Algebra & Number Theory

Volume 20
2026
No. 8

Supersymmetric monoidal categories

Steven V Sam and Andrew Snowden

msp

ALGEBRA AND NUMBER THEORY 20:8 (2026)
https://doi.org/10.2140/ant.2026.20.1479

Supersymmetric monoidal categories

Steven V Sam and Andrew Snowden

We develop the idea of a supersymmetric monoidal supercategory, following ideas of Kapranov. Roughly, this is a monoidal category in which the objects and morphisms are $\mathbb{Z}/2$-graded, equipped with isomorphisms $X \otimes Y \to Y \otimes X$ of parity $|X||Y|$ on homogeneous objects. There are two fundamental examples: the groupoid of spin-sets, and the category of isomeric vector spaces equipped with the half tensor product; other important examples can be derived from these (such as the category of linear spin species). There are also two general constructions. The first is the exterior algebra of a supercategory (due to Ganter and Kapranov). The second is a construction we introduce, called Clifford eversion. This defines an equivalence between a certain 2-category of supersymmetric monoidal supercategories and a corresponding 2-category of symmetric monoidal supercategories. We use our theory to better understand some aspects of the isomeric superalgebra, such as certain factors of $\sqrt{2}$ in the theory of Q-symmetric functions and Schur–Sergeev duality.

1. Introduction 1479
2. Supercategories 1486
3. Monoidal structures 1489
4. Some 2-category theory 1503
5. Spin-symmetric groups 1511
6. Linear spin-species 1517
7. Isomeric vector spaces 1522
8. Clifford eversion 1526
9. Schur–Sergeev duality 1536
Appendix. Bott periodicity 1540
References 1541

1. Introduction

The categorical analog of a commutative algebra is a symmetric (or braided) linear monoidal category. These categories have played a fundamental role in category theory and representation theory, among other places. The purpose of this paper is to develop a categorical analog of supercommutative superalgebras, following ideas of Kapranov [2018, Section 3.4]; we call these supersymmetric (or superbraided) monoidal supercategories. In the rest of this introduction, we explain the basic idea behind the definition, some of

Sam was supported by NSF grant DMS-1812462. Snowden was supported by NSF grant DMS-1453893.
MSC2020: primary 18M05; secondary 05E10.
Keywords: superalgebra, supercategories.

© 2026 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

our results and applications, and compare our contributions to the existing discussions of supercategories in the literature.

1.1. ***The definition.*** Recall that a supercommutative superalgebra is a $\mathbb{Z}/2$-graded algebra such that $xy = (-1)^{|x||y|}yx$ holds for all homogeneous elements x and y. Here we write $|x| \in \mathbb{Z}/2$ for the degree of a homogeneous element x. Thus, we expect that, in a superbraided monoidal category, there should be even and odd homogeneous objects, and there should be an isomorphism between $X \otimes Y$ and $(-1)^{|X||Y|}$ "times" $Y \otimes X$ for homogeneous objects X and Y.

The first puzzle here is what it means to negate an object. This has been well understood for some time. Let V be a finite-dimensional super-vector space. (Recall that a super-vector space is simply a $\mathbb{Z}/2$-graded vector space.) The superdimension of V is defined to be $\mathrm{sdim}(V) = \dim(V_0) - \dim(V_1)$. Let $\Pi(V)$ be the super-vector space given by $\Pi(V)_i = V_{i+1}$, i.e., $\Pi(V)$ is obtained from V by interchanging the two graded pieces. Then $\mathrm{sdim}(\Pi(V)) = -\mathrm{sdim}(V)$. Thus, $\Pi(V)$ can be thought of as the negative of V.

The identity map $V \to \Pi(V)$ is an odd-degree isomorphism. This map characterizes $\Pi(V)$ uniquely up to even isomorphism: if W is any super-vector space equipped with an odd-degree isomorphism $V \to W$, then there is a unique even isomorphism $W \to \Pi(V)$ making the triangle commute. We can thus speak of $\Pi(V)$ for any object V in a category in which the morphisms are $\mathbb{Z}/2$-graded. Precisely, we define a *supercategory* to be a category enriched over super-vector spaces, so that all Hom spaces are super-vector spaces. Given an object V in a supercategory, we define $\Pi(V)$ to be an object equipped with an odd-degree isomorphism $V \to \Pi(V)$ if such an object and morphism exist. (We assume in what follows that they always do.) We can thus regard $\Pi(V)$ as the negative of V.

Following our initial intuition, suppose that we have a $\mathbb{Z}/2$-supercategory (i.e., one in which the objects are $\mathbb{Z}/2$-graded) equipped with a monoidal structure $\otimes$. Then we see that a superbraiding should be a natural isomorphism

$$\beta_{X,Y} : X \otimes Y \to \Pi^{|X||Y|}(Y \otimes X),$$

defined for all homogeneous objects X and Y.

Of course, as with ordinary braidings, our superbraiding should satisfy some version of the hexagon axioms. It turns out that the usual hexagon axioms are inconsistent in this setting due to the signs that arise from composition in supercategories, but one can attempt to resolve the inconsistencies by allowing the

0-category	1-category
vector space	supercategory
commutative algebra	symmetric monoidal supercategory
super-vector space	$\mathbb{Z}/q$-supercategory
commutative superalgebra	symmetric monoidal $\mathbb{Z}/q$-supercategory
supercommutative superalgebra	supersymmetric monoidal $\mathbb{Z}/q$-supercategory

Table 1. Some analogies relevant to this paper. Here q should be even.

hexagons to commute up to scalars. This ends up working if we assume the ground field has an eighth root of unity; however, the scalars that come up are not at all obvious. By working with $\mathbb{Z}/4$-supercategories (or $\mathbb{Z}/q$-supercategories for any q divisible by 4), one can get by with much nicer systems of scalars that only involve signs (and thus make sense over general fields). In both cases, we introduce a notion of factor system (Definition 3.5) to axiomatize the conditions on the scalars that appear in the hexagon axioms. One of the main contributions of this article is identifying these conditions and verifying them in important examples of interest.

We have thus arrived at the main definition: a superbraiding is a natural isomorphism β as above such that the hexagon axioms hold up to scalars determined by a choice of factor system. See Sections 3.5 and 3.6 for details.

1.2. ***The 2-categorical perspective.*** One can define the notion of supercommutative superalgebra in elementary terms, as we did above. There is also a more abstract approach: one endows the monoidal category of super-vector spaces with a new symmetry τ (namely $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$), and realizes supercommutative superalgebras as the *commutative* algebra objects in this category. The more abstract approach has several benefits; for example, any properties of commutative algebras in symmetric tensor categories can immediately be applied to supercommutative superalgebras, and it also enables one to easily consider similar structures, such as Lie superalgebras.

In our discussion of superbraided supercategories above, we have taken the elementary approach. We now explain a version of the abstract approach. Let $\mathscr{C}$ be the 2-category of $\mathbb{Z}/q$-supercategories, with q even. This 2-category has a natural monoidal structure $\boxtimes$, given by a standard product construction for supercategories. Given two $\mathbb{Z}/q$-supercategories $\mathcal{A}$ and $\mathcal{B}$, we define $\mathrm{T} : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{B} \boxtimes \mathcal{A}$ to be the equivalence given by $X \boxtimes Y \mapsto \Pi^{|X||Y|}(Y) \boxtimes X$, where X and Y are homogeneous objects of $\mathcal{A}$ and $\mathcal{B}$. One should think of the equivalence T as analogous to the isomorphism τ from the previous paragraph. We show that T (together with a choice of factor system) defines a symmetry on $\mathscr{C}$ in Section 4. One can then regard supersymmetric monoidal supercategories as the *symmetric* monoidal objects in the symmetric monoidal 2-category $(\mathscr{C}, \mathrm{T})$.

1.3. ***Spin-symmetric groups.*** The most fundamental example of a symmetric monoidal category is the groupoid of finite sets under disjoint union. A skeletal version $\mathcal{S}$ of this category can be constructed explicitly, as follows. There is one object $[n]$ for each natural number n. We put $\mathrm{End}_{\mathcal{S}}([n]) = \mathfrak{S}_n$ and $\mathrm{Hom}_{\mathcal{S}}([n], [m]) = \varnothing$ for $n \neq m$. The monoidal operation $\sqcup$ is defined on objects by $[n] \sqcup [m] = [n+m]$ and on morphisms using the natural homomorphism $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$. Finally, the symmetry isomorphism $[n] \sqcup [m] \to [m] \sqcup [n]$ is given by the element $\tau_{n,m} \in \mathfrak{S}_{n+m}$ defined by

$$\tau_{n,m}(i) = \begin{cases} i+m & \text{if } 1 \le i \le n, \\ i-n & \text{if } n+1 \le i \le n+m. \end{cases}$$

The hexagon axioms amount to some identities satisfied by the $\tau_{n,m}$ elements.

The above picture has an analog in the setting of supersymmetric monoidal categories in which the symmetric groups are replaced by spin-symmetric groups. Recall that the n-th spin-symmetric group $\widetilde{\mathfrak{S}}_n$ is a certain central extension of $\mathfrak{S}_n$ by $\mathbb{Z}/2$; see Section 5.2. We define a skeletal category $\tilde{\mathcal{S}}$ as follows. There is again one object $[n]$ for each natural number n. We put $\mathrm{End}_{\tilde{\mathcal{S}}}([n]) = \boldsymbol{k}[\widetilde{\mathfrak{S}}_n]/(c+1)$, where $c \in \widetilde{\mathfrak{S}}_n$ is the distinguished central element, and define $\mathrm{Hom}_{\tilde{\mathcal{S}}}([n],[m]) = 0$ if $n \neq m$. The monoidal operation $\sqcup\!\!\sqcup$ on $\tilde{\mathcal{S}}$ is defined on objects by $[n] \sqcup\!\!\sqcup [m] = [n+m]$, and on morphisms using the natural map $\widetilde{\mathfrak{S}}_n \times \widetilde{\mathfrak{S}}_m \to \widetilde{\mathfrak{S}}_{n+m}$. Finally, the supersymmetry is induced by a particular choice of lift $\tilde{\tau}_{n,m} \in \widetilde{\mathfrak{S}}_{n+m}$ of the element $\tau_{n,m}$ defined above. In Section 5.4, we verify that the axioms of a supersymmetric monoidal category hold in this case; this amounts to establishing some identities satisfied by the $\tilde{\tau}_{n,m}$ elements.

The definition and study of the elements $\tilde{\tau}_{n,m}$ (in Section 5.3), while totally elementary, is one of the most important parts of this paper. For a particular choice of n and m, there are two lifts of $\tau_{n,m}$ to $\widetilde{\mathfrak{S}}_{n+m}$, and neither is preferred. We find, however, that the lifts $\tilde{\tau}_{n,m}$ we define have important compatibility properties as n and m vary; in other words, there is a preferred *system* of lifts (up to the action of a single $\mathbb{Z}/2$). This strikes us as an important discovery. In fact, as discussed in Section 1.6 below, we show that essentially every supersymmetry can, in a sense, be constructed from these elements.

The category $\mathcal{S}$ is not merely the simplest example of a symmetric monoidal category: it is also universal. Indeed, suppose that $\mathcal{C}$ is a symmetric monoidal category. Given an object X of $\mathcal{C}$, the symmetric group $\mathfrak{S}_n$ naturally acts on $X^{\otimes n}$. This construction induces a symmetric monoidal functor $\mathcal{S} \to \mathcal{C}$ via $[n] \mapsto X^{\otimes n}$. This observation shows that $(\mathcal{S}, [1])$ is the universal symmetric monoidal category equipped with an object, in the sense that giving a symmetric monoidal functor $\mathcal{S} \to \mathcal{C}$ is equivalent to giving an object of $\mathcal{C}$.

An analogous result holds for $\tilde{\mathcal{S}}$. Indeed, if $\mathcal{C}$ is a supersymmetric monoidal $\mathbb{Z}$-category and X is a degree 1 object in $\mathcal{C}$, then we show that the spin-symmetric group $\widetilde{\mathfrak{S}}_n$ naturally acts on $X^{\otimes n}$. This leads to the following theorem:

Theorem 1.1. $(\tilde{\mathcal{S}}, [1])$ *is the universal supersymmetric monoidal $\mathbb{Z}$-supercategory equipped with an object of degree* 1.

Another important example of a supersymmetric category comes from the representation theory of spin-symmetric groups. A *linear spin-species* is a sequence $(V_n)_{n\geq 0}$ where V_n is a representation of $\widetilde{\mathfrak{S}}_n$ (that interacts with superstructures appropriately). The category of such sequences, denoted by $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$, admits a monoidal structure defined by inducing from Young subgroups. The $\tilde{\tau}_{n,m}$ elements can be used to construct a supersymmetry on this category. The details are carried out in Section 6.

1.4. ***The half tensor product.*** The second most fundamental example of a symmetric monoidal category is the category **Vec** of vector spaces under tensor product. The category **SVec** of super-vector spaces is *not* supersymmetric: it is a "plain" symmetric monoidal supercategory since the isomorphism $V \otimes W \to W \otimes V$ has even degree. (The same applies to any category where the tensor product and symmetry are inherited from **SVec**, such as the category of representations of a Lie superalgebra.) However, there is a supersymmetric analog to **Vec**, which we now discuss.

Let V be a super-vector space. An *isomeric structure* on V is an odd-degree automorphism $\nu : V \to V$ that squares to the identity. Suppose that U and V are super-vector spaces equipped with isomeric structures μ and ν. Then $\mu \otimes \nu$ is an even-degree automorphism of $U \otimes V$ that squares to -1. The *half tensor product* of U and V, denoted by $2^{-1}(U \otimes V)$, is defined to be the ζ_4-eigenspace of $\mu \otimes \nu$. (Here $\zeta_4 = \sqrt{-1}$.) The choice of ζ_4 does not matter much, as $\mu \otimes 1$ (or $1 \otimes \nu$) interchanges the $+\zeta_4$ and $-\zeta_4$ eigenspaces.

This construction appears frequently in the representation theory of superalgebras. Suppose that A is a $\mathbb{C}$-superalgebra and M is a finite-dimensional simple A-module. The super version of Schur's lemma states that $\mathrm{End}_A(M)$ is either 1-dimensional ("type M") or 2-dimensional ("type Q"); in the latter case, the algebra $\mathrm{End}_A(M)$ is generated over $\mathbb{C}$ by a isomeric structure. Now suppose that B is a second $\mathbb{C}$-superalgebra and N is a finite-dimensional simple B-module. If either M or N has type M, then $M \otimes N$ is a simple $(A \otimes B)$-module. If both M and N have type Q, then $2^{-1}(M \otimes N)$ is a simple module, and $M \otimes N \cong 2^{-1}(M \otimes N) \otimes \mathbb{C}^{1|1}$. (See [Cheng and Wang 2012, Section 3.1.3] for details.)

Returning to the general situation, let (U, μ) and (V, ν) be as above. A simple computation shows that the natural symmetry $\tau_{U,V} : U \otimes V \to V \otimes U$ carries the ζ_4-eigenspace of $\mu \otimes \nu$ into the $(-\zeta_4)$-eigenspace of $\nu \otimes \mu$. In particular, $\tau_{U,V}$ does *not* map $2^{-1}(U \otimes V)$ into $2^{-1}(V \otimes U)$. However, the composition $(\nu \otimes 1) \circ \tau_{U,V}$ does map $2^{-1}(U \otimes V)$ to $2^{-1}(V \otimes U)$. This is an odd-degree isomorphism, which suggests a lurking supersymmetry.

In Section 7, we define a $\mathbb{Z}/2$-supercategory **Ismr** by taking the degree 0 objects to be super-vector spaces and the degree 1 objects to be super-vector spaces equipped with an isomeric structure. We define a monoidal operation $\odot$ on this category by using the half tensor product on two degree 1 objects and the ordinary tensor product on other pairs of objects. We show that this does indeed define a monoidal structure; this is not obvious, as the associator maps are nontrivial. We further show that the construction in the previous paragraph (scaled by an appropriate root of unity) defines a supersymmetry on this monoidal structure. Thus, **Ismr** is an example of a supersymmetric monoidal $\mathbb{Z}/2$-supercategory. We believe this places the half tensor product construction in its proper theoretical framework.

1.5. ***Exterior algebras.*** Let $\mathcal{A}$ be a $\mathbb{C}$-linear category. One can then define the n-th symmetric power of $\mathcal{A}$, denoted by $\mathrm{Sym}^n(\mathcal{A})$, to be the category of $\mathfrak{S}_n$-equivariant objects in the n-th power of $\mathcal{A}$ (for an appropriate notion of power). The sum of these categories, denoted by $\mathrm{Sym}(\mathcal{A})$, is naturally a symmetric monoidal category, and is the universal such category to which $\mathcal{A}$ maps.

Now suppose that $\mathcal{A}$ is a supercategory. Ganter and Kapranov [2014, Section 3.6] defined the n-th exterior power of $\mathcal{A}$, denoted by $\bigwedge^n(\mathcal{A})$, to be the category of appropriately $\widetilde{\mathfrak{S}}_n$-equivariant objects in the n-th power of $\mathcal{A}$. We show that the sum of these categories, denoted by $\bigwedge(\mathcal{A})$, is naturally a supersymmetric monoidal supercategory, and has a universal property similar to the symmetric power. This construction provides a large source of supersymmetric categories. See Section 6.8 for details.

1.6. ***Clifford eversion.*** Let $\mathcal{A}$ be a nice $\mathbb{Z}/q$-supercategory, with q even; "nice" means that $\mathcal{A}$ is additive, Karoubian and admits a Π structure. In Section 8, we define a new nice $\mathbb{Z}/q$-supercategory $\mathrm{Cl}(\mathcal{A})$, called

the *Clifford eversion* of $\mathcal{A}$, by considering certain kinds of Clifford modules in $\mathcal{A}$. (This construction is essentially a generalization of the category **Ismr** discussed above; see Section 7.5.) If $\mathcal{A}$ has a symmetric monoidal structure, then we show that $\mathrm{Cl}(\mathcal{A})$ naturally has a supersymmetric monoidal structure; this construction again hinges on the $\tilde{\tau}_{n,m}$ elements. Moreover, we sketch a proof for:

Theorem 1.2. *Clifford eversion defines an equivalence between the* 2*-category of nice symmetric monoidal* $\mathbb{Z}/q$*-supercategories and the* 2*-category of nice supersymmetric monoidal* $\mathbb{Z}/q$*-supercategories.*

Let $\mathscr{C}$ be the 2-category of nice $\mathbb{Z}/q$-supercategories. This category has a natural monoidal structure that admits an obvious symmetry Σ; it also admits the less obvious symmetry T discussed in Section 1.2. In fact, Clifford eversion induces an equivalence of symmetric monoidal 2-categories $(\mathscr{C}, \Sigma) \to (\mathscr{C}, \mathrm{T})$. This recovers the above theorem, but is a more fundamental statement. However, we only partially prove this statement since the notion of equivalence of symmetric monoidal 2-categories is rather complicated.

Let $\mathcal{A}$ be nice symmetric monoidal $\mathbb{Z}/q$-supercategory with an exact structure, and let $\mathrm{K}_+(\mathcal{A})$ be the quotient of the Grothendieck group by the relations $[\Pi(X)] = [X]$. We show (Proposition 8.21) that there is a natural isomorphism of rings

$$\mathrm{K}_+(\mathcal{A})\big[\tfrac{1}{\sqrt{2}}\big] \cong \mathrm{K}_+(\mathrm{Cl}(\mathcal{A}))\big[\tfrac{1}{\sqrt{2}}\big].$$

This helps explain the appearance of certain factors of $\sqrt{2}$ in the theory of Q-symmetric functions.

1.7. ***Sergeev–Schur duality.*** (We take the base field to be $\mathbb{C}$ in this section.) Classical Schur–Weyl duality connects the representation theory of the general linear group and the symmetric groups. Let $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ be the category of polynomial representations of the infinite general linear group; this is the tensor category generated by the standard representation $\boldsymbol{V}$. The symmetric group $\mathfrak{S}_n$ acts on $\boldsymbol{V}^{\otimes n}$ by permuting tensor factors, and so we obtain a functor $\mathrm{Rep}(\mathfrak{S}_n) \to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ by $M \mapsto M \otimes_{\mathbb{C}[\mathfrak{S}_n]} \boldsymbol{V}^{\otimes n}$. Let $\mathrm{Rep}(\mathfrak{S}_*)$ be the category of sequences $(V_n)_{n\geq 0}$ where V_n is a representation of the symmetric group $\mathfrak{S}_n$; such sequences are sometimes called *linear species*. Then one formulation of Schur–Weyl duality states that the above functor extends to an equivalence $\mathrm{Rep}(\mathfrak{S}_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$. This equivalence respects various additional structures on the two categories. In particular, it is an equivalence of symmetric monoidal categories, where $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ is endowed with the usual tensor product and $\mathrm{Rep}(\mathfrak{S}_*)$ is given the induction tensor product, defined by

$$(V \otimes W)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_n} (V_i \otimes W_j).$$

This equivalence is extremely useful when studying algebra within the tensor category $\mathrm{Rep}(\mathfrak{S}_*)$, as it provides a bridge to more classical theory.

Sergeev–Schur duality is an analog of Schur–Weyl duality in which the general linear group is replaced by the isomeric Lie superalgebra. Precisely, let $\boldsymbol{V} = \mathbb{C}^{\infty|\infty}$ and fix an odd-degree isomorphism $\alpha : \boldsymbol{V} \to \boldsymbol{V}$ squaring to 2. The infinite isomeric Lie superalgebra $\mathfrak{q}$ is defined to be the subalgebra of $\mathfrak{gl}(\boldsymbol{V})$ that preserves α. The symmetric group $\mathfrak{S}_n$ acts on $\boldsymbol{V}^{\otimes n}$ by permuting tensor factors. However, there are additional natural endomorphisms that commute with $\mathfrak{q}$; namely, we can apply α to any tensor factor. To

make this more precise, let Cl_n denote the n-th Clifford algebra; this has n anticommuting generators that each square to 2. Define the Hecke–Clifford algebra $\mathcal{H}_n$ to be the semidirect product $\mathfrak{S}_n \ltimes \mathrm{Cl}_n$. Then $\mathcal{H}_n$ acts on $\boldsymbol{V}^{\otimes n}$ and commutes with $\mathfrak{q}$. As in the previous paragraph, we obtain an induced functor $\mathrm{Rep}(\mathcal{H}_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$. One formulation of Sergeev–Schur duality is that this is an equivalence (and respects the natural symmetric monoidal structures).

There is a close relationship between the algebra $\mathcal{H}_n$ and the spin-symmetric groups: we have an isomorphism $\mathcal{H}_n \cong \boldsymbol{k}[\widetilde{\mathfrak{S}}_n]/(c+1) \otimes \mathrm{Cl}_n$. Thus, an $\mathcal{H}_n$-module can be viewed as a Clifford module in the category $\mathrm{Rep}(\widetilde{\mathfrak{S}}_n)$ of linear spin species. We thus see that $\mathrm{Rep}(\mathcal{H}_*)$ can be identified with the Clifford eversion of the category $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ of sequences of $\widetilde{\mathfrak{S}}_n$-representations. We can thus rephrase the above discussion as follows:

Theorem 1.3. *The category* $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$ *is equivalent to the Clifford eversion of the category* $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ *of linear spin species.*

We show that this equivalence respects natural symmetric monoidal structures on the two categories. We view the above theorem as an elegant expression of Sergeev–Schur duality, as it provides a direct link between the isomeric algebra and the spin-symmetric groups. See Section 9 for details.

The Grothendieck ring of $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$ is naturally identified with the ring Γ of odd symmetric functions. This ring has a basis consisting of the Schur Q-functions Q_λ. However, they do not represent classes of simple objects in general: one must adjust them by appropriate powers of $\sqrt{2}$ (which depend on λ as well as which category is being used). One of our goals was to better understand the nature of these powers of $\sqrt{2}$, and the effect of Clifford eversion on Grothendieck rings is one piece of that puzzle. Unfortunately, there is one more piece which is not explained by our general framework: some simple modules of $\boldsymbol{k}[\widetilde{\mathfrak{S}}_n]/(c+1)$ carry an action of Cl_1, while others do not, and this also factors into the powers of $\sqrt{2}$ that appear.

1.8. ***Relation to previous work.*** Kapranov [2018, Section 3.4] proposed a sketch for the definition of a supersymmetric monoidal supercategory. A large portion of this article is devoted to providing the details for this sketch.

Versions of (monoidal) supercategories have been considered in [Brundan and Ellis 2017; Kang et al. 2016; Ellis and Lauda 2016]. However, these sources do not treat the notion of (super)braiding or (super)symmetry for a monoidal supercategory, which, as far as we know, is treated in detail for the first time here. The Clifford twist in [Kang et al. 2016, Section 2.1] is similar to our notion of Clifford eversion, but is not the same (for instance, the Clifford twist is an involution even if the coefficient field does not contain $\sqrt{-1}$).

1.9. ***Open problems.*** We mention a few open problems:

(a) In Remark 3.16, we note a connection between even factor systems and degree 3 group cohomology. However, this connection is not complete, and it does not apply to odd factor systems (which is the relevant case for supersymmetries). It would be helpful to have a better understanding of these issues.

(b) Is there a conceptual a priori explanation for why the 2-categories of symmetric and supersymmetric monoidal supercategories are equivalent?

(c) Is there a "natural" construction of the groupoid of spin-sets? That is, is there a natural category where the objects are some kind of "enhanced" finite sets whose automorphism groups are spin-symmetric groups? This might clarify a number of our constructions (such as the $\tilde{\tau}_{n,m}$ elements). Is there a nongroupoid version?

(d) We have provided explanations for most of the proof of Theorem 8.16, but it remains incomplete.

1.10. ***Notation.*** We fix a field $\boldsymbol{k}$ throughout, which we assume does not have characteristic 2. One could allow $\boldsymbol{k}$ to be a commutative $\mathbb{Z}\left[\frac{1}{2}\right]$-algebra, but we stick to fields for simplicity. We fix a compatible system of primitive 2-power roots of unity ζ_{2^k} in $\bar{\boldsymbol{k}}$; by compatible, we mean that $\zeta_{2^{k+1}}^2 = \zeta_{2^k}$.

In this article, "superalgebra" means an algebra with a $\mathbb{Z}/2$-grading and a "commutative superalgebra" refers to a commutative algebra with a $\mathbb{Z}/2$-grading (which does not affect the definition of commutative). This is sometimes used in the literature to mean a skew-commutative algebra; we reserve the phrase "supercommutative superalgebra" for this, i.e., an algebra A with a $\mathbb{Z}/2$-grading such that $xy = (-1)^{|x||y|}yx$ for homogeneous elements $x, y \in A$.

2. Supercategories

2.1. ***Super-vector spaces.*** A *super-vector space* over $\boldsymbol{k}$ is a $\mathbb{Z}/2$-graded $\boldsymbol{k}$-vector space. Given a super-vector space V, we write V_i for the subspace of homogeneous elements of degree i. For $v \in V$, we write $|v| = i$ to indicate that v is a homogeneous element of degree i. For $n \in \mathbb{Z}$, we let $V[n]$ be the super-vector space with $V[n]_i = V_{n+i}$.

Let V and W be two super-vector spaces and let $f : V \to W$ be a linear map. We say that f is *homogeneous* of parity n if $f(V_i) \subseteq W_{i+n}$. We write $\mathrm{Hom}(V, W)$ for the set of all linear maps, and $\mathrm{Hom}(V, W)_i$ for those of parity i. Every linear map decomposes uniquely as a sum of even and odd maps, and so we have a decomposition

$$\mathrm{Hom}(V, W) = \mathrm{Hom}(V, W)_0 \oplus \mathrm{Hom}(V, W)_1.$$

In this way, we regard $\mathrm{Hom}(V, W)$ itself as a super-vector space. We let **SVec** be the category of super-vector spaces and all linear maps, and $\mathbf{SVec}^\circ$ the category of super-vector spaces and even linear maps.

The tensor product of two super-vector spaces is just the tensor product of the underlying vector spaces, graded in the usual manner. Given homogeneous linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, we define $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ to be the linear map given on homogeneous elements by the formula

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|v_1||f_2|} f_1(v_1) \otimes f_2(v_2).$$

We caution that $\otimes$ is *not* a functor $\mathbf{SVec} \times \mathbf{SVec} \to \mathbf{SVec}$; this will be explained in detail in Section 3.1. However, $\otimes$ is a functor $\mathbf{SVec}^\circ \times \mathbf{SVec}^\circ \to \mathbf{SVec}^\circ$, and naturally endows $\mathbf{SVec}^\circ$ with the structure of a monoidal category.

Let V and W be super-vector spaces. We define isomorphisms $\sigma, \tau : V \otimes W \to W \otimes V$ for homogeneous elements $v \in V$ and $w \in W$ by

$$\sigma(v \otimes w) = w \otimes v, \quad \tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

Both σ and τ define symmetric structures on $\otimes$. However, the symmetric structure σ is uninteresting in that it does not interact with the grading. In what follows, we regard $\mathbf{SVec}^\circ$ as a symmetric monoidal category using τ, unless explicitly said otherwise.

2.2. ***Supercategories.*** A *supercategory* is a category $\mathcal{A}$ enriched in the category $\mathbf{SVec}^\circ$ of super-vector spaces. Thus, $\mathcal{A}$ is a category and $\mathrm{Hom}_\mathcal{A}(X, Y)$ has the structure of a super-vector space for all objects X and Y such that composition defines an even map

$$\mathrm{Hom}_\mathcal{A}(Y, Z) \otimes \mathrm{Hom}_\mathcal{A}(X, Y) \to \mathrm{Hom}_\mathcal{A}(X, Z), \quad f \otimes g \mapsto f \circ g.$$

Given a supercategory $\mathcal{A}$, we let $\mathcal{A}^\circ$ be the category with the same collection of objects but only the even morphisms; that is, $\mathrm{Hom}_{\mathcal{A}^\circ}(X, Y) = \mathrm{Hom}_\mathcal{A}(X, Y)_0$.

Let Λ be an abelian group (or, for now, any set). A *homogeneous* Λ*-supercategory* is a collection $\mathcal{A} = \{\mathcal{A}_i\}_{i \in \Lambda}$, where each $\mathcal{A}_i$ is a supercategory. We refer to $\mathcal{A}_i$ as the category of homogeneous objects of degree i. Informally, an inhomogeneous Λ-supercategory is a supercategory $\mathcal{A}$ equipped with subcategories $\mathcal{A}_i$ for $i \in \mathcal{A}$ such that $\mathcal{A}$ is a direct sum of the $\mathcal{A}_i$'s, in some sense. This concept will appear in some examples, but we do not give a rigorous definition to avoid details of what direct sum means here (there are some subtleties if, e.g., $\mathcal{A}$ is nonadditive or Λ is infinite). In the inhomogeneous case, any construction will be determined by what it does on homogeneous objects, so we can typically restrict our attention to them. In what follows, "Λ-supercategory" will by default refer to the homogeneous version, and we almost always take Λ to be $\mathbb{Z}$ or $\mathbb{Z}/q$.

Suppose that $\mathcal{A}$ is a Λ-supercategory and we have a function $f : \Lambda \to \Lambda'$. Then we can convert $\mathcal{A}$ into a Λ'-supercategory by defining $\mathcal{A}_i = \coprod_{j \in f^{-1}(i)} \mathcal{A}_j$ for $i \in \Lambda'$. Here the coproduct is taken in the sense of linear categories: the Hom space between objects in different factors is 0. In particular, we can convert a $\mathbb{Z}$-supercategory into a $\mathbb{Z}/q$-supercategory, or a $\mathbb{Z}/q$-supercategory into a $\mathbb{Z}/q'$-supercategory if q' divides q.

Example 2.1. Let G be a supergroup over $\mathbb{C}$ containing a central $\boldsymbol{G}_m$. Then $\mathcal{A} = \mathrm{Rep}(G)$ is an (inhomogeneous) $\mathbb{Z}$-supercategory, by taking $\mathcal{A}_i$ to be the subcategory of objects where $\boldsymbol{G}_m$ acts with weight i. □

2.3. ***Superfunctors.*** Let $\mathcal{A}$ and $\mathcal{B}$ be supercategories. A *superfunctor* $F : \mathcal{A} \to \mathcal{B}$ is a functor (in the usual sense) such that the maps

$$F : \mathrm{Hom}_\mathcal{A}(X, Y) \to \mathrm{Hom}_\mathcal{B}(F(X), F(Y))$$

are even maps of super-vector spaces. If $\mathcal{A}$ and $\mathcal{B}$ are Λ-supercategories, then a Λ-*superfunctor* is a collection of superfunctors $F = \{F_i : \mathcal{A}_i \to \mathcal{B}_i\}_{i \in \Lambda}$.

Let F and G be two superfunctors from $\mathcal{A}$ to $\mathcal{B}$. A *homogeneous natural transformation* $\eta : F \to G$ of parity $p \in \mathbb{Z}/2$ is a rule that assigns to every object X of $\mathcal{A}$ a homogeneous morphism $\eta_X : F(X) \to G(X)$ of degree p in $\mathcal{B}$ such that, if $f : X \to Y$ is a homogeneous morphism in $\mathcal{A}$, then $G(f) \circ \eta_X = (-1)^{p|f|}\eta_Y \circ F(f)$. A general natural transformation is a sum of an even and an odd natural transformation. The definition is the same for Λ-superfunctors: the grading does not affect the definition at all.

2.4. Π-*structures.* Let $\mathcal{A}$ be a supercategory. A Π-*structure* on an object X of $\mathcal{A}$ is an odd-degree isomorphism $\xi_X : X \to \Pi(X)$ for some object $\Pi(X)$. A Π-*structure* on $\mathcal{A}$ is a choice of Π-structure on each object of $\mathcal{A}$. Such a structure need not exist. Note that, if $\mathcal{A}$ is Λ-graded and X is homogeneous, then $\Pi(X)$ is necessarily homogeneous of the same degree.

Suppose we have a Π-structure on $\mathcal{A}$. For a homogeneous morphism $f : X \to Y$ in $\mathcal{A}$, define $\Pi(f)$ to be the unique morphism making the following diagram commute up to $(-1)^{|f|}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \xi_X}\downarrow & & \downarrow{\scriptstyle \xi_Y} \\ \Pi(X) & \xrightarrow{\Pi(f)} & \Pi(Y) \end{array}$$

By uniqueness, it is clear that Π is a superfunctor $\mathcal{A} \to \mathcal{A}$, and that $\xi : \mathrm{id}_{\mathcal{A}} \to \Pi$ is an odd natural isomorphism. (If $\mathcal{A}$ is a Λ-supercategory, then Π is a Λ-superfunctor.) Furthermore, taking $f = \xi_X$, we find $\Pi(\xi_X) = -\xi_{\Pi(X)}$.

For a nonnegative integer n, we let Π^n be the n-fold iterate of Π. For $n < m$, we define $\xi_X^{n,m} : \Pi^n(X) \to \Pi^m(X)$ to be the composition $\xi_{\Pi^{m-1}(X)} \circ \cdots \circ \xi_{\Pi^{n+1}(X)} \circ \xi_{\Pi^n(X)}$; for $n > m$, we define $\xi_X^{n,m} = (\xi_X^{m,n})^{-1}$; and for $n = m$, we let $\xi_X^{n,m}$ be the identity. The map $\xi_X^{n,m}$ is an isomorphism of parity $m - n$. In particular, $\xi^{0,2} : \mathrm{id}_{\mathcal{A}} \to \Pi^2$ is an even-degree isomorphism. For q even, we define Π^n for $n \in \mathbb{Z}/q$ to be the colimit of the system

$$\cdots \to \Pi^{n-q} \xrightarrow{\xi^{n-q,n}} \Pi^n \xrightarrow{\xi^{n+q,n}} \Pi^{n+q} \to \cdots .$$

This colimit exists since all the maps are even isomorphisms.

Suppose that (Π, ξ) and (Π', ξ') are two Π-structures on $\mathcal{A}$. Define $i_X : \Pi(X) \to \Pi'(X)$ to be the morphism $\xi'_X \circ \xi_X^{-1}$. Then i_X is an even-degree isomorphism, and defines an even-degree isomorphism of functors $i : \Pi \to \Pi'$ compatible with ξ and ξ'. We thus see that any two Π-structures are canonically (and, in fact, uniquely) isomorphic by an even-degree natural transformation.

Proposition 2.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a superfunctor between supercategories with Π-structures. Then there is a canonical even isomorphism $F \circ \Pi = \Pi \circ F$.*

Proof. Let X be an object of $\mathcal{A}$. Then $\xi_{F(X)}F(\xi_X)^{-1}$ gives an even-degree isomorphism $F(\Pi(X)) \cong \Pi(F(X))$. We now check that this is a natural isomorphism. Let $f : X \to Y$ be a homogeneous morphism. Then

$$\xi_Y f = (-1)^{|f|}\Pi(f)\xi_X, \quad \xi_{F(Y)}F(f) = (-1)^{|f|}\Pi(F(f))\xi_{F(X)}.$$

We apply F to the first equation and rearrange to get $F(f)F(\xi_X)^{-1} = (-1)^{|f|}F(\xi_Y)^{-1}F(\Pi(f))$. Post-compose this equation with $\xi_{F(Y)}$ and use the second equation to get

$$\xi_{F(Y)}F(\xi_Y)^{-1}F(\Pi(f)) = \Pi(F(f))\xi_{F(X)}F(\xi_X)^{-1},$$

which shows that we have defined a natural isomorphism $F \circ \Pi \cong \Pi \circ F$. □

2.5. ***Product categories.*** Let $\mathcal{A}$ and $\mathcal{B}$ be two supercategories. We let $\mathcal{A} \check{\boxtimes} \mathcal{B}$ be the product category, in the sense of enriched categories. Precisely, the objects of $\mathcal{A} \check{\boxtimes} \mathcal{B}$ are pairs (A, B) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$; we write $A \boxtimes B$ in place of (A, B). The Hom spaces are defined by

$$\mathrm{Hom}_{\mathcal{A}\check{\boxtimes}\mathcal{B}}(A \boxtimes B, A' \boxtimes B') = \mathrm{Hom}_{\mathcal{A}}(A, A') \otimes \mathrm{Hom}_{\mathcal{B}}(B, B').$$

The composition of homogeneous morphisms $f \otimes g : A \boxtimes B \to A' \boxtimes B'$ and $f' \otimes g' : A' \boxtimes B' \to A'' \boxtimes B''$ is defined by

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|f||g'|}(f' \circ f) \otimes (g' \circ g).$$

The sign here comes from the symmetry of the tensor product in **SVec**. (To define the product of enriched categories in general, the enriching category needs a symmetric monoidal structure, and here we are using the symmetry τ on $\mathbf{SVec}^\circ$.)

Let $\mathcal{A}$ and $\mathcal{B}$ be two homogeneous Λ-supercategories. We define $\mathcal{A} \boxtimes \mathcal{B}$ to be the homogeneous Λ-supercategory with

$$(\mathcal{A} \boxtimes \mathcal{B})_i = \coprod_{j+k=i} \mathcal{A}_j \check{\boxtimes} \mathcal{B}_k.$$

We note that $\check{\boxtimes}$ does not distribute over direct sums of supercategories (even finite direct sums), and so does not give the expected result on inhomogeneous supercategories; this is why we have used a different symbol in the Λ-graded case.

If either $\mathcal{A}$ or $\mathcal{B}$ admits a Π-structure, then so does $\mathcal{A} \boxtimes \mathcal{B}$. Indeed, suppose $\mathcal{A}$ does, and let $A \boxtimes B$ be a given object of $\mathcal{A} \boxtimes \mathcal{B}$. If $\xi : A \to \Pi(A)$ is an odd-degree isomorphism, then $\xi \otimes \mathrm{id}_B : A \boxtimes B \to \Pi(A) \boxtimes B$ is an odd-degree isomorphism, and so the claim is verified.

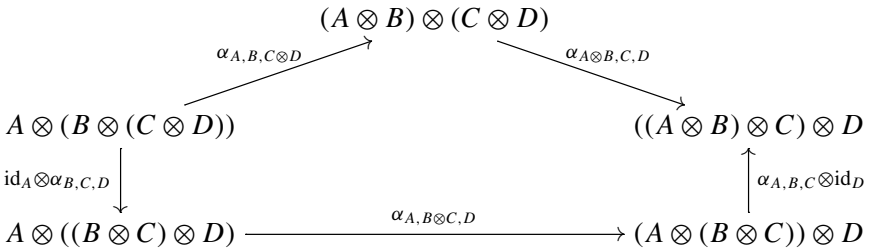
2.6. ***Grothendieck groups.*** Let $\mathcal{A}$ be a supercategory equipped with an exact structure on $\mathcal{A}^\circ$. We can then consider the Grothendieck group $\mathrm{K}(\mathcal{A}^\circ)$ of $\mathcal{A}^\circ$. For $\varepsilon \in \{+, -\}$, we define $\mathrm{K}_\varepsilon(\mathcal{A})$ to be the quotient of $\mathrm{K}(\mathcal{A}^\circ)$ by the relations $[X] = \varepsilon[Y]$ whenever there is an odd-degree isomorphism $X \to Y$. If $\mathcal{A}$ admits a Π-structure, then $\mathrm{K}_\varepsilon(\mathcal{A})$ is simply the quotient of $\mathrm{K}(\mathcal{A}^\circ)$ by the relations $[\Pi(X)] = \varepsilon[X]$. If $\mathcal{A}$ admits a Λ-grading, then $\mathrm{K}(\mathcal{A}^\circ) = \bigoplus_{i \in \Lambda} \mathrm{K}(\mathcal{A}_i^\circ)$, and so $\mathrm{K}(\mathcal{A}^\circ)$ is canonically Λ-graded. This induces a Λ-grading on $\mathrm{K}_\varepsilon(\mathcal{A})$.

3. Monoidal structures

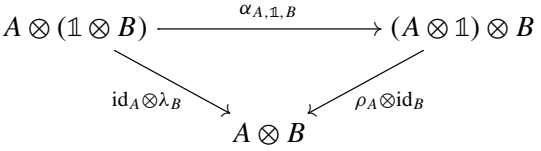
3.1. ***Monoidal structures.*** Let $\mathcal{A}$ be a Λ-supercategory. A *monoidal structure* on $\mathcal{A}$ consists of

- a Λ-superfunctor $\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$;

- a unit object $\mathbb{1}$;
- an even natural isomorphism $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ satisfying the pentagon axiom, i.e., the diagram



  is commutative for all objects A, B, C, D;
- even natural isomorphisms $\lambda_X : \mathbb{1} \otimes X \to X$ and $\rho_X : X \otimes \mathbb{1} \to X$ satisfying the triangle axiom, i.e., the diagram



  is commutative for all objects A, B.

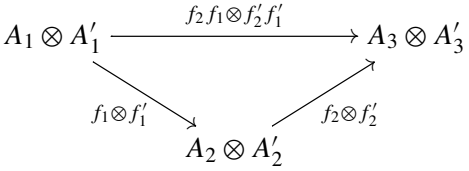
We note that, if A and B are homogeneous objects of $\mathcal{A}$, then $A \otimes B$ is homogeneous and $|A \otimes B| = |A| + |B|$. A *monoidal* Λ*-supercategory* is a Λ-supercategory equipped with a monoidal structure.

Example 3.1. The supercategory **SVec** is monoidal under tensor product. (Note here that we are using all of **SVec**, and not just the subcategory **SVec**$^\circ$.) □

Remark 3.2. Suppose that $\mathcal{A}$ is a monoidal supercategory and we have homogeneous morphisms

$$f_1 : A_1 \to A_2, \quad f_2 : A_2 \to A_3, \quad f_1' : A_1' \to A_2', \quad f_2' : A_2' \to A_3'.$$

Then the diagram



commutes up to $(-1)^{|f_1||f_2'|}$, that is, we have the identity

$$(f_2 \otimes f_2') \circ (f_1 \otimes f_1') = (-1)^{|f_1||f_2'|}(f_2 f_1 \otimes f_2' f_1').$$

This follows from the definition of the composition law in the category $\mathcal{A} \boxtimes \mathcal{A}$. From this, we see that $\otimes$ does *not* define a functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (in general). However, it does define a functor $\mathcal{A}^\circ \times \mathcal{A}^\circ \to \mathcal{A}^\circ$, and $\mathcal{A}^\circ$ is naturally a monoidal category. □

Let $\mathcal{A}$ be a monoidal Λ-supercategory admitting a Π-structure. Let A and B be objects of $\mathcal{A}$. Then we have odd-degree isomorphisms $\xi_{A\otimes B} : A \otimes B \to \Pi(A \otimes B)$ and $\xi_A \otimes \mathrm{id}_B : A \otimes B \to \Pi(A) \otimes B$. We thus have a canonical even-degree isomorphism $\Pi(A \otimes B) \to \Pi(A) \otimes B$. Similarly, we have a canonical even-degree isomorphism $\Pi(A \otimes B) \to A \otimes \Pi(B)$. We thus have canonical even-degree isomorphisms $\Pi(A) \cong \Pi(\mathbb{1}) \otimes A \cong A \otimes \Pi(\mathbb{1})$.

Definition 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal Λ-supercategories. A *monoidal Λ-superfunctor* $F : \mathcal{A} \to \mathcal{B}$ is a Λ-superfunctor F equipped with even isomorphisms $\Phi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ and $\varphi : F(\mathbb{1}_{\mathcal{A}}) \to \mathbb{1}_{\mathcal{B}}$ such that the following diagrams commute for all objects X, Y, Z of $\mathcal{A}$:

$$\begin{array}{ccccc}
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{\Phi_{X,Y}\otimes 1} & F(X\otimes Y)\otimes F(Z) & \xrightarrow{\Phi_{X\otimes Y,Z}} & F((X\otimes Y)\otimes Z) \\
\downarrow{\scriptstyle \alpha_{F(X),F(Y),F(Z)}} & & & & \downarrow{\scriptstyle F(\alpha_{X,Y,Z})} \\
F(X)\otimes(F(Y)\otimes F(Z)) & \xrightarrow{1\otimes\Phi_{Y,Z}} & F(X)\otimes F(Y\otimes Z) & \xrightarrow{\Phi_{X,Y\otimes Z}} & F(X\otimes(Y\otimes Z))
\end{array}$$

$$\begin{array}{ccc}
\mathbb{1}\otimes F(X) & \xrightarrow{\lambda_{F(X)}} & F(X) \\
\downarrow{\scriptstyle \varphi\otimes 1} & & \uparrow{\scriptstyle F(\lambda_X)} \\
F(\mathbb{1})\otimes F(X) & \xrightarrow{\Phi_{\mathbb{1},X}} & F(\mathbb{1}\otimes X)
\end{array}
\qquad
\begin{array}{ccc}
F(X)\otimes\mathbb{1} & \xrightarrow{\rho_{F(X)}} & F(X) \\
\downarrow{\scriptstyle 1\otimes\varphi} & & \uparrow{\scriptstyle F(\rho_X)} \\
F(X)\otimes F(\mathbb{1}) & \xrightarrow{\Phi_{X,\mathbb{1}}} & F(X\otimes\mathbb{1})
\end{array}$$

We will generally suppress Φ and φ from the notation. □

Definition 3.4. Given two monoidal Λ-superfunctors (F, Φ, φ) and (G, Γ, γ) between Λ-supercategories $\mathcal{A}$ and $\mathcal{B}$, a *monoidal natural transformation* $\eta : F \to G$ is an even natural transformation η such that the following two diagrams commute for all objects X, Y of $\mathcal{A}$:

$$\begin{array}{ccc}
F(X)\otimes F(Y) & \xrightarrow{\eta_X\otimes\eta_Y} & G(X)\otimes G(Y) \\
\downarrow{\scriptstyle \Phi_{X,Y}} & & \downarrow{\scriptstyle \Gamma_{X,Y}} \\
F(X\otimes Y) & \xrightarrow{\eta_{X\otimes Y}} & G(X\otimes Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\gamma} & G(\mathbb{1}) \\
\downarrow{\scriptstyle \varphi} & \nearrow{\scriptstyle \eta_{\mathbb{1}}} & \\
F(\mathbb{1}) & &
\end{array}$$

We thus have a 2-category of monoidal supercategories and, in particular, a notion of monoidal equivalence. □

3.2. ***The equivalences* $\boldsymbol{\Sigma}$ *and* $\mathbf{T}$.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{Z}/q$-graded supercategories, with q even. Define an equivalence

$$\Sigma_{\mathcal{A},\mathcal{B}} : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{B} \boxtimes \mathcal{A}$$

as follows. On homogeneous objects, we put $\Sigma_{\mathcal{A},\mathcal{B}}(A \boxtimes B) = B \boxtimes A$. Given homogeneous morphisms $f : A \to A'$ and $g : B \to B'$ between objects of $\mathcal{A}$ and $\mathcal{B}$, we define $\Sigma_{\mathcal{A},\mathcal{B}}(f \otimes g) = (-1)^{|f||g|} g \otimes f$.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ admit Π-structures. We define an equivalence

$$\mathrm{T}_{\mathcal{A},\mathcal{B}} : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{B} \boxtimes \mathcal{A}$$

as follows. On homogeneous objects, we put $\mathrm{T}_{\mathcal{A},\mathcal{B}}(A \boxtimes B) = \Pi^{|A||B|}(B) \boxtimes A$. Given homogeneous morphisms $f : A \to A'$ and $g : B \to B'$ of homogeneous objects, we put $\mathrm{T}_{\mathcal{A},\mathcal{B}}(f \otimes g) = (-1)^{|f||g|}\Pi^{|A||B|}(g) \otimes f$.

We regard Σ and T as analogous to the two symmetries σ and τ we defined on **SVec** in Section 2.1. They will give rise to the notion of symmetric and supersymmetric monoidal supercategories, just as σ and τ give rise to the notion of commutative and supercommutative superalgebras.

3.3. ***Factor systems.*** We next want to define braidings and symmetries, and their super analogs. Braidings will be required to satisfy a version of the hexagon axioms (see (H1) and (H2) in Section 3.4 below) but these axioms are only consistent if we allow the diagrams to commute up to certain scalars, depending on the degrees of the objects involved. These "factor systems" are defined below, but the definition is rather involved, so we first give a bit of explanation.

We will have two functions $(\mathbb{Z}/q)^3 \to \boldsymbol{k}^\times$ recording the scalars we allow for the two hexagons (H1) and (H2); we have a $(\mathbb{Z}/q)^3$ here since these diagrams involve three objects. These scalars are themselves subject to a number of constraints, coming from compatibilities of multiple instances of the hexagon axioms (explained in Sections 3.8 and 4). Moreover, the form of these compatibilities depends on whether we are working with braidings or superbraidings, which is recorded in the parity p below. In the symmetric case, the braiding will square to a scalar multiple of the identity (depending on the degrees of the objects), so we require an additional function $(\mathbb{Z}/q)^2 \to \boldsymbol{k}^\times$ recording these scalars.

With this explanation in hand, we now give the formal definition:

Definition 3.5. Let q be an even integer and let $p \in \mathbb{Z}/2$. A *B-factor system* of parity p is a pair $\omega = (\omega_1, \omega_2)$ of functions

$$\omega_1, \omega_2 : (\mathbb{Z}/q)^3 \to \boldsymbol{k}^\times$$

satisfying the following conditions for all $a, b, c, d \in \mathbb{Z}/q$ (in what follows, the semicolon in, e.g., $\omega_1(a; b, c)$, indicates that the b and c parameters play similar roles, while the a parameter is somewhat different):

$$\omega_1(a; b, c)\omega_1(a; b + c, d) = \omega_1(a; b, c + d)\omega_1(a; c, d), \tag{3.5a}$$

$$\omega_2(a + b, c; d)\omega_2(a, b; d) = \omega_2(b, c; d)\omega_2(a, b + c; d), \tag{3.5b}$$

$$\omega_1(b; c, d)\omega_1(a; c, d)\omega_1(a + b; c, d)^{-1} = (-1)^{abcdp}\omega_2(a, b; c)\omega_2(a, b; d)\omega_2(a, b; c + d)^{-1}, \tag{3.5c}$$

$$\omega_1(a; b, c)\omega_1(a; c, b)^{-1} = \omega_2(b, a; c)\omega_2(a, b; c)^{-1}. \tag{3.5d}$$

An *S-factor system* of parity p is a triple $\omega = (\omega_1, \omega_2, \omega_\sharp)$ where (ω_1, ω_2) is a B-factor system of parity p and $\omega_\sharp : (\mathbb{Z}/q)^2 \to \boldsymbol{k}^\times$ is a function satisfying

$$\omega_1(a; b, c)\omega_2(b, c; a) = \omega_\sharp(a, b)\omega_\sharp(a, c)\omega_\sharp(a, b + c)^{-1}, \tag{3.5e}$$

$$\omega_1(c; a, b)\omega_2(a, b; c) = \omega_\sharp(a, c)\omega_\sharp(b, c)\omega_\sharp(a + b, c)^{-1}. \tag{3.5f}$$

We say that ω is *symmetric* if $\omega_\sharp(a, b) = \omega_\sharp(b, a)$. □

Remark 3.6. (a) The "B" and "S" prefixes stand for "braiding" and "syllepsis" (see Section 4 for more).

(b) The parity p of the factor system only intervenes in (3.5c).

(c) Let Γ_p be the set of all S-factor systems of parity p for $\mathbb{Z}/q$. Then $\Gamma = \Gamma_0 \oplus \Gamma_1$ is a $\mathbb{Z}/2$-graded abelian group. □

Example 3.7. Let ω_1, ω_2 and $\omega_\sharp$ be the constant functions taking the value 1. Then ω is an even S-factor system. We refer to this as the *trivial factor system*. □

Example 3.8. Suppose that q is divisible by 4. Then $n \mapsto \binom{n}{2}$ is a well-defined function $\mathbb{Z}/q \to \mathbb{Z}/2$. Put

$$\mathtt{A}_1(a; b, c) = 1, \quad \mathtt{A}_2(a, b; c) = (-1)^{\binom{c}{2}ab}, \quad \mathtt{A}_\sharp(a, b) = (-1)^{\binom{a}{2}\binom{b}{2}}.$$

Then $\mathtt{A}$ is an odd symmetric S-factor system. This is perhaps the most important example for this paper. □

Example 3.9. Suppose that $\zeta_4 \in \boldsymbol{k}$. Define

$$\mathtt{B}_1(a; b, c) = 1, \qquad \mathtt{B}_2(a, b; c) = \begin{cases} \zeta_4 & \text{if } a, b \text{ and } c \text{ are all odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $(\mathtt{B}_1, \mathtt{B}_2)$ is an odd B-factor system. Now suppose that $\zeta_8 \in \boldsymbol{k}$, and put

$$\mathtt{B}_\sharp(a, b) = \begin{cases} \zeta_8 & \text{if } a \text{ and } b \text{ are both odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $(\mathtt{B}_1, \mathtt{B}_2, \mathtt{B}_\sharp)$ is an odd symmetric S-factor system. This factor system has the advantage over the S-factor system $\mathtt{A}$ in that it does not require q to be divisible by 4. □

Example 3.10. Define

$$\mathtt{C}_1(r; s, t) = (-1)^{rst}, \quad \mathtt{C}_2(r, s; t) = (-1)^{rst}, \quad \mathtt{C}_\sharp(r, s) = 1.$$

Then $\mathtt{C}$ is an even symmetric S-factor system. □

Example 3.11. Define

$$\mathtt{D}_1(r; s, t) = (-1)^{rst}, \quad \mathtt{D}_2(r, s; t) = (-1)^{rst}, \quad \mathtt{D}_\sharp(r, s) = (-1)^{rs}.$$

Then $\mathtt{D}$ is an even symmetric S-factor system. □

Definition 3.12. Let $\varphi : (\mathbb{Z}/q)^2 \to \boldsymbol{k}^\times$ be an arbitrary function. Put

$$\begin{aligned} \omega_1(r; s, t) &= \varphi(r, s+t)\varphi(r, s)^{-1}\varphi(r, t)^{-1}, \\ \omega_2(r, s; t) &= \varphi(r+s, t)\varphi(r, s)^{-1}\varphi(s, t)^{-1}, \\ \omega_\sharp(r, s) &= \varphi(r, s)\varphi(s, r). \end{aligned}$$

Then $(\omega_1, \omega_2, \omega_\sharp)$ is a symmetric even S-factor system, called the *coboundary* of φ. We denote this system by $\partial(\varphi)$. □

Example 3.13. Suppose $4 \mid q$. Then $\mathtt{C}$ is the coboundary of the function $\mathtt{c}$ given by

$$\mathtt{c}(r, s) = (-1)^{\binom{rs}{2}}.$$ □

Example 3.14. Suppose q is an arbitrary even integer and $\zeta_4 \in \boldsymbol{k}$. Then $\mathtt{D}$ is the coboundary of the function $\mathtt{d}$ given by

$$\mathtt{d}(x, y) = \begin{cases} 1 & \text{if } x \text{ or } y \text{ is even,} \\ \zeta_4 & \text{if } x \text{ and } y \text{ are odd.} \end{cases}$$

□

We now give a somewhat more involved example of a coboundary.

Proposition 3.15. *Set $\epsilon(n) = -1$ if $n \equiv 3 \pmod 4$ and $\epsilon(n) = 1$ otherwise.*

(a) *Suppose that $\zeta_4 \in \boldsymbol{k}$ and that q is divisible by 8. Define*

$$\mathtt{a}(n, m) = \epsilon(n)^m \zeta_4^{-\binom{n}{2}m}.$$

Then $\mathtt{A} = \mathtt{B} \cdot \partial(\mathtt{a})$ holds as B-factor systems.

(b) *Suppose that $\zeta_{16} \in \boldsymbol{k}$ and that q is divisible by 16. Define*

$$\mathtt{a}'(n, m) = \epsilon(n)^m \zeta_4^{-\binom{n}{2}m} \zeta_{16}^{nm}.$$

Then $\mathtt{A} = \mathtt{B} \cdot \partial(\mathtt{a}')$ holds as S-factor systems.

Proof. (a) The first component of the equation $\mathtt{A} = \mathtt{B} \cdot \partial(\mathtt{a})$ amounts to the identity

$$\epsilon(a)^{b+c} \zeta_4^{-\binom{a}{2}(b+c)} = \epsilon(a)^b \zeta_4^{-\binom{a}{2}b} \epsilon(a)^c \zeta_4^{-\binom{a}{2}c},$$

which is clear. The second component amounts to

$$\mathtt{B}_2(a, b; c)\epsilon(a+b)^c \zeta_4^{-\binom{a+b}{2}c} = (-1)^{\binom{c}{2}ab} \epsilon(b)^c \zeta_4^{-\binom{b}{2}c} \epsilon(a)^c \zeta_4^{-\binom{a}{2}c}.$$

The quotient of the left side by the right side is

$$(-1)^{\binom{c}{2}ab} \mathtt{B}_2(a, b; c)\epsilon(a+b)^c \epsilon(a)^c \epsilon(b)^c \zeta_4^{-abc},$$

which now only depends on a, b, c modulo 4. We will show it is 1 in all cases. If c is even, this reduces to $((-1)^{\binom{c}{2}} \zeta_4^{-c})^{ab} = 1$. If c is odd, then this simplifies to $\mathtt{B}_2(a, b; 1)\epsilon(a+b)\epsilon(a)\epsilon(b)\zeta_4^{-ab}$. If $a = 0$, we get $\epsilon(b)^2 = 1$. If $a = 1$, we get $\mathtt{B}_2(1, b; 1)\epsilon(b+1)\epsilon(b)\zeta_4^{-b}$. If $a = 2$, we get $\epsilon(b+2)\epsilon(b)(-1)^b$. Finally, if $a = 3$, we get $-\mathtt{B}_2(1, b; 1)\epsilon(b-1)\epsilon(b)\zeta_4^b$. In all cases, we see that the expressions are 1 for all choices of b by inspection.

(b) Since $\mathtt{a}'$ is equal to $\mathtt{a}$ times a quantity that is bilinear, the first two components of $\partial(\mathtt{a}')$ agree with those of $\partial(\mathtt{a})$. In particular, the equation $\mathtt{A} = \mathtt{B} \cdot \partial(\mathtt{a}')$ holds on the first two components by part (a). On the $\sharp$ component, it amounts to the identity

$$\epsilon(b)^a \zeta_4^{-\binom{b}{2}a} \zeta_{16}^{ab} \epsilon(a)^b \zeta_4^{-\binom{a}{2}b} \zeta_{16}^{ab} (-1)^{\binom{a}{2}\binom{b}{2}} = \mathtt{B}_\sharp(a, b).$$

The quotient of the left side by the right side is

$$\epsilon(b)^a \zeta_4^{-\binom{b}{2}a - \binom{a}{2}b} \zeta_8^{ab} \epsilon(a)^b (-1)^{\binom{a}{2}\binom{b}{2}} \mathtt{B}_\sharp(a, b)^{-1}.$$

We will show this is always 1. Suppose that $a = 2c$ is even (we will assume that a, b are integers). Then the above simplifies to

$$(-1)^{\binom{b}{2}c}\zeta_4^{-2bc^2+bc}\zeta_4^{bc}(-1)^{\binom{b}{2}c} = 1.$$

Since the expression is symmetric in a and b, it remains to prove that the expression is 1 when $a = 2c+1$ and $b = 2d+1$ are odd. In that case, it simplifies to

$$\begin{aligned}\epsilon(b)\zeta_4^{-(2d^2+d)a}\zeta_4^{-(2c^2+c)b}(-1)^{cd}\zeta_4^{c+d}\zeta_8\epsilon(a)(-1)^{cd}\zeta_8^{-1} &= \epsilon(b)(-1)^d\zeta_4^{-ad}(-1)^c\zeta_4^{-bc}\zeta_4^{c+d}\epsilon(a)\\ &= \epsilon(a)(-1)^c\epsilon(b)(-1)^d = 1.\end{aligned}$$

□

Remark 3.16. Let $\mathrm{C}^r(G, A)$ denote the group of r-cochains of a group G with values in the abelian group A. Note that $\mathrm{C}^0(G, A) = A$ is the set of functions from a single point to A. These groups form a cochain complex that computes the group cohomology of G. Let $\mathrm{C}^{r,s} = \mathrm{C}^r(\mathbb{Z}/q, \boldsymbol{k}^\times) \otimes \mathrm{C}^s(\mathbb{Z}/q, \mathbb{Z})$. We have a natural map

$$i : \mathrm{C}^{r,s} \to \mathrm{C}^{r+s}(\mathbb{Z}/q, \boldsymbol{k}^\times)$$

given on pure tensors by

$$i(c \otimes c')(g_1, \ldots, g_{r+s}) = c(g_1, \ldots, g_r)^{c'(g_{r+1},\ldots,g_{r+s})}.$$

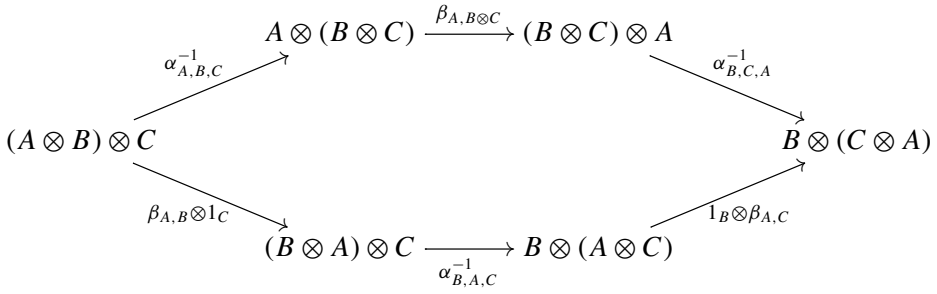
One easily sees that i is an isomorphism (e.g., by considering cochains supported at a single point). Thus, $\mathrm{C}^{1,2} \oplus \mathrm{C}^{2,1}$ can be identified with the set of pairs (ω_1, ω_2) of functions $(\mathbb{Z}/q)^3 \to \boldsymbol{k}^\times$. For such a pair to form an even B-factor system, it must be killed by the differential of the total complex (this accounts for the first three conditions in the definition of factor system). This suggests that even B-factor systems are closely related to $\mathrm{H}^3(\mathbb{Z}/q \times \mathbb{Z}/q, \boldsymbol{k}^\times)$, and that there could be a more fundamental definition including $\mathrm{C}^{0,3}$ and $\mathrm{C}^{3,0}$. We note that 3-cocycles of $\mathbb{Z}/q$ can be used to twist the associativity constraint in a monoidal $\mathbb{Z}/q$-supercategory, which might be relevant. We expect that even S-factor systems are similarly related to third abelian cohomology (as defined in [MacLane 1952]). It is not clear to us exactly how odd factor systems relate to cohomology. □

3.4. ***Braidings.*** Let $\mathcal{A}$ be a monoidal $\mathbb{Z}/q$-supercategory with q even (we allow $q = 0$). Consider an even isomorphism of superfunctors $\beta : \otimes \to \otimes \circ \Sigma$. Explicitly, β assigns to every pair (A, B) of objects an even isomorphism $\beta_{A,B} : A \otimes B \to B \otimes A$ such that, if $f : A \to A'$ and $g : B \to B'$ are homogeneous morphisms, then the diagram

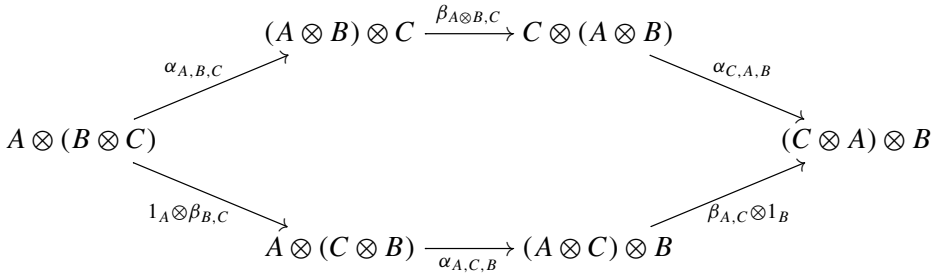
$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta_{A,B}} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ A' \otimes B' & \xrightarrow{\beta_{A',B'}} & B' \otimes A' \end{array}$$

commutes up to $(-1)^{|f||g|}$.

Consider the two hexagons



(H1)



(H2)

Definition 3.17. We say that β is a *braiding* with respect to the even B-factor system ω if, in (H1), the bottom path is equal to $\omega_1(|A|; |B|, |C|)$ times the top path, and, in (H2), the bottom path is equal to $\omega_2(|A|, |B|; |C|)$ times the top path whenever A, B and C are homogeneous objects. □

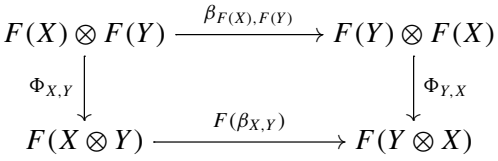
Definition 3.18. We say that β is a *symmetry* with respect to the even S-factor system ω if it is a braiding and satisfies the condition $\beta_{B,A} \circ \beta_{A,B} = \omega_\sharp(|A|, |B|)\,\mathrm{id}_{A\otimes B}$ for all homogeneous objects A and B. □

Remark 3.19. By taking ω to be the trivial factor system (Example 3.7), one recovers the usual definition of symmetry: the diagrams (H1) and (H2) commute, and $\beta_{B,A}\beta_{A,B} = \mathrm{id}_{A\otimes B}$. □

Remark 3.20. Suppose β is a symmetry with respect to ω, and let $\varphi : (\mathbb{Z}/q)^2 \to \boldsymbol{k}^\times$ be an arbitrary function. Define β' by $\beta'_{X,Y} = \varphi(|X|, |Y|)\beta_{X,Y}$. Then β' is a symmetry with respect to $\omega \cdot \partial\varphi$, where $\partial(\varphi)$ is as in Definition 3.12. □

Remark 3.21. The reason that even factor systems are the appropriate systems of scalars will be made clear in Section 3.8. □

Definition 3.22. Let $\mathcal{A}$ and $\mathcal{B}$ be braided monoidal $\mathbb{Z}/q$-supercategories. A *braided monoidal* Λ-*superfunctor* $F : \mathcal{A} \to \mathcal{B}$ is a monoidal Λ-superfunctor (F, Φ, φ) such that the following diagram commutes for all objects X, Y of $\mathcal{A}$:



A natural transformation of braided monoidal superfunctors is simply one of monoidal superfunctors, i.e., the braiding is not used in the definition. □

3.5. ***Type I superbraidings.*** Suppose $\mathcal{A}$ is a monoidal $\mathbb{Z}/q$-supercategory, with q even (we allow $q = 0$), admitting a Π-structure. Consider an even isomorphism of functors $\beta : \otimes \to \otimes \circ \mathrm{T}$. Explicitly, β assigns to every pair (A, B) of homogeneous objects an even isomorphism $\beta_{A,B} : A \otimes B \to \Pi^{|A||B|}(B) \otimes A$ such that, if $f : A \to A'$ and $g : B \to B'$ are nonzero homogeneous morphisms of homogeneous objects, then the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta_{A,B}} & \Pi^{|A||B|}(B) \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle \Pi^{|A||B|}(g)\otimes f} \\ A' \otimes B' & \xrightarrow{\beta_{A',B'}} & \Pi^{|A||B|}(B') \otimes A' \end{array}$$

commutes up to $(-1)^{|f||g|}$. Note that the existence of nonzero f, g forces $|A| = |A'|$ and $|B| = |B'|$; if f or g is 0, then the power of Π may need to change, but the resulting diagram will commute for trivial reasons.

The hexagons (H1) and (H2) admit obvious modifications (H1′) and (H2′) in this setting, as follows. Let A, B and C be homogeneous objects. Then (H1′) is the diagram

$$\begin{array}{ccccc} & & A \otimes (B \otimes C) & \xrightarrow{\psi_1} & (\Pi^{b+c}(B) \otimes C) \otimes A & & \\ & {\scriptstyle \alpha^{-1}}\nearrow & & & & \searrow{\scriptstyle \alpha^{-1}} & \\ (A \otimes B) \otimes C & & & & & & \Pi^{b+c}(B) \otimes (C \otimes A) \\ & {\scriptstyle \beta\otimes \mathrm{id}}\searrow & & & & \nearrow{\scriptstyle \psi_2} & \\ & & (\Pi^b(B) \otimes A) \otimes C & \xrightarrow[\alpha^{-1}]{} & \Pi^b(B) \otimes (A \otimes C) & & \end{array} \qquad (\mathrm{H1}')$$

Here $b = |A||B|$, $c = |A||C|$, ψ_1 is the composition

$$A \otimes (B \otimes C) \xrightarrow{\beta_{A,B\otimes C}} \Pi^{b+c}(B \otimes C) \otimes A \xrightarrow{\sim} (\Pi^{b+c}(B) \otimes C) \otimes A,$$

ψ_2 is the composition

$$\Pi^b(B) \otimes (A \otimes C) \xrightarrow{\mathrm{id}\otimes\beta_{A,C}} \Pi^b(B) \otimes (\Pi^c(C) \otimes A) \xrightarrow{\xi\otimes\xi\otimes\mathrm{id}} \Pi^{b+c}(B) \otimes (C \otimes A),$$

and we have omitted indices for readability; recall $\xi = \xi^{n,m}$ is the isomorphism $\Pi^n \to \Pi^m$. The diagram (H2′) is

$$\begin{array}{ccccc} & & (A \otimes B) \otimes C & \xrightarrow{\beta} & \Pi^{a+b}(C) \otimes (A \otimes B) & & \\ & {\scriptstyle \alpha}\nearrow & & & & \searrow{\scriptstyle \alpha} & \\ A \otimes (B \otimes C) & & & & & & (\Pi^{a+b}(C) \otimes A) \otimes B \\ & {\scriptstyle \mathrm{id}\otimes\beta}\searrow & & & & \nearrow{\scriptstyle \beta\otimes\mathrm{id}} & \\ & & A \otimes (\Pi^b(C) \otimes B) & \xrightarrow[\alpha]{} & (A \otimes \Pi^b(C)) \otimes B & & \end{array} \qquad (\mathrm{H2}')$$

where $a = |A||C|$ and $b = |B||C|$, and we have again omitted indices.

It is not correct to ask for the above hexagons to commute: in fact, this leads to inconsistencies (see Section 3.8). Instead, we make the following definition:

Definition 3.23. We say that β is a *type I superbraiding* with respect to the odd B-factor system ω if, in (H1′), the bottom path is equal to $\omega_1(|A|; |B|, |C|)$ times the top path, and, in (H2′), the bottom path is equal to $\omega_2(|A|, |B|; |C|)$ times the top path. □

Definition 3.24. We say that β is a *type I supersymmetry* with respect to the odd S-factor system ω if it is a type I superbraiding with respect to ω and, for all homogeneous objects A and B, the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta_{A,B}} & \Pi^{ab}(B) \otimes A \\ & \searrow^{\omega_\sharp(a,b)\xi\otimes\xi} & \downarrow{\scriptstyle \beta_{\Pi^{ab}(B),A}} \\ & & \Pi^{ab}(A) \otimes \Pi^{ab}(B) \end{array}$$

commutes, where $a = |A|$ and $b = |B|$. □

Remark 3.25. If β satisfies the conditions to be a supersymmetry, then the two hexagon axioms are equivalent conditions: starting from either hexagon, invert all of the arrows and relabel the objects to get the other hexagon. □

Example 3.26. We define a *type Ia superbraiding* (resp. *supersymmetry*) to be a type I superbraiding (resp. supersymmetry) with respect to the factor system $\mathtt{A}$ from Example 3.8. □

Example 3.27. We define a *type Ib superbraiding* (resp. *supersymmetry*) to be a type I superbraiding (resp. supersymmetry) with respect to the factor system $\mathtt{B}$ from Example 3.9. □

Remark 3.28. One can convert between type Ia and type Ib superbraidings when $\mathtt{BA}^{-1}$ is a coboundary, in the sense of Definition 3.12 (see Remark 3.20). Proposition 3.15 gives conditions that ensure $\mathtt{BA}^{-1}$ is a coboundary. □

Definition 3.29. Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal $\mathbb{Z}/q$-supercategories equipped with type I superbraidings with respect to the same factor systems. A *type I superbraided monoidal superfunctor* is a monoidal superfunctor $F : \mathcal{A} \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccccc} F(X) \otimes F(Y) & & \xrightarrow{\beta_{F(X),F(Y)}} & & \Pi^{|X||Y|}(F(Y) \otimes F(X)) \\ {\scriptstyle \Phi_{X,Y}}\downarrow & & & & \downarrow{\scriptstyle \Pi^{|X||Y|}(\Phi_{Y,X})} \\ F(X \otimes Y) & \xrightarrow{F(\beta_{X,Y})} & F(\Pi^{|X||Y|}(Y \otimes X)) & = & \Pi^{|X||Y|}(F(Y \otimes X)) \end{array}$$

commutes, where the equality is the canonical isomorphism coming from Proposition 2.2. A natural transformation of such functors is simply one of monoidal superfunctors. □

3.6. ***Type II superbraidings.*** The definition of type I superbraiding followed the general procedure of replacing Σ by T, and therefore has some conceptual clarity. However, these superbraidings can be somewhat inconvenient due to the Π's that arise; for instance, if X is an odd-degree object, then a type I superbraiding does not induce an endomorphism of $X \otimes X$, but rather a map $X \otimes X \to \Pi(X) \otimes X$. Of course, one can convert an even-degree map $X \otimes Y \to \Pi^{|X||Y|}(Y) \otimes X$ into a map $X \otimes Y \to Y \otimes X$ of degree $|X||Y|$. This leads to an alternative formulation of the concept of superbraiding, in which there are no Π's, but the maps may not be even. We now give the details.

Let $\mathcal{A}$ be a monoidal $\mathbb{Z}/q$-supercategory, with q even (we allow $q = 0$); we do not require a Π-structure. Consider an isomorphism $\beta : \otimes \to \otimes \circ \Sigma$ such that, if A and B are homogeneous, then $\beta_{A,B}$ is homogeneous of degree $|A||B|$. Explicitly, β assigns to every pair of homogeneous objects (A, B) an isomorphism $\beta_{A,B} : A \otimes B \to B \otimes A$ of degree $|A||B|$ such that, if $f : A \to A'$ and $g : B \to B'$ are homogeneous morphisms of homogeneous objects, then the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta_{A,B}} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ A' \otimes B' & \xrightarrow{\beta_{A',B'}} & B' \otimes A' \end{array}$$

commutes up to $(-1)^{|A||B|(|f|+|g|)+|f||g|}$. The first part of the sign, namely $(-1)^{|A||B|(|f|+|g|)}$, comes from the condition that β is a natural transformation, while the second part of the sign, namely $(-1)^{|f||g|}$, comes from the definition of Σ.

Definition 3.30. We say that β is a *type II superbraiding* with respect to the B-factor system ω if, in (H1), the bottom path is equal to $\omega_1(|A|; |B|, |C|)$ times the top path, and, in (H2), the bottom path is equal to $\omega_2(|A|, |B|; |C|)$ times the top path. □

Definition 3.31. We say that β is a *type II supersymmetry* with respect to the S-factor system ω if it is a type II superbraiding with respect to ω and, for all homogeneous objects A and B, we have $\beta_{B,A}\beta_{A,B} = \omega_\sharp(|A|, |B|) \cdot \mathrm{id}_{A\otimes B}$ □

Remark 3.32. We define type IIa/IIb superbraidings/supersymmetries analogously to the type I case (see Examples 3.26 and 3.27). □

Definition 3.33. Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal $\mathbb{Z}/q$-supercategories equipped with type II superbraidings with respect to the same factor systems. A *type II superbraided monoidal functor* is a monoidal superfunctor $F : \mathcal{A} \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{\beta_{F(X),F(Y)}} & F(Y) \otimes F(X) \\ {\scriptstyle \Phi_{X,Y}}\downarrow & & \downarrow{\scriptstyle \Phi_{Y,X}} \\ F(X \otimes Y) & \xrightarrow{F(\beta_{X,Y})} & F(Y \otimes X) \end{array}$$

commutes for all objects X and Y. A natural transformation of such functors is simply one of monoidal superfunctors. □

3.7. ***Equivalence between type I and type II superbraidings.*** Fix a monoidal $\mathbb{Z}/q$-supercategory (with q even) $\mathcal{A}$ with a Π-structure. The following proposition shows how to convert between the two types of superbraidings:

Proposition 3.34. *Suppose that β is a type II superbraiding with respect to the odd B-factor system ω. For homogeneous objects A and B, define $\beta'_{A,B}$ to be the composition*

$$A \otimes B \xrightarrow{\beta_{A,B}} B \otimes A \xrightarrow{\xi_B^{0,|A||B|} \otimes \mathrm{id}_A} \Pi^{|A||B|}(B) \otimes A.$$

Then β' is a type I superbraiding with respect to the odd B-factor system $\mathtt{C}\omega$, where $\mathtt{C}$ is as in Example 3.10. Moreover, this construction defines a bijection between the two types of superbraidings.

Now suppose that β is a type II supersymmetry with respect to the odd S-factor system ω. Then β' is a type I supersymmetry with respect to the odd S-factor system $\mathtt{D}\omega$, where $\mathtt{D}$ is as in Example 3.11. Again, this construction is bijective.

We note that $\mathtt{C} = \mathtt{D}$ as B-factor systems, and so one could use $\mathtt{D}$ in the first part of the proposition as well.

Proof. First we check that β' is a natural transformation, i.e., that the square

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta'_{A,B}} & \Pi^{|A||B|}(B) \otimes A \\ {\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle \Pi^{|A||B|}(g) \otimes f} \\ A' \otimes B' & \xrightarrow{\beta'_{A',B'}} & \Pi^{|A||B|}(B') \otimes A' \end{array}$$

commutes up to $(-1)^{|f||g|}$ for all nonzero homogeneous morphisms $f : A \to A'$ and $g : B \to B'$ between homogeneous objects. By definition, this expands to the diagram

$$\begin{array}{ccccc} A \otimes B & \xrightarrow{\beta_{A,B}} & B \otimes A & \xrightarrow{\xi \otimes 1} & \Pi^{|A||B|}(B) \otimes A \\ {\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle g \otimes f} & & \downarrow{\scriptstyle \Pi^{|A||B|}(g) \otimes f} \\ A' \otimes B' & \xrightarrow{\beta_{A',B'}} & B' \otimes A' & \xrightarrow{\xi \otimes 1} & \Pi^{|A||B|}(B') \otimes A' \end{array}$$

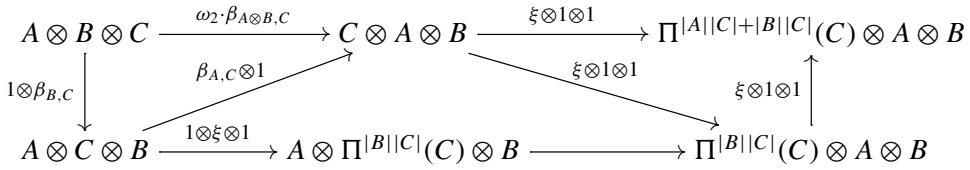
Since β is a type II superbraiding, the left square commutes up to $(-1)^{|A||B|(|f|+|g|)+|f||g|}$. By definition of how Π acts on morphisms, the right square commutes up to $(-1)^{|A||B|(|f|+|g|)}$, and so the original square commutes up to $(-1)^{|f||g|}$, as desired.

Now we check that the hexagon axiom (H1′) holds. Consider the diagram (for simplicity we ignore the associators)

$$\begin{array}{ccccc} A \otimes B \otimes C & \xrightarrow{\omega_1 \cdot \beta_{A,B\otimes C}} & B \otimes C \otimes A & \xrightarrow{\xi \otimes 1} & \Pi^{|A||B|+|A||C|}(B \otimes C) \otimes A \\ {\scriptstyle \beta_{A,B}\otimes 1}\downarrow & \nearrow{\scriptstyle 1 \otimes \beta_{A,C}} & & & \downarrow{\scriptstyle \cong} \\ B \otimes A \otimes C & & {\scriptstyle \xi\otimes 1\otimes 1}\downarrow & \searrow{\scriptstyle \xi\otimes\xi\otimes 1} & \Pi^{|A||B|+|A||C|}(B) \otimes C \otimes A \\ {\scriptstyle \xi\otimes 1\otimes 1}\downarrow & & & & \uparrow{\scriptstyle \cong} \\ \Pi^{|A||B|}(B) \otimes A \otimes C & \xrightarrow{1\otimes\beta_{A,C}} & \Pi^{|A||B|}(B) \otimes C \otimes A & \xrightarrow{1\otimes\xi\otimes 1} & \Pi^{|A||B|}(B) \otimes \Pi^{|A||C|}(C) \otimes A \end{array}$$

where the ω_1 in the top left arrow is $\omega_1(|A|;|B|,|C|)$. On the left side, the upper triangle commutes by the first hexagon axiom for β. The bottom quadrilateral commutes up to $(-1)^{|A||B||C|}$. On the right side, the bottom triangle commutes up to $(-1)^{|A||B||C|}$ by the sign rule when we compose the bottom path. The top triangle commutes: we can take $\Pi(X\otimes Y)=\Pi(X)\otimes Y$, and $\xi : X\otimes Y\to\Pi(X\otimes Y)$ to be $\xi_X\otimes \mathrm{id}_Y$, and we use this convention for the $\Pi(B\otimes C)$ above. Then the top right vertical map is the identity map; the bottom right vertical map is $\xi\otimes\xi\otimes 1$. Hence, we see that (H1′) holds for β' by following the outer paths.

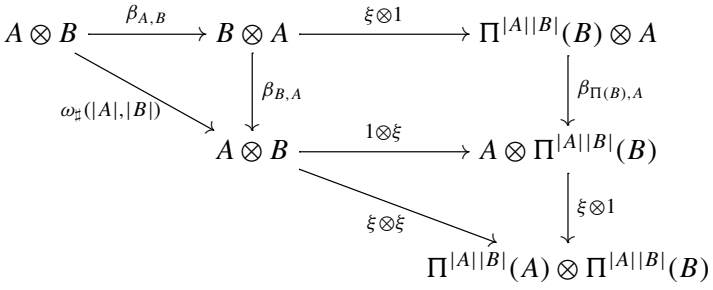
Finally, we check that the hexagon axiom (H2′) holds for β' by considering the diagram (again we ignore associators for simplicity)



where the δ in the top left arrow is $\omega_2(|A|,|B|;|C|)$, and the bottom right map is $\beta_{A,\Pi^{|B||C|}(C)}\otimes 1$. The left triangle commutes by the second hexagon axiom for β. The middle quadrilateral commutes up to $(-1)^{|A||B||C|}$ and the right triangle commutes. Hence, by considering the outer paths, we see that β' satisfies the second hexagon axiom.

Finally, the diagrams above can be used to show that β is a type II superbraiding if β' is a type I superbraiding and hence the stated construction gives a bijection between type I and type II superbraidings.

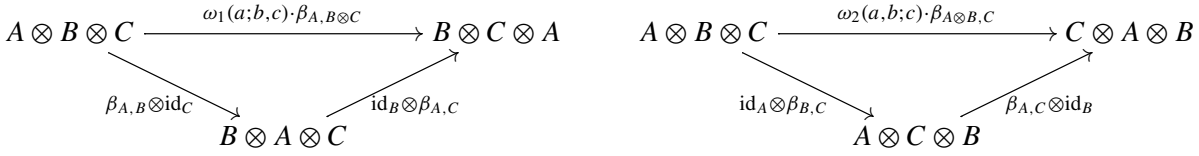
Now we handle the last statement, and suppose that β is a type I supersymmetry. Consider the diagram



The upper left triangle commutes since β is a type II supersymmetry, and the bottom triangle commutes by definition of composition. The square commutes up to $(-1)^{|A||B|}$ since β is a type II superbraiding, so we conclude that β' is a superbraiding with respect to $\mathtt{D}\omega$. □

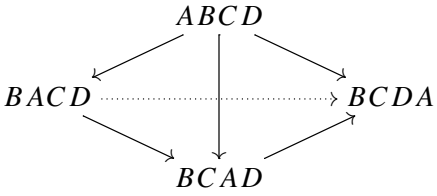
3.8. ***Explanation of factor systems.*** We now explain the rationale behind the definition of B-factor systems. We work with type II superbraidings here, but this discussion can be translated to type I superbraidings or ordinary braidings as well. Let $\mathcal{A}$ be a monoidal $\mathbb{Z}/q$-supercategory with q even. We assume, for simplicity, that the monoidal structure is strict. Suppose that we have an isomorphism $\beta_{X,Y} : \otimes\to\otimes\circ\Sigma$ such that $\beta_{X,Y}$ is homogeneous of degree $|X||Y|$ on homogeneous objects, and that (H1) and (H2) commute up to scalars. Precisely, for homogeneous objects A, B and C of degrees a, b

and c, we suppose that the diagrams



commute, where ω_1 and ω_2 are nonzero scalars. Note that these triangles are the hexagon axioms since we have assumed that the associators are the identity.

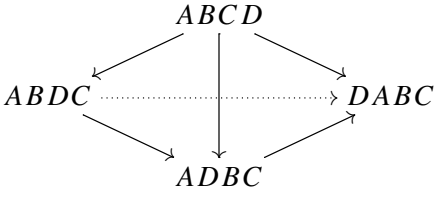
Let A, B, C and D be four homogeneous objects of degrees a, b, c and d. Consider the tetrahedron



(3.35a)

Here we have omitted the $\otimes$ symbols, and each arrow is the unique β map that makes sense. This tetrahedron has four faces that each commute up to an ω_1-factor. This yields the identity (3.5a).

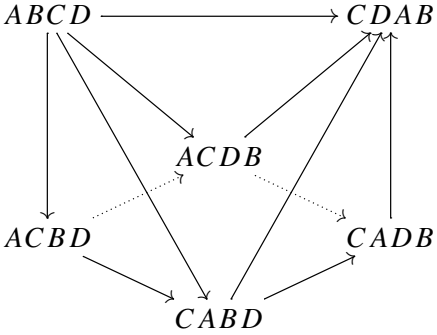
Similarly, we have the commuting tetrahedron



(3.35b)

which yields the identity (3.5b).

Now consider the diagram

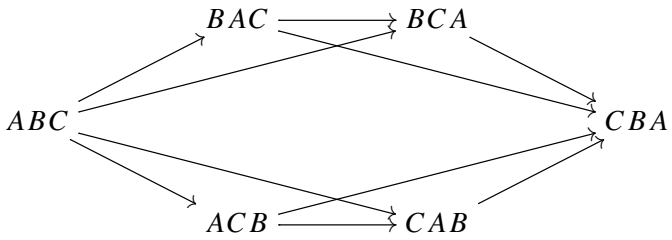


(3.35c)

This diagram has six triangles and one square (composed of the bottom four vertices). If a, b, c and d are all odd, then both $\beta_{A,C}$ and $\beta_{B,D}$ are odd, and the square anticommutes; this is the one place where the specifics of supercategories intervene. Otherwise, at least one of $\beta_{A,C}$ and $\beta_{B,D}$ are even, and the square

commutes. Thus, in general, the square commutes up to $(-1)^{abcd}$, and we obtain (3.5c) with $p = 1$. (In the case of ordinary braidings, this sign would not be present, and so we would obtain (3.5c) with $p = 0$.)

Finally, consider the diagram



(3.35d)

This diagram has four triangles: two that commute up to ω_1-factors and two that commute up to ω_2-factors. It also has two squares that commute by the naturality of β. For example, one of the squares is

$$\begin{array}{ccc} ABC & \xrightarrow{\beta_{A,BC}} & BCA \\ {\scriptstyle \mathrm{id}\otimes\beta_{B,C}}\downarrow & & \downarrow{\scriptstyle \beta_{B,C}\otimes\mathrm{id}} \\ ACB & \xrightarrow{\beta_{A,CB}} & CBA \end{array}$$

and this commutes up to $(-1)^{a(b+c)bc} = 1$. This diagram yields (3.5d).

We thus see that the four equations in the definition of a B-factor system are necessary for consistency of the hexagon axioms. Of course, one might wonder if there are additional equations that are needed that we have overlooked. In fact, this is not the case: this is best explained by the 2-categorical considerations in Section 4. (The discussion there will also explain the equations for S-factor systems.)

3.9. ***Grothendieck groups.*** Let $\mathcal{A}$ be a monoidal $\mathbb{Z}/q$-supercategory with q even. Suppose that $\mathcal{A}^\circ$ has an exact structure and $\otimes$ is exact. Then $\otimes$ induces a multiplication on $\mathrm{K}(\mathcal{A}^\circ)$ and $\mathrm{K}_\pm(\mathcal{A})$ and gives them each the structure of a superring. If the monoidal structure admits a braiding, then all three of these superrings are commutative. If the monoidal structure admits a superbraiding (of any kind), then $\mathrm{K}_+(\mathcal{A})$ is commutative while $\mathrm{K}_-(\mathcal{A})$ is supercommutative. Thus, via K_-, we can view supersymmetric monoidal supercategories as a categorification of the notion of a supercommutative superalgebra.

4. Some 2-category theory

In this section, we explain the 2-categorical underpinnings for supersymmetries and superbraidings. The main point is to show that the equivalence T (introduced in Section 3.2), together with a B-factor system, allows us to define a braiding on the 2-category of $\mathbb{Z}/q$-supercategories; the braided objects will then be our superbraided categories. There is a similar story involving S-factor systems and also the equivalence Σ.

We do not intend for this discussion to be perfectly rigorous, and we allow ourselves to take some shortcuts and make some simplifications. In Sections 4.1–4.3, we review the theory of monoidal 2-categories in general. We have mostly followed [Crans 1998], and refer there for more details; see also

[Baez and Neuchl 1996; Kapranov and Voevodsky 1994]. From Section 4.4 onwards, we specialize to our case of interest, the 2-category of $\mathbb{Z}/q$-supercategories.

4.1. ***Monoidal 2-categories.*** Let $(\mathscr{C}, \otimes)$ be a semistrict monoidal 2-category. We refer to [Crans 1998, Section 2.1] for the precise definition, and make a few remarks here. We require composition of 1-morphisms in $\mathscr{C}$ to be strictly associative (this is part of what "2-category" means). We also require $\otimes$ to be strictly associative (this is part of what "semistrict" means). We ignore the unit of the monoidal structure in our discussion. An important piece of the monoidal structure is that the square

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\mathrm{id}_A \otimes g} & A \otimes B' \\ {\scriptstyle f \otimes \mathrm{id}_B}\downarrow & & \downarrow{\scriptstyle f \otimes \mathrm{id}_{B'}} \\ A' \otimes B & \xrightarrow{\mathrm{id}_{A'} \otimes g} & A' \otimes B' \end{array}$$

commutes up to a given 2-isomorphism $i_{f,g}$; precisely,

$$i_{f,g} : (f \otimes \mathrm{id}_{B'}) \circ (\mathrm{id}_A \otimes g) \xrightarrow{\sim} (\mathrm{id}_{A'} \otimes g) \circ (f \otimes \mathrm{id}_B).$$

In what follows, we omit tensor symbols (i.e., we write AB in place of $A \otimes B$), and simply write A for the identity 1-morphism on an object A of $\mathscr{C}$.

4.2. ***Braidings on monoidal 2-categories.*** A *braiding* on $\mathscr{C}$ consists of a pseudonatural equivalence $R_{A,B} : AB \to BA$ together with 2-isomorphisms

$$\rho_{A|B,C} : (BR_{A,C}) \circ (R_{A,B}C) \to R_{A,BC}, \quad \rho_{A,B|C} : (R_{A,C}B) \circ (AR_{B,C}) \to R_{AB,C}$$

satisfying the five axioms below (we have inverted the convention from [Crans 1998, Section 2.2] using ρ in place of ρ^{-1}). Note that the above two 2-isomorphisms make the diagrams

$$\begin{array}{ccc} ABC & \xrightarrow{R_{A,BC}} & BCA \\ & {\scriptstyle R_{A,B}C} \searrow \quad \nearrow {\scriptstyle BR_{A,C}} & \\ & BAC & \end{array} \qquad \begin{array}{ccc} ABC & \xrightarrow{R_{AB,C}} & CAB \\ & {\scriptstyle AR_{B,C}} \searrow \quad \nearrow {\scriptstyle R_{A,C}B} & \\ & ACB & \end{array}$$

commute up to isomorphism, and are essentially the hexagon axioms. (Remember that the associators are identity morphisms, and can thus be omitted.) We recall that, if $f : A \to A'$ and $g : B \to B'$ are 1-morphisms, then there is a given 2-isomorphism

$$R_{f,g} : (gf)R_{A,B} \xrightarrow{\sim} R_{A',B'}(fg);$$

this is part of the data of the pseudonatural equivalence R. This isomorphism makes the square

$$\begin{array}{ccc} AB & \xrightarrow{R_{A,B}} & BA \\ {\scriptstyle fg}\downarrow & & \downarrow{\scriptstyle gf} \\ A'B' & \xrightarrow{R_{A',B'}} & B'A' \end{array}$$

commute up to isomorphism.

Axiom 1. The tetrahedron in (3.35a) commutes for all objects A, B, C and D, where the edges are the obvious 1-morphisms built from R and the faces are the 2-morphisms provided by ρ. Here "commutes" is taken in the 2-categorical sense. Explicitly, this means that the diagram

$$\begin{array}{ccc}
(BCR_{A,D})(BR_{A,C}D)(R_{A,B}CD) & \xrightarrow{B\rho_{A|C,D}\circ 1} & (BR_{A,CD})(R_{A,B}CD) \\
\downarrow{\scriptstyle 1\circ(\rho_{A|B,C}D)} & & \downarrow{\scriptstyle \rho_{A|B,CD}} \\
(BCR_{A,D})(R_{A,BC}D) & \xrightarrow{\rho_{A|BC,D}} & R_{A,BCD}
\end{array}$$

commutes.

Axiom 2. The tetrahedron in (3.35b) commutes. This means that the diagram

$$\begin{array}{ccc}
(R_{A,D}BC)(AR_{B,D}C)(ABR_{C,D}) & \xrightarrow{(\rho_{A,B|D}C)\circ 1} & (R_{AB,D}C)(ABR_{C,D}) \\
\downarrow{\scriptstyle 1\circ(A\rho_{B,C|D})} & & \downarrow{\scriptstyle \rho_{AB,C|D}} \\
(R_{A,D}BC)(AR_{BC,D}) & \xrightarrow{\rho_{A,BC|D}} & R_{ABC,D}
\end{array}$$

commutes.

Axiom 3. The diagram in (3.35c) commutes; the 2-isomorphism in the square is given by i. We write this out explicitly using three diagrams. First, we require that the diagram

$$\begin{array}{ccc}
(CR_{A,D}B)(R_{A,C}DB)(ACR_{B,D})(AR_{B,C}D) & \xrightarrow{1\circ 1\circ A\rho_{B|C,D}} & (CR_{A,D}B)(R_{A,C}DB)(AR_{B,CD}) \\
\downarrow{\scriptstyle \rho_{A|C,D}B\circ 1\circ 1} & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle \rho_{A|C,D}B\circ 1} \\
(R_{A,CD}B)(ACR_{B,D})(AR_{B,C}D) & \xrightarrow{1\circ A\rho_{B|C,D}} & (R_{A,CD}B)(AR_{B,CD})
\end{array}$$

commute, and define φ to be the indicated arrow. Second, we require that the diagram

$$\begin{array}{ccc}
(CR_{A,D}B)(CAR_{B,D})(R_{A,C}BD)(AR_{B,C}D) & \xrightarrow{C\rho_{A,B|D}\circ 1\circ 1} & (CR_{AB,D})(R_{A,C}BD)(AR_{B,C}D) \\
\downarrow{\scriptstyle 1\circ 1\circ\rho_{A,B|C}D} & \overset{\psi}{\searrow} & \downarrow{\scriptstyle 1\circ\rho_{A,B|C}D} \\
(CR_{A,D}B)(CAR_{B,D})(R_{AB,C}D) & \xrightarrow{C\rho_{A,B|D}\circ 1} & (CR_{AB,D})(R_{AB,C}D)
\end{array}$$

commute, and again define ψ to be the indicated arrow. Finally, we require that the diagram

$$\begin{array}{ccccc}
(CR_{A,D}B)(R_{A,C}DB)(ACR_{B,D})(AR_{B,C}D) & & \xrightarrow{1\circ i\circ 1} & & (CR_{A,D}B)(CAR_{B,D})(R_{A,C}BD)(AR_{B,C}D) \\
\downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle \psi} \\
(R_{A,CD}B)(AR_{B,CD}) & \xrightarrow{\rho_{A,B|CD}} & R_{AB,CD} & \xleftarrow{\rho_{AB|C,D}} & (CR_{AB,D})(R_{AB,C}D)
\end{array}$$

commute, where $i = i_{f,g}$ with $f = R_{A,C}$ and $g = R_{B,D}$.

Axiom 4. The diagram in (3.35d) commutes; the 2-isomorphisms in the four triangles come from ρ, while the 2-isomorphisms in the two squares come from R. Explicitly, this means that the diagram

$$\begin{array}{ccccc}
(R_{BA,C})(R_{A,B}C) & \xleftarrow{\rho_{B,A|C}\circ 1} & (R_{B,C}A)(BR_{A,C})(R_{A,B}C) & \xrightarrow{1\circ\rho_{A|B,C}} & (R_{B,C}A)(R_{A,BC}) \\
\downarrow{\scriptstyle R_{R_{A,B},\mathrm{id}_C}} & & & & \downarrow{\scriptstyle R_{\mathrm{id}_A,R_{B,C}}} \\
(CR_{A,B})(R_{AB,C}) & \xrightarrow{1\circ\rho^{-1}_{A,B|C}} & (CR_{A,B})(R_{A,C}B)(AR_{B,C}) & \xleftarrow{\rho^{-1}_{A|C,B}\circ 1} & (R_{A,CB})(AR_{B,C})
\end{array}$$

commutes.

Axiom 5. A number of conditions regarding the unit object hold; see [Crans 1998, Section 2.2].

4.3. ***Symmetric and sylleptic structures.*** There are two "levels" of commutativity for monoidal 1-categories: braided and symmetric. For monoidal 2-categories, there are three levels: braided, sylleptic and symmetric. We now recall what the latter two mean.

Fix a braiding R on $\mathscr{C}$. A *syllepsis* is a 2-isomorphism (we have inverted the convention from [Crans 1998, Section 4], using v in place of v^{-1})

$$v_{A,B} : R_{B,A} \circ R_{A,B} \to \mathrm{id}_{A\otimes B}$$

such that the following diagrams commute (as before, the dotted arrows denote the definition of the common composition in the first and third diagrams):

$$\begin{array}{ccc}
(R_{B,A}C)(BR_{C,A})(BR_{A,C})(R_{A,B}C) & \xrightarrow{1\circ 1\circ\rho_{A|B,C}} & (R_{B,A}C)(BR_{C,A})(R_{A,BC}) \\
\downarrow{\scriptstyle \rho_{B,C|A}\circ 1\circ 1} & \overset{\varphi}{\cdots\cdots\to} & \downarrow{\scriptstyle \rho_{B,C|A}\circ 1} \\
(R_{BC,A})(BR_{A,C})(R_{A,B}C) & \xrightarrow{1\circ\rho_{A|B,C}} & (R_{BC,A})(R_{A,BC})
\end{array}$$

$$\begin{array}{ccc}
(R_{B,A}C)(BR_{C,A})(BR_{A,C})(R_{A,B}C) & \xrightarrow{\varphi} & (R_{BC,A})(R_{A,BC}) \\
\downarrow{\scriptstyle 1\circ Bv_{A,C}\circ 1} & & \downarrow{\scriptstyle v_{A,BC}} \\
(R_{B,A}C)(R_{A,B}C) & \xrightarrow{v_{A,B}C} & \mathrm{id}_{ABC}
\end{array}$$

$$\begin{array}{ccc}
(AR_{C,B})(R_{C,A}B)(R_{A,C}B)(AR_{B,C}) & \xrightarrow{1\circ 1\circ\rho_{A,B|C}} & (AR_{C,B})(R_{C,A}B)(R_{AB,C}) \\
\downarrow{\scriptstyle \rho_{C|A,B}\circ 1\circ 1} & \overset{\psi}{\cdots\cdots\to} & \downarrow{\scriptstyle \rho_{C|A,B}\circ 1} \\
(R_{C,AB})(R_{A,C}B)(AR_{B,C}) & \xrightarrow{1\circ\rho_{A,B|C}} & (R_{C,AB})(R_{AB,C})
\end{array}$$

$$\begin{array}{ccc}
(AR_{C,B})(R_{C,A}B)(R_{A,C}B)(AR_{B,C}) & \xrightarrow{\psi} & (R_{C,AB})(R_{AB,C}) \\
\downarrow{\scriptstyle 1\circ v_{A,C}B\circ 1} & & \downarrow{\scriptstyle v_{AB,C}} \\
(AR_{C,B})(AR_{B,C}) & \xrightarrow{Av_{B,C}} & \mathrm{id}_{ABC}
\end{array}$$

There are also conditions involving the unit object that we omit (see [Crans 1998, Section 4]). A syllepsis is a *symmetry* if $1 \circ v_{A,B} = v_{B,A} \circ 1$ agree as isomorphisms

$$R_{A,B}R_{B,A}R_{A,B} \to R_{A,B}.$$

4.4. ***The 2-category of supercategories.*** Fix an even positive integer q. For the remainder of this section, we let $\mathscr{C}$ be the 2-category whose objects are $\mathbb{Z}/q$-supercategories equipped with a Π-functor. The 1-morphisms of $\mathscr{C}$ are superfunctors, and the 2-morphisms are even natural transformations. This is a 2-category in the sense that we have been using it, as composition of 1-morphisms in $\mathscr{C}$ is strictly associative. We denote supercategories by $A, B, \ldots$ and objects of supercategories by $a, b, \ldots$, to maintain consistency with the rest of this section.

Let A and B be objects of $\mathscr{C}$. We let $AB = A \otimes B$ be the supercategory $A \boxtimes B$ defined in Section 2.5. For objects $a \in A$ and $b \in B$, we write $a \otimes b$, or simply ab, instead of $a \boxtimes b$ in this section. We equip $A \otimes B$ with the Π-structure $\Pi(a \otimes b) = \Pi(a) \otimes b$. Thus, $A \otimes B$ is again an object of $\mathscr{C}$. This defines a monoidal structure on $\mathscr{C}$. However, it is *not* semistrict, since this product is not strictly associative. We will ignore this detail, and simply treat $\otimes$ as if it were semistrict. This is the main reason for the disclaimer at the start of this section that our discussion would not be entirely rigorous.

In the remainder of this section, we investigate how Σ and T (as defined in Section 3.2) define braidings (or syllepses or symmetries) on $\mathscr{C}$. We focus on the T case since it is most relevant to this paper, but the Σ case is very similar. To maintain consistency with the rest of this section, we write R in place of T. Thus, for objects A and B of $\mathscr{C}$, we have the equivalence $R_{A,B} : A \otimes B \to B \otimes A$ defined on objects by $R_{A,B}(a \otimes b) = \Pi^{|a||b|}(b) \otimes a$ and on morphisms by $R_{A,B}(f \otimes g) = (-1)^{|f||g|}\Pi^{|a||b|}(g) \otimes f$.

4.5. ***Braiding.*** To give R the structure of a braiding, we first must define the ρ isomorphisms. Fix functions $\omega_1, \omega_2 : (\mathbb{Z}/q)^3 \to \boldsymbol{k}^\times$. Let A, B and C be objects of $\mathscr{C}$, and let a, b and c be homogeneous objects of A, B and C. We have

$$((BR_{A,C})(R_{A,B}C))(a \otimes b \otimes c) = \Pi^{|a||b|}(b) \otimes \Pi^{|a||c|}(c) \otimes a$$

and

$$R_{A,BC}(a \otimes b \otimes c) = \Pi^{|a|(|b|+|c|)}(b) \otimes c \otimes a.$$

We define

$$\rho_{A|B,C} : \Pi^{|a||b|}(b) \otimes \Pi^{|a||c|}(c) \otimes a \to \Pi^{|a|(|b|+|c|)}(b) \otimes c \otimes a$$

by

$$\rho_{A|B,C} = \omega_1(a; b, c) \cdot \xi \otimes \xi \otimes 1,$$

where we write $\omega_1(a; b, c)$ in place of $\omega_1(|a|; |b|, |c|)$. Here $\xi^{n,m} : \Pi^n \to \Pi^m$ is the isomorphism defined in Section 2.4, and we omit the indices to ξ when they are clear. Similarly, we have

$$((R_{A,C}B)(AR_{B,C}))(a \otimes b \otimes c) = \Pi^{|a||c|}(\Pi^{|b||c|}(c)) \otimes a \otimes b$$

and

$$R_{AB,C}(a \otimes b \otimes c) = \Pi^{|c|(|a|+|b|)}(c) \otimes a \otimes b.$$

We define $\rho_{A,B|C}$ to be $\omega_2(a,b;c)$ times the identity map; note that $\Pi^{|a||c|} \circ \Pi^{|b||c|} = \Pi^{|c|(|a|+|b|)}$. We have thus defined the necessary data for a braiding on $\mathscr{C}$. We now examine the axioms.

Axiom 1: The diagram becomes

$$\begin{array}{ccc} \Pi^{|a||b|}(b)\Pi^{|a||c|}(c)\Pi^{|a||d|}(d)a & \xrightarrow{\omega_1(a;c,d)1\otimes\xi\otimes\xi\otimes 1} & \Pi^{|a||b|}(b)\Pi^{|a|(|c|+|d|)}(c)da \\ \downarrow{\scriptstyle \omega_1(a;b,c)\xi\otimes\xi\otimes 1\otimes 1} & & \downarrow{\scriptstyle \omega_1(a;b,cd)\xi\otimes\xi\otimes 1\otimes 1} \\ \Pi^{|a|(|b|+|c|)}(b)c\Pi^{|a||d|}(d)a & \xrightarrow{\omega_1(a;bc,d)\xi\otimes 1\otimes\xi\otimes 1} & \Pi^{|a|(|b|+|c|+|d|)}(b)cda \end{array}$$

The top composition is $\omega_1(a;b,cd)\omega_1(a;c,d)\xi\otimes\xi\otimes\xi\otimes 1$ while the bottom composition is

$$\omega_1(a;b,c)\omega_1(a;bc,d)\xi\otimes\xi\otimes\xi\otimes 1.$$

Hence, the diagram commutes if and only if

$$\omega_1(a;b,c)\omega_1(a;bc,d) = \omega_1(a;b,cd)\omega_1(a;c,d),$$

which is equivalent to (3.5a) (note we have multiplicative notation as opposed to additive notation here since we are using the objects rather than their degrees as input).

Axiom 2: The diagram becomes

$$\begin{array}{ccc} \Pi^{|a||d|}\big(\Pi^{|b||d|}(\Pi^{|c||d|}(d))\big)abc & \xrightarrow{\omega_2(a,b;d)} & \Pi^{(|a|+|b|)|d|}(\Pi^{|c||d|}(d))abc \\ \downarrow{\scriptstyle \omega_2(b,c;d)} & & \downarrow{\scriptstyle \omega_2(ab,c;d)} \\ \Pi^{|a||d|}(\Pi^{(|b|+|c|)|d|}(d))abc & \xrightarrow{\omega_2(a,bc;d)} & \Pi^{(|a|+|b|+|c|)|d|}(d)abc \end{array}$$

where each map is a scalar times the identity. Thus, commutativity of this diagram is equivalent to

$$\omega_2(ab,c;d)\omega_2(a,b;d) = \omega_2(b,c;d)\omega_2(a,bc;d),$$

which is (3.5b).

Axiom 3: The first diagram comes out to

$$\begin{array}{ccc} \Pi^{|a||c|}(\Pi^{|b||c|}(c))\Pi^{|a||d|}(\Pi^{|b||d|}(d))ab & \xrightarrow{①} & \Pi^{|a||c|}(\Pi^{|b|(|c|+|d|)}(c))\Pi^{|a||d|}(d)ab \\ \downarrow{\scriptstyle ②} & \overset{\varphi}{\cdots\!\!\rightarrow} & \downarrow{\scriptstyle ③} \\ \Pi^{|a|(|c|+|d|)}(\Pi^{|b||c|}(c))\Pi^{|b||d|}(d)ab & \xrightarrow{④} & \Pi^{|a|(|c|+|d|)}(\Pi^{|b|(|c|+|d|)}(c))dab \end{array}$$

The four numbered maps are given by

$$\begin{aligned} ① &= (-1)^{|a||b||d|(|c|+1)}\omega_1(b;c,d)\xi\otimes\xi\otimes 1\otimes 1, \\ ② &= \omega_1(a;c,d)\xi\otimes\xi\otimes 1\otimes 1, \\ ③ &= \omega_1(a;c,d)\xi\otimes\xi\otimes 1\otimes 1, \\ ④ &= (-1)^{|a||b||d|(|c|+1)}\omega_1(b;c,d)\xi\otimes\xi\otimes 1\otimes 1. \end{aligned}$$

To compute the signs, we use the identities $\Pi^k(\xi^{n,m}) = (-1)^{k(n+m)}\xi^{n+k,m+k}$ and $\xi^{n,m}_{\Pi^k} = \xi^{n+k,m+k}$. We thus see that the square commutes, and

$$\varphi = (-1)^{|a||b||c||d|}\omega_1(b;c,d)\omega_1(a;c,d)\xi\otimes\xi\otimes 1\otimes 1.$$

Note that one has to apply the sign rule when computing the compositions, and this cancels the $(-1)^{|a||b||d|}$ part of the sign. The second diagram is

$$\begin{array}{ccc}
\Pi^{|a||c|}(\Pi^{|b||c|}(c))\Pi^{|a||d|}(\Pi^{|b||d|}(d))ab & \xrightarrow{\omega_2(a,b;d)} & \Pi^{|a||c|}(\Pi^{|b||c|}(c))\Pi^{|d|(|a|+|b|)}(d)ab \\
\Big\downarrow{\scriptstyle \omega_2(a,b;c)} & \overset{\psi}{\searrow} & \Big\downarrow{\scriptstyle \omega_2(a,b;c)} \\
\Pi^{|c|(|a|+|b|)}(c)\Pi^{|a||d|}(\Pi^{|b||d|}(d))ab & \xrightarrow{\omega_2(a,b;d)} & \Pi^{|c|(|a|+|b|)}(c)\Pi^{|d|(|a|+|b|)}(d)ab
\end{array}$$

All maps are scalar multiples of the identity, so the diagram commutes and

$$\psi = \omega_2(a,b;d)\omega_2(a,b;c)\cdot\mathrm{id}.$$

Finally, we have the third diagram. It is

$$\begin{array}{ccc}
\Pi^{|a||c|}(\Pi^{|b||c|}(c))\Pi^{|a||d|}(\Pi^{|b||d|}(d))ab & \xrightarrow{\quad\mathrm{id}\quad} & \Pi^{|a||c|}(\Pi^{|b||c|}(c))\Pi^{|a||d|}(\Pi^{|b||d|}(d))ab \\
\Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \psi} \\
\Pi^{|a|(|c|+|d|)}(\Pi^{|b|(|c|+|d|)}(c))dab & & \Pi^{|c|(|a|+|b|)}(c)\Pi^{|d|(|a|+|b|)}(d)ab \\
& {\scriptstyle \omega_2(a,b;cd)\,\mathrm{id}}\searrow \quad \swarrow{\scriptstyle \omega_1(ab;c,d)\xi\xi 11} & \\
& \Pi^{(|a|+|b|)(|c|+|d|)}(c)dab &
\end{array}$$

(We have omitted $\otimes$ symbols in the bottom right map.) This diagram commutes if and only if (3.5c) holds.

Axiom 4: The diagram becomes

$$\begin{array}{ccccc}
\Pi^{|c|(|a|+|b|)}(c)\Pi^{|a||b|}(b)a & \xleftarrow{\omega_2(b,a;c)} & \Pi^{|b||c|}(\Pi^{|a||c|}(c))\Pi^{|a||b|}(b)a & \longrightarrow & \Pi^{|b||c|}(c)\Pi^{|a|(|b|+|c|)}(b)a \\
\| & & & & \Big\downarrow{\scriptstyle (-1)^{|a||b||c|}\xi_c\otimes\xi\otimes 1} \\
\Pi^{(|a|+|b|)|c|}(c)\Pi^{|a||b|}(b)a & \xrightarrow{\omega_2(a,b;c)^{-1}} & \Pi^{|a||c|}(\Pi^{|b||c|}(c))\Pi^{|a||b|}(b)a & \longleftarrow & \Pi^{|a|(|b|+|c|)}(\Pi^{|b||c|}(c))ba
\end{array}$$

where the top right map is $(-1)^{|a||c|(|b|+1)}\omega_1(a;b,c)\xi_c\otimes\xi\otimes 1$ and the bottom right map is

$$(-1)^{|a||b|(|c|+1)}\omega_1(a;c,b)^{-1}\xi_c^{-1}\otimes\xi^{-1}\otimes 1.$$

Taking into account signs from composing tensor products of maps, the right-side composition is $\omega_1(a;b,c)\omega_1(a;c,b)^{-1}$ times the identity, and hence the commutativity of this diagram is equivalent to (3.5d).

We thus see that R and ρ define a braiding if and only if (ω_1,ω_2) is a B-factor system. In particular, we obtain braidings using the systems defined in Examples 3.26 and 3.27.

4.6. ***Symmetry.*** Now we attempt to define a syllepsis on R. Fix a function $\omega_\sharp : (\mathbb{Z}/q)^2 \to \boldsymbol{k}^\times$. We have

$$R_{B,A}(R_{A,B}(a \otimes b)) = \Pi^{|a||b|}(a) \otimes \Pi^{|a||b|}(b).$$

We define $v_{A,B} : R_{B,A}R_{A,B} \to \mathrm{id}$ to be the isomorphism $\omega_\sharp(a, b)\xi \otimes \xi$ on an object $a \otimes b$. We check the commutativity of the four diagrams in order. The first is

$$\begin{array}{ccc}
\Pi^{|a||b|}(\Pi^{(|a||c|)}(a))\Pi^{|a||b|}(b)\Pi^{|a||c|}(c) & \xrightarrow{\omega_1(a;b,c)1\otimes\xi\otimes\xi} & \Pi^{|a||b|}(\Pi^{|a||c|}(a))\Pi^{|a|(|b|+|c|)}(b)c \\
\downarrow{\scriptstyle \omega_2(b,c;a)\,\mathrm{id}} & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle \omega_2(b,c;a)\,\mathrm{id}} \\
\Pi^{|a|(|b|+|c|)}(a)\Pi^{|a||b|}(b)\Pi^{|a||c|}(c) & \xrightarrow{\omega_1(a;b,c)1\otimes\xi\otimes\xi} & \Pi^{|a|(|b|+|c|)}(a)\Pi^{|a|(|b|+|c|)}(b)c
\end{array}$$

This formally commutes, and so

$$\varphi = \omega_1(a; b, c)\omega_2(b, c; a)1 \otimes \xi \otimes \xi.$$

The second diagram is

$$\begin{array}{ccc}
\Pi^{|a||b|}(\Pi^{|a||c|}(a))\Pi^{|a||b|}(b)\Pi^{|a||c|}(c) & \xrightarrow{\varphi} & \Pi^{|a|(|b|+|c|)}(a)\Pi^{|a|(|b|+|c|)}(b)c \\
\downarrow{\scriptstyle (-1)^{|a||b||c|}\omega_\sharp(a,c)\xi\otimes1\otimes\xi} & & \downarrow{\scriptstyle \omega_\sharp(a,bc)\xi\otimes\xi\otimes1} \\
\Pi^{|a||b|}(a)\Pi^{|a||b|}(b)c & \xrightarrow{\omega_\sharp(a,b)\xi\otimes\xi\otimes1} & abc
\end{array}$$

This diagram commutes if and only if

$$\omega_1(a; b, c)\omega_2(b, c; a)\omega_\sharp(a, bc) = \omega_\sharp(a, b)\omega_\sharp(a, c),$$

which is equivalent to (3.5e). The third diagram is

$$\begin{array}{ccc}
\Pi^{|a||c|}(a)\Pi^{|b||c|}(b)\Pi^{|a||c|}(\Pi^{|b||c|}(c)) & \xrightarrow{\omega_2(a,b;c)\,\mathrm{id}} & \Pi^{|a||c|}(a)\Pi^{|b||c|}(b)\Pi^{(|a|+|b|)|c|}(c) \\
\downarrow{\scriptstyle \omega_1(c;a,b)\xi\otimes\xi\otimes1} & \overset{\psi}{\searrow} & \downarrow{\scriptstyle \omega_1(c;a,b)\xi\otimes\xi\otimes1} \\
\Pi^{(|a|+|b|)|c|}(a)b\Pi^{|a||c|}(\Pi^{|b||c|}(c)) & \xrightarrow{\omega_2(a,b;c)\,\mathrm{id}} & \Pi^{(|a|+|b|)|c|}(a)b\Pi^{(|a|+|b|)|c|}(c)
\end{array}$$

This formally commutes, and we find that

$$\psi = \omega_1(c; a, b)\omega_2(a, b; c)\xi \otimes \xi \otimes 1.$$

The fourth diagram is

$$\begin{array}{ccc}
\Pi^{|a||c|}(a)\Pi^{|b||c|}(b)\Pi^{|a||c|}(\Pi^{|b||c|}(c)) & \xrightarrow{\psi} & \Pi^{(|a|+|b|)|c|}(a)b\Pi^{(|a|+|b|)|c|}(c) \\
\downarrow{\scriptstyle \omega_\sharp(a,c)\xi\otimes1\otimes\xi} & & \downarrow{\scriptstyle \omega_\sharp(ab,c)\xi\otimes1\otimes\xi} \\
a\Pi^{|b||c|}(b)\Pi^{|b||c|}(c) & \xrightarrow{\omega_\sharp(b,c)1\otimes\xi\otimes\xi} & abc
\end{array}$$

This diagram commutes if and only if

$$\omega_1(c; a, b)\omega_2(a, b; c)\omega_\sharp(ab, c) = \omega_\sharp(a, c)\omega_\sharp(b, c),$$

which is equivalent to (3.5f). Finally, the two isomorphisms

$$(R_{A,B}R_{B,A}R_{A,B})(ab) = \Pi^{2|a||b|}(b)\Pi^{|a||b|}(a) \to \Pi^{|a||b|}(b)a = R_{A,B}(ab)$$

are

$$1 \circ v_{A,B} = \omega_\sharp(a, b)\xi \otimes \xi, \quad v_{B,A} \circ 1 = \omega_\sharp(b, a)\xi \otimes \xi.$$

These agree if and only if $\omega_\sharp(a, b) = \omega_\sharp(b, a)$, i.e., γ is symmetric. We thus see that v defines a syllepsis if and only if $(\omega_1, \omega_2, \omega_\sharp)$ is an S-factor system, and it defines a symmetry if and only if, additionally, $\omega_\sharp$ is symmetric.

We thus obtain symmetries from the systems in Examples 3.26 and 3.27.

Remark 4.1. A variant of the above discussion shows that even factor systems can be used to endow Σ with a braided, sylleptic or symmetric structure. □

5. Spin-symmetric groups

5.1. ***S-sets and s-groups.*** An *s-set* is a triple $(X, |\cdot|, c)$ where X is a set, $|\cdot| : X \to \mathbb{Z}/2$ is a function (the degree function), and c is an involution of X such that $|cx| = |x|$ for all $x \in X$. We write X_i for the set of elements of parity i.

Let X and Y be s-sets. A *homogeneous map* $X \to Y$ of degree n is a function that maps X_i into Y_{i+n} and is compatible with c. We let $\mathrm{Hom}(X, Y)_i$ denote the set of maps of degree i, and let $\mathrm{Hom}(X, Y) = \mathrm{Hom}(X, Y)_0 \amalg \mathrm{Hom}(X, Y)_1$, which is naturally an s-set. We thus have a category **SSet** of s-sets. We let **SSet**$^\circ$ be the subcategory where the morphisms are even (degree 0).

Let X and Y be s-sets. We define the *product* of X and Y, denoted by $X \ast\!\!\ast Y$, as follows. The underlying set is the quotient of the usual cartesian product $X \times Y$ by the involution $c \times c$. That is, elements of $X \ast\!\!\ast Y$ are represented by ordered pairs (x, y), and the pairs (x, y) and (cx, cy) represent the same element. The degree function is given by $|(x, y)| = |x| + |y|$. The involution c is given by $c(x, y) = (cx, y) = (x, cy)$. We define an isomorphism $X \ast\!\!\ast Y \to Y \ast\!\!\ast X$ by $(x, y) \mapsto c^{|x||y|}(y, x)$. The product $\ast\!\!\ast$ gives **SSet**$^\circ$ the structure of a symmetric monoidal category.

An *s-group* is a group object in the monoidal category **SSet**$^\circ$. Explicitly, an s-group is a triple $(G, |\cdot|, c)$ where G is a group, $|\cdot| : G \to \mathbb{Z}/2$ is a group homomorphism, and $c \in G$ is a central element satisfying $c^2 = 1$ and $|c| = 0$. A *homomorphism* of s-groups $f : G \to H$ is a group homomorphism such that $f(c) = c$ and $|f(g)| = |g|$ for all $g \in G$.

If G and H are s-groups, then $G \ast\!\!\ast H$ is naturally an s-group, with multiplication given by $(g, h)(g', h') = c^{|h||g'|}(gg', hh')$. Two elements g and h of an s-group G are said to *supercommute* if $gh = c^{|h||g|}hg$. Similarly, two subsets A and B of G are said to supercommute if every element of A supercommutes with

every element of B. If H_1 and H_2 are supercommuting s-subgroups of G, then there is a homomorphism $H_1 \ltimes\!\!\!\ltimes H_2 \to G$ given by $(h_1, h_2) \mapsto h_1 h_2$.

Example 5.1. Let X be an object in a supercategory, and let $\mathrm{Aut}^h(X)$ be the group of all homogeneous automorphisms of X. Then $\mathrm{Aut}^h(X)$ is an s-group, taking $c = -1$ and the degree map to be the usual degree map. For a super-vector space V, we write $\mathbf{GL}^h(V)$ in place of $\mathrm{Aut}^h(V)$. □

5.2. ***Spin-symmetric groups.*** Recall that the braid group $\mathfrak{B}_n$ is generated by elements $s_1, \dots, s_{n-1}$ modulo

- the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \le i \le n-2$; and
- the commutativity relations $s_i s_j = s_j s_i$ for $|i-j| \ge 2$.

The symmetric group $\mathfrak{S}_n$ is isomorphic to the quotient of $\mathfrak{B}_n$ by the relations $s_i^2 = 1$; the isomorphism takes s_i to the transposition $(i\ i{+}1)$.

For $\delta \in \mathbb{Z}/2$, we define $\widetilde{\mathfrak{B}}_n^\delta$ to be the group generated by elements $\tilde{s}_1, \dots, \tilde{s}_{n-1}, c$ modulo

- the element c is central and $c^2 = 1$;
- the braid relations $\tilde{s}_i \tilde{s}_{i+1} \tilde{s}_i = \tilde{s}_{i+1} \tilde{s}_i \tilde{s}_{i+1}$ for $1 \le i \le n-2$; and
- the (super)commutativity relations $\tilde{s}_i \tilde{s}_j = c^\delta \tilde{s}_j \tilde{s}_i$ for $|i-j| \ge 2$.

For $\epsilon \in \mathbb{Z}/2$, define $\widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)}$ to be the quotient of $\widetilde{\mathfrak{B}}_n^\delta$ by the relations $\tilde{s}_i^2 = c^\epsilon$. It is well known that $c \ne 1$ in $\widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)}$ (for $\delta = 1$ and $\epsilon = 0$, this follows from Proposition 8.1, for instance), and thus $c \ne 1$ in $\widetilde{\mathfrak{B}}_n^\delta$ as well. We therefore have central extensions

$$1 \to \langle c \rangle \to \widetilde{\mathfrak{B}}_n^\delta \to \mathfrak{B}_n \to 1 \quad \text{and} \quad 1 \to \langle c \rangle \to \widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)} \to \mathfrak{S}_n \to 1,$$

where $\langle c \rangle$ is cyclic of order two.

We define the degree function on $\widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)}$ to be the composition $\widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)} \to \mathfrak{S}_n \xrightarrow{\ell} \mathbb{Z}/2$, where ℓ is the length modulo 2 of a permutation, i.e., 0 for even permutations and 1 for odd permutations. In this way, $\widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)}$ has the structure of an s-group. We let $\widetilde{\mathfrak{S}}_n = \widetilde{\mathfrak{S}}_n^{(1,0)}$, and refer to this as the *spin-symmetric group*.

Remark 5.2. For $n = 0, 1$, the central extension $\widetilde{\mathfrak{S}}_n^{(\delta,\epsilon)}$ is trivial for all ϵ and δ, while, for $n = 2, 3$, it is independent of δ. For $n \ge 4$, the four central extensions are nonisomorphic, and the group $\mathrm{H}^2(\mathfrak{S}_n, \mathbb{Z}/2)$ of central extensions is isomorphic to $(\mathbb{Z}/2)^2$ via (δ, ϵ).

The calculation of $\mathrm{H}^2(\mathfrak{S}_n, \mathbb{Z}/2)$ can be done as follows. First, $\mathrm{H}_1(\mathfrak{S}_n, \mathbb{Z})$ is the abelianization of $\mathfrak{S}_n$, which is just $\mathbb{Z}/2$ for $n \ge 2$ and trivial otherwise. Next, $\mathrm{H}_2(\mathfrak{S}_n, \mathbb{Z})$ is dual to $\mathrm{H}^2(\mathfrak{S}_n, \mathbb{C}^\times)$, which is $\mathbb{Z}/2$ for $n \ge 4$ and trivial otherwise [Hoffman and Humphreys 1992, Theorems 2.7 and 2.9]. Our earlier claim then follows from the universal coefficients theorem. □

5.3. ***The $\tilde{\tau}$ elements.*** Let $\tau_{n,m} \in \mathfrak{S}_{n+m}$ be the permutation defined by

$$\tau_{n,m}(i) = \begin{cases} i+m & \text{if } 1 \le i \le n, \\ i-n & \text{if } n+1 \le i \le n+m. \end{cases}$$

In a sense, $\tau_{n,m}$ induces the natural bijection of sets $[n] \amalg [m] \to [m] \amalg [n]$. Let $\sigma_{n,i} \in \mathfrak{S}_{n+m}$ be the $(n+1)$-cycle $(i\ i{+}1\ \cdots\ i{+}n)$. Then

$$\sigma_{n,i} = s_i s_{i+1} \cdots s_{i+n-1}, \quad \tau_{n,m} = \sigma_{n,m} \cdots \sigma_{n,1}.$$

We define

$$\tilde{\sigma}_{n,i} = \tilde{s}_i \tilde{s}_{i+1} \cdots \tilde{s}_{i+n-1}, \quad \tilde{\tau}_{n,m} = \tilde{\sigma}_{n,m} \cdots \tilde{\sigma}_{n,1}$$

as elements of $\widetilde{\mathfrak{S}}_{n+m}$. The $\tilde{\tau}_{n,m}$ elements are very important, and are essentially responsible for all examples of supersymmetric monoidal structures that we write down.

Proposition 5.3. *We have* $\tilde{\tau}_{n,m}\tilde{\tau}_{m,n} = c^N$ *with* $N = \binom{n}{2}\binom{m}{2}$.

Proof. For any i, j, k, we have

$$\tilde{\sigma}_{i,k}\tilde{\sigma}_{j,i+k-1} = (\tilde{s}_k \tilde{s}_{k+1} \cdots \tilde{s}_{i+k-1})(\tilde{s}_{i+k-1}\tilde{s}_{i+k} \cdots \tilde{s}_{i+j+k-2}) = c^{(i-1)(j-1)}\tilde{\sigma}_{j-1,i+k}\tilde{\sigma}_{i-1,k},$$

where we have canceled the $\tilde{s}_{i+k-1}$ and commuted the resulting $\tilde{\sigma}_{i-1,k}$ past $\tilde{\sigma}_{j-1,i+k}$. By applying this relation successively, this implies that

$$\tilde{\sigma}_{i,k}\tilde{\sigma}_{j,i+k-1}\tilde{\sigma}_{j,i+k-2} \cdots \tilde{\sigma}_{j,i} = c^{\binom{i}{2}(j-1)}\tilde{\sigma}_{j-1,i+k}\tilde{\sigma}_{j-1,i+k-1} \cdots \tilde{\sigma}_{j-1,i+1}.$$

Finally, we successively use this relation to get

$$\begin{aligned}
\tilde{\tau}_{n,m}\tilde{\tau}_{m,n} &= (\tilde{\sigma}_{n,m}\tilde{\sigma}_{n,m-1} \cdots \tilde{\sigma}_{n,1})(\tilde{\sigma}_{m,n}\tilde{\sigma}_{m,n-1} \cdots \tilde{\sigma}_{m,1}) \\
&= c^{(m-1)\binom{n}{2}}(\tilde{\sigma}_{n,m}\tilde{\sigma}_{n,m-1} \cdots \tilde{\sigma}_{n,2})(\tilde{\sigma}_{m-1,n+1}\tilde{\sigma}_{m-1,n} \cdots \tilde{\sigma}_{m-1,1}) \\
&\ \ \vdots \\
&= c^{\binom{m}{2}\binom{n}{2}}.
\end{aligned}$$

□

There is a natural injective homomorphism of s-groups

$$j_{n,m} : \widetilde{\mathfrak{S}}_n \ltimes\!\!\!\gg \widetilde{\mathfrak{S}}_m \to \widetilde{\mathfrak{S}}_{n+m} \tag{5.4}$$

defined by $(\tilde{s}_i, 1) \mapsto \tilde{s}_i$ and $(1, \tilde{s}_i) \mapsto \tilde{s}_{n+i}$. (To see that this map is injective, note that it is injective modulo $\langle c \rangle$, and so the kernel is contained in $\langle c \rangle$, but c itself is not in the kernel.)

Proposition 5.5. *For* $g \in \widetilde{\mathfrak{S}}_n$ *and* $h \in \widetilde{\mathfrak{S}}_m$, *we have*

$$\tilde{\tau}_{n,m}\, j_{n,m}(g,h)\, \tilde{\tau}_{n,m}^{-1} = c^{nm|g|+nm|h|+|g||h|}\, j_{m,n}(h,g).$$

Proof. First consider the case $h = 1$, $m = 1$ and $g = \tilde{s}_i$ with $1 \le i \le n-1$. Then $\tilde{\tau}_{n,1} = \tilde{\sigma}_{n,1} = \tilde{s}_1\tilde{s}_2 \cdots \tilde{s}_n$ and hence

$$\tilde{\tau}_{n,1}\tilde{s}_i\tilde{\tau}_{n,1}^{-1} = \tilde{s}_1\tilde{s}_2 \cdots \tilde{s}_n\tilde{s}_i\tilde{s}_n\tilde{s}_{n-1} \cdots \tilde{s}_1 = c^{n-i-1}\tilde{s}_1\tilde{s}_2 \cdots (\tilde{s}_{i+1}\tilde{s}_i\tilde{s}_{i+1}) \cdots \tilde{s}_1 = c^{n-2}\tilde{s}_{i+1} = c^n\tilde{s}_{i+1},$$

where in the second equality we used the commuting relations $\tilde{s}_j\tilde{s}_i = c\tilde{s}_i\tilde{s}_j$ for $j = n, n-1, \ldots, i+2$, and in the third equality we used the braid relation $\tilde{s}_{i+1}\tilde{s}_i\tilde{s}_{i+1} = \tilde{s}_i\tilde{s}_{i+1}\tilde{s}_i$ and then again the commuting relations for $j = i-1, i-2, \ldots, 1$. Since conjugation is a group homomorphism, it follows immediately

that $\tilde{\tau}_{n,1} j_{n,m}(g,1)\tilde{\tau}_{n,1}^{-1} = c^{n|g|} j_{m,n}(1,g)$ for any $g \in \widetilde{\mathfrak{S}}_n$ and hence $\tilde{\tau}_{n,m} j_{n,m}(g,1)\tilde{\tau}_{n,m}^{-1} = c^{nm|g|} j_{m,n}(1,g)$. Similarly, $\tilde{\tau}_{n,m} j_{n,m}(1,h)\tilde{\tau}_{n,m}^{-1} = c^{nm|h|} j_{m,n}(h,1)$. Finally, we have

$$\begin{aligned}\tilde{\tau}_{n,m} j_{n,m}(g,h)\tilde{\tau}_{n,m}^{-1} &= \tilde{\tau}_{n,m} j_{n,m}(g,1) j_{n,m}(1,h)\tilde{\tau}_{n,m}^{-1} \\ &= c^{nm|g|+nm|h|} j_{m,n}(1,g) j_{m,n}(h,1) \\ &= c^{nm|g|+nm|h|+|g||h|} j_{m,n}(h,g). \qquad \square\end{aligned}$$

Proposition 5.6. *For any m, n, p, we have $j_{n,m+p}(1,\tilde{\tau}_{m,p}) j_{n+m,p}(\tilde{\tau}_{m,n},1) = \tilde{\tau}_{m,n+p}$.*

Proof. By definition, we have $j_{n,m+p}(1,\tilde{\sigma}_{m,i}) = \tilde{\sigma}_{m,n+i}$, so

$$\begin{aligned}j_{n,m+p}(1,\tilde{\tau}_{m,p}) j_{n+m,p}(\tilde{\tau}_{m,n},1) &= j_{n,m+p}(1,\tilde{\sigma}_{m,p}\cdots\tilde{\sigma}_{m,1}) j_{n+m,p}(\tilde{\sigma}_{m,n}\cdots\tilde{\sigma}_{m,1},1) \\ &= \tilde{\sigma}_{m,n+p}\cdots\tilde{\sigma}_{m,n+1}\tilde{\sigma}_{m,n}\cdots\tilde{\sigma}_{m,1} = \tilde{\tau}_{m,n+p}. \qquad \square\end{aligned}$$

5.4. ***The category $\tilde{\mathbb{S}}$.*** We define a homogeneous $\mathbb{Z}$-supercategory $\tilde{\mathbb{S}}$ as follows. There is a single homogeneous object $[n]$ of degree n for each $n \geq 0$, and no homogeneous object of negative degree. We put $\mathrm{End}_{\tilde{\mathbb{S}}}([n]) = \boldsymbol{k}[\widetilde{\mathfrak{S}}_n]/(c+1)$ and $\mathrm{Hom}_{\tilde{\mathbb{S}}}([n],[m]) = 0$ for $n \neq m$. We now define a monoidal structure $\otimes$ on $\tilde{\mathbb{S}}$. On objects, we define $[n]\otimes[m] = [n+m]$. We define

$$\otimes : \mathrm{End}_{\tilde{\mathbb{S}}}([n]) \otimes \mathrm{End}_{\tilde{\mathbb{S}}}([m]) \to \mathrm{End}_{\tilde{\mathbb{S}}}([n+m])$$

to be the linear map induced by $j_{n,m}$. The unit object is $[0]$. The monoidal structure is strict (e.g., the associator maps are the identities). One readily verifies that this satisfies the requisite axioms for a monoidal $\mathbb{Z}$-supercategory.

For $n, m \in \mathbb{N}$, define $\beta_{n,m} : [n]\otimes[m] \to [m]\otimes[n]$ to be $\tilde{\tau}_{n,m}$.

Proposition 5.7. *β defines a type IIa supersymmetry on $\tilde{\mathbb{S}}$.*

Proof. Proposition 5.5 exactly shows that β is a natural transformation (see the first paragraph of Section 3.6). Proposition 5.6 exactly shows that the hexagon (H1) commutes. Proposition 5.3 shows that β is symmetric, i.e., $\beta_{m,n}\beta_{n,m} = (-1)^k$ with $k = \binom{n}{2}\binom{m}{2}$, which also implies the second hexagon axiom. $\square$

Now suppose that $\zeta_{16} \in \boldsymbol{k}$. For $n \in \mathbb{Z}$, define $\epsilon(n) = -1$ if $n \equiv 3 \pmod 4$ and $\epsilon(n) = 1$ otherwise. Define $\beta'_{n,m} : [n+m] \to [n+m]$ to be the map $\epsilon(n)^m \zeta_4^{-\binom{n}{2}m} \zeta_{16}^{nm} \tilde{\tau}_{n,m}$.

Proposition 5.8. *β' defines a type IIb supersymmetry on $\tilde{\mathbb{S}}$.*

Proof. This follows from Proposition 3.15. $\square$

Let q be a positive integer. We can then regard $\tilde{\mathbb{S}}$ as $\mathbb{Z}/q$-graded, and the monoidal structure carries over. If q is divisible by 4, then β defines a type IIa supersymmetry, while, if q is any even integer, then β' defines a type IIb supersymmetry (the key point here is that the scalar $\beta'_{n,m}\beta'_{m,n}$ only depends on the parities of n and m; this is the purpose of the somewhat complicated scalar factor in the definition of β').

Remark 5.9. In the category $\tilde{\mathbb{S}}$, the object $[n]$ admits an odd-degree automorphism if and only if $n \geq 2$. Thus, $\tilde{\mathbb{S}}$ does not admit a Π-structure, since the objects $[0]$ and $[1]$ do not. $\square$

Remark 5.10. In our discussion of supersymmetric monoidal categories, we have worked exclusively with linear categories since the examples we care about are linear. However, this is not necessary. One can define a "nonlinear supercategory" to be a category enriched in **SSet**$^\circ$, and then go on to define nonlinear analogs of many of the other notions we have discussed. The category $\tilde{\mathcal{S}}$ defined above is in fact the linearization of what we would call the groupoid of spin-sets, which is likely an object of fundamental importance. □

5.5. ***Actions of spin-symmetric groups.*** Let $\mathcal{A}$ be a type IIa supersymmetric monoidal $\mathbb{Z}/q$-supercategory with $4 \mid q$. In what follows, we assume that $\mathcal{A}$ is strict (i.e., the associativity isomorphisms are identity maps) for simplicity. (In fact, this comes at no loss of generality due to a version of Mac Lane's coherence theorem in this setting; see the discussion following Definition 1.4 in [Brundan and Ellis 2017].) Let X be a homogeneous object of degree $p \in \mathbb{Z}/q$ and fix a positive integer n. For $1 \le i \le n-1$, define $\sigma_i : X^{\otimes n} \to X^{\otimes n}$ to be the map

$$X^{\otimes n} = X^{\otimes(i-1)} \otimes X^{\otimes 2} \otimes X^{\otimes(n-i-1)} \xrightarrow{\mathrm{id}\otimes\beta_{X,X}\otimes\mathrm{id}} X^{\otimes(i-1)} \otimes X^{\otimes 2} \otimes X^{\otimes(n-i-1)} = X^{\otimes n}.$$

Note that σ_i has degree p (since $p^2 = p$ modulo 2). We then have:

Proposition 5.11. *The automorphisms σ satisfy the following relations*:

(a) $\sigma_i\sigma_j = (-1)^p\sigma_j\sigma_i$ *for* $|i-j| > 1$.

(b) $\sigma_i\sigma_{i+1}\sigma_i = (-1)^p\sigma_{i+1}\sigma_i\sigma_{i+1}$.

(c) $\sigma_i^2 = (-1)^{\binom{p}{2}}$.

Proof. Relation (a) follows immediately from how composition works in product categories and the fact that each σ_i has degree p. We now verify (b). It suffices to treat the case $n = 3$. We must show that the following diagram commutes up to $(-1)^p$:

$$\begin{array}{ccccc} X\otimes X\otimes X & \xrightarrow{\sigma_2} & X\otimes X\otimes X & \xrightarrow{\sigma_1} & X\otimes X\otimes X \\ \downarrow{\scriptstyle\sigma_1} & & & & \downarrow{\scriptstyle\sigma_2} \\ X\otimes X\otimes X & \xrightarrow{\sigma_2} & X\otimes X\otimes X & \xrightarrow{\sigma_1} & X\otimes X\otimes X \end{array}$$

By the first hexagon axiom, $\sigma_1\sigma_2 = \beta_{X\otimes X,X}$. It thus suffices to show that the following diagram commutes up to $(-1)^p$:

$$\begin{array}{ccc} X\otimes X\otimes X & \xrightarrow{\beta_{X\otimes X,X}} & X\otimes X\otimes X \\ \downarrow{\scriptstyle\beta_{X,X}\otimes\mathrm{id}_X} & & \downarrow{\scriptstyle\mathrm{id}_X\otimes\beta_{X,X}} \\ X\otimes X\otimes X & \xrightarrow{\beta_{X\otimes X,X}} & X\otimes X\otimes X \end{array}$$

This follows from the definition of β (see Section 3.6) and the fact that $|X| = |\beta_{X,X}| = p$ and $|\mathrm{id}_X| = 0$. Finally, (c) follows from the definition of supersymmetry. □

We introduce the notation

$$\widetilde{\mathfrak{S}}_n^0 = \widetilde{\mathfrak{S}}_n^{(0,0)}, \quad \widetilde{\mathfrak{S}}_n^1 = \widetilde{\mathfrak{S}}_n^{(1,0)}, \quad \widetilde{\mathfrak{S}}_n^2 = \widetilde{\mathfrak{S}}_n^{(0,1)}, \quad \widetilde{\mathfrak{S}}_n^3 = \widetilde{\mathfrak{S}}_n^{(1,1)}.$$

The index p in $\widetilde{\mathfrak{S}}_n^p$ is regarded as an element of $\mathbb{Z}/4$. The above proposition shows that $\widetilde{\mathfrak{S}}_n^p$ acts on $X^{\otimes n}$ if we let the generator $\tilde{s}_i$ act by $(-1)^{ip}$ and c act by $(-1)^p$. In fact:

Proposition 5.12. *The group $\widetilde{\mathfrak{S}}_n^p$ naturally acts on the functor $\mathcal{A}_p \to \mathcal{A}_{pn}$ given by $X \mapsto X^{\otimes n}$.*

Proof. We just need to verify naturality. Given a homogeneous morphism $f : X \to Y$ between homogeneous objects, we get a morphism $f^{\otimes n} : X^{\otimes n} \to Y^{\otimes n}$, and we check that it supercommutes with the action of σ_i. We have

$$\begin{aligned}
\sigma_i f^{\otimes n} &= (1 \otimes \beta_{Y,Y} \otimes 1)(f^{\otimes i} \otimes f^{\otimes 2} \otimes f^{\otimes(i-2)}) \\
&= (-1)^{i|f||Y|}(f^{\otimes i} \otimes \beta_{Y,Y} f^{\otimes 2} \otimes f^{\otimes(i-2)}) \\
&= (-1)^{i|f||Y|+|f|}(f^{\otimes i} \otimes f^{\otimes 2}\beta_{X,X} \otimes f^{\otimes(i-2)}) \\
&= (-1)^{|f|}(f^{\otimes i} \otimes f^{\otimes 2} \otimes f^{\otimes(i-2)})(1 \otimes \beta_{X,X} \otimes 1) \\
&= (-1)^{|f|} f^{\otimes n}\sigma_i.
\end{aligned}$$

□

Remark 5.13. There are similar results to the above in the superbraiding case and the type IIb case. □

5.6. ***Universality of $\tilde{\mathbb{S}}$.*** Fix an integer q divisible by 4 (we allow $q = 0$). In this section, we regard the category $\tilde{\mathbb{S}}$ as a type IIa supersymmetric monoidal $\mathbb{Z}/q$-supercategory via the supersymmetry β defined in Section 5.4. The following theorem shows that $\tilde{\mathbb{S}}$ is the universal supersymmetric monoidal $\mathbb{Z}/q$-supercategory equipped with an object of degree 1:

Theorem 5.14. *Let $\mathcal{A}$ be a type IIa supersymmetric monoidal $\mathbb{Z}/q$-supercategory. Let $\mathcal{B}$ be the category of supersymmetric monoidal $\mathbb{Z}/q$-superfunctors $\tilde{\mathbb{S}} \to \mathcal{A}$. Then the functor $\Phi : \mathcal{B} \to \mathcal{A}_1^\circ$ given by $\Phi(F) = F([1])$ is an equivalence of $\boldsymbol{k}$-linear categories.*

Proof. Suppose that X is a degree 1 object of $\mathcal{A}$. We have seen that $\widetilde{\mathfrak{S}}_n$ naturally acts on $X^{\otimes n}$, and so we can build a functor $\Psi_X : \tilde{\mathbb{S}} \to \mathcal{A}$ by $\Psi_X([n]) = X^{\otimes n}$, which is easily seen to be a supersymmetric monoidal $\mathbb{Z}/q$-functor. The assignment $X \mapsto \Psi_X$ is functorial by Proposition 5.12. The resulting functor $\Psi : \mathcal{A}_1^\circ \to \mathcal{B}$ is easily seen to be quasi-inverse to Φ. □

Remark 5.15. Given $p \in \mathbb{Z}/q$, one can build a supersymmetric monoidal $\mathbb{Z}/q$-supercategory $\tilde{\mathbb{S}}^p$ using the $\widetilde{\mathfrak{S}}^p$ groups, specializing to $\tilde{\mathbb{S}}$ when $p = 1$. The category $\tilde{\mathbb{S}}^p$ is the universal supersymmetric monoidal $\mathbb{Z}/q$-category with an object of degree p. □

5.7. ***Type IIa′ supersymmetries.*** Let $\mathcal{A}$ be a type IIa monoidal $\mathbb{Z}/4$-supercategory. We have seen that $\mathfrak{S}_n^p$ acts on $X^{\otimes n}$ when X is homogeneous of degree $p \in \mathbb{Z}/4$. This may seem strange since it identifies $\mathrm{H}^2(\mathfrak{S}_n, \mathbb{Z}/2)$ with $\mathbb{Z}/4$ (as sets). In fact, this is due to a choice we made, namely the choice of factor system.

To make this clear, define a type IIa′ supersymmetry to be a type II supersymmetry with respect to ACD, where A, C and D are as in Examples 3.8, 3.10 and 3.11. Suppose $\mathcal{A}$ is a type IIa′ supersymmetric monoidal $\mathbb{Z}/4$-category and X is an object of degree $p \in \mathbb{Z}/4$. Then one can show that $\widetilde{\mathfrak{S}}_n^{-p}$ acts on $X^{\otimes n}$.

We note that, if $\zeta_4 \in \boldsymbol{k}$, then one can pass between type IIa and type IIa′ superbraidings; indeed, if β is a type IIa superbraiding, then, putting $\beta'_{X,Y} = \zeta_4^{|X||Y|}\beta_{X,Y}$, one finds that β' is a type IIa′ superbraiding. (The ζ_4 factor in this expression realizes CD as a coboundary.)

6. Linear spin-species

6.1. ***S-representations.*** Let G be an s-group. An *s-representation* of G on a superspace V is a homomorphism of s-groups $\rho : G \to \mathbf{GL}^h(V)$. More explicitly, an s-representation ρ associates to each $g \in G$ an automorphism $\rho(g)$ of V such that

(a) $\rho(c) = -1$,

(b) $\rho(g)$ is homogeneous of degree $\deg(g)$, and

(c) $\rho(gh) = \rho(g)\rho(h)$.

Note that an s-representation is simply a representation of G on V in the usual sense satisfying certain conditions. A *morphism* of s-representations $f : V \to W$ of parity p is a map of super-vector spaces of parity p such that $f(gv) = (-1)^{p|g|}gf(v)$ for all $v \in V$. We thus have an **SVec**-category $\mathrm{Rep}(G)$ of representations of G, which is abelian and admits a Π-structure (as we see below).

Let G and H be s-groups and let V and W be s-representations of them. We define an s-representation of $G \ltimes\!\!\!\gg H$ on $V \otimes W$ by

$$(g, h)(v \otimes w) = (-1)^{|v||h|}gv \otimes hw.$$

Let $j : G \ltimes\!\!\!\gg H \to H \ltimes\!\!\!\gg G$ be the usual isomorphism of s-groups, and regard $W \otimes V$ as an s-representation of $G \ltimes\!\!\!\gg H$ via j; explicitly,

$$(g, h)(w \otimes v) = (-1)^{|g||h|+|w||g|}hw \otimes gv.$$

Then the usual isomorphism $\tau : V \otimes W \to W \otimes V$ (as defined in Section 2.1) is an isomorphism of $G \ltimes\!\!\!\gg H$ s-representations.

Remark 6.1. When V and W are two s-representations of G, there is no natural s-representation of G on $V \otimes W$, due to the lack of a diagonal morphism $G \to G \ltimes\!\!\!\gg G$. However, one can regard $V \otimes W$ as a representation of the ordinary group G (or even $G/\langle c\rangle$) by $g(v \otimes w) = gv \otimes gw$. Note that there is no sign here, even if g and v are both of odd degree. This action of $G/\langle c\rangle$ is by even transformations. □

6.2. **Π-*structures.*** Let V be a super-vector space, and let $\pi \in \boldsymbol{k}[1]$ be the element $1 \in \boldsymbol{k}$. There are (at least) three natural choices of Π-structure on V:

- Let $\Pi_n(V) = V[1]$, and let $\xi_n : V \to \Pi_n(V)$ be the map $v \mapsto v$.
- Let $\Pi_\ell(V) = \boldsymbol{k}[1] \otimes V$, and let $\xi_\ell : V \to \Pi_\ell(V)$ be the map $v \mapsto \pi \otimes v$.
- Let $\Pi_r(V) = V \otimes \boldsymbol{k}[1]$, and let $\xi_r : V \to \Pi_r(V)$ be the map $v \mapsto (-1)^{|v|}v \otimes \pi$.

We refer to Π_ℓ and Π_r as the "left" and "right" Π-structures, and Π_n as the "neutral" Π-structure (since it does not favor left or right). The neutral Π-structure may seem to be the most natural choice, but there is a practical issue with it: if v is an element of V then it is also an element of $V[1]$, and thus the symbol v becomes ambiguous. The reason for the sign in the right Π-structure is so that the following diagram commutes:

$$\begin{array}{ccc} V & = & V \otimes \boldsymbol{k} \\ {\scriptstyle \xi_r}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\xi_r} \\ \Pi_r(V) & = & V \otimes \Pi_r(\boldsymbol{k}) \end{array}$$

Here $\xi_r : \boldsymbol{k} \to \Pi_r(\boldsymbol{k}) = \boldsymbol{k}[1]$ is the map $1 \mapsto \pi$. Since this is an odd map, $(\mathrm{id}\otimes\xi_r)(v\otimes 1) = (-1)^{|v|} v \otimes \pi$. The sign in ξ_r also ensures that the canonical even-degree isomorphism $\Pi_\ell(V) \to \Pi_r(V)$ is just the symmetry τ of the tensor product.

Suppose that V is an s-representation of an s-group G. We define s-representations of G on $\Pi_\ell(V) = \boldsymbol{k}[1]\otimes V$ and $\Pi_r(V) = V\otimes\boldsymbol{k}[1]$ by $g(\pi\otimes v) = (-1)^{|g|}\pi\otimes gv$ and $g(v\otimes\pi) = gv\otimes\pi$. One readily verifies that the maps $\xi_r : V \to \Pi_r(V)$ and $\xi_\ell : V \to \Pi_\ell(V)$ are odd-degree isomorphisms of s-representations. Furthermore, the canonical even-degree isomorphism $\Pi_\ell(V) \to \Pi_r(V)$ is one of s-representations. (We do not bother defining Π_n for s-representations.) We thus have two Π-structures for s-representations: Π_r and Π_ℓ.

6.3. ***Induction.*** Let $H \subseteq G$ be s-groups and let V be an s-representation of H. Consider the induced representation $W = \mathrm{Ind}_H^G(V) = \boldsymbol{k}[G] \otimes_{\boldsymbol{k}[H]} V$. We give W the structure of a super-vector space by declaring $g \otimes v$ to be homogeneous of degree $|g| + |v|$ whenever $v \in V$ is homogeneous. In this way, W is an s-representation of G. As usual, induction is the left adjoint of restriction.

Consider the following situation:

- H_1 and H_2 are s-subgroups of G.
- V_1 (resp. V_2) is an s-representation of H_1 (resp. H_2).
- τ is an element of G such that $\tau H_1 \tau^{-1} = H_2$.
- $i : V_1 \to V_2^\tau$ is an even-degree isomorphism of s-representations of H_1, where V_2^τ is the s-representation of H_1 on V_2 defined by $h \cdot v = (\tau h \tau^{-1})v$. In other words, $i(hv) = (\tau h \tau^{-1}) i(v)$ for $h \in H_1$.

Define

$$\varphi : \mathrm{Ind}_{H_1}^G(V_1) \to \Pi_r^{|\tau|} \mathrm{Ind}_{H_2}^G(V_2), \quad \varphi(g\otimes v) = g\tau^{-1} \otimes i(v) \otimes \pi^{|\tau|}.$$

We then have:

Proposition 6.2. *The map φ is an even-degree isomorphism of s-representations of G.*

Proof. We first check that φ is well defined, i.e., that $\varphi(gh\otimes v) = \varphi(g\otimes hv)$ for $h \in H_1$. Omitting the factors of π for economy, we have

$$\varphi(gh\otimes v) = gh\tau^{-1}\otimes i(v) = g\tau^{-1}\otimes \tau h\tau^{-1} i(v) = g\tau^{-1}\otimes i(hv) = \varphi(g\otimes hv).$$

Thus, φ is well defined. For $g' \in G$,

$$g'\varphi(g \otimes v) = g'(g\tau^{-1} \otimes i(v) \otimes \pi^{|\tau|}) = \varphi(g'g \otimes v),$$

and so φ is a map of s-representations. Finally, interchanging the roles of τ and τ^{-1} gives a map in the opposite direction that is the inverse of φ. Thus, φ is an isomorphism. □

6.4. ***The category of linear spin-species.*** A *representation of* $\widetilde{\mathfrak{S}}_*$ is a sequence $(V_n)_{n\geq 0}$, where V_n is an s-representation of the s-group $\widetilde{\mathfrak{S}}_n$ over $\boldsymbol{k}$. A *morphism* of $\widetilde{\mathfrak{S}}_*$-representations $V \to W$ is a sequence $f = (f_n)_{n\geq 0}$ where $f_n : V_n \to W_n$ is a (not necessarily homogeneous) morphism of $\widetilde{\mathfrak{S}}_n$-representations. We say that an $\widetilde{\mathfrak{S}}_*$-representation V is *even* (resp. *odd*) if $V_n = 0$ for all odd (resp. even) values of n. This gives $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ the structure of a supercategory. For an integer q, we can regard $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ to be an (inhomogeneous) $\mathbb{Z}/q$-supercategory by declaring an object $(V_n)_{n\geq 0}$ to have degree $d \in \mathbb{Z}/q$ if $V_n = 0$ for all $n \not\equiv d \pmod q$. Every object decomposes as a direct sum of homogeneous objects, and this decomposition is finite if $q \neq 0$.

6.5. ***Monoidal structure.*** Let n and m be nonnegative integers. Suppose that V and W are two representations of $\widetilde{\mathfrak{S}}_*$. We define $V \otimes W$ to be the representation of $\widetilde{\mathfrak{S}}_*$ given by

$$(V \otimes W)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{\widetilde{\mathfrak{S}}_i \rtimes \widetilde{\mathfrak{S}}_j}^{\widetilde{\mathfrak{S}}_n} (V_i \otimes W_j),$$

where here $\widetilde{\mathfrak{S}}_i \rtimes \widetilde{\mathfrak{S}}_j$ is regarded as a subgroup of $\widetilde{\mathfrak{S}}_n$ via the map $j_{n,m}$ defined in (5.4). Given a third $\widetilde{\mathfrak{S}}_*$-representation U, we have a natural identification

$$((U \otimes V) \otimes W)_n = \bigoplus_{i+j+k=n} \mathrm{Ind}_{\widetilde{\mathfrak{S}}_i \rtimes \widetilde{\mathfrak{S}}_j \rtimes \widetilde{\mathfrak{S}}_k}^{\widetilde{\mathfrak{S}}_n} (U_i \otimes V_j \otimes W_k),$$

and similarly for $U \otimes (V \otimes W)$. This yields a natural associativity isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ that satisfies the pentagon axiom. The $\widetilde{\mathfrak{S}}_*$-representation that is $\boldsymbol{k}$ in degree 0 and 0 in other degrees is naturally a unit object for $\otimes$. Thus, $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ has the structure of a monoidal $\mathbb{Z}/q$-supercategory.

6.6. ***Supersymmetry.*** In this section, we assume q is divisible by 4 and construct a supersymmetry on $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$. We begin with the following observation:

Proposition 6.3. *Let V and W be s-representations of $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathfrak{S}}_m$. Regard $\widetilde{\mathfrak{S}}_n \rtimes \widetilde{\mathfrak{S}}_m$ as a subgroup of $\widetilde{\mathfrak{S}}_{n+m}$ via $j_{n,m}$, and similarly with n and m reversed. Then*

$$i_{V,W} : V \otimes W \to (W \otimes V)^{\tilde{\tau}_{n,m}}, \quad v \otimes w \mapsto (-1)^{|v||w|+nm|v|+nm|w|} w \otimes v,$$

is an isomorphism of representations of $\widetilde{\mathfrak{S}}_n \rtimes \widetilde{\mathfrak{S}}_m$, where the superscript denotes the conjugate representation. Precisely, this means

$$i_{V,W}(j_{n,m}(g,h)(v \otimes w)) = \tilde{\tau}_{n,m} j_{n,m}(g,h) \tilde{\tau}_{n,m}^{-1} i_{V,W}(v \otimes w)$$

for all $v \in V$ and $w \in W$ and all $g \in \widetilde{\mathfrak{S}}_n$ and $h \in \widetilde{\mathfrak{S}}_m$.

Proof. We have

$$\begin{aligned} i_{V,W}(j_{n,m}(g,h)(v\otimes w)) &= (-1)^{|h||v|} i_{V,W}(gv\otimes hw) \\ &= (-1)^{|h||v|+(|g|+|v|)(|h|+|w|)+nm(|g|+|v|)+nm(|h|+|w|)} hw\otimes gv \\ &= (-1)^{|g||h|+nm|g|+nm|h|+|v||w|+nm|v|+nm|w|+|g||w|} hw\otimes gv \\ &= \tilde{\tau}_{n,m} j_{n,m}(g,h)\tilde{\tau}_{n,m}^{-1} i_{V,W}(v\otimes w). \end{aligned}$$

In the last equality, we used Proposition 5.5 and that c acts by -1. □

As discussed in Section 6.3, we have an isomorphism

$$\operatorname{Ind}_{\widetilde{\mathfrak{S}}_n \rtimes \widetilde{\mathfrak{S}}_m}^{\widetilde{\mathfrak{S}}_{n+m}} (V\otimes W) \to \Pi_r^{nm} \operatorname{Ind}_{\widetilde{\mathfrak{S}}_m \rtimes \widetilde{\mathfrak{S}}_n}^{\widetilde{\mathfrak{S}}_{n+m}} (W\otimes V) \tag{6.4}$$

coming from conjugation by $\tilde{\tau}_{n,m}$, and the isomorphism $i_{V,W}$ (maintaining the notation from the proposition).

Definition 6.5. Given homogeneous objects V and W of $\operatorname{Rep}(\widetilde{\mathfrak{S}}_*)$, we define

$$\beta_{V,W} : V\otimes W \to \Pi_r^{|V||W|}(W\otimes V)$$

to be the isomorphism induced by the isomorphisms (6.4) using the elements $\tilde{\tau}_{n,m}$ and the isomorphisms i. Explicitly, for $v\in V_n$ and $w\in W_m$ and $g\in \widetilde{\mathfrak{S}}_{n+m}$, we have

$$\beta_{V,W}(g\otimes v\otimes w) = (-1)^{nm(|v|+|w|)+|v||w|} g\tilde{\tau}_{n,m}^{-1}\otimes w\otimes v\otimes \pi^{nm},$$

where here $g\otimes v\otimes w$ denotes an element of $\operatorname{Ind}_{\widetilde{\mathfrak{S}}_n \rtimes \widetilde{\mathfrak{S}}_m}^{\widetilde{\mathfrak{S}}_{n+m}} (V\otimes W)$. □

Proposition 6.6. *β defines a type Ia supersymmetry on $\otimes$.*

Proof. First, we check that β is a natural transformation. Let $f: V_n\to V_n'$ and $g: W_m\to W_m'$ be nonzero homogeneous morphisms. Pick $v\in V_n$, $w\in W_m$ and $\sigma\in\widetilde{\mathfrak{S}}_{n+m}$. Then

$$\begin{aligned} &\Pi^{|V||W|}(g\otimes f)\beta_{V_n,W_m}(\sigma\otimes v\otimes w) \\ &\qquad = (-1)^{nm(|v|+|w|)+|v||w|+|f|(|\sigma|+nm+|w|)+|g|(|\sigma|+nm)} \sigma\tilde{\tau}_{n,m}^{-1}\otimes g(w)\otimes f(v)\otimes \pi^{nm}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\beta_{V_n',W_m'}(f\otimes g)(\sigma\otimes v\otimes w) \\ &\qquad = \beta_{V_n',W_m'}((-1)^{|g|(|\sigma|+|v|)+|f||\sigma|}\sigma\otimes f(v)\otimes g(w)) \\ &\qquad = (-1)^{|g|(|\sigma|+|v|)+|f||\sigma|+nm(|f|+|v|+|g|+|w|)+(|f|+|v|)(|g|+|w|)} \sigma\tilde{\tau}_{n,m}^{-1}\otimes g(w)\otimes f(v)\otimes \pi^{nm}. \end{aligned}$$

This agrees with the first expression up to $(-1)^{|f||g|}$, which is what we wanted to show.

Next, it follows from Proposition 5.3 that $\beta_{V,W}\beta_{W,V} = (-1)^{\binom{n}{2}\binom{m}{2}}$. Hence, we only need to check that one of the hexagon axioms hold, and we check (H1′). So pick nonnegative integers m, n, p and let U, V, W be representations of $\widetilde{\mathfrak{S}}_m$, $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathfrak{S}}_p$, respectively. We need to check that $(1_V\otimes\beta_{U,W})(\beta_{U,V}\otimes 1_W) = \beta_{U,V\otimes W}$. (Here we identify U with the $\mathfrak{S}_*$-representation that is U in degree m and 0 in other degrees, and similarly for V and W. We note that it suffices to check the axiom on such objects.)

The upper path in the hexagon is the composition

$$\begin{aligned}\mathrm{Ind}_{m+n,p}^{m+n+p}(\mathrm{Ind}_{m,n}^{m+n}(U\otimes V)\otimes W) &\xrightarrow{\alpha^{-1}_{U,V,W}} \mathrm{Ind}_{m,n+p}^{m+n+p}(U\otimes \mathrm{Ind}_{n,p}^{n+p}(V\otimes W))\\ &\xrightarrow{\beta_{U,V\otimes W}} \mathrm{Ind}_{n+p,m}^{m+n+p}(\mathrm{Ind}_{n,p}^{n+p}(V\otimes W)\otimes U)\otimes \boldsymbol{k}[m(n+p)]\\ &\xrightarrow{\alpha^{-1}_{V,W,U}} \mathrm{Ind}_{n,m+p}^{m+n+p}(V\otimes \mathrm{Ind}_{p,m}^{p+m}(W\otimes U))\otimes \boldsymbol{k}[m(n+p)].\end{aligned}$$

Here we have written n in place of $\widetilde{\mathfrak{S}}_n$ in the inductions for readability. We compute the composition explicitly. Let $g\in\widetilde{\mathfrak{S}}_{m+n+p}$ and let $u\in U$, $v\in V$ and $w\in W$ be homogeneous elements. We have

$$\begin{aligned}g\otimes(1\otimes u\otimes v)\otimes w &\mapsto g\otimes u\otimes(1\otimes v\otimes w)\\ &\mapsto (-1)^{m(n+p)(|u|+|v|+|w|)+|u|(|v|+|w|)} g\tilde{\tau}^{-1}_{m,n+p}\otimes(1\otimes v\otimes w)\otimes u\otimes\pi^{m(n+p)}\\ &\mapsto (-1)^{m(n+p)(|u|+|v|+|w|)+|u|(|v|+|w|)} g\tilde{\tau}^{-1}_{m,n+p}\otimes v\otimes(1\otimes w\otimes u)\otimes\pi^{m(n+p)}.\end{aligned}$$

The lower path in the hexagon is the composition

$$\begin{aligned}\mathrm{Ind}_{m+n,p}^{m+n+p}(\mathrm{Ind}_{m,n}^{m+n}(U\otimes V)\otimes W) &\xrightarrow{\beta_{U,V}\otimes\mathrm{id}_W} \mathrm{Ind}_{m+n,p}^{m+n+p}(\mathrm{Ind}_{n,m}^{m+n}(V\otimes W)\otimes \boldsymbol{k}[mn]\otimes U)\\ \xrightarrow{\sim} \mathrm{Ind}_{m+n,p}^{m+n+p}(\mathrm{Ind}_{n,m}^{m+n}(V\otimes W)\otimes U)\otimes\boldsymbol{k}[mn] &\xrightarrow{\alpha^{-1}_{V,W,U}} \mathrm{Ind}_{n,m+p}^{m+n+p}(V\otimes\mathrm{Ind}_{n,p}^{n+p}(U\otimes W))\otimes\boldsymbol{k}[mn]\\ &\xrightarrow{\mathrm{id}_V\otimes\beta_{U,W}} \mathrm{Ind}_{n,m+p}^{m+n+p}(V\otimes\mathrm{Ind}_{p,m}^{m+p}(W\otimes U)\otimes\boldsymbol{k}[mp])\otimes\boldsymbol{k}[mn]\\ &\xrightarrow{\sim} \mathrm{Ind}_{n,m+p}^{m+n+p}(V\otimes\mathrm{Ind}_{p,m}^{m+p}(W\otimes U))\otimes\boldsymbol{k}[m(n+p)].\end{aligned}$$

We have

$$\begin{aligned}&g\otimes(1\otimes u\otimes v)\otimes w\\ &\quad\mapsto (-1)^{mn(|u|+|v|)+|u||v|} g\otimes(\tau^{-1}_{m,n}\otimes v\otimes u\otimes\pi^{mn})\otimes w\\ &\quad\mapsto (-1)^{mn(|u|+|v|)+|u||v|+mn|w|} g\tilde{\tau}^{-1}_{m,n}\otimes(1\otimes v\otimes u)\otimes w\otimes\pi^{mn}\\ &\quad\mapsto (-1)^{mn(|u|+|v|)+|u||v|+mn|w|} g\tilde{\tau}^{-1}_{m,n}\otimes v\otimes(1\otimes u\otimes w)\otimes\pi^{mn}\\ &\quad\mapsto (-1)^{mn(|u|+|v|)+|u||v|+mn|w|+mp(|u|+|w|)+|u||w|} g\tilde{\tau}^{-1}_{m,n}\otimes v\otimes(\tilde{\tau}^{-1}_{m,p}\otimes w\otimes u\otimes\pi^{mp})\otimes\pi^{mn}\\ &\quad\mapsto (-1)^{mn(|u|+|v|)+|u||v|+mn|w|+mp(|u|+|w|)+|u||w|+mp|v|} g\tilde{\tau}^{-1}_{m,n} j(1,\tilde{\tau}^{-1}_{m,p})\otimes v\otimes(1\otimes w\otimes u)\otimes\pi^{m(n+p)},\end{aligned}$$

where here $j=j_{n,m+p}$. In the final step, we used the identity

$$v\otimes(\tilde{\tau}^{-1}_{m,p}\otimes w\otimes u)=(-1)^{mp|v|}(1,\tilde{\tau}^{-1}_{m,p})(v\otimes(1\otimes w\otimes u))$$

in $V\otimes\mathrm{Ind}_{p,m}^{m+p}(W\otimes U)$, and then moved $(1,\tilde{\tau}^{-1}_{m,p})$ into the first tensor factor via j. Since $\tilde{\tau}^{-1}_{m,n} j(1,\tilde{\tau}^{-1}_{m,p})=\tilde{\tau}^{-1}_{m,n+p}$ (Proposition 5.6) and the signs in the two compositions agree, the result follows. □

6.7. ***Universal property.*** Let $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)^{\mathrm{pf}}$ be the full subcategory of $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ on finite-length projective objects. If $\mathrm{char}(\boldsymbol{k})=0$, then the projective condition is automatic. The category $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)^{\mathrm{pf}}$ is "nice"

in the sense defined in Section 8.1: it is additive, Karoubian and admits a Π-structure. Let $\boldsymbol{V}$ be the "standard" representation of $\widetilde{\mathfrak{S}}_*$: it is $\boldsymbol{k}$ in degree 1 and 0 in other degrees.

Theorem 6.7. *The pair* $(\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)^{\mathrm{pf}}, \boldsymbol{V})$ *is the universal nice type Ia supersymmetric monoidal* $\mathbb{Z}$*-supercategory with an object of degree* 1*, in the following sense. Suppose* $\mathcal{C}$ *is a nice supersymmetric monoidal* $\mathbb{Z}$*-supercategory and* M *is an object of* $\mathcal{C}$ *of degree* 1*. Then there exists a supersymmetric monoidal superfunctor* $\Phi : \mathrm{Rep}(\widetilde{\mathfrak{S}}_*)^{\mathrm{pf}} \to \mathcal{C}$ *and an isomorphism* $\iota : \Phi(\boldsymbol{V}) \to M$*. Furthermore,* (Φ, ι) *is unique up to isomorphism.*

Proof. One can prove this directly; however, we simply note that $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)^{\mathrm{pf}}$ is the "nice envelope" of $\tilde{\mathfrak{S}}$, and so inherits the universal property of $\tilde{\mathfrak{S}}$ (Theorem 5.14). □

6.8. ***Exterior powers of categories.*** One can generalize the above discussion to use coefficients in a supercategory $\mathcal{A}$. This leads to the notion of the exterior power of $\mathcal{A}$, as defined in [Ganter and Kapranov 2014]. The following discussion is intended as an extended comment, and so we omit many details.

Let $\mathcal{A}$ be a nice supercategory (see Section 8.1), and let $\mathcal{A}^{\hat{\boxtimes} n}$ be its n-fold self-product with respect to the tensor product $\hat{\boxtimes}$ defined in Section 8.6 below. The symmetric group $\mathfrak{S}_n$ acts in a natural way on $\mathcal{A}^{\hat{\boxtimes} n}$, and so $\widetilde{\mathfrak{S}}_n$ does too through the quotient map $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$. A Sym^n*-object* of $\mathcal{A}$ is an object M of $\mathcal{A}^{\hat{\boxtimes} n}$ equipped with even isomorphisms $\alpha_\sigma : \sigma^*(M) \to M$ for $\sigma \in \mathfrak{S}_n$ satisfying the usual cocycle conditions. A $\bigwedge^n$*-object* of $\mathcal{A}$ is an object M of $\mathcal{A}^{\hat{\boxtimes} n}$ equipped with isomorphisms $\beta_\sigma : \sigma^*(M) \to M$ for $\sigma \in \widetilde{\mathfrak{S}}_n$ satisfying the cocycle conditions and such that $\beta_c = -1$ and β_σ is homogeneous of the same degree as σ. We let $\mathrm{Sym}^n(\mathcal{A})$ and $\bigwedge^n(\mathcal{A})$ denote the categories of Sym^n- and $\bigwedge^n$-objects of $\mathcal{A}$. We define $\mathrm{Sym}(\mathcal{A})$ and $\bigwedge(\mathcal{A})$ to be the $\mathbb{Z}$-supercategories with these homogeneous pieces.

One can endow $\mathrm{Sym}(\mathcal{A})$ with a symmetric monoidal structure and $\bigwedge(\mathcal{A})$ with a supersymmetric monoidal structure, similar to the definitions used above for $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$. We note that $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ itself can be viewed as $\bigwedge(\mathbf{SVec})$. These categories satisfy natural universal properties; e.g., given a nice supersymmetric monoidal $\mathbb{Z}$-supercategory $\mathcal{B}$, giving a supersymmetric monoidal $\mathbb{Z}$-functor $\bigwedge(\mathcal{A}) \to \mathcal{B}$ is the same as giving a superfunctor $\mathcal{A} \to \mathcal{B}$.

7. Isomeric vector spaces

We assume $\boldsymbol{k}$ *contains* ζ_{16} *throughout this section, and put* $\sqrt{2} = \zeta_8(1 - \zeta_4)$.

7.1. ***Isomeric vector spaces.*** Let V be a super-vector space. An *isomeric structure* on V is an odd isomorphism $\nu : V \to V$ such that $\nu^2 = 1$. An *isomeric vector space* is a vector space equipped with an isomeric structure. Suppose that (U, μ) and (V, ν) are isomeric vector spaces. A homogeneous morphism of isomeric spaces $(U, \mu) \to (V, \nu)$ is a homogeneous linear map $f : U \to V$ such that $f\mu = (-1)^{|f|}\nu f$. In this way, we have a supercategory **QVec** of isomeric vector spaces.

7.2. ***The half tensor product.*** Let (U, μ) and (V, ν) be two isomeric vector spaces. Then $\mu \otimes \nu$ defines an even automorphism of $U \otimes V$ squaring to -1. We define the *half tensor product* of U and V, denoted

by $2^{-1}(U \otimes V)$, to be the ζ_4-eigenspace of $\mu \otimes \nu$. We note that the $(-\zeta_4)$-eigenspace is isomorphic to the ζ_4-eigenspace, via $\mu \otimes 1$ or $1 \otimes \nu$. The half tensor product is an ordinary super-vector space; it does not come with a preferred isomeric structure.

7.3. ***The isomeric monoidal category.*** Motivated by the above discussion, we define the *isomeric category* **Ismr** to be the $\mathbb{Z}/2$-supercategory with $\mathbf{Ismr}_0 = \mathbf{SVec}$ and $\mathbf{Ismr}_1 = \mathbf{QVec}$. We now define a monoidal operation $\odot$ on **Ismr**. We define $\odot$ on objects as follows:

- Given $U, V \in \mathbf{Ismr}_0$, we let $U \odot V = U \otimes V$.
- Let $U \in \mathbf{Ismr}_0$ and $V \in \mathbf{Ismr}_1$, and let ν be the isomeric structure on V. Then $U \otimes V$ carries the isomeric structure $1 \otimes \nu$. We define $U \odot V$ to be the isomeric vector space $(U \otimes V, 1 \otimes \nu)$. We similarly define $V \odot U$.
- Given $U, V \in \mathbf{Ismr}_1$, we define $U \odot V$ to be the half tensor product $2^{-1}(U \otimes V)$.

The definition of $\odot$ on morphisms is straightforward. In this way, we have defined a superfunctor

$$\odot : \mathbf{Ismr} \boxtimes \mathbf{Ismr} \to \mathbf{Ismr}.$$

The unit object for this monoidal structure is the unit object of $\mathbf{Ismr}_0$, i.e., the super-vector space $\boldsymbol{k}$. The isomorphisms λ and ρ are similar to **SVec**, and we omit the details.

We now discuss the associativity constraint α for $\odot$. For this, we will require the following result:

Proposition 7.1. *Let (U, μ), (V, ν) and (W, σ) be isomeric vector spaces. Then $\mu\sigma - 1$ is an automorphism of $U \otimes V \otimes W$ that maps $U \otimes 2^{-1}(V \otimes W)$ isomorphically to $2^{-1}(U \otimes V) \otimes W$.*

Proof. Since $(\mu\sigma)^2 = -1$, the inverse of $\mu\sigma - 1$ is $-\frac{1}{2}(\mu\sigma + 1)$. Now suppose that $x \in U \otimes 2^{-1}(V \otimes W)$. Thus, $\nu\sigma x = \zeta_4 x$; applying σ to each side, we find $\nu x = -\zeta_4 \sigma x$. We thus have

$$\mu\nu(\mu\sigma - 1)x = -\nu\sigma x - \mu\nu x = -\zeta_4 x + \zeta_4 \mu\sigma x = \zeta_4(\mu\sigma - 1)x,$$

which shows that $(\mu\sigma - 1)x$ belongs to $2^{-1}(U \otimes V) \otimes W$. By symmetry, $\sigma\mu - 1$ maps $2^{-1}(U \otimes V) \otimes W$ to $U \otimes 2^{-1}(V \otimes W)$. Since $\mu\sigma - 1$ and $\sigma\mu - 1$ are inverses (up to a factor of 2), these maps are isomorphisms. □

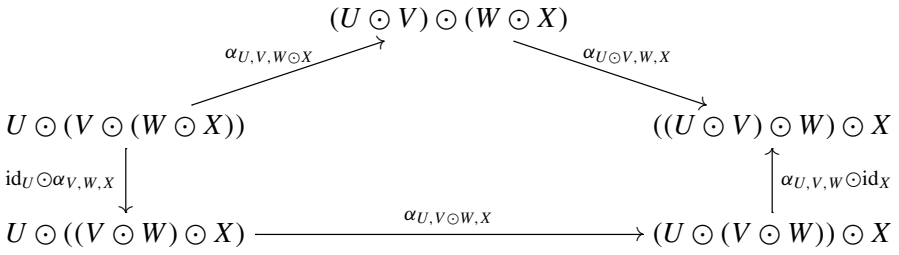
Let $U, V, W \in \mathbf{Ismr}$ be homogeneous objects. We define an isomorphism

$$\alpha_{U,V,W} : U \odot (V \odot W) \to (U \odot V) \odot W$$

as follows. We identify the domain and target above with subspaces of $U \otimes V \otimes W$. If at least one of U, V or W has degree 0, then the domain and target are identified with the same subspace, and we take α to be the identity map. Suppose now that U, V and W are each degree 1, and let μ, ν and σ be the isomeric structures on them. We then take $\alpha_{U,V,W} = \frac{1}{\sqrt{2}}(\mu\sigma - 1)$, which is an appropriate isomorphism by Proposition 7.1.

Proposition 7.2. *The pentagon axiom holds.*

Proof. Let $U, V, W, X \in \mathbf{Ismr}$ be homogeneous objects. We let μ, ν, σ, τ denote isomeric structures on these objects, respectively, whenever they have degree 1. We must show that the diagram



commutes. We identify each space above with a subspace of $U \otimes V \otimes W \otimes X$. If at most two of U, V, W and X are degree 1, then all maps in the diagram are the identity map, and so it commutes. Now suppose exactly three of the objects have degree 1. Then, in each of the two paths in the pentagon, exactly one α is not the identity, and these two α's are equal; for example, if $|U| = 0$, then $\alpha_{U\odot V,W,X} = \mathrm{id}_U \odot \alpha_{V,W,X} = \frac{1}{\sqrt{2}}(\nu\tau - 1)$. Finally, suppose that all four objects have degree 1. Then all edges in the pentagon except the bottom left and bottom right are the identity. We must show $(\mu\sigma - 1)(\nu\tau - 1) = 2$ on $U \odot (V \odot (W \odot X))$. Suppose x belongs to this space. Then $\mu\nu x = \zeta_4 x$, and so $\nu x = \zeta_4 \mu x$; similarly, $\sigma\tau x = \zeta_4 x$, and so $\tau x = \zeta_4 \sigma x$. We thus find

$$\nu\tau x = \zeta_4 \nu\sigma x = -\zeta_4 \sigma\nu x = \sigma\mu x = -\mu\sigma x.$$

Thus,

$$(\mu\sigma - 1)(\nu\tau - 1)x = (\mu\sigma - 1)(-\mu\sigma - 1)x = 2x,$$

since $(\mu\sigma)^2 = -1$. □

7.4. ***Supersymmetry.*** We now aim to define a supersymmetry on $\odot$. We let τ be the standard symmetry on **SVec**. We begin with the following observation:

Proposition 7.3. *Let (U, μ) and (V, ν) be isomeric vector spaces. Then the diagram*

$$\begin{array}{ccc} U \otimes V & \xrightarrow{\mu\otimes\nu} & U \otimes V \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ V \otimes U & \xrightarrow{\nu\otimes\mu} & V \otimes U \end{array}$$

commutes up to -1. In particular, τ maps the ζ_4-eigenspace of $\mu \otimes \nu$ on $U \otimes V$ to the $(-\zeta_4)$-eigenspace of $\nu \otimes \mu$ on $V \otimes U$.

Proof. The first statement follows from the fact that τ is a natural transformation and μ and ν are odd (see the first paragraph of Section 3.4). The second statement follows from the first. □

Let $U, V \in \mathbf{Ismr}$. We define an isomorphism

$$\beta_{U,V} : U \odot V \to V \odot U$$

as follows. If at least one of U or V has degree 0, then $\beta_{U,V} = \tau_{U,V}$. Now suppose that both U and V have degree 1, and let ν be the isomeric structure on V. Then $\beta_{U,V} = \zeta_{16}^{-1}(\nu \otimes 1)\tau_{U,V}$. This maps $U \odot V$ into $V \odot U$ by Proposition 7.3 and properties of the half tensor product (see Section 7.2).

Proposition 7.4. *β defines a type IIb supersymmetry on $\odot$.*

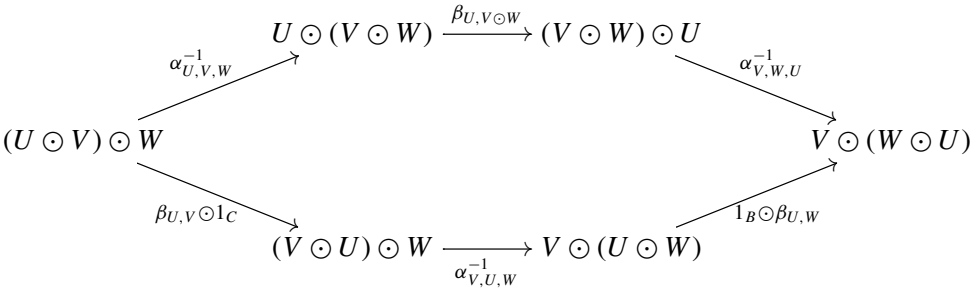
Proof. One easily sees that β is a natural transformation, and it is evident that $\beta_{U,V}$ has degree $|U||V|$ when U and V are homogeneous objects. We will use the following convention in the remainder of the proof: for an endomorphism μ of a super-vector space U, we write μ still for the induced endomorphism of $U \otimes V$ or $V \otimes U$. Note that, under this convention, μ commutes with $\tau_{U,V}$.

Now let $U, V \in \mathbf{Ismr}$ be homogeneous objects. If either U or V has degree 0, then $\beta_{V,U}\beta_{U,V}$ is the identity. Now suppose that both U and V have degree 1, and let μ and ν be the isomeric structures. Then

$$\beta_{V,U}\beta_{U,V} = \zeta_8^{-1}\mu\tau_{V,U}\nu\tau_{U,V} = \zeta_8^{-1}\mu\nu\tau_{V,U}\tau_{U,V} = \zeta_8^{-1}\mu\nu = \zeta_8.$$

In the final step, we used the fact that $\mu\nu = \zeta_4$ on $U \odot V$. We thus see that $\beta_{V,U}\beta_{U,V} = \mathtt{B}_\sharp(|U|, |V|)\,\mathrm{id}_{U\otimes V}$ in all cases, which verifies the requisite symmetry condition for β.

We now verify the hexagon axiom (H1). Let $U, V, W \in \mathbf{Ismr}$ be homogeneous objects, and let μ, ν, σ be the isomeric structures on the objects of degree 1. We must show that the diagram



commutes. (Recall that $\mathtt{B}_1 = 1$ identically, which is why the diagram must commute exactly.) If at most one of the objects has degree 1, then this is the usual hexagon in **SVec**, which commutes. Now suppose exactly two objects have degree 1. Then all the associators are the identity (or, more precisely, induced by the standard associator on the ordinary tensor products). If $|U| = 0$, then each of the β's is just a τ, and the diagram commutes. Otherwise, one of $\beta_{U,V}$ or $\beta_{U,W}$ is the usual τ, and the other involves the same isomeric structure as $\beta_{U,V\odot W}$; for instance, if $|V| = 0$, then

$$\beta_{U,V\odot W} = \zeta_{16}^{-1}\sigma\tau_{U,V\odot W}, \quad \beta_{U,V} = \tau_{U,V}, \quad \beta_{U,W} = \zeta_{16}^{-1}\sigma\tau_{U,W}.$$

Thus, the diagram commutes (the isomeric structure can be moved past all the τ's and then one has the ordinary hexagon axiom in **SVec**).

Finally, suppose that all objects have degree 1. The top path is (induced by) the map

$$\tfrac{1}{2}(\mu\nu - 1)\tau_{U,V\otimes W}(\sigma\mu - 1) = \tfrac{1}{2}\tau_{U,V\otimes W}(\mu\nu - 1)(\sigma\mu - 1).$$

The bottom path is (induced by) the map

$$\tfrac{1}{\sqrt{2}}\zeta_8^{-1}\sigma\tau_{U,W}(\sigma\nu-1)\nu\tau_{U,V} = \tfrac{1}{\sqrt{2}}\zeta_8^{-1}\tau_{U,W}\tau_{U,V}\sigma(\sigma\nu-1)\nu.$$

Since $\tau_{U,V\otimes W} = \tau_{U,W}\tau_{U,V}$, it suffices to show that

$$\tfrac{1}{2}(\mu\nu-1)(\sigma\mu-1) = \tfrac{1}{\sqrt{2}}\zeta_8^{-1}\sigma(\sigma\nu-1)\nu$$

as functions on $(U\odot V)\odot W$. This equation is equivalent to

$$\nu\sigma-\mu\nu-\sigma\mu+1 = \sqrt{2}\zeta_8^{-1}(1-\sigma\nu).$$

Since $\mu\nu=\zeta_4$ on the source, we have $\mu=-\zeta_4\nu$, and so the left side of the above equation is

$$-\sigma\nu-\zeta_4+\zeta_4\sigma\nu+1 = (1-\zeta_4)(1-\sigma\nu),$$

and so the result follows, as $1-\zeta_4=\sqrt{2}\zeta_8^{-1}$.

Since β satisfies the symmetry condition, the second hexagon axiom is a consequence of the first one. □

7.5. ***Further comments.*** The pentagon and hexagon axioms in **Ismr** are rather complicated. In fact, there is an alternative way of defining the category **Ismr** that makes these axioms more transparent. Recall that the Clifford algebra Cl_n is generated by n supercommuting odd elements that each square to 1. One formulation of Bott periodicity (discussed in the Appendix) states that Cl_n and Cl_{n+2} are Morita-equivalent. We can identify **QVec** with the category of Cl_1-modules. The half tensor product is then the composition

$$\mathrm{Mod}_{\mathrm{Cl}_1}\times\mathrm{Mod}_{\mathrm{Cl}_1}\to\mathrm{Mod}_{\mathrm{Cl}_2}\to\mathbf{SVec},$$

where the first functor takes (V, W) to $V\otimes W$, with obvious Cl_2-structure, and the second functor is the Morita equivalence provided by Bott periodicity. It is the second step that introduces the complications.

To obtain a "cleaner" version of the isomeric category, we can simply omit the Morita equivalence above. More precisely, we could redefine $\mathbf{Ismr}_i$ to be the 2-colimit of the categories $\mathrm{Mod}_{\mathrm{Cl}_n}$ taken over n of the same parity as i. The tensor product would then be more natural and the axioms would be more transparent.

This new approach also points the way to a vast generalization: one can start with an arbitrary $\mathbb{Z}/2$-supercategory $\mathcal{A}$ and construct an "isomeric version" by considering $\mathrm{Cl}_{\mathrm{even}}$-modules in $\mathcal{A}_0$ and $\mathrm{Cl}_{\mathrm{odd}}$-modules in $\mathcal{A}_1$. In fact, one can also work over more general base fields where the periodicity in Clifford algebras may be 4 or 8 instead of 2. We call this generalization "Clifford eversion", and carry out the details in the next section.

8. Clifford eversion

8.1. ***Nice supercategories.*** Let $\mathcal{C}$ be a supercategory. We say that $\mathcal{C}$ is *additive* if all finite direct sums exist. (The direct sum of two objects is an object equipped with *even* inclusion and projection maps satisfying the usual conditions.) We say that $\mathcal{C}$ is *Karoubian* if every *even* idempotent $e : A\to A$ in $\mathcal{C}$

has an image. We say that $\mathcal{C}$ is *nice* if it is additive, Karoubian and admits a Π-structure. We say that a Λ-supercategory is nice if each graded piece is. We will mostly be concerned with nice supercategories in this section. We note that, if M is an object in a nice supercategory and V is a finite-dimensional super-vector space, then we can make sense of $V \otimes M$, e.g., $\boldsymbol{k}^{r|s} \otimes M = M^{\oplus r} \oplus \Pi(M)^{\oplus s}$. (In fact, this construction does not require the Karoubian condition.)

8.2. ***The Clifford algebra.*** The *Clifford algebra* Cl_n is the $\boldsymbol{k}$-superalgebra generated by odd-degree variables $\alpha_1, \dots, \alpha_n$ subject to the relations $\alpha_i^2 = 2$ and $\alpha_i \alpha_j = -\alpha_j \alpha_i$ for $i \neq j$. If 2 is a square in $\boldsymbol{k}$, then one can renormalize α_i so that it squares to 1 (as we did in Section 7.5), which is a more common convention. We use our convention so that the following proposition works without need of $\sqrt{2}$:

Proposition 8.1. *There is a group homomorphism* $\widetilde{\mathfrak{S}}_n \to \mathrm{Cl}_n^\times$ *given by*

$$\tilde{s}_i \mapsto \tfrac{1}{2}(\alpha_{i+1} - \alpha_i), \quad c \mapsto -1.$$

Proof. Let $\sigma_i = \frac{1}{2}(\alpha_{i+1} - \alpha_i)$. We have

$$\sigma_i^2 = \tfrac{1}{4}(\alpha_{i+1}^2 - \alpha_{i+1}\alpha_i - \alpha_i \alpha_{i+1} + \alpha_i^2) = 1,$$

as the middle terms cancel and the outer terms are each equal to 2. It is clear that σ_i and σ_j anticommute if $|i - j| > 1$. Finally,

$$\sigma_i \sigma_{i+1} \sigma_i = \tfrac{1}{2}(\alpha_i - \alpha_{i+2}) = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

so the braid relation holds. We thus see that the σ's satisfy the relations defining the spin-symmetric group, and the claim follows. □

8.3. ***Clifford modules.*** Let $\mathcal{A}$ be a supercategory. A Cl_n*-module* in $\mathcal{A}$ is a pair (M, ρ) consisting of an object M of $\mathcal{A}$ and a morphism of $\boldsymbol{k}$-superalgebras $\rho : \mathrm{Cl}_n \to \mathrm{End}(M)$. Explicitly, a Cl_n-module consists of an object M together with odd-degree supermorphisms $\alpha_1, \dots, \alpha_n : M \to M$ such that $\alpha_i^2 = 2$ and $\alpha_i \alpha_j = -\alpha_j \alpha_i$ for $i \neq j$. Suppose that M and N are two Cl_n-modules in $\mathcal{A}$. A *morphism* $f : M \to N$ of parity p is a morphism in $\mathcal{A}$ of parity p compatible with the Cl_n-structures, i.e., such that $f\alpha_i = (-1)^p \alpha_i f$. We let $\mathrm{Cl}_n(\mathcal{A})$ be the supercategory of Cl_n-modules in $\mathcal{A}$.

One easily sees that, if $\mathcal{A}$ is additive or Karoubian, then so is $\mathrm{Cl}_n(\mathcal{A})$. Moreover, if $\mathcal{A}$ admits a Π-structure, then so does $\mathrm{Cl}_n(\mathcal{A})$, via

$$\Pi(M; \alpha_1, \dots, \alpha_n) = (\Pi(M); \Pi(\alpha_1), \dots, \Pi(\alpha_n)).$$

We thus see that, if $\mathcal{A}$ is nice, then so is $\mathrm{Cl}_n(\mathcal{A})$. The following statement is clear, and we omit its proof:

Proposition 8.2. *We have an equivalence* $\mathrm{Cl}_m(\mathrm{Cl}_n(\mathcal{A})) \to \mathrm{Cl}_{n+m}(\mathcal{A})$ *defined by*

$$((M; \alpha_1, \dots, \alpha_n); \alpha_1', \dots, \alpha_m') \mapsto (M; \alpha_1, \dots, \alpha_n, \alpha_1', \dots, \alpha_m').$$

Given a Cl_n-module $(M; \alpha_1, \dots, \alpha_n)$ in $\mathcal{A}$ and an element $\sigma \in \mathfrak{S}_n$, put

$$\sigma_*(M; \alpha_1, \dots, \alpha_n) = (M; \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}).$$

This construction defines a left action of $\mathfrak{S}_n$ on the category $\mathrm{Cl}_n(\mathcal{A})$. We let $\widetilde{\mathfrak{S}}_n$ act via the homomorphism $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$. The following proposition shows that every object of $\mathrm{Cl}_n(\mathcal{A})$ is canonically equivariant for the $\widetilde{\mathfrak{S}}_n$-action:

Proposition 8.3. *Let $(M; \alpha_1, \dots, \alpha_n)$ be a Cl_n-module. Then $\widetilde{\mathfrak{S}}_n$ naturally acts on M, and permutes the α's via the surjection $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$. In particular, for $\sigma \in \widetilde{\mathfrak{S}}_n$, we have a natural isomorphism of Cl_n-modules*

$$\sigma : (M; \alpha_1, \dots, \alpha_n) \to (M; \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$$

that has the same parity as σ.

Proof. The action comes from Proposition 8.1 using the homomorphism $\varphi : \widetilde{\mathfrak{S}}_n \to \mathrm{Cl}_n^\times$. We claim that $\varphi(\sigma)\alpha_i = (-1)^{|\sigma|}\alpha_{\sigma^{-1}(i)}\varphi(\sigma)$ for all $\sigma \in \widetilde{\mathfrak{S}}_n$. It is enough to check this when $\sigma = \tilde{s}_j$. Then

$$\begin{aligned}
\varphi(\tilde{s}_j)\alpha_j &= \tfrac{1}{2}(\alpha_{j+1}-\alpha_j)\alpha_j = \tfrac{1}{2}\alpha_{j+1}\alpha_j - 1 = -\tfrac{1}{2}\alpha_{j+1}(\alpha_{j+1}-\alpha_j) = -\alpha_{j+1}\varphi(\tilde{s}_j),\\
\varphi(\tilde{s}_j)\alpha_{j+1} &= \tfrac{1}{2}(\alpha_{j+1}-\alpha_j)\alpha_{j+1} = 1 - \tfrac{1}{2}\alpha_j\alpha_{j+1} = -\tfrac{1}{2}\alpha_j(\alpha_{j+1}-\alpha_j) = -\alpha_j\varphi(\tilde{s}_j),\\
\varphi(\tilde{s}_j)\alpha_k &= \tfrac{1}{2}(\alpha_{j+1}-\alpha_j)\alpha_k = -\tfrac{1}{2}\alpha_k(\alpha_{j+1}-\alpha_j) = -\alpha_k\varphi(\tilde{s}_j) \qquad (k \notin \{j, j+1\}).
\end{aligned}$$

□

8.4. ***Periodicity.*** Throughout this section, we fix a value p such that Cl_p is isomorphic as a superalgebra to a matrix superalgebra. (By Theorem A.4, we know that $p = 8$ is possible for any $\boldsymbol{k}$ and that we can take $p = 2$ if $\zeta_4 \in \boldsymbol{k}$.) Let U_p be a Cl_p-module such that the functor $\mathbf{SVec} \to \mathrm{Mod}_{\mathrm{Cl}_p}$ given by $V \mapsto V \otimes U_p$ is an equivalence, and let $\epsilon_p \in \mathrm{Cl}_p$ be an even idempotent such that the functor $\mathrm{Mod}_{\mathrm{Cl}_p} \to \mathbf{SVec}$ given by $M \mapsto \epsilon_p M$ is a quasi-inverse.

Let $\mathcal{A}$ be a nice supercategory. The following is clear:

Proposition 8.4. *The functor $\mathcal{A} \to \mathrm{Cl}_p(\mathcal{A})$ given by $M \mapsto M \otimes U_p$ is an equivalence.*

We let $\Phi_n : \mathrm{Cl}_n(\mathcal{A}) \to \mathrm{Cl}_{n+p}(\mathcal{A})$ be the equivalence induced by the identification $\mathrm{Cl}_{n+p}(\mathcal{A}) = \mathrm{Cl}_p(\mathrm{Cl}_n(\mathcal{A}))$. For a nonnegative integer i, define $\mathrm{Cl}_{(i;p)}(\mathcal{A})$ to be the 2-colimit of the system

$$\mathrm{Cl}_i(\mathcal{A}) \xrightarrow{\Phi_i} \mathrm{Cl}_{i+p}(\mathcal{A}) \xrightarrow{\Phi_{i+p}} \mathrm{Cl}_{i+2p}(\mathcal{A}) \to \cdots.$$

Explicitly, an object of $\mathrm{Cl}_{(i;p)}(\mathcal{A})$ is a Cl_n-module in $\mathcal{A}$ for some integer $n \geq 0$ such that $n \equiv i \pmod p$. If M is a Cl_m-module and N is a Cl_n-module, with $m \equiv n \equiv i \pmod p$, then

$$\mathrm{Hom}_{\mathrm{Cl}_{(i;p)}(\mathcal{A})}(M, N) = \varinjlim \mathrm{Hom}_{\mathrm{Cl}_r(\mathcal{A})}(\Phi^{(r-m)/p}(M), \Phi^{(r-n)/p}(N)),$$

where the colimit is taken as $r \to \infty$ over integers r such that $r \equiv i \pmod p$. It is clear that the natural functor $\mathrm{Cl}_i(\mathcal{A}) \to \mathrm{Cl}_{(i;p)}(\mathcal{A})$ is an equivalence, and so, in a sense, these limit categories are nothing new. However, the categories $\mathrm{Cl}_{(i;p)}(\mathcal{A})$ will sometimes be easier to work with, especially when dealing with products.

8.5. ***Clifford eversion.*** As before, let p be such that Cl_p is a matrix superalgebra, and fix an integer q divisible by p (we allow $q = 0$). Note that q is necessarily even.

Definition 8.5. The *Clifford eversion* of a nice $\mathbb{Z}/q$-supercategory $\mathcal{A}$, denoted by $\mathrm{Cl}(\mathcal{A})$, is the nice $\mathbb{Z}/q$-supercategory given by $\mathrm{Cl}(\mathcal{A})_i = \mathrm{Cl}_{(i;p)}(\mathcal{A}_i)$. □

Remark 8.6. We defined $\mathrm{Cl}(\mathcal{A})_i$ to be $\mathrm{Cl}_{(i;p)}(\mathcal{A}_i)$. In some ways, it is more natural to use $\mathrm{Cl}_{(i;q)}(\mathcal{A}_i)$, but there is one problem with this: it is not defined if $q = 0$ and $i < 0$. □

Remark 8.7. The motivation for the definition of Clifford eversion comes from the isomeric category; see Section 7.5. □

We let Cl^k be the k-fold iterate of Cl. To study this, it is convenient to introduce a slight variant. Define $\mathrm{Cl}^{(k)}(\mathcal{A})$ to be the $\mathbb{Z}/q$-supercategory given by $\mathrm{Cl}^{(k)}(\mathcal{A})_i = \mathrm{Cl}_{(ki;p)}(\mathcal{A})$. Thus, $\mathrm{Cl}^{(1)} = \mathrm{Cl}$. The equivalence in Proposition 8.2 induces an equivalence $\mathrm{Cl}_{(i;p)}(\mathrm{Cl}_{(j;p)}(\mathcal{B})) \to \mathrm{Cl}_{(i+j;p)}(\mathcal{B})$ for any nice supercategory $\mathcal{B}$. It follows that there is a natural equivalence $\mathrm{Cl}^{(k)}(\mathrm{Cl}^{(\ell)}(\mathcal{A})) \to \mathrm{Cl}^{(k+\ell)}(\mathcal{A})$. In particular, there is a natural equivalence $\mathrm{Cl}^k(\mathcal{A}) \to \mathrm{Cl}^{(k)}(\mathcal{A})$ for any $k \geq 0$.

Proposition 8.8. *We have a canonical equivalence* $\mathcal{A} \cong \mathrm{Cl}^p(\mathcal{A})$.

Proof. Combine the natural equivalences $\mathcal{A} \to \mathrm{Cl}^{(p)}(\mathcal{A})$ and $\mathrm{Cl}^p(\mathcal{A}) \to \mathrm{Cl}^{(p)}(\mathcal{A})$. □

Suppose $\mathcal{A}$ is a nice monoidal $\mathbb{Z}/q$-supercategory. We define a natural monoidal structure on $\mathrm{Cl}(\mathcal{A})$ as follows. Let $r, s \in \mathbb{Z}/q$ and let r' and s' be nonnegative integers congruent to r and s modulo p. We have a functor $\mathrm{Cl}_{r'}(\mathcal{A}_r) \boxtimes \mathrm{Cl}_{s'}(\mathcal{A}_s) \to \mathrm{Cl}_{r'+s'}(\mathcal{A}_{r+s})$ given by

$$(M; \mu_1, \dots, \mu_{r'}) \otimes (N; \nu_1, \dots, \nu_{s'}) \mapsto (M \otimes N; \mu_1, \dots, \mu_{r'}, \nu_1, \dots, \nu_{s'}).$$

Here μ_i is shorthand for $\mu_i \otimes 1$, and similarly for ν_j. Note that M is an object of $\mathcal{A}_r$ and N is an object of $\mathcal{A}_s$, so $M \otimes N$ is an object of $\mathcal{A}_{r+s}$. Thus, the above object does indeed belong to $\mathrm{Cl}(\mathcal{A})_{r+s}$. The diagram

$$\begin{array}{ccc} \mathrm{Cl}_{r'}(\mathcal{A}_r) \boxtimes \mathrm{Cl}_{s'}(\mathcal{A}_s) & \xrightarrow{1 \otimes \Phi_{s'}} & \mathrm{Cl}_{r'}(\mathcal{A}_r) \boxtimes \mathrm{Cl}_{s'+p}(\mathcal{A}_s) \\ \downarrow & & \downarrow \\ \mathrm{Cl}_{r'+s'}(\mathcal{A}_{r+s}) & \xrightarrow{\Phi_{r'+s'}} & \mathrm{Cl}_{r'+s'+p}(\mathcal{A}_{r+s}) \end{array}$$

commutes, since in both cases the extra endomorphisms for Cl_p are appended to the end of the list. The diagram

$$\begin{array}{ccc} \mathrm{Cl}_{r'}(\mathcal{A}_r) \boxtimes \mathrm{Cl}_{s'}(\mathcal{A}_s) & \xrightarrow{\Phi_{s'} \otimes 1} & \mathrm{Cl}_{r'+p}(\mathcal{A}_r) \boxtimes \mathrm{Cl}_{s'}(\mathcal{A}_s) \\ \downarrow & & \downarrow \\ \mathrm{Cl}_{r'+s'}(\mathcal{A}_{r+s}) & \xrightarrow{\Phi_{r'+s'}} & \mathrm{Cl}_{r'+s'+p}(\mathcal{A}_{r+s}) \end{array}$$

does not commute, but does commute up to canonical even isomorphism. Precisely, let $(M; \mu_\bullet)$ and $(N; \nu_\bullet)$ be objects of $\mathrm{Cl}_{r'}(\mathcal{A}_r)$ and $\mathrm{Cl}_{s'}(\mathcal{A}_s)$, and let $(U_q; \lambda_1, \dots, \lambda_p)$ be the Cl_p-module inducing periodicity. Then the top path in the above diagram yields $(M \otimes N; \mu_\bullet, \lambda_\bullet, \nu_\bullet)$, while the bottom path yields $(M \otimes N; \mu_\bullet, \nu_\bullet, \lambda_\bullet)$. These two objects are isomorphic via Proposition 8.3, using that $j_{r',p+s'}(1, \tilde{\tau}_{p,s'}) \in \widetilde{\mathfrak{S}}_{r'+s'+p}$. Note that this isomorphism is even since p is even.

It follows from the above discussion that we have a well-defined functor $\mathrm{Cl}(\mathcal{A}) \boxtimes \mathrm{Cl}(\mathcal{A}) \to \mathrm{Cl}(\mathcal{A})$. The associativity and unit constraints on $\mathrm{Cl}(\mathcal{A})$ are defined in the obvious manner, and one readily verifies that the axioms of a monoidal $\mathbb{Z}/q$-supercategory are satisfied.

We now investigate how the above construction interacts with braidings. In what follows, we assume that $4 \mid q$. Suppose we have an isomorphism $\beta : \otimes \to \otimes \circ \Sigma$. Given objects $(M; \alpha_\bullet)$ and $(N; \alpha'_\bullet)$ as above, we define

$$\beta'_{(M;\alpha_\bullet),(N;\alpha'_\bullet)} : (M; \alpha_\bullet) \otimes (N; \alpha'_\bullet) \to (N; \alpha'_\bullet) \otimes (M; \alpha_\bullet)$$

to be the composition

$$(M \otimes N; \alpha_\bullet, \alpha'_\bullet) \xrightarrow{\beta_{M,N}} (N \otimes M; \alpha_\bullet, \alpha'_\bullet) \xrightarrow{\tilde{\tau}_{r,s}^{-1}} (N \otimes M; \alpha'_\bullet, \alpha_\bullet).$$

The second map uses the action of $\widetilde{\mathfrak{S}}_{r+s}$ on $N \otimes M$, as in Proposition 8.3. We note that $|\beta'| = |\beta| + rs$. For the following, we use the notation from Section 3.3.

Proposition 8.9. *Maintain the above notation.*

(a) *If β is a symmetry (or braiding) with respect to $\omega \in \Gamma_0$, then β' is a type II supersymmetry (or superbraiding) with respect to $\mathtt{DA}\omega$.*

(b) *If β is a type II supersymmetry (or superbraiding) with respect to $\omega \in \Gamma_1$, then β' is a symmetry (or braiding) with respect to $\mathtt{A}\omega$.*

Proof. Suppose that β is a braiding with respect to $\omega = (\omega_1, \omega_2)$. We show that β' is a type II superbraiding with respect to $\mathtt{A}\omega$. We have already observed that $\beta'_{X,Y}$ is homogeneous of degree $|X||Y|$ if X and Y are homogeneous objects. We now verify that β' is a natural transformation. Suppose that $f : (M; \mu_1, \dots, \mu_r) \to (M'; \mu'_1, \dots, \mu'_r)$ and $g : (N; \nu_1, \dots, \nu_s) \to (N'; \nu'_1, \dots, \nu'_s)$ are homogeneous morphisms of homogeneous objects of $\mathrm{Cl}(\mathcal{A})$. Consider the diagram

$$\begin{array}{ccccc}
(M \otimes N; \mu_\bullet, \nu_\bullet) & \xrightarrow{\beta_{M,N}} & (N \otimes M; \mu_\bullet, \nu_\bullet) & \xrightarrow{\tilde{\tau}_{r,s}^{-1}} & (N \otimes M; \nu_\bullet, \mu_\bullet) \\
\downarrow{\scriptstyle f\otimes g} & & \downarrow{\scriptstyle g\otimes f} & & \downarrow{\scriptstyle g\otimes f} \\
(M' \otimes N'; \mu'_\bullet, \nu'_\bullet) & \xrightarrow{\beta_{M',N'}} & (N' \otimes M'; \mu'_\bullet, \nu'_\bullet) & \xrightarrow{\tilde{\tau}_{r,s}^{-1}} & (N' \otimes M'; \nu'_\bullet, \mu'_\bullet)
\end{array}$$

The first square commutes up to $(-1)^{|f||g|}$, as this is simply what it means that β is a natural transformation, while the second square commutes up to $(-1)^{(|f|+|g|)rs}$. This shows that β' is natural.

We now look at the hexagon axioms. Let $(M; \mu_\bullet)$, $(N; \nu_\bullet)$ and $(P, \pi_\bullet)$ be homogeneous objects of $\mathrm{Cl}(\mathcal{A})$ of degrees r, s and t. Consider the diagram

$$\begin{array}{ccccc}
(MNP; \mu, \nu, \pi) & \xrightarrow{\beta_{M,NP}} & (NPM; \mu, \nu, \pi) & \xrightarrow{x} & (NPM; \nu, \pi, \mu) \\
\downarrow{\scriptstyle \beta_{N,M}} & & & & \uparrow{\scriptstyle z} \\
(NMP; \mu, \nu, \pi) & & & & (NPM; \nu, \mu, \pi) \\
& \searrow^{y} & & \nearrow_{\beta_{M,P}} & \\
& & (NMP; \nu, \mu, \pi) & &
\end{array}$$

Here we have omitted $\otimes$ and $\bullet$ symbols, and x, y and z are the appropriate elements of $\widetilde{\mathfrak{S}}_*$ appearing in the definition of β'. This is the hexagon (H1) for β', where we have omitted the associators. Put

$$(\alpha_1, \dots, \alpha_{r+s+t}) = (\mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s, \pi_1, \dots, \pi_t).$$

We regard each α_i as acting on MNP, or any of the other tensor products in the above diagram, in the natural manner. We let $\widetilde{\mathfrak{S}}_{r+s+t}$ act on MNP via the α's (as in Proposition 8.1). Then

$$x = \tilde{\tau}_{r,s+t}^{-1}, \quad y = j_{r+s,t}(\tilde{\tau}_{r,s}^{-1}, 1), \quad z = (-1)^{rst} \cdot y \cdot j_{s,r+t}(1, \tilde{\tau}_{r,t}^{-1}) \cdot y^{-1}.$$

The expressions for x and y are clear. By definition, z is $\mathrm{id}_N \otimes \tilde{\tau}_{r,t}^{-1}$, but where this $\tilde{\tau}_{r,t}$ element is built from the μ's and π's, i.e., the elements $\alpha_1, \dots, \alpha_r$ and $\alpha_{r+s+1}, \dots, \alpha_{r+s+t}$. This agrees with the above formula; note that

$$y\alpha_i y^{-1} = (-1)^{rs} \alpha_{\tau_{r,s}^{-1}(i)}$$

by Proposition 8.3. One readily verifies that $\beta_{M,P}$ commutes with the α_i's (in the evident sense), and thus with y as well. Since $\beta_{M,P}\beta_{N,M} = \omega_1(r; s, t)\beta_{M,NP}$ (this is the first hexagon axiom for β) and $j_{s,r+t}(1, \tilde{\tau}_{r,t}) \cdot j_{r+s,t}(\tilde{\tau}_{r,s}, 1) = \tilde{\tau}_{r,s+t}$ (Proposition 5.6), the above diagram commutes up to $(-1)^{rst}\omega_1(r; s, t)$, which verifies the first hexagon axiom for β' with respect to $\mathtt{DA}\omega$. The second hexagon axiom is similar. We thus see that β' is a type II superbraiding.

Now suppose that β is symmetric, and let us show that β' is supersymmetric. Maintaining the above notation, we have

$$\beta'_{(N;\nu),(M;\mu)}\beta'_{(M;\mu),(N;\nu)} = y\beta_{N,M}x\beta_{M,N},$$

where $x = \tilde{\tau}_{r,s}^{-1}$, and y is defined like $\tilde{\tau}_{s,r}^{-1}$ but with respect to $\alpha_{r+1}, \dots, \alpha_{r+s}, \alpha_1, \dots, \alpha_r$; in other words, $y = (-1)^{rs} x \tilde{\tau}_{s,r}^{-1} x^{-1} = (-1)^{rs} \tilde{\tau}_{s,r}^{-1}$. Since $\beta_{N,M}$ and x commute, we have

$$\beta'_{(N;\nu),(M;\mu)}\beta'_{(M;\mu),(N;\nu)} = (-1)^{rs} \tilde{\tau}_{s,r}^{-1} \tilde{\tau}_{r,s}^{-1} \beta_{N,M}\beta_{M,N} = (-1)^{\binom{r}{2}\binom{s}{2}+rs} \omega_\sharp(|M|, |N|)\, \mathrm{id}_{M\otimes N},$$

where we have used Proposition 5.3. We thus see that β' is a supersymmetry with respect to $\mathtt{DA}\omega$.

We have thus proved statement (a). Statement (b) is similar, though some additional signs come in: in the hexagon axiom we pick up an additional $(-1)^{rst}$ when commuting $\beta_{M,P}$ and y, and in the symmetry axiom we pick up an additional $(-1)^{rs}$ when commuting $\beta_{N,M}$ and $\tilde{\tau}_{r,s}$. □

Remark 8.10. If one only cares about (super)braidings, one can phrase the above proposition using B-factor systems instead of S-factor systems. □

The above proposition can be summarized pictorially as follows: for $\omega \in \Gamma_0$, Clifford eversion acts by

$$\omega \to \mathtt{DA}\omega \to \mathtt{D}\omega \to \mathtt{A}\omega \to \omega \to \cdots.$$

That is, given a (super)symmetric monoidal category for one factor system, the Clifford eversion will be a (super)symmetric monoidal category for the next factor system. The list of factor systems repeats with periodicity 4, while Clifford eversion itself repeats with period p (which is a divisor of 8).

We have seen that there is a canonical equivalence $\mathcal{A} \to \mathrm{Cl}^p(\mathcal{A})$. If $\mathcal{A}$ is (super)braided or (super-)symmetric, then so is $\mathrm{Cl}^p(\mathcal{A})$, and with respect to the same factor system (recall that q is divisible by 4). We now show that this equivalence is appropriately monoidal.

Proposition 8.11. *The equivalence $\mathcal{A} \to \mathrm{Cl}^p(\mathcal{A})$ is monoidal, (super)braided monoidal or (super-) symmetric monoidal, depending on what structure $\mathcal{A}$ has.*

Proof. The definition of the monoidal structure and (super)braiding on $\mathrm{Cl}(\mathcal{A})$ generalizes in the obvious manner to $\mathrm{Cl}^{(k)}(\mathcal{A})$ for any k. The equivalences $\mathrm{Cl}^{(k)}(\mathrm{Cl}^{(\ell)}(\mathcal{A})) \to \mathrm{Cl}^{(k+\ell)}(\mathcal{A})$ are easily seen to be compatible with these structures; it follows that the equivalence $\mathrm{Cl}^k(\mathcal{A}) \to \mathrm{Cl}^{(k)}(\mathcal{A})$ is as well. The equivalence $\mathcal{A} \to \mathrm{Cl}^{(p)}(\mathcal{A})$ induced by Clifford periodicity is also compatible with these structures. □

Combining the above results, we find:

Proposition 8.12. *Suppose q is divisible by 4 and let $\omega \in \Gamma_0$. Then the 2-category of symmetric monoidal $\mathbb{Z}/q$-supercategories with respect to ω is equivalent to the 2-category of type II supersymmetric monoidal $\mathbb{Z}/q$-supercategories with respect to $\mathtt{A}\omega$.*

Specializing to the canonical choice $\omega = 1$, we find:

Corollary 8.13. *The 2-category of symmetric monoidal $\mathbb{Z}/q$-supercategories is equivalent to the 2-category of type IIa supersymmetric monoidal $\mathbb{Z}/q$-supercategories, via Clifford eversion.*

Remark 8.14. We have assumed $4 \mid q$ throughout the above discussion. There are analogous results if $p = 2$ and $2 \mid q$, assuming one has sufficiently many roots of unity in $\boldsymbol{k}$. □

8.6. ***Envelopes.*** Let $\mathcal{C}$ be a supercategory. We define its *additive envelope* $\mathcal{C}^{\mathrm{add}}$ as follows. The objects are symbols (formal direct sums) $A_1 \oplus \cdots \oplus A_r$ where the A_i are objects of $\mathcal{C}$. A morphism $f : A_1 \oplus \cdots \oplus A_r \to B_1 \oplus \cdots \oplus B_s$ is an element $(f_{i,j}) \in \prod_{i,j} \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$, thought of as a matrix. We say that f is homogeneous of parity p if each $f_{i,j}$ is homogeneous of parity p. Given another morphism $g : B_1 \oplus \cdots \oplus B_s \to C_1 \oplus \cdots \oplus C_t$, the composition gf is defined by $(gf)_{i,j} = \sum_k g_{k,j} f_{i,k}$. One readily verifies that $\mathcal{C}^{\mathrm{add}}$ is an additive supercategory. We have a fully faithful functor $i : \mathcal{C} \to \mathcal{C}^{\mathrm{add}}$ that satisfies a universal property: given a superfunctor $j : \mathcal{C} \to \mathcal{A}$ to an additive supercategory $\mathcal{A}$, there is a unique (up to isomorphism) superfunctor $j' : \mathcal{C}^{\mathrm{add}} \to \mathcal{A}$ such that $j = j'i$. In particular, i is an equivalence if $\mathcal{C}$ is additive.

The *Karoubian envelope* of a supercategory $\mathcal{A}$, denoted by $\mathcal{A}^{\mathrm{kar}}$, consists of objects (A, e) where A is an object of $\mathcal{C}$ and e is an even idempotent of A. A morphism $(A, e) \to (B, f)$ in $\mathcal{A}^{\mathrm{kar}}$ is a morphism $\varphi : A \to B$ such that $f\varphi = \varphi = \varphi e$. (The identity of (A, e) is given by $e : A \to A$.) One easily sees that $\mathcal{A}^{\mathrm{kar}}$ is a Karoubian supercategory. We have a fully faithful superfunctor $i : \mathcal{A} \to \mathcal{A}^{\mathrm{kar}}$ defined by $i(A) = (A, 1)$ that satisfies a universal property: given an superfunctor $j : \mathcal{A} \to \mathcal{B}$ to a Karoubian supercategory $\mathcal{B}$, there is a unique (up to isomorphism) superfunctor $j' : \mathcal{A}^{\mathrm{kar}} \to \mathcal{B}$ such that $j = j'i$. In particular, i is an equivalence if $\mathcal{A}$ is Karoubian.

Lemma 8.15. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful superfunctor of supercategories. Then $F^{\mathrm{add}} : \mathcal{A}^{\mathrm{add}} \to \mathcal{B}^{\mathrm{add}}$ and $F^{\mathrm{kar}} : \mathcal{A}^{\mathrm{kar}} \to \mathcal{B}^{\mathrm{kar}}$ are also fully faithful.*

Proof. This is clear for F^{add} by the explicit description for morphisms given above. Similarly, a morphism $(A, e) \to (A', e')$ in the Karoubian envelope is just a morphism $A \to A'$ satisfying some conditions which are preserved by F, so F^{kar} is also fully faithful. □

Let $\mathcal{A}$ and $\mathcal{B}$ be nice $\mathbb{Z}/q$-supercategories. The product $\mathcal{A}\boxtimes\mathcal{B}$ will typically not be additive or Karoubian, and thus not nice. We define $\mathcal{A}\,\hat{\boxtimes}\,\mathcal{B}$ to be the Karoubian envelope of the additive envelope of $\mathcal{A}\boxtimes\mathcal{B}$. This is again a nice $\mathbb{Z}/q$-supercategory, and $\hat{\boxtimes}$ endows the 2-category of nice $\mathbb{Z}/q$-supercategories with a monoidal structure. We note that, if $\mathcal{A}$ is a nice monoidal $\mathbb{Z}/q$-supercategory, then the monoidal product $\mathcal{A}\boxtimes\mathcal{A} \to \mathcal{A}$ canonically extends to a superfunctor $\mathcal{A}\,\hat{\boxtimes}\,\mathcal{A} \to \mathcal{A}$, by formal properties of envelopes.

8.7. 2-*categorical aspects.* We now explain some 2-categorical aspects of Clifford eversion. As in Section 4, we do not intend this discussion to be completely rigorous. Fix a positive integer q divisible by 4, and let $\mathscr{C}$ be the 2-category of nice $\mathbb{Z}/q$-supercategories, which we regard as monoidal under the product $\hat{\boxtimes}$. Let $\omega \in \Gamma_0$ be an even S-factor system. Then Σ and ω induce a symmetric monoidal structure on $\mathscr{C}$, just as T and an odd S-factor system do (see Section 4). We write $(\mathscr{C}, \Sigma, \omega)$ for the resulting symmetric monoidal 2-category. Of course, we similarly get a symmetric monoidal 2-category from T and an odd S-factor system. The following is the main statement we want to explain, though we do not give a complete proof:

Theorem 8.16. *Clifford eversion defines an equivalence of symmetric monoidal* 2*-categories*

$$\mathrm{Cl} : (\mathscr{C}, \Sigma, \omega) \to (\mathscr{C}, \mathrm{T}, \mathtt{A}\omega).$$

Lemma 8.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be nice $\mathbb{Z}/q$-supercategories. We have a canonical equivalence*

$$i_{\mathcal{A},\mathcal{B}} : \mathrm{Cl}(\mathcal{A})\,\hat{\boxtimes}\,\mathrm{Cl}(\mathcal{B}) \to \mathrm{Cl}(\mathcal{A}\,\hat{\boxtimes}\,\mathcal{B}).$$

Proof. Let

$$j^{n,m}_{\mathcal{A},\mathcal{B}} : \mathrm{Cl}_n(\mathcal{A}) \boxtimes \mathrm{Cl}_m(\mathcal{B}) \to \mathrm{Cl}_{n+m}(\mathcal{A}\,\hat{\boxtimes}\,\mathcal{B})$$

be the functor defined by

$$(M; \mu_1, \dots, \mu_n) \boxtimes (N; \nu_1, \dots, \nu_m) \mapsto (M \boxtimes N; \mu_1, \dots, \mu_n, \nu_1, \dots, \nu_m),$$

where here we write μ_i in place of $\mu_i \otimes \mathrm{id}$ on the right side, and similarly for ν_j.

We claim that $j^{n,m}_{\mathcal{A},\mathcal{B}}$ is fully faithful. Faithfulness is clear. Let A, A' be objects of $\mathrm{Cl}_n(\mathcal{A})$ and let B, B' be objects of $\mathrm{Cl}_m(\mathcal{B})$. A homogeneous morphism $A \otimes B \to A' \otimes B'$ in $\mathrm{Cl}_{n+m}(\mathcal{A}\,\hat{\boxtimes}\,\mathcal{B})$ is an element $f = \sum_k g_k \otimes h_k \in \mathrm{Hom}_{\mathcal{A}}(A, A') \otimes \mathrm{Hom}_{\mathcal{B}}(B, B')$, where the g_k are homogeneous and linearly independent, and similarly for the h_k, such that

$$f(\mu_i \otimes 1) = (-1)^{|f|}(\mu_i' \otimes 1)f \quad \text{and} \quad f(1 \otimes \nu_j) = (-1)^{|f|}(1 \otimes \nu_j')f$$

for all i, j. The first condition becomes $\sum_k (-1)^{|h_k|} g_k \mu_i \otimes h_k = \sum_k (-1)^{|f|} \mu_i' g_k \otimes h_k$, so, by linear independence of the h_k, we deduce that $g_k \mu_i = (-1)^{|g_k|} \mu_i' g_k$ for all i, k. (Note $|f| = |g_k| + |h_k|$.) Similarly, we deduce that $h_k \nu_j = (-1)^{|h_k|} \nu_j' h_k$ for all j, k. Hence, $j^{n,m}_{\mathcal{A},\mathcal{B}}$ is full. Thus, each g_k defines a morphism

$A \to A'$ in $\mathrm{Cl}_n(\mathcal{A})$ and each h_k defines a morphism $B \to B'$ in $\mathrm{Cl}_m(\mathcal{B})$, and so f is in the image of $j^{n,m}_{\mathcal{A},\mathcal{B}}$. This proves the claim.

One easily sees (using an argument we have already made) that the $j^{n,m}_{\mathcal{A},\mathcal{B}}$ are compatible with the periodicity equivalences. They therefore induce a superfunctor

$$j_{\mathcal{A},\mathcal{B}} : \mathrm{Cl}(\mathcal{A}) \boxtimes \mathrm{Cl}(\mathcal{B}) \to \mathrm{Cl}(\mathcal{A} \hat{\boxtimes} \mathcal{B})$$

that is also fully faithful. Let

$$i_{\mathcal{A},\mathcal{B}} : \mathrm{Cl}(\mathcal{A}) \hat{\boxtimes} \mathrm{Cl}(\mathcal{B}) \to \mathrm{Cl}(\mathcal{A} \hat{\boxtimes} \mathcal{B})$$

be the functor induced by $j_{\mathcal{A},\mathcal{B}}$ by taking envelopes; the target does not change since it is already additive and Karoubian. By Lemma 8.15, $i_{\mathcal{A},\mathcal{B}}$ is fully faithful. Iterating this construction, we obtain a fully faithful functor

$$i^k_{\mathcal{A},\mathcal{B}} : \mathrm{Cl}^k(\mathcal{A}) \hat{\boxtimes} \mathrm{Cl}^k(\mathcal{B}) \to \mathrm{Cl}^k(\mathcal{A} \hat{\boxtimes} \mathcal{B})$$

for any $k \geq 0$. The diagram

$$\begin{array}{ccc} \mathrm{Cl}^{p-1}(\mathrm{Cl}(\mathcal{A})) \hat{\boxtimes} \mathrm{Cl}^{p-1}(\mathrm{Cl}(\mathcal{B})) & \xrightarrow{\quad i^p_{\mathcal{A},\mathcal{B}} \quad} & \mathrm{Cl}^{p-1}(\mathrm{Cl}(\mathcal{A} \hat{\boxtimes} \mathcal{B})) \\ & {\scriptstyle i^{p-1}_{\mathrm{Cl}(\mathcal{A}),\mathrm{Cl}(\mathcal{B})}} \searrow \quad \nearrow {\scriptstyle \mathrm{Cl}^{p-1}(i_{\mathcal{A},\mathcal{B}})} & \\ & \mathrm{Cl}^{p-1}(\mathrm{Cl}(\mathcal{A}) \hat{\boxtimes} \mathrm{Cl}(\mathcal{B})) & \end{array}$$

commutes, essentially by definition. The two categories in the top row are each identified with $\mathcal{A} \hat{\boxtimes} \mathcal{B}$ by Proposition 8.8, and $i^p_{\mathcal{A},\mathcal{B}}$ is easily seen to be identified with the identity functor; thus, the top line is an equivalence. Since the two diagonal functors are fully faithful and their composite is an equivalence, each is an equivalence. Since Cl is an equivalence of 2-categories, again by Proposition 8.8, and $\mathrm{Cl}^{p-1}(i_{\mathcal{A},\mathcal{B}})$ is an equivalence, it follows that $i_{\mathcal{A},\mathcal{B}}$ is an equivalence. □

Lemma 8.18. *The diagram*

$$\begin{array}{ccc} \mathrm{Cl}(\mathcal{A}) \hat{\boxtimes} \mathrm{Cl}(\mathcal{B}) & \xrightarrow{i_{\mathcal{A},\mathcal{B}}} & \mathrm{Cl}(\mathcal{A} \hat{\boxtimes} \mathcal{B}) \\ {\scriptstyle \mathrm{T}_{\mathrm{Cl}(\mathcal{A}),\mathrm{Cl}(\mathcal{B})}} \downarrow & & \downarrow {\scriptstyle \mathrm{Cl}(\Sigma_{\mathcal{A},\mathcal{B}})} \\ \mathrm{Cl}(\mathcal{B}) \hat{\boxtimes} \mathrm{Cl}(\mathcal{A}) & \xrightarrow{i_{\mathcal{B},\mathcal{A}}} & \mathrm{Cl}(\mathcal{B} \hat{\boxtimes} \mathcal{A}) \end{array}$$

commutes up to a canonical even isomorphism.

Proof. Let $(M; \mu_\bullet)$ be $(N; \nu_\bullet)$ be objects of $\mathrm{Cl}_m(\mathcal{A})$ and $\mathrm{Cl}_n(\mathcal{B})$. Then

$$i_{\mathcal{A},\mathcal{B}}((M; \mu_\bullet) \otimes (N; \nu_\bullet)) = (M \otimes N; \mu_\bullet, \nu_\bullet),$$

where here we have written μ_i in place of $\mu_i \otimes 1$ and ν_j in place of $1 \otimes \nu_j$. The functor $\Sigma_{\mathcal{A},\mathcal{B}} : \mathcal{A} \hat{\boxtimes} \mathcal{B} \to \mathcal{B} \hat{\boxtimes} \mathcal{A}$ induces a functor $\mathrm{Cl}_{m+n}(\Sigma_{\mathcal{A},\mathcal{B}}) : \mathrm{Cl}_{m+n}(\mathcal{A} \hat{\boxtimes} \mathcal{B}) \to \mathrm{Cl}_{m+n}(\mathcal{B} \hat{\boxtimes} \mathcal{A})$ that takes the above object to $(N \otimes M; \mu_\bullet, \nu_\bullet)$, where now μ_i means $1 \otimes \mu_i$ and ν_j means $\nu_j \otimes 1$. Note that the order of the μ and ν

does not change, since $\mathrm{Cl}_{m+n}(\Sigma_{\mathcal{A},\mathcal{B}})$ simply applies $\Sigma_{\mathcal{A},\mathcal{B}}$ to all components of the input. On the other hand,

$$\mathrm{T}((M;\mu_\bullet)\otimes(N;\nu_\bullet)) = \Pi^{mn}(N;\nu_\bullet)\otimes(M;\mu_\bullet).$$

Applying $i_{\mathcal{B},\mathcal{A}}$ to this yields

$$\Pi^{mn}(N\otimes M;\nu_\bullet,\mu_\bullet).$$

We have a natural even-degree isomorphism

$$(N\otimes M;\mu_\bullet,\nu_\bullet)\to\Pi^{mn}(N\otimes M;\nu_\bullet,\mu_\bullet)$$

coming from Proposition 8.3 using the element $\tilde{\tau}_{m,n}^{-1}$ defined in Section 5.3; note that $\tilde{\tau}_{m,n}^{-1}$ has parity mn. □

We know that Clifford eversion is an equivalence of 2-categories $\mathscr{C}\to\mathscr{C}$. The main content of Theorem 8.16 is that it admits a symmetric monoidal structure. A definition of braided monoidal 2-functor is given in [Baez and Neuchl 1996, Section 4], though this definition is not the most general possible, and only concerns braidings (not symmetries or syllepses). The above lemmas define much of the structure required for Cl to be a braided monoidal 2-functor (though we have not discussed the modification α in [ibid., Definition 16]). To complete the proof that Cl is braided monoidal, we would need to verify the quite complicated conditions in [loc. cit.]. We leave this (together with the problem of upgrading from braided monoidal to symmetric monoidal) as an open problem. We note that the particular factor systems appearing in Theorem 8.16 have not appeared in the above discussion; however, these factor systems must be the correct ones due to Proposition 8.9: precisely, if $\mathcal{A}$ is a symmetric monoidal object in $(\mathscr{C},\Sigma,\omega)$, i.e., a symmetric monoidal supercategory with respect to ω, then, by Proposition 8.9, $\mathrm{Cl}(\mathcal{A})$ is a type II supersymmetric monoidal supercategory with respect to $\mathtt{AD}\omega$, and therefore (can be converted to) a type I supersymmetric monoidal supercategory with respect to $\mathtt{A}\omega$ (via Proposition 3.34), i.e., a symmetric monoidal object of $(\mathscr{C},\mathrm{T},\mathtt{A}\omega)$.

8.8. ***Grothendieck groups and eversion.*** Fix a Cl_1-module $(\boldsymbol{k}^{1|1},\xi)$ in **SVec**. Explicitly, let e_0 be a basis vector for $\boldsymbol{k}^{1|0}$ and let e_1 be a basis vector for $\boldsymbol{k}^{0|1}$, then let ξ be the map given by $e_0\mapsto e_1$ and $e_1\mapsto 2e_0$.

Let $\mathcal{A}$ be a nice supercategory. For an object M of $\mathcal{A}$, let $F(M)$ be the Cl_1-module $(M\otimes\boldsymbol{k}^{1|1},1\otimes\xi)$ in $\mathcal{A}$; this defines a functor $F:\mathcal{A}\to\mathrm{Cl}_1(\mathcal{A})$. Let $G:\mathrm{Cl}_1(\mathcal{A})\to\mathcal{A}$ be the forgetful functor, i.e., $G(M,\alpha)=M$.

Proposition 8.19. *We have natural even isomorphisms* $G\circ F=\mathrm{id}\oplus\Pi$ *and* $F\circ G=\mathrm{id}\oplus\Pi$.

Proof. The statement for $G\circ F$ is clear. Suppose that (M,α) is a Cl_1-module in $\mathcal{A}$. Then $M\otimes\boldsymbol{k}^{1|1}$ is naturally a Cl_2-module via $\alpha=\alpha\otimes 1$ and $\xi=1\otimes\xi$. One easily verifies that $2-\xi\alpha$ defines an even isomorphism $(M\otimes\boldsymbol{k}^{1|1},\alpha)\to(M\otimes\boldsymbol{k}^{1|1},\xi)$ of Cl_1-modules; this is a variant of Proposition 8.3. Since $(M\otimes\boldsymbol{k}^{1|1},\alpha)$ is naturally isomorphic to $(M,\alpha)\oplus\Pi(M,\alpha)$ in $\mathrm{Cl}_1(\mathcal{A})$ by an even-degree isomorphism, the result follows. □

Now suppose that $\mathcal{A}$ is equipped with an exact structure. Then $\mathrm{Cl}_n(\mathcal{A})$ has a natural exact structure as well.

Proposition 8.20. *The forgetful functor* $\mathrm{Cl}_1(\mathcal{A}) \to \mathcal{A}$ *induces an isomorphism*

$$\mathrm{K}_+(\mathrm{Cl}_1(\mathcal{A}))\big[\tfrac{1}{2}\big] \to \mathrm{K}_+(\mathcal{A})\big[\tfrac{1}{2}\big].$$

Proof. This follows from Proposition 8.19 since $\mathrm{id} \oplus \Pi$ induces multiplication by 2 on K_+. □

Now suppose that $\mathcal{A}$ is a $\mathbb{Z}/2$-supercategory with an exact structure and that $\zeta_4 \in \boldsymbol{k}$. We take $p = 2$ in Clifford periodicity, so that the Clifford eversion $\mathrm{Cl}(\mathcal{A})$ is also a $\mathbb{Z}/2$-supercategory. We define a $\mathbb{Z}\big[\frac{1}{\sqrt{2}}\big]$-linear map

$$i : \mathrm{K}_+(\mathrm{Cl}(\mathcal{A})) \otimes \mathbb{Z}\big[\tfrac{1}{\sqrt{2}}\big] \to \mathrm{K}_+(\mathcal{A}) \otimes \mathbb{Z}\big[\tfrac{1}{\sqrt{2}}\big]$$

as follows. If A is an object of $\mathrm{Cl}(\mathcal{A})_0 = \mathcal{A}_0$, then $i([A]) = [A]$. If (A, α) is an object of $\mathrm{Cl}_1(\mathcal{A}) = \mathrm{Cl}_1(\mathcal{A}_1)$, then we define $i\big([(A, \alpha)]\big) = \frac{1}{\sqrt{2}}[A]$. Thus, on the odd piece we are just scaling the map induced by the forgetful functor by $\frac{1}{\sqrt{2}}$. It follows from Proposition 8.20 that i is an isomorphism of abelian groups. The following proposition explains the reason for the $\sqrt{2}$ factor:

Proposition 8.21. *If* $\mathcal{A}$ *has a monoidal structure, then* i *is a ring isomorphism.*

Proof. Let A, A' be objects of $\mathcal{A}_0$ and let (M, α), (M', α') be objects of $\mathrm{Cl}_1(\mathcal{A}_1)$. Then $A \otimes A'$ is the same when computed in $\mathrm{Cl}(\mathcal{A})$ or $\mathcal{A}$, so $i([A][A']) = i([A])i([A'])$. Similarly, $A \otimes (M, \alpha) = (A \otimes M, 1 \otimes \alpha)$, so again $i\big([A][(M, \alpha)]\big) = i([A])i\big([(M, \alpha)]\big)$.

Finally, $(M, \alpha) \otimes (M', \alpha')$ is identified via $\mathrm{Cl}_2(\mathcal{A}) \cong \mathcal{A}$ with the ζ_4-eigenspace N of $\alpha \otimes \alpha'$. The ζ_4- and $(-\zeta_4)$-eigenspaces of $M \otimes M'$ are interchanged by $\alpha \otimes 1$, and hence $(M, \alpha) \otimes (M', \alpha') \cong N \oplus \Pi(N)$, so

$$i\big([(M, \alpha)][(M', \alpha')]\big) = [N] = \tfrac{1}{2}[M][M'] = i\big([(M, \alpha)]\big)i\big([(M', \alpha')]\big).$$ □

9. Schur–Sergeev duality

We work over $\boldsymbol{k} = \mathbb{C}$ *in this section for simplicity.*

9.1. ***Polynomial representations.*** For an integer $n \geq 0$, we let $\mathfrak{q}(n) \subset \mathfrak{gl}(n|n)$ denote the super-Lie subalgebra of matrices of the form $\left[\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right]$ where A, B are $n \times n$ matrices. This is the *isomeric Lie superalgebra*. Alternatively, if V_n is a vector space of dimension n, $\mathfrak{q}(V_n)$ is the commutator of the Cl_1-module structure on $V_{n|n} = V_n \otimes \boldsymbol{k}^{1|1}$ in $\mathfrak{gl}(V_{n|n})$. There are natural embeddings $V_{n|n} \to V_{n+1|n+1}$ which induce embeddings $\mathfrak{q}(n) \to \mathfrak{q}(n+1)$, and we define $\mathfrak{q}(\infty)$ to be the direct limit.

Let $\boldsymbol{V} = \varinjlim_n V_{n|n}$; this is the *standard representation* of $\mathfrak{q}(\infty)$. The tensor powers $V_{n|n}^{\otimes d}$ are semisimple, and the centralizer of $\mathfrak{q}(n)$ is the Hecke–Clifford algebra $\mathcal{H}_d$ (defined below) when $n \gg d$ [Wan and Wang 2012, Theorem 4.7]. The inclusion maps $V_{n|n}^{\otimes d} \to V_{n+1|n+1}^{\otimes d}$ are $\mathcal{H}_d$-linear and hence the tensor powers $\boldsymbol{V}^{\otimes d}$ are semisimple. The simple constituents of this representation are naturally indexed by strict partitions of d; we let L_λ denote the simple corresponding to the partition λ [ibid., Theorem 4.8]. We say that a representation of $\mathfrak{q}(\infty)$ is *polynomial* if it is a direct sum of L_λ's. The category of polynomial representations is denoted by $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$. It is naturally a symmetric monoidal $\mathbb{Z}$-supercategory. The object L_λ is homogeneous of degree $|\lambda|$.

Remark 9.1. Even though $\Pi(\boldsymbol{V}) \cong \boldsymbol{V}$, we do not have $\Pi(V) \cong V$ for all objects V in $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$. More precisely, the endomorphism algebra of the irreducible L_λ, where λ is a strict partition, is Cl_1 if and only if $\ell(\lambda)$ is odd [Cheng and Wang 2012, Lemma 1.40]. □

9.2. ***The Hecke–Clifford algebra.*** The symmetric group $\mathfrak{S}_n$ acts on the algebra Cl_n by $g \cdot \alpha_i = \alpha_{g(i)}$. The *Hecke–Clifford algebra* is $\mathcal{H}_n = \mathbb{C}[\mathfrak{S}_n] \ltimes \mathrm{Cl}_n$. This is a superalgebra via the grading $\deg(g) = 0$ for $g \in \mathfrak{S}_n$ and $\deg(\alpha_i) = 1$. We define a category $\mathrm{Rep}(\mathcal{H}_*)$ whose objects are sequences $(V_n)_{n\geq 0}$ where V_n is a finite-dimensional super-vector space with a homogeneous action of $\mathcal{H}_n$ such that $V_n = 0$ for $n \gg 0$. We define $\Pi : \mathrm{Rep}(\mathcal{H}_*) \to \mathrm{Rep}(\mathcal{H}_*)$ by $(\Pi V)_n = \Pi(V_n)$. There is a natural identification of $\mathcal{H}_{n,m} = \mathcal{H}_n \otimes \mathcal{H}_m$ with a subalgebra of $\mathcal{H}_{n+m}$, and we define a monoidal structure on $\mathrm{Rep}(\mathcal{H}_*)$ via

$$(V \otimes W)_n = \bigoplus_{i=0}^{n} \mathcal{H}_n \otimes_{\mathcal{H}_{i,n-i}} (V_i \boxtimes W_{n-i}).$$

We define a symmetry β' on this tensor product by

$$a \otimes v \otimes w \mapsto (-1)^{|v||w|} a\tau_{i,n-i}^{-1} \otimes w \otimes v,$$

where $a \in \mathcal{H}_n$, $v \in V_i$ and $w \in W_{n-i}$. We omit the straightforward check that this is well defined.

The Hecke–Clifford algebra is closely related to the isomeric superalgebra. Let $\xi : \boldsymbol{V} \to \boldsymbol{V}$ be the given odd endomorphism. The algebra Cl_n has a right action on $\boldsymbol{V}^{\otimes n}$ via

$$(v_1 \otimes \cdots \otimes v_n)\alpha_i = (-1)^{|v_{i+1}|+\cdots+|v_n|}(v_1 \otimes \cdots \otimes \xi(v_i) \otimes \cdots \otimes v_n),$$

where the v_i are homogeneous elements of $\boldsymbol{V}$. The symmetric group $\mathfrak{S}_n$ also has a right action on $\boldsymbol{V}^{\otimes n}$, in the usual manner. Together, these define a right action of the Hecke–Clifford algebra $\mathcal{H}_n$ on $\boldsymbol{V}^{\otimes n}$. The following is a typical formulation of Schur–Sergeev duality:

Proposition 9.2. *The functor* $\mathrm{Rep}(\mathcal{H}_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$ *given by* $(N_n)_n \mapsto \bigoplus_n \boldsymbol{V}^{\otimes n} \otimes_{\mathcal{H}_n} N_n$ *is an equivalence of symmetric monoidal* $\mathbb{Z}$*-supercategories.*

Proof. Let Φ denote the functor under discussion. The explicit decomposition of $\boldsymbol{V}^{\otimes n}$ given in [Wan and Wang 2012, Theorem 4.8] shows that Φ is an equivalence of abelian categories. Let M and N be $\mathcal{H}_m$- and $\mathcal{H}_n$-modules, respectively. Then

$$\begin{aligned}
\Phi(M \otimes N) &= \boldsymbol{V}^{\otimes(m+n)} \otimes_{\mathcal{H}_{m+n}} (\mathcal{H}_{m+n} \otimes_{\mathcal{H}_{m,n}} (M \otimes N)) \\
&\cong (\boldsymbol{V}^{\otimes m} \otimes \boldsymbol{V}^{\otimes n}) \otimes_{\mathcal{H}_{m,n}} (M \otimes N) \\
&\cong (\boldsymbol{V}^{\otimes m} \otimes_{\mathcal{H}_m} M) \otimes (\boldsymbol{V}^{\otimes n} \otimes_{\mathcal{H}_n} N) \\
&= \Phi(M) \otimes \Phi(N),
\end{aligned}$$

which shows that Φ is monoidal. Finally, we verify the symmetry. Let β be the symmetry on $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$. Pick $v \otimes x \in \Phi(M)$ and $v' \otimes y \in \Phi(N)$, where $x \in M$, $y \in N$, $v \in \boldsymbol{V}^{\otimes m}$ and $v' \in \boldsymbol{V}^{\otimes n}$ are homogeneous.

Then

$$\begin{aligned}\beta((v\otimes x)\otimes(v'\otimes y)) &= (-1)^{(|v|+|x|)(|v'|+|y|)}(v'\otimes y)\otimes(v\otimes x)\\ &= (-1)^{|v||v'|+|x||v'|+|x||y|}(v'\otimes v)\otimes(y\otimes x).\end{aligned}$$

On the other hand, under the identification $\Phi(M)\otimes\Phi(N)\cong\Phi(M\otimes N)$, $(v\otimes x)\otimes(v'\otimes y)$ maps to $(-1)^{|v'||x|}(v\otimes v')\otimes(1\otimes x\otimes y)$, where 1 is the unit of $\mathcal{H}_{m+n}$. If we apply the symmetry β' in $\mathrm{Rep}(\mathcal{H}_*)$ to the second factor, then

$$(-1)^{|v'||x|+|x||y|}(v\otimes v')\otimes(\tau_{m,n}^{-1}\otimes y\otimes x) = (-1)^{|v'||x|+|x||y|+|v||v'|}(v'\otimes v)\otimes(1\otimes y\otimes x),$$

which agrees with the previous expression, so Φ is symmetric. □

9.3. ***Sergeev–Schur duality.*** Let β be the type II supersymmetry on $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ obtained from the type I supersymmetry described in Section 6.6 by the general process in Section 3.7. Thus, explicitly, if M is a representation of $\widetilde{\mathfrak{S}}_m$ and N is a representation of $\widetilde{\mathfrak{S}}_n$, then

$$\beta_{M,N}:\mathrm{Ind}_{\widetilde{\mathfrak{S}}_m\rtimes\widetilde{\mathfrak{S}}_n}^{\widetilde{\mathfrak{S}}_{n+m}}(M\otimes N)\to\mathrm{Ind}_{\widetilde{\mathfrak{S}}_n\rtimes\widetilde{\mathfrak{S}}_m}^{\widetilde{\mathfrak{S}}_{n+m}}(N\otimes M)$$

is given by

$$g\otimes v\otimes w\mapsto(-1)^{|v||w|+mn|g|+mn}g\tilde{\tau}_{m,n}^{-1}\otimes w\otimes v.$$

By results in the aforementioned sections, β is a type II supersymmetry with respect to the factor system $\mathtt{DA}$. Define β^* by

$$\beta^*_{M,N}=(-1)^{\binom{mn}{2}}\zeta_4^{mn}\beta_{M,N},$$

where $m=|M|$ and $n=|N|$. Note that $(-1)^{\binom{mn}{2}}\zeta_4^{mn}=\mathtt{d}(m,n)$, where $\mathtt{d}$ is as defined in Example 3.14, and so β^* is a type II supersymmetry with respect to $\mathtt{A}$.

The following is our reformulation of Sergeev–Schur duality, and the main result of this section:

Theorem 9.3. *We have a natural equivalence of symmetric monoidal $\mathbb{Z}$-supercategories*

$$\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})\cong\mathrm{Cl}(\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)),$$

where here $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ *is equipped with the supersymmetry* β^*.

The proof will take the rest of the section. We omit the proof of the following fact, which is a straightforward calculation:

Lemma 9.4. *We have an isomorphism of superalgebras*

$$\mathbb{C}[\widetilde{\mathfrak{S}}_n]/(c+1)\otimes\mathrm{Cl}_n\to\mathcal{H}_n$$

given by $1\otimes\alpha_i\mapsto\alpha_i$ *and* $\tilde{s}_j\otimes 1\mapsto\frac{1}{2}\zeta_4 s_j(\alpha_j-\alpha_{j+1})$.

Proposition 9.5. *We have an equivalence of symmetric monoidal $\mathbb{Z}$-supercategories*

$$\mathrm{Cl}(\mathrm{Rep}(\widetilde{\mathfrak{S}}_*))\cong\mathrm{Rep}(\mathcal{H}_*).$$

Proof. Via the isomorphism in Lemma 9.4, an $\mathcal{H}_n$-module M is the same thing as a Cl_n-module in $\mathrm{Rep}(\widetilde{\mathfrak{S}}_n)$. This gives an equivalence $\Phi : \mathrm{Rep}(\mathcal{H}_*) \to \mathrm{Cl}(\mathrm{Rep}(\widetilde{\mathfrak{S}}_*))$ which preserves the $\mathbb{Z}$-grading. We claim that this is a monoidal equivalence. Let M be an $\mathcal{H}_m$-module and let N be an $\mathcal{H}_n$-module. We have $\mathrm{Cl}_m \otimes \mathrm{Cl}_n \cong \mathrm{Cl}_{m+n}$, so $\mathcal{H}_{m+n} \otimes_{\mathcal{H}_{m,n}} (M \otimes N)$ is isomorphic, as a vector space, to $\mathrm{Ind}_{\widetilde{\mathfrak{S}}_m \wr \widetilde{\mathfrak{S}}_n}^{\widetilde{\mathfrak{S}}_{m+n}} (M \otimes N)$. The action of Cl_m is via M and the action of Cl_n is via N, so that the action of Cl_{m+n} is via the tensor product action, and hence Φ is a monoidal equivalence.

We now check symmetry. We have homomorphisms

$$\varphi : \widetilde{\mathfrak{S}}_{n+m} \to \mathcal{H}_{n+m}, \quad \tilde{s}_i \mapsto \tfrac{1}{2}\zeta_4 s_i(\alpha_{i+1} - \alpha_i),$$
$$\psi : \widetilde{\mathfrak{S}}_{n+m} \to \mathcal{H}_{n+m}, \quad \tilde{s}_i \mapsto \tfrac{1}{2}(\alpha_{i+1} - \alpha_i).$$

The existence of ψ comes from Proposition 8.1, while the existence of φ comes from Lemma 9.4. Observe that $\varphi(\tilde{s}_i)$ skew-commutes with α_j, for all i and j, and so $\varphi(\tilde{s}_i)$ and $\psi(\tilde{s}_j)$ skew-commute for all i and j. We also have $\varphi(\tilde{s}_i) = \zeta_4 s_i \psi(\tilde{s}_i)$, and so $\varphi(\tilde{s}_i)\psi(\tilde{s}_i) = \zeta_4 s_i$. It follows that

$$\tau_{m,n} = (-1)^{\binom{mn}{2}} \zeta_4^{mn} \varphi(\tilde{\tau}_{m,n}) \psi(\tilde{\tau}_{m,n}).$$

The vector space isomorphism

$$\mathrm{Ind}_{\widetilde{\mathfrak{S}}_m \wr \widetilde{\mathfrak{S}}_n}^{\widetilde{\mathfrak{S}}_{m+n}} (M \otimes N) \to \mathcal{H}_{m+n} \otimes_{\mathcal{H}_{m,n}} (M \otimes N)$$

is given by $g \otimes v \otimes w \mapsto \varphi(g) \otimes v \otimes w$. We thus have

$$\begin{aligned}
\beta'_{M,N}(\varphi(g) \otimes v \otimes w) &= (-1)^{|v||w|} \varphi(g) \tau_{m,n}^{-1} \otimes w \otimes v \\
&= (-1)^{|v||w| + \binom{mn}{2}} \zeta_4^{-mn} \varphi(g) \psi(\tilde{\tau}_{m,n})^{-1} \varphi(\tilde{\tau}_{m,n})^{-1} \otimes w \otimes v \\
&= (-1)^{|v||w| + \binom{mn}{2} + mn|g|} \zeta_4^{-mn} \psi(\tilde{\tau}_{m,n})^{-1} \varphi(g \tilde{\tau}_{m,n}^{-1}) \otimes w \otimes v \\
&= \psi(\tilde{\tau}_{m,n})^{-1} \beta^*(g \otimes v \otimes w).
\end{aligned}$$

The expression on the final line is exactly the symmetry on $\mathrm{Cl}(\mathrm{Rep}(\widetilde{\mathfrak{S}}_*))$, and so the result follows. □

Theorem 9.3 now follows from Propositions 9.5 and 9.2.

9.4. ***Symmetric functions.*** In this section, $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ denotes an integer partition, $\ell(\lambda)$ denotes the number of nonzero entries of λ, and $|\lambda| = \sum_i \lambda_i$.

Let Λ be the ring of symmetric functions in an alphabet $x_1, x_2, \ldots$. Define symmetric functions $q_k(x)$ by the identity

$$\sum_{k \geq 0} q_k(x) t^k = \prod_{i \geq 1} \frac{1 + x_i t}{1 - x_i t}.$$

Let Γ be the subring of Λ generated by $q_1, q_2, \ldots$. They satisfy the relations, for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i q_i q_{n-i} = 0.$$

So $\Gamma \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ is generated by $q_1, q_3, q_5, \ldots$, which are algebraically independent. For a, b, define

$$Q_{(a,b)} = q_a q_b + 2\sum_{i=1}^{b} (-1)^i q_{a+i} q_{b-i}$$

(with the convention that $q_i = 0$ if $i < 0$). Note that $Q_{(b,a)} = -Q_{(a,b)}$. Let $\lambda = (\lambda_1 > \cdots > \lambda_{2r} \geq 0)$ be a strict partition (here $\ell(\lambda) = 2r$ if $\lambda_{2r} > 0$ and otherwise $\ell(\lambda) = 2r - 1$); we define

$$Q_\lambda = \mathrm{Pf}(Q_{(\lambda_i, \lambda_j)})_{1 \leq i,j \leq 2r},$$

where Pf denotes the Pfaffian of a skew-symmetric matrix. The Q_λ form a $\mathbb{Z}$-basis for Γ [Macdonald 1995, (8.9)] and $\Gamma \otimes \mathbb{Z}\left[\frac{1}{\sqrt{2}}\right]$ is a model for the Grothendieck groups $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$, $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$ and $\mathrm{Rep}(\mathcal{H}_*)$ (in each case, the ring structure comes from the monoidal product) as we now explain. In particular, in each case, the simple objects are indexed by strict partitions.

For $\mathrm{Rep}^{\mathrm{pol}}(\mathfrak{q})$ and $\mathrm{Rep}(\mathcal{H}_*)$, the classes of the simple objects L_λ are $2^{-\lfloor \ell(\lambda)/2 \rfloor} Q_\lambda$ [Cheng and Wang 2012, Theorem 3.48], and, for $\mathrm{Rep}(\widetilde{\mathfrak{S}}_*)$, the classes of the simple objects $\mathbb{N}_\lambda$ are $2^{(\varepsilon(\lambda) - \ell(\lambda))/2} Q_\lambda$, where $\varepsilon(\lambda) \in \{0, 1\}$ is the reduction modulo 2 of $|\lambda| - \ell(\lambda)$ [Hoffman and Humphreys 1992, Theorem 8.6].

In particular, when $|\lambda|$ is even, $[L_\lambda] = [\mathbb{N}_\lambda]$, and, when $|\lambda|$ is odd,

$$[L_\lambda] = \begin{cases} \frac{1}{\sqrt{2}}[\mathbb{N}_\lambda] & \text{if } \ell(\lambda) \text{ is even,} \\ \sqrt{2}[\mathbb{N}_\lambda] & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

Recall from Proposition 9.5 that we have an equivalence $\mathrm{Rep}(\mathcal{H}_n) \simeq \mathrm{Cl}_1(\mathrm{Rep}(\widetilde{\mathfrak{S}}_n))$ when n is odd. Hence, from Proposition 8.21, we have an isomorphism

$$\mathrm{K}_+\big(\mathrm{Cl}_1(\mathrm{Rep}(\widetilde{\mathfrak{S}}_n))\big) \otimes \mathbb{Z}\left[\tfrac{1}{\sqrt{2}}\right] \to \mathrm{K}_+(\mathrm{Rep}(\widetilde{\mathfrak{S}}_n)) \otimes \mathbb{Z}\left[\tfrac{1}{\sqrt{2}}\right], \quad [(A, \alpha)] \mapsto \tfrac{1}{\sqrt{2}}[A].$$

To match this with the Q-functions, we note that $\mathbb{N}_\lambda$ is an isomeric object if and only if $|\lambda| - \ell(\lambda)$ is odd [Józefiak 1991, Corollary 4.19]. Hence, when $|\lambda|$ is odd, the simple objects of $\mathrm{Cl}_1(\mathrm{Rep}(\widetilde{\mathfrak{S}}_n))$ are

$$\{\mathbb{N}_\lambda \mid \ell(\lambda) \text{ is even}\} \cup \{\mathbb{N}_\lambda \otimes \boldsymbol{k}^{1|1} \mid \ell(\lambda) \text{ is odd}\}$$

and the correspondence becomes

$$[L_\lambda] \mapsto \begin{cases} \frac{1}{\sqrt{2}}[\mathbb{N}_\lambda] & \text{if } \ell(\lambda) \text{ is even,} \\ \frac{1}{\sqrt{2}}[\mathbb{N}_\lambda \otimes \boldsymbol{k}^{1|1}] & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

Appendix: Bott periodicity

Let $\boldsymbol{k}$ be a field of characteristic different from 2. In this paper, the Clifford algebra Cl_k is the $\boldsymbol{k}$-superalgebra generated by odd elements $\alpha_1, \ldots, \alpha_k$ subject to the relations $\alpha_i^2 = 2$ for all i and $\alpha_i \alpha_j = -\alpha_j \alpha_i$ for all $i \neq j$. Our goal here is to establish a version of Bott periodicity (Theorem A.4) for these Clifford algebras. While this is well known, we have provided the details since most references work over $\boldsymbol{k} = \mathbb{R}$ and our definition of the algebra is not standard. Here we follow the proof in [Trimble 2005].

Given a superalgebra A, the opposite superalgebra A^{op} has the same underlying vector space but multiplication of homogeneous elements is given by $a \cdot_{\mathrm{op}} b = (-1)^{|a||b|} b \cdot a$. In particular, the presentation of $\mathrm{Cl}_k^{\mathrm{op}}$ is the same as Cl_k except that the α_i square to -2 instead of 2. Also, note that $\mathrm{Cl}_k \otimes \mathrm{Cl}_\ell \cong \mathrm{Cl}_{k+\ell}$, where we emphasize that we use the tensor product of superalgebras, so that, on homogeneous elements, $(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{|x_2||y_1|} x_1 x_2 \otimes y_1 y_2$.

Lemma A.1. *We have an isomorphism of superalgebras* $\mathrm{Cl}_1 \otimes \mathrm{Cl}_1^{\mathrm{op}} \cong \mathrm{End}(\boldsymbol{k}^{1|1})$.

Proof. Let e_0, e_1 be a homogeneous basis for $\boldsymbol{k}^{1|1}$. We represent $\alpha_1 \otimes 1$ by the operator $e_0 \mapsto e_1$, $e_1 \mapsto 2e_0$, and $1 \otimes \alpha_1$ by the operator $e_0 \mapsto e_1$, $e_1 \mapsto -2e_0$. □

Let H denote the quaternion algebra concentrated in degree 0, i.e., H is the associative $\boldsymbol{k}$-algebra generated by even elements $\boldsymbol{i}$ and $\boldsymbol{j}$ such that $\boldsymbol{ij} = -\boldsymbol{ji}$ and $\boldsymbol{i}^2 = \boldsymbol{j}^2 = -1$.

Lemma A.2. *We have an isomorphism of superalgebras* $\mathrm{Cl}_3 \cong H \otimes \mathrm{Cl}_1^{\mathrm{op}}$.

Proof. We define $H \otimes \mathrm{Cl}_1^{\mathrm{op}} \to \mathrm{Cl}_3$ by $\boldsymbol{i} \otimes 1 \mapsto \frac{1}{2}\alpha_1\alpha_2$, $\boldsymbol{j} \otimes 1 \mapsto \frac{1}{2}\alpha_1\alpha_3$ and $1 \otimes \alpha_1 \mapsto \frac{1}{2}\alpha_1\alpha_2\alpha_3$. It is readily verified to be surjective and both algebras have dimension 8 and hence it is an isomorphism. □

Lemma A.3. *We have an isomorphism of superalgebras* $\mathrm{Cl}_4 \cong \mathrm{Cl}_4^{\mathrm{op}}$.

Proof. By Lemmas A.1 and A.2,

$$\mathrm{Cl}_4 \cong \mathrm{Cl}_3 \otimes \mathrm{Cl}_1 \cong H \otimes \mathrm{Cl}_1^{\mathrm{op}} \otimes \mathrm{Cl}_1 \cong H \otimes \mathrm{End}(\boldsymbol{k}^{1|1}).$$

Next, $H \cong H^{\mathrm{op}}$ via $\boldsymbol{i} \mapsto -\boldsymbol{i}$ and $\boldsymbol{j} \mapsto -\boldsymbol{j}$ and $\mathrm{End}(\boldsymbol{k}^{1|1}) \cong \mathrm{End}(\boldsymbol{k}^{1|1})^{\mathrm{op}}$ via $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a & -c \\ b & d \end{smallmatrix}\right]$, where the matrices are given with respect to a homogeneous basis of $\boldsymbol{k}^{1|1}$. □

Theorem A.4 (Bott periodicity). *We have an isomorphism of superalgebras* $\mathrm{Cl}_8 \cong \mathrm{End}(\boldsymbol{k}^{8|8})$.

Furthermore, if $\boldsymbol{k}$ *contains a primitive fourth root of unity* ζ_4, *then* $\mathrm{Cl}_2 \cong \mathrm{End}(\boldsymbol{k}^{1|1})$.

Proof. By Lemmas A.1 and A.3,

$$\mathrm{Cl}_8 \cong \mathrm{Cl}_4 \otimes \mathrm{Cl}_4 \cong \mathrm{Cl}_4 \otimes \mathrm{Cl}_4^{\mathrm{op}} \cong \mathrm{End}(\boldsymbol{k}^{1|1})^{\otimes 4} \cong \mathrm{End}(\boldsymbol{k}^{8|8}).$$

The second statement follows from Lemma A.1 since $\mathrm{Cl}_1 \cong \mathrm{Cl}_1^{\mathrm{op}}$ via $\alpha_1 \mapsto \zeta_4\alpha_1$. □

The periodicity comes when we consider Morita equivalence. Since $\mathrm{End}(\boldsymbol{k}^{m|n})$ is a simple superalgebra for any m, n, its category of (super)modules is equivalent to the category of super-vector spaces, and hence $\mathrm{Mod}_{\mathrm{Cl}_{k+8}} \simeq \mathrm{Mod}_{\mathrm{Cl}_k}$ for any k and $\mathrm{Mod}_{\mathrm{Cl}_{k+2}} \simeq \mathrm{Mod}_{\mathrm{Cl}_k}$ when $\zeta_4 \in \boldsymbol{k}$.

References

[Baez and Neuchl 1996] J. C. Baez and M. Neuchl, "Higher-dimensional algebra, I: Braided monoidal 2-categories", *Adv. Math.* **121**:2 (1996), 196–244. MR

[Brundan and Ellis 2017] J. Brundan and A. P. Ellis, "Monoidal supercategories", *Comm. Math. Phys.* **351**:3 (2017), 1045–1089. MR

[Cheng and Wang 2012] S.-J. Cheng and W. Wang, *Dualities and representations of Lie superalgebras*, Graduate Studies in Math. **144**, Amer. Math. Soc., Providence, RI, 2012. MR

[Crans 1998] S. E. Crans, "Generalized centers of braided and sylleptic monoidal 2-categories", *Adv. Math.* **136**:2 (1998), 183–223. MR

[Ellis and Lauda 2016] A. P. Ellis and A. D. Lauda, "An odd categorification of $U_q(\mathfrak{sl}_2)$", *Quantum Topol.* **7**:2 (2016), 329–433. MR

[Ganter and Kapranov 2014] N. Ganter and M. Kapranov, "Symmetric and exterior powers of categories", *Transform. Groups* **19**:1 (2014), 57–103. MR

[Hoffman and Humphreys 1992] P. N. Hoffman and J. F. Humphreys, *Projective representations of the symmetric groups: Q-functions and shifted tableaux*, Clarendon, New York, 1992. MR

[Józefiak 1991] T. Józefiak, "Schur Q-functions and cohomology of isotropic Grassmannians", *Math. Proc. Cambridge Philos. Soc.* **109**:3 (1991), 471–478. MR

[Kang et al. 2016] S.-J. Kang, M. Kashiwara, and S. Tsuchioka, "Quiver Hecke superalgebras", *J. Reine Angew. Math.* **711** (2016), 1–54. MR

[Kapranov 2018] M. Kapranov, "Supergeometry in mathematics and physics", preprint, 2018. arXiv 1512.07042v2

[Kapranov and Voevodsky 1994] M. M. Kapranov and V. A. Voevodsky, "2-categories and Zamolodchikov tetrahedra equations", pp. 177–259 in *Algebraic groups and their generalizations: quantum and infinite-dimensional methods* (University Park, PA, 1991), edited by W. J. Haboush and B. J. Parshall, Proc. Sympos. Pure Math. **56.2**, Amer. Math. Soc., Providence, RI, 1994. MR

[Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Clarendon, New York, 1995. MR

[MacLane 1952] S. MacLane, "Cohomology theory of abelian groups", pp. 8–14 in *Proceedings of the International Congress of Mathematicians* (Cambridge, MA, 1950), vol. II, edited by L. M. Graves et al., Amer. Math. Soc., Providence, RI, 1952. MR

[Trimble 2005] T. Trimble, "The super Brauer group and super division algebras", webpage, 2005, available at http://math.ucr.edu/home/baez/trimble/superdivision.html.

[Wan and Wang 2012] J. Wan and W. Wang, "Lectures on spin representation theory of symmetric groups", *Bull. Inst. Math. Acad. Sin. (N.S.)* **7**:1 (2012), 91–164. MR

Communicated by Antoine Chambert-Loir
Received 2021-04-15 Revised 2024-04-05 Accepted 2025-08-05

ssam@ucsd.edu *Department of Mathematics, University of California, San Diego, La Jolla, CA, United States*
http://math.ucsd.edu/~ssam/

asnowden@umich.edu *Department of Mathematics, University of Michigan, Ann Arbor, MI, United States*
http://www-personal.umich.edu/~asnowden/

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR

Antoine Chambert-Loir
Université Paris-Diderot
France

EDITORIAL BOARD CHAIR

David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Jason P. Bell	University of Waterloo, Canada	Philippe Michel	École Polytechnique Fédérale de Lausanne
Olivier Benoist	Ecole Normale Supérieure, France	Martin Olsson	University of California, Berkeley, USA
Bhargav Bhatt	University of Michigan, USA	Irena Peeva	Cornell University, USA
Frank Calegari	University of Chicago, USA	Jonathan Pila	University of Oxford, UK
J-L. Colliot-Thélène	CNRS, Université Paris-Saclay, France	Anand Pillay	University of Notre Dame, USA
Brian D. Conrad	Stanford University, USA	Bjorn Poonen	Massachusetts Institute of Technology, USA
Samit Dasgupta	Duke University, USA	Victor Reiner	University of Minnesota, USA
Hélène Esnault	Freie Universität Berlin, Germany	Peter Sarnak	Princeton University, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Benjamin Schraen	Université Claude Bernard Lyon 1, France
Sergey Fomin	University of Michigan, USA	Michael Singer	North Carolina State University, USA
Edward Frenkel	University of California, Berkeley, USA	Vasudevan Srinivas	SUNY Buffalo, USA
Wee Teck Gan	National University of Singapore	Shunsuke Takagi	University of Tokyo, Japan
Andrew Granville	Université de Montréal, Canada	Pham Huu Tiep	Rutgers University, USA
Ben J. Green	University of Oxford, UK	Ravi Vakil	Stanford University, USA
Christopher Hacon	University of Utah, USA	Akshay Venkatesh	Institute for Advanced Study, USA
Roger Heath-Brown	Oxford University, UK	Melanie Matchett Wood	Harvard University, USA
János Kollár	Princeton University, USA	Shou-Wu Zhang	Princeton University, USA
Michael J. Larsen	Indiana University Bloomington, USA		

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2026 is US $590/year for the electronic version, and $865/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 2000 Allston Way #59, Berkeley, CA 94701-4004, is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2026 Mathematical Sciences Publishers

Algebra & Number Theory

Volume 20 No. 8 2026

Supersymmetric monoidal categories 1479
STEVEN V SAM and ANDREW SNOWDEN

Rational curves and the Hilbert property on Jacobian Kummer varieties 1543
DAMIÁN GVIRTZ-CHEN and ZHIZHONG HUANG

Homological regularities and concavities 1559
ELLEN KIRKMAN, ROBERT WON and JAMES J. ZHANG

Transcendence properties of the Artin–Hasse exponential modulo p 1597
JOE KRAMER-MILLER

Anticyclotomic Euler system over biquadratic fields 1615
KIM TUAN DO

Explicit Hecke descent for special cycles 1649
SYED WAQAR ALI SHAH