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Local square mean in the hyperbolic circle problem

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Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a finite volume Fuchsian group. The hyperbolic circle problem is the estimation of the number of elements of the Γ-orbit of z in a hyperbolic circle around w of radius R, where z and w are given points of the upper half-plane and R is a large number. An estimate with error term $e^{2R/3}$ is known, and this has not been improved for any group. Petridis and Risager proved that in the special case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ taking $z = w$ and averaging over z locally the error term can be improved to $e^{(7/12+\epsilon)R}$. Here we show such an improvement for the local L^2-norm of the error term. Our estimate is $e^{(9/14+\epsilon)R}$, which is better than the pointwise bound $e^{2R/3}$ but weaker than the bound of Petridis and Risager for the local average.

1. Introduction

1.1. ***Statement of the main result.*** Let $\mathbb{H}$ be the upper half-plane. For $z, w \in \mathbb{H}$ let

$$u(z,w) = \frac{|z-w|^2}{4\,\mathrm{Im}\,z\,\mathrm{Im}\,w}; \tag{1-1}$$

this is closely related to the hyperbolic distance $\rho(z,w)$ of z and w, namely we have $1 + 2u = \cosh\rho$. The elements of the group $\mathrm{PSL}_2(\mathbb{R})$ act on $\mathbb{H}$ by linear fractional transformations, which are isometries of the hyperbolic plane. Let $d\mu_z = dx\,dy/y^2$; this measure is invariant with respect to the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For $z \in \mathbb{H}$ and $X > 2$ define

$$N(z,X) := \bigl|\{\gamma \in \Gamma : 4u(\gamma z, z) + 2 \le X\}\bigr|;$$

this is the number of points γz in the hyperbolic circle around z of radius $\cosh^{-1}(X/2)$, so the estimation of this quantity is called the hyperbolic circle problem. We know that

$$|N(z,X) - 3X| = O_z(X^{2/3}); \tag{1-2}$$

this is an unpublished theorem of Selberg, but it is proved also in [L-P]; see also [I, Theorem 12.1]. Let $\mathcal{F}$ be the closure of the standard fundamental domain of Γ, i.e.,

$$\mathcal{F} = \bigl\{z \in \mathbb{C} : \mathrm{Im}\,z > 0, -\tfrac{1}{2} \le \mathrm{Re}\,z \le \tfrac{1}{2},\ |z| \ge 1\bigr\}. \tag{1-3}$$

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The goal of this paper is to prove the following theorem.

Theorem 1.1. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, let $\mathcal{F}$ be as in (1-3), and let $\Omega \subseteq \mathcal{F}$ be a compact set. For any $\epsilon > 0$, we have*

$$\left(\int_\Omega (N(z,X) - 3X)^2 \, d\mu_z\right)^{1/2} = O_{\Omega,\epsilon}(X^{9/14+\epsilon}).$$

Remark 1.1. The significance of the theorem is that the estimate is better on average than the pointwise bound $X^{2/3}$.

Remark 1.2. Let f be a smooth nonnegative function that is compactly supported on $\mathcal{F}$ and let $\epsilon > 0$. It was proved in [P-R] that $\int_{\mathcal{F}} f(z)(N(z,X) - 3X)\, d\mu_z = O_{f,\epsilon}(X^{7/12+\epsilon})$.

Remark 1.3. The bound (1-2) remains valid if we take any finite-volume Fuchsian group (a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting discontinuously on $\mathbb{H}$ and having a fundamental domain of finite volume with respect to $d\mu_z$) in place of $\mathrm{PSL}_2(\mathbb{Z})$, provided the main term is defined including all small Laplace eigenvalues. The analogue of the theorem of [P-R] mentioned in Remark 1.2 was proved in [B1] for any finite volume Fuchsian group with exponent $\frac{5}{8}$ in place of $\frac{7}{12}$.

Remark 1.4. It would be interesting to extend Theorem 1.1 for any finite volume Fuchsian group in place of $\mathrm{PSL}_2(\mathbb{Z})$ with some exponent smaller than $\frac{2}{3}$, similarly as the theorem of [P-R] was extended in [B1]. Our present proof uses arithmetic tools, so it might be extended only for groups similar to $\mathrm{PSL}_2(\mathbb{Z})$.

Remark 1.5. Several other kinds of average results in the hyperbolic circle problem were proved in [C] and [C-R].

1.2. ***Outline of the proof.*** We take an integer $J \geq 2$, which will be fixed to be large enough in terms of ϵ. We also take a parameter d, which will tend to ∞ together with X, and we assume $X^{2/3} \leq d = X^{1-\delta}$ with some fixed $\delta > 0$. We take the sum

$$N_{d,J}(z,X) := \sum_{j=0}^{J} (-1)^j \binom{J}{j} \int_1^2 \eta_0(\tau) N(z, X - jd\tau)\, d\tau,$$

where η_0 is a given nonnegative smooth function on $(0,\infty)$ such that $\eta_0(\tau) = 0$ for $\tau \notin [1,2]$, and $\int_1^2 \eta_0(\tau)\, d\tau = 1$. Then the $j = 0$ term equals $N(z,X)$, but the terms $j \neq 0$ are smoothed versions of $N(z,X)$. It can be proved by spectral methods that for $z \in \Omega$ the $j \neq 0$ terms can be replaced by their main terms with an error term $O_\Omega(X/\sqrt{d})$. One gets from these spectral estimates that

$$N_{d,J}(z,X) = N(z,X) - 3X + O_\Omega(X/\sqrt{d}) \tag{1-4}$$

for $z \in \Omega$. If we take d larger than $X^{2/3}$, this error term will be smaller than $X^{2/3}$. One can also see easily that the contribution of the nonhyperbolic $\gamma \in \Gamma$ to $N_{d,J}(z,X)$ is $O_{\Omega,\epsilon}(X^{1/2+\epsilon})$. Therefore, for the proof of Theorem 1.1 it is enough to estimate

$$\int_{\mathcal{F}} (N_{d,J,\mathrm{hyp}}(z,X))^2 \, d\mu_z, \tag{1-5}$$

where $N_{d,J,\mathrm{hyp}}(z,X)$ is the contribution of the hyperbolic $\gamma \in \Gamma$ to $N_{d,J}(z,X)$. We will give an expression for (1-5) whose most essential part will be an expression of the form

$$\sum_{\substack{t_1,t_2 \\ f^2 \neq (t_1^2-4)(t_2^2-4)}} h(t_1^2-4, t_2^2-4, f) \sum_{j_1,j_2=0}^{J} (-1)^{j_1+j_2} \binom{J}{j_1}\binom{J}{j_2} F_{X,d}(t_1,t_2,f,j_1,j_2), \tag{1-6}$$

where $t_1, t_2 > 2$ and f run over the integers, the factor $F_{X,d}(t_1,t_2,f,j_1,j_2)$ is an analytic expression, and $h(t_1^2-4, t_2^2-4, f)$ has the following arithmetic meaning. If $d_1, d_2, t \in \mathbb{Z}$, then $h(d_1,d_2,t)$ denotes the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of pairs (Q_1, Q_2) of quadratic forms $Q_i(X,Y) = A_iX^2 + B_iXY + C_iY^2$ with integer coefficients satisfying that the discriminant of Q_i is d_i, and the codiscriminant $B_1B_2 - 2A_1C_2 - 2A_2C_1$ of Q_1 and Q_2 is t.

Now, (1-6) can be estimated in the following way. For certain ranges of the parameters t_1, t_2 and f we will show that if these three parameters are fixed, then the summation over j_1, j_2 will be negligibly small. This will follow simply from the mean-value theorem of differential calculus, using that J is large enough. For those ranges of t_1, t_2 and f where this reasoning does not work, we estimate every term of the summation separately. In this way we get an upper bound for (1-6) of size $d^{5/2}X^{-1/2+\epsilon}$. Balancing it with the square of the error term in (1-4) we get the theorem choosing $d = X^{5/7}$.

We note that $h(d_1,d_2,t)$ was studied in the papers [H-W] and [M]. They gave explicit formulas for $h(d_1,d_2,t)$ but only under restrictive conditions for the parameters, so we cannot apply their results. Therefore we prove a general upper bound for $h(d_1,d_2,t)$ and apply it in the proof of Theorem 1.1. It would be interesting to investigate in the future whether it is possible to improve the estimate in Theorem 1.1 using an explicit formula instead of our upper bound.

1.3. ***Structure of the paper.*** In Section 2 we give a general formula for the inner product of two automorphic functions $\sum_{\gamma\in\Gamma_{t_i}} m_i(u(z,\gamma z))$, where the m_i are test functions, $t_1, t_2 > 2$ are integers, and Γ_{t_i} is the set of elements of $\mathrm{SL}_2(\mathbb{Z})$ with trace t_i. The class numbers $h(t_1^2-4, t_2^2-4, f)$ occur in that formula. In Section 3 we give an upper bound for $h(t_1^2-4, t_2^2-4, f)$, and in Section 4 we investigate the special functions appearing in the formula of Section 2 in the case when m_i are characteristic functions as in the circle problem. In Section 5 we begin the proof of Theorem 1.1 by giving the spectral estimate and bounding the contribution of nonhyperbolic elements. In Section 6 we complete the proof by estimating the square integral (1-5).

2. Inner product of automorphic functions and class numbers of pairs of quadratic forms

Our main goal in this section is to prove Lemma 2.2, which relates the inner product of two automorphic functions of a special kind to class numbers of pairs of quadratic forms. Before that we give the necessary definitions and prove an easy lemma to be used later.

2.1. ***Definitions and an upper bound.*** We start by taking a positive discriminant s and introducing the set $\mathcal{Q}_s$ of quadratic forms with discriminant s. Let s be a positive integer with $s \equiv 0, 1$ modulo 4 and define

$$\mathcal{Q}_s := \{Q(X,Y) = AX^2 + BXY + CY^2 : A, B, C \in \mathbb{Z},\ B^2 - 4AC = s\}.$$

If $\tau = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and Q is a quadratic form, let us define the quadratic form Q^τ by $Q^\tau(X,Y) = Q(aX + bY, cX + dY)$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts in this way on $\mathcal{Q}_s$. When $s = t^2 - 4$, the set $\mathcal{Q}_s$ can be identified with elements in $\mathrm{SL}_2(\mathbb{Z})$ with trace t. Indeed, if $t > 2$ is an integer, let

$$\Gamma_t = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a + d = t\}.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on Γ_t by conjugation. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_t$, let $Q_\gamma(X,Y) = cX^2 + (d-a)XY - bY^2$. It is easy to see (see [B2], p. 119) that the map $\gamma \mapsto Q_\gamma$ is a one-to-one correspondence between Γ_t and $\mathcal{Q}_s$ with $s = t^2 - 4$, and also between the conjugacy classes of Γ_t over $\mathrm{SL}_2(\mathbb{Z})$ and the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{Q}_s$. More precisely: If $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma \in \Gamma_t$, then $Q_{\tau^{-1}\gamma\tau} = Q_\gamma^\tau$. The fixed points of γ on $\mathbb{R}$ are exactly the roots of the quadratic polynomial $Q_\gamma(X, 1)$.

For $d_1, d_2, t \in \mathbb{Z}$, let $\mathcal{Q}_{d_1,d_2,t}$ be the subset of $\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2}$ consisting of those pairs (Q_1, Q_2) of quadratic forms having codiscriminant t. In other words, writing

$$Q_1(X,Y) = A_1X^2 + B_1XY + C_1Y^2, \quad Q_2(X,Y) = A_2X^2 + B_2XY + C_2Y^2, \tag{2-1}$$

we require that the discriminant of Q_j be d_j $(j = 1, 2)$ and that

$$B_1B_2 - 2A_1C_2 - 2A_2C_1 = t. \tag{2-2}$$

It is easy to check that if $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $(Q_1, Q_2) \in \mathcal{Q}_{d_1,d_2,t}$, then $(Q_1^\tau, Q_2^\tau) \in \mathcal{Q}_{d_1,d_2,t}$. Hence $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_{d_1,d_2,t}$. Let us denote by $h(d_1, d_2, t)$ the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{Q}_{d_1,d_2,t}$.

If $t_1 > 2$, $t_2 > 2$ are integers, let $\mathcal{R}_{t_1,t_2}$ be the subset of $\mathcal{Q}_{t_1^2-4} \times \mathcal{Q}_{t_2^2-4}$ consisting of those pairs (Q_1, Q_2) of quadratic forms satisfying

$$Q_1 = \lambda Q_2 \quad \text{for } some \lambda \in \mathbb{Q}. \tag{2-3}$$

Note that $\mathcal{R}_{t_1,t_2}$ is empty unless $(t_1^2 - 4)/(t_2^2 - 4) \in \mathbb{Q}^2$. It is easy to check that if $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$, then $(Q_1^\tau, Q_2^\tau) \in \mathcal{R}_{t_1,t_2}$. Hence $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{R}_{t_1,t_2}$. Let $\mathcal{R}^*_{t_1,t_2}$ denote a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{R}_{t_1,t_2}$.

If $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$, we can define a nonnegative real number $n(Q_1, Q_2)$ in the following way. Using the bijection $\gamma \mapsto Q_\gamma$ defined earlier, let $\gamma_i \in \Gamma_{t_i}$ be such that $Q_{\gamma_i} = Q_i$ for $i = 1, 2$. The γ_i are uniquely determined. The fixed points on $\mathbb{R}$ of the hyperbolic transformations γ_1 and γ_2 are the same by (2-3), since they are the roots of the polynomial $Q_1(X, 1) = \lambda Q_2(X, 1)$. Denoting the centralizer of a hyperbolic element γ in $\mathrm{SL}_2(\mathbb{Z})$ by $C(\gamma)$ it is well-known and easily proved that

$$C(\gamma) = \{\tau \in \mathrm{SL}_2(\mathbb{Z}) : \tau z_1 = z_1,\ \tau z_2 = z_2\}, \tag{2-4}$$

where z_1 and z_2 are the fixed points of γ. Therefore $C(\gamma_1) = C(\gamma_2)$. The image of $C(\gamma_1)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is infinite cyclic, i.e., there is a $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$C(\gamma_1) = \{\pm\gamma_0^l \in \mathrm{SL}_2(\mathbb{Z}) : l \in \mathbb{Z}\}. \tag{2-5}$$

Let $N(\gamma)$ denote the norm of a hyperbolic transformation γ, see p. 19 of [I]. Let us define $n(Q_1, Q_2) := \left|\log N(\gamma_0)\right|$; this quantity is well-defined. It can be seen that if $\tau \in \mathrm{SL}_2(\mathbb{Z})$, then $n\left(Q_1^\tau, Q_2^\tau\right) = n(Q_1, Q_2)$. Finally, if $t_1 > 2$, $t_2 > 2$ are integers, let us define

$$E_{t_1,t_2} := \sum_{(Q_1,Q_2)\in\mathcal{R}^*_{t_1,t_2}} n(Q_1, Q_2). \tag{2-6}$$

The following lemma will be enough for handling E_{t_1,t_2} during the proof of Theorem 1.1.

Lemma 2.1. *If $2 < t_1 \le t_2$ are integers, then $E_{t_1,t_2} \ll_\epsilon t_2^{1+\epsilon}$ for every $\epsilon > 0$.*

Proof. If $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$ and $\gamma_i \in \Gamma_{t_i}$ is such that $Q_{\gamma_i} = Q_i$ for $i = 1, 2$, then for γ_0 satisfying (2-5) we clearly have $\left|\log N(\gamma_0)\right| \le \left|\log N(\gamma_2)\right| \ll \log t_2$. So it is enough to show that the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{R}_{t_1,t_2}$ is $\ll_\epsilon t_2^{1+\epsilon}$. If $Q_2 \in \mathcal{Q}_{t_2^2-4}$ is given, then there are at most two possibilities for Q_1 to have $(Q_1, Q_2) \in \mathcal{R}_{t_1,t_2}$, so it is enough to show that the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathcal{Q}_{t_2^2-4}$ is $\ll_\epsilon t_2^{1+\epsilon}$. This follows from [Bu, Proposition 3.3 and formula (3.1)]. The lemma is proved. □

2.2. *The formula for the inner product.* If $t > 2$ is an integer and m is a compactly supported bounded function on $[0, \infty)$, then for $z, w \in \mathbb{H}$ write

$$m(z, w) = m(u(z, w)) \tag{2-7}$$

by an abuse of notation; see (1-1) for $u(z, w)$. For $z \in \mathbb{H}$ define

$$M_{t,m}(z) = \sum_{\gamma\in\Gamma_t} m(z, \gamma z). \tag{2-8}$$

The main result of this subsection, Lemma 2.2, expresses the inner product of two such functions M_{t_1,m_1}, M_{t_2,m_2} in terms of the quantities E_{t_1,t_2} and $h(t_1^2-4, t_2^2-4, f)$ defined above. We need also the following definitions to state the lemma.

Let $t_1, t_2 > 2$ be real numbers and let m_1, m_2 be compactly supported bounded functions on $[0, \infty)$. Let us write

$$\mathcal{J}(t_1, t_2, m_1, m_2) := \int_{-\pi/2}^{\pi/2} m_1\left(\frac{t_1^2-4}{4\cos^2\theta}\right) m_2\left(\frac{t_2^2-4}{4\cos^2\theta}\right) \frac{d\theta}{\cos^2\theta}, \tag{2-9}$$

and for every real F with $|F| \ne 1$ let us write

$$\mathcal{I}(t_1, t_2, F, m_1, m_2) := \iint \frac{m_1\left(\frac{1}{4}(t_1^2-4)(1+S^2)\right) m_2\left(\frac{1}{4}(t_2^2-4)(1+T^2)\right)}{\sqrt{S^2+T^2+2FTS+1-F^2}}\, dS\, dT, \tag{2-10}$$

where we integrate over the set

$$\{(S, T) \in \mathbb{R}^2 : S^2+T^2+2FTS+1-F^2 > 0\}. \tag{2-11}$$

Remark 2.1. The integral (2-9) is absolutely convergent, because m_1 and m_2 are compactly supported and bounded. The absolute convergence of the integral in (2-10) is trivial in the case $|F| < 1$, because then we always have $S^2 + T^2 + 2FTS \geq 0$. In the case $|F| > 1$ we use the linear substitution $u = S + \left(F + \sqrt{F^2 - 1}\right)T$, $v = S + \left(F - \sqrt{F^2 - 1}\right)T$. Since m_1 and m_2 are compactly supported, we have in (2-10) that $|S|$ and $|T|$ are bounded from above. But then we have also $|u|, |v| < C$ with some $C > 0$. The condition in (2-11) reads as $uv \geq F^2 - 1$, therefore we have also $|u|, |v| > c$ with some $c > 0$. Now, integrating over the set defined by the conditions $c < |u|,\ |v| < C$ and $uv \geq F^2 - 1$, we clearly have $\iint (1/\sqrt{uv + 1 - F^2})\, du\, dv < \infty$. Thus (2-10) is absolutely convergent also for $|F| > 1$.

Lemma 2.2. *Let $t_1, t_2 > 2$ be integers and let m_1, m_2 be compactly supported bounded functions on $[0, \infty)$. Then, using equation (2-8), the integral*

$$\int_{\mathcal{F}} M_{t_1,m_1}(z) M_{t_2,m_2}(z)\, d\mu_z \tag{2-12}$$

equals the sum of

$$\mathcal{J}(t_1, t_2, m_1, m_2) E_{t_1,t_2} \tag{2-13}$$

and

$$\sum_{\substack{f \in \mathbb{Z} \\ f^2 \neq (t_1^2-4)(t_2^2-4)}} h(t_1^2 - 4, t_2^2 - 4, f)\, \mathcal{I}\left(t_1, t_2, \frac{f}{\sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}}, m_1, m_2\right). \tag{2-14}$$

(*See equations* (2-1) *and* (2-2), *with the subsequent paragraph, for $h(t_1^2 - 4, t_2^2 - 4, f)$, equation* (2-6) *for E_{t_1,t_2}, and* (2-9), (2-10) *for the functions $\mathcal{J}$ and $\mathcal{I}$.*)

Remark 2.2. We will see later that the class numbers $h(t_1^2 - 4, t_2^2 - 4, f)$ are finite, see Lemma 3.1 and Remark 3.1. The sum (2-14) is actually finite, because for large enough $|f|$ the function $\mathcal{I}$ vanishes there. This can be seen easily from (2-10) and (2-11).

For the proof of Lemma 2.2 we need a few preliminary lemmas. To state the first one, we give some definitions using the notations of Lemma 2.2.

Write $G := \Gamma_{t_1} \times \Gamma_{t_2}$, and let G_0 be the set of those elements $(\gamma_1, \gamma_2) \in G$ for which the set of fixed points on $\mathbb{R}$ of γ_1 and of γ_2 are the same. If (γ_1, γ_2), $(\gamma_1^*, \gamma_2^*) \in G$, we say that (γ_1, γ_2) and (γ_1^*, γ_2^*) are $\mathrm{SL}_2(\mathbb{Z})$-equivalent if there is an element $\tau \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau^{-1}\gamma_i\tau = \gamma_i^*$ for $i = 1, 2$. We denote by G_0^* a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of G_0, and by $(G \setminus G_0)^*$ a complete set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $G \setminus G_0$.

Lemma 2.3. *Let $t_1, t_2 > 2$ be integers and let m_1, m_2 be compactly supported bounded functions on $[0, \infty)$. Recall equation* (2-8). *Then*

$$\int_{\mathcal{F}} M_{t_1,m_1}(z) M_{t_2,m_2}(z)\, d\mu_z \tag{2-15}$$

equals the sum of

$$\sum_{(\gamma_1,\gamma_2) \in G_0^*} \int_{C(\gamma_1)\backslash\mathbb{H}} m_1(z, \gamma_1 z) m_2(z, \gamma_2 z)\, d\mu_z \quad \textit{and} \quad \sum_{(\gamma_1,\gamma_2) \in (G \setminus G_0)^*} \int_{\mathbb{H}} m_1(z, \gamma_1 z) m_2(z, \gamma_2 z)\, d\mu_z.$$

Remark 2.3. To avoid confusion we emphasize that $G \setminus G_0$ denotes set difference, while $C(\gamma_1)\backslash\mathbb{H}$ denotes quotient on the left.

Proof. An element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\operatorname{tr}\gamma > 2$ determines a hyperbolic transformation of $\mathbb{H}$; see Section 1.5 of [I]. Hence γ has two different fixed points on $\mathbb{R}$. Assume that $\gamma_1 \in \Gamma_{t_1}$, $\gamma_2 \in \Gamma_{t_2}$, $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\tau^{-1}\gamma_1\tau = \gamma_1, \quad \tau^{-1}\gamma_2\tau = \gamma_2. \tag{2-16}$$

It is clear by (2-4) that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then (2-16) is true if and only if $\tau = \pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. If $(\gamma_1, \gamma_2) \in G_0$, then by (2-4) we see that $C(\gamma_1) = C(\gamma_2)$, and (2-16) is true if and only if $\tau \in C(\gamma_1)$.

From the definitions we see that (2-15) equals

$$\sum_{\gamma_1\in\Gamma_{t_1}} \sum_{\gamma_2\in\Gamma_{t_2}} \int_{\mathcal{F}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z.$$

We partition G into $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes. Since for $\tau \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\int_{\mathcal{F}} m_1(z,\tau^{-1}\gamma_1\tau z)m_2(z,\tau^{-1}\gamma_2\tau z)\,d\mu_z = \int_{\tau\mathcal{F}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z,$$

our considerations above give the lemma. □

Lemma 2.4. *Let $\gamma_1 = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ be hyperbolic elements of $\mathrm{SL}_2(\mathbb{R})$, and assume that the set of fixed points of γ_1 and the set of fixed points of γ_2 are disjoint. Let*

$$F := F(\gamma_1,\gamma_2) = \frac{(d-a)(D-A)+2bC+2Bc}{\sqrt{(d+a)^2-4}\sqrt{(D+A)^2-4}}. \tag{2-17}$$

Let us write $t_1 = a+d$, $t_2 = A+D$, and assume $t_1, t_2 > 2$. Let m_1, m_2 be compactly supported bounded functions on $[0,\infty)$ and use (2-7). Then we have

$$\int_{\mathbb{H}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z = \mathcal{I}(t_1,t_2,F,m_1,m_2), \tag{2-18}$$

where the function $\mathcal{I}$ is defined in (2-10) and (2-11).

Proof. It is easy to check that $F(\gamma_1,\gamma_2) = F(\tau^{-1}\gamma_1\tau, \tau^{-1}\gamma_2\tau)$ for $\tau \in \mathrm{SL}_2(\mathbb{R})$. Since (2-18) also remains the same if we write $\tau^{-1}\gamma_1\tau$ and $\tau^{-1}\gamma_2\tau$ in place of γ_1 and γ_2, and since we can choose τ in such a way that $\tau^{-1}\gamma_1\tau$ is diagonal, for the proof of the lemma we may assume that γ_1 is diagonal.

So assume that $b = c = 0$. Then by the conditions we have $BC \neq 0$. It can be easily computed by the definitions that

$$u(z,\gamma_1 z) = \frac{(d-a)^2|z|^2}{4\operatorname{Im}^2 z}, \quad u(z,\gamma_2 z) = \frac{\left|Cz^2+(D-A)z-B\right|^2}{4\operatorname{Im}^2 z}.$$

Hence, if $z = x+iy$, using $ad = 1$ and $AD - BC = 1$ we get after some computations that

$$u(z,\gamma_1 z) = \frac{(a+d)^2-4}{4} + \frac{(d-a)^2x^2}{4y^2}, \tag{2-19}$$

$$u(z,\gamma_2 z) = \frac{(A+D)^2-4}{4} + \frac{(Cx^2+(D-A)x-B+Cy^2)^2}{4y^2}. \tag{2-20}$$

Since $m(z,w)$ is defined through the function u, we will be able to compute the left-hand side of (2-18) using (2-19) and (2-20).

Let us use the substitution

$$q := \frac{x}{y}, \quad r := \frac{Cx^2 + (D-A)x - B + Cy^2}{y}. \tag{2-21}$$

Then the determinant of the Jacobi matrix $\frac{dq\,dr}{dx\,dy}$ is

$$\det\begin{pmatrix} 1/y & (2Cx+(D-A))/y \\ -x/y^2 & C-(Cx^2+(D-A)x-B)/y^2 \end{pmatrix} = \frac{B+Cx^2+Cy^2}{y^3}.$$

It is not hard to check that

$$\frac{B + Cx^2 + Cy^2}{y} = r - (D-A)q + \frac{2B}{y} \tag{2-22}$$

and

$$\frac{B}{y^2} + \frac{r-(D-A)q}{y} - (C + Cq^2) = 0. \tag{2-23}$$

From (2-22) and (2-23) we get

$$\left(\frac{B+Cx^2+Cy^2}{y}\right)^2 = ((D-A)q - r)^2 + 4B(C + Cq^2).$$

Hence if we want to compute the left-hand side of (2-18) by the substitution (2-21), then on the one hand we see that for q and r we have the condition

$$((D-A)q - r)^2 + 4B(C + Cq^2) \geq 0. \tag{2-24}$$

On the other hand, in the case $BC > 0$ we see from the quadratic equation (2-23) that for every real q and r satisfying (2-24) there is exactly one $y > 0$ and real x satisfying (2-21). Similarly, in the case $BC < 0$ we see from the quadratic equation (2-23) that $((D-A)q - r)/B > 0$, i.e., combined with (2-24) we must have

$$\frac{(D-A)q - r}{B} \geq 2\sqrt{-\frac{C + Cq^2}{B}}. \tag{2-25}$$

If the left side of (2-25) is larger than the right side, then we have two positive solutions of (2-23) in $1/y$. If (2-25) holds with equality, then we have a double root.

Putting everything together we see that the left-hand side of (2-18) equals

$$\int_{-\infty}^{\infty} \int_{A_q} f(r,q)\,dr\,dq \tag{2-26}$$

for $BC > 0$, and the left-hand side of (2-18) equals

$$2\int_{-\infty}^{\infty} \int_{A_q^+} f(r,q)\,dr\,dq \tag{2-27}$$

for $BC < 0$, where

$$f(r,q) := \frac{m_1\big(\frac{1}{4}((a+d)^2-4)+\frac{1}{4}(d-a)^2q^2\big)m_2\big(\frac{1}{4}((A+D)^2-4)+\frac{1}{4}r^2\big)}{\sqrt{((D-A)q-r)^2+4B(C+Cq^2)}},$$

$$A_q := \{r \in \mathbb{R} : ((D-A)q-r)^2+4B(C+Cq^2) \geq 0\},$$

and for $BC < 0$ we write

$$A_q^+ := \left\{ r \in \mathbb{R} : \frac{(D-A)q-r}{B} \geq 2\sqrt{-\frac{C+Cq^2}{B}} \right\}.$$

For $BC < 0$ we define also

$$A_q^- := \left\{ r \in \mathbb{R} : \frac{(D-A)q-r}{B} \leq -2\sqrt{-\frac{C+Cq^2}{B}} \right\}.$$

We see that for $BC < 0$ we have $-A_{-q}^+ = A_q^-$, so, since $f(r,q) = f(-r,-q)$, for $BC < 0$ we have that (2-27) equals

$$\int_{-\infty}^{\infty}\int_{A_q^+} f(r,q)\,dr\,dq + \int_{-\infty}^{\infty}\int_{A_q^-} f(r,q)\,dr\,dq.$$

Since A_q is the disjoint union of A_q^+ and A_q^- for $BC < 0$, we finally get that the left-hand side of (2-18) equals (2-26) also in the case $BC < 0$. Apply the substitution

$$S = q, \quad T = \frac{-r}{\sqrt{(D+A)^2-4}}$$

in (2-26). Recalling (2-17), $b = c = 0$ and $BC \neq 0$ we have that

$$(a+d)^2-4 = (a-d)^2, \quad (D-A)^2+4BC = (D+A)^2-4, \quad F^2 = \frac{(D-A)^2}{(D+A)^2-4} \neq 1.$$

Taking into account that $\mathcal{I}(t_1,t_2,F,m_1,m_2)$ is even in F, we get (2-18) for the case $b = c = 0$. But we have seen that then the lemma is completely proved. □

Lemma 2.5. *Let $\gamma_1 \in \Gamma_{t_1}$, $\gamma_2 \in \Gamma_{t_2}$, where $t_i > 2$ for $i = 1,2$. Assume that γ_1 and γ_2 have the same fixed points. Then*

$$\int_{C(\gamma_1)\backslash\mathbb{H}} m_1(z,\gamma_1 z)m_2(z,\gamma_2 z)\,d\mu_z = \mathcal{J}(t_1,t_2,m_1,m_2)\left|\log N(\gamma_0)\right|, \tag{2-28}$$

where $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ is a generator of the centralizer $C(\gamma_1)$; see (2-5). (The function $\mathcal{J}$ is defined in (2-9).)

Proof. We may assume that $N(\gamma_0) > 1$. We can choose $\tau \in \mathrm{SL}_2(\mathbb{R})$ in such a way that $\tau^{-1}\gamma_i\tau z = \lambda_i z$ for every $z \in \mathbb{H}$ and $0 \leq i \leq 2$ with $\lambda_0 = N(\gamma_0)$ and $\lambda_i = N(\gamma_i)^{\epsilon_i}$ for $i = 1,2$, where $\epsilon_i \in \{-1,1\}$. The fundamental domain of the group $\tau^{-1}C(\gamma_1)\tau$ in $\mathbb{H}$ is the subset $\{1 \leq |z| < N(\gamma_0)\}$. Then the left-hand side of (2-28) equals

$$\int_{\{z\in\mathbb{H}:\,1\leq|z|<N(\gamma_0)\}} m_1(z,\tau^{-1}\gamma_1\tau z)m_2(z,\tau^{-1}\gamma_2\tau z)\,d\mu_z. \tag{2-29}$$

By the substitution $z = re^{i(\frac{\pi}{2}+\theta)}$ with $1 \le r < N(\gamma_0)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ we get using $d\mu_z = \frac{dr\, d\theta}{r \cos^2\theta}$ that (2-29) equals

$$\int_{-\pi/2}^{\pi/2} \int_1^{N(\gamma_0)} m_1\left(\frac{\lambda_1 + \lambda_1^{-1} - 2}{4\cos^2\theta}\right) m_2\left(\frac{\lambda_2 + \lambda_2^{-1} - 2}{4\cos^2\theta}\right) \frac{dr\, d\theta}{r\cos^2\theta}.$$

It is clear that $\lambda_1^{1/2} + \lambda_1^{-1/2} = t_1$, $\lambda_2^{1/2} + \lambda_2^{-1/2} = t_2$, since the trace is invariant under conjugation. The lemma is proved. □

Proof of Lemma 2.2. We use the one-to-one correspondence between Γ_{t_i} and $\mathcal{Q}_{t_i^2-4}$ for $i = 1, 2$.

Using the notation of Lemma 2.3, we see that if $(\gamma_1, \gamma_2) \in G$, then $(\gamma_1, \gamma_2) \in G_0$ holds if and only if $Q_{\gamma_1} = \lambda Q_{\gamma_2}$ with some $\lambda \in \mathbb{Q}$. Indeed, $Q_{\gamma_1} = \lambda Q_{\gamma_2}$ holds with some $\lambda \in \mathbb{Q}$ if and only if the polynomials $Q_{\gamma_1}(X, 1)$ and $Q_{\gamma_2}(X, 1)$ have the same roots, i.e., if and only if γ_1 and γ_2 have the same fixed points.

It is clear that $(\gamma_1, \gamma_2), (\gamma_1^*, \gamma_2^*) \in G$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent if and only if there is a $\tau \in \mathrm{SL}_2(\mathbb{Z})$ such that $(Q_{\gamma_1}^\tau, Q_{\gamma_2}^\tau) = (Q_{\gamma_1^*}, Q_{,\gamma_2^*})$. We note also that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then for the quantity $F(\gamma_1, \gamma_2)$ defined in (2-17) we have $F(\gamma_1, \gamma_2) = /(f)\sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}$, with

$$Q_{\gamma_1}(X, Y) = A_1X^2 + B_1XY + C_1Y^2, \quad Q_{\gamma_2}(X, Y) = A_2X^2 + B_2XY + C_2Y^2, \tag{2-30}$$

$$f = B_1B_2 - 2A_1C_2 - 2A_2C_1. \tag{2-31}$$

We show that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then $f^2 \ne (t_1^2 - 4)(t_2^2 - 4)$. Indeed, writing $d_i := t_i^2 - 4$ and using $d_i = B_i^2 - 4A_iC_i$ for $i = 1, 2$, we easily get from (2-30) and (2-31) that

$$d_2^2A_1^2 - 2fd_2A_1A_2 + d_1d_2A_2^2 = d_2(B_2A_1 - B_1A_2)^2,$$
$$d_2^2C_1^2 - 2fd_2C_1C_2 + d_1d_2C_2^2 = d_2(B_2C_1 - B_1C_2)^2.$$

Assume $f^2 = d_1d_2$. Then the left-hand sides above are squares, and since $d_2 = t_2^2 - 4$ cannot be a square, we get $B_2A_1 - B_1A_2 = 0$, $B_2C_1 - B_1C_2 = 0$. One has the identity

$$(A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1) = \frac{1}{4}\left(f^2 - \prod_{i=1}^{2}(B_i^2 - 4A_iC_i)\right), \tag{2-32}$$

with f defined in (2-31). We also get $A_1C_2 - A_2C_1 = 0$ from (2-32). It follows that the vectors (A_1, B_1, C_1) and (A_2, B_2, C_2) are linearly dependent, hence $(\gamma_1, \gamma_2) \in G_0$, which is a contradiction.

We note finally that if $(\gamma_1, \gamma_2) \in G \setminus G_0$, then γ_1 and γ_2 have no fixed points in common.

With these considerations, applying Lemmas 2.3, 2.4, 2.5 and the definitions we obtain the lemma. □

3. Estimates on the number of equivalence classes of quadratic forms

Recall the definitions of $\mathcal{Q}_{d_1,d_2,t}$ and $h(d_1, d_2, t)$ from (2-1), (2-2) and the subsequent paragraph. In this section we will give several upper bounds for $h(d_1, d_2, t)$ itself and for certain sums containing $h(d_1, d_2, t)$.

Let $\gcd(n_1, n_2, \ldots, n_r)$ be the greatest common divisor of the integers $n_1, n_2, \ldots, n_r$. The integer part of a real number x is denoted by $[x]$.

3.1. ***A general upper bound for $h(d_1, d_2, t)$.*** Our aim in this subsection is to prove Lemma 3.1. The upper bound we give for $h(d_1, d_2, t)$ will be smaller than $(1 + |d_1 d_2 t|)^{\epsilon}$ for any fixed $\epsilon > 0$ in many cases. This is not always true, but the exceptions are rare.

For any finite set of integers $n_1, n_2, \ldots, n_r$ we write

$$S(n_1, n_2, \ldots, n_r) = \max\{k \geq 1 : k^2 \mid \gcd(n_1, n_2, \ldots, n_r)\}. \tag{3-1}$$

Denote by $\tau(n)$ the number of divisors and by $\omega(n)$ the number of distinct prime divisors of a nonzero integer n.

Lemma 3.1. *Assume that $d_1, d_2, t \in \mathbb{Z}$ and d_i is not a square of an integer $(i = 1, 2)$. Assume also that $t^2 - d_1 d_2 \neq 0$. Then*

$$h(d_1, d_2, t) \leq C \cdot 2^{\omega(t^2 - d_1 d_2)} \tau(t^2 - d_1 d_2) S(d_1, d_2, t^2), \tag{3-2}$$

where $C > 0$ is an absolute constant.

Remark 3.1. Even the finiteness of $h(d_1, d_2, t)$ is not completely obvious. For a short proof of this fact using the theory of algebraic groups see Appendix I of [M].

For the case $d_i = t_i^2 - 4$ for $i = 1, 2$ with integers $t_i > 2$, which is our primary interest, one can give a trivial upper bound for $h(d_1, d_2, t)$ using Lemma 2.2. For simplicity let us consider the case when $t_1^2 - 4$, $t_2^2 - 4$ and t all lie in $[X, 2X]$ with a large real number X. Then choosing m_1 and also m_2 in Lemma 2.2 to be the characteristic function of the interval $[0, CX]$ with a suitable absolute constant C, one can show the trivial bound $h(t_1^2 - 4,\ t_2^2 - 4,\ t) \ll X$. Indeed, the coefficient of $h(t_1^2 - 4,\ t_2^2 - 4,\ t)$ is bounded from below by a positive constant, every term in (2-14) and (2-13) is nonnegative, and one can prove that (2-12) equals $O(X)$ in this case. The estimate (3-2) gives better than $h(t_1^2 - 4,\ t_2^2 - 4,\ t) \ll X$ even in the worst case, when $S(t_1^2 - 4,\ t_2^2 - 4,\ t^2)$ is as large as $\sqrt{X}$. But the S-function is often much smaller than $\sqrt{X}$, so the bound given in Lemma 3.1 is much stronger than the trivial bound.

To prepare the proof of Lemma 3.1 we need two preliminary lemmas. We introduce the notation

$$C_{d_1,d_2,t} := \{(x, y) \in \mathbb{R}^2 : d_2 x^2 + d_1 y^2 - 2txy = 1\}.$$

In the first lemma we prove general statements for any two different points of $C_{d_1,d_2,t}$. In the second one we show that if we have any element of $\mathcal{Q}_{d_1,d_2,t}$, then we can parametrize the rational points of $C_{d_1,d_2,t}$.

Lemma 3.2. *Let d_1, d_2, t be as in Lemma 3.1, and assume that $(x_i, y_i) \in C_{d_1,d_2,t}$ for $i = 1, 2$ and $(x_1, y_1) \neq (x_2, y_2)$. Then*

$$d_2(x_1 - x_2)^2 + d_1(y_1 - y_2)^2 - 2t(x_1 - x_2)(y_1 - y_2) \neq 0 \tag{3-3}$$

and

$$(d_1 y_1 - t x_1)(y_1 - y_2) + (d_2 x_1 - t y_1)(x_1 - x_2) \neq 0. \tag{3-4}$$

Proof. Let S_1 and S_2 be the quantities appearing in (3-3) and (3-4), respectively. One can check that

$$S_2 = \begin{pmatrix} y_1 & x_1 \end{pmatrix} \begin{pmatrix} d_1 & -t \\ -t & d_2 \end{pmatrix} \begin{pmatrix} y_1 - y_2 \\ x_1 - x_2 \end{pmatrix} \tag{3-5}$$

and

$$2S_2 + \sum_{i=1}^{2} (-1)^i (d_2 x_i^2 + d_1 y_i^2 - 2t x_i y_i) = S_1.$$

Since $(x_i, y_i) \in C_{d_1,d_2,t}$ for $i = 1, 2$, we have $S_1 = 2S_2$. Hence it is enough to show that $S_1 \neq 0$. Assume for a contradiction that $S_1 = 0$. Then the right-hand side of (3-5) is 0, but this is true also by exchanging the role of (x_1, y_1) and (x_2, y_2), so we get

$$\begin{pmatrix} y_1 & x_1 \\ y_2 & x_2 \end{pmatrix} \begin{pmatrix} d_1 & -t \\ -t & d_2 \end{pmatrix} \begin{pmatrix} y_1 - y_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $\left(\begin{smallmatrix} y_1 - y_2 \\ x_1 - x_2 \end{smallmatrix}\right)$ is nonzero and $\det\left(\begin{smallmatrix} d_1 & -t \\ -t & d_2 \end{smallmatrix}\right) \neq 0$ since $t^2 - d_1 d_2 \neq 0$, so we must have $\det\left(\begin{smallmatrix} y_1 & x_1 \\ y_2 & x_2 \end{smallmatrix}\right) = 0$. Hence $(x_2, y_2) = \lambda(x_1, y_1)$ with some constant $\lambda \neq 1$, so $S_1 = (1-\lambda)^2 \neq 0$ by our assumptions. This is a contradiction, and the lemma is proved. □

Lemma 3.3. *Let d_1, d_2, t be as in Lemma 3.1, and let $Q_i(X, Y) = A_i X^2 + B_i XY + C_i Y^2$ $(i = 1, 2)$ be such that $(Q_1, Q_2) \in \mathcal{Q}_{d_1,d_2,t}$. Assume that $A_1 B_2 - A_2 B_1 \neq 0$. Define*

$$R(X, Y) = (A_1 B_2 - A_2 B_1) X^2 + 2XY(A_1 C_2 - A_2 C_1) + Y^2 (B_1 C_2 - B_2 C_1).$$

Then for $x, y \in \mathbb{Q}$ the following two statements are equivalent.

(i) $(x, y) \in C_{d_1,d_2,t}$.

(ii) *There are $a, b \in \mathbb{Q}$ such that $R(a, b) \neq 0$ and writing* $x_{a,b} := \dfrac{Q_1(a,b)}{R(a,b)}$, $y_{a,b} := \dfrac{Q_2(a,b)}{R(a,b)}$ *we have* $x = x_{a,b}$, $y = y_{a,b}$.

Proof. By a straightforward computation using the definitions we get the identity

$$d_2 (Q_1(a,b))^2 + d_1 (Q_2(a,b))^2 - 2t Q_1(a,b) Q_2(a,b) = (R(a,b))^2. \tag{3-6}$$

Introduce the abbreviations

$$a_1 = \frac{A_1}{A_1 B_2 - A_2 B_1}, \quad a_2 = \frac{A_2}{A_1 B_2 - A_2 B_1}. \tag{3-7}$$

Note that writing $a = 1$, $b = 0$ in (3-6) we get $(a_1, a_2) \in C_{d_1,d_2,t}$. We first assume (ii). Then (i) follows at once from (3-6).

We now assume (i). If $(x, y) = (a_1, a_2)$, then we can take $a = 1$, $b = 0$. So let us assume that $(x, y) \neq (a_1, a_2)$. It is easy to see that if $a, b \in \mathbb{Q}$, then

$$Q_1(a,b) - a_1 R(a,b) = \frac{b(a\alpha + b\beta)}{(A_1 B_2 - A_2 B_1)}, \quad Q_2(a,b) - a_2 R(a,b) = \frac{b(\gamma a + \delta b)}{(A_1 B_2 - A_2 B_1)} \tag{3-8}$$

with

$$\alpha := B_1(A_1B_2 - A_2B_1) + 2A_1(A_2C_1 - A_1C_2) = tA_1 - d_1A_2, \tag{3-9}$$

$$\beta := C_1(A_1B_2 - A_2B_1) + A_1(C_1B_2 - C_2B_1), \tag{3-10}$$

$$\gamma := B_2(A_1B_2 - A_2B_1) + 2A_2(A_2C_1 - A_1C_2) = -tA_2 + d_2A_1, \tag{3-11}$$

$$\delta := C_2(A_1B_2 - A_2B_1) + A_2(C_1B_2 - C_2B_1). \tag{3-12}$$

Let us write $g := \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$. One can compute that

$$\det g = 2(A_2B_1 - A_1B_2)\big((A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1)\big).$$

The last factor, in larger parentheses, equals $\frac{1}{4}(t^2 - d_1d_2)$ by (2-32) and (2-31). Hence $t^2 - d_1d_2 \neq 0$ and $A_1B_2 - A_2B_1 \neq 0$ imply $\det g \neq 0$.

Let us take $a, b \in \mathbb{Q}$ in the following way:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}. \tag{3-13}$$

By (3-8) and (3-13) we then easily get

$$\begin{pmatrix} Q_1(a,b) - a_1R(a,b) \\ Q_2(a,b) - a_2R(a,b) \end{pmatrix} = \frac{b \det g}{A_1B_2 - A_2B_1}\begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}. \tag{3-14}$$

So this is true if $(x, y) \neq (a_1, a_2)$, and a, b are defined by (3-9)–(3-13).

Assume that $b = 0$. Then by (3-9), (3-11) and (3-13) we get

$$(d_1A_2 - tA_1)(y - a_2) + (-tA_2 + d_2A_1)(x - a_1) = 0.$$

By (3-7), $(a_1, a_2), (x, y) \in C_{d_1,d_2,t}$ and $(x, y) \neq (a_1, a_2)$. This contradicts (3-4). So we have $b \neq 0$.

Assume that $R(a, b) = 0$. Then (3-6) and (3-14) imply that

$$d_2(x - a_1)^2 + d_1(y - a_2)^2 - 2t(x - a_1)(y - a_2) = 0.$$

But this contradicts (3-3). So we have $R(a, b) \neq 0$.

Then (3-14) clearly implies

$$\begin{pmatrix} x_{a,b} - a_1 \\ y_{a,b} - a_2 \end{pmatrix} = \frac{b \det g}{R(a,b)(A_1B_2 - A_2B_1)}\begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix}.$$

Hence $(x_{a,b}, y_{a,b}) \neq (a_1, a_2)$, since we assumed $(x, y) \neq (a_1, a_2)$. We would like to show that $(x, y) = (x_{a,b}, y_{a,b})$. If this is false, then (a_1, a_2), $(x_{a,b}, y_{a,b})$ and (x, y) are three distinct points lying on a line, all belonging to $C_{d_1,d_2,t}$. Hence the equation

$$d_2(a_1 + q(x - a_1))^2 + d_1(a_2 + q(y - a_2))^2 - 2t(a_1 + q(x - a_1))(a_2 + q(y - a_2)) = 1$$

has three different real solutions in q. The coefficient of q^2 is nonzero by (3-3), so this is a contradiction. The lemma is proved. □

For the next proof we need the following notation. If p is a prime and $n \neq 0$ is an integer, let us denote by $\nu_p(n)$ the largest nonnegative integer such that $p^{\nu_p(n)}$ divides n.

Proof of Lemma 3.1. If $\mathcal{Q}_{d_1,d_2,t}$ is empty, then $h(d_1,d_2,t)=0$ and there is nothing to prove. So assume in the sequel that $\mathcal{Q}_{d_1,d_2,t} \neq \varnothing$. We divide the proof into four parts, which we formulate as claims.

Claim A. *Let $Q_i(X,Y)=A_iX^2+B_iXY+C_iY^2$ $(i=1,2)$ be quadratic forms such that $(Q_1,Q_2) \in \mathcal{Q}_{d_1,d_2,t}$. Then replacing (Q_1,Q_2) by an element in its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class we can achieve $B_2A_1-B_1A_2 \neq 0$.*

Proof. We first show the weaker statement that replacing (Q_1,Q_2) by an element in its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class we may assume that $(B_1,B_2) \neq (0,0)$. If $\tau = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$Q_i^{\tau}(X,Y)=Q_i(X+bY,Y)=A_iX^2+(B_i+2A_ib)XY+C_i^{*}Y^2 \tag{3-15}$$

for $i=1,2$ with some C_i^{*}. If $(B_1+2A_1b,\, B_2+2A_2b)=(0,0)$ for every integer b, then $A_i=B_i=0$ for $i=1,2$. But this is impossible, since this would imply $d_1=d_2=t=0$, but this contradicts $t^2-d_1d_2 \neq 0$.

Hence we may assume that $(B_1,B_2) \neq (0,0)$. Let $B_1 \neq 0$, say. Assume for a contradiction that $B_2A_1-B_1A_2=0$ and $B_2C_1-B_1C_2=0$. Then $(A_2,C_2)=\lambda(A_1,C_1)$ with $\lambda=B_2/B_1$, hence $C_2A_1-C_1A_2=0$. So the matrix $\left(\begin{smallmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{smallmatrix}\right)$ has rank 1, hence its lines are linearly dependent. But this contradicts $t^2-d_1d_2 \neq 0$.

So we may assume that $B_2A_1-B_1A_2 \neq 0$ or $B_2C_1-B_1C_2 \neq 0$. But applying the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we can exchange the roles of A_i and C_i. Claim A follows. □

Claim B. $C_{d_1,d_2,t} \neq \varnothing$.

Proof. By Claim A there is an element $(Q_1,Q_2) \in \mathcal{Q}_{d_1,d_2,t}$ whose coefficients satisfy $B_2A_1-B_1A_2 \neq 0$. Taking $a=1$, $b=0$ in Lemma 3.3(ii) we see by that lemma that there are numbers $x,y \in \mathbb{Q}$ such that $(x,y) \in C_{d_1,d_2,t}$. Claim B is proved. □

Claim C. *There exists a subset $\mathcal{A}$ of $\mathbb{Z}^2$ of size $2\tau(t^2-d_1d_2)$ such that every $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of $\mathcal{Q}_{d_1,d_2,t}$ contains an element (Q_1,Q_2) with coefficients $Q_i(X,Y)=A_iX^2+B_iXY+C_iY^2$ $(i=1,2)$ such that $(A_1,A_2) \in \mathcal{A}$.*

Proof. Fix $x,y \in \mathbb{Q}$ such that $(x,y) \in C_{d_1,d_2,t}$, this is possible by Claim B. We also fix relatively prime integers m and n and a nonzero $s \in \mathbb{Q}$ such that $x=sm$, $y=sn$. Let us take an arbitrary $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of $\mathcal{Q}_{d_1,d_2,t}$. We know by Claim A that we can take an element (Q_1,Q_2) in this equivalence class such that we have $B_2A_1-B_1A_2 \neq 0$ for their coefficients. Then it follows from Lemma 3.3 that there are $a,b \in \mathbb{Q}$ such that $Q_1(a,b)=qx$, $Q_2(a,b)=qy$ with some $q \in \mathbb{Q}$, $q \neq 0$. We may clearly assume here that $a,b \in \mathbb{Z}$ and $(a,b)=1$. Taking $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with some suitable c and d we then see that replacing (Q_1,Q_2) by an element in its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class we may assume that for their coefficients we have $A_1=rx$, $A_2=ry$ with some $r \in \mathbb{Q}$, $r \neq 0$. Observe also that

$$\gcd(A_1,A_2) \mid (t^2-d_1d_2) \tag{3-16}$$

follows from the definition of d_1, d_2 and t. Then recalling $x = sm$, $y = sn$ we see that $A_1 = \delta m$, $A_2 = \delta n$ with some integer δ dividing $t^2 - d_1 d_2$. Claim C is proved. □

Claim D. *Let the integers A_1 and A_2 be given and assume that there are N inequivalent elements in $\mathcal{Q}_{d_1,d_2,t}$ of the form $Q_i(X,Y) = A_i X^2 + B_i XY + C_i Y^2$ having these fixed coefficients A_1 and A_2. Recall (3-1). Then we have, with an absolute implied constant,*

$$N = O\big(2^{\omega(t^2-d_1 d_2)} S(d_1, d_2, t^2)\big). \tag{3-17}$$

Proof. We can assume $A_i \neq 0$ for $i = 1, 2$ by the assumption that d_i is not a square. Also, by the identity (3-6) with $a = 1$, $b = 0$ there are only two possibilities for $B_2 A_1 - B_1 A_2$. So we can fix D such that there are at least $N/2$ inequivalent forms having the fixed coefficients A_1, A_2 and having $B_2 A_1 - B_1 A_2 = D$. But we see from (3-15) that the residue of B_1 modulo $2A_1$ determines the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of (Q_1, Q_2) once A_1, A_2 and $B_2 A_1 - B_1 A_2$ is given. Therefore it is enough to estimate the possible values of B_1 modulo A_1 (for given A_1, A_2 and $D = B_2 A_1 - B_1 A_2$) by the right-hand side of (3-17).

We have

$$B_i^2 \equiv d_i \pmod{A_i} \tag{3-18}$$

for $i = 1, 2$. Let p be a prime, and let us denote $\nu_p(A_1)$ by α and $\nu_p(A_2)$ by β. We consider two cases.

Case $\nu_p(d_1) < \alpha$: We use (3-18) with $i = 1$. There is a solution only if $\nu_p(d_1) = 2k$ for some integer k, and then we must have $\nu_p(B_1) \geq k$ and $(B_1/p^k)^2 \equiv (d_1/p^{2k})$ modulo $p^{\alpha-2k}$. Since $\alpha - 2k > 0$, we get from this congruence that there are at most $2(1 + \nu_p(2))$ possibilities for B_1/p^k modulo $p^{\alpha-2k}$. Hence we finally get that there are at most $2(1 + \nu_p(2)) p^{[\nu_p(d_1)/2]}$ possibilities for B_1 modulo p^α.

Case $\nu_p(d_1) \geq \alpha$: We use again (3-18) with $i = 1$, and we see that $\nu_p(B_1) \geq \alpha/2$. So there are at most $p^{[\alpha/2]}$ possibilities for B_1 modulo p^α.

Hence in both cases there are at most $2(1 + \nu_p(2)) p^{\min([\frac{1}{2}\nu_p(d_1)],[\frac{1}{2}\alpha])}$ possibilities for B_1 modulo p^α. Similarly, there are at most $2(1 + \nu_p(2)) p^{\min([\frac{1}{2}\nu_p(d_2)],[\frac{1}{2}\beta])}$ possibilities for B_2 modulo p^β. Since $D = B_2 A_1 - B_1 A_2$ is fixed, we see that if B_2 is given modulo p^β, then $D + B_1 A_2 = B_2 A_1$ is given modulo $p^{\alpha+\beta}$, hence B_1 is given modulo p^α. Taking into account (3-16) we finally get that for every prime p there are at most

$$2(1 + \nu_p(2)) \min\big(p^{[\nu_p(d_1)/2]}, p^{[\nu_p(d_2)/2]}, p^{[\nu_p(t^2-d_1 d_2)/2]}\big)$$

possibilities for B_1 modulo $p^{\nu_p(A_1)}$. We apply this to every prime divisor p of A_1 that also divides $t^2 - d_1 d_2$. If $p \mid A_1$ but p does not divide $t^2 - d_1 d_2$, then by (3-16) p does not divide A_2, and so $D = B_2 A_1 - B_1 A_2$ implies that B_1 is determined modulo $p^{\nu_p(A_1)}$. So for the number of possible values of B_1 modulo A_1 we have the upper bound

$$C\Big(\prod_{p \mid t^2-d_1 d_2} 2\Big) \prod_p \min\big(p^{[\nu_p(d_1)/2]}, p^{[\nu_p(d_2)/2]}, p^{[\nu_p(t^2-d_1 d_2)/2]}\big) \tag{3-19}$$

with an absolute constant C. Formula (3-19) proves Claim D. □

Claims C and D imply Lemma 3.1 at once. □

3.2. ***Upper bounds for certain special averages of $h(d_1, d_2, t)$.*** When we apply Lemma 3.1, we will have numbers d_i of special form $d_i = t_i^2 - 4$, and we will have certain triple sums of $S(t_1^2-4, t_2^2-4, f^2)$, where t_1, t_2, f run over integers. We will use the trivial upper bounds $S(t_1^2-4, t_2^2-4, f^2) \le S(t_1^2-4, t_2^2-4)$ and $S(t^2-4, t^2-4, f^2) \le S(t^2-4, f^2)$, and we will use Lemmas 3.5, 3.6 and 3.7 below. We need a preliminary lemma.

Lemma 3.4. *Let $t_1, t_2 > 2$ be integers, $t_1 \ne t_2$, and let $E = \gcd(t_1^2-4, t_2^2-4)$.*

(i) *There is a divisor e of E such that*

$$e \ge c\sqrt{E}, \quad e \,|\, (t_1 - \delta t_2).$$

with an absolute constant $c > 0$ and with some $\delta \in \{-1, 1\}$.

(ii) $E \le |t_1 - t_2|(t_1 + t_2)$.

Proof. Part (ii) follows at once from the fact that E divides $t_1^2 - t_2^2$, so it remains to show part (i). Let $p \,|\, E$ be a prime. Then $p^{\nu_p(E)}$ divides $(t_1-t_2)(t_1+t_2)$, so writing $\alpha := \nu_p(t_1-t_2)$ and $\beta := \nu_p(t_1+t_2)$ we have $\nu_p(E) \le \alpha+\beta$. If $m = \min(\alpha, \beta)$, then $m \le \nu_p(2t_1)$, so $2m \le \nu_p(4t_1^2)$. But $0 < \nu_p(E) \le \nu_p(4t_1^2-16)$, so if $m > 0$, then we must have $p = 2$. If $p = 2$ and $m > 2$, then we have $\nu_2(4t_1^2) > 4$, and so $\nu_2(4t_1^2-16) = 4$, hence $\nu_2(E) \le 4$. It follows for every prime p that $p^{\nu_p(E)}$ divides either $16(t_1 - t_2)$ or $16(t_1 + t_2)$. Then there is a decomposition $E = e_1 e_2$ such that $\gcd(e_1, e_2) = 1$, and e_1 divides $16(t_1 - t_2)$, e_2 divides $16(t_1 + t_2)$. The lemma is proved. □

Lemma 3.5. *Let $3 \le a < b \le c \le 2a$ be integers. Recall the definition (3-1). For any $\epsilon > 0$ we have*

$$\sum_{t_1=a}^{c-1} \sum_{\substack{a \le t_2 \le c-1 \\ 0<|t_2-t_1| \le b-a}} S(t_1^2-4, t_2^2-4) \ll_\epsilon a^{1/2+\epsilon}(c-a)(b-a)^{1/2} \tag{3-20}$$

$$\sum_{t_1=a}^{c-1} \sum_{\substack{a \le t_2 \le c-1 \\ 0<|t_2-t_1|}} \frac{S(t_1^2-4, t_2^2-4)}{\sqrt{|t_1-t_2|}} \ll_\epsilon a^{1/2+\epsilon}(c-a). \tag{3-21}$$

Remark 3.2. With the very crude estimate $S(t_1^2-4, t_2^2-4) \ll a$ the trivial bound in (3-20) would be $(c-a)(b-a)a$, so in (3-20) we save roughly $\sqrt{(b-a)a}$.

Proof. Inequality (3-21) follows from (3-20) using a dyadic subdivision. To prove (3-20) we may assume $b = c$, since the general case follows by dividing the summation over t_1 into $O\left(\frac{c-a}{b-a}\right)$ subsums.

So let $b = c$. By Lemma 3.4(ii), we have $\gcd(t_1^2-4, t_2^2-4) \ll (b-a)b$. Then by Lemma 3.4(i) and because $S(t_1^2-4, t_2^2-4) \le \sqrt{\gcd(t_1^2-4, t_2^2-4)}$ the left-hand side of (3-20) is

$$\ll \sum_{t_1=a}^{b-1} \Bigg(\sum_{\substack{E \,|\, t_1^2-4 \\ E \ll (b-a)b}} \sqrt{E} \sum_{\substack{e \,|\, E \\ e \ge c\sqrt{E}}} \sum_{\delta \in \{-1,1\}} \sum_{\substack{a \le t_2 < b \\ e \,|\, t_2 - \delta t_1}} 1 \Bigg),$$

and for a given t_1 the quantity in parentheses is

$$\ll \sum_{\substack{E \mid t_1^2-4 \\ E \ll (b-a)b}} \sqrt{E} \sum_{\substack{e \mid E \\ e \geq c\sqrt{E}}} \left(1 + \frac{b-a}{e}\right) \ll b^{\epsilon}\left(\sqrt{(b-a)b} + b - a\right).$$

The lemma follows. □

Lemma 3.6. *Let $3 \leq a < b \leq 2a$ be integers. For every $\epsilon > 0$ we have*

$$\sum_{t=a}^{b-1} \max\{k \geq 1 : k^2 \mid t^2 - 4\} \ll_{\epsilon} a^{1+\epsilon}\sqrt{b-a}, \tag{3-22}$$

$$\sum_{t_1=a}^{b-1} \sum_{t_2=a}^{b-1} S(t_1^2 - 4, t_2^2 - 4) \ll_{\epsilon} a^{\epsilon}\left(a\sqrt{b-a} + a^{1/2}(b-a)^{3/2}\right). \tag{3-23}$$

Remark 3.3. Estimating every summand by $O(a)$ in (3-22) the trivial bound in (3-22) would be $(b-a)a$, so in (3-20) we save roughly $\sqrt{b-a}$.

Proof. Statement (3-23) follows at once from Lemma 3.5 and (3-22), so we deal only with (3-22). It is enough to show that for any integer $1 \leq K \leq 2a$ we have

$$K \sum_{t=a}^{b-1} \sum_{k=K}^{2K} \sum_{\substack{d>1 \\ t^2-4=dk^2}} \mu^2(d) \ll_{\epsilon} a^{\epsilon} a\sqrt{b-a}. \tag{3-24}$$

A trivial upper bound for the left-hand side of (3-24) is $K(b-a)$.

Let d be fixed and assume $t^2 - 4 = dk^2$. Then $\alpha := \frac{1}{2}(t + k\sqrt{d})$ is an algebraic integer, since it is a root of the equation $x^2 - tx + 1$. We also see that α is a unit in the ring R of algebraic integers of the real quadratic field $\mathbb{Q}(\sqrt{d})$. By the Dirichlet unit theorem, there is a unit $1 < \epsilon \in R$ such that every unit of R has the form $\pm\epsilon^l$ with integer l. One has $\epsilon = \frac{1}{2}(a + b\sqrt{d})$ with integers a, b, where $b \neq 0$. Then $\epsilon^{-1} = \delta \cdot \frac{1}{2}(a - b\sqrt{d})$ with $\delta \in \{-1, 1\}$, hence $\epsilon = b\sqrt{d} + \delta\epsilon^{-1}$, so $\epsilon > \sqrt{d} - 1 \geq \sqrt{2} - 1$. But $\alpha = \epsilon^l$ for some positive integer l and $\alpha \leq t \leq 2a$. So we have proved that for a fixed d there are at most $C \log a$ possibilities for the pair (t, k), with an absolute constant C. We have $d \ll a^2/K^2$ in (3-24); thus the left side of (3-24) is $\ll_{\epsilon} a^{2+\epsilon}/K$, and hence $\ll_{\epsilon} a^{\epsilon} \min(K(b-a), a^2/K)$, given the observation made after (3-24). This minimum here is clearly $\ll a\sqrt{b-a}$, and the lemma is proved. □

Lemma 3.7. *Let $t > 2$ be an integer and let $1 \leq A \ll t^2$. For any $\epsilon > 0$,*

$$\sum_{\substack{f \in \mathbb{Z} \\ t^2-4-A \leq |f| < t^2-4}} \frac{S(t^2-4, f^2)}{\sqrt{t^2-4-|f|}} \ll_{\epsilon} t^{\epsilon}\sqrt{A}.$$

Remark 3.4. We save roughly t here with respect to the trivial bound.

Proof. The left-hand side is at most

$$\sum_{k^2 \mid t^2-4} k \sum_{\substack{g \in \mathbb{Z} \\ 0 < \frac{t^2-4}{k} - |g| \le \frac{A}{k}}} \frac{1}{\sqrt{k}\sqrt{\frac{t^2-4}{k} - |g|}},$$

and the inner sum here is $\ll (1/\sqrt{k})\sqrt{A/k} = \sqrt{A}/k$. We used here that the inner sum is empty if $A < k$. The lemma is proved. □

4. Identities and estimates for special functions

In this section we consider the functions $\mathcal{I}(t_1, t_2, F, m_1, m_2)$ and $\mathcal{J}(t_1, t_2, m_1, m_2)$ defined in (2-10) and (2-9) for the special case when the functions m_i are characteristic functions of some intervals $[0, x_i]$ for $i = 1, 2$. This case will be important in our application. In the first subsection we prove an identity for this $\mathcal{I}$-function for every $1 \neq F > 0$; in the second and third subsections we use it to give estimates for the cases $F > 1$ and $F < 1$, respectively. In the last subsection we compute $\mathcal{J}(t_1, t_2, m_1, m_2)$ for the above-mentioned special case.

4.1. ***Computing $\mathcal{I}(t_1, t_2, F, m_1, m_2)$ when the m_i are characteristic functions.*** For $S_0, T_0, F > 0$, $F \neq 1$ define

$$Z(S_0, T_0, F) := \iint \frac{1}{\sqrt{S^2 + T^2 + 2FTS + 1 - F^2}}\, dS\, dT, \tag{4-1}$$

where we integrate over the set

$$\left\{(S, T) \in \mathbb{R}^2 : |S| \le S_0, |T| \le T_0,\ S^2 + T^2 + 2FTS + 1 - F^2 > 0\right\}.$$

By the reasoning of Remark 2.1 we see that (4-1) is absolutely convergent. One can see that (4-1) is divergent for $F = 1$, but we do not need that case. It is clear that the function $\mathcal{I}$ can be expressed in terms of the function Z in the case when the m_i are characteristic functions.

Lemma 4.1. *Let $S_0, T_0, F > 0$, $F \neq 1$. We have*

$$Z(S_0, T_0, F) = 2J(S_0, T_0, F) + 2J(T_0, S_0, F),$$

where we write

$$J(S_0, T_0, F) := \int_{\substack{|y| \le T_0/S_0 \\ (1+y^2+2Fy)S_0^2 > F^2 - 1}} \frac{\sqrt{(1 + y^2 + 2Fy)S_0^2 + 1 - F^2}}{1 + y^2 + 2Fy}\, dy \tag{4-2}$$

in the case $F > 1$, and

$$J(S_0, T_0, F) := \int_{|y| \le T_0/S_0} \frac{\sqrt{(1 + y^2 + 2Fy)S_0^2 + 1 - F^2}}{1 + y^2 + 2Fy}\, dy - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sqrt{1 - F^2}}{1 + y^2 + 2Fy}\, dy$$

in the case $F < 1$.

Proof. It is clear by the substitution $(S,T)\mapsto(-S,-T)$ that the $S<0$ and $S>0$ parts of the integral (4-1) have the same value. For $S>0$ we make the substitution $y=T/S$, and we get

$$Z(S_0,T_0,F)=2\int_{-\infty}^{\infty}\int\frac{S}{\sqrt{S^2+y^2S^2+2FyS^2+1-F^2}}\,dS\,dy,$$

where the inner integral is taken over the set

$$\{S\in\mathbb{R}:0\le S\le S_0,|Sy|\le T_0,\ S^2+y^2S^2+2FyS^2+1-F^2>0\}. \tag{4-3}$$

If $F<1$, the last condition is always true. If $F>1$, then $1+y^2+2Fy>0$ should hold, otherwise (4-3) is empty. For a fixed y we integrate in S over the interval

$$\sqrt{\frac{F^2-1}{1+y^2+2Fy}}\le S\le\min(S_0,T_0/|y|)$$

in the case $F>1$, and we integrate over

$$0\le S\le\min(S_0,T_0/|y|)$$

in the case $F<1$. We consider separately the cases $|y|\le T_0/S_0$ and $|y|\ge T_0/S_0$. We can compute the S-integral in each case. Making the substitution $y\mapsto 1/y$ in the case $|y|\ge T_0/S_0$ we obtain the lemma. □

4.2. ***The case $F>1$.*** In Lemma 4.3 we express the function $J(S_0,T_0,F)$ defined in Lemma 4.1 in terms of a simple function in the case $F>1$. Then we give estimates for this simple function in Lemma 4.4 and for higher derivatives of a variant of this function in Lemma 4.6.

Lemma 4.2. *Let $S_0,T_0>0$ and $F>1$. For the function $J(S_0,T_0,F)$ defined in (4-2) we have*

$$J(S_0,T_0,F)=\int_{H(S_0,T_0,F)}\frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}}{1+y^2+2Fy}\,dy, \tag{4-4}$$

where

$$H(S_0,T_0,F):=\begin{cases}[-T_0/S_0,T_0/S_0] & \text{if } T_0/S_0\le 1 \text{ and } 1<F\le A(S_0,T_0),\\ [-T_0/S_0,C(F,S_0)]\cup[D(F,S_0),T_0/S_0] & \text{if } T_0/S_0\ge 1 \text{ and } 1<F\le A(S_0,T_0),\\ [D(F,S_0),T_0/S_0] & \text{if } A(S_0,T_0)\le F\le B(S_0,T_0),\\ \varnothing & \text{if } F\ge B(S_0,T_0),\end{cases} \tag{4-5}$$

where we write

$$A(S_0,T_0):=\sqrt{(1+S_0^2)(1+T_0^2)}-S_0T_0,\quad B(S_0,T_0):=\sqrt{(1+S_0^2)(1+T_0^2)}+S_0T_0, \tag{4-6}$$

$$C(F,S_0):=-F-\sqrt{F^2-1}\sqrt{1+S_0^2}/S_0,\quad D(F,S_0):=-F+\sqrt{F^2-1}\sqrt{1+S_0^2}/S_0. \tag{4-7}$$

We mean every statement in such a way that if we write an interval $[a,b]$, this implicitly means that $a\le b$.

Proof. One can check that $(1+y^2+2Fy)S_0^2 > F^2-1$ holds if and only if $y<C(F,S_0)$ or $y>D(F,S_0)$. The following three claims can be checked by direct computation. For the proof of Claim 2 we use the obvious fact that $F-S_0T_0 > -\sqrt{(1+S_0^2)(1+T_0^2)}$.

Claim 1. *The sign of*

$$\frac{\sqrt{F^2-1}\sqrt{1+S_0^2}}{S_0} - \left|\frac{T_0}{S_0} - F\right| \tag{4-8}$$

equals the sign of $F-A(S_0,T_0)$.

Claim 2. *The sign of*

$$\frac{\sqrt{F^2-1}\sqrt{1+S_0^2}}{S_0} - \left(\frac{T_0}{S_0} + F\right) \tag{4-9}$$

equals the sign of $F-B(S_0,T_0)$.

Claim 3. *The sign of* T_0/S_0-1 *equals the sign of* $T_0/S_0-A(S_0,T_0)$.

We get from Claim 1 that if $T_0/S_0\le 1$ and $1<F\le A(S_0,T_0)$, then $D(F,S_0)\le -T_0/S_0$, and this gives the first branch of (4-5).

If $\frac{T_0}{S_0}\ge 1$ and $1<F\le A(S_0,T_0)$, we get from Claims 3 and 1 that

$$F + \frac{\sqrt{F^2-1}\sqrt{1+S_0^2}}{S_0} \le \frac{T_0}{S_0}, \tag{4-10}$$

and this implies the second branch.

If $A(S_0,T_0)\le F\le B(S_0,T_0)$, then $-T_0/S_0\le D(F,S_0)\le T_0/S_0$ by Claims 1 and 2, and $C(F,S_0)\le -T_0/S_0$ by Claim 1. This proves the third branch.

If $F\ge B(S_0,T_0)$, then $D(F,S_0)\ge T_0/S_0$ by Claim 2, and $C(F,S_0)\le -T_0/S_0$ by Claim 1. This gives the last branch, and the lemma is proved. ☐

Lemma 4.3. *Use the notation of Lemma 4.2. Write* $\sigma = 1+1/S_0^2$, *and for* $0<y<1$ *let*

$$\Phi(y) = \Phi(S_0,y) := \int_0^y \frac{\sigma r^2}{(1-r^2)(r^2+\sigma-1)}\,dr. \tag{4-11}$$

Then for $1<F\le B(S_0,T_0)$ *we have* $J(S_0,T_0,F) = S_0(\Phi(y_1)+\epsilon\Phi(y_2))$, *where*

$$y_1 = y_1(S_0,T_0,F) := \sqrt{1-\frac{(1+S_0^2)(F^2-1)}{(T_0+S_0F)^2}}, \quad y_2 = y_2(S_0,T_0,F) := \sqrt{1-\frac{(1+S_0^2)(F^2-1)}{(T_0-S_0F)^2}},$$

and $\epsilon=\epsilon(S_0,T_0,F)$ *is defined by*

$$\epsilon = \begin{cases} 1 & \text{if } T_0/S_0>1 \text{ and } 1<F\le A(S_0,T_0), \\ -1 & \text{if } T_0/S_0<1 \text{ and } 1<F\le A(S_0,T_0), \\ 0 & \text{if } A(S_0,T_0)<F\le B(S_0,T_0). \end{cases}$$

Assuming $1 < F \le B(S_0, T_0)$ *for* $i = 1$ *and* $1 < F \le A(S_0, T_0)$ *for* $i = 2$ *we have, for* $i = 1, 2$,

$$0 \le y_i \le 1 - c_1 \frac{F-1}{(1+S_0)^{c_2}(1+T_0)^{c_2}} \tag{4-12}$$

with some positive absolute constants c_1, c_2.

Proof. Note that $1 < F \le A(S_0, T_0)$ implies $T_0/S_0 \ne 1$ by Claim 3, so ϵ is well-defined. We get the statement $0 \le y_i \le 1$ by Claims 1 and 2 above. Then (4-12) follows by easy estimates using the conditions $S_0, T_0 > 0$ and $1 < F \le B(S_0, T_0)$.

To compute $J(S_0, T_0, F)$ we use (4-4). This is the same formula as (4-2), but the integration set is given there explicitly. Use the substitution

$$r = r(y) = \sqrt{1 - \frac{\sigma(F^2-1)}{(y+F)^2}} = \sqrt{\frac{1 + y^2 + 2Fy + (F^2-1)(1-\sigma)}{(y+F)^2}}. \tag{4-13}$$

We have a positive number under the square root by (4-2). It is clear from the conditions and the definitions of $C(F, S_0)$ and $D(F, S_0)$ that the sign of $y + F$ is constant on each of the four intervals which are present in (4-5); hence r is well-defined and strictly monotone on each of those intervals. It is easy to check that

$$\frac{\sqrt{1 + y^2 + 2Fy + (F^2-1)(1-\sigma)}}{1 + y^2 + 2Fy} \left| \frac{dy}{dr} \right| = \frac{\sigma r^2}{\left(1 - r^2\right)\left(r^2 + \sigma - 1\right)}. \tag{4-14}$$

We have $r(C(F, S_0)) = r(D(F, S_0)) = 0$, hence by Lemma 4.2 we get the present lemma. □

Lemma 4.4. *Let* $S_0 > 0$ *and* $0 < y < 1$. *For the function* $\Phi(S_0, y)$ *of Lemma 4.3 we have the estimates*

$$S_0 \Phi(S_0, y) \ll \begin{cases} S_0^3 y^3 & \text{if } S_0 \ge 1 \text{ and } 0 < y \le 1/(2S_0), \\ S_0 y & \text{if } S_0 \ge 1 \text{ and } 1/(2S_0) \le y \le \frac{1}{2}, \\ S_0 y^3 & \text{if } S_0 \le 1 \text{ and } y \le \frac{1}{2}. \end{cases} \tag{4-15}$$

Finally we have in every case

$$\Phi(S_0, y) \ll \log \frac{1}{1-y}. \tag{4-16}$$

The implied constants are absolute in formulas (4-15) and (4-16).

Proof. We have from the definitions that

$$S_0 \Phi(S_0, y) = S_0 \int_0^y \frac{S_0^2 r^2 + r^2}{(1-r^2)(S_0^2 r^2 + 1)} \, dr.$$

Every estimate follows easily. □

We recall Faà di Bruno's formula. If F and G are smooth functions and $H(x) = F(G(x))$, then for every $j \ge 1$ we have

$$H^{(j)}(x) = \sum_{l=1}^{j} \sum_{k=(k_1,\dots,k_j) \in H_{j,l}} a_{j,l,k} F^{(l)}(G(x)) \prod_{i=1}^{j} (G^{(i)}(x))^{k_i}, \tag{4-17}$$

with some constants $a_{j,l,k}$, where

$$H_{j,l}=\Big\{(k_1,\dots,k_j)\in\mathbb{Z}^j : k_i\ge 0,\ \sum_{i=1}^{j}k_i=l,\ \sum_{i=1}^{j}ik_i=j\Big\}. \tag{4-18}$$

This can be seen by induction using the chain rule.

Lemma 4.5. *Let $S_0>0$ and $\sigma=1+1/S_0^2$.*

(i) *Recall the definition of $\Phi(y)$ from (4-11). Write $\phi(t)=\Phi(1/t)$. For every $j\ge 1$ and $t>1$ we have*

$$\phi^{(j)}(t)\ll_j \frac{1}{t(t-1)^j}$$

uniformly in S_0.

(ii) *For $Y>\sqrt{\sigma}$ let $G(Y)=Y/\sqrt{Y^2-\sigma}$. Then for every $j\ge 1$ we have*

$$G^{(j)}(Y)\ll_j \begin{cases}\left(\frac{Y}{Y^2-\sigma}\right)^j\frac{Y}{\sqrt{Y^2-\sigma}} & \text{for } \sqrt{\sigma}<Y\le 2\sqrt{\sigma},\\ \frac{\sigma}{Y^{j+2}} & \text{for } Y\ge 2\sqrt{\sigma},\end{cases} \tag{4-19}$$

uniformly in S_0.

(iii) *For $Y>\sqrt{\sigma}$ let $H(Y)=\phi(G(Y))$. Then for every $j\ge 1$ and $Y>\sqrt{\sigma}$ we have*

$$H^{(j)}(Y)\ll_j \frac{\sqrt{Y^2-\sigma}}{Y}\left(\frac{Y}{Y^2-\sigma}\right)^j \tag{4-20}$$

uniformly in S_0.

Proof. By (4-11) and the substitution $r\mapsto 1/r$ we have

$$\phi(t)=\int_t^\infty \frac{\sigma}{(r^2-1)(1+r^2(\sigma-1))}\,dr.$$

The integrand here equals $\frac{1}{r^2-1}-\frac{1}{r^2+S_0^2}$. Considering separately the cases $t\ge 2$ and $1<t\le 2$ we obtain (i) easily.

For (ii), note that if $j\ge 1$, the left-hand side of (4-19) is a linear combination of terms of the form $Y^l/\big(\sqrt{Y^2-\sigma}\big)^{j+l}$, where $0\le l\le j+1$ and $j+l$ is odd. Here clearly $l=j+1$ gives the largest term, and we get the first branch of (4-19). The second branch follows easily from the Taylor expansion

$$G(Y)=\frac{1}{\sqrt{1-\sigma Y^{-2}}}=1+\frac{\sigma}{2Y^2}+\sum_{m=2}^{\infty}a_m\frac{\sigma^m}{Y^{2m}}, \tag{4-21}$$

where the a_m are absolute constants such that $\sum_{m=1}^{\infty}|a_m|\,r^m<\infty$ for every $0<r<1$.

For the proof of (iii) we use Faà di Bruno's formula (4-17), and we see that it is enough to estimate terms of the form

$$\phi^{(l)}(G(Y))\prod_{i=1}^{j}(G^{(i)}(Y))^{k_i}, \tag{4-22}$$

where $1\le l\le j$ and the k_i satisfy the conditions in (4-18).

If $Y \geq 2\sqrt{\sigma}$, then $1 < G(Y) \leq 2/\sqrt{3}$, and we get from (i) and (4-19) that (4-22) is

$$\ll_j \frac{1}{(G(Y)-1)^l} \prod_{i=1}^{j} \left(\frac{\sigma}{Y^{i+2}}\right)^{k_i} = \left(\frac{\sigma}{Y^2(G(Y)-1)}\right)^l Y^{-j},$$

where we used the conditions in (4-18). We see from (4-21) that $1 \ll \frac{\sigma}{Y^2(G(Y)-1)} \ll 1$, so we get (iii) for $Y \geq 2\sqrt{\sigma}$.

If $\sqrt{\sigma} < Y \leq 2\sqrt{\sigma}$, then $G(Y) \geq 2/\sqrt{3}$, and we get from (i) and (4-19) that (4-22) is

$$\ll_j \frac{1}{(G(Y))^{l+1}} \prod_{i=1}^{j} \left(\left(\frac{Y}{Y^2-\sigma}\right)^i \frac{Y}{\sqrt{Y^2-\sigma}}\right)^{k_i} = \frac{1}{G(Y)} \left(\frac{Y/G(Y)}{\sqrt{Y^2-\sigma}}\right)^l \left(\frac{Y}{Y^2-\sigma}\right)^j,$$

where we used the conditions in (4-18). By the definition of $G(Y)$ we get (iii) also for this case. The lemma is proved. ☐

Lemma 4.6. *Let $S_0 > 0$, $F > 1$, $t \geq 3$, $\tau \in \{-1, 1\}$ be given. If $x > t^2 - 4$, write*

$$T_0 = T_0(x) := \sqrt{\frac{x}{t^2-4} - 1} \quad \text{and} \quad R(x) := \frac{|\tau F S_0 + T_0(x)|}{S_0\sqrt{F^2-1}}. \tag{4-23}$$

Let the number σ and the function H be defined as in Lemma 4.5, and let us define $K(x) = H(R(x))$ for $x \in H_{S_0,F,t,\tau}$, where

$$H_{S_0,F,t,1} := \{x > t^2-4 : F < B(S_0, T_0(x))\} \quad \text{and} \quad H_{S_0,F,t,-1} := \{x > t^2-4 : F < A(S_0, T_0(x))\}$$

(see (4-6)). *Then K is well-defined. If $\tau = -1$, then $K(x)$ is a smooth function for $\{x \in H_{S_0,F,t,-1} : T_0(x) < S_0\}$ and also for $\{x \in H_{S_0,F,t,-1} : T_0(x) > S_0\}$. For every $j \geq 1$ and every x satisfying the above conditions we have*

$$K^{(j)}(x) \ll_j (x-t^2+4)^{-j} \max\left(1, \left(\frac{T_0|T_0 - FS_0|}{\sqrt{(S_0^2+1)(T_0^2+1)(A(S_0,T_0)-F)}}\right)^j\right) \quad \textit{for } \tau = -1,$$

$$K^{(j)}(x) \ll_j (x-t^2+4)^{-j} \max\left(1, \left(\frac{T_0(T_0 + FS_0)}{\sqrt{(S_0^2+1)(T_0^2+1)}\,(B(S_0,T_0)-F)}\right)^j\right) \quad \textit{for } \tau = 1.$$

Proof. From Claims 1 and 2 we have $R(x) > \sqrt{\sigma}$; hence $K(x)$ is well-defined. Also, for $\tau = -1$ we have $|-FS_0 + T_0| = FS_0 - T_0$ in the case $S_0 > T_0$, and $|-FS_0 + T_0| = T_0 - FS_0$ in the case $S_0 < T_0$. This follows from the conditions, using Claim 3. We cannot have $T_0 = S_0$ if $\tau = -1$, because $T_0 = S_0$ implies $A(S_0, T_0) = 1$, so $1 < F < A(S_0, T_0)$ is impossible. Hence if $\tau = -1$, then $R(x)$ is indeed a smooth function for $T_0 < S_0$, and also for $T_0 > S_0$, so we can speak about the derivatives of K.

We see from Faà di Bruno's formula (4-17) that it is enough to estimate terms of the form

$$H^{(l)}(R(x)) \prod_{i=1}^{j} (R^{(i)}(x))^{k_i}, \tag{4-24}$$

where $1 \leq l \leq j$ and the k_i satisfy the conditions in (4-18). It is clear from the definitions that

$$R^{(i)}(x) \ll_i \frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{F^2-1}}. \quad \text{for } i \geq 1.$$

Hence, using also Lemma 4.5(iii) we get that (4-24) is

$$\ll_j \frac{\sqrt{(R(x))^2-\sigma}}{R(x)} \left(\frac{R(x)}{(R(x))^2-\sigma}\right)^l \prod_{i=1}^{j} \left(\frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{F^2-1}}\right)^{k_i}.$$

By the conditions in (4-18) and the definition of T_0 in (4-23), this equals

$$\frac{\sqrt{(R(x))^2-\sigma}}{R(x)} \left(\frac{R(x)T_0}{\left((R(x))^2-\sigma\right) S_0\sqrt{F^2-1}}\right)^l (x-t^2+4)^{-j}.$$

Note that $\frac{\sqrt{(R(x))^2-\sigma}}{R(x)} \leq 1$. From the definition of R in (4-23) is easy to compute, using also $\sigma = 1+1/S_0^2$, that

$$S_0\sqrt{F^2-1}\,\frac{(R(x))^2-\sigma}{R(x)} = \frac{(S_0^2+1)(T_0^2+1)-(F-\tau S_0T_0)^2}{|\tau F S_0+T_0|}.$$

The lemma is proved. □

4.3. *The case F < 1.* In Lemma 4.7 we give a new expression for the function $J(S_0, T_0, F)$ defined in Lemma 4.1 in the case $F < 1$, and we also give upper bounds for the new expression. In Lemma 4.8 we give another new expression for $J(S_0, T_0, F)$, expressing it in terms of a simple function. Then in Lemma 4.10 we give estimates for higher derivatives of a variant of this simple function.

Lemma 4.7. *Let* $0 < S_0$, $0 < T_0 < T_0^*$, $0 < F < 1$. *We have*

$$J(S_0, T_0, F) + J(T_0, S_0, F) = K(S_0, T_0, F) + K(T_0, S_0, F), \tag{4-25}$$

where

$$K(S_0, T_0, F) := \int_{|y|\leq T_0/S_0} \frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}-\sqrt{1-F^2}}{1+y^2+2Fy}\,dy.$$

We also have

$$K(S_0, T_0, F) \ll \frac{S_0T_0}{\sqrt{1-F^2}}, \qquad K(S_0, T_0^*, F) - K(S_0, T_0, F) \ll \frac{S_0(T_0^*-T_0)}{\sqrt{1-F^2}}. \tag{4-26}$$

Proof. The first statement follows from Lemma 4.1 and

$$\int_{|y|\leq T_0/S_0} \frac{dy}{1+y^2+2Fy} + \int_{|y|\leq S_0/T_0} \frac{dy}{1+y^2+2Fy} = \int_{-\infty}^{\infty} \frac{dy}{1+y^2+2Fy},$$

which follows by the substitution $y \to 1/y$. We have

$$\frac{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}-\sqrt{1-F^2}}{1+y^2+2Fy} = \frac{S_0^2}{\sqrt{(1+y^2+2Fy)S_0^2+1-F^2}+\sqrt{1-F^2}} \leq \frac{S_0^2}{\sqrt{1-F^2}}.$$

The lemma follows. □

Lemma 4.8. *Let $S_0, T_0 > 0$ and $F < 1$. Write $\sigma = 1 + 1/S_0^2$, and for $-1 < t < 1$ let*

$$V(t) = V(S_0, t) := \int_0^t \frac{\sigma}{(1-r^2)(1+(\sigma-1)r^2)}\,dr. \tag{4-27}$$

Then

$$J(S_0, T_0, F) = S_0(V(s_1) - V(s_2)) - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sqrt{1-F^2}}{1+y^2+2Fy}\,dy,$$

where

$$s_1 = s_1(S_0, T_0, F) := \frac{S_0 F + T_0}{\sqrt{(S_0 F + T_0)^2 + (1+S_0^2)(1-F^2)}},$$

$$s_2 = s_2(S_0, T_0, F) := \frac{S_0 F - T_0}{\sqrt{(S_0 F - T_0)^2 + (1+S_0^2)(1-F^2)}}.$$

Proof. We use the substitution

$$r = r(y) = \frac{y+F}{\sqrt{(y+F)^2 + \sigma(1-F^2)}}$$

in the first integral in the definition of $J(S_0, T_0, F)$. It is easy to check that

$$\frac{\sqrt{1+y^2+2Fy+(1-F^2)(\sigma-1)}}{1+y^2+2Fy}\,dy = \frac{\sigma}{(1-r^2)(1+(\sigma-1)r^2)}dr,$$

and the lemma follows. □

Lemma 4.9. *Let $S_0 > 0$, $\sigma = 1 + 1/S_0^2$, and let V be as in (4-27).*

(i) *For every $j \geq 1$ and $-1 < t < 1$ we have*

$$V^{(j)}(t) \ll_j (1-|t|)\left(\frac{1}{|t|+S_0} + \frac{1}{1-|t|}\right)^{j+1}$$

uniformly in S_0.

(ii) *For $-\infty < Y < \infty$ let $g(Y) = Y/\sqrt{Y^2+\sigma}$. Then for every $j \geq 1$ we have*

$$g^{(j)}(Y) \ll_j \left(\frac{1}{\sqrt{\sigma}}\right)^j \quad \text{for } |Y| \leq 2\sqrt{\sigma}, \tag{4-28}$$

$$g^{(j)}(Y) \ll_j \frac{\sigma}{|Y|^{j+2}} \quad \text{for } |Y| \geq 2\sqrt{\sigma}, \tag{4-29}$$

uniformly in S_0.

(iii) *For $-\infty < Y < \infty$ let $h(Y) = V(g(Y))$. Then for every $j \geq 1$ and $Y > 0$ we have*

$$h^{(j)}(Y) \ll_j \sqrt{\sigma}(1+|Y|)^{-j}$$

uniformly in S_0.

Proof. For the proof of (i) note that

$$\frac{\sigma}{(1-r^2)(1+(\sigma-1)r^2)}=\frac{1}{1-r^2}+\frac{1}{S_0^2+r^2}=\frac{1}{1-r^2}+\frac{1/2S_0}{S_0+ir}+\frac{1/2S_0}{S_0-ir}.$$

Considering first the case $|t|\geq\frac{1}{2}$, and then if $|t|\leq\frac{1}{2}$, then considering separately $|t|\leq S_0$ and $|t|\geq S_0$ we get (i). In (ii) we can assume $Y\geq 0$, and then the proof is completely similar to that of Lemma 4.5(ii).

For the proof of (iii) we use Faà di Bruno's formula (4-17), and we see that it is enough to estimate terms of the form

$$V^{(l)}(g(Y))\prod_{i=1}^{j}(g^{(i)}(Y))^{k_i}, \tag{4-30}$$

where $1\leq l\leq j$ and the k_i satisfy the conditions in (4-18).

If $|Y|\geq 2\sqrt{\sigma}$, then $|g(Y)|\geq 2/\sqrt{5}$, and we get from (i) and (4-29) that (4-30) is

$$\ll_j \frac{1}{(1-|g(Y)|)^l}\prod_{i=1}^{j}\left(\frac{\sigma}{|Y|^{i+2}}\right)^{k_i}=\left(\frac{\sigma}{|Y|^2\,(1-|g(Y)|)}\right)^{l}|Y|^{-j},$$

where we used the conditions in (4-18). It is easy to see that $1\ll(\sigma/|Y|^2)/(1-|g(Y)|)\ll 1$, so taking into account $\sigma\geq 1$ we get (iii) for $|Y|\geq 2\sqrt{\sigma}$. If $|Y|\leq 2\sqrt{\sigma}$, then $|g(Y)|\leq 2/\sqrt{5}$, and we get from (i) and (4-28) that (4-30) is

$$\ll_j\left(\frac{1}{|g(Y)|+S_0}+1\right)^{l+1}\prod_{i=1}^{j}\left(\left(\frac{1}{\sqrt{\sigma}}\right)^{i}\right)^{k_i}=\left(\frac{1}{|g(Y)|+S_0}+1\right)^{l+1}\left(\frac{1}{\sqrt{\sigma}}\right)^{j},$$

where we used the conditions in (4-18). If $S_0\gg 1$, then $1\ll\sigma\ll 1$, and we get (iii). If $S_0\ll 1$, then $1/\sqrt{\sigma}\ll S_0\ll 1/\sqrt{\sigma}$, $|Y|/\sqrt{\sigma}\ll|g(Y)|\ll|Y|/\sqrt{\sigma}$, and

$$\left(\frac{1}{|g(Y)|+S_0}+1\right)^{l+1}\left(\frac{1}{\sqrt{\sigma}}\right)^{j}\ll\left(\frac{1}{|g(Y)|+S_0}\right)\left(\frac{1}{|g(Y)|\sqrt{\sigma}+S_0\sqrt{\sigma}}\right)^{j}.$$

The lemma follows. □

Lemma 4.10. *Let $S_0>0$, $F<1$, $t\geq 3$, $\tau\in\{-1,1\}$ be given. If $x>t^2-4$, write*

$$T_0=T_0(x):=\sqrt{\frac{x}{t^2-4}-1}\quad\text{and}\quad r(x):=\frac{FS_0+\tau T_0}{S_0\sqrt{1-F^2}}. \tag{4-31}$$

Let the number σ and the function h be defined as in Lemma 4.9, and let us define $k(x)=h(r(x))$ for every x satisfying $x>t^2-4$. Then for every $j\geq 1$ and every $x>t^2-4$ we have

$$k^{(j)}(x)\ll_j\sqrt{\sigma}(x-t^2+4)^{-j}\max\left(1,\left(\frac{T_0}{S_0\sqrt{1-F^2}+|\tau FS_0+T_0|}\right)^{j}\right). \tag{4-32}$$

Proof. We see from Faà di Bruno's formula (4-17) that it is enough to estimate terms of the form

$$h^{(l)}(r(x))\prod_{i=1}^{j}(r^{(i)}(x))^{k_i}, \tag{4-33}$$

where $1\le l\le j$ and the k_i satisfy the conditions in (4-18). It is clear from the definitions that for $i\ge 1$ we have

$$r^{(i)}(x)\ll_i \frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{1-F^2}}.$$

Hence, using also Lemma 4.9(iii), we get that (4-33) is

$$\ll_j \sqrt{\sigma}\,(1+|r(x)|)^{-l}\prod_{i=1}^{j}\left(\frac{(x-t^2+4)^{1/2-i}}{S_0\sqrt{t^2-4}\sqrt{1-F^2}}\right)^{k_i}.$$

Using the conditions in (4-18) and the relations (4-31) we obtain the lemma. □

4.4. ***Computing $\mathcal{J}(t_1,t_2,m_1,m_2)$ when the m_i are characteristic functions.*** For $x>0$ introduce the notation

$$k_x(y)=1 \text{ for } 0\le y\le x,\quad k_x(y)=0 \text{ for } y>x. \tag{4-34}$$

Lemma 4.11. *Let $t_i>2$ and $x_i>0$ for $i=1,2$. Then $\mathcal{J}\left(t_1,t_2,k_{x_1/4},k_{x_2/4}\right)$ is nonzero only if $x_i>t_i^2-4$ for $i=1,2$. Assuming that this is true, we have*

$$\mathcal{J}\left(t_1,t_2,k_{x_1/4},k_{x_2/4}\right)=2\frac{\sqrt{1-m}}{\sqrt{m}},$$

where $m:=\max\left(\frac{t_1^2-4}{x_1},\frac{t_2^2-4}{x_2}\right)$.

Proof. The statement is trivial for $m\ge 1$, so let us assume $m<1$. Then by definition we have

$$\mathcal{J}(t_1,t_2,k_{x_1/4},k_{x_2/4})=\int_{-\arccos\sqrt{m}}^{\arccos\sqrt{m}}\frac{d\theta}{\cos^2\theta}=2\frac{\sin\left(\arccos\sqrt{m}\right)}{\cos\left(\arccos\sqrt{m}\right)}.$$

The lemma is proved. □

5. First steps of the proof of Theorem 1.1

Let η_0 be a nonnegative smooth function on $(0,\infty)$ vanishing outside of $\tau\in[1,2]$ and such that

$$\int_1^2\eta_0(\tau)\,d\tau=1. \tag{5-1}$$

Recall the definition of $k_x(y)$ from (4-34). For $x>0$, $D>0$, define

$$k_{x,D}(y):=\frac{1}{D}\int_D^{2D}\eta_0\left(\frac{\tau}{D}\right)k_x(y+\tau)\,d\tau \tag{5-2}$$

for $y\ge 0$. We will use also the notations of Theorem 1.1.

5.1. *A spectral estimate.* Our aim is to prove Lemma 5.2, whose result will show that for a smoothed version of the hyperbolic circle problem one can give a good estimate by spectral methods. We first need some notation.

The hyperbolic Laplace operator is denoted by $\Delta := y^2\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)$. Let $\{u_j(z) : j \geq 0\}$ be a complete orthonormal system of Maass forms for $\mathrm{PSL}_2(\mathbb{Z})$ (the function $u_0(z)$ being constant), and let $\Delta u_j = \left(-\frac{1}{4} - t_j^2\right)u_j$, where $t_0 = \frac{i}{2}$ and t_j is real for $j > 0$.

If m is a compactly supported bounded function on $[0, \infty)$, let (see [I, (1.62)])

$$g_m(a) = 2q_m\left(\frac{e^a + e^{-a} - 2}{4}\right), \quad \text{where } q_m(v) = \int_0^\infty \frac{m(v+\tau)}{\sqrt{\tau}}\,d\tau, \tag{5-3}$$

and for any complex r let

$$h_m(r) = \int_{-\infty}^{\infty} g_m(a)e^{ira}\,da. \tag{5-4}$$

For simplicity introduce the abbreviations $h_x = h_{k_x}$ and $h_{x,D} = h_{k_{x,D}}$.

Lemma 5.1. *Assume* $1 < D < x/10$. *For every integer* $j \geq 0$ *and all* $r \geq 1$ *we have*

$$h_{x,D}(r) \ll_j \frac{x^{1/2}}{r^{3/2}} \min\left(1, \frac{x}{Dr}\right)^j + \frac{x^{1/2}}{r^{5/2}}. \tag{5-5}$$

We also have for every real r

$$h_{x,D}(r) \ll x^{1/2}\log x. \tag{5-6}$$

Furthermore, we have

$$h_{x,D}\left(\frac{i}{2}\right) = 4\pi x - 4\pi D\int_1^2 \eta_0(\tau)\tau\,d\tau. \tag{5-7}$$

Proof. It is easy to see from (5-3), (5-4) and (5-2) that

$$h_{x,D}(r) = \frac{1}{D}\int_D^{2D} \eta_0\left(\frac{\tau}{D}\right) h_{x-\tau}(r)\,d\tau \tag{5-8}$$

for every complex r. Now we apply Lemma 2.4 of [C] for the function $h_{x-\tau}(r)$ choosing $R = R(\tau)$ in that lemma in such a way that

$$\tfrac{1}{2}\cosh R(\tau) - \tfrac{1}{2} = x - \tau. \tag{5-9}$$

Applying part (d) of that lemma we see that $h_{x-\tau}\left(\frac{i}{2}\right) = 4\pi(x-\tau)$, and taking into account (5-1) we get (5-7). The estimate (5-6) follows from a trivial estimation of (2.6) of [C]. Finally, for the proof of (5-5) we apply part (a) of Lemma 2.4 of [C]. Applying it in (5-8) we get for $r \geq 1$ that

$$h_{x,D}(r) = \frac{2\sqrt{2\pi}}{r^{3/2}D}\int_D^{2D} \eta_0\left(\frac{\tau}{D}\right)\sqrt{\sinh R(\tau)}\cos\left(rR(\tau) - \tfrac{3}{4}\pi\right)d\tau + O\left(\frac{x^{1/2}}{r^{5/2}}\right).$$

Through the substitution $R = R(\tau)$ and use of (5-9) this equals

$$\frac{\sqrt{2\pi}}{r^{3/2}D}\int_{R_1}^{R_2} \eta_0\left(\frac{1 + 2x - \cosh R}{2D}\right)(\sinh R)^{3/2}\cos\left(rR - \tfrac{3}{4}\pi\right)dR + O\left(\frac{x^{1/2}}{r^{5/2}}\right),$$

where $\cosh R_1 = 1 + 2x - 4D$ and $\cosh R_2 = 1 + 2x - 2D$. Repeated partial integration gives (5-5). The lemma is proved. □

Lemma 5.2. *Assume* $1 < D < x/10$ *and* $z \in \Omega$. *Then*

$$\sum_{\gamma\in\Gamma} k_{x,D}(u(\gamma z, z)) = 12x - 12D \int_1^2 \eta_0(\tau)\tau\, d\tau + O_\Omega\left(\frac{x}{\sqrt{D}} + x^{1/2}\log x\right). \tag{5-10}$$

Proof. It is clear by (5-3) and (5-4) that the function $h_{x,D}(r)$ satisfies [I, condition (1.63)], i.e., it is even, it is holomorphic in the strip $|\operatorname{Im} r| \le \frac{1}{2} + \epsilon$ and $h_{k_{x,D}}(r) = O((1+|r|)^{-2-\epsilon})$ in this strip for some $\epsilon > 0$. Then we get from Theorem 7.4 of [I], using again the abbreviation $h_{x,D} = h_{k_{x,D}}$, that the left-hand side of (5-10) equals

$$\sum_{j=0}^{\infty} h_{x,D}(t_j)|u_j(z)|^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_{x,D}(r)\left|E\left(z, \tfrac{1}{2} + ir\right)\right|^2 dr$$

for any $z \in \mathbb{H}$, where $E(z, s)$ is the Eisenstein series for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$; see [I, Chapter 3]. For the fundamental domain $\mathcal{F}$ defined in (1-3) we have $|\mathcal{F}| = \pi/3$ by [I, (6.33) and (3.26)]; hence $|u_0(z)|^2$ equals $3/\pi$ for every z. By Lemma 5.1 above and by [I, Proposition 7.2], we get the lemma. □

5.2. *Nonhyperbolic elements.* We give an easy estimate for the contribution of the nonhyperbolic elements in the hyperbolic circle problems.

Lemma 5.3. *Let* $z \in \Omega$ *and* $X > 2$. *Then for every* $\epsilon > 0$ *we have*

$$\left|\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) : |\operatorname{tr}\gamma| \le 2,\ 4u(\gamma z, z) \le X - 2\}\right| \ll_{\Omega,\epsilon} X^{1/2+\epsilon}. \tag{5-11}$$

Proof. Write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. By [I, (1.9), (1.11)] we have

$$4u(\gamma z, z) = \frac{|cz^2 + (d-a)z - b|^2}{\operatorname{Im}^2 z}.$$

It is easy to compute that if $z = x + iy$, then

$$\operatorname{Im}\left(cz^2 + (d-a)z - b\right) = 2cxy + (d-a)y \quad \text{and} \quad \operatorname{Re}\left(cz^2 + (d-a)z - b\right) = c(x^2 - y^2) + (d-a)x - b.$$

Hence if $z \in \Omega$, then the second inequality in (5-11) gives

$$2cx + d - a \ll_\Omega \sqrt{X}, \tag{5-12}$$

$$c(x^2 + y^2) + b \ll_\Omega \sqrt{X}. \tag{5-13}$$

By the first inequality in (5-11) we get from (5-12) that

$$d = -cx + O_\Omega(\sqrt{X}), \quad a = cx + O_\Omega(\sqrt{X}),$$

and from these relations and (5-13) we get

$$1 = ad - bc = -c^2x^2 + c^2(x^2 + y^2) + O_\Omega\left(\sqrt{X}\,(\sqrt{X} + |c|)\right).$$

This implies $c = O_\Omega(\sqrt{X})$, and so (5-12) gives $d - a \ll_\Omega \sqrt{X}$. Then there are $O_\Omega(\sqrt{X})$ possibilities for the pair (a, d). If a and d are given with $ad \neq 1$, then $bc = ad - 1$ implies that there are $O_\epsilon(X^\epsilon)$ possibilities for the pair (b, c). Finally, if $ad = 1$, then $bc = 0$, and so (5-13) implies that there are $\ll_\Omega \sqrt{X}$ possibilities for the pair (b, c). The lemma is proved. □

5.3. ***Reduction to the estimation of a square integral on the fundamental domain.*** Let us take an integer $J \geq 2$; it will be fixed but we will choose it sufficiently large. Let d be a parameter that will be chosen optimally later, at the moment we assume that $d \geq 100$ and $100Jd \leq X$.

Let us define

$$N_{d,J}(z, X) := \sum_{j=0}^{J} (-1)^j \binom{J}{j} \int_1^2 \eta_0(\tau) N(z, X - jd\tau)\, d\tau. \tag{5-14}$$

Then using (5-1) we see that $N_{d,J}(z, X)$ equals

$$N(z, X) + \sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{\gamma \in \Gamma} \int_1^2 \eta_0(\tau) k_{\frac{X-2}{4}}\bigl(u(\gamma z, z) + \tfrac{1}{4} jd\tau\bigr)\, d\tau,$$

which equals

$$N(z, X) + \sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{\gamma \in \Gamma} k_{\frac{X-2}{4}, \frac{jd}{4}}(u(\gamma z, z))$$

by (5-2). Applying Lemma 5.2 this equals

$$N(z, X) + O_{\Omega, J}\left(\frac{X}{\sqrt{d}} + X^{1/2} \log X\right) + \sum_{j=1}^{J} (-1)^j \binom{J}{j} \Bigl(3X - 3jd \int_1^2 \eta_0(\tau) \tau\, d\tau\Bigr)$$

for $z \in \Omega$. Now, it follows from the binomial theorem, taking into account that $j\binom{J}{j} = J\binom{J-1}{j-1}$ for $1 \leq j \leq J$, that

$$\sum_{j=1}^{J} (-1)^j \binom{J}{j} = -1 \quad \text{and} \quad \sum_{j=1}^{J} (-1)^j j \binom{J}{j} = 0.$$

Hence we have proved that for $z \in \Omega$

$$N_{d,J}(z, X) = N(z, X) - 3X + O_{\Omega, J}\left(\frac{X}{\sqrt{d}} + X^{1/2} \log X\right). \tag{5-15}$$

Recalling the notation $M_{t,m}$ from (2-8) we get from Lemma 5.3 that

$$N(z, X) = O_{\Omega, \epsilon}(X^{1/2+\epsilon}) + \sum_{t>2} M_{t, k_{\frac{X-2}{4}}}(z)$$

for $z \in \Omega$, $X > 2$ and for any $\epsilon > 0$. Hence by (5-14) we see that

$$N_{d,J}(z, X) = O_{\Omega, \epsilon, J}(X^{1/2+\epsilon}) + \int_1^2 \eta_0(\tau) \Bigl(\sum_{t>2} \sum_{j=0}^{J} (-1)^j \binom{J}{j} M_{t, k_{\frac{X-jd\tau-2}{4}}}(z) \Bigr)\, d\tau$$

for $z \in \Omega$, $\epsilon > 0$. By Cauchy–Schwarz we have that

$$\left(\int_1^2 \eta_0(\tau)\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)\,d\tau\right)^2$$

is

$$\ll \int_1^2\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)^2\,d\tau.$$

Hence, using also (5-15) we finally get that

$$\int_\Omega (N(z,X)-3X)^2\,d\mu_z \tag{5-16}$$

is

$$O_{\Omega,\epsilon,J}\left(\frac{X^2}{d}+X^{1+\epsilon}+\int_1^2\int_{\mathcal{F}}\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)^2\,d\mu_z\,d\tau\right) \tag{5-17}$$

if $\epsilon > 0$, $d \geq 100$ and $100Jd \leq X$.

We will show in Section 6 that if $\epsilon > 0$ is given and the integer $J \geq 2$ is fixed to be large enough in terms of ϵ, then we have

$$\int_{\mathcal{F}}\Big(\sum_{t>2}\sum_{j=0}^{J}(-1)^j\binom{J}{j}M_{t,k_{\frac{X-jd\tau-2}{4}}}(z)\Big)^2\,d\mu_z \ll_\epsilon X^\epsilon\frac{d^{5/2}}{\sqrt{X}} \tag{5-18}$$

uniformly for $1 \leq \tau \leq 2$ and

$$X^{2/3} \leq d \leq X^{99/100}. \tag{5-19}$$

Assume that (5-18) is true. Then we see from (5-16) and (5-17) that (5-16) equals

$$O_{\Omega,\epsilon}\left(\frac{X^2}{d}+X^\epsilon\frac{d^{5/2}}{\sqrt{X}}\right) \tag{5-20}$$

for any d satisfying (5-19). Note that we choose J in terms of ϵ, so we do not have to denote the dependence on J in (5-20). Choosing $d = X^{5/7}$ we obtain Theorem 1.1. So it is enough to show the estimate (5-18).

6. Conclusion

The goal of this section is to prove the estimate (5-18).

6.1. ***Application of Lemma 2.2 and basic observations.*** It is easy to see that if $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and the trace of γ is $t > 2$, then we have $u(\gamma z, z) \geq \frac{1}{4}(t^2-4)$ for every $z \in \mathbb{H}$. Therefore, the contribution of the terms $t > \sqrt{X+2}$ to the sum (5-18) is 0. We can take integers $1 \leq I \ll \log X$ and

$$3 = a_1 < a_2 < \cdots < a_I < 1+\sqrt{X+2} \leq a_{I+1} < 2+\sqrt{X+2} \tag{6-1}$$

such that

$$a_{i+1} \leq \tfrac{3}{2}a_i, \quad 3+\sqrt{X+2}-a_i \leq 2\big(3+\sqrt{X+2}-a_{i+1}\big) \tag{6-2}$$

for $1 \le i \le I$. By the Cauchy–Schwarz inequality we get for every $1 \le \tau \le 2$ that

$$\int_{\mathcal{F}} \Big(\sum_{t>2} \sum_{j=0}^{J} (-1)^j \binom{J}{j} M_{t,k_{\frac{X-jd\tau-2}{4}}}(z) \Big)^2 d\,\mu_z \ll \log X \sum_{i=1}^{I} U_i \tag{6-3}$$

with

$$U_i = U_i(\tau) := \int_{\mathcal{F}} \Big(\sum_{t=a_i}^{a_{i+1}-1} \sum_{j=0}^{J} (-1)^j \binom{J}{j} M_{t,k_{\frac{X-jd\tau-2}{4}}}(z) \Big)^2 d\mu_z. \tag{6-4}$$

By Lemma 2.2 we have for every $1 \le i \le I$ that U_i equals the sum of

$$\sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} E_{t_1,t_2} S_{t_1,t_2} \tag{6-5}$$

with the two complementary pieces

$$\sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \sum_{\substack{f \in \mathbb{Z} \\ f^2 < (t_1^2-4)(t_2^2-4)}} h(t_1^2-4, t_2^2-4, f) R_{t_1,t_2,f}, \tag{6-6}$$

and

$$\sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \sum_{\substack{f \in \mathbb{Z} \\ f^2 > (t_1^2-4)(t_2^2-4)}} h(t_1^2-4, t_2^2-4, f) R_{t_1,t_2,f}, \tag{6-7}$$

with the abbreviations

$$a_{j_1,j_2} := (-1)^{j_1+j_2} \binom{J}{j_1} \binom{J}{j_2}, \tag{6-8}$$

$$S_{t_1,t_2} := \sum_{j_1=0}^{J} \sum_{j_2=0}^{J} a_{j_1,j_2} \mathcal{J}\Big(t_1, t_2, k_{\frac{X-j_1 d\tau-2}{4}}, k_{\frac{X-j_2 d\tau-2}{4}}\Big) \tag{6-9}$$

$$R_{t_1,t_2,f} := \sum_{j_1=0}^{J} \sum_{j_2=0}^{J} a_{j_1,j_2} \mathcal{I}\Big(t_1, t_2, \frac{f}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}}, k_{\frac{X-j_1 d\tau-2}{4}}, k_{\frac{X-j_2 d\tau-2}{4}}\Big). \tag{6-10}$$

By (2-10) and Lemma 4.11 we see that the functions $\mathcal{I}$ and $\mathcal{J}$ from (6-9) and (6-10) can be nonzero only in the case

$$t_1^2 - 4 \le X - j_1 d\tau - 2, \quad t_2^2 - 4 \le X - j_2 d\tau - 2. \tag{6-11}$$

If (6-11) is true, then by (2-10) and (4-1) we have

$$\mathcal{I}\Big(t_1, t_2, \frac{f}{\sqrt{t_1^2-4}\sqrt{t_2^2-4}}, k_{\frac{X-j_1 d\tau-2}{4}}, k_{\frac{X-j_2 d\tau-2}{4}}\Big) = Z(S_0, T_0, F), \tag{6-12}$$

and by Lemma 4.11 we have

$$\mathcal{J}\Big(t_1, t_2, k_{\frac{X-j_1 d\tau-2}{4}}, k_{\frac{X-j_2 d\tau-2}{4}}\Big) = 2\min(S_0, T_0), \tag{6-13}$$

where we use the abbreviations

$$S_0 = S_0(j_1, t_1) = \sqrt{\frac{X-j_1 d\tau-2}{t_1^2-4} - 1}, \quad T_0 = T_0(j_2, t_2) = \sqrt{\frac{X-j_2 d\tau-2}{t_2^2-4} - 1}, \tag{6-14}$$

and

$$F = F(t_1, t_2, f) = \left| \frac{f}{\sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}} \right|. \tag{6-15}$$

We now consider the sum (6-7). Assume that (6-11) holds. Then by Lemma 4.1 and the last line of (4-5) we see that (6-12) can be nonzero only if

$$|f| < B(S_0, T_0)\sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}. \tag{6-16}$$

If (6-11) is true and (6-16) holds for some f in (6-7), then we have

$$\frac{|f|}{\sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}} - 1 \geq \frac{1}{\sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}\,\left(|f| + \sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4}\right)} \gg X^{-c} \tag{6-17}$$

with some absolute constant c. If (6-11) holds, then we determine (6-12) by Lemmas 4.1 and 4.3. In some cases we will apply the upper bounds of Lemma 4.4. By (6-17) and (4-12) we see that when we apply these lemmas for the estimation of (6-12) we always have $\log \frac{1}{1-y} \ll \log X$. So assuming (6-11) we get that for any f in (6-7) the value of (6-12) is

$$\ll \left(\sqrt{\frac{X - j_1 d\tau - 2}{t_1^2 - 4} - 1} + \sqrt{\frac{X - j_2 d\tau - 2}{t_2^2 - 4} - 1} \right) \log X. \tag{6-18}$$

We note finally that if $a_i \leq t_1, t_2 < a_{i+1}$ for some i, then

$$(t_1 t_2 - 5)^2 + \tfrac{1}{6} t_1 t_2 < (t_1^2 - 4)(t_2^2 - 4) \leq (t_1 t_2 - 4)^2, \tag{6-19}$$

since by the assumption $a_{i+1} \leq \frac{3}{2} a_i$ made in (6-2) we have $\frac{2}{3} \leq t_2/t_1 \leq \frac{3}{2}$, and we also have $t_1 t_2 \geq 9$. So there is an absolute constant $c_0 > 0$ such that if $a_i \leq t_1, t_2 < a_{i+1}$, then

$$(t_1 t_2 - 5) + c_0 \leq \sqrt{(t_1^2 - 4)(t_2^2 - 4)} \leq t_1 t_2 - 4. \tag{6-20}$$

6.2. *The case of very large a_i.* Assume that we have

$$\sqrt{X + 2} - a_i = O\left(\frac{d}{\sqrt{X}} X^{\delta} \right) \tag{6-21}$$

for some $\delta > 0$ chosen small enough in terms of ϵ. Consider first (6-7). Since it is easy to see that $B(S, T) - 1 \leq S^2 + T^2$ for any $S, T \geq 0$, the number of integers f in (6-7) satisfying (6-16) is

$$\ll 1 + \sqrt{t_1^2 - 4}\sqrt{t_2^2 - 4} \left(\left(\frac{X - j_1 d\tau - 2}{t_1^2 - 4} - 1 \right) + \left(\frac{X - j_2 d\tau - 2}{t_2^2 - 4} - 1 \right) \right) \ll d X^{\delta},$$

where in the last step we used (6-21), (6-1) and (6-2). On the other hand, we get by (6-11), (6-12), (6-18), (6-21) and (6-2) that (6-10) is always $\ll_\delta (\sqrt{d}/\sqrt{X}) X^{\delta}$ for every such f. Hence for i satisfying (6-21) we have, applying also Lemma 3.1, (3-23) and (6-21) that (6-7) is

$$\ll_\delta X^{3\delta} \frac{d^{3/2}}{\sqrt{X}} \sum_{t_1 = a_i}^{a_{i+1} - 1} \sum_{t_2 = a_i}^{a_{i+1} - 1} S(t_1^2 - 4, t_2^2 - 4) \ll_\delta X^{5\delta} d^{3/2} \left(\left(\frac{d}{\sqrt{X}} \right)^{1/2} + \frac{(d/\sqrt{X})^{3/2}}{X^{1/4}} \right),$$

where we used (6-1), (6-2). By (6-3) we see that its contribution is acceptable in (5-18).

We now consider (6-6). We get by (6-11), (6-12), Lemma 4.1, (4-25), the first relation in (4-26) and (6-21) that (6-10) is $\ll_\delta X^\delta d/\sqrt{(t_1^2-4)\left(t_2^2-4\right)-f^2}$. Hence for i satisfying (6-21) we have, applying also Lemma 3.1, that (6-6) is

$$\ll_\delta X^{2\delta} d \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \sum_{\substack{f\in\mathbb{Z} \\ f^2<(t_1^2-4)(t_2^2-4)}} \frac{S(t_1^2-4,\,t_2^2-4,\,f^2)}{\sqrt{(t_1^2-4)(t_2^2-4)-f^2}}. \tag{6-22}$$

In the $t_1 \neq t_2$ part we use $S(t_1^2-4, t_2^2-4, f^2) \leq S(t_1^2-4, t_2^2-4)$, and we easily see that the sum over f in (6-22) is $\ll S(t_1^2-4, t_2^2-4)$. Then applying (3-20) with $b=c$ and (6-21) we see that the $t_1 \neq t_2$ part of (6-22) is

$$\ll_\delta X^{3\delta} d\sqrt{a_i}\left(\frac{d}{\sqrt{X}}X^\delta\right)^{3/2} \ll_\delta X^{5\delta}\frac{d^{5/2}}{\sqrt{X}},$$

which is acceptable in (5-18). For the $t_1=t_2$ part of (6-22) we use Lemma 3.7 and we get that the sum over f in (6-22) is $\ll_\delta X^\delta$, and so the $t_1=t_2$ part of (6-22) is $\ll_\delta X^{3\delta} d\left(\frac{d}{\sqrt{X}}X^\delta\right)$, which is smaller than our estimate for the $t_1 \neq t_2$ part. So we have proved that assuming (6-21) the contribution of (6-6) is also acceptable in (5-18).

Assuming (6-21) in (6-5) we clearly have $\min(S_0, T_0) \ll_\delta X^\delta\sqrt{d}/\sqrt{X}$; hence using also (6-9), (6-13) and Lemma 2.1 we get that (6-5) is

$$\ll_\delta \frac{X^{2\delta}\sqrt{d}}{\sqrt{X}}\left(\frac{d}{\sqrt{X}}X^\delta\right)^2\sqrt{X} = \frac{X^{4\delta}d^{5/2}}{X}.$$

Hence in the case of (6-21) we get that the contribution of U_i in (5-18) is acceptable. We may assume from now on that

$$\sqrt{X+2}-a_i > X^{\delta_0}\frac{d}{\sqrt{X}} \tag{6-23}$$

with some $\delta_0>0$, which is fixed in terms of ϵ.

6.3. *Easy consequences of* (6-23). Observe that (6-23) implies the following relations, for all real numbers $0 \leq j_1, j_2, j \leq J$ and integers $a_i \leq t_1, t_2, t \leq a_{i+1}$:

$$\frac{X^{1/4}\sqrt{\sqrt{X}-a_i}}{a_i} \ll S_0(j,t_1), T_0(j,t_2) \ll \frac{X^{1/4}\sqrt{\sqrt{X}-a_i}}{a_i}, \tag{6-24}$$

$$\frac{X}{a_i^2} \ll 1+S_0^2(j,t_1), 1+T_0^2(j,t_2) \ll \frac{X}{a_i^2}, \tag{6-25}$$

$$T_0(j_1,t)-T_0(j_2,t) = O\left(\frac{d}{X^{1/4}a_i\sqrt{\sqrt{X}-a_i}}\right) = S_0(j_1,t)-S_0(j_2,t), \tag{6-26}$$

$$S_0(j_1,t_1)-T_0(j_2,t_2) = \frac{X(t_2^2-t_1^2)\left((t_1^2-4)(t_2^2-4)\right)^{-1}}{S_0(j_1,t_1)+T_0(j_2,t_2)} + O\left(\frac{dX^{-1/4}}{\sqrt{\sqrt{X}-a_i}a_i}\right) \tag{6-27}$$

We have in general that $\frac{d}{dT}\big(\sqrt{1+S^2}\sqrt{1+T^2}+ST\big)=\frac{\sqrt{1+S^2}T}{\sqrt{1+T^2}}+S$ and

$$\frac{d}{dT}\big(\sqrt{1+S^2}\sqrt{1+T^2}-ST\big)=\frac{\sqrt{1+S^2}\,T}{\sqrt{1+T^2}}-S=\frac{(T-S)(T+S)}{\sqrt{1+T^2}\big(\sqrt{1+S^2}T+\sqrt{1+T^2}S\big)};$$

hence we get from (6-24)–(6-27) and the mean-value theorem that

$$B\big(S_0(j_1,t_1),T_0(j,t_2)\big)-B\big(S_0(j_1,t_1),T_0(0,t_2)\big)\ll d/a_i^2 \tag{6-28}$$

and

$$A\big(S_0(j_1,t_1),T_0(j,t_2)\big)-A\big(S_0(j_1,t_1),T_0(0,t_2)\big)\ll\left(\frac{|t_2-t_1|}{a_i}+\frac{d}{X}\right)\frac{d}{X-a_i^2} \tag{6-29}$$

for all real numbers $0\le j_1,j\le J$ and integers $a_i\le t_1,t_2\le a_{i+1}$.

We will also need later the easily proved general identity

$$\frac{T_0}{S_0}-A(S_0,T_0)=\frac{(1+S_0^2)(T_0^2-S_0^2)}{S_0T_0(1+S_0^2)+S_0^2\sqrt{(1+S_0^2)(1+T_0^2)}}. \tag{6-30}$$

This implies by (6-24) and (6-25) that assuming (6-23) we have

$$\left|\frac{T_0}{S_0}-1\right|\ll\left|\frac{T_0}{S_0}-A\left(S_0,T_0\right)\right|\ll\left|\frac{T_0}{S_0}-1\right| \tag{6-31}$$

for every choice $S_0=S_0(j_1,t_1)$, $T_0=T_0\left(j_2,t_2\right)$ with any real numbers $0\le j_1,j_2\le J$ and integers $a_i\le t_1,t_2\le a_{i+1}$. We also see from (6-30) that the signs of $T_0/S_0-A(S_0,T_0)$ and T_0/S_0-1 are the same. Therefore we get from (6-31) that assuming (6-23) we have

$$\left|\frac{T_0(j_2,t_2)}{S_0(j_1,t_1)}-1\right|\ll\left|\frac{T_0(j_2,t_2)}{S_0(j_1,t_1)}-F\right|\ll\left|\frac{T_0(j_2,t_2)}{S_0(j_1,t_1)}-1\right| \tag{6-32}$$

for any real numbers $0\le j_1,j_2\le J$ and any $1<F\le A\big(S_0(j_1,t_1),T_0(j_2,t_2)\big)$.

Assuming (6-23) we see from (6-2) that (6-11) is always true. Then it follows by (6-9) and (6-13) that (6-5) equals

$$2\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{t_2=a_i}^{a_{i+1}-1}E_{t_1,t_2}\sum_{j_1=0}^{J}\sum_{j_2=0}^{J}a_{j_1,j_2}\min\big(S_0(j_1,t_1),\,T_0(j_2,t_2)\big). \tag{6-33}$$

We also see that (6-12) always holds. Then by Lemma 4.1 and (6-10) we get for any f that

$$R_{t_1,t_2,f}=2\sum_{j_1=0}^{J}\sum_{j_2=0}^{J}a_{j_1,j_2}\big(J(S_0,T_0,F)+J(T_0,S_0,F)\big). \tag{6-34}$$

By the change of variables $j_1\mapsto j_2$, $t_1\mapsto t_2$ we see that substituting (6-34) into (6-7) and (6-6) the contributions of $J(S_0,T_0,F)$ and $J(T_0,S_0,F)$ in (6-7) are the same.

6.4. ***Estimating* (6-5).** Assume besides (6-23) that we have

$$|t_2-t_1|>\frac{da_i}{X}X^{\delta} \tag{6-35}$$

for some $\delta > 0$ which is fixed in terms of ϵ. Then we see from (6-27) and (6-24) that the sign of $S_0(j_1,t_1) - T_0(j_2,t_2)$ is the same for every pair $0 \le j_1, j_2 \le J$. But then that part of (6-33) where (6-35) holds is 0, since $\sum_{j_1=0}^{J} a_{j_1,j_2} = 0$ for every j_2, and $\sum_{j_2=0}^{J} a_{j_1,j_2} = 0$ for every j_1 by (6-8). So we may assume in (6-33) that

$$|t_2 - t_1| \ll \frac{da_i}{X} X^{\delta} \tag{6-36}$$

for some $\delta > 0$ which is chosen small enough in terms of ϵ. Then we see using (6-24) and Lemma 2.1 that (6-5) is $\ll_\delta X^{2\delta} X^{1/4} \sqrt{\sqrt{X} - a_i}\, a_i (1 + da_i/X) \ll_\delta X^{2\delta} \sqrt{X} d$, which is acceptable in (5-18).

6.5. ***Estimating* (6-6).** Assume besides (6-23) that in (6-6) we have (6-36) and

$$1 - \left(\frac{d}{X - a_i^2}\right)^2 X^{\delta} < F < 1 \tag{6-37}$$

for some $\delta > 0$ which is chosen small enough in terms of ϵ. By (6-34) and (4-25) we have

$$R_{t_1,t_2,f} = 2 \sum_{j_1=0}^{J} \sum_{j_2=0}^{J} a_{j_1,j_2} \big(K(S_0, T_0, F) + K(T_0, S_0, F)\big) \tag{6-38}$$

in the case $F < 1$, and by the substitutions $j_1 \mapsto j_2$, $t_1 \mapsto t_2$ we see that the contributions of $K(S_0, T_0, F)$ and $K(T_0, S_0, F)$ in (6-6) are the same. Hence it is enough to consider the contribution of $K(S_0, T_0, F)$. Applying the second relation in (4-26) we see for fixed t_1, t_2, f and j_1 that

$$\sum_{j_2=0}^{J} (-1)^{j_2} \binom{J}{j_2} K\left(S_0, T_0(j_2,t_2), F\right) \ll \max_{0 \le j_2 < J} \frac{|S_0| \big|T_0(j_2+1,t_2) - T_0(j_2,t_2)\big|}{\sqrt{1-F^2}}.$$

The parameters are written here only in the case of T_0, since only this variable depends on j_2. By (6-24) and (6-26) this is $\ll d/(a_i^2 \sqrt{1-F^2})$. Hence using (6-8), (6-38) and Lemma 3.1 we get that that part of (6-6) where (6-36) and (6-37) hold is

$$\ll_\delta \frac{dX^{\delta}}{a_i^2} \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{\substack{a_i \le t_2 < a_{i+1} \\ |t_2 - t_1| \ll da_i X^{\delta-1}}} \sum_{\substack{f \in \mathbb{Z} \\ 0 < 1-F < \left(\frac{d}{X-a_i^2}\right)^2 X^{\delta}}} \frac{S(t_1^2-4, t_2^2-4, f^2)}{\sqrt{1-F^2}}. \tag{6-39}$$

From (6-20) we see that the innermost sum is

$$\ll a_i \sum_f \frac{S(t_1^2-4, t_2^2-4, f^2)}{\sqrt{(t_1t_2-4) - |f|}}, \tag{6-40}$$

where the sum ranges over all f such that

$$t_1t_2 - 5 - \sqrt{(t_1^2-4)(t_2^2-4)} \left(\frac{d}{X-a_i^2}\right)^2 X^{\delta} \le |f| \le t_1t_2 - 5.$$

In the case $t_1 \neq t_2$ we use $S(t_1^2-4, t_2^2-4, f^2) \leq S(t_1^2-4, t_2^2-4)$, and we see that (6-40) is

$$\ll_\delta X^\delta a_i \left(1+\frac{da_i}{X-a_i^2}\right) S(t_1^2-4, t_2^2-4).$$

So, applying also (3-20) we get that the $t_1 \neq t_2$ part of (6-39) is

$$\ll_\delta X^{3\delta}\left(\frac{d}{a_i}+\frac{d^2}{X-a_i^2}\right)\left(\min(a_i, \sqrt{X}-a_i)\left(a_i\frac{da_i}{X}\right)^{1/2}\right) \ll_\delta X^{3\delta}\frac{d^{5/2}}{\sqrt{X}},$$

which is acceptable in (5-18). In the case $t_1 = t_2$ we estimate (6-40) by Lemma 3.7 and we get that (6-40) is

$$\ll_\delta X^\delta a_i \left(1+(t_1^2-4)\left(\frac{d}{X-a_i^2}\right)^2\right)^{1/2}.$$

So the $t_1 = t_2$ part of (6-39) is

$$\ll_\delta X^{2\delta}\frac{d}{a_i}\min(a_i, \sqrt{X}-a_i)\left(1+a_i\frac{d}{X-a_i^2}\right) \ll_\delta X^{2\delta}\frac{d^2}{\sqrt{X}}.$$

Hence that part of (6-6) where (6-36) and (6-37) hold is acceptable in (5-18).

So it is enough to consider that part of (6-6) where at least one of the conditions (6-36) and (6-37) is false. We prove that this part is negligible. We use (6-34), and we recall that the contributions of $J(S_0, T_0, F)$ and $J(T_0, S_0, F)$ in (6-6) are the same. By Lemma 4.8 and $\sum_{j_2=0}^{J} a_{j_1,j_2} = 0$ we see that it is enough to show that for fixed t_1, t_2, f, j_1 the sum

$$\sum_{j_2=0}^{J}(-1)^{j_2}\binom{J}{j_2}V\left(s_i\left(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f)\right)\right) \tag{6-41}$$

is negligibly small for $i = 1, 2$. Observe that by the notation of Lemma 4.10, using $t = t_2$, $S_0 = S_0(j_1, t_1)$, $F = F(t_1, t_2, f)$ and $\tau = 1$ for $i = 1$, $\tau = -1$ for $i = 2$ we have

$$V\big(s_i(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f)\big) = k(X - j_2 d\tau - 2).$$

By Taylor's formula with remainder (see Theorem 7.6 of [A], for example), the value of (6-41) is $\ll d^J \max_{X-2-Jd\tau\leq x\leq X-2}\left|k^{(J)}(x)\right|$. Then by Lemma 4.10 and (6-24) we see that (6-41) is

$$\ll \sqrt{1+\frac{1}{S_0(j_1,t_1)^2}}\,d^J(X-a_i^2)^{-J}\max\left(1, \left(\sqrt{1-F^2}+\left|\tau F+\frac{T_0(j,t_2)}{S_0(j_1,t_1)}\right|\right)^{-J}\right), \tag{6-42}$$

with some real number $0 \leq j \leq J$. We see that if (6-37) is false, then this is negligibly small, since J is fixed to be large enough. So we can assume that (6-37) is true but (6-36) is false. We show that (6-42) is negligibly small. If

$$\left|\tau F+\frac{T_0(j,t_2)}{S_0(j_1,t_1))}\right| \gg X^\delta\frac{d}{X-a_i^2}, \tag{6-43}$$

then this is true. So we may assume that (6-43) is false. But then using also (6-37) and the triangle inequality, taking into account (6-23) we get

$$\left|\tau + \frac{T_0(j, t_2)}{S_0(j_1, t_1)}\right| \ll X^\delta \frac{d}{X - a_i^2}. \tag{6-44}$$

This is impossible for $\tau = 1$ for small δ by (6-23), so we may assume $\tau = -1$. Hence, using (6-27) and (6-44) with $\tau = -1$, applying also (6-24) we get $|t_2 - t_1| \ll X^\delta da_i/X$. But this is a contradiction, since we assumed that (6-36) is false. So that part of (6-6) where at least one of the conditions (6-36) and (6-37) is false is also negligibly small. Consequently (6-6) is acceptable in (5-18).

6.6. ***A new expression for* (6-7).** Recall that substituting (6-34) into (6-7) the contributions of $J(S_0, T_0, F)$ and $J(T_0, S_0, F)$ in (6-7) are the same. Hence, applying also Lemma 4.3 we get that (6-7) equals

$$4 \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} \left(A_{t_1,t_2} + B_{t_1,t_2} - C_{t_1,t_2}\right) \tag{6-45}$$

with

$$A_{t_1,t_2} := \sum_{\substack{0 \le j_1, j_2 \le J \\ f \in \mathbb{Z} \\ 1 < F \le B(S_0, T_0)}} h(t_1^2 - 4, t_2^2 - 4, f) a_{j_1,j_2} S_0 \Phi(y_1(S_0, T_0, F)), \tag{6-46}$$

$$B_{t_1,t_2} := \sum_{\substack{0 \le j_1, j_2 \le J \\ f \in \mathbb{Z}, T_0 \ge S_0 \\ 1 < F \le A(S_0, T_0)}} h(t_1^2 - 4, t_2^2 - 4, f) a_{j_1,j_2} S_0 \Phi\left(y_2(S_0, T_0, F)\right), \tag{6-47}$$

$$C_{t_1,t_2} := \sum_{\substack{0 \le j_1, j_2 \le J \\ f \in \mathbb{Z}, T_0 \le S_0 \\ 1 < F \le A(S_0, T_0)}} h(t_1^2 - 4, t_2^2 - 4, f) a_{j_1,j_2} S_0 \Phi\left(y_2(S_0, T_0, F)\right). \tag{6-48}$$

6.7. ***The contribution of B_{t_1,t_2} and C_{t_1,t_2}.*** Assume besides (6-23) that

$$1 < |F| \le A(S_0, T_0) \tag{6-49}$$

and (6-36) holds with some $\delta > 0$ which is chosen small enough in terms of ϵ. Applying (6-17), (4-12) and (4-16) we get that the terms $\Phi\left(y_2(S_0, T_0, F)\right)$ in (6-47), (6-48) are always $O(\log X)$. Using

$$A(S_0, T_0) - 1 = \frac{(S_0 - T_0)^2}{\sqrt{(1 + S_0^2)(1 + T_0^2)} + S_0 T_0 + 1} \le \frac{(S_0 - T_0)^2}{\sqrt{(1 + S_0^2)(1 + T_0^2)}},$$

(6-27), (6-25), (6-24) and (6-36) we see that the number of integers f satisfying (6-49) and (6-15) is $\ll 1 + X^{2\delta}(d^2 a_i^2/X)/(X - a_i^2)$. So we get, applying also Lemma 3.1 that the contribution to (6-45) of the terms B_{t_1,t_2}, C_{t_1,t_2} satisfying (6-49) and (6-36) is

$$\ll_\delta X^{3\delta}\left(\frac{\sqrt{X-a_i^2}}{a_i}+\frac{d^2a_i/X}{\sqrt{X-a_i^2}}\right)\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2\le a_{i+1}-1\\ |t_2-t_1|\ll a_i\frac{d}{X}X^\delta}} S(t_1^2-4,t_2^2-4).$$

By (3-20) and (3-22) this is

$$\ll_\delta X^{4\delta}\left(\frac{\sqrt{X-a_i^2}}{a_i}+\frac{d^2a_i/X}{\sqrt{X-a_i^2}}\right)\left(a_i\sqrt{\sqrt{X}-a_i}+(\sqrt{X}-a_i)a_i\left(\frac{d}{X}\right)^{1/2}\right),$$

which in turn is $\ll_\delta X^{4\delta}d^{5/2}/\sqrt{X}$ by (6-1), (6-2) and (5-19). Hence we have proved that the contribution to (6-45) of the terms B_{t_1,t_2}, C_{t_1,t_2} satisfying (6-49) and (6-36) is acceptable in (5-18).

Now consider the contribution of those terms B_{t_1,t_2}, C_{t_1,t_2} to (6-45) for which the inequalities

$$1<F\le A\big(S_0(j_1,t_1),T_0(0,t_2)\big)-\frac{d\,|t_2-t_1|}{a_i(X-a_i^2)}X^\delta \tag{6-50}$$

and (6-35) hold for some $\delta>0$ which is fixed in terms of ϵ. We want to prove that this contribution is negligibly small. We first show that for fixed t_1,t_2, j_1 and f the conditions in the summations in (6-47) and (6-48) are independent of $0\le j_2\le J$. It is enough to see that we have $F\le A\big(S_0(j_1,t_1),T_0(j,t_2)\big)$ for every $0\le j\le J$, and the sign of $S_0(j_1,t_1)-T_0(j,t_2)$ is the same for every $0\le j\le J$. These statements follow easily from (6-29), (6-27) and (6-24). Hence for fixed t_1,t_2, j_1 and f satisfying $a_i\le t_1,t_2<a_{i+1}$, $0\le j_1\le J$ and the conditions (6-50), (6-35) we have that either each $0\le j_2\le J$ satisfies the conditions of the summations in (6-47), or each $0\le j_2\le J$ satisfies the conditions of the summations in (6-48). Consequently, recalling (6-8) we see that it is enough to show that for every fixed t_1,t_2, j_1 and f satisfying the conditions just mentioned the sum

$$\sum_{j_2=0}^{J}(-1)^{j_2}\binom{J}{j_2}\Phi\big(y_2(S_0(j_1,t_1),T_0(j_2,t_2),F(t_1,t_2,f))\big) \tag{6-51}$$

is negligibly small. In the notation of Lemma 4.6, using $\tau=-1$ and $t=t_2$, $S_0=S_0(j_1,t_1)$, $F=F(t_1,t_2,f)$ there, we have

$$\Phi\big(y_2(S_0(j_1,t_1),T_0(j_2,t_2),F(t_1,t_2,f))\big)=K(X-j_2d\tau-2).$$

Theorem 7.6 of [A] shows that (6-51) is $\ll d^J\max_{X-2-Jd\tau\le x\le X-2}\big|K^{(J)}(x)\big|$. By Lemma 4.6, (6-32), (6-27), (6-24), (6-23), (6-1), (6-2) this is

$$\ll d^J(X-a_i^2)^{-J}\max\left(1,\left(\frac{|t_2-t_1|}{a_i\,(A(S_0,T_0)-F)}\right)^J\right).$$

From (6-50), (6-29) and (6-23) we see that this is negligibly small, since J is fixed to be large enough in terms of ϵ. Hence we have proved that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-50) and (6-35) hold is negligibly small.

Consider the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35) holds and we have

$$A\left(S_0(j_1,t_1),T_0(0,t_2)\right)-\frac{d\,|t_2-t_1|}{a_i(X-a_i^2)}X^{\delta}<F\le A\left(S_0(j_1,t_1),T_0(j_2,t_2)\right) \tag{6-52}$$

for some $\delta>0$ which is small enough in terms of ϵ. Using also (6-29) this shows that the number of possible values of the integers f satisfying (6-52) is

$$\ll 1+\frac{d\,|t_2-t_1|\,a_i}{X-a_i^2}X^{\delta}. \tag{6-53}$$

It is easy to compute that

$$y_2(S_0,T_0,F)=\sqrt{\frac{(B(S_0,T_0)+F)(A(S_0,T_0)-F)}{(T_0-S_0F)^2}},$$

so using (6-24), (6-25), (6-27), (6-32) and (6-52) we get

$$y_2(S_0,T_0,F)\ll_\delta X^{\delta}\sqrt{\frac{da_i}{X\,|t_2-t_1|}},\qquad S_0y_2(S_0,T_0,F)\ll_\delta X^{\delta}\sqrt{\frac{d(\sqrt{X}-a_i)}{a_i\sqrt{X}\,|t_2-t_1|}}. \tag{6-54}$$

Now, if $a_i\ge\sqrt{X}/2$, then we have $S_0\ll 1$ by (6-24), and so by Lemma 4.4, (4-12) and (6-17) we get

$$S_0\Phi\left(y_2(S_0,T_0,F)\right)\ll_\delta S_0y_2^3(S_0,T_0,F)X^{\delta}\ll_\delta X^{4\delta}\sqrt{\frac{d^3(\sqrt{X}-a_i)}{X^2\,|t_2-t_1|^3}}. \tag{6-55}$$

If $a_i\le\sqrt{X}/2$, then we have $S_0\gg 1$ by (6-24), and by the second relation in (6-54), Lemma 4.4, (4-12), (6-17) we see that if $|t_2-t_1|\ll d/a_i$, then

$$S_0\Phi\left(y_2(S_0,T_0,F)\right)\ll_\delta X^{\delta}S_0y_2(S_0,T_0,F)\ll_\delta X^{2\delta}\sqrt{\frac{d}{a_i\,|t_2-t_1|}}, \tag{6-56}$$

while if $|t_2-t_1|\gg d/a_i$, then

$$S_0\Phi\left(y_2(S_0,T_0,F)\right)\ll_\delta X^{\delta}S_0^3y_2^3(S_0,T_0,F)\ll_\delta X^{4\delta}\sqrt{\frac{d^3}{a_i^3\,|t_2-t_1|^3}}. \tag{6-57}$$

If $a_i\ge\sqrt{X}/2$, then by (6-35) and (5-19) we see that the second term is larger than the first one in (6-53). Then by (6-53) and (6-55) we see that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35) and (6-52) hold is

$$\ll_\delta X^{6\delta}\frac{d^{5/2}}{X^{3/4}\sqrt{X-a_i^2}}\sum_{t_1=a_i}^{a_{i+1}-1}\ \sum_{\substack{a_i\le t_2<a_{i+1}\\ |t_2-t_1|\ge da_iX^{\delta-1}}}\frac{S(t_1^2-4,t_2^2-4)}{\sqrt{|t_2-t_1|}} \tag{6-58}$$

in the case $a_i\ge\frac{1}{2}\sqrt{X}$. By (3-21) we have that the sum over t_1,t_2 is $\ll_\delta X^{\delta}(\sqrt{X}-a_i)X^{1/4}$; hence (6-58) is acceptable in (5-18).

If $a_i \le \frac{1}{2}\sqrt{X}$, then by (6-53) and (6-56) the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35), (6-52) and $|t_2 - t_1| \ll d/a_i$ hold is $\ll_\delta$ than the sum of

$$X^{4\delta}\frac{d^2}{X}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\0<|t_2-t_1|\ll d/a_i}} S(t_1^2-4, t_2^2-4) \tag{6-59}$$

and

$$X^{4\delta}\sqrt{\frac{d}{a_i}}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\0<|t_2-t_1|}} \frac{S(t_1^2-4, t_2^2-4)}{\sqrt{|t_2-t_1|}}. \tag{6-60}$$

By (3-20) we get that (6-59) is $\ll_\delta X^{5\delta-1}d^2a_i^{3/2}(d/a_i)^{1/2} \ll_\delta X^{5\delta}d^{5/2}/\sqrt{X}$. We estimate (6-60) by (3-21); the result is an upper bound $X^{5\delta}(d/a_i)^{1/2}a_i^{3/2} \ll X^{5\delta}\sqrt{d}\sqrt{X}$, which is smaller than $d^{5/2}/\sqrt{X}$ by (5-19).

If $a_i \le \frac{1}{2}\sqrt{X}$ and $|t_2 - t_1| \gg d/a_i$, then by (5-19) the second term in (6-53) is larger than the first one. Hence by (6-53) and (6-57) we see in the case $a_i \le \frac{1}{2}\sqrt{X}$ that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35), (6-52) and $|t_2 - t_1| \gg d/a_i$ hold is

$$\ll_\delta X^{6\delta}\frac{d^{5/2}}{a_i^{1/2}X}\sum_{t_1=a_i}^{a_{i+1}-1}\sum_{\substack{a_i\le t_2<a_{i+1}\\0<|t_2-t_1|}} \frac{S(t_1^2-4, t_2^2-4)}{\sqrt{|t_2-t_1|}} \ll_\delta X^{7\delta}\frac{d^{5/2}a_i}{X},$$

where in the last step we used (3-21). This is again acceptable in (5-18).

We examined every case, so we proved that the contribution to (6-45) of those terms B_{t_1,t_2}, C_{t_1,t_2} for which (6-35) and (6-52) hold is acceptable in (5-18). Using also the previous estimates we see that the whole contributions of B_{t_1,t_2} and C_{t_1,t_2} in (6-45) is acceptable in (5-18).

6.8. *The contribution of A_{t_1,t_2}.* Now consider that part of the contribution of A_{t_1,t_2} in (6-45) where

$$1 < F \le B(S_0(j_1,t_1), T_0(0,t_2)) - \frac{d}{a_i^2}X^\delta \tag{6-61}$$

for some $\delta > 0$ which is fixed in terms of ϵ. We show that for fixed t_1, t_2, j_1 and f the condition in the summation in (6-46) is independent of $0 \le j_2 \le J$. It is enough to see that we have $F \le B\big(S_0(j_1,t_1), T_0(j,t_2)\big)$ for every $0 \le j \le J$, and this follows by (6-28). Hence for fixed t_1, t_2, j_1 and f satisfying $a_i \le t_1, t_2 < a_{i+1}$, $0 \le j_1 \le J$ and (6-61) each $0 \le j_2 \le J$ satisfies the conditions of the summation in (6-46). Recalling (6-8) we then see that if we can show that for every fixed t_1, t_2, j_1 and f satisfying the above-mentioned conditions the sum

$$\sum_{j_2=0}^{J}(-1)^{j_2}\binom{J}{j_2}\Phi\big(y_1(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f))\big) \tag{6-62}$$

is negligibly small, then we will get that that part of the contribution of A_{t_1,t_2} in (6-45) where (6-61) is true is negligibly small. Observe that by the notations of Lemma 4.6, using $\tau = 1$ and $t = t_2$, $S_0 = S_0(j_1,t_1)$,

$F = F(t_1, t_2, f)$ there we have

$$\Phi\big(y_1(S_0(j_1,t_1), T_0(j_2,t_2), F(t_1,t_2,f))\big) = K(X - j_2 d\tau - 2). \tag{6-63}$$

Theorem 7.6 of [A] gives that (6-62) is $\ll d^J \max_{X-Jd\tau-2\le x\le X-2} |K^{(J)}(x)|$. By Lemma 4.6 and (6-24) this is

$$\ll d^J \max\Big((X-a_i^2)^{-J}, \Big(\frac{1}{a_i^2(B(S_0,T_0)-F)}\Big)^J\Big).$$

By (6-61), (6-28) and (6-23) this is negligibly small, since J is fixed to be large enough in terms of ϵ. Hence that part of the contribution of A_{t_1,t_2} in (6-45) where (6-61) holds is negligibly small.

Now consider that part of the contribution of A_{t_1,t_2} in (6-45) where

$$B\big(S_0(j_1,t_1), T_0(0,t_2)\big) - \frac{d}{a_i^2} X^\delta < F \le B\big(S_0(j_1,t_1), T_0(j_2,t_2)\big) \tag{6-64}$$

for some $\delta > 0$ chosen small enough in terms of ϵ. It is easy to compute that

$$y_1(S_0,T_0,F) = \sqrt{\frac{(B(S_0,T_0)-F)(A(S_0,T_0)+F)}{(T_0+S_0F)^2}}. \tag{6-65}$$

By (6-25), (6-64) and (5-19) we see that $X/a_i^2 \ll B(S_0,T_0) \ll X/a_i^2$ and $B(S_0,T_0) - F = o(X/a_i^2)$. So

$$S_0 y_1(S_0,T_0,F) \ll \frac{\sqrt{B(S_0,T_0)-F}}{\sqrt{B(S_0,T_0)}} = o(1). \tag{6-66}$$

Then applying Lemma 4.4 (note that the middle case of (4-15) cannot hold by (6-66)), and also using (4-12) and (6-17), we get in every case that

$$S_0\Phi\big(S_0, y_1(S_0,T_0,F)\big) \ll_\delta X^\delta (S_0+S_0^3) y_1^3(S_0,T_0,F) \ll_\delta \frac{X^{1+\delta}(B(S_0,T_0)-F)^{3/2}}{a_i^2 S_0^2 (B(S_0,T_0))^{3/2}};$$

the second inequality follows from (6-25) and (6-66). By (6-24), (6-25) and (6-64) this gives

$$S_0\Phi\big(S_0, y_1(S_0,T_0,F)\big) \ll_\delta X^{3\delta} \frac{d^{3/2}}{X(\sqrt{X}-a_i)}. \tag{6-67}$$

The number of possible values of the integers f satisfying (6-64) is $\ll_\delta X^\delta d$. Therefore, using also Lemma 3.1, we get that that part of the contribution of A_{t_1,t_2} in (6-45) where (6-64) holds is

$$\ll_\delta X^{5\delta} \frac{d^{5/2}}{X(\sqrt{X}-a_i)} \sum_{t_1=a_i}^{a_{i+1}-1} \sum_{t_2=a_i}^{a_{i+1}-1} S(t_1^2-4, t_2^2-4) \ll_\delta \frac{X^{6\delta} d^{5/2}}{X\big(\sqrt{X}-a_i\big)} (\sqrt{X}-a_i) a_i, \tag{6-68}$$

where in the last step we used (3-23), noting that $a(b-a)$ is an upper bound there for both terms. This estimate is again acceptable in (5-18). So we proved that the contribution of A_{t_1,t_2} in (6-45) is acceptable in (5-18), hence the whole sum (6-45) is acceptable. The proof of (5-18) is now complete, so Theorem 1.1 is also proved.

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