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The role of primes of good reduction in the Brauer–Manin obstruction

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We discuss the role of primes of good reduction in the existence of the Brauer–Manin obstruction to weak approximation for varieties defined over number fields. Following Bright and Newton, we give conditions on the ramification index of primes involved in the Brauer–Manin obstruction. To support the results, we give many examples of transcendental nature on K3 surfaces.

1. Introduction

Let k be a number field and $\boldsymbol{A}_k$ be the ring of adèles of k, i.e., the restricted product of k_v for all places v of k, taken with respect to the rings of integers $\mathcal{O}_v \subseteq k_v$. Let V be a smooth, proper, geometrically irreducible variety over k. We say that V satisfies *weak approximation* if the image of $V(k)$ in the set of the adèlic points $V(\boldsymbol{A}_k)$ is dense. Manin [1971] introduced the use of the Brauer group to study this image, and Colliot-Thélène and Sansuc [1977, Proposition 5] used the same idea to study weak approximation. More precisely, they showed that there exists a pairing

$$\langle -, - \rangle : \mathrm{Br}(V) \times V(\boldsymbol{A}_k) \to \mathbb{Q}/\mathbb{Z}$$

such that the closure of the rational points of V lies in the right kernel of the pairing, denoted by $V(\boldsymbol{A}_k)^{\mathrm{Br}}$. If $V(\boldsymbol{A}_k)^{\mathrm{Br}}$ is not equal to the whole $V(\boldsymbol{A}_k)$, we say that there is a *Brauer–Manin obstruction to weak approximation* on V.

For every nonarchimedean place $\mathfrak{p}$ and for every element $\mathcal{A} \in \mathrm{Br}(V)$, we denote by $\mathrm{ev}_{\mathcal{A}}$ the map $\langle \mathcal{A}, - \rangle|_{V(k_\mathfrak{p})} : V(k_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z}$.

Definition 1.1. We say that a prime $\mathfrak{p}$ of k *plays a role* in the Brauer–Manin obstruction to weak approximation on V if there exists an element $\mathcal{A} \in \mathrm{Br}(V)$ such that the corresponding evaluation map $\mathrm{ev}_{\mathcal{A}} : V(k_\mathfrak{p}) \to \mathrm{Br}(k_\mathfrak{p})$ is nonconstant.

We will address the following question.

Question 1.2. Assume $\mathrm{Pic}(\overline{V})$ to be torsion-free and finitely generated. Which primes can play a role in the Brauer–Manin obstruction to weak approximation on V?

This question is inspired by one asked by Swinnerton-Dyer [Colliot-Thélène and Skorobogatov 2013, Question 1]. He asked whether only primes of bad reduction and archimedean places can play a role in

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the Brauer–Manin obstruction to weak approximation. Bright and Newton [2023] prove the following theorem, which leads to a negative answer to Swinnerton-Dyer's question.

Theorem [Bright and Newton 2023, Theorem C]. *Assume that* $\mathrm{H}^0(V, \Omega_V^2) \neq 0$. *Let* $\mathfrak{p}$ *be a prime at which* V *has good ordinary reduction, with residue characteristic* p. *Then there exists a finite field extension* k'/k, *a prime* $\mathfrak{p}'$ *of* k' *lying over* $\mathfrak{p}$, *and an element* $\mathcal{A} \in \mathrm{Br}(V_{k'})\{p\}$ *such that the corresponding evaluation map is nonconstant on* $V(k'_{\mathfrak{p}'})$.

Hence, under the assumptions of [loc. cit.], up to a base change to a finite field extension k'/k we can always find a prime $\mathfrak{p}'$ of good reduction that plays a role in the Brauer–Manin obstruction to weak approximation on $V_{k'}$.

The main focus of this paper is on the field extension k'/k appearing in Bright and Newton's result, with a particular emphasis on K3 surfaces. We analyse how the reduction type and the absolute ramification index are involved in the possibility for a prime of good reduction to play a role in the Brauer–Manin obstruction to weak approximation.

Theorem 1.3. *Let* $\mathfrak{p}$ *be a prime of good ordinary reduction for* V *of residue characteristic* p. *Assume that the special fibre* $\mathcal{V}(\mathfrak{p})$ *at* $\mathfrak{p}$ *has no nontrivial global* 1*-forms,* $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ *and* $(p-1) \nmid e_{\mathfrak{p}}$. *Then the prime* $\mathfrak{p}$ *does not play a role in the Brauer–Manin obstruction to weak approximation on* V.

As pointed out in [Bright and Newton 2023, Remark 11.5], if V is a K3 surface, then, for every prime of good reduction $\mathfrak{p}$, the special fibre is still a K3 surface and hence has no nontrivial global 1-forms and satisfies $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$. It follows from Theorem 1.3 that, for K3 surfaces over number fields, in order for a prime of good ordinary reduction to be involved in the Brauer–Manin obstruction to weak approximation it is necessary that $(p-1) \mid e_{\mathfrak{p}}$. In the case of K3 surfaces, we analyse also what happens for primes of good nonordinary reduction.

Theorem 1.4. *Let* V *be a K*3 *surface and* $\mathfrak{p}$ *be a prime of good nonordinary reduction for* V *with* $e_{\mathfrak{p}} \leq p-1$. *Then the prime* $\mathfrak{p}$ *does not play a role in the Brauer–Manin obstruction to weak approximation on* V.

Furthermore, in Section 3.2 we improve the result of Theorem C under an extra assumption on the crystalline cohomology of the special fibre at a prime $\mathfrak{p}$ of good ordinary reduction. In particular, we show that it is always possible to find (over a finite field extension k'/k) an element whose order is *exactly* p and whose evaluation map is nonconstant; see Theorem 3.6.

We offer several examples in support of the conditions on the ramification index found in Theorems 1.3 and 1.4. In particular, we exhibit K3 surfaces V over number fields such that:

(a) V has good ordinary reduction at a prime $\mathfrak{p}$ with ramification index $e_{\mathfrak{p}} = p-1$ and there is an element $\mathcal{A} \in \mathrm{Br}(V)[p]$ whose evaluation map is nonconstant on $V(k_{\mathfrak{p}})$; see Theorem 6.2 with $\alpha = 1$.

(b) V has good ordinary reduction at a prime $\mathfrak{p}$ with $e_{\mathfrak{p}} = p-1$ and $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation; see Example 4.11.

(c) V has good nonordinary reduction at a prime $\mathfrak{p}$ with $e_{\mathfrak{p}} = p$ and there is an element $\mathcal{A} \in \mathrm{Br}(V)[p]$ whose evaluation map is nonconstant on $V(k_{\mathfrak{p}})$; see Theorem 6.2 with $\alpha = \sqrt{2}$.

More precisely: (a) shows the condition $(p-1)\nmid e_{\mathfrak{p}}$ in Theorem 1.3 is necessary; (b) shows the converse of Theorem 1.3 does not hold in general; (c) shows the inequality in Theorem 1.4 cannot be refined.

Finally, we point out that in all the examples of K3 surfaces in which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation, the corresponding element in the Brauer group is of *transcendental* nature, i.e., it does not belong to the algebraic Brauer group, which is defined as the kernel of the natural map from $\mathrm{Br}(V)$ to $\mathrm{Br}(\overline{V})$, where $\overline{V}$ is the base change of V to an algebraic closure of k (see Corollary 2.31). The first example of a transcendental element in the Brauer group of a K3 surface defined over a number field was given in [Wittenberg 2004]. Other examples of transcendental elements that obstruct weak approximation can be found in [Hassett et al. 2011; Ieronymou 2010; Preu 2013; Newton 2016; Berg and Várilly-Alvarado 2020; Ieronymou and Skorobogatov 2015]. In all these articles, the obstruction to weak approximation comes from the fact that the transcendental algebra has nonconstant evaluation either at the place at infinity or at a prime of bad reduction. The first example of a K3 surface in which a prime of good reduction plays a role in the Brauer–Manin obstruction to weak approximation can be found in [Pagano 2022].

1.1. ***Notation.*** If G is an abelian group and n a positive integer, then $G[n]$ denotes the kernel of multiplication by n on G. If p is a prime, then $G\{p\}$ denotes the p-power torsion subgroup of G. For any smooth scheme X over a field k, we denote by $\mathbb{Z}/n\mathbb{Z}(r)$ the element of the bounded derived category $D^b(X_{\text{ét}})$ defined by

$$\mathbb{Z}/n\mathbb{Z}(r) := \begin{cases} \mu_n^{\otimes r} & \text{if } k \text{ has characteristic } 0, \\ \mu_m^{\otimes r} \oplus W_s\Omega^r_{X,\log}[-r] & \text{if } k \text{ has characteristic } p>0 \text{ and } n=mp^s,\ p\nmid m. \end{cases}$$

For the definition of $W_s\Omega^r_{X,\log}$, see [Illusie 1979].

Finally, if R is a k-algebra and either n is invertible in k or R is smooth over k, we define

$$\mathrm{H}^q_n(R) := \mathrm{H}^q(R_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(q-1)) \quad \text{and} \quad \mathrm{H}^q(R) := \varinjlim_n \mathrm{H}^q_n(R). \tag{1}$$

We denote by k any number field and by $\mathcal{O}_k$ its ring of integers. Let Ω_k be the set of places of k; for any finite subset $S\subset\Omega_k$, we define $\mathcal{O}_{k,S}$ as the intersection in k of the local rings $\mathcal{O}_v$ for $v\in S$. We denote by V any proper, smooth and geometrically integral variety over a number field k. Let $\mathcal{V}$ be an $\mathcal{O}_{k,S}$-scheme of finite type such that there exists an isomorphism between $\mathcal{V}\times_{\mathcal{O}_{k,S}}\mathrm{Spec}(k)$ and V (i.e., $\mathcal{V}$ is an $\mathcal{O}_{k,S}$ model for the scheme V); for any finite place $\mathfrak{p}$ not in S, we denote by $\mathcal{V}(\mathfrak{p})$ the special fibre at $\mathfrak{p}$ (i.e., the base change of $\mathcal{V}$ to $\mathrm{Spec}(k(\mathfrak{p}))$), by $\mathcal{V}_{\mathfrak{p}}$ the base change of $\mathcal{V}$ to $\mathcal{O}_{\mathfrak{p}}$, and by $V_{\mathfrak{p}}$ the base change of V to $k_{\mathfrak{p}}$.

Let p be a prime number. We denote by L any p-adic field (i.e., a finite field extension of the field of p-adic numbers $\mathbb{Q}_p$), with ring of integers $\mathcal{O}_L$ and residue field ℓ, which is a finite field of positive characteristic. We denote by X any smooth and geometrically integral variety over L and by $\mathcal{X}$ any smooth model over $\mathcal{O}_L$ with geometrically integral fibre, i.e., $\mathcal{X}$ is such that there exists an isomorphism between $\mathcal{X}\times_{\mathcal{O}_L}\mathrm{Spec}(L)$ and X. Moreover, we denote by Y the special fibre of $\mathcal{X}$, which is the base change of $\mathcal{X}$ to ℓ and is a smooth, proper and geometrically integral variety over the finite field ℓ, since smoothness and properness are stable under base change.

Outline. Section 2 is devoted to introducing the notion and some properties of the refined Swan conductor. In Section 3 we prove Theorem 1.3, and we prove a stronger version of [Bright and Newton 2023, Theorem C] for K3 surfaces. Section 4 is devoted to the explanation of several examples of K3 surfaces in which primes of good ordinary reduction are (or are not) involved in the Brauer–Manin obstruction to weak approximation. In Section 5 we give a proof of Theorem 1.4. Finally, in Section 6 we exhibit a family of examples where, depending on a parameter α, both ordinary and nonordinary reduction occur and we analyse what happens to the Brauer–Manin obstruction to weak approximation depending on the parameter α.

2. The refined Swan conductor

Let V be a smooth, proper and geometrically integral variety over a number field k. In order to study the role of a prime $\mathfrak{p}$ in the Brauer–Manin obstruction, we need to understand, for a given element $\mathcal{A} \in \mathrm{Br}(V)$, the behaviour of the corresponding evaluation map

$$\mathrm{ev}_{\mathcal{A}} : V(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}.$$

If we denote by $\mathrm{res} : \mathrm{Br}(V) \to \mathrm{Br}(V_{\mathfrak{p}})$ the natural restriction map, then, for every point $P \in V(k_{\mathfrak{p}})$, we have a corresponding point $P_{\mathfrak{p}} \in V_{\mathfrak{p}}(k_{\mathfrak{p}})$ and

$$\mathrm{ev}_{\mathrm{res}(\mathcal{A})}(P_{\mathfrak{p}}) = \mathrm{ev}_{\mathcal{A}}(P).$$

Hence, if the evaluation map attached to $\mathcal{A}$ is nonconstant on $V(k_{\mathfrak{p}})$, then also $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(V_{\mathfrak{p}})$ has nonconstant evaluation map on $V_{\mathfrak{p}}(k_{\mathfrak{p}})$.

2.1. ***p-adic setting.*** Let p be a prime number and L a finite field extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformiser π and residue field ℓ. Let X be a smooth and geometrically irreducible L-variety having good reduction (i.e., there exists a smooth, proper $\mathcal{O}_L$-scheme $\mathcal{X}$ whose generic fibre is isomorphic to X). We assume furthermore the special fibre $Y := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\ell)$ is geometrically irreducible:

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \mathcal{X} & \xleftarrow{\ i\ } & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) & \longleftarrow & \mathrm{Spec}(\ell) \end{array} \tag{2}$$

Bright and Newton [2023] define the *evaluation filtration* on the Brauer group of X by

$$\mathrm{Ev}_n \mathrm{Br}\, X := \{\mathcal{B} \in \mathrm{Br}(X) \mid \mathrm{ev}_{\mathcal{B}} \text{ is constant on } B(P, e_{L'/L}(n+1)) \text{ for all } L'/L \text{ finite and } P \in \mathcal{X}(\mathcal{O}_{L'})\} \quad (n \geq 0),$$

$$\mathrm{Ev}_{-1} \mathrm{Br}\, X := \{\mathcal{B} \in \mathrm{Br}(X) \mid \mathrm{ev}_{\mathcal{B}} \text{ is constant on } \mathcal{X}(\mathcal{O}_{L'}) \text{ for all } L'/L \text{ finite}\},$$

$$\mathrm{Ev}_{-2} \mathrm{Br}\, X := \{\mathcal{B} \in \mathrm{Br}(X) \mid \mathrm{ev}_{\mathcal{B}} \text{ is zero on } \mathcal{X}(\mathcal{O}_{L'}) \text{ for all } L'/L \text{ finite}\}.$$

For every positive integer m, we denote by $\mathrm{Ev}_n \mathrm{Br}(X)[m]$ the restriction of $\mathrm{Ev}_n \mathrm{Br}(X)$ to $\mathrm{Br}(X)[m]$, i.e., $\mathrm{Ev}_n \mathrm{Br}(X)[m] := \mathrm{Ev}_n \mathrm{Br}(X) \cap \mathrm{Br}(X)[m]$.

If $(m, p) = 1$, Colliot-Thélène and Skorobogatov [2013] and Bright [2015] have proven $\mathrm{Ev}_0 \mathrm{Br}(X)[m] = \mathrm{Br}(X)[m]$. Moreover, there exists a map ∂_m, called the *residue map*, that fits in the sequence [Colliot-Thélène and Skorobogatov 2021, Chapter 3, (3.17)]

$$0 \to \mathrm{Br}(\mathcal{X})[m] \to \mathrm{Br}(X)[m] \xrightarrow{\partial_m} H^1(Y, \mathbb{Z}/m\mathbb{Z}).$$

The residue map is such that

$$\mathrm{Ev}_{-1} \mathrm{Br}(X)[m] = \{\mathcal{A} \in \mathrm{Br}(X)[m] \mid \partial_m(\mathcal{A}) \in H^1(\ell, \mathbb{Z}/m\mathbb{Z})\},$$
$$\mathrm{Ev}_{-2} \mathrm{Br}(X)[m] = \{\mathcal{A} \in \mathrm{Br}(X)[m] \mid \partial_m(\mathcal{A}) = 0\};$$

see [Colliot-Thélène and Saito 1996] for a proof.

In order to give a description of the evaluation filtration also on the p-power torsion part of $\mathrm{Br}(X)$, Bright and Newton introduce the filtration $\{\mathrm{fil}_n \mathrm{Br}(X)\}_{n \geq 0}$ on $\mathrm{Br}(X)$, defined through the notion (introduced by Kato [1989]) of Swan conductor on discrete Henselian valuation fields; see Section 2.2. The interaction between the two filtrations is described in the theorem that follows.

Theorem 2.1. *There exists a residue map*

$$\partial : \mathrm{fil}_0 \mathrm{Br}(X) \to \varinjlim_n H^1(Y, \mathbb{Z}/n\mathbb{Z}) =: H^1(Y, \mathbb{Q}/\mathbb{Z})$$

such that:

(a) $\mathrm{fil}_0 \mathrm{Br}(X)$ *coincides with* $\mathrm{Ev}_0 \mathrm{Br}(X)$.

(b) $\mathrm{Ev}_{-1} \mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \partial(\mathcal{A}) \in H^1(\ell, \mathbb{Q}/\mathbb{Z})\}$.

(c) $\mathrm{Ev}_{-2} \mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{Br}(X) \mid \partial(\mathcal{A}) = 0\}$.

For every $n \geq 1$, *there is a map*

$$\mathrm{rsw}_{n,\pi} : \mathrm{fil}_n \mathrm{Br}(X) \to \mathrm{H}^0(Y, \Omega_Y^2) \oplus \mathrm{H}^0(Y, \Omega_Y^1),$$

called the refined Swan conductor, *such that*

$$\mathrm{Ev}_n \mathrm{Br}(X) = \{\mathcal{A} \in \mathrm{fil}_{n+1} \mathrm{Br}(X) \mid \mathrm{rsw}_{n+1,\pi}(\mathcal{A}) \in \mathrm{H}^0(Y, \Omega_Y^2) \oplus 0\}.$$

Proof. This is a reformulation of [Bright and Newton 2023, Theorem A]. □

It is clear that to understand the evaluation filtration on the Brauer group we need to understand the residue map ∂ and the refined Swan conductor maps $\mathrm{rsw}_{n,\pi}$.

2.2. ***Definition of the refined Swan conductor.*** We denote by K a Henselian discrete valuation field of characteristic zero with ring of integers $\mathcal{O}_K$ and residue field F of characteristic p. Let π be a uniformiser in $\mathcal{O}_K$ and $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_K$. Let A be a ring over $\mathcal{O}_K$, $R := A/\mathfrak{m}A$ and i, j the inclusions of the special and generic fibres into $\mathrm{Spec}(A)$:

$$\mathrm{Spec}(A \otimes_{\mathcal{O}_K} K) \xrightarrow{j} \mathrm{Spec}(A) \xleftarrow{i} \mathrm{Spec}(R).$$

We define

$$V_n^q(A) := \mathrm{H}^q((R)_{\text{ét}}, i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1))$$

and $V^q(A) := \varinjlim_n V_n^q(A)$.

Remark 2.2. The natural map in $D^b(A_{\text{ét}})$,

$$\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1) \to i_*i^*\mathrm{R}j_*\mathbb{Z}/n\mathbb{Z}(q-1),$$

induces a natural map $\mathrm{H}^q_n(A \otimes_{\mathcal{O}_K} K) \to V_n^q(A)$ for all n, q. Gabber [1994] proved that this map is an isomorphism if $(A, \mathfrak{m}A)$ is a Henselian pair. In this case, using the Kummer map $(A \otimes_{\mathcal{O}_K} K)^\times \to \mathrm{H}^1(A \otimes_{\mathcal{O}_K} K, \mathbb{Z}/n\mathbb{Z}(1))$ and the cup product, we define a product

$$V_n^q(A) \times ((A \otimes_{\mathcal{O}_K} K)^\times)^{\oplus r} \to V_n^{q+r}(A), \quad (\chi, a_1, \dots, a_r) \mapsto \{\chi, a_1, \dots, a_r\}.$$

For a general A, the isomorphism [Pagano 2024, Remark 1.2.13]

$$V_n^q(A) \simeq V_n^q(A^{(h)}) \tag{3}$$

allows us to extend the product above. From now on we identify $V_n^q(A)$ and $V_n^q(A^{(h)})$. In particular, if $A = \mathcal{O}_K[T]$, then all the polynomials of the form $1 + \pi^n p(T)$ are invertible in $A^{(h)}$; see [Stacks 2005–, Tag 0EM7] for an overview on the Henselisation of (not necessarily local) rings.

Definition 2.3. The increasing filtration $\{\mathrm{fil}_n \mathrm{H}^q(K)\}_{n\geq 0}$ on $\mathrm{H}^q(K)$ is defined by

$$\chi \in \mathrm{fil}_n \mathrm{H}^q(K) \iff \{\chi, 1+\pi^{n+1}T\} = 0 \text{ in } V^{q+1}(\mathcal{O}_K[T]).$$

We say that $\chi \in \mathrm{H}^q(K)$ has *Swan conductor* n if $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ and $\chi \notin \mathrm{fil}_{n-1} \mathrm{H}^q(K)$. By [Kato 1989, Lemma 2.2], $\mathrm{H}^q(K) = \bigcup_n \mathrm{fil}_n \mathrm{H}^q(K)$; hence, the Swan conductor is defined for every element $\chi \in \mathrm{H}^q(K)$.

In [Kato 1989; Bright and Newton 2023], certain maps $\lambda_\pi : \mathrm{H}^q_n(R) \oplus \mathrm{H}^{q-1}_n(R) \to V_n^q(A)$ are defined under additional assumptions on the $\mathcal{O}_K$-algebra A. We provide an overview of these maps, which, with a slight abuse of notation, we will always refer to as λ_π.

Remark 2.4.

- Kato [1989, Section 1.4] defines for every n an injective map

$$\lambda_\pi : \mathrm{H}^q_n(F) \oplus \mathrm{H}^{q-1}_n(F) \to \mathrm{H}^q_n(K).$$

  This collection of maps induces an injective map

$$\lambda_\pi : \mathrm{H}^q(F) \oplus \mathrm{H}^{q-1}(F) \to \mathrm{H}^q(K).$$

- Kato [1989, Section 1.9] extends the definition of λ_π to any smooth $\mathcal{O}_K$-algebra A. In particular, he defines a map

$$\lambda_\pi : \mathrm{H}^q_p(R) \oplus \mathrm{H}^{q-1}_p(R) \to V_p^q(A).$$

- Finally, Bright and Newton [2023, Section 2.2] generalise the previous map by defining

$$\lambda_\pi : \mathrm{H}^q_{p^r}(R) \oplus \mathrm{H}^{q-1}_{p^r}(R) \to V^q_{p^r}(A)$$

for any $r \geq 1$.

Kato [1989, Proposition 6.1] proves that the image of

$$\lambda_\pi : \mathrm{H}^q_n(F) \oplus \mathrm{H}^{q-1}_n(F) \to \mathrm{H}^q_n(K)$$

coincides with $\mathrm{fil}_0 \mathrm{H}^q_n(K)$, and also [ibid., Section 1.3] that, for every $r \geq 1$, there is a surjection δ_r from $W_r\Omega^q_R$ to $\mathrm{H}^q_{p^r}(R)$. Following [Kato 1989; Bright and Newton 2023], we sometimes use λ_π also to denote the composition

$$W_r\Omega^q_R \oplus W_r\Omega^{q-1}_R \xrightarrow{\delta_r} V^q_{p^r}(R) \oplus V^{q-1}_{p^r}(R) \xrightarrow{\lambda_\pi} V^q_{p^r}(A). \tag{4}$$

The following theorem allows Kato to define the refined Swan conductor of $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$.

Theorem 2.5. *Let $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ with $n \geq 1$; then there exists a unique pair (α, β) in $\Omega^q_F \oplus \Omega^{q-1}_F$ such that*

$$\{\chi, 1+\pi^n T\} = \lambda_\pi(T\alpha, T\beta) \quad \text{in } V^{q+1}_p(\mathcal{O}_K[T]). \tag{5}$$

Proof. See [Kato 1989, Section 5]. □

The uniqueness of the pair (α, β) in the previous theorem implies, for every $n \geq 1$, the existence of a homomorphism

$$\mathrm{rsw}_{n,\pi} : \mathrm{fil}_n \mathrm{H}^q(K) \to \Omega^q_F \oplus \Omega^{q-1}_F$$

whose kernel is $\mathrm{fil}_{n-1} \mathrm{H}^q(K)$. The pair (α, β) is called the *refined Swan conductor* of $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$.

We end this section with the definition of the residue map ∂ from $\mathrm{fil}_0 \mathrm{H}^q(K)$ to $\mathrm{H}^{q-1}(F)$ (see [Bright and Newton 2023, Section 2.5; Kato 1989, Section 7.5]).

Definition 2.6. The *residue map*

$$\partial : \mathrm{fil}_0 \mathrm{H}^q(K) \to \mathrm{H}^{q-1}(F)$$

is defined as the projection on the second component of the inverse of the isomorphism λ_π from $\mathrm{H}^q(F) \oplus \mathrm{H}^{q-1}(F)$ to $\mathrm{fil}_0 \mathrm{H}^q(K)$.

2.3. ***The image of the refined Swan conductor.*** To further describe the refined Swan conductor maps, we need to recall some properties of differential forms over fields of positive characteristic.

Let ℓ be a perfect field of characteristic $p > 0$ and R be an ℓ-algebra. Let

$$Z^q_R := \ker(d : \Omega^q_R \to \Omega^{q+1}_R) \quad \text{and} \quad B^q_R := \mathrm{im}(d : \Omega^{q-1}_R \to \Omega^q_R).$$

Lemma 2.7 (inverse Cartier operator). *Assume R to be regular; then there exists a unique homomorphism of groups*

$$C^{-1}_R : \Omega^1_R \to \Omega^1_R / B^1_R$$

satisfying

- $C^{-1}_R(da) = a^{p-1}da \mod B^1_R$ *for all* $a \in R$;

- $C_R^{-1}(\lambda\omega) = \lambda^p C_R^{-1}(\omega)$ *for all* $\lambda \in R$;
- $d \circ C_R^{-1} = 0$.

Moreover, C^{-1} *induces an isomorphism from* Ω_R^1 *to* Z_R^1/B_R^1.

Proof. See [Brion and Kumar 2005, Theorem 1.3.4]. □

Remark 2.8. The subgroup B_R^1 has a natural structure of an R-module, which is given by $\alpha \cdot d\beta = \alpha^p d\beta = d(\alpha^p \beta)$. If we denote by ${}^p\Omega_R^1$ the R-module structure on Ω_R^1 given by $\alpha \cdot \omega = \alpha^p \omega$, then the condition $C_R^{-1}(\lambda\omega) = \lambda^p C_F^{-1}(\omega)$ is equivalent to asking that $C_R^{-1} : \Omega_R^1 \to {}^p\Omega_R^1/{}^pB_R^1$ be a morphism of R-modules.

We can extend the definition of C_R^{-1} to higher differential forms by setting

$$C_R^{-1}(\omega_1 \wedge \cdots \wedge \omega_q) := C_R^{-1}(\omega_1) \wedge \cdots \wedge C_R^{-1}(\omega_q).$$

Theorem 2.9. *Let* R *be regular*; *then the morphism*

$$C_R^{-1} : \Omega_R^q \to Z_R^q/B_R^q$$

is an isomorphism for all $q \geq 0$. *We denote by* C_R *its inverse*, *which is called the Cartier operator.*

Proof. See [Brion and Kumar 2005, Theorem 1.3.4]. □

The following corollary gives a way to characterise exact differential forms in terms of the Cartier operator.

Corollary 2.10. *Let* R *be regular. A* q*-form* $\omega \in \Omega_R^q$ *is exact if and only if* $d(\omega) = 0$ *and* $C_R(\omega) = 0$.

The last object we need to define is the subgroup of logarithmic q-differential forms on R, which will play a crucial role in this paper.

Definition 2.11. The *logarithmic* q-differential forms on R, denoted by $\Omega_{R,\log}^q$, are defined as the kernel of the map

$$C_R^{-1} - \mathrm{id} : \Omega_R^q \to \Omega_R^q/B_R^q.$$

We have the following explicit description of logarithmic forms when R is a field.

Theorem 2.12. *Let* F *be a field*, *finitely generated over a perfect field* ℓ. *The logarithmic differential* q*-forms* $\Omega_{F,\log}^q$ *is the subgroup of* Ω_F^q *generated by elements of the form*

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \quad \text{with } y_i \in F^\times.$$

Proof. This follows from the surjectivity in the Bloch–Gabber–Kato theorem [Gille and Szamuely 2017, Theorem 9.5.2]. □

If R is smooth over a field of positive characteristic, we can identify $\mathrm{H}^q_p(R) = \mathrm{H}^1(R_{\text{ét}}, \Omega^{q-1}_{R,\log})$ with the cokernel of

$$C_R^{-1} - 1 : \Omega_R^{q-1} \to \Omega_R^{q-1}/B_R^{q-2};$$

see [Kato 1989, Section 1.3]. We denote by δ_1 both the map from Ω_R^{q-1}/B_R^{q-2} to $\mathrm{H}^q_p(R)$ and its composition with the natural map $\mathrm{H}^q_p(R) \to \mathrm{H}^q(R)$.

Let $e := \mathrm{ord}_K(p)$ be the absolute ramification index of K and $e' := ep(p-1)^{-1}$.

Lemma 2.13. *Let χ be an element in $\mathrm{fil}_n \mathrm{H}^q(K)$ with*

$$\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta) \in \Omega^2_F \oplus \Omega^1_F.$$

Then $d\alpha = 0$ and $d\beta = (-1)^q n\alpha$.

Proof. See [Bright and Newton 2023, Lemma 2.17]. □

Remark 2.14. (1) If $p \mid n$, then $d\alpha = 0$ and $d\beta = 0$, meaning that $(\alpha, \beta) \in Z^q_F \oplus Z^{q-1}_F$.

(2) If $p \nmid n$, then $\alpha = \bar{n}^{-1} d\beta$, with $\bar{n}$ the residue of n modulo π. In particular, the composition

$$\mathrm{fil}_n \mathrm{H}^q(K) \xrightarrow{\mathrm{rsw}_{n,\pi}} \Omega^q_F \oplus \Omega^{q-1}_F \xrightarrow{\mathrm{pr}_2} \Omega^{q-1}_F$$

also has kernel equal to $\mathrm{fil}_{n-1} \mathrm{H}^q(K)$.

Bright and Newton [2023, Lemma 2.19] link the refined Swan conductor of $\chi \in \mathrm{fil}_n \mathrm{H}^q(K)$ to that of $p \cdot \chi$ whenever $n \geq e'$. More precisely, assume that $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$ and let $\bar{u}$ be the reduction modulo π of $p \cdot \pi^{-e}$; then $p \cdot \chi \in \mathrm{fil}_{n-e} \mathrm{H}^q(K)$ and

$$\mathrm{rsw}_{n-e,\pi}(p \cdot \chi) = \begin{cases} (\bar{u}\alpha, \bar{u}\beta) & \text{if } n > e', \\ (\bar{u}\alpha + C(\alpha), \bar{u}\beta + C(\beta)) & \text{if } n = e'. \end{cases} \tag{6}$$

We prove a result analogous to the one proven by Bright and Newton for elements $\chi \in \mathrm{fil}_{np} \mathrm{H}^q(K)$ when $np < e'$.

Lemma 2.15. *Let $\chi \in \mathrm{fil}_{np} \mathrm{H}^q(K)$, with $np < e'$. Then $p \cdot \chi \in \mathrm{fil}_n \mathrm{Br}(K)$ and, if $\mathrm{rsw}_{np,\pi}(\mathcal{A}) = (\alpha, \beta)$, then $d\alpha = 0$, $d\beta = 0$ and*

$$\mathrm{rsw}_{n,\pi}(p \cdot \chi) = (C(\alpha), C(\beta)).$$

Proof. From Remark 2.14(1), we know that $(\alpha, \beta) \in Z^q_F \oplus Z^{q-1}_F$, since clearly $p \mid np$. The condition $e' > np$ implies

$$e - np + n > 0. \tag{7}$$

Let $u \in \mathcal{O}_K^\times$ be such that $p = u \cdot \pi^e$. We have

$$\{p \cdot \chi, 1 + \pi^{n+1} T\} = \{\chi, (1 + \pi^{n+1} T)^p\} = \{\chi, 1 + \pi^{np+1} b(T)\},$$

where

$$b(T) = \frac{(1 + \pi^{n+1} T)^p - 1}{\pi^{np+1}}.$$

We can rewrite $b(T)$ as

$$\sum_{k=1}^{p-1} \pi^{e+(n+1)k-(np+1)} a_k T^k + \pi^{p(n+1)-(np+1)} T^p \quad \text{for some } a_k \in \mathbb{Z}.$$

For every $1 \le k \le p$, the inequality (7) gives

$$e + (n+1)k - np - 1 = (e - np) + (n+1)k - 1 \ge 0.$$

Therefore, $b(T) \in \mathcal{O}_K[T]$. Now, since by assumption $\chi \in \mathrm{fil}_{np} \mathrm{H}^q(K)$, we have $\{\chi, 1 + \pi^{np+1} b(T)\} = 0$ for all $b(T) \in \mathcal{O}_K[T]$; thus, $p \cdot \chi \in \mathrm{fil}_n \mathrm{H}^q(K)$. In a similar way,

$$\{p \cdot \chi, 1 + \pi^n T\} = \{\mathcal{A}, (1 + \pi^n T)^p\} = \{\chi, 1 + \pi^{np} c(T)\},$$

where

$$c(T) = \frac{(1 + \pi^n T)^p - 1}{\pi^{np}}.$$

We can rewrite $c(T)$ as

$$\sum_{k=1}^{p-1} \pi^{e+nk-np} a_k T^k + T^p \quad \text{for some } a_k \in \mathbb{Z}.$$

In this case, again, the inequality (7) together with $1 \le k \le p-1$ implies $e + nk - np > 0$. Therefore, $c(T) \in \mathcal{O}_K[T]$ and its reduction modulo π is equal to T^p. It follows from [Kato 1989, (6.3.1)] that

$$\{\chi, 1 + \pi^{np} c(T)\} = \lambda_\pi(\bar{c}(T)\alpha, \bar{c}(T)\beta) = \lambda_\pi(T^p \alpha, T^p \beta) = \lambda_\pi(TC(\alpha), TC(\beta)),$$

where the last equality follows from [Bright and Newton 2023, Lemma 2.18(2)]. □

Remark 2.16. For every nonnegative integer d, we denote by $\mathrm{fil}_n \mathrm{H}^q_d(K)$ the intersection of $\mathrm{fil}_n \mathrm{H}^q(K)$ with $\mathrm{H}^q_d(K)$. Kato [1989, Proposition 6.1] proves that, for every nonnegative integer d prime to p, $\mathrm{H}^q_d(K) = \mathrm{fil}_0 \mathrm{H}^q_d(K)$. We now prove some useful properties of the refined Swan conductor on p-power order elements:[1]

(1) The filtration $\mathrm{fil}_n \mathrm{H}^q_{p^m}(K)$ is finite.

Assume first e' to be an integer, i.e., $(p-1) \mid e$. For $m = 1$, Kato [1989, Proposition 4.1] proves that $\mathrm{H}^q_p(K) = \mathrm{fil}_{e'} \mathrm{H}^q_p(K)$. This is equivalent to saying that, for all $n > e'$ and $\chi \in \mathrm{fil}_n \mathrm{H}^q_p(K)$, we have $\mathrm{rsw}_{n,\pi}(\chi) = (0, 0)$. Assume that $m > 1$, $\chi \in \mathrm{fil}_n \mathrm{H}^q_{p^m}(K)$ with $n > e' + (m-1)e$, and $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha, \beta)$. From (6) we know that $\mathrm{rsw}_{n-e,\pi}(p \cdot \chi) = (\bar{u}\alpha, \bar{u}\beta)$; hence, working by induction on m, we get $\mathrm{rsw}_{n,\pi}(\chi) = (0, 0)$. This result is essentially already proven in [Ieronymou 2022, Proposition 17].

If e' is not an integer, then we take the field extension $K(\zeta)/K$, with ζ a primitive p-root of unity ζ. We denote by e_K and $e_{K(\zeta)}$ the absolute ramification indices of K and $K(\zeta)$, respectively, and by e'_K and $e'_{K(\zeta)}$ the products $e_K \cdot p \cdot (p-1)^{-1}$ and $e_{K(\zeta)} \cdot p \cdot (p-1)^{-1}$, respectively. Let $n > e'_K + (m-1) \cdot e_K$ and $\chi \in \mathrm{fil}_n \mathrm{H}^q_{p^m}(K)$; then

$$e_{K(\zeta)/K} \cdot n > e'_{K(\zeta)} + (m-1) e_{K(\zeta)}.$$

[1] Parts (2) and (3) are already mentioned by Kato [1989, Sections 4 and 5].

Since $(p-1) \mid e'_{K(\zeta)}$, we get (from what we said above) that

$$\mathrm{rsw}_{e_{K(\zeta)/K}\cdot n,\pi}(\mathrm{res}(\chi)) = (0,0),$$

where res is the natural map from $\mathrm{H}^q_{p^m}(K) \to \mathrm{H}^q_{p^m}(K(\zeta))$. It follows from [Bright and Newton 2023, Lemma 2.16] that

$$\mathrm{rsw}_{e_{K(\zeta)/K}\cdot n,\pi}(\mathrm{res}(\chi)) = (\bar{a}^{-n}(\alpha + \beta \wedge d\log \bar{a}), \bar{a}^{-n} e_{K(\zeta)/K}\beta)$$

with $\bar{a}$ invertible in the residue field of $K(\zeta)$ (see Section 2.6). Hence, since $p \nmid e_{K(\zeta)/K}$, we can conclude that $\mathrm{rsw}_{n,\pi}(\chi) = (0,0)$.

(2) Let $\chi \in \mathrm{fil}_{np}\,\mathrm{H}^q_p(K)$ with $np < e'$ and $\mathrm{rsw}_{np,\pi}(\chi) = (\alpha,\beta)$. We know from Lemma 2.15 that $(\alpha,\beta) \in Z^q_F \oplus Z^{q-1}_F$; since χ has order p,

$$(C(\alpha), C(\beta)) = \mathrm{rsw}_{n,\pi}(p\cdot\chi) = (0,0).$$

Equivalently, Corollary 2.10 implies that, for $np < e'$, the refined Swan conductor on the p-torsion takes image in $B^q_F \oplus B^{q-1}_F$.

(3) Assume e' to be an integer, $\chi \in \mathrm{fil}_{e'}\,\mathrm{H}^q_p(K)$ and $\mathrm{rsw}_{e',\pi}(\chi) = (\alpha,\beta)$. By (6),

$$-\bar{u}\alpha = C(\alpha) \quad \text{and} \quad -\bar{u}\beta = C(\beta).$$

Let ζ be a primitive p-root of unity and $c = (\zeta-1)^p \pi^{-e'}$; then $c = (c_1)^p$ with $c_1 = (\zeta-1)\pi^{-e/(p-1)}$. By the properties of the Cartier operator, we have

$$C(\bar{c}\alpha) = \bar{c}_1 C(\alpha) = -\bar{u}\bar{c}_1\alpha.$$

Since $(\zeta-1)^{p-1} \equiv -p \bmod \pi^{e+1}$,[2] we have $(\zeta-1)^{p-1} = -p + \pi^{e+1}v$ for some $v \in \mathcal{O}_K$. Hence,

$$c = \frac{\zeta-1}{\pi^{e/(p-1)}} \cdot \frac{(\zeta-1)^{p-1}}{\pi^e} = c_1 \cdot \left(\frac{-p+\pi^{e+1}v}{\pi^e}\right) = c_1 \cdot (-u + \pi v).$$

Thus, $\bar{c} = -\bar{u}\bar{c}_1$ and

$$\mathrm{mult}_{\bar{c}} \circ \mathrm{rsw}_{e',\pi}(\mathrm{H}^q_p(K)) \subseteq \Omega^q_{F,\log} \oplus \Omega^{q-1}_{F,\log},$$

where $\mathrm{mult}_{\bar{c}}$ is the map from Ω^q_F to Ω^q_F sending a q-form ω to $\bar{c}\cdot\omega$.

2.4. ***Swan conductor on*** $\mathbf{Br}(X)$**.** Let K^h be the field of fractions of the Henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},Y}$. Note that

$$\mathrm{H}^2(K^h) = \varinjlim_n \mathrm{H}^2(K^h_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(1)) = \mathrm{Br}(K^h).$$

Bright and Newton [2023] define the filtration $\{\mathrm{fil}_n\,\mathrm{Br}(X)\}_{n\geq 0}$ on $\mathrm{Br}(X)$ as the pullback via the natural map $\mathrm{Br}(X) \to \mathrm{Br}(K^h)$ of the filtration $\{\mathrm{fil}_n\,\mathrm{Br}(K^h)\}_{n\geq 0}$ on $\mathrm{Br}(K^h)$. Therefore, it is possible to extend the

[2] In fact, $1 = ((\zeta-1)+1)^p = (\zeta-1)^p + p(\zeta-1)^{p-1} + \binom{p}{2}(\zeta-1)^{p-2} + \cdots + p\cdot(\zeta-1) + 1$. Since $\zeta \neq 1$, we get $(\zeta-1)^{p-1} + p(\zeta-1)^{p-2} + \binom{p}{2}(\zeta-1)^{p-3} + \cdots + p = 0$. Now the congruence follows from the fact that $\mathrm{val}((\zeta-1)\cdot p) \geq 1+e$.

definitions of the residue map and the refined Swan conductor to elements in $\mathrm{Br}(X)$ simply as the residue map and refined Swan conductor of the image of $\mathcal{A}$ in $\mathrm{Br}(K^h)$. A priori, these maps take values in $\mathrm{H}^1(F)$ and $\Omega^2_F \oplus \Omega^1_F$, respectively. However, Bright and Newton prove that the residue map of an element in $\mathrm{fil}_0 \mathrm{Br}(X)$ takes values in $\mathrm{H}^1(Y, \mathbb{Q}/\mathbb{Z}) \subseteq \mathrm{H}^1(F)$ (see [ibid., Proposition 3.1(1)]) and that the refined Swan conductor on $\mathrm{fil}_n \mathrm{Br}(X)$ takes image in $\mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \subseteq \Omega^2_F \oplus \Omega^1_F$ (a proof following that of [Kato 1989, Theorem 7.1] can be found in [Bright and Newton 2023, Theorem B]).

The aim of this subsection is to transfer the results we got in Section 2.3 for the refined Swan conductor on $\mathrm{Br}(K^h)$ to the refined Swan conductor on $\mathrm{Br}(X)$.

It is possible to generalise the definition of inverse Cartier operator on a smooth and proper variety Y defined over a perfect field ℓ of positive characteristic. Following [Illusie 1979], we denote by F_Y the absolute Frobenius endomorphism of Y and by $Y^{(p)}$ the base change of Y via the absolute Frobenius σ_ℓ of the base field ℓ, namely

$$\begin{array}{ccccc} Y & \xrightarrow{F_{Y/\ell}} & Y^{(p)} & \xrightarrow{W} & Y \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\ell) & \xrightarrow{\sigma_\ell} & \mathrm{Spec}(\ell) \end{array}$$

where $W \circ F_{Y/\ell} = F_Y$; we call $F_{Y/\ell}$ the relative Frobenius of Y over ℓ and denote by $\Omega^\bullet_{Y/\ell}$ the de Rham complex of Y/ℓ.

Remark 2.17. Illusie [1979] works more generally with S-schemes, where the base S is a scheme of positive characteristic. Here, we are only interested in varieties over perfect fields, and in this case the absolute de Rham complex $\Omega^\bullet_Y$ coincides with the relative de Rham complex $\Omega^\bullet_{Y/\ell}$ (since ℓ perfect implies $\Omega^q_\ell = 0$).

For every $q \geq 0$, we define

$$Z^q_Y := \ker(d : \Omega^q_Y \to \Omega^{q+1}_Y) \quad \text{and} \quad B^q_Y := \mathrm{im}(d : \Omega^{q-1}_Y \to \Omega^q_Y).$$

The differentials $d : \Omega^q_Y \to \Omega^{q+1}_Y$ are $\mathcal{O}_{Y^{(p)}}$-linear; hence, the sheaves

$$(F_{Y/\ell})_* Z^q_Y \quad \text{and} \quad (F_{Y/\ell})_* B^q_Y$$

are $\mathcal{O}_{Y^{(p)}}$-modules and the sheaf of abelian groups $\mathcal{H}^q((F_{Y/\ell})_* \Omega^\bullet_Y)$ is also a sheaf of $\mathcal{O}_{Y^{(p)}}$-modules.

Definition 2.18. For every $q \geq 0$, there is a morphism of $\mathcal{O}_Y$-modules, called the *inverse Cartier operator*,

$$C^{-1}_Y : \Omega^q_Y \to W_* \mathcal{H}^q((F_{Y/\ell})_* \Omega^\bullet_Y).$$

Remark 2.19. Since $\Omega^q_{Y^{(p)}} = W^* \Omega^q_Y$, by adjunction we get a morphism of $\mathcal{O}_{Y^{(p)}}$-modules

$$C^{-1}_{Y/\ell} : \Omega^q_{Y^{(p)}} \to \mathcal{H}^q((F_{Y/\ell})_* \Omega^\bullet_Y).$$

Theorem 2.20. *If Y is a smooth variety over ℓ, then $C^{-1}_{Y/\ell}$ is an isomorphism of $\mathcal{O}_{Y^{(p)}}$-modules.*

Proof. See [Illusie 1979, Theorem 0.2.1.9]. □

From now on we assume Y to be smooth and proper over ℓ. In this case, we denote by $C_{Y/\ell}$ the inverse of $C_{Y/\ell}^{-1}$.

Definition 2.21. For every nonnegative integer q, define

$$\Omega^q_{Y,\log} := \ker(W^* - C_{Y/\ell} : Z^q_Y \to \Omega^q_{Y^{(p)}}).$$

The sheaf $\Omega^q_{Y,\log}$ is called the sheaf of *logarithmic* q-forms on Y.

Theorem 2.22. *The sheaf $\Omega^q_{Y,\log}$ is the subsheaf of Ω^q_Y generated étale-locally by the logarithmic differentials, i.e., the sections of the form*

$$\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \quad \text{with } y_i \in \mathcal{O}_Y^*.$$

Proof. See [Illusie 1979, Theorem 0.2.4.2]. □

The Cartier operator induces an exact sequence

$$0 \to \mathrm{H}^0(Y, B^q_Y) \to \mathrm{H}^0(Y, Z^q_Y) \xrightarrow{C_Y} \mathrm{H}^0(Y, \Omega^q_Y).$$

We have the following lemma.

Lemma 2.23. *Let $\omega \in \mathrm{H}^0(Y, \Omega^q_Y)$; then*:

(1) *$\omega \in \mathrm{H}^0(Y, \Omega^q_{Y,\log})$ if and only if the image of ω in Ω^q_F lies in $\Omega^q_{F,\log}$.*

(2) *$\omega \in \mathrm{H}^0(Y, B^q_Y)$ if and only if the image of ω in Ω^q_F lies in B^q_F.*

Proof. The group $\Omega^q_{F,\log}$ can also be computed as the kernel of

$$\mathrm{id} - C : Z^q_F \to \Omega^q_F;$$

see [Milne 1976, Lemma 1.5]. Moreover, the natural inclusion of $\mathrm{H}^0(Y, \Omega^q_Y)$ in Ω^q_F is compatible with the differential maps and with the Cartier operator. The proof of (1) follows from the definition of logarithmic forms, while (2) is an immediate consequence of Corollary 2.10. □

Corollary 2.24. *Let $u = p\pi^{-e} \in \mathcal{O}_L^\times$; then*:

(1) *If $p \nmid n$, then*

$$\mathrm{pr}_2 \circ \mathrm{rsw}_{n,\pi} : \mathrm{fil}_n \mathrm{Br}(X) \to \mathrm{H}^0(Y, \Omega^1_Y)$$

has kernel equal to $\mathrm{fil}_{n-1} \mathrm{Br}(X)$.

(2) *If $p \mid n$ and $n < e'$, write $n = mp$; then (α, β) lies in $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ and the diagram*

$$\begin{array}{ccc} \mathrm{fil}_n \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\ {\scriptstyle (-)^{\otimes p}}\downarrow & & \downarrow{\scriptstyle C} \\ \mathrm{fil}_m \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{m,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \end{array}$$

commutes.

(3) *If $n = e'$, then (α, β) lies in $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ and the diagram*

$$\begin{array}{ccc} \mathrm{fil}_{e'} \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\ {\scriptstyle (-)^{\otimes p}}\downarrow & & \downarrow{\scriptstyle \mathrm{mult}_{\bar{u}}+C} \\ \mathrm{fil}_{e'-e} \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n-e,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \end{array}$$

commutes. Moreover, let $\bar{c} \in \ell^\times$ be the reduction of $c = (\zeta - 1)^p \pi^{-e'} \in \mathcal{O}_L^\times$; then $\bar{c}$ is such that

$$\mathrm{mult}_{\bar{c}}(\mathrm{rsw}_{e',\pi} \mathrm{fil}_{e'} \mathrm{Br}(X)[p]) \subseteq \mathrm{H}^0(Y, \Omega^2_{Y,\log}) \oplus \mathrm{H}^0(Y, \Omega^1_{Y,\log}).$$

(4) *If $p \mid n$ and $n > e'$, then (α, β) lies in $\mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y)$ and the diagram*

$$\begin{array}{ccc} \mathrm{fil}_{n} \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, Z^2_Y) \oplus \mathrm{H}^0(Y, Z^1_Y) \\ {\scriptstyle (-)^{\otimes p}}\downarrow & & \downarrow{\scriptstyle \mathrm{mult}_{\bar{u}}} \\ \mathrm{fil}_{n-e} \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n-e,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \end{array}$$

commutes.

Proof. All these are consequences of Section 2.3. More precisely: (1) follows from Remark 2.14(2); (2) follows from Lemma 2.15; (3) and (4) are a direct consequence of [Bright and Newton 2023, Lemma 2.19]. □

Remark 2.25. For $\mathcal{F} = \Omega^q_Y$, Z^q_Y or B^q_F, the cohomology group $\mathrm{H}^q(Y, \mathcal{F})$ in the Zariski or small étale site are isomorphic, and hence we will not specify over which site we are computing them.

2.5. ***Refined Swan conductor on cyclic algebras.*** This subsection is needed only to understand the computations in the examples appearing in this paper, it is not used to prove the main theorems.

Let K be a Henselian field of characteristic 0 with ring of integers $\mathcal{O}_K$, uniformiser π and residue field F of positive characteristic p. Here we assume the field K to contain a primitive p-root of unity ζ. It follows from the Merkurjev–Suslin theorem [Gille and Szamuely 2017, Theorem 8.6.5] that, in this case, $\mathrm{Br}(K)[p]$ is generated by the classes of *cyclic algebras*. The study of the p-primary part of the Galois cohomology of a complete discrete valuation field K in terms of its residue field F, which is not necessarily perfect, goes back to [Kato 1982] (see [Colliot-Thélène 1999] for a survey on the topic).

Definition 2.26. Let $a, b \in K^\times$; then the K-algebra

$$(x, y)_p := \langle a, b \mid a^p = x,\ b^p = y,\ ab = \zeta ba \rangle$$

is a central simple algebra; see [Gille and Szamuely 2017, Section 2.5]. With an abuse of notation, we also denote by $(x, y)_p$ the corresponding equivalence class in $\mathrm{Br}(K)[p]$.

It is also possible to realise $(x, y)_p$ as the cup product of an element $\chi_x \in \mathrm{H}^1_p(K)$ with $\delta(y)$, where δ is the boundary map $K^\times \to \mathrm{H}^1(K, \mathbb{Z}/p\mathbb{Z}(1))$ coming from the Kummer sequence; see the proof of [ibid., Proposition 4.7.1].

2.5.1. *Bloch–Kato filtration.* The map from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}/p\mathbb{Z}(1)$ sending 1 to ζ induces an isomorphism

$$\mathrm{Br}(K)[p] \simeq \mathrm{H}^2(K, \mathbb{Z}/p\mathbb{Z}(2)) =: h^2(K). \tag{8}$$

For any two nonzero elements $x, y \in K$, we denote by $\{x, y\} \in h^2(K)$ the cup product of $\delta(x)$ with $\delta(y)$.

Bloch and Kato [1986] define a decreasing filtration $\{U^m h^2(K)\}_{m\geq 0}$ on $h^2(K)$ as follows: $U^0 h^2(K) = h^2(K)$ and, for $m \geq 1$, $U^m h^2(K)$ is the subgroup of $h^2(K)$ generated by symbols of the form

$$\{1 + \pi^m x, y\} \quad \text{for } x \in \mathcal{O}_K \text{ and } y \in K^\times.$$

They describe the graded factors of the filtration

$$\mathrm{gr}^m := \frac{U^m h^2(K^h)}{U^{m+1} h^2(K^h)}$$

in terms of differential forms on the residue field F.

Proposition 2.27. *We have the following description of the graded factors* gr^m:

(1) $U^m h^2(K^h) = \{0\}$ *for* $m > e'$, *and* $U^{e'} h^2(K)$ *coincides with the image of the injective map*

$$\begin{aligned} \lambda_\pi : \mathrm{H}^2_p(F) \oplus \mathrm{H}^1_p(F) &\to \mathrm{gr}^{e'} = h^2(K), \\ \delta_1[\bar{x} \cdot d\log \bar{y}] &\mapsto \{1 + (\zeta - 1)^p x, y\}, \\ \delta_1[\bar{x}] &\mapsto \{1 + (\zeta - 1)^p x, \pi\}, \end{aligned}$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K *and* $\delta_1[\omega]$ *as in* (4).

(2) *Let* $0 < m < e'$ *and* $p \nmid m$. *Then we have an isomorphism*

$$\rho_m : \Omega^1_F \xrightarrow{\simeq} \mathrm{gr}^m, \quad \bar{x} \cdot d\log \bar{y} \mapsto \{1 + \pi^m x, y\},$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K.

(3) *Let* $0 < m < e'$ *with* $p \mid m$. *Then we have an isomorphism*

$$\rho_m : \Omega^1_F / Z^1_F \oplus \Omega^0_F / Z^0_F \xrightarrow{\simeq} \mathrm{gr}^m, \qquad ([\bar{x} \cdot d\log \bar{y}], 0) \mapsto \{1 + \pi^m x, y\}, \quad (0, [\bar{x}]) \mapsto \{1 + \pi^m x, \pi\},$$

where x *and* y *are any lifts of* $\bar{x}$ *and* $\bar{y}$ *to* K, *and* $[\omega]$ *is the image of* $\omega \in \Omega^i_F$ *in* Ω^i_F / Z^i_F *for* $i = 0, 1$.

(4) *We have an isomorphism*

$$\rho_0 : \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \xrightarrow{\simeq} \mathrm{gr}^0, \qquad (d\log \bar{y}_1 \wedge d\log \bar{y}_2, 0) \mapsto \{y_1, y_2\}, \quad (0, d\log \bar{y}) \mapsto \{y, \pi\},$$

where y, y_1 *and* y_2 *are any lifts of* $\bar{y}$, $\bar{y}_1$ *and* $\bar{y}_2$ *to* K.

Proof. See [Bloch and Kato 1986, Section 5]. More precisely, [ibid., Lemma 5.1] proves (1). However, instead of λ_π (see Remark 2.4) they have the morphism $\rho_{e'}$, and they then prove [ibid., equation (5.15.1)] that $\rho_{e'}$ induces the map λ_π. The rest is in [ibid., Lemmas 5.2 and 5.3]. □

Note that since, by assumption, K contains a primitive p-root of unity ζ, the ramification index $e = \mathrm{ord}_K(p)$ is divisible by $p-1$, and hence $e' = ep(p-1)^{-1}$ is divisible by p. The key result of this section is the following proposition, which allows us to use Bloch–Kato filtration to compute the refined Swan conductor.

Proposition 2.28. *For every $0 \le m \le e'$, the isomorphism (8) induces an isomorphism*

$$\mathrm{fil}_{e'-m}\,\mathrm{Br}(K)[p] \simeq U^m h^2(K).$$

Let $\bar{c}$ be the reduction modulo π of $c := \pi^{-e'}(\zeta-1)^p$. For $m < e'$, the compositions $\mathrm{rsw}_{e'-m} \circ \rho_m$ are

$$\begin{aligned}
\mathrm{rsw}_{e',\pi}(\rho_0(\alpha,\beta)) &= (\bar{c}\alpha, \bar{c}\beta),\\
\mathrm{rsw}_{e'-m,\pi}(\rho_m(\beta)) &= (\bar{c}d\beta, (e'-m)\bar{c}\beta) \quad \text{if } p \nmid m,\\
\mathrm{rsw}_{e'-m,\pi}(\rho_m(\alpha,\beta)) &= (\bar{c}d\alpha, \bar{c}d\beta) \quad \text{if } p \mid m,
\end{aligned}$$

where ρ_m is the map defined in the previous proposition.

Proposition 2.28 is a rephrasing of [Kato 1989, Lemma 4.3].[3]

2.6. ***Refined Swan conductor and extension of the base field.*** In this section we analyse the behaviour of the refined Swan conductor under field extension of the base field. Bright and Newton [2023] prove the following result.

Lemma 2.29. *Let K'/K be a finite extension of Henselian discrete valuation fields of ramification index e. Let π' be a uniformiser in K', F' be the residue field of K', and define $\bar{a} \in F'$ to be the reduction of $\pi(\pi')^{-e}$. Let $\chi \in \mathrm{fil}_n \mathrm{Br}(K)$, and let*

$$\mathrm{res} : \mathrm{Br}(K) \to \mathrm{Br}(K')$$

be the restriction map. Then $\mathrm{res}(\chi) \in \mathrm{fil}_{en}\mathrm{Br}(K')$ and, if $\mathrm{rsw}_{n,\pi}(\chi) = (\alpha,\beta)$, then

$$\mathrm{rsw}_{en,\pi'}(\mathrm{res}(\chi)) = \big(\bar{a}^{-n}(\alpha + \beta \wedge d\log(\bar{a})), \bar{a}^{-n}e\beta\big).$$

Proof. See [Bright and Newton 2023, Lemma 2.7]. □

We return to the general setting of Section 2.1. The aim of this subsection is to prove the following lemma.

Lemma 2.30. *Let L'/L be a finite field extension, with ramification index $e_{L'/L}$. Let π' be a uniformiser in L' and ℓ' its residue field. Let $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$ and let*

$$\mathrm{res} : \mathrm{Br}(X) \to \mathrm{Br}(X_{L'})$$

be the restriction map. Then $\mathrm{res}(\mathcal{A}) \in \mathrm{fil}_{e_{L'/L}n}\mathrm{Br}(X_{L'})$ and, if $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha,\beta) \in \mathrm{H}^0(Y,\Omega^2_Y) \oplus \mathrm{H}^0(Y,\Omega^1_Y)$, then

$$\mathrm{rsw}_{e_{L'/L}n,\pi'}(\mathrm{res}(\mathcal{A})) = (\bar{a}^{-n}\alpha, \bar{a}^{-n}e_{L'/L}\beta) \in \mathrm{H}^0(Y_{\ell'},\Omega^2_{Y_{\ell'}}) \oplus \mathrm{H}^0(Y_{\ell'},\Omega^1_{Y_{\ell'}})$$

with $\bar{a} \in \ell'$ the reduction of $\pi(\pi')^{-e_{L'/L}}$.

[3] A (quite technical) proof of it with some extra details can be found in [Pagano 2024, Section 4.2.1].

By definition of the refined Swan conductor on $\mathrm{Br}(X)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{fil}_n \mathrm{Br}(X) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \mathrm{H}^0(Y, \Omega^2_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \\ \downarrow & & \downarrow \\ \mathrm{fil}_n \mathrm{Br}(K^h) & \xrightarrow{\mathrm{rsw}_{n,\pi}} & \Omega^2_F \oplus \Omega^1_F \end{array}$$

We recall the construction of K^h: let η be the generic point of $Y \subseteq \mathcal{X}$; then we define R as the Henselisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X},\eta}$ and K^h as the fraction field of R. The construction of $\mathcal{O}_{\mathcal{X},\eta}$ (and hence of K^h) is local on $\mathcal{X}$. Therefore, we can assume $\mathcal{X} = \mathrm{Spec}(A)$, with A a smooth $\mathcal{O}_L$-algebra and $Y = \mathrm{Spec}(A/\pi A)$; hence, $\eta = (\pi) \in \mathrm{Spec}(A)$ and $\mathcal{O}_{\mathcal{X},\eta} = A_{(\pi)}$. We can rewrite diagram (2) as

$$\begin{array}{ccccc} A \otimes_{\mathcal{O}_L} L & \longleftarrow & A & \longrightarrow & A/\pi A \\ \uparrow & & \uparrow & & \uparrow \\ L & \longleftarrow & \mathcal{O}_L & \longrightarrow & \ell \end{array} \tag{9}$$

We denote by X', $\mathcal{X}'$ and Y' the base change of X, $\mathcal{X}$ and Y to $\mathrm{Spec}(L')$, $\mathrm{Spec}(\mathcal{O}_{L'})$ and $\mathrm{Spec}(\ell')$, respectively. Let $(K')^h$ be the fraction field of R', where R' is the Henselianisation of the discrete valuation ring $\mathcal{O}_{\mathcal{X}',Y'}$. In this setting,

$$\frac{A}{\pi A} \otimes_\ell \ell' = (A \otimes_{\mathcal{O}_L} \ell) \otimes_\ell \ell' = (A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}) \otimes_{\mathcal{O}_{L'}} \ell' = \frac{A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}}{(1 \otimes \pi')}.$$

Thus, the generic point η' of Y' is the ideal generated by $1 \otimes \pi'$, so $\mathcal{O}_{\mathcal{X}',Y'}$ is the ring $(A \otimes_{\mathcal{O}_L} \mathcal{O}_{L'})_{(1\otimes\pi')}$. Using [Stacks 2005–, Lemmas 04GH and 05WP], $(K')^h/K^h$ is finite with ramification index $e_{L'/L}$

Proof of Lemma 2.30. This follows immediately from the observation above together with Lemma 2.29 and the fact that since $\bar{a} \in \ell'$, which is a finite field, $d\log(\bar{a}) = 0$. □

Lemma 2.30 gives a way to detect whether an element $\mathcal{A} \in \mathrm{Br}(X)$ is transcendental.

Corollary 2.31. *Assume that $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$ for some $n \geq 1$ is such that $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$ with $\alpha \neq 0$; then $\mathcal{A} \notin \mathrm{Br}_1(X)$, i.e., $\mathcal{A}$ is a transcendental element in the Brauer group of X.*

Proof. Assume $\mathcal{A}$ to be in $\mathrm{Br}_1(X)$; then, by definition of $\mathrm{Br}_1(X)$, there is a finite field extension L'/L such that $\mathrm{res}(\mathcal{A}) = 0$ in $\mathrm{Br}(X_{L'})$, where res is the restriction map from $\mathrm{Br}(X)$ to $\mathrm{Br}(X_{L'})$. Let $e_{L'/L}$ be the ramification index of the extension, π' be a uniformiser of L', and ℓ' be its residue field. We know from Lemma 2.30 that

$$\mathrm{rsw}_{e(L'/L)n,\pi_{L'}}(\mathrm{res}(\mathcal{A})) = (\bar{a}^{-n} \cdot \alpha, \bar{a}^{-n} e_{L'/L} \cdot \beta),$$

where $\bar{a}^{-n} \in (\ell')^\times$. Hence, $\mathrm{rsw}_{e(L'/L)n,\pi_{L'}}(\mathrm{res}(\mathcal{A})) \neq (0, 0)$ and, therefore, $\mathrm{res}(\mathcal{A})$ cannot be the trivial element. □

We end this section with a lemma that shows how the evaluation map behaves under base change without the assumption of good reduction for X.

Lemma 2.32. *Let X be a variety over a p-adic field L, not necessarily having good reduction, and let $\mathcal{A} \in \mathrm{Br}(X)\{p\}$ be such that $\mathrm{ev}_{\mathcal{A}} : X(L) \to \mathrm{Br}(L)$ is nonconstant. Then, for every field extension L'/L with order coprime to p, $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_{L'})$ also has a nonconstant evaluation map $\mathrm{ev}_{\mathrm{res}(\mathcal{A})} : X_{L'}(L') \to \mathrm{Br}(L')$.*

Proof. Let $P, Q \in X(L)$ be such that

$$\mathrm{ev}_{\mathcal{A}}(P) \neq \mathrm{ev}_{\mathcal{A}}(Q).$$

Denote by P' and Q' the base change of P and Q to L'. Then, since L'/L has degree coprime to p, the restriction map $\mathrm{Br}(L) \to \mathrm{Br}(L')$ is injective on the p-primary torsion subgroup; hence,

$$\mathrm{ev}_{\mathrm{res}(A)}(P') \neq \mathrm{ev}_{\mathrm{res}(A)}(Q'). \qquad \square$$

3. Ordinary good reduction

Let V be a smooth, proper and geometrically integral k-variety, and $\mathcal{V}$ an $\mathcal{O}_k$-model of V. The main result of this section is the following theorem.

Theorem 3.1. *Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Assume that the special fibre $\mathcal{V}(\mathfrak{p})$ has no nontrivial global 1-forms, $\mathrm{H}^1(\overline{\mathcal{V}(\mathfrak{p})}, \mathbb{Z}/p\mathbb{Z}) = 0$ and $(p-1) \nmid e_{\mathfrak{p}}$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Let X be the base change to $k_{\mathfrak{p}}$ of V, $\mathcal{X}$ the base change of $\mathcal{V}$ to $\mathcal{O}_{\mathfrak{p}}$, and Y the special fibre $\mathcal{V}(\mathfrak{p})$. As already explained in Section 2.1, if we are able to show that, for all elements $\mathcal{A} \in \mathrm{Br}(X)$, the evaluation map $\mathrm{ev}_{\mathcal{A}} : X(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant, then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.

We start with the definition of an ordinary variety.

Definition 3.2. Let Y be a smooth, proper and geometrically integral variety over a perfect field ℓ of positive characteristic. We say that Y is *ordinary* if $\mathrm{H}^i(Y, B_Y^q) = 0$ for every i, q.

As pointed out in [Bloch and Kato 1986, Example 7.4], if Y is an abelian variety, then we recover the usual definition of ordinary, namely Y is ordinary if and only if there is an isomorphism between $Y(\bar{\ell})[p]$ and $(\mathbb{Z}/p\mathbb{Z})^{\dim(Y)}$. In general, the definition can be extended in many different equivalent ways; see for example [ibid., Proposition 7.3]. In particular, this notion is related to the slopes of the Newton polygon of the F-crystal $\mathrm{H}^q_{\mathrm{cris}}(Y/W)$. Roughly speaking, the role played by the Dieudonné module for abelian varieties is replaced by the F-crystals $\mathrm{H}^q_{\mathrm{cris}}(Y/W)$ with $q \geq 0$. However, since for the aim of this section we can avoid introducing crystalline cohomology and the de Rham–Witt complex, we prefer not to add too many technicalities and use the definition of ordinary variety requiring the fewest new concepts.

3.1. ***Proof of Theorem 3.1.*** Throughout this section, we assume that X is such that its special fibre Y is smooth, *ordinary* and such that both $\mathrm{H}^0(Y, \Omega^1_Y)$ and $\mathrm{H}^1(\overline{Y}, \mathbb{Z}/p\mathbb{Z})$ are trivial. In this case, the Cartier operator gives a bijection between the global closed q-forms and the global q-forms on Y. In

fact, for every q, the short exact sequence $0 \to B_Y^q \to Z_Y^q \xrightarrow{C} \Omega_Y^q \to 0$ induces a long exact sequence in cohomology

$$0 \to \mathrm{H}^0(Y, B_Y^q) \to \mathrm{H}^0(Y, Z_Y^q) \xrightarrow{C} \mathrm{H}^0(Y, \Omega_Y^2) \to \mathrm{H}^1(Y, B_Y^q) \to \cdots .$$

The condition that Y is ordinary assures the vanishing of $\mathrm{H}^0(Y, B_Y^q)$ and $\mathrm{H}^1(Y, B_Y^q)$ and hence the bijectivity of the Cartier operator $C : \mathrm{H}^0(Y, Z_Y^q) \to \mathrm{H}^0(Y, \Omega_Y^q)$.

Lemma 3.3. *Let n be an integer such that $0 < n < e'$ with $e' = ep(p-1)^{-1}$. Then, for every $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$,*

$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0.$$

Proof. The assumption $\mathrm{H}^0(Y, \Omega_Y^1) = 0$ together with Corollary 2.24(1) assures us that, if $p \nmid n$, then $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0$. Moreover, as already pointed out in Section 2.3, if $\mathcal{A} \in \mathrm{Br}(X)$ has order coprime to p, then $\mathcal{A} \in \mathrm{fil}_0 \mathrm{Br}(X)$. Therefore, we are reduced to the case $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)[p^r]$ with $r \geq 1$ and $p \mid n$. We prove the lemma by induction on r. If $r = 1$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$, then, by Corollary 2.24(2),

$$(0, 0) = \mathrm{rsw}_{n/p,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha), C(\beta)).$$

By Lemma 2.23(2), it follows that $(\alpha, \beta) \in \mathrm{H}^0(Y, B_Y^2) \oplus \mathrm{H}^0(Y, B_Y^1)$, which are zero since Y is ordinary. Assume the result to be true for $r-1$ and let $\mathcal{A} \in \mathrm{Br}(X)[p^r]$; then, again by Corollary 2.24(2) together with the induction hypothesis,

$$(0, 0) = \mathrm{rsw}_{n/p,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha), C(\beta)),$$

and, once again, $(\alpha, \beta) \in \mathrm{H}^0(Y, B_Y^2) \oplus \mathrm{H}^0(Y, B_Y^1)$ and therefore α and β have to be zero. □

Lemma 3.4. *Assume that $(p-1) \nmid e$; then, for every $n \geq 1$ and for every $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$, we have*

$$\mathrm{rsw}_{n,\pi}(\mathcal{A}) = 0.$$

Proof. Let $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)[p^r]$. By the previous lemma, we already know that the result is true if $n < e'$, and we know that the result is true if $p \nmid n$; see Corollary 2.24(1). Again, we work by induction on r. For $r = 1$, we know from Remark 2.16(1) that, if $n > e'$, then $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$. Hence, we can assume $n \leq e'$. Since by assumption e' is not an integer, $n \leq e'$ is equivalent to $n < e'$; we can therefore conclude by applying the previous lemma. Let $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)[p^r]$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta)$. By assumption, we know that, if $n \geq e'$, then $n > e'$. By Corollary 2.24(3),

$$\mathrm{rsw}_{n-e}(\mathcal{A}^{\otimes p}) = (\bar{u}\alpha, \bar{u}\beta)$$

with $\bar{u} \in \ell^\times$, and the result follows from the induction hypothesis. □

Summarising, we have proved the following theorem.

Theorem 3.5. *Assume that Y is an ordinary variety with no global 1-forms. If $(p-1) \nmid e$, then $\mathrm{Br}(X) = \mathrm{fil}_0 \mathrm{Br}(X)$.*

The assumption $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$ implies that $\mathrm{Ev}_{-1} \mathrm{Br}(X)\{p\} = \mathrm{fil}_0 \mathrm{Br}(X)\{p\}$; see [Bright and Newton 2023, Lemma 11.3]. Combining this result with Theorem 3.5, we get the proof of Theorem 3.1.

3.2. ***Existence of Brauer–Manin obstruction over field extensions.*** In this section, we prove a stronger version of [Bright and Newton 2023, Theorem C] under the additional assumption that the crystalline cohomology groups on the special fibre are torsion-free. We show that, with this extra assumption, it is always possible (after a finite field extension) to find an element of order *exactly* p with nonconstant evaluation map.

Theorem 3.6. *Let V be a smooth, proper and geometrically integral variety over a number field k and $\mathfrak{p}$ be a prime of good ordinary reduction for V. Assume that*

- *the special fibre $\mathcal{V}(\mathfrak{p})$ at $\mathfrak{p}$ has $\mathrm{H}^0(\mathcal{V}(\mathfrak{p}), \Omega^2_{\mathcal{V}(\mathfrak{p})}) \neq \{0\}$;*
- *the crystalline cohomology groups $\mathrm{H}^i(\mathcal{V}(\mathfrak{p})/W)$ are torsion-free for every $i \geq 0$.*

Then there exists a finite field extension k'/k and an element $\mathcal{A} \in \mathrm{Br}(V_{k'})[p]$ that obstructs weak approximation on $V_{k'}$.

For the proof, we first change setting and work over the p-adic field $L := k_{\mathfrak{p}}$. As usual, we denote by X and $\mathcal{X}$ the base change of V and $\mathcal{X}$ the base change and $\mathcal{V}$ to $k_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{p}}$, respectively. Moreover, we denote by Y the base change of $\mathcal{V}$ to $\ell := k(\mathfrak{p})$.

Without loss of generality, we can assume that L contains a primitive p-root of unity. We fix, for every q, an isomorphism on $X_{\text{ét}}$ between $\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbb{Z}/p\mathbb{Z}(q)$. Define

$$\mathcal{M}_1^q := i^* \mathrm{R}^q j_*(\mathbb{Z}/p\mathbb{Z}(1)),$$

where i and j denote the inclusions of the generic and special fibre (X and Y, respectively) in $\mathcal{X}$; see Section 2.4.

Remark 3.7. The main reference for this section is [Bloch and Kato 1986], which links a "twisted" version of the sheaves $\mathcal{M}_1^q$ with the sheaf of logarithmic forms on Y. Block and Kato work with the sheaves $M_1^q := i^* \mathrm{R}^q j_*(\mathbb{Z}/p\mathbb{Z}(q))$. However, since for every q we fixed an isomorphism between $\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbb{Z}/p\mathbb{Z}(q)$, the results apply also to the sheaves $\mathcal{M}_1^q$.

The sheaves $\mathcal{M}_1^q$ fit in the spectral sequence of vanishing cycles [ibid., (0.2)]

$$E_2^{s,t} := \mathrm{H}^s(Y, \mathcal{M}_1^t) \Rightarrow \mathrm{H}^{s+t}(X, \mathbb{Z}/p\mathbb{Z}(1)).$$

By comparing the spectral sequence of vanishing cycles for X and K^h, we get the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) & \xrightarrow{f} & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\ \downarrow & & \downarrow g \\ \mathrm{Br}(K^h)[p] & \xrightarrow{\text{res}} & \mathrm{H}^0(K^h, \mathrm{Br}(K^h_{\mathrm{nr}})[p]) \end{array}$$

where by K^h_{nr} we mean the maximal unramified extension of K^h inside a fixed separable closure. Here we are using Hilbert's Theorem 90 [Colliot-Thélène and Skorobogatov 2021, Theorem 1.3.2] to identify $\mathrm{Br}(K^h)[p] = \mathrm{H}^2(K^h, \mathbb{Z}/p\mathbb{Z}(1))$. Bright and Newton [2023, Lemma 3.4] prove that the map g is injective.

The map f is defined as the composition of the projection $\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) \twoheadrightarrow E_\infty^{0,2}$ with the inclusion map $E_\infty^{0,2} \hookrightarrow E_2^{0,2}$.

The injectivity of g gives a map $f_1 : \mathrm{Br}(X)[p] \to \mathrm{H}^0(Y, \mathcal{M}_1^2)$ such that the diagram

$$\begin{array}{ccc} \mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1)) & \xrightarrow{f} & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\ \downarrow & \nearrow_{f_1} & \\ \mathrm{Br}(X)[p] & & \end{array}$$

commutes, where the vertical arrow comes from the Kummer exact sequence [Colliot-Thélène and Skorobogatov 2021, Section 1.3.4]. In fact, every element in $\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}(1))$ coming from an element of $\mathrm{Pic}(X)$ is sent to 0 by f, since its image in $\mathrm{Br}(K^h)[p]$ comes from an element in $\mathrm{Pic}(K^h)$, and the map $\mathrm{Pic}(K^h) \to \mathrm{Br}(K^h)[p]$ is trivial.

By [Kato 1989, Proposition 6.1], we also know that $\ker(\mathrm{res}) = \mathrm{fil}_0 \mathrm{Br}(K^h)[p]$; hence, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{fil}_0 \mathrm{Br}(X)[p] & \longrightarrow & \mathrm{Br}(X)[p] & \xrightarrow{f_1} & \mathrm{H}^0(Y, \mathcal{M}_1^2) \\ \downarrow & & \downarrow & & \downarrow g \\ \mathrm{fil}_0 \mathrm{Br}(K^h)[p] & \longrightarrow & \mathrm{Br}(K^h)[p] & \xrightarrow{\mathrm{res}} & \mathrm{H}^0(K^h, \mathrm{Br}(K^h_{\mathrm{nr}})[p]) \end{array} \tag{10}$$

Note that we can build a diagram analogous to (10) for every finite field extension L'/L.

We are left to prove the existence (over a field extension L' of L) of an element $\mathcal{A}$ in $\mathrm{Br}(X_{L'})[p]$ such that $f_{L'}(\mathcal{A}) \neq 0$. In order to do this, we analyse the spectral sequence of vanishing cycles over an algebraic closure $\bar{L}$ of L. Let Λ be the integral closure of $\mathcal{O}_L$ in $\bar{L}$, and $\bar{\ell}$ the residue field of Λ. Let $\overline{X}$, $\overline{\mathcal{X}}$ and $\overline{Y}$ be the base change of X, $\mathcal{X}$ and Y to $\bar{L}$, Λ and $\bar{\ell}$ respectively. We define $\overline{\mathcal{M}}_1^q := \bar{\imath}^* \mathrm{R}^q \bar{\jmath}_*(\mathbb{Z}/p\mathbb{Z}(1))$, where $\bar{\imath}$ and $\bar{\jmath}$ denote the inclusions of the generic and special fibre ($\overline{X}$ and $\overline{Y}$, respectively) in $\overline{\mathcal{X}}$. These sheaves build a spectral sequence of vanishing cycles

$$\overline{E}_2^{s,t} := \mathrm{H}^s(\overline{Y}, \overline{\mathcal{M}}_1^t) \Rightarrow \mathrm{H}^{s+t}(\overline{X}, \mathbb{Z}/p\mathbb{Z}(1)).$$

We are interested in the map $\bar{f}_1 : \mathrm{Br}(\overline{X})[p] \to \overline{E}_\infty^{0,2}$, defined in the same way as the map f_1.

Combining Theorem 8.1, the short exact sequence (8.0.1) and Proposition 7.3 of [Bloch and Kato 1986], if Y is ordinary, then

$$\mathrm{H}^n(\overline{Y}, \overline{\mathcal{M}}_1^q) \simeq \mathrm{H}^n(\overline{Y}, \Omega^q_{\overline{Y},\log}) \quad \text{and} \quad \mathrm{H}^n(\overline{Y}, \Omega^q_{\overline{Y},\log}) \otimes_{\mathbb{F}_p} \bar{\ell} \xrightarrow{\sim} \mathrm{H}^n(\overline{Y}, \Omega^q_{\overline{Y}}) \tag{11}$$

for every q, n.

The assumption on the special fibre Y assures us that the spectral sequence of vanishing cycles $\overline{E}_2^{p,q}$ degenerates at the second page; see [Bloch and Esnault 1996, page 307]. Thus,

$$\overline{E}_\infty^{0,2} = \ker(\overline{E}_2^{0,2} \to \overline{E}_2^{2,1}) = \overline{E}_2^{0,2}$$

and hence (by construction) the map $\bar{f}_1 : \mathrm{Br}(\overline{X})[p] \to \mathrm{H}^0(\overline{Y}, \overline{\mathcal{M}}_1^2)$ is surjective. Finally, by (11), the existence of a nontrivial global 2-form on Y assures that $\mathrm{H}^0(\overline{Y}, \overline{\mathcal{M}}_1^2)$ is not trivial.

Proof of Theorem 3.6. Without loss of generality, we can assume that k contains a primitive p-root of unity ζ. Let $\mathfrak{p}$ be a prime of good ordinary reduction for V of residue characteristic p. Let L be the p-adic field $k_\mathfrak{p}$ and X be the base change of V to L. Let $\mathcal{A} \in \mathrm{Br}(\overline{X})[p]$ be such that $\bar{f}_1(\mathcal{A}) \neq 0$; then there is a finite field extension L'/L such that $\mathcal{A}$ is defined over L' (i.e., $\mathcal{A} \in \mathrm{Br}(X_{L'})[p]$) and $f_{1,L'}(\mathcal{A}) \neq 0$. Then, by what we have said above, $\mathcal{A} \notin \mathrm{fil}_0 \mathrm{Br}(X_{L'})[p]$.

Since $\mathrm{Br}(\overline{V})[p] \simeq \mathrm{Br}(V \times_k \bar{k}_\mathfrak{p})[p]$, there exists $\mathcal{A} \in \mathrm{Br}(\overline{V})$ such that $\bar{f}_1(\mathcal{A}) \neq 0$. Hence, if k'/k is a finite field extension such that $\mathcal{A}$ is defined over k', it follows from what we have shown above that there is a prime $\mathfrak{p}'$ above $\mathfrak{p}$ such that $\mathrm{ev}_\mathcal{A}$ is nonconstant on $X_{k'}(k'_{\mathfrak{p}'})$. □

4. Ordinary case: examples

In this section we give examples on K3 surfaces defined over number fields, for which primes of good reduction play a role in the Brauer–Manin obstruction. We prove that, if X is a Kummer K3 surface coming from the product of elliptic curves defined over $\mathbb{Q}$ with good ordinary reduction at the prime 2 and full 2-torsion defined over $\mathbb{Q}_2$, then $\mathrm{Br}(X)[2] = \mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$ (see Theorem 4.10). This theorem proves what was already predicted by Ieronymou [2023, Remark 2.6] after some computational evidence.

4.1. ***The case of K3 surfaces.*** For a smooth variety, we write ω_X for the canonical sheaf of X.

Definition 4.1. An algebraic K3 surface is a smooth projective two-dimensional variety over a field k such that $\omega_X \simeq \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.[4]

More precisely, we work with two kinds of K3 surface (see [Huybrechts 2016, Section 1.1] for more details):

(1) X a smooth quartic surface in $\mathbb{P}^3_k$;

(2) $X = \mathrm{Kum}(A)$, where A is an abelian surface over a field of characteristic different from 2 and X is obtained by resolving singularities from the quotient A/ι, with $\iota : A \to A$ the involution given by $x \mapsto -x$.

The special fibre of a K3 surface with good reduction is still a K3 surface; see for example [Bright and Newton 2023, Remark 11.5]. We start by stating the following well-known result.

Lemma 4.2. *Let p be a prime number and Y a K3 surface over the finite field $\mathbb{F}_{p^n}$ for some nonnegative n. Then Y is ordinary if and only if $|Y(\mathbb{F}_{p^n})| \not\equiv 1 \bmod p$.*

Proof. This is an almost immediate consequence of [Bogomolov and Zarhin 2009, Section 1].[5] □

Given a K3 surface X defined over a local field L of residue characteristic p with $(p-1) \mid e$, we are looking for elements in $\mathrm{Br}(X)$ such that the corresponding evaluation map is nonconstant. For K3 surfaces, $\mathrm{H}^1(\overline{Y}, \mathbb{Z}/p\mathbb{Z}) = 0$, so, by [Bright and Newton 2023, Lemma 11.3], $\mathrm{fil}_0 \mathrm{Br}(X) = \mathrm{Ev}_0 \mathrm{Br}(X) = \mathrm{Ev}_{-1} \mathrm{Br}(X)$.

[4] Here k can be any field.

[5] For more details, see for example [Pagano 2024].

This means that we are looking for $\mathcal{A} \notin \mathrm{Ev}_0 \mathrm{Br}(X)$, namely such that there is $n \geq 1$ with $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$ and $\mathrm{rsw}_{n,\pi}(\mathcal{A}) \neq (0,0)$.

Lemma 4.3. *Let ℓ be a field and $f(x_0, x_1, x_2, x_3) \in \ell[x_0, x_1, x_2, x_3]$ be a homogeneous polynomial of degree 4. Assume that the corresponding projective variety Y is smooth. Then Y is a K3 surface and the one-dimensional ℓ-vector space of global 2-forms is generated by the 2-form*

$$\omega = \frac{d(x_1/x_0) \wedge d(x_2/x_0)}{(1/x_0^3) \cdot \partial f/\partial x_3}.$$

Proof. This result is most likely well known. Since we were not able to find a reference in the literature, we detailed a proof in [Pagano 2024, Lemma 4.3.3]. □

We start by recalling the following example, which is the central result of [Pagano 2022].

Example 4.4 [Pagano 2022]. Let $V \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the projective K3 surface defined by

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \tag{12}$$

Then V has good ordinary reduction at 2 and the class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \mathrm{Br}\, \mathbb{Q}(V)$$

defines an element in $\mathrm{Br}(V)$. The evaluation map $\mathrm{ev}_{\mathcal{A}} : V(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is nonconstant, and therefore gives an obstruction to weak approximation on X.

In this case, $\mathrm{H}^0(Y, \Omega^2_Y)$ is a one-dimensional $\mathbb{F}_2$-vector space. Let ω be the only nontrivial element; then $C(\omega) = 0$ or $C(\omega) = \omega$. However, Y being ordinary implies that $\mathrm{H}^0(Y, B^2_Y) = 0$; hence, by Lemma 2.23 and Corollary 2.10, $C(\omega) = \omega$ and $\mathrm{H}^0(Y, \Omega^2_{Y,\log})$ is a one-dimensional $\mathbb{F}_2$-vector space. By Lemma 4.3, the nonzero global logarithmic 2-form ω can be written (locally) as

$$\omega = \frac{d((z^3 + w^2x + xyz)/x^3)}{(z^3 + w^2x + xyz)/x^3} \wedge \frac{d(z/x)}{z/x}.$$

If we denote by f and g the functions $(z^3 + w^2x + xyz)/x^3$ and z/x seen as elements in the function field F of Y, then we see that the two functions appearing in the definition of $\mathcal{A}$ are lifts to characteristic 0 of f and g, and hence, from Proposition 2.27,

$$\rho_0(\omega, 0) = \left[\left\{ \frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right\} \right] \in \mathrm{gr}^0.$$

Using Proposition 2.28,

$$\mathrm{rsw}_{2,\pi}(\mathcal{A}) = (\omega, 0) \neq (0,0)$$

and $\mathcal{A} \notin \mathrm{fil}_1 \mathrm{Br}(X)[2] \supseteq \mathrm{Ev}_{-1} \mathrm{Br}(X)[2]$.

4.2. ***Kummer K3 surfaces over 2-adic fields.*** Let L be a 2-adic field. Lazda and Skorobogatov [2023] give a way to detect whether the Kummer K3 surface attached to an abelian variety A/L with good

(non-supersingular) reduction has good (non-supersingular) reduction.[6] Skorobogatov and Zarhin [2012] link the transcendental part of the Brauer group of a Kummer K3 surface to that of the underlying abelian variety (see Section 4.2.1). Combining these results makes it possible to construct examples of K3 surfaces with good reduction at the prime 2 for which we are able to study the Brauer group.

In this subsection, we prove that, for every pair of elliptic curves E_1, E_2 over $\mathbb{Q}$ with good ordinary reduction at $p = 2$ and full 2-torsion defined over $\mathbb{Q}_2$, the 2-torsion elements in the Brauer group of the corresponding Kummer K3 surface do not play a role in the Brauer–Manin obstruction to weak approximation. In particular, this shows that the field extension in Theorem 3.6 is needed. We then use these computations to exhibit an example of a K3 surface X over $\mathbb{Q}_2$ with good ordinary reduction and such that $\mathrm{Br}(X) = \mathrm{Ev}_{-1}\,\mathrm{Br}(X)$, showing that the converse of Theorem 1.3 does not hold in general.

4.2.1. *Brauer group of Kummer K3 surfaces.* Let A be an abelian surface over a field k of characteristic different from 2 and $V = \mathrm{Kum}(A)$ the corresponding Kummer surface. Skorobogatov and Zarhin [2012] prove that there is a well-defined map

$$\pi^* : \mathrm{Br}(V) \to \mathrm{Br}(A)$$

that induces an injection of $\mathrm{Br}(V)/\mathrm{Br}_1(V)$ into $\mathrm{Br}(A)/\mathrm{Br}_1(A)$. They also prove that this injection is an isomorphism on the p-torsion for all odd primes p; see [ibid., Theorem 2.4]. We say that an element $\mathcal{A} \in \mathrm{Br}(A)$ *descends* to $\mathrm{Br}(V)$ if there exists $\mathcal{C} \in \mathrm{Br}(V)$ such that $\pi^*(\mathcal{C}) = \mathcal{A}$.

Lemma 4.5. *Let $V = \mathrm{Kum}(A)$, $\mathcal{C} \in \mathrm{Br}(V)$ and $\mathcal{B} := \pi^*(\mathcal{C}) \in \mathrm{Br}(A)$. Let $\mathfrak{p}$ be a prime in $\mathcal{O}_k$; if the image of $\mathcal{C}$ in $\mathrm{Br}(V_\mathfrak{p})$ lies in $\mathrm{Ev}_{-1}\,\mathrm{Br}(V_\mathfrak{p})$ then the image of $\mathcal{B}$ in $\mathrm{Br}(A_\mathfrak{p})$ lies in $\mathrm{Ev}_{-1}\,\mathrm{Br}(A_\mathfrak{p})$.*

Proof. The result follows from the definition of Ev_{-1} and the fact that, for any finite field extension $M/k_\mathfrak{p}$ and $P \in A(M)$, we have

$$\mathrm{ev}_{\mathcal{B}}(P) = \mathrm{ev}_{\pi^*(\mathcal{C})}(P) = \mathrm{ev}_{\mathcal{C}}(\pi(P)). \qquad \square$$

Skorobogatov and Zarhin [2012] show that, given two elliptic curves E_1 and E_2 with Weierstrass equations

$$E_1 : v_1^2 = u_1 \cdot (u_1 - \gamma_{1,1}) \cdot (u_1 - \gamma_{1,2}), \quad E_2 : v_2^2 = u_2 \cdot (u_2 - \gamma_{2,1}) \cdot (u_2 - \gamma_{2,2}),$$

the quotient $\mathrm{Br}(E_1 \times E_2)[2]/\mathrm{Br}_1(E_1 \times E_2)[2]$ is generated by the classes of the four Azumaya algebras

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = ((u_1 - \epsilon_1)(u_1 - \gamma_{1,2}), (u_2 - \epsilon_2)(u_2 - \gamma_{2,2})) \quad \text{with } \epsilon_i \in \{0, \gamma_{i,1}\}.$$

Finally, if M is the matrix

$$M = \begin{pmatrix} 1 & \gamma_{1,1}\cdot\gamma_{1,2} & \gamma_{2,1}\cdot\gamma_{2,2} & -\gamma_{1,1}\cdot\gamma_{2,1} \\ \gamma_{1,1}\cdot\gamma_{1,2} & 1 & \gamma_{1,1}\cdot\gamma_{2,1} & \gamma_{2,1}\cdot(_{2,1}-_{2,2}) \\ \gamma_{2,1}\cdot\gamma_{2,2} & \gamma_{1,1}\cdot\gamma_{2,1} & 1 & \gamma_{1,1}\cdot(\gamma_{1,1}-\gamma_{1,2}) \\ -\gamma_{1,1}\cdot\gamma_{2,1} & \gamma_{2,1}\cdot(\gamma_{2,1}-\gamma_{2,2}) & \gamma_{1,1}\cdot(\gamma_{1,1}-\gamma_{1,2}) & 1 \end{pmatrix},$$

[6] This was already known for K3 surfaces over p-adic fields with $p \neq 2$, for the supersingular case in characteristic 2; see [Matsumoto 2023].

then, by [ibid., Lemma 3.6]:

(1) $\mathcal{A}_{\gamma_{1,1},\gamma_{2,1}}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the first row of M are all squares.

(2) $\mathcal{A}_{\gamma_{1,1},0}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the second row of M are all squares.

(3) $\mathcal{A}_{0,\gamma_{2,1}}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the third row of M are all squares.

(4) $\mathcal{A}_{0,0}$ descends to $\mathrm{Br}(V)$ if and only if the entries of the last row of M are all squares.

4.2.2. *Product of elliptic curves with good reduction at* 2 *and full* 2*-torsion.* In order to use the results just summarised, we need to analyse what the 2-torsion points of an elliptic curve with good ordinary reduction at 2 look like. Let $E/\mathbb{Q}$ be an elliptic curve having good reduction at 2 and such that the 2-torsion of E is defined over $\mathbb{Q}_2$, i.e., $E(\mathbb{Q}_2)[2] = E(\overline{\mathbb{Q}}_2)[2]$. Without loss of generality, we can assume $E/\mathbb{Q}$ to be defined by the minimal Weierstrass equation[7]

$$y^2 + xy = x^3 + ax^2 + bx + c \tag{13}$$

with $a, b, c \in \mathbb{Z}_2$. Let $\alpha_i, \beta_i \in \mathbb{Q}_2$ be such that

$$E(\mathbb{Q}_2)[2] = \{\mathcal{O}, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\},$$

with $\mathcal{O}$ the point at infinity of E.

Lemma 4.6. *Assume that* $\beta_1, \beta_2, \beta_3$ *are ordered as*

$$\mathrm{ord}_2(\beta_1) \le \mathrm{ord}_2(\beta_2) \le \mathrm{ord}_2(\beta_3).$$

Then $\mathrm{ord}_2(\alpha_1) = -2$ *and* $\alpha_2, \alpha_3 \in \mathbb{Z}_2$.

Proof. The 2-torsion points on E can be computed through the 2-division polynomial of E, which is $\psi_2(x, y) = 2y + x$. In particular, $\alpha_i = -2\beta_i$ with β_i a solution of

$$\Phi(y) := y^2 - 2y^2 - ((-2y)^3 + a(-2y)^2 + b(-2y) + c).$$

The polynomial $\Phi(y)$ can be rewritten as

$$\Phi(y) = 8y^3 - (1 + 4a)y^2 + 2by - c. \tag{14}$$

Looking at the coefficients of $\Phi(y)$, we get

$$\begin{aligned} \mathrm{ord}_2(\beta_1 + \beta_2 + \beta_3)U &= \mathrm{ord}_2(1 + 4a) - \mathrm{ord}_2(8), \\ \mathrm{ord}_2(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)U &= \mathrm{ord}_2(2b) - \mathrm{ord}_2(8), \\ \mathrm{ord}_2(\beta_1\beta_2\beta_3)U &= \mathrm{ord}_2(c) - \mathrm{ord}_2(8). \end{aligned}$$

From the first equation, we get $\mathrm{ord}_2(\beta_1) \le -3$, which, combined with [Silverman 1986, Theorem VIII.7.1], tells us that $\mathrm{ord}_2(\beta_1) = -3$. From the third equation, we get $\mathrm{ord}_2(\beta_2) + \mathrm{ord}_2(\beta_3) \ge 0$. If $\mathrm{ord}_2(\beta_2) = \mathrm{ord}_2(\beta_3)$,

[7] If $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is a minimal Weierstrass equation over $\mathbb{Z}_2$, then $a_1 \in \mathbb{Z}_2^\times$ (since the reduction modulo 2 is ordinary). The change of variables $(x, y) \mapsto (a_1^2x' - a_3/a_1, a_1^3y' + (1 - a_1^5)/(2a_1^4)x')$ gives another minimal Weierstrass equation of the form $y^2 + xy = x^3 + ax^2 + bx + c$.

then both valuations are nonnegative. Otherwise, $\mathrm{ord}_2(\beta_2) < \mathrm{ord}_2(\beta_3)$, and, from the second equation, we get $\mathrm{ord}_2(\beta_2) \geq 1$, which implies that also in this case both β_2 and β_3 have nonnegative 2-adic valuation. The result now follows from the fact that $\alpha_i = -2\beta_i$. □

Lemma 4.7. *The change of variables given by*

$$u = 4x - 4\alpha_1, \quad v = 4(2y + x) \tag{15}$$

induces an isomorphism between E and the elliptic curve given by

$$v^2 = u(u - \gamma_1)(u - \gamma_2), \tag{16}$$

where $\gamma_1 = 4 \cdot (\alpha_2 - \alpha_1)$ and $\gamma_2 = 4 \cdot (\alpha_3 - \alpha_1)$.

Proof. The change of variables

$$u_1 = 4x, \quad v_1 = 4(2y + x)$$

sends the elliptic curve given by

$$v_1^2 = u_1^3 + (4a + 1)u_1^2 + 16c \tag{17}$$

to the elliptic curve given by (13). Moreover, the 2-division polynomial of E is given by $2y + x$. Hence, the nontrivial 2-torsion points on the elliptic curve given by (17) are sent to nontrivial 2-torsion points on E. The extra translation $u = u_1 - 4\alpha_1$ and $v = v_1$ gives the desired equation. □

Let E_1 and E_2 be two elliptic curves with equations of the form (13). We denote by (a_i, c_i) the parameters that determine the equation attached to E_i, by $(\alpha_{i,j}, \beta_{i,j})$ for $j \in \{1, 2, 3\}$ the nontrivial 2-torsion points of E_i, and by A the abelian surface given by the product of E_1 with E_2. We denote by $\langle \mathrm{Ev}_{-1}\,\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2] \rangle$ the subgroup of $\mathrm{Br}(A)[2]$ generated by $\mathrm{Ev}_{-1}\,\mathrm{Br}(A)[2]$ and $\mathrm{Br}_1(A)[2]$, where $\mathrm{Br}_1(A)[2]$ is the algebraic Brauer group of A.

Lemma 4.8. *Assume that ϵ_1 and ϵ_2 are as in Section 4.2.1; then the class of the quaternion algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ lies in $\langle \mathrm{Ev}_{-1}\,\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2] \rangle$ if and only if at least one of ϵ_1 and ϵ_2 is different from 0.*

Proof. We fix $\pi = 2$ as a uniformiser and $\xi = -1$ as a primitive 2-root of unity. We start by assuming that at least one among ϵ_1 and ϵ_2 is different from 0. By the symmetry of the statement, we can assume without loss of generality that $\epsilon_1 \neq 0$. Then

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = ((u_1 - \gamma_{1,1}) \cdot (u_1 - \gamma_{1,2}), (u_2 - \epsilon_2) \cdot (u_2 - \gamma_{2,2})) = (u_1, f_{\epsilon_2}(u_2)),$$

where $f_{\epsilon_2}(u_2) = (u_2 - \epsilon_2) \cdot (u_2 - \gamma_{2,2})$. The quaternion algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ corresponds via the change of variables of Lemma 4.7 to

$$\mathcal{A}_{\epsilon_1,\epsilon_2} = (4 \cdot (x_1 - \alpha_{1,1}), f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})) = (x_1 - \alpha_{1,1}, f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})).$$

We define

$$g_{\epsilon_2}(x_2) := \begin{cases} (x_2 - \alpha_{2,2}) \cdot (x_2 - \alpha_{2,3}) & \text{if } \epsilon_2 = \gamma_{2,1}, \\ (x_2 - \alpha_{2,1}) \cdot (x_2 - \alpha_{2,3}) & \text{if } \epsilon_2 = 0. \end{cases}$$

Then $16 \cdot g_{\epsilon_2}(x_2) = f_{\epsilon_2}(4x_2 - 4\alpha_{2,1})$. Thus, we can rewrite $\mathcal{A}_{\epsilon_1,\epsilon_2}$ as

$$(-\alpha_{1,1}, g_{\epsilon_2}(x_2)) \otimes (1 + (-\alpha_{1,1}^{-1}) \cdot x_1, g_{\epsilon_2}(x_2)).$$

Since $(-\alpha_{1,1}, g_{\epsilon_2}(x_2))$ lies in $\mathrm{Br}_1(A)[2]$, we are left to show that the class of the quaternion algebra $(1 + (-\alpha_{1,1})^{-1} \cdot x_1, g_{\epsilon_2}(x_2))$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$. By Lemma 4.6, we know that $\mathrm{ord}_2(\alpha_{1,1}^{-1}) = 2$ and therefore, by Proposition 2.28,

$$(1 + (-\alpha_{1,1}^{-1}) \cdot x_1, g_{\epsilon_2}(x_2)) \in \mathrm{fil}_0 \mathrm{Br}(A)[2].$$

By [Bright and Newton 2023, Theorem C], in order to establish whether $(1 + (-\alpha_{1,1})^{-1} \cdot x_1, g_{\epsilon_2}(x_2))$ belongs to $\mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$, we need to compute $\partial(1 + (-\alpha_{1,1}^{-1}) \cdot x_1, g_{\epsilon_2}(x_2))$. We have $g_{\epsilon_2}(x_2) \not\equiv 0 \bmod 2$ and, from Proposition 2.27(1) together with Proposition 2.28, we get

$$\lambda_\pi\big(\bar{x}_1 \cdot d\log(\bar{g}_{\epsilon_2}(\bar{x}_2)), 0\big) = (1 + (-\alpha_{1,1}^{-1}) \cdot x_1, g_{\epsilon_2}(x_2))$$

since $1 + (-\alpha_{1,1}^{-1}) \cdot x_1 = 1 + 4 \cdot (s^{-1} \cdot x_1)$ with $s = -4 \cdot \alpha_{1,1} \in \mathbb{Z}_2^\times$ and hence $s^{-1} \cdot x_1$ is a lift to characteristic 0 of $\bar{x}_1$. Therefore, by definition of the residue map ∂,

$$\partial\big((1 + (-\alpha_{1,1}^{-1}) \cdot x_1, g_{\epsilon_2}(x_2))\big) = 0,$$

which, by Theorem 2.1, implies that

$$(1 + (-\alpha_{1,1}^{-1}) \cdot x_1, g_{\epsilon_2}(x_2)) \in \mathrm{Ev}_{-2}\mathrm{Br}(A)[2] \subseteq \mathrm{Ev}_{-1}\mathrm{Br}(A)[2].$$

It remains to show that $\mathcal{A}_{0,0} \notin \langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2] \rangle$. The change of variables of Lemma 4.7 sends the class of the quaternion algebra

$$\mathcal{A}_{0,0} = (u_1 \cdot (u_1 - \gamma_{1,2}), u_2 \cdot (u_2 - \gamma_{2,2})) = (u_1 - \gamma_{1,1}, u_2 - \gamma_{2,1})$$

to the class of the quaternion algebra

$$(4 \cdot (x_1 - \alpha_{1,2}), 4 \cdot (x_2 - \alpha_{2,2})) = (x_1 + 2\beta_{1,2}, x_2 + 2\beta_{2,2}).$$

From Proposition 2.27(4), the latter is such that

$$\rho_0\left(\frac{d(\bar{x}_1)}{\bar{x}_1} \wedge \frac{d(\bar{x}_2)}{\bar{x}_2}\right) = [\{x_1 + 2\beta_{2,1}, x_2 + 2\beta_{2,2}\}] \in \mathrm{gr}^0.$$

In fact, $x_1 + 2\beta_{2,1}$ and $x_2 + 2\beta_{2,2}$ are lifts to characteristic 0 of $\bar{x}_1$ and $\bar{x}_2$, respectively. Note that $d(\bar{x}_1)/\bar{x}_1 \wedge d(\bar{x}_2)/\bar{x}_2$ comes from a global 2-form on the special fibre Y of A, and hence it is nonzero in its function field. Finally, using Proposition 2.28,

$$\mathrm{rsw}_{2,\pi}((x_1 + 2\beta_{2,1}, x_2 + 2\beta_{2,2})) = \left(\frac{d(\bar{x}_1)}{\bar{x}_1} \wedge \frac{d(\bar{x}_2)}{\bar{x}_2}, 0\right) \neq (0, 0)$$

and hence $(x_1 + 2\beta_{1,2}, x_2 + 2\beta_{2,2}) \notin \mathrm{fil}_1 \mathrm{Br}(A)[2] \supseteq \mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$. Moreover, by Corollary 2.31, $\mathcal{A}_{0,0} \notin \langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2] \rangle$. In fact, otherwise, there would be an element $\mathcal{A}_1 \in \mathrm{Ev}_{-1}\mathrm{Br}(A)[2]$ such that $\mathcal{A}_{0,0} \otimes \mathcal{A}_1 \in \mathrm{Br}_1(A)[2]$, but $\mathcal{A}_{0,0} \otimes \mathcal{A}_1$ has the same refined Swan conductor as $\mathcal{A}_{0,0}$. □

Remark 4.9. We will later use a slightly stronger version of the lemma above. Let $L/\mathbb{Q}_2$ be any field extension. Then the image of $\mathcal{A}_{\epsilon_1,\epsilon_2}$ in $\mathrm{Br}(A_L)[2]$ lies in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2], \mathrm{Br}_1(A_L)[2]\rangle$ if and only if at least one among ϵ_1 and ϵ_2 is different from 0.

Clearly, $\mathcal{A}_{\epsilon_1,\epsilon_2}$ in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A)[2], \mathrm{Br}_1(A)[2]\rangle$ implies $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2})$ in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2], \mathrm{Br}_1(A_L)[2]\rangle$. Moreover, we have proven that the first component of $\mathrm{rsw}_{2,\pi}(\mathcal{A}_{0,0})$ is different from 0, and hence, using Lemma 2.30, $\mathrm{rsw}_{e_{L'/L}2,\pi}(\mathrm{res}(\mathcal{A}_{0,0})) \neq (0,0)$; hence, the image of $\mathcal{A}_{0,0}$ in $\mathrm{Br}(A_L)[2]$ does not lie in $\langle \mathrm{Ev}_{-1}\mathrm{Br}(A_L)[2], \mathrm{Br}_1(A_L)[2]\rangle$.

4.2.3. *Brauer–Manin obstruction from 2-torsion elements.* Let V be $\mathrm{Kum}(A_{\mathbb{Q}})$. We denote by A and X the base changes of the abelian surface $A_{\mathbb{Q}}$ and the corresponding Kummer surface X to $\mathbb{Q}_2$. By Section 4.2.1, we know that $\mathcal{A}_{0,0}$ descends to X if and only if

$$[-\gamma_{1,1}\cdot\gamma_{2,1}, \gamma_{2,1}(\gamma_{2,1}-\gamma_{2,2}), \gamma_{1,1}(\gamma_{1,1}-\gamma_{1,2}), 1] \in (\mathbb{Q}_2^{\times 2})^4.$$

By construction, $\gamma_{1,1} = 4\cdot(\alpha_{1,2}-\alpha_{1,1}) = 8\beta_{1,1} - 8\beta_{1,2}$ and therefore

$$\gamma_{1,1} \equiv 8\beta_{1,1} \equiv 1 + 4a_1 \mod 8.$$

In fact, from Lemma 4.6 and, more precisely, from (14), we know that $8\beta_{1,1} + 8\beta_{1,2} + 8\beta_{1,3} = 1 + 4a_1$ and $\mathrm{ord}_2(\beta_{1,2}) = \mathrm{ord}_2(\beta_{1,3}) \geq 0$. Similarly,

$$\gamma_{2,1} \equiv 8\beta_{2,1} \equiv 1 + 4a_2 \mod 8.$$

In particular, both $\gamma_{1,1}$ and $\gamma_{2,1}$ are either 1 or 5 modulo 8; hence, $-\gamma_{1,1}\cdot\gamma_{2,1}$ is either -1 or 3 and therefore it is never a square.

In summary, we just showed that $\mathcal{A}_{0,0} \in \mathrm{Br}(A)$ never descends to $\mathrm{Br}(X)$. We are ready to prove the main theorem of this section.

Theorem 4.10. *Let $X = \mathrm{Kum}(A)$, with $A = E_1 \times E_2$ as in Section 4.2.1. Then $\mathrm{Br}(X)[2] = \mathrm{Ev}_{-1}\mathrm{Br}(X)[2]$.*

Proof. We recall that, if $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)[2]$, we denote by $\mathcal{C}_{\epsilon_1,\epsilon_2}$ the corresponding element in $\mathrm{Br}(X)[2]$, i.e., $\mathcal{C}_{\epsilon_1,\epsilon_2}$ is such that $\pi^*(\mathcal{C}_{\epsilon_1,\epsilon_2}) = \mathcal{A}_{\epsilon_1,\epsilon_2}$. We need to prove that, if $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)$, then $\mathcal{C}_{\epsilon_1,\epsilon_2}$ lies in $\mathrm{Ev}_{-1}\mathrm{Br}(X)$. Since we have already shown at the beginning of this section that $\mathcal{A}_{0,0}$ never descends to $\mathrm{Br}(X)$, we are left to show it for $(\epsilon_1,\epsilon_2) \neq (0,0)$.

Let $L/\mathbb{Q}_2$ be such that all elements appearing in the matrix M of Section 4.2.1 are squares, i.e., the injective map

$$\pi^* : \frac{\mathrm{Br}(X)[2]}{\mathrm{Br}_1(X)[2]} \hookrightarrow \frac{\mathrm{Br}(A)[2]}{\mathrm{Br}_1(A)[2]}$$

is an isomorphism.

With abuse of notation, we denote by res both

$$\mathrm{res} : \mathrm{Br}(A) \to \mathrm{Br}(A_L) \quad \text{and} \quad \mathrm{res} : \mathrm{Br}(X) \to \mathrm{Br}(X_L).$$

We denote by $(\mathcal{C}_{\epsilon_1,\epsilon_2})_L$ the preimage of $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2}) \in \mathrm{Br}(A_L)[2]$.

From [Lazda and Skorobogatov 2023, Theorem 2], we know that the reduction of X is an ordinary K3 surface. Let e be the ramification index of $L/\mathbb{Q}_2$ and π_L a uniformiser of $\mathcal{O}_L$; then $\mathrm{fil}_n \mathrm{Br}(X_L)[p] = \mathrm{fil}_0 \mathrm{Br}(X_L)[p] = \mathrm{Ev}_{-1} \mathrm{Br}(X_L)[p]$ if $n < e' := 2e$ and $\mathrm{fil}_{e'} \mathrm{Br}(X)[p] = \mathrm{Br}(X_L)[p]$; see Corollary 2.24(1). Hence, using Corollary 2.24(3), we have an injection

$$\mathrm{mult}_{\bar{c}} \circ \mathrm{rsw}_{e',\pi_L} : \frac{\mathrm{Br}(X_L)[2]}{\mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2]} \hookrightarrow \mathrm{H}^0(Y_\ell, \Omega^2_{Y_\ell,\log}),$$

where ℓ is the residue field of L.

Since X (and hence all its base changes) has good ordinary reduction, we know that $\mathrm{H}^0(\overline{Y}, \Omega^2_{\overline{Y},\log}) \otimes_{\mathbb{F}_2} \bar{\ell}$ is a one-dimensional $\bar{\ell}$-vector space (see (11)) and hence $\mathrm{Br}(X_L)[2]/\mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2]$ is a vector space of dimension at most 1 over $\mathbb{F}_2$.

By Lemma 4.8, $\mathcal{A}_{0,0} \notin \mathrm{fil}_1 \mathrm{Br}(A)[2]$. Applying Lemma 2.30 gives $\mathrm{res}(\mathcal{A}_{0,0}) \notin \mathrm{fil}_e \mathrm{Br}(A_L)[2]$, and, by Lemma 4.5, $(\mathcal{C}_{0,0})_L \notin \mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2]$. Therefore,

$$\langle [(\mathcal{C}_{0,0})_L] \rangle = \frac{\mathrm{Br}(X_L)[2]}{\mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2]}.$$

Assume that there exists $(\epsilon_1, \epsilon_2) \neq (0,0)$ such that $\mathcal{A}_{\epsilon_1,\epsilon_2}$ descends to $\mathrm{Br}(X)$ and the corresponding element $\mathcal{C}_{\epsilon_1,\epsilon_2}$ does not lie in $\mathrm{Ev}_{-1} \mathrm{Br}(X)[2]$. By Lemma 2.30, $\mathrm{res}(\mathcal{C}_{\epsilon_1,\epsilon_2})$ does not lie in $\mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2]$ and therefore, since the quotient $\mathrm{Br}(X_L)[2]/\mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2]$ is a one-dimensional $\mathbb{F}_2$-vector space, also the product $\mathrm{res}(\mathcal{C}_{\epsilon_1,\epsilon_2}) \otimes (\mathcal{C}_{0,0})_L$ lies in $\mathrm{Ev}_{-1} \mathrm{Br}(X)[2]$. This implies that also the corresponding element in $\mathrm{Br}(A_L)[2]$, $\mathrm{res}(\mathcal{A}_{\epsilon_1,\epsilon_2} \otimes \mathcal{A}_{0,0})$ lies in $\mathrm{Ev}_{-1} \mathrm{Br}(A_L)$. However, since, by Remark 4.9, $\mathrm{res}(A_{\epsilon_1,\epsilon_2})$ lies in $\langle \mathrm{Ev}_{-1} \mathrm{Br}(X_L)[2], \mathrm{Br}_1(X_L)[2] \rangle$, also $\mathrm{res}(\mathcal{A}_{0,0})$ has to lie in $\langle \mathrm{Ev}_{-1} \mathrm{Br}(A_L)[2], \mathrm{Br}_1(A_L)[2] \rangle$, which gives us the desired contradiction. □

We end this subsection with an example of a K3 surface over $\mathbb{Q}$ with good ordinary reduction at 2 and such that $\mathrm{Br}(X) = \mathrm{Ev}_{-1} \mathrm{Br}(X)$. The existence of such an example shows that the converse of Theorem 1.3 is not true, i.e., it is not enough to have $(p-1) \mid e$ in order to find an element in $\mathrm{Br}(X)$ that does not lie in $\mathrm{Ev}_{-1} \mathrm{Br}(X)$.

Example 4.11. Let $A = E \times E$, where E is the elliptic curve given by the minimal Weierstrass equation

$$y^2 + xy = x^3 - 3x^2 - 4x + 11.$$

With the same notation as in the previous subsections, $\beta_1 = -\frac{11}{8}$, $\beta_2 = -1$ and $\beta_3 = 1$. Hence,

$$\alpha_1 = \tfrac{11}{4}, \quad \alpha_2 = 2, \quad \alpha_3 = -2 \quad \text{and} \quad \gamma_1 = -3, \quad \gamma_2 = -19.$$

The matrix M is of the form

$$\begin{pmatrix} 1 & 3\cdot 19 & 3\cdot 19 & -9 \\ 3\cdot 19 & 1 & 9 & -3\cdot 18 \\ 3\cdot 19 & 9 & 1 & -3\cdot 18 \\ -9 & -3\cdot 18 & -3\cdot 18 & 1 \end{pmatrix}.$$

In particular, all the rows of M have at least one term which does not lie in $\mathbb{Q}_2^{\times 2}$. Moreover, using [Skorobogatov and Zarhin 2012, Proposition 3.7], we can compute the dimension as an $\mathbb{F}_2$-vector space of the quotient of $\mathrm{Br}(X)[2]$ by $\mathrm{Br}(\mathbb{Q}_2)[2]$, and, in this case,

$$\dim_{\mathbb{F}_2}\left(\frac{\mathrm{Br}(X)[2]}{\mathrm{Br}(\mathbb{Q}_2)[2]}\right)=0.$$

We want to show that $\mathrm{Br}(X)\{2\}=\mathrm{Br}(\mathbb{Q}_2)\{2\}$. We work by induction on n. Let $\mathcal{A}$ be in $\mathrm{Br}(X)[2^n]$; then

$$\mathcal{A}^{\otimes 2^{n-1}}\in \mathrm{Br}(X)[2]=\mathrm{Br}(\mathbb{Q}_2)[2].$$

In particular, given $P\in X(\mathbb{Q}_2)$,

$$(\mathcal{A}\otimes \mathrm{ev}_{\mathcal{A}}(P))^{\otimes 2^{n-1}}=\mathcal{A}^{\otimes 2^{n-1}}\otimes \mathrm{ev}_{\mathcal{A}^{\otimes 2^{n-1}}}(P)=0;$$

hence, $\mathcal{A}\otimes \mathrm{ev}_{\mathcal{A}}(P)\in \mathrm{Br}(X)[2^{n-1}]$, and, by the induction hypothesis, $\mathcal{A}\in \mathrm{Br}(\mathbb{Q}_2)[2^n]$.

4.3. ***Examples of Brauer–Manin obstruction.*** We continue by giving new examples of primes of good reduction that play a role in the Brauer–Manin obstruction to weak approximation in the cases $p=3$ and $p=5$.

Example 4.12. Let $L=\mathbb{Q}_3(\zeta)$ with ζ a primitive 3-root of unity. Let π be a uniformiser of $\mathcal{O}_L$; then $e=e(L/\mathbb{Q}_3)=2$ and the residue field ℓ is equal to $\mathbb{F}_3$.

We define X to be the Kummer K3 surface over L attached to the abelian surface $A=E\times E$, with E the elliptic curve over L defined by the Weierstrass equation

$$y^2=x^3+4\cdot x^2+3\cdot x+1.$$

The elliptic curve E (and hence A and X) has good ordinary reduction at the prime $\mathfrak{p}=(\pi)$. We denote by $\{x, y, z\}$ and $\{u, v, w\}$ the variables corresponding to the embedding of, respectively, the first and the second copy of E in $\mathbb{P}^2_L$. We define the cyclic algebra

$$\mathcal{A}:=\left(\frac{v-u}{w},\frac{y-x}{z}\right)_{\zeta}\in \mathrm{Br}(L(A))[3].$$

Claim 1. *$\mathcal{A}$ belongs to* $\mathrm{Br}(A)[3]$.

Proof. First of all, from $y^2z=x^3+4x^2z+3xz^2+z^3$ we get

$$z(y-x)(y+x)=(x+z)^3 \quad \text{and} \quad z(y^2-4x^2-3xz-z^2)=x^3.$$

Then:

- If $z=0$, then $x=0$ and therefore $y^2-4x^2-3xz-z^2\neq 0$ and $x\neq y$, and, from the equation above the claim, $\mathcal{A}$ is equivalent to

$$\left(\frac{v-u}{(v^2-4u^2-3uw-w^2)^{-1}},\frac{y-x}{(y^2-4x^2-3xz-z^2)^{-1}}\right)_{\zeta}.$$

- If $x = y$, then $x \neq -y$ and $z \neq 0$ (since $z = 0$ implies $x \neq y$), and, from the equation above, $\mathcal{A}$ is equivalent to
$$\left(\frac{w}{v+u}, \frac{z}{x+y}\right)_\zeta.$$

Thus, we see that, given a divisor D over which $\mathcal{A}$ is not well defined, we are able to find an Azumaya algebra which is equivalent to $\mathcal{A}$ on an open set and well defined along D. Hence, $\mathcal{A}$ defines an element in $\mathrm{Br}(A)[3]$. □

Claim 2. *$\mathcal{A}$ does not lie in* $\mathrm{Ev}_{-1}\,\mathrm{Br}(A)[3]$.

Proof. The regular global 1-form on the reduction of E modulo $\mathfrak{p}$ is given by the (local) formula
$$\frac{dx}{2y} = -\frac{1}{2}\cdot\frac{dx\cdot(x-y)}{y(y-x)} = -\frac{1}{2}\cdot\frac{dx\cdot x/y - dx}{y-x} = \frac{d(y-x)}{y-x},$$
where the last equality follows from the fact that, on the special fibre, $dx/y = dy/x$ and, since we are in characteristic 3, $\frac{1}{2} = -1$. Hence, if we denote by Y the reduction modulo $\mathfrak{p}$ of A, the global 2-form on Y is given by
$$\omega = \frac{d((v-u)/w)}{(v-u)/w} \wedge \frac{d((y-x)/z)}{(y-x)/z}.$$

Finally, $\rho_0(\omega, 0) = [\{(v-u)/w, (y-x)/z\}]$ and hence, again by Proposition 2.28,
$$\mathrm{rsw}_{3,\pi}(\mathcal{A}) \neq (\bar{c}^{-1}\cdot\omega, 0)$$
and therefore $\mathcal{A} \notin \mathrm{fil}_2\,\mathrm{Br}(A)[3] \supseteq \mathrm{Ev}_{-1}\,\mathrm{Br}(A)[3]$. □

Claim 3. *The cyclic algebra $\mathcal{A}$ in* $\mathrm{Br}(A)[3]$ *is not algebraic, i.e.,* $\mathcal{A} \notin \mathrm{Br}_1(A)[3]$.

Proof. This follows directly from Corollary 2.31. □

Finally, from [Skorobogatov and Zarhin 2012, Theorem 2.4], the map
$$\pi^* : \frac{\mathrm{Br}(X)[3]}{\mathrm{Br}_1(X)[3]} \hookrightarrow \frac{\mathrm{Br}(A)[3]}{\mathrm{Br}_1(A)[3]}$$
is an isomorphism. Let $\mathcal{B} \in \mathrm{Br}(X)[3]$ be such that $\pi^*(\mathcal{B}) = \mathcal{A} \in \mathrm{Br}(A)[3]$. Then, by Lemma 4.5, $\mathcal{B} \notin \mathrm{Ev}_{-1}\,\mathrm{Br}(X)[3]$, so (since $\mathrm{Ev}_0\,\mathrm{Br}(X) = \mathrm{Ev}_{-1}\,\mathrm{Br}(X)$ for K3 surfaces; see Section 4.1) the corresponding evaluation map on $X(L)$ is nonconstant.

Example 4.13. Let X be the diagonal quartic surface over $\mathbb{Q}_5$ defined by
$$5x^4 - 4y^4 = z^4 + w^4.$$

Skorobogatov and Ieronymou [2015, Theorem 1.1 and Proposition 5.12] prove that there exists an element $\mathcal{A} \in \mathrm{Br}(X)[5]$ with surjective evaluation map. Let $L = \mathbb{Q}_5(\sqrt[4]{5})$, $e(L/\mathbb{Q}_5) = 4$ and $\alpha \in L$ be such that $\alpha^4 = 5$. Then the change of variables
$$(x, y, z, w) \mapsto \left(\frac{x_1}{\alpha}, y_1, z_1, w_1\right)$$

sends X_L to the diagonal quartic $\widetilde{X}/L$ given by

$$x_1^4 - 4y_1^4 = z_1^4 + w_1^4.$$

The surface $\widetilde{X}$ has good ordinary reduction over L; indeed, its special fibre is given by the Fermat quartic over $\mathbb{F}_5$, which, using Lemma 4.2, can be checked to be ordinary. Finally, by Lemma 2.32, we know that $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_{L'}) = \mathrm{Br}(\widetilde{X})$ has nonconstant evaluation map.

Remark 4.14. We have $e' = p = 5$ and, by Corollary 2.24, $\mathcal{A}$ has to lie in $\mathrm{fil}_p \mathrm{Br}(X)[5] \setminus \mathrm{fil}_{p-1} \mathrm{Br}(X)[5]$. In fact, since X is ordinary, we have $\mathrm{H}^0(Y, B_Y^2) = \mathrm{H}^0(Y, B_Y^1) = 0$ and therefore the image of $\mathrm{rsw}_{n,\pi}$ is zero on the 5-torsion elements for $n < e' = 5$.

5. Nonordinary good reduction

In this section, we analyse what happens for primes of good nonordinary reduction in the case of K3 surfaces. We are going to prove the following theorem.

Theorem 5.1. *Let V be a K3 surface and $\mathfrak{p}$ be a prime of good nonordinary reduction for V with $e_\mathfrak{p} \le p-1$. Then the prime $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation on V.*

Like in the previous section, let X be the base change to $k_\mathfrak{p}$ of V; then we denote by $\mathcal{X}$ the base change of $\mathcal{V}$ to $\mathcal{O}_\mathfrak{p}$, and by Y the special fibre $\mathcal{V}(\mathfrak{p})$. As already explained, it is enough to show that for every element $\mathcal{A} \in \mathrm{Br}(X)$ the evaluation map $\mathrm{ev}_\mathcal{A} : X(k_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z}$ is constant. We begin with the following lemma.

Lemma 5.2. *Assume that the special fibre Y of the $\mathcal{O}_L$-model $\mathcal{X}$ is such that the Cartier operator acts trivially on $\mathrm{H}^0(Y, Z_Y^2)$ and there are no global 1-forms. Then, if the absolute ramification index satisfies $e \le p-1$, we have $\mathrm{Br}(X) = \mathrm{fil}_0 \mathrm{Br}(X)$.*

Proof. If $e < p-1$, the result is proven by Bright and Newton [2023, Lemma 11.2] without any extra assumption on Y. If $e = p-1$, let $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}(X)$ with $\mathrm{rsw}_{n,\pi} = (\alpha, \beta)$. In order to prove the lemma, it is enough to show that $(\alpha, \beta) = (0, 0)$.

- If $n < e'$, then $p \nmid n$, and hence, by Corollary 2.24(1) and the assumption that $\mathrm{H}^0(Y, \Omega_Y^1) = 0$, we get $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$.
- If $n > e'$, then $p \nmid n$ or $p \nmid n - e$, and hence either $\mathrm{rsw}_{n,\pi}(\mathcal{A})$ or $\mathrm{rsw}_{n-e,\pi}(\mathcal{A}^{\otimes p})$ is zero. Therefore, by Corollary 2.24(2), $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (0, 0)$.
- If $n = e'$, then, by Corollary 2.24(3), we have

$$\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = (C(\alpha) - \bar{u}\alpha, C(\beta) - \bar{u}\beta).$$

However, by assumption, $C(\alpha) = 0$ on Y. Hence,

$$\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = (-\bar{u}\alpha, 0).$$

Finally, $e' - e < e'$ and hence $\mathrm{rsw}_{e'-e,\pi}(\mathcal{A}^{\otimes p}) = 0$, which implies that $\alpha = 0$ and thus also in this case $\mathrm{rsw}_{e',\pi}(\mathcal{A}) = (0, 0)$. □

Proof of Theorem 5.1. If V is a K3 surface and $\mathfrak{p}$ is a prime of good nonordinary reduction, then the special fibre at $\mathfrak{p}$ is a K3 surface having good nonordinary reduction. It is proven in [van der Geer and Katsura 2000, proof of Corollary 9.5] that the natural inclusion $B_Y^2 \hookrightarrow Z_Y^2$ induces an isomorphism $\mathrm{H}^0(Y, B_Y^2) \simeq \mathrm{H}^0(Y, Z_Y^2)$. The short exact sequence

$$0 \to B_Y^2 \to Z_Y^2 \xrightarrow{C} \Omega_Y^2 \to 0$$

implies that the Cartier operator is trivial on $\mathrm{H}^0(Y, Z_Y^2)$. Hence, we are in the setting of the previous lemma. The result now follows from the fact that, for K3 surfaces, $\mathrm{fil}_0 \mathrm{Br}(V_\mathfrak{p}) = \mathrm{Ev}_{-1} \mathrm{Br}(V_\mathfrak{p})$. □

5.1. ***An example.*** This example was suggested by Evis Ieronymou.

Ieronymou and Skorobogatov [2015] prove that, for the diagonal quartic surface $X/\mathbb{Q}_3$ defined by

$$x^4 - y^4 = 12 \cdot z^4 - 9 \cdot w^4, \tag{18}$$

there exists an element $\mathcal{A} \in \mathrm{Br}(X)[3]$ such that the corresponding evaluation map is nonconstant on $X(\mathbb{Q}_3)$. In analogy with Example 4.13, X has potentially good reduction. Let $L = \mathbb{Q}_3(\zeta_4, \alpha)$ with $\alpha^4 = 3$; then $e(L/\mathbb{Q}_3) = 4$ and the change of variables

$$(x, y, z, w) \mapsto \left(x_1, y_1, \frac{z_1}{\alpha}, \frac{w_1}{\alpha^2}\right)$$

sends X_L to the diagonal quartic $\widetilde{X}/L$ given by

$$x_1^4 - y_1^4 = 4 \cdot z_1^4 - w_1^4.$$

Using Lemma 4.2, it is possible to check that the K3 surface $\widetilde{X}$ has good nonordinary reduction over L. The special fibre is Fermat's quartic over $\mathbb{F}_3$, which, by [Shioda 1975], is known to be supersingular. By Lemma 2.32, $\mathrm{res}(\mathcal{A}) \in \mathrm{Br}(X_L) = \mathrm{Br}(\widetilde{X})$ has nonconstant evaluation map. Hence, $\widetilde{X}$ is a K3 surface over a 3-adic field having ramification index 4 and good nonordinary reduction, and being such that $\mathrm{Ev}_{-1} \mathrm{Br}(\widetilde{X})[3] \subsetneq \mathrm{Br}(\widetilde{X})[3]$. Moreover, $e' = 6$ and we proved that $\mathrm{rsw}_{6,\pi} = 0$ on the 3-torsion of $\mathrm{Br}(\widetilde{X})$. Hence, $\mathrm{res}(\mathcal{A})$ has to lie in $\mathrm{fil}_3 \mathrm{Br}(\widetilde{X})[3]$ and be such that $\mathrm{rsw}_{3,\pi}(\mathrm{res}(\mathcal{A})) \neq (0, 0)$.

6. A family of examples

We end this paper by giving an example of a family of K3 surfaces.

Let $\alpha \in \overline{\mathbb{Q}}$ be such that $\alpha^2 \in \mathbb{Z}$ and let V_α be the K3 surface over $k := \mathbb{Q}(\alpha)$ defined by

$$x^3y + y^3z + z^3w - w^4 + \alpha^2 \cdot xyzw - 2 \cdot \alpha^{-1} \cdot xzw^2 = 0. \tag{19}$$

Lemma 6.1. *The class of the quaternion algebra*

$$\mathcal{A} := \left(\frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right) \in \mathrm{Br}(k(V_\alpha))$$

lies in $\mathrm{Br}(V_\alpha)[2]$.

Proof. Let $f := z^2 + \alpha^2 xy$, and C_x, C_z and C_f be the closed subsets of V_α defined by the equations $x = 0$, $z = 0$ and $f = 0$, respectively. The quaternion algebra $\mathcal{A}$ defines an element in $\mathrm{Br}(U)$, where $U := V_\alpha \setminus (C_x \cup C_z \cup C_f)$. The purity theorem for the Brauer group [Colliot-Thélène and Skorobogatov 2021, Theorem 3.7.2] assures the existence of the exact sequence

$$0 \to \mathrm{Br}(V_\alpha)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_D} \bigoplus_D \mathrm{H}^1(k(D), \mathbb{Z}/2), \tag{20}$$

where D ranges over the irreducible divisors of V_α with support in $X \setminus U$ and $k(D)$ denotes the residue field at the generic point of D.

In order to use the exact sequence (20), we need to understand what the prime divisors of V_α with support in $V_\alpha \setminus U = C_x \cup C_z \cup C_f$ look like. Using `Magma` [Bosma et al. 1997], it is possible to check the following:

- C_x has one irreducible component D_1 defined by the set of equations $\{x = 0,\ y^3 z + z^3 w + w^4 = 0\}$.
- C_z has one irreducible component D_2, defined by $\{z = 0,\ x^3 y - w^4 = 0\}$.
- C_f has one irreducible component D_3, defined by

$$\{\alpha^2 xy + z^2 = 0,\ x^3 z^2 + y^2 z^3 + 2\alpha x^2 z w^2 + \alpha^2 x w^4 = 0,\ \alpha^2 y^3 z - x^2 z^2 - 2\alpha x z w^2 - \alpha^2 w^4 = 0\}.$$

Therefore, we can rewrite (20) as

$$0 \to \mathrm{Br}(V_\alpha)[2] \to \mathrm{Br}(U)[2] \xrightarrow{\oplus \partial_{D_i}} \bigoplus_{i=1}^{3} H^1(k(D_i), \mathbb{Z}/2). \tag{21}$$

Moreover, we have an explicit description of the residue map on quaternion algebras: for an element $(a, b) \in \mathrm{Br}(U)[2]$,

$$\partial_{D_i}(a, b) = \left[(-1)^{v_i(a)v_i(b)} \frac{a^{v_i(b)}}{b^{v_i(a)}}\right] \in \frac{k(D_i)^\times}{k(D_i)^{\times 2}} \simeq H^1(k(D_i), \mathbb{Z}/2), \tag{22}$$

where v_i is the valuation associated to the prime divisor D_i. This follows from the definition of the tame symbols in Milnor K-theory together with the compatibility of the residue map with the tame symbols given by the Galois symbols (see [Gille and Szamuely 2017, Proposition 7.5.1]).

We can proceed with the computation of the residue maps ∂_{D_i} for $i = 1, \ldots, 3$:

(1) $v_1(x) = 1$ and $v_1(f) = v_1(z) = 0$. Hence,

$$\partial_{D_1}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[\left(\frac{f}{z^2}\right)^{-1}\right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^{\times 2}},$$

where the last equality follows from the fact that $x = 0$ on D_1; thus, $f|_{D_1} = z^2$.

(2) $v_2(z) = 1$ and $v_2(f) = v_2(x) = 0$. Hence,

$$\partial_{D_2}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[\left(\frac{f}{z^2}\right)\left(-\frac{z}{x}\right)^2\right] = \left[\frac{f}{x^2}\right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^{\times 2}},$$

where the last equality follows from the fact that $z = 0$ and $x^3y = w^4$ on D_2; thus, $f|_{D_2} = \alpha^2 xy$ and $\alpha^2 y/x = \alpha^2(w/x)^4 = (\alpha w^2/x^2)^2$.

(3) $v_3(f) = 1$ and $v_3(x) = v_3(z) = 0$. On D_3, we have

$$y^3z = w^4 + \frac{2}{\alpha}xzw^2 + \frac{x^2z^2}{\alpha^2} = \left(w^2 + \frac{xz}{\alpha}\right)^2,$$
$$y^3z = (xy)^3\frac{z}{x}\frac{1}{x^2} = -\frac{z}{x}\left(\frac{z^2}{\alpha^2}\right)^3\frac{1}{x^2} = \frac{-z}{x}\left(\frac{z^3}{\alpha^3 x}\right)^2 \quad \text{using } xy = -\frac{1}{\alpha^2}z^2,$$

and thus

$$\partial_{D_3}\left(\frac{f}{z^2}, -\frac{z}{x}\right) = \left[-\frac{z}{x}\right] = \left[\left(\frac{\alpha^3 x}{z^3}\left(w^2 + \frac{xz}{\alpha}\right)\right)^2\right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^{\times 2}}.$$

Therefore, $\partial_{D_i}(\mathcal{A}) = 0$ for all $i \in \{1, 2, 3\}$, and hence $\mathcal{A} \in \mathrm{Br}(V_\alpha)$. □

Let $\mathfrak{p}$ be a prime above 2 and $\mathcal{O}_\mathfrak{p}$ be the valuation ring of $k_\mathfrak{p}$. Note that $2 \cdot \alpha^{-1} \in \mathcal{O}_\mathfrak{p}$ if and only if $\alpha^2 \not\equiv 0 \bmod 8$; we can define $\mathcal{X}_\alpha$ to be the $\mathcal{O}_\mathfrak{p}$-scheme defined by (19). If $\alpha^2 \not\equiv 0 \bmod 8$, then $\mathcal{X}_\alpha$ is smooth and hence V_α has good reduction at $\mathfrak{p}$; we denote by X_α the base change of V_α to $k_\mathfrak{p}$.

Theorem 6.2. *Assume that $\alpha^2 \not\equiv 0 \bmod 8$. Then $\mathcal{X}_\alpha$ has good ordinary reduction if and only if $\alpha^2 \equiv 1 \bmod 2$. The evaluation map attached to $\mathcal{A}$,*

$$\mathrm{ev}_\mathcal{A} : X_\alpha(k_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z},$$

is nonconstant if and only if

$$\alpha^2 \not\equiv 0 \mod 4.$$

Proof. For K3 surfaces, the evaluation map attached to $\mathcal{A}$ is nonconstant if and only if $\mathcal{A} \notin \mathrm{fil}_0 \mathrm{Br}(X_\alpha)$ (see Section 4.1).

- If $\alpha^2 \equiv 1 \bmod 2$, then the special fibre Y_α is defined by

$$x^3y + y^3z + z^3w + w^4 + xyzw = 0.$$

By Lemma 4.2, Y_α is an ordinary K3 surface. Lemma 4.3 says that a generator (as a $k(\mathfrak{p})$-vector space) of $\mathrm{H}^0(Y_\alpha, \Omega^2_{Y_\alpha})$ is given by the global 2-form ω that can be written (locally) as

$$\frac{d(y/x) \wedge d(z/x)}{(z^3 + xyz)/x^3} = \frac{x^2}{z^2 + xy} d\left(\frac{y}{x}\right) \wedge d\left(\frac{z}{x}\right)\frac{x}{z} = \left(\frac{x^2}{z^2 + xy}\right) \cdot d\left(\frac{z^2 + xy}{x^2}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right),$$

where the last equality follows from $d((z^2 + xy)/x^2) \wedge d(z/x) = d(y/x) \wedge d(z/x)$. Hence, we can write ω as $df/f \wedge dg/g$, with $f = (z^2 + xy)/x^2$ and $g = z/x$. Finally, we see that, by Proposition 2.27,

$$[\mathcal{A}] = \left[\left\{\frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\}\right] = \left[\left\{\frac{z^2 + \alpha^2 \cdot xy}{x^2}, -\frac{z}{x}\right\}\right] = \rho_0(\omega, 0)$$

since $(z^2 + \alpha^2 \cdot xy)/x^2$ and $-z/x$ are lifts to characteristic 0 of f and g, respectively. Hence, using Proposition 2.28, $\mathrm{rsw}_{e',\pi}(\mathcal{A}) \neq (0, 0)$ and $\mathcal{A} \notin \mathrm{fil}_{e'-1} \mathrm{Br}(X_\alpha) \supseteq \mathrm{fil}_0 \mathrm{Br}(X_\alpha)$.

- If $\alpha^2 \equiv 2 \bmod 4$, then the special fibre Y_α is defined by

$$x^3y + y^3z + z^3w + w^4 = 0.$$

By Lemma 4.2, Y_α is a nonordinary K3 surface over $k(\mathfrak{p})$. Lemma 4.3 says that $\mathrm{H}^0(Y_\alpha, \Omega^2_{Y_\alpha})$ is generated (as a $k(\mathfrak{p})$-vector space) by the 2-form ω that can be written (locally) as

$$\frac{d(y/x) \wedge d(z/x)}{z^3/x^3} = \left(\frac{x^2}{z^2}\right) \cdot d\left(\frac{y}{x}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right) = d\left(\frac{xy}{z^2}\right) \wedge \left(\frac{x}{z}\right) \cdot d\left(\frac{z}{x}\right),$$

where the last equality follows from the fact that, since we are working over a field of characteristic 2, $(x/z)^2 d(y/x) = d(xy/z^2)$. Hence, in this case, we can write ω as $d(f \cdot dg/g)$, with $f = xy/z^2$ and $g = z/x$. Since $\alpha^2 \equiv 2 \bmod 4$, the prime ideal (2) is ramified in the field extension $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ and $\pi = \alpha$ is a uniformiser. By Proposition 2.27,

$$\left[\left\{\frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\}\right] = \left[\left\{1 + \alpha^2 \frac{xy}{z^2}, -\frac{z}{x}\right\}\right] = \rho_2\left(f \cdot \frac{dg}{g}\right)$$

since xy/z^2 and $-z/x$ are two lifts to characteristic 0 of f and g, respectively. Hence, by Proposition 2.28, $\mathrm{rsw}_{2,\pi}(\mathcal{A}) \neq (0, 0)$. Therefore, $\mathcal{A} \notin \mathrm{fil}_1 \mathrm{Br}(X_\alpha)$, and thus $\mathcal{A} \notin \mathrm{fil}_0 \mathrm{Br}(X_\alpha) \subseteq \mathrm{fil}_1 \mathrm{Br}(X_\alpha)$.

- If $\alpha^2 \equiv 0 \bmod 4$, then the special fibre Y_α is defined by

$$x^3y + y^3z + z^3w + w^4 + xzw^2 = 0.$$

Again, by Lemma 4.2, Y_α is a nonordinary K3 surface over $k(\mathfrak{p})$. If $\mathbb{Q}_2(\alpha)/\mathbb{Q}_2$ is unramifie,d then we are done by Theorem 1.4. If the field extension is ramified and π is a uniformiser, then, since $\alpha^2 \equiv 0 \bmod 4$ and $\alpha^2 \not\equiv 0 \bmod 8$, we have $\alpha^2 = \pi^4 \beta$ with $\beta \in \mathcal{O}_\mathfrak{p}^\times$. Hence, if we look at the corresponding element in $k^2(K^h)$ via the isomorphism of (8), we see

$$\mathcal{A} \mapsto \left\{\frac{z^2 + \alpha^2 \cdot xy}{z^2}, -\frac{z}{x}\right\} = \left\{1 + \pi^4 \cdot \frac{\beta \cdot xy}{z^2}, -\frac{z}{x}\right\} \in U^4 k^2(K^h).$$

Hence, using again Proposition 2.28, $\mathcal{A} \in \mathrm{fil}_0 \mathrm{Br}(X_\alpha)[2]$. □

Remark 6.3. The case $\alpha^2 \equiv 2 \bmod 4$ proves that the bound appearing in Theorem 1.4 is optimal. In fact, we are able to find examples of K3 surfaces over a quadratic field extension of $\mathbb{Q}$ such that there is a prime above 2 whose ramification index is $e(\mathfrak{p}/2) = 2$ and which plays a role in the Brauer–Manin obstruction to weak approximation.

Finally, when $\alpha^2 \equiv 0 \bmod 4$ and $e(\mathfrak{p}/2) = 1$, we know from Theorem 1.4 that $\mathrm{Ev}_{-1} \mathrm{Br}(X_\alpha) = \mathrm{Br}(X_\alpha)$. However, if $e(\mathfrak{p}/2) = 2$, we just showed that the element $\mathcal{A}$ of the previous theorem lies in $\mathrm{Ev}_{-1} \mathrm{Br}(X_\alpha)[2]$ and not that the whole Brauer group of X_α is contained in $\mathrm{Ev}_{-1} \mathrm{Br}(X_\alpha)$.

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