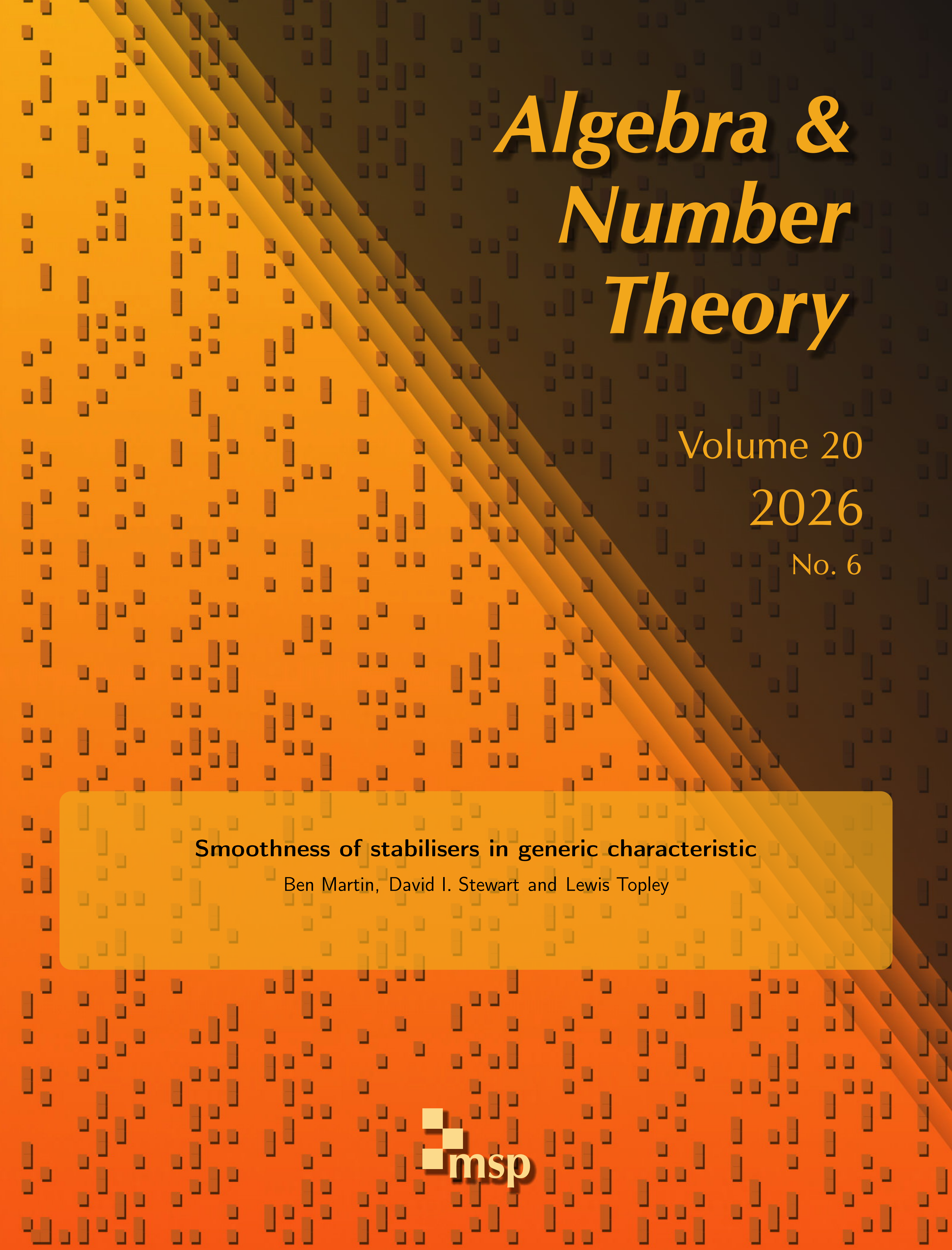

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Smoothness of stabilisers in generic characteristic

Ben Martin, David I. Stewart and Lewis Topley

Let R be a commutative unital ring. Given a finitely presented affine R-group scheme G acting on a finitely presented separated scheme X over R, we show that there is a prime p_0 such that for any R-algebra k that is a field of characteristic $p \geq p_0$, the centraliser in G_k of any closed subscheme of X_k is smooth. When X is not necessarily separated we show similarly that for any closed finitely presented subscheme $Y \subseteq X$ there is a p_1 depending on Y such that when k has characteristic $p \geq p_1$, the normaliser of Y_k in G_k is smooth. For the proof, we may assume k is algebraically closed, whence we prove these results using the Lefschetz principle together with careful application of Gröbner basis techniques, and using a suitable notion of the complexity of an action.

We apply our results to demonstrate that the Kostant–Kirillov–Souriau theorem holds for Lie algebras of algebraic groups in large positive characteristics: the coadjoint module of every such Lie algebra decomposes as a disjoint union of symplectic varieties, each of which is a coadjoint orbit.

1. Introduction

Let R be a commutative unital ring. By an *R-field* we mean an R-algebra that is also a field and by an *algebraic R-group* we mean a finitely presented affine R-group scheme. We prove the following.

Theorem 1.1. *Let G be an algebraic R-group and let X be a finitely presented separated G-scheme over R. Then there exists $p_0 \in \mathbb{N}$ such that whenever k is an R-field of characteristic $p \geq p_0$, the centraliser $C_{G_k}(Y)$ is smooth for every closed subscheme Y of X_k.*

Theorem 1.2. *Let G be an algebraic R-group and let X be a G-scheme of finite type over R. Let Y be a finitely presented closed subscheme of X. Then there exists $p_1 \in \mathbb{N}$ such that whenever k is an R-field of characteristic $p \geq p_1$, the normaliser $N_{G_k}(Y_k)$ is smooth.*

If we assume that R is noetherian then we can remove some of the hypotheses on X and Y: see Remark 4.15.

Theorem 1.1 uses the hypothesis that X is separated in order to infer that the centralisers are closed subschemes of G (see [Jan03, I.2.6]). Note also that the lower bound of p_0 in Theorem 1.1 for centralisers does not depend on Y; the bound in Theorem 1.2 for normalisers does, however, depend on Y, and this dependence cannot be removed: see Remark 4.12.

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Natural examples of algebraic groups over rings abound, of course. The split reductive groups are all $\mathbb{Z}$-defined [Dem63]; and so too are the subgroups of reductive groups normalised by a split maximal torus — so-called *subsystem subgroups*. This class includes all parabolic subgroups of reductive groups.

There are several known special cases of the theorem. One of the most influential in Lie theory occurs when G is split reductive and X is either G itself or its Lie algebra $\mathfrak{g}$, on which G acts by the relevant adjoint action. Then it is well known that the group G, the algebra $\mathfrak{g}$, and the adjoint action are defined over $\mathbb{Z}$ (see [Jan03, II.1.1]) and the centralisers of single elements of $X(\bar{k})$ are smooth whenever p is a very good prime for G. This follows (see [BMRT10, Theorem 1.3(a)]) from work of Richardson [Ric67], who used the notion of a reductive pair to give an elegant proof for very good p that the number of unipotent and nilpotent orbits of $G_{\bar{k}}$ is finite. This smoothness result was generalised in [BMRT10, Theorem 1.2] to cover arbitrary subgroups of $G_{\bar{k}}$ and subalgebras of $\mathrm{Lie}(G_{\bar{k}})$. The hypotheses were further weakened in [Her13]. Normalisers, while much less well-behaved, were thoroughly considered in [HS16], where it was shown that (necessarily large) bounds on the characteristic exist, depending on the root system, which ensure the normalisers of subspaces of the Lie algebras of reductive groups are smooth. Through the classification of nilpotent and unipotent orbits, these results have found applications in developing the subgroup structure of simple algebraic groups and subalgebra structure of their Lie algebras; recent examples include [LT18] and [PS19].

One new feature of our work is that we move beyond the affine case: our results apply not just to affine varieties but to quasi-projective varieties and other varieties of finite type. Here is an application. The second author computed explicitly in [Ste16] the orbits of exceptional groups on their Lie algebras, determining when centralisers and stabilisers of lines are smooth for minimal induced (or dual-Weyl) modules; non-smoothness occurs only in characteristics 2 or 3. This motivated [op. cit., Question 1.4], to which our theorem provides the following strong answer.

Corollary 1.3. *Let G be an algebraic R-group and let V be a G-module which is finitely generated and projective as an R-module. Then there is a prime p_V such that whenever k is an R-field of characteristic $p \geq p_V$, the centraliser $C_{G_k}(W)$ and normaliser $N_{G_k}(W)$ are smooth for any k-subspace W of V_k.*

The smoothness of centralisers follows immediately from Theorem 1.1. To prove smoothness of normalisers one can apply Theorem 1.2. Alternatively, one can apply Theorem 1.1 to $X = \mathrm{Gr}_m(V)$, the Grassmannian of m-generated submodules of V: the idea is that the stabiliser of a subspace is the centraliser of the corresponding point in the Grassmannian — for details, see Section 4.6. The same conclusion does not hold if we weaken the hypothesis and consider all G_k-modules V of bounded dimension (see Remark 4.13), but it does if G_k is reductive and we bound the weights of the G_k-module V in an appropriate sense: see Proposition 4.17.

In the final section of this paper we apply our main theorem to prove a modular analogue of the Kostant–Kirillov–Souriau (KKS) theorem from symplectic geometry. Their theorem states that if G is a complex algebraic group with Lie algebra $\mathfrak{g}$ then the symplectic leaves of the Poisson variety $\mathfrak{g}^*$ are precisely the coadjoint orbits. When G is semisimple, this result leads to a classification of symplectic homogeneous G-varieties, since they all arise as finite covers of coadjoint orbits. We refer the reader to [GS77, § IV.7] for a detailed background of the theory. When k is an algebraically closed field of characteristic $p > 0$ one can ask whether the Poisson variety $\mathfrak{g}^*$ decomposes into a disjoint union of symplectic G-homogeneous subvarieties. In general the answer is negative, however we show that the KKS theorem holds whenever the characteristic is sufficiently large (Theorem 5.2).

Let us say some words on the proofs of our main results. The central idea is to combine the Lefschetz principle from first-order model theory with Gröbner basis techniques. This approach was suggested in [Sch00] as a method to solve certain problems in algebraic geometry, but to our knowledge the current paper is the first place it has been carried out in practice. Roughly speaking, the Lefschetz principle says that a first-order property that holds for algebraically closed fields k of characteristic 0 also holds for algebraically closed fields of large enough characteristic. It is well known that an algebraic k-group in characteristic 0 is smooth; we show that smoothness can be expressed in a first-order way, then use the Lefschetz principle to deduce smoothness for algebraic k-groups in sufficiently large characteristic. In order to do this, one needs suitable notions of complexity for schemes, morphisms and group actions. This allows us to work with honest first-order sentences without needing parameters from the field: rather than considering a fixed G and a fixed X, we quantify over all G and X of bounded complexity (a similar trick was used in [MST19]). Two crucial tools are Lemma 3.9, which allows us to give a bound on complexities arising from certain ideal membership problems, and Lemma 3.8. We find, surprisingly, that although certain complexities need to be constrained, others do not: for example, we prove variations of Theorems 1.1 and 1.2 which hold for schemes X that are not of finite type (see Corollaries 4.4 and 4.8). See also Remarks 4.15 and 4.18.

Outline. In Section 2 we recall the language of Hopf algebras and the Lefschetz principle from model theory, which we use to pass information from characteristic zero to large positive characteristics. Section 3 is the heart of the paper. Here we recall the theory of Gröbner bases which allows us to express a criterion for smoothness in terms of first-order sentences in the language of rings. We recall the definition of complexity for schemes and morphisms (Section 3.2). We introduce the notion of a d-bounded Hopf quadruple, and in Theorem 3.19 we prove that such Hopf algebras correspond to smooth algebraic groups in large characteristics $p > p_0(d)$. In Section 4 we provide the proofs of the main theorems, using the functor-theoretic descriptions of centralisers and normalisers and the notions of G-complexity (Definition 4.1) and (G, Δ)-complexity (Definition 4.5). Finally in Section 5 we provide a short proof of our modular version of the KKS theorem from symplectic geometry.

2. Preliminaries

Throughout, we consider a fixed commutative unital ring R. The closure of a field k is denoted by $\bar{k}$.

2.1. *Schemes, group schemes and Hopf algebras.*

We take the functorial approach to schemes, as per [DG70] and [Jan03]. Thus for an R-algebra Q we think of $\mathrm{Spec}_R(Q)$ as the functor $\mathrm{Hom}_{R\text{-Alg}}(Q, -) : \underline{R\text{-Alg}} \to \underline{\mathrm{Set}}$. An *$R$-functor* is a functor $X : \underline{R\text{-Alg}} \to \underline{\mathrm{Set}}$; we call an R-functor X *affine* if it is isomorphic to $\operatorname{Spec}_R(R[X])$ for some R-algebra $R[X]$. If Q is an R-algebra then X_Q denotes the Q-functor obtained from X by base change. We say a subfunctor Y of an R-functor X is *open* if for every R-algebra Q and natural transformation $\beta : \operatorname{Spec}_R Q \to X$, the subfunctor $\beta^{-1}(Y)$ of $\operatorname{Spec}_R Q$ is an open subfunctor of $\operatorname{Spec}_R Q$, and is *closed* if for every R-algebra Q and natural transformation $\beta : \operatorname{Spec}_R Q \to X$, the subfunctor $\beta^{-1}(Y)$ of $\operatorname{Spec}_R Q$ is a closed subfunctor of $\operatorname{Spec}_R Q$. A closed subfunctor of an affine functor is affine. Then X is a *scheme* (or an *R-scheme*) if it is *local* in the sense of [Jan03, I.1.8] and admits a decomposition $X = \bigcup_{i \in \mathbb{I}} X_i$ for some indexing set $\mathbb{I}$, where the X_i are open affine subfunctors of X.

If Q is an R-algebra and X is an R-scheme then X_Q is a Q-scheme [Jan03, I.1.10]. We say X is of *finite type* if $\mathbb{I}$ is finite and each $R[X_i]$ is finitely generated over R; in this case we say X is *finitely presented* if each $R[X_i]$ is a finitely presented R-algebra. If R is Noetherian (e.g., a field) then any finitely generated R-algebra is finitely presented, so in this case any R-scheme of finite type is finitely presented. We do not assume that all schemes are separated; recall that a scheme is *separated* if the diagonal map $D : X \to X \times X$ is an embedding. By an embedding of schemes we mean a closed immersion.

An *affine R-group scheme* G is a functor from $\underline{R\text{-Alg}}$ to $\underline{\text{Grp}}$ which, as a functor to $\underline{\text{Set}}$, is naturally equivalent to one of the form $\operatorname{Spec}_R(R[G])$ for some finitely generated R-algebra $R[G]$. We consider only the case where $R[G]$ is finitely presented; in keeping with [Jan03, I.2.1], we then call G an *algebraic R-group*. We do not assume that algebraic R-groups are smooth. There is a natural notion of a closed subgroup of G (loc. cit.). The archetypal example of an algebraic R-group is GL_d, which is also an example of a split reductive group.

A Hopf R-algebra consists of data $(R[G], \Delta, \sigma, \epsilon)$ where $R[G]$ is an R-algebra, and there are R-algebra homomorphisms $\Delta : R[G] \to R[G] \otimes_R R[G]$, $\sigma : R[G] \to R[G]$ and $\epsilon : R[G] \to R$ satisfying the dual of the group axioms [Jan03, I.2.3(1–3)]:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{2-1}$$

$$(\epsilon \mathbin{\overline{\otimes}} \mathrm{id}) \circ \Delta = (\mathrm{id} \mathbin{\overline{\otimes}} \epsilon) \circ \Delta, \tag{2-2}$$

$$(\sigma \mathbin{\overline{\otimes}} \mathrm{id}) \circ \Delta = \bar{\epsilon} = (\mathrm{id} \mathbin{\overline{\otimes}} \sigma) \circ \Delta. \tag{2-3}$$

Here the symbol $\varphi \mathbin{\overline{\otimes}} \psi$ denotes the tensor product of the maps φ, ψ followed by the natural multiplication map $R[G] \otimes R[G] \to R[G]$, and $\bar{\epsilon}$ denotes ϵ followed by the inclusion of R in $R[G]$. Hence by definition, the category of algebraic R-groups is the opposite category to the category of finitely presented Hopf algebras over R.

The Lie algebra $\operatorname{Lie}(G)$ of an algebraic R-group G is defined to be the R-module of all R-linear maps $I/I^2 \to R$ where $I = \operatorname{Ker}(\epsilon)$; in other words it is $\ker\big(G(R[\varepsilon]/(\varepsilon^2)) \to G(R)\big)$ where $R[\varepsilon]/(\varepsilon^2)$ is the algebra of dual numbers and the map takes ϵ to 0. Following [Jan03, I.7.7(3)] one obtains a natural R-linear Lie bracket on $\operatorname{Lie}(G)$ induced by the comultiplication Δ. Every morphism of algebraic R-groups induces a natural R-linear homomorphism of their Lie algebras.

When G is an algebraic R-group and k is any R-algebra, we can consider the base change G_k, which is an algebraic k-group. To see this concretely, suppose $R[G] \cong R[x_1, \dots, x_n]/(g_1, \dots, g_m)$. We have an obvious map $\omega : R[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$ and we see that

$$G_k \cong \operatorname{Spec}_k\big(k[x_1, \dots, x_n]/(\omega(g_1), \dots, \omega(g_m))\big). \tag{2-4}$$

An action of G on an R-scheme X is a natural transformation $\alpha : G \times X \to X$ with the property that $\alpha(Q) : G(Q) \times X(Q) \to X(Q)$ is a group action for all R-algebras Q. If X is an R-module (see [Jan03, I.2.2]) and $G(Q)$ acts through Q-linear transformations of $X(Q)$, then we say X is a G-module. If X is affine, then we get a coaction map of R-algebras $\Delta_X : R[X] \to R[X] \otimes R[G]$. In case X is a G-module, this is the *comodule map* of [Jan03, I.2.8].

Let V be a G-module such that V is finitely generated and projective as an R-module. We may regard V as an affine G-scheme of finite type over R with co-ordinate ring the symmetric algebra $S(V^*)$ [Jan03, I.2.2 and I.2.7]. This construction commutes with base change.

2.2. ***Model theory and the Lefschetz principle.*** In this paper we use the Lefschetz principle to deduce statements about algebras over algebraically closed fields of large positive characteristic from the corresponding statements in characteristic zero. In doing so we pursue a theme from our earlier work [MST19], in which we proved a version of the first Kac–Weisfeiler theorem for representations of modular Lie algebras using the Lefschetz principle. A detailed introduction to the principle can be read in [Mar02] for example, and a concise overview may be found in [MST19, §2.1].

For future reference we state the Lefschetz principle, in a version taken from [MST19, Corollary 2.4]. The *language of rings* $\mathcal{L}_{\mathrm{ring}}$ is the collection of first-order formulas that can be built from the symbols $\{\forall, \exists, \vee, \wedge, \neg, +, -, \times, 0, 1, =\}$ along with an arbitrary choice of variables. A *sentence* is a formula containing no free variables. For example, the formula $(\exists y) y^2 = x$ is not a sentence because the variable x is free, but by quantifying over x we can obtain a sentence: e.g., $(\forall x)(\exists y) y^2 = x$. Given a ring R, a sentence is either true or false for that ring: for instance, the sentence $(\exists y_1)(\exists y_2)\big((y_1 \neq y_2) \wedge (y_1^2 = y_2^2)\big)$ is true in every field of characteristic $p > 2$, but is false in every reduced ring of characteristic 2.

A *theory* is a set of sentences. A ring $\mathfrak{R}$ is a *model* of a theory T if every sentence belonging to T is true in $\mathfrak{R}$: for instance, if p is either 0 or prime then AC_p is the set of sentences that are true in every algebraically closed field of characteristic p, and any algebraically closed field of characteristic p is a model of AC_p.

Theorem 2.1 (Lefschetz principle). *Let ϕ be a sentence in $\mathcal{L}_{\mathrm{ring}}$.*

(1) *If ϕ is true in some model of AC_p, where $p \geq 0$, then ϕ is true in every model of AC_p.*

(2) *If ϕ is true in some model of AC_0, then there exists $p_0 \in \mathbb{N}$ such that ϕ is true in any model of AC_p for $p > p_0$.*

2.3. ***Ideals in tensor products of algebras.*** Suppose that k is a field and that A, B are finitely generated k-algebras. Let K be an ideal of $A \otimes_k B$ with generators $f_1, \dots, f_n$ (note that K is finitely generated as $A \otimes_k B$ is noetherian). Fix a basis $\Xi = \{c_\lambda \mid \lambda \in \Lambda\}$ for B over k. We can write $f_j = \sum_m a_{j,m} \otimes b_{j,m}$ for elements $a_{j,m} \in A$ and $b_{j,m} \in \Xi$. Without loss of generality we can assume the elements $\{b_{j,m} \mid m\}$

are distinct for each fixed j, and under this assumption the ideal $K' \subseteq A$ generated by $\{a_{j,m} \mid j, m\}$ is uniquely determined by K, as we see from the next lemma.

Lemma 2.2. *K' is the smallest ideal of A such that $K \subseteq K' \otimes_k B$.*

Proof. Certainly $K \subseteq K' \otimes_k B$ and so it suffices to take an ideal $L \subseteq A$ such that $K \subseteq L \otimes_k B$ and show that $K' \subseteq L$. Observe that $A \otimes_k B$ is a free A-module with basis $\{1 \otimes c_\lambda \mid \lambda \in \Lambda\}$. Furthermore $L \otimes_k B = \bigoplus_{\lambda \in \Lambda} L \otimes c_\lambda$ and so if $f_j \in L \otimes_k B$ then $a_{j,m} \in L$ for all m appearing in the expression $f_j = \sum_m a_{j,m} \otimes b_{j,m}$. It follows that $K' \subseteq L$ as required. $\square$

3. Smoothness of centralisers and Gröbner bases

3.1. ***Bounded polynomials and Gröbner bases.*** Throughout this subsection we fix $n \in \mathbb{N}$ and k denotes a field. We will want to quantify over all k-algebras of bounded presentation, equipped with the structure of a Hopf algebra of bounded presentation. Here "bounded" means that the lengths and degrees of the polynomial expressions appearing in the defining ideal of the underlying affine algebra, together with the comultiplication, antipode and counit, are bounded. To do so, we need to formulate statements to say that the Hopf algebra axioms are satisfied. Our main tool to this end will be to quantify over all Gröbner bases of bounded degree. We refer to [Eis95, Chapter 15] for a hearty introduction to Gröbner bases, but for our purposes we collect a simplified version here.

The basic principle is to provide a process for reduction of elements of $S := k[x_1, \ldots x_n]$ by elements of an ideal, which will terminate in a finite number of steps. Hence one wants to know when the size of an expression is reduced by an operation, and for this one first needs to choose a total order on monomials. This order needs to be *admissible* in the sense that $m_1 > m_2$ if and only if $m_1 m_3 > m_2 m_3 > m_2$ for any monomials m_1, m_2, m_3 such that $m_3 \neq 1$.

We will demand of the order that for any monomial $m \in S$, there are only finitely many $m' < m$. For instance, we may use the *homogeneous* (or *graded*) *lexicographic ordering*, in which

$$m := x_1^{a_1} \ldots x_n^{a_n} > m' := x_1^{b_1} \ldots x_n^{b_n} \iff \deg m > \deg m' \text{ or if } \deg m = \deg m' \text{ then } a_i > b_i \text{ for the first index } i \text{ with } a_i \neq b_i.$$

Thus the set of monomials is isomorphic to $\mathbb{N}$ as a totally ordered set. We define m_r to be the *r-th monomial in S*. If m is a monomial then we define $k^* \cdot m$ to be $\{\lambda m \mid \lambda \in k^*\}$ and we call elements of this form *terms*; every polynomial can be written uniquely as a sum of terms. We extend $>$ to terms by defining $\lambda m_i > \mu m_j$ if $i > j$ and $0 \neq \lambda, \mu \in k$, and we define the *initial term* $\mathrm{in}(f)$ to be the greatest term appearing in f with respect to $>$ (taking $\mathrm{in}(0) = 0$). For an ideal $I \subseteq S$, we define $\mathrm{in}(I)$ to be the ideal generated by the elements $\mathrm{in}(f)$ for all $f \in I$. If g and h are terms then there is a unique monomial m such that m divides g and h, and any other monomial dividing g and h also divides m: we define $\gcd(g, h) = m$.

Definition 3.1. A *Gröbner basis*[1] with respect to $>$ is an ordered list of elements $(g_1, \dots, g_t) \in S^t$ for some t such that if I is the ideal of S generated by $g_1, \dots, g_t$, then $\mathrm{in}(g_1), \dots, \mathrm{in}(g_t)$ generate $\mathrm{in}(I)$. Note that the g_i need not be distinct and that although we work with ordered lists, the property of being a Gröbner basis does not depend on the ordering of the g_i.

Fix $d \in \mathbb{N}$. We wish to view a polynomial in $S = k[x_1, \dots, x_n]$ as a finite list of its coefficients, where we will ultimately be quantifying over all possible lists of those coefficients. We define the degree $\deg(f)$ of a polynomial $f \in S$ to be the total degree of $\mathrm{in}(f)$. According to our chosen (homogeneous) monomial order, $\deg(f)$ is the highest total degree of any term in f. Let $\mathcal{X}_d \subset S$ be the set of monomials of degree at most d, and let $\ell_d := |\mathcal{X}_d|$. By homogeneity of the monomial order again, this means $\mathcal{X}_d = \{\mathsf{m}_1, \dots, \mathsf{m}_{\ell_d}\}$. Furthermore, we may identify the set S_d of polynomials of degree at most d with the Cartesian product k^{ℓ_d}: the polynomial $\sum_{i=1}^{\ell_d} \lambda_i \mathsf{m}_i$ corresponds to $(\lambda_1, \lambda_2, \dots, \lambda_{\ell_d}) \in k^{\ell_d}$.

Definition 3.2. Let $S = R[x_1, \dots, x_n]$ be a polynomial ring over a ring R. Let S_d denote the polynomials in S of degree at most d. We say that an ordered list $\mathscr{B}$ of polynomials in R is *d-bounded* if $|\mathscr{B}| = \ell_d$ and $\mathscr{B}$ consists of elements of S_d.

If $R = k$ and $\mathscr{B}$ is also a Gröbner basis, we say $\mathscr{B}$ is a *d-bounded Gröbner basis*.

We identify the set of d-bounded lists of elements of S with $S_d^{\ell_d} = k^{\ell_d^2}$.

Remarks 3.3. (i) Any Gröbner basis of length greater than ℓ_d consisting of polynomials of degree at most d can be reduced to a Gröbner basis of cardinality at most ℓ_d. For if there are at least $\ell_d + 1$ elements then two, f and g say, must have the same leading monomial. So for some $\lambda \in k$, $g - \lambda f$ has a lower leading monomial and replacing g by $g - \lambda f$ we still have a Gröbner basis, directly from Definition 3.1. Inductively we may assume g is zero, thus it can be removed to produce a smaller Gröbner basis.

(ii) Conversely, any finite list of polynomials (resp. Gröbner basis) can be embedded into a d-bounded list of polynomials (resp. d-bounded Gröbner basis) for some d by appending an appropriate number of zero polynomials to the end of the list.

Lemma 3.4. *Let $d \in \mathbb{N}$ and let $1 \le e \le \ell_d$. Then there is a first-order formula $\phi_{e,d}$ in the language $\mathcal{L}_{\mathrm{ring}}$ of rings with ℓ_d free variables such that for any polynomial $f \in S$ of degree at most d,*

$$\phi_{e,d}(f) \text{ holds} \iff \mathrm{in}(f) \in k^* \cdot \mathsf{m}_e.$$

Proof. After identifying the set S_d with the space k^{ℓ_d}, so that the polynomial $f = \sum_{i=1}^{\ell_d} \lambda_i \mathsf{m}_i$ identifies with $(\lambda_1, \dots, \lambda_{\ell_d})$, the required formula is

$$(\lambda_e \neq 0) \wedge (\lambda_{e+1} = 0) \wedge (\lambda_{e+2} = 0) \wedge \cdots \wedge (\lambda_{\ell_d} = 0). \qquad \square$$

Given a d-bounded list of polynomials, we need to check with a first-order formula that it forms a Gröbner basis. For this, we appeal to Buchberger's criterion [Eis95, Theorem 15.8], which we reproduce here.

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[1] In contrast to [BW93], but consistently with [Eis95], we allow elements of Gröbner bases to be zero.

Let c be any integer and let $\mathscr{B} = (g_1, \dots, g_c) \in S^c$. For each pair of indices $1 \leq i, j \leq c$, we define

$$m_{ij} = \frac{\mathrm{in}(g_i)}{\gcd(\mathrm{in}(g_i), \mathrm{in}(g_j))} \in S.$$

(We interpret this as 0 if $g_i = 0$ or $g_j = 0$.) Then it follows from the division algorithm [Eis95, Proposition 15.6] that there exist $f_u^{(ij)} \in S$ with $\mathrm{in}(m_{ji}g_i - m_{ij}g_j) \geq \mathrm{in}(f_u^{(ij)}g_u)$ for each $1 \leq u \leq c$, and remainders $h_{ij} \in S$, none of whose terms is in $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_c))$, such that

$$m_{ji}g_i - m_{ij}g_j = \left(\sum_u f_u^{(ij)} g_u\right) + h_{ij}. \tag{3-1}$$

We call an expression (3-1) a *standard expression* for $m_{ji}g_i - m_{ij}g_j$.

Theorem 3.5 (Buchberger's criterion). *The set $\mathscr{B}$ is a Gröbner basis if and only if there exist standard expressions* (3-1) *such that $h_{ij} = 0$ for all $1 \leq i, j \leq c$.*

Lemma 3.6. *If g_i and g_j have no variables in common, there is a standard expression for $m_{ji}g_i - m_{ij}g_j$ such that $h_{ij} = 0$.*

Proof. Since $m_{ij} = \mathrm{in}(g_i)$ and $m_{ji} = \mathrm{in}(g_j)$, (3-1) becomes

$$m_{ji}g_i - m_{ij}g_j = \underbrace{-(g_j - \mathrm{in}(g_j))}_{f_i^{(ij)}}\, g_i + \underbrace{(g_i - \mathrm{in}(g_i))}_{f_j^{(ij)}}\, g_j.$$

Set $f_u^{(ij)} = 0$ for $u \neq i, j$. Write $g_i = \mathrm{in}(g_i) + \widetilde{\mathrm{in}(g_i)} + g_i'$ with $\widetilde{\mathrm{in}(g_i)}$ the initial term of $g_i - \mathrm{in}(g_i)$, and likewise for g_j. Then $m_{ji}g_i - m_{ij}g_j = \mathrm{in}(g_j)\,\widetilde{\mathrm{in}(g_i)} + \mathrm{in}(g_j)g_i' - \mathrm{in}(g_i)\,\widetilde{\mathrm{in}(g_j)} - \mathrm{in}(g_i)g_j'$ has as initial term whichever is the larger of $-\mathrm{in}(g_j)\,\widetilde{\mathrm{in}(g_i)}$ and $\mathrm{in}(g_i)\,\widetilde{\mathrm{in}(g_j)}$ (these two terms cannot cancel each other as g_i and g_j have no variables in common). But the initial terms of $f_i^{(ij)}$ and $f_j^{(ij)}$ are $-\widetilde{\mathrm{in}(g_j)}\,\mathrm{in}(g_i)$ and $\widetilde{\mathrm{in}(g_i)}\,\mathrm{in}(g_j)$, respectively, so $\mathrm{in}(m_{ji}g_i - m_{ij}g_j) \geq \mathrm{in}(f_u^{(ij)}g_u)$ for $u = i$ and $u = j$. Hence we get $h_{ij} = 0$, as required. □

Lemma 3.7. *Let $d \in \mathbb{N}$. There is a first-order formula β_d in the language of rings with ℓ_d^2 free variables such that if $\mathscr{B}$ is a d-bounded list of elements of S, then*

$$\beta_d(\mathscr{B}) \text{ holds } \iff \mathscr{B} \text{ is a Gröbner basis.}$$

Proof. Suppose $\mathscr{B} = (g_1, \dots, g_{\ell_d}) \in S_d^{\ell_d}$. We will produce a first-order formula that detects whether there exist expressions (3-1) for each pair (g_i, g_j), with $h_{i,j} = 0$. Suppose $\mathrm{in}(g_i) \in k^* \cdot \mathsf{m}_a$ and $\mathrm{in}(g_j) \in k^* \cdot \mathsf{m}_b$ and that there is a formula $\chi_{a,b}$ such that $\chi_{a,b}(g_i, g_j)$ is true if and only if there exist expressions (3-1) such that $h_{ij} = 0$. Then using Lemma 3.4 we set $\beta_d(\mathscr{B})$ to be the formula

$$\bigwedge_{1 \leq i, j \leq \ell_d} \Bigl(\bigvee_{1 \leq a, b \leq \ell_d} \bigl(\chi_{a,b}(g_i, g_j) \wedge \phi_{a,d}(g_i) \wedge \phi_{b,d}(g_j)\bigr)\Bigr),$$

and we see that $\beta_d(\mathscr{B})$ is true if and only if $\mathscr{B}$ satisfies the necessary and sufficient criterion of Buchberger's criterion to deduce that $\mathscr{B}$ is a Gröbner basis.

Thus we have reduced the problem, without loss of generality, to showing the existence of $\chi_{a,b}(g_1, g_2)$. For fixed a and b, $\mathsf{m}_{e'} := \gcd(\mathsf{m}_a, \mathsf{m}_b)$ is also fixed, depending just on the bijection between $\mathbb{N}$ and the monomials in S, hence so are the monomials m_{12} and m_{21}. Now, the highest monomial appearing in the left-hand side of (3-1) is at most the d'-th, where d' is given by $\mathsf{m}_{d'+1} = (\mathsf{m}_a\mathsf{m}_b/\mathsf{m}_{e'})$. Suppose $\operatorname{in}(m_{ji}g_i - m_{ij}g_j) = \mathsf{m}_e$. Then there is a finite set of pairs $P = \{(g_{a_b}, \mathsf{m}_{a_b})\}_{1\le b\le p}$ such that $\operatorname{in}(g_{a_b}\mathsf{m}_{a_b}) \le \mathsf{m}_e$. Hence, setting $\chi_{e,a,b}(g_1, g_2)$ to be the formula

$$(\exists\lambda_1)\dots(\exists\lambda_p)(m_{21}g_1 - m_{12}g_2 - \sum_{1\le b\le p} \lambda_b g_{a_b}\mathsf{m}_{a_b} = 0),$$

we see that $\chi_{e,a,b}(g_1, g_2)$ holds if and only if there is an expression of the form (3-1) for $m_{21}g_1 - m_{12}g_2$ with $h_{i,j} = 0$ (given that $\operatorname{in}(m_{ji}g_i - m_{ij}g_j) = \mathsf{m}_e$). Lastly, let $\chi_{a,b}(g_1, g_2)$ be the formula

$$\bigvee_{e=1}^{d'} \big(\phi_{e,d}(m_{2,1}g_1 - m_{1,2}g_2) \wedge \chi_{e,a,b}(g_1, g_2)\big);$$

this will do. □

Another important thing we need to be able to encode with a first-order statement is the dimension $\dim(I) = \dim(\operatorname{Spec}_k(S/I))$ of the scheme determined by an ideal $I \subseteq S = k[x_1, \dots, x_n]$. If $I = (g_1, \dots, g_{\ell_d})$ then in general it is not easy to read off $\dim I$ from the elements $\{g_1, \dots, g_{\ell_d}\}$. However, when $\{g_1, \dots, g_{\ell_d}\}$ form a Gröbner basis for I there is a simple method: the dimension is the maximal size of a subset $X \subseteq \{x_1, \dots, x_n\}$ such that $\operatorname{in}(g_1), \dots, \operatorname{in}(g_n)$ depend only on the elements of $\{x_1, \dots, x_n\} \setminus X$ [BW93, Definition 9.22 and Corollary 9.28]. Using this fact along with Lemma 3.4, we can determine dimension with a first-order formula.

Lemma 3.8. *Let $d \in \mathbb{N}$ and $0 \le e \le n$. Then there is a first-order formula $\delta_{e,d}$ in the language $\mathcal{L}_{\text{ring}}$ of rings, with ℓ_d^2 free variables, such that if $\mathscr{B}$ is any d-bounded Gröbner basis with $I = (\mathscr{B})$, then*

$$\delta_{e,d}(\mathscr{B}) \text{ holds} \iff \dim(I) = e.$$

Proof. There is obviously a finite collection of lists of monomials that could play the role of initial terms of the elements of a d-bounded Gröbner basis defining an ideal of dimension e. More formally, there is a set $\mathcal{X}_e = \{X_j \mid j \in T\}$, where T is some finite index set, and where each X_j is a d-bounded list of monomials in S satisfying: (i) there are distinct $i_1, \dots, i_e \in \{1, \dots, , n\}$ such that each $m \in X_j$ does not involve $x_{i_1}, \dots, x_{i_e}$; (ii) for any distinct $i_1, \dots, i_{e+1} \in \{1, \dots, n\}$, there is $m \in X_j$ depending on x_{i_k} for some $1 \le k \le e + 1$. For convenience we assume that the X_j are ordered sets and identify the monomials with their ordinal via the bijection of monomials of S with $\mathbb{N}$. Then we may set $\delta_{e,d}(\mathscr{B})$ to be the formula

$$\bigvee_{X_j=(a_1,\dots,a_{\ell_d})\in\mathcal{X}_e} \big(\phi_{a_1,d}(g_1) \wedge \phi_{a_2,d}(g_2) \wedge \dots \wedge \phi_{a_{\ell_d},d}(g_{\ell_d})\big).$$

□

The next lemma uses the ideal membership algorithm for Gröbner bases to write a first-order formula whose truth determines whether an element is in an ideal. If $\mathscr{B}$ is a d-bounded Gröbner basis and $f \in S_d$ then we may identify $(\mathscr{B}, f)$ with an element of $k^{{\ell_d}^2+\ell_d}$ in the usual manner.

Lemma 3.9. *Let $d \in \mathbb{N}$. Then there is a first-order formula ι_d in $\mathcal{L}_{\text{ring}}$ with $\ell_d^2+\ell_d$ free variables, such that for any $f \in S_d$ and d-bounded Gröbner basis $\mathscr{B}$ with $I := (\mathscr{B})$, we have*

$$\iota_d(\mathscr{B}, f) \textit{ holds} \iff f \in I.$$

Proof. Let $(g_1, \dots, g_{\ell_d})$ be a d-bounded Gröbner basis and let $f \in S_d$. Since the elements of $\mathscr{B}$ have bounded total degree d, and $<$ is a homogeneous order, there are only finitely many monomials m such that $\operatorname{in}(mg_i) \le \operatorname{in}(f)$ for some $1 \le i \le \ell_d$, where this number depends only on d. Let $\mathsf{m}_{d'}$ be the greatest such monomial. Thus we set $\iota_d(\mathscr{B}, f)$ to be the formula

$$(\exists \lambda_{i,j})_{1\le i\le d', 1\le j\le d} \quad f = g_1\biggl(\sum_{i=0}^{d'} \lambda_{i,1}\mathsf{m}_i\biggr) + g_2\biggl(\sum_{i=0}^{d'} \lambda_{i,2}\mathsf{m}_i\biggr) + \dots + g_d\biggl(\sum_{i=0}^{d'} \lambda_{i,d}\mathsf{m}_i\biggr). \tag{†}$$

We claim that $\iota_d(\mathscr{B}, f)$ is true if and only if $f \in I$. This follows by induction on e where $\operatorname{in}(f) = \mathsf{m}_e$: since $\mathscr{B}$ is a Gröbner basis, by [BW93, 5.35(vii)], f is *top-reducible* by some g_i or is not in I. In the former case, this means that there is a term m such that $\operatorname{in}(f - g_i m) < \operatorname{in}(f)$. By the inductive hypothesis, $\iota_d(\mathscr{B}, f - g_i m)$ is true whenever $f - g_i m \in I$, which is the case if and only if $f \in I$. If $f - g_i m \in I$ this says that there is an expression of the form (†) with f replaced by $f - g_i m$; moving $g_i m$ to the other side of the equation, this says that there is also one for f. □

3.2. ***Complexity of schemes and their morphisms.*** We recall some terminology, now reasonably common in the literature, to describe the boundedness of affine schemes; see [Sch00, Definition 4.1] for example. It is closely related to the notion of d-boundedness above.

Definition 3.10. (a) For $n \in \mathbb{N}$, we say that an ideal I of $S = R[x_1, \dots, x_n]$ *has complexity at most d* if I can be generated by polynomials of degree at most d; in this case we also say that the affine R-scheme X defined by the vanishing locus of I has complexity at most d.

(b) If in addition, $m \in \mathbb{N}$ we say that a homomorphism of R-algebras $\varphi : S \to T := R[y_1, \dots, y_m]$ has *degree at most d* if $\varphi(x_i)$ is a polynomial in the y_j of degree at most d.

More generally, we say that a morphism of affine schemes $f : Y \to X$ has *complexity at most d* if there are embeddings $X \subseteq \mathbb{A}^n_R$ and $Y \subseteq \mathbb{A}^m_R$ determined by ideals I of $S = R[x_1, \dots, x_n]$ and J of $T = R[y_1, \dots, y_m]$, such that the comorphism $f^* : S \to T$ applied to each x_i is represented modulo J by a polynomial in the y_j of degree at most d.

In particular, when $m = n$, $I = 0$ and f is the embedding $Y \hookrightarrow X = \mathbb{A}^n$, then f^* is just the quotient of S by J and has complexity at most 1.

Note that our definition differs slightly from that in [Sch00]: we regard n as fixed but we do not require that $n \le d$, as this is not necessary for our purposes.

By an *affine embedding* of an R-scheme X, we mean an embedding of X in $\mathbb{A}^n_R$ for some $n \in \mathbb{N}$. Below when we speak of the complexity of an R-scheme, we mean it to be taken with respect to a fixed affine embedding, and likewise for morphisms of R-schemes; we will not mention the affine embedding explicitly unless it is necessary. If we are given an affine embedding of X then we use the same embedding for any closed subscheme of X. If G is an algebraic R-group then we pick an affine embedding arising from a Hopf quadruple in the sense of Definition 3.13 below. If we are given affine embeddings of R-schemes X_1 and X_2 then we take our affine embedding of $X_1 \times X_2$ to be the product embedding.

Remarks 3.11. (i) If I has complexity at most d then a generating set of polynomials of degree at most d can be transformed into a d-bounded list as per Remarks (ii)(i) and (ii). This allows us to apply the results from Section 3.1 replacing hypotheses involving boundedness with hypotheses involving bounded complexity.

(ii) Let X, X' and X'' be affine schemes corresponding to ideals I of S, I' of S' and I'' of S'', respectively. Let $f : X \to X'$ and $g : X' \to X''$ be maps of complexity at most r and s, respectively. It follows immediately from the definitions that $g \circ f$ has complexity at most rs. Similarly, if Y' is a closed subscheme of X' and Y' has complexity at most d then $f^{-1}(Y')$ has complexity at most dr — note that this bound does not depend on the complexity of X and X'.

(iii) Suppose $\{X_i \mid i \in I\}$ is a family of affine R-schemes given by ideals I_i of S. It is immediate that if each X_i has complexity at most d then $\bigcap_{i \in I} X_i$ also has complexity at most d.

(iv) The notion of complexity behaves well under base change in the following sense. Suppose X is an affine R-scheme with a given embedding in some affine space $\mathbb{A}^n_R$. Let k be an R-field. Then base change gives an affine k-scheme X_k with an embedding in $\mathbb{A}^n_k$. If $f_1, \dots, f_t \in R[x_1, \dots, x_n]$ generate the vanishing ideal of X in $\mathbb{A}^n_R$ then the images of the f_i in $k[x_1, \dots, x_n]$ generate the vanishing ideal of X_k in $\mathbb{A}^n_k$. Hence if X has complexity at most d then X_k also has complexity at most d. The analogous result holds for morphisms.

(v) Suppose X and Y are both affine R-schemes, say

$$R[X] = R[x_1, \dots, x_r]/I \quad \text{and} \quad R[Y] = R[y_1, \dots, y_s]/J$$

with

$$I = (f_1, \dots, f_t) \quad \text{and} \quad J = (g_1, \dots, g_u).$$

Then

$$R[X \times Y] \cong R[X] \otimes R[Y] \cong R[x_1, \dots, x_r, y_1, \dots y_s]/K,$$

where K is the ideal generated by the concatenation of the f_i and g_i (after extending the domain of f_i and g_i to be trivial functions of the y_j and x_j respectively). Then one sees that if the complexity of X is d_1 and of Y is d_2, we get a presentation of $R[X \times Y]$ which shows its complexity is at most $\max\{d_1, d_2\}$. In particular, arguing by induction one shows that if S/I has complexity at most d, then $(S/I)^{\otimes r}$ has complexity at most d.

Definition 3.12. Let G be an algebraic R-group and let X be an affine G-scheme with action map $\alpha : G \times X \to X$. We say that *the action of G on X has complexity at most d* if α does.

Definition 3.13. Recall that $S = R[x_1, \dots, x_n]$. Let $H := (\mathscr{B}, \Delta, \sigma, \epsilon)$ be a quadruple with $(\mathscr{B}) = I \trianglelefteq S$; and $\Delta : S \to S^{\otimes 2}$, $\sigma : S \to S$ and $\epsilon : S \to R$ being R-algebra homomorphisms satisfying $\Delta(I) \subseteq I \otimes S + S \otimes I$, $S(I) \subseteq I$ and $\epsilon(I) = 0$. Then we say H is a *Hopf quadruple* if S/I equipped with the maps Δ, σ and ϵ forms a Hopf algebra. We say that a Hopf quadruple is of *complexity at most d* if $\mathscr{B}$ consists of polynomials of degree at most d and the complexity of Δ, σ and ϵ are at most d (this is automatic for ϵ, as ϵ has complexity 0). Dually, if an affine algebraic R-group G is described by the data $(\operatorname{Spec}(S/(\mathscr{B})), \Delta^*, \sigma^*, \epsilon^*)$, where $(\mathscr{B}, \Delta, \sigma, \epsilon)$ is a Hopf quadruple of complexity at most d, then we say that G is an *algebraic group of complexity at most d*. If G' is a closed subgroup of G then we may represent G' by the Hopf quadruple $(\mathscr{B}', \Delta, \sigma, \epsilon)$, where $\mathscr{B}' \supseteq \mathscr{B}$; if the polynomials in $\mathscr{B}'$ all have degree at most d then G' also has complexity at most d.

When R is a field, we call such a quadruple a *GroHo quadruple* if $\mathscr{B}$ happens to be a Gröbner basis.

We now work over a field k. The above definition invites us to consider the complexity of homomorphisms from S to $S' = S^{\otimes r}$ for various r. To that end, write $\ell_{d,r}$ for the total number of monomials of degree at most d in S'. Let $I = (\mathscr{B}) \subseteq S$ be an ideal generated by a d-bounded list $\mathscr{B} = \{f_1, \dots, f_{\ell_d}\}$ of elements of S. Write $f_i = f_i(x_1, \dots, x_n)$ and define $f_{i,j} = f_i(x_{j,1}, \dots, x_{j,n})$ for $1 \le j \le r$. Let $J_r = (\mathscr{B}_r)$, where $\mathscr{B}_r = \{f_{1,1}, \dots, f_{1,\ell_d}, \dots, f_{r,1}, \dots, f_{r,\ell_d}\}$. Then there is an isomorphism

$$\varphi_r : (S/I)^{\otimes r} \to k[x_{1,1}, \dots, x_{1,n}, \dots, x_{r,1}, \dots, x_{r,n}]/J_r. \tag{3-2}$$

It follows that the scheme corresponding to $(S/I)^{\otimes r}$ also has complexity at most d.

Lemma 3.14. *Let $d, r \in \mathbb{N}$. Then there is a first-order formula $\zeta_{d,r}$ in $\mathcal{L}_{\text{ring}}$ with $n\ell_{d,r} + \ell_d^2$ free variables such that if $\Lambda : S \to S^{\otimes r}$ is a homomorphism of complexity at most d and $\mathscr{B} = \{f_1, \dots, f_{\ell_d}\}$ is any d-bounded Gröbner basis for an ideal I of S,*

$$\zeta_{d,r}(\mathscr{B}, \Lambda) \text{ holds} \iff \Lambda \text{ factors to a homomorphism } S/I \to (S/I)^{\otimes r}.$$

Proof. Recall that $S = k[x_1, \dots, x_n]$ and $I = (\mathscr{B})$. With reference to Remark 3.11(v), we may take a presentation of $(S/I)^{\otimes r}$ as

$$(S/I)^{\otimes r} \cong k[x_{i,j}]/K$$

where $1 \le i \le n$, $1 \le j \le r$ and K is the ideal generated by the disjoint union $\mathscr{B}_r := \{f_{i,j}\}$, with $f_{i,j}$ acting on $x_{l,m}$ as zero if $j \ne m$ and otherwise acting as f_i acts on the x_l.

To deploy our Gröbner basis formulae from earlier, we need to specify a monomial order on the $x_{i,j}$'s. Define first an order on the (i, j) for $1 \le i \le n$ and $1 \le j \le r$ by $(i, j) < (i', j')$ if $j < j'$ or $j = j'$ and $i < i'$. Now define an order on monomials by $m := \prod_{i,j} x_{i,j}^{a_{i,j}} < m' := \prod_{i,j} x_{i,j}^{b_{i,j}}$ if and only if $\deg m < \deg m'$ or if $\deg m = \deg m'$ and $a_{i,j} < b_{i,j}$ for the greatest pair (i, j) such that $a_{i,j} \ne b_{i,j}$. We claim that $\mathscr{B}_r$ as above is a Gröbner basis for the homogeneous lexicographic monomial order on the monomials in $x_{i,j}$. Since our order extends the monomial orders on the subalgebras $k[x_{1,i}, \dots, x_{n,i}]$ for any fixed i, we see that Buchberger's criterion (Theorem 3.5) holds for all pairs $(f_{i,j}, f_{i',j'})$; furthermore if $j \neq j'$ then $f_{i,j}$ and $f_{i',j'}$ have no variables in common, so Buchberger's criterion holds for $(f_{i,j}, f_{i',j'})$ by Lemma 3.6. This proves the claim.

Now since Λ has complexity at most d and $\mathscr{B}_r$ is d-bounded, we may appeal to Lemma 3.9 to get first-order formulas $\iota_{d,r}$ such that $\iota_{d,r}(\mathscr{B}_r, \varphi_r(\Lambda(x_i)))$ holds if and only if $\varphi_r(\Lambda(x_i)) \in J_r$. Hence we set $\zeta_{d,r}$ to be the formula

$$\bigwedge_{i=1}^{n} \iota_{d,r}(\mathscr{B}_r, \varphi_r(\Lambda(x_i))). \qquad \square$$

Recall the axioms of a Hopf algebra, listed in (2-1)–(2-3).

Lemma 3.15. *Let $d \in \mathbb{N}$. There is a formula $\eta_d \in \mathcal{L}_{\mathrm{ring}}$ with $\ell_d^2 + n(\ell_{d,2} + \ell_d + 1)$ free variables such that if $\mathscr{B}$ is any d-bounded Gröbner basis, with $I = (\mathscr{B})$ and $\Delta : S \to S^{\otimes 2}$, $\sigma : S \to S$ and $\epsilon : S \to k$ any d-bounded homomorphisms, then*

$$\eta_d(\mathscr{B}, \Delta, \sigma, \epsilon) \textit{ holds} \iff (S/I, \Delta, \sigma, \epsilon) \textit{ is a Hopf algebra.}$$

Proof. Assume Δ, σ and ϵ factor as $S/I \to (S/I)^{\otimes r}$. We must find formulas $\eta_d^{(1)}$ (resp. $\eta_d^{(2)}, \eta_d^{(3)}$) which hold if and only if (2-1) (resp. (2-2), (2-3)) are satisfied. Since the constructions are almost identical for each formula, we give the details for $\eta_d^{(1)}$. To see that (2-1) holds, it clearly suffices to check that $(\Delta \otimes \mathrm{id}) \circ \Delta(x_i + I) - (\mathrm{id} \otimes \Delta) \circ \Delta(x_i + I) = 0 \in (S/I)^{\otimes 3} \cong S^{\otimes 3}/J_3$ for each $1 \le i \le n$, where $J_3 = (\mathscr{B}_3)$ as above. This amounts to checking that $(\Delta \otimes \mathrm{id}) \circ \Delta(x_i) - (\mathrm{id} \otimes \Delta) \circ \Delta(x_i) \in J$. Since Δ is d-bounded, $\varphi_2(\Delta(x_i))$ is a d-bounded polynomial in $S^{\otimes 2}$; similarly $f_i := \varphi_3((\Delta \otimes \mathrm{id}) \circ (\Delta(x_i)))$ is a d^2-bounded polynomial in $S^{\otimes 3}$. Likewise $\mathscr{B}_3$ is d-bounded. Thus we may set $\eta_d^{(1)}(\mathscr{B}, \Delta, \sigma, \epsilon)$ to be the formula

$$\bigwedge_{i=1}^{n} \iota_{d^2}(\mathscr{B}_3, f_i).$$

Finally, we set $\eta_d(\mathscr{B}, \Delta, \sigma, \epsilon)$ to be the formula

$$\zeta_{d,2}(\mathscr{B}, \Delta) \wedge \zeta_{d,1}(\mathscr{B}, \sigma) \wedge \zeta_{d,0}(\mathscr{B}, \epsilon) \wedge \eta_d^{(1)}(\mathscr{B}, \Delta, \sigma, \epsilon) \wedge \eta_d^{(2)}(\mathscr{B}, \Delta, \sigma, \epsilon) \wedge \eta_d^{(3)}(\mathscr{B}, \Delta, \sigma, \epsilon),$$

where the $\zeta_{d,r}$ are as in Lemma 3.14. $\square$

3.3. ***Generic smoothness of algebraic groups of bounded complexity.*** Here we invoke the Lefschetz principle and our first-order formulas to show that algebraic groups of bounded complexity are generically smooth. We use the fact that when k is a field, an algebraic k-group G is smooth if and only if $\dim(G) = \dim(\mathrm{Lie}(G))$ [Jan03, I.7.17]. Note also that if k'/k is a field extension then G is smooth if and only if $G_{k'}$ is smooth.

As a k-vector space, the Lie algebra $\mathrm{Lie}(G)$ is the tangent space $T_{\epsilon^*}(G)$, where ϵ^* is the identity element of G. Thus its dimension is the nullity of the $\ell_d \times n$ matrix $\mathscr{J}$ where $\mathscr{J}_{kl} = \epsilon(\partial f_k / \partial x_l)$.

Lemma 3.16. *Let $d \in \mathbb{N}$, and $0 \le e \le d$. There is a first-order formula $\tau_{e,d}$ in $\mathcal{L}_{\text{ring}}$ with ℓ_d^2 free variables such that for any GroHo quadruple $(\mathscr{B}, \Delta, \sigma, \epsilon)$ of complexity at most d that describes the algebraic k-group G we have*

$$\tau_{e,d}(\mathscr{B}) \text{ holds} \iff \dim \operatorname{Lie}(G) = e.$$

Proof. As we identify each $f_i \in \mathscr{B}$ with the set of its ℓ_d coefficients λ_{ij}, partial differentiation by $\partial/\partial x_i$ gives a linear map $k^{\ell_d} \to k^{\ell_d}$. Composing with ϵ is then a linear map $k^{\ell_d} \to k$. Hence each $\mathscr{J}_{kl}$ is a fixed linear combination of the λ_{ij}'s. The statement that the nullity of $\mathscr{J}$ is e is equivalent to the statement that there are e linearly independent vectors $v_1, \dots, v_e \in k^{\ell_d}$ satisfying $\mathscr{J} \cdot v_i = 0$ and given any $v_{e+1} \in k^{\ell_d}$ such that $v_1, \dots, v_{e+1} \in k^{\ell_d}$ is linearly independent, there exists $v \in \langle v_1, \dots, v_{e+1}\rangle$ such that $\mathscr{J} \cdot v \neq 0$. This statement can be given as a formula in $\mathcal{L}_{\text{ring}}$ in an obvious way (see [MST19, Example 2.1(i)] for example). □

Lemma 3.17. *Let $d \in \mathbb{N}$. Then there is a first-order formula θ_d with ℓ_d^2 free variables such that for any GroHo quadruple $H := (\mathscr{B}, \Delta, \sigma, \epsilon)$ of complexity at most d,*

$$\theta_d(\mathscr{B}) \text{ holds} \iff H \text{ describes a smooth } k\text{-group}.$$

Proof. The k-group G described by H is a subscheme of $\operatorname{Spec}(S) \cong \mathbb{A}^n$, so $0 \le \dim G \le n$. Then invoking Lemmas 3.16 and 3.8 we may set $\theta_d(\mathscr{B}, \Delta, \sigma, \epsilon)$ to be the following formula:

$$\bigvee_{e=0}^{n} \left(\delta_{e,d}(\mathscr{B}) \wedge \tau_{e,d}(\mathscr{B})\right). \qquad \square$$

We wish to apply the above results to obtain statements about the set of algebraic groups of complexity at most d. This amounts to statements about the Hopf quadruples of complexity at most d. The following theorem of Dubé guarantees that any algebraic group of complexity at most d can in fact be described by a GroHo quadruple of complexity at most D, where D depends just on d and n.

Theorem 3.18 [Dub90]. *If $\mathscr{B}' \subset S$ is a set of polynomials of degree at most d, then $(\mathscr{B}')$ has a Gröbner basis $\mathscr{B}$ consisting of polynomials of degree at most*

$$2\left(\frac{d^2}{2} + d\right)^{2^{n-1}}.$$

Theorem 3.19. *Let $d \in \mathbb{N}$. Then there is a prime $p_0 = p_0(n, d)$ such that whenever* $\operatorname{char}(k) \ge p_0$, *any algebraic k-group of complexity at most d is smooth.*

Proof. By Theorem 3.18, each affine algebraic group of complexity at most d is described by a GroHo quadruple of complexity at most D, where D depends just on d and n; here n has been fixed. So it suffices to prove that there is a $p_0(D)$ such that any GroHo quadruple of complexity at most D over a field k of characteristic $p \ge p_0$ describes a smooth algebraic k-group.

First suppose that k is algebraically closed. Let $H = (\mathscr{B}, \Delta, \sigma, \epsilon)$ be a GroHo quadruple of complexity at most D. Recall we identify H with a string of $\ell_D^2 + n(\ell_{D,2} + 2\ell_D + 1)$ coefficients in k, which we write $(\lambda_i)_{i=1}^{\ell_D^2+n(\ell_{D,2}+2\ell_D+1)}$.Then invoking Lemmas 3.7, 3.15 and 3.17, the following formula Φ_D is a *sentence* in $\mathcal{L}_{\mathrm{ring}}$ that is true if and only if all GroHo quadruples of complexity at most D describe smooth algebraic groups:

$$(\forall\lambda_1)\cdots(\forall\lambda_{\ell_D^2+n(\ell_{D,2}+2\ell_D+1)})(\beta_D(\mathscr{B}) \wedge \eta_D(\mathscr{B}, \Delta, \sigma, \epsilon) \wedge \theta_D(\mathscr{B})).$$

By Cartier's theorem [Jan03, I.7.17(2)], Φ is true for all fields of characteristic 0. Therefore the Lefschetz principle (Theorem 2.1) guarantees the existence of a prime p_0 such that the same is true for all algebraically closed fields of characteristic $p \geq p_0$.

Now let k be arbitrary and let G be an algebraic k-group of complexity at most d. Suppose $p \geq p_0$. We can choose a GroHo quadruple $H = (\mathscr{B}, \Delta, \sigma, \epsilon)$ of complexity at most D such that G is the corresponding algebraic k-group. By changing base from k to $\bar{k}$, we may regard H as a GroHo quadruple that defines $G_{\bar{k}}$. Recall that complexity does not increase under base change, so the complexity of this GroHo quadruple is still at most d. So $G_{\bar{k}}$ is smooth by the algebraically closed case, which implies that G is smooth as well. This proves the theorem. $\square$

4. Normalisers and centralisers

We now prove our main results Theorems 1.1 and 1.2. We will work with non-affine schemes, so we need some preliminary material. We use the definitions and terminology of [Jan03, Chapter I]. Although we work mainly over k, we need to consider arbitrary R and study the behaviour of some of the constructions below under base change to k. Note that every R-scheme is locally free in the sense of [Jan03, I.1.15] if R is a field. We fix an R-scheme X and an open covering $\{X_i \mid i \in \mathbb{I}\}$ of X such that each X_i is affine. For each i, we fix an affine embedding of X_i in some $\mathbb{A}_R^{n_i}$. This allows us to talk about the complexity of certain schemes and maps involving the X_i.

We also fix a (not necessarily smooth) algebraic R-group G described by a Hopf quadruple $(S/I, \Delta, \sigma, \epsilon)$. We fix an action $\alpha : G \times X \to X$ of G on X.

4.1. ***The functor $\mathfrak{Mor}(-, -)$.*** We review some basic constructions of algebraic geometry: see [Jan03, I.1.15, I.2.6]. Recall that for R-schemes Z and W, we get an R-functor

$$\mathfrak{Mor}(Z, W) : \underline{R\text{-Alg}} \to \underline{\text{Set}}$$

given by $\mathfrak{Mor}(Z, W)(Q) = \mathrm{Mor}(Z_Q, W_Q)$ for any R-algebra Q; if $\varphi : Q \to Q'$ is a homomorphism of R-algebras then $\mathfrak{Mor}(Z, W)(\varphi) : \mathrm{Mor}(Z_Q, W_Q) \to \mathrm{Mor}(Z_{Q'}, W_{Q'})$ is given by base change. Given another R-scheme C and a morphism $\alpha : C \times Z \to W$ of R-schemes, we get a map of R-functors $\nu : C \to \mathfrak{Mor}(Z, W)$ such that for an R-algebra Q and $c \in C(Q)$, ν maps c to $\alpha(c, -) \in \mathrm{Mor}(Z_Q, W_Q)$.

Now let W' be a closed subscheme of W. If Z is locally free as an R-scheme then by [Jan03, I.1.15] we may regard $\mathfrak{Mor}(Z, W')$ as a closed R-subfunctor of $\mathfrak{Mor}(Z, W)$. So by [Jan03, I.1.15(3)] we obtain a closed subscheme $\nu^{-1}(\mathfrak{Mor}(Z, W'))$ of C.

4.2. ***G-complexity.*** Next we describe the precise boundedness condition that we need. With the notation of the last subsection, let Y be a closed subscheme of X. Set $Y_i = Y \cap X_i$ for each i. Recall $A := R[G] \cong S/I$ for $S = R[x_1, \ldots, x_n]$ and let $B_i = R[Y_i]$. Let $\alpha_i : G \times Y_i \to X$ be the restriction of α to $G \times Y_i$. Then $\alpha_i^{-1}(Y)$ is a closed subscheme of $G \times Y_i$ [Jan03, I.1.12(2)], so it corresponds to an ideal K_i of $R[G \times Y_i] = A \otimes_R B_i$. We can write $K_i = (f_j^{(i)} \mid j \in \mathbb{J}_i)$, where $\mathbb{J}_i$ is some (possibly infinite) indexing set, and where each $f_j^{(i)}$ has the form

$$f_j^{(i)} = \sum_m a_{mj}^{(i)} \otimes b_{mj}^{(i)} \tag{4-1}$$

for some $a_{mj}^{(i)} \in A$ and $b_{mj}^{(i)} \in B_i$.

Definition 4.1. Let G, X and Y be as above. We say that Y *has G-complexity at most d* if there exist $f_j^{(i)}$ as above such that each $a_{mj}^{(i)}$ has a representative $\widetilde{a}_{mj}^{(i)}$ in S of degree at most d.

Note that we do not place any restrictions on the degrees of the $b_{mj}^{(i)}$ in the definition, and we do not require the K_i to be finitely generated.

Remark 4.2. Let G, X and Y be as above. The G-complexity condition in Definition 4.1 can be hard to verify, but here is a useful special case. Let $X \subseteq \mathbb{A}^s$ be affine; let J be an ideal of $R[\mathbb{A}^s] = R[t_1, \ldots, t_s]$ of complexity at most e defining a closed subscheme Y of X. Furthermore, recalling Definition 3.12, suppose the action $\alpha : G \times X \to X$ has complexity at most e'. Then we claim Y has G-complexity at most ee'.

To see this, let $\widetilde{\alpha}$ be the restriction of α to $G \times Y$, and note that $\widetilde{\alpha}^{-1}(Y)$ is the closed subscheme in $R[G] \otimes_R R[Y]$ defined by the vanishing of $\widetilde{\alpha}^*(J)$. So since J is generated by polynomials in the t_i of degree at most e, then applying the algebra homomorphism $\widetilde{\alpha}^*$ to them leads to polynomials of degree at most ee' (see Remark 3.11(ii)).

Much of the time we can do better than this; if the image of $\widetilde{\alpha}$ is Y — so that G normalises Y — then $\widetilde{\alpha}^{-1}(Y) = G \times Y$ is the closed subscheme of $G \times Y$ defined by the zero ideal. In that case, Y has G-complexity 0.

Now let Q be an R-algebra. Change of base yields an algebraic Q-group G_Q acting on a Q-scheme X_Q with an open covering by affine schemes $(X_i)_Q$. If Y is a closed subscheme of X then Y_Q is a closed subscheme of X_Q. The various constructions in Section 4.2 are well-behaved with respect to base change, so we see that if Y has G-complexity at most d then Y_Q has G_Q-complexity at most d.

4.3. ***Normalisers.*** We start by recalling the scheme-theoretic definition of the normaliser (see [Jan03, I.2.6] for details). In this subsection we assume the ground ring is a field k. This implies that the local freeness condition used in Section 4.1 holds. Let Y be a closed subscheme of X. Then the *normaliser of Y* (denoted $N_G(Y)$) is the k-subgroup functor of G given by

$$N_G(Y)(Q) = \{g \in G(Q) \mid \alpha(g, h) \in Y(Q') \text{ for all } h \in Y(Q') \text{ and all } Q\text{-algebras } Q'\}.$$

It is clear from the definition that if k' is a k-algebra then $N_{G_{k'}}(Y_{k'}) = (N_G(Y))_{k'}$. We will need the explicit description of $N_G(Y)$ afforded by the k-functor $\mathfrak{Mor}(-,-)$. The action $\alpha : G \times X \to X$ gives rise to a map $\nu : G \to \mathfrak{Mor}(X, X)$ as described in Section 4.1. We have a map $\mathfrak{Mor}(X, X) \to \mathfrak{Mor}(Y, X)$ given by restriction, and we let $\gamma : G \to \mathfrak{Mor}(Y, X)$ be the composition with ν. Then $N_G(Y) = \gamma^{-1}(\mathfrak{Mor}(Y, Y)) \cap i_G(\gamma^{-1}(\mathfrak{Mor}(Y, Y)))$, where $i_G : G \to G$ is the inversion map. Since Y is locally free, $\mathfrak{Mor}(Y, Y)$ is closed in $\mathfrak{Mor}(Y, X)$ and it follows that $N_G(Y)$ is a closed subgroup functor of G. Now $N_G(Y)$ is an algebraic k-group since k is noetherian.

Theorem 4.3. *Let $d \in \mathbb{N}$ and let G, X and Y be as above. Suppose G has complexity at most d and Y has G-complexity at most d. Then $N_G(Y)$ is a closed subscheme of G of complexity at most d^2.*

Proof. We have $N_G(Y) = \gamma^{-1}(\mathfrak{Mor}(Y, Y)) \cap i_G(\gamma^{-1}(\mathfrak{Mor}(Y, Y)))$. By Remark 3.11(ii), it is enough to prove that $\gamma^{-1}(\mathfrak{Mor}(Y, Y))$ has complexity at most d (since i_G has complexity at most d).

Each Y_i is a closed subscheme of X_i [Jan03, I.1.13, Lemma], and the Y_i form an open cover of Y. Let γ_i be the composition $G \to \gamma\mathfrak{Mor}(Y, X) \to \mathfrak{Mor}(Y_i, X)$, where the second map is given by restriction. By [Jan03, I.1.15(2)], $\gamma^{-1}(\mathfrak{Mor}(Y, Y)) = \bigcap_{i \in \mathbb{I}} \gamma_i^{-1}(\mathfrak{Mor}(Y_i, Y))$. Each $\gamma_i^{-1}(\mathfrak{Mor}(Y_i, Y))$ is a closed subscheme of G by an argument similar to the one for $\gamma^{-1}(\mathfrak{Mor}(Y, Y))$. In the proof of [Jan03, I.1.15(3)], one considers a map f from $\mathrm{Spec}_k(R)$ to $\mathfrak{Mor}(Y_i, X)$, where R is an arbitrary k-algebra; one obtains a map f' from $\mathrm{Spec}_k(R) \times Y_i$ to X. We apply this construction when $R = A = k[G]$, $B_i = k[Y_i]$ and $f = \gamma_i$; then f' is just the map $\alpha_i : G \times Y_i \to X$. Let K_i be the ideal of $k[G \times Y_i] = A \otimes_k B_i$ corresponding to $\alpha_i^{-1}(Y)$ as before, and let K_i' be the ideal of A corresponding to $\gamma_i^{-1}(\mathfrak{Mor}(Y_i, Y))$. By the argument of loc. cit., K_i' is the smallest ideal of A such that $K_i' \otimes B_i$ contains K_i.

Fix $i \in \mathbb{I}$. Choose $f_j^{(i)}$, $a_{mj}^{(i)}$, $\widetilde{a}_{mj}^{(i)}$ and $b_{mj}^{(i)}$ as in Definition 4.1, where each $\widetilde{a}_{mj}^{(i)}$ has degree at most d. Rewriting and expanding as necessary, we may assume each $b_{mj}^{(i)}$ is a member of a fixed k-basis of B_i; evidently this does not affect the bound on the degree of the $\widetilde{a}_{mj}^{(i)}$. Thanks to Lemma 2.2 we see that K_i' is generated by the $a_{mj}^{(i)}$, thus has complexity at most d.

Let $I_0 = I + \sum_{i \in \mathbb{I}} K_i'$ be the ideal of S corresponding to $\gamma^{-1}(\mathfrak{Mor}(Y, Y))$. Since each K_i' has complexity at most d, so does I_0. The result now follows. ☐

Corollary 4.4. *Let $d, n \in \mathbb{N}$. Then there is a prime $p_1 = p_1(n, d)$ such that if:*

- *k is any field of characteristic $p \geq p_1$;*
- *G is any affine algebraic k-group of complexity at most d;*
- *X is any k-scheme on which G acts;*
- *Y is any closed subscheme of X of G-complexity at most d;*

then $N_G(Y)$ is a smooth closed subscheme of G.

Proof. This follows from Theorem 3.19 and Theorem 4.3. We need to be working with a fixed n as well as a fixed d in order to apply Theorem 3.19; but this is the case here as we are considering subgroups of a fixed G. ☐

4.4. ***(G, Δ)-complexity.*** Once again we work over an arbitrary ring R. We need a slightly different boundedness condition for Theorem 4.7. Here we assume again that $X = \bigcup_{i \in \mathbb{I}} X_i$ is an R-scheme with Y a closed subscheme. This time, we also assume that X is separated; this means that the diagonal Δ_X is closed in $X \times X$. Set $Y_i = Y \cap X_i$ for each i as before. Let $A = R[G]$ and let $B_i = R[Y_i]$ for each i.

Define $\beta : G \times X \to X \times X$ by $\beta = \alpha \times \mathrm{pr}_2$, where $\mathrm{pr}_2 : G \times X \to X$ is the projection; let β_i, β_Y and β_{Y_i} be its restriction to $G \times X_i$, $G \times Y$ and $G \times Y_i$, respectively. Thanks to [Jan03, I.1.12(2)] we see that $\beta_{Y_i}^{-1}(\Delta_X)$ is a closed subscheme of $G \times Y_i$, so it corresponds to an ideal K_i of $R[G \times Y_i] = A \otimes_R B_i$. We can write $K_i = (f_j^{(i)} \mid j \in \mathbb{J}_i)$, where each $f_j^{(i)}$ has the form

$$f_j^{(i)} = \sum_m a_{mj}^{(i)} \otimes b_{mj}^{(i)} \tag{4-2}$$

for some $a_{mj}^{(i)} \in A$ and some $b_{mj}^{(i)} \in B_i$.

Definition 4.5. Let G, X and Y be as above. We say that Y *has* (G, Δ)*-complexity at most d* if there exist $f_j^{(i)}$ as above such that each $a_{mj}^{(i)}$ has a representative $\tilde{a}_{mj}^{(i)}$ in S of degree at most d.

Lemma 4.6. *Let G, X and Y be as above. Suppose X has (G, Δ)-complexity at most d. Then Y has (G, Δ)-complexity at most d.*

Proof. Fix $i \in \mathbb{I}$. By hypothesis, $\beta_i^{-1}(\Delta_X)$ is the closed subscheme of $G \times X_i$ defined by elements of the form $f_j^{(i)} = \sum_m a_{mj}^{(i)} \otimes b_{mj}^{(i)}$ for j in some indexing set $\mathbb{J}_i$, where each $a_{mj}^{(i)} \in A$ has a representative $\tilde{a}_{mj}^{(i)}$ in S of degree at most d and each $b_{mj}^{(i)} \in B_i$. Let $L_i = (h_\ell^{(i)} \mid \ell \in \mathbb{L}_i)$ be the ideal cutting out Y_i in X_i, where $\mathbb{L}$ is a (possibly infinite) indexing set. Now, we have $\beta_{Y_i}^{-1}(\Delta_X) = \beta_i^{-1}(\Delta_X) \cap (G \times Y_i)$, so the ideal cutting out $\beta_{Y_i}^{-1}(\Delta_X)$ is generated by the $f_j^{(i)}$ for $j \in \mathbb{J}_i$ together with the elements $1 \otimes h_\ell^{(i)}$ for $\ell \in \mathbb{L}$. As the constant polynomial 1 has complexity 0, the result now follows. ☐

Now let Q be an R-algebra. We see that if X has (G, Δ)-complexity at most d then X_Q has (G_Q, Δ_Q)-complexity at most d.

4.5. ***Centralisers.*** The argument is similar to the one for normalisers, and we recommence with the notation from Section 4.3. In particular, our ground ring is a field k. We assume additionally that the k-scheme X is separated. Recall the definition of the centraliser $C_G(Y)$ from [Jan03, I.2.6]: it is the k-subgroup functor of G whose Q-points are

$$C_G(Y)(Q) = \{g \in G(Q) \mid \alpha(g, y) = y \text{ for all } y \in Y(Q') \text{ and all } Q\text{-algebras } Q'\},$$

where Q is any k-algebra. It is clear from the definition that if k' is a k-algebra then $C_{G_{k'}}(Y_{k'}) = (C_G(Y))_{k'}$.

The maps β and β_Y from Section 4.2 give rise to maps

$$\delta : G \to \mathfrak{Mor}(X, X \times X) \quad \text{and} \quad \delta_Y : G \to \mathfrak{Mor}(Y, X \times X),$$

as described in Section 4.1. We have a map $\mathfrak{Mor}(X, X \times X) \to \mathfrak{Mor}(Y, X \times X)$ given by restriction, and it is easily seen that δ_Y is the composition $G \to \delta\mathfrak{Mor}(X, X \times X) \to \mathfrak{Mor}(Y, X \times X)$.

Since X is separated, Δ_X is closed in $X \times X$. Then $C_G(Y) = \delta^{-1}(\mathfrak{Mor}(Y, \Delta_X))$, and this is a closed subgroup functor of G; in particular, $C_G(Y)$ is an algebraic k-group.

Theorem 4.7. *Let $d \in \mathbb{N}$ and let G, X and Y be as above. Suppose G has complexity at most d and X has (G, Δ)-complexity at most d. Then $C_G(Y)$ is a closed subscheme of G of complexity at most d.*

Proof. Let δ_i be the composition $G \to \gamma\mathfrak{Mor}(Y, X \times X) \to \mathfrak{Mor}(Y_i, X \times X)$, where the second map is given by restriction. We have $C_G(Y) = \delta^{-1}(\mathfrak{Mor}(Y, \Delta_X))$. By [Jan03, I.1.15(2)], $\delta^{-1}(\mathfrak{Mor}(Y, \Delta_X)) = \bigcap_{i \in \mathbb{I}} \delta_i^{-1}(\mathfrak{Mor}(Y_i, \Delta_X))$. Each $\delta_i^{-1}(\mathfrak{Mor}(Y_i, \Delta_X))$ is a closed subscheme of G. In the proof of [Jan03, I.1.15(3)], one considers a map f from $\mathrm{Spec}_k(R)$ to $\mathfrak{Mor}(Y_i, X)$, where R is an arbitrary k-algebra; one obtains a map f' from $\mathrm{Spec}_k(R) \times Y_i$ to X. We apply this construction when $R = A = k[G]$, $B_i = k[Y_i]$ and $f = \delta_i$; then f' is just the map $\beta_i : G \times Y_i \to X \times X$. Let K_i be the ideal of $k[G \times Y_i] = A \otimes_k B_i$ corresponding to $\beta_{Y_i}^{-1}(\Delta_X)$, and K_i' the ideal of A corresponding to $\delta_i^{-1}(\mathfrak{Mor}(Y_i, \Delta_X))$. By the argument of loc. cit., K_i' is the smallest ideal of A such that $K_i' \otimes B_i$ contains K_i.

Since Y has (G, Δ)-complexity at most d by Lemma 4.6, K_i is generated by elements of the form $f_j^{(i)} = \sum_m a_{mj}^{(i)} \otimes b_{mj}^{(i)}$ for some $a_{mj}^{(i)} \in A$ and some $b_{mj}^{(i)} \in B_i$, where each $a_{mj}^{(i)}$ has a representative $\widetilde{a}_{mj}^{(i)}$ in S of degree at most d. As in the proof of Theorem 4.3, we know K_i' is generated by the elements $a_{mj}^{(i)}$, and has complexity at most d. Hence $C_G(Y)$ has complexity at most d, and we are done. ☐

Corollary 4.8. *Let $d, n \in \mathbb{N}$. Then there is a prime $p_0 = p_0(n, d)$ such that if:*

- *k is any field of characteristic $p \geq p_0$;*
- *G is any affine algebraic k-group of complexity at most d;*
- *X is any separated k-scheme of (G, Δ)-complexity at most d;*

then for any closed subscheme Y of X, the centraliser $C_G(Y)$ is a smooth closed subscheme of G.

Proof. This follows from Theorem 3.19 and Theorem 4.7. As in Corollary 4.4, we are working with subgroups of a fixed G, so there is no problem with applying Theorem 3.19. ☐

Remark 4.9. Because of Lemma 4.6, the complexity hypotheses in Corollary 4.8 do not involve Y; therefore we get better smoothness results for centralisers than for normalisers (cf. Remark 4.12). The complexity hypothesis on X is difficult to check in general, but it clearly holds if X is of finite type, since then we can take I to be finite. Here is a useful special case. Suppose X is finitely presented and affine and suppose the action $\alpha : G \times X \to X$ has complexity at most d. By assumption, X corresponds to an ideal I of $R[x_1, \ldots, x_n]$, where I is generated by polynomials $f_1, \ldots, f_t$ of $R[x_1, \ldots, x_n]$ for some $n, t \in \mathbb{N}$. Then X has complexity at most d', where d' is the maximum of the degrees of the f_i. We claim that X has (G, Δ)-complexity at most dd'. To see this, observe that Δ_X corresponds to the ideal J of $R[x_1, \ldots, x_n] \otimes_R R[x_1, \ldots, x_n]$ generated by the polynomials $f_i \otimes 1 - 1 \otimes f_i$ for $1 \leq i \leq t$, so Δ_X has complexity at most d'. The map $\beta : G \times X \to X \times X$ has complexity at most d, since α has complexity at most d and pr_2 has complexity 1. Hence $\beta^{-1}(\Delta_X)$ has complexity at most dd' by Remark 3.11(ii), and the claim follows.

Remark 4.10. Let G, X and Y be as above (defined over arbitrary R). Suppose G has complexity at most d. Then for any R-algebra Q and any $y \in Y_Q(Q)$, the singleton $\{y\}$ has G_Q-complexity at most d. In case $Q = k$ is algebraically closed, suppose Y_k is reduced; thus its k-points $Y_k(k)$ are dense. It follows that $C_{G_k}(Y_k)$ is equal to the closed subgroup $\bigcap_{y \in Y(k)} N_{G_k}(\{y\})$. Each $N_{G_k}(\{y\})$ has G-complexity at most d^2 by Theorem 4.3, so $C_{G_k}(Y_k)$ has complexity at most d^2. Hence by Theorem 3.19 there exists $p_0 = p_0(n, d)$ such that if $\operatorname{char}(k) \geq p_0$ then we have that $C_{G_k}(Y_k)$ is smooth. Note that by translating the problem to normalisers of points, we do need to assume that X is separated.

4.6. ***Proof of main theorems and examples.*** We can now give a quick proof of our main results.

Proof of Theorems 1.1 and 1.2. Let G and X be as in the statement of Theorem 1.1. Since G is affine and finitely presented, there exists $d_1 \in \mathbb{N}$ such that G has complexity at most d_1. Since X is finitely presented, there exists $d_2 \in \mathbb{N}$ such that X has (G, Δ)-complexity at most d_2; note that there are only finitely many ideals K_i that we need consider in the definition of (G, Δ)-complexity for X, and each K_i is finitely generated. Hence for any R-field k, we see from Remark 3.11(iv) that G_k has complexity at most d and X_k has (G_k, Δ)-complexity at most d, where $d = \max\{d_1, d_2\}$. Now Theorem 1.1 follows from Corollary 4.8.

Now let G, X and Y be as in the statement of Theorem 1.2. Since X is of finite type and Y is finitely presented, we can choose a finite cover of X by open affine subsets X_i such that $Y_i := Y \cap X_i$ is finitely presented for each i. There exists $d_2 \in \mathbb{N}$ such that Y has G-complexity at most d_2 (we need consider only finitely many finitely generated ideals K_i in the definition of G-complexity). We finish the proof as we did in the centraliser case, using Corollary 4.4 in place of Corollary 4.8. □

Proof of Corollary 1.3. The G-module V is a finitely presented affine G-scheme over R. The proof for centralisers of subspaces follows immediately from Theorem 1.1 applied to G and V.

For normalisers we will use Corollary 4.4. The dual of a finitely generated projective module is finitely generated and projective, so we can choose generators $\alpha_1, \ldots, \alpha_n$ for V^* as an R-module. The map $\alpha_1 \times \cdots \times \alpha_n$ gives an R-linear embedding of V as a subspace of $\mathbb{A}^n_R$. There exists $d \in \mathbb{N}$ such that G has complexity at most d and the map $G \times V \to V$ given by the action has complexity at most d. Now let k be an R-field. Then G_k has complexity at most d and the map $G_k \times V_k \to V_k$ given by the action has complexity at most d, by Remark 3.11(iv). Let W be a subspace of V_k. Then W has complexity at most 1 with respect to the embedding of V_k in $\mathbb{A}^n_k$, so W has G-complexity at most d by Remark 4.2. The result now follows from Corollary 4.4. □

Remark 4.11. We sketch another proof of the normaliser part of Corollary 1.3. Let k be an R-field and let W be a subspace of V_k. Set $r = \dim(W)$. Then $N_{G_k}(W) = C_{G_k}(x)$, where x is the element of $\mathbb{P}(\Lambda^r(V_k))$ corresponding to the line $\Lambda^r(W)$ in the exterior power $\Lambda^r(V_k)$, so we can deduce the result by applying Theorem 1.1 to the G-module $\mathbb{P}(\Lambda^r(V))$. We leave the details to the reader.

Remark 4.12. Any hope to extend Theorem 1.2 to deal with normalisers of arbitrary closed subschemes of X will fail without first imposing some further hypotheses. For instance, [HS16, Lemma 11.11] gives for each prime p a smooth subgroup H_p of GL_3 over an algebraically closed field of characteristic p such that the normaliser of H_p is nonsmooth. Here we can take $G = X = \mathrm{GL}_3$ over $R = \mathbb{Z}$ with G acting on X by conjugation; Theorem 1.2 does not apply because our closed subschemes H_p are not of the form Y_k for any fixed closed subscheme Y of X.

Remark 4.13. Likewise, Theorem 1.1 fails without some kind of complexity hypothesis. For example, let G be a split simple and simply connected group over $\mathbb{Z}$ and $N = V_G(\lambda)$ the Weyl module for G with minuscule highest weight λ. Then N_k is irreducible for each algebraically closed field k. When $\operatorname{char} k = p > 0$, let $M_k = (N_k)^{[1]}$ be the Frobenius twist of N_k through $F : G_k \to G_k^{(1)}$; as $G_k \cong F(G_k) = G_k^{(1)}$ we have M_k irreducible too. By irreducibility, $C_{G_k}(m) \subsetneqq G_k$ for any $0 \neq m \in M_k$. The k-group G being connected and smooth it follows that $\dim_k(C_{G_k}(m)) < \dim G_k$, yet $\mathrm{Lie}(G_k)$ is in the kernel of the action on M_k. Thus $\dim_k \mathrm{Lie}(C_{G_k}(m)) = \dim G_k$; it follows that $C_{G_k}(m)$ is not smooth. Note that X is of finite type here and G and X are fixed, but the action is not.

Remark 4.14. Here is a closely related example of the limits of Theorem 1.1; this time X, G and the G-action on X are fixed but X is not of finite type. Let $R = \mathbb{Z}$, $G = \mathrm{SL}_2$ and X be the G-scheme not of finite type which is the disjoint union of the G-modules $H^0(p) = \mathrm{Ind}_B^G(p)$ for every prime p — here B is a Borel subgroup of G and the integer p denotes a free $\mathbb{Z}$-module of rank 1 on which B acts with weight p through the quotient map to a maximal torus. When $\operatorname{char} k = p$ and $k = \bar{k}$, the simple socle of the G_k-module $H^0(p)_k$ is isomorphic to a Frobenius twist $L(1)_k^{[1]}$ of the natural 2-dimensional G_k-module $L(1)_k$. If v is a point in the socle of $H^0(p)_k$ then its centraliser is a proper subgroup of G_k, but v is centralised by the whole Lie algebra. We conclude that the centraliser of v is not smooth. In this instance, the action map $\alpha : G \times X \to X$ is not d-bounded for any d.

Remark 4.15. If we assume that R is noetherian then we obtain some variations on Theorems 1.1 and 1.2, as follows. First consider normalisers. Let R be noetherian, let G be an algebraic R-group and let X be a G-scheme. Let Y be a closed subscheme of X. We claim that there exists $p_1 \in \mathbb{N}$ such that whenever k is an R-field of characteristic $p \geq p_1$, $N_{G_k}(Y_k)$ is smooth.

To see this, choose an open covering of X by affine schemes X_i for i in some indexing set $\mathbb{I}$. Set $Y_i = Y \cap X_i$, let $\alpha_i : G \times Y_i \to X$ be the restriction of the action and let K_i be the ideal of $A \otimes_R B_i$ corresponding to $\alpha_i^{-1}(Y_i)$, where $A := R[G]$ and $B_i := R[Y_i]$. Choose generators $f_j^{(i)}$ for K_i, where j runs over some indexing set $\mathbb{J}_i$. For each $i \in \mathbb{I}$ and each $j \in \mathbb{J}_i$, we can write $f_j^{(i)} = \sum_m a_{mj}^{(i)} \otimes b_{mj}^{(i)}$, where each $a_{mj}^{(i)}$ belongs to A and each $b_{mj}^{(i)}$ belongs to B_i. Now let K_i' be the ideal of A generated by the $a_{mj}^{(i)}$, and let $K' = \sum_{i \in \mathbb{I}} K_i'$. Since G is algebraic, A is finitely generated. It follows that K' is finitely generated as R is noetherian: so some finite subset F of the $a_{mj}^{(i)}$ generates K'.

Now let k be an R-field. We obtain X_k, Y_k, $(Y_i)_k$, $(\alpha_i)_k$ by changing base. The ideal of $(\alpha_i)_k^{-1}(Y_k)$ is $K_i \otimes_R k$, and we have $\sum_{i \in \mathbb{I}} K_i' \otimes_R k = K' \otimes_R k$. But $K' \otimes_R k$ is generated by the elements $a_{mj}^{(i)} \otimes 1$, where the $a_{mj}^{(i)}$ run over the elements of F. It follows that $K' \otimes_R k$ has complexity bounded by some d which is independent of k. Applying Theorem 3.19 yields the result.

By a very similar argument we obtain the following result for centralisers. Let R be noetherian, let G be an algebraic R-group and let X be a separated G-scheme over R. Let Y be a closed subscheme of X. Then there exists $p_0 \in \mathbb{N}$ such that whenever k is an R-field of characteristic $p \geq p_0$, $C_{G_k}(Y_k)$ is smooth.

Note that in these results we did not need X or Y to be finitely presented, or even locally of finite type. On the other hand, Remark 4.14 shows that Theorem 1.1 becomes false if we remove the assumption that X is of finite type, even when R is noetherian.

4.7. ***Modules for reductive groups.*** Let k be a field of characteristic $p > 0$ and let G be a split reductive k-group (which by convention means that it is connected). We recall some of the basic representation theory of G as found in [Jan03, II.1, II.2]. Let B be a Borel subgroup of G, containing a split maximal torus T of G. This choice defines a set of simple roots of the (reduced) root lattice Φ of G, and a subset of dominant weights $X(T)^+$ of the character lattice $X(T) = \operatorname{Hom}(T, \mathbb{G}_m)$. Moreover, there is a 1–1 correspondence between the dominant weights $\lambda \in X(T)^+$ and the simple G-modules $L(\lambda)$. Let k_λ denote the 1-dimensional k-module on which T acts with weight λ; this is also a B-module via the canonical projection $B \to T$. The induced representation $H^0(\lambda) := \operatorname{Ind}_B^G(k_\lambda)$ is finite-dimensional and contains $L(\lambda)$ as its unique simple submodule; i.e., as its socle.

There is a natural pairing of $X(T)$ with the cocharacter lattice $Y(T) = \operatorname{Hom}(\mathbb{G}_m, T)$ denoted

$$\langle\ ,\ \rangle : X(T) \times Y(T) \to \mathbb{Z};\ (\lambda, \varphi) \mapsto \lambda \circ \phi.$$

One can show that any root $\alpha \in \Phi \subseteq X(T)$ gives rise to a coroot $\alpha^\vee \in Y(T)$. The next result follows from [Jan03, V.5.6].

Lemma 4.16. *Let ρ denote half the sum of the positive roots.*

(i) *Let λ be a dominant weight. If $\langle \lambda + \rho, \alpha^\vee \rangle \leq p$ for all $\alpha \in \Phi^+$ then $H^0(\lambda) = L(\lambda)$.*

(ii) *If V is a G-module such that $\langle \lambda + \rho, \alpha^\vee \rangle \leq p$ for all $\alpha \in \Phi^+$ and all dominant weights λ in V, then V is semisimple.*

Say a dominant weight λ is *d-bounded* if $\langle \lambda + \rho, \alpha^\vee \rangle \leq d$ for all $\alpha \in \Phi^+$.

Proposition 4.17. *Let Φ be a* (*reduced*) *root system and fix $d \in \mathbb{N}$. There are primes p_2 and p_3 with the following properties. If k is a field of characteristic $p \geq p_2$* (*resp. $p \geq p_3$*), *if G is any connected reductive k-group such that $G_{\bar{k}}$ has root system Φ, and if V is a finite-dimensional G-module such that the dominant weights of $V_{\bar{k}}$ are all d-bounded, then the centralisers of all closed subschemes of V in G* (*resp., the normalisers of all closed subschemes of V of complexity at most d*) *are smooth.*

Proof. Since centralisers, normalisers and smoothness are well-behaved under field extensions, we may assume without loss that $k = \bar{k}$. We may assume $p_2, p_3 \geq d$. Then by Lemma 4.16 and local finiteness [Jan03, I.2.13–14],

$$V \cong \bigoplus_{\lambda \in \Lambda} L(\lambda) \cong \bigoplus_{\lambda \in \Lambda} H^0(\lambda),$$

for some indexing set Λ of dominant weights. Recall that the group G is defined over $\mathbb{Z}$ in the sense that there is an algebraic $\mathbb{Z}$-group $\boldsymbol{G}$ such that $\boldsymbol{G}_k \cong G$ [Jan03, II.1.17]; furthermore $\boldsymbol{G}$ has a Borel subgroup $\boldsymbol{B}$ containing a split maximal torus $\boldsymbol{T}$ and such that $\boldsymbol{B}_k$ is a Borel subgroup of G and $\boldsymbol{T}_k$ is a split maximal torus of G. All Borel subgroups of G are $G(k)$-conjugate and all split maximal tori of B are $B(k)$-conjugate, so without loss of generality, $B = \boldsymbol{B}_k$ and $T = \boldsymbol{T}_k$. Since $\boldsymbol{G}$ is split, there is a character $\boldsymbol{\lambda}$ of $\boldsymbol{T}$ such that $\lambda = \boldsymbol{\lambda}_k$. Now induction commutes with flat base change [Jan03, I.3.5(3)], which implies $H^0(\lambda) = \mathrm{Ind}_{\boldsymbol{B}}^{\boldsymbol{G}}(\mathbb{Z}_{\boldsymbol{\lambda}})_k$. Thus V is isomorphic to the base change to k of a $\boldsymbol{G}$-module $\boldsymbol{V}$, which is free as a $\mathbb{Z}$-module. Moreover as there are only finitely many possible isomorphism classes of the direct summands of $\boldsymbol{V}$ — there are only finitely many d-bounded weights — the complexity of $\boldsymbol{G}$, the complexity of $\boldsymbol{V}$ and the complexity of the action map of $\boldsymbol{G}$ on $\boldsymbol{V}$ are bounded by a function $f = f(\Phi, d)$ depending just on Φ and d, being that of the maximal summand. (To obtain the bound on the complexity of $\boldsymbol{V}$ we used Remark 3.11(v).) Hence by Remark 3.11(iv) the complexity of G, the complexity of V and the complexity of the action map of G on V are bounded by a function $f = f(\Phi, d)$. It follows from Remark 4.9 that V has (G, Δ)-complexity at most $f(\Phi, d)^2$, and it follows from Remark 4.2 that if Y is a closed subscheme of V of complexity at most d then Y has G-complexity at most $df(\Phi, d)$. Now we are done by an application of Corollaries 4.8 and 4.4. ☐

Remark 4.18. Note that we do not need any bound on $\dim(V)$ in Proposition 4.17.

5. Application: the Kostant–Kirillov–Souriau theorem in characteristic *p*

A foundational result in the theory of smooth complex Poisson varieties in Weinstein's symplectic foliation theorem, which states that every such variety decomposes into a disjoint union of its symplectic leaves. In general these are not complex submanifolds and, even when they are, they usually fail to be locally closed for the Zariski topology (see [BG03, Remark 3.6(1)] for example). Nevertheless there is a large class of Poisson varieties which admit locally closed symplectic leaves. If G is a complex algebraic group with Lie algebra $\mathfrak{g}$ then $\mathfrak{g}^*$ carries a natural Poisson structure. A fundamental result in symplectic geometry states that the coadjoint orbits coincide with the symplectic leaves of $\mathfrak{g}^*$; this is known as the Kostant–Kirillov–Souriau theorem. For semisimple groups this construction is exhaustive in a precise sense: every symplectic homogeneous space is a finite covering of such an orbit (see [GS77, §IV.7]).

The theory of symplectic varieties in positive characteristic is still in its early stages, although the foundations have been carefully lain in the landmark work of Bezrukavnikov and Kaledin [BK08], where they classified Frobenius constant quantisations of smooth symplectic varieties. Another notable result is the proof of the formal version of Weinstein's splitting theorem [Tik18], which decomposes a restricted Poisson variety into a product of a symplectic subvariety and a transverse Poisson slice, in a formal neighbourhood of a point.

Although one cannot define symplectic leaves in positive characteristic, many Poisson varieties decompose into a disjoint union of quasi-affine symplectic subvarieties. This seems to be the most natural replacement for the symplectic foliation in this setting.

When G is an algebraic group over an algebraically closed field of positive characteristic we can ask whether the coadjoint orbits are symplectic subvarieties of $\mathfrak{g}^*$. In general the answer is negative, and the failure can be traced back to the fact that the quotient map from a group to an orbit is not always separable. We now demonstrate that the Kostant–Kirillov–Souriau theorem holds for algebraic groups over algebraically closed fields of sufficiently large positive characteristics, using Theorem 1.1. First, we require a preparatory lemma. If G is an algebraic R-group and k is an algebraically closed R-field then we write $\mathfrak{g}_k$ for $\mathrm{Lie}(G_k)$.

Proposition 5.1. *Let G be an algebraic R-group. There exists a prime $p_4 \in \mathbb{N}$ with the property that if k is any algebraically closed field of characteristic $p \geq p_4$, then for all $x \in \mathfrak{g}_k$ and all $\chi \in \mathfrak{g}_k^*$, the subgroups $C_{G_k}(x)$ and $C_{G_k}(\chi)$ are smooth.*

Proof. Since G is finitely presented, we have $A = R[G] = R[x_1, \dots, x_n]/I$ with $I = (f_1, \dots, f_r)$, and suppose G has complexity at most d — which implies we can choose the f_i to have degree at most d. We may assume G is nontrivial and so $d \geq 1$. The vanishing ideal I_1 at the identity is then finitely generated of complexity at most d and so the Lie algebra $\mathrm{Lie}(G) \cong (I_1/I_1^2)^*$ and the adjoint action $G \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ have bounded complexity as a function just of d; see [Jan03, I.2.4(8)] for a formula. (Certainly d^4 will suffice.) Moreover, if Q is any R-algebra then $Q[G_Q] \cong R[G] \otimes_R Q$ has complexity at most d and the same formula implies the adjoint action has complexity at most d^4 also. A similar argument yields the bounded complexity of the co-adjoint action. The result now follows from Remark 4.9 and Corollary 4.8. ☐

Let G be an algebraic R-group and let p_4 be as Proposition 5.1. Pick an algebraically closed R-field k of characteristic p. The following is a version of the Kostant–Kirillov–Souriau theorem.

Theorem 5.2. *If $p \geq p_4$ then the induced Poisson structure on coadjoint orbits in $\mathfrak{g}_k^*$ is symplectic. Hence $\mathfrak{g}_k^*$ decomposes into a disjoint union of locally closed symplectic subvarieties.*

Proof. Since $p \geq p_4$ it follows from Proposition 5.1 that the coadjoint stabilisers in $\mathfrak{g}_k$ are all smooth. For the proof we fix $\chi \in \mathfrak{g}_k^*$ and write Ω for the coadjoint orbit of χ, write ad^* for the coadjoint representation of $\mathfrak{g}_k$ on $\mathfrak{g}_k^*$ and write $\mathfrak{g}_k^\chi$ for the stabiliser of χ. Thanks to [Jan04, 2.1], the natural bijective morphism $G_k/C_{G_k}(\chi) \to \Omega$ is separable and so we have isomorphisms

$$T_\chi \Omega \xrightarrow{\sim} \mathrm{ad}^*(\mathfrak{g}_k)\chi \xrightarrow{\sim} \mathfrak{g}_k/\mathfrak{g}_k^\chi. \tag{5-1}$$

Let $I \subseteq k[\mathfrak{g}_k]$ be the defining ideal of $\overline{\Omega}$. Since Ω is G-stable, I is G-stable and hence $\mathfrak{g}_k$-stable. Hence I is a Poisson ideal and $k[\overline{\Omega}]$ inherits a Poisson structure. Let $x_1, \dots, x_n$ be a basis for $\mathfrak{g}_k$. The rank of the Poisson structure at χ is the rank of the matrix π^χ such that $\pi^\chi_{i,j} = \chi([x_i, x_j])$. However π^χ is nothing other than the matrix of the linear form $\bigwedge^2 \mathfrak{g}_k \to k$ given by $(x, y) \mapsto \chi([x, y])$. The radical of this form is $\mathfrak{g}_k^\chi$ and so we conclude that the rank of the Poisson structure on $\overline{\Omega}$ at χ is $\dim \mathfrak{g}_k/\mathfrak{g}_k^\chi$. It follows from (5-1) that the rank coincides with $\dim \Omega$. Since G acts by Poisson automorphisms and Ω is homogeneous we conclude that the Poisson structure on $\overline{\Omega}$ has full rank at every point of Ω, as desired. ☐

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References

[BG03] K. A. Brown and I. Gordon, "Poisson orders, symplectic reflection algebras and representation theory", *J. Reine Angew. Math.* **559** (2003), 193–216. MR

[BK08] R. Bezrukavnikov and D. Kaledin, "Fedosov quantization in positive characteristic", *J. Amer. Math. Soc.* **21**:2 (2008), 409–438. MR

[BMRT10] M. Bate, B. Martin, G. Röhrle, and R. Tange, "Complete reducibility and separability", *Trans. Amer. Math. Soc.* **362**:8 (2010), 4283–4311. MR

[BW93] T. Becker and V. Weispfenning, *Gröbner bases: a computational approach to commutative algebra*, Graduate Texts in Mathematics **141**, Springer, 1993. MR

[Dem63] M. Demazure, "Fibrés tangents, algèbres de Lie", fascicle 1, exposé 2, 40 pp. in *Schémas en groupes* (IHES, 1963), Inst. Hautes Études Sci., Paris, 1963. MR

[DG70] M. Demazure and P. Gabriel, *Groupes algébriques*, *I*: *Géométrie algébrique*, *généralités*, *groupes commutatifs*, Masson, Paris, 1970. MR

[Dub90] T. W. Dubé, "The structure of polynomial ideals and Gröbner bases", *SIAM J. Comput.* **19**:4 (1990), 750–775. MR

[Eis95] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995. MR

[GS77] V. Guillemin and S. Sternberg, *Geometric asymptotics*, Mathematical Surveys **14**, Amer. Math. Soc., 1977. MR

[Her13] S. Herpel, "On the smoothness of centralizers in reductive groups", *Trans. Amer. Math. Soc.* **365**:7 (2013), 3753–3774. MR

[HS16] S. Herpel and D. I. Stewart, "On the smoothness of normalisers, the subalgebra structure of modular Lie algebras, and the cohomology of small representations", *Doc. Math.* **21** (2016), 1–37. MR

[Jan03] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs **107**, Amer. Math. Soc., 2003. MR

[Jan04] J. C. Jantzen, "Nilpotent orbits in representation theory", pp. 1–211 in *Lie theory*, Progr. Math. **228**, Birkhäuser, Boston, 2004. MR

[LT18] A. J. Litterick and A. R. Thomas, "Complete reducibility in good characteristic", *Trans. Amer. Math. Soc.* **370**:8 (2018), 5279–5340. MR

[Mar02] D. Marker, *Model theory*: *an introduction*, Graduate Texts in Mathematics **217**, Springer, 2002. MR

[MST19] B. Martin, D. Stewart, and L. Topley, "A proof of the first Kac–Weisfeiler conjecture in large characteristics", *Represent. Theory* **23** (2019), 278–293. MR

[PS19] A. Premet and D. I. Stewart, "Classification of the maximal subalgebras of exceptional Lie algebras over fields of good characteristic", *J. Amer. Math. Soc.* **32**:4 (2019), 965–1008. MR

[Ric67] R. W. Richardson, Jr., "Conjugacy classes in Lie algebras and algebraic groups", *Ann. of Math.* (2) **86** (1967), 1–15. MR

[Sch00] H. Schoutens, "Uniform bounds in algebraic geometry and commutative algebra", pp. 43–93 in *Connections between model theory and algebraic and analytic geometry*, Quad. Mat. **6**, Dept. Math., Seconda Univ. Napoli, Caserta, 2000. MR

[Ste16] D. I. Stewart, "On the minimal modules for exceptional Lie algebras: Jordan blocks and stabilizers", *LMS J. Comput. Math.* **19**:1 (2016), 235–258. MR

[Tik18] A. Tikaradze, "On the local structure of quantizations in characteristic *p*", *Bull. Lond. Math. Soc.* **50**:1 (2018), 159–165. MR

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