent series in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. The left side is of the form $\sum_{m_1 \geq -M_1} q_v^{m_1\lambda_i}$, while the right side is of the form $\sum_{m_2 \geq -M_2} q_v^{-m_2\lambda_i}$. Since they are equal, they must both be polynomials in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$. This proves Claim 7.2, and with it Proposition 7.1. □

One will see that Proposition 7.1 is insufficient for our continuation in next few sections. Hence we need to compute

$$R_v(s, W_{1,v}, W_{2,v}; \lambda) := \frac{\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1, \tag{7-6}$$

more explicitly. We will do principal series case below since this is the only case we need for the meromorphic continuation of Rankin–Selberg convolutions– it suffices to consider the partial Euler factors which corresponds to the principal series at all but finitely many places.

Lemma 7.3. *Let v be a nonarchimedean place of F. Let π_v be induced from $B(F_v)$ by characters $\chi_{v,1}, \chi_{v,2}, \ldots, \chi_{v,n}$. Assume that π_v is right K_v-finite. Let $\alpha \in T(F_v)$ and let $W_v(\alpha, \lambda)$ be a Whittaker function associated to $\pi_{v,\lambda}$ and α. Then $W_v(\alpha, \lambda)$ is of the form $\mathcal{B}_v(\alpha, \lambda)\mathcal{L}_v(\lambda)$, where $\mathcal{B}_v(\alpha, \lambda)$ is a holomorphic function of λ, and*

$$\mathcal{L}_v(\lambda) = \prod_{1\leq i<r} \prod_{i<j\leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{v,i}\overline{\chi}_{v,j})^{-1}.$$

Proof. Starting with $n = 2$, we may assume that $\chi_{1,2} = \chi_{v,1}\chi_{v,2}^{-1}$ is unramified. Otherwise, the local L-function $L(s, \chi_1\overline{\chi}_2)$ is trivial, and Lemma 7.3 follows from Part (a) of Proposition 7.1. In view of

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ u^{-1} & 1 \end{pmatrix}\begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}\begin{pmatrix} u^{-1} & 1 \\ 1 & 0 \end{pmatrix}$$

and the K_v-finiteness condition, one has

$$W_v(\alpha, \lambda) = \sum_{j\in J} \sum_{l=1}^{\infty} c_j \int_{\varpi_v^{-l}\mathcal{O}_v^\times} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)\, du + W_v^\circ(\alpha, \lambda),$$
$$W_v^\circ(\alpha, \lambda) = \sum_{j\in J} c_j \int_{\mathcal{O}_v} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)\, du = \sum_{j\in J} c_j \int_{\mathcal{O}_v} \theta(\alpha u)\, du,$$

where j runs over a finite set J and the c_j's are constants; both J and the c_j's depend only on the K_v-type of π_v. For $u \in F_v^\times$, write $u = u^\circ\varpi_v^l$, where $u^\circ \in \mathcal{O}_v^\times = \mathcal{O}_v^\times$, and $l \in \mathbb{Z}$. Write $\alpha = \alpha^\circ\varpi_v^k$, where $\alpha^\circ \in \mathcal{O}_v^\times$. Recall that by definition the one sees that the conductor of θ_v is precisely the inverse different of F_v, which is $\mathfrak{D}_{F_v}^{-1} = \{x_v \in F_v : \mathrm{tr}_{F_v/\mathbb{Q}_p}(x_v) \in \mathbb{Z}_p\}$, where p is the characteristic of residue field of $\mathcal{O}_v$. Since $\mathfrak{D}_{F_v}^{-1}$ is a $\mathbb{Z}_p$-module of F_v, it has the representation $\mathfrak{D}_{F_v}^{-1} = \varpi_v^{-d}\mathcal{O}_v$, where $d \in \mathbb{N}_{\geq 0}$. Hence

$$I = \int_{\mathcal{O}_v} \theta(\alpha u)\, du = \int_{\mathcal{O}_v} \theta(\alpha^\circ u\varphi_v^k)\, du = \int_{\mathcal{O}_v} \theta(u\varpi_v^k)\, du$$

is vanishing if $k \leq -d - 1$. Clearly $I = 1$ if $k \geq -d$. Note that

$$\int_{\varpi_v^{-l}\mathcal{O}_v^\times} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)\, du = \chi_{12}(\varpi_v)^l|\varpi_v|_v^{(1+\lambda_1-\lambda_2)l} \int_{\varpi_v^{-l}\mathcal{O}_v^\times} \theta(\alpha u)\, du$$

is vanishing if $l \geq k+d+2$. Let $q_v = |\varpi_v|_v^{-1}$. Then one sees that

$$W_v(\alpha, \lambda) = C + C \sum_{l=1}^{k+d} (1 - q_v^{-1}) \chi_{12}(\varpi_v)^l q_v^{-(\lambda_1-\lambda_2)l} + C \cdot W_{re}, \tag{7-7}$$

where C is a constant depending only on F and K_v-type of ϕ_v and

$$W_{re} = \chi_{12}(\varpi_v)^{k+d+1} q_v^{-(k+d+1)(1+\lambda_1-\lambda_2)} \int_{\varpi_v^{-k-d-1}\mathcal{O}_v^\times} \theta(u\varpi_v^k)\, du. \tag{7-8}$$

Since θ is nontrivial on $\varpi_v^{-d-1}\mathcal{O}_v$, then $\int_{\varpi_v^{-k-d-1}\mathcal{O}_v} \theta(u\varpi_v^k)\, du = 0$. Note that $\varpi_v^{-k-d-1}\mathcal{O}_v^\times = \varpi_v^{-k-d-1}\mathcal{O}_v \setminus \varpi_v^{-k-d}\mathcal{O}_v$. Then

$$\int_{\varpi_v^{-k-d-1}\mathcal{O}_v^\times} \theta(u\varpi_v^k)\, du = \int_{\varpi_v^{-k-d-1}\mathcal{O}_v} \theta(u\varpi_v^k)\, du - \int_{\varpi_v^{-k-d}\mathcal{O}_v} \theta(u\varpi_v^k)\, du = -q_v^{k+d}.$$

Then it follows form (7-7) and (7-8) that $W_v(\alpha, \lambda)$ is equal to C multiplying

$$L = 1 + \sum_{l=1}^{k+d} (1 - q_v^{-1}) \chi_{12}(\varpi_v)^l q_v^{-(\lambda_1-\lambda_2)l} - \chi_{12}(\varpi_v)^{k+d+1} q_v^{-(k+d+1)(\lambda_1-\lambda_2)-1}.$$

An elementary computation leads to the identity

$$L = (1 - \chi_{12}(\varpi_v) q_v^{-(1+\lambda_1-\lambda_2)}) \cdot P(\chi_{12}(\varpi_v) q_v^{-(\lambda_1-\lambda_2)}), \tag{7-9}$$

where $P(z) = (1 - z^{k+d+1}) \cdot (1-z)^{-1} = 1 + z + \cdots + z^{k+d} \in \mathbb{C}[z]$.

Therefore, one has $W_v(\alpha, \lambda) = CQ(\chi_{12}(\varpi_v) q_v^{-(\lambda_1-\lambda_2)}) \cdot L_v(1+\lambda_1-\lambda_2, \chi_{12})$, where $Q(z) = P(z)$ if $k \geq -d$ and $Q(z) \equiv 0$ otherwise. Taking $\mathcal{B}_v(\alpha, \lambda)$ to be the function $CQ(\chi_{12}(\varpi_v) q_v^{-(\lambda_1-\lambda_2)})$, we then obtain Lemma 7.3 in $n = 2$ case. The general case follows from the recursion property of Whittaker functions [37, §2.3] and induction, since the integral with respect to $\chi_{l,j}$ is exactly the same as above, $1 \leq l < j \leq n$. □

Proposition 7.4. *Let v be a nonarchimedean place of F. Let π_v be induced from $B(F_v)$ by characters $\chi_{1,v}, \chi_{2,v}, \ldots, \chi_{n,v}$. Assume that π_v is right K_v-finite. Then the function $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is of the form $Q_v(s, \lambda)\mathcal{L}_v(\lambda)$, where the function $Q_v(s, \lambda)$ lies in $\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq n]$ and*

$$\mathcal{L}_v(\lambda) := \prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{i,v}\overline{\chi}_{j,v})^{-1} \cdot L_v(1 - \lambda_i + \lambda_j, \overline{\chi}_{i,v}\chi_{j,v})^{-1}.$$

Proof. By Lemma 7.3 the function

$$W_v(x_v; \phi_{1,v}, \lambda) \prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{i,v}\overline{\chi}_{j,v})$$

is in $\mathbb{C}[q_v^{\pm\lambda_j} : 1 \leq j \leq n]$. Applying the expansions in [17] and changing orders of summations we see that

$$R_v(s, W_{1,v}, W_{2,v}; \lambda) \prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{i,v}\overline{\chi}_{j,v}) L_v(1 - \lambda_i + \lambda_j, \overline{\chi}_{i,v}\chi_{j,v})$$

lies in $\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq n]$. □

In conjunction with the Langlands–Shahidi method, we have:

Corollary 7.5. *Let $v \in \Sigma_{F,\mathrm{fin}}$ be a finite place such that π_v is unramified and $\Phi_v = \Phi_v^\circ$ is the characteristic function of $G(\mathcal{O}_{F,v})$. Assume that $\phi_{1,v} = \phi_{2,v} = \phi_v^\circ$ is the unique $G(\mathcal{O}_{F,v})$-fixed vector in the space of π_v such that $\phi_v^0(I_n) = 1$. Then $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is equal to*

$$\prod_{1\le i<r}\prod_{i<j\le r} L_v(1+\lambda_i-\lambda_j, \pi_{i,v}\times\widetilde{\pi}_{j,v})^{-1}\cdot L_v(1-\lambda_i+\lambda_j, \widetilde{\pi}_{i,v}\times\pi_{j,v})^{-1}. \tag{7-10}$$

In particular, $R_v(s,\lambda)$ is independent of s.

Proof. Fix $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$. Let $W_{j,v}^\circ$ be the $G(\mathcal{O}_{F,v})$-invariant vectors such that $W_{j,v}^\circ(I_n) = 1$, $1\le j\le 2$. Then by the computation from [17], we know that $\Psi_v(s, W_{1,v}^\circ, W_{2,v}^\circ; \lambda, \Phi_v^\circ)/L_v(s, \pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$ is equal to (7-10), where Φ_v° is the characteristic function of $\mathcal{O}_{F,v}^n$. Then Corollary 7.5 follows from induction and unramified computations of nonconstant Fourier coefficients of Eisenstein series (see [31], Chapter 7). □

Corollary 7.5 will be used in Section 9 to investigate the meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ for general τ, e.g., Theorem G.

Now we move to the archimedean case. In the current state of affairs the local L-functions

$$L_\infty(s, \pi_\lambda\times\tau\times\widetilde{\pi}_{-\lambda}) = \prod_{v\mid\infty} L_v(s, \pi_{\lambda,v}\times\tau_v\times\widetilde{\pi}_{-\lambda,v})$$

are not defined intrinsically through the integrals as in the nonarchimedean case, but rather extrinsically through the Langlands correspondence and then related to the integrals. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2^{1-s}\pi^{-s}\Gamma(s)$. Then by Langlands classification, each archimedean L-function $L_v(s, \pi_{\lambda,v}\times\tau_v\times\pi_{-\lambda,v})$ is of the form

$$\prod_{i\in I}\Gamma_{\mathbb{R}}(s+\mu_i)\prod_{j\in J}\Gamma_{\mathbb{C}}(s+\mu_j'), \tag{7-11}$$

where I and J are finite set of integers, and $\mu_i, \mu_j' \in \mathbb{C}$.

Combining results from [15] and well known estimates on archimedean Satake parameters (see [23], for example) one concludes the following result.

Proposition 7.6 (archimedean case). *Let notation be as before. Let $v \in \Sigma_{F,\infty}$ be an archimedean place.*

(a) *$\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v)$ converges absolutely and normally in the right half plane $\{s\in\mathbb{C} : \mathrm{Re}(s) > 1-2/(n^2+1)\}$, uniformly in $\lambda \in i\mathfrak{a}_Q/i\mathfrak{a}_G$. Moreover, it is bounded at infinity in any strip of finite width.*

(b) *The function $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is holomorphic in s and λ. Hence, $\Psi_v(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v) = R_v(s, W_{1,v}, W_{2,v}; \lambda)L_v(s, \pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$ admits a meromorphic continuation to the whole complex plane.*

(c) *We have the local functional equation*

$$\frac{\Psi_v(1-s,\widetilde{W}_{1,v},\widetilde{W}_{2,v};-\bar{\lambda},\widehat{\Phi}_v)}{L_v(1-s,\widetilde{\pi}_{-\bar{\lambda},v}\otimes\bar{\tau}_v\times\pi_{\lambda,v})}=\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)\cdot\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})},$$

where $\varepsilon_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v},\theta_v)$ *is a holomorphic function.*

Remark 7.7. It follows from Lemma 5.4 in [15] that if both $\pi_{1,v}$ and $\pi_{2,v}$ are tempered, then the Rankin–Selberg convolution $\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)$ converges absolutely and normally in the right half-plane $\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$, uniformly in $\lambda\in i\mathfrak{a}_Q/i\mathfrak{a}_G$.

7C. ***Global theory for*** $\boldsymbol{\Psi(s,W_1,W_2;\lambda)}$. In this section, we shall compute the global integral representation $\Psi(s,W_1,W_2;\lambda,\Phi)$ defined via (7-4).

Let $\widetilde{\pi}_{\lambda,v}$ be the contragredient of $\pi_{\lambda,v}$. Let ϖ_v be a uniformizer of $\mathcal{O}_{F,v}$, the ring of integers of F_v. Let $q_v=N_{F_v/\mathbb{Q}_p}(\varpi_v)$, where p is the rational prime such that v is above p. Define

$$R(s,W_1,W_2;\lambda):=\prod_{v\in\Sigma_F}\frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})},\quad \mathrm{Re}(s)>1. \tag{7-12}$$

Then $R(s,W_1,W_2;\lambda)$ is holomorphic for any $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$. Putting together the local computations in the last section, we get:

Proposition 7.8 (global case). *Let notation be as before. Let $s\in\mathbb{C}$ be such that* $\mathrm{Re}(s)>1$.*Then*

(a) *The integral* $\Psi(s,W_1,W_2;\lambda,\Phi)$ *converges absolutely in* $\mathrm{Re}(s)>1$, *and it is bounded at infinity in any strip of finite width.*

(b) *We have the global functional equation for* $\mathrm{Re}(s)>1$:

$$\Psi(1-s,\widetilde{W}_1,\widetilde{W}_2;-\lambda,\tau^{-1},\widehat{\Phi})=\Psi(s,W_1,W_2;\lambda,\tau,\Phi).$$

(c) *For any fixed* $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, $R(s,W_1,W_2;\lambda)$ *can be continued to an entire function.*

7D. ***A refinement of Theorem E.*** Let notation be as before. For $\lambda\in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$ and $\phi_2\in\mathfrak{B}_{Q,\chi}$, define

$$R_\varphi(s,\lambda;Q,\chi)=\sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}}\frac{\langle\mathcal{I}_Q(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})},\quad \mathrm{Re}(s)>1, \tag{7-13}$$

where

$$\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda}):=\prod_{v\in\Sigma_F}L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$$

is the complete L-function. According to Propositions 7.1 and 7.6 the function

$$\Psi(s,W_1,W_2;\lambda)\,\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})^{-1}$$

is entire.

Theorem F. *Let notation be as before. Let* $0 < \mathrm{Re}(s) < 1$ *or* $\mathrm{Re}(s) > 1$. *Then*

$$I_{\mathrm{Whi}}(s,\tau) = \sum_{\chi} \sum_{Q \in \mathcal{Q}} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} R_\varphi(s,\lambda; Q,\chi)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})\,d\lambda \tag{7-14}$$

converges absolutely and uniformly on every compact subsets. In particular, Theorem E holds with $c_\varphi = 1$.

Proof. Fix a proper parabolic subgroup $Q \in \mathcal{Q}$ of type $(n_1, n_2, \dots, n_r)$. Let $\mathfrak{X}_Q$ be the subset of cuspidal data $\chi = \{(M,\sigma)\}$ such that $M = M_Q$. Set

$$J_Q(s) = \sum_{\chi \in \mathfrak{X}_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} R_\varphi(s,\lambda; Q,\chi)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})\,d\lambda.$$

Let $M_Q = \mathrm{diag}(M_1, M_2, \dots, M_r)$, where $M_i \in \mathrm{GL}(n_i)$, $1 \le i \le r$. We may write $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_r)$, where $\sigma_i \in \mathcal{A}_0(M_i(F)\backslash M_i(\mathbb{A}_F))$. By the K-finiteness of φ, each σ_i has a fixed finite type, so its arithmetic conductor is bounded uniformly (depending only on φ). Write φ as a finite sum of convolutions $\varphi_\alpha * \varphi_\beta$. So

$$R_\varphi(s,\lambda; Q,\chi) = \sum_\alpha \sum_\beta \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \frac{\Psi(s, W_\alpha, W_\beta; \lambda)}{\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})},$$

where $W_\alpha(x;\lambda) = W(x, \mathcal{I}_P(\lambda,\varphi_\alpha)\phi;\lambda)$ and $W_\beta(x;\lambda) = W(x, \mathcal{I}_P(\lambda,\varphi_\beta)\phi;\lambda)$. For $v \in \Sigma_F$, we define

$$R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi) := \frac{\Psi_v(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v})}{L_v(s,\pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1/2,$$

as the local component. By the calculations at the unramified places in Section 7, we have $R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi) = 1$ for almost all v, and $R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$ is holomorphic at the finitely many remaining places. As a consequence,

$$\frac{\Psi(s, W_\alpha, W_\beta; \lambda)}{\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})} = \prod_{v \in \Sigma_F} R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$$

is a finite product of holomorphic functions. Here we have identified $R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$ with its holomorphic continuation.

Write $i\mathfrak{a}_Q^*/i\mathfrak{a}_G^* \ni \lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 + \dots + \lambda_n = 0$. Let $s = \mu + i\gamma$ with $0 < \mu < 1$ and $\gamma \in \mathbb{R}$. Set $s' = \mu' + i\gamma$, where $\mu' = 100 + |\beta| + c_\varphi$. Here c_φ is the constant defined in Theorem E. Let

$$V(s,\lambda) = s^n (s-1)^n \prod_{i,j} (s - \lambda_i + \lambda_j)^n (s - 1 - \lambda_i + \lambda_j)^n.$$

Then $V(s,\lambda)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is entire. As a consequence,

$$J_{\alpha,\beta}(s) := V(s,\lambda)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \prod_{v \in \Sigma_F} R_{\varphi,v}^{\alpha,\beta}(s,\lambda;\phi)$$

is entire. By Proposition 7.8 we have $\lim_{\gamma \to \infty} J_{\alpha,\beta}(s) = 0$. So one can apply the maximum principle to get $|J_{\alpha,\beta}(s)| \le \max\{|J_{\alpha,\beta}(1-s')|, |J_{\alpha,\beta}(s')|\}$. By the functional equation and the estimate on the

ε-factor, whose size is a power of the norm of the arithmetic conductor, we obtain

$$|J_{\alpha,\beta}(1-s')| \ll |J_{\alpha,\beta}(s')| \cdot M^{s'} \prod_{p\mid\infty} \left| \frac{L_v(1-s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)}{L_v(s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)} \right|,$$

where M is an integer determined by the K-type of φ (i.e., the finite places v such that φ_v is *not* $G(\mathcal{O}_v)$-invariant) and the implied constant is absolute. Note that $\lambda_j \in i\mathbb{R}$ and $\mathrm{Re}(s) \neq 0, 1$. So $V(s,\lambda) \gg_s 1$, with the implied constant depending on s. By Stirling's formula concerning gamma functions we obtain

$$\frac{|V(s',\lambda)|}{|V(s,\lambda)|} \cdot M^{s'} \prod_{p\mid\infty} \left| \frac{L_v(1-s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)}{L_v(s', \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda}, v)} \right| \ll_{s,\varphi} 1$$

where the implied constant depends on s and φ. Then

$$|R_\varphi(s,\lambda;Q,\chi)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})| \ll \sum_\alpha \sum_\beta \left| \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \Psi(s', W_\alpha, W_\beta; \lambda) \right|,$$

where the implied constant depends only on φ. So the absolute convergence of $J_Q(s)$ follows from Theorem E. The above argument also works for $1 < \mu \le c_\varphi$. So Theorem F follows. □

Corollary 7.10. *Let notation be as before. Assume τ is such that $\tau^k \neq \mathbf{1}$ for all $1 \le k \le n$. Then*

$$I_{\mathrm{Whi}}(s,\tau) = \sum_\chi \sum_{Q \in \mathcal{Q}} \frac{1}{c_Q} \int_{i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*} R_\varphi(s,\lambda;Q,\chi)\,\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})\,d\lambda$$

admits a holomorphic continuation to the whole s-plane. The function $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ is also entire.

Proof. Since $\tau^k \neq \mathbf{1}$ for all $1 \le k \le n$, we have $\pi_\lambda \otimes \tau \not\simeq \pi_\lambda$ for all λ. Then $\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is entire. Hence the arguments in the proof of Theorem F (with $V(s,\lambda) \equiv 1$) work here for all $\mathrm{Re}(s) > 0$. Then the first part of Corollary 7.10 follows from the functional equation Proposition 7.8.

Note that $\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is a product of $\Lambda(s,\tau)^n$ and some other Rankin–Selberg L-functions on $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$, $n_1, n_2 < n$. Hence, $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ is entire if $\tau^k \neq \mathbf{1}$ for all $1 \le k \le n$. □

We say τ is *exceptional* if $\tau^k = \mathbf{1}$ for some $1 \le k \le n$. For nonexceptional τ, Corollary 7.10 gives a holomorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to $\mathrm{Re}(s) > 0$. However, if τ is exceptional, the holomorphic functions defined by (7-14) in $0 < \mathrm{Re}(s) < 1$ and $\mathrm{Re}(s) > 1$ are *not* compatible, i.e., they do not give a natural continuation of $I_{\mathrm{Whi}}(s,\tau)$. For example, when τ is trivial, (7-14) diverges for *all* s with $\mathrm{Re}(s) = 1$. A meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ in these (finitely many) exceptional cases will be given in Section 9.

8. The Dedekind conjecture: proof of main results

8A. ***Proof of Theorem A.*** From Theorems C, D, E and F, we conclude the first part of Theorem A, obtaining (1-5), namely, for $\mathrm{Re}(s) > 1$,

$$I_0^\varphi(s,\tau) = I^\varphi_{\mathrm{Geo,Reg}}(s,\tau) + I^\varphi_{P,\mathrm{Reg}}(s,\tau) + I^\varphi_{\mathrm{Sing}}(s,\tau) + I_{\mathrm{Whi}}(s,\tau).$$

Moreover, $I_0^{\varphi}(s,\tau)$, $I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau)$, and $I_{P,\mathrm{Reg}}^{\varphi}(s,\tau)$ admit meromorphic continuation to the whole s-plane. Consequently, $I_{\mathrm{Sing}}^{\varphi}(s,\tau)$ can be continued to a meromorphic function on $\mathbb{C}$.

Assume τ is such that $\tau^k \neq \mathbf{1}$, $1 \le k \le n$. By Corollary 7.10 the function $I_{\mathrm{Whi}}(s,\tau)$ has a meromorphic continuation to $\mathrm{Re}(s) > 0$. By the functional equation of Eisenstein series, we conclude that $I_{\mathrm{Whi}}(s,\tau)$ has a meromorphic continuation to the whole s-plane. Then Theorem A follows.

8B. *Proof of Theorem B.* In this section, our objective is to construct a suitable test function φ in Theorem A to prove Theorem B, i.e., holomorphy of adjoint L-functions for GL(n) implies the Dedekind conjecture for degree n.

8B1. *Auxiliary results.* We establish several important auxiliary results: Lemmas 8.1–8.5, which address key points essential to our analysis:

- distribution of the fractional part $\{\alpha \log p\}$ of $\alpha \log p$ as primes p traverse arithmetic progressions;
- description of conjugacy classes supported in compact sets;
- construction of the local test function as the matrix coefficient of a suitable supercuspidal representation;
- nonvanishing of the local integrals of $I_{\mathrm{Geo,Reg}}(s,\tau)$.

These results play a crucial role in our subsequent analysis of the nonvanishing of the geometric side (see Section 8B2).

Lemma 8.1. *Let $\alpha \in \mathbb{R}_{>0}$. Let $n, m \in \mathbb{Z}_{\ge 1}$. There are infinitely many rational primes p such that $p \equiv 1 \pmod m$ and $\{\alpha \log p\} \le n^{-1} p^{-1/6}$, where $\{\cdot\}$ is the fractional part function.*

Proof. If $\alpha = 0$, then Lemma 8.1 boils down to Dirichlet's theorem.

Suppose $\alpha \neq 0$ henceforth. Let $k > 100n + 100\alpha\,|\log 2\alpha| + 10m\alpha$. By [28, §VIII.14a, p. 290] there exists a prime $p \in (e^{k/\alpha}, e^{k/\alpha} + e^{4k/(5\alpha)})$ such that $p \equiv 1 \pmod m$. Write $p^{\alpha} = e^k + \beta$, where $0 \le \beta \le (e^{k/\alpha} + e^{4k/(5\alpha)})^{\alpha} - e^k$. Then

$$\alpha \log p - k = \log(1 + \beta e^{-k}) \le (1 + e^{-k/(5\alpha)})^{\alpha} - 1 \le 2\alpha e^{-k/(5\alpha)} \le n^{-1} p^{-1/6}.$$

Hence, Lemma 8.1 follows. □

For $v \in \Sigma_{F,\mathrm{fin}}$, denote by $e_v(\cdot)$ the standard evaluation on F_v normalized as $e_v(\varpi_v) = 1$. For $x_v = (x_{i,j}) \in G(F_v)$, we set

$$e_{\min}(x_v) = \min_{1 \le i,j \le n} e_v(x_{i,j}).$$

Let Ω be a compact set in $G(F_v)$. Set

$$\Omega^{-1} := \{x^{-1} : x \in \Omega\},$$
$$e_{\min}(\Omega) := \min_{x_v \in \Omega \cup \Omega^{-1}} e_{\min}(x_v) \le 0.$$

Lemma 8.2. *Let notation be as before. Let $v \in \Sigma_{F,\mathrm{fin}}$. Let Ω be a compact set in $G(F_v)$. Let*

$$\gamma_0 = \begin{pmatrix} 0 & & \cdots & -c_0 \\ 1 & 0 & \cdots & -c_1 \\ & \ddots & \ddots & \vdots \\ & & 1 & -c_{n-1} \end{pmatrix} \in G(F) \hookrightarrow G(F_v). \tag{8-1}$$

Let $x_v = tuk \in G(F_v)$, where $t = \mathrm{diag}(t_1, \dots, t_n) \in T(F_v)$, $k \in G(\mathcal{O}_v)$, and $u = (u_{i,j}) \in N(F_v)$ satisfies that, for $1 \le i < j \le n$, either $e_v(u_{i,j}) < 0$ or $u_{i,j} = 0$. Suppose that $\det x_v \in \mathcal{O}_v^\times$ *and $x_v^{-1}\gamma_0 x_v \in \Omega$. There exists a polynomial $p(n) > 0$ of n, with coefficients determined by $e_v(c_i)$, $0 \le i \le n-1$, such that*

$$\begin{cases} p(n)e_{\min}(\Omega) \le e_v(t_i) \le -p(n)e_{\min}(\Omega) & \text{if } 1 \le i \le n, \\ p(n)e_{\min}(\Omega) \le e_v(u_{i,j}) \le -1 & \text{if } u_{i,j} \ne 0 \text{ and } 1 \le i < j \le n. \end{cases} \tag{8-2}$$

Proof. Write $u^{-1} = (u'_{i,j})$. Then $e_v(u'_{i,j}) < 0$ if $u'_{i,j} \ne 0$, $1 \le i < j \le n$. By a straightforward calculation we have

$$kx_v^{-1}\gamma_0 x_v k^{-1} = \begin{pmatrix} u'_{1,2} & u_{1,2}u'_{1,2} + t_1^{-1}t_2u'_{1,2} & \cdots & * \\ t_1t_2^{-1} & t_1t_2^{-1}u_{1,2} + u'_{2,2} & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & t_{n-1}t_n^{-1} & * \end{pmatrix} \in k\Omega k^{-1}. \tag{8-3}$$

Upon analyzing the corresponding elements in (8-3) individually, we obtain

$$\begin{cases} e_v(t_i) - e_v(t_{i+1}) \ge e_{\min}(\Omega) & \text{if } 1 \le i \le n-1, \\ e_v(u_{i,j}) \ge p_1(n)e_{\min}(\Omega) & \text{if } 1 \le i, j \le n, \end{cases} \tag{8-4}$$

where $p_1(n) > 0$ is an explicit polynomial of n, whose coefficients depends only on $e_v(c_i)$, $0 \le i \le n-1$. Note that

$$\gamma_0^{-1} = \begin{pmatrix} -c_0^{-1}c_1 & -c_0^{-1}c_2 & \cdots & -c_0^{-1} \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \in G(F).$$

Taking the transpose inverse of (8-3) we then obtain $M_1M_2 \in k\Omega^{-1}k^{-1}$, where

$$M_1 := \begin{pmatrix} t_1 & & & \\ t_1u_{1,2} & t_2 & & \\ \vdots & \ddots & \ddots & \\ t_1u_{1,n-1} & \cdots & t_{n-1}u_{n-1,n} & t_n \end{pmatrix},$$

and the matrix M_2 is defined by

$$M_2 := \begin{pmatrix} -c_0^{-1}c_1 & -c_0^{-1}c_2 & \cdots & -c_0^{-1} \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} t_1^{-1} & & & \\ t_2^{-1}u'_{1,2} & t_2^{-1} & & \\ \vdots & \ddots & \ddots & \\ t_n^{-1}u'_{1,n-1} & \cdots & t_n^{-1}u'_{n-1,n} & t_n^{-1} \end{pmatrix}.$$

For $2 \leq i \leq n$, denote by m_i the $(i, i-1)$-th entry of $M_1 M_2 \in k\Omega^{-1}k^{-1}$. Then

$$m_i = t_1 u_{1,i-1}(-c_0^{-1}c_{i-1}t_{i-1}^{-1} + \cdots - c_0^{-1}t_n^{-1}u'_{i-2,i-1}) + \cdots + t_i t_{i-1}^{-1}.$$

Here we extend the notation by setting $u'_{i-2,i} = 1$ if $i = 2$.

In the expansion of $m_i - t_i t_{i-1}^{-1}$, if a term contains $t_l^{-1}t_j$, then $l \leq j$. Consequently, according to (8-4), there exists a polynomial $p_2(n)$, whose coefficients depend solely on $e_v(c_i)$ with $0 \leq i \leq n-1$, such that

$$e_v(m_i - t_i t_{i-1}^{-1}) \geq p_2(n) e_{\min}(\Omega).$$

Note that $e_v(m_i) \geq e_{\min}(\Omega)$ and $e_{\min}(\Omega) \leq 0$. So

$$e_v(t_i t_{i-1}^{-1}) \geq \min\{e_v(m_i), p_2(n)e_{\min}(\Omega)\} \geq (p_2(n)+1)e_{\min}(\Omega). \tag{8-5}$$

In addition, it follows from $\det x_v \in \mathcal{O}_v^\times$ that $e_v(t_1) + \cdots + e_v(t_n) = 0$. Hence, combining (8-4) and (8-5) we obtain (8-2). ☐

Remark 8.3. By performing some elementary calculations (albeit tedious), it is possible to explicitly determine the polynomial $p(n)$. However, for our specific purpose, it suffices to note that $p(n)$ depends solely on the value of n and the characteristic polynomial of γ_0.

Lemma 8.4. *Let j be a fixed odd integer coprime to n. Let $\gamma_0 \in G(F)$. Let $v \in \Sigma_{F,\mathrm{fin}}$ be a finite place such that $q_v = \#(\mathcal{O}_v/\mathfrak{p}_v)$ is sufficiently large, where $\mathfrak{p}_v$ is the maximal ideal. Let $J^1 := (1+\mathfrak{p}_v)(1+\mathcal{B}^{\frac{j+1}{2}})$, where $\mathcal{B} = \{b = (b_{i,j}) \in G(\mathcal{O}_v) : b_{i,j} \in \varpi_v\mathcal{O}_v,\ 1 \leq j \leq i \leq n\}$. Then there exists a supercuspidal representation σ of $G(F_v)$ with the following properties*:

- *σ has depth j and trivial central character.*
- *There exists a matrix coefficient m_σ of σ such that*

$$m_\sigma(x_v^{-1}\gamma_{0,v}x_v) = c(\gamma_{0,v}, \sigma)\mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v), \tag{8-6}$$

  where $\gamma_{0,v}$ is the embedding of γ_0 into $G(F_v)$, and
  - *$c(\gamma_{0,v}, \sigma)$ is a nonzero number depending only on F_v, $\gamma_{0,v}$ and σ;*
  - *Ω is a compact set with $-e_{\min}(\Omega) \ll 1$. Here the implied constant depends only on j and n.*

Proof. A construction for such a σ has been presented in [7, §1]. In this context, we follow the reinterpretation outlined in [12, §3] to emphasize the specific properties mentioned in equation (8-6).

Following the notations in [12, §3]. Let β be an $n \times n$ matrix with $\min_{i\in\mathbb{Z}}\{\beta \in \mathcal{B}^i\} = -j$ such that $L = F_v[\beta]$ is a totally ramified field extension of degree n with the property that L^* normalizes $\mathfrak{A} = \{b = (b_{i,j}) \in G(\mathcal{O}_v) : b_{i,j} \in \varpi_v\mathcal{O}_v,\ 1 \leq j < i \leq n\}$. We may take β to be "minimal" in the sense of [7, (1.4.14)]. Let σ be the supercuspidal representation associated with β constructed in loc. cit. (or [12, §3]). Then σ has depth j and trivial central character.

Since q_v is sufficiently large, we can define the unique simple character θ on J^1 as $\theta(x) = \psi \circ \mathrm{Tr}(\beta(x-1))$, where ψ is a fixed unramified additive character of F_v. See [7, §3] or [12, Definition 3.6].

Then there is a unique vector $\xi \in \sigma$ such that $g^{-1}J^1 g$ acts on $\pi(g)\xi$ by θ^g for all $g \in G(F_v)$, where $\theta^g(x) := \theta(g^{-1}xg)$. We define a matrix coefficient by

$$m_\sigma(x) := \langle \sigma(x)\sigma(\beta^{-1})\xi, \xi\rangle \mathbf{1}_{x\in J^1\beta},$$

which is a slight variant of the matrix coefficient defined in [12, (3.23)]. Then by loc. cit. we have

$$m_\sigma(x) = \psi \circ \mathrm{Tr}(\beta(x\beta^{-1} - 1))\langle \xi, \xi\rangle = \psi \circ \mathrm{Tr}(x - \beta)\langle \xi, \xi\rangle.$$

Taking $x = x_v^{-1}\gamma_{0,v}x_v$, we obtain

$$m_\sigma(x_v^{-1}\gamma_{0,v}x_v) = \psi \circ \mathrm{Tr}(\beta(x_v^{-1}\gamma_{0,v}x_v\beta^{-1} - 1))\langle \xi, \xi\rangle = \psi \circ \mathrm{Tr}(\gamma_{0,v} - \beta)\langle \xi, \xi\rangle.$$

Then Lemma 8.4 follows with $c(\gamma_{0,v}, \sigma) = \psi \circ \mathrm{Tr}(\gamma_{0,v} - \beta) \neq 0$ and $\Omega = J^1\beta$. □

Lemma 8.5. *Let $s_0 = \sigma_0 + it_0$ with $\sigma_0 \geq 1/2$ and $t_0 \geq 0$. Let $\gamma_0 \in G(F)$ be regular elliptic. Then there exists a finite place $v \in \Sigma_F$, a real-valued Schwartz function Φ_v on F_v^n, and a compactly supported smooth function φ_v on $\mathrm{PGL}_n(F_v)$ such that $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0) \neq 0$, where $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0)$ is defined by*

$$\int_{G_{\gamma_0}(F_v)\backslash G(F_v)} \varphi_v(x_v^{-1}\gamma_{0,v}x_v) \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau_v(\det t x_v)\, |\det t x_v|_v^{s_0}\, dt\, dx_v.$$

Proof. Set $\alpha = (2\pi)^{-1}t_0$. Let $m \geq 1$ be the norm of the arithmetic conductor of τ. Fix a sufficiently large rational p as in Lemma 8.1. Then $\{\alpha \log p\} \leq n^{-1}p^{-1/6}$. Let v be a place above p. Then $\{\alpha \log q_v\} \leq q_v^{-1/(6n)}$, where q_v is the cardinality of the residue field of F_v.

We may assume γ_0 is given in its companion matrix form (8-1). Since p is sufficiently large, $e_v(c_i) = 0$, $0 \leq i \leq n-1$. Here c_i's are coefficients of the characteristic polynomial of γ_0. Let j be a *fixed* odd integer with $(j, n) = 1$. We will adopt the notation in Lemma 8.4. Take $\varphi_v = m_\sigma$ constructed therein. Then

$$\varphi_v(x_v^{-1}\gamma_{0,v}x_v) = c(\gamma_{0,v}, \sigma)\mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v), \quad c(\gamma_{0,v}, \sigma) \neq 0.$$

Take $\Phi_v = \mathbf{1}_{\mathcal{O}_v^n}$. The integral $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0)$ becomes

$$c(\gamma_{0,v}, \sigma) \int_{G_{\gamma_0}(F_v)\backslash G(F_v)} \mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v) \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det t x_v)\, |\det t x_v|_v^{s_0}\, dt\, dx_v.$$

For each $x_v \in G_{\gamma_0}(F_v)\backslash G(F_v)$, we may assume $\det x_v \in \mathcal{O}_v^\times$. For all x_v such that $x_v^{-1}\gamma_{0,v}x_v \in \Omega$ and $\det x_v \in \mathcal{O}_v^\times$, by Lemma 8.2, we have uniformly that

$$\mathbf{1}_{\varpi_v^{e_1}\mathcal{O}_v}(\eta t) \leq \Phi_v(\eta t x_v) \leq \mathbf{1}_{\varpi_v^{e_2}\mathcal{O}_v}(\eta t), \tag{8-7}$$

where $e_1 \leq e_2$ are two integers depending only on n and j.

Note that $p \equiv 1 \pmod{m}$, which yields $\tau_v(\det t x_v) = 1$. Hence,

$$\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det t x_v)\, |\det t x_v|_v^{s_0} dt = \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\, |\det t x_v|_v^{s_0} dt.$$

Write $|\det tx_v|_v = q_v^{-l}$. By (8-7) and Lemma 8.2 we have $l \geq e_3$, where e_3 is a constant determined only by n and j. Note that

$$\mathrm{Re}(|\det tx_v|_v^{s_0}) = q_v^{-\sigma_0 l}\cos(t_0 l \log q_v) = q_v^{-\sigma_0 l}\cos(2\pi l\{\alpha \log q_v\}).$$

Since $\{\alpha \log q_v\} \leq q_v^{-1/(6n)}$, we have $\cos(2\pi l\{\alpha \log q_v\}) \geq 1/2$ for $l \leq 100^{-1} q_v^{1/(6n)}$. So

$$\mathrm{Re}(|\det tx_v|_v^{s_0}) \geq q_v^{-\sigma_0 l}/2, \quad l \leq 100^{-1} q_v^{1/(6n)}.$$

Therefore, we obtain

$$\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\,\mathrm{Re}(|\det tx_v|_v^{s_0})\,dt \gg 1 + O(\mathcal{E}), \tag{8-8}$$

where the implied constants depend only on n and j, and the "tail" $\mathcal{E}$ is defined by

$$\mathcal{E} := \sum_{l > 100^{-1} q_v^{\delta}} q_v^{-l} \ll q_v^{-q_v^{\delta}}, \quad \delta = \frac{1}{6n}.$$

By taking q_v sufficiently large we see that the right-hand side of (8-8) is positive. Hence,

$$\mathrm{Re}\Big(\int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det tx_v)\,|\det tx_v|_v^{s_0}\,dt\Big) \geq c_{n,j},$$

where $c_{n,j} > 0$ is a constant determined by n and j. So

$$\mathrm{Re}\Big(\int_{G_{\gamma_0}(F_v)\backslash G(F_v)} \mathbf{1}_\Omega(x_v^{-1}\gamma_{0,v}x_v) \int_{G_{\gamma_0}(F_v)} \Phi_v(\eta t x_v)\tau(\det tx_v)\,|\det tx_v|_v^{s_0}\,dt\,dx_v\Big) > 0,$$

namely, $\mathrm{Re}(\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0)/c(\gamma_{0,v}, \sigma)) \neq 0$. Hence, $\mathcal{I}_v(s_0, \varphi_v, \Phi_v, \gamma_0) \neq 0$. □

8B2. *Proof of Theorem B.* Fix a field extension E/F of degree n. Let $s_0 \in \mathbb{C} - \{1\}$ be such that $\mathrm{Re}(s) \geq 1/2$. Let $\gamma_0 \in G(F)$ be such that $F[\gamma_0]^\times = E$. Although such γ_0 are not unique, we fix one γ_0. Consider the continuous map

$$\sigma : G(F) \to F^n, \quad \gamma \mapsto (a_{n-1}(\gamma), \dots, a_1(\gamma), a_0(\gamma)),$$

where the $a_i(\gamma)$ are the coefficients of the characteristic polynomial f_γ of γ, i.e., $f_\gamma(t) = \det(tI_n - \gamma) = t^n + a_{n-1}(\gamma)t^{n-1} + \cdots + a_1(\gamma)t + a_0(\gamma)$. Then σ extends to a continuous function $G(\mathbb{A}_F) \to \mathbb{A}_F^n$.

As γ ranges over $G(F)$, the image $\sigma(\gamma)$ becomes discrete in $\mathbb{A}_F^n$. Consequently, there exists a small neighborhood U_0 centered at the identity $I_n \in G(\mathbb{A}_F)$ such that the set $\{x^{-1}\gamma_0 x : x \in G_{\gamma_0}(\mathbb{A}_F)U_0\}$ does not contain any other rational γ distinct from γ_0. By appropriately shrinking the neighborhood, we can ensure that $\tau(\det x) = 1$ and $|\det x| = 1$ for $x \in U_0$. Denote by

$$T(s, x) = \int_{\mathbb{A}_E^\times} \Phi(\eta t x)\tau(\det tx)\,|\det tx|^s\,d^\times t.$$

Then, by Tate's thesis, $T(s, x)$ is an integral representation for $\Lambda(s, \tau \circ N_{E/F})$. So $T(s, x) = Q(s, x)\Lambda(s, \tau \circ N_{E/F})$, where $Q(s, x)$ is a function holomorphic in s and smooth in x, depending on Φ, τ, and E. Moreover, by Tate's thesis, one can choose Φ such that $Q(s,x)\equiv 1$ when $x=1$. Fix that choice of $\Phi=\bigotimes_v \Phi_v$. By continuity there exists a small neighborhood U_1 of $I_n\in G(\mathbb{A}_F)$ such that

$$\operatorname{Re}(Q(s_0,x))>1/2,\quad x\in U_1. \tag{8-9}$$

Let $C:=\{x^{-1}\gamma_0 x : x\in U_0\cap U_1\}$ be a compact subset of $G(\mathbb{A}_F)$. Write $C=\bigotimes_v C_v$. For $v\in\Sigma_F$, define $\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)$ by

$$\int_{G_{\gamma_0}(F_v)\backslash G(F_v)}\varphi_v(x_v^{-1}\gamma_{0,v}x_v)\int_{G_{\gamma_0}(F_v)}\Phi_v(\eta t x_v)\tau_v(\det t x_v)\,|\det t x_v|_v^{s_0}\,dt\,dx_v.$$

This notation has been used in Lemma 8.5. We will construct φ_v and Φ_v as follows.

Let v_* be the finite place defined in Lemma 8.5. In that lemma, we construct functions φ_{v_*} and $\Phi_{v_*}=\mathbf{1}_{\mathcal{O}_{v_*}^n}$. It follows that

$$\mathcal{I}_{v_*}(s_0,\varphi_{v_*},\Phi_{v_*},\gamma_0)\neq 0. \tag{8-10}$$

Note that q_{v_*} is sufficiently large, the v_*-th component of Φ coincides with $\mathbf{1}_{\mathcal{O}_{v_*}^n}$. Therefore, the notation Φ_{v_*}, which refers to both the function constructed in Lemma 8.5 and the v_*-th component of Φ, is consistent and compatible.

For $v\in\Sigma_F-\{v_*\}$, we take $\varphi_v=\mathbf{1}_{C_v}$. Let $\varphi=\bigotimes_{v\in\Sigma_F}\varphi_v$. By construction, there exists a finite set S of places, such that $C_v=K_v$ and $\Phi_v=\mathbf{1}_{\mathcal{O}_v^n}$ for $v\notin S$. Then the local component $Q_v(s_0,x_v)\equiv 1$ for $x_v\in C_v$, $v\notin S$. As a consequence, we obtain, for $v\notin S$, that

$$\prod_{v\notin S}\frac{\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)}{L_v(s_0,\tau_v\circ N_{E_v/F_v})}=\prod_{v\notin S}\int_{G_{\gamma_0}(F_v)\backslash G(F_v)}\varphi_v(x_v^{-1}\gamma_{0,v}x_v)\,dx_v>0. \tag{8-11}$$

By (8-9) and (8-10), $\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)\neq 0$, for $v\in S$. Hence,

$$\prod_{v\in S}\frac{\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)}{L_v(s_0,\tau_v\circ N_{E_v/F_v})}\neq 0. \tag{8-12}$$

It the follows from (8-11) and (8-12) that

$$\int_{G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_0 x)Q(s_0,x)\,dx=\prod_{v\in\Sigma_F}\frac{\mathcal{I}_v(s_0,\varphi_v,\Phi_v,\gamma_0)}{L_v(s_0,\tau_v\circ N_{E_v/F_v})}\neq 0. \tag{8-13}$$

Substituting this choice of φ into Theorem A we obtain

$$\frac{I_0^{\varphi}(s,\tau)}{\Lambda(s,\tau)}=\frac{\Lambda(s,\tau\circ N_{E/F})}{n\Lambda(s,\tau)}\int_{G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_0 x)Q(s_0,x)\,dx. \tag{8-14}$$

Assume that the twisted adjoint L-function $L(s,\pi,\mathrm{Ad}\otimes\tau)$ is holomorphic outside $s\in\{0,1\}$ for all $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\mathbf{1})$, which is the subset of cuspidal representations with a supercuspidal component. By spectral expansion (2-5), the function $I_0^{\varphi}(s,\tau)/\Lambda(s,\tau)$ is holomorphic at $s=s_0$. Therefore, it follows from (8-13) and (8-14) that the meromorphic function $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is holomorphic at $s=s_0$.

Since s_0 is arbitrary with $\mathrm{Re}(s_0) \geq 1/2$, $\Lambda(s, \tau \circ N_{E/F})/\Lambda(s, \tau)$ is holomorphic in $\mathrm{Re}(s) \geq 1/2$ and $s \neq 1$. Utilizing the functional equation, we thus conclude that $\Lambda(s, \tau \circ N_{E/F})/\Lambda(s, \tau)$ is holomorphic at all $s \in \mathbb{C} - \{0, 1\}$. So the τ-twisted Dedekind conjecture holds. Then Theorem B follows.

Remark 8.6. It is conjectured in [18; 19] that the reverse direction also holds, namely, the τ-twisted Dedekind conjecture for all field extensions E/F of degree n should imply holomorphy of the τ-twisted adjoint L-functions. This is proved in [36] for $n \leq 4$.

Proof of Corollary 1.6. Let E be a field extension of F of degree n, such that $\zeta_E(1/2) \neq 0$. By the proof of Theorem B, one can choose some test function φ such that

$$\int_{G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(x^{-1}\gamma_0 x) Q(1/2, x)\, dx \neq 0.$$

By (8-14) we have $I_0^{\varphi}(1/2, \tau) \neq 0$.

Corollary 1.6 then follows from the spectral expansion (2-5) of the cuspidal kernel function $\mathrm{K}_0(x, x)$. □

9. Meromorphic continuation: the exceptional case

In Sections 6 and 7, we established the holomorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ to the complex plane, except in the case where τ is an exceptional representation, meaning $\tau^k = \mathbf{1}$ for some $1 \leq k \leq n$. It is important to note that the number of exceptional τ is finite.

The main focus of this section is to derive a specific meromorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ beyond the region $\mathrm{Re}(s) > 1$, specifically for these exceptional representations. However, it is important to mention that for general n, we lack a symmetrical description of this continuation process. Thus, we will restrict our analysis to cases where $n \leq 4$ in this paper.

To illustrate with a simplified example, let's focus on the case when $n = 2$ and make certain oversimplifications. We can approximate the expression as follows:

$$I_{\mathrm{Whi}}(s, \tau) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{(s-1-\lambda)(s-1+\lambda)}\, d\lambda, \quad \mathrm{Re}(s) > 1.$$

Now we fix s such that $1 < \mathrm{Re}(s) < 1 + \epsilon/2$, for a small $\epsilon > 0$. By shifting the contour, as explained in [18], we have:

$$I_{\mathrm{Whi}}(s, \tau) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{(s-1-\lambda)(s-1+\lambda)}\, d\lambda - \frac{1}{2(s-1)}. \tag{9-1}$$

The right side of (9-1) defines a meromorphic function in the region $1 - \epsilon/2 < \mathrm{Re}(s) < 1 + \epsilon/2$ with a simple pole at $s = 1$. Therefore, we obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s, \tau)$ to the region $\mathrm{Re}(s) > 1 - \epsilon/2$. By repeating this process, we can achieve a meromorphic continuation to the entire complex plane with an explicit description of the poles.

Just as the above prototype, the genuine situation admits the same idea of continuation, but with more delicate techniques required, since $I_{\mathrm{Whi}}(s, \tau)$ is typically infinitely many sums of such integrals.

Details will be provided in the following subsections. Moreover, we find all possible explicit poles of the continuation of each such integral as well, and show they cancel with each other except for $s = 1/2$, where $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ has at most a simple pole if $\tau^2 = 1$.

9A. *Notation.* We fix an *exceptional* unitary character τ, e.g., $\tau = \mathbf{1}$. Let $\mathcal{D}_\tau$ be a standard (open) zero-free region of $L(s,\tau)$ (see, for example, [6]). Fix such a $\mathcal{D}_\tau$ once and for all. We thus can form a domain

$$\mathcal{R}(1/2;\tau)^- := \{s \in \mathbb{C} : 2s \in \mathcal{D}_\tau\} \supsetneq \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 1/2\}. \tag{9-2}$$

In Section 9C, we will obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to the region $\mathcal{R}(1/2;\tau)^-$. In conjunction with the functional equation we then obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to the whole complex plane. The zero-free region plays a role as the strip $\mathrm{Re}(s) > 1-\epsilon/2$ in (9-1).

Let Q be a standard parabolic subgroup of G of type $(n_1, n_2, \ldots, n_r)$. Let $M = \mathrm{diag}(M_1, M_2, \ldots, M_r)$ be the Levi of Q. Let $\mathfrak{X}_Q$ be the subset of cuspidal data $\chi = \{(M,\sigma)\}$. We may write $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$, where $\sigma_i \in \mathcal{A}_0(M_i(F)\backslash M_i(\mathbb{A}_F))$. Let π be a representation induced from $\chi = \{(M,\sigma)\}$.

For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in i\mathfrak{a}_Q^*/i\mathfrak{a}_G^* \simeq (i\mathbb{R})^{r-1}$, satisfying that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 0$, we let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_r) \in \mathbb{C}^{r-1}$ be such that

$$\begin{aligned} \kappa_j &= \lambda_j - \lambda_{j+1}, \ 1 \leq j \leq r-1, \\ \kappa_r &= \lambda_1 - \lambda_r = \kappa_1 + \kappa_2 + \cdots + \kappa_{r-1}. \end{aligned} \tag{9-3}$$

Then we have a bijection $i\mathfrak{a}_Q^*/i\mathfrak{a}_G^* \leftrightarrow i\mathfrak{a}_Q^*/i\mathfrak{a}_G^*$, $\lambda \mapsto \boldsymbol{\kappa}$ given by (9-3), which induces a change of coordinates with $d\lambda = m_Q d\boldsymbol{\kappa}$, where m_Q is an absolute constant (the determinant of the transform (9-3)). So that we can write $\lambda = \lambda(\boldsymbol{\kappa})$. Recall that

$$R_\varphi(s,\lambda;Q,\chi) = \sum_{\phi_1,\phi_2 \in \mathfrak{B}_{Q,\chi}} \frac{\langle \mathcal{I}_Q(\lambda,\varphi)\phi_1,\phi_2\rangle \cdot \Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})}, \quad \mathrm{Re}(s) > 1, \tag{7-13}$$

where $\Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is the complete L-function. Then we can write

$$R_\varphi(s,\lambda;Q,\chi) = R_\varphi(s,\boldsymbol{\kappa};Q,\chi), \quad \Lambda(s,\pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) = \Lambda(s,\pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}}).$$

Let v be a place. Recall from Section 7B the definition

$$R_v(s,W_{1,v},W_{2,v};\lambda) := \frac{\Psi_v(s,W_{1,v},W_{2,v};\lambda,\Phi_v)}{L_v(s,\pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1. \tag{7-6}$$

If π_v is unramified, Φ_v is the characteristic function of $G(\mathcal{O}_{F,v})$, and $W_{1,v} = W_{2,v}$ is the normalized spherical vector in the Whittaker model; then, by Corollary 7.5, $R_v(s,W_{1,v},W_{2,v};\boldsymbol{\kappa}) := R_v(s,W_{1,v},W_{2,v};\lambda)$ is equal to

$$\prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1+\boldsymbol{\kappa}_{i,j}, \sigma_{i,v} \times \tilde{\sigma}_{j,v})^{-1} \cdot L_v(1-\boldsymbol{\kappa}_{i,j}, \tilde{\sigma}_{i,v} \times \sigma_{j,v})^{-1}, \tag{9-4}$$

where $\boldsymbol{\kappa}_{i,j} = \kappa_i + \cdots + \kappa_{j-1}$. Since φ is K-finite, then there exists a finite set $S_{\varphi,\tau,\Phi}$ of finite places such that for *any* π from a cuspidal datum $\chi \in \mathfrak{X}_Q$ in the spectral side, $R_v(s,W_{1,v},W_{2,v};\boldsymbol{\kappa})$ is equal to the formula in (9-4) if $v \notin S_{\varphi,\tau,\Phi}$. In particular, $R_v(s, W_{1,v}, W_{2,v}; \boldsymbol{\kappa})$ is independent of s for all but finitely many places v. Moreover, $R_\varphi(s, \boldsymbol{\kappa}; Q, \chi) L(\boldsymbol{\kappa}, \pi, \widetilde{\pi})$ is holomorphic as a function of s and $\boldsymbol{\kappa}$, where

$$L(\boldsymbol{\kappa}, \pi, \widetilde{\pi}) := \prod_{1\le i<r} \prod_{i<j\le r} \Lambda(1+\boldsymbol{\kappa}_{i,j}, \sigma_i \times \tilde{\sigma}_j) \cdot \Lambda(1-\boldsymbol{\kappa}_{i,j}, \tilde{\sigma}_i \times \sigma_j).$$

Let $1 \le m, m' \le n$ be two integers. Let $\sigma \in \mathcal{A}_0(\mathrm{GL}_m(F)\backslash \mathrm{GL}_m(\mathbb{A}_F))$ and $\sigma' \in \mathcal{A}_0(\mathrm{GL}_{m'}(F)\backslash \mathrm{GL}_{m'}(\mathbb{A}_F))$. Fix $\epsilon_0 > 0$. For any $c' > 0$, let $\mathcal{D}_{c'}(\sigma, \sigma')$ be

$$\left\{\kappa = \beta + i\gamma : \beta \ge 1 - c' \cdot \left[\frac{(C(\sigma)C(\sigma'))^{-2(m+m')}}{(|\gamma|+3)^{2mm'[F:\mathbb{Q}]}}\right]^{\frac{1}{2}+\frac{1}{2(m+m')}-\epsilon_0}\right\} \tag{9-5}$$

if $\sigma' \not\simeq \tilde{\sigma}$; and let $\mathcal{D}_{c'}(\sigma, \sigma')$ denote the region

$$\left\{\kappa = \beta + i\gamma : \beta \ge 1 - c' \cdot \left[\frac{(C(\sigma))^{-8m}}{(|\gamma|+3)^{2mm'[F:\mathbb{Q}]}}\right]^{-\frac{7}{8}+\frac{5}{8m}-\epsilon_0}\right\} \tag{9-6}$$

if $\sigma' \simeq \tilde{\sigma}$. According to [6] and the Appendix of [22], there exists a constant $c_{m,m'} > 0$ depending only on m and m', such that $L(s, \sigma \times \sigma') \ne 0$ in $s \in \mathcal{D}_{c_{m,m'}}(\sigma, \sigma')$.

We may assume that c (depending only on n) is small such that $1 \pm 2\boldsymbol{\kappa}_{i,j} \in \mathcal{D}_c(\sigma_i, \tilde{\sigma}_j)$, and the boundary of $\mathcal{D}_c(\sigma_i, \tilde{\sigma}_j)$ lies in the strip $1 - 1/(n+4) < \mathrm{Re}(\kappa_j) < 1$, $1 \le i \le j \le r$. Let $\mathcal{D}_\chi = \bigcap_{1\le i\le j\le r} \mathcal{D}_c(\sigma_i, \tilde{\sigma}_j)$. Then $L(\boldsymbol{\kappa}, \pi, \widetilde{\pi}) \ne 0$ if $\boldsymbol{\kappa} \in \mathcal{D}_\chi^{r-1} = \mathcal{D}_\chi \times \cdots \times \mathcal{D}_\chi$.

Let $\mathcal{C}_\chi$ be the boundary of $\mathcal{D}_\chi$. For $\epsilon \in \{0, 1\}$, we define

$$\mathcal{C}_\chi(\epsilon) := \begin{cases} \mathcal{C}_\chi & \text{if } \epsilon = 1, \\ i\mathbb{R}, & \text{if } \epsilon = 0. \end{cases}$$

Set $\mathcal{C}_\chi(\boldsymbol{\epsilon}) = \mathcal{C}_\chi(\epsilon_1) \times \cdots \times \mathcal{C}_\chi(\epsilon_{r-1})$, $\epsilon_l \in \{0, 1\}$, $1 \le l \le r-1$.

Let $\mathrm{Re}(s) > 1$. For any $\phi \in \mathfrak{B}_{Q,\chi}$ and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_{r-1}) \in \{0, 1\}^{r-1}$, let

$$J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\boldsymbol{\epsilon})) = \int_{\mathcal{C}_\chi(\boldsymbol{\epsilon})} R_\varphi(s, \boldsymbol{\kappa}; Q, \chi) \Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}})\, d\boldsymbol{\kappa}. \tag{9-7}$$

which is well defined because $J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\boldsymbol{\epsilon})) = J_{Q,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0}))$ (by the Cauchy integral formula), where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^{r-1}$. Therefore, according to Theorem E, for any s with $\mathrm{Re}(s) > 1$,

$$\sum_P \frac{1}{c_Q} \sum_{\chi \in \mathfrak{X}_Q} \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \left| \int_{\mathcal{C}_\chi(\boldsymbol{\epsilon})} R_\varphi(s, \boldsymbol{\kappa}; Q, \chi) \Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}})\, d\boldsymbol{\kappa} \right| < \infty.$$

For any $\beta \ge 1/2$, we set

$$\mathcal{R}(\beta; \chi) = \{s \in \beta - 1 + \mathcal{D}_\chi\} \cap \{s \in \beta + 1 - \mathcal{D}_\chi\}. \tag{9-8}$$

9B. ***Meromorphic continuation via the zero-free region.*** Let $s \in \mathcal{D}_\chi$ with $\mathrm{Re}(s) > 1$. Define

$$\mathcal{F}(\boldsymbol{\kappa}; s) = \mathcal{F}(\boldsymbol{\kappa}; s, Q, \chi) = R_\varphi(s, \boldsymbol{\kappa}; Q, \chi) \Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}}) \tag{9-9}$$

if χ is fixed in the context. Then by Proposition 7.4 in Section 5B we see that $\mathcal{F}(\boldsymbol{\kappa};s)$ is equal to a holomorphic function multiplying

$$\prod_{k=1}^{r}\Lambda(s,\sigma_k\otimes\tau\times\tilde{\sigma}_k)\prod_{j=1}^{r-1}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\sigma_i\otimes\tau\times\tilde{\sigma}_{j+1})\Lambda(s-\kappa_{i,j},\sigma_{j+1}\otimes\tau\times\tilde{\sigma}_i)}{\Lambda(1+\kappa_{i,j},\sigma_i\times\tilde{\sigma}_{j+1})\Lambda(1-\kappa_{i,j},\sigma_{j+1}\times\tilde{\sigma}_i)}.$$

Let $\mathcal{G}(\boldsymbol{\kappa};s)=\mathcal{G}(\boldsymbol{\kappa};s,Q,\chi)$ denote the above product. The denominators of $\mathcal{G}(\boldsymbol{\kappa};s)$ are L-functions from Langlands–Shahidi method, and they will play an important role in the meromorphic continuation across the critical line $\mathrm{Re}(s)=1$.

Let $\mathcal{C}$ denote the boundary $\mathcal{C}_\chi(1)$ and (0) represent the imaginary axis. Analyzing the potential poles of $\mathcal{G}(\boldsymbol{\kappa};s)$, along with contour shifting, yields an expression for the integral $J_{Q,\chi}(s;\phi,\mathcal{C}_\chi(\mathbf{0}))=J_{Q,\chi}(s;\phi,\mathcal{C})-\mathcal{J}_\chi(s)$, where $\mathcal{J}_\chi(s)$ is defined by

$$2\pi i\sum_{j=1}^{r-1}\sum_{i=1}^{j}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}.$$

Here $\mathrm{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)$ vanishes identically unless $\sigma_i\otimes\tau\simeq\sigma_{j+1}$, in which case one must have $n_i=n_{j+1}$. To obtain a meromorphic continuation of $J_{Q,\chi}(s;\phi,\mathcal{C})$ to the critical strip $0<\mathrm{Re}(s)<1$, we start with the following initial step:

Proposition 9.1. *Let notation be as before. Let $\chi\in\mathfrak{X}_Q$. Let $s\in\mathcal{R}(\beta;\chi)$ and* $\mathrm{Re}(s)>1$. *Then*

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C}_\chi(\mathbf{0}))=\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C})-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\mathcal{J}(s;\phi,\mathcal{C}),\tag{9-10}$$

where $\mathcal{C}=\mathcal{C}_\chi(1)$, and the summand $\mathcal{J}(s;\phi,\mathcal{C})$ is defined to be

$$\sum_{m=1}^{r-1}\sum_{\substack{j_m,j_{m-1},\dots,j_1\\1\le j_m<\cdots<j_1\le r-1}}\!\!\!\cdots\sum c_{j_1,\dots,j_m}\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{j_m}=s-1}\cdots\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}},$$

where the expression $d\kappa_{r-1}\cdots d\kappa_1/(d\kappa_{j_m}\cdots d\kappa_{j_1})$ denotes omitting $d\kappa_{j_m},\dots,d\kappa_{j_1}$ from $d\kappa_{r-1}\cdots d\kappa_1$, and the coefficients $c_{j_1,\dots,j_m}$ are explicit integers. Each term in (9-10) *converges absolutely within $\mathcal{R}(1;\chi)\setminus 1$, where $\mathcal{R}(1;\chi)$ is defined in* (9-8)*. Consequently,* (9-10) *provides a meromorphic continuation of the function*

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C}_\chi(\mathbf{0}))$$

to $\mathcal{R}(1;\chi)$, potentially having a pole at $s=1$.

Proof. For any $1\le j\le r-1$ and $1\le i\le j$, if $n_i=n_{j+1}$, we can make the following change of variables to simplify the integral of $\mathrm{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)$:

$$\lambda_i'=\lambda_j,\qquad\lambda_j'=\lambda_i,\qquad\lambda_l'=\lambda_l\ \text{ for } l\neq i,j;$$
$$\sigma_i'=\sigma_j,\qquad\sigma_j'=\sigma_i,\qquad\sigma_l'=\sigma_l\ \text{ for } l\neq i,j.$$

Let $\kappa_l=\lambda_l-\lambda_{l+1}$, $\kappa_l'=\lambda_l'-\lambda_{l+1}'$, $1\leq l\leq r-1$; and $\kappa_{l,m}'=\kappa_l'+\cdots+\kappa_m'$, $1\leq l\leq m\leq r-1$. To describe the relation between $\{\kappa_l : 1\leq l\leq r-1\}$ and $\{\kappa_l' : 1\leq l\leq r-1\}$, we argue as follows:

Case $i=j-1$: A direct computation shows that

$$\kappa_{i-1}=\kappa_{i-1,i}',\qquad \kappa_i=-\kappa_i',\qquad \kappa_{i+1}=\kappa_{i,i+1}',\qquad \kappa_l=\kappa_l',\ \ 1\leq l\leq r-1,\ l\neq i-1,i,i+1.$$

Hence, $\mathrm{Re}(\kappa_l)=0$, $1\leq l\leq i=j-1$ amounts to $\mathrm{Re}(\kappa_l')=0$, $1\leq l\leq i=j-1$. The determinant of transition matrix is $\det\{\partial\kappa_l/\partial\kappa_m'\}_{1\leq l,m\leq r-1}=-1$. Note that $\kappa_j'=\kappa_{i,j}=s-1$, leading to

$$\begin{aligned}&\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}\\ &\qquad=-\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}'\cdots d\kappa_1'\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_j'=s-1}\mathcal{F}(\boldsymbol{\kappa};s,Q,\chi')\,d\kappa_{r-1}'\cdots d\kappa_{j+1}',\end{aligned}$$

where χ' is the cuspidal datum attached to the representations $(\sigma_1',\ldots,\sigma_r')$. Hence $\chi'=\chi$ as an equivalent class.

Case $i\leq j-2$: A direct computation leads to

$$\begin{gathered}\kappa_{i-1}=\kappa_{i-1,j-1}',\qquad \kappa_i=-\kappa_{i+1,j-1}',\qquad \kappa_{j-1}=-\kappa_{i,j-2}',\qquad \kappa_j=\kappa_{i,j}',\\ \kappa_l=\kappa_l',\ \ 1\leq l\leq r-1,\ l\neq i-1,i,j-1,j.\end{gathered}$$

In addition, $\mathrm{Re}(\kappa_l)=0$, $1\leq l\leq i=j-1$ amounts to $\mathrm{Re}(\kappa_l')=0$, $1\leq l\leq i=j-1$. Again we have $\det\{\partial\kappa_l/\partial\kappa_m'\}_{l,m}=-1$, and $\kappa_j'=\kappa_{i,j}$. Therefore,

$$\begin{aligned}&\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}\\ &\qquad=-\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}'\cdots d\kappa_1'\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\mathrm{Res}_{\kappa_j=s-1}\,\mathcal{F}(\boldsymbol{\kappa};s,Q,\chi')\,d\kappa_{r-1}'\cdots d\kappa_{j+1}'.\end{aligned}$$

If $n_i\neq n_{j+1}$, then $\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)=0$. In all, we have

$$\begin{aligned}&\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}\\ &\qquad=-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\mathrm{Res}_{\kappa_j=s-1}\,\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1}.\end{aligned}$$

Therefore, $\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C}_\chi(\mathbf{0}))-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}J_{Q,\chi}(s;\phi,\mathcal{C})$ is equal to

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\sum_{j=1}^{r-1}c_j'\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\mathrm{Res}_{\kappa_j=s-1}\,\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j+1},$$

where the c_j's are some explicit constants, depending only on the type of Q. Consider

$$\int_{(0)}\cdots\int_{(0)}d\kappa_{j_1-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\,d\kappa_{r-1}\cdots d\kappa_{j_1+1},\quad 1\leq j_1\leq r-1.$$

By the Cauchy integral formula we can write this as the sum of

$$\int_{\mathcal{C}}\cdots\int_{\mathcal{C}} d\kappa_{j_1-1}\cdots d\kappa_1 \int_{\mathcal{C}}\cdots\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_1}=s-1} \mathcal{F}(\boldsymbol{\kappa};s)\, d\kappa_{r-1}\cdots d\kappa_{j_1+1}$$

and

$$\sum_{j_2=1}^{j_1-1}\sum_{i_2=1}^{j_2} c'_{i_2,j_2}\int_{(0)}\cdots\int_{(0)} d\kappa_{j_2-1}\cdots d\kappa_1 \int_{\mathcal{C}}\cdots\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{i_2,j_2}=s-1}\ \operatorname*{Res}_{\kappa_{j_1}=s-1} \mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_{j_2+1}}{d\kappa_{j_1}},$$

where the c'_{i_2,j_2} are some explicit integers depending only on the type of Q.

One can do a similar analysis to replace $\kappa_{i_2,j_2}=s-1$ with $\kappa_{j_2}=s-1$. By induction (or just continuing this process until $m=r-1$) we obtain (9-10). Recall that by the definition in (9-9), we have

$$\mathcal{F}(\boldsymbol{\kappa};s)=\sum_{\phi_1,\phi_2\in\mathfrak{B}_{Q,\chi}}\langle\mathcal{I}_Q(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\Psi(s,W_1,W_2;\lambda).$$

Then $\mathcal{F}(\boldsymbol{\kappa};s)$ is a Schwartz function of $\boldsymbol{\kappa}$. Hence all the above integrals converge absolutely, and the proof is completed. □

Let notation be as in Proposition 9.1. Denote by $\mathcal{I}_0(s;\chi)$ the summand of the first term of the right-hand side of (9-10), i.e.,

$$\mathcal{I}_{0,\chi}(s)=\sum_{\phi\in\mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s;\phi,\mathcal{C}_\chi),\quad s\in\mathcal{R}(1;\chi),\ \operatorname{Re}(s)>1.$$

Proposition 9.2. *Let notation be as before. Let $s\in\mathcal{R}(1;\chi)$ and* $\operatorname{Re}(s)>1$. *Then*

$$\mathcal{I}_{0,\chi}(s)=\sum_{\phi\in\mathfrak{B}_{Q,\chi}} J_{Q,\chi}(s;\phi,\mathcal{C}_\chi)+\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\mathcal{J}^0_\chi(s),\tag{9-11}$$

where the summand $\mathcal{J}^0_\chi(s)$ is defined by

$$\sum_{m=1}^{r-1}\ \sum_{\substack{j_m,j_{m-1},\dots,j_1\\ 1\le j_m<\cdots<j_1\le r-1}}\!\!\cdots\sum \tilde{c}_{j_1,\dots,j_m}\int_{(0)}\cdots\cdots\int_{(0)} \operatorname*{Res}_{\kappa_{j_m}=1-s}\cdots\operatorname*{Res}_{\kappa_{j_1}=1-s}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}},$$

where the $\tilde{c}_{j_1,\dots,j_m}$ are some explicit integers, depending only on Q. Moreover, the terms in (9-11) *converges absolutely and normally inside any bounded strip.*

We omit the proof, which is similar to that of Proposition 9.1.

9C. ***Meromorphic continuation inside the critical strip.*** Let $s\in\mathcal{R}(1;\chi)$ and $1\le m\le r-1$. Let $j_1,j_2,\dots,j_m$ be m integers such that $1\le j_m<\cdots<j_1\le r-1$. Consider the summand in the second term of (9-10):

$$\mathcal{I}_{m,\chi}(s):=\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\cdots\cdots\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_m}=s-1}\cdots\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}}.$$

Then each $\mathcal{I}_{m,\chi}(s)$ is naturally meromorphic in $\mathcal{R}(1;\chi)$ with a possible at $s=1$.

Theorem G. *Let $n \leq 4$ and consider the notation as previously defined. Take $\chi \in \mathfrak{X}_Q$. Assume that the adjoint L-function $L(s, \sigma, \mathrm{Ad} \otimes \tau)$ is holomorphic within the strip $0 < \mathrm{Re}(s) < 1$ for all cuspidal representations $\sigma \in \mathcal{A}_0(\mathrm{GL}_k(\mathbb{A}_F))$, where $1 \leq k \leq n-1$. Then, for any $0 \leq m \leq r-1$, the function*

$$\sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{I}_{m,\chi}(s), \quad s \in \mathcal{R}(1; \chi),$$

admits a meromorphic continuation to the area $\mathcal{R}(1/2; \tau)^-$, with simple poles possible only at $s \in \left\{\frac{1}{2}, \frac{2}{3}, \dots, 1 - \frac{1}{n}, 1\right\}$, where $\mathcal{R}(1/2; \tau)^-$ is defined in (9-2). *For any $3 \leq k \leq n$, if $L((k-1)/k, \tau) = 0$, then $s = (k-1)/k$ is not a pole.*

Remark 9.4. If F is a function field, the number of cuspidal data χ's appearing in the spectral decomposition for a fixed test function is finite. As a consequence, Theorem G provides the continuation of the entire spectral side in the function field scenario.

Remark 9.5. As can be seen from the proof, when $n \leq 3$ we can continue the functions $\sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{I}_{m,\chi}(s)$ to $\mathrm{Re}(s) > 1/3$. When $n = 4$, we can only continue $\sum_{\phi \in \mathfrak{B}_{Q,\chi}} \mathcal{I}_{m,\chi}(s)$ to $\mathcal{R}(1/2; \tau)^-$, an open region just containing the right half-plane $\mathrm{Re}(s) \geq 1/2$. This is because some of its components involve $\Lambda(2s, \tau^2)^{-1}$ as a factor. The key ingredient is that $\mathcal{R}(1/2; \tau)^-$ is uniform with respect to $\chi \in \mathfrak{X}_Q$.

Let notation be as before. To simplify our computations below, we shall write, for any $\beta \in \mathbb{R}$, that $\mathcal{R}(\beta) = \mathcal{R}(\beta; \chi)$, $\mathcal{R}(\beta)^- = \mathcal{R}(\beta; \chi) \cap \{s : \mathrm{Re}(s) < \beta\}$, and $\mathcal{R}(\beta)^+ = \mathcal{R}(\beta; \chi) \cap \{s : \mathrm{Re}(s) > \beta\}$. We also use $S_{(a,b)}$ to denote the strip $a < \mathrm{Re}(s) < b$, for any $a < b$.

9C1. *Proof of Theorem G when $n = 3$.*

Proof. Let $n = 3$. Then there are two possibilities: $r = 2$ or $r = 3$. If $r = 2$, then the parabolic subgroup Q is maximal, and any associated cuspidal datum is of the form $\chi \simeq (\sigma_1, \sigma_2)$, where σ_1 is a cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_F)$ and σ_2 is a Hecke character on $\mathbb{A}_F^\times$. In this case, $\mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to an entire function multiplying

$$\frac{\Lambda(s + \kappa_1, \sigma_1 \otimes \tau \times \tilde{\sigma}_2)\Lambda(s - \kappa_1, \sigma_2 \otimes \tau \times \tilde{\sigma}_1)}{\Lambda(1 + \kappa_1, \sigma_1 \times \tilde{\sigma}_2)\Lambda(1 - \kappa_1, \sigma_2 \times \tilde{\sigma}_1)} \cdot \prod_{k=1}^{2} \Lambda(s, \sigma_k \otimes \tau \times \tilde{\sigma}_k). \tag{9-12}$$

Since each completed L-functions in (9-12) is entire inside $S_{(0,1)}$, then $\mathcal{F}(\boldsymbol{\kappa}, s)$ is holomorphic (after continuation) when $0 < \mathrm{Re}(s) < 1$. On the other hand, $\mathcal{F}(\boldsymbol{\kappa}, s)$ vanishes when $\mathrm{Im}(\kappa_1) \to \infty$. Let $\mathrm{Re}(s) > 1$. By the Cauchy integral formula,

$$J_{Q,\chi}(s; \phi, \mathcal{C}(\mathbf{0})) = \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s)\, d\kappa_1 = \sum_{\phi \in \mathfrak{B}_{Q,\chi}} \int_{\mathcal{C}} \mathcal{F}(\boldsymbol{\kappa}, s)\, d\kappa_1,$$

which gives holomorphic continuation to $\mathrm{Re}(s) > 1 - \epsilon_1$, for some $\epsilon_1 > 0$. Hence we obtain holomorphic continuation of $J_{Q,\chi}(s; \phi, \mathcal{C}(\mathbf{0}))$ to $\mathrm{Re}(s) > 0$.

Now we handle the more complicated case where $r = 3$. In this case, cuspidal data χ correspond to (χ_1, χ_2, χ_3), where χ_i's are unitary Hecke characters such that $\chi_1\chi_2\chi_3 = \omega$, the fixed central character.

Then $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to

$$\mathcal{H}(s,\boldsymbol{\kappa})\,\Lambda(s,\tau)^3\prod_{j=1}^{2}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\tau\chi_i\overline{\chi}_{j+1})\,\Lambda(s-\kappa_{i,j},\tau\chi_{j+1}\overline{\chi}_i)}{\Lambda(1+\kappa_{i,j},\chi_i\overline{\chi}_{j+1})\,\Lambda(1-\kappa_{i,j},\chi_{j+1}\overline{\chi}_i)},\tag{9-13}$$

where $\mathcal{H}(s,\boldsymbol{\kappa})$ is an entire function and $\Lambda(s,\chi')$ is the completed Hecke L-function associated to the unitary Hecke character χ' over F. Let $\sum_\phi$ denote the sum over $\phi\in\mathfrak{B}_{Q,\chi}$. Then, by Proposition 9.1,

$$\begin{aligned}J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\,d\kappa_1-c_{1,2}\sum_\phi\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\-c_1\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1,\end{aligned}$$

for some integers c_1, c_2 and $c_{1,2}$; and $s\in\mathcal{D}_\chi$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the right-hand side of the above equality. Then $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $s\in\mathcal{R}(1)$. In particular, we get a meromorphic continuation inside $\mathcal{R}(1)^-$ with a possible pole at $s=1$.

Recall that, for meromorphic functions $A(s)$ and $B(s)$, we write $A(s)\propto B(s)$ if there exists some holomorphic function $C(s)$ such that $A(s)=C(s)B(s)$. By (9-13),

$$\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s-\kappa_2,\chi_1\overline{\chi}_2\tau)\,\Lambda(2s-1+\kappa_2,\chi_2\overline{\chi}_1\tau^2)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(1+\kappa_2,\chi_2\overline{\chi}_1)\,\Lambda(2-s-\kappa_2,\chi_1\overline{\chi}_2\tau^{-1})\,\Lambda(2-s,\tau^{-1})},$$

$$\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s-\kappa_1,\chi_2\overline{\chi}_1\tau)\,\Lambda(2s-1+\kappa_1,\chi_1\overline{\chi}_2\tau^2)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_1\overline{\chi}_2)\,\Lambda(2-s-\kappa_1,\chi_2\overline{\chi}_1\tau^{-1})\,\Lambda(2-s,\tau^{-1})},$$

$$\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.$$

Hence by the Cauchy integral formula we have, for $s\in\mathcal{R}(1)^-$,

$$\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2=\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-2\pi i\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s),\tag{9-14}$$

where the right-hand side is holomorphic inside $1/2<\mathrm{Re}(s)<1$, since

$$\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.\tag{9-15}$$

From (9-15) we see $\int_{\mathcal{C}}\mathrm{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2$ has a potential pole at $s=2/3$ when $\tau^3=1$. Likewise, we have the continuation for $\int_{\mathcal{C}}\mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$:

$$\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1=\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-2\pi i\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s),\tag{9-16}$$

where the right-hand side is holomorphic inside $1/2<\mathrm{Re}(s)<1$, since

$$\mathrm{Res}_{\kappa_1=2-2s}\,\mathrm{Res}_{\kappa_2=s-1}\,\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.\tag{9-17}$$

From (9-17) we see $\int_{\mathcal{C}}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$ has a potential pole at $s=2/3$ when $\tau^3=1$. Now we deal with the remaining term, $\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$. By Proposition 9.2, for $s\in\mathcal{R}(1)^-$, there are integers c_1, c_2 and $c_{1,2}$ such that

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1=\sum_{\phi}\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\sum_{\phi}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$$
$$-c'_1\sum_{\phi}\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c'_2\sum_{\phi}\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1.$$

Using (9-13), one can compute the partial residues of $\mathcal{F}(\boldsymbol{\kappa},s)$:

$$\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_1,\chi_1\overline{\chi}_2\tau)\,\Lambda(2s-1-\kappa_1,\chi_2\overline{\chi}_1\tau^2)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(1-\kappa_1,\chi_2\overline{\chi}_1)\,\Lambda(2-s+\kappa_1,\chi_1\overline{\chi}_2\tau^{-1})\,\Lambda(2-s,\tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_2,\chi_2\overline{\chi}_1\tau)\,\Lambda(2s-1-\kappa_2,\chi_1\overline{\chi}_2\tau^2)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(1-\kappa_2,\chi_1\overline{\chi}_2)\,\Lambda(2-s+\kappa_2,\chi_2\overline{\chi}_1\tau^{-1})\,\Lambda(2-s,\tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-2,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})}.$$

Combining these formulas with the analytic behavior of the function $\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ we conclude that $\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ admits a meromorphic continuation to $1/2<\operatorname{Re}(s)<1$, with a possible pole at $s=2/3$ when $\tau^3=1$. Denote by $J^{(1/2,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ this continuation. Now we continue our meromorphic continuation to some open set containing $\operatorname{Re}(s)\geq 1/2$. Let $s\in\mathcal{R}(1/2)^+$. Then one can plug (9-14) and (9-16) into formulas for $\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ and $\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ and shift contours to see that $J^{(1/2,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c'_1\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2$$
$$-c'_2\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_1\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$$
$$+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$$
$$=\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c'_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2$$
$$-c'_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$$
$$+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$$
$$+c'_1\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c'_2\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s),$$

where the right side of the equality has a natural meromorphic continuation to the domain $\mathcal{R}(1/2)$. Denote by $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the last expression. Note that

$$\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)},$$

$$\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)}.$$

Also, when $s\in\mathcal{R}(1/2)$, $2-2s$ lies in the zero-free region of $L(s,\tau^{-2})$, then $\Lambda(2-2s,\tau^{-2})\neq 0$. So the last two terms of $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $\mathcal{R}(1/2)$ with a possible simple pole at $s=1/2$ when $\tau^2=1$. Hence, we have a meromorphic continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ to the region $\mathcal{R}(1/2)$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

Now consider $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$, where $s\in\mathcal{R}(1/2)^-$. Using Cauchy's formula to determine the analytic behaviors of $\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ and $\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$, we obtain that $J^{(1/3,1/2)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\begin{aligned}&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1-c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c'_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\quad-c'_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_1\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2-c_2\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\\
&\quad+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&\quad+c'_1\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c'_2\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c_1\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&\quad+c_2\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s).\end{aligned}$$

Denote by $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the last expression. We have

$$\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)},$$

$$\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(3s-1,\tau^3)\,\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\,\Lambda(2-s,\tau^{-1})\,\Lambda(1+s,\tau)},$$

so $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ has a holomorphic continuation to $1/3<\operatorname{Re}(s)<1/2$. In all, we obtain the meromorphic continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ to $S_{(1/3,1)}\cup\mathcal{R}(1)$:

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\begin{cases}J^{1}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1),\\ J^{(1/2,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in S_{(1/2,1)},\\ J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1/2),\\ J^{(1/3,1/2)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{S}(1/3,1/2).\end{cases}$$

In particular, $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$ and $s=2/3$ when $\tau^2=1$ and $\tau^3=1$, respectively. □

9C2. *Proof of Theorem G when $n=4$.* The case $n=4$ presents additional complexities compared to $n=3$, although they share a common underlying idea. Consequently, the proof follows a similar approach. However, the main challenge as n increases lies in determining the partial residues of each continuation. There are approximately $O(n^2)$ such multiple residues, but a straightforward and systematic description of them is currently unavailable. Therefore, we provide a proof by explicitly addressing all possible cases. Further computations and continuations can be found in the appendix of [35].

Proof. Let $n=4$. There are three possibilities: $r=2$, $r=3$ or $r=4$.

Case $r=2$: Here the parabolic subgroup Q is of type $(2,2)$, and any associated cuspidal datum is of the form $\chi\simeq(\sigma_1,\sigma_2)$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}(2,\mathbb{A}_F)$. In this case, $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to an entire function multiplying

$$\frac{\Lambda(s+\kappa_1,\sigma_1\otimes\tau\times\tilde{\sigma}_2)\Lambda(s-\kappa_1,\sigma_2\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(1+\kappa_1,\sigma_1\times\tilde{\sigma}_2)\Lambda(1-\kappa_1,\sigma_2\times\tilde{\sigma}_1)}\cdot\prod_{k=1}^{2}\Lambda(s,\sigma_k\otimes\tau\times\tilde{\sigma}_k). \tag{9-18}$$

Let $s\in\mathcal{R}(1)^+$. Since $\mathcal{F}(\boldsymbol{\kappa},s)$ vanishes when $\mathrm{Im}(\kappa_1)\to\infty$, by the Cauchy integral formula, we have

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-2\pi i\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s). \tag{9-19}$$

The term $\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_1\simeq\sigma_2\otimes\tau$. Hence

$$\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(2s-1,\sigma_1\otimes\tau^2\times\tilde{\sigma}_1)\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(2-s,\sigma_1\otimes\tau^{-1}\times\tilde{\sigma}_1)}.$$

So $\mathrm{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation in $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$. Now the right-hand side of (9-19) is meromorphic inside $\mathcal{R}(1)$, with a possible pole at $s=1$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ in $\mathcal{R}(1)$. By Cauchy's formula,

$$\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1=\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1+2\pi i\,\mathrm{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s), \tag{9-20}$$

where $s\in\mathcal{R}(1)^-$. By (9-18), $\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$ is holomorphic inside $S_{(1/2,1)}$; also, $\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_2\simeq\sigma_1\otimes\tau$, in which case

$$\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(2s-1,\sigma_2\otimes\tau^2\times\tilde{\sigma}_2)\Lambda(s,\sigma_2\otimes\tau\times\tilde{\sigma}_2)}{\Lambda(2-s,\sigma_2\otimes\tau^{-1}\times\tilde{\sigma}_2)}.$$

So $\mathrm{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation to $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$. Substituting this with (9-20) into (9-19) we conclude that $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ admits a meromorphic continuation to the domain $\mathcal{R}(1/2)\cup S_{(1/2,1)}$. Denote by $J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ this continuation. Hence we have

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\begin{cases}J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1),\\ J^{1/2}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in S_{(0,1)}.\end{cases}$$

By assumption, $\Lambda(s,\sigma_2\otimes\tau\times\tilde{\sigma}_2)\Lambda(s,\tau)^{-1}$ is holomorphic in $S_{(0,1)}$, so from the expressions above we see that $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))\,\Lambda(s,\tau)^{-1}$ admits a meromorphic continuation in $s\in S_{(1/3,1)}$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

Case $r=3$: In this case, the parabolic subgroup Q is of type $(2,1,1)$, and any associated cuspidal datum is of the form $\chi\simeq(\sigma_1,\chi_2,\chi_3)$, where σ_1 is a cuspidal representations of $\mathrm{GL}(2,\mathbb{A}_F)$; and χ_2, χ_3 are unitary Hecke characters on $\mathbb{A}_F^{\times}$. Since $\Lambda(s,\sigma_1\otimes\tau\times\chi_i)$ is entire, $2\le i\le 3$, then $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to an entire function $\mathcal{H}(\boldsymbol{\kappa},s)$ multiplying

$$\frac{\Lambda(s+\kappa_2,\chi_2\overline{\chi}_3\tau)\,\Lambda(s-\kappa_2,\chi_3\overline{\chi}_2\tau)\,\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)\,\Lambda(s,\tau)^2}{\Lambda(1+\kappa_2,\chi_2\overline{\chi}_3\tau)\,\Lambda(1-\kappa_2,\chi_3\overline{\chi}_2\tau)}. \tag{9-21}$$

Let $s\in\mathcal{R}(1)^{+}$. Since $\mathcal{F}(\boldsymbol{\kappa},s)$ vanishes as $\mathrm{Im}(\kappa_1)\to\infty$, by the Cauchy integral formula, $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\,d\kappa_2-\sum_{\phi\in\mathfrak{B}_{Q,\chi}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1. \tag{9-22}$$

The term $\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\chi_1\simeq\chi_2\otimes\tau$. Hence

$$\mathrm{Res}_{\kappa_2=s-1}\,\mathcal{F}(\boldsymbol{\kappa},s)=\mathcal{H}(s,\kappa_1)\frac{\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)\,\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(2-s,\tau^{-1})},$$

where $\mathcal{H}(s,\kappa_1)$ is an holomorphic function. So $\mathrm{Res}_{\kappa_2=s-1}\,\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation in $S_{(0,1)}$, with possible simple poles at $s=1/2$. Now the right side of (9-22) is meromorphic inside $\mathcal{R}(1)$, with a possible pole at $s=1$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the continuation of $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ in $\mathcal{R}(1)$. Apply Cauchy's formula again to get

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1\,d\kappa_2=\int_{\mathcal{C}}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1+\int_{\mathcal{C}}\mathrm{Res}_{\kappa_2=1-s}\,\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1, \tag{9-23}$$

where $s\in\mathcal{R}(1)^{-}$. By (9-21), $\int_{\mathcal{C}}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2 d\kappa_1$ is holomorphic inside $S_{(1/3,1)}$; also, $\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_2\simeq\sigma_1\otimes\tau$, in which case one has

$$\mathrm{Res}_{\kappa_2=1-s}\,\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(2s-1,\tau^2)\,\Lambda(s,\tau)\,\Lambda(s,\sigma_1\otimes\tau\times\tilde{\sigma}_1)}{\Lambda(2-s,\tau^{-1})}.$$

So $\int_{\mathcal{C}}\mathrm{Res}_{\kappa_2=1-s}\,\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1$ admits a meromorphic continuation to $S_{(1/3,1)}$, with possible simple poles at $s=1/2$. Substituting this and (9-23) into (9-19) we conclude that $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ admits a meromorphic continuation to the domain $S_{(1/3,1)}$. Denote by $J^{(1/3,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ this continuation. Hence invoking the above discussion we have

$$J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))=\begin{cases}J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in\mathcal{R}(1);\\ J^{(1/3,1)}_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) & \text{if } s\in S_{(1/3,1)}.\end{cases}$$

By assumption $\Lambda(s,\sigma_2\otimes\tau\times\tilde{\sigma}_2)\,\Lambda(s,\tau)^{-1}$ is holomorphic in $S_{(0,1)}$, then from the expressions above we see that $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))\,\Lambda(s,\tau)^{-1}$ admits a meromorphic continuation in $s\in S_{(1/3,1)}$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

Case $r=4$: Here the parabolic subgroup Q is of type $(1,1,1,1)$, and any associated cuspidal datum is of the form $\chi\simeq(\chi_1,\chi_2,\chi_3,\chi_4)$, where χ_i's are unitary Hecke characters on $\mathbb{A}_F^\times$ such that $\chi_1\chi_2\chi_3\chi_4=\omega$. Then there exists an entire function $\mathcal{H}(s,\boldsymbol{\kappa})$ such that $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to

$$\mathcal{H}(s,\boldsymbol{\kappa})\Lambda(s,\tau)^4\prod_{j=1}^{3}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\tau\chi_i\bar{\chi}_{j+1})\Lambda(s-\kappa_{i,j},\tau\chi_{j+1}\bar{\chi}_i)}{\Lambda(1+\kappa_{i,j},\chi_i\bar{\chi}_{j+1})\Lambda(1-\kappa_{i,j},\chi_{j+1}\bar{\chi}_i)},\tag{9-24}$$

where $\Lambda(s,\chi')$ is the completed Hecke L-function associated to the unitary Hecke character χ' over F. Then by Proposition 9.1, when $s\in\mathcal{R}(1)^+$, $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is equal to

$$\begin{aligned}
&\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2d\kappa_1-c_1\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2\\
&\quad-c_2\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_1-c_3\sum_\phi\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2d\kappa_1\\
&\quad-c_{1,2}\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3-c_{1,3}\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\quad-c_{2,3}\sum_\phi\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c_{1,2,3}\sum_\phi\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s),
\end{aligned}$$

where the coefficients c_1, c_2, c_3, $c_{1,2}$, $c_{1,3}$, $c_{2,3}$ and $c_{1,2,3}$ are some absolute integers; and the sum with respect to ϕ in taken over $\phi\in\mathfrak{B}_{Q,\chi}$.

Due to the finiteness of $\mathfrak{B}_{Q,\chi}$ and rapidly decay of $\mathcal{F}(\boldsymbol{\kappa},s)$ as a function of $\boldsymbol{\kappa}$, each term in the above expression converges absolutely and locally uniformly. Hence we only need to consider each summand in the above expression. Denote by $\chi_{ij}=\chi_i\bar{\chi}_j$, $1\le i,j\le 4$. By (9-24) we have

$$\begin{aligned}
\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto{}&\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(s-\kappa_{12},\chi_{31}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1-\kappa_1,\chi_{21})\Lambda(1+\kappa_2,\chi_{23})\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})}\\
&\times\frac{\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1+\kappa_{12},\chi_{13}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1+\kappa_{12},\chi_{13})\Lambda(2-s-\kappa_{12},\chi_{31}\tau^{-1})\Lambda(2-s,\tau^{-1})},\\
\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto{}&\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s+\kappa_{13},\chi_{14}\tau)\Lambda(s-\kappa_{13},\chi_{41}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1+\kappa_3,\chi_{34})\Lambda(1+\kappa_{13},\chi_{14})\Lambda(2-s-\kappa_1,\chi_{21}\tau^{-1})}\\
&\times\frac{\Lambda(2s-1+\kappa_3,\chi_{34}\tau^2)\Lambda(2s-1+\kappa_1,\chi_{12}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{13},\chi_{41})\Lambda(2-s-\kappa_3,\chi_{43}\tau^{-1})\Lambda(2-s,\tau^{-1})},\\
\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto{}&\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(s-\kappa_{23},\chi_{42}\tau)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_3,\chi_{34})\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})}\\
&\times\frac{\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1+\kappa_{23},\chi_{24}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1+\kappa_{23},\chi_{24})\Lambda(2-s-\kappa_{23},\chi_{42}\tau^{-1})\Lambda(2-s,\tau^{-1})}.
\end{aligned}$$

From these expressions we see that $\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function times

$$\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-2+\kappa_1,\chi_{12}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(3-2s-\kappa_1,\chi_{21}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Likewise, $\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is a holomorphic multiple of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(2-s,\tau^{-1})^{-2}$ times

$$\frac{\Lambda(1-\kappa_2,\chi_{31})\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(3s-2+\kappa_2,\chi_{23}\tau^3)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})\Lambda(3-2s-\kappa_2,\chi_{32}\tau^{-2})},$$

and $\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the function

$$\frac{\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(3s-2+\kappa_3,\chi_{34}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_3,\chi_{34})\Lambda(3-2s-\kappa_3,\chi_{43}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

One can continue the computation to see that

$$\operatorname{Res}_{\kappa_1=s-1}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

The above shows that $J_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ admits a meromorphic continuation to $s\in\mathcal{S}(1)$. Denote by $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ the continuation. Then clearly $J^1_{Q,\chi}(s;\phi,\mathcal{C}(\mathbf{0}))$ is holomorphic when $s\in\mathcal{R}(1)^-$.

Let $s\in\mathcal{R}(1)^-$. Let $L(s,\tau)$ be the finite part of Hecke L-function with respect to τ. Then by the Cauchy integral formula (see the Appendix of [35]), we have:

Claim 9.6. *The integrals*

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_j},\quad \int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_j=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_j},$$

for $j\in\{1,2,3\}$, *and the integrals*

$$\int_{\mathcal{C}}\operatorname{Res}_{\kappa_i=s-1}\operatorname{Res}_{\kappa_j=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_i\,d\kappa_j},\quad \int_{(0)}\operatorname{Res}_{\kappa_i=1-s}\operatorname{Res}_{\kappa_j=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_i\,d\kappa_j},$$

for $i,j\in\{1,2,3\}$, $i\neq j$, *admit a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, *this continuation has at most simple poles at possibly* $s=3/4$, $s=2/3$ *and* $s=1/2$. *If* $L(3/4,\tau)=0$, *then* $s=3/4$ *is not a pole*; *if* $L(2/3,\tau)=0$, *then* $s=2/3$ *is not a pole.*

By Proposition 9.2, for $s\in\mathcal{R}(1)^-$, there are absolute integers $c'_1,c'_2,c'_3,c'_{1,2},c'_{1,3},c'_{2,3},c'_{1,2,3}$ such that

$$\begin{aligned}
&\sum_{\phi}\int_{\mathcal{C}}\int_{\mathcal{C}}\int_{\mathcal{C}}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2d\kappa_1\\
&\quad=\sum_{\phi}\int_{(0)}\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2d\kappa_1-c'_1\sum_{\phi}\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_2\\
&\qquad-c'_2\sum_{\phi}\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3d\kappa_1-c'_3\sum_{\phi}\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2d\kappa_1\\
&\qquad-c'_{1,2}\sum_{\phi}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_3-c'_{1,3}\sum_{\phi}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_2\\
&\qquad-c'_{2,3}\sum_{\phi}\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\,d\kappa_1-c'_{1,2,3}\sum_{\phi}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s),
\end{aligned}$$

where the sums are taken over $\phi\in\mathfrak{B}_{Q,\chi}$.

Due to the finiteness of $\mathfrak{B}_{Q,\chi}$ and the rapid decay of $\mathcal{F}(\boldsymbol{\kappa},s)$ as a function of $\boldsymbol{\kappa}$ (see [31], for example), each term in the above expression converges absolutely and locally normally. Hence we only need to consider each summand in that expression. According to (9-24), we have

$$\operatorname*{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(s+\kappa_{12},\chi_{13}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1-\kappa_1,\chi_{21})\Lambda(1-\kappa_2,\chi_{32})\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})}\times\frac{\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)\Lambda(2s-1-\kappa_{12},\chi_{31}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{12},\chi_{31})\Lambda(2-s+\kappa_{12},\chi_{13}\tau^{-1})\Lambda(2-s,\tau^{-1})},$$

$$\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s+\kappa_{13},\chi_{14}\tau)\Lambda(s-\kappa_{13},\chi_{41}\tau)}{\Lambda(1-\kappa_1,\chi_{21})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_{13},\chi_{14})\Lambda(2-s+\kappa_1,\chi_{12}\tau^{-1})}\times\frac{\Lambda(2s-1-\kappa_3,\chi_{43}\tau^2)\Lambda(2s-1-\kappa_1,\chi_{21}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{13},\chi_{41})\Lambda(2-s+\kappa_3,\chi_{34}\tau^{-1})\Lambda(2-s,\tau^{-1})},$$

$$\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(s+\kappa_{23},\chi_{24}\tau)}{\Lambda(1-\kappa_2,\chi_{32})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_3,\chi_{34})\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})}\times\frac{\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)\Lambda(2s-1-\kappa_{23},\chi_{42}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{23},\chi_{42})\Lambda(2-s+\kappa_{23},\chi_{24}\tau^{-1})\Lambda(2-s,\tau^{-1})}.$$

From these expressions we see that $\operatorname*{Res}_{\kappa_2=1-s}\operatorname*{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(3s-2-\kappa_1,\chi_{21}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_1,\chi_{21})\Lambda(3-2s+\kappa_1,\chi_{12}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Likewise, $\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(2-s,\tau^{-1})^{-2}$ and

$$\frac{\Lambda(1+\kappa_2,\chi_{13})\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)\Lambda(3s-2-\kappa_2,\chi_{32}\tau^3)}{\Lambda(1-\kappa_2,\chi_{32})\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})\Lambda(3-2s+\kappa_2,\chi_{23}\tau^{-2})},$$

and $\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the function

$$\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(3s-2-\kappa_3,\chi_{43}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_3,\chi_{43})\Lambda(3-2s+\kappa_3,\chi_{34}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Moreover, one can continue the computation to see that

$$\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\operatorname*{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\propto\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Let $s\in\mathcal{R}(1)^-$. The desired meromorphic continuation of

$$\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_j=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_j},\quad j\in\{1,2,3\},$$

and

$$\int_{(0)}\operatorname{Res}_{\kappa_i=1-s}\operatorname{Res}_{\kappa_j=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\frac{d\kappa_1\,d\kappa_2\,d\kappa_3}{d\kappa_i\,d\kappa_j},\qquad i,j\in\{1,2,3\},\ i\neq j,$$

follows from Claim 9.6. Then Theorem G follows. □

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