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Rational curves and the Hilbert property on Jacobian Kummer varieties

Damián Gvirtz-Chen and Zhizhong Huang

Let K be a finitely generated field of characteristic zero. A conjecture by Corvaja and Zannier predicts that smooth, projective, simply connected varieties over K with Zariski dense set of rational points have the Hilbert property. We confirm this conjecture for all Kummer surfaces associated to the Jacobian of a genus 2 curve over K, and in general for all Jacobian Kummer varieties associated to a hyperelliptic curve of genus ≥ 3 of odd degree over K.

1. Introduction

The aim of this article is to introduce the use of rational curves into the study of the Hilbert property for nonunirational varieties.

1.1. ***Background for the Hilbert property.*** Let X be a quasiprojective integral variety of dimension at least one over a field K. Recall after [Serre 2008, §3] (see also [Colliot-Thélène 2020, §2]) that a subset $A \subset X(K)$ is *thin* if there exists a K-morphism $f : Y \to X$ from a K-variety Y to X such that

- $A \subset f(Y(K))$;
- the fibre of f over the generic point of X is finite, and f does not have any rational sections.

Definition 1.1 [Colliot-Thélène and Sansuc 1987, p. 189]. We say that X has the *Hilbert property* (or X is *of Hilbert type*) if $X(K)$ is not a thin set.

The field K is called *Hilbertian* if the projective line $\mathbb{P}^1_K$ satisfies the Hilbert property. Not every base field is Hilbertian. The field of rational numbers, and, more generally, all number fields, are classically known to be Hilbertian by the Hilbert irreducibility theorem [1892] (see [Serre 2008, Theorem 3.4.3]). We will assume, unless explicitly stated to the contrary, that K is a finitely generated field of characteristic 0. These fields are known to be Hilbertian (see, e.g., [Lang 1983, Chapter 9; Fried and Jarden 2008, §13.4]).

It should be pointed out that Definition 1.1 looks different from the one introduced later in [Corvaja and Zannier 2017, §1.1, Definition; Corvaja et al. 2022, Definition 1.1]. We shall provide a proof (see Proposition 2.1 and Corollary 2.2 below) that all these definitions are actually equivalent.

It is clear from the definition that Zariski density of $X(K)$ in X is a necessary condition for the Hilbert property, and that the Hilbert property is a birational invariant. Moreover, if X satisfies the Hilbert property,

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then it is geometrically integral (see [Serre 2008, p. 20]). The Hilbert property is a specialisation of weak weak approximation (see [Serre 2008, §3.5, Corollary 3.5.4, Definition 3.5.6] and also [Colliot-Thélène 2020, §2 (AFF), Théorème 2.3]) and provides an approach to solving the inverse Galois problem (see for example [Serre 2008], notably Theorem 3.5.9).

There is a well-known topological obstruction to the Hilbert property which goes back to the Chevalley–Weil theorem. Namely, a projective variety X cannot have the Hilbert property if it is not algebraically simply connected (or, equivalently for normal X, if its fundamental group has a proper subgroup of finite index); see [Corvaja and Zannier 2017, Theorem 1.4]. They conjecture that the converse is true:

Conjecture 1.2 [Corvaja and Zannier 2017, §2, Question–Conjecture 1]. *Let X be a smooth, projective and algebraically simply connected variety over K. If $X(K)$ is Zariski dense, then X has the Hilbert property.*[1]

While there have been quite a number of results confirming the weak weak approximation property for unirational varieties (they are simply connected by [Serre 1959]), as conjectured by Colliot-Thélène (see [Colliot-Thélène and Sansuc 1987; Serre 2008, Conjecture 3.5.8]), progress towards the Hilbert property for intermediate type varieties has been very slow. In particular, Corvaja and Zannier [2017, Appendix 1] ask for a proof that the Hilbert property holds for Kummer surfaces with a Zariski dense set of rational points. (See [Harpaz and Skorobogatov 2016, p. 817] for a stronger conjecture of weak weak approximation on K3 surfaces and [Holmes and Pannekoek 2015] for a consequence on the rank of quadratic twists of abelian varieties.)

1.2. ***Main results and discussion.*** Let H be a smooth, projective hyperelliptic curve of genus $g \geq 2$ over a finitely generated field K of characteristic 0. Let $\mathrm{Jac}(H)$ be its Jacobian variety and let $\mathrm{Kum}(\mathrm{Jac}(H))$ be the associated Kummer quotient, i.e., the quotient of $\mathrm{Jac}(H)$ by the natural involution. The minimal desingularisation of $\mathrm{Kum}(\mathrm{Jac}(H))$ is a simply connected variety of dimension g, called a *Jacobian Kummer variety* (see Section 3.1 for details).

In dimension two, all principally polarised abelian surfaces over K are isomorphic to either the Jacobian of a genus 2 curve or, up to a quadratic extension, the product of two elliptic curves (see [González et al. 2005, Theorem 3.1]). The Jacobian case is the generic one, in the sense of forming a Zariski dense open subvariety of the corresponding moduli space. Our main theorem confirms Conjecture 1.2 in the Jacobian case.

Theorem 1.3. *Let H be a smooth, projective genus 2 curve over K. Then the variety* $\mathrm{Kum}(\mathrm{Jac}(H))$ *has the Hilbert property.*[2]

The study of the Hilbert property for K3 surfaces was first addressed by Corvaja and Zannier [2017], and they verified it for the diagonal quartic surface $x^4 + y^4 = z^4 + w^4$ [ibid., Theorem 1.6] using multiple

[1] In [Corvaja and Zannier 2017, §2, Question–Conjecture 1bis], one finds an affine analogue of this conjecture concerning integral points. However, we are informed by Prof. Corvaja that in some cases of the analogue a finite field extension is necessary. Whether this is also the case for Conjecture 1.2 is unclear to the authors.

[2] Note that any smooth genus 2 curve is hyperelliptic.

elliptic fibrations. The same method was formalised and applied by Demeio [2021, Proposition 4.2.1]. He establishes the Hilbert property for the Kummer surface $\mathrm{Kum}(E_1 \times E_2)$ associated to the product of two elliptic curves E_1, E_2 over K, under the assumption that rational points on $\mathrm{Kum}(E_1 \times E_2)$ are Zariski dense. (Demeio's theorem is stated over a number field and contains the stronger assumption that the elliptic curves have positive Mordell–Weil ranks. In fact, the proof of [ibid., Theorem 1.0.2] works for any finitely generated field K of characteristic 0 and only assumes that the set of rational points is Zariski dense. This is known to hold unconditionally unless $j(E_1) = j(E_2) \in \{0, 1728\}$, thanks to [Kuwata and Wang 1993].)

Huang [2021] studied various Kummer surfaces associated to torsors under the product of two elliptic curves, though the Hilbert property for such surfaces remains still open. More recently, Gvirtz-Chen and Mezzedimi [2023] have shown that the Hilbert property holds for Kummer surfaces if one allows a finite field extension of the base field K.

The elliptic fibrations (modulo automorphisms) of a generic Jacobian Kummer surface (i.e., $\mathrm{Jac}(H)$ has no extra endomorphisms) over an algebraically closed field have been classified by Kumar [2014]. Unfortunately, in general none of these fibrations descend to K and, thus, it is not possible to apply directly the method of [Demeio 2021] over K. Instead, the novelty in our proof of Theorem 1.3 is, amongst other things, the use of rational curves on $\mathrm{Kum}(\mathrm{Jac}(H))$. To our knowledge, this is the first time that rational curves have been used to prove the Hilbert property for nonunirational varieties.

In fact, our approach allows us to settle Conjecture 1.2 for a family of higher-dimensional varieties. We are not aware of any previous results on the Hilbert property for nonunirational varieties of dimension > 2, apart from naive constructions like product varieties.

Theorem 1.4. *Let H be a smooth, projective hyperelliptic curve of genus $g \geq 3$ and of odd degree over K, i.e., H is defined by an affine model $y^2 = f(x)$ with f separable of degree $2g + 1$. Then the variety* $\mathrm{Kum}(\mathrm{Jac}(H))$ *has the Hilbert property.*

1.3. ***Strategy and outline.*** In order to reduce the a priori unknown Hilbert property of a higher-dimensional variety X to the known Hilbert property of (possibly singular) rational subcurves on X, their behaviour under pullbacks along covers of X has to be understood. This is the criterion of Proposition 5.1, which we formulate in a way tailored to our application to Jacobian Kummer varieties.

Let $\mathrm{Alb}^1(H)$ be the (first) Albanese torsor of the hyperelliptic curve H, and let $\mathrm{Kum}(\mathrm{Alb}^1(H))$ be the associated Kummer quotient, i.e., the quotient of $\mathrm{Alb}^1(H)$ by the natural involution. See Section 3.2 for details. In general, $\mathrm{Kum}(\mathrm{Alb}^1(H))$ is a nonisomorphic Galois twist of $\mathrm{Kum}(\mathrm{Jac}(H))$. In practice, it turns out that working with $\mathrm{Kum}(\mathrm{Alb}^1(H))$ instead of $\mathrm{Kum}(\mathrm{Jac}(H))$ is crucial for our purposes. The most general form of our main theorem is the following, from which Theorems 1.3 and 1.4 follow as special cases:

Theorem 1.5. *Let H be a smooth, projective hyperelliptic curve over K. Then the variety* $\mathrm{Kum}(\mathrm{Alb}^1(H))$ *has the Hilbert property.*

The variety $\mathrm{Kum}(\mathrm{Alb}^1(H))$ always possesses a countable family of rational curves whose union is Zariski dense. To show that the conditions in the criterion of Proposition 5.1 are satisfied for $\mathrm{Kum}(\mathrm{Alb}^1(H))$, we prove pullback lemmas that deal with the three occurring types of covers of $\mathrm{Kum}(\mathrm{Alb}^1(H))$:

(1) covers that factor rationally through $\mathrm{Alb}^1(H)$;

(2) covers that arise as Kummer quotients of étale covers of $\mathrm{Alb}^1(H)$;

(3) covers that pull back to ramified covers of $\mathrm{Alb}^1(H)$.

While type (1) covers are relatively easy to treat, for the other types we rely on two separate methods. Type (2) covers are dealt with through a fundamental group argument supplemented by a *coprimality trick*. As for type (3) covers, we make use of the recently established *weak Hilbert property for abelian varieties* due to Corvaja, Demeio, Javanpeykar, Lombardo and Zannier [Corvaja et al. 2022], which guarantees the existence of many rational points avoiding these covers. To control the pullbacks of rational curves on $\mathrm{Kum}(\mathrm{Alb}^1(H))$, we apply this result not to $\mathrm{Alb}^1(H)$ itself but instead to the base change of $\mathrm{Alb}^1(H)$ to the function field of H.

Finally, for any given finite collection of these covers, we need to carefully match these methods together, so as to extract a subfamily of rational curves which simultaneously have no rational sections along any one of them. Thus, a fortiori, these curves contribute to many rational points not coming from these covers, thereby achieving the Hilbert property for $\mathrm{Kum}(\mathrm{Alb}^1(H))$. The deduction of the Hilbert property for $\mathrm{Kum}(\mathrm{Jac}(H))$ in the case $g = 2$ is based on the fact (due to Cassels and Flynn; see Theorem 3.5) that $\mathrm{Kum}(\mathrm{Alb}^1(H))$ and $\mathrm{Kum}(\mathrm{Jac}(H))$ are birational to each other over K. However, in light of the lack of an analogue of Theorem 3.5 for $g > 2$, we cannot extend Theorem 1.4 to the setting where H has no K-rational Weierstrass point (or, slightly weaker, when H has no zero-cycle of degree 1 fixed under τ_H — the existence of such a zero-cycle is, for example, guaranteed if f factors into polynomials of coprime degrees). It would be interesting to remove the assumption on the degree of f in future investigations.

The article is structured as follows. After notation and background in Section 2, in Section 3 we construct and prove properties of rational curves on $\mathrm{Kum}(\mathrm{Jac}(H))$ and $\mathrm{Kum}(\mathrm{Alb}^1(H))$. In Section 4 the crucial pullback lemmas are formulated for covers of types (1)–(3). These are applied in Section 5 to prove the main theorems.

2. Notation, conventions and preliminaries

A *variety* over K is a separated reduced K-scheme of dimension > 0 and of finite type. To simplify exposition, throughout the rest of this article all varieties are assumed to be integral. We write $K(X)$ for the function field of a K-variety X.

Let $\phi : Y \to X$ be a morphism of varieties and let $X' \subseteq X$ be a subvariety. We say that X' *has a rational section along* ϕ if the morphism $\phi|_{\phi^{-1}(X')} : \phi^{-1}(X') \to X'$ has a rational section, i.e., there exists an open dense subset $U' \subset X'$ and $\psi : U' \to Y$ such that $\phi \circ \psi$ is the identity map on U'.

A K-variety birationally equivalent to $\mathbb{P}^n_K$ is called *K-rational*. By a *rational curve* on a K-variety X, we mean a closed (possibly singular) subvariety on X which is birationally equivalent to $\mathbb{P}^1_K$. By a *hyperelliptic curve* over K, we mean a smooth, projective curve over K with a finite morphism to $\mathbb{P}^1_K$ of degree 2. (Note that a curve may be geometrically hyperelliptic but not hyperelliptic over K, the image of the canonical map being a conic without a K-rational point. This does not happen when g is even.)

Assume that a K-variety V is equipped with an involution $\tau : V \to V$, usually taken to be implicit from the context. Then we define the *Kummer quotient* $\mathrm{Kum}(V)$ as the (possibly singular) quotient of V by the $\mathbb{Z}/2\mathbb{Z}$-action for which the group generator acts by τ.

Let $(G, +)$ be an abelian group. A *coset* $\mathscr{C}$ of G is of the form $g + H$, where $g \in G$ and $H \subset G$ is a subgroup.

Our next proposition says that proving the Hilbert property boils down to a specific family of maps.

Proposition 2.1. *Let X be a quasiprojective variety over a field K. Then a subset $M \subset X(K)$ is thin if and only if there exists a finite collection $(\phi_i : Y_i \to X)$, where Y_i are normal and geometrically integral varieties over K, and ϕ_i are finite surjective K-morphisms of degree $[K(Y_i) : K(X)] > 1$, such that $M \setminus \bigcup_{i=1}^r \phi_i(Y_i(K))$ is not Zariski dense.*

Proof. Upon considering the morphism from the variety Y, as the disjoint union of Y_i with a Zariski closed subset of X, to X, the "if" part follows just by definition.

We now prove the "only if" part. Namely, for every thin set $M \subset X(K)$, we want to construct such a finite collection $\phi_i : Y_i \to X$ as above such that $M \setminus \bigcup_{i=1}^r \phi_i(Y_i(K))$ is contained in a proper Zariski closed subset. Assume that $M \subset f(Z(K))$, where Z is a K-variety and the morphism f is generically finite and has no rational section. Upon considering the irreducible components of Z, it suffices to assume that $M = \phi(Y(K))$, where Y is an integral K-variety with $\dim Y = \dim X$, and $\phi : Y \to X$ is a dominant K-morphism with degree $[K(Y) : K(X)] > 1$. Let us fix a projective model $\overline{X}$ of X in what follows.

By the Nagata compactification theorem (see [Lütkebohmert 1993, Theorem 3.2]), ϕ extends to a proper morphism $\bar{\phi} : \overline{Y} \to \overline{X}$, where $\overline{Y}$ is a proper model of Y. Using the Stein factorisation [Hartshorne 1977, III, Corollary 11.5], we can decompose into $\bar{\phi} = g_2 \circ g_1 : \overline{Y} \to W \to \overline{X}$, where $g_1 : \overline{Y} \to W$ is proper with connected fibres and $g_2 : W \to \overline{X}$ is finite. It follows that g_1 is birational, and g_2 is proper by [ibid., Exercise II.4.1] and hence it is surjective (since ϕ is dominant). Let Z denote the nonnormal locus of W and let $h : \widehat{W} \to W$ be the normalisation, which is finite birational. We may assume that $\widehat{W}(K) \neq \varnothing$ and hence $\widehat{W}$ is geometrically integral (see [Serre 2008, p. 20]). Otherwise M is contained in the Zariski closed subset $g_2(Z)$. We let $\widetilde{W} = \widehat{W} \setminus (g_2 \circ h)^{-1}(\overline{X} \setminus X)$. We claim that the morphism $g_2 \circ h|_{\widetilde{W}} : \widetilde{W} \to X$ meets our need. Indeed, $g_2 \circ h$ is finite surjective of degree $[K(W) : K(X)] = [K(Y) : K(X)] > 1$, and $M \setminus \big((g_2 \circ h)(\widetilde{W}(K))\big)$ is contained in the Zariski closed subset $g_2(Z)$. □

The proposition above motivates the following convention.

A *cover* $\phi : Y \to X$ between K-varieties is a finite surjective morphism with Y being normal and geometrically integral. The *degree* of ϕ is $[K(Y) : K(X)]$.

Therefore, Proposition 2.1 may be rephrased as follows:

Corollary 2.2. *Let X be a quasiprojective variety over a Hilbertian field K. Then X has the Hilbert property over K if and only if, for any finite family of covers $\phi_i : Y_i \to X$ for $i = 1, \dots, r$ with $\deg(\phi_i) > 1$, $X(K) \setminus \bigcup_{i=1}^{r} \phi_i(Y_i(K))$ is Zariski dense in X.*

3. Rational curves on Jacobian Kummer varieties

Let H be a hyperelliptic curve of genus g over K. Recall the following classical facts about H, its Jacobian and Albanese varieties, and their Kummer quotients.

The hyperelliptic involution $\tau_H : H \to H$ is defined over K. The Kummer quotient $C = \mathrm{Kum}(H)$ is noncanonically isomorphic to $\mathbb{P}^1_K$, and we write $\pi_H : H \to C$ for the quotient morphism. The set $W \subset H(\overline{K})$ of Weierstrass points of H, i.e., the fixed points under τ_H, forms a zero-dimensional reduced K-subscheme of degree $2g + 2$.

3.1. ***The Jacobian variety and rational curves on its Kummer quotient.*** The Jacobian $J = \mathrm{Jac}(H)$ is a principally polarised abelian variety over K, even when $H(K) = \varnothing$. See, e.g., [Milne 1986b, Theorem 1.1, Summary 6.11]. For $n \in \mathbb{Z}$, write $[n_J] : J \to J$ for the multiplication-by-n morphism, and for a closed point $x \in J$, write $t_x : J \to J$ for the translation-by-x morphism. The fixed points of the involution $\tau_J = [(-1)_J]$ consist precisely of the 2-torsion $J[2]$, which is a zero-dimensional subgroup scheme of degree 2^{2g}. Let $X = \mathrm{Kum}(J)$ and $\pi : J \to X$ be the quotient morphism. Clearly the morphisms $[n_J]$ and τ_J commute, so $[n_J]$ descends to X and we denote the resulting morphism by $[n_X]$. The morphism π is étale outside the images of $J[2]$, which form the singular locus of X. Blowing up these 2^{2g} canonical singularities gives a minimal resolution $\widetilde{X} \to X$, where $\widetilde{X}$ is known as the *Kummer variety* of J. It is a smooth, projective and simply connected variety (see [Spanier 1956, Theorem 1]).

For every $x_0 \in H(K)$, there exists an Abel–Jacobi closed K-embedding

$$\iota_{x_0} : H \hookrightarrow J, \quad x \mapsto \langle x_0 - x \rangle,$$

where $\langle \cdot \rangle$ denotes the class of a divisor. By the Riemann–Roch theorem, the closed points on H generate J (see [Hartshorne 1977, IV, Remark 4.10.9] or [Milne 1986b, §5]).

We assume throughout the rest of Section 3.1 that H has a K-rational Weierstrass point ∞. We consider H as a subvariety of J via the Abel–Jacobi embedding ι_∞. Since the restriction of the map τ_J to H is just τ_H, the image of H under π is a smooth rational curve, which we denote by C. For every $n \in \mathbb{Z}_{>0}$, let $H_n = [n_J](H)$, the image of H under the map $[n_J]$, and let $C_n = \pi(H_n)$. The curves C_n are rational for all $n \in \mathbb{Z}_{>0}$. (See, e.g., [Huybrechts 2016, Chapter 13, Example 0.4].)

Lemma 3.1. *The family $(H_n)_{n\in\mathbb{N}}$ consists of pairwise distinct curves. Consequently, $(C_n)_{n\in\mathbb{N}}$ are pairwise distinct rational curves on X.*

Proof. Assume $H_m = H_n$ for some $0 < m < n$. The m^{2g} translates $t_x(H)$ for $x \in J[m]$ are pairwise distinct. Since $[m_J]$ is étale of degree m^{2g}, it follows that $[m_J]|_H : H \to H_m$ is of degree 1, i.e., a birational map.

The same holds for $[n_J]|_H$. Thus, there is an automorphism σ of H such that, as rational maps,

$$\sigma = [m_J]|_H \circ ([n_J]|_H)^{-1} = ([n_J]|_H)^{-1} \circ [m_J]_H.$$

Because σ has finite order, after replacing m and n by powers thereof, we can assume $[m_J]_H = [n_J]_H$. It follows that $H \subset \mathrm{Ker}([(n-m)_J]) \subset J$, which is impossible since the closed points of H generate J. □

Remark 3.2. There is an oversight in [Huybrechts 2016, p. 274], where it is stated that, when $g = 2$, the rational curve C_n is smooth for all n. In fact this is false for $n > 1$ and it is for this reason that we will need the coprimality argument in Proposition 4.4. If C_n were smooth, then, since it is an ample geometrically irreducible divisor of J, its pullback along $[n_J]$ would also be geometrically irreducible by [Birkenhake and Lange 2004, Corollary 4.3.4]. But this is clearly not the case since it is (geometrically) a sum of translates of C. See also Remark 4.3 below.

More concretely, let us take a look at the curve C_2. Since H contains six points of $J[2]$, $H_2 = [2_J](H)$ passes through $0 \in J$ with multiplicity at least 6. Hence, C_2 has multiplicity at least 3 at $\pi(0)$. In general, the singular locus of H_n (resp. C_n) is the closed subscheme of all points $[n_J]x$ (resp. $[n_X]\pi(x)$) for $x \in H$ such that there exists $y \in H$ different from x with $[n_J](x-y) = 0$. In other words, x is in a *torsion package* of size > 1 in the sense of [Baker and Poonen 2001]. We omit the proof since this fact will not be needed.

Lemma 3.3. *Let $\eta_H \in J_{K(H)}$ be the image of the generic point of H under the embedding $\iota_\infty : H \hookrightarrow J$. Then η_H is nondegenerate in $J_{K(H)}$, i.e., $\mathbb{Z}\eta_H$ is Zariski dense in $J_{K(H)}$.*

Proof. Assume instead that η_H is contained in a proper abelian subvariety of $J_{K(H)}$. Then, by the Poincaré irreducibility theorem, there exists a nonzero endomorphism $e \in \mathrm{End}(J_{\overline{K(H)}})$ (actually in $\mathrm{End}(J_{K(H)})$) such that $e\eta_H = 0$. Because $\mathrm{End}(J_{\overline{K(H)}}) \simeq \mathrm{End}(J_{\overline{K}})$ — see [Chai et al. 2014, Lemma 1.2.1.2] — it follows that there exists $e' \in \mathrm{End}(J_{\overline{K}})$ such that $e's(x) = e'x = 0$ for all $x \in H$. Thus, H would have to be contained in a proper abelian subvariety of J, which is impossible again since the closed points of H generate J. □

Corollary 3.4. *For every finite-index coset $\mathcal{N} \subset \mathbb{Z}$, $\bigcup_{n\in\mathcal{N}} H_n$ is Zariski dense in J and $\bigcup_{n\in\mathcal{N}} C_n$ is Zariski dense in X.*

Proof. Write $\mathcal{N} = a\mathbb{Z}+b$, where a, b are integers with $a \neq 0$. Since the morphism $t_{b\eta_H} \circ [a_{J_{K(H)}}] : J_{K(H)} \to J_{K(H)}$ is dominant, the conclusion follows directly from Lemma 3.3. □

3.2. ***The Albanese torsor $\mathbf{Alb}^1(H)$ and rational curves on its Kummer quotient.*** The zero-cycles of degree 1 (up to linear equivalence) on H fixed under τ_H form a torsor T under $J[2]$, which is trivial if H has a K-rational Weierstrass point. Let $c \in \mathrm{H}^1(K, J[2])$ be the associated cohomology class. Then, for every fixed $\overline{K}$-Weierstrass point $\infty \in W$, c is given by the cocycle $\gamma \mapsto \langle \gamma(\infty) - \infty \rangle$. The twist of J by the image of c in $\mathrm{H}^1(K, J)$, i.e., the quotient of $J \times T$ by the diagonal action of $J[2]$, is isomorphic to the Albanese torsor $J^1 = \mathrm{Alb}^1(H)$.

In general, we denote by $J^i = \mathrm{Alb}^i(H)$ the i-th Albanese torsor, which parametrises zero-cycles of degree i on H. There is an isomorphism $J^i \simeq J^{i+2}(H)$ for all $i \geq 0$, given by adding $\langle 2\infty \rangle$, and $J^0 \simeq J$. The variety J^1 is an initial object within the category of torsors V under J with a morphism

$\iota_V : H \to V$ (see [Wittenberg 2008, Appendix A]). The involution τ_H extends formally to an involution τ_{J^1} on J^1 in a way that is compatible with the involution τ_J on J and the J-torsor structure. We write $\pi^1 : J^1 \to X^1 = \mathrm{Kum}(J^1)$ for the corresponding Kummer quotient morphism.

In the case $g = 2$, the surfaces X and X^1 are projective duals of each other as quartic surfaces in $\mathbb{P}^3$. (By [Cassels and Flynn 1996, Chapter 4], they are isomorphic if and only if at least one of the translates of $C \subset X$ by $J[2](\overline{K})$ — the so-called *tropes* — is defined over K.) However, Cassels and Flynn [1996, Chapter 16] point out (somehow surprisingly) that:

Theorem 3.5 (Cassels and Flynn). *The minimal desingularisations of the surfaces X and X^1 are isomorphic over K.*

Proof. We offer a more conceptional and simplified proof here. Indeed, consider the involutions

$$\sigma_1 : H \times H \to H \times H, \quad (x_1, x_2) \mapsto (\tau_H(x_2), \tau_H(x_1)),$$
$$\sigma_2 : H \times H \to H \times H, \quad (x_1, x_2) \mapsto (x_2, x_1).$$

We define $\mathrm{Inv}^2(H)$ (resp. $\mathrm{Sym}^2(H)$) to be the quotient of $H \times H$ by σ_1 (resp. by σ_2). For any point $(x_1, x_2) \in H \times H$, we write $\{x_1, x_2\}$ for the corresponding quotient image.

We now define

$$\beta : \mathrm{Inv}^2(H) \to J^0 = J, \quad \{x_1, x_2\} \mapsto \langle x_1 - x_2 \rangle,$$
$$\beta' : \mathrm{Sym}^2(H) \to J^2, \quad \{x_1, x_2\} \mapsto \langle x_1 + x_2 \rangle.$$

Then, by the Riemann–Roch theorem (see also [Baker and Poonen 2001, Proposition 4(2)]), outside the image of the diagonal $(\mathrm{id}_H \times \mathrm{id}_H)(H)$ (resp. the antidiagonal $(\mathrm{id}_H \times \tau_H)(H)$), β (resp. β') is an isomorphism. Hence, both β, β' are birational.

Let us define the rational maps b, b' over K by

$$b : X \dashrightarrow X^1, \quad \{z, \tau_J(z)\} \mapsto \{z + \langle x_2 \rangle, z + \langle \tau_H(x_1) \rangle\},$$

for $z \in J \setminus J[2]$, where $\{x_1, x_2\}$ is the preimage of $2z$ under β, and

$$b' : X^1 \dashrightarrow X, \quad \{z^1, \tau_{J^1}(z^1)\} \mapsto \{z^1 - \langle x \rangle, z^1 - \langle x' \rangle\},$$

for $z^1 \in J^1 \setminus T$, where $\{x, x'\}$ is the preimage of $2z^1$ under β'. Then b and b' are inverse to each other, and hence are birational maps. Blowing up the singular loci of X and X' gives the desired isomorphism. □

Remark 3.6. Alternatively, in terms of cocycles, one can check that the image of c in $\mathrm{H}^1(K, \mathrm{Aut}_{\overline{K}}\, \overline{K}(X))$ is the coboundary of the automorphism of $\overline{K}(X)$ which is induced by the birational map

$$\sigma : X_{\overline{K}} \dashrightarrow X_{\overline{K}}, \quad \pi(z) \mapsto \pi(z + \langle x_2 - \infty \rangle);$$

thus, c vanishes in $\mathrm{H}^1(K, \mathrm{Aut}_{\overline{K}}\, \overline{K}(X))$.

The image H^1 under the natural embedding $\iota_{J^1} : H \to J^1$ is stable under the involution τ_{J^1} and thus yields a smooth rational subcurve $C^1 = \pi^1(H^1) \subset X^1$. For any *odd* integer $n \in \mathbb{Z}_{>0}$, we denote by $[n_{J^1}]$ the composition of the formal multiplication-by-n map (of zero-cycles of degree 1) $J^1 \to J^n$ and the

previously defined isomorphism $J^n \simeq J^1$. Because $[n_{J^1}]$ commutes with involution, it descends to a morphism $[n_{X^1}]$. We let $H_n^1 = [n_{J^1}](H^1)$, the image of H^1 under the map $[n_{J^1}]$, and let $C_n^1 = \pi^1(H_n^1)$. Since

$$C_n^1 = (\pi^1 \circ [n_{J^1}])(H^1) = ([n_{X^1}] \circ \pi^1)(H^1) = [n_{X^1}](C^1),$$

all the curves C_n^1 are rational.

If H has a K-rational Weierstrass point ∞, then the K-morphism

$$\alpha_J : J^1 \to J, \quad x \mapsto x - \langle \infty \rangle,$$

commutes with the involutions τ_J, τ_{J^1}, so it descends to a K-morphism $\alpha_X : X^1 \to X$ and yields a commutative diagram

$$\begin{array}{ccc} J^1 & \xrightarrow{\pi^1} & X^1 \\ \downarrow{\scriptstyle \alpha_J} & & \downarrow{\scriptstyle \alpha_X} \\ J & \xrightarrow{\pi} & X \end{array} \tag{$\dagger$}$$

where the vertical arrows are K-isomorphisms. It is easy to see that H_n^1 maps to H_n (as a subvariety of J) under α_J, and C_n^1 maps to C_n under α_X for all odd $n \in \mathbb{Z}_{>0}$.

4. Pullbacks along covers

Throughout this section, K denotes a finitely generated field of characteristic zero and $X = \mathrm{Kum}(J)$ denotes the Kummer quotient associated to the Jacobian J of a genus g hyperelliptic curve H over K with a K-rational Weierstrass point ∞. We continue to consider H embedded into J via ι_∞.

4.1. *Pullback along covers factoring through the Jacobian.*

Lemma 4.1. *Let $\phi : Y \to X$ be a cover of degree > 1. Suppose that the base change variety $Y \times_X J$ is not geometrically integral. Then, for every finite-index coset $\mathcal{N} \subset \mathbb{Z}$, there exists a subcoset $\mathcal{N}' \subset \mathcal{N}$ of finite index such that none of the rational curves $(C_n)_{n \in \mathcal{N}'}$ have a rational section along ϕ over $\overline{K}$.*

Proof. We may base change everything to $\overline{K}$ and we fix an algebraic closure Ω of $\overline{K}(X)$. We have that $Y_{\overline{K}} \times_{X_{\overline{K}}} J_{\overline{K}}$ is not geometrically integral if and only if $\overline{K}(Y) \otimes_{\overline{K}(X)} \overline{K}(J)$ is not a field, or, equivalently, the field extensions $\overline{K}(Y)$ and $\overline{K}(J)$ in Ω are not linearly disjoint over $\overline{K}(X)$. Since $[\overline{K}(Y) \cdot \overline{K}(J) : \overline{K}(Y)] \leq [\overline{K}(J) : \overline{K}(X)] = 2$, this happens if and only if $[\overline{K}(Y) \cdot \overline{K}(J) : \overline{K}(Y)] < 2$, or equivalently $\overline{K}(J) \subseteq \overline{K}(Y)$. It follows that, over $\overline{K}$, $\phi : Y \to X$ factors as a rational map $\phi_1 : Y \dashrightarrow J$ then to X.

By Lemma 3.3, we fix a closed point $P \in H$ which is nondegenerate in J. By [Corvaja et al. 2022, Lemma 4.6], there exists a finite-index subcoset $\mathcal{N}' \subset \mathcal{N}$ such that the set $\{nP : n \in \mathcal{N}'\}$ does not intersect with the indeterminacy locus, say $Z \subset J$, of ϕ_1. This implies that none of the curves $(H_n)_{n \in \mathcal{N}'}$ is contained in Z. If a rational curve $(C_n)_{\overline{K}} \subset X_{\overline{K}}$ for $n \in \mathcal{N}'$ has a rational section along ϕ, say $\tilde{\phi}$, then it has a rational section $\phi_1 \circ \tilde{\phi}$ along π. This is impossible since there are no nonconstant rational maps from the projective line to abelian varieties (see [Milne 1986a, Corollary 3.8]). □

4.2. *Pullbacks along unramified covers.*

Lemma 4.2. *Let D be a smooth, projective curve of genus ≥ 1 over a separably closed field. Fix a point x in D inducing the Abel–Jacobi embedding $\iota_x : D \hookrightarrow \mathrm{Jac}(D)$. Let $\pi_1(D, x)$ (resp. $\pi_1(\mathrm{Jac}(D), 0)$) be the pointed étale fundamental group of D (resp. $\mathrm{Jac}(D)$). Then the map*

$$(\iota_x)_* : \pi_1(D, x) \to \pi_1(\mathrm{Jac}(D), 0)$$

induced by ι_x is surjective. Consequently, every pullback via ι_x of a connected finite étale covering of $\mathrm{Jac}(D)$ remains connected as a finite étale covering over D.

Proof. This is explained in [Milne 1986b, Proposition 9.1]. Indeed, $(\iota_x)_*$ induces an isomorphism

$$\pi_1(D, x)^{\mathrm{ab}} \to \pi_1(\mathrm{Jac}(D), 0),$$

where $\pi_1(D, x)^{\mathrm{ab}}$ is the maximal abelian quotient of $\pi_1(D, x)$, so, in particular, $(\iota_x)_*$ is surjective. Thus, by the theory of Galois categories, all connected finite étale coverings of $\mathrm{Jac}(D)$ have abelian Galois group, and the surjectivity of $(\iota_x)_*$ implies that all connected finite abelian coverings of D are obtained precisely by pulling back the connected finite étale coverings of J. See also [Szamuely 2009, Proposition 5.5.4(2)]. □

Remark 4.3. If $g = 2$, one may also prove Lemma 4.2 by using the fact that the pullback of a geometrically irreducible, unibranch, ample divisor on an abelian variety along an isogeny is again geometrically irreducible. (See [Birkenhake and Lange 2004, Corollary 4.3.4]. Note that the statement is missing the crucial unibranch — or at the least smoothness — assumption.)

Lemma 4.2 applies to $H \hookrightarrow J$, while an analogous statement for H_n with $n > 1$ fails because of the nonsmoothness. However, one can remedy the situation by restricting to a subfamily:

Proposition 4.4 ("coprimality trick"). *Let $\phi : J' \to J$ be an unramified cover. Then $\phi^{-1}(H_n)$ are geometrically irreducible for all n coprime to* $\deg(\phi)$.

Proof. We may base change everything to $\overline{K}$, without affecting the geometrical irreducibility. By the Lang–Serre theorem [Serre 1988, Chapter I, §2], J' is an abelian variety and ϕ can be made into an isogeny. The case where $\deg(\phi) = 1$ is clear and so we assume from now on that $\deg(\phi) > 1$.

If $n = 1$, then the claim follows from Lemma 4.2. In particular, $\mathrm{Ker}(\phi) \subset \phi^{-1}(H_1)$. In general, if n is coprime to $\deg(\phi) = \deg(\mathrm{Ker}(\phi))$, then $[n_{J'}]|_{\mathrm{Ker}\,\phi}$ is an isomorphism. Thus,

$$\phi^{-1}(H_n) = \phi^{-1}([n_J](H_1)) = [n_{J'}](\phi^{-1}(H_1)) + \mathrm{Ker}(\phi) = [n_{J'}](\phi^{-1}(H_1) + \mathrm{Ker}(\phi)) = [n_{J'}](\phi^{-1}(H_1)),$$

and hence $\phi^{-1}(H_n)$ is geometrically irreducible. □

Corollary 4.5. *Let $\phi_i : Y_i \to X$ for $i \in I$ be any finite collection of covers of X with $\deg(\phi_i) > 1$ such that $Y_i \times_X J$ are all geometrically integral, and the base change morphisms $\tilde{\phi}_i : Y_i \times_X J \to J$ are all unramified. Then the $\phi_i^{-1}(C_n)$ are all geometrically irreducible for all n coprime to $\prod_i \deg(\phi_i)$.*

Proof. Since H_n is stable under the involution, $\phi_i^{-1}(C_n)$ is geometrically irreducible if and only if $\tilde{\phi}_i^{-1}(H_n)$ is. Since $Y_i \times_X J$ are all geometrically integral, we have $\deg(\tilde{\phi}_i) = \deg(\phi_i)$. We then apply Proposition 4.4 to $\tilde{\phi}_i : Y_i \times_X J \to J$. □

4.3. ***Pullbacks along ramified covers.*** The following result is essentially Theorem 1.3 of [Corvaja et al. 2022]:

Theorem 4.6 (Corvaja, Demeio, Javanpeykar, Lombardo and Zannier). *Let A be an abelian variety over a finitely generated field K of characteristic 0 with a nondegenerate point $x \in A(K)$. Let $f_i : B_i \to A$ be a finite collection of ramified covers. Then, for any finite-index coset $\mathcal{P} \subset \mathbb{Z}x$, there exists a finite-index subcoset of $\mathcal{P}' \subset \mathcal{P}$ such that $(f_i^{-1}(x'))(K) = \varnothing$ for every $x' \in \mathcal{P}'$.*

Proof. It suffices to work with a single ramified cover $f : B \to A$ by successively analysing the finite-index coset obtained at each step. Write $\mathcal{P} = \Pi + bx$, where $b \in \mathbb{Z}$ and Π is a finite-index subgroup of $\mathbb{Z}x$ and $P \in A(K)$. Define $f' = t_{-bx} \circ f$, where $t_{-bx} : A \to A$ is the translation-by-$(-bx)$ morphism. Then we apply [Corvaja et al. 2022, Theorem 1.3] to Π and get a finite-index coset $\Pi' \subset \Pi$ for which any K-fibre of f' has no K-point. Then we may take $\mathcal{P}' = \Pi' + bx$ to obtain the same conclusion for f. □

Corollary 4.7. *Let $\phi_i : Y_i \to X$ for $i \in I$ be any finite collection of covers such that all the base change morphisms $\tilde{\phi}_i : Y_i \times_X J \to J$ are ramified with $Y_i \times_X J$ geometrically integral. Then, for any finite-index coset $\mathcal{N} \subset \mathbb{Z}$, there exists a finite-index subcoset $\mathcal{N}' \subset \mathcal{N}$ such that, for any $n \in \mathcal{N}'$, the rational curve C_n does not have a rational section along ϕ_i for all $i \in I$.*

Proof. Consider the base change

$$\tilde{\phi}_{i,K(H)} : (Y_i \times_X J)_{K(H)} \to J_{K(H)}$$

to the function field $K(H)$ of H. This is still a ramified morphism with $(Y_i \times_X J)_{K(H)}$ being geometrically integral.

By Lemma 3.3, the $K(H)$-rational point $\eta_H \in J_{K(H)}$ induced by $\iota_\infty : H \hookrightarrow J$ is nondegenerate. By Theorem 4.6 applied to $\mathcal{N}\eta_H$ over the finitely generated field $K(H)$, there exists a finite-index coset $\mathcal{N}' \subset \mathcal{N}$ such that, for any $\eta'_H \in \mathcal{N}'\eta_H \subset J_{K(H)}$,

$$\eta'_H \notin \bigcup_{i \in I} \tilde{\phi}_{i,K(H)}\big((Y_i \times_X J)_{K(H)}(K(H))\big).$$

Consider the cartesian diagram

$$\begin{array}{ccccc} (Y_i \times_X J)_{K(H)} & \longrightarrow & Y_{i,K(H)} & \longrightarrow & Y_{i,K(C)} \\ \downarrow{\scriptstyle \tilde{\phi}_{i,K(H)}} & & \downarrow{\scriptstyle \phi_{i,K(H)}} & & \downarrow{\scriptstyle \phi_{i,K(C)}} \\ J_{K(H)} & \longrightarrow & X_{K(H)} & \longrightarrow & X_{K(C)} \end{array}$$

where the rightmost square is the base change from $K(C)$ to $K(H)$. Let $\eta_C \in X_{K(C)}$ be the image of η_H under $J_{K(H)} \to X_{K(C)}$. Then, for any $\eta'_C \in \mathcal{N}'\eta_C \subset X_{K(C)}$,

$$\eta'_C \notin \bigcup_{i \in I} \phi_{i,K(C)}\big(Y_{i,K(C)}(K(C))\big).$$

Equivalently, none of the curves C_n for $n \in \mathcal{N}'$ has a rational section along ϕ_i for all $i \in I$. □

5. Hilbert property for Jacobian Kummer varieties

Proposition 5.1. *Let X be a quasiprojective variety over a Hilbertian field K. Assume that there exists an infinite set $\mathscr{C}$ of (not necessarily smooth) K-rational subvarieties on X satisfying the following condition*:

For every finite family of covers $\phi_i : Y_i \to X$ for $i = 1, \dots, r$ with $\deg(\phi_i) > 1$, there exists a subset $\mathscr{C}' \subset \mathscr{C}$ such that $\bigcup_{C \in \mathscr{C}'} C$ is Zariski dense in X and, for all $C \in \mathscr{C}'$ and $i = 1, \dots, r$, C does not have a rational section along ϕ_i.

Then X has the Hilbert property.

Proof. Let us fix a finite family of covers $\phi_i : Y_i \to X$ for $i = 1, \dots, r$ with $\deg(\phi_i) > 1$. Let $\mathscr{C}' \subset \mathscr{C}$ be as stated in the proposition. Because the union of the $C \in \mathscr{C}'$ is Zariski dense, upon replacing $\mathscr{C}'$ by a subset if necessary, we may assume that, for every subvariety $C \in \mathscr{C}'$, C is not contained in the ramification locus of any ϕ_i for $i = 1, \dots, r$. In view of Proposition 2.1, to show the Hilbert property for X, it suffices to show that

$$\left(\bigcup_{C \in \mathscr{C}'} C(K) \right) \setminus \left(\bigcup_{i=1}^{r} \phi_i(Y_i(K)) \right)$$

is not contained in any proper Zariski closed subset of X.

Assume that there exists $Z \subset X$ proper Zariski closed containing all the K-points above. We fix $C_0 \in \mathscr{C}'$ such that $C_0 \not\subset Z$. For every $i \in \{1, \dots, r\}$, let C_{ij} for $j = 1, \dots, k_i$ be the irreducible components of $\phi_i^{-1}(C_0)$. The maps $\phi_i|_{C_{ij}} : C_{ij} \to C_0$ are finite morphisms because C_0 does not have a rational section along ϕ_i, and, since C_0 is not contained in the ramification locus of ϕ_i by assumption, if $\dim(C_{ij}) = \dim(C_0)$, then $\phi_i|_{C_{ij}}$ is dominant, and hence has degree > 1. It thus follows from the Hilbert property for rational varieties (see [Serre 2008, p. 30, Examples]) that $C_0(K)$ is not thin, whence

$$C_0(K) \setminus \left((Z \cap C_0)(K) \cup \left(\bigcup_{i=1}^{r} \bigcup_{j=1}^{k_i} \phi_i(C_{ij}(K)) \right) \right) \neq \varnothing,$$

which is a contradiction. □

We now turn to the proofs of the main theorems. We will apply Proposition 5.1 only in the case where $\mathscr{C}$ is a countable set of subcurves.

Proof of Theorem 1.5. Consider the rational K-subcurves $\{C_n^1\}_{n \in 1+2\mathbb{Z}}$ on X^1. Let $\phi_i : Y_i \to X^1$ for $i = 1, \dots, r$ be any fixed finite family of covers with $\deg(\phi_i) > 1$. We want to construct a finite-index subcoset $\mathcal{N}' \subset 1 + 2\mathbb{Z}$ of odd integers such that, for $\{C_n^1\}_{n \in \mathcal{N}'}$, all conditions of Proposition 5.1 are satisfied. Let us take a finite field extension L of K such that H has an L-rational Weierstrass point ∞. We base change everything to L (and we omit the subscript L). In view of the diagram (†), we shall replace X^1 by the isomorphic X and C_n^1 by C_n in the following.

Let $\tilde{\phi}_i : \widetilde{Y}_i \to J$ be the base change of ϕ_i along the Kummer quotient $\pi : J \to X$. It may be assumed that, for certain r_1, all $\widetilde{Y}_i$ for $1 \leq i < r_1$ are not geometrically integral and the remaining $\widetilde{Y}_i$ for $r_1 \leq i \leq r$ are geometrically integral. For $r_1 \leq i \leq r$, we may moreover assume that, for certain $r_2 \geq r_1$, the morphisms $\tilde{\phi}_i$ for $r_1 \leq i \leq r_2$ are all unramified, while the remaining $\tilde{\phi}_i$ for $r_2 < i \leq r$ are all ramified.

We start with

$$\mathscr{C} = \{C_n : n \in 1 + 2\mathbb{Z}\}.$$

If $r_2 > r_1$, i.e., certain $\tilde{\phi}_i$ are unramified, we consider

$$d = \operatorname{lcm}(\deg \phi_i : i = r_1, \dots, r_2), \quad \mathcal{N}^{\text{ét}} = 1 + 2d\mathbb{Z}.$$

Then $\mathcal{N}^{\text{ét}}$ is a finite-index coset of $\mathbb{Z}$, and, by Corollary 4.5, any C_n with $n \in \mathcal{N}^{\text{ét}}$ does not have a rational section along ϕ_i for all $r_1 \le i \le r_2$. Otherwise, if there are no such unramified covers, we just take $\mathcal{N}^{\text{ét}} = 1 + 2\mathbb{Z}$.

By Corollary 4.7, one obtains a finite-index subcoset $\mathcal{N}^{\text{geo}} \subset \mathcal{N}^{\text{ét}}$ such that any C_n with $n \in \mathcal{N}^{\text{geo}}$ does not have a rational section along ϕ_i for all $i = r_2 + 1, \dots, r$. Then, on applying Lemma 4.1, one obtains a further finite-index subcoset $\mathcal{N}' \subset \mathcal{N}^{\text{geo}}$ such that any C_n with $n \in \mathcal{N}'$ does not have a rational section along ϕ_i for all $i = 1, \dots, r_1 - 1$ (still over the field extension L of K).

Any such C_n^1, $n \in \mathcal{N}'$ cannot have a rational section along any ϕ_i over the base field K as well. Thus, by using again Corollary 3.4, all conditions of Proposition 5.1 are satisfied for the family of rational curves

$$\mathscr{C}' = \{C_n : n \in \mathcal{N}'\},$$

from which we conclude the Hilbert property. □

Proof of Theorem 1.3. This follows from Theorem 1.5 together with Theorem 3.5 using the birational invariance of the Hilbert property. □

Proof of Theorem 1.4. This follows immediately from Theorem 1.5 since H has, by assumption, a K-rational Weierstrass point (at infinity), and thus $X \cong X^1$ over K in view of the diagram (†). □

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