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Homological regularities and concavities

Ellen Kirkman, Robert Won and James J. Zhang

This paper concerns homological notions of regularity for noncommutative algebras. Properties of an algebra A are reflected in the regularities of certain (complexes of) A-modules. We study the classical Tor regularity and Castelnuovo–Mumford regularity, which were generalized from the commutative setting to the noncommutative setting by Jørgensen, Dong and Wu. We also introduce two new numerical homological invariants: concavity and Artin–Schelter regularity. Artin–Schelter regular algebras occupy a central position in noncommutative algebra and noncommutative algebraic geometry, and we use these invariants to establish criteria which can be used to determine whether a noetherian connected graded algebra is Artin–Schelter regular.

0. Introduction

Let $\Bbbk$ be a base field. An $\mathbb{N}$-graded algebra $A := \bigoplus_{i=0}^{\infty} A_i$ is called *connected graded* if $A_0 = \Bbbk$. Throughout let A denote a connected $\mathbb{N}$-graded $\Bbbk$-algebra unless otherwise stated. The Hilbert series $h_A(t)$ of A will be recalled in Definition 1.1.

A seminal result due to Stanley [1978, Theorem 4.4] states:

Theorem 0.1. *Let A be a commutative finitely generated connected graded Cohen–Macaulay domain. Then A is Gorenstein if and only if its Hilbert series $h_A(t)$ satisfies*

$$h_A(t^{-1}) = \pm t^{\ell} h_A(t) \tag{0.1.1}$$

for some integer ℓ.

The above theorem provides a surprising and effective criterion which is equivalent to the Gorenstein property. For example, if G is a finite group acting linearly on the polynomial ring $B := \Bbbk[x_1, \ldots, x_n]$, then the Hilbert series of the fixed subring B^G can be calculated using Molien's theorem [Kirkman 2016, Proposition 1.6]. One can then use (0.1.1) to determine whether or not B^G is Gorenstein. In general, verifying (0.1.1) is much easier than verifying the Gorenstein property directly. A noncommutative version of Stanley's criterion was proved in [Jørgensen and Zhang 2000, Theorems 6.1 and 6.2].

Note that a commutative connected graded algebra A is regular (i.e., has finite global dimension) if and only if A is isomorphic to a polynomial ring $\Bbbk[x_1, \ldots, x_n]$ with $\deg x_i > 0$ for all i. Hence, in the commutative case, checking whether or not A is regular is relatively easy. However, there are many more noncommutative connected graded regular algebras, namely the so-called Artin–Schelter regular

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algebras, which we now define. For a connected graded algebra A, let $\mathfrak{m} = A_{\geq 1}$. The trivial graded A-bimodule $A/\mathfrak{m}$ is also denoted by $\Bbbk$.

Definition 0.2 [Artin and Schelter 1987, p. 171]. A connected graded algebra T is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

(a) T has injective dimension $d < \infty$ on the left and on the right.

(b) $\operatorname{Ext}^i_T({}_T\Bbbk, {}_TT) = \operatorname{Ext}^i_T(\Bbbk_T, T_T) = 0$ for all $i \neq d$.

(c) $\operatorname{Ext}^d_T({}_T\Bbbk, {}_TT) \cong \operatorname{Ext}^d_T(\Bbbk_T, T_T) \cong \Bbbk(\mathfrak{l})$ for some integer $\mathfrak{l}$. Here $\mathfrak{l}$ is called the *AS index* of T.

In this case, we say T is of type $(d, \mathfrak{l})$. If, in addition,

(d) T has finite global dimension, and

(e) T has finite Gelfand–Kirillov dimension,

then T is called *Artin–Schelter regular* (or *AS regular*, for short) of dimension d.

In this paper we generally reserve the letters S and T for AS regular algebras. Artin–Schelter regular algebras play an important role in noncommutative algebraic geometry and many other subjects [Artin and Schelter 1987; Artin et al. 1990; 1991; Reyes et al. 2014]. Therefore, it would be very useful to have reasonably verifiable criteria for the AS regular property.

Recall that, for a graded left A-module M, we say that M *has a linear resolution* (or simply say that M *is linear*) if M has a minimal free A-resolution of the form

$$\cdots \to A(-i)^{\beta_i} \to A(-i+1)^{\beta_{i-1}} \to \cdots \to A(-1)^{\beta_1} \to A^{\beta_0} \to M \to 0 \tag{0.2.1}$$

for some integers β_i, or, equivalently, $\operatorname{Tor}^A_i(\Bbbk, M)_j = 0$ for all $j \neq i$. In this paper it is necessary to deal with (cochain) complexes of graded A-modules instead of graded A-modules. A complex X of graded left A-modules is naturally graded by homological and internal degrees. By examining certain complexes, properties of A can be reflected in the relationships between these degrees. For example, A is *Koszul* if the trivial graded A-module $\Bbbk$ has a linear resolution, or, equivalently if the *Tor regularity* (Definition 0.4) of $\Bbbk$ is 0. Here is one of our main results:

Theorem 0.3 (Theorem 4.6). *Let A be a noetherian connected graded s-Cohen–Macaulay algebra. Suppose that there is a graded algebra map $f : T \to A$, where T is a Koszul Artin–Schelter regular algebra, such that the induced modules A_T and ${}_TA$ are both finitely generated. Then the following are equivalent*:

(a) *A is Artin–Schelter regular.*

(b) *The Hilbert series $h_A(t)$ of A satisfies* $\deg h_A(t) = -s$.

In the commutative case, the existence of the desired map $f : T \to A$ is automatic (assuming A is generated in degree 1). Similar to Theorem 0.1, the combinatorial property in Theorem 0.3(b) is much easier to check than the definition of AS regularity and so Theorem 0.3 provides an important criterion

which can be used to determine whether or not A is AS regular. In the rest of paper, we will prove other criteria for determining AS regularity by using more sophisticated numerical homological invariants.

We now recall or introduce the following homological invariants, which will be studied throughout the paper:

- Tor regularity,
- Castelnuovo–Mumford regularity,
- (numerical) Artin–Schelter regularity,
- concavities associated to the Tor and Castelnuovo–Mumford regularities.

Definition 0.4 [Jørgensen 2004; 1999; Dong and Wu 2009]. The *Tor regularity* of a nonzero complex X of graded left A-modules is defined to be

$$\operatorname{Torreg}(X) = \sup_{i,j\in\mathbb{Z}} \{j - i \mid \operatorname{Tor}_i^A(\Bbbk, X)_j \neq 0\}. \tag{0.4.1}$$

It is clear that $\operatorname{Torreg}(X)$ provides a measure of the growth of the degrees of generators of the free modules in a minimal free resolution of X.

Since $\operatorname{Torreg}({}_A\Bbbk) \geq 0$ and the equality holds if and only if A is Koszul, $\operatorname{Torreg}({}_A\Bbbk)$ can be regarded as an invariant that measures how far A is from being Koszul, at least in the noncommutative setting. When A is a commutative algebra generated in degree one, $\operatorname{Torreg}({}_A\Bbbk)$ takes on only the values of 0 or ∞ [Avramov and Peeva 2001], so it indicates only whether or not A is Koszul. However, in the noncommutative case, $\operatorname{Torreg}({}_A\Bbbk)$ can be any value in $\mathbb{N} \cup \{+\infty\}$ (see Example 2.4(4) and Lemma 5.6).

Next we recall the definition of Castelnuovo–Mumford regularity, which was first introduced and studied in the noncommutative setting in [Jørgensen 2004; 1999; Dong and Wu 2009]. Recall that the *i-th local cohomology* of a complex X of graded left A-modules is defined to be

$$H^i_{\mathfrak{m}}(X) = \lim_{n\to\infty} \operatorname{Ext}^i_A(A/\mathfrak{m}^n, X). \tag{0.4.2}$$

See [Jørgensen 1997; 2004] and Section 1 for more details.

Definition 0.5 [Jørgensen 2004; 1999; Dong and Wu 2009]. Let A be a noetherian connected graded algebra and let X be a nonzero complex of graded left A-modules. The *Castelnuovo–Mumford regularity* (or *CM regularity*, for short) of X is defined to be

$$\operatorname{CMreg}(X) = \sup_{i,j\in\mathbb{Z}} \{j + i \mid H^i_{\mathfrak{m}}(X)_j \neq 0\}.$$

The notions of regularity defined in Definitions 0.4 and 0.5 are natural generalizations of the classical Tor and Castelnuovo–Mumford regularities in the commutative case. For a finitely generated graded A-module M, the relationship between the regularities $\operatorname{Torreg}(M)$ and $\operatorname{CMreg}(M)$ has been studied. When A is a polynomial ring generated in degree 1, $\operatorname{Torreg}(M) = \operatorname{CMreg}(M)$ for any finitely generated graded A-module M [Eisenbud and Goto 1984]. Other relationships between these invariants were established in the commutative case [Römer 2008] and were extended to the noncommutative case in [Jørgensen 2004;

1999; Dong and Wu 2009]. In this paper we provide further relationships between these invariants in the noncommutative case.

Now we introduce a new numerical homological invariant associated to every noetherian connected graded algebra A.

Definition 0.6. Let A be a noetherian connected graded algebra. The *Artin–Schelter regularity* (or *AS regularity*) of A is defined to be

$$\mathrm{ASreg}(A) = \mathrm{Torreg}(\Bbbk) + \mathrm{CMreg}(A).$$

We remark that, in the literature, the phrase "AS regularity of A" often refers to A possessing the Artin–Schelter regular property given in Definition 0.2. Here we use the phrase "AS regularity of A" to refer to the numerical invariant of A defined in Definition 0.6, which is related to the AS regular property. Indeed, by [Jørgensen 2004, Theorems 2.5 and 2.6] or Corollary 2.6(2), $\mathrm{ASreg}(A) \geq 0$; and, by Theorem 0.8, equality holds if and only if A is AS regular. Hence, $\mathrm{ASreg}(A)$ can be considered as an invariant that measures how far A is from being AS regular.

By Example 2.4(3), if T is AS regular of type (d, ℓ), then

$$\mathrm{CMreg}(T) = d - \ell = -\mathrm{Torreg}(\Bbbk), \tag{0.6.1}$$

which will appear in several places in this paper. As a consequence, $\mathrm{ASreg}(T) = 0$. Some further computations of regularities in the non-Koszul case are provided in Example 2.4(3).

The next theorem is another main result of the paper, which was announced in [Dong and Wu 2009, Proposition 5.6] without proof. It is also an extension of [Römer 2008, Theorem 4.2]. See Corollary 3.4 for a related result.

Theorem 0.7 (Theorem 2.8). *Let A be a noetherian connected graded algebra with balanced dualizing complex. Let X be a nonzero object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$ with finite projective dimension. Then*

$$\mathrm{CMreg}(X) = \mathrm{Torreg}(X) + \mathrm{CMreg}(A).$$

As a companion to Theorem 0.7, we generalize two very nice results of [Dong and Wu 2009, Theorems 4.10 and 5.4] to the not-necessarily Koszul setting.

Theorem 0.8. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent*:

(a) *A is AS regular.*

(b) $\mathrm{ASreg}(A) = 0$.

Another main goal of this paper is to introduce a new notion of regularity, which we call the *concavity* of a graded algebra. If $f : A \to B$ is a graded algebra homomorphism between two $\mathbb{N}$-graded algebras. We say that f is a *finite map* if the modules ${}_AB$ and B_A are finitely generated.

Definition 0.9. Let $\mathcal{P}$ be a numerical invariant that is defined on all AS regular algebras. Let A be a locally finite $\mathbb{N}$-graded algebra (that is not necessarily connected graded).

(1) The $\mathcal{P}$-*concavity* of A is defined to be

$$c_{\mathcal{P}}(A) := \inf\{\mathcal{P}(T) \mid T \text{ is AS regular and there is a finite map } f : T \to A\}.$$

If no such T exists, we define $c_{\mathcal{P}}(A) = \infty$. A similar convention is used in the other parts of this definition.

If $\mathcal{P}$ is a numerical invariant that is defined on locally finite $\mathbb{N}$-graded algebras and if $\mathcal{P}(A)$ is finite, then we define the *normalized* $\mathcal{P}$-*concavity* of A to be

$$c_{\mathcal{P},-}(A) := c_{\mathcal{P}}(A) - \mathcal{P}(A).$$

Below are some special cases.

(2) If we take $\mathcal{P} = \mathrm{Torreg}$, then the Torreg-*concavity* of A is defined to be

$$c_{\mathrm{Torreg}}(A) := \inf\{\mathrm{Torreg}(T) \mid T \text{ is AS regular and there is a finite map } f : T \to A\}.$$

(3) If we take $\mathcal{P} = -\mathrm{CMreg}$, then the *concavity* of A is defined to

$$c(A) := \inf\{-\mathrm{CMreg}(T) \mid T \text{ is AS regular and there is a finite map } f : T \to A\}.$$

The *normalized concavity* of A is defined to be

$$c_{-}(A) := c(A) + \mathrm{CMreg}(A).$$

In this paper we focus primarily on the concavity and the normalized concavity of the invariant $\mathcal{P} = -\mathrm{CMreg}$. By (0.6.1), for any AS regular algebra T, $\mathrm{CMreg}(T) \leq 0$, and so, by definition, for any locally finite $\mathbb{N}$-graded algebra A, we have $c(A) \geq 0$. If S is a noetherian connected graded AS regular algebra, then $c(S) = 0$ if and only if S is Koszul (Theorem 0.10(2)). Let us explain the motivation behind the terminology "concavity". By analogy with the commutative case, for noncommutative T, we can imagine $\mathrm{Spec}\, T$ as a noncommutative space associated to T. If T is Koszul, then we should consider $\mathrm{Spec}\, T$ to be a flat space, as the minimal free resolution of the trivial module is linear (0.2.1). If $T_1 \to T_2$ is a finite map between two noetherian AS regular algebras (or, by analogy, if there is a finite map $\mathrm{Spec}\, T_2 \to \mathrm{Spec}\, T_1$), then we can show that $c(T_2) \leq c(T_1)$ and so the concavity of T_2 is bounded by the concavity of T_1. As a consequence, if T_1 is Koszul, then so is T_2 (Theorem 0.10(3)). Hence, in some sense, "concavity" measures how far away a noncommutative space is from being "flat".

By definition, the concavity of A depends on all finite maps f from AS regular algebras T to A. However, when A itself is AS regular, this invariant can be calculated by taking f to be the identity map, as shown in the following theorem, which can be considered as an extension of Theorem 0.3 in the case that $A = S$ is AS regular:

Theorem 0.10. *Let T and S be noetherian AS regular algebras.*

(1) $c(S) = -\mathrm{CMreg}(S)$.

(2) *S is Koszul if and only if $c(S) = 0$.*

(3) *Suppose $f : T \to S$ is a finite map. Then $c(T) \geq c(S) \geq 0$. If further $c(T) = c(S)$, then f is surjective and ${}_TS$ and S_T are linear Cohen–Macaulay modules over T.*

As a corollary to the above results, we see that, if there is a finite map between AS regular algebras $f : T \to S$, then the Koszul property of S is controlled by the Koszul property of T.

Corollary 0.11 (Corollary 4.4). *Let T and S be noetherian AS regular algebras and let $f : T \to S$ be a finite map.*

(1) *If T is Koszul (or, equivalently, $\mathrm{Torreg}({}_T\Bbbk) = 0$), then S is Koszul. Further, f is surjective and ${}_TS$ and S_T are linear Cohen–Macaulay modules over T.*

(2) *Suppose $\mathrm{Torreg}({}_T\Bbbk) \leq \deg(\Bbbk \otimes_T S)$. Then S is Koszul.*

(3) *Suppose $\mathrm{Torreg}({}_T\Bbbk) = 1$. If f is not surjective, then S is Koszul. As a consequence, if T is a proper subalgebra of S, then S is Koszul.*

We also show that (normalized) concavity is related to the AS regular property.

Theorem 0.12. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Let $c_-(A)$ be the normalized concavity defined in Definition 0.9(3). Then the following hold*:

(1) $c_-(A) \geq 0$.

(2) $c_-(A) = 0$ *if and only if A is AS regular.*

By Theorem 0.12, $c_-(A)$ can be viewed as an invariant, similar to $\mathrm{ASreg}(A)$, that measures how far away A is from being AS regular. Here is a list of numerical invariants that qualify as indicators of the AS regular property for different special classes of algebras:

(i) $\deg_t h_A(t)$ (Theorem 0.3),

(ii) $\mathrm{ASreg}(A)$ (Theorem 0.8),

(iii) $c_-(A)$ (Theorem 0.12),

(iv) $\mathrm{CMreg}(A)$ (Theorem 4.5), and

(v) $\mathrm{Torreg}({}_TA)$ (Remark 5.10).

Remark 0.13. As noted in [Stanley 1979, p. 506] (or [Kirkman et al. 2015, p. 256]) we have the following hierarchy of homological properties for commutative graded algebras:

$$\text{regular} \Longrightarrow \text{hypersurface} \Longrightarrow \text{complete intersection} \Longrightarrow \text{Gorenstein} \Longrightarrow \text{Cohen–Macaulay}.$$

In this paper, we mainly study the "regular" property. It would be very interesting if new numerical (and hopefully computable) invariants could be found to detect or characterize the other properties in the above

diagram. Of course, Stanley's theorem (Theorem 0.1) for the Gorenstein property is our model. Some necessary conditions to be a "noncommutative complete intersection" are given in [Kirkman et al. 2015].

To conclude the introduction, we mention one application of concavity to noncommutative invariant theory. Kirkman et al. [2022] used CMreg to bound the degrees of generators of invariant subrings T^H when a semisimple Hopf algebra H acts on an AS regular algebra T homogeneously. The following proposition is an easy consequence of [ibid., Theorem 0.8]. For any connected graded algebra A, let

$$\beta_i(A) = \deg \operatorname{Tor}_i^A(\Bbbk, \Bbbk).$$

Then $\beta_1(A)$ is the largest degree of an element in a minimal generating set of A and $\beta_2(A)$ is the largest degree of an element in a minimal relation set of A. We have the following lower bounds for $c(T^H)$ in terms of the degrees of the generators and the relations of T^H:

Proposition 0.14. *Let H be a semisimple Hopf algebra acting on a noetherian AS regular algebra T homogeneously. Let $R = T^H$ denote the invariant subring of this action. Then the following hold*:

(1) $c(R) \geq \beta_1(R) - 1$.

(2) $c(R) \geq \min\{\frac{1}{2}\beta_2(R) - \operatorname{CMreg}(T), \frac{1}{2}(\beta_2(R) - \operatorname{CMreg}(T) - 1), \beta_2(R) - 2\}$.

By [Kirkman et al. 2009, Lemma 3.2(b)], T^H is Cohen–Macaulay with balanced dualizing complex. Now, if we are given a noetherian Cohen–Macaulay domain A with balanced dualizing complex (or even a noetherian AS Gorenstein domain), it is generally very difficult to determine if there exist (T, H) such that A is isomorphic to T^H. The simple inequality in Proposition 0.14(1) provides an easy way of showing that a noetherian connected graded algebra A cannot be isomorphic to any invariant subring T^H of an AS regular algebra T under a semisimple Hopf algebra H action (if $c(A) < \beta_1(A) - 1$). For example, let

$$A = \Bbbk[x_1, \ldots, x_n][t]/(t^2 = f(x_1, \ldots, x_n)),$$

where $\deg x_i = 1$, $\deg t \geq 2$ and f is an irreducible homogeneous polynomial in x_i of degree $2 \deg t$. It follows from the definition that

$$0 = c(A) < 1 \leq \deg t - 1 = \beta_1(A) - 1.$$

Therefore, A cannot be isomorphic to T^H by Proposition 0.14(1). See Example 4.2 for further examples.

The paper is organized as follows. Section 1 recalls some basic definitions and properties from homological algebra (including local cohomology). Section 2 gives the definitions of invariants such as Castelnuovo–Mumford regularity, Ext or Tor regularity, and basic inequalities and equalities relating these regularities. Theorem 0.7 is proved in Section 2. In Section 3 we define AS regularity and prove Theorem 0.8. Section 4 concerns concavity associated to the Castelnuovo–Mumford regularity. We prove Theorems 0.10 and 0.12 and Proposition 0.14 on concavity in Section 4. Finally, Section 5 contains some examples, questions and remarks.

1. Preliminaries

For an $\mathbb{N}$-graded $\Bbbk$-algebra A, we let A-Gr denote the category of $\mathbb{Z}$-graded left A-modules. When convenient, we identify the graded right A-modules with graded left A^{op}-modules and denote the category by A^{op}-Gr. The derived category of complexes of graded A-modules is denoted by $\mathsf{D}(A\text{-Gr})$. We use the standard notation $\mathsf{D}^+(A\text{-Gr})$, $\mathsf{D}^-(A\text{-Gr})$ and $\mathsf{D}^{\mathrm{b}}(A\text{-Gr})$ for the full subcategories of complexes which are bounded below, bounded above and bounded, respectively. We use the subscript fg to denote the full subcategories consisting of complexes with finitely generated cohomology, e.g., $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. We use the standard convention that a left A-module M can be viewed as complex concentrated in position 0.

Let ℓ be an integer. For a graded A-module M, the shifted A-module $M(\ell)$ is defined by

$$M(\ell)_m = M_{m+\ell}$$

for all $m \in \mathbb{Z}$. For a cochain complex $X = (X^n, d^n : X^n \to X^{n+1})$, we define two notions of shifting: $X(\ell)$ shifts the degrees of each graded vector space $X^i(\ell)_m = X^i_{m+\ell}$ for all $i, m \in \mathbb{Z}$; and $X[\ell]$ shifts the complex $X^i[\ell] = X^{i+\ell}$ for all $i \in \mathbb{Z}$.

Definition 1.1. Let $A := \bigoplus_{i\geq 0} A_i$ be an $\mathbb{N}$-graded locally finite algebra. The *Hilbert series of* A is defined to be

$$h_A(t) = \sum_{i\in\mathbb{N}} (\dim_{\Bbbk} A_i) t^i.$$

Similarly, if $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a $\mathbb{Z}$-graded A-module (or $\mathbb{Z}$-graded vector space), the *Hilbert series of* M is defined to be

$$h_M(t) = \sum_{i\in\mathbb{Z}} (\dim_{\Bbbk} M_i) t^i.$$

We say that M is *locally finite* if $\dim_{\Bbbk} M_d < \infty$ for all $d \in \mathbb{Z}$. Define the *degree* of M to be the maximal degree of the nonzero homogeneous elements in M, namely

$$\deg(M) = \inf\{d \mid (M)_{\geq d} = 0\} - 1 = \sup\{d \mid (M)_d \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}. \tag{1.1.1}$$

By convention, we define $\deg(0) = -\infty$. Similarly, we define

$$\operatorname{ged}(M) = \sup\{d \mid (M)_{\leq d} = 0\} + 1 = \inf\{d \mid (M)_d \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}. \tag{1.1.2}$$

By convention, we define $\operatorname{ged}(0) = \infty$.

For a nonzero cochain complex X in $\mathsf{D}(A\text{-Gr})$, the *degree* of X is defined to be

$$\deg(X) = \sup_{m,n\in\mathbb{Z}} \{m + n \mid H^n(X)_m \neq 0\} = \sup_{n\in\mathbb{Z}} \{\deg H^n(X) + n\}, \tag{1.1.3}$$

where $H^d(X)$ is the d-th homology of the complex X. Similarly, the ged of X is defined to be

$$\operatorname{ged}(X) = \inf_{m,n\in\mathbb{Z}} \{m + n \mid H^n(X)_m \neq 0\} = \inf_{n\in\mathbb{Z}} \{\operatorname{ged} H^n(X) + n\}. \tag{1.1.4}$$

We also define

$$\sup(X) = \sup\{d \mid H^d(X) \neq 0\}$$

and

$$\inf(X) = \inf\{d \mid H^d(X) \neq 0\}.$$

Let A be a connected graded algebra. Recall that $\mathfrak{m}$ denotes the graded Jacobson radical (or maximal graded ideal) $A_{\geq 1}$ and that $\Bbbk$ denotes the graded A-bimodule $A/\mathfrak{m}$. For a graded left A-module M, let

$$t_i^A({}_AM) = \deg \operatorname{Tor}_i^A(\Bbbk, M). \tag{1.1.5}$$

If M is a graded right A-module, let

$$t_i^A(M_A) = \deg \operatorname{Tor}_i^A(M, \Bbbk). \tag{1.1.6}$$

It is clear that $t_i^A({}_A\Bbbk) = t_i^A(\Bbbk_A)$. If the context is clear, we will use $t_i^A(M)$ instead of $t_i^A({}_AM)$ (or $t_i^A(M_A)$).

For each graded left A-module M, we define

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1\} = \lim_{n\to\infty} \operatorname{Hom}_A(A/A_{\geq n}, M)$$

and call this the $\mathfrak{m}$-*torsion submodule* of M. It is standard that the functor $\Gamma_{\mathfrak{m}}(-)$ is a left exact functor $A\text{-Gr} \to A\text{-Gr}$. Since this category has enough injectives, the i-th right derived functors, denoted by $H^i_{\mathfrak{m}}$ or $\mathrm{R}^i\Gamma_{\mathfrak{m}}$, are defined and called the *local cohomology functors*; see [Artin and Zhang 1994; Van den Bergh 1997; Jørgensen 1997; 2004] for more details. For a complex X of graded left A-modules, the i-th local cohomology group of X is given by (0.4.2). For example, if M is a graded left A-module, then

$$H^i_{\mathfrak{m}}(M) = \mathrm{R}^i\Gamma_{\mathfrak{m}}(M) := \lim_{n\to\infty} \operatorname{Ext}^i_A(A/A_{\geq n}, M). \tag{1.1.7}$$

Definition 1.2. Let A be a connected graded noetherian graded algebra. Let M be a finitely generated graded left A-module. We call M *s-Cohen–Macaulay* or simply *Cohen–Macaulay* if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq s$ and $H^s_{\mathfrak{m}}(M) \neq 0$. We say A is *Cohen–Macaulay* if ${}_AA$ is Cohen–Macaulay.

Throughout the rest of this paper, when we need a dualizing complex we assume the following hypothesis; we refer the reader to [Yekutieli 1992] for the definitions of a dualizing complex and a balanced dualizing complex.

Hypothesis 1.3. Let A be a noetherian connected graded algebra with balanced dualizing complex. In this case, by [Van den Bergh 1997, Theorem 6.3], the balanced dualizing complex will be given by $\mathrm{R}\Gamma_{\mathfrak{m}}(A)'$, where $'$ denotes the graded vector space dual.

The local cohomological dimension of a graded A-module M is defined to be

$$\operatorname{lcd}(M) := \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M) \neq 0\}$$

and the cohomological dimension of $\Gamma_{\mathfrak{m}}$ is defined to be

$$\operatorname{cd}(\Gamma_{\mathfrak{m}}) = \sup_{M\in A\text{-Gr}} \{\operatorname{lcd}(M)\}.$$

We will use the following *local duality theorem* of Van den Bergh several times.

Theorem 1.4 [Van den Bergh 1997, Theorem 5.1]. *Let A be a noetherian connected graded $\Bbbk$-algebra with $\operatorname{cd}(\Gamma_{\mathfrak{m}}) < \infty$ and let C be a connected graded $\Bbbk$-algebra. Then, for any $X \in \mathsf{D}((A \otimes C^{\mathrm{op}})\text{-Gr})$, there is an isomorphism*

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X)' \cong \mathrm{RHom}_A(X, \mathrm{R}\Gamma_{\mathfrak{m}}(A)')$$

in $\mathsf{D}((C \otimes A^{\mathrm{op}})\text{-Gr})$.

2. Equalities and inequalities

In this section we study the relationships between the regularities defined in the previous sections, recalling and generalizing results of [Jørgensen 2004; 1999; Dong and Wu 2009]. Throughout this section, we assume Hypothesis 1.3.

Recall, from Definition 0.5, that the Castelnuovo–Mumford regularity of cochain complex X of left A-modules is defined to be

$$\mathrm{CMreg}(X) = \deg(\mathrm{R}\Gamma_{\mathfrak{m}}(X)) = \sup_{i,j \in \mathbb{Z}} \{j + i \mid H^i_{\mathfrak{m}}(X)_j \neq 0\}.$$

As noted in [Jørgensen 2004, Observation 2.3], if $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$, then $\mathrm{R}\Gamma_{\mathfrak{m}}(X)' \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A^{\mathrm{op}}\text{-Gr})$ and $\mathrm{R}\Gamma_{\mathfrak{m}}(X)' \ncong 0$. It follows that $\mathrm{CMreg}(X)$ is finite. In particular, if A is a finitely generated commutative algebra, then $\mathrm{CMreg}(X)$ is finite for all $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. In Example 5.1, we show that there exists a noetherian connected domain which does not satisfy Hypothesis 1.3 with $\mathrm{GKdim}\, A = 2$ and $\mathrm{CMreg}(A) = \infty$.

Example 2.1. Assume Hypothesis 1.3.

(1) If M is a finite-dimensional nonzero graded left A-module, then $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq 0$ and $H^0_{\mathfrak{m}}(M) = M$, so

$$\mathrm{CMreg}(M) = \deg(M). \tag{2.1.1}$$

A more general case is considered in part (4), as a finite-dimensional module is 0-Cohen–Macaulay.

(2) Let A be an AS Gorenstein algebra of type (d, ℓ). Then $\mathrm{CMreg}(A) = d - \ell$. This is a well-known fact, which is a consequence of [Artin and Zhang 1994, Theorem 8.1(3)].

(3) Let A be an AS regular algebra of type (d, ℓ). Recall that, when regarded as a rational function, $\deg_t h_A(t) = -\ell$ [Stephenson and Zhang 1997, Proposition 3.1(4)]. Hence,

$$\mathrm{CMreg}(A) = d - \ell = \mathrm{gldim}\, A + \deg_t h_A(t).$$

By the second statement in [Stephenson and Zhang 1997, Proposition 3.1(4)], $d \leq \ell$. As a consequence,

$$\mathrm{CMreg}(A) = d - \ell \leq 0. \tag{2.1.2}$$

(4) If M is s-Cohen–Macaulay, then, by definition,

$$\mathrm{CMreg}(M) = s + \deg(H^s_{\mathfrak{m}}(M)). \tag{2.1.3}$$

Recall that the *Tor regularity* of a nonzero complex X of graded left A-modules was defined in Definition 0.4, namely

$$\operatorname{Torreg}(X) = \deg(\Bbbk \otimes_A^L X) = \sup_{i,j\in\mathbb{Z}} \{j - i \mid \operatorname{Tor}_i^A(\Bbbk, X)_j \neq 0\}.$$

Definition 2.2. Let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. The Ext *regularity* of X is defined to be

$$\operatorname{Extreg}(X) = -\operatorname{ged}(\mathrm{R}\operatorname{Hom}_A(X, \Bbbk)) = -\inf_{i\in\mathbb{Z}}\{\operatorname{ged}(\operatorname{Ext}_A^i(X, \Bbbk)) + i\}.$$

By [Dong and Wu 2009, Remark 4.5], if X has a finitely generated minimal free resolution over A, then $\operatorname{Extreg}(X) = \operatorname{Torreg}(X)$, and we will not distinguish between $\operatorname{Extreg}(X)$ and $\operatorname{Torreg}(X)$ in this case. The following easy lemma is useful for computing Torreg of modules over tensor products. Throughout this paper, $\otimes$ means $\otimes_{\Bbbk}$.

Lemma 2.3. *Let A and B be connected graded algebras. Let P (resp. Q) be a nonzero object in $\mathsf{D}^-_{\mathrm{fg}}(A\text{-Gr})$ (resp. $\mathsf{D}^-_{\mathrm{fg}}(B\text{-Gr})$). Then* $\operatorname{Torreg}({}_{A\otimes B}P \otimes Q) = \operatorname{Torreg}({}_A P) + \operatorname{Torreg}({}_B Q)$.

Proof. Replacing P by its (minimal) free resolution, we can assume that each term P^i is a finitely generated free A-module. The same applies to Q. Let X and Y denote the complexes given by tensoring P and Q with $\Bbbk_A$ and $\Bbbk_B$, respectively, on the left. Then $\operatorname{Tor}_i^A(\Bbbk, P)$ and $\operatorname{Tor}_i^B(\Bbbk, Q)$ can be computed by taking homology of X and Y, respectively. Further, $\operatorname{Tor}_i^{A\otimes B}(\Bbbk, P \otimes Q)$ can be computed by taking homology of the complex $X \otimes Y$. By the Künneth formula (see, e.g., [Rotman 1979, Theorem 10.8.1]), we have

$$\bigoplus_{p+q=n} \operatorname{Tor}_p^A(\Bbbk, P) \otimes \operatorname{Tor}_q^B(\Bbbk, Q) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \cong H_n(X \otimes Y) \cong \operatorname{Tor}_n^{A\otimes B}(\Bbbk, P \otimes Q).$$

Therefore, using the convention that $\deg(0) = -\infty$,

$$\begin{aligned}
\operatorname{Torreg}({}_{A\otimes B}P \otimes Q) &= \sup_{n\in\mathbb{Z}}\{\deg(\operatorname{Tor}_n^{A\otimes B}(\Bbbk, P \otimes Q)) - n\} \\
&= \sup_{p,q\in\mathbb{Z}} \{\deg(\operatorname{Tor}_p^A(\Bbbk, P)) + \deg(\operatorname{Tor}_q^B(\Bbbk, Q)) - (p+q)\} \\
&= \sup_{p\in\mathbb{Z}}\{\deg(\operatorname{Tor}_p^A(\Bbbk, P)) - p\} + \sup_{q\in\mathbb{Z}}\{\deg(\operatorname{Tor}_q^A(\Bbbk, Q)) - q\} \\
&= \operatorname{Torreg}({}_A P) + \operatorname{Torreg}({}_B Q),
\end{aligned}$$

as desired. □

Example 2.4. Assume Hypothesis 1.3.

(1) If $M \in A\text{-Gr}$ and $r = \operatorname{Torreg}(M)$, then

$$t_i^A({}_A M) := \deg(\operatorname{Tor}_i^A(\Bbbk, M)) \leq (r + i) \tag{2.4.1}$$

for all i.

(2) $\operatorname{Extreg}(A) = \operatorname{Torreg}(A) = 0$.

(3) Let T be any noetherian AS regular algebra of type (d, ℓ). It is well known that

$$\operatorname{Torreg}({}_T\Bbbk) = \ell - d \quad (\text{which is equal to } -\operatorname{CMreg}(T)). \tag{2.4.2}$$

This assertion follows from [Stephenson and Zhang 1997, Proposition 3.1(3) and equation (3.2)].

Next we give an explicit example. Let T be a non-Koszul AS regular algebra of global dimension 3 that is generated in degree 1. By [Artin and Schelter 1987], T is generated by two elements that satisfy cubic relations. So the minimal free resolution of ${}_T\Bbbk$ has the form

$$0 \to T(-4) \to T(-3)^{\oplus 2} \to T(-1)^{\oplus 2} \to T \to \Bbbk \to 0.$$

Therefore, T is of type $(3, 4)$ and

$$t_i^T(\Bbbk) = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ 3 & \text{if } i = 2, \\ 4 & \text{if } i = 3, \\ -\infty & \text{if } i > 3. \end{cases}$$

By Example 2.1(3), $\operatorname{CMreg}(T) = -4 + 3 = -1$ and it is easy to check that

$$\operatorname{Torreg}(\Bbbk) = \max\{0, 1-1, 3-2, 4-3, -\infty\} = 1.$$

As a consequence, $\operatorname{Torreg}(\Bbbk) = -\operatorname{CMreg}(T)$, or, equivalently,

$$\operatorname{ASreg}(T) = 0.$$

Of course, this equation always holds for all AS regular algebras by (2.4.2).

(4) Let n be a fixed positive integer. Let A be any finitely generated commutative Koszul algebra (but not regular) and let T be the algebra in part (3). Let $B = A \otimes T^{\otimes n}$. By a similar argument to [Kirkman et al. 2023, Lemma 2.7], using the Künneth formula, one can easily check that $\operatorname{Torreg}({}_B\Bbbk) = n$ (Lemma 2.3) and that B is neither AS regular nor Koszul.

The following result of Jørgensen plays an important role in this paper:

Theorem 2.5 [Jørgensen 2004, Theorems 2.5 and 2.6]. *Let A be a noetherian connected graded algebra with a balanced dualizing complex, and let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.*

(1) $\operatorname{Torreg}(X) \le \operatorname{CMreg}(X) + \operatorname{Torreg}(\Bbbk)$.

(2) $\operatorname{CMreg}(X) \le \operatorname{Torreg}(X) + \operatorname{CMreg}(A)$.

We have the following immediate consequences:

Corollary 2.6. *Let A be a noetherian connected graded algebra with a balanced dualizing complex and let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.*

(1) *If* $\operatorname{Torreg}(\Bbbk)$ *is finite, then so is* $\operatorname{Torreg}(X)$.

(2) $\operatorname{ASreg}(A) \ge 0$.

(3) *If* $\mathrm{ASreg}(A) = 0$, *then equality holds in both parts of Theorem 2.5. Namely*,

$$\mathrm{Torreg}(X) = \mathrm{CMreg}(X) + \mathrm{Torreg}(\Bbbk).$$

Proof. (1) By Theorem 2.5(1),

$$\mathrm{Torreg}(X) \leq \mathrm{CMreg}(X) + \mathrm{Torreg}(\Bbbk),$$

and, since $\mathrm{CMreg}(X)$ is finite, the result follows.

(2) The statement follows by taking the sum of two inequalities in Theorem 2.5.

(3) If $\mathrm{ASreg}(A) = 0$, then $\mathrm{Torreg}(\Bbbk) = -\mathrm{CMreg}(A)$. Hence, using both inequalities in Theorem 2.5, we have

$$\mathrm{Torreg}(X) \leq \mathrm{CMreg}(X) + \mathrm{Torreg}(\Bbbk) = \mathrm{CMreg}(X) - \mathrm{CMreg}(A) \leq \mathrm{Torreg}(X)$$

and so equality holds throughout. □

We will show that if X has finite projective dimension, then Theorem 2.5(2) becomes an equality. We will need the following straightforward lemma. Recall that, for a cochain complex X and $\ell \in \mathbb{Z}$, the complexes $X(\ell)$ and $X[\ell]$ were defined at the beginning of Section 1.

Lemma 2.7. *Let A be a noetherian connected graded algebra and X be a nonzero complex of graded left A-modules. Suppose* $\deg(X)$ *is finite. Then*

$$\deg(X[1]) = \deg(X) - 1 \quad \textit{and} \quad \deg(X(1)) = \deg(X) - 1.$$

Similar equations hold for $\mathrm{ged}(X)$, $\mathrm{CMreg}(X)$, $\mathrm{Extreg}(X)$ *and* $\mathrm{Torreg}(X)$.

For a cochain complex

$$X = (\cdots \to X^{s-1} \to X^s \to X^{s+1} \to \cdots),$$

we denote the *brutal truncations* of X by

$$X^{\geq s} := (\cdots \to 0 \to \cdots \to 0 \to X^s \to X^{s+1} \to \cdots)$$

and

$$X^{\leq s} := (\cdots \to X^{s-1} \to X^s \to 0 \to \cdots \to 0 \to \cdots).$$

(We remark that the notation $X^{\geq s}$ and $X^{\leq s}$ may mean different truncations in other papers.)

Theorem 2.8. *Let A be a noetherian connected graded algebra with a balanced dualizing complex. Let X be a nonzero object in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$ *with finite projective dimension. Then*

$$\mathrm{CMreg}(X) = \mathrm{Torreg}(X) + \mathrm{CMreg}(A).$$

Proof. By Lemma 2.7, we may assume that $X^n = 0$ for all $n \geq 1$. Let F be a minimal free resolution of X, which we write as

$$F := (\cdots \to 0 \to F^{-s} \xrightarrow{d^{-s}} \cdots \to F^{-1} \xrightarrow{d^{-1}} F^0 \to 0 \to \cdots)$$

for some $s \geq 0$. We will prove the assertion by induction on s, which is (an upper bound on) the projective dimension of X.

For the initial step, we assume that $s = 0$, or $X = F^0 = \bigoplus_i A(-a_i)$ for some integers a_i. In this case, it is clear that

$$\operatorname{Torreg}(X) = \operatorname{Torreg}(F^0) = \max_i\{a_i\} =: a.$$

By Lemma 2.7,

$$\begin{aligned}\operatorname{CMreg}(X) &= \operatorname{CMreg}\left(\bigoplus_i A(-a_i)\right) = \max_i\{\operatorname{CMreg}(A(-a_i))\}\\ &= \operatorname{CMreg}(A) + \max_i\{a_i\} = \operatorname{CMreg}(A) + \operatorname{Torreg}(X),\end{aligned}$$

so the assertion holds for $X = F^0$, as required.

For the inductive step, assume that $s > 0$. Let $F^{\leq -1}$ be the brutal truncation of the complex F,

$$F^{\leq -1} := (\cdots \to 0 \to F^{-s} \to \cdots \to F^{-1} \to 0 \to 0 \to \cdots),$$

which is obtained by replacing F^0 by 0. We have a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$,

$$F^0 \xrightarrow{f} F \to F^{\leq -1} \to F^0[1], \tag{2.8.1}$$

where F^0 is viewed as a complex concentrated at position 0 and f is the inclusion. Let G be the complex $F^{\leq -1}[-1]$, which is a minimal free complex concentrated in position $\{-(s-1), \ldots, 0\}$. Then we have a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$,

$$G \xrightarrow{\phi_2} F^0 \xrightarrow{f} F \to G[1], \tag{2.8.2}$$

obtained by rotating (2.8.1). By the induction hypothesis, the assertion holds for both G and F^0. We need to show that the assertion holds for X, or, equivalently, for F, as $F \cong X$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. By Theorem 2.5(2), the assertion is equivalent to

$$\operatorname{CMreg}(F) \geq \operatorname{Torreg}(F) + \operatorname{CMreg}(A). \tag{2.8.3}$$

We fix the temporary notation

$$a = \operatorname{Torreg}(F^0), \quad b = \operatorname{Torreg}(G), \quad c = \operatorname{Torreg}(F) = \operatorname{Torreg}(X),$$

and

$$\alpha = \operatorname{CMreg}(F^0), \quad \beta = \operatorname{CMreg}(G), \quad \gamma = \operatorname{CMreg}(F) = \operatorname{CMreg}(X).$$

By definition and the minimality of F, we have

$$c = \max\{\operatorname{Torreg}(F^0), \operatorname{Torreg}(F^{\leq -1})\} = \max\{a, b-1\}.$$

By the above equation, we have

$$a \leq c \quad \text{and} \quad b - 1 \leq c, \tag{2.8.4}$$

and c must equal to either a or $b-1$. There are two cases:

(1) $c = a$ and $a \geq b$.

(2) $c = b-1$,

Case 1: Suppose that $c = a$ and $a \geq b$. By the definition of a, we have $F^0 = A(-a) \oplus C^0$, where C^0 is a graded free left A-module. Let $\phi_1 : F^0 \to A(-a)$ be the corresponding projection. By the definition of $\alpha := \mathrm{CMreg}(F^0)$, there is an integer $j \in \mathbb{Z}$ such that $H^j_{\mathfrak{m}}(F^0)_{\alpha-j} \neq 0$ and the induced projection

$$\tau_1 := H^j_{\mathfrak{m}}(\phi_1)_{\alpha-j} : H^j_{\mathfrak{m}}(F^0)_{\alpha-j} \to H^j_{\mathfrak{m}}(A(-a))_{\alpha-j}$$

is nonzero. The triangle (2.8.2) gives rise to a long exact sequence

$$\cdots \to H^j_{\mathfrak{m}}(G)_{\alpha-j} \to H^j_{\mathfrak{m}}(F^0)_{\alpha-j} \to H^j_{\mathfrak{m}}(F)_{\alpha-j} \to H^{j+1}_{\mathfrak{m}}(G)_{\alpha-j} \to \cdots . \tag{2.8.5}$$

If

$$\tau_2 := H^j_{\mathfrak{m}}(\phi_2)_{\alpha-j} : H^j_{\mathfrak{m}}(G)_{\alpha-j} \to H^j_{\mathfrak{m}}(F^0)_{\alpha-j}$$

is not surjective, then (2.8.5) implies that $H^j_{\mathfrak{m}}(F)_{\alpha-j} \neq 0$. By definition, the assumption that $a = c$, and the induction hypothesis, we have

$$\begin{aligned}
\mathrm{CMreg}(F) &\geq \alpha - j + j = \alpha \\
&= a + \mathrm{CMreg}(A) && \text{(induction hypothesis)} \\
&= c + \mathrm{CMreg}(A) && \text{(assumption in Case 1)} \\
&= \mathrm{Torreg}(F) + \mathrm{CMreg}(A),
\end{aligned}$$

as desired.

It remains to show the claim that τ_2 is not surjective. Assume to the contrary that τ_2 is surjective. Then so is the composed map

$$\tau_3 := \tau_1 \circ \tau_2 : H^j_{\mathfrak{m}}(G)_{\alpha-j} \to H^j_{\mathfrak{m}}(A(-a))_{\alpha-j}.$$

In particular, τ_3 is not a zero map. Note that

$$\tau_3 = \tau_1 \circ \tau_2 = H^j_{\mathfrak{m}}(\phi_1)_{\alpha-j} \circ H^j_{\mathfrak{m}}(\phi_2)_{\alpha-j} = H^j_{\mathfrak{m}}(\phi_1 \circ \phi_2)_{\alpha-j},$$

which implies that $\phi_3 := \phi_1 \circ \phi_2$ is nonzero in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. Consider F as the cone of the map $\phi_2 : G \to F^0$; it is clear that ϕ_2 is the map from the top row G to the middle row F^0 in the diagram

$$\begin{array}{ccccccccc}
F^{-s} & \longrightarrow & \cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{-1}=\phi_2} & & \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & F^0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi_1} & & \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A(-a) & \longrightarrow & 0
\end{array}$$

Since $b \leq a$, F^{-1}, which is the zeroth term in the minimal free resolution of G, is generated in degree $\leq a$. Since F is a minimal free resolution, $\operatorname{im}\phi_2 = \operatorname{im} d^{-1} \subseteq \mathfrak{m}F^0$, and, consequently, $\operatorname{im}\phi_3 \subseteq \mathfrak{m}A(-a)$. For every generator x in F^{-1} which has degree $\leq a$, the image $\phi_3(x)$ lies in $\mathfrak{m}A(-a)$, which has degree at least $a+1$. Therefore, $\phi_3(x) = 0$. This implies that $\phi_3(F^{-1}) = 0$, yielding a contradiction. So we have proved the claim and finished the proof in Case 1.

Case 2: Suppose $c = b - 1$. By the definition of $\beta := \operatorname{CMreg}(G)$, there is an integer $j \in \mathbb{Z}$ such that $H^j_{\mathfrak{m}}(G)_{\beta-j} \neq 0$. The triangle (2.8.2) gives rise to a long exact sequence

$$\cdots \to H^{j-1}_{\mathfrak{m}}(F)_{\beta-j} \to H^j_{\mathfrak{m}}(G)_{\beta-j} \to H^j_{\mathfrak{m}}(F^0)_{\beta-j} \to \cdots. \tag{2.8.6}$$

By the induction hypothesis, the assumption that $c < b$, (2.8.4), and the definitions of α, β, a and b, we have

$$\begin{aligned}
\beta &= \operatorname{CMreg}(G) \\
&= \operatorname{Torreg}(G) + \operatorname{CMreg}(A) && \text{(induction hypothesis)} \\
&= b + \operatorname{CMreg}(A) > c + \operatorname{CMreg}(A) \\
&\geq a + \operatorname{CMreg}(A) = \operatorname{CMreg}(F^0) && \text{(induction hypothesis)} \\
&= \alpha,
\end{aligned}$$

and thus $H^j_{\mathfrak{m}}(F^0)_{\beta-j} = 0$. Since $H^j_{\mathfrak{m}}(G)_{\beta-j} \neq 0$ by definition, (2.8.6) implies that $H^{j-1}_{\mathfrak{m}}(F)_{\beta-j} \neq 0$. By definition, $\operatorname{CMreg}(F) \geq \beta - j + (j-1) = \beta - 1$. This inequality implies that

$$\begin{aligned}
\operatorname{CMreg}(F) \geq \beta - 1 &= \operatorname{CMreg}(G) - 1 \\
&= \operatorname{Torreg}(G) + \operatorname{CMreg}(A) - 1 && \text{(induction hypothesis)} \\
&= b + \operatorname{CMreg}(A) - 1 = (b-1) + \operatorname{CMreg}(A) \\
&= c + \operatorname{CMreg}(A) \\
&= \operatorname{Torreg}(F) + \operatorname{CMreg}(A),
\end{aligned}$$

as desired; see (2.8.3).

Combining these two cases completes the proof. □

We conclude this section with the following remark:

Remark 2.9. Let A and B be noetherian connected graded algebras with balanced dualizing complexes. Let P (resp. Q) be a nonzero object in $\mathsf{D}^b_{\mathrm{fg}}(A\text{-Gr})$ (resp. $\mathsf{D}^b_{\mathrm{fg}}(B\text{-Gr})$). Then

$$\operatorname{CMreg}({}_{(A\otimes B)}P \otimes Q) = \operatorname{CMreg}({}_A P) + \operatorname{CMreg}({}_B Q).$$

The proof of the above equality is similar to the proof of Lemma 2.3 using a version of the Künneth formula for local cohomology, so it is omitted.

3. Artin–Schelter regularities

In this section we prove results that are related to AS regular algebras. Recall from Definition 0.6 that *the AS regularity* of A is defined to be

$$\mathrm{ASreg}(A) = \mathrm{Torreg}(\Bbbk) + \mathrm{CMreg}(A).$$

In general, $\mathrm{ASreg}(A)$ can be any positive integer, as the next example shows.

Example 3.1. Let $d \geq 2$ be an integer and let $B = \Bbbk[x]/(x^d)$ with $\deg x = 1$. By Example 2.1(2), $\mathrm{CMreg}(B) = d - 1$. By an easy computation,

$$\deg \mathrm{Tor}_n^B(\Bbbk, \Bbbk) = \begin{cases} 0 & \text{if } n = 0, \\ \lfloor \frac{1}{2} n \rfloor d & \text{if } n > 0 \text{ is even}, \\ 1 + \lfloor \frac{1}{2} n \rfloor d & \text{if } n > 0 \text{ is odd}. \end{cases}$$

As a consequence, if $d > 2$, then $\mathrm{Torreg}({}_B\Bbbk) = \infty$ and $\mathrm{ASreg}(B) = \infty$. If $d = 2$, then $\mathrm{Torreg}({}_B\Bbbk) = 0$ and $\mathrm{ASreg}(B) = d - 1 = 1$.

Now let $d = 2$ so $B = \Bbbk[x]/(x^2)$ and let C be the algebra $B^{\otimes m}$ for a positive integer m. Then, by Example 2.1(2), $\mathrm{CMreg}(C) = m$. Since C is Koszul, $\mathrm{Torreg}({}_C\Bbbk) = 0$. Therefore, $\mathrm{ASreg}(C) = m$.

The goal of this section is to prove Theorem 0.8. We begin with a generalization of a nice result of Dong and Wu [2009, Theorem 4.10] that provides the first step towards to the proof of Theorem 0.8.

Theorem 3.2. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent*:

(i) *A is AS regular.*

(ii) *A is Cohen–Macaulay and* $\mathrm{ASreg}(A) = 0$.

When A is Koszul, [Dong and Wu 2009, Theorem 4.10] can be recovered from Theorem 3.2 since standard AS Gorenstein algebras satisfy (ii) in the above theorem.

Proof. We first prove that (i) implies (ii). Suppose that A is AS regular of type $(d, \mathfrak{l})$. It is well known that A is Cohen–Macaulay. By (2.4.2), $\mathrm{ASreg}(A) = 0$.

We now show that (ii) implies (i). Let A be noetherian connected graded with balanced dualizing complex. If $\mathrm{projdim}\, {}_A\Bbbk < \infty$, then A has finite global dimension. Since A is noetherian, if it has finite global dimension, then it has finite GK dimension. By [Zhang 1997, Theorem 0.3], A is AS Gorenstein and so A is AS regular by definition. Hence, it suffices to show that $\mathrm{projdim}\, \Bbbk < \infty$.

Let

$$F := (\cdots \to F_i \to \cdots \to F_0 \to \Bbbk \to 0)$$

be a minimal free resolution of the trivial left A-module $\Bbbk$. Since A is Cohen–Macaulay, by [Van den Bergh 1997, Theorem 6.3], the balanced dualizing complex over A is

$$R := \mathrm{R}\Gamma_{\mathfrak{m}}(A)' \cong \omega[d],$$

where ω is a dualizing A-bimodule and $d := \operatorname{lcd}(A)$. By the local duality theorem (Theorem 1.4), for every complex X of graded left A-modules,

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X)' \cong \mathrm{R}\operatorname{Hom}_A(X, R) \cong \mathrm{R}\operatorname{Hom}_A(X, \omega[d]). \tag{3.2.1}$$

Since the dualizing complex has finite injective dimension, we obtain that

$$\operatorname{injdim}(\omega) = d < \infty.$$

By the consequence of the local duality theorem (3.2.1), $\Gamma_{\mathfrak{m}}$ has cohomological dimension d.

For each $j \geq 0$, let $Z_j(F)$ denote the j-th syzygy of the complex F. We will show that $Z_j(F) = 0$ for $j \gg 0$, which implies that $\operatorname{projdim} \Bbbk < \infty$, as desired. Assume to the contrary that $Z_j(F) \neq 0$ for all $j \geq 0$. Note that

$$\cdots \to F_{j+2} \to F_{j+1} \to Z_j(F) \to 0$$

is a minimal free resolution of $Z_j(F)$.

Claim. *For all $j \geq d$, $t^A_{j+1}(\Bbbk) \leq t^A_j(\Bbbk)$.*

Proof. By the balanced condition, $\operatorname{Ext}^i_A(\Bbbk, \omega) = 0$ for all $i \neq d = \operatorname{injdim} \omega$. By induction on syzygies, we have $\operatorname{Ext}^i_A(Z_{d-1}(F), \omega) = 0$ for all $i \neq 0$. Further, by induction, one sees that $\operatorname{Ext}^i_A(Z_{j-1}(F), \omega) = 0$ for all $i \neq 0$ and all $j \geq d$. From now on, we fix $j \geq d$. By local duality (3.2.1), we obtain that $H^i_{\mathfrak{m}}(Z_{j-1}(F)) = 0$ for all $i \neq d$. Since A is Cohen–Macaulay, $H^i_{\mathfrak{m}}(F_j) = 0$ for all $i \neq d$. Applying $\mathrm{R}\Gamma_{\mathfrak{m}}(-)$ to the short exact sequence

$$0 \to Z_j(F) \to F_j \to Z_{j-1}(F) \to 0,$$

we obtain a long exact sequence, which has only three nonzero terms, yielding a short exact sequence

$$0 \to H^d_{\mathfrak{m}}(Z_j(F)) \to H^d_{\mathfrak{m}}(F_j) \to H^d_{\mathfrak{m}}(Z_{j-1}(F)) \to 0.$$

This short exact sequence implies that $\deg H^d_{\mathfrak{m}}(Z_j(F)) \leq \deg H^d_{\mathfrak{m}}(F_j)$. By definition, $\operatorname{CMreg}(Z_j(F)) \leq \operatorname{CMreg}(F_j)$. Since $\operatorname{ASreg}(A) = 0$, Corollary 2.6(3), for any $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$,

$$\operatorname{Torreg}(X) = \operatorname{CMreg}(X) + c,$$

where $c = \operatorname{Torreg}({}_A\Bbbk) = -\operatorname{CMreg}(A)$. Then

$$\begin{aligned} t^A_{j+1}(\Bbbk) &= t^A_0(F_{j+1}) = t^A_0(Z_j(F)) = t^A_0(Z_j(F)) - 0 \\ &\leq \operatorname{Torreg}(Z_j(F)) = \operatorname{CMreg}(Z_j(F)) + c \\ &\leq \operatorname{CMreg}(F_j) + c = \operatorname{Torreg}(F_j) = \sup\{t^A_i(F_j) - i \mid i \in \mathbb{Z}\} \\ &= t^A_0(F_j) - 0 = t^A_0(F_j) \\ &= t^A_j(\Bbbk), \end{aligned}$$

as desired. This finishes the proof of the claim.

Since $t_j^A(\Bbbk) \geq j$ when $F_j \neq 0$, for $j \gg 0$ the claim contradicts the fact that $t_j^A(\Bbbk) \geq j$. Therefore, we obtain a contradiction, and hence $\operatorname{projdim} \Bbbk < \infty$, as required. □

Next we work to complete the proof of Theorem 0.8, which is a generalization of [Dong and Wu 2009, Theorem 5.4]. We need the following lemma [ibid., Lemma 5.3]. Let R be a nonzero object in $\mathsf{D}^{\mathrm{b}}(A\text{-Gr})$, which will be a balanced dualizing complex over A in the proof of Theorem 0.8. Let $f : F \to R$ be a minimal free resolution of R. Since $R \in \mathsf{D}^{\mathrm{b}}(A\text{-Gr})$, each term in F is a finitely generated free graded left A-module. Set $s = \inf R$. Then f naturally induces a morphism from the truncated complex $F^{\geq s} \to R$, denoted by $\bar{f}$.

Lemma 3.3. *Retain the above notation.*

(1) [Dong and Wu 2009, Lemma 5.3] *There is a quasi-isomorphism* $g : F^{\leq s-1} \to \operatorname{cone}(\bar{f})$.

(2) $H^i(\operatorname{cone}(\bar{f})) = \begin{cases} F^{s-1}/\operatorname{im}(d_X^{s-2}) & \text{if } i = s-1, \\ 0 & \text{if } i \neq s-1. \end{cases}$

(3) $F^{\leq s-1}$ *is a minimal free resolution of* $\operatorname{cone}(\bar{f})$.

Proof. Part (1) is a consequence of [Dong and Wu 2009, Lemma 5.3] as f is a quasi-isomorphism.

Parts (2) and (3) follow immediately from part (1). □

Proof of Theorem 0.8. By (2.4.2), (i) implies (ii).

We now show that (ii) implies (i). Let $c = -\operatorname{CMreg}(A) = \operatorname{Torreg}(\Bbbk)$. By Corollary 2.6(3),

$$\operatorname{Extreg}(X) = \operatorname{Torreg}(X) = \operatorname{CMreg}(X) + c$$

for any nonzero object X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.

Let R be a balanced dualizing complex over A. Then, by [Van den Bergh 1997, Theorem 6.3], $R \cong \mathrm{R}\Gamma_{\mathfrak{m}}(A)'$. By assumption, $\operatorname{CMreg}(A) = -c$, so, for all $i \in \mathbb{Z}$, we have $H^i_{\mathfrak{m}}(A)_{>-(i+c)} = 0$. Hence, for all integers i, $H^i(R)_{<-i+c} = 0$, or $H^i(R(c))_{<-i} = 0$, and so, by [Dong and Wu 2009, Lemma 5.2], $\operatorname{Ext}^i_A(R(c), \Bbbk)_{>-i} = 0$. By a degree shift, we obtain that $\operatorname{Ext}^i_A(R, \Bbbk)_{>-i-c} = 0$ for all $i \in \mathbb{Z}$.

Since R is a balanced dualizing complex over A, by definition, $\mathrm{R}\Gamma_{\mathfrak{m}}(R) \cong A'$, which implies that $\operatorname{CMreg}(R) = 0$. Hence, by the first paragraph, $\operatorname{Extreg}(R) = \operatorname{Torreg}(R) = \operatorname{CMreg}(R) + c = c$. By definition of $\operatorname{Extreg}(R)$, for all integers i, we have $\operatorname{Ext}^i_A(R, \Bbbk)_{<-i-c} = 0$. Combining this with the last paragraph, we obtain that

$$\operatorname{Ext}^i_A(R, \Bbbk)_j \neq 0 \quad \text{if and only if} \quad j = -i - c. \tag{3.3.1}$$

Since $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$, R has a minimal free resolution $F \xrightarrow{\cong} R$ such that each term in F is a finitely generated free graded left A-module. Set

$$s = \inf R = \inf\{j \mid H^j(R) \neq 0\}.$$

By [Van den Bergh 1997, Theorem 6.3], $R \cong \mathrm{R}\Gamma_{\mathfrak{m}}(A)'$, which implies that

$$s = -\sup\{j \mid H^j_{\mathfrak{m}}(A) \neq 0\}.$$

Since both brutal truncations $F^{\geq s}$ and $F^{\leq s-1}$ of F are minimal free complexes, (3.3.1) implies that

$$\operatorname{Extreg}(F^{\geq s}) = \begin{cases} c & \text{if } F^{\geq s} \text{ is not acyclic,} \\ -\infty & \text{if } F^{\geq s} \text{ is acyclic,} \end{cases} \tag{3.3.2}$$

and

$$\operatorname{Extreg}(F^{\leq s-1}) = \begin{cases} c & \text{if } F^{\leq s-1} \text{ is not acyclic,} \\ -\infty & \text{if } F^{\leq s-1} \text{ is acyclic.} \end{cases} \tag{3.3.3}$$

By the choice of s, $F^s \neq 0$ and $\operatorname{Ext}_A^{-s}(F^{\geq s}, \Bbbk) = \operatorname{Hom}_A(F^s, \Bbbk) \neq 0$. Hence, $F^{\geq s}$ is not acyclic, and it follows that $\operatorname{Extreg}(F^{\geq s}) = c$ by (3.3.2). Let f be the quasi-isomorphism from $F \to R$. Then f naturally induces a morphism from $F^{\geq s}$ to R, denoted by $\bar{f}$. Hence, we are in the setting of Lemma 3.3.

We claim that $\bar{f}$ is a quasi-isomorphism. Suppose on the contrary that $\bar{f}$ is not a quasi-isomorphism. Then $Y := \operatorname{cone}(\bar{f}) \ncong 0$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. Consider the distinguished triangle

$$F^{\geq s} \xrightarrow{\bar{f}} R \to Y (:= \operatorname{cone}(\bar{f})) \to F^{\geq s}[1].$$

Since $\mathrm{R}\Gamma_{\mathfrak{m}}(R) \cong A'$ in $\mathrm{D}(A^e\text{-Gr})$, we have the exact sequences

$$0 \to H^{-1}_{\mathfrak{m}}(Y) \to H^0_{\mathfrak{m}}(F^{\geq s}) \to H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(Y) \to H^1_{\mathfrak{m}}(F^{\geq s}) \to 0, \tag{3.3.4}$$

and the isomorphisms

$$H^{j-1}_{\mathfrak{m}}(Y) \cong H^j_{\mathfrak{m}}(F^{\geq s}), \quad j \neq 0, 1. \tag{3.3.5}$$

By Lemma 3.3(2), $Y \cong H^{s-1}(Y)[1-s]$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. By the local duality theorem (Theorem 1.4), we have isomorphisms

$$\mathrm{R}\Gamma_{\mathfrak{m}}(Y)' \cong \mathrm{RHom}_A(Y, R) \cong \mathrm{RHom}_A(H^{s-1}(Y)[1-s], R) \cong \mathrm{RHom}_A(H^{s-1}(Y), R)[s-1].$$

By taking the zeroth cohomology, we obtain the isomorphism

$$H^0_{\mathfrak{m}}(Y)' \cong \operatorname{Ext}^{s-1}(H^{s-1}(Y), R).$$

Since $H^j(R) = 0$ for all $j < s$, one sees that

$$H^0_{\mathfrak{m}}(Y) \cong \operatorname{Ext}^{s-1}(H^{s-1}(Y), R)' = 0.$$

Since $\operatorname{CMreg}(F^{\geq s}) = \operatorname{Extreg}(F^{\geq s}) - c = c - c = 0$, $H^{j-1}_{\mathfrak{m}}(Y)_{>-j} = 0$ for $j \neq 1$ by (3.3.4) and (3.3.5). By definition, $\operatorname{CMreg}(Y) \leq -1$. However, Lemma 3.3(3) says that $F^{\leq s-1}$ is a minimal free resolution of Y. Hence, (3.3.3) implies that $\operatorname{Extreg}(Y) = c$, and consequently, by the first paragraph, $\operatorname{CMreg}(Y) = c - c = 0$. This yields a contradiction. Therefore, $\bar{f}$ is a quasi-isomorphism and $Y \cong 0$.

Since $F^{\leq s-1}$ is a minimal free resolution of Y, $F^j = 0$ for all $j \leq s-1$, which means that the projective dimension of R is finite. By [Dong and Wu 2009, Theorem 3.6], A is AS Gorenstein. As a consequence, A is Cohen–Macaulay. It follows from Theorem 3.2 that A is AS regular. □

Corollary 3.4. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent*:

(a) *A is AS regular.*

(b) $\mathrm{Torreg}(X) = \mathrm{CMreg}(X) + \mathrm{Torreg}(\Bbbk)$ *for every nonzero* $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.

(c) $\mathrm{CMreg}(X) = \mathrm{Torreg}(X) + \mathrm{CMreg}(A)$ *for every nonzero* $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.

(d) *There is a constant c such that* $\mathrm{CMreg}(X) = \mathrm{Torreg}(X) + c$ *for every nonzero* $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.

(e) *There is a constant c such that* $\mathrm{CMreg}(M) = \mathrm{Torreg}(M) + c$ *for every nonzero finitely generated graded left A-module M.*

Proof. By Theorem 0.8 and Corollary 2.6(3), (a) implies both (b) and (c). It is clear that either (b) or (c) implies (d), and also that (d) implies (e). Hence, we need only show that (e) implies (a).

Suppose there is a constant c such that $\mathrm{CMreg}(M) = \mathrm{Torreg}(M) + c$ for every nonzero finitely generated graded left A-module M. Since $\mathrm{Torreg}(A) = 0$ (Example 2.4(2)), it follows by setting $M = A$ that $c = \mathrm{CMreg}(A)$. Since $\mathrm{CMreg}(\Bbbk) = 0$ (Example 2.1(1)), setting $M = \Bbbk$ implies that $c = -\mathrm{Torreg}(\Bbbk)$. Therefore,

$$\mathrm{ASreg}(A) = \mathrm{CMreg}(A) + \mathrm{Torreg}(\Bbbk) = c - c = 0.$$

Hence, by Theorem 0.8, A is AS regular. □

We remark that Corollary 3.4 is a generalization of [Römer 2008, Theorem 4.1].

4. Concavity

In this section we use the letters A and B for connected graded noetherian algebras, S and T for connected graded noetherian AS regular algebras, and F and G for general locally finite $\mathbb{N}$-graded noetherian algebras.

For a locally finite $\mathbb{N}$-graded noetherian algebra F, let

$$\Phi(F) := \{T \mid \text{there is a finite map } \phi : T \to F\}, \tag{4.0.1}$$

where T ranges over all connected graded noetherian AS regular algebras.

Now we recall Definition 0.9. Let $\mathcal{P}$ be any numerical invariant that is defined for locally finite $\mathbb{N}$-graded noetherian rings (or a subclass of such algebras).

(1) The *$\mathcal{P}$-concavity* of F is defined to be

$$c_{\mathcal{P}}(F) := \inf_{T \in \Phi(F)} \{\mathcal{P}(T)\}.$$

If no such T exists, we write $c_{\mathcal{P}}(F) = \infty$. The *normalized $\mathcal{P}$-concavity* of F is defined to be

$$c_{\mathcal{P},-}(F) := c_{\mathcal{P}}(F) - \mathcal{P}(F).$$

(2) The *concavity* of F is defined to

$$c(F) := \inf_{T \in \Phi(F)} \{-\mathrm{CMreg}(T)\}$$

and the *normalized concavity* of F is defined to be

$$c_-(F) := c(F) + \mathrm{CMreg}(F).$$

Proposition 4.1. *Let $f : F \to G$ be a finite map of locally finite graded noetherian algebras. Then $c_{\mathcal{P}}(F) \geq c_{\mathcal{P}}(G)$. As a consequence, the following hold*:

(1) *If F is a subalgebra of G such that ${}_FG$ and G_F are finitely generated, then $c_{\mathcal{P}}(F) \geq c_{\mathcal{P}}(G)$.*

(2) *If H is a finite dimensional semisimple Hopf algebra acting on F homogeneously, then $c_{\mathcal{P}}(F^H) \geq c_{\mathcal{P}}(F) \geq c_{\mathcal{P}}(F \# H)$.*

(3) $c_{\mathcal{P}}(F) \geq c_{\mathcal{P}}(M_n(F))$.

(4) *Let t be a commutative indeterminate of degree 1. Then $c(F[t]) = c(F)$.*

Proof. By definition, $c_{\mathcal{P}}(F) = \inf_{T \in \Phi(F)}\{\mathcal{P}(T)\}$. If $T \in \Phi(F)$, then, by definition, there is a finite map $\phi : T \to F$. But, since $f : F \to G$ is a finite map, the composition $f \circ \phi : T \to G$ is a finite map. Hence, $T \in \Phi(G)$, so $\Phi(F) \subseteq \Phi(G)$, which implies that $c_{\mathcal{P}}(F) \geq c_{\mathcal{P}}(G)$.

Parts (1)–(3) are immediate consequences of the main assertion.

For part (4), note that there is a finite map $F[t] \to F$ given by sending t to 0. Hence, taking $\mathcal{P} = -\mathrm{CMreg}$ in the main assertion, we have $c(F[t]) \geq c(F)$.

Fix a real number $\epsilon > 0$. By definition of $c(F)$, there is a noetherian AS regular algebra T of type $(d, \mathfrak{l})$ and a finite map $\phi : T \to F$ such that $-\mathrm{CMreg}(T) \leq c(F) + \epsilon$. Then $T[t] \to F[t]$ is a finite map. Hence,

$$\begin{aligned} c(F[t]) &\leq -\mathrm{CMreg}(T[t]) = -((d+1) - (\mathfrak{l} + \deg t)) \\ &= -(d - \mathfrak{l}) - (1 - \deg t) = -(d - \mathfrak{l}) \\ &= -\mathrm{CMreg}(T) \leq c(F) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we obtain that $c(F[t]) \leq c(F)$. Combined with the previous paragraph, we conclude that $c(F[t]) = c(F)$. □

We do not have any examples with strict inequalities $c_{\mathcal{P}}(F) > c_{\mathcal{P}}(F \# H)$ and $c_{\mathcal{P}}(F) > c_{\mathcal{P}}(M_n(F))$. Proposition 0.14 can be used to provide many examples with $c_{\mathcal{P}}(F^H) > c_{\mathcal{P}}(F)$. We now prove Proposition 0.14. For the rest of the paper we consider only connected graded noetherian algebras.

Proof of Proposition 0.14. Let H be a semisimple Hopf algebra acting on a noetherian AS regular algebra T homogeneously and let $R = T^H$ denote the invariant subring.

(1) By [Kirkman et al. 2022, Theorem 0.8(1)], if $S \in \Phi(R)$, then

$$\beta_1(R) \leq \max\{\beta_1(S), \mathrm{CMreg}(T) - \mathrm{CMreg}(S)\}.$$

It is clear that $\mathrm{CMreg}(T) - \mathrm{CMreg}(S) \leq -\mathrm{CMreg}(S) < -\mathrm{CMreg}(S) + 1$. By definition and Theorem 0.7,

$$\beta_1(S) = t_1^S(\Bbbk) \leq \mathrm{Torreg}({}_S\Bbbk) + 1 = \mathrm{CMreg}({}_S\Bbbk) - \mathrm{CMreg}(S) + 1 = -\mathrm{CMreg}(S) + 1,$$

which implies that $\beta_1(R) \leq -\mathrm{CMreg}(S)+1$, or, equivalently, $-\mathrm{CMreg}(S) \geq \beta_1(R)-1$. Since $S \in \Phi(R)$ was arbitrary, we obtain that $c(R) \geq \beta_1(R)-1$.

(2) By [Kirkman et al. 2022, Theorem 0.8(2)], if $S \in \Phi(R)$, then

$$\beta_2(R) \leq \max\{2(\mathrm{CMreg}(T)-\mathrm{CMreg}(S)), \mathrm{CMreg}(T)-\mathrm{CMreg}(S)+\beta_1(S), \beta_2(S)\}.$$

If $\beta_2(R) \leq 2(\mathrm{CMreg}(T)-\mathrm{CMreg}(S))$, then $-\mathrm{CMreg}(S) \geq \frac{1}{2}\beta_2(R)-\mathrm{CMreg}(T)$. Since $S \in \Phi(R)$ was arbitrary, we obtain that

$$c(R) \geq \tfrac{1}{2}\beta_2(R)-\mathrm{CMreg}(T).$$

If $\beta_2(R) \leq \mathrm{CMreg}(T)-\mathrm{CMreg}(S)+\beta_1(S)$, then $\beta_2(R) \leq \mathrm{CMreg}(T)-\mathrm{CMreg}(S)+(-\mathrm{CMreg}(S)+1)$, which implies that

$$c(R) \geq \tfrac{1}{2}(\beta_2(R)-\mathrm{CMreg}(T)-1).$$

Finally, if $\beta_2(R) \leq \beta_2(S)$, then

$$\beta_2(R) \leq \beta_2(S) \leq \mathrm{Torreg}(\Bbbk)+2 \leq \mathrm{CMreg}(\Bbbk)-\mathrm{CMreg}(S)+2 = -\mathrm{CMreg}(S)+2,$$

which implies that $c(R) \geq \beta_2(R)-2$.

Combining these three cases, we obtain that

$$c(R) \geq \min\left\{\tfrac{1}{2}\beta_2(R)-\mathrm{CMreg}(T), \tfrac{1}{2}(\beta_2(R)-\mathrm{CMreg}(T)-1), \beta_2(R)-2\right\},$$

as desired. □

The inequalities in Proposition 0.14 can be used to test if an algebra is isomorphic to the invariant subring of a semisimple Hopf action on an AS regular algebra.

Example 4.2. Let A be a connected graded algebra that is not generated in degree 1 (or, equivalently, $\beta_1(A) \geq 2$). Suppose there is a noetherian Koszul AS regular algebra $S \in \Phi(A)$. Then $c(A)=0$ by definition, and so, by Proposition 0.14(1), A cannot be isomorphic to an invariant subring T^H.

A specific example is the graded algebra

$$A = \Bbbk[x_1, \ldots, x_n, t_1, \ldots, t_m]/(t_j^2 - f_j(x_1, \ldots, x_n) \mid j = 1, \ldots, m),$$

where $\deg x_i = 1$ for all $i = 1, \ldots, n$, $\deg t_{j_0} > 1$ for some j_0, and each f_j is a homogeneous polynomial in the x_i of degree equal to $2 \deg t_j$. Then $c(A)=0$, as the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ is in $\Phi(A)$. It is clear that $\beta_1(A) = \max_{1 \leq j \leq m}\{\deg t_j\} > 1$.

Our next aim is to prove Theorems 0.10 and 0.12. We begin by proving the following lemma. As usual, let T denote a noetherian AS regular algebra.

Lemma 4.3. *Let A be a connected graded noetherian algebra, let $\phi : T \to A$ be a finite map, and let M be a finitely generated graded left A-module. For parts* (2)–(5)*, we further assume that ${}_TA$ is linear.*

(1) $\mathrm{Torreg}({}_TM) \leq \mathrm{Torreg}({}_TA) + \mathrm{Torreg}({}_AM)$.

(2) *For all* $s \geq 0$, *we have* $t_s^T(\Bbbk) - s \leq \max_{0\leq i\leq s}\{t_i^A(\Bbbk) - i\}$.

(3) *For all* $0 \leq s \leq \operatorname{gldim} T$, *we have* $\max_{0\leq i\leq s}\{t_i^A(\Bbbk) - i\} = \max_{0\leq i\leq s}\{t_i^T(\Bbbk) - i\}$.

(4) *For all* $j > \operatorname{gldim} T$, *we have*

$$t_j^A(\Bbbk) - j \leq \operatorname{Torreg}({}_T\Bbbk) = \max_{0\leq i\leq \operatorname{gldim} T}\{t_i^T(\Bbbk) - i\}.$$

(5) $\operatorname{Torreg}({}_A\Bbbk) = \operatorname{Torreg}({}_T\Bbbk)$. *As a consequence*, A *is Koszul if and only if* T *is Koszul.*

Proof. We will use the change of rings spectral sequence given in [Rotman 1979, Theorem 10.60], namely

$$E^2_{p,q} := \operatorname{Tor}^A_p(\operatorname{Tor}^T_q(\Bbbk_T, A), {}_AM) \Rightarrow \operatorname{Tor}^T_{p+q}(\Bbbk_T, {}_TM). \tag{4.3.1}$$

The E^2-page of the spectral sequence is similar to the one given after [Kirkman et al. 2022, Lemma 5.1].

(1) We have

$$\begin{aligned}
\operatorname{Torreg}({}_TM) &= \operatorname{CMreg}({}_TM) + \operatorname{Torreg}({}_T\Bbbk) && \text{(Corollary 2.6)}\\
&= \operatorname{CMreg}({}_TM) - \operatorname{CMreg}(T) && \text{(Example 2.4(3))}\\
&= \operatorname{CMreg}({}_TM) + \operatorname{Torreg}({}_TA) - \operatorname{CMreg}({}_TA) && \text{(Corollary 3.4(c))}\\
&= \operatorname{CMreg}({}_TM) + \operatorname{Torreg}({}_TA) - \operatorname{CMreg}(A)\\
&= \operatorname{Torreg}({}_TA) + \operatorname{CMreg}({}_AM) - \operatorname{CMreg}(A)\\
&\leq \operatorname{Torreg}({}_TA) + \operatorname{Torreg}({}_AM) && \text{(Theorem 2.5(2))}.
\end{aligned}$$

(2) For the remainder of the proof, we assume that ${}_TA$ is linear; that is, we assume for all q that there exists some $n_q \geq 0$ such that $\operatorname{Tor}^T_q(\Bbbk_T, A) = \Bbbk(-q)^{n_q}$. We claim that the map $f : T \to A$ is surjective. Since ${}_TA$ is linear, $\Bbbk \otimes_T A = \Bbbk^{\oplus n}$ for some positive integer n. So ${}_TA$ is generated by elements of degree 0. But A has only one element in degree 0 (up to a scalar). So ${}_TA$ is generated by a single element of degree 0. This implies that the map $\phi : T \to A$ is surjective. Note that $n_q = 0$ for all $q > \operatorname{projdim}{}_TA$. By taking $M = \Bbbk$ in (4.3.1), we have

$$\begin{aligned}
t_n^T(\Bbbk) &= \deg \operatorname{Tor}^T_n(\Bbbk_T, {}_T\Bbbk)\\
&\leq \max_{p+q=n}\{\deg E^2_{p,q}\} = \max_{p+q=n}\{\deg \operatorname{Tor}^A_p(\operatorname{Tor}^T_q(\Bbbk_T, A), {}_A\Bbbk)\}\\
&\leq \max_{p+q=n}\{\deg \operatorname{Tor}^A_p(\Bbbk_A, {}_A\Bbbk) + q\} = \max_{p+q=n}\{t_p^A(\Bbbk) + q\}.
\end{aligned} \tag{4.3.2}$$

Therefore, the assertion in (2) follows.

(3) We use induction on s from 0 to $g := \operatorname{gldim} T$. Since $t_0^T(\Bbbk) = t_0^A(\Bbbk) = 0$, the assertion holds for $s = 0$. Now assume that $s > 0$ and assume the assertion holds for $s - 1$. Part (2) above implies that $\max_{0\leq i\leq s}\{t_i^A(\Bbbk) - i\} \geq \max_{0\leq i\leq s}\{t_i^T(\Bbbk) - i\}$. Hence, we need only show that $\max_{0\leq i\leq s}\{t_i^A(\Bbbk) - i\} \leq \max_{0\leq i\leq s}\{t_i^T(\Bbbk) - i\}$. By the induction hypothesis, it suffices to show that

$$t_s^A(\Bbbk) - s \leq \max_{0\leq i\leq s}\{t_i^T(\Bbbk) - i\}. \tag{4.3.3}$$

By [Stephenson and Zhang 1997, (3-4), p. 1600], $\{t_0^T(\Bbbk), \dots, t_g^T(\Bbbk)\}$ is strictly increasing, or, equivalently,

$$\{t_0^T(\Bbbk), t_1^T(\Bbbk) - 1, \dots, t_{g-1}^T(\Bbbk) - (g-1), t_g^T(\Bbbk) - g\} \quad \text{is nondecreasing.} \tag{4.3.4}$$

So (4.3.3) is equivalent to

$$t_s^A(\Bbbk) - s \le t_s^T(\Bbbk) - s, \quad \text{or} \quad t_s^A(\Bbbk) \le t_s^T(\Bbbk). \tag{4.3.5}$$

Let $\alpha = t_s^T(\Bbbk)$ and $\beta = t_s^A(\Bbbk)$. Using the spectral sequence (4.3.1) (and an inequality in (4.3.2)), we see that, for each $p < s$,

$$\begin{aligned}\deg E^\infty_{p,s-p} \le \deg E^2_{p,s-p} &\le \max_{p+q=s, p<s}\{t_p^A(\Bbbk) + q\} \\ &= \max_{0\le p\le s-1}\{t_p^A(\Bbbk) - p\} + s = \max_{0\le i\le s-1}\{t_i^T(\Bbbk) - i\} + s \\ &\le t_s^T(\Bbbk) - s + s = t_s^T(\Bbbk) = \alpha.\end{aligned}$$

Hence, if $\beta > \alpha$, then $(E^\infty_{p,s-p})_\beta = 0$ for all $p < s$. Similarly, for all $r \ge 2$ and for all $p + q < s$, we can show $\deg E^r_{p,q} \le t^T_{p+q}(\Bbbk) \le \alpha$. Observe that, if $r \ge 2$, the incoming differentials to $E^r_{s,0}$ are all zero, and the outgoing differentials map to $E^r_{s-r,r-1}$ with $\deg E^r_{s-r,r-1} \le \alpha$. Hence, $0 \ne (E^2_{s,0})_\beta$ is in the kernel of all outgoing differentials and so $(E^2_{s,0})_\beta$ survives on the ∞-page. Thus, by (4.3.1), $\operatorname{Tor}^T_s(\Bbbk, \Bbbk)_\beta \ne 0$, which contradicts that $\deg \operatorname{Tor}^T_s(\Bbbk, \Bbbk) = \alpha < \beta$. Therefore, $\beta \le \alpha$, which is (4.3.5). This completes the inductive step and the proof.

(4) We prove this by induction on $j > \operatorname{gldim} T$. The initial step and the inductive step are similar, so we treat them together. Let $\gamma = \operatorname{Torreg}({}_T\Bbbk)$. Similar to the proof of part (3), one sees that, for every $r \ge 2$,

$$\deg E^r_{j-r,r-1} \le \max_{p\le j-1}\{t_p^A(\Bbbk) - p + (j-1)\} \le \gamma + (j-1),$$

where the second inequality is the inductive step when $j > \operatorname{gldim} T + 1$, and is part (3) when $j = \operatorname{gldim} T + 1$. Since $E^\infty_{j,0} = 0$ as $j > \operatorname{gldim} T$, the "outgoing differential" argument in the proof of part (3) shows that

$$\deg E^2_{j,0} \le \max_{r\ge 2}\{\deg E^r_{j-r,r-1}\} \le \gamma + j - 1.$$

This is equivalent to

$$t_j^A(\Bbbk) \le \gamma + j - 1 < \gamma + j,$$

and therefore the assertion holds.

(5) The equation follows from parts (3) and (4). The consequence follows from the fact that A is Koszul if and only if $\operatorname{Torreg}({}_A\Bbbk) = 0$. □

Now we are ready to prove Theorems 0.10 and 0.12.

Proof of Theorem 0.10. Fix a noetherian AS regular algebra S. We first remark that, by [Artin and Zhang 1994, Theorem 8.3(3)], if there is a finite map $T \to S$, then, for any finitely generated graded S-module M, we have $\operatorname{CMreg}({}_T M) = \operatorname{CMreg}({}_S M)$. In particular, $\operatorname{CMreg}({}_T S) = \operatorname{CMreg}({}_S S)$.

(1) Suppose T is any noetherian AS regular algebra and suppose $T \to S$ is a finite map. By Theorem 0.7 and the fact that $\mathrm{Torreg}({}_T S) \geq 0$, we obtain that

$$\mathrm{CMreg}(T) = \mathrm{CMreg}({}_T S) - \mathrm{Torreg}({}_T S) \leq \mathrm{CMreg}(S),$$

and hence $-\mathrm{CMreg}(T) \geq -\mathrm{CMreg}(S)$. Therefore, $c(S) \geq -\mathrm{CMreg}(S)$. By definition, it is clear that $c(S) \leq -\mathrm{CMreg}(S)$, and so we have equality, as desired.

(2) If S is Koszul, then $c(S) = -\mathrm{CMreg}(S) = \mathrm{Torreg}({}_S\Bbbk) = 0$. Conversely, if $c(S) = 0$, then $\mathrm{Torreg}({}_S\Bbbk) = -\mathrm{CMreg}(S) = c(S) = 0$, which implies that S is Koszul.

(3) Suppose T is a noetherian AS regular algebra and $f : T \to S$ is a finite map. By Theorem 0.7, we have

$$\mathrm{CMreg}(S)(= \mathrm{CMreg}({}_S S)) = \mathrm{CMreg}({}_T S) = \mathrm{Torreg}({}_T S) + \mathrm{CMreg}(T).$$

This implies that $-\mathrm{CMreg}(T) \geq -\mathrm{CMreg}(S)$, as $\mathrm{Torreg}({}_T S) \geq 0$ by definition. The first assertion now follows from part (1).

If $c(T) = c(S)$, then, by part (1), we have $\mathrm{CMreg}(T) = \mathrm{CMreg}(S) = \mathrm{CMreg}({}_T S)$. Hence, by Theorem 0.7, $\mathrm{Torreg}({}_T S) = 0$, which means that ${}_T S$ is linear. Similarly, S_T is linear. By the proof of Lemma 4.3(2), the map $f : T \to S$ is surjective.

Since the local cohomology of S can be computed as a left S-module or as a left T-module, $H^i_{\mathfrak{m}}({}_T S)$ is zero unless $i = \mathrm{gldim}\, S$. Hence, ${}_T S$ is Cohen–Macaulay. □

Proof of Theorem 0.12. (1) We have

$$\begin{aligned} c_-(A) &= c(A) + \mathrm{CMreg}(A) = \inf_{T \in \Phi(A)} \{-\mathrm{CMreg}(T) + \mathrm{CMreg}({}_T A)\} \\ &= \inf_{T \in \phi(A)} \{\mathrm{Torreg}({}_T A)\} \geq 0. \end{aligned}$$

(2) If A is AS regular, by letting $T = A$, we obtain that $c(A) \leq -\mathrm{CMreg}(A)$. As a consequence, $c_-(A) \leq 0$. By part (1), $c_-(A) = 0$.

Conversely, suppose $c_-(A) = 0$, or, equivalently, $c(A) = -\mathrm{CMreg}(A)$. By definition and the fact that $c(A)$ is an integer, there is a noetherian AS regular algebra T with a finite map $\phi : T \to A$ such that $-\mathrm{CMreg}(T) = c(A) = -\mathrm{CMreg}(A)$. By Theorem 0.7,

$$\mathrm{Torreg}({}_T A) = \mathrm{CMreg}(A) - \mathrm{CMreg}(T) = 0.$$

Then ${}_T A$ is a linear left T-module (i.e., $t_i^T({}_T A) = i$ if $\mathrm{Tor}_i^T(\Bbbk, {}_T A) \neq 0$).

By Lemma 4.3(5), $\mathrm{Torreg}({}_A\Bbbk) = \mathrm{Torreg}({}_T\Bbbk)$. We know that $\mathrm{CMreg}(A) = \mathrm{CMreg}(T)$ and, since T is AS regular, we have $\mathrm{CMreg}(T) = -\mathrm{Torreg}({}_T\Bbbk)$. Combining these equations, we obtain that

$$\mathrm{ASreg}(A) = \mathrm{CMreg}(A) + \mathrm{Torreg}({}_A\Bbbk) = \mathrm{CMreg}(T) + \mathrm{Torreg}({}_T\Bbbk) = \mathrm{ASreg}(T) = 0.$$

Finally, the assertion follows from Theorem 0.8. □

We collect some criteria for Koszulness of AS regular algebras that follow from our previous results.

Corollary 4.4. *Let T and S be noetherian AS regular algebras and $f : T \to S$ be a finite map.*

(1) *If T is Koszul (or, equivalently,* $\operatorname{Torreg}({}_T\Bbbk) = 0$*), then so is S. Further, f is surjective, and ${}_TS$ and S_T are linear Cohen–Macaulay modules over T.*

(2) *Suppose* $\operatorname{Torreg}({}_T\Bbbk) \leq \deg(\Bbbk \otimes_T S)$. *Then S is Koszul.*

(3) *Suppose* $\operatorname{Torreg}({}_T\Bbbk) = 1$. *If f is not surjective, then S is Koszul. As a consequence, if T is a proper subalgebra of S, then S is Koszul.*

Proof. (1) Suppose T is a Koszul noetherian AS regular algebra and suppose $f : T \to S$ is a finite map. By Theorem 0.10(3) and the fact that T is Koszul, we have

$$0 = -\operatorname{CMreg}(T) \geq -\operatorname{CMreg}(S) \geq 0.$$

Hence, $\operatorname{CMreg}(S) = \operatorname{CMreg}(T) = 0$. The result follows from parts (2) and (3) of Theorem 0.10 (the Koszulness of S also follows from Lemma 4.3(5)).

(2) Suppose S and T are AS regular. By Theorem 0.7, we have

$$\operatorname{CMreg}(S) = \operatorname{CMreg}({}_TS) = \operatorname{Torreg}({}_TS) + \operatorname{CMreg}(T).$$

By Theorem 0.8, $\operatorname{CMreg}(S) = -\operatorname{Torreg}({}_S\Bbbk)$ and $\operatorname{CMreg}(T) = -\operatorname{Torreg}({}_T\Bbbk)$, so

$$\begin{aligned} 0 \leq \operatorname{Torreg}({}_S\Bbbk) &= -\operatorname{CMreg}(S) = -\operatorname{CMreg}(T) - \operatorname{Torreg}({}_TS) \\ &= \operatorname{Torreg}({}_T\Bbbk) - \operatorname{Torreg}({}_TS) \leq \operatorname{Torreg}({}_T\Bbbk) - \deg(\Bbbk \otimes_T S). \end{aligned}$$

Hence, if $\operatorname{Torreg}({}_T\Bbbk) \leq \deg(\Bbbk \otimes_T S)$, then $\operatorname{Torreg}({}_S\Bbbk) = 0$, whence S is Koszul.

(3) If f is not surjective, then $\deg(\Bbbk \otimes_T S) \geq 1$ and so, by part (2), S is Koszul. □

Theorem 4.5. *Let A be a noetherian connected graded algebra. Suppose that there is a finite map $f : T \to A$, where T is a noetherian Koszul AS regular algebra. Then the following hold*:

(1) $\operatorname{CMreg}(A) \geq 0$ *and* $\operatorname{CMreg}(A) = 0$ *if and only if A is AS regular (and Koszul).*

(2) *If A is commutative and generated in degree* 1, *then* $\operatorname{CMreg}(A) \geq 0$, *and A is a polynomial ring if and only if* $\operatorname{CMreg}(A) = 0$.

Part (2) is an improvement of [Römer 2008, Theorem 4.1(iv)$\Longleftrightarrow$(v)].

Proof of Theorem 4.5. (1) Since the map $\phi : T \to A$ is finite and T is Koszul, by Theorem 0.7,

$$\operatorname{CMreg}(A) = \operatorname{CMreg}({}_TA) = \operatorname{Torreg}({}_TA) + \operatorname{CMreg}(T) = \operatorname{Torreg}({}_TA) \geq 0.$$

Since $T \in \Phi(A)$ and T is Koszul, by definition,

$$0 \leq c(A) \leq -\operatorname{CMreg}(T) = 0,$$

which implies that $c(A) = 0$. Consequently, by the definition of $c_-(A)$, we have $c_-(A) = \operatorname{CMreg}(A)$. Now, by Theorem 0.12(2), A is AS regular if and only if $\operatorname{CMreg}(A) = 0$. If this happens, $\operatorname{CMreg}(A) = 0$ implies that $\operatorname{Torreg}({}_A\Bbbk) = 0$ by Theorem 0.8, and hence A is Koszul.

(2) Let $T = \Bbbk[A_1]$. Then $A = T/J$ for some ideal J. Since T is noetherian, Koszul and AS regular, the assertion follows from part (1). □

Note that part (2) of the above theorem fails when A is not commutative (Example 5.4(ii)). Now we are ready to prove Theorem 0.3. Recall that the Hilbert series of a graded A-module M, denoted by $h_M(t)$, is given in Definition 1.1. If $h_M(t)$ is a rational function, then the *a-invariant of* M, denoted by $a(M)$, is defined to be the t-degree of the rational function $h_M(t)$.

Theorem 4.6. *Let A be a noetherian connected graded s-Cohen–Macaulay algebra. Suppose that there is a finite map $f : T \to A$, where T is a noetherian Koszul AS regular algebra. Then the following are equivalent*:

(a) *A is AS regular.*

(b) *A is AS regular and Koszul.*

(c) $\deg h_A(t) = -s$.

Proof. Since there is a finite map to A from an AS regular algebra T, by taking the minimal free resolution of the graded T-module A, it follows that the Hilbert series $h_A(t)$ is rational. Then, by the proof of [Kirkman et al. 2022, Theorem 4.7(2)], we have

$$\mathrm{CMreg}(A) = s + a(A) = s + \deg h_A(t).$$

Now the assertion follows from Theorem 4.5(1). □

The following simple corollary is useful in practice:

Corollary 4.7. *Let A and B be algebras satisfying Hypothesis 1.3. Assume that there is a finite map $f : A \to B$ such that ${}_AB$ is linear of finite projective dimension. Then A is AS regular if and only if B is AS regular.*

Proof. Suppose B is AS regular. By the change of rings spectral sequence given in [Rotman 1979, Theorem 10.60] (or see (4.3.1)), we have

$$\mathrm{gldim}\, A = \mathrm{projdim}\, {}_A\Bbbk \leq \mathrm{projdim}\, {}_B\Bbbk + \mathrm{projdim}\, {}_AB < \infty.$$

By [Zhang 1997, Theorem 0.3], A is AS regular.

Now we assume A is AS regular. By Lemma 4.3(5), $\mathrm{Torreg}({}_A\Bbbk) = \mathrm{Torreg}({}_B\Bbbk)$. By Theorem 0.7,

$$\mathrm{CMreg}(B) = \mathrm{CMreg}({}_AB) = \mathrm{Torreg}({}_AB) + \mathrm{CMreg}(A) = \mathrm{CMreg}(A).$$

Then

$$\mathrm{ASreg}(B) = \mathrm{Torreg}({}_B\Bbbk) + \mathrm{CMreg}(B) = \mathrm{Torreg}({}_A\Bbbk) + \mathrm{CMreg}(A) = \mathrm{ASreg}(A) = 0.$$

The assertion follows from Theorem 0.8. □

5. Examples and remarks

This section contains some examples and remarks. To save space we will omit some nonessential details. Our first example shows that it is necessary to assume Hypothesis 1.3 for most of the results in this paper.

Example 5.1. We refer to [Artin and Zhang 1994; Stafford and Zhang 1994] for the definition of the χ-*condition*. Let A be the noetherian connected graded domain of GK-dimension 2 given in [Stafford and Zhang 1994, Theorem 2.3], which does not satisfy the χ-condition. By [Van den Bergh 1997, Theorem 6.3], A does not admit a balanced dualizing complex. Further, by [Stafford and Zhang 1994, Theorem 2.3], $\mathrm{Ext}^1_A(\Bbbk, A)$ is not bounded above. Since $\mathrm{Hom}_A(\Bbbk, A) = 0$, $H^1_{\mathfrak{m}}(A)$ contains $\mathrm{Ext}^1_A(\Bbbk, A)$ as a graded $\Bbbk$-subspace. Therefore, $\deg H^1_{\mathfrak{m}}(A) = +\infty$, and, consequently, $\mathrm{CMreg}(A) = +\infty$.

If there is a finite map $T \to A$ for some noetherian AS regular algebra T, then [Artin and Zhang 1994, Theorem 8.1(1) and Lemma 8.2(4)] implies that A satisfies the χ-condition. Since A does not satisfy the χ-condition, we conclude that there does not exist a finite map from a noetherian AS regular algebra to A. As a consequence, $c(A) = +\infty$ by definition.

Rogalski and Sierra [2012] provide a family of examples of noetherian Koszul algebras of global dimension 4 for which the χ-condition fails. By similar arguments, these algebras have infinite CM regularity and concavity.

Before we give more examples, we prove some well-known lemmas.

Lemma 5.2. *Let A be a noetherian connected graded algebra and let Ω be a regular normal element of degree* 1 *or* 2. *Let $B = A/(\Omega)$. Then* $\mathrm{Torreg}({}_B\Bbbk) \leq \mathrm{Torreg}({}_A\Bbbk)$. *Consequently, the following hold*:

(1) *If A has finite global dimension, then* $\mathrm{Torreg}({}_B\Bbbk) < \infty$.

(2) [Polishchuk 2006, Theorem 1.2] *If A is Koszul, then so is B.*

(3) *Suppose* $\deg \Omega = 1$. *If A has finite global dimension, then so does B.*

Proof. The first part of the proof is copied from the proof of [Kirkman et al. 2015, Theorem 1.11]. Since B is a factor ring of A, there is a graded version of the change of rings spectral sequence given in [Rotman 1979, Theorem 10.71],

$$E^2_{p,q} := \mathrm{Tor}^B_p(\Bbbk, \mathrm{Tor}^A_q(B, \Bbbk)) \Rightarrow \mathrm{Tor}^A_n(\Bbbk, \Bbbk). \tag{5.2.1}$$

Let $a = \deg \Omega$. Since $B = A/(\Omega)$ and Ω is a regular normal element, we have $\mathrm{Tor}^A_0(B, \Bbbk) = \Bbbk$, $\mathrm{Tor}^A_1(B, \Bbbk) = \Bbbk(-a)$ and $\mathrm{Tor}^A_i(B, \Bbbk) = 0$ for $i > 1$. Hence, the E^2-page of the spectral sequence (5.2.1) has only two possibly nonzero rows, namely

$$\begin{aligned} q = 0: &\quad \mathrm{Tor}^B_p(\Bbbk, \Bbbk) &&\text{for } p = 0, 1, 2, \ldots, \\ q = 1: &\quad \mathrm{Tor}^B_p(\Bbbk, \Bbbk(-a)) &&\text{for } p = 0, 1, 2, \ldots. \end{aligned}$$

Since (5.2.1) converges, we have $\mathrm{Tor}_0^B(\Bbbk,\Bbbk)=\mathrm{Tor}_0^A(\Bbbk,\Bbbk)=\Bbbk$ and a long exact sequence

$$\begin{aligned}\cdots\to \mathrm{Tor}_4^B(\Bbbk,\Bbbk)&\to \mathrm{Tor}_2^B(\Bbbk,\Bbbk)(-a)\\ \to \mathrm{Tor}_3^A(\Bbbk,\Bbbk)&\to \mathrm{Tor}_3^B(\Bbbk,\Bbbk)\to \mathrm{Tor}_1^B(\Bbbk,\Bbbk)(-a)\\ \to \mathrm{Tor}_2^A(\Bbbk,\Bbbk)&\to \mathrm{Tor}_2^B(\Bbbk,\Bbbk)\to \mathrm{Tor}_0^B(\Bbbk,\Bbbk)(-a)\\ \to \mathrm{Tor}_1^A(\Bbbk,\Bbbk)&\to \mathrm{Tor}_1^B(\Bbbk,\Bbbk)\to 0.\end{aligned}$$

By the k-th row from the bottom in the above exact sequence, we have

$$\begin{aligned}\deg(\mathrm{Tor}_k^B(\Bbbk,\Bbbk))-k&\le \max\{\deg(\mathrm{Tor}_k^A(\Bbbk,\Bbbk))-k,\deg(\mathrm{Tor}_{k-2}^B(\Bbbk,\Bbbk)(-a))-k\}\\ &=\max\{\deg(\mathrm{Tor}_k^A(\Bbbk,\Bbbk))-k,\deg(\mathrm{Tor}_{k-2}^B(\Bbbk,\Bbbk))-(k-2)-(2-a)\}\\ &\le \max\{\mathrm{Torreg}({}_A\Bbbk),\mathrm{Torreg}({}_A\Bbbk)-(2-a)\}\\ &\le \mathrm{Torreg}({}_A\Bbbk),\end{aligned}$$

where we use $\mathrm{Tor}_{k-2}^B(\Bbbk,\Bbbk)-(k-2)\le\mathrm{Torreg}({}_A\Bbbk)$ by the induction hypothesis and $2-a\ge 0$ as $a=1$ or 2. The assertion follows from the definition.

Parts (1) and (2) of this lemma follow immediately from the main assertion.

(3) For each $k>\mathrm{gldim}\,A+1$, we have an exact sequence

$$(\mathrm{Tor}_k^A(\Bbbk,\Bbbk)=)\ \ 0\to \mathrm{Tor}_k^B(\Bbbk,\Bbbk)\to \mathrm{Tor}_{k-2}^B(\Bbbk,\Bbbk)(-1)\to 0\ \ (=\mathrm{Tor}_{k-1}^A(\Bbbk,\Bbbk)).$$

So $\mathrm{ged}\,\mathrm{Tor}_k^B(\Bbbk,\Bbbk)=\mathrm{ged}\,\mathrm{Tor}_{k-2}^B(\Bbbk,\Bbbk)+1$.

If $\mathrm{Tor}_k^B(\Bbbk,\Bbbk)\ne 0$, B being connected graded implies that $\mathrm{ged}\,\mathrm{Tor}_k^B(\Bbbk,\Bbbk)\ge\mathrm{ged}\,\mathrm{Tor}_{k-2}^B(\Bbbk,\Bbbk)+2$ by using the minimal free resolution of $\Bbbk$. This yields a contradiction, and thus $\mathrm{Tor}_k^B(\Bbbk,\Bbbk)=0$, or $\mathrm{gldim}\,B<k<\infty$. □

See Lemma 4.3 and Corollary 4.7 for related results. Note that the inequality $\mathrm{Torreg}({}_B\Bbbk)\le\mathrm{Torreg}({}_A\Bbbk)$ in Lemma 5.2 can be strict; see Example 5.4(iv). Part (3) of Lemma 5.2 is a special case of Corollary 4.7 if A satisfies Hypothesis 1.3.

Lemma 5.3. *Let A be a noetherian connected graded AS Gorenstein algebra of type (d,ℓ). Let Ω be a regular normal element of degree a.*

(1) *If $a\ge\ell-d+2$, then $A/(\Omega)$ is not AS regular.*

(2) *Suppose A is generated in degree 1 and is not Koszul. If $a\ge\max\{3,\ell-d+1\}$, then $A/(\Omega)$ is not AS regular.*

Proof. By the Rees lemma, $B:=A/(\Omega)$ is also AS Gorenstein (of injective dimension $d-1$). Applying $H^i_{\mathfrak{m}}(-)$ to the short exact sequence of left A-modules

$$0\to A(-a)\to A\to B\to 0,$$

we obtain that $H^i_{\mathfrak{m}}(B) = 0$ for all $i \neq d-1$ and

$$0 \to H^{d-1}_{\mathfrak{m}}(B) \to H^d_{\mathfrak{m}}(A(-a)) \to H^d_{\mathfrak{m}}(A) \to 0$$

is an exact sequence. Since $H^d_{\mathfrak{m}}(A)$ is nonzero and bounded above,

$$\deg H^{d-1}_{\mathfrak{m}}(B) = \deg H^d_{\mathfrak{m}}(A(-a)) = \deg H^d_{\mathfrak{m}}(A) + a = -\mathfrak{l} + a.$$

This implies that

$$\mathrm{CMreg}(B) = (d-1) + (-\mathfrak{l} + a) = (d - \mathfrak{l}) + (a-1). \tag{5.3.1}$$

As a consequence, B is of type $(d-1, \mathfrak{l}-a)$.

(1) If $a \geq \mathfrak{l} - d + 2$, then, by (5.3.1), $\mathrm{CMreg}(B) \geq 1$. By (2.1.2), B is not AS regular.

(2) Part (1) takes care of the case when $a \geq \mathfrak{l} - d + 2$. It remains to consider the case $a = \mathfrak{l} - d + 1 \geq 3$. By (5.3.1), $\mathrm{CMreg}(B) = a - (\mathfrak{l} - d + 1) = 0$. If B is AS regular, then $\mathrm{Torreg}({}_B\Bbbk) = -\mathrm{CMreg}(B) = 0$ or B is Koszul. Since A is isomorphic to $\Bbbk\langle x_1, \ldots, x_n\rangle/(R)$ for some x_i of degree 1, B is isomorphic to $\Bbbk\langle x_1, \ldots, x_n\rangle/(R, \Omega)$. Since $\deg \Omega \geq 3$, B is not quadratic. So B is not Koszul, yielding a contradiction. Therefore, B is not AS regular. □

Example 5.4(iv) shows that, if $a = \max\{2, \mathfrak{l} - d + 1\}$, then $A/(\Omega)$ may be AS regular.

Example 5.4. Let T be a noetherian AS regular algebra. Then $\mathrm{Torreg}({}_T\Bbbk) < \infty$. Let B be a factor ring $T/(\Omega)$, where Ω is a regular normal element of degree a. The algebra B is s-Cohen–Macaulay, where $s = \mathrm{gldim}\, T - 1 = d - 1$ and, by (5.3.1),

$$\mathrm{CMreg}(B) = \mathrm{CMreg}(T) + (a-1). \tag{5.4.1}$$

If a is 1 or 2, by Lemma 5.2(1), $\mathrm{Torreg}({}_B\Bbbk) < \infty$.

Now we give some explicit examples:

(i) Let $A = \Bbbk[x]/(x^a)$ with $a \geq 2$. If $a \geq 3$, $\mathrm{Torreg}({}_A\Bbbk) = \infty$ by Example 3.1. This means that the assertion in Lemma 5.2(1) fails if $\deg \Omega > 2$.

(ii) Let T be the algebra $\Bbbk\langle x, y\rangle/(x^2y - yx^2, xy^2 - y^2x)$, which is a noetherian AS regular algebra of type $(3, 4)$ (a special case of the algebra given in Example 2.4(3)). By Example 2.4(3), $\mathrm{Torreg}({}_T\Bbbk) = 1$.

It is easy to see that $\Omega := x^2$ is a normal regular element of T. Let $B = T/(\Omega)$. Then, by Lemma 5.2(1), $\mathrm{Torreg}({}_B\Bbbk) \leq \mathrm{Torreg}({}_T\Bbbk) = 1$. Since B is isomorphic to $\Bbbk\langle x, y\rangle/(x^2, xy^2 - y^2x)$, which is not Koszul, $\mathrm{Torreg}({}_B\Bbbk) \geq 1$. Therefore, $\mathrm{Torreg}({}_B\Bbbk) = 1$. By (5.4.1),

$$\mathrm{CMreg}(B) = \mathrm{CMreg}(T) + (2-1) = -\mathrm{Torreg}({}_T\Bbbk) + 1 = -1 + 1 = 0.$$

Hence, $\mathrm{ASreg}(B) = 1$ and B is not AS regular. This example shows that Theorem 4.5(2) fails without the commutativity assumption.

Let $A = B^{\otimes n}$ for any $n \geq 1$. Then it is easy to check that $\mathrm{CMreg}(A) = 0$ and $\mathrm{Torreg}({}_A\Bbbk) = \mathrm{ASreg}(A) = n$. As a consequence, A is not AS regular. By Theorem 4.5(1), there does not exist a finite map from a Koszul AS regular algebra to A (for any $n \geq 1$).

(iii) Let B be the algebra in part (ii). Since $T \to B$ is a surjective map, $c(B) \leq -\mathrm{CMreg}(T) = \mathrm{Torreg}({}_T\Bbbk) = 1$. We claim that $c(B) = 1$. First, by definition, $c(B) \geq 0$, and hence $c(B)$ is either 0 or 1. Now, since $\mathrm{CMreg}(B) = 0$, we have $c_-(B) = c(B) + \mathrm{CMreg}(B) = c(B)$, which implies that $c_-(B)$ is either 0 or 1. Since B is not AS regular by part (ii), by Theorem 0.12, $c_-(B) = 1$. Therefore, $c(B) = c_-(B) = 1$ and we have proved the claim.

It is easy to see that

$$1 \leq c(B^{\otimes n}) = c_-(B^{\otimes n}) \leq n.$$

It would be interesting to work out the exact value of $c(B^{\otimes n})$.

(iv) Let T be as in part (ii) and let Ω be $xy - yx$. Then Ω is a regular normal element of T such that $B := T/(\Omega)$ is the commutative polynomial ring $\Bbbk[x, y]$. So

$$\mathrm{Torreg}({}_B\Bbbk) = 0 < 1 = \mathrm{Torreg}({}_T\Bbbk),$$

where the last equation is given in part (ii).

Since T is of type $(3, 4)$, $a = 2 = \max\{2, \ell - d + 1\}$. Therefore, Lemma 5.3(2) fails if the hypothesis $a \geq \max\{3, \ell - d + 1\}$ is replaced by $a \geq \max\{2, \ell - d + 1\}$.

Throughout this paper, we were mainly interested in five numerical homological invariants:

$$\mathrm{Torreg}({}_A\Bbbk), \quad \mathrm{CMreg}(A), \quad c(A), \quad \mathrm{ASreg}(A) \quad \text{and} \quad c_-(A).$$

By definition (Definitions 0.6 and 0.9(3)), the last two invariants are dependent on the first three. We now seek to understand what values are possible for the first three invariants.

Definition 5.5. Let A be a noetherian connected graded algebra with balanced dualizing complex.

(1) The $\mathfrak{ta}$-*pair associated to* A is defined to be

$$\mathfrak{ta}(A) := (\mathrm{Torreg}({}_A\Bbbk), \mathrm{ASreg}(A)) \in (\mathbb{N} \times \mathbb{N}) \cup \{(+\infty, +\infty)\}.$$

(2) The $\mathfrak{tc}$-*pair* associated to A is defined to be

$$\mathfrak{tc}(A) := (\mathrm{Torreg}({}_A\Bbbk), \mathrm{CMreg}(A)) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{Z}.$$

Note that $\mathrm{CMreg}(A) \geq -\mathrm{Torreg}({}_A\Bbbk)$ for every A.

In the next lemma, let D denote a noetherian connected graded algebra with balanced dualizing complex.

Lemma 5.6. (1) *For every* $t \in \mathbb{N} \cup \{+\infty\}$ *and* $c \in \mathbb{Z}$ *with* $c \geq -t$, *there is an algebra* D *such that* $\mathfrak{tc}(D) = (t, c)$.

(2) *For every* $t \in \mathbb{N}$ *and* $c \in \mathbb{Z}$ *with* $c \geq -t$, *there is an algebra* D *such that* $\mathfrak{tc}(D) = (t, c)$ *and* $c(D) = t$.

(3) *For every pair* $(t, a) \in (\mathbb{N} \times \mathbb{N}) \cup \{(+\infty, +\infty)\}$, *there is an algebra* D *such that* $\mathfrak{ta}(D) = (t, a)$.

(4) *For every pair* $(t, b) \in \mathbb{N} \times \mathbb{N}$, *there is an algebra* D *such that*

$$(\operatorname{Torreg}({}_A\Bbbk), c_-(D)) = (c(D), c_-(D)) = (t, b).$$

Proof. (1) First assume $t = \infty$. Let T be the algebra in Example 5.4(ii). Then $\operatorname{CMreg}(T) = -1$. Let $A = \Bbbk[x]/(x^3)$. Then $\operatorname{CMreg}(A) = 2$. For every integer c, there are nonnegative integers p, q such that $c = (-1)p + 2q$. Let $D = T^{\otimes p} \otimes A^{\otimes q}$. One can calculate

$$\operatorname{CMreg}(D) = p \operatorname{CMreg}(T) + q \operatorname{CMreg}(A) = p(-1) + q(2) = c,$$

as desired.

Now we assume that $t \in \mathbb{N}$. Let T be the algebra in Example 5.4(ii). Then $\operatorname{Torreg}({}_T\Bbbk) = 1$ and $\operatorname{Torreg}({}_{T^{\otimes t}}\Bbbk) = t$ by Lemma 2.3. Let $a = t + c$, which is a nonnegative integer by the hypothesis. Let E be the algebra $\Bbbk[x]/(x^2)$, which has $\operatorname{CMreg}(E) = 1$ and $\operatorname{Torreg}({}_E\Bbbk) = 0$. Let $D = T^{\otimes t} \otimes E^{\otimes a}$. Then

$$\operatorname{CMreg}(D) = t \operatorname{CMreg}(T) + a \operatorname{CMreg}(E) = -t + a = c$$

and

$$\operatorname{Torreg}(D) = t \operatorname{Torreg}(T) + a \operatorname{Torreg}(E) = t + a0 = t,$$

as desired.

(2) We use the second half of the proof of (1) and note that

$$c(D) = c(T^{\otimes t} \otimes E^{\otimes a}) = c(T^{\otimes t}) = t,$$

as desired.

(3) Since $\operatorname{ASreg}(A) = \operatorname{Torreg}({}_A\Bbbk) + \operatorname{CMreg}(A)$, the assertion follows easily from part (1).

(4) Since $c_-(A) = c(A) + \operatorname{CMreg}(A)$, the assertion follows easily from part (2). □

Note that the algebras A in the above proof are generated in degree 1 and are module-finite over their centers. We conclude the paper with the following remarks:

Remark 5.7. Assume Hypothesis 1.3.

(1) It would be interesting to work out the range of $c(A)$ (resp. $c_-(A)$) for every fixed $\mathfrak{tc}$-pair (t, c). When $c = -t$, by Theorems 0.8 and 0.12(1), $c(A)$ must be $-c$. However, computing the concavity $c(A)$ and normalized concavity $c_-(A)$ is generally much harder than computing $\operatorname{Torreg}({}_A\Bbbk)$ and $\operatorname{CMreg}(A)$.

(2) Further it would be interesting to study algebras with $\mathfrak{ta}$ near $(0, 0)$, for example $\mathfrak{ta}(A) = (1, 0)$ or $\mathfrak{ta}(A) = (0, 1)$. It is not clear to us whether an algebra A with $\mathfrak{ta}(A) = (0, 1)$ is AS Gorenstein, although we do not have any counterexample to this.

Remark 5.8. It is an open question whether the tensor product of two noetherian AS regular algebras (or two algebras satisfying Hypothesis 1.3) is always noetherian. If we ignore this issue, we can consider the behavior of regularities with respect to the tensor product.

(1) (Lemma 2.3) $\mathrm{Torreg}({}_A\Bbbk)$ is additive in the following sense:

$$\mathrm{Torreg}({}_{A\otimes B}\Bbbk) = \mathrm{Torreg}({}_A\Bbbk) + \mathrm{Torreg}({}_B\Bbbk).$$

(2) (Remark 2.9) $\mathrm{CMreg}(A)$ is additive in the following sense:

$$\mathrm{CMreg}(A \otimes B) = \mathrm{CMreg}(A) + \mathrm{CMreg}(B).$$

(3) It follows from parts (1) and (2), $\mathrm{ASreg}(A)$ is additive in the following sense:

$$\mathrm{ASreg}(A \otimes B) = \mathrm{ASreg}(A) + \mathrm{ASreg}(B).$$

(4) We do not know if $c(A)$ (resp. $c_-(A)$) is additive. It follows from the definition that

$$c(A \otimes B) \leq c(A) + c(B).$$

The same is true for $c_-(A)$.

Remark 5.9. It follows immediately from Theorem 0.10(3) that, if S and T are noetherian AS regular algebras and $f : T \to S$ and $g : S \to T$ are finite maps, then both f and g are isomorphisms of graded algebras.

The above statement fails if we assume that S and T are only AS Gorenstein. Let $S = T = \Bbbk[x]/(x^2)$, which is noetherian AS Gorenstein. Let $f(= g) : T \to T$ be defined by $a + bx \mapsto a$ for all $a, b \in \Bbbk$. Then f is a finite map that is not an isomorphism of graded algebras.

Remark 5.10. Let T be a noetherian AS regular Koszul algebra and let A be a connected graded algebra. Suppose $f : T \to A$ is a finite map. It is trivial that:

(1) $\mathrm{Torreg}({}_T A) \geq 0$.

By Theorem 0.7,

$$\mathrm{Torreg}({}_T A) = \mathrm{Torreg}({}_T A) + \mathrm{CMreg}(T) = \mathrm{CMreg}({}_T A) = \mathrm{CMreg}(A).$$

Now it follows from Theorem 4.5(1) that:

(2) $\mathrm{Torreg}({}_T A) = 0$, or, equivalently, ${}_T A$ is linear, if and only if A is AS regular (and Koszul).

Therefore, $\mathrm{Torreg}({}_T A)$ qualifies as an indicator of the AS regular property in this case.

We can say more in terms of the Hilbert series of the algebras.

Remark 5.11. We continue Remark 5.10 and consider a special case when $h_T(t) = (1-t)^{-d}$, where $d = \mathrm{gldim}\, T$. Let $f : T \to A$ be a finite map.

(1) Then A is AS regular if and only if there is an integer $0 \leq d' \leq d$ such that the i-th term in the minimal free resolution of the T-module ${}_T A$ is isomorphic to $T(-i)^{\oplus \binom{d'}{i}}$ for all i. If this happens, A is Koszul and $h_A(t) = (1-t)^{-d+d'}$.

(2) If A is known to be Cohen–Macaulay, then A is AS regular if and only if $h_A(t) = (1-t)^{-d+d'}$ for an integer $0 \leq d' \leq d$. If this happens, all properties in part (1) hold.

Remark 5.12. Let A and B be two connected graded algebras. We say that A and B are *finite-equivalent* if there are finite graded algebra homomorphisms $f : A \to B$ and $g : B \to A$. It is easy to see that being finite-equivalent is an equivalence relation. For example, $\Bbbk$ and $\Bbbk[x]/(x^a)$ with $a \geq 2$ are finite-equivalent. It follows from Proposition 4.1 that:

(1) If A and B are finite equivalent, then $c(A) = c(B)$.

The concavity $c(A)$ also has the following properties:

(2) If A is commutative and generated in degree 1, then $c(A) = 0$.

(3) (Proposition 4.1(4)) If x is a commutative indeterminate of degree 1, then $c(A[x]) = c(A)$.

By Remark 5.9, if S and T are finite-equivalent noetherian AS regular algebras, then $S \cong T$. Properties (1) and (2) fail for other invariants that we have defined in this paper. See Remark 5.8 for property (3).

Remark 5.13. The relationship between $\mathrm{Torreg}({}_A\Bbbk)$ and $\mathrm{CMreg}(A)$ (resp. $c(A)$ and $\mathrm{CMreg}(A)$) is given in Lemma 5.6(1,2). It seems that there is a mysterious (maybe closer) connection between $\mathrm{Torreg}({}_A\Bbbk)$ and $c(A)$. By Jørgensen's theorem (Theorem 2.5),

$$\mathrm{Torreg}({}_A\Bbbk) \geq -\mathrm{CMreg}(A),$$

and, by Theorem 0.12(1) and the definition,

$$c(A) = c_-(A) - \mathrm{CMreg}(A) \geq -\mathrm{CMreg}(A).$$

By Lemma 5.6(2), for each $t \geq 0$, there is an algebra D such that $\mathrm{Torreg}({}_A\Bbbk) = c(A) = t$.

Based on this very limited evidence, we ask the following two questions:

Question 5.14. Suppose both $c(A)$ and $\mathrm{Torreg}({}_A\Bbbk)$ are finite. Then is $\mathrm{Torreg}({}_A\Bbbk)$ uniformly bounded by a function of $c(A)$ (or vice versa)?

A special case of the above question is related to a nice result in [Avramov and Peeva 2001] (also see [Römer 2008, Theorem 2.3]). Is there a noncommutative version of this result?

Question 5.15. Suppose $c(A) = 0$ and $\mathrm{Torreg}({}_A\Bbbk) < \infty$. Is then $\mathrm{Torreg}({}_A\Bbbk) = 0$ (or, equivalently, is A Koszul)?

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