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Anticyclotomic Euler system over biquadratic fields

Kim Tuan Do

We construct a new Euler system (anticyclotomic, in the sense of Jetchev–Nekovář–Skinner) for the Galois representation $V_{f,\chi}$ attached to a newform f of weight $k \geq 2$ twisted by an anticyclotomic Hecke character χ defined over an imaginary biquadratic field K_0. We then show some arithmetic applications of the constructed Euler system, including results on the Bloch–Kato conjecture and a divisibility towards the Iwasawa–Greenberg main conjecture for $V_{f,\chi}$.

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1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N_f))$ be an elliptic newform of even weight $k = 2r \geq 2$, and let $p \nmid 6N_f$ be a prime. Let $K_0/\mathbb{Q}$ be an imaginary biquadratic field in which p splits. This means that K_0 contains two distinct imaginary quadratic subfields K_1, K_2 together with one real quadratic subfield K_3. Let L be a number field containing K_0 and the Fourier coefficients of f, and let $\mathfrak{P}$ be a prime of L above p at which f is ordinary, i.e., $v_{\mathfrak{P}}(a_p) = 0$. Let χ be an anticyclotomic Hecke character of K_0 with infinity type $(-a, a, -b, b)$, where $a \geq b \geq 0$,[1] that satisfies the decomposition hypothesis, Assumption 6.1, i.e., χ can be factored as

$$\chi = \tilde{\psi}_1 \tilde{\psi}_2 \mathbf{N}^{(k_1+k_2-2)/2}.$$

Here, for $i \in \{1, 2\}$, ψ_i is a Hecke character of K_i of infinity type $(1 - k_i, 0)$ and modulus $\mathfrak{f}_i$; and $\tilde{\psi}_i$ is the Hecke character of K_0, obtained by composing $\mathbb{A}_{K_0}^{\times} \xrightarrow{\mathbb{N}_{K_0/K_i}} \mathbb{A}_{K_i}^{\times} \xrightarrow{\psi_i} \mathbb{C}$. Note that, if this happens, we must have $k_1 = a - b + 1$ and $k_2 = a + b + 1$. We then focus on the conjugate self-dual

[1] By either using $L(f/K_0, \chi, r) = L(f/K_0, \chi^{\boldsymbol{c}}, r)$, where $\chi^{\boldsymbol{c}}$ is the composition of χ with the action of complex conjugation, or swapping the order of K_1 and K_2, we would be able to cover other cases of a and b.

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$G_{K_0} = \mathrm{Gal}(\overline{\mathbb{Q}}/K_0)$-representation

$$V_{f,\chi} := V_f^\vee(1-r) \otimes \chi^{-1},$$

where $V_f^\vee$ is the contragredient of Deligne's $\mathfrak{P}$-adic Galois representation associated to f.

For the remainder of this section, we assume

- f is ordinary and non-Eisenstein at $\mathfrak{P}$;
- p splits completely in K_0;
- $p \nmid h_{K_0}$, where h_{K_0} is the class number of K_0.

For every integral ideal μ_3 of $\mathcal{O}_{K_3}$, let $K_0[\mu_3]$ be the maximal p-subextension of the ring class field of K_0 of conductor μ_3. Denote by $\mathcal{N}$ the set of squarefree products of primes $\mu_3 \subset \mathcal{O}_{K_3}$, where $m = N_{K_3/\mathbb{Q}}(\mu_3)$ is squarefree, prime to p, and split in K_0.

Theorem A (Theorem 3.3.1). *There exists a collection of Iwasawa cohomology classes*

$$\boldsymbol{z}_{f,\chi,\mu_3} \in H^1_{\mathrm{Iw}}(K_0[\mu_3 p^\infty], T_{f,\chi}),$$

indexed by the ideals $\mu_3 \in \mathcal{N}$ with $m = N_{K_3/\mathbb{Q}}(\mu_3)$, where $T_{f,\chi}$ is a certain G_K-stable $\mathcal{O}$-lattice inside $V_{f,\chi}$, such that, for every prime $\lambda_3 \in \mathcal{N}$ of norm ℓ with $(\ell, mp) = 1$, we have the norm relation

$$\mathrm{Norm}_{K_0[\mu_3]}^{K_0[\mu_3\lambda_3]}(\boldsymbol{z}_{f,\chi,\mu_3\lambda_3}) = P_{\mathcal{L}_4}(\mathrm{Frob}_{\mathcal{L}_4})(\boldsymbol{z}_{f,\chi,\mu_3}),$$

where $P_{\mathcal{L}_4}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathcal{L}_4} | (T_{f,\chi})^\vee(1))$, and $\mathrm{Frob}_{\mathcal{L}_4}$ is the geometric Frobenius.

Remark. Jetchev, Nekovář and Skinner [Jetchev et al. 2024] have developed a theory of "split" anticyclotomic Euler systems attached to conjugate self-dual representations over CM fields, where classes are defined over ring class extensions of CM fields (indexed by ideals of their totally real subfields). Our construction fits within their framework. Furthermore, the condition that $m = N_{K_3/\mathbb{Q}}(\mu_3)$ splits in K_0 does exclude the setting where m is inert in K_3 and μ_3 splits in K_0. Nevertheless, this does not affect the application of the Jetchev–Nekovář–Skinner machinery (see some details for the imaginary quadratic case in [Do 2022, Section 4.3]).

Due to its geometric origin, if we let

$$\kappa_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(\boldsymbol{z}_{f,\chi,(1)}),$$

then it will land in a Selmer subgroup of $H^1(K_0, V_{f,\chi})$ with "nice" local conditions (see Section 6.3). Then, feeding Theorem A to the general Euler system machinery of [Jetchev et al. 2024], we deduce the following cases of the Bloch–Kato conjecture in analytic rank 0.

Theorem B (Theorem 6.5.1). *Let $f \in S_k(\Gamma_0(N_f))$ be a newform. Let χ be an anticyclotomic Hecke character of K_0 of infinity type $(-a, a, -b, b)$ satisfying Assumption 6.1. Assume further that*:

(1) *Either $k \geq 2a + 2$ or $2b \geq k$.*

(2) $N_f\mathcal{O}_{K_3} = \mathfrak{n}^+\mathfrak{n}^-$, *where* $\mathfrak{n}^+$ (*resp.* $\mathfrak{n}^-$) *is divisible only by primes which are split* (*resp. inert*) *in* K_0/K_3 *and* $\mathfrak{n}^-$ *is a squarefree product of an even number of primes.*

(3) $\bar{\rho}_f$ *is absolutely irreducible.*

(4) $(pN_f, \mathrm{Norm}_{K_1/\mathbb{Q}}(\mathfrak{f}_1)\mathrm{Norm}_{K_2/\mathbb{Q}}(\mathfrak{f}_2)D_{K_0}) = 1$.

Then

$$L(f/K_0, \chi, r) \neq 0 \implies \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) = 0,$$

and hence the Bloch–Kato conjecture for $V_{f,\chi}$ *holds in this case.*

Note that the first two conditions of Theorem B imply that the sign of the functional equation of $V_{f,\chi}$ is equal to $+1$; see also Remark 6.5.2. This puts us in an ideal situation for the nonvanishing of central L-values generically.

Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. We say that f has big image if, for a certain Galois stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$, the image of $G_{\mathbb{Q}}$ in $\mathrm{Aut}_{\mathcal{O}}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbb{Z}_p)$. Under this assumption, we also have results towards the Bloch–Kato conjecture in the analytic rank 1 case.

Theorem C (Theorem 6.7.1). *Let the hypotheses be as in Theorem B, and assume in addition that*:

(1) $\bar{\rho}_f$ *is p-distinguished.*

(2) f *has big image.*

(3) $p > k - 2$.

If $2a \geq k \geq 2b + 2$ (*which implies* $L(f/K_0, \chi, r) = 0$), *then*

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) \geq 1.$$

Finally, we note that these results also include the proof of a divisibility towards the anticyclotomic Iwasawa main conjecture for $V_{f,\chi}$; see Theorem 6.6.1.

1.1. ***Relation to previous works.*** When χ is an anticyclotomic Hecke character over an imaginary quadratic field K, the arithmetic of $V_{f,\chi}$ has been studied intensively via the Euler system of Heegner points pioneered in [Gross and Zagier 1986; Kolyvagin 1988] (see also [Zhang 1997; Tian 2003; Nekovář 2007]), and generalized Heegner cycles by Bertolini, Darmon and Prasanna [Bertolini et al. 2013]. In particular, these objects have direct implications towards the Bloch–Kato conjecture in analytic rank 0 for $V_{f,\chi}$ either by varying the generalized Heegner cycles in p-adic families like in [Castella and Hsieh 2018] (see also [Castella 2020]), or by the "level-raising" method like in [Bertolini and Darmon 2005] (see also [Longo and Vigni 2010; Chida and Hsieh 2015; Chida 2017]). In the same vein as [Bertolini and Darmon 2005], Nekovář [2012] and Wang [2023] proved results towards the rank 0 Bloch–Kato conjecture when f is a cuspidal Hilbert modular eigenform over a totally real field F of parallel weight 2 and higher weights, respectively, where χ is a finite-order character; see also [Tamiozzo 2021].

Outside of the Heegner realm, it is worthwhile to mention that the Euler system of Beilinson–Flach classes constructed by Lei, Loeffler and Zerbes [Lei et al. 2014; 2015] and Kings, Loeffler and Zerbes

[Kings et al. 2017; 2020] can be applied to obtain similar rank 0 results. Relying on this, Lamplugh [2016] constructed Euler systems for $\mathrm{Ind}_{K_0}^{K_1}\mathcal{O}(\chi\rho)$ over K_1 (where ρ is an auxiliary character) and used that to bound the associated Selmer group over K_0 via Rubin's machinery [2000].

The anticyclotomic Euler system over K_0 that we will describe in this paper is more comparable with the anticyclotomic diagonal Euler system [Do 2022; Castella and Do 2025] over K (an imaginary quadratic field) and comes together with application towards the Bloch–Kato conjecture in analytic rank 0. The construction of the cohomology classes, similar to [Do 2022; Castella and Do 2025], is based on a generalization of the diagonal cycles pioneered in [Gross and Kudla 1992; Gross and Schoen 1995], and improved in Bertolini–Seveso–Venerucci and Darmon–Rotger [Bertolini et al. 2022a]. Despite the fact that it was addressed later, the imaginary biquadratic case is actually a generic case (where $K_1 \neq K_2$), while the imaginary quadratic case is a degenerate situation (where $K = K_1 = K_2$).

In future work, we intend to construct a bipartite Euler system over a biquadratic field as well as investigate the case where p does not split completely in K_0.

2. Preliminaries

2.1. ***Galois representations associated to newforms.*** In this section, we follow [Castella and Do 2025, Section 1.1] and introduce some important notation and results. Let $f \in S_k(\Gamma_1(N_f), \chi_f)$ be a normalized newform of weight $k \geq 2$ and let $\sum_{n=1}^{\infty} a_n q^n$ be its q-expansion. Let $p \nmid N_f$ be a prime. Fix embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $L/\mathbb{Q}$ be the coefficient field of f, that is, L contains all values $i_\infty^{-1}(a_n)$ and $i_\infty^{-1} \circ \chi_f$. Let $\mathfrak{P}$ be the prime of L above p with respect to i_p. Let $S = \{\text{primes } \ell \mid pN_f\} \cup \{\infty\}$. Then Eichler and Shimura (for $k = 2$) and Deligne (for $k > 2$) construct a p-adic Galois representation associated to f,

$$\rho_{f,\mathfrak{P}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(L_{\mathfrak{P}}),$$

such that, for all primes $\ell \notin S$,

- $\mathrm{trace}(\rho_{f,\mathfrak{P}}(\mathrm{Frob}_\ell)) = i_p(a_\ell)$;
- $\det(\rho_{f,\mathfrak{P}}(\mathrm{Frob}_\ell)) = i_p(\chi_f(\ell)\ell^{k-1})$;
- $\rho_{f,\mathfrak{P}}$ is irreducible, and hence absolutely irreducible.

Here Frob_ℓ is the geometric Frobenius.

As in [loc. cit.], one obtains the geometric realization V_f of $\rho_{f,\mathfrak{P}}$ defined as the subspace of

$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbb{Q}}}, \mathscr{S}_{k-2}) \otimes L_{\mathfrak{P}}.$$

Dually, $V_f^\vee = \mathrm{Hom}(V_f, L_{\mathfrak{P}})$ can be interpreted as the maximal quotient of

$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbb{Q}}}, \mathscr{L}_{k-2}(1)) \otimes L_{\mathfrak{P}}$$

on which the dual Hecke operator T'_ℓ acts as multiplication by a_ℓ for all $\ell \nmid N_f p$ and $\langle d \rangle = \langle d \rangle^*$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbb{Z}/N_f\mathbb{Z})^\times$.

Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. There exists a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$ defined as the image of $H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbb{Q}}}, \mathscr{L}_{k-2}(1)) \otimes \mathcal{O}$ in $V_f^\vee$.

If f is ordinary at p (which means $i_p(a_p) \in \mathcal{O}^\times$), then the restriction of V_f to $G_{\mathbb{Q}_p}$ is reducible. This leads us to an exact sequence of $L_{\mathfrak{P}}[G_{\mathbb{Q}_p}]$-modules

$$0 \to V_f^+ \to V_f \to V_f^- \to 0,$$

where $\dim_{L_{\mathfrak{P}}} V_f^\pm = 1$. Dually, we also obtain an exact sequence for the restriction of $V_f^\vee$ to $G_{\mathbb{Q}_p}$,

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0, \tag{2-1}$$

where $V_f^{\vee,+} \simeq (V_f^-)^\vee(1-k)(\chi_f^{-1})$, and the $G_{\mathbb{Q}_p}$-action on the quotient $V_f^{\vee,-}$ is given by the unramified character sending the arithmetic Frobenius Frob_p^{-1} to α_p, which is the unit root of $x^2 - a_p x + \chi_f(p)p^{k-1}$.

2.2. ***Patched CM Hecke modules.*** Here, we recall the conventions on Hecke characters and the construction of certain patched CM Hecke modules from [Castella and Do 2025, Section 1.3; Lei et al. 2015].

2.2.1. *Hecke characters and theta series.* Let K be an imaginary quadratic field. Let $p = \mathfrak{p}\bar{\mathfrak{p}}$ be a prime that splits in K with $\mathfrak{p}$, the prime of K above p, induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We say that a Hecke character $\psi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ has infinity type (m, n), where m, n are integers, if $\psi_\infty(x_\infty) = x_\infty^m \bar{x}_\infty^n$.

Let $\mathrm{rec}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ be the geometrically normalized Artin reciprocity map. Following [Castella and Do 2025, Section 1.3.1], given $g \in G_K$, we take $x \in \mathbb{A}_K^\times$ such that $\mathrm{rec}_K(x) = g|_{K^{\mathrm{ab}}}$ and define

$$\psi_{\mathfrak{P}}(g) = i_p \circ i_\infty^{-1}(\psi(x) x_\infty^{-m} \bar{x}_\infty^{-n}) x_{\mathfrak{p}}^m x_{\bar{\mathfrak{p}}}^n.$$

Such a $\psi_{\mathfrak{P}}$ will be called the p-adic avatar of ψ. We shall also use ψ to denote its p-adic avatar if the context makes this usage reasonable.

Attached to ψ, a Hecke character of K of infinity type $(-1, 0)$ with conductor $\mathfrak{f}$ that takes values in a finite extension L/K, is the theta series

$$\theta_\psi = \sum_{(\mathfrak{a}, \mathfrak{f})=1} \psi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in S_2(\Gamma_1(N_\psi), \chi_\psi \epsilon_K),$$

where $N_\psi = N_{K/\mathbb{Q}}(\mathfrak{f}) \operatorname{disc}(K/\mathbb{Q})$, χ_ψ is the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$ such that $\psi((n)) = n\chi_\psi(n)$ for all $n \in \mathbb{Z}$ with $(n, N_{K/\mathbb{Q}}(\mathfrak{f})) = 1$, and ϵ_K is the quadratic Dirichlet character attached to K. The cuspform θ_ψ is new of level $N_\psi = N_{K/\mathbb{Q}}(\mathfrak{f}) \cdot \operatorname{disc}(K/\mathbb{Q})$ by [Miyake 1989]. One obtains the description of the $\mathfrak{P}$-adic representation of θ_ψ

$$V_{\theta_\psi}^\vee \cong \mathrm{Ind}_K^{\mathbb{Q}} L_{\mathfrak{P}}(\psi^{-1}),$$

where $\mathfrak{P}$ is the prime of L above p with respect to i_p.

2.2.2. *Hecke algebras and norm maps.* We keep the notation of the previous section and follow [Castella and Do 2025, Section 1.3.1]. Let $\mathfrak{n} \subset \mathcal{O}_K$ be an ideal divisible by $\mathfrak{f}$ and let $N = N_{K/\mathbb{Q}}(\mathfrak{n}) \operatorname{disc}(K/\mathbb{Q})$. Let $K_{\mathfrak{n}}$ be the ray class field of K with conductor $\mathfrak{n}$. Let $H_{\mathfrak{n}} = \mathrm{Gal}(K_{\mathfrak{n}}/K)$ be the ray class group of K modulo $\mathfrak{n}$. Let $K(\mathfrak{n})$ be the largest p-subextension of K contained in $K_{\mathfrak{n}}$, i.e., $\mathrm{Gal}(K(\mathfrak{n})/K) \cong H_{\mathfrak{n}}^{(p)}$ is

the largest p-power quotient of $H_{\mathfrak{n}}$. Given an ideal $\mathfrak{k}$ of K that is coprime to $\mathfrak{n}$, let $[\mathfrak{k}]$ be the class of $\mathfrak{k}$ in $H_{\mathfrak{n}}$. Let $\mathbb{T}'(N)$ be the subalgebra of $\mathrm{End}_{\mathbb{Z}}\big(H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})\big)$ generated by $\langle d\rangle'$ and T'_ℓ for all primes ℓ; then one can prove that:

Proposition 2.2.1 [Lei et al. 2015, Proposition 3.2.1]. *There exists a homomorphism $\phi_{\mathfrak{n}} : \mathbb{T}'(N) \to \mathcal{O}[H_{\mathfrak{n}}]$ defined by*

$$\phi_{\mathfrak{n}}(T'_\ell) = \sum_{\substack{\mathfrak{l}\subset\mathcal{O}_K, \mathfrak{l}\nmid\mathfrak{n}\\ N_{K/\mathbb{Q}}(\mathfrak{l})=\ell}} [\mathfrak{l}]\psi(\mathfrak{l}), \quad \phi_{\mathfrak{n}}(\langle d\rangle') = \chi_\psi(d)\epsilon_K(d)[(d)].$$

For $\mathfrak{m} = \mathfrak{n}\mathfrak{l}$ with $\mathfrak{l}$ a prime ideal and $(\mathfrak{m}, p) = 1$, put $M = N_{K/\mathbb{Q}}(\mathfrak{m})\,\mathrm{disc}(K/\mathbb{Q})$; then one has the map

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}} : \mathcal{O}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathbb{T}'(M)\otimes\mathbb{Z}_p,\phi_{\mathfrak{m}}} H^1_{\text{ét}}(Y_1(M)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \to \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)).$$

This norm map is defined explicitly by splitting into three cases (see [Castella and Do 2025, Section 1.1.2] for the definition of the degeneracy map):

- If $\mathfrak{l} \mid \mathfrak{n}$, then
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}} = 1 \otimes \mathrm{pr}_{1*}.$$
- If $\mathfrak{l}\nmid\mathfrak{n}$ is split or ramified in K and $N_{K/\mathbb{Q}}(\mathfrak{l}) = \ell$, then
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell} \otimes \mathrm{pr}_{\ell*}.$$
- If $\mathfrak{l}\nmid\mathfrak{n}$ is inert in K, say $\mathfrak{l} = (\ell)$, then
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell^2} \otimes \mathrm{pr}_{\ell\ell*}.$$

Note that one can extend the definition of $\mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}}$ to any pair of ideals $\mathfrak{n} \mid \mathfrak{m}$ by composition.

Following [ibid., Section 1.3.2], if p splits in K and $(p, \mathfrak{f}) = 1$ then, for any ideal $\mathfrak{n} \subset \mathcal{O}_K$ divisible by $\mathfrak{f}$ such that $(\mathfrak{n}, \bar{\mathfrak{p}}) = 1$, the maximal ideal of $\mathbb{T}'(N)$ defined by the kernel of the composition

$$\mathbb{T}'(N) \xrightarrow{\phi_{\mathfrak{n}}} \mathcal{O}[H_{\mathfrak{n}}] \xrightarrow{\text{aug}} \mathcal{O} \to \mathcal{O}/\mathfrak{P},$$

is non-Eisenstein, p-ordinary and p-distinguished.

We finish this section by extracting a crucial result in [Lei et al. 2015] in the case where p splits in K. This will be used later to prove the norm relation of our Euler system.

Theorem 2.2.2 [Lei et al. 2015, Corollary 5.2.6]. *Assume that $(p, \mathfrak{f}) = 1$. Let $\mathcal{A}$ be the set of ideals $\mathfrak{m} \subset \mathcal{O}_K$ with $(\mathfrak{m}, \bar{\mathfrak{p}}) = 1$, and put $\mathcal{A}_{\mathfrak{f}} = \{\mathfrak{f}\mathfrak{m} : \mathfrak{m} \in \mathcal{A}\}$. Given $\mathfrak{n} \in \mathcal{A}_{\mathfrak{f}}$, there is a $G_{\mathbb{Q}}$-equivariant isomorphism of $\mathcal{O}[H_{\mathfrak{n}}^{(p)}]$-modules*

$$\nu_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\cong} \mathrm{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}}\, \mathcal{O}(\psi_{\mathfrak{P}}^{-1}).$$

Furthermore, for any $\mathfrak{n}, \mathfrak{m} \in \mathcal{A}_{\mathfrak{f}}$ *with* $\mathfrak{n} \mid \mathfrak{m}$, *the diagram*

$$\begin{array}{ccc} \mathcal{O}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathbb{T}'(M)\otimes\mathbb{Z}_p,\phi_{\mathfrak{m}}} H^1_{\text{ét}}(Y_1(M)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\cong]{\nu_{\mathfrak{m}}} & \mathrm{Ind}_{K(\mathfrak{m})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \Big\downarrow \mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}} & & \Big\downarrow \mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{m}} \\ \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\cong]{\nu_{\mathfrak{n}}} & \mathrm{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \end{array}$$

commutes, *where* $\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{m}}$ *is the natural norm map of the induced representations.*

3. The construction

For a newform f and two Hecke characters ψ_1, ψ_2 of two distinct imaginary quadratic fields K_1, K_2, respectively, using the results from [Castella and Do 2025; Bertolini et al. 2022b; Lei et al. 2015] recalled in the preceding section, we construct a family of cohomology classes for $f \otimes \tilde{\psi}_1\tilde{\psi}_2$ defined over ring class field extensions of K_0, which is the compositum of K_1 and K_2, and prove that they satisfy the norm relations of an anticyclotomic Euler system. Following [Castella and Do 2025, Section 2], we first give the construction and show the tame norm relations in the case where $(f, \theta_{\psi_1}, \theta_{\psi_2})$ have weights $(2, 2, 2)$. Then, by varying the diagonal cycle classes in Hida families, we extend the construction to more general weights and prove the wild norm relations.

Throughout this section we consider the following setup:

(1) Let $f \in S_k(\Gamma_0(N_f))$ be a newform of weight $k \geq 2$.

(2) Let $K_1/\mathbb{Q}$ be an imaginary quadratic field of discriminant D_1 coprime to N_f. Let ψ_1 be a Hecke character of K_1 of infinity type $(1-k_1, 0)$, with $k_1 \geq 1$, and modulus $\mathfrak{f}_1$.

(3) Let $K_2/\mathbb{Q}$ be an imaginary quadratic field of discriminant $D_2 \neq D_1$ and coprime to N_f. Let ψ_2 be a Hecke character of K_2 of infinity type $(1-k_2, 0)$, with $k_2 \geq 1$, and modulus $\mathfrak{f}_2$.

(4) Denote by ϵ_{K_i} the quadratic character attached to the quadratic field K_i for $i \in \{1, 2\}$.

(5) Let K_0 be the compositum of K_1 and K_2. Since K_0 is a biquadratic field, we can consider K_3, the unique real quadratic field inside K_0.

(6) Let $\tilde{\psi}_i$ be the Hecke character of K_0, obtained by composing $\mathbb{A}_{K_0}^{\times} \xrightarrow{\mathbb{N}_{K_0/K_i}} \mathbb{A}_{K_i}^{\times} \xrightarrow{\psi_i} \mathbb{C}$ for $i \in \{1, 2\}$.

(7) Denote by

$$\theta_{\psi_i} \in S_{k_i}(N_{\psi_i}, \chi_{\psi_i}\epsilon_{K_i})$$

the associated theta series, where $N_{\psi_i} = N_{K_i/\mathbb{Q}}(\mathfrak{f}_i) \cdot \mathrm{disc}(K_i/\mathbb{Q})$ and χ_{ψ_i} is the Dirichlet character modulo $N_{K_i/\mathbb{Q}}(\mathfrak{f}_i)$ defined by $\psi_i((n)) = n^{k_i-1}\chi_{\psi_i}(n)$ for all integers n prime to $N_{K_i/\mathbb{Q}}(\mathfrak{f}_i)$ for $i \in \{1, 2\}$.

(8) We assume the self-duality condition

$$\chi_{\psi_1}\epsilon_{K_1}\chi_{\psi_2}\epsilon_{K_2} = 1. \tag{3-1}$$

Let L/K_0 be a finite extension containing the Fourier coefficients of f, θ_{ψ_1} and θ_{ψ_2}. Let $p \geq 5$ be a prime that splits in K_0 such that $(p, N_f N_{\psi_1} N_{\psi_2}) = 1$, and let $\mathfrak{P} \mid \mathfrak{p}$ be the prime of L/K_0 above p determined by a fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Finally, let $L_{\mathfrak{P}}$ be the completion of L at $\mathfrak{P}$, and denote by $\mathcal{O}$ the ring of integers of $L_{\mathfrak{P}}$.

3.1. ***Digression to primes decomposition and the left superscript notation.*** Let ℓ be a split prime in K_0, i.e., $(\ell) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$. We can write $(\ell) = \lambda_1\bar{\lambda}_1$ and $(\ell) = \lambda_2\bar{\lambda}_2$ in K_1 and K_2, respectively. Note that ℓ also splits in K_3 as $\lambda_3\tilde{\lambda}_3$, where the tilde corresponds to the nontrivial element generating the Galois group $\mathrm{Gal}(K_3/\mathbb{Q})$.

Let τ_i be the generator of $\mathrm{Gal}(K_0/K_i)$ for $i = \{1, 2, 3\}$; then $\tau_3 = \tau_1\tau_2$ (this is the complex conjugation on K_0/K_3). Due to the Galois group action on primes lying above ℓ, we can further assume that

$$\lambda_1 = \mathcal{L}_1\mathcal{L}_4, \quad \bar{\lambda}_1 = \mathcal{L}_3\mathcal{L}_2, \quad \lambda_2 = \mathcal{L}_1\mathcal{L}_3, \quad \bar{\lambda}_2 = \mathcal{L}_2\mathcal{L}_4,$$
$$\lambda_3 = \mathcal{L}_4\mathcal{L}_3 \quad (\text{so } \lambda_3 \mid \lambda_1\lambda_2) \quad \text{and} \quad \tilde{\lambda}_3 = \mathcal{L}_1\mathcal{L}_2,$$

where

$$\mathcal{L}_4 = \tau_1\mathcal{L}_1, \quad \mathcal{L}_3 = \tau_2\mathcal{L}_1, \quad \mathcal{L}_2 = \tau_3\mathcal{L}_1 = \tau_1\tau_2\mathcal{L}_1.$$

Denote by $\mathcal{L}$ the set of primes $\lambda_3 \subset \mathcal{O}_{K_3}$, where $\ell = N_{K_3/\mathbb{Q}}(\lambda_3)$ is prime to p and ℓ splits in K_0. Let $\mathcal{N}$ be the set of squarefree products of primes inside $\mathcal{L}$ whose norm down to $\mathbb{Q}$ is still squarefree. For such λ_3, we can choose $\lambda_1 \subset \mathcal{O}_{K_1}$ and $\lambda_2 \subset \mathcal{O}_{K_2}$ as above such that $\lambda_3 \mid \lambda_1\lambda_2$.

Let $\mu_3 \in \mathcal{N}$ and $N_{K_3/\mathbb{Q}}(\mu_3) = m$. Then its norm $m = \prod_i \ell_i$ will be a product of split primes ℓ_i in K_0. Similarly, we can decompose $(m) = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\mathcal{M}_4$, $(m) = \mu_1\bar{\mu}_1$, $(m) = \mu_2\bar{\mu}_2$ and $(m) = \mu_3\tilde{\mu}_3$ inside K_0, K_1, K_2 and K_3, respectively, where we can assume

$$\mu_1 = \mathcal{M}_1\mathcal{M}_4, \quad \bar{\mu}_1 = \mathcal{M}_3\mathcal{M}_2, \quad \mu_2 = \mathcal{M}_1\mathcal{M}_3, \quad \bar{\mu}_2 = \mathcal{M}_2\mathcal{M}_4,$$
$$\mu_3 = \mathcal{M}_4\mathcal{M}_3 \quad (\text{so } \mu_3 \mid \mu_1\mu_2) \quad \text{and} \quad \tilde{\mu}_3 = \mathcal{M}_1\mathcal{M}_2.$$

Here, for every i, $\ell_i = \prod_j \mathcal{L}_{j,i}$, and $\mathcal{M}_j = \prod_i \mathcal{L}_{j,i}$ for $1 \leq j \leq 4$.

For each $i \in \{0, 1, 2\}$, we define ${}^iK_{\mathfrak{n}_i}$ as the ray class field of K_i with conductor $\mathfrak{n}_i$ (an integral ideal inside $\mathcal{O}_{K_i}$), and let ${}^iH_{\mathfrak{n}_i}$ be the ray class group of K_i modulo $\mathfrak{n}_i$. Let $K_i(\mathfrak{n}_i)$ be the largest p-subextension of K_i contained in ${}^iK_{\mathfrak{n}_i}$, so $\mathrm{Gal}(K_i(\mathfrak{n}_i)/K_i) \cong {}^iH_{\mathfrak{n}_i}^{(p)}$ is the largest p-power quotient of ${}^iH_{\mathfrak{n}_i}$.

3.2. ***Construction in weight*** **(2, 2, 2)** ***and tame norm relation.*** Suppose in this section that $(k, k_1, k_2) = (2, 2, 2)$. Let $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$. Following [Castella and Do 2025, Section 2.1], which is based on the diagonal classes in the triple product of modular curves [Bertolini et al. 2022b, Section 3], we have cohomology classes

$$\mathcal{Z}_m^{(1)} := \tilde{\kappa}_m^{(3)} \in H^1\big(\mathbb{Q}, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))(-1)\big)$$

for every positive integer m. One then chooses a test vector $\check{f} \in S_2(N)[f]$. As noted in [loc. cit.], the maps used to construct $\mathcal{Z}_m^{(1)}$ are compatible with correspondences. This allows one to tensor them with $\mathcal{O}$

and obtain

$$\mathcal{Z}_{\mu_3}^{(1)} \in H^1\big(\mathbb{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1}] \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2}](-1)\big). \quad (3\text{-}2)$$

Here, the chosen $\check{f}$ is used to take the image under the projection $H^1_{\text{ét}}(Y_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \to T_f^\vee$ in the first factor. The tensor products are taken from Proposition 2.2.1,

$$\phi_{\mathfrak{f}_1\mu_1} : \mathbb{T}'(N_{\psi_1}m) \to \mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1}], \quad \phi_{\mathfrak{f}_2\mu_2} : \mathbb{T}'(N_{\psi_2}m) \to \mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2}],$$

with respect to two distinct imaginary quadratic fields K_1 and K_2, respectively.

Via the isomorphisms from Theorem 2.2.2 with respect to two distinct imaginary quadratic fields,

$$\nu_{\mathfrak{f}_1\mu_1} : H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1}] \xrightarrow{\sim} \operatorname{Ind}^{\mathbb{Q}}_{K_1(\mathfrak{f}_1\mu_1)} \mathcal{O}(\psi_1^{-1}),$$
$$\nu_{\mathfrak{f}_2\mu_2} : H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2}] \xrightarrow{\sim} \operatorname{Ind}^{\mathbb{Q}}_{K_2(\mathfrak{f}_2\mu_2)} \mathcal{O}(\psi_2^{-1}),$$

one then obtains a cohomology class in

$$H^1\big(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K_1(\mathfrak{f}_1\mu_1)} \mathcal{O}(\psi_1^{-1}) \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K_2(\mathfrak{f}_2\mu_2)} \mathcal{O}(\psi_2^{-1})(-1)\big),$$

which, under the maps ${}^1H_{\mathfrak{f}_1\mu_1} \twoheadrightarrow {}^1H_{\mu_1}$ and ${}^2H_{\mathfrak{f}_2\mu_2} \twoheadrightarrow {}^2H_{\mu_2}$, can be projected to a class

$$\mathcal{Z}_{\mu_3}^{(2)} \in H^1\big(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K_1} \mathcal{O}_{\psi_1^{-1}}[{}^1H^{(p)}_{\mu_1}] \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K_2} \mathcal{O}_{\psi_2^{-1}}[{}^2H^{(p)}_{\mu_2}](-1)\big). \quad (3\text{-}3)$$

Note that the group cohomology can be rewritten as

$$H^1\big(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K_0} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[{}^1H^{(p)}_{\mu_1} \times {}^2H^{(p)}_{\mu_2}](-1)\big),$$

which, by Shapiro's lemma, gives us elements

$$\mathcal{Z}_{\mu_3}^{(3)} \in H^1\big(K_0, T_f^\vee \otimes_{\mathcal{O}} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[{}^1H^{(p)}_{\mu_1} \times {}^2H^{(p)}_{\mu_2}](-1)\big). \quad (3\text{-}4)$$

3.2.1. *Projection to ring class groups.* Recall the fundamental exact sequence for ray class groups

$$\mathcal{O}_{K_i}^\times \to (\mathcal{O}_{K_i}/\mu_i\mathcal{O}_{K_i})^\times \to {}^iH_{\mu_i} \to {}^iH_1 \to 1$$

for $i \in \{1, 2\}$. Assume that $p \nmid 6h_{K_0}$, where h_{K_0} is the class number of K_0. Note that, by [Fröhlich and Taylor 1993, Theorem 74], we have

$$p \nmid h_{K_i},$$

which is the class number of K_i for $i \in \{1, 2, 3\}$. Taking the p-primary parts of the above exact sequence induces two isomorphisms

$${}^1H^{(p)}_{\mu_1} \xrightarrow{\simeq} (\mathcal{O}_{K_1}/\mu_1\mathcal{O}_{K_1})^{\times,(p)} \xrightarrow{\simeq} (\mathcal{O}_{K_0}/\mathcal{M}_4\mathcal{O}_{K_0})^{\times,(p)},$$
$${}^2H^{(p)}_{\mu_2} \xrightarrow{\simeq} (\mathcal{O}_{K_2}/\mu_2\mathcal{O}_{K_2})^{\times,(p)} \xrightarrow{\simeq} (\mathcal{O}_{K_0}/\mathcal{M}_3\mathcal{O}_{K_0})^{\times,(p)},$$

and hence the projection

$$ {}^1H_{\mu_1}^{(p)} \times {}^2H_{\mu_2}^{(p)} \xrightarrow{\simeq} (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^{\times,(p)} \twoheadrightarrow {}^0H_{\mu_3}^{(p)}. \tag{3-5} $$

Recall that K_3 is the totally real field sitting inside a CM field K_0. Given an integral ideal $\mathfrak{n}$ of K_3, let $H[\mathfrak{n}]$ be the ring class group of K_0 of conductor $\mathfrak{n}$, so $H[\mathfrak{n}] \simeq \mathrm{Pic}(\mathcal{O}_\mathfrak{n})$ under the Artin reciprocity map, where $\mathcal{O}_\mathfrak{n} = \mathcal{O}_{K_3} + \mathfrak{n}\mathcal{O}_{K_0}$ is the order of K_0 of conductor $\mathfrak{n}$. Let $H[\mathfrak{n}]^{(p)}$ be the maximal p-power quotient of $H[\mathfrak{n}]$, and denote by $K_0[\mathfrak{n}]$ the maximal p-extension inside the ring class field of K_0 of conductor $\mathfrak{n}$, i.e., $H[\mathfrak{n}]^{(p)} = \mathrm{Gal}(K_0[\mathfrak{n}]/K_0)$. Note that, for the ring class groups and fields of K_0, we drop the left superscript 0 from the notation.

Proposition 3.2.1. *Suppose $p \nmid 6h_{K_0}$ and μ_3 is a squarefree ideal of $\mathcal{O}_{K_3}$ of norm m, where m is divisible only by primes that are split in K_0. Using* (3-5)*, we have the short exact sequence*

$$ 1 \to (\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \xrightarrow{\Delta} {}^1H_{\mu_1}^{(p)} \times {}^2H_{\mu_2}^{(p)} \xrightarrow{e\circ w} H[\mu_3]^{(p)} \to 1. $$

Here, the map Δ uses the identifications

$$ (\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \simeq (\mathcal{O}_{K_0}/\mathcal{M}_4\mathcal{O}_{K_0})^{\times,(p)}, \quad (\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \simeq (\mathcal{O}_{K_0}/\mathcal{M}_3\mathcal{O}_{K_0})^{\times,(p)}. $$

Moreover, if $(\ell) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$ is a prime that splits in K_0 and is coprime to m, the projection $e\circ w$ (defined in the proof below) is given by

$$ [\lambda_1] \times [\bar{\lambda}_2] \mapsto \mathrm{Frob}_{\mathcal{L}_4/\lambda_3}, $$

where $\mathrm{Frob}_{\mathcal{L}_4/\lambda_3}$ *is the Frobenius element of $\mathcal{L}_4$ in $H[\mu_3]^{(p)}$.*

Proof. We have the exact diagram

$$ \begin{array}{ccccccccc} \mathcal{O}_{K_0}^\times & \longrightarrow & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^\times & \longrightarrow & {}^0H_{\mu_3} & \longrightarrow & {}^0H_1 & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{O}_{K_0}^\times/\mathcal{O}_{K_3}^\times & \longrightarrow & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^\times/(\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^\times & \longrightarrow & H[\mu_3] & \longrightarrow & {}^0H_1 & \longrightarrow & 1 \end{array} $$

Taking the p-part of this, using the assumption that $p \nmid 6h_{K_0}$ together with the fact that $|\mathcal{O}_{K_0}^\times/\mathcal{O}_{K_3}^\times|$ is a power of 2, we obtain the diagram

$$ \begin{array}{ccccccc} \mathcal{O}_{K_0}^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^{\times,(p)} & \xrightarrow{w} & {}^0H_{\mu_3}^{(p)} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow e & & \\ 1 & \longrightarrow & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^{\times,(p)}/(\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} & \longrightarrow & H[\mu_3]^{(p)} & \longrightarrow & 1 \end{array} $$

Using the middle arrow and the identification (3-5), we can show the first exact sequence.

One can show the second part by noting that $\mathcal{L}_4$ is a prime of K_0 lying above both λ_1 (a prime of K_1) and $\bar{\lambda}_2$ (a prime of K_2). □

In the same setting as Proposition 3.2.1, we can consider the image of $\mathcal{Z}^{(3)}_{\mu_3}$ from (3-4) under the composition $e \circ w$. This results in the class

$$\mathcal{Z}^{(4)}_{\mu_3} \in H^1\big(K_0, T_f^\vee \otimes_{\mathcal{O}} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3]^{(p)}](-1)\big).$$

By Shapiro's lemma, its image under the isomorphism

$$H^1\big(K_0, T_f^\vee \otimes_{\mathcal{O}} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3]^{(p)}](-1)\big) \simeq H^1\big(K_0[\mu_3], T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1)\big)$$

then defines

$$\mathcal{Z}^{(5)}_{\mu_3} \in H^1(K_0[\mu_3], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)).$$

The next lemma is in the same vein as [Castella and Do 2025, Lemma 2.1.3]:

Lemma 3.2.2. *Let λ_3 be a split prime in K_3 of norm ℓ coprime to mp, where ℓ splits in K_0. The diagram*

$$\begin{array}{ccc} \mathrm{Ind}_{K_1}^{\mathbb{Q}}\mathcal{O}_{\psi_1^{-1}}[{}^1H^{(p)}_{\mu_1\lambda_1}] \otimes_{\mathcal{O}} \mathrm{Ind}_{K_2}^{\mathbb{Q}}\mathcal{O}_{\psi_2^{-1}}[{}^2H^{(p)}_{\mu_2\lambda_2}] & \longrightarrow & \mathrm{Ind}_{K_0}^{K_3}\mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3\lambda_3]^{(p)}] \\ \Big\downarrow{\scriptstyle \mathrm{Norm}_{\mu_1^{\lambda_1}}\otimes\mathrm{Norm}_{\mu_2^{\lambda_2}}} & & \Big\downarrow{\scriptstyle \mathrm{Norm}_{\mu_3^{\lambda_3}}} \\ \mathrm{Ind}_{K_1}^{\mathbb{Q}}\mathcal{O}_{\psi_1^{-1}}[{}^1H^{(p)}_{\mu_1}] \otimes_{\mathcal{O}} \mathrm{Ind}_{K_2}^{\mathbb{Q}}\mathcal{O}_{\psi_2^{-1}}[{}^2H^{(p)}_{\mu_2}] & \longrightarrow & \mathrm{Ind}_{K_0}^{K_3}\mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3]^{(p)}] \end{array}$$

commutes, where the norm maps are the natural ones and the horizontal arrows are given by the composition $e \circ w$ in Proposition 3.2.1.

Using Lemma 3.2.2, one can show the following proposition. Similar to [Castella and Do 2025, Proposition 2.2.1], this is the key result for the construction of our anticyclotomic Euler system for $T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1)$ over the biquadratic field K_0.

Proposition 3.2.3. *The family $\{\mathcal{Z}^{(5)}_{\mu_3}\}$ satisfies the norm relation*

$$\begin{aligned}\mathrm{Norm}^{K_0[\mu_3\lambda_3]}_{K_0[\mu_3]}(\mathcal{Z}^{(5)}_{\mu_3\lambda_3}) = (\ell-1)\Bigg(a_\ell(f) - \frac{\psi_1(\lambda_1)\psi_2(\bar{\lambda}_2)}{\ell}([\lambda_1]\times[\bar{\lambda}_2]) - \frac{\psi_1(\bar{\lambda}_1)\psi_2(\lambda_2)}{\ell}([\bar{\lambda}_1]\times[\lambda_2]) \\ + (1-\ell)\frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])\Bigg)(\mathcal{Z}^{(5)}_{\mu_3}).\end{aligned}$$

Proof. As in the proof of [loc. cit.], one has

$$\begin{aligned}&(1\otimes\mathcal{N}_{\mu_1}^{\mu_1\lambda_1}\otimes\mathcal{N}_{\mu_2}^{\mu_2\lambda_2})(\mathcal{Z}^{(5)}_{\mu_3\lambda_3}) \\ &= (\ell-1)\Bigg((T_\ell,1,1) - \frac{\psi_1(\lambda_1)[\lambda_1]}{\ell}(1,1,T'_\ell) - \frac{\psi_2(\lambda_2)[\lambda_2]}{\ell}(1,T'_\ell,1) \\ &\qquad + \frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])(\ell+1)\Bigg)(\mathcal{Z}^{(5)}_{\mu_3}) \\ &= (\ell-1)\Bigg(a_\ell(f) - \frac{\psi_1(\lambda_1)[\lambda_1]}{\ell}(\psi_2(\lambda_2)[\lambda_2] + \psi_2(\bar{\lambda}_2)[\bar{\lambda}_2]) \\ &\qquad - (\psi_1(\lambda_1)[\lambda_1] + \psi_1(\bar{\lambda}_1)[\bar{\lambda}_1])\frac{\psi_2(\lambda_2)[\lambda_2]}{\ell} + \frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])(\ell+1)\Bigg)(\mathcal{Z}^{(5)}_{\mu_3})\end{aligned}$$

$$= (\ell - 1)\Bigg(a_\ell(f) - \frac{\psi_1(\lambda_1)\psi_2(\bar{\lambda}_2)}{\ell}([\lambda_1] \times [\bar{\lambda}_2]) - \frac{\psi_1(\bar{\lambda}_1)\psi_2(\lambda_2)}{\ell}([\bar{\lambda}_1] \times [\lambda_2]) + (1-\ell)\frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1] \times [\lambda_2])\Bigg)(\mathcal{Z}^{(5)}_{\mu_3}).$$

This implies the result via combining Theorem 2.2.2 and Lemma 3.2.2. □

Following [Castella and Do 2025, Section 2.2] verbatim, which borrows ideas from [Darmon and Rotger 2017, Section 1.4], we can strip out the $\ell - 1$ factor by quotienting out the diamond operators action and obtain modified classes

$$\mathcal{Z}^{(6)}_{\mu_3} \in H^1(K_0[\mu_3], T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1)).$$

Then the term in the right-hand side of Proposition 3.2.3 can be massaged to agree with the local Euler factor at $\mathcal{L}_2$ of the Galois representation $[T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1)]^\vee(1) = T_f(\tilde{\psi}_1\tilde{\psi}_2)(2)$, giving the correct norm relations:

Theorem 3.2.4. *Suppose $p \nmid 6h_{K_0}$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mu_3 \in \mathcal{N}$ be a squarefree ideal of $\mathcal{O}_{K_3}$. Then there exists a collection of cohomology classes*

$$\mathcal{Z}_{\mu_3} \in H^1(K_0[\mu_3], T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1))$$

such that, for every split prime $\lambda_3 \in \mathcal{L}$ of $\mathcal{O}_{K_3}$ of norm ℓ with $(\ell, m) = 1$, we have the norm relation

$$\mathrm{Norm}^{K_0[\mu_3\lambda_3]}_{K_0[\mu_3]}(\mathcal{Z}_{\mu_3\lambda_3}) = P_{\mathcal{L}_4}(\mathrm{Frob}_{\mathcal{L}_4/\lambda_3})(\mathcal{Z}_{\mu_3}),$$

where $P_{\mathcal{L}_4}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathcal{L}_4/\lambda_3} | T_f(\tilde{\psi}_1\tilde{\psi}_2)(2))$.

Proof. The proof is parallel to the one of [Castella and Do 2025, Theorem 2.2.7]. First note that $[\lambda_1] \times [\bar{\lambda}_2]$ corresponds to $\mathrm{Frob}_{\mathcal{L}_4/\lambda_3} \in H[\mu_3]^{(p)}$ under the map $e \circ w$ of Proposition 3.2.1. One then multiplies the class $\mathcal{Z}^{(6)}_{\mu_3\lambda_3}$ with $-\psi_1(\lambda_1)\psi_2(\bar{\lambda}_2)([\lambda_1] \times [\bar{\lambda}_2])$. From $\psi_1\psi_2((\ell)) = \chi_{\psi_1}\chi_{\psi_2}(\ell)\ell^2 = \epsilon_{K_1}(\ell)^{-1}\epsilon_{K_2}(\ell)^{-1}\ell^2 = \ell^2$ since ℓ splits in K_0, $\psi_1(\lambda_1)\psi_2(\bar{\lambda}_2) = \tilde{\psi}_1(\mathcal{L}_4)\tilde{\psi}_2(\mathcal{L}_4)$, and the fact that $[\ell] \times [\ell]$ maps to the identity element inside the ring class group, together with [Rubin 2000, Lemmas 9.6.1 and 9.6.3], the result follows from the explicit formula of Proposition 3.2.3. □

3.3. ***Construction for general weights and wild norm relations.*** We now extend the above construction to other weights $(k, k_1, k_2) \in \mathbb{Z}^3_{\geq 1}$ following [Castella and Do 2025, Section 2.3]. Then we show that the constructed cohomology classes also satisfy the wild norm relations for the anticyclotomic $\mathbb{Z}_p^2$-extension of K_0.

First, we assume that $p \nmid 6h_{K_0}$. Assume further that p splits in K_0, i.e.,

$$(p) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4 \text{ in } K_0,$$

and $\mathcal{P}_2 = \tau_3\mathcal{P}_1$, $\mathcal{P}_3 = \tau_2\mathcal{P}_1$, $\mathcal{P}_4 = \tau_1\mathcal{P}_1$. Hence,

$$(p) = \mathfrak{p}_1\bar{\mathfrak{p}}_1 \quad \text{in } K_1, \qquad (p) = \mathfrak{p}_2\bar{\mathfrak{p}}_2 \quad \text{in } K_2,$$

with $\mathcal{P}_1$ the prime of K_0 above p induced by our fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and $\mathcal{P}_1$ lies above $\mathfrak{p}_i$, the prime of K_i, for $i \in \{1, 2\}$. Note that the numbering here is parallel to our convention in Section 3.1.

Let $\Gamma_{\mathfrak{p}_i}^{K_i}$ be the Galois group of the unique $\mathbb{Z}_p$-extension of K_i unramified outside $\mathfrak{p}_i$. There exists a unique Hecke character $\psi_{0,i}$ of K_i of infinity type $(-1, 0)$ and conductor $\mathfrak{p}_i$ such that its p-adic avatar factors through $\Gamma_{\mathfrak{p}_i}^{K_i}$. The character ψ_i fixed at the beginning of this section can be decomposed as

$$\psi_i = \xi_i \psi_{0,i}^{k_i - 1},$$

where ξ_i is a ray class character of K of conductor dividing $\mathfrak{f}_i \mathfrak{p}_i$. Noting that $\Gamma_{\mathfrak{p}_i}^{K_i}$ is a quotient of ${}^iH_{\mathfrak{f}_i \mathfrak{p}_i^\infty}$ allows us to view $\psi_{0,i}$ and ξ_i as characters of ${}^iH_{\mathfrak{f}_i \mathfrak{p}_i^\infty}$. The formal q-expansion

$$\boldsymbol{\theta}_{\xi_i}(q) = \sum_{(\mathfrak{a}, \mathfrak{f}_i \mathfrak{p}_i) = 1} \xi_i \psi_{0,i}(\mathfrak{a})[\mathfrak{a}] q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \Lambda_{\mathfrak{p}_i}[\![q]\!],$$

where $\Lambda_{\mathfrak{p}_i} = \mathcal{O}[\![\Gamma_{\mathfrak{p}_i}^{K_i}]\!] \simeq \mathcal{O}[\![\Gamma]\!]$ and $\Gamma = 1 + p\mathbb{Z}_p$, is the Hida family passing through θ_{ψ_i} (the specialization of $\boldsymbol{\theta}_{\xi_i}$ at weight k_i and trivial character recovers the ordinary p-stabilization of θ_{ψ_i}). Here we identify $\Gamma_{\mathfrak{p}_i}^{K_i}$ with $1 + p\mathbb{Z}_p$ via the (geometrically normalized) local Artin map.

Let $\boldsymbol{f}$ be the Hida family associated to f. Let $\boldsymbol{g} = \boldsymbol{\theta}_{\xi_1}$ and $\boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}$ be the CM Hida families associated to ψ_1 and ψ_2, respectively. Denote by κ_f, κ_g and κ_h the Dirichlet characters modulo p giving the p-part of the tame characters of $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, respectively.

Under the assumption that $\xi_i \psi_{0,i} \not\equiv \omega \pmod{\mathfrak{P}}$ for $i \in \{1, 2\}$, following equation (2.17) from [Castella and Do 2025, Section 2.3] and its notation, we have the $G_{\mathbb{Q}}$-equivariant maps

$$\begin{aligned} H^1(\Gamma(m, p), \mathcal{D}'_{\kappa_1}) \otimes \mathcal{O}[\![{}^1H^{(p)}_{\mathfrak{f}_1 \mu_1 \mathfrak{p}_1^\infty}]\!] &\to \mathrm{Ind}_{K_1}^{\mathbb{Q}} \mathcal{O}_{(\xi_1 \psi_{0,1})^{-1}}[{}^1H^{(p)}_{\mu_1}][\![\Gamma_{\mathfrak{p}_1}^{K_1}]\!], \\ H^1(\Gamma(m, p), \mathcal{D}'_{\kappa_2}) \otimes \mathcal{O}[\![{}^2H^{(p)}_{\mathfrak{f}_2 \mu_2 \mathfrak{p}_2^\infty}]\!] &\to \mathrm{Ind}_{K_2}^{\mathbb{Q}} \mathcal{O}_{(\xi_2 \psi_{0,2})^{-1}}[{}^2H^{(p)}_{\mu_2}][\![\Gamma_{\mathfrak{p}_2}^{K_2}]\!], \end{aligned} \tag{3-6}$$

where $\Gamma(m, p) = \Gamma_1(Nm) \cap \Gamma_0(p)$ is a congruence subgroup. Focusing on the class $\boldsymbol{\kappa}_m^{(2)}$ in equation (2.15) of [loc. cit.], we first tensor it with $\mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1 \mu_1 \mathfrak{p}_1^r}]$ and $\mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2 \mu_2 \mathfrak{p}_2^r}]$, let $r \to \infty$, and then arrive at

$$\boldsymbol{\mathcal{Z}}_{\mu_3}^{(1)} \in H^1\Big(\mathbb{Q}, H^1(\Gamma(1, p), \mathcal{D}'_{\kappa_f}) \hat{\otimes}_{\mathcal{O}} \big(H^1(\Gamma(m, p), \mathcal{D}'_{\kappa_g}) \otimes \mathcal{O}[\![{}^1H^{(p)}_{\mathfrak{f}_1 \mu_1 \mathfrak{p}_1^\infty}]\!]\big) \hat{\otimes}_{\mathcal{O}[D_m]} \big(H^1(\Gamma(m, p), \mathcal{D}'_{\kappa_h}) \otimes \mathcal{O}[\![{}^2H^{(p)}_{\mathfrak{f}_2 \mu_2 \mathfrak{p}_2^\infty}]\!]\big)(2 - \kappa^*_{\boldsymbol{fgh}})\Big).$$

Now choose a level-N test vector for f, denoted by $\check{f}$. It comes with a specialization map

$$\pi_f : H^1(\Gamma(1, p), \mathcal{D}'_{\kappa_f})(1) \to T_f^\vee. \tag{3-7}$$

Under the natural maps induced by (3-6) and (3-7), the image of $\boldsymbol{\mathcal{Z}}_{\mu_3}^{(1)}$ is then $\boldsymbol{\mathcal{Z}}_{\mu_3}^{(2)}$ in

$$H^1\big(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} (\mathrm{Ind}_{K_1}^{\mathbb{Q}} \mathcal{O}_{(\xi_1 \psi_{0,1})^{-1}}[{}^1H^{(p)}_{\mu_1}][\![\Gamma_{\mathfrak{p}_1}^{K_1}]\!]) \hat{\otimes}_{\mathcal{O}[D_m]} (\mathrm{Ind}_{K_2}^{\mathbb{Q}} \mathcal{O}_{(\xi_2 \psi_{0,2})^{-1}}[{}^2H^{(p)}_{\mu_2}][\![\Gamma_{\mathfrak{p}_2}^{K_2}]\!])(-1 - \kappa^*_{\boldsymbol{fgh}})\big).$$

We first follow (3-3) and then apply the diagonal map $e \circ w$ in Proposition 3.2.1. This induces the class

$$\boldsymbol{\mathcal{Z}}_{\mu_3}^{(3)} \in H^1\big(K_0, T_f^\vee\big(1 - \tfrac{1}{2}k\big) \otimes_{\mathcal{O}} \mathrm{Ind}_{K_0[\mu_3]}^{K_0} \Lambda_{\mathcal{O}}(\tilde{\psi}_1^{-1} \tilde{\psi}_2^{-1} \tilde{\kappa}_{\mathrm{ac},1}^{(k_1-2)/2} \tilde{\kappa}_{\mathrm{ac},2}^{(k_2-2)/2} \boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\big(1 - \tfrac{1}{2}(k_1 + k_2)\big)\big). \tag{3-8}$$

Here, for $i \in \{1, 2\}$, we identify $\Gamma_i^- = \mathrm{Gal}(K_{i,\infty}^-/K_i)$ with the antidiagonal in $(1+p\mathbb{Z}_p)\times(1+p\mathbb{Z}_p) \simeq \mathcal{O}_{K_i,\mathfrak{p}_i}^{(1)} \times \mathcal{O}_{K_i,\bar{\mathfrak{p}}_i}^{(1)}$ via the geometric normalized Artin map, and define

$$\kappa_{\mathrm{ac},i} : \Gamma_i^- \to \mathbb{Z}_p^\times, \quad ((1+p)^{-1/2}, (1+p)^{1/2}) \mapsto (1+p);$$

compare this with [Castella and Do 2025, equation (2.19)]. Note that here we identify the anticyclotomic $\mathbb{Z}_p$-extension of K_i with the unique $\mathbb{Z}_p$-extension of K_i unramified outside $\mathfrak{p}_i$ (projection to the anticyclotomic part introduces a square root, $\gamma \mapsto \gamma^{(1-c)/2}$; see also [Hida 2015, page 636]).

We then identify the anticyclotomic $\mathbb{Z}_p^2$-extension $\Gamma^- = \mathrm{Gal}(K_{0,\infty}^-/K_0)$ of K_0 with $\Gamma_{\mathfrak{p}_1}^{K_1} \times \Gamma_{\mathfrak{p}_2}^{K_2}$ via the diagram

$$\begin{array}{ccccc}
(\mathcal{O}^{(1)}_{K_0,\mathcal{P}_1}\times\mathcal{O}^{(1)}_{K_0,\mathcal{P}_2})\times(\mathcal{O}^{(1)}_{K_0,\mathcal{P}_3}\times\mathcal{O}^{(1)}_{K_0,\mathcal{P}_4}) & \twoheadrightarrow & \dfrac{(1+p\mathbb{Z}_p)\times(1+p\mathbb{Z}_p)}{\mathrm{diag}}\times\dfrac{(1+p\mathbb{Z}_p)\times(1+p\mathbb{Z}_p)}{\mathrm{diag}} & \xrightarrow{\simeq} & \mathbb{Z}_p^2 \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{O}^{(1)}_{K_1,\mathfrak{p}_1}\times\mathcal{O}^{(1)}_{K_1,\bar{\mathfrak{p}}_1})\times(\mathcal{O}^{(1)}_{K_2,\mathfrak{p}_2}\times\mathcal{O}^{(1)}_{K_2,\bar{\mathfrak{p}}_2}) & \twoheadrightarrow & \dfrac{(1+p\mathbb{Z}_p)\times(1+p\mathbb{Z}_p)}{\mathrm{diag}}\times\dfrac{(1+p\mathbb{Z}_p)\times(1+p\mathbb{Z}_p)}{\mathrm{diag}} & \xrightarrow{\simeq} & \mathbb{Z}_p\times\mathbb{Z}_p
\end{array} \tag{3-9}$$

Let $\Lambda^- = \mathbb{Z}_p[\![\Gamma^-]\!]$ and define further $\boldsymbol{\kappa}_{\mathrm{ac}} : \Gamma^- \to \Lambda^\times$, where

$$((1+p)^{-1/2}, (1+p)^{1/2}, (1+p)^{-1/2}, (1+p)^{1/2}) \mapsto [(1+p), (1+p)].$$

Given an $\mathcal{O}$-lattice T inside a G_{K_0}-representation V, Shapiro's lemma allows us to write

$$H^1(K_0, T \hat{\otimes}_{\mathcal{O}} \Lambda_{\mathcal{O}}^-(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})) \simeq H^1_{\mathrm{Iw}}(K_0[p^\infty], T),$$

where $H^1_{\mathrm{Iw}}(K_0[p^\infty], T) := \varprojlim_{r,s} H^1(K_0[\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s], T)$ with limit under the corestriction maps. Then the image of $\boldsymbol{\mathcal{Z}}_{\mu_3}^{(3)}$ in (3-8) under Shapiro's lemma is an Iwasawa cohomology class

$$\boldsymbol{\mathcal{Z}}_{\mu_3} \in H^1_{\mathrm{Iw}}\big(K_0[\mu_3 p^\infty], T_f^\vee\big(1-\tfrac{1}{2}k\big)\otimes\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\tilde{\kappa}_{\mathrm{ac},1}^{(k_1-2)/2}\tilde{\kappa}_{\mathrm{ac},2}^{(k_2-2)/2}\big(1-\tfrac{1}{2}(k_1+k_2)\big)\big) \tag{3-10}$$

for the conjugate self-dual representation $T_f^\vee\big(1-\frac{1}{2}k\big)$ twisted by the Hecke character

$$\chi^{-1} = \tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\mathbf{N}^{1-(k_1+k_2)/2}$$

(up to an anticyclotomic twist). Here χ is anticyclotomic and of infinity type (corresponding to the order $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$ or $(1, \tau_3, \tau_2, \tau_1)$)

$$\big(\tfrac{1}{2}(2-k_1-k_2), \tfrac{1}{2}(k_1+k_2-2), \tfrac{1}{2}(k_1-k_2), \tfrac{1}{2}(k_2-k_1)\big).$$

Let

$$T_{f,\chi} = T_f^\vee\big(1-\tfrac{1}{2}k\big)\otimes\chi^{-1}. \tag{3-11}$$

Following the proof of Theorem 3.2.4 and invoking [Rubin 2000, Theorem 6.4.1], we can obtain a collection of Iwasawa cohomology classes for anticyclotomic twists (to eliminate $\tilde{\kappa}_{\mathrm{ac},1}^{(k_1-2)/2}\tilde{\kappa}_{\mathrm{ac},2}^{(k_2-2)/2}$). We thus arrive at the proof for the wild norm relation, which is formulated inside the following theorem.

Theorem 3.3.1. *Suppose $p \nmid 6h_{K_0}$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mu_3 \in \mathcal{N}$ and $m = N_{K_3/\mathbb{Q}}(\mu_3)$. Then there exists a collection of Iwasawa cohomology classes*

$$z_{f,\chi,\mu_3} \in H^1_{\mathrm{Iw}}(K_0[\mu_3 p^\infty], T_{f,\chi})$$

such that, for every split prime λ_3 of $\mathcal{O}_{K_3}$ of norm ℓ, where ℓ splits in K_0, with $(\ell, mp) = 1$, we have the norm relation

$$\mathrm{Norm}_{K_0[\mu_3]}^{K_0[\mu_3\lambda_3]}(z_{f,\chi,\mu_3\lambda_3}) = P_{\mathcal{L}_4}(\mathrm{Frob}_{\mathcal{L}_4})(z_{f,\chi,\mu_3}),$$

where $P_{\mathcal{L}_4}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathcal{L}_4} | (T_{f,\chi})^\vee(1))$.

4. Selmer groups

In this section, we show that the classes constructed in Theorem 3.3.1 land in certain Selmer groups defined in [Greenberg 1994]. Keeping the setup at the start of Section 3, we further assume that f is a p-ordinary newform of even weight $k \geq 2$ with $p \nmid N_f$.

Let χ be an anticyclotomic Hecke character of K_0 of infinity type $(-a, a, -b, b)$ for some integers $a, b \geq 0$. We will focus on the conjugate self-dual G_{K_0}-representation

$$V_{f,\chi} := V_f^\vee\big(1 - \tfrac{1}{2}k\big) \otimes \chi^{-1}.$$

Definition 4.1. For each prime $\mathcal{P} \in \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\}$ of K_0 above p, we fix a $G_{K_{0,\mathcal{P}}}$-stable subspace $\mathscr{F}_\mathcal{P}^+(V_{f,\chi}) \subset V_{f,\chi}$ and let

$$\mathscr{F}_\mathcal{P}^-(V_{f,\chi}) = V_{f,\chi}/\mathscr{F}_\mathcal{P}^+(V_{f,\chi}).$$

Let L be a finite extension of K_0. The *Greenberg Selmer group* $\mathrm{Sel}_\mathscr{F}(L, V_{f,\chi})$ attached to $\mathscr{F} = \{\mathscr{F}_\mathcal{P}^+(V_{f,\chi})\}_{\mathcal{P} \mid p}$ is defined by

$$\mathrm{Sel}_\mathscr{F}(L, V_{f,\chi}) := \ker\left\{ H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_\mathscr{F}(L_w, V_{f,\chi})} \right\}, \tag{4-1}$$

where w runs over the finite primes of L and the local conditions are given by

$$H^1_\mathscr{F}(L_w, V_{f,\chi}) = \begin{cases} \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\} & \text{if } w \nmid p, \\ \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, \mathscr{F}_\mathcal{P}^-(V_{f,\chi}))\} & \text{if } w \mid \mathcal{P} \mid p. \end{cases}$$

We fix a lattice $T_{f,\chi} \subset V_{f,\chi}$. Let $H^1_\mathscr{F}(L_w, T_{f,\chi})$ be the inverse image of $H^1_\mathscr{F}(L_w, V_{f,\chi})$ under the natural map

$$H^1(L_w, T_{f,\chi}) \to H^1(L_w, V_{f,\chi}).$$

This then defines $\mathrm{Sel}_\mathscr{F}(L, T_{f,\chi})$ as in (4-1). For any $\mathbb{Z}_p^2$-extension $L_\infty = \bigcup_{r,s} L_{r,s}$ of L, we put

$$\mathrm{Sel}_\mathscr{F}(L_\infty, T_{f,\chi}) := \varprojlim_{r,s} \mathrm{Sel}_\mathscr{F}(L_{r,s}, T_{f,\chi}),$$

where the inverse limit is taken with respect to the corestriction map. We also put $\mathrm{Sel}_\mathscr{F}(L_\infty, V_{f,\chi}) := \mathrm{Sel}_\mathscr{F}(L_\infty, T_{f,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that this group is independent of the chosen lattice $T_{f,\chi}$.

Definition 4.2. We also define the *Bloch–Kato Selmer group* $\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi})$, following [Bloch and Kato 1990], as

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) := \ker\left\{ H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_f(L_w, V_{f,\chi})} \right\},$$

where the local conditions are given by

$$H^1_f(L_w, V_{f,\chi}) = \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\}$$

at primes $w \nmid p$, and the crystalline condition at primes $w \mid p$,

$$H^1_f(L_w, V_{f,\chi}) = \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, V_{f,\chi} \otimes \boldsymbol{B}_{\mathrm{cris}})\},$$

with $\boldsymbol{B}_{\mathrm{cris}}$ being Fontaine's crystalline period ring. The local conditions $H^1_f(L_w, T_{f,\chi}) \subset H^1(L_w, T_{f,\chi})$ are defined by propagation similarly.

Besides the crystalline condition, there are three local conditions at primes $\mathcal{P} \mid p$ that we will be interested in:

(1) The *strict* condition
$$\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) = 0.$$

(2) The *relaxed* condition
$$\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) = V_{f,\chi}.$$

(3) The *ordinary* condition, corresponding to the fact that the restriction of $V_{f,\chi}$ to $G_{\mathbb{Q}_p}$ is reducible (see (2-1)),
$$\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) = V^+_{f,\chi} := V_f^{\vee,+}\big(1 - \tfrac{1}{2}k\big) \otimes \chi^{-1}.$$

Definition 4.3. Denote by $\mathrm{Sel}_{\alpha,\beta,\gamma,\delta}(K_0, V)$ the subgroup of $H^1(K_0, V)$ where classes are unramified at all primes $v \nmid p$ and satisfy the conditions α, β, γ, δ at $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, respectively, where $\alpha, \beta, \gamma, \delta \in$ {rel, str, ord}, which correspond to the relaxed, strict, and ordinary condition, respectively.

We will now compute the explicit local conditions for the Bloch–Kato Selmer group. Here we shall adopt the convention that the p-adic cyclotomic character has Hodge–Tate weight -1. Thus, since χ has infinity type $(-a, a, -b, b)$, the p-adic avatar of χ has Hodge–Tate weight $a, -a, b, -b$ at $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, respectively.

Lemma 4.4. *Assume that $a \geq b$. For any finite extension L of K_0, we have*

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) = \begin{cases} \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(L, V_{f,\chi}) & \text{if } k \geq 2a+2, \\ \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(L, V_{f,\chi}) & \text{if } 2b+2 \leq k < 2a+2, \\ \mathrm{Sel}_{\mathrm{rel,str,rel,str}}(L, V_{f,\chi}) & \text{if } k < 2b+2. \end{cases}$$

Proof. By the Panchiskin condition [Bloch and Kato 1990, Theorem 4.1(ii)] (see also [Nekovář 2000, (3.1)–(3.2); Flach 1990, Lemma 2, page 125]), for every prime $w \mid \mathcal{P} \mid p$ of $L/K_0/\mathbb{Q}$, we have

$$H^1_f(L_w, V_{f,\chi}) = \mathrm{im}\{H^1(L_w, \mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi})) \to H^1(L_w, V_{f,\chi})\},$$

where $\mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi}) \subset V_{f,\chi}$ is a $G_{K_{\mathcal{P}}}$-stable subspace (assuming it exists) such that the Hodge–Tate weights of $\mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi})$ (resp. $V_{f,\chi}/\mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi})$) are all < 0 (resp. ≥ 0).

We compute the Hodge–Tate weights table of $V^+_{f,\chi}$ and $V^-_{f,\chi} := V_{f,\chi}/V^+_{f,\chi}$ at the primes of K_0 above p:

	$V^+_{f,\chi}$	$V^-_{f,\chi}$
HT weight at $\mathcal{P}_1$	$-a-\frac{1}{2}k$	$-a-1+\frac{1}{2}k$
HT weight at $\mathcal{P}_2$	$a-\frac{1}{2}k$	$a-1+\frac{1}{2}k$
HT weight at $\mathcal{P}_3$	$-b-\frac{1}{2}k$	$-b-1+\frac{1}{2}k$
HT weight at $\mathcal{P}_4$	$b-\frac{1}{2}k$	$b-1+\frac{1}{2}k$

This obtains the equalities in the lemma. □

Fix a choice of Galois stable subgroups $\mathscr{F} = \{\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi})\}_{\mathcal{P}\,|\,p}$ and let

$$A_{f,\chi} := \mathrm{Hom}_{\mathbb{Z}_p}(T_{f,\chi}, \mu_{p^\infty}).$$

Define the associated *dual Selmer group* $\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi})$ by

$$\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi}) := \ker\left\{ H^1(L, A_{f,\chi}) \to \prod_w \frac{H^1(L_w, A_{f,\chi})}{H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})} \right\},$$

where $H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})$ is the orthogonal complement of $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ under local Tate pairing

$$H^1(L_w, T_{f,\chi}) \times H^1(L_w, A_{f,\chi}) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

One can then compute the following:

(1) The dual Selmer group of $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(L, T_{f,\chi})$ consists of classes that are unramified outside p and have the strict, relaxed, ordinary, ordinary condition at $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, respectively. Compatibly with Definition 4.3, this can be written as $\mathrm{Sel}_{\mathrm{str,rel,ord,ord}}(L, A_{f,\chi})$.

(2) The dual Selmer group of $\mathrm{Sel}_{\mathrm{rel,str,rel,str}}(L, T_{f,\chi})$ consists of classes that are unramified outside p and have the strict, relaxed, strict, relaxed condition at $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, respectively. Compatibly with Definition 4.3, this can be written as $\mathrm{Sel}_{\mathrm{str,rel,str,rel}}(L, A_{f,\chi})$.

(3) The dual Selmer group of $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(L, T_{f,\chi})$ consists of classes that are unramified outside p, and land in the image of the natural map

$$H^1(L_w, \mathscr{F}^+_{\mathcal{P}}(A_{f,\chi})) \to H^1(L_w, A_{f,\chi}), \quad \mathscr{F}^+_{\mathcal{P}}(A_{f,\chi}) := \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{F}^-_{\mathcal{P}}(T_{f,\chi}), \mu_{p^\infty}),$$

for $w \,|\, \mathcal{P} \,|\, p$. Compatibly with Definition 4.3, this can be written as $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(L, A_{f,\chi})$.

5. Triple product p-adic L-function and Selmer group

Here, we will recall some conventions on Hida families, triple product p-adic L-functions ($\boldsymbol{f}$-unbalanced), and Selmer groups (balanced and $\boldsymbol{f}$-unbalanced) following [Hsieh 2021].

5.1. ***Hida families.*** We follow the convention of [Hsieh 2021, Section 3.1]. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$. Let $\mathbb{I}$ be a normal domain, finite flat over the Iwasawa algebra

$$\Lambda := \mathcal{O}[\![1 + p\mathbb{Z}_p]\!].$$

Let N be a positive integer prime to p and $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character. Denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and branch character χ.

Let $\mathfrak{X}_{\mathbb{I}}^+ \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbb{Q}}_p)$ be the set of arithmetic points of $\mathbb{I}$, which consists of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbb{Q}}_p$ such that, for some $k_Q \in \mathbb{Z}_{\geq 2}$, called the weight of Q, and $\epsilon_Q(z) \in \mu_{p^\infty}$,

$$Q|_{1+p\mathbb{Z}_p} : z \mapsto z^{k_Q - 1}\epsilon_Q(z).$$

We say that $\boldsymbol{f} = \sum_{n=1}^\infty a_n(\boldsymbol{f})q^n \in S^o(N, \chi, \mathbb{I})$ is a primitive Hida family if the specialization $\boldsymbol{f}_Q$ for every $Q \in \mathfrak{X}_{\mathbb{I}}^+$ gives the q-expansion of an ordinary p-stabilized newform of weight k_Q and tame conductor N. Let $\mathfrak{X}_{\mathbb{I}}^{\mathrm{cls}} \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbb{Q}}_p)$ be the set of ring homomorphisms Q as above, with $k_Q \in \mathbb{Z}_{\geq 1}$, such that $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form.

Given $\boldsymbol{f}$ a primitive Hida family of tame conductor N, one can associate a Galois representation

$$\rho_{\boldsymbol{f}} : G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathbb{I}}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{I}),$$

where the determinant of $\rho_{\boldsymbol{f}}$ is $\chi_{\mathbb{I}} \cdot \varepsilon_{\mathrm{cyc}}$; see [Hsieh 2021, Section 3.2]. By [Wiles 1988, Theorem 2.2.2], the restriction of $V_{\boldsymbol{f}}$ to $G_{\mathbb{Q}_p}$ is reducible and one has a short exact sequence

$$0 \to V_{\boldsymbol{f}}^+ \to V_{\boldsymbol{f}} \to V_{\boldsymbol{f}}^- \to 0.$$

Here the quotient $V_{\boldsymbol{f}}^-$ is free of rank one over $\mathbb{I}$, with $G_{\mathbb{Q}_p}$ acting via the unramified character sending an arithmetic Frobenius Frob_p^{-1} to $a_p(\boldsymbol{f})$. Let $\mathbb{T}(N, \mathbb{I})$ be the Hecke algebra acting on $\bigoplus_\chi S^o(N, \chi, \mathbb{I})$, where χ runs over the characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. There is a $\mathbb{I}$-algebra homomorphism attached to $\boldsymbol{f}$,

$$\lambda_{\boldsymbol{f}} : \mathbb{T}(N, \mathbb{I}) \to \mathbb{I},$$

that factors through a local component $\mathbb{T}_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal containing $\ker \lambda_{\boldsymbol{f}}$. Following [Hida 1988], we define the congruence ideal $C(\boldsymbol{f})$ of $\boldsymbol{f}$ by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\mathrm{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\ker \lambda_{\boldsymbol{f}})) \subset \mathbb{I}.$$

Under the assumption that the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished, Wiles [1995] and Hida [1988] prove that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}} \in \mathbb{I}$.

5.2. ***CM Hida families revisited.*** We explicitly construct CM Hida families, following the exposition in [Hsieh 2021, Section 8.1]. Let K be an imaginary quadratic field of discriminant $-D_K < 0$, and suppose that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

Let K_∞ be the $\mathbb{Z}_p^2$-extension of K. Let $K(\mathfrak{p}^\infty)$ be the maximal subfield of K_∞ unramified outside $\mathfrak{p}$. Put

$$\Gamma_\infty := \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^2, \quad \Gamma_{\mathfrak{p}} := \mathrm{Gal}(K(\mathfrak{p}^\infty)/K) \simeq \mathbb{Z}_p.$$

For every ideal $\mathfrak{c} \subset \mathcal{O}_K$, recall that $K_\mathfrak{c}$ is the ray class field of K of conductor $\mathfrak{c}$. Using our notation, $K(\mathfrak{p}^\infty)$ is the maximal $\mathbb{Z}_p$-extension of K inside $K_{\mathfrak{p}^\infty}$. Denote by $\mathrm{Art}_\mathfrak{p}$ the restriction of the Artin map to $K_\mathfrak{p}^\times$, with geometric normalization. Then $\mathrm{Art}_\mathfrak{p}$ induces an embedding $1 + p\mathbb{Z}_p \to \Gamma_\mathfrak{p}$, where we identified $\mathbb{Z}_p^\times$ and $\mathcal{O}_{K_\mathfrak{p}}^\times$ via ι_p. Let $\gamma_\mathfrak{p}$ be the image of $1 + p$; hence, it will be a topological generator of $\Gamma_\mathfrak{p}$.

For each variable S, let $\Psi_S : \Gamma_\infty \to \mathcal{O}[\![S]\!]^\times$ be the universal character given by

$$\Psi_S(\sigma) = (1+S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbb{Z}_p$ is such that $\sigma|_{K(\mathfrak{p}^\infty)} = \gamma_\mathfrak{p}^{l(\sigma)}$. Now assume that $\mathfrak{c}$ is prime to p. Given a finite-order character $\xi : G_K \to \mathcal{O}^\times$ of conductor dividing $\mathfrak{c}$, let

$$\boldsymbol{\theta}_\xi(S)(q) = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{c})=1} \xi(\sigma_\mathfrak{a})\Psi^{-1}_{(1+S)/(1+p)-1}(\sigma_\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \mathcal{O}[\![S]\!][\![q]\!],$$

where $\sigma_\mathfrak{a} \in \mathrm{Gal}(K_{\mathfrak{c}\mathfrak{p}^\infty}/K)$ is the Artin symbol of $\mathfrak{a}$. Then $\boldsymbol{\theta}_\xi(S)$ is a Hida family defined over $\mathcal{O}[\![S]\!]$ of tame level $N_{K/\mathbb{Q}}(\mathfrak{c})D_K$ and tame character $(\xi \circ \mathscr{V})\epsilon_K\omega^{-1}$, where $\mathscr{V} : G_\mathbb{Q}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ is the transfer map and ϵ_K is the quadratic character corresponding to $K/\mathbb{Q}$.

5.3. ***Triple products of Hida families.*** Let

$$\boldsymbol{f} \in S^o(N_f, \chi_f, \mathbb{I}_f), \quad \boldsymbol{g} \in S^o(N_g, \chi_g, \mathbb{I}_g), \quad \boldsymbol{h} \in S^o(N_h, \chi_h, \mathbb{I}_h)$$

be three primitive Hida families such that

$$\chi_f\chi_g\chi_h = \omega^{2a} \quad \text{for some } a \in \mathbb{Z}, \tag{5-1}$$

where ω is the Teichmüller character. Let

$$\mathcal{R} = \mathbb{I}_f \hat{\otimes}_\mathcal{O} \mathbb{I}_g \hat{\otimes}_\mathcal{O} \mathbb{I}_h$$

be a finite extension of the three-variable Iwasawa algebra $\Lambda \hat{\otimes}_\mathcal{O} \Lambda \hat{\otimes}_\mathcal{O} \Lambda$.

Let $\mathfrak{X}_\mathcal{R}^+ \subset \mathrm{Spec}\,\mathcal{R}(\overline{\mathbb{Q}}_p)$ be the weight space of $\mathcal{R}$ given by

$$\mathfrak{X}_\mathcal{R}^+ := \{\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathbb{I}_f}^+ \times \mathfrak{X}_{\mathbb{I}_g}^{\mathrm{cls}} \times \mathfrak{X}_{\mathbb{I}_h}^{\mathrm{cls}} : k_{Q_1} + k_{Q_2} + k_{Q_3} \equiv 0 \ (\mathrm{mod}\, 2)\}.$$

One can then partition $\mathfrak{X}_\mathcal{R}^+ = \mathfrak{X}_\mathcal{R}^{\mathrm{bal}} \sqcup \mathfrak{X}_\mathcal{R}^{\boldsymbol{f}} \sqcup \mathfrak{X}_\mathcal{R}^{\boldsymbol{g}} \sqcup \mathfrak{X}_\mathcal{R}^{\boldsymbol{h}}$ into

(1) the set of *balanced* weights

$$\mathfrak{X}_\mathcal{R}^{\mathrm{bal}} := \{\underline{Q} \in \mathfrak{X}_\mathcal{R}^+ : k_{Q_1} + k_{Q_2} + k_{Q_3} > 2k_{Q_i} \text{ for all } i \in \{1,2,3\}\},$$

(2) the set of $\boldsymbol{f}$-unbalanced weights

$$\mathfrak{X}_\mathcal{R}^{\boldsymbol{f}} := \{\underline{Q} \in \mathfrak{X}_\mathcal{R}^+ : k_{Q_1} \geq k_{Q_2} + k_{Q_3}\},$$

(3) the set of $\boldsymbol{g}$-unbalanced weights

$$\mathfrak{X}_\mathcal{R}^{\boldsymbol{g}} := \{\underline{Q} \in \mathfrak{X}_\mathcal{R}^+ : k_{Q_2} \geq k_{Q_1} + k_{Q_3}\},$$

(4) the set of $\boldsymbol{h}$-unbalanced weights

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{h}} := \{\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{+} : k_{Q_3} \geq k_{Q_1} + k_{Q_2}\}.$$

Let $\boldsymbol{V} = V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. By (5-1), one can decompose the determinant of $\boldsymbol{V}$ as $\det \boldsymbol{V} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$. Put

$$\boldsymbol{V}^{\dagger} := \boldsymbol{V} \otimes \mathcal{X}^{-1}. \tag{5-2}$$

This is a self-dual twist of $\boldsymbol{V}$. For any $\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, denote by $\boldsymbol{V}_{\underline{Q}}^{\dagger}$ the corresponding specialization.

For each prime ℓ, let $\varepsilon_{\ell}(\boldsymbol{V}_{\underline{Q}}^{\dagger})$ be the epsilon factor attached to the local representation $\boldsymbol{V}_{\underline{Q}}^{\dagger}|_{G_{\mathbb{Q}_\ell}}$ (see [Tate 1979, page 21]). We assume that, for some $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, we have

$$\varepsilon_{\ell}(\boldsymbol{V}_{\underline{Q}}^{\dagger}) = +1 \quad \text{for all prime factors } \ell \text{ of } N_f N_g N_h. \tag{5-3}$$

Note that condition (5-3) is independent of $\underline{Q}$ (see [Hsieh 2021, Section 1.2]). Furthermore, it implies that the sign of the functional equation for the triple product L-function (with center at $s = 0$)

$$L(\boldsymbol{V}_{\underline{Q}}^{\dagger}, s)$$

is $+1$ (resp. -1) for all $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} \cup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{g}} \cup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{h}}$ (resp. $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\mathrm{bal}}$).

Theorem 5.3.1 [Hsieh 2021, Theorem A]. *Let $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ be three primitive Hida families satisfying conditions* (5-1) *and* (5-3). *Assume also that* $\gcd(N_f, N_g, N_h)$ *is squarefree, and the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished. Fix a generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$. Then there exists a unique element*

$$\mathscr{L}_p^{\boldsymbol{f}, \eta_{\boldsymbol{f}}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}$$

such that, for all $\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ of weight (k_1, k_2, k_3) with $\epsilon_{Q_1} = 1$, we have

$$(\mathscr{L}_p^{\boldsymbol{f}, \eta_{\boldsymbol{f}}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(\underline{Q}))^2 = \Gamma_{\boldsymbol{V}_{\underline{Q}}^{\dagger}}(0) \cdot \frac{L(\boldsymbol{V}_{\underline{Q}}^{\dagger}, 0)}{(\sqrt{-1})^{2k_1} \cdot \Omega_{\boldsymbol{f}_{Q_1}}^2} \cdot \mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\boldsymbol{V}_{\underline{Q}}^{\dagger})) \cdot \prod_{q \in \Sigma_{\mathrm{exc}}} (1 + q^{-1})^2,$$

where:

- $\Gamma_{\boldsymbol{V}_{\underline{Q}}^{\dagger}}(0) = 16(2\pi)^{-2k_1}\Gamma(w_{\underline{Q}})\Gamma(w_{\underline{Q}} + 2 - k_2 - k_3)\Gamma(w_{\underline{Q}} + 1 - k_2)\Gamma(w_{\underline{Q}} + 1 - k_3)$ *and* $w_{\underline{Q}} = \frac{1}{2}(k_1 + k_2 + k_3 - 2)$.
- $\Omega_{\boldsymbol{f}_{Q_1}}$ *is the Hida canonical period*

$$\Omega_{\boldsymbol{f}_{Q_1}} := (-2\sqrt{-1})^{k_1+1} \cdot \frac{\|\boldsymbol{f}_{Q_1}^{\circ}\|_{\Gamma_0(N_f)}^2}{\eta_{\boldsymbol{f}_{Q_1}}} \cdot \left(1 - \frac{\chi_f'(p)p^{k_1-1}}{\alpha_{Q_1}^2}\right)\left(1 - \frac{\chi_f'(p)p^{k_1-2}}{\alpha_{Q_1}^2}\right),$$

with $\boldsymbol{f}_{Q_1}^{\circ} \in S_{k_1}(\Gamma_0(N_f))$ the newform of conductor N_f associated with $\boldsymbol{f}_{Q_1}$, χ_f' the prime-to-p part of χ_f, and α_{Q_1} the specialization of $a_p(\boldsymbol{f}) \in \mathbb{I}_f^{\times}$ at Q_1.

- $\mathcal{E}_p(\mathscr{F}_p^f(\boldsymbol{V}_{\underline{Q}}^\dagger))$ *is the modified* p*-Euler factor and* Σ_{exc} *is an explicitly defined subset of the prime factors of* $N_f N_g N_h$ [Hsieh 2021, page 416].

5.4. ***Triple product Selmer groups.*** Recall from (5-2) that $\boldsymbol{V}^\dagger = \boldsymbol{V} \otimes \mathcal{X}^{-1}$ is the self-dual twist of the Galois representation associated to a triple of primitive Hida families $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ given by (5-1).

Definition 5.4.1. Let

$$\mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}^\dagger) := (V_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}}^+ \otimes V_{\boldsymbol{h}}^+ + V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}^+ + V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}}^+ \otimes V_{\boldsymbol{h}}) \otimes \mathcal{X}^{-1},$$

and define the balanced local condition $\mathrm{H}^1_{\mathrm{bal}}(\mathbb{Q}_p, \boldsymbol{V}^\dagger)$ by

$$\mathrm{H}^1_{\mathrm{bal}}(\mathbb{Q}_p, \boldsymbol{V}^\dagger) := \mathrm{im}\big(\mathrm{H}^1(\mathbb{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}^\dagger)) \to \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}^\dagger)\big).$$

Similarly, let

$$\mathscr{F}_p^{\boldsymbol{f}}(\boldsymbol{V}^\dagger) = (V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}) \otimes \mathcal{X}^{-1}, \quad \mathscr{F}_p^{\boldsymbol{h}}(\boldsymbol{V}^\dagger) = (V_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}^+) \otimes \mathcal{X}^{-1}.$$

Define the $\boldsymbol{f}$-unbalanced local condition $\mathrm{H}^1_{\boldsymbol{f}}(\mathbb{Q}_p, \boldsymbol{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{f}}(\mathbb{Q}_p, \boldsymbol{V}^\dagger) := \mathrm{im}\big(\mathrm{H}^1(\mathbb{Q}_p, \mathscr{F}_p^{\boldsymbol{f}}(\boldsymbol{V}^\dagger)) \to \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}^\dagger)\big)$$

and the $\boldsymbol{h}$-unbalanced local condition $\mathrm{H}^1_{\boldsymbol{h}}(\mathbb{Q}_p, \boldsymbol{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{h}}(\mathbb{Q}_p, \boldsymbol{V}^\dagger) := \mathrm{im}\big(\mathrm{H}^1(\mathbb{Q}_p, \mathscr{F}_p^{\boldsymbol{h}}(\boldsymbol{V}^\dagger)) \to \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}^\dagger)\big).$$

Note that the maps appearing in these definitions are injective, so we can identify $\mathrm{H}^1_\star(\mathbb{Q}_p, \boldsymbol{V}^\dagger)$ with $\mathrm{H}^1(\mathbb{Q}_p, \mathscr{F}_p^\star(\boldsymbol{V}^\dagger))$ for $\star \in \{\mathrm{bal}, \boldsymbol{f}, \boldsymbol{h}\}$.

Definition 5.4.2. Let $\star \in \{\mathrm{bal}, \boldsymbol{f}, \boldsymbol{h}\}$. Define the Selmer group $\mathrm{Sel}^\star(\mathbb{Q}, \boldsymbol{V}^\dagger)$ by

$$\mathrm{Sel}^\star(\mathbb{Q}, \boldsymbol{V}^\dagger) := \ker\left\{\mathrm{H}^1(\mathbb{Q}, \boldsymbol{V}^\dagger) \to \frac{\mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}^\dagger)}{\mathrm{H}^1_\star(\mathbb{Q}_p, \boldsymbol{V}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbb{Q}_v^{\mathrm{nr}}, \boldsymbol{V}^\dagger)\right\}.$$

We call $\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger)$ the balanced Selmer group, $\mathrm{Sel}^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{V}^\dagger)$ the $\boldsymbol{f}$-unbalanced Selmer group, and $\mathrm{Sel}^{\boldsymbol{h}}(\mathbb{Q}, \boldsymbol{V}^\dagger)$ the $\boldsymbol{h}$-unbalanced Selmer group.

Definition 5.4.3. Let $\boldsymbol{A}^\dagger = \mathrm{Hom}_{\mathbb{Z}_p}(\boldsymbol{V}^\dagger, \mu_{p^\infty})$ and let $\star \in \{\mathrm{bal}, \boldsymbol{f}, \boldsymbol{h}\}$. Define $\mathrm{H}^1_\star(\mathbb{Q}_p, \boldsymbol{A}^\dagger) \subset \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_\star(\mathbb{Q}_p, \boldsymbol{V}^\dagger)$ under the local Tate pairing

$$\mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}^\dagger) \times \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{A}^\dagger) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Similarly as above, we then define the balanced, the $\boldsymbol{f}$-unbalanced and the $\boldsymbol{h}$-unbalanced Selmer groups with coefficients in $\boldsymbol{A}^\dagger$ by

$$\mathrm{Sel}^\star(\mathbb{Q}, \boldsymbol{A}^\dagger) := \ker\left\{\mathrm{H}^1(\mathbb{Q}, \boldsymbol{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{A}^\dagger)}{\mathrm{H}^1_\star(\mathbb{Q}_p, \boldsymbol{A}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbb{Q}_v^{\mathrm{nr}}, \boldsymbol{A}^\dagger)\right\}.$$

Let

$$X^\star(\mathbb{Q}, \boldsymbol{A}^\dagger) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}^\star(\mathbb{Q}, \boldsymbol{A}^\dagger), \mathbb{Q}_p/\mathbb{Z}_p)$$

be the Pontryagin dual of $\mathrm{Sel}^\star(\mathbb{Q}, \boldsymbol{A}^\dagger)$.

6. Arithmetic applications

Finally, we obtain some arithmetic results from our constructed Euler system through identifying our classes as an anticyclotomic Euler system in the sense of [Jetchev et al. 2024] and the explicit reciprocity law.

6.1. ***Reciprocity law and Greenberg–Iwasawa main conjectures.*** Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families as in Section 5.1 satisfying (5-1). Let $N = \mathrm{lcm}(N_f, N_g, N_h)$. The big diagonal class constructed in [Bertolini et al. 2022b, Section 8.1],

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{H}^1(\mathbb{Q}, \boldsymbol{V}^\dagger(N)), \tag{6-1}$$

where $\boldsymbol{V}^\dagger(N)$ (this is $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ using the notation of [loc. cit.]) is a free $\mathcal{R}$-module isomorphic to finitely many copies of $\boldsymbol{V}^\dagger$, can be identified with classes $\tilde{\boldsymbol{\kappa}}_m^{(1)}$, $\boldsymbol{\kappa}_m^{(2)}$ in equations (2.14) and (2.15), respectively, of [Castella and Do 2025]. The definition of the Selmer groups in Section 5.4 extends to $\boldsymbol{V}^\dagger(N)$, and, by [Bertolini et al. 2022b, Corollary 8.2], we have $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger(N))$. Now we choose level-$N$ test vectors $(\check{\boldsymbol{f}}, \check{\boldsymbol{g}}, \check{\boldsymbol{h}})$, provided by [Hsieh 2021, Theorem A], to project the classes from $\mathbb{V}^\dagger(N)$ to $\mathbb{V}^\dagger$.

We define more $G_{\mathbb{Q}_p}$-invariant subspaces of $\boldsymbol{V}^\dagger$,

$$\begin{aligned}
\mathscr{F}_p^3(\boldsymbol{V}^\dagger) &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\
\boldsymbol{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} &= V_{\boldsymbol{f}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\
\boldsymbol{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\
\boldsymbol{V}_{\boldsymbol{h}}^{\boldsymbol{fg}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^- \otimes \mathcal{X}^{-1},
\end{aligned} \tag{6-2}$$

and obtain

$$\mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}^\dagger)/\mathscr{F}_p^3(\boldsymbol{V}^\dagger) \cong \boldsymbol{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} \oplus \boldsymbol{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} \oplus \boldsymbol{V}_{\boldsymbol{h}}^{\boldsymbol{fg}}. \tag{6-3}$$

Assume that the congruence ideal $C(\boldsymbol{f}) \subset \mathbb{I}_f$ is principal, generated by the nonzero $\eta_{\boldsymbol{f}} \in \mathbb{I}_f$ (this will be satisfied when the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished). One can deduce from [Kings et al. 2017] the construction of an injective three-variable p-adic regulator map with pseudonull cokernel,

$$\mathrm{Log}^{\eta_f} : \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}) \to \mathcal{R}; \tag{6-4}$$

see the explicit map in [Castella and Do 2025, Section 4.3.1] and the explanation in [Bertolini et al. 2022b, Section 7.3].

Let $\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}}$ be the image of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under the natural composition of maps

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger) \xrightarrow{\mathrm{res}_p} \mathrm{H}^1(\mathbb{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}^\dagger)) \to \mathrm{H}^1(\mathbb{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}^\dagger)/\mathscr{F}_p^3(\boldsymbol{V}^\dagger)) \to \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}), \tag{6-5}$$

where we first restrict at p and then project onto the first direct summand in (6-3). The following result is an explicit reciprocity law that relates diagonal cycles with the triple product p-adic L-functions.

Theorem 6.1.1 [Bertolini et al. 2022b, Theorem A]. *Let* $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ *be a triple of primitive Hida families as in Theorem 5.3.1. Then*

$$\mathrm{Log}^{\eta_f}\big(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_f\big) = \mathscr{L}_p^{\boldsymbol{f},\eta_f}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}).$$

Assume that the associated ring $\mathcal{R}$ is regular. Similar to [Alonso et al. 2025, Section 7.3], the following result can be seen as the equivalence between two different formulations of the Iwasawa main conjecture in the style of [Greenberg 1994] for the p-adic deformation $\boldsymbol{V}^\dagger$, relating the $\boldsymbol{f}$-unbalanced Selmer group to the balanced one (or one with $\mathscr{L}_p^{\boldsymbol{f}}$ and another "without p-adic L-functions").

Proposition 6.1.2 [Castella and Do 2025, Proposition 4.3.3]. *The following are equivalent*:

(I) $\mathscr{L}_p^{\boldsymbol{f},\eta_f}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ *is nonzero, the modules* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{V}^\dagger)$ *and* $X^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{A}^\dagger)$ *are both* $\mathcal{R}$*-torsion, and*

$$\mathrm{char}_{\mathcal{R}}(X^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{A}^\dagger)) = (\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^2)$$

in $\mathcal{R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(II) $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ *is not* $\mathcal{R}$*-torsion, the modules* $\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger)$ *and* $X^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{A}^\dagger)$ *both have* $\mathcal{R}$*-rank one, and*

$$\mathrm{char}_{\mathcal{R}}(X^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{A}^\dagger)_{\mathrm{tors}}) = \mathrm{char}_{\mathcal{R}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger)}{\mathcal{R} \cdot \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})}\right)^2$$

in $\mathcal{R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, *where the subscript* tors *denotes the* $\mathcal{R}$*-torsion submodule.*

6.2. ***The setup.*** Let $f \in S_{2r}(pN_f)$ be a p-stabilized newform, and suppose the residual representation $\bar{\rho}_f$ is absolutely irreducible and p-distinguished. By Hida theory, f is the specialization of a unique primitive Hida family $\boldsymbol{f} \in S^o(N_f, \mathbb{I})$ at an arithmetic point $Q_1 \in \mathcal{X}_{\mathbb{I}}^+$ of weight $2r$. For $i \in \{1, 2\}$, let $\mathfrak{f}_i \subset \mathcal{O}_{K_i}$ be an ideal coprime to pN_f, and ξ_i be ray class characters of K_i of conductors dividing $\mathfrak{f}_i$. Let χ_{ξ_i} be the central character of ξ_i. We assume that

$$\chi_{\xi_1}\epsilon_{K_1}\chi_{\xi_2}\epsilon_{K_2} = 1, \tag{6-6}$$

and let

$$\boldsymbol{g}_1 = \boldsymbol{\theta}_{\xi_1}(S_1) \in \mathcal{O}[\![S_1]\!][\![q]\!], \quad \boldsymbol{g}_2 = \boldsymbol{\theta}_{\xi_2}(S_2) \in \mathcal{O}[\![S_2]\!][\![q]\!] \tag{6-7}$$

be the CM Hida families attached to ξ_1 and ξ_2, respectively.

The triple $(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2)$ satisfies condition (5-1) and the associated $\boldsymbol{f}$-unbalanced triple product p-adic L-function $\mathscr{L}_p^{\boldsymbol{f},\eta_f}(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2)$ is an element in $\mathcal{R} = \mathbb{I} \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![S_1]\!] \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![S_2]\!] \simeq \mathbb{I}[\![S_1, S_2]\!]$. Let

$$\mathscr{L}_p^{\boldsymbol{f},\eta_f}(f, \boldsymbol{g}_1, \boldsymbol{g}_2) \in \mathcal{O}[\![S_1, S_2]\!] \tag{6-8}$$

be its image under the natural map $\mathbb{I}[\![S_1, S_2]\!] \to \mathcal{O}[\![S_1, S_2]\!]$ defined by Q_1.

Write $\boldsymbol{V}_{Q_1}^\dagger$ for the specialization of $\boldsymbol{V}^\dagger$ at Q_1. Let $V_f^\vee$ be the Galois representation associated to f, and recall that $\det(V_f^\vee) = \varepsilon_{\mathrm{cyc}}^{2r-1}$ in our conventions. Setting $T_i = \boldsymbol{v}^{-1}(1 + S_i) - 1$ for $i \in \{1, 2\}$, we have

$\det(V_{\boldsymbol{g}_{T_1}} \otimes V_{\boldsymbol{h}_{T_2}}) = \Psi_{T_1}\Psi_{T_2} \circ \mathscr{V}$, and so

$$\begin{aligned} \boldsymbol{V}_{Q_1}^\dagger &\simeq T_f^\vee \otimes (\mathrm{Ind}_K^{\mathbb{Q}}\, \xi_1^{-1}\Psi_{T_1}) \otimes (\mathrm{Ind}_K^{\mathbb{Q}}\, \xi_2^{-1}\Psi_{T_2}) \otimes \varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2} \circ \mathscr{V}) \\ &\simeq T_f^\vee(1-r) \otimes \mathrm{Ind}_{K_0}^{\mathbb{Q}}\, \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}, \end{aligned} \tag{6-9}$$

where $T_f^\vee$ is a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice inside $V_f^\vee$. In particular, we get

$$\mathrm{H}^1(\mathbb{Q}, \boldsymbol{V}_{Q_1}^\dagger) \simeq \mathrm{H}^1(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \tag{6-10}$$

by Shapiro's lemma.

6.3. ***Local conditions at p of the Euler system.*** Recall from Theorem 3.3.1 that we have constructed classes

$$\boldsymbol{z}_{f,\chi,\mu_3} \in H^1_{\mathrm{Iw}}(K_0[\mu_3 p^\infty], T_{f,\chi}),$$

where $T_{f,\chi} = T_f^\vee(1-r) \otimes \chi^{-1}$ and $\chi^{-1} = \tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\mathbf{N}^{1-(k_1+k_2)/2}$.

Proposition 6.3.1. *Suppose $p \nmid 6h_{K_0}$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mu_3 \in \mathcal{N}$ (taken from Section 3.1) run over squarefree product of prime ideals of $\lambda_3 \in \mathcal{L}$ with $m = N_{K_3/\mathbb{Q}}(\mu_3)$ coprime to p. The class $\boldsymbol{z}_{f,\chi,\mu_3}$ of Theorem 3.3.1 satisfies*

$$\boldsymbol{z}_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}).$$

Proof. By [Bertolini et al. 2022b, Corollary 8.2; Castella and Do 2025, Section 3.2], the class $\boldsymbol{z}_{f,\chi,\mu_3}$ lands in the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger)$, where the balanced local condition at p when specialized to f is given by

$$\begin{aligned} \mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}_{Q_1}^\dagger) \simeq (T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \oplus (T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-\boldsymbol{c}}\tilde{\Psi}_{T_2}^{\boldsymbol{c}}) \\ \oplus (T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-\boldsymbol{c}}\tilde{\Psi}_{T_1}^{\boldsymbol{c}}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}). \end{aligned} \tag{6-11}$$

Put $\widetilde{\boldsymbol{V}}_{Q_1}^\dagger = T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}$; then Shapiro's lemma tells us that $H^1(\mathbb{Q}, \boldsymbol{V}_{Q_1}^\dagger) \simeq H^1(K_0, \widetilde{\boldsymbol{V}}_{Q_1}^\dagger)$; see (6-10). Following [Castella and Do 2025, Section 5.3], the local condition $\mathscr{F}_p^{\mathrm{bal}}(\boldsymbol{V}_{Q_1}^\dagger)$ cutting out the specialized balanced Selmer group at p corresponds to

$$\begin{aligned} \mathscr{F}_{\mathcal{P}_1}^{\mathrm{bal}}(\widetilde{\boldsymbol{V}}_{Q_1}^\dagger|_{G_{K_0}}) &= T_f^\vee\big(1-\tfrac{1}{2}k\big) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}, \\ \mathscr{F}_{\mathcal{P}_2}^{\mathrm{bal}}(\widetilde{\boldsymbol{V}}_{Q_1}^\dagger|_{G_{K_0}}) &= 0, \\ \mathscr{F}_{\mathcal{P}_3}^{\mathrm{bal}}(\widetilde{\boldsymbol{V}}_{Q_1}^\dagger|_{G_{K_0}}) &= T_f^{\vee,+}\big(1-\tfrac{1}{2}k\big) \otimes \tilde{\xi}_1^{-\boldsymbol{c}}\tilde{\Psi}_{T_1}^{\boldsymbol{c}}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}, \\ \mathscr{F}_{\mathcal{P}_4}^{\mathrm{bal}}(\widetilde{\boldsymbol{V}}_{Q_1}^\dagger|_{G_{K_0}}) &= T_f^{\vee,+}\big(1-\tfrac{1}{2}k\big) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-\boldsymbol{c}}\tilde{\Psi}_{T_2}^{\boldsymbol{c}}. \end{aligned} \tag{6-12}$$

Hence, the class $\boldsymbol{z}_{f,\chi,\mu_3}$ satisfies the relaxed-strict-ordinary-ordinary condition at the primes above p.

On the other hand, at the primes $w \nmid p$, because $V_{f,\chi}$ is conjugate self-dual and pure of weight -1, we see that

$$H^0(K_0[\mu_3\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s]_w, V_{f,\chi}) = H^2(K_0[\mu_3\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s]_w, V_{f,\chi}) = 0$$

for all r, s, and therefore

$$H^1(K_0[\mu_3\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s]_w, V_{f,\chi}) = 0$$

by Tate's local Euler characteristic formula. This implies the torsionness of $H^1(K_0[\mu_3\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s]_w, T_{f,\chi})$, and one has the inclusion

$$\mathrm{res}_w(\mathfrak{z}_{f,\chi,\mu_3}) \in \varprojlim_{r,s} H^1_f(K_0[\mu_3\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s]_w, T_{f,\chi}). \qquad \square$$

Proposition 6.3.2. *Via the isomorphism* (6-10):

(1) *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger_{Q_1})$ *can be rewritten as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{V}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}).$$

(2) *The* $\boldsymbol{f}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{V}^\dagger_{Q_1})$ *can be rewritten as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{V}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}).$$

(3) *The* $\boldsymbol{h}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{h}}(\mathbb{Q}, \boldsymbol{V}^\dagger_{Q_1})$ *can be rewritten as*

$$\mathrm{Sel}^{\boldsymbol{h}}(\mathbb{Q}, \boldsymbol{V}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{rel,str,rel,str}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}).$$

Proof. For the balanced case, see Proposition 6.3.1. For the $\boldsymbol{f}$-unbalanced case, note that

$$\begin{aligned}\mathscr{F}_p^{\boldsymbol{f}}(\boldsymbol{V}^\dagger_{Q_1}) \simeq (T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \oplus (T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-\boldsymbol{c}}\tilde{\Psi}^{\boldsymbol{c}}_{T_2}) \\ \oplus (T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-\boldsymbol{c}}\tilde{\Psi}^{\boldsymbol{c}}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \oplus (T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-\boldsymbol{c}}\tilde{\Psi}^{\boldsymbol{c}}_{T_1}\tilde{\xi}_2^{-\boldsymbol{c}}\tilde{\Psi}^{\boldsymbol{c}}_{T_2}), \quad (6\text{-}13)\end{aligned}$$

and the result follows.

The $\boldsymbol{h}$-unbalanced case can be obtained in a similar manner, where

$$\mathscr{F}_p^{\boldsymbol{h}}(\boldsymbol{V}^\dagger_{Q_1}) \simeq (T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \oplus (T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-\boldsymbol{c}}\tilde{\Psi}^{\boldsymbol{c}}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}). \qquad \square$$

As a consequence, we also obtain the following isomorphisms for the Selmer groups with coefficients in $\boldsymbol{A}^\dagger_{Q_1} = \mathrm{Hom}_{\mathbb{Z}_p}(\boldsymbol{V}^\dagger_{Q_1}, \mu_{p^\infty})$ by local Tate duality. Let $A_f(r) = \mathrm{Hom}_{\mathbb{Z}_p}(T_f^\vee(1-r), \mu_{p^\infty})$.

Corollary 6.3.3. *We can identify the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{A}^\dagger_{Q_1})$ *as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbb{Q}, \boldsymbol{A}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{str,rel,ord,ord}}(K_0, A_f(r) \otimes \tilde{\xi}_1\tilde{\Psi}_{T_1}^{-1}\tilde{\xi}_2\tilde{\Psi}_{T_2}^{-1}),$$

the $\boldsymbol{f}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{A}^\dagger_{Q_1})$ *as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbb{Q}, \boldsymbol{A}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_0, A_f(r) \otimes \tilde{\xi}_1\tilde{\Psi}_{T_1}^{-1}\tilde{\xi}_2\tilde{\Psi}_{T_2}^{-1}),$$

and the $\boldsymbol{h}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{h}}(\mathbb{Q}, \boldsymbol{A}^\dagger_{Q_1})$ *as*

$$\mathrm{Sel}^{\boldsymbol{h}}(\mathbb{Q}, \boldsymbol{A}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{str,rel,str,rel}}(K_0, A_f(r) \otimes \tilde{\xi}_1\tilde{\Psi}_{T_1}^{-1}\tilde{\xi}_2\tilde{\Psi}_{T_2}^{-1}).$$

6.4. ***Applying the general machinery.*** We show some arithmetic applications by invoking the general Euler system machinery of [Jetchev et al. 2024]; see some details for the imaginary quadratic case in [Do 2022, Section 4.3; Alonso et al. 2025, Section 8]. These results will be used to deduce the Bloch–Kato conjecture and the anticyclotomic Iwasawa main conjecture by exploiting the relation between our Euler system classes and special values of complex and p-adic L-functions via an explicit reciprocity law.

For every ideal $\mu_3 \in \mathcal{N}$, denote by

$$z_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3], T_{f,\chi})$$

the image of z_{f,χ,μ_3} from Theorem 3.3.1 under the projection

$$\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}) \to \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3], T_{f,\chi}). \tag{6-14}$$

Define the base class

$$z_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(z_{f,\chi,1}) \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi}).$$

Note that since we assume $p \nmid h_{K_0}$, $K_0[1]$ is actually the same with K_0 (recall that $K_0[\mathfrak{n}]$ is the maximal p-extension inside the ring class field of K_0 of conductor $\mathfrak{n}$). Therefore, $z_{f,\chi} = z_{f,\chi,1}$.

Theorem 6.4.1. *Assume that f is not of CM type, non-Eisenstein at $\mathfrak{P}$, and that $p \nmid 6h_{K_0}$. Then*

$$z_{f,\chi} \text{ is nontorsion } \Longrightarrow \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, V_{f,\chi}) \text{ is one-dimensional.}$$

Proof. Combining Theorem 3.3.1 and Proposition 6.3.1, the system of classes

$$\{z_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3], T_{f,\chi}) : \mu_3 \in \mathcal{N}\} \tag{6-15}$$

forms an anticyclotomic Euler system in the sense of [Jetchev et al. 2024] for the relaxed-strict-ordinary-ordinary Greenberg Selmer group.

Under the assumption that f is not of CM type, the following properties follow from Momose's big image results [1981] as in [Lei et al. 2015, Proposition 7.1.4]:

(i) $V_{f,\chi}$ is absolutely irreducible.

(ii) There is an element $\sigma \in G_{K_0}$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_{K_0}^\times)^{1/p^\infty})$ such that $V_{f,\chi}/(\sigma - 1)V_{f,\chi}$ is one-dimensional.

(iii) There is an element $\gamma \in G_{K_0}$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_{K_0}^\times)^{1/p^\infty})$ such that $V_{f,\chi}^{\gamma=1} = 0$.

Hence, the fact that $z_{f,\chi}$ is nontorsion implies the one-dimensionality of $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, V_{f,\chi})$ by the general machinery of [Jetchev et al. 2024]. □

Recall that $K_{0,\infty}^-$ is the anticyclotomic $\mathbb{Z}_p^2$-extension over K_0 and $\Lambda_{K_0}^- = \mathbb{Z}_p[\![\mathrm{Gal}(K_{0,\infty}^-/K_0)]\!]$. Let $\boldsymbol{z}_{f,\chi,1}$ be the $\Lambda_{K_0}^-$-adic class of Theorem 3.3.1 of conductor $\mu_3 = (1)$, and define the Iwasawa-theoretic base class

$$\boldsymbol{z}_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(\boldsymbol{z}_{f,\chi,(1)}).$$

Again $z_{f,\chi} = z_{f,\chi,(1)}$ from the assumption on the class number of K_0. Note that, by Proposition 6.3.1, one has

$$z_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi}).$$

Definition 6.4.2. We say that f has big image at $\mathfrak{P}$ if the image of $G_{\mathbb{Q}}$ in $\mathrm{Aut}_{\mathcal{O}}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbb{Z}_p)$.

Remark 6.4.3. By a theorem of Ribet [1985], if f is not of CM type then it has big image for all but finitely many primes of L.

Let

$$X_{\mathrm{str,rel,ord,ord}}(K_{0,\infty}^-, A_{f,\chi}) = \mathrm{Hom}_{\mathbb{Z}_p}\big(\varinjlim \mathrm{Sel}_{\mathrm{str,rel,ord,ord}}(K_0[\mathfrak{p}_3^r\tilde{\mathfrak{p}}_3^s], A_{f,\chi}), \mathbb{Q}_p/\mathbb{Z}_p\big).$$

One then has a divisibility towards an anticyclotomic Iwasawa main conjecture "without p-adic L-functions" as follows:

Theorem 6.4.4. *Assume that f is not of CM type, has big image at $\mathfrak{P}$, and that $p \nmid 6h_{K_0}$. If $z_{f,\chi}$ is nontorsion, then*:

(1) $X_{\mathrm{str,rel,ord,ord}}(K_{0,\infty}^-, A_{f,\chi})$ *and* $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})$ *both have $\Lambda_{K_0}^-$-rank one.*

(2) *We have the divisibility*

$$\mathrm{char}_{\Lambda_{K_0}^-}(X_{\mathrm{str,rel,ord,ord}}(K_{0,\infty}^-, A_{f,\chi})_{\mathrm{tors}}) \supset \mathrm{char}_{\Lambda_{K_0}^-}\left(\frac{\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})}{\Lambda_{K_0}^- \cdot z_{f,\chi}}\right)^2$$

in $\Lambda_{K_0}^-$.

Here, the subscript tors *denotes the $\Lambda_{K_0}^-$-torsion submodule.*

Proof. Combining Theorem 3.3.1 and Proposition 6.3.1, the system of classes

$$\{z_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}) : \mu_3 \in \mathcal{N}\} \tag{6-16}$$

forms a $\Lambda_{K_0}^-$-adic anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner for the relaxed-strict-ordinary-ordinary Selmer group.

Under the assumption that f has big image at $\mathfrak{P}$, we have (see [Lei et al. 2015, Proposition 7.1.6]):

(i) $\overline{T}_{f,\chi} := T_{f,\chi}/\mathfrak{P}T_{f,\chi}$ is absolutely irreducible.

(ii) There is an element $\sigma \in G_K$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $T_{f,\chi}/(\sigma - 1)T_{f,\chi}$ is free of rank 1 over $\mathcal{O}$.

(iii) There is an element $\gamma \in G_{K_0}$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and acting as multiplication by a scalar $a_\gamma \neq 1$ on $\overline{T}_{f,\chi}$.

Thus, the nontorsionness of $z_{f,\chi}$ implies the conclusions by the general machinery of [Jetchev et al. 2024]. □

6.5. ***On the Bloch–Kato conjecture in rank* 0.** Our first application is the Bloch–Kato conjecture in analytic rank zero for the conjugate self-dual G_{K_0}-representation $V_{f,\chi} = V_f^\vee(1-r) \otimes \chi^{-1}$.

Assumption 6.1. We assume that the anticyclotomic Hecke character χ over K_0 can be decomposed as

$$\chi = \tilde{\psi}_1 \tilde{\psi}_2 \mathbf{N}^{(k_1+k_2-2)/2},$$

where:

(1) ψ_1 is a Hecke character of K_1 of infinity type $(1-k_1, 0)$, with $k_1 \geq 1$, and modulus $\mathfrak{f}_1$.

(2) ψ_2 is a Hecke character of K_2 of infinity type $(1-k_2, 0)$, with $k_2 \geq 1$ and modulus $\mathfrak{f}_2$.

(3) $\tilde{\psi}_i$ is the Hecke character of K_0, obtained by composing $\mathbb{A}_{K_0}^\times \xrightarrow{\mathbb{N}_{K_0/K_i}} \mathbb{A}_{K_i}^\times \xrightarrow{\psi_i} \mathbb{C}$ for each $i \in \{1, 2\}$.

(4) By swapping K_1 and K_2, we may assume that $k_2 \geq k_1$.

In this scenario, the infinity type of χ (corresponding to the order $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$ or $(1, \tau_3, \tau_2, \tau_1)$) is

$$\big(\tfrac{1}{2}(2-k_1-k_2), \tfrac{1}{2}(k_1+k_2-2), \tfrac{1}{2}(k_1-k_2), \tfrac{1}{2}(k_2-k_1)\big).$$

Theorem 6.5.1. *Let $f \in S_k(\Gamma_0(pN_f))$ be a p-ordinary newform of weight $k = 2r \geq 2$. Let χ be an anticyclotomic Hecke character of K_0 as in Assumption 6.1. Assume that*:

(1) *Either $k \geq k_1 + k_2$ or $k_2 - k_1 \geq k$.*

(2) *$N_f \mathcal{O}_{K_3} = \mathfrak{n}^+\mathfrak{n}^-$, where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is divisible only by primes which are split (resp. inert) in K_0/K_3 and $\mathfrak{n}^-$ is a squarefree product of an even number of primes.*

(3) *$\bar{\rho}_f$ is absolutely irreducible.*

(4) *$(pN_f, \mathrm{Norm}_{K_1/\mathbb{Q}}(\mathfrak{f}_1)\mathrm{Norm}_{K_2/\mathbb{Q}}(\mathfrak{f}_2)D_{K_0}) = 1$.*

(5) *$p \nmid 6h_{K_0}$, the class number of K_0.*

Then

$$L(f/K_0, \chi, r) \neq 0 \implies \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) = 0.$$

In other words, the Bloch–Kato conjecture holds in analytic rank zero for $V_{f,\chi}$.

Proof. We consider the CM Hida families

$$\boldsymbol{g} = \boldsymbol{\theta}_{\xi_1}(S_1), \quad \boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}(S_2)$$

that pass through θ_{ψ_1} and θ_{ψ_2}, respectively. Note that the triple $(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2)$ satisfies (5-3). Then the isomorphism (6-9) of the associated $\boldsymbol{V}_{\mathcal{Q}_1}^\dagger$ together with the specialization $\mathcal{Q}_1$ corresponding to θ_{ψ_1} and θ_{ψ_2} show that

$$L(\mathbb{V}_{\mathcal{Q}_1}^\dagger, 0) = L(f/K_0, \chi, r).$$

By Theorem 6.1.1, we then have

$$L(f/K_0, \chi, r) \neq 0 \implies \mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}, \mathcal{Q}_1} \neq 0.$$

From our construction and Proposition 6.3.2, the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_{\mathcal{Q}_1} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, V_{f,\chi})$ is the base class of the anticyclotomic Euler system

$$\{z_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[m], T_{f,\chi}) : \mu_3 \in \mathcal{N}\}$$

of (6-15). Recall again that $K_0[1] = K_0$ from the assumption $p \nmid h_{K_0}$. By Theorem 6.4.1, we conclude that the Selmer group $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, V_{f,\chi})$ is one-dimensional, spanned by

$$z_{f,\chi} = \mathrm{Norm}_{K_0}^{K_0[1]}(z_{f,\chi,1}) = \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_{\mathcal{Q}_1}.$$

If $k \geq k_1 + k_2$, we observe that the composition of maps in (6-5) corresponds to the composition

$$\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \xrightarrow{\mathrm{res}_{\mathcal{P}_1}} \mathrm{H}^1(K_{0,\mathcal{P}_1}, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2})$$
$$\to \mathrm{H}^1(K_{0,\mathcal{P}_1}, T_f^{\vee,-}(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2})$$

by (6-10). Hence, $\mathrm{res}_{\mathcal{P}_1}(z_{f,\chi}) \neq 0$ by the reciprocity law of Theorem 6.1.1. Then the vanishing of $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_0, V_{f,\chi})$ follows by a standard argument using Poitou–Tate duality (see [Do 2022, Section 5.1.1]). This yields the result by using Lemma 4.4 for $k \geq k_1 + k_2$ to identify the latter group with $\mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi})$.

If $k_2 - k_1 \geq k$, similarly, by using $\mathrm{res}_{\mathcal{P}_2}(z_{f,\chi}) \neq 0$, we obtain the vanishing of $\mathrm{Sel}_{\mathrm{rel,str,rel,str}}(K_0, V_{f,\chi})$, which is again the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi})$ by Lemma 4.4 for $k_2 - k_1 \geq k$. □

Remark 6.5.2. Define $\epsilon(f, \chi)$ to be the sign of the functional equation for $V_{f,\chi}$. Then $\epsilon(f, \chi) = \prod \epsilon\big(\pi_{K_{0,v}} \otimes \chi_v, \frac{1}{2}\big)$ over places v of K_0 as a product of local root numbers. If $v \mid \mathfrak{n}^+$, then $\epsilon\big(\pi_{K_{0,v}} \otimes \chi_v, \frac{1}{2}\big) = +1$ and, if $v \mid \mathfrak{n}^-$, then $\epsilon\big(\pi_{K_{0,v}} \otimes \chi_v, \frac{1}{2}\big) = -1$. Therefore, the contribution from the local places is $+1$ due to assumption (2). At the infinity places,

$$\epsilon_\infty\big(\pi_{K_0} \otimes \chi, \tfrac{1}{2}\big) = i^{|k-1+(k_1+k_2-2)|+|k-1-(k_1+k_2-2)|+|k-1+(k_2-k_1)|+|k-1-(k_2-k_1)|}$$
$$= \begin{cases} +1 & \text{if } k > (k_1+k_2-2), \\ -1 & \text{if } k_2 - k_1 < k \leq k_1+k_2-2, \\ +1 & \text{if } k \leq k_2 - k_1. \end{cases}$$

Hence, conditions (1) and (2) of Theorem 6.5.1 then imply that $\epsilon(f, \chi) = 1$.

6.6. ***On the Iwasawa main conjecture.*** Our second application is evidence towards the anticyclotomic Iwasawa main conjecture for modular forms. Recall that we have an eigenform f of weight $k = 2r \geq 2$ with trivial nebentypus and an anticyclotomic character χ satisfying Assumption 6.1. Let

$$A_{f,\chi} = \mathrm{Hom}_{\mathbb{Z}_p}(T_f^\vee(1-r) \otimes \chi^{-1}, \mu_{p^\infty}).$$

Theorem 6.6.1. *Under the same assumption as in Theorem 6.5.1, we assume further that*:

(1) $\bar{\rho}_f$ *is p-distinguished.*

(2) f *has big image.*

(3) $p > k - 2$.

If $k \geq k_1 + k_2$, then $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, A_{f,\chi})$ is cotorsion over $\Lambda_{K_0}^-$. Furthermore, inside $\Lambda_{K_0}^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, we have the inclusion

$$\mathrm{char}_{\Lambda_{K_0}^-}(\mathrm{Sel}_{\mathrm{ord},\mathrm{ord},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, A_{f,\chi})^\vee) \supset (\mathscr{L}_p^{\boldsymbol{f},\eta_f}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)^2).$$

Proof. Recall from Corollary 6.3.3 that

$$\mathrm{Sel}^f(\mathbb{Q}, \mathbb{A}^\dagger) \simeq \mathrm{Sel}_{\mathrm{ord},\mathrm{ord},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, A_f(r) \otimes \chi), \tag{6-17}$$

where $\mathbb{A}^\dagger = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{V}^\dagger, \mu_{p^\infty})$.

Note that, from (6-8), $\mathscr{L}_p^{\boldsymbol{f},\eta_f}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)$ is an element of $\mathcal{O}[\![S_1, S_2]\!]$. We then identify $\Lambda_{K_0}^- \simeq \mathcal{O}[\![S_1, S_2]\!]$ via the diagram (3-9). The p-adic L-function $\mathscr{L}_p^{\boldsymbol{f},\eta_f}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)$ is nonzero by [Hung 2017, Theorem C]. Note that our assumption that $k \geq k_1 + k_2$ ensures that we are in the critical specializations, i.e., $-\frac{1}{2}k_\sigma < m_\sigma < \frac{1}{2}k_\sigma$ for all $\sigma \in \Sigma$, following the notation of [loc. cit.]. Hence, Theorem 6.1.1 and the proof of Theorem 6.5.1 implies that the class

$$\kappa(f, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2})$$

is nontorsion. By construction, we can treat $\kappa(f, \boldsymbol{g}, \boldsymbol{h})$ as the base class of the $\Lambda_{K_0}^-$-adic anticyclotomic Euler system

$$\{z_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}) : \mu_3 \in \mathcal{N}\}$$

in (6-16). Via the isomorphism (6-10), the result follows immediately from Theorem 6.4.4 applied to

$$z_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(z_{f,\chi,(1)}) = \kappa(f, \boldsymbol{g}, \boldsymbol{h}), \tag{6-18}$$

the equivalence between two different formulation of the Iwasawa main conjecture in Proposition 6.1.2, and the Selmer group isomorphism (6-17). □

Remark 6.6.2. Within Theorem 6.6.1, the right-hand side can be compared to the p-adic L-function of [Wan 2015, Theorem 86; Hung 2017] (under the assumptions of [Fujiwara 2006, Theorems 11.1–11.2; Wan 2015, Theorem 103]). The author then expects that the full Iwasawa main conjecture, which means an equality of Theorem 6.6.1, will follow by combining Theorem 6.6.1 with the opposite inclusion of [Wan 2015] and the vanishing of the μ-invariant of the p-adic L-function [Hung 2017] (which are generalizations of [Skinner and Urban 2014; Vatsal 2003]).

Remark 6.6.3. One expects a similar result that, if $k_2 - k_1 \geq k$, then $\mathrm{Sel}_{\mathrm{str},\mathrm{rel},\mathrm{str},\mathrm{rel}}(K_\infty^-, A_{f,\chi})$ is cotorsion over $\Lambda_{K_0}^-$ together with the inclusion, inside $\Lambda_{K_0}^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,

$$\mathrm{char}_{\Lambda_{K_0}^-}(\mathrm{Sel}_{\mathrm{str},\mathrm{rel},\mathrm{str},\mathrm{rel}}(K_{0,\infty}^-, A_{f,\chi})^\vee) \supset (\mathscr{L}_p^{\boldsymbol{h},\eta_h}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)^2).$$

The only missing ingredient is the nonvanishing of the p-adic L-function in this region.

6.7. ***On the Bloch–Kato conjecture in rank* 1.** Our last application is extracted from the proof of Theorem 6.6.1. It provides a result towards the Bloch–Kato conjecture in rank 1.

Theorem 6.7.1. *Under the same assumption as in Theorem 6.6.1, if* $k_1+k_2-2 \geq k \geq k_2-k_1+2$ *(which induces* $L(f/K_0, \chi, r)=0$*), then*

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) \geq 1.$$

Proof. The class $\boldsymbol{z}_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})$ is nontorsion via the proof of Theorem 6.6.1. Furthermore, $\boldsymbol{z}_{f,\chi}$ is the base of a $\Lambda_{K_0}^-$-adic anticyclotomic Euler system as in (6-18) for the relaxed-strict-ordinary-ordinary Selmer group. Theorem 6.4.4 then implies that $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})$ has $\Lambda_{K_0}^-$-rank 1. The natural map (compare with the projection (6-14))

$$\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})/(\gamma_{1,-}-1, \gamma_{2,-}-1) \to \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi}) \tag{6-19}$$

is injective (see also [Mazur and Rubin 2004, Proposition 5.3.14; Greenberg 1999, page 72]).

Hence, the Selmer group $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi})$ has a positive $\mathcal{O}$-rank. The theorem then follows by Lemma 4.4, which computes the local conditions of the Bloch–Kato Selmer group explicitly. □

Remark 6.7.2. By letting $z_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi})$ be the image of $\boldsymbol{z}_{f,\chi}$ under the map (6-19), such a class $z_{f,\chi} \in \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi})$ satisfies

$$z_{f,\chi} \neq 0 \implies \dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) = 1$$

by Theorem 6.4.1.

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References

[Alonso et al. 2025] R. Alonso, F. Castella, and O. Rivero, "An anticyclotomic Euler system for adjoint modular Galois representations", *Ann. Inst. Fourier* (*Grenoble*) **75**:1 (2025), 291–329. MR

[Bertolini and Darmon 2005] M. Bertolini and H. Darmon, "Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions", *Ann. of Math.* (2) **162**:1 (2005), 1–64. MR

[Bertolini et al. 2013] M. Bertolini, H. Darmon, and K. Prasanna, "Generalized Heegner cycles and p-adic Rankin L-series", *Duke Math. J.* **162**:6 (2013), 1033–1148. MR

[Bertolini et al. 2022a] M. Bertolini, H. Darmon, V. Rotger, M. A. Seveso, and R. Venerucci, *Heegner points, Stark–Heegner points, and diagonal classes*, Astérisque **434**, Soc. Math. France, Paris, 2022. MR

[Bertolini et al. 2022b] M. Bertolini, M. A. Seveso, and R. Venerucci, "Reciprocity laws for balanced diagonal classes", pp. 77–174 in *Heegner points*, *Stark–Heegner points*, *and diagonal classes*, Astérisque **434**, Soc. Math. France, Paris, 2022. MR

[Bloch and Kato 1990] S. Bloch and K. Kato, "L-functions and Tamagawa numbers of motives", pp. 333–400 in *The Grothendieck Festschrift*, *I*, edited by P. Cartier et al., Progr. Math. **86**, Birkhäuser, Boston, MA, 1990. MR

[Castella 2020] F. Castella, "On the p-adic variation of Heegner points", *J. Inst. Math. Jussieu* **19**:6 (2020), 2127–2164. MR

[Castella and Do 2025] F. Castella and K. T. Do, "Diagonal cycles and anticyclotomic Iwasawa theory of modular forms", *J. Eur. Math. Soc.* (online publication November 2025).

[Castella and Hsieh 2018] F. Castella and M.-L. Hsieh, "Heegner cycles and p-adic L-functions", *Math. Ann.* **370**:1-2 (2018), 567–628. MR

[Chida 2017] M. Chida, "Selmer groups and central values of L-functions for modular forms", *Ann. Inst. Fourier* (*Grenoble*) **67**:3 (2017), 1231–1276. MR

[Chida and Hsieh 2015] M. Chida and M.-L. Hsieh, "On the anticyclotomic Iwasawa main conjecture for modular forms", *Compos. Math.* **151**:5 (2015), 863–897. MR

[Darmon and Rotger 2017] H. Darmon and V. Rotger, "Diagonal cycles and Euler systems, II: The Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-functions", *J. Amer. Math. Soc.* **30**:3 (2017), 601–672. MR

[Do 2022] K. T. Do, *Construction of an anticyclotomic Euler system with applications*, Ph.D. thesis, Princeton University, 2022, available at https://www.proquest.com/docview/2720964424. MR

[Flach 1990] M. Flach, "A generalisation of the Cassels–Tate pairing", *J. Reine Angew. Math.* **412** (1990), 113–127. MR

[Fröhlich and Taylor 1993] A. Fröhlich and M. J. Taylor, *Algebraic number theory*, Cambridge Studies in Advanced Mathematics **27**, Cambridge Univ. Press, 1993. MR

[Fujiwara 2006] K. Fujiwara, "Deformation rings and Hecke algebras in the totally real case", preprint, 2006. arXiv math/0602606

[Greenberg 1994] R. Greenberg, "Iwasawa theory and p-adic deformations of motives", pp. 193–223 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55.2**, Amer. Math. Soc., Providence, RI, 1994. MR

[Greenberg 1999] R. Greenberg, "Iwasawa theory for elliptic curves", pp. 51–144 in *Arithmetic theory of elliptic curves* (Cetraro, 1997), edited by C. Viola, Lecture Notes in Math. **1716**, Springer, 1999. MR

[Gross and Kudla 1992] B. H. Gross and S. S. Kudla, "Heights and the central critical values of triple product L-functions", *Compositio Math.* **81**:2 (1992), 143–209. MR

[Gross and Schoen 1995] B. H. Gross and C. Schoen, "The modified diagonal cycle on the triple product of a pointed curve", *Ann. Inst. Fourier* (*Grenoble*) **45**:3 (1995), 649–679. MR

[Gross and Zagier 1986] B. H. Gross and D. B. Zagier, "Heegner points and derivatives of L-series", *Invent. Math.* **84**:2 (1986), 225–320. MR

[Hida 1988] H. Hida, "Modules of congruence of Hecke algebras and L-functions associated with cusp forms", *Amer. J. Math.* **110**:2 (1988), 323–382. MR

[Hida 2015] H. Hida, "Big Galois representations and p-adic L-functions", *Compos. Math.* **151**:4 (2015), 603–664. MR

[Hsieh 2021] M.-L. Hsieh, "Hida families and p-adic triple product L-functions", *Amer. J. Math.* **143**:2 (2021), 411–532. MR

[Hung 2017] P.-C. Hung, "On the non-vanishing mod ℓ of central L-values with anticyclotomic twists for Hilbert modular forms", *J. Number Theory* **173** (2017), 170–209. MR

[Jetchev et al. 2024] D. Jetchev, J. Nekovář, and C. Skinner, "Split Euler systems for conjugate-dual Galois representations", preprint, 2024.

[Kings et al. 2017] G. Kings, D. Loeffler, and S. L. Zerbes, "Rankin–Eisenstein classes and explicit reciprocity laws", *Camb. J. Math.* **5**:1 (2017), 1–122. MR

[Kings et al. 2020] G. Kings, D. Loeffler, and S. L. Zerbes, "Rankin–Eisenstein classes for modular forms", *Amer. J. Math.* **142**:1 (2020), 79–138. MR

[Kolyvagin 1988] V. A. Kolyvagin, "Finiteness of $E(\mathbb{Q})$ and Ш$(E, \mathbb{Q})$ for a subclass of Weil curves", *Izv. Akad. Nauk SSSR Ser. Mat.* **52**:3 (1988), 522–540. In Russian; translated in *Math. USSR-Izv.* **32**:3 (1989), 523–541. MR

[Lamplugh 2016] J. Lamplugh, "An Euler system for characters over an imaginary biquadratic field", 2016. arXiv 1611.05702

[Lei et al. 2014] A. Lei, D. Loeffler, and S. L. Zerbes, "Euler systems for Rankin–Selberg convolutions of modular forms", *Ann. of Math.* (2) **180**:2 (2014), 653–771. MR

[Lei et al. 2015] A. Lei, D. Loeffler, and S. L. Zerbes, "Euler systems for modular forms over imaginary quadratic fields", *Compos. Math.* **151**:9 (2015), 1585–1625. MR

[Longo and Vigni 2010] M. Longo and S. Vigni, "On the vanishing of Selmer groups for elliptic curves over ring class fields", *J. Number Theory* **130**:1 (2010), 128–163. MR

[Mazur and Rubin 2004] B. Mazur and K. Rubin, *Kolyvagin systems*, Mem. Amer. Math. Soc. **799**, Amer. Math. Soc., Providence, RI, 2004. MR

[Miyake 1989] T. Miyake, *Modular forms*, Springer, 1989. MR

[Momose 1981] F. Momose, "On the l-adic representations attached to modular forms", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28**:1 (1981), 89–109. MR

[Nekovář 2000] J. Nekovář, "p-adic Abel–Jacobi maps and p-adic heights", pp. 367–379 in *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), edited by B. B. Gordon et al., CRM Proc. Lecture Notes **24**, Amer. Math. Soc., Providence, RI, 2000. MR

[Nekovář 2007] J. Nekovář, "The Euler system method for CM points on Shimura curves", pp. 471–547 in *L-functions and Galois representations*, edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge Univ. Press, 2007. MR

[Nekovář 2012] J. Nekovář, "Level raising and anticyclotomic Selmer groups for Hilbert modular forms of weight two", *Canad. J. Math.* **64**:3 (2012), 588–668. MR

[Ribet 1985] K. A. Ribet, "On l-adic representations attached to modular forms, II", *Glasgow Math. J.* **27** (1985), 185–194. MR

[Rubin 2000] K. Rubin, *Euler systems*, Annals of Math. Studies **147**, Princeton Univ. Press, 2000. MR

[Skinner and Urban 2014] C. Skinner and E. Urban, "The Iwasawa main conjectures for GL_2", *Invent. Math.* **195**:1 (2014), 1–277. MR

[Tamiozzo 2021] M. Tamiozzo, "On the Bloch–Kato conjecture for Hilbert modular forms", *Math. Z.* **299**:1-2 (2021), 427–458. MR

[Tate 1979] J. Tate, "Number theoretic background", pp. 3–26 in *Automorphic forms, representations and L-functions* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **XXXIII.2**, Amer. Math. Soc., Providence, RI, 1979. MR

[Tian 2003] Y. Tian, *Euler systems of CM points on Shimura curves*, Ph.D. thesis, Columbia University, 2003, available at https://www.proquest.com/docview/305333682. MR

[Vatsal 2003] V. Vatsal, "Special values of anticyclotomic L-functions", *Duke Math. J.* **116**:2 (2003), 219–261. MR

[Wan 2015] X. Wan, "The Iwasawa main conjecture for Hilbert modular forms", *Forum Math. Sigma* **3** (2015), art. id. e18. MR

[Wang 2023] H. Wang, "On the anticyclotomic Iwasawa main conjecture for Hilbert modular forms of parallel weights", *Ann. Math. Qué.* **47**:1 (2023), 195–248. MR

[Wiles 1988] A. Wiles, "On ordinary λ-adic representations associated to modular forms", *Invent. Math.* **94**:3 (1988), 529–573. MR

[Wiles 1995] A. Wiles, "Modular elliptic curves and Fermat's last theorem", *Ann. of Math.* (2) **141**:3 (1995), 443–551. MR

[Zhang 1997] S. Zhang, "Heights of Heegner cycles and derivatives of L-series", *Invent. Math.* **130**:1 (1997), 99–152. MR

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