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Derived isogenies and isogenies for abelian surfaces

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We study twisted Fourier–Mukai partners of abelian surfaces. Following Huybrechts (2019), we introduce twisted derived equivalences (also called derived isogenies) between abelian surfaces. We show that there is a twisted derived Torelli theorem for abelian surfaces over algebraically closed fields with characteristic $\neq 2, 3$.

Our approach involves extending to rational Hodge structures, ℓ-adic Tate modules and F-crystals a trick introduced by Shioda in the context of integral Hodge structures. Using this trick, we can confirm the Tate conjecture in a special case. Then we make use of Tate's isogeny theorem to give a characterization of derived isogenies between abelian surfaces via so-called principal isogenies. As a consequence, we show the two abelian surfaces are principally isogenous if and only if they are derived isogenous.

1. Introduction

1.1. ***Background.*** In the study of abelian varieties, a natural question is to classify their Fourier–Mukai partners. Thanks to Orlov and Polishchuk's *derived Torelli theorem* for abelian varieties [56; 58], there is a geometric/cohomological classification of derived equivalences between them. More generally, one can consider the notion of *twisted derived equivalence* or *derived isogeny* between abelian varieties, in the spirit of [31]:

Definition 1.1.1. Two abelian varieties X and Y are derived isogenous if they can be connected by derived equivalences between twisted abelian varieties, i.e., if there exist twisted abelian varieties (X_i, α_i) and

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(X_i, β_i) such that there is a sequence of derived equivalences

$$
\begin{aligned}
\mathrm{D}^b(X,\alpha) \xrightarrow{\simeq} \mathrm{D}^b(X_1,\beta_1),& \\
\mathrm{D}^b(X_1,\alpha_2) \xrightarrow{\simeq} \mathrm{D}^b(X_2,\beta_2),& \\
\vdots& \\
\mathrm{D}^b(X_n,\alpha_{n+1}) \xrightarrow{\simeq} \mathrm{D}^b(Y,\beta_n),&
\end{aligned} \tag{1.1.1}
$$

where $\mathrm{D}^b(X,\alpha)$ is the bounded derived category of α-twisted coherent sheaves on X.

In [66] (see especially Theorem 1.2 there) proved that derived isogenous complex abelian surfaces are isogenous, using the Kuga–Satake varieties associated to their transcendental lattices. The converse is not true: there are isogenous abelian surfaces that are not derived isogenous [66, Remark 4.4(ii)]. In this paper we give a cohomological and geometric classification of derived isogenies between abelian surfaces over algebraically closed fields of arbitrary characteristic.

1.2. ***A twisted derived Torelli theorem for abelian surfaces in characteristic zero.*** We first classify derived isogenies between abelian surfaces in term of isogenies. For this purpose, we introduce a new kind of isogeny: two abelian surfaces X and Y are *principally isogenous* if there is an isogeny f from X to Y of square degree. For example, X and its dual abelian variety $\hat{X}$ are principally isogenous since any polarization $\mathcal{L}$ on X induces an isogeny $f_{\mathcal{L}}: X \to \hat{X}$ of degree $\chi(\mathcal{L})^2$.

The first main result is this:

Theorem 1.2.1. *Let X and Y be two abelian surfaces over $k = \bar{k}$ with* $\mathrm{char}\, k = 0$. *The following statements are equivalent.*

(i) *X and Y are derived isogenous.*

(ii) *X and Y are principally isogenous.*

A notable fact for abelian surfaces is that besides their first cohomology groups, their second cohomology groups also carry rich structures. In the untwisted case, Mukai and Orlov [49; 56] have shown that

$$\mathrm{D}^b(X) \cong \mathrm{D}^b(Y) \iff \widetilde{\mathrm{H}}(X,\mathbb{Z}) \cong_{\mathrm{Hdg}} \widetilde{\mathrm{H}}(Y,\mathbb{Z}) \iff \mathrm{T}(X) \cong_{\mathrm{Hdg}} \mathrm{T}(Y),$$

where $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ and $\widetilde{\mathrm{H}}(Y,\mathbb{Z})$ are Mukai lattices, $\mathrm{T}(X) \subseteq \mathrm{H}^2(X,\mathbb{Z})$ and $\mathrm{T}(Y) \subseteq \mathrm{H}^2(Y,\mathbb{Z})$ are transcendental lattices, and $\cong_{\mathrm{Hdg}}$ stands for an integral Hodge isometry (see[12, Theorem 5.1]). The next statement can be viewed as a generalization of Mukai and Orlov's result.

Corollary 1.2.2. *Statements* (i) *and* (ii) *of Theorem 1.2.1 are also equivalent to the each of the following conditions*:

(iii) *The associated Kummer surfaces* $\mathrm{Km}(X)$ *and* $\mathrm{Km}(Y)$ *are derived isogenous.*

(iv) *The Chow motives are isomorphic as exterior algebras*: $\mathfrak{h}(X) \cong \mathfrak{h}(Y)$. *Their even degree parts are isomorphic as Frobenius algebras*: $\mathfrak{h}^{\mathrm{even}}(X) \cong \mathfrak{h}^{\mathrm{even}}(Y)$.

(v) *The even degree Chow motives are isomorphic as Frobenius algebras*: $\mathfrak{h}^{\mathrm{even}}(X) \cong \mathfrak{h}^{\mathrm{even}}(Y)$.

When $k = \mathbb{C}$, *these conditions are also equivalent to each of the following*:

(vi) $\mathrm{H}^2(X, \mathbb{Q}) \cong \mathrm{H}^2(Y, \mathbb{Q})$ *is a rational Hodge isometry.*

(vii) $\widetilde{\mathrm{H}}(X, \mathbb{Q}) \cong \widetilde{\mathrm{H}}(Y, \mathbb{Q})$ *is a rational Hodge isometry.*

(viii) $\mathrm{T}(X) \otimes \mathbb{Q} \cong \mathrm{T}(Y) \otimes \mathbb{Q}$ *is a rational Hodge isometry.*

Here, the motive $\mathfrak{h}(X)$ admits a canonical motivic decomposition

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4} \mathfrak{h}^i(X) \tag{1.2.1}$$

à la Deninger and Murre [18], such that $\mathrm{H}^*(\mathfrak{h}^i(X)) \cong \mathrm{H}^i(X)$ for any Weil cohomology $\mathrm{H}^*(-)$. It satisfies $\mathfrak{h}^i(X) = \bigwedge^i \mathfrak{h}^1(X)$ for all i, $\mathfrak{h}^4(X) \simeq \mathbb{1}(-4)$ and $\bigwedge^i \mathfrak{h}^1(X) = 0$ for $i > 4$ (see [37]). The motive $\mathfrak{h}(X)$ is an exterior algebra object in the category of Chow motives over k and the even-degree part

$$\mathfrak{h}^{\mathrm{even}}(X) = \bigoplus_{k=0}^{2} \bigwedge^{2k} \mathfrak{h}^1(X) \tag{1.2.2}$$

forms a Frobenius algebra object in the sense of [23].

Equivalences (i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) are motivic realizations of derived isogenies between abelian surfaces, which can be viewed as an analogue of the motivic global Torelli theorem on K3 surfaces (compare [31, Conjecture 0.3] and [23, Theorem 1]). Equivalences (i) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (viii) can be viewed as a generalization of [66, Theorem 1.2]. The Hodge-theoretic realization (i) $\Longleftrightarrow$ (vi) follows a strategy similar to that of [31, Theorem 0.1], which makes use of Shioda's period map and the Cartan–Dieudonné decomposition of a rational isometry. Equivalences (vi) $\Longleftrightarrow$ (vii) $\Longleftrightarrow$ (viii) follow from the Witt cancellation theorem (see page 1225).

1.3. *Shioda's trick.* The proof of Theorem 1.2.1 involves a new ingredient, which we call the *rational Shioda's trick on abelian surfaces.* The original Shioda's trick in [63] plays a key role in the proof of Shioda's global Torelli theorem for abelian surfaces, which links the weight-1 integral Hodge structure of an abelian surface to its weight-2 integral Hodge structure. This is its generalization, proved in Section 4:

Theorem 1.3.1 (Shioda's trick). *Let* X *and* Y *be complex abelian surfaces. For any admissible Hodge isometry*

$$\psi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q}),$$

we can find an isogeny $f : Y \to X$ *of degree* d^2 *such that* $\psi = f^*/d$.

As an application, the generalized Shioda's trick gives the algebraicity of some cohomological cycles. For any integer d, one can consider a Hodge similitude of degree d,

$$\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q}),$$

called a *Hodge isogeny of degree d*. From the Hodge conjecture on products of abelian surfaces, we know that every Hodge isogeny is algebraic. Our generalized Shioda's trick actually shows that Hodge isogenies between abelian surfaces are induced by certain isogenies. We also prove ℓ-adic and p-adic Shioda's tricks, which lead to a proof of the Tate conjecture for isometries between second cohomology groups (as either Galois modules or crystals) of abelian surfaces over finitely generated fields. See Corollary 4.6.3 for details.

1.4. *Results in positive characteristic.* The second part of this paper investigates the twisted derived Torelli theorem over positive characteristic fields. Because no satisfactory global Torelli theorem exists, one cannot follow the argument in characteristic zero directly. Instead, we need some input from p-adic Hodge theory:

Theorem 1.4.1. *Let X and Y be abelian surfaces over $k=\bar{k}$ with* $\operatorname{char} k = p > 3$. *The following statements are equivalent*:

(i′) *X and Y are prime-to-p derived isogenous.*

(ii′) *X and Y are prime-to-p principally isogenous.*

If X is supersingular, then Y is derived isogenous to X if and only if Y is supersingular.

Here, we say a derived isogeny as (1.1.1) is *prime-to-p* if its crystalline realization is integral (see Definition 3.1.1 for details), which is a somewhat technical condition. The main ingredient in the proof of Theorem 1.4.1 is the lifting-specialization technique, which works well for prime-to-p derived isogenies. Actually, our method shows that there is an implication (i′) $\Rightarrow$ (ii′) for derived isogenies which are not necessarily prime-to-p (see Theorem 6.3.1). Conversely, we believe that the existence of quasiliftable isogenies will imply the existence of a derived isogeny (see Conjecture 6.3.2). The only obstruction is the existence of the specialization of non-prime-to-p derived isogenies between abelian surfaces. See Remark 6.2.2.

Another natural question is whether two abelian surfaces are derived isogenous if and only if their associated Kummer surfaces are derived isogenous over positive characteristic fields. We cannot prove this equivalence, but we provide a partial solution to the question in Theorem 6.4.1.

Similarly, one may ask whether such results also hold for K3 surfaces. Let $\mathbb{F}_q$ be a finite field, with $q=p^r$. Two K3 surfaces S and S' over $\mathbb{F}_q$ are (geometrically) isogenous in the sense of Yang [69] if there exists an algebraic correspondence Γ that induces an isometry of $\operatorname{Gal}(\bar{\mathbb{F}}_p/k)$-modules

$$\Gamma_\ell^* : \mathrm{H}^2_{\text{ét}}(S_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(S'_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$$

for all $\ell \nmid p$ and an isometry of isocrystals

$$\Gamma_p^* : \mathrm{H}^2_{\text{crys}}(S_k/K) \xrightarrow{\sim} \mathrm{H}^2_{\text{crys}}(S'_k/K),$$

where $k/\mathbb{F}_q$ is a finite extension and K is the fraction field of the Cohen ring $W=W(k)$. More generally, we can take a finitely generated field k over $\mathbb{F}_p$ and a Cohen ring of k. Then we say that the isogeny is prime-to-p if the isometry Γ_p^* is integral, i.e., $\Gamma_p^*(\mathrm{H}^2_{\mathrm{crys}}(S_k/W)) = \mathrm{H}^2_{\mathrm{crys}}(S'_k/W)$. This leads us to a formulation of the twisted derived Torelli conjecture for K3 surfaces.

Conjecture 1.4.2. *For K3 surfaces S and S' over a finitely generated field k, the following statements are equivalent.*

(a) *There exists a derived isogeny $\mathrm{D}^b(S) \sim \mathrm{D}^b(S')$.*

(b) *There exists an isogeny between S and S'.*

The implication (a) $\Longrightarrow$ (b) is clear, while the converse remains open if $\operatorname{char} k > 0$. In the case of Kummer surfaces, our results provide some evidence of Conjecture 1.4.2. We mention that Bragg and Yang have studied derived isogenies between K3 surfaces over positive characteristic fields, proving a weaker version of Conjecture 1.4.2 (see [10, Theorem 1.2]).

Outline. The next two sections review some known constructions and facts; specifically, Section 2 contains computations for the Brauer group of abelian surfaces using the Kummer construction. This will allow us to prove the lifting lemma for twisted abelian surfaces of finite height.

In Section 3, we collect knowledge on derived isogenies between abelian surfaces and their cohomological realizations, which include the motivic realization, $\boldsymbol{B}$-field theory, twisted Mukai lattices, and a filtered Torelli theorem and its relation to the moduli space of twisted sheaves. At the end of the section, we follow Bragg and Lieblich's twistor line argument in [8] to conclude the supersingular case of Theorem 1.4.1.

In Section 4, we review Shioda's work and extend it to rational Hodge isogenies. This is the key ingredient for proving Theorem 1.2.1. Then, after introducing admissible ℓ-adic and p-adic bases, we prove the ℓ-adic and p-adic Shioda's tricks for admissible isometries on abelian surfaces. In an application, we prove the algebraicity of these isometries on abelian surfaces over finitely generated fields.

Sections 5 and 6 are devoted to proving Theorems 1.2.1 and 1.4.1, the first of which is restated as (essentially) Theorems 5.1.3 and 5.3.4. The proof of Theorem 1.4.1 is much more subtle. We establish the lifting and specialization theorem for prime-to-p derived isogeny. Then we can conclude (i′) $\Longleftrightarrow$ (ii′) from Theorem 1.2.1 for abelian surfaces of finite heights.

Notation and conventions.

(1) Throughout, k will denote a field. If k is a perfect field and $\operatorname{char} k = p > 0$, we write $W := W(k)$ for the ring of Witt vectors in k, which is equipped with a morphism $\sigma : W \to W$ induced by the Frobenius map on k. If k is not perfect, we consider the Cohen ring W with $W/pW = k$. Inside the ring of Witt vectors in a fixed algebraic closure $\bar{k}$ of k, we get a fixed Frobenius lift $\sigma : W \to W$ of k.

(2) Let X be a smooth projective variety over k. We denote by $\mathrm{H}^\bullet_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell)$ the ℓ-adic étale cohomology group of $X_{\bar{k}}$. The $\mathbb{Z}_\ell$-module $\mathrm{H}^\bullet_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell)$ is endowed with a canonical $G_k = \mathrm{Gal}(\bar{k}/k)$-action. We use $\mathrm{H}^i_{\mathrm{crys}}(X/W)$ to denote the i-th crystalline cohomology group of X over the p-adic base $W \twoheadrightarrow k$, which is a W-module.

(3) For any abelian group G and integer n, we denote by $G[n]$ the n-torsion subgroup of G and by $G\{n\}$ the union of all n-power torsion elements. For a lattice L in $\mathbb{Z}$ or $\mathbb{Q}$ and an integer n, we use $L(n)$ for the lattice twisted by n, that is, $L = L(n)$ as a $\mathbb{Z}$ or $\mathbb{Q}$-module, but with

$$\langle x, y\rangle_{L(n)} = n\langle x, y\rangle_L.$$

The reader should not confuse this with the Tate twist.

(4) Let X and Y be abelian surfaces. Here is a list of the various notions of isogeny between X and Y.

- An *isogeny* is a surjective homomorphism $X \to Y$ with finite kernel.
- A *quasi-isogeny* is a $\mathbb{Q}$-isogeny.
- A *prime-to-ℓ quasi-isogeny* is a $\mathbb{Z}_{(\ell)}$-isogeny.
- A *principal quasi-isogeny* is a quasi-isogeny whose degree is a square.
- A *derived isogeny* is a chain of twisted derived equivalences from X to Y.
- A *prime-to-ℓ derived isogeny* is a derived isogeny whose cohomological realization is prime-to-ℓ.

2. Twisted abelian surfaces

In this section, we give some preliminary results in the theory of twisted abelian surfaces, especially in positive characteristic. Many of them are well-known to experts.

2.1. ***Gerbes on abelian surfaces.*** Let X be a smooth projective variety over a field k and let $\mathscr{X} \to X$ be a μ_n-gerbe over X. This corresponds to a pair (X, α) for some $\alpha \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n)$, where the cohomology group is with respect to the fppf topology. Since μ_n is commutative, there is a bijection of sets

$$\mathrm{H}^2_{\mathrm{fl}}(X, \mu_n) \xrightarrow{\sim} \{\mu_n\text{-gerbes on } X\}/\simeq,$$

where $\simeq$ is the μ_n-equivalence defined as in [25, IV.3.1.1]. We may write $\alpha = [\mathscr{X}]$. For any integer m, let $\mathscr{X}^{(m)}$ be the gerbe corresponding to the cohomological class $m[\mathscr{X}] \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n)$.

The Kummer exact sequence induces a surjective map

$$\mathrm{H}^2_{\mathrm{fl}}(X, \mu_n) \to \mathrm{Br}(X)[n], \tag{2.1.1}$$

where the right-hand side is the *cohomological Brauer group* $\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. There is an associated $\mathbb{G}_m$-gerbe on X via the map (2.1.1), denoted by $\mathscr{X}_{\mathbb{G}_m}$. Let $[\mathscr{X}_{\mathbb{G}_m}]$ denote the corresponding class in $\mathrm{Br}(X)[n]$. If $[\mathscr{X}_{\mathbb{G}_m}] = 0$, we will call $\mathscr{X}$ an *essentially trivial* μ_n-gerbe.

Following [39, §2], one can define twisted coherent sheaves and their twisted derived category in terms of gerbes.

Definition 2.1.1. Let $\mathscr{X} \to X$ be a μ_n-gerbe or $\mathbb{G}_m$-gerbe over X. Let $\mathtt{Coh}^{(m)}(\mathscr{X})$ be the abelian category of $\mathscr{X}^{(m)}$*-twisted coherent sheaves*, consisting of m-fold coherent sheaves on the stack $\mathscr{X}$. We define $\mathrm{D}^{(m)}(\mathscr{X})$ as the bounded derived category of $\mathtt{Coh}^{(m)}(\mathscr{X})$.

As shown in [39, Propositions 2.1.2.6 and 2.1.3.3], there are natural equivalences

$$\mathtt{Coh}^{(1)}(\mathscr{X}) \simeq \mathtt{Coh}^{(1)}(\mathscr{X}_{\mathbb{G}_m}) \simeq \mathtt{Coh}(X, [\mathscr{X}_{\mathbb{G}_m}]),$$

the last of which is the abelian category of twisted sheaves defined by Căldăraru [14]. Throughout this paper, we mainly use Lieblich's terminology.

For two G-gerbes $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$, we denote by $\mathscr{X} \wedge_{i,j} \mathscr{Y}$ the G-gerbe on $X \times Y$ given by the image of $G \times G$-gerbe $\mathscr{X} \times \mathscr{Y}$ under the map

$$\mathrm{H}^2_{\mathrm{fl}}(X \times Y, G \times G) \to \mathrm{H}^2_{\mathrm{fl}}(X \times Y, G)$$

induced by the multiplication $G \times G \to G$, $(g_1, g_2) \mapsto (g_1^i g_2^j)$. There is an equivalence

$$\mathtt{Coh}^{(1)}(\mathscr{X} \wedge_{i,j} \mathscr{Y}) \xrightarrow{\sim} \mathtt{Coh}^{(i,j)}(\mathscr{X} \times \mathscr{Y}),$$

where the right side is the subcategory of (i, j)-fold coherent sheaves on $\mathscr{X} \times \mathscr{Y}$ [28, Corollary 2.3.2]. When $i = j = 1$, we simply write $\mathscr{X} \wedge \mathscr{Y}$ for $\mathscr{X} \wedge_{1,1} \mathscr{Y}$.

A *derived equivalence* means a k-linear exact equivalence between triangulated categories in the form

$$\Phi : \mathrm{D}^{(1)}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y}).$$

If Φ is of the form

$$\Phi^{\mathcal{P}}(\mathcal{E}) = \boldsymbol{R}q_*(p^*\mathcal{E} \otimes \mathcal{P}),$$

we call it a Fourier–Mukai transform with a kernel $\mathcal{P} \in \mathrm{D}^{(-1,1)}(\mathscr{X} \times \mathscr{Y})$ and projections $p : \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$, $q : \mathscr{X} \times \mathscr{Y} \to \mathscr{Y}$, and $\mathscr{X}$, $\mathscr{Y}$ are called a pair of Fourier–Mukai partners. If these gerbes are (essentially) trivial, then by Orlov's result, any k-linear exact equivalence between bounded derived categories of smooth projective varieties is of this form.

Similarly to Orlov's theorem, Canonaco and Stellari showed that any twisted derived equivalence is also of Fourier–Mukai type:

Proposition 2.1.2 [15]. *Any derived equivalence* $\mathrm{D}^{(1)}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y})$ *can be written uniquely* (*up to isomorphism*) *as a Fourier–Mukai transform*

$$\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y}),$$

whose kernel $\mathcal{P}$ *is a perfect complex in* $\mathrm{D}^{(-1,1)}(\mathscr{X} \times \mathscr{Y})$.

2.2. *Kummer construction.* If k has characteristic $p \neq 2$, there is an associated Kummer surface $\mathrm{Km}(X)$ constructed as follows:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\sigma}} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathrm{Km}(X) & \xrightarrow{\sigma} & X/\iota \end{array} \tag{2.2.1}$$

where ι is the involution of X given by sending x to $-x$, σ is the crepant resolution of quotient singularities, $\tilde{\sigma}$ is the blow-up of X along the closed subscheme $X[2] \subset X$. Its birational inverse is denoted by $\tilde{\sigma}^{-1}$.

Let $E \subset \tilde{X}$ be the exceptional locus of $\tilde{\sigma}$. For a classical cohomology theory $\mathrm{H}^\bullet(-)$ (such as Betti, étale and crystalline) with coefficients in R, if 2 is invertible in R, we have a canonical decomposition

$$\mathrm{H}^2(\mathrm{Km}(X)) \cong \mathrm{H}^2(X) \oplus \pi_* \Sigma_X, \tag{2.2.2}$$

where Σ_X is the summand in $\mathrm{H}^2(\tilde{X})$ generated by irreducible components of E.

Moreover, we have a composition of the sequence of morphisms

$$(\tilde{\sigma}^{-1})^* : \mathrm{Br}(\tilde{X}) \to \mathrm{Br}(\tilde{X} \setminus E) \cong \mathrm{Br}(X \setminus X[2]) \cong \mathrm{Br}(X).$$

Here, the last isomorphism $\mathrm{Br}(X) \to \mathrm{Br}(X \setminus X[2])$ is due to Grothendieck's purity theorem (see [26; 16]).

Proposition 2.2.1. *When $k = \bar{k}$ and $p \neq 2$, the $(\tilde{\sigma}^{-1})^* \pi^*$ induces an isomorphism between cohomological Brauer groups*

$$\Theta : \mathrm{Br}(\mathrm{Km}(X)) \to \mathrm{Br}(X). \tag{2.2.3}$$

In particular, when X is supersingular over $\bar{k}$, then $\mathrm{Br}(X)$ *is isomorphic to the additive group $\bar{k}$.*

Proof. For torsions of (2.2.3) whose orders are coprime to p, the proof is essentially the same as that of [65, Proposition 1.3], by the Hochschild–Serre spectral sequence and the fact that $\mathrm{H}^2(\mathbb{Z}/2\mathbb{Z}, k^*) = 0$ as the characteristic p is not 2 [68, Proposition 6.1.10]. See also [66, Lemma 4.1] for the case $k = \mathbb{C}$. For p-primary torsion part, we have

$$\mathrm{Br}(\mathrm{Km}(X))\{p\} \cong \mathrm{Br}(X)^\iota\{p\}$$

from the Hochschild–Serre spectral sequence, where $\mathrm{Br}(X)^\iota$ is the ι-invariant subgroup. Hence, it suffices to prove that ι acts trivially on $\mathrm{Br}(X)$. This is well-known to experts and works for any abelian varieties over an algebraically closed field. (See the proof of [53, Lemma 8.1], for example.)

In fact, $\mathrm{H}^2_{\mathrm{fl}}(X, \mu_p)$ can be ι-equivariantly embedded to $\mathrm{H}^2_{\mathrm{dR}}(X/k)$ by de Rham–Witt theory (see [51, Proposition 1.2]). The action of ι on $\mathrm{H}^2_{\mathrm{dR}}(X/k) = \bigwedge^2 \mathrm{H}^1_{\mathrm{dR}}(X/k)$ is the identity, as its action on $\mathrm{H}^1_{\mathrm{dR}}(X/k)$ is given by $x \mapsto -x$. Thus the involution on $\mathrm{H}^2_{\mathrm{fl}}(X, \mu_p)$ is trivial. Then by the exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}/p \to \mathrm{H}^2_{\mathrm{fl}}(X, \mu_p) \to \mathrm{Br}(X)[p] \to 0,$$

we can deduce that $\mathrm{Br}(X)[p]$ is invariant under the involution. For p^n-torsions with $n \geq 2$, we can proceed by induction on n. Assume that all elements in $\mathrm{Br}(X)[p^d]$ are ι-invariant if $1 \leq d < n$. By abuse of notation, we still use ι to denote the induced map $\mathrm{Br}(X) \to \mathrm{Br}(X)$. For $\alpha \in \mathrm{Br}(X)[p^n]$, $p\alpha \in \mathrm{Br}(X)[p^{n-1}]$ is ι-invariant. This gives

$$p\alpha = \iota(p\alpha) = p\iota(\alpha),$$

which implies $\alpha - \iota(\alpha) \in \mathrm{Br}(X)[p]$. Applying ι to $\alpha - \iota(\alpha)$, we obtain

$$\alpha - \iota(\alpha) = \iota(\alpha) - \alpha.$$

This implies that $\alpha - \iota(\alpha)$ is also a 2-torsion element. Since p is coprime to 2, we conclude that $\alpha = \iota(\alpha)$.

If X is supersingular, then $\mathrm{Km}(X)$ is also supersingular. By [2], the Brauer group of a supersingular K3 surface is isomorphic to k. Thus, $\mathrm{Br}(X) \cong k$. □

Remark 2.2.2. In the case where X is supersingular, the method of [2] cannot be applied directly to show that $\mathrm{Br}(X) = k$ as $\mathrm{H}^1_{\mathrm{fl}}(X, \mu_{p^n})$ is not trivial in general for an abelian surface X.

2.3. *A lifting lemma.* In [7], Bragg showed that a twisted K3 surface can be lifted to characteristic 0. Though his method cannot be applied directly to twisted abelian surfaces, one can still obtain a lifting result for twisted abelian surfaces via the Kummer construction. The following result will be used frequently in this paper.

Lemma 2.3.1. *Let $\mathscr{X}_0 \to X_0$ be a $\mathbb{G}_m$-gerbe on an abelian surface X_0 over $k = \bar{k}$. Suppose* $\mathrm{char}\, k > 2$ *and X has finite height. Then there exists a complete discrete valuation ring V with residue field k, fraction field K and the following properties*:

- *There is a smooth projective abelian scheme $\mathscr{X}_V \to X_V$ over* $\mathrm{Spec}\, V$ *whose special fiber is isomorphic to $\mathscr{X}_0 \to X_0$.*
- *There is a sequence of isomorphisms*

$$\mathrm{NS}(X_{\overline{K}}) \xleftarrow{\sim} \mathrm{NS}(X_V) \xrightarrow{\sim} \mathrm{NS}(X_0).$$

Here $\mathrm{NS}(X_V)$ *is the group or Cartier divisors on X_V modulo numerical equivalence over V, and the morphisms are given by pullback.*

Proof. The existence of such a lifting is ensured by [7, Theorem 7.10; 38, Lemma 3.9] and Proposition 2.2.1. Generally speaking, let $\mathscr{S}_0 \to \mathrm{Km}(X_0)$ be the associated twisted Kummer surface via the isomorphism (2.2.3). Then [7, Theorem 7.10] (by taking $\mathrm{Pic}(\mathrm{Km}(X_0))$ as a saturated sublattice of itself) asserts that there exists some discrete valuation ring V and a projective family of K3 surfaces

$$\begin{array}{ccc} \mathscr{S}_V & \longrightarrow & S_V \\ & \searrow & \downarrow \\ & & \mathrm{Spec}\, V \end{array}$$

such that the special fiber is $\mathscr{S}_0 \to \mathrm{Km}(X_0)$ and the specialization map $\mathrm{NS}(S_{\overline{K}}) \to \mathrm{NS}(\mathrm{Km}(X_0))$ of Néron–Severi lattices is an isomorphism, where $K = \mathrm{Frac}(V)$. Now we can apply [38, Lemma 3.9] to get a lifting $X_V \to \mathrm{Spec}\, V$ of X such that $\mathrm{Km}(X_V) \cong S_V$ over $\mathrm{Spec}\, V$.

We have an isomorphism $\mathrm{NS}(X_V) \cong \mathrm{NS}(X_K)$ since X_V is regular. Consider the commutative diagram

$$\mathrm{NS}(X_{\overline{K}}) \longleftarrow \mathrm{NS}(X_K) \cong \mathrm{NS}(X_V) \longrightarrow \mathrm{NS}(X_0), \qquad \mathrm{sp}\colon \mathrm{NS}(X_{\overline{K}}) \to \mathrm{NS}(X_0) \tag{2.3.1}$$

(see [44, Proposition 3.3] and its proof). The morphism $\mathrm{NS}(X_V) \to \mathrm{NS}(X_0)$ is injective by Proposition 3.6 of [44] since $\mathrm{NS}(X_K)$ is torsion-free. The morphism $\mathrm{NS}(X_K) \to \mathrm{NS}(X_{\bar{K}})$ is a primitive embedding since $\mathrm{Br}(V) = 0$. Thus, it is sufficient to see that the specialization map sp is an isomorphism. The relative Kummer construction $\mathrm{Km}(X_V) \cong S_V$ canonically identifies $\mathrm{NS}(X_K)$ (resp. $\mathrm{NS}(X_0)$) with a sublattice of $\mathrm{NS}(S_{\bar{K}})$ (resp. $\mathrm{NS}(\mathrm{Km}(X_0))$) after division by 2 (see [51, Lemma 7.11] or [64, Proposition 3.1]). The identification is compatible under specialization. We conclude using the isomorphism $\mathrm{NS}(S_{\bar{K}}) \cong \mathrm{NS}(\mathrm{Km}(X_0))$.

Lifting the $\mathbb{G}_m$-gerbe $\mathscr{X}_0 \to X_0$ to Spec V is equivalent to finding find a Brauer class in $\mathrm{Br}(X_V)$ whose restriction to X_0 is $[\mathscr{X}_0]$. Analogously to the proof of Proposition 2.2.1, there is a canonical map between the cohomological Brauer groups

$$\Theta = (\tilde{\sigma}^{-1})^* \pi^* : \mathrm{Br}(\mathrm{Km}(X_V)) \to \mathrm{Br}(X_V)$$

as in (2.2.3). The image $\Theta([\mathscr{S}_V]) \in \mathrm{Br}(X_V)$ is the desired lifting of $[\mathscr{X}_0]$. □

2.4. ***Flat cohomology of abelian surfaces.*** Finally, we consider the representability of the flat cohomology of abelian surfaces. Let $f : X \to S$ be a flat and proper morphism of algebraic spaces of finite type over k. Consider the sheaf of the abelian groups $\mathrm{R}^i f_* \mu_p$ on the big fppf site $(\mathtt{Sch}/S)_{\mathrm{fl}}$, which can be expressed as the fppf sheafification of

$$S' \mapsto \mathrm{H}^i_{\mathrm{fl}}(X_{S'}, \mu_p)$$

for any S-scheme S'. The representability of $\mathrm{R}^i f_* \mu_p$ is difficult to determine due to the complexity of flat cohomology with p-torsion coefficients. We will prove it for abelian surfaces.

Proposition 2.4.1. *Let $f : X \to S$ be an abelian S-scheme of relative dimension 2. Then $\mathrm{R}^1 f_* \mu_p \cong \hat{X}[p]$ is a finite flat S-group scheme.*

Proof. It suffices to check the statement affine locally on the base. Assume S is an affine scheme of finite type over k. Taking the Stein factorization, we can further assume $f_* \mathcal{O}_X \cong \mathcal{O}_S$. Then $f_* \mu_p \cong \mu_p$ also holds universally. Under this assumption, we have an exact sequence of fppf-sheaves by Kummer theory:

$$0 \to \mathrm{R}^1 f_* \mu_p \to \mathrm{R}^1 f_* \mathbb{G}_m \to \mathrm{R}^1 f_* \mathbb{G}_m. \tag{2.4.1}$$

Since $\mathrm{R}^1 f_* \mathbb{G}_m$ computes the relative Picard scheme $\mathrm{Pic}_{X/S}$ and the Néron–Severi group of X is torsion-free, we get

$$\mathrm{R}^1 f_* \mu_p \cong \ker\big(\mathrm{Pic}_{X/S} \xrightarrow{\cdot p} \mathrm{Pic}_{X/S}\big) \cong \ker\big(\mathrm{Pic}^0_{X/S} \xrightarrow{\cdot p} \mathrm{Pic}^0_{X/S}\big).$$

But $\mathrm{Pic}^0_{X/S}$ is representable by the dual abelian S-scheme $\hat{X}$, by [50, Corollary 6.8]. Thus, $\mathrm{R}^1 f_* \mu_p$ is representable by the commutative finite group S-scheme $\hat{X}[p]$. □

Proposition 2.4.2. *Let $f : \mathcal{X} \to S$ be a proper smooth family of abelian surfaces over an algebraic space S. Then $\mathrm{R}^2 f_* \mu_p$ is representable by an algebraic space, which is separated and locally of finite presentation over S.*

Proof. This follows from [9, Corollary 1.11 and Example 11.5], because $\mathrm{R}^1 f_* \mu_p$ is representable by Proposition 2.4.1. ☐

Remark 2.4.3. The case in which $X \to S = \mathrm{Spec}(k)$ is a smooth surface for some field k is claimed by Artin in [2, Theorem 3.1] without proof. Bragg and Olsson provided a proof (Corollary 1.6 in [9]). For relative K3 surfaces, there is a moduli-theoretic proof given by Bragg and Lieblich using the stack of Azumaya algebras (see [8, Theorem 2.1.6]). Their proof cannot be used directly for relative abelian surfaces as the essential assumption $\mathrm{R}^1 f_* \mu_p = 0$ fails in the fppf site $(\mathtt{Sch}/S)_{\mathrm{fl}}$.

Remark 2.4.4. An alternative proof for Proposition 2.4.2 consists in applying Artin's representability criterion [1, Theorem 5.3]. The most technical part is to see the separatedness.

The following observation is essential in the construction of the twistor space of supersingular abelian or K3 surfaces.

Corollary 2.4.5 [8, Proposition 2.2.4]. *Suppose that each geometric fiber of $f : \mathcal{X} \to S$ is supersingular. The connected components of any geometric fiber of* $\mathrm{R}^2 f_* \mu_p \to S$ *are isomorphic to the additive group scheme* $\mathbb{G}_a$.

Proof. The completion of each geometric fiber of $\mathrm{R}^2 f_* \mu_p$ at $\bar{s} \in S$, along the identity section, is isomorphic to the formal Brauer group $\widehat{\mathrm{Br}}_{X_{\bar{s}}/k(\bar{s})}$, which is isomorphic to $\widehat{\mathbb{G}}_a$. The only smooth connected p-torsion group scheme at $k(\bar{s})$ with this property is $\mathbb{G}_a$. ☐

3. Cohomological realizations of derived isogeny

We next provide a summary of the derived isogenies on the cohomology groups of abelian surfaces and introduce the notion of prime-to-ℓ derived isogenies. This action can be described in two ways: via

(1) the motivic realization, which provides rational isomorphisms on the cohomology groups; or

(2) the realization on the integral twisted Mukai lattices.

Following [27; 40], we then extend the filtered Torelli theorem to twisted abelian surfaces over an algebraically closed field k with $\mathrm{char}\, k \neq 2$. As a corollary, we show in Theorem 3.5.3 that any Fourier–Mukai partner of a twisted abelian surface is isomorphic to a moduli space of stable twisted sheaves.

3.1. ***Motivic realization of derived isogeny on cohomology groups.*** It is known that (twisted) derived equivalent smooth projective surfaces over a field k have isomorphic Chow motives (see [30, §2.4] and [23, §1.2], for example). We record these results for convenience, focusing on abelian surfaces over k for concreteness.

For any algebraic surface X over a field k, one may consider idempotent correspondences $\pi^2_{\mathrm{alg},X}$ and $\pi^2_{\mathrm{tr},X}$ in $\mathrm{CH}^2(X \times X)_{\mathbb{Q}}$ defined as

$$\pi^2_{\mathrm{alg},X} := \sum_{i=1}^{\rho} \frac{1}{\deg(E_i \cdot E_i)} E_i \times E_i, \quad \pi^2_{\mathrm{tr},X} = \pi^2_X - \pi^2_{\mathrm{alg},X},$$

where π_X^2 is the idempotent correspondence given by the Chow–Künneth decomposition (1.2.1) and the E_i are divisors generating the Néron–Severi group $\mathrm{NS}(X_{k^s})$ such that $E_i \cdot E_i \neq 0$ and $E_i \cdot E_j = 0$ for any $i \neq j$. Consider the decomposition

$$\mathfrak{h}^2(X) = \mathfrak{h}^2_{\mathrm{alg}}(X) \oplus \mathfrak{h}^2_{\mathrm{tr}}(X)$$

given by $\pi^2_{\mathrm{alg},X}$ and $\pi^2_{\mathrm{tr},X}$. It is not hard to see that $\mathfrak{h}^2_{\mathrm{alg}}(X)$ is a Tate motive after base change to the separable closure k^s, whose Chow realization is

$$\mathrm{CH}^*_{\mathbb{Q}}(\mathfrak{h}^2_{\mathrm{alg}}(X_{k^s})) \cong \mathrm{NS}(X_{k^s})_{\mathbb{Q}}.$$

Let $\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y})$ be a derived equivalence between two twisted abelian surfaces over k. Consider the cycle class

$$\mathrm{ch}_{\mathscr{X}^{(-1)} \wedge \mathscr{Y}}(\mathcal{P}) \cdot \sqrt{\mathrm{td}_{X\times Y}} = \mathrm{ch}_{\mathscr{X}^{(-1)} \wedge \mathscr{Y}}(\mathcal{P}) \in \mathrm{CH}^*(X \times Y)_{\mathbb{Q}}. \tag{3.1.1}$$

Here $\mathrm{ch}_{\mathscr{X}^{(-1)} \wedge \mathscr{Y}}(-)$ is the twisted Chern character defined as in (3.3.2); this provides an isomorphism

$$\mathfrak{h}(X) \xrightarrow{\sim} \mathfrak{h}(Y),$$

which preserves the even-degree parts:

$$\mathfrak{h}^{\mathrm{even}}(-) := \bigoplus_{k=0}^{2} \mathfrak{h}^{2k}(-) \cong \bigoplus_{k=0}^{2} \bigwedge^{2k} \mathfrak{h}^1(-)$$

(cf. [23, §§1.2.3]). For a Weil cohomology theory H, its cohomological realization

$$\mathrm{H}^{\mathrm{even}}(X) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}(Y) \tag{3.1.2}$$

preserves the Mukai pairing. The cohomological realization (3.1.2) is not integral in general. We can introduce the prime-to-ℓ derived isogeny via integral cohomological realizations, which will be used in the rest of the paper.

Definition 3.1.1. Let ℓ be a prime and $\mathrm{char}\, k = p$. When $\ell \neq p$, the derived isogeny $\mathrm{D}^b(X) \sim \mathrm{D}^b(Y)$ given by (1.1.1) is called *prime-to-ℓ* if each cohomological realization in the sequence

$$\tilde{\varphi}_\ell : \mathrm{H}^{\mathrm{even}}_{\mathrm{\acute{e}t}}(X_{i-1,\bar{k}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}_{\mathrm{\acute{e}t}}(X_{i,\bar{k}}, \mathbb{Q}_\ell)$$

is integral, i.e., $\tilde{\varphi}_\ell \left(\mathrm{H}^{\mathrm{even}}_{\mathrm{\acute{e}t}}(X_{\bar{k}}, \mathbb{Z}_\ell)\right) = \mathrm{H}^{\mathrm{even}}_{\mathrm{\acute{e}t}}(Y_{\bar{k}}, \mathbb{Z}_\ell)$. In the case $\ell = p$, it is called *prime-to-p* if each $\tilde{\varphi}_p : \mathrm{H}^{\mathrm{even}}_{\mathrm{crys}}(X_{i-1}/K) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}_{\mathrm{crys}}(X_i/K)$ is integral.

Remark 3.1.2. The correspondence (3.1.1) does not necessarily preservecohomological degrees. However, it admits a modification that is an isomorphism between degree-two parts: indeed, the cycle class $[\Gamma_{\mathrm{tr}}] \in \mathrm{CH}^2(X \times Y)_{\mathbb{Q}}$ given by the codimension-two component of (3.1.1) induces an isomorphism of transcendental motives by a weight argument

$$[\Gamma_{\mathrm{tr}}]_2 := \pi^2_{\mathrm{tr},Y} \circ [\Gamma_{\mathrm{tr}}] \circ \pi^2_{\mathrm{tr},X} : \mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y).$$

It extends to an isomorphism $\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$ since their algebraic parts are abstractly isomorphic, as X and Y have the same Picard number. This supports the implication (v) $\Longrightarrow$ (vii) in Corollary 1.2.2.

3.2. *Mukai lattices and B-fields.* Let k be an algebraically closed field with $\operatorname{char} k \neq 2$. Let X be an abelian surface over k. When $k = \mathbb{C}$, the *Mukai lattice* of X is defined as

$$\widetilde{\mathrm{H}}(X, \mathbb{Z}) := \mathrm{H}^0(X, \mathbb{Z}(-1)) \oplus \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathrm{H}^4(X, \mathbb{Z}(1))$$

with the Mukai pairing

$$\langle (r_1, b_1, s_1), (r_2, b_2, s_2) \rangle := b_1 \cdot b_2 - r_1 s_2 - r_2 s_1 \tag{3.2.1}$$

and a pure $\mathbb{Z}$-Hodge structure of weight 2. In general, we have the following notion of Mukai lattices, taken from [40, §2]. (The definition there is for K3 surfaces, but in fact it works well for any smooth surface with trivial canonical bundle.)

- Let $\widetilde{\mathrm{N}}(X)$ be the *extended Néron–Severi lattice*, defined as $\widetilde{\mathrm{N}}(X) := \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$, with Mukai pairing
$$\langle (r_1, c_1, s_1), (r_2, c_2, s_2) \rangle = c_1 \cdot c_2 - r_1 s_2 - r_2 s_1.$$
The Chow realization of
$$\mathfrak{h}^0(X)(-1) \oplus \mathfrak{h}^2_{\mathrm{alg}}(X) \oplus \mathfrak{h}^4(X)(1)$$
can be identified with $\widetilde{\mathrm{N}}(X)_{\mathbb{Q}}$.
- If $\operatorname{char} k = 0$ or if $\operatorname{char} k = p > 0$ and $\ell \neq \operatorname{char} k$ is a prime, the ℓ-adic Mukai lattice is defined on the even degrees of the integral ℓ-adic cohomology of X by
$$\mathrm{H}^0_{\text{ét}}(X, \mathbb{Z}_\ell(-1)) \oplus \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell) \oplus \mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}_\ell(1)),$$
with $\widetilde{\mathrm{H}}(X, \mathbb{Z}_\ell)$ a Mukai pairing defined analogously to (3.2.1).
- If $\operatorname{char} k = p > 0$, the p-adic Mukai lattice $\widetilde{\mathrm{H}}(X, W)$ is defined on the even degrees of the crystalline cohomology of X with coefficients in $W(k)$ by
$$\mathrm{H}^0_{\mathrm{crys}}(X/W(k))(-1) \oplus \mathrm{H}^2_{\mathrm{crys}}(X/W(k)) \oplus \mathrm{H}^4_{\mathrm{crys}}(X/W(k))(1),$$
where the twist (i) is given by replacing the Frobenius by $F \mapsto p^{-i} F$, and the Mukai pairing is given similarly to formula (3.2.1).

Hodge B-fields. Assume $k = \mathbb{C}$. For any $\mathbb{G}_m$-gerbe $\mathscr{X} \to X$, one can find a lift $B \in \mathrm{H}^2(X, \mathbb{Q})$ of $[\mathscr{X}] \in \mathrm{Br}(X)$ from the exponential sequence. Such a B is called a ***B***-field lift of α. We define the *twisted Mukai lattice* of $\mathscr{X}$ as

$$\widetilde{\mathrm{H}}(X, \mathbb{Z}; B) := \exp(B) \cdot \widetilde{\mathrm{H}}(X, \mathbb{Z}) \subset \widetilde{\mathrm{H}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which is isomorphic to $\widetilde{\mathrm{H}}(X, \mathbb{Z})$. For simplicity of notation, we still use (r, c, s) to denote the vector $\exp(B)(r, c, s)$. There is an induced pure Hodge structure of weight 2 on $\widetilde{\mathrm{H}}(X, \mathbb{Z}; B)$ given by

$$\widetilde{\mathrm{H}}^{0,2}(X; B) = \exp(B)\widetilde{\mathrm{H}}^{0,2}(X),$$

(see [32, Definition 2.3]). It is clear that a different choice of lift B' satisfies $B - B' \in \mathrm{H}^2(X, \mathbb{Z})$, and thus there is a Hodge isometry

$$\exp(B - B') : \widetilde{\mathrm{H}}(X, \mathbb{Z}; B') \xrightarrow{\sim} \widetilde{\mathrm{H}}(X, \mathbb{Z}; B).$$

This means that, up to isomorphisms, $\widetilde{\mathrm{H}}(X, \mathbb{Z}; B)$ is independent of the choice of the ***B***-field lifting and can also be denoted by $\widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z})$.

As shown in [71, Corollary 4.4], for any derived equivalence $\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y})$ between two twisted abelian surfaces, the Fourier–Mukai kernel induces a Hodge isometry

$$\tilde{\varphi} = \varphi_{B,B'} : \widetilde{\mathrm{H}}(X, \mathbb{Z}; B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y, \mathbb{Z}; B') \tag{3.2.2}$$

for suitable ***B***-field lifts B, B'. It provides the cohomological realization as in (3.1.2) rationally.

ℓ-adic and crystalline B-fields. Let us recall the generalized notions of ***B***-fields in ℓ-adic cohomology [42, §3.2] and crystalline cohomology [6, §3], as analogues in Betti cohomology. Full considerations for the cases ℓ-adic and p-adic are given in [10, §2], and are applicable to both K3 and abelian surfaces. Therefore, we omit some technical details here.

For a prime $\ell \neq p$ and $n \in \mathbb{N}$, the Kummer sequence of étale sheaves

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{(\cdot)^{\ell^n}} \mathbb{G}_m \to 1 \tag{3.2.3}$$

induces a long exact sequence

$$\cdots \to \mathrm{Pic}(X) \xrightarrow{\cdot l^n} \mathrm{Pic} X \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \mathrm{Br}(X)[\ell^n] \to 0.$$

Taking the inverse limit over n, we get a map

$$\pi_\ell : \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) = \varprojlim_n \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \twoheadrightarrow \mathrm{Br}(X)[\ell^n].$$

Lemma 3.2.1. *The map π_ℓ is surjective.*

Proof. By [45, Chapter V, Lemma 1.11], we have a short exact sequence

$$0 \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))[\ell^n] \to 0.$$

Since $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))$ is torsion-free for any abelian surface X, we have an isomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n \cong \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}).$$

Therefore, the reduction morphism $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n})$ can be identified with

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \twoheadrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n,$$

which is surjective. The assertion follows. □

For any $\alpha \in \mathrm{Br}(X)[\ell^n]$ such that $\ell \neq p$, let $B_\ell(\alpha) := \pi_\ell^{-1}(\alpha)$, which is nonempty by Lemma 3.2.1.

For a Brauer class $\alpha \in \mathrm{Br}(X)[p^n]$, we need the following commutative diagram arising from de Rham–Witt theory [34, I.3.2, II.5.1, théorème 5.14]:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^2(X, \mathbb{Z}_p(1)) & \longrightarrow & \mathrm{H}^2_{\mathrm{crys}}(X/W) & \xrightarrow{p-F} & \mathrm{H}^2_{\mathrm{crys}}(X/W) \\ & & \downarrow & & \downarrow{\scriptstyle p_n := (\otimes W_n)} & & \\ & & \mathrm{H}^2_{\mathrm{fl}}(X, \mu_{p^n}) & \xrightarrow{d\log} & \mathrm{H}^2_{\mathrm{crys}}(X/W_n) & & \end{array} \tag{3.2.4}$$

Here $\mathrm{H}^2(X, \mathbb{Z}_p(1)) := \varprojlim_n \mathrm{H}^2_{\mathrm{fl}}(X, \mu_{p^n})$. The map $d\log$ is injective by flat duality [51, Proposition 1.2]. Since the crystalline cohomology groups of an abelian surface are torsion-free, the mod p^n reduction map p_n is surjective. Consider the canonical surjective map

$$\pi_p : \mathrm{H}^2_{\mathrm{fl}}(X, \mu_{p^n}) \twoheadrightarrow \mathrm{Br}(X)[p^n]$$

induced by the Kummer sequence. We set

$$B_p(\alpha) := \left\{ b \in \mathrm{H}^2_{\mathrm{crys}}(X/W) \mid p_n(b) = d\log(t) \text{ for some } t \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_{p^n}) \text{ such that } \pi_p(t) = \alpha \right\}.$$

Following [10, Definitions 2.16 and 2.17], we can introduce (mixed) $\boldsymbol{B}$-fields for twisted abelian surfaces.

Definition 3.2.2. Let $\mathscr{X} \to X$ be a μ_n-gerbe and $[\mathscr{X}_{\mathbb{G}_m}] \in \mathrm{Br}(X)[n]$.

- If $n = \ell^t$ for some prime ℓ, an *ℓ-adic $\boldsymbol{B}$-field lift* of $\mathscr{X} \to X$ is an element $B = b/\ell^t$, where $b \in B_\ell([\mathscr{X}_{\mathbb{G}_m}])$. When $\ell = p$, it is also called a *crystalline $\boldsymbol{B}$-field* lift.
- In general, a mixed $\boldsymbol{B}$-field lift of $\mathscr{X} \to X$ is a collection $B = \{B_\ell\}$ consisting of a choice of an ℓ-adic $\boldsymbol{B}$-field lift B_ℓ of $[\mathscr{X}_{\mathbb{G}_m}^{(n\ell^{-t_\ell})}]$ for all prime factors $\ell \mid n$, where t_ℓ is the ℓ-adic valuation of n.

Remark 3.2.3. Not all elements in $\mathrm{H}^2_{\mathrm{crys}}(X/W)\left[\frac{1}{p}\right]$ are crystalline $\boldsymbol{B}$-fields, since the map $d\log$ is not surjective. From the first row in the diagram (3.2.4), we can see $B \in \mathrm{H}^2_{\mathrm{crys}}(X/W)\left[\frac{1}{p}\right]$ is a $\boldsymbol{B}$-field lift of some Brauer class if and only if $F(B) = pB$.

3.3. ***Twisted Mukai lattice over arbitrary fields.*** Let $\pi : \mathscr{X} \to X$ be a μ_n-gerbe with $\mathrm{ord}([\mathscr{X}_{\mathbb{G}_m}]) = n$, and let $B = \{B_\ell\}$ be a mixed $\boldsymbol{B}$-field lift of $[\mathscr{X}_{\mathbb{G}_m}]$. We define the ℓ-adic twisted Mukai lattice as

$$\widetilde{\mathrm{H}}(X, B_\ell) = \begin{cases} \exp(B_\ell)\widetilde{\mathrm{H}}(X, \mathbb{Z}_\ell) & \text{if } \ell \neq p, \\ \exp(B_\ell)\widetilde{\mathrm{H}}(X, W) & \text{if } \ell = p, \end{cases} \tag{3.3.1}$$

endowed with the Mukai pairing (3.2.1), where $\exp(B_\ell) = 1 + B_\ell + \frac{1}{2}B_\ell^2$.

Up to isomorphisms, the twisted Mukai lattice $\widetilde{\mathrm{H}}(X, B_\ell)$ is independent of the choice of the $\boldsymbol{B}$-field lift. We may use $\widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell)$ or $\widetilde{\mathrm{H}}(\mathscr{X}, W)$ to denote the twisted Mukai lattices to highlight the coefficients, irrespective of the choice of the $\boldsymbol{B}$-field lift.

Definition 3.3.1. Let $\mathrm{K}_0^{(1)}(\mathscr{X})$ be the Grothendieck group of $\mathtt{Coh}^{(1)}(\mathscr{X})$. The *twisted Chern character map* is the unique additive group homomorphism

$$\mathrm{ch}_{\mathscr{X}} : \mathrm{K}_0^{(1)}(\mathscr{X}) \to \widetilde{\mathrm{N}}(X)_{\mathbb{Q}}$$

such that for any locally free $\mathscr{X}$-twisted sheaf $\mathcal{E}$ on $\mathscr{X}$ with positive rank we have

$$\mathrm{ch}_{\mathscr{X}}(\mathcal{E}) = \sqrt[n]{\pi_*(\mathcal{E}^{\otimes n})} \in \widetilde{\mathrm{N}}(X)_{\mathbb{Q}}, \tag{3.3.2}$$

where $\sqrt[n]{-}$ means a choice of n-roots such that the 0-codimension component of $\mathrm{ch}_{\mathscr{X}}(\mathcal{E})$ is equal to rank $\mathcal{E}$.

Denote by $\widetilde{\mathrm{N}}(\mathscr{X})$ the image of $\mathrm{K}_0^{(1)}(\mathscr{X})$ in $\widetilde{\mathrm{N}}(X)_{\mathbb{Q}}$ under the twisted Chern character map; we call it the *extended twisted Néron–Severi lattice*. For $\mathcal{E} \in \mathrm{D}^{(1)}(\mathscr{X})$, we define $v(\mathcal{E}) = \mathrm{ch}_{\mathscr{X}}([\mathcal{E}]) \in \widetilde{\mathrm{N}}(\mathscr{X})$ to be the Mukai vector of $\mathcal{E}$.

One can also define the twisted Chern character map to a cohomological twisted Mukai lattice

$$\mathrm{ch}_B : \mathrm{K}_0^{(1)}(\mathscr{X}) \to \widetilde{\mathrm{H}}(X, B_\ell);$$

see [42, §3.3] and [6, Appendix A3] for the ℓ-adic and crystalline cases, respectively. For any mixed $\boldsymbol{B}$-field lift B of $[\mathscr{X}_{\mathbb{G}_m}]$, the twisted Chern character ch_B factors through $\widetilde{\mathrm{N}}(\mathscr{X})$:

$$\begin{array}{ccc} \mathrm{K}_0^{(1)}(\mathscr{X}) & \xrightarrow{\ \mathrm{ch}_{B_\ell}\ } & \widetilde{\mathrm{H}}(X, B_\ell) \\ & {\scriptstyle \mathrm{ch}_{\mathscr{X}}} \searrow \quad \nearrow {\scriptstyle \exp(B_\ell)\,\mathrm{cl}_{\mathrm{H}}} & \\ & \widetilde{\mathrm{N}}(\mathscr{X}) & \end{array}$$

where cl_{H} is the cycle class map to the cohomology theory $\mathrm{H}(-)$. The following result is essentially proved in [10].

Proposition 3.3.2 [10, Proposition 3.5]. *Let B be a mixed $\boldsymbol{B}$-field lift of $[\mathscr{X}_{\mathbb{G}_m}] \in \mathrm{Br}(X)$. Then*

$$\widetilde{\mathrm{N}}(\mathscr{X}) \cong \bigcap_{\ell} \left(\widetilde{\mathrm{N}}(X) \otimes \mathbb{Z}\left[\tfrac{1}{\ell}\right] \cap \widetilde{\mathrm{H}}(X, B_\ell) \right),$$

where the intersection $\widetilde{\mathrm{N}}(X) \otimes \mathbb{Z}\left[\frac{1}{\ell}\right] \cap \widetilde{\mathrm{H}}(X, B_\ell)$ is taken in $\widetilde{\mathrm{N}}(X) \otimes \mathbb{Q}_\ell$ and the intersection $\bigcap_\ell$ is taken in $\widetilde{\mathrm{N}}(\tilde{X}) \otimes \mathbb{Q}$. The lattice $\widetilde{\mathrm{N}}(\mathscr{X})$ only depends on the associated $\mathbb{G}_m$-gerbe $\mathscr{X}_{\mathbb{G}_m}$, up to a lattice isomorphism.

Similarly, one can define the relative extended twisted Mukai lattice on smooth projective families of twisted abelian surfaces.

3.4. *A filtered Torelli theorem.* In [40; 41], Lieblich and Olsson introduced the notion of filtered derived equivalence and demonstrated that K3 surfaces with such equivalence are isomorphic. We will present an analogous result for (twisted) abelian surfaces. The proof is simpler than for K3 surfaces, as the bounded derived category of a (twisted) abelian surface corresponds to a generic K3 category [33].

Let $\mathscr{X} \to X$ be a μ_n-gerbe. The rational numerical Chow ring $\mathrm{CH}^*_{\mathrm{num}}(\mathscr{X})_{\mathbb{Q}} \cong \mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ is equipped with a codimension filtration

$$\mathrm{Fil}^i \mathrm{CH}^*_{\mathrm{num}}(\mathscr{X})_{\mathbb{Q}} := \bigoplus_{k \geq i} \mathrm{CH}^k_{\mathrm{num}}(\mathscr{X})_{\mathbb{Q}}.$$

Since X is a surface, we have a natural identification $\widetilde{\mathrm{N}}(\mathscr{X})_{\mathbb{Q}} \cong \mathrm{CH}^*_{\mathrm{num}}(\mathscr{X})_{\mathbb{Q}}$.

Definition 3.4.1. Let $\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{X}) \to \mathrm{D}^{(1)}(\mathscr{Y})$ be a Fourier–Mukai transform. The derived equivalence $\Phi^{\mathcal{P}}$ is called *filtered* if its induced isomorphism $\Phi^{\mathcal{P}}_{\mathrm{CH}} : \widetilde{\mathrm{N}}(\mathscr{X}) \xrightarrow{\sim} \widetilde{\mathrm{N}}(\mathscr{Y})$ preserves the induced codimension filtrations.

Since the isomorphism $\widetilde{\mathrm{N}}(\mathscr{X}) \xrightarrow{\sim} \widetilde{\mathrm{N}}(\mathscr{Y})$ preserves the Mukai pairing, it is not hard to see that $\Phi^{\mathcal{P}}$ is filtered if and only if it sends the Mukai vector $(0, 0, 1)$ to $(0, 0, \pm 1)$. At the cohomological level, the codimension filtration on $\widetilde{\mathrm{H}}(X)\big[\frac{1}{\ell}\big]$ (the prime ℓ depending on the choice of ℓ-adic or crystalline twisted Mukai lattice) is given by $F^i = \oplus_{r \geq i} \mathrm{H}^{2r}(X)\big[\frac{1}{\ell}\big]$. The filtration on $\widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell)$ is defined by

$$F^i \widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell) = \widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell) \cap F^i \widetilde{\mathrm{H}}(X, \mathbb{Z}_\ell)\big[\tfrac{1}{\ell}\big].$$

If we choose a $\boldsymbol{B}$-field lift B_ℓ, a direct computation shows that the graded pieces of $F^\bullet$ are

$$\begin{aligned}
\mathrm{Gr}^0_F \widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell) &= \big\{\big(r, rB_\ell, \tfrac{1}{2} r B_\ell^2\big) \mid r \in \mathrm{H}^0(X, \mathbb{Z}_\ell(-1))\big\}, \\
\mathrm{Gr}^1_F \widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell) &= \big\{(0, b, b \cdot B_\ell) \mid b \in \mathrm{H}^2(X, \mathbb{Z}_\ell)\big\} \cong \mathrm{H}^2(X, \mathbb{Z}_\ell), \\
\mathrm{Gr}^2_F \widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell) &= \big\{(0, 0, s) \mid s \in \mathrm{H}^4(X, \mathbb{Z}_\ell(1))\big\} \cong \mathrm{H}^4(X, \mathbb{Z}_\ell(1)).
\end{aligned} \tag{3.4.1}$$

Lemma 3.4.2. *A Fourier–Mukai transform $\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{X}) \to \mathrm{D}^{(1)}(\mathscr{Y})$ is filtered if and only if its cohomological realization is filtered for all $\boldsymbol{B}$-field liftings.*

Proof. A Fourier–Mukai transform that is filtered is necessarily cohomologically filtered. This is because the map

$$\exp(B_\ell) \cdot \mathrm{cl}_H : \widetilde{\mathrm{N}}(\mathscr{X}) \to \widetilde{\mathrm{H}}(\mathscr{X}, \mathbb{Z}_\ell)$$

preserves the filtrations for any $\boldsymbol{B}$-field lift B of $[\mathscr{X}_{\mathbb{G}_m}]$.

For the converse, notice that $\Phi^{\mathcal{P}}$ is filtered if and only if the induced map $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ takes the vector $(0, 0, 1)$ to $(0, 0, \pm 1)$. As $\Phi^{\mathcal{P}}$ is cohomologically filtered for B, the cohomological realization of $\Phi^{\mathcal{P}}$ preserves the graded piece Gr^2_F in (3.4.1). This implies that $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ takes $(0, 0, 1)$ to $(0, 0, \pm 1)$. □

Proposition 3.4.3 (filtered Torelli theorem for twisted abelian surfaces). *Suppose $k = \bar{k}$ is such that* $\mathrm{char}\, k \neq 2$. *Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be μ_n-gerbes on abelian surfaces. The following statements are equivalent.*

(1) *There is an isomorphism between the associated $\mathbb{G}_m$-gerbes $\mathscr{X}_{\mathbb{G}_m}$ and $\mathscr{Y}_{\mathbb{G}_m}$.*

(2) *There is a filtered Fourier–Mukai transform $\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{X}) \to \mathrm{D}^{(1)}(\mathscr{Y})$.*

Proof. For untwisted case, i.e., $\mathscr{X} = X$ and $\mathscr{Y} = Y$, this is exactly [27, Proposition 3.1]. Here we extend it to the twisted case. As one direction is obvious, it suffices to show that (2) implies (1).

Firstly, we claim that all semirigid objects in $\mathrm{D}^{(1)}(\mathscr{Y})$ are in $\mathtt{Coh}^{(1)}(\mathscr{Y})$ up to a shift. According to Remark 3.13 in [33], it is sufficient to show that there are no stable spherical sheaves in $\mathtt{Coh}(\mathscr{Y}^{(1)})$. If $\mathcal{E}$ is a spherical $\mathscr{Y}^{(1)}$-twisted sheaf with rank $\mathcal{E} = 0$, then $c_1(\mathcal{E})^2 = -\chi(\mathcal{E}, \mathcal{E}) = -2$, which is impossible for the abelian surface. Suppose that there is a stable spherical $\mathscr{Y}$-twisted sheaf $\mathcal{E}$ with Mukai vector $v = (r, c, s)$ such that $r > 0$. Choose a polarization $H \in \mathrm{Pic}(Y)$ so that $\mathcal{E}$ is H-semistable. Let $M_H(\mathscr{Y}, v)$ be the moduli space of H-semistable $\mathscr{Y}$-twisted sheaves on Y. Then $M_H(\mathscr{Y}, v)$ is nonempty. Consider the determinant morphism to the Picard stack of invertible $\mathscr{Y}^{(r)}$-twisted sheaves

$$\mathbf{det} : \mathscr{M}_H(\mathscr{Y}, v) \to \mathrm{Pic}(\mathscr{Y}^{(r)}).$$

For any $\mathcal{L} \in \mathrm{Pic}^0(Y)$ and $\mathcal{E} \in \mathscr{M}_H(\mathscr{Y}, v)$, the tensor product $\mathcal{E} \otimes \mathcal{L}$ is still a stable $\mathscr{Y}$-twisted sheaf with the Mukai vector v. Thus, the map **det** dominates the component of $\mathrm{Pic}(\mathscr{Y}^{(r)})$ containing $\det(\mathcal{E})$, which is of dimension 2. Therefore, the deformation theory of twisted coherent sheaf implies

$$\dim_k \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \geq \dim \mathscr{M}_H(\mathscr{Y}, v) \geq 2,$$

contradicting the assumption that $\mathcal{E}$ is spherical.

Let $\Phi^{\mathcal{P}} : \mathrm{D}^b(\mathscr{X}^{(1)}) \to \mathrm{D}^b(\mathscr{Y}^{(1)})$ be a Fourier–Mukai transform. For a closed point $x \in X$, denote

$$\mathcal{P}_x := \Phi^{\mathcal{P}}(k(x)) = \mathcal{P}|_{\{x\} \times \mathscr{Y}},$$

by image of the skyscraper sheaf $k(x)$. Since $k(x)$ is semirigid, $\mathcal{P}_x$ is also semirigid. The previous discussion implies that there is an integer m such that $\mathcal{H}^i(\mathcal{P}_x) = 0$ for any $i \neq m$ and closed point $x \in \mathcal{Y}$. Therefore, there is a $\mathscr{X}^{(-1)} \wedge \mathscr{Y}$-twisted sheaf $\mathcal{E} \in \mathtt{Coh}(\mathscr{X}^{(-1)} \times \mathscr{Y})$ such that $\mathcal{P} \cong \mathcal{E}[m]$.

Suppose $\Phi^{\mathcal{P}}$ is filtered. Composing it with the shift functor $\mathcal{F} \mapsto \mathcal{F}[1]$ if necessary, we may assume that the cohomological realization of $\Phi^{\mathcal{P}}$ sends $(0, 0, 1)$ to $(0, 0, 1)$. In this case, $\mathcal{E}_x$ is just a skyscraper sheaf on $\{x\} \times Y$. The same argument as in [15, Corollary 5.3] or [29, Corollary 5.22, 5.23] shows that there is an isomorphism $f : X \to Y$ such that $f^*([\mathscr{Y}_{\mathbb{G}_m}]) = [\mathscr{X}_{\mathbb{G}_m}]$. ☐

3.5. ***Twisted FM partners via moduli space of twisted sheaves.*** In the rest of this section, we will assume that $k = \bar{k}$ and $\operatorname{char} k = p \neq 2$. Let $\mathscr{X} \to X$ be a twisted abelian surface over k.

Definition 3.5.1 [70, Definition 0.1]. Let $v = (r, c, s) \in \widetilde{\mathrm{N}}(\mathscr{X})$ be a primitive Mukai vector such that $v^2 = 0$. If

(1) $r > 0$, or

(2) $r = 0$, c is effective and $s \neq 0$, or

(3) $r = c = 0$ and $s > 0$,

then v is called *positive*.

We denote by $\mathscr{M}_H(\mathscr{X}, v)$ the moduli stack of H-semistable $\mathscr{X}$-twisted sheaves with the Mukai vector $v \in \widetilde{\mathrm{N}}(\mathscr{X})$, where H is a v-generic ample divisor on X. Here, we record a well-known nonemptiness criterion for $\mathscr{M}_H(\mathscr{X}, v)$ when X is not supersingular. We will extend this result to the supersingular case in Theorem 3.6.6, using the theory of supersingular twistor space.

Proposition 3.5.2 (Minamide, Yanagida and Yoshioka; Bragg and Lieblich). *Suppose X is an abelian surface over k that is not supersingular. If v is positive with $v^2 = 0$, then for any v-generic polarization H, the coarse moduli space $M_H(\mathscr{X}, v)$ is an abelian surface, and the moduli stack $\mathscr{M}_H(\mathscr{X}, v)$ is a $\mathbb{G}_m$-gerbe on $M_H(\mathscr{X}, v)$.*

Proof. In characteristic 0, this is [71, Theorem 3.16].

When $\operatorname{char} k = p > 2$, the nonemptiness can be seen through a lifting argument, as shown in [8, Proposition 4.1.20] and [46, Proposition A.2.1]. Since X is of finite height when $\operatorname{char} k = p > 0$, Lemma 2.3.1 implies there exists a DVR V with residue field k and a projective lifting $\mathscr{X}_V \to X_V$ of $\mathscr{X} \to X$ over $\operatorname{Spec} V$, together with an extension $v_V \in \widetilde{\mathrm{N}}(\mathscr{X}_V)$ and a polarization $H_V \in \mathrm{NS}(X_V)$ such that $H_V|_{\operatorname{Spec} k} = H$. Consider the relative moduli space of twisted sheaves $\mathscr{M}_{H_V}(\mathscr{X}_V, v_V)$ over $\operatorname{Spec} V$. Its (geometric) generic fiber is a moduli space of twisted sheaves with positive Mukai vector in characteristic zero, which is nonempty by Yoshioka's result. Thus its special fiber, which is isomorphic $\mathscr{M}_H(\mathscr{X}, v)$, is also nonempty by Langton's semistable reduction theorem. □

The following is an extension of [27, Theorem 1.2].

Theorem 3.5.3. *Assume $k = \bar{k}$ with $\operatorname{char} k \neq 2$. Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be $\mathbb{G}_m$-gerbes over an abelian surface defined over k. Then $\mathrm{D}^{(1)}(\mathscr{X}) \simeq \mathrm{D}^{(1)}(\mathscr{Y})$ if and only if $\mathscr{Y}^{(-1)} \to Y$ is isomorphic to the moduli stack $\mathscr{M}_H(\mathscr{X}, v) \to M_H(\mathscr{X}, v)$ for some $v \in \widetilde{\mathrm{N}}(\mathscr{X})$ and v-generic polarization H.*

Proof. For the "if" part, note that the universal family of twisted sheaves on $\mathscr{M}_H(\mathscr{X}, v) \times \mathscr{X}$ induces a derived equivalence.

For the other direction, suppose $\mathrm{D}^{(1)}(\mathscr{X}) \simeq \mathrm{D}^{(1)}(\mathscr{Y})$ are equivalent. Let

$$\Phi^{\mathcal{P}} : \mathrm{D}^{(1)}(\mathscr{Y}) \to \mathrm{D}^{(1)}(\mathscr{X})$$

be a Fourier–Mukai transform. Let $v \in \widetilde{\mathrm{N}}(\mathscr{X})$ be the image of $(0, 0, 1) \in \widetilde{\mathrm{N}}(\mathscr{Y})$ under $\Phi^{\mathcal{P}}$. Up to a shift, we can assume that v is a positive vector. By Proposition 3.5.2, $M_H(\mathscr{X}, v)$ is an abelian surface and $\mathscr{M}_H(\mathscr{X}, v) \to M_H(\mathscr{X}, v)$ is a $\mathbb{G}_m$-gerbe over it.

Let $\mathcal{E}$ be a universal $\mathscr{X}$-twisted sheaf on $\mathscr{M}_H(\mathscr{X}, v) \times \mathscr{X}$ that is a $(1, 1)$-fold twisted sheaf and induces a derived equivalence

$$\Phi^{\mathcal{E}} : \mathrm{D}^{(-1)}(\mathscr{M}_H(\mathscr{X}, v)) \to \mathrm{D}^{(1)}(\mathscr{X}),$$

whose cohomological realization maps the Mukai vector $(0, 0, 1)$ to v. Composing $\Phi^{\mathcal{E}}$ with the derived equivalence

$$(\Phi^{\mathcal{P}})^{-1} \simeq \Phi^{\mathcal{P}^\vee}[2] : \mathrm{D}^{(1)}(\mathscr{X}) \to \mathrm{D}^{(1)}(\mathscr{Y}),$$

we obtain a filtered derived equivalence from $\mathscr{M}_H(\mathscr{X}, v)^{(-1)}$ to $\mathscr{Y}$, which induces an isomorphism from $\mathscr{M}_H(\mathscr{X}, v)^{(-1)}$ to $\mathscr{Y}$ by Proposition 3.4.3. □

3.6. ***Supersingular twisted abelian surfaces.*** Finally, we discuss the case of supersingular twisted abelian surfaces. In this part, we extend the construction the supersingular twistor space as [8] via the Ogus crystalline Torelli theorem for supersingular abelian surfaces (see [51, §2]).

Definition 3.6.1. Let p be a prime $\neq 2$. Let Λ be an indefinite p-elementary even lattice, meaning that $\operatorname{disc}(\Lambda \otimes \mathbb{Q}) = -1$ and $\Lambda^\vee/\Lambda$ is p-torsion. Then $|\Lambda^\vee/\Lambda| = p^{2\sigma_0(\Lambda)}$ for $1 \leq \sigma_0(\Lambda) \leq \frac{1}{2}n$ and the integer $\sigma_0(\Lambda)$ is called the *Artin invariant* of Λ. We define M_Λ to be the Ogus *moduli space of characteristic subspaces* of $p\Lambda^\vee/p\Lambda$.

When Λ has signature $(1, n-1)$, $n \geq 2$, as shown in [61, Section 1], Λ is uniquely determined by its Artin invariant. When $n = 6$, we call it a *supersingular abelian surface lattice*, because for every supersingular abelian surface X, the Néron–Severi lattice $\operatorname{NS}(X)$ is a supersingular abelian surface lattice [52, (1.6)].

From now on, let us assume that Λ is a supersingular abelian surface lattice. For simplicity, denote by σ_0 the Artin invariant $\sigma_0(\Lambda)$. We set

$$\tilde{\Lambda} = \Lambda \oplus U(p),$$

where $U(p)$ is the twisted hyperbolic plane generated by the vectors e and f such that $e^2 = f^2 = 0$ and $e \cdot f = -p$. Let $M^{\langle e\rangle}_{\tilde{\Lambda}} \subseteq M_{\tilde{\Lambda}}$ be the moduli space of characteristic subspaces of $p\tilde{\Lambda}^\vee/p\tilde{\Lambda}$ that do not contain e.

Proposition 3.6.2 [8, §3]. *The moduli stacks $M^{\langle e\rangle}_{\tilde{\Lambda}}$ and M_Λ are representable by schemes over $\mathbb{F}_p$, which are smooth of dimensions σ_0 and $\sigma_0 - 1$, respectively. There is a smooth morphism*

$$\pi_e : M^{\langle e\rangle}_{\tilde{\Lambda}} \to M_\Lambda$$

whose fiber at a closed point is isomorphic to a group scheme with connected components $\mathbb{A}^1$.

Proof. The first assertion is given in [52, Proposition 4.6]. Let us sketch the construction of π_e. Given any $\widetilde{\mathcal{K}} \in M^{\langle e\rangle}_{\tilde{\Lambda}}(T)$ over an $\mathbb{F}_p$-scheme T, a characteristic subspace

$$\mathcal{K} \subseteq (p\Lambda^\vee/p\Lambda) \otimes \mathcal{O}_T$$

can be formed as the image of $\widetilde{\mathcal{K}} \cap (e^\perp \otimes \mathcal{O}_T)$ in $(e^\perp/e) \otimes \mathcal{O}_T$ (see [8, Lemma 3.1.9]). Consequently, the map $\widetilde{\mathcal{K}} \mapsto \mathcal{K}$ defines a morphism

$$\pi_e : M^{\langle e\rangle}_{\tilde{\Lambda}} \to M_\Lambda.$$

The rest of the assertion is a consequence of [8, Lemma 3.1.15]. □

Definition 3.6.3. The *twistor line* in $M_{\tilde{\Lambda}} \otimes_{\mathbb{F}_p} k$ is an affine line $\mathbb{A}^1 \subset M_{\tilde{\Lambda}} \otimes_{\mathbb{F}_p} k$ that is a connected component of a fiber of π_e over a k-point of $M_\Lambda(k)$ for some isotropic vector $e \in \tilde{\Lambda}$.

The moduli functor S_Λ of Λ-marked supersingular abelian surfaces is representable by a locally separated and smooth algebraic space of dimension $\sigma_0 - 1$ over k, by the crystalline Torelli theorem [51, Theorem 7.3] together with the argument in [52, Theorem 2.7]. Consider the universal family of supersingular abelian surfaces

$$u : \mathcal{X} \to S_\Lambda,$$

which is smooth with relative dimension 2. By Proposition 2.4.2, the higher direct image $\mathrm{R}^2 u_* \mu_p$ is representable by an algebraic group space over S_Λ, denoted by

$$\pi : \mathscr{S}_\Lambda \to S_\Lambda.$$

The connected component of the identity $\mathscr{S}_\Lambda^o \subset \mathscr{S}_\Lambda$ parameterizes the μ_p-gerbes which are not essentially trivial except the identity, on each Λ-marked supersingular abelian surface in $S_\Lambda(k)$. Then there are (twisted) period morphisms following the approach in [52, §3].

Proposition 3.6.4. *There are (twisted) period morphisms*

$$\rho : S_\Lambda \to \overline{M}_\Lambda := M_\Lambda \otimes_{\mathbb{F}_p} k \quad \textit{and} \quad \tilde{\rho} : \mathscr{S}_\Lambda^o \to \overline{M}_{\tilde{\Lambda}}^{\langle e \rangle} := M_{\tilde{\Lambda}}^{\langle e \rangle} \otimes_{\mathbb{F}_p} k$$

such that the commutative diagram

$$\begin{array}{ccc} \mathscr{S}_\Lambda^o & \xrightarrow{\pi|_{\mathscr{S}_\Lambda^o}} & S_\Lambda \\ \downarrow{\scriptstyle \tilde{\rho}_\Lambda} & & \downarrow{\scriptstyle \rho_\Lambda} \\ \overline{M}_{\tilde{\Lambda}}^{\langle e \rangle} & \xrightarrow{\pi_e} & \overline{M}_\Lambda \end{array} \tag{3.6.1}$$

is Cartesian. Moreover, ρ and $\tilde{\rho}$ are étale surjective when $p > 2$.

Proof. This was proved by Bragg and Lieblich in the case of supersingular K3 surfaces [8, §3 and§5]. But everything works for supersingular abelian surfaces as well. We mention that one can also use the Kummer construction to deduce the statement from the K3 case.

For reference, let us sketch the construction of $\tilde{\rho}_\Lambda$ and ρ_Λ. Let (X, η) be a Λ-marked supersingular abelian surface. The K3-crystal $\mathrm{H}^2_{\mathrm{crys}}(X/W)$ determines a characteristic subspace

$$\mathcal{K}_{\mathrm{H}^2(X)} := \ker(\mathrm{NS}(X) \otimes k \to \mathrm{H}^2_{\mathrm{crys}}(X/W) \otimes k).$$

Then $\rho_\Lambda(X, \eta)$ is the characteristic subspace $\eta^{-1}(\mathcal{K}_{\mathrm{H}^2(X)})$ in $(p\Lambda^\vee / p\Lambda) \otimes_{\mathbb{F}_p} k$. Suppose $\mathscr{X} \to X$ is a μ_p-gerbe. We define

$$\mathcal{K}_{\widetilde{\mathrm{H}}(\mathscr{X})} := \ker(\widetilde{\mathrm{N}}(\mathscr{X}) \otimes k \to \widetilde{\mathrm{H}}(\mathscr{X}, W) \otimes_W k) \subset \frac{p\widetilde{\mathrm{N}}(\mathscr{X})^\vee}{p\widetilde{\mathrm{N}}(\mathscr{X})} \otimes k$$

as the strictly characteristic subspace of $\widetilde{\mathrm{H}}(\mathscr{X}, W)$. Note that there is an extended map of K3 crystals,

$$\tilde{\Lambda} \otimes \mathbb{Z}_p \xrightarrow{\tilde{\eta}} \widetilde{\mathrm{N}}(\mathscr{X}) \otimes \mathbb{Z}_p \to \widetilde{\mathrm{H}}(\mathscr{X}, W),$$

where $\tilde{\eta}$ is given by $e \mapsto (0, 0, 1)$, $f \mapsto (p, 0, 0)$ and $c \mapsto \eta(c)$ for all $c \in \Lambda$. Then $\tilde{\rho}_\Lambda(\mathscr{X}, \eta) = \tilde{\eta}^{-1}(\mathcal{K}_{\widetilde{\mathrm{H}}(\mathscr{X})})$ is the characteristic subspace of $(p\tilde{\Lambda}^\vee / p\tilde{\Lambda}) \otimes k$. □

Remark 3.6.5. In one view, the moduli space M_Λ is a crystalline analog of the classical period domain. Let H be a supersingular K3 crystal. The associated Tate module $\mathrm{T}_H \subseteq H$ is a supersingular K3 $\mathbb{Z}_p$-lattice in the sense of Ogus (see [51, 3.13]). According to [51, Theorem 3.20], the functor

$$H \rightsquigarrow (T_H, \mathcal{K}_H),$$

where $\mathcal{K}_H = \ker(T_H \otimes k \to H \otimes k)$, defines an equivalence between the category of supersingular K3 crystals and the category of strictly characteristic subspaces of a supersingular K3 $\mathbb{Z}_p$-lattice.

Using the twisted period map, we obtain:

Theorem 3.6.6. *Let $\mathscr{X} \to X$ be a μ_p-gerbe over a supersingular abelian surface X over k.*

(1) *If a primitive vector $v \in \widetilde{\mathrm{N}}(\mathscr{X})$ is positive and isotropic, the coarse moduli space $M_H(\mathscr{X}, v)$ is an abelian surface.*

(2) *If $\mathscr{Y} \to Y$ is another twisted abelian surface, we have $\mathrm{D}^{(1)}(\mathscr{X}) \simeq \mathrm{D}^{(1)}(\mathscr{Y})$ if and only if there is an isomorphism*

$$\widetilde{\mathrm{H}}(\mathscr{X}, W) \cong \widetilde{\mathrm{H}}(\mathscr{Y}, W)$$

of K3 crystals.

(3) *There is a derived equivalence*

$$\mathrm{D}^{(1)}(\mathscr{X}_0) \simeq \mathrm{D}^b(X),$$

where $\mathscr{X}_0 \to X_0$ is a μ_p-gerbe over the unique superspecial abelian surface X_0.

Proof. For (1), if $\mathscr{X} \to X$ is an essentially trivial μ_p-gerbe over a supersingular abelian surface X, this can be proved by a standard lifting argument (see also [22, Proposition 6.9]). When $\mathscr{X} \to X$ is nontrivial, we can take the universal family of μ_p-gerbes

$$f : \mathfrak{X} \to \mathbb{A}^1$$

on the connected component $\mathbb{A}^1 \subset \mathrm{R}^2 u_* \mu_p$ that contains $\mathscr{X}$ (see Corollary 2.4.5). The fibers of f contain $\mathscr{X} \to X$ and the trivial μ_p-gerbe over X. By taking the relative moduli space of twisted sheaves (with a suitable v-generic polarization) on $\mathfrak{X} \to \mathbb{A}^1$, one obtains the nonemptiness of $M_H(\mathscr{X}, v)$ from the case of essentially trivial gerbes.

For the proof of the forward direction of (2), we notice that by Remark 3.6.5, it is sufficient to find an isomorphism between pairs

$$(\widetilde{\mathrm{N}}(\mathscr{X}), \mathcal{K}_{\widetilde{\mathrm{H}}(\mathscr{X})}) \xrightarrow{\sim} (\widetilde{\mathrm{N}}(\mathscr{Y}), \mathcal{K}_{\widetilde{\mathrm{H}}(\mathscr{Y})});$$

this is provided by the de Rham realization of the derived equivalence $\mathrm{D}^{(1)}(\mathscr{X}) \simeq \mathrm{D}^{(1)}(\mathscr{Y})$. The other direction is handled like the case of K3 surfaces proved in [6, Theorem 3.5.5]. The key is that if $\widetilde{\mathrm{H}}(\mathscr{X}, W) \cong \widetilde{\mathrm{H}}(\mathscr{Y}, W)$, then there exists $v \in \widetilde{\mathrm{N}}(\mathscr{X})$ such that the induced isomorphism

$$\widetilde{\mathrm{H}}(\mathscr{M}_H(\mathscr{X}, v)^{(-1)}, W) \cong \widetilde{\mathrm{H}}(\mathscr{Y}, W)$$

of K3 crystals sends $(0, 0, 1)$ to $(0, 0, 1)$. The assertion then essentially follows from the Ogus crystalline Torelli theorem for supersingular abelian surfaces (Theorem 7.3 of [51]), as in [6, Theorem 3.5.2]. We omit details.

For (3), due to (2), it suffices to find a μ_p-gerbe $\mathscr{X}_0 \to X_0$ such that there is a supersingular K3 crystal isomorphism

$$\widetilde{\mathrm{H}}(\mathscr{X}_0, W) \cong \widetilde{\mathrm{H}}(X, W).$$

By Remark 3.6.5, this is equivalent to finding $\mathscr{X}_0 \to X_0$ and an isometry $\widetilde{\mathrm{N}}(\mathscr{X}_0) \otimes \mathbb{Z}_p \cong \widetilde{\mathrm{N}}(X) \otimes \mathbb{Z}_p$ sending $\mathcal{K}_{\widetilde{\mathrm{H}}(\mathscr{X}_0)}$ to $\mathcal{K}_{\widetilde{\mathrm{H}}(X)}$.

Let us give an explicit construction of $\mathscr{X}_0 \to X_0$ via the twisted period map. If X is superspecial, no further proof is necessary. Suppose X is not superspecial. Then $\sigma_0(\mathrm{NS}(X)) = 2$ by Proposition 3.7 of [64]. Let Λ be the supersingular abelian surface lattice with Artin invariant 2 and let $\eta : \Lambda \xrightarrow{\sim} \mathrm{NS}(X)$ be a Λ-marking. As shown in [61, Section 2] (see also [22, Proposition 6.1]), $\Lambda = U(p) \oplus \Lambda'$ contains $U(p)$ as a direct summand and the image of $U(p)$ in $(p\Lambda^\vee/p\Lambda) \otimes k$ is not contained in the strictly characteristic subspace $\rho_\Lambda(X, \eta)$.

Note that the lattice $\Lambda_0 = U \oplus \Lambda'$ is a supersingular abelian lattice with Artin invariant 1. There is a natural isomorphism

$$\widetilde{\mathrm{N}}(X) \xrightarrow{\eta \oplus \mathrm{id}} \Lambda \oplus U \cong \Lambda_0 \oplus U(p) = \tilde{\Lambda}_0 \tag{3.6.2}$$

and we can identify $\mathcal{K}_{\widetilde{\mathrm{H}}(X)}$ with $\rho_\Lambda(X, \eta)$ via the isometry $\eta \oplus \mathrm{id}$. Let

$$\mathcal{K} \subseteq (p\tilde{\Lambda}_0^\vee / p\tilde{\Lambda}_0) \otimes k$$

be the image of $\mathcal{K}_{\widetilde{\mathrm{H}}(X)}$ through the map induced by (3.6.2). By our assumption, $\mathcal{K}$ does not contain the image of some isotropic vector $e \in U(p)$ and therefore can be viewed as a point in $M^{\langle e \rangle}_{\tilde{\Lambda}_0}(k)$. As $\tilde{\rho}_{\Lambda_0}$ is surjective, there is a Λ_0-marked supersingular abelian surface $(\mathscr{X}_0 \to X_0, \eta_0)$ such that $\tilde{\rho}_{\Lambda_0}(\mathscr{X}_0, \eta_0) = \mathcal{K}$. It is easy to see that $\mathscr{X}_0 \to X_0$ is as desired. ☐

4. Shioda's Torelli theorem for abelian surfaces

In [63], Shioda discovered that there is a way, now called Shioda's trick, to extract information about the first cohomology of a complex abelian surface from its second cohomology. This established a global Torelli theorem for complex abelian surfaces via second cohomology, which is also a key step in Piatetskii-Shapiro and Shafarevich's proof of the Torelli theorem for K3 surfaces (see [57, §5, Lemma 4 and Theorem 1]).

The aim of this section is to generalize Shioda's method to all fields and establish an isogeny theorem for abelian surfaces via the second cohomology. We will deal with Shioda's trick for Betti cohomology, étale cohomology and crystalline cohomology separately.

4.1. ***Recap of Shioda's trick for Hodge isometry.*** We first recall Shioda's construction. Suppose X is a complex abelian surface. Its singular cohomology ring $\mathrm{H}^\bullet(X, \mathbb{Z})$ is canonically isomorphic to the exterior algebra $\bigwedge^\bullet \mathrm{H}^1(X,\mathbb{Z})$. Let V be a free $\mathbb{Z}$-module of rank 4. We denote by Λ the lattice $\left(\bigwedge^2 \mathrm{V}, q\right)$, where $q : \bigwedge^2 \mathrm{V} \times \bigwedge^2 \mathrm{V} \to \bigwedge^4 \mathrm{V} \cong \mathbb{Z}$ is the wedge product. After choosing a $\mathbb{Z}$-basis $\{v_i\}_{1\le i\le 4}$ for $\mathrm{H}^1(X,\mathbb{Z})$, we have an isometry of $\mathbb{Z}$-lattices $\Lambda \xrightarrow{\sim} \mathrm{H}^2(X,\mathbb{Z})$. The set of vectors

$$\{v_{ij} := v_i \wedge v_j\}_{0\le i<j\le 4}$$

clearly forms a basis of $\mathrm{H}^2(X,\mathbb{Z})$, which will be called an *admissible basis* of A for its second singular cohomology. For another complex abelian surface Y, a Hodge isometry

$$\varphi : \mathrm{H}^2(Y,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(X,\mathbb{Z})$$

will be called *admissible* if $\det(\varphi) = 1$ with respect to some admissible bases on X and Y. It is clear that the admissibility of a morphism is independent of the choice of admissible bases.

In terms of admissible bases, we can view φ as an element in $\mathrm{SO}(\Lambda)$. On the other hand, we have the exact sequence of groups

$$1 \to \{\pm 1\} \to \mathrm{SL}_4(\mathbb{Z}) \xrightarrow{\bigwedge^2} \mathrm{SO}(\Lambda). \tag{4.1.1}$$

Shioda observed that the image of $\mathrm{SL}_4(\mathbb{Z})$ in $\mathrm{SO}(\Lambda)$ is a subgroup of index two and does not contain $-\mathrm{id}_\Lambda$. From this, he proved:

Theorem 4.1.1 (Shioda [63, Theorem 1]). *For any admissible integral Hodge isometry ψ, there is an isomorphism of integral Hodge structures*

$$\psi : \mathrm{H}^1(Y,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(X,\mathbb{Z})$$

such that $\bigwedge^2(\psi) = \varphi$ or $-\varphi$.

This is what we call "Shioda's trick". As we can assume that a Hodge isometry is admissible after possibly taking the dual abelian variety for one of them (see Example 4.2.3 below), we can obtain the Torelli theorem for complex abelian surfaces by using weight-two Hodge structures — that is, X is isomorphic to Y or its dual $\hat{Y}$ if and only if there is an integral Hodge isometry $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(Y,\mathbb{Z})$ [63, Theorem 1].

4.2. *Admissible bases.* To extend Shioda's work to arbitrary fields, we must define admissibility for different cohomology theories (e.g., étale and crystalline cohomology).

Let k be a field with $\mathrm{char}\, k = p \ge 0$. Suppose X is an abelian surface over k and $\ell \nmid p$ is a prime. For simplicity of notations, we will denote $\mathrm{H}^\bullet(-)_R$ for one of the following cohomology theories:

(1) if $k \hookrightarrow \mathbb{C}$ and $R = \mathbb{Z}$ or any number field E, then $\mathrm{H}^\bullet(X)_R = \mathrm{H}^\bullet(X(\mathbb{C}), R)$ the singular cohomology.

(2) if $R = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, then $\mathrm{H}^\bullet(X)_R = \mathrm{H}^\bullet_{\text{ét}}(X_{\bar{k}}, R)$, the ℓ-adic étale cohomology.

(3) if $\mathrm{char}\, k = p > 0$, then we can take $R = W$ a Cohen ring of k or the fraction field K of W, then $\mathrm{H}^\bullet(X)_R = \mathrm{H}^\bullet_{\text{crys}}(X/W)$ or $\mathrm{H}^\bullet_{\text{crys}}(X/W) \otimes K$, the crystalline cohomology.

There is an isomorphism between the cohomology ring $\mathrm{H}^\bullet(X)_R$ and the exterior algebra $\bigwedge^\bullet \mathrm{H}^1(X)_R$. We denote by $\mathrm{tr}_X : \mathrm{H}^4(X)_R \xrightarrow{\sim} R$ the corresponding trace map. The Poincaré pairing $\langle - , - \rangle$ on $\mathrm{H}^2(X)_R$ can be realized as

$$\langle \alpha, \beta \rangle = \mathrm{tr}_X(\alpha \wedge \beta).$$

Analogously to Section 4.1, an R-basis $\{v_i\}$ of $\mathrm{H}^1(X)_R$ will be called a *d-admissible basis* if it satisfies

$$\mathrm{tr}_X(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = d$$

for some $d \in R^*$. When $d = 1$, we speak of an *admissible basis*. For any d-admissible (resp. admissible) basis $\{v_i\}$, the associated R-basis $\{v_{ij} := v_i \wedge v_j\}_{i<j}$ of $\mathrm{H}^2(X)_R$ will also be called d-admissible (resp. admissible).

Example 4.2.1. Let $\{v_1, v_2, v_3, v_4\}$ be an R-linear basis of $\mathrm{H}^1(X)_R$. Suppose

$$\mathrm{tr}_X(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = t \in R^*.$$

For any $d \in R^*$, there is a natural d-admissible R-linear basis $\{(d/t)v_1, v_2, v_3, v_4\}$.

Definition 4.2.2. Let X and Y be abelian surfaces over k.

- An R-linear isomorphism $\psi : \mathrm{H}^1(X)_R \to \mathrm{H}^1(Y)_R$ is d-admissible if it takes an admissible basis to a d-admissible basis.
- An R-linear isomorphism $\varphi : \mathrm{H}^2(X)_R \to \mathrm{H}^2(Y)_R$ is d-admissible if

$$\mathrm{tr}_Y \circ \textstyle\bigwedge^2(\varphi) = d\,\mathrm{tr}_X$$

  for some $d \in R^*$, or equivalently, if it sends an admissible basis to a d-admissible basis. When $d = 1$, it will also be called admissible.

The set of d-admissible isomorphisms is denoted by $\mathrm{Isom}^{\mathrm{ad},(d)}(\mathrm{H}^i(X)_R, \mathrm{H}^i(Y)_R)$ accordingly.

For any isomorphism $\varphi : \mathrm{H}^2(X)_R \xrightarrow{\sim} \mathrm{H}^2(Y)_R$, let $\det(\varphi)$ be the determinant of the matrix with respect to some admissible bases. It is not hard to see $\det(\varphi)$ is independent of the choice of admissible bases, and φ is admissible if and only if $\det(\varphi) = 1$.

Example 4.2.3. Let $\{v_i\}$ be an admissible basis of $\mathrm{H}^1(X)_R$. For the dual abelian surface $\hat{X}$, the dual basis $\{v_i^*\}$ with respect to the Poincaré pairing naturally forms an admissible basis of $\hat{X}$, under the identification $\mathrm{H}^1(X)_R^\vee \cong \mathrm{H}^1(\hat{X})_R$. Let

$$\varphi^{\mathcal{P}} : \mathrm{H}^2(X)_R \to \mathrm{H}^2(\hat{X})_R$$

be the isomorphism induced by the Poincaré bundle $\mathcal{P}$ on $X \times \hat{X}$. A direct computation (see Lemma 9.3 of [29], for instance) shows that $\varphi^{\mathcal{P}}$ is nothing but

$$-\mathrm{D} : \mathrm{H}^2(X)_R \xrightarrow{\sim} \mathrm{H}^2(X)_R^\vee \cong \mathrm{H}^2(\hat{X})_R,$$

where D is Poincaré duality. For an admissible basis $\{v_i\}$ of X, its R-linear dual $\{v_i^*\}$ with respect to

Poincaré pairing forms an admissible basis of $\hat{X}$. By our construction, we see that

$$\mathrm{D}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) = (v_{34}^*, -v_{24}^*, v_{23}^*, v_{14}^*, -v_{13}^*, v_{12}^*),$$

which implies that D is of determinant -1 under these admissible bases. Thus the determinant of $\varphi^{\mathcal{P}}$ is not admissible.

Example 4.2.4. Let $f : X \to Y$ be an isogeny of degree d between two abelian surfaces, for some positive integer d. If d is coprime to ℓ, then it will induce an isomorphism

$$f^* : \mathrm{H}^2(Y)_{\mathbb{Z}_\ell} \xrightarrow{\sim} \mathrm{H}^2(X)_{\mathbb{Z}_\ell},$$

that is d-admissible. If, in addition, $d = n^2$, then f^*/n will be an admissible $\mathbb{Z}_\ell$-integral isometry with respect to the Poincaré pairing. If $d = k^4$, then f/k will be a $\mathbb{Z}_{(\ell)}$-isogeny such that its pullback is admissible integral.

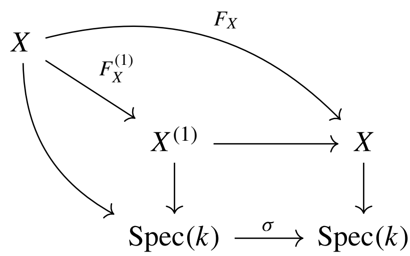
Example 4.2.5. Suppose X is an abelian surface over a perfect field k with $\operatorname{char} k = p > 0$. The F-crystal $\mathrm{H}^1(X)_W$, together with the trace map

$$\mathrm{tr}_X : \mathrm{H}^4(X)_W \xrightarrow{\sim} W,$$

forms an abelian crystal (of genus 2) in the sense of [51, §6]. We can see that an isomorphism of F-crystals $\mathrm{H}^1(X)_W \xrightarrow{\sim} \mathrm{H}^1(Y)_W$ is admissible if and only if it is an isomorphism between abelian crystals, i.e., it is compatible with trace maps.

4.3. ***More on admissible bases of F-crystals.*** In contrast to ℓ-adic étale cohomology, the semilinear structure on crystalline cohomology from its Frobenius is trickier to work with. Therefore, we spend some words on the interaction of Frobenius with admissible bases.

Suppose k is a perfect field with $\operatorname{char} k = p > 0$. We have the following Frobenius pullback diagram:



Via the natural identification $\mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W) \cong \mathrm{H}^1_{\mathrm{crys}}(X/W) \otimes_\sigma W$, the σ-linearization of Frobenius action on $\mathrm{H}^1_{\mathrm{crys}}(X/W)$ can be viewed as the injective W-linear map

$$F^{(1)} := (F_X^{(1)})^* : \mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W) \hookrightarrow \mathrm{H}^1_{\mathrm{crys}}(X/W).$$

If k is not perfect, then after passing to $W(\bar{k})$ or equivalently choosing a Frobenius lift on the Cohen ring W, we also get a Frobenius action on $\mathrm{H}^1_{\mathrm{crys}}(X/W)$, whose linearization is given by the relative Frobenius morphism.

There is a decomposition $\mathrm{H}^1_{\mathrm{crys}}(X/W) = H_0(X) \oplus H_1(X)$ such that

$$F^{(1)}\big(\mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W)\big) \cong H_0(X) \oplus pH_1(X), \tag{4.3.1}$$

and $\mathrm{rank}_W H_i = 2$ for $i = 0, 1$, which is related to the Hodge decomposition of the de Rham cohomology of X/k by Mazur's theorem; see [3, §8, Theorem 8.26].

The Frobenius map can be expressed in terms of admissible bases. We can choose an admissible basis $\{v_i\}$ of $\mathrm{H}^1_{\mathrm{crys}}(X/W)$ such that

$$v_1, v_2 \in H_0(X) \quad \text{and} \quad v_3, v_4 \in H_1(X).$$

Then $\{p^{\alpha_i} v_i\} := \{v_1, v_2, pv_3, pv_4\}$ forms an admissible basis of $\mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W)$ under the identification (4.3.1), since $\mathrm{tr}_{X^{(1)}} \circ \bigwedge^4 F^{(1)} = p^2 \sigma_W \circ \mathrm{tr}_X$. In term of these bases, the Frobenius map can be written as

$$F^{(1)}(p^{\alpha_i} v_i) = \sum_j c_{ij} p^{\alpha_j} v_j, \tag{4.3.2}$$

where the c_{ij} form an invertible 4×4-matrix $C_X = (c_{ij})$ with coefficients in W.

Suppose Y is another abelian surface over k, $\psi : \mathrm{H}^1_{\mathrm{crys}}(X/W) \to \mathrm{H}^1_{\mathrm{crys}}(Y/W)$ is an admissible map, and $\psi^{(1)}$ is the induced map $\psi \otimes_\sigma W : \mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W) \to \mathrm{H}^1_{\mathrm{crys}}(Y^{(1)}/W)$. Denote by M and M' the matrices of ψ and $\psi^{(1)}$, respectively, with respect to the chosen admissible bases.

Lemma 4.3.1. *The map ψ commutes with Frobenius if and only if $C_Y M' C_X^{-1} = M$.*

Proof. By definition, ψ commutes with Frobenius if and only if $(F_Y^{(1)})^* \circ \psi^{(1)} = \psi \circ (F_X^{(1)})^*$. The statement is then clear from (4.3.2). ☐

4.4. *Generalizing Shioda's trick.* Let us review some basic properties of the special orthogonal group scheme over an integral domain. Our main reference is [17, Appendix C].

Let Λ be an even $\mathbb{Z}$-lattice of rank $2n$. We can associate with it a vector bundle $\underline{\Lambda}$ on $\mathrm{Spec}(\mathbb{Z})$ with constant rank $2n$ equipped with a quadratic form q over $\mathrm{Spec}(\mathbb{Z})$ obtained from Λ. The functor

$$A \mapsto \{g \in \mathrm{GL}(\Lambda_A) \,|\, q_A(g \cdot x) = q_A(x) \text{ for all } x \in \Lambda_A\}$$

is representable by a $\mathbb{Z}$-subscheme of $\mathrm{GL}(\Lambda)$, denoted by $\mathrm{O}(\Lambda)$. There is a homomorphism of $\mathbb{Z}$-group schemes

$$D_\Lambda : \mathrm{O}(\Lambda) \to \underline{\mathbb{Z}/2\mathbb{Z}},$$

called the Dickson morphism (see [17, p. 313] for the definition). Roughly speaking,

$$D_\Lambda(g) = \begin{cases} 0 & \text{if } g \text{ is a product of an even number of reflections,} \\ 1 & \text{if } g \text{ is a product of an odd number of reflections,} \end{cases}$$

for a point $g \in \mathrm{O}(\Lambda)$ over a field in characteristic zero. The Dickson morphism is surjective since Λ is even, and its construction is compatible with any base change [17, Proposition C.2.8]. The *special* *orthogonal group scheme* over $\mathbb{Z}$ with respect to Λ is defined as the kernel of D_Λ and denoted by $\mathrm{SO}(\Lambda)$. We have

$$\mathrm{SO}(\Lambda)_{\mathbb{Z}[1/2]} \cong \ker\left(\det : \mathrm{O}(\Lambda) \to \mathbb{G}_m\right)_{\mathbb{Z}[1/2]}.$$

By [17, Proposition C.2.11], for example, $\mathrm{SO}(\Lambda) \to \mathrm{Spec}(\mathbb{Z})$ is smooth in relative dimension $\frac{1}{2}n(n-1)$ and has connected fibers.

For any ℓ, the special orthogonal group scheme

$$\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}) \cong \mathrm{SO}(\Lambda)_{\mathbb{Z}_\ell}$$

is smooth over $\mathbb{Z}_\ell$ with connected fibers, which implies that its generic fiber $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell})$ is connected. Thus, $\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is connected as a group scheme over $\mathbb{Z}_\ell$, since $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell}) \subset \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is dense.

The special orthogonal group scheme admits a universal covering

$$\mathrm{Spin}(\Lambda) \to \mathrm{SO}(\Lambda),$$

i.e., a simply connected central isogeny; see [17, Section C.4] for the construction.

Lemma 4.4.1. *Let* V *be free* $\mathbb{Z}$*-module of rank* 4 *and* $\Lambda = \bigwedge^2 \mathrm{V}$. *Let* R *be a ring of coefficients as listed in Section 4.2. There is an exact sequence of smooth* R*-group schemes*

$$1 \to \mu_{2,R} \to \mathrm{SL}(\mathrm{V})_R \xrightarrow{\bigwedge^2(-)_R} \mathrm{SO}(\Lambda)_R \to 1$$

(*as fppf-sheaves if* $\frac{1}{2} \notin R$). *Moreover, there is an exact sequence*

$$1 \to \{\pm \mathrm{id}_4\} \to \mathrm{SL}(\mathrm{V})(R) \xrightarrow{\bigwedge^2(-)_R} \mathrm{SO}(\Lambda)(R) \to R^*/(R^*)^2. \tag{4.4.1}$$

Proof. For the first statement, it suffices to assume $R = \mathrm{Spec}(\bar{k})$ for an algebraically closed field $\bar{k}$, where it is clear from a computation. Note that we have an exact sequence on rational points (see [25, Proposition 3.2.2])

$$1 \to \mu_2(R) \to \mathrm{SL}(\mathrm{V})(R) \to \mathrm{SO}(\Lambda)(R) \to \mathrm{H}^1(\mathrm{Spec}(R), \mu_2).$$

Notice that for the rings of coefficients listed in Section 4.2, we have $\mathrm{Pic}(R)[2] = 0$. Therefore,

$$\mathrm{H}^1_{\mathrm{fl}}(\mathrm{Spec}(R), \mu_2) \cong R^*/(R^*)^2$$

from the Kummer sequence for μ_2.

For the last statement, notice that there is an isomorphism of R-group schemes $\mathrm{SL}(\mathrm{V})_R \xrightarrow{\sim} \mathrm{Spin}(\Lambda)_R$ such that the diagram

$$\begin{array}{ccc} \mathrm{SL}(V)(R) & \xrightarrow{\sim} & \mathrm{Spin}(\Lambda)(R) \\ & \searrow \quad \swarrow & \\ & \mathrm{SO}(\Lambda)(R) & \end{array}$$

commutes. The group scheme $\mathrm{SL}(\mathrm{V})$ is simply connected (its geometric fibers form a semisimple algebraic group of type A_3). Thus, the central isogeny $\mathrm{SL}(\mathrm{V})_R \to \mathrm{SO}(\Lambda)_R$ forms the universal covering of $\mathrm{SO}(\Lambda)_R$, which induces an isomorphism $\mathrm{SL}(\mathrm{V})_R \xrightarrow{\sim} \mathrm{Spin}(\Lambda)_R$ by using the isomorphism theorem over a general ring (see [17, Theorems 6.1.16 and 6.1.17], for example). □

Remark 4.4.2. When $R = \mathbb{Z}_\ell$, we have

$$\mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2 \cong \begin{cases} \{\pm 1\} & \text{if } \ell \neq 2, \\ \{\pm 1\} \times \{\pm 5\} & \text{if } \ell = 2. \end{cases}$$

Thus the image of $\mathrm{SL}(\mathrm{V})(\mathbb{Z}_\ell)$ is a finite-index subgroup in $\mathrm{SO}(\Lambda)(\mathbb{Z}_\ell)$.

Remark 4.4.3. When $R = W(k)$, we have

$$W(k)^*/(W(k)^*)^2 \cong \begin{cases} \{1, \epsilon\} & \text{if } k = \mathbb{F}_{p^s} \text{ for } p > 2, s \geq 1, \\ \{1\} & \text{if } k = \bar{k} \text{ or } k^s = k, \operatorname{char} k > 2, \end{cases}$$

where $\epsilon \in \mathbb{Z}$ is such that $\epsilon \not\equiv y^2 \mod p^s$ for an integer y, as $W(k)$ is Henselian. Thus, the wedge map $\mathrm{SL}(\mathrm{V})(W) \to \mathrm{SO}(\Lambda)(W)$ is surjective when $k = \bar{k}$.

Let X and Y be abelian surfaces over k. Let $V_R = \mathrm{H}^1(X)_R$. We can see the set

$$\mathrm{Isom}^{\mathrm{ad},(d)}(\mathrm{H}^1(X)_R, \mathrm{H}^1(Y)_R)$$

is a (right) $\mathrm{SL}(V_R)$-torsor if it is nonempty. The wedge product provides a natural map

$$\textstyle\bigwedge^2 : \mathrm{Isom}^{\mathrm{ad},(d)}\big(\mathrm{H}^1(X)_R, \mathrm{H}^1(Y)_R\big) \to \mathrm{Isom}^{\mathrm{ad},(d)}\big(\mathrm{H}^2(X)_R, \mathrm{H}^2(Y)_R\big).$$

Let $\{v_i\}$ be an admissible basis of $\mathrm{H}^1(X)_R$ and let $\{v_i'\}$ be a d-admissible basis of $\mathrm{H}^1(Y)_R$. There is a d-admissible isomorphism $\psi_0 \in \mathrm{Isom}^{\mathrm{ad},(d)}(\mathrm{H}^1(X)_R, \mathrm{H}^1(Y)_R)$ such that $\psi_0(v_i) = v_i'$. For a d-admissible isometry $\varphi : \mathrm{H}^2(X, R) \to \mathrm{H}^2(Y, R)$, we can see that

$$\varphi = \textstyle\bigwedge^2(\psi_0) \circ g \quad \text{for some } g \in \mathrm{SO}(\Lambda_R).$$

In this way, any d-admissible isomorphism φ can be identified with the (unique) element $g \in \mathrm{SO}(\Lambda)(R)$ when the admissible bases are fixed. This allows us to deal with d-admissible isomorphisms group-theoretically. In particular, we have the following notion:

Definition 4.4.4. The *spinor norm* of the d-admissible isomorphism φ is the image $\mathrm{SN}(\varphi)$ of g under $\mathrm{SN} : \mathrm{SO}(\Lambda)(R) \to R^*/(R^*)^2$.

Lemma 4.4.5. *The spinor norm* $\mathrm{SN}(\varphi)$ *is independent of the choice of admissible bases.*

Proof. With a different choice of admissible bases, the new element satisfies $\tilde{g} = KgK^{-1}$, for some $K \in \mathrm{SO}(\Lambda_R)$. Therefore, $\mathrm{SN}(\tilde{g}) = \mathrm{SN}(g)$. □

Remark 4.4.6. When R is a field, the spinor norm can be computed by the Cartan–Dieudonné decomposition. That is, we can write any $g \in \mathrm{SO}(\Lambda)(R)$ as a composition of reflections

$$\boldsymbol{R}_{b_n} \circ \boldsymbol{R}_{b_{n-1}} \circ \cdots \circ \boldsymbol{R}_{b_1}$$

for some nonisotropic vectors $b_1, \ldots, b_n \in \Lambda_R$, and $\mathrm{SN}(g) = \big[(b_1)^2 \cdots (b_{n-1})^2 (b_n)^2\big]$.

Lemma 4.4.7. *A d-admissible isomorphism φ is a wedge of some d-admissible isomorphism ψ : $\mathrm{H}^1(X, R) \to \mathrm{H}^1(Y, R)$ if and only if* $\mathrm{SN}(\varphi) = 1$.

Proof. The exact sequence (4.4.1) shows that if $\mathrm{SN}(\varphi) = \mathrm{SN}(g) = 1$, there is some $h \in \mathrm{SL}(V_R)$ such that $\bigwedge^2(h) = g$. Thus, we can take $\psi = \psi_0 \circ h$ when $\mathrm{SN}(\varphi) = 1$, and see that

$$\bigwedge^2(\psi) = \bigwedge^2(\psi_0) \circ \bigwedge^2(h) = \varphi.$$

The converse is clear. □

4.4.1. *The isogeny category.* Recall that the isogeny category of abelian varieties $\mathtt{AV}_{\mathbb{Q},k}$ has as objects all abelian varieties over a field k, and as arrows the sets of homomorphisms

$$\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},k}}(X, Y) := \mathrm{Hom}_{\mathtt{AV}_k}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\mathrm{Hom}_{\mathtt{AV}_k}(X, Y)$ is the abelian group of homomorphisms from X to Y with the natural addition. We may also write $\mathrm{Hom}^0(X, Y)$ for $\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},k}}(X, Y)$ if there is no confusion about k.

Definition 4.4.8. Let R be a commutative ring with unit. An *R-isogeny* from X to Y is an invertible element $f \in \mathrm{Hom}_{\mathtt{AV}_k}(X, Y) \otimes R$ i.e., there is $g \in \mathrm{Hom}_{\mathtt{AV}_k}(Y, X) \otimes R$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

A $\mathbb{Q}$-isogeny is called a quasi-isogeny, while a $\mathbb{Z}_{(\ell)}$-isogeny is called a *prime-to-ℓ quasi-isogeny*. For any quasi-isogeny (resp. prime-to-ℓ quasi-isogeny) f, we can find a minimal integer n (resp. one with $\ell \nmid n$) such that

$$nf : X \to Y$$

is an isogeny (resp. of degree prime-to-ℓ).

When $k = \mathbb{C}$, with the uniformization of complex abelian varieties, we have a canonical bijection

$$\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},\mathbb{C}}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Hdg}}\big(\mathrm{H}^1(Y, \mathbb{Q}), \mathrm{H}^1(X, \mathbb{Q})\big),$$

where the right-hand side is the set of $\mathbb{Q}$-linear morphisms that preserve Hodge structures. Then the integer n for f is also the minimal integer such that $(nf)^*(\mathrm{H}^1(Y, \mathbb{Z})) \subseteq \mathrm{H}^1(X, \mathbb{Z})$.

4.5. ***Shioda's trick for Hodge isogenies.*** Suppose $k = \mathbb{C}$. Let d be an integer. A *Hodge isogeny of degree d* is an isomorphism of $\mathbb{Q}$-Hodge structures

$$\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$$

such that

$$\langle x, y\rangle = d\langle \varphi(x), \varphi(y)\rangle.$$

When $d = 1$, we recover the classical Hodge isometry. Clearly, a d-admissible rational Hodge isomorphism is a Hodge isogeny of degree d. In terms of spinor norms, we can generalize Shioda's theorem (Theorem 4.1.1) to admissible rational Hodge isogenies.

Proposition 4.5.1 (Shioda's trick on admissible Hodge isogenies).

(1) *A d-admissible Hodge isogeny of degree d*

$$\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$$

is a wedge of some rational Hodge isomorphism $\psi : \mathrm{H}^1(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^1(Y, \mathbb{Q})$ *if its spinor norm is trivial. In this case, the Hodge isogeny is induced by a quasi-isogeny of degree* d^2.

(2) *When* $d = 1$, *any admissible Hodge isometry* $\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$ *is induced by an isogeny* $f : Y \to X$ *of degree* n^2 *for some integer n such that* $\varphi = f^*/n$.

Proof. Under the assumption of part (1), we can find a d-admissible isomorphism ψ by applying Lemma 4.4.7. It remains to prove that ψ preserves the Hodge structure; this is done essentially as for [63, Theorem 1].

For (2), we suppose the spinor norm $\mathrm{SN}(\varphi)$ equals $n\mathbb{Q}^{*2} \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. Let $E = \mathbb{Q}(\sqrt{n})$. We can see that the base change $\mathrm{H}^2(X, E) \xrightarrow{\sim} \mathrm{H}^2(Y, E)$ is a Hodge isometry with coefficients in E such that $\mathrm{SN}(\varphi) = 1 \in E^*/(E^*)^2$. By applying Lemma 4.4.7, we obtain an admissible Hodge isomorphism $\psi : \mathrm{H}^1(X, E) \xrightarrow{\sim} \mathrm{H}^1(Y, E)$ (for some admissible bases for $\mathrm{H}^1(X, \mathbb{Q})$ and $\mathrm{H}^1(Y, \mathbb{Q})$). Let

$$\sigma : a + b\sqrt{n} \rightsquigarrow a - b\sqrt{n}$$

be the generator of $\mathrm{Gal}(E/\mathbb{Q})$. As we have fixed the $\mathbb{Q}$-linear admissible bases, the wedge map

$$\mathrm{SL}_4(E) \xrightarrow{\bigwedge^2} \mathrm{SO}(\Lambda)(E)$$

is defined over $\mathbb{Q}$, and so is σ-equivariant. Let g be the element in $\mathrm{SL}_4(E)$ that corresponds to ψ. Since $\bigwedge^2(g) \in \mathrm{SO}(\Lambda) \subset \mathrm{SO}(\Lambda_E)$, we see that

$$\bigwedge^2(\sigma(g)) = \sigma\big(\bigwedge^2(g)\big) = \bigwedge^2(g).$$

which implies that $\sigma(g)g^{-1} = \pm\mathrm{id}_4$ since $\ker \bigwedge^2 = \{\pm\mathrm{id}_4\}$. If $\sigma(g) = g$, then $g \in \mathrm{SL}_4(\mathbb{Q})$ and the statement is trivially valid. If $\sigma(g) = -g$, then $g_0 = \sqrt{n}g$ lies in $\mathrm{GL}_4(\mathbb{Q})$. Let

$$\psi_0 : \mathrm{H}^1(X, \mathbb{Q}) \to \mathrm{H}^1(Y, \mathbb{Q})$$

be the element corresponding to g_0 in $\mathrm{Isom}^{\mathrm{ad},(n^2)}\big(\mathrm{H}^1(X, \mathbb{Q}), \mathrm{H}^1(Y, \mathbb{Q})\big)$. As $\bigwedge^2\psi_0 = n\varphi$ is a Hodge isogeny, part (1) then implies that ψ_0 is also a Hodge isomorphism. Thus, ψ_0 extends to a quasi-isogeny $f_0 : Y \to X$ and we have

$$\varphi = \bigwedge^2(\psi) = \frac{f_0^*}{n} : \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(Y, \mathbb{Q}).$$

Replacing f_0 by the product mf_0 for some integer m, we get an isogeny of degree $(m^2n)^2$. □

Remark 4.5.2. If a Hodge isometry $\psi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$ is not admissible, that is, if its determinant is -1 with respect to some admissible bases, then we can take its composition with the isometry $\psi^{\mathcal{P}}$

induced by the Poincaré bundle as in Example 4.2.3. After that, we see that $\psi^{\mathcal{P}} \circ \psi$ is admissible and is induced by an isogeny $f : \hat{Y} \to X$.

4.6. *ℓ-adic and p-adic Shioda's tricks.* For the integral ℓ-adic étale cohomology, we have the following statement similar to Shioda's trick for integral Betti cohomology.

Proposition 4.6.1 (ℓ-adic Shioda's trick). *Suppose $\ell \neq 2$. For any d-admissible $\mathbb{Z}_\ell$-linear isomorphism*

$$\varphi_\ell : \mathrm{H}^2_{\acute{e}t}(Y_{k^s}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\acute{e}t}(X_{k^s}, \mathbb{Z}_\ell),$$

there are an integer u and a $(u^2 d)$-admissible $\mathbb{Z}_\ell$-isomorphism ψ_ℓ such that $\bigwedge^2(\psi_\ell) = u\varphi_\ell$. If φ_ℓ is $\mathrm{Gal}(k^s/k)$*-equivariant, then ψ_ℓ is also* $\mathrm{Gal}(k^s/k)$*-equivariant after replacing k with some finite extension.*

Proof. One can choose an element $u \in (\mathbb{Z} \setminus \{0\}) \cap \mathbb{Z}_\ell^*$ that is not a square in $\mathbb{Z}_\ell$, e.g., one satisfying the equation $u^{\frac{\ell-1}{2}} \equiv -1 \mod \ell$ as $\ell \neq 2$. Since $\mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2 \cong \{\pm 1\}$ for any $\ell \neq 2$, either φ_ℓ or $u\varphi_\ell$ has spinor norm one. Then the first statement follows from Lemma 4.4.7.

Suppose φ_ℓ is $\mathrm{Gal}(k^s/k)$-equivariant. We may assume $\bigwedge^2(\psi_\ell) = \varphi_\ell$ for simplicity. For any $g \in \mathrm{Gal}(k^s/k)$, we have

$$\textstyle\bigwedge^2(g^{-1}\psi_\ell g) = g^{-1}\bigwedge^2(\psi_\ell) g = \bigwedge^2(\psi_\ell).$$

Therefore, $g^{-1}\psi_\ell g = \pm\psi_\ell$. By passing to a finite extension k'/k, we always have $g^{-1}\psi_\ell g = \psi_\ell$ for all $g \in \mathrm{Gal}(k^s/k')$, which proves the assertion. ☐

For F-crystals attached to abelian surfaces, we have another variant of Shioda's trick.

Proposition 4.6.2 (p-adic Shioda's trick). *Let k be a finite field or an algebraically closed field such that* $\mathrm{char}\, k = p > 2$*. For any d-admissible W-linear isomorphism*

$$\varphi_p : \mathrm{H}^2_{\mathrm{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{crys}}(X/W),$$

there exist an integer u and a $(u^2 d)$-admissible W-linear isomorphism $\psi_p : \mathrm{H}^1_{\mathrm{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{crys}}(X/W)$ such that $\bigwedge^2(\psi_p) = u\varphi_p$. If k is algebraically closed, we can take $u = 1$.

Moreover, if φ_p is compatible with Frobenius and $\mathbb{F}_{p^2} \subseteq k$, then there is $\xi \in \mathbb{Z}_{p^2}^* \subseteq W(k)$ such that $\xi\psi_p$ is compatible with Frobenius and $\xi^2 \in \mathbb{Z}_p^*$.

Proof. The first statement follows from a similar reasoning as Proposition 4.6.1, since $W^*/(W^*)^2 \subseteq \{1, \epsilon\}$ (see Remark 4.4.3).

For the second statement, we assume $\bigwedge^2(\psi_p) = \varphi_p$. If φ_p commutes with the Frobenius action, then

$$\textstyle\bigwedge^2(C_X^{-1} \cdot \psi_p^{(1)} \cdot C_Y) = \varphi_p.$$

as in Section 4.3. Thus $C_X^{-1} \cdot \psi_p^{(1)} \cdot C_Y = \pm\psi_p^{(1)}$, which implies

$$\psi_p \circ F_X^{(1)} = \pm F_Y^{(1)} \circ \psi_p^{(1)}$$

by Lemma 4.3.1.

If $F_X^{(1)} \circ \psi_p^{(1)} = \psi_p \circ F_Y^{(1)}$, we need do nothing. If $F_X^{(1)} \circ \psi_p^{(1)} = -\psi_p \circ F_Y^{(1)}$, we can take $\xi \in \mathbb{Z}_{p^2}^* \subseteq W(k)$ such that $\xi^{p-1} = -1$. This implies

$$F_X^{(1)} \circ (\xi \psi_p)^{(1)} = \xi^p F_X^{(1)} \circ \psi = (\xi \psi_p) \circ F_Y^{(1)}.$$

Note that $\xi^2 \in \mathbb{Z}_p^*$ as $\sigma(\xi^2) = \xi^2$ and $\xi^{2p+2} = 1$. Therefore, we can conclude. □

Combining this with Tate's isogeny theorem, we have the following consequences of Propositions 4.6.1 and 4.6.2. They include a special case of Tate's conjecture.

Corollary 4.6.3. *Suppose k is a finitely generated field over $\mathbb{F}_p$ with $p > 2$. Let $\ell \neq 2$ be a prime not equal to p.*

(1) *For any admissible isometry of* $\mathrm{Gal}(k^s/k)$*-modules*

$$\varphi_\ell : \mathrm{H}^2_{ét}(Y_{k^s}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{ét}(X_{k^s}, \mathbb{Z}_\ell),$$

we can find a $\mathbb{Z}_\ell$-isogeny $f_\ell \in \mathrm{Hom}_{k'}(X_{k'}, Y_{k'}) \otimes \mathbb{Z}_\ell$, for some finite extension k'/k, that induces $u\varphi_\ell$ for some integer u prime-to-ℓ. In particular, φ_ℓ is algebraic.

(2) *If k is finite, then for any admissible isometry*

$$\varphi_p : \mathrm{H}^2_{\mathrm{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{crys}}(X/W)$$

that is compatible with Frobenius, we can find a $\mathbb{Z}_p$-isogeny $f_p \in \mathrm{Hom}_{k'}(X_{k'}, Y_{k'}) \otimes \mathbb{Z}_p$ over some finite extension k'/k, such that

$$\epsilon f_p^*|_{\mathrm{H}^2_{\mathrm{crys}}(Y/W)} = u\varphi_p$$

for some prime-to-p integer u and $\epsilon \in \mathbb{Z}_p^*$. In particular, φ_p is algebraic.

Proof. For (1), Proposition 4.6.1 implies that there is an isomorphism

$$\psi_\ell : \mathrm{H}^1_{ét}(Y_{k^s}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{ét}(X_{k^s}, \mathbb{Z}_\ell),$$

that induces $u\varphi_\ell$ and is $\mathrm{Gal}(k^s/k)$-equivariant after a finite extension of k. Then f_ℓ exists thanks to the canonical bijection

$$\mathrm{Hom}^0(X, Y) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(k^s/k)}\left(\mathrm{H}^1_{ét}(Y_{k^s}, \mathbb{Z}_\ell), \mathrm{H}^1_{ét}(X_{k^s}, \mathbb{Z}_\ell)\right)$$

(see [72] and [20, VI.3, Theorem 1]).

For (2), we may assume that $\mathbb{Z}_{p^2} \subseteq W(k)$, after taking a finite extension of k. Then Proposition 4.6.2 implies that there is an isomorphism

$$\psi_p : \mathrm{H}^1_{\mathrm{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{crys}}(X/W)$$

that induces $u\varphi_p$, and a $\xi \in \mathbb{Z}_{p^2}^*$ such that $\xi\psi_p$ is compatible with Frobenius.

Since k a finite field, there are canonical isomorphisms

$$\mathrm{Hom}^0(X, Y) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{Hom}_k\big(X[p^\infty], Y[p^\infty]\big) \xrightarrow{\sim} \mathrm{Hom}_{F,V}\big(\mathrm{H}^1_{\mathrm{crys}}(Y/W), \mathrm{H}^1_{\mathrm{crys}}(X/W)\big). \tag{4.6.1}$$

Here the first isomorphism is from p-adic Tate's isogeny theorem [36, Theorem 2.6] and the second from the faithfulness of Dieudonné functor over W [35, Theorem]. The canonical bijection (4.6.1) implies that $\xi\psi_p$ is induced by a $\mathbb{Z}_p$-isogeny $f_p \in \mathrm{Hom}^0(X, Y) \otimes \mathbb{Z}_p$. Therefore

$$f_p^*|_{\mathrm{H}^2_{\mathrm{crys}}(Y/W)} = \xi^2 u \varphi_p.$$

The $\mathbb{Z}_p$-isogeny f_p is what we require. □

Remark 4.6.4. In [73], Zarhin introduced the notion of *almost isomorphisms*. Two abelian varieties over k are called almost isomorphic if their Tate modules T_ℓ are isomorphic as Galois modules (replaced by p-divisible groups when $\ell = p$). Propositions 4.6.1 and 4.6.2 imply that it is possible to characterize almost isomorphic abelian surfaces by their second cohomology groups.

5. Derived isogeny in characteristic zero

In this section, we follow [23] and [31] to prove the twisted Torelli theorem for abelian surfaces over algebraically closed fields of characteristic zero.

5.1. ***Over $\mathbb{C}$: Hodge isogeny versus derived isogeny.*** Let X and Y be complex abelian surfaces. Throughout this section, let $\Lambda = U^{\oplus 3}$ be the direct sum of three hyperbolic lattices.

Definition 5.1.1. A rational Hodge isometry $\varphi : \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(Y, \mathbb{Q})$ is called *reflective* if it is a reflection on Λ along a nonisotropic vector $b \in \Lambda$:

$$\boldsymbol{R}_b : \Lambda_{\mathbb{Q}} \xrightarrow{\sim} \Lambda_{\mathbb{Q}}, \quad x \mapsto x - \frac{2(x, b)}{(b, b)} b,$$

after one chooses markings $\mathrm{H}^2(X, \mathbb{Z}) \cong \Lambda$ and $\mathrm{H}^2(Y, \mathbb{Z}) \cong \Lambda$.

Lemma 5.1.2. *Any reflective Hodge isometry φ induces a Hodge isometry on twisted Mukai lattices*

$$\tilde{\varphi} : \widetilde{\mathrm{H}}(X, \mathbb{Z}; B) \to \widetilde{\mathrm{H}}(Y, \mathbb{Z}; B'),$$

for some $B \in \mathrm{H}^2(X, \mathbb{Q})$ and $B' = -\varphi(B)$ such that the restriction of $\tilde{\varphi}_{\mathbb{Q}} : \widetilde{\mathrm{H}}(X, \mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y, \mathbb{Q})$ on $\mathrm{H}^2(X, \mathbb{Q})$ is equal to φ.

Proof. We use [31, §1.2]. The discussion there involves a purely linear-algebraic argument for twisted Mukai lattices, so it works for abelian surfaces without changes. Let us recall the construction of $\tilde{\varphi}$. By definition, there are markings $f : \mathrm{H}^2(X, \mathbb{Z}) \cong \Lambda$ and $g : \mathrm{H}^2(Y, \mathbb{Z}) \cong \Lambda$ such that the composition

$$\Lambda_{\mathbb{Q}} \xrightarrow{f^{-1}} \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\varphi} \mathrm{H}^2(Y, \mathbb{Q}) \xrightarrow{g} \Lambda_{\mathbb{Q}}$$

is a reflection $\boldsymbol{R}_b$, with $b \in \Lambda$ a primitive vector.

Let $B = f^{-1}(b)/n \in \mathrm{H}^2(X, \mathbb{Q})$ and $B' = g^{-1}(b)/n \in \mathrm{H}^2(Y, \mathbb{Q})$, where $n = \frac{1}{2}b^2$. The map

$$\tilde{\varphi} : \widetilde{\mathrm{H}}(X, \mathbb{Z}; B) \to \widetilde{\mathrm{H}}(Y, \mathbb{Z}; B'),$$

defined by sending a vector (r, c, s) to $\big(n(B, c)-r-ns,\ \varphi(c)-n((B, c)-s)B',\ -s\big)$ is a Hodge isometry. We have $\tilde{\varphi}((0, c, (B, c))) = (0, \varphi(c), (B', \varphi(c)))$ and $\tilde{\varphi}((0, 0, 1)) = (-n, -nB', -1)$, which gives the last assertion. □

The following result characterizes reflective Hodge isometries between abelian surfaces. The idea of the proof is based on [31, Theorem 1.1], with some necessary modifications for abelian surfaces.

Theorem 5.1.3. *Let X and Y be two complex abelian surfaces. If there is a reflective Hodge isometry*

$$\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q}),$$

then, up to sign, φ is induced (in the sense of Section 3.1) *by a derived isogeny*

$$\mathrm{D}^b(X) \sim \mathrm{D}^b(Y). \tag{5.1.1}$$

Proof. According to Lemma 5.1.2, there is a Hodge isometry

$$\tilde{\varphi} : \widetilde{\mathrm{H}}(X, \mathbb{Z}; B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y, \mathbb{Z}; B')$$

whose restriction on $\mathrm{H}^2(X, \mathbb{Q})$ is just φ. Let $v_{B'} = (-n, -nB', -1)$ be the image of the Mukai vector $(0, 0, 1)$ under $\tilde{\varphi}$. From our construction, the Mukai vector

$$v = \exp(-B') \cdot v_{B'} = (-n, 0, 0) \in \widetilde{\mathrm{H}}(Y, \mathbb{Z})$$

satisfies $v_{B'} = \exp(B') \cdot v$. We can assume that v is positive (see Definition 3.5.1) up to a shift of $\mathrm{D}^{(1)}(\mathscr{Y})$.

Let $\mathscr{Y} \to Y$ be a $\mathbb{G}_m$-gerbe that admits a $\boldsymbol{B}$-field lift B'. For some v-generic polarization H, the moduli stack $\mathscr{M}_H(\mathscr{Y}, v)$ of $\mathscr{Y}$-twisted sheaves on Y with Mukai vector v forms a $\mathbb{G}_m$-gerbe on its coarse moduli space $M_H(\mathscr{Y}, v)$. Let $\mathcal{E}$ be a universal $(1, 1)$-twisted sheaf on $\mathscr{Y} \times \mathscr{M}_H(\mathscr{Y}, v)$. It induces a twisted Fourier–Mukai transform

$$\Phi^{\mathcal{E}} : \mathrm{D}^{(-1)}(\mathscr{M}_H(\mathscr{Y}, v)) \to \mathrm{D}^{(1)}(\mathscr{Y})$$

(see [71, Theorem 4.3]) and a Hodge isometry

$$\varphi^{\mathcal{E}} : \widetilde{\mathrm{H}}(M_H(\mathscr{Y}, v), \mathbb{Z}; B'') \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y, \mathbb{Z}; B'),$$

where B'' is a $\boldsymbol{B}$-field lift of $\mathscr{M}_H(\mathscr{Y}, v)^{(-1)} \to M_H(\mathscr{Y}, v)$. The composition

$$(\varphi^{\mathcal{E}})^{-1} \circ \tilde{\varphi} : \widetilde{\mathrm{H}}(X, \mathbb{Z}; B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(M_H(\mathscr{Y}, v), \mathbb{Z}; B''), \tag{5.1.2}$$

defines a Hodge isometry that maps the Mukai vector $(0, 0, 1)$ to $(0, 0, 1)$ and preserves the Mukai pairing; it also sends $(1, 0, 0)$ to $(1, b, \frac{b^2}{2})$ for some $b \in \mathrm{H}^2(Y, \mathbb{Z})$. Changing B'' to $B'' + b$, one obtains a Hodge isometry that maps $(1, 0, 0)$ to $(1, 0, 0)$ and $(0, 0, 1)$ to $(0, 0, 1)$. This restricts to a Hodge isometry

$$\mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(M_{H'}(\mathscr{Y}, v), \mathbb{Z}). \tag{5.1.3}$$

If the Hodge isometry (5.1.3) is admissible, we can apply Shioda's Torelli theorem for abelian surfaces (Theorem 4.1.1) to conclude that there is an isomorphism

$$f : M_{H'}(\mathscr{Y}, v) \xrightarrow{\sim} X$$

such that $(\varphi^{\mathcal{E}})^{-1} \circ \tilde{\varphi} = f^*$ up to sign. Take $\mathscr{X} \to X$ as the $\mathbb{G}_m$-gerbe $\mathscr{M}_{H'}(\mathscr{Y}, v)^{(-1)} \to M_{H'}(\mathscr{Y}, v)$. Then the Hodge realization of the derived equivalence

$$\Phi^{\mathcal{E}} \circ f^* : \mathrm{D}^{(1)}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y}) \tag{5.1.4}$$

is $\tilde{\varphi}$ up to sign.

Otherwise, the composition

$$\mathrm{H}^2(\hat{X}, \mathbb{Z}) \xrightarrow{-\mathrm{D}} \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(M_H(\mathscr{Y}, v), \mathbb{Z})$$

is admissible, as explained in Example 4.2.3, and it can be realized as the pullback under an isomorphism $f : M_H(\mathscr{Y}, v) \xrightarrow{\sim} \hat{X}$ up to sign. Thus, the Hodge realization of the derived equivalence $f^* \circ \Phi^{\mathcal{P}}$: $\mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(M_H(\mathscr{Y}, v))$ yields the Hodge isometry (5.1.3), where $\mathcal{P}$ is the Poincaré bundle. We can consider the derived isogeny

$$\begin{aligned} \mathrm{D}^b(X) \xrightarrow{f^* \circ \Phi^{\mathcal{P}}} & \mathrm{D}^b(M_H(\mathscr{Y}, v)), \\ & \mathrm{D}^{(-1)}(\mathscr{M}_H(\mathscr{Y}, v)) \xrightarrow{\Phi^{\mathcal{E}}} \mathrm{D}^{(1)}(\mathscr{Y}). \end{aligned} \tag{5.1.5}$$

From the construction, its rational Hodge realization on second cohomology yields φ up to sign. □

Remark 5.1.4. If φ is induced from a reflection of a vector with norm $2n$, let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be the equivalent twisted abelian surfaces obtained in Theorem 5.1.3. Then we have

$$[\mathscr{X}]^n = \exp(nB) = 1 \in \mathrm{Br}(X),$$

which implies $[\mathscr{X}] \in \mathrm{Br}(X)[n]$. Similarly, the order of $[\mathscr{Y}]$ divides n.

Next we show that any rational Hodge isometry can be decomposed into a chain of reflective Hodge isometries. This is a special case of the Cartan–Dieudonné theorem that says that any element $g \in \mathrm{SO}(\Lambda_{\mathbb{Q}})$ can be decomposed as a product of reflections:

$$g = \boldsymbol{R}_{b_1} \circ \boldsymbol{R}_{b_2} \circ \cdots \circ \boldsymbol{R}_{b_n}, \tag{5.1.6}$$

such that $b_i \in \Lambda$ and $(b_i)^2 \neq 0$ for each i. From the surjectivity of the period map [63, Theorem II], for any rational Hodge isometry $\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$, we can find a sequence of abelian surfaces $\{X_i\}$ with Λ-markings and Hodge isometries

$$\varphi_i : \mathrm{H}^2(X_{i-1}, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_i, \mathbb{Q}),$$

where $X_0 = X$ and $X_n = Y$, such that φ_i is induced by some reflection $\boldsymbol{R}_{b_i} \in \mathrm{O}(\Lambda \otimes \mathbb{Q})$. We can arrange them as in (1.1.1):

$$\begin{aligned} &\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\varphi_1} \mathrm{H}^2(X_1, \mathbb{Q}), \\ &\qquad\qquad \mathrm{H}^2(X_1, \mathbb{Q}) \xrightarrow{\varphi_2} \mathrm{H}^2(X_2, \mathbb{Q}), \\ &\qquad\qquad\qquad\qquad \vdots \\ &\qquad\qquad\qquad\qquad \mathrm{H}^2(X_{n-1}, \mathbb{Q}) \xrightarrow{\varphi_n} \mathrm{H}^2(Y, \mathbb{Q}). \end{aligned} \tag{5.1.7}$$

Corollary 5.1.5. *If there is a rational Hodge isometry* $\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$, *then there is a derived isogeny from X to Y that φ up to sign as in (5.1.7).*

Remark 5.1.6. One application of Corollary 5.1.5 is that any rational Hodge isometry between abelian surfaces is algebraic, which is a special case of the Hodge conjecture for products of two abelian surfaces. Unlike the case of K3 surfaces, the Hodge conjecture for products of abelian surfaces has been known for a long time; see, for example, [60, Theorem 3.15].

Corollary 5.1.7. *There is a rational Hodge isometry* $\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$ *if and only if there is a derived isogeny from* $\mathrm{Km}(X)$ *to* $\mathrm{Km}(Y)$.

Proof. Any rational Hodge isometry induces a rational isometry of Néron–Severi lattices $\mathrm{NS}(X)_{\mathbb{Q}} \simeq \mathrm{NS}(Y)_{\mathbb{Q}}$. Let $\mathrm{T}(-)$ be the transcendental part of $\mathrm{H}^2(-)$. Applying Witt's cancellation theorem, we get

$$\mathrm{H}^2(X, \mathbb{Q}) \simeq \mathrm{H}^2(Y, \mathbb{Q}) \iff \mathrm{T}(X)_{\mathbb{Q}} \simeq \mathrm{T}(Y)_{\mathbb{Q}}$$

as Hodge isometries. According to [31, Theorem 0.1], $\mathrm{Km}(X)$ is derived isogenous to $\mathrm{Km}(Y)$ if and only if there is a Hodge isometry $\mathrm{T}(\mathrm{Km}(X))_{\mathbb{Q}} \simeq \mathrm{T}(\mathrm{Km}(Y))_{\mathbb{Q}}$. Then the statement is clear from the fact that there is a canonical integral Hodge isometry $\mathrm{T}(X)(2) \simeq \mathrm{T}(\mathrm{Km}(X))$, by [48, Proposition 4.3(i)]. □

5.2. *Prime-to-ℓ Hodge isometries.*

Definition 5.2.1. We say that a rational Hodge isometry

$$\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$$

is *prime-to-ℓ* if it descends to an isometry $\mathrm{H}^2(X, \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Z}_{(\ell)})$.

Lemma 5.2.2. *Assume* $\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$ *is a reflective Hodge isometry, induced by a primitive vector $b \in \Lambda$. Then φ is prime-to-ℓ if and only if* $\ell \nmid n = \frac{(b)^2}{2}$.

Proof. One direction is obvious. For the other, suppose φ is prime-to-ℓ. By definition, there are markings $\mathrm{H}^2(X, \mathbb{Z}) \cong \Lambda$ and $\mathrm{H}^2(X, \mathbb{Z}) \cong \Lambda$ such that the isometry

$$\Lambda \otimes \mathbb{Q} \cong \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\varphi} \mathrm{H}^2(Y, \mathbb{Q}) \cong \Lambda \otimes \mathbb{Q}$$

is the reflection $\boldsymbol{R}_b \in \mathrm{O}(\Lambda \otimes \mathbb{Q})$. As φ is prime-to-ℓ, the reflection $\boldsymbol{R}_b$ lies in $\mathrm{O}(\Lambda \otimes \mathbb{Z}_{(\ell)})$.

If $\ell \mid n$, one must have $\ell \mid (x, b)$ for any $x \in \Lambda$. However, this is contradictory, as Λ is unimodular and any primitive vector has divisibility 1. □

Another useful tool is as follows.

Lemma 5.2.3 (prime-to-ℓ Cartan–Dieudonné decomposition). *Let Λ be an integral lattice over $\mathbb{Z}$ whose reduction mod ℓ is still nondegenerate. Any orthogonal matrix $A \in \mathrm{O}(\Lambda)(\mathbb{Z}_{(\ell)}) \subset \mathrm{O}(\Lambda)(\mathbb{Q})$, with $\ell > 2$, can be decomposed into a sequence of prime-to-ℓ reflections.*

Proof. We follow the proof of [62] to refine the Cartan–Dieudonné decomposition for any field of characteristic $\neq 2$. In general, if Λ_k is a quadratic space on a field k of characteristic $\neq 2$ with Gram matrix G, let I be the identity matrix.

The proof of the Cartan–Dieudonné decomposition in [62] relies on the following facts: for any element $A \in \mathrm{O}(\Lambda_k)$, we have:

(i) A is a reflection if $\mathrm{rank}(A - I) = 1$ (see [62, Lemma 2]).

(ii) Suppose that $\mathrm{rank}(A - I) > 1$. If $S = G(A - I)$ is not skew-symmetric, then there exists $a \in \Lambda$ satisfying $a^t Sa \neq 0$ and

$$S + S^t \neq \frac{1}{a^t Sa}(Sa \cdot a^t S + S^t a \cdot a^t S^t).$$

In this case $\mathrm{rank}(A\boldsymbol{R}_b - I) = \mathrm{rank}(A - I) - 1$ and $G(A\boldsymbol{R}_b - I)$ is not skew-symmetric with $b = (A - I)a$ satisfying $b^2 = -2a^t Sa$ (see [62, Lemmas 4 and 5]).

(iii) If $S = G(A - I)$ is skew-symmetric, then there exists $b \in \Lambda$ such that $G(A\boldsymbol{R}_b - I)$ is not skew-symmetric (see the proof of [62, Theorem 2]).

Then we can decompose A as a series of reflections using (ii) repeatedly. In our case, it suffices to show that if $k = \mathbb{Q}$ and A is coprime to ℓ, i.e., nA is integral for some n coprime to ℓ, then:

(i′) A is a prime-to-ℓ reflection if $\mathrm{rank}(A - I) = 1$.

(ii′) Suppose that $\mathrm{rank}(A - I) > 1$. If the matrix $S = G(A - I)$ modulo ℓ is not skew-symmetric, then there exists a vector $a \in \Lambda$ satisfying $\ell \nmid a^t Sa$, and

$$S + S^t \neq \frac{1}{a^t Sa}(Sa \cdot a^t S + S^t a \cdot a^t S^t).$$

In this case, $\boldsymbol{R}_b$ is prime-to-ℓ with $b = (A - I)a$, $\mathrm{rank}(A\boldsymbol{R}_b - I) = \mathrm{rank}(A - I) - 1$ and $G(A\boldsymbol{R}_b - I)$ is not skew-symmetric.

(iii′) If the matrix $S = G(A - I)$ modulo ℓ is skew-symmetric, then there exists $b \in \Lambda$ such that AR_b is coprime to ℓ and the modulo ℓ reduction of $G(AR_b - I)$ is not skew-symmetric.

For (i′), this is obvious.

For (ii′), if the modulo ℓ reduction $\bar{G}(\bar{A}-\bar{I})$ of $G(A-I)$ is not skew-symmetric, we can apply (ii) to the matrix $\bar{A} \in \mathrm{O}(\Lambda_{\mathbb{F}_\ell})$ to obtain a nonzero vector $\bar{a} \in \Lambda_{\mathbb{F}_\ell}$ such that $\bar{a}^t \bar{S}\bar{a} \neq 0 \in \mathbb{F}_\ell$ and

$$\bar{S} + \bar{S}^t \neq \frac{1}{\bar{a}^t \bar{S}\bar{a}}(\bar{S}\bar{a} \cdot \bar{a}^t \bar{S} + \bar{S}^t \bar{a} \cdot \bar{a}^t \bar{S}^t). \tag{5.2.1}$$

Let $a \in \Lambda$ be a lifting of $\bar{a}$. It is easy to see that this is as desired.

For (iii′), the argument is similar to (ii′). □

As a result, we get the following.

Theorem 5.2.4. *Let $\ell > 2$ be a prime. If there is a prime-to-ℓ rational Hodge isometry*

$$\varphi : \mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q}),$$

then there exists a prime-to-ℓ derived isogeny from X to Y that induces φ up to sign. If X and Y are prime-to-ℓ derived isogenous, then there is a prime-to-ℓ derived isogeny in which the orders of $\mathbb{G}_m$-gerbes are all prime-to-ℓ.

Proof. By using the prime-to-ℓ Cartan–Dieudonné decomposition given in Lemma 5.2.3, one can decompose the Hodge isometry

$$\varphi : \mathrm{H}^2(X, \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Z}_{(\ell)}),$$

into a chain of prime-to-ℓ reflective Hodge isometries. Then Lemma 5.2.2 implies that the lift $\tilde{\varphi}$ extends to an integral isometry

$$\widetilde{\mathrm{H}}(X, \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y, \mathbb{Z}_{(\ell)})$$

In the first case of the proof in Theorem 5.1.3, the derived isogeny (5.1.4) induces $\tilde{\varphi}$ up to sign, and is thus prime-to-ℓ. In the second case, the derived isogeny (5.1.5) is also prime-to-ℓ, since the Poincaré dual

$$\widetilde{\mathrm{H}}(X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(\hat{X}, \mathbb{Z})$$

is integral and switches $(0, 0, 1)$ and $(1, 0, 0)$.

If X and Y are prime-to-ℓ derived isogenous, there is an isometry $\mathrm{T}(X) \otimes \mathbb{Z}_{(\ell)} \cong \mathrm{T}(Y) \otimes \mathbb{Z}_{(\ell)}$. Since $\ell > 2$, there is a prime-to-ℓ rational Hodge isometry $\mathrm{H}^2(X, \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Z}_{(\ell)})$ by [47, Theorem 3.2]. We can use the prime-to-ℓ Cartan–Dieudonné decomposition again to obtain a derived isogeny in which all the reflexive Hodge isometries are prime-to-ℓ. Then we can conclude the assertion by Lemma 5.2.2 and Remark 5.1.4. □

5.3. *Isogeny versus derived isogeny.* Let us now describe derived isogenies through suitable isogenies.

The functor $\underline{\mathrm{Hom}}(X, Y)$ of group homomorphisms from X to Y (not just as scheme morphisms) is representable by an étale group scheme over k (see [19, (7.14)], for example). Therefore, via Galois descent, we have

$$\mathrm{Hom}_{\mathtt{AV}_{\bar{k}}}(X_{\bar{k}}, Y_{\bar{k}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathtt{AV}_{\bar{K}}}(X_{\bar{K}}, Y_{\bar{K}}), \tag{5.3.1}$$

for any algebraically closed field $\bar{K} \supset k$. A similar statement holds for derived isogenies.

Lemma 5.3.1. *Let X and Y be abelian surfaces defined over k with* $\operatorname{char} k = 0$. *Let $\overline{K} \supseteq k$ be an algebraically closed field containing k. Let $\bar{k}$ be the algebraic closure of k in $\overline{K}$. Then if $X_{\overline{K}}$ and $Y_{\overline{K}}$ are twisted derived equivalent, so are $X_{\bar{k}}$ and $Y_{\bar{k}}$.*

Proof. As $X_{\overline{K}}$ is twisted derived equivalent to $Y_{\overline{K}}$, by Theorem 3.5.3, there exist finitely many abelian surfaces $X_0, X_1, \ldots, X_n$ defined over $\overline{K}$ with $X_0 = X_{\overline{K}}$ and

$$X_i \cong M_{H_i}(\mathscr{X}_{i-1}, v_i) \quad Y_{\overline{K}} \cong M_{H_n}(\mathscr{X}_n, v_n)$$

for some $[\mathscr{X}_{i-1}] \in \operatorname{Br}(X_{i-1})[r]$. Let us construct abelian surfaces over $\bar{k}$ to connect $X_{\bar{k}}$ and $Y_{\bar{k}}$ as follows:

Set $X'_0 = X_{\bar{k}}$, then we take $X'_1 = M_{H'_1}(\mathscr{X}'_0, v'_1)$ where $\mathscr{X}'_0$, H'_1 and v'_1 are the descent of $\mathscr{X}_0$, H_1 and v through the isomorphisms $\operatorname{Br}(X_{\overline{K}})[r] \cong \operatorname{Br}(X_{\bar{k}})[r]$, $\operatorname{NS}(X_{\overline{K}}) \cong \operatorname{NS}(X_{\bar{k}})$ and $\widetilde{\mathrm{H}}(X_{\overline{K}}) \cong \widetilde{\mathrm{H}}(X_{\bar{k}})$. The invariance of Brauer group and (ℓ-adic) Mukai lattice under extension $\bar{k} \subseteq \overline{K}$ is from the smooth base change theorem. For Néron–Severi groups, see [44, Proposition 3.1]. Then inductively, we can define X'_i as the moduli space of twisted sheaves $M_{H'_i}(\mathscr{X}'_{i-1}, v'_i)$ (or its dual, respectively) over $\bar{k}$. Note that we have natural isomorphisms

$$(M_{H'_i}(\mathscr{X}'_{i-1}, v'_i))_{\overline{K}} \cong M_{H_i}(\mathscr{X}_{i-1}, v_i)$$

over $\overline{K}$. In particular, $(M_{H'_i}(\mathscr{X}'_n, v'_i))_{\overline{K}} \cong Y_{\overline{K}}$. It follows that $M_{H'_i}(\mathscr{X}'_n, v'_i) \cong Y_{\bar{k}}$. □

For any abelian surface $X_{\mathbb{C}}$ over $\mathbb{C}$, the spreading-out argument shows that there is a finitely generated field $k \subset \mathbb{C}$ and an abelian surface X over k such that $X \times_k \mathbb{C} \cong X_{\mathbb{C}}$. We have the Artin comparison

$$\mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \cong \mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell, \tag{5.3.2}$$

for any $i \in \mathbb{Z}$ and ℓ a prime. Suppose Y is another abelian surface defined over k. Suppose $f : Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is a prime-to-ℓ quasi-isogeny. By definition, it induces an isomorphism of $\mathbb{Z}_{(\ell)}$-modules

$$f^* : \mathrm{H}^1(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \mathrm{H}^1(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)}$$

such that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} & \xrightarrow{\sim} & \mathrm{H}^i(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \\ \downarrow & & \downarrow \\ \mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) & \xrightarrow{\sim} & \mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \end{array}$$

for any i, under the comparison (5.3.2). For the converse, we have the following simple fact given by a faithfully flat descent of modules along $\mathbb{Z}_{(\ell)} \hookrightarrow \mathbb{Z}_\ell$ and the ℓ-adic Shioda thick.

Lemma 5.3.2. *A (quasi-)isogeny $f : Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is prime-to-ℓ if and only if it induces an isomorphism of integral ℓ-adic realizations*

$$f^* : \mathrm{H}^2_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{ét}(Y_{\bar{k}}, \mathbb{Z}_\ell).$$

Inspired by Shioda's trick for Hodge isogenies (Proposition 4.5.1), we introduce the following notions.

Definition 5.3.3. Let X and Y be g-dimensional abelian varieties over k.

- X and Y are (prime-to-ℓ) *principally isogenous* if there is a (prime-to-ℓ) isogeny f from X to Y of square degree, that is, $\deg(f) = d^2$ for some $d \in \mathbb{Z}$. This f is called a *principal isogeny*.
- An isogeny $f : X \to Y$ is *quasiliftable* if f can be written as the composition of finitely many isogenies that are liftable to characteristic zero.

Now, we can state the main result in this section, which yields in particular Theorem 1.2.1.

Theorem 5.3.4. *Suppose* $\operatorname{char} k = 0$. *Let* $\ell > 2$ *be a prime. The following statements are equivalent*:

(1) *X is* (*prime-to-ℓ*) *principally isogenous to Y over* $\bar{k}$.

(2) *X and Y are* (*prime-to-ℓ*) *derived isogenous over* $\bar{k}$.

Proof. (1) $\Longrightarrow$ (2): we can assume that $f : X \to Y$ is a principal isogeny defined over a finitely generated field k'. By embedding k' into $\mathbb{C}$, two complex abelian surfaces $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are derived isogenous since there is a rational Hodge isometry

$$(f^*/n) \otimes \mathbb{Q} : \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q} \cong \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q}$$

where $\deg(f) = n^2$. By Lemma 5.3.1, one concludes that $X_{\bar{k}}$ and $Y_{\bar{k}}$ are derived isogenous, with rational Hodge realization $(f^*/n) \otimes \mathbb{Q}$.

If f is a prime-to-ℓ isogeny, the map f^*/n restricts to an isomorphism

$$\mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)}.$$

The assertion then follows from Theorem 5.2.4.

(2) $\Longrightarrow$ (1): We may assume X and Y are derived isogenous over a finitely generated field k'. Embedding k' into $\mathbb{C}$, $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are derived isogenous as well by Lemma 5.3.1. According to Remark 3.1.2, there is a Hodge isometry

$$\varphi : \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q}). \tag{5.3.3}$$

According to Example 4.2.3, we can assume φ is admissible after replacing X by its dual $\hat{X}$. By Proposition 4.5.1, they are principally isogenous over $\mathbb{C}$. It follows that X and Y are principally isogenous over $\bar{k}$ by (5.3.1).

If $\mathrm{D}^b(X) \sim \mathrm{D}^b(Y)$ is prime-to-ℓ, then we can choose a motive isomorphism $\mathfrak{h}^2(X) \simeq \mathfrak{h}^2(Y)$ whose ℓ-adic realization φ_ℓ is integral by the cancellation theorem over $\mathbb{Z}_\ell$ (see [54, Theorem 92:3]). The principal isogeny that induces φ is prime-to-ℓ by Lemma 5.3.2. This proves the assertion. □

Proof of Corollary 1.2.2. Let us summarize all the results that lead to Corollary 1.2.2. Using an argument similar to the one in Theorem 5.3.4, we can reduce them to the case $k = \mathbb{C}$.

(i) $\Longleftrightarrow$ (ii): This is Theorem 5.3.4.

(i) $\Longleftrightarrow$ (vi): This is Corollary 5.1.5.

(vi) $\iff$ (vii) $\iff$ (viii): This follows from the Witt cancellation theorem.

(i) $\iff$ (iii): This is Corollary 5.1.7.

(ii) $\implies$ (iv) $\implies$ (v): This is from the computation in [23, Proposition 4.6]. In fact, one may take the correspondence

$$\Gamma := \bigoplus_i \Gamma_{2i} : \mathfrak{h}^{\mathrm{even}}(X) \xrightarrow{\sim} \mathfrak{h}^{\mathrm{even}}(Y),$$

where

$$\Gamma_{2i} := (f^*/n^i) \circ \pi_X^{2i} : \mathfrak{h}^{2i}(X) \to \mathfrak{h}^{2i}(Y),$$

and $f : X \to Y$ is the given principal isogeny.

(v) $\implies$ (ii): Let $\Gamma : \mathfrak{h}^{\mathrm{even}}(X) \xrightarrow{\sim} \mathfrak{h}^{\mathrm{even}}(Y)$ be an isomorphism of Frobenius algebra objects. The Betti realization of its second component is a Hodge isometry by the Frobenius condition [23, Theorem 3.3]. Thus, X and Y are derived isogenous by Corollary 5.1.5, and hence are principally isogenous. □

6. Derived isogeny in positive characteristic

In this section, we prove the twisted derived Torelli theorem for abelian surfaces over odd characteristic fields. The primary strategy is to lift everything to characteristic zero. Throughout this section, we let k denote an algebraically closed field with characteristic $p > 3$.

6.1. *Lifting of derived isogenies and quasi-isogenies.* Let us start with a lifting result for derived isogenies, which is the only place we may require $p > 3$.

Proposition 6.1.1. *Let $\mathscr{X}_0 \to X_0$ and $\mathscr{Y}_0 \to Y_0$ be twisted abelian surfaces over k, which are of finite height. If there is a derived equivalence $\Phi_0 : \mathrm{D}^{(1)}(\mathscr{X}_0) \to \mathrm{D}^{(1)}(\mathscr{Y}_0)$, then there exists a discrete valuation ring V whose residue field is k and twisted abelian surfaces*

$$\begin{array}{ccc} \mathscr{X}_V & \longrightarrow & X_V \\ & \searrow & \downarrow \\ & & \operatorname{Spec} V \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathscr{Y}_V & \longrightarrow & Y_V \\ & \searrow & \downarrow \\ & & \operatorname{Spec} V \end{array}$$

over V with the following properties:

- *The special fibers are geometrically isomorphic to $\mathscr{X}_0 \to X_0$ and $\mathscr{Y}_0 \to Y_0$ respectively.*
- *There is a Fourier–Mukai transform $\Phi_V : \mathrm{D}^{(1)}(\mathscr{X}_V) \to \mathrm{D}^{(1)}(\mathscr{Y}_V)$ whose Fourier–Mukai kernel restricted to $\mathscr{X} \times \mathscr{Y}$ induces Φ_0.*

Moreover, if Φ_0 is prime-to-p and $p > 3$, the derived equivalence $\Phi_K : \mathrm{D}^{(1)}(\mathfrak{X}_K) \to \mathrm{D}^{(1)}(\mathfrak{Y}_K)$ on the generic fiber is also prime-to-p where K is the fraction field of V.

Proof. The proof proceeds like that of [10, Theorem 5.8], which ensures the existence of liftings of derived isogenies between K3 surfaces. By Theorem 3.5.3, we know that

$$\mathscr{X}_0^{(-1)} \cong \mathscr{M}_H(\mathscr{Y}_0, v)$$

is a moduli stack of $\mathscr{Y}_0$-twisted coherent sheaves for some vector $v \in \widetilde{\mathrm{N}}(\mathscr{Y}_0)$. By Lemma 2.3.1, we can find a DVR V and a projective lift $\mathscr{Y}_V \to Y_V$ over V such that $\mathrm{NS}(Y_V) \cong \mathrm{NS}(Y_0)$. Let H_V be the element in $\mathrm{NS}(Y_V)$ that extends H. Following the description of twisted extended Néron–Severi lattices as in Proposition 3.3.2, we can see that $\widetilde{\mathrm{N}}(\mathscr{Y}_V) \cong \widetilde{\mathrm{N}}(\mathscr{Y}_0)$ and hence the twisted Mukai vector v can be extended over V, still denoted by v.

Let $\mathscr{X}_V^{(-1)} = \mathscr{M}_{H_V}(\mathscr{Y}_V, v)$ be the relative moduli stack of $\mathscr{X}_V$-twisted coherent sheaves. The universal object in $\mathrm{D}^{(-1,1)}(\mathscr{X}_V \times \mathscr{Y}_V)$ induces a derived equivalence $\Phi_V : \mathrm{D}^{(1)}(\mathscr{X}_V) \to \mathrm{D}^{(1)}(\mathscr{Y}_V)$ as desired.

For the last assertion, we need to prove that the p-adic realization of Φ_K is integral. This can be deduced from a similar argument as in the proof of Theorem 1.5 in [10], based on Cais and Liu's crystalline cohomological description for the integral p-adic Hodge theory [13]. Let us sketch the proof. As Φ is prime-to-p, its cohomological realization restricts to an isometry of F-crystals

$$\tilde{\varphi}_p : \mathrm{H}^{\mathrm{even}}_{\mathrm{crys}}(X_0/W) \simeq \mathrm{H}^{\mathrm{even}}_{\mathrm{crys}}(Y_0/W)$$

by our definition. The base extension $\tilde{\varphi}_p \otimes K$ can be identified with the de Rham cohomological realization of Φ_K:

$$\tilde{\varphi}_K : \mathrm{H}^{\mathrm{even}}_{\mathrm{dR}}(X_K/K) \simeq \mathrm{H}^{\mathrm{even}}_{\mathrm{dR}}(Y_K/K),$$

by the Berthelot–Ogus comparison (see [4, Corollary 2.5] or [24, Theorem B.3.1]). It also preserves Hodge filtrations. Let S be the p-completion of the divided power envelope of the pair $\big(W[\![u]\!], \ker(W[\![u]\!] \to \mathcal{O}_K)\big)$. Then the map

$$\tilde{\varphi}_p \otimes_W S : \mathrm{H}^{\mathrm{even}}_{\mathrm{crys}}(X_0/S) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}_{\mathrm{crys}}(Y_0/S) \tag{6.1.1}$$

is an isomorphism of strongly divisible S-lattices (see [13, §4]). If $p > 3$, according to [13, Theorem 5.4], one can apply Breuil's functor on (6.1.1) to see that ϕ_K restricts to an $\mathbb{Z}_p$-integral $\mathrm{Gal}(\overline{K}/K)$-equivariant isometry $\mathrm{H}^{\mathrm{even}}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}_{\text{ét}}(Y_{\overline{K}}, \mathbb{Z}_p)$. □

Remark 6.1.2. The technical requirement that $p > 3$ is needed in [13, Theorem 4.3(3) and (4)]. When $\mathcal{O}_K = W(k)$ is unramified, this condition can be weakened to $p > 2$ by using Fontaine's Theorem 2(iii) in [21]. In general, when $p = 3$, a possible approach is to prove Shioda's trick as in Section 4 for strongly divisible S-lattices (see [11, Definition 2.1.1]), which can reduce the statement to crystalline Galois representations of Hodge–Tate weight one.

Next, one can lift separable isogenies between abelian surfaces:

Proposition 6.1.3. *Let $f : X_0 \to Y_0$ be a separable isogeny between two abelian surfaces over k. Let $W = W(k)$ be the ring of Witt vectors. Then there exist liftings $X_W \to \mathrm{Spec}\, W$ and $Y_W \to \mathrm{Spec}\, W$ such that the isogeny f can be lifted to an isogeny $f_W : X_W \to Y_W$ such that $\deg f = \deg f_W$. Thus, every prime-to-p isogeny can be lifted to a prime-to-p isogeny.*

Proof. According to [55, Proposition 11.1], there is a projective lifting $X_W \to \operatorname{Spec} W$ of X_0. Given that f is separable, $\ker f \subset X_0$ constitutes a finite étale group scheme over k, which is liftable. Choosing a lifting $G_W \subset X_W$ of $\ker f$, we obtain an isogeny

$$f_W : X_W \to Y_W := X_W/G_W,$$

which serves as a lifting of f. If f is prime-to-p, then we have $\ker f_W \subseteq X_W[n]$ for some n that is coprime to p. Consequently, f_W is also prime-to-p. □

6.2. *Specialization of prime-to-p derived isogenies.* Next, we shall show that prime-to-p geometrically derived isogenies are preserved under reduction. The idea is to show that the specialization of a moduli space of stable twisted sheaves on an abelian surface or K3 surface remains a moduli space.

Theorem 6.2.1. *Let V be a DVR with residue field k and $K = \operatorname{Frac}(V)$. Let $X_V \to \operatorname{Spec} V$ and $Y_V \to \operatorname{Spec} V$ be projective abelian surfaces or K3 surfaces over $\operatorname{Spec} V$ satisfying*

$$\mathrm{NS}(X_{\bar{K}}) \cong \mathrm{NS}(X_k), \tag{6.2.1}$$

where X_k is the special fiber of $X_V \to \operatorname{Spec} V$. If their generic fibers X_K and Y_K are (geometrically) prime-to-p derived isogenies, so are the special fibers X_k and Y_k.

Proof. With Theorem 5.2.4, it is sufficient to consider the case where there is a derived equivalence

$$\Phi_V : \mathrm{D}^{(1)}(\mathscr{X}_{\bar{K}}) \xrightarrow{\sim} \mathrm{D}^{(1)}(\mathscr{Y}_{\bar{K}})$$

for some prime-to-p $\mathbb{G}_m$-gerbes $\mathscr{X}_K \to X_K$ and $\mathscr{Y}_K \to Y_K$. From Theorem 3.5.3, we know that there is an isomorphism

$$\mathscr{Y}_{\bar{K}} \cong \mathscr{M}_H(\mathscr{X}_{\bar{K}}, v_K)^{(-1)},$$

for some twisted Mukai vector $v_K \in \widetilde{\mathrm{N}}(\mathscr{X}_K)$ and $H_K \in \mathrm{NS}(X_{\bar{K}})$ being v-generic. Up to taking a finite extension, we may assume that everything can be defined over K.

We claim that there exists a $\mathbb{G}_m$ gerbe $\mathscr{X}_V \to X_V$ whose restriction to $\operatorname{Spec} K$ is $\mathscr{X}_K \to X_K$. It suffices to show that the class $[\mathscr{X}_K] \in \mathrm{Br}(X_K)$ can be extended to an element in $\mathrm{Br}(X_V)$. By the Chinese remainder theorem, we may assume $\operatorname{ord}([\mathscr{X}_K]) = \ell^n$ for some prime $\ell \neq p$. For each prime $\ell \neq p$, from the Kummer sequence, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Pic}(X_V)/\ell^n & \to & \mathrm{H}^1_{\text{ét}}(X_V, \mu_{\ell^n}) & \to & \mathrm{Br}(X_V)[\ell^n] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Pic}(X_K)/\ell^n & \to & \mathrm{H}^1_{\text{ét}}(X_K, \mu_{\ell^n}) & \to & \mathrm{Br}(X_K)[\ell^n] & \to & 0 \end{array}$$

The second vertical morphism is an isomorphism by smooth and proper base change. Therefore, $\mathrm{Br}(X_V)[\ell^n] \to \mathrm{Br}(X_K)[\ell^n]$ is surjective, which proves the claim.

By our assumption (6.2.1), we can pick extensions $v_V \in \widetilde{\mathrm{N}}(\mathscr{X}_V)$ and $H_V \in \mathrm{Pic}(X_V)$ of v_K and H_K. Let $\mathscr{M}_{H_V}(X_V, v_V) \to \operatorname{Spec} V$ be the relative moduli space of H_V-stable twisted sheaves. Then we have the commutative diagram

$$\begin{array}{ccccccc} M_{H_V}(\mathscr{X}_V, v_V) & \longleftarrow & M_{H_K}(\mathscr{X}_K, v_K) & \xrightarrow{\cong} & Y_K & \longrightarrow & Y_V \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} V & \longleftarrow & \operatorname{Spec}(K) & \longrightarrow & \operatorname{Spec}(K) & \longrightarrow & \operatorname{Spec} V \end{array}$$

According to Matsusaka and Mumford [43, Theorem 1], the isomorphism between the generic fiber can be extended to Spec V. Thus Y_k is isomorphic to $M_{H_k}(\mathscr{X}_k, v_k)$, where $v_k = v_V|_{\operatorname{Spec} k}$ and $H_k = H_V|_{\operatorname{Spec} k}$. It follows that there is a prime-to-p derived equivalence $\mathrm{D}^{(1)}(\mathscr{X}_k) \simeq \mathrm{D}^{(-1)}(\mathscr{M}_{H_k}(\mathscr{X}_k, v_k))$. ☐

Remark 6.2.2. Our proof fails when the twisted derived equivalence is not prime-to-p. This is because if the associated Brauer class α has order p^n, the map

$$\mathrm{Br}(X_V)[p^n] \to \mathrm{Br}(X_K)[p^n]$$

may not be surjective (see [59, 6.8.2]).

Proof of Theorem 1.4.1. When X or Y is supersingular, the assertion follows from Theorem 3.6.6(2). So we can assume that X and Y both have finite height.

(i′) $\Longrightarrow$ (ii′): By Proposition 6.1.1, we can find projective liftings $X_V \to \operatorname{Spec} V$ and $Y_V \to \operatorname{Spec} V$ of X and Y over some DVR V such that there is a prime-to-p twisted derived equivalence between generic fibers X_K and Y_K.

By Theorem 5.3.4, the generic fibers X_K and Y_K are geometrically prime-to-p principally isogenous. Up to a finite extension of K, we can find a prime-to-p principal isogeny $f_K : X_K \to Y_K$. The Néron extension property of smooth models X_V, Y_V [5, §7.3, Proposition 6] ensures that f_K can be extended to an isogeny

$$f_V : X_V \to Y_V.$$

The restriction $f_k : X \to Y$ over the special fibers is still a principal isogeny and we can conclude that f_k is prime-to-p by using Tate's spreading theorem for p-divisible groups (see [67, Theorem 4]).

(i′) $\Longrightarrow$ (ii′): Suppose that there is an isogeny $f : X \to Y$ that is prime-to-p of degree d^2. By Proposition 6.1.3, we can lift it to a prime-to-p isogeny of degree d^2 over W:

$$f_W : X_W \to Y_W.$$

Set $K = \mathrm{Frac}(W)$. The induced isogeny f_K between the generic fibers is a prime-to-p principal isogeny, which induces a G_K-equivariant isometry

$$\frac{f_K^*}{d} : \mathrm{H}^2_{\text{ét}}(Y_{\bar{K}}, \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p).$$

By Theorem 5.3.4, there exists a prime-to-p derived isogeny $\mathrm{D}^b(X_{\overline{K}}) \sim \mathrm{D}^b(Y_{\overline{K}})$ whose p-adic cohomological realization is f_K^*/d. The assertion follows from Theorem 6.2.1. □

6.3. *Further remarks.* From the proof of the implication (i′) ⇒ (ii′) of Theorem 1.4.1, we can see that the lifting-specialization argument also works for non-prime-to-p derived isogenies. Thus we have

Theorem 6.3.1. *Suppose X_0 and Y_0 are abelian surfaces over k with finite height. If X_0 and Y_0 are derived isogenous, then they are quasiliftable principally isogenous.*

Moreover, we believe that the converse of Theorem 6.3.1 also holds.

Conjecture 6.3.2. *Two abelian surfaces X_0 and Y_0 are derived isogenous over k if and only if they are quasiliftable principally isogenous.*

For this conjecture, our approach remains valid provided that there is a specialization theorem for non-prime-to-p derived isogenies. According to the proof of Theorem 6.2.1, it suffices to establish the existence of specialization of Brauer classes of order p. Adhering to the notations in Theorem 6.2.1, this requires the restriction map

$$\mathrm{Br}(X_V) \to \mathrm{Br}(X_K)$$

to be surjective. See Remark 6.2.2 for further details.

6.4. *Derived isogeny for Kummer surfaces.* We now explore the interrelations between the derived isogenies of abelian surfaces and their associated Kummer surfaces. Using the lifting argument, the following theorem is an immediate consequence of the result in characteristic 0.

Theorem 6.4.1. *Assume $p > 2$. If X_0 and Y_0 are prime-to-p derived isogenous abelian surfaces over k, then the associated Kummer surfaces $\mathrm{Km}(X_0)$ and $\mathrm{Km}(Y_0)$ are prime-to-p derived isogenous. If there is a derived equivalence*

$$\mathrm{D}^b(\mathrm{Km}(X_0), \alpha_0) \simeq \mathrm{D}^b(\mathrm{Km}(Y_0), \beta_0) \tag{6.4.1}$$

with $\mathrm{ord}(\alpha_0)$ and $\mathrm{ord}(\beta_0)$ prime-to-p, then X and Y are prime-to-p derived isogenous.

Proof. For the first assertion, as before, we can quasi-lift the prime-to-p derived isogeny between X and Y to characteristic 0. By Theorem 1.4.1 and Proposition 6.1.1, their liftings are geometrically prime-to-p derived isogenous. According to [66, Corollary 4.3], the associated Kummer surfaces are prime-to-p derived isogenous. It follows from Theorem 6.2.1 that $\mathrm{Km}(X_0)$ and $\mathrm{Km}(Y_0)$ are prime-to-p derived isogenous.

For the last assertion, if X_0 and Y_0 are supersingular, then α_0 and β_0 are trivial under our assumptions. In this case, the result follows from [38, Theorem 1.2]. Suppose X_0 or Y_0 is of finite height (then both are of finite height). According to [10, Theorem 5.8], we can find a DVR V with residue field k and projective twisted K3 surfaces over V

$$(S_V, \alpha_V) \to \operatorname{Spec} V \quad \text{and} \quad (S'_V, \beta_V) \to \operatorname{Spec} V$$

satisfying the following conditions:

- The special fibers are $(\mathrm{Km}(X_0), \alpha_0)$ and $(\mathrm{Km}(Y_0), \beta_0)$ respectively.
- The generic fibers (S_K, α_K) and (S'_K, β_K) are geometrically derived equivalent.
- $\mathrm{NS}(S_{\bar{K}}) \cong \mathrm{NS}(\mathrm{Km}(X_0))$ and $\mathrm{NS}(S'_{\bar{K}}) \cong \mathrm{NS}(\mathrm{Km}(Y_0))$.

Note that $\mathrm{NS}(S_K)$ and $\mathrm{NS}(S'_K)$ contain Kummer lattices. As seen in the proof of Lemma 2.3.1, this implies that there exist projective liftings of X_0 and Y_0, denoted by $X_V \to \operatorname{Spec} V$ and $Y_V \to \operatorname{Spec} V$, such that

$$S_{\bar{K}} \cong \mathrm{Km}(X_{\bar{K}}) \quad \text{and} \quad S'_{\bar{K}} \cong \mathrm{Km}(Y_{\bar{K}}).$$

Choose an embedding $K \hookrightarrow \mathbb{C}$, set $X_{\mathbb{C}} = X_K \otimes_K \mathbb{C}$ and $Y_{\mathbb{C}} = Y_K \otimes_K \mathbb{C}$. Then we have a prime-to-p Hodge isometry

$$\mathrm{H}^2(\mathrm{Km}(X_{\mathbb{C}}), \mathbb{Z}_{(p)}) \to \mathrm{H}^2(\mathrm{Km}(Y_{\mathbb{C}}), \mathbb{Z}_{(p)}) \tag{6.4.2}$$

induced from the prime-to-p derived equivalence. Based on the Kummer construction, for any abelian surface $X_{\mathbb{C}}$, as $p > 2$, there is a natural Hodge isometry

$$\mathrm{H}^2(\mathrm{Km}(X_{\mathbb{C}}), \mathbb{Z}_{(p)}) \cong \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}_{(p)}) \oplus (\Sigma_{X_{\mathbb{C}}} \otimes \mathbb{Z}_{(p)}),$$

where $\Sigma_{X_{\mathbb{C}}} \cong \bigoplus_{i=1}^{16} \mathbb{Z}e_i$ with $(e_i, e_j) = -2\delta_{ij}$ being the Kummer lattice. Then one obtains a Hodge isometry

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}_{(p)}) \to \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z}_{(p)})$$

from (6.4.2) through the Witt cancellation procedure. By Theorem 5.3.4, X_K and Y_K are geometrically prime-to-p derived isogenous. The assertion follows from Theorem 6.2.1. □

Remark 6.4.2. It is natural to consider if one can apply the lifting method to prove the converse of Theorem 6.4.1. Specifically, one may wonder if $\mathrm{Km}(X_0)$ and $\mathrm{Km}(Y_0)$ are prime-to-p derived isogenous, as are X and Y.

However, the issue is that the derived isogeny between $\mathrm{Km}(X_0)$ and $\mathrm{Km}(Y_0)$ is merely quasiliftable, not known to be liftable. In other words, although we can lift every derived equivalence between twisted abelian surfaces or K3 surface to characteristic 0, we cannot necessarily find liftings of X_0 and Y_0 respectively such that the generic fibers of their associated Kummer surfaces are prime-to-p geometrically derived isogenous.

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