Algebra & Number Theory

Volume 20
2026
No. 8

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ALGEBRA AND NUMBER THEORY 20:8 (2026)
https://doi.org/10.2140/ant.2026.20.1649

Explicit Hecke descent for special cycles

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We derive an explicit formula for the action of a geometric Hecke correspondence on special cycles on a Shimura variety in terms of such cycles at a fixed neat level and compare it with another closely related expression that is valid for modular curves. We provide evidence that the two formulas do not agree in general.

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1. Introduction

The study of special cycles on Shimura varieties often entails aspects of smooth representation theory. Kudla [1997] considered certain weighted linear combinations of special cycles on orthogonal Shimura varieties whose equivariance properties are neatly captured by Schwartz spaces admitting a smooth group action. Similar constructions have appeared in [Cornut 2018; Jetchev 2014; Li and Liu 2021; Lai and Skinner 2024] and several other recent works. One motivation for studying such equivariant labeling of cycles is to facilitate the study of Hecke actions on them. Although conventions differ from one author to another, it seems to us that such descriptions can give rise to two fundamentally different ways of defining the said Hecke action, and only one of these a priori agrees with the geometric action of Hecke correspondences in all situations. Unfortunately, the "nongeometric" action seems to be occasionally used in the literature and assumed to be compatible with the geometric one.

The purpose of this note is twofold. First, we derive a general formula for the action of a Hecke correspondence on a given irreducible special cycle in terms of such cycles that are all at a fixed (finite) level, assuming some mild conditions on the level and the Shimura data. This formula also generalizes to any abstract pushforward construction involving such cycles, e.g., Gysin maps in étale cohomology. Second, we address the aforementioned compatibility of the two actions on the group of equidimensional algebraic cycles and provide evidence that they only agree under very special circumstances.

MSC2020: 11G18, 14C25, 14G35.
Keywords: special cycles, Hecke correspondences, Shimura varieties, descent.

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1.1. ***Motivation.*** We motivate our question in the familiar setting of modular curves. Let K be a compact–open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ that is contained in a principal congruence subgroup of level $N \geq 3$.[1] Let Y_K denote the modular curve of level K in the sense of [Deligne 1971]. It is a smooth quasiprojective algebraic curve defined over $\mathbb{Q}$ with complex points given by

$$Y_K(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathscr{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K),$$

where $\mathscr{H}^{\pm} := \mathbb{C} \setminus \mathbb{R}$ and $\mathbb{A}_f$ denotes the ring of finite rational adeles. For $(x, g) \in \mathscr{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)$, we denote by $[x, g]_K \in Y_K(\mathbb{C})$ the class of (x, g). Let $\mathscr{C}(K) := \mathbb{Z}[Y_K(\mathbb{C})]$ denote the group of complex divisors on $Y_K(\mathbb{C})$. For $\sigma \in \mathrm{GL}_2(\mathbb{A}_f)$, we write $K_\sigma := K \cap \sigma K \sigma^{-1}$ and $K^\sigma := K \cap \sigma^{-1} K \sigma$ for brevity. Consider the diagram of $\mathbb{Q}$-varieties

$$Y_K \xleftarrow{\ p\ } Y_{K_\sigma} \xrightarrow[\sim]{[\sigma]} Y_{K^\sigma} \xrightarrow{\ q\ } Y_K$$

in which p, q are the natural degeneracy maps induced by the inclusions $K_\sigma \hookrightarrow K$, $K^\sigma \hookrightarrow K$, respectively, and $[\sigma]$ is the twisting isomorphism given on $\mathbb{C}$-points by $[x, g]_{K_\sigma} \mapsto [x, g\sigma]_{K^\sigma}$. The maps p and q are finite étale and induce pushforwards and pullbacks on divisors with the expected properties. The $\mathbb{Z}$-linear map $[K\sigma K]_* = q_* \circ [\sigma]_* \circ p^* : \mathscr{C}(K) \to \mathscr{C}(K)$ is called the *covariant Hecke correspondence*[2] induced by σ. It is easily seen that

$$[K\sigma K]_* \cdot [x, g]_K = \sum_{\gamma \in K\sigma K/K} [x, g\gamma]_K. \tag{1.1}$$

On the other hand, one can construct the inductive limit

$$\widehat{\mathscr{C}} := \varinjlim_L \mathscr{C}(L)$$

over all (sufficiently small) levels L of $\mathrm{GL}_2(\mathbb{A}_f)$, where the transition maps for the limit are given by pullbacks along the degeneracy maps $Y_{L'} \to Y_L$ induced by the inclusions $L' \hookrightarrow L$. This comes equipped with a smooth left action

$$\mathrm{GL}_2(\mathbb{A}_f) \times \widehat{\mathscr{C}} \to \widehat{\mathscr{C}}$$

given by pullbacks along the twisting isomorphisms. Explicitly, if $z \in \widehat{\mathscr{C}}$ can be represented by $[x, g]_L \in \mathscr{C}(L)$, then $h \cdot z$ for $h \in \mathrm{GL}_2(\mathbb{A}_f)$ is the class of $[x, gh^{-1}]_{hLh^{-1}}$ in $\widehat{\mathscr{C}}$. Since divisors satisfy étale descent, one has $\mathscr{C}(K) = \widehat{\mathscr{C}}^K$, obtained via the inclusion $\mathscr{C}(K) \hookrightarrow \widehat{\mathscr{C}}^K$. Under this identification, the pushforward $\mathscr{C}(L) \to \mathscr{C}(K)$ along the degeneracy map induced by the inclusion of any (not necessarily normal) subgroup L of K corresponds to the norm map $\sum_{\gamma \in K/L} \gamma$. In particular, q_* is identified with $\sum_{K/K^\sigma} \gamma$ and $[K\sigma K]_*$ with $\sum_{\gamma \in K/K^\sigma} \gamma\sigma^{-1}$. Thus,

$$[K\sigma K]_* \cdot [x, g]_K = \sum_{\delta \in K\backslash K\sigma K} [x, g\delta]_{\delta^{-1}K\delta}, \tag{1.2}$$

[1] See [Shah 2023, Section 2.4] for a treatment of the case of a general compact–open subgroup K.

[2] This is sometimes referred to as the "Albanese" convention for Hecke correspondences [Ribet 1990, p. 443]. See also [Lei et al. 2015, Remarks 4.1.1 and 4.3.2] and [Lei et al. 2018, Section 2.6] for a discussion.

where the right-hand side is interpreted as an element of $\widehat{\mathscr{C}}^K$. Note that individual points in this expression live on modular curves of *different* levels.

The right-hand side of (1.2) computing the action of $[K\sigma K]_*$ is formal in nature and remains valid, for instance, in the cohomology of any Shimura variety and, more generally, any locally symmetric space. The right-hand side of (1.1) is, however, more useful, since it explicitly gives a divisor in terms of points that all share the original level K. It is also what one usually finds in the literature on Shimura curves; see, for example, [Cornut and Vatsal 2007, Section 3.4; Zhang 2001, Section 1.4]. We say that (1.1) provides an *explicit descent formula* for the formal K-invariant expression given by (1.2).

1.2. ***Main problem.*** We may make similar considerations for special cycles on Shimura varieties. Say E is an imaginary quadratic field and $x = x_\iota$ is defined as the image of "pt" under the morphism of Shimura data

$$\iota : (\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m, \{\mathrm{pt}\}) \hookrightarrow (\mathrm{GL}_2, \mathscr{H}^{\pm}) \tag{1.3}$$

obtained by considering E as a $\mathbb{Q}$-vector space. Then the points $z_L(g) := [x_\iota, g]_L \in Y_L(\mathbb{C})$ for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ are algebraic and referred to as special points (or CM points). One may equivalently describe $z_L(g)$ as the point obtained by taking the distinguished geometric connected component "[1]" of the zero-dimensional Shimura variety for $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ of level $(\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)(\mathbb{A}_f) \cap gLg^{-1}$, embedding it into $Y_{gLg^{-1}}$, and taking its image under the twisting isomorphism $[g] : Y_{gLg^{-1}} \to Y_L$. See [Shah 2023, Section 2] for details on this setup. In this paradigm, we may replace (1.3) by an arbitrary embedding of Shimura data

$$\iota : (\boldsymbol{H}, X_{\boldsymbol{H}}) \hookrightarrow (\boldsymbol{G}, X_{\boldsymbol{G}})$$

and construct, for any compact–open subgroup L of $\boldsymbol{G}(\mathbb{A}_f)$, *irreducible special cycles* $z_L(g)$ (defined over the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$) of a fixed codimension n on $\mathrm{Sh}_{\boldsymbol{G}}(L)$ for any $g \in \boldsymbol{G}(\mathbb{A}_f)$ in an analogous fashion. Let $\mathscr{C}^n(L)$ denote the free $\mathbb{Z}$-module on codimension n irreducible $\overline{\mathbb{Q}}$-subvarieties on $\mathrm{Sh}_{\boldsymbol{G}}(L)$. One may then ask for the analog of (1.1) for the Hecke action on the special cycle $z_L(g)$ in the space $\mathscr{C}^n(L)$.

More precisely, let K denote a fixed compact–open subgroup of $\boldsymbol{G}(\mathbb{A}_f)$. To avoid pathologies arising from degrees of degeneracy maps, we assume that K is *neat* [Pink 1989, Section 0.1]. Fix a $\sigma \in \boldsymbol{G}(\mathbb{A}_f)$ and write $K_\sigma := K \cap \sigma K \sigma^{-1}$ and $K^\sigma := K \cap \sigma^{-1} K \sigma$ as before. We define the *covariant Hecke correspondence*

$$[K\sigma K]_* : \mathscr{C}^n(K) \xrightarrow{\mathrm{pr}^*} \mathscr{C}^n(K_\sigma) \xrightarrow{[\sigma]_*} \mathscr{C}^n(K^\sigma) \xrightarrow{\mathrm{pr}_*} \mathscr{C}^n(K),$$

where pr^* (resp. pr_*) denotes the flat pullback (resp. proper pushforward) of cycles along the degeneracy map induced by the inclusion $K_\sigma \hookrightarrow K$ (resp. $K^\sigma \hookrightarrow K$) and $[\sigma]_*$ is the isomorphism induced by pushforward along the twisting map[3] $[\sigma] : \mathrm{Sh}_{\boldsymbol{G}}(K_\sigma) \to \mathrm{Sh}_{\boldsymbol{G}}(K^\sigma)$. See also [Noot 2006, Section 6.2; Esnault and Harris 2017, Section 1.2]. In analogy with (1.2), it is not hard to show that, for any $g \in \boldsymbol{G}(\mathbb{A}_f)$,

[3]Throughout, the action of $\sigma \in \boldsymbol{G}(\mathbb{A}_f)$ on the tower of Shimura varieties is denoted by $[\sigma]$ and gives a *right action* of $\boldsymbol{G}(\mathbb{A}_f)$, as in [Milne 2005, Definition 5.14].

we have

$$[K\sigma K]_* \cdot z_K(g) = \sum_{\delta \in K \backslash K\sigma K} z_{\delta^{-1}K\delta}(g\delta),$$

where the right-hand side is again viewed as an element in the inductive limit of $\mathcal{C}^n(L)$ over all (sufficiently small) levels L. A natural question that arises at this point is the following.

Question 1.4. *Is there a $\mathbb{Z}$-linear combination of irreducible special cycles $z_K(-) \in \mathcal{C}^n(K)$ that equals $[K\sigma K]_* z_K(g)$? Equivalently, is it possible to write the K-invariant limit expression $\sum_{\delta \in K\backslash K\sigma K} z_{\delta^{-1}K\delta}(g\delta)$ in terms of such cycles? If so, can one give a formula for it?*

A first guess suggested by (1.1) would be that

$$[K\sigma K]_* \cdot z_K(g) \stackrel{?}{=} \sum_{\gamma \in K\sigma K/K} z_K(g\gamma). \tag{1.5}$$

Unfortunately, this is not necessarily always the case. While the answer to Question 1.4 is affirmative, the correct expression is given by Corollary 4.20 below and involves various nontrivial coefficients. In fact, the number of irreducible special cycles in our expression seems to be far smaller in general than what is prescribed by (1.5). See Section 5 for computations. This discrepancy continues to exist if one replaces $z_K(g)$ with the full fundamental cycle of the source Shimura variety. Analogs of (1.5) seem to be used in [Boumasmoud et al. 2020, Section 3.1; Li and Liu 2021, p. 846] and various other writings.

On the other hand, we show that, for the cases of zero cycles and codimension zero cycles, our expression does agree with (1.5). This is of course in agreement with (1.1). For a simple group-theoretic explanation of this phenomenon, see Section 6.

Our approach to Question 1.4 is based on an elementary but useful idea: that explicit distribution relations among objects associated with $\mathrm{Sh}_{\boldsymbol{H}}$ (such as fundamental cycles of $\mathrm{Sh}_{\boldsymbol{H}}$) can be parlayed into corresponding relations on $\mathrm{Sh}_{\boldsymbol{G}}$ via a gadget we refer to as the *mixed Hecke correspondence* [Shah 2024, Section 2]. This approach utilizes the built-in functoriality properties of pushforwards and pullbacks of cycles on schemes. Consequently, our methods readily apply in the more general settings of pushforwards of cycle classes into cohomology with coefficients in local systems on $\mathrm{Sh}_{\boldsymbol{G}}$, such as the one studied in [Graham and Shah 2023]. This is a nontrivial extension, since cohomology with integral coefficients does not satisfy Galois descent in general. We establish this formula in Theorem 4.12 for abstract pushforwards using the language of RIC functors developed in [Shah 2024, Section 2], which we briefly recall in Section 3.2. We then derive the expression asked for in Question 1.4 in Corollary 4.20. We anticipate that our formula will find applications in establishing Euler system norm relations when one works with more general coefficient systems, as, for instance, in the approach originally envisioned by Cornut [2018].

1.3. ***Outline.*** This note is structured as follows. In Section 2, we collect relevant facts about Shimura varieties needed in the derivation of our formula. In Section 3, we study the special cycles $z_K(-)$ from the introduction and their analogs for abstract pushforwards. In Section 4, we derive our main formula and provide some simple examples. In Section 5, we furnish two sets of counterexamples to (1.5). In

Section 6, we investigate an analogous question in the setting of function spaces that provides another perspective on the issue. This section may be read independently of the rest of this note.

2. Recollections

In this section, we recall some basic facts about Shimura varieties that we will need later on. We also generalize a result from [Loeffler et al. 2022], which provides a simple criterion for checking when maps between Shimura varieties are closed immersions. It is, however, only needed in Section 5.

2.1. ***Shimura varieties.*** Suppose that

$$\iota : (\boldsymbol{H}, X_{\boldsymbol{H}}) \hookrightarrow (\boldsymbol{G}, X_{\boldsymbol{G}}) \tag{2.1}$$

is an embedding of Shimura data, i.e., $\boldsymbol{H}$, $\boldsymbol{G}$ are connected reductive algebraic groups over $\mathbb{Q}$, the pairs $(\boldsymbol{H}, X_{\boldsymbol{H}})$, $(\boldsymbol{G}, X_{\boldsymbol{G}})$ satisfy the Milne–Deligne axioms (SV1)–(SV3) [Milne 2005, Definition 5.5], and $\iota : \boldsymbol{H} \to \boldsymbol{G}$ is an embedding preserving the said datum. Denote by $\mathbb{A}_f$ the ring of finite rational adeles and set

$$H := \boldsymbol{H}(\mathbb{A}_f), \quad G := \boldsymbol{G}(\mathbb{A}_f).$$

For each neat [Pink 1989, Section 0.1] compact–open subgroup K of G, there is a smooth equidimensional quasiprojective Shimura variety $\mathrm{Sh}_{\boldsymbol{G}}(K)$ defined over a canonically defined subfield of $\mathbb{C}$, called the reflex field of $\boldsymbol{G}$, and similarly for $\boldsymbol{H}$. We will, however, exclusively consider all Shimura varieties as objects in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$, the category of schemes over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

The $\mathbb{C}$-points of $\mathrm{Sh}_{\boldsymbol{G}}(K)$ are given by $\boldsymbol{G}(\mathbb{Q})\backslash(X_{\boldsymbol{G}} \times G/K)$ and a general point is denoted as $[x, g]_K$, where $x \in X_{\boldsymbol{G}}$ and $g \in G$. Similar notation and conventions will be used for corresponding objects associated with $\boldsymbol{H}$. We set

$$n := \dim \mathrm{Sh}_{\boldsymbol{G}}(K) - \dim \mathrm{Sh}_{\boldsymbol{H}}(U) \tag{2.2}$$

for some choice of compact–open subgroups $K \subset G$ and $U \subset H$. Then n is independent of K and U. In what follows, we think of H as a closed subgroup of G. Then, for any compact–open subgroup K of G, the intersection $H \cap K := H \cap \iota^{-1}(K)$ is a compact–open subgroup of H, which is neat if K is.

2.2. ***Maps.*** For any $g \in G$ and any neat compact–open subgroup K of G, there is a *twisting isomorphism*

$$[g] = [g]_{K, g^{-1}Kg} : \mathrm{Sh}_{\boldsymbol{G}}(K) \xrightarrow{\sim} \mathrm{Sh}_{\boldsymbol{G}}(g^{-1}Kg) \tag{2.3}$$

given on $\mathbb{C}$-points by $[x, \gamma]_K \mapsto [x, \gamma g]_{g^{-1}Kg}$ for all $x \in X_{\boldsymbol{G}}$ and $\gamma \in G$. Let $\boldsymbol{Z}_{\boldsymbol{G}}$ (resp. $\boldsymbol{Z}_{\boldsymbol{H}}$) denote the center of $\boldsymbol{G}$ (resp. $\boldsymbol{H}$). For any compact–open subgroup L contained in K that satisfies

$$K \cap \boldsymbol{Z}_{\boldsymbol{G}}(\mathbb{Q}) = L \cap \boldsymbol{Z}_{\boldsymbol{G}}(\mathbb{Q}), \tag{2.4}$$

the natural surjection $\mathrm{pr}_{L,K} : \mathrm{Sh}_{\boldsymbol{G}}(L) \to \mathrm{Sh}_{\boldsymbol{G}}(K)$ given by $[x, g]_L \mapsto [x, g]_K$ is finite étale of degree $[K : L]$, by [Shah 2024, Lemma 2.7.1]. When (SV5) of [Milne 2005] holds for $(\boldsymbol{G}, X_{\boldsymbol{G}})$, the condition (2.4) is

automatic by neatness of K. One may also consider the limit

$$\mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C}) := \varprojlim_{L} \mathrm{Sh}_{\boldsymbol{G}}(L)(\mathbb{C})$$

taken over all neat compact–open subgroups L of $\boldsymbol{G}(\mathbb{A}_f)$ along the degeneracy maps induced by the inclusions $L' \hookrightarrow L$. Then $\mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C})$ inherits a continuous right action of $G = \boldsymbol{G}(\mathbb{A}_f)$ and $\mathrm{Sh}_{\boldsymbol{G}}(K)(\mathbb{C}) = \mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C})/K$ for any K. If $(\boldsymbol{G}, X_{\boldsymbol{G}})$ satisfies (SV5), any neat compact–open subgroup acts freely on $\mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C})$.

We also have maps between Shimura varieties of $\boldsymbol{H}$ and $\boldsymbol{G}$. If K is a neat compact–open subgroup of G and U is a compact–open subgroup of $H \cap K$, there is a finite unramified morphism

$$\iota_{U,K} : \mathrm{Sh}_{\boldsymbol{H}}(U) \to \mathrm{Sh}_{\boldsymbol{G}}(K) \tag{2.5}$$

of smooth schemes in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$, which is given on $\mathbb{C}$-points by $[y, h]_U \mapsto [y, h]_K$ for all $y \in X_{\boldsymbol{H}}$ and $h \in H$. We can always choose a compact–open subgroup $L = L_U$ of G which contains U such that $\mathrm{Sh}_{\boldsymbol{H}}(U) \to \mathrm{Sh}_{\boldsymbol{G}}(L)$ is a closed immersion [Deligne 1971, Proposition 1.15]. We may assume that $L \subset K$ by replacing L with its intersection with K if necessary, since $\mathrm{Sh}_{\boldsymbol{H}}(U) \to \mathrm{Sh}_{\boldsymbol{G}}(L)$ factors as

$$\mathrm{Sh}_{\boldsymbol{H}}(U) \to \mathrm{Sh}_{\boldsymbol{G}}(L \cap K) \to \mathrm{Sh}_{\boldsymbol{G}}(L)$$

and [Stacks 2005–, Lemma 07RK] implies that $\mathrm{Sh}_{\boldsymbol{H}}(U) \to \mathrm{Sh}_{\boldsymbol{G}}(L \cap K)$ is also a closed immersion. We may moreover assume that (2.4) holds for L. Indeed, if we let $L' := L\Gamma$, where $\Gamma := \boldsymbol{Z}_{\boldsymbol{G}}(\mathbb{Q}) \cap K$, we have $L \subset L' \subset K$, and it is elementary to show that $\mathrm{Sh}_{\boldsymbol{G}}(L) = \mathrm{Sh}_{\boldsymbol{G}}(L')$. In any case, $\iota_{U,K}$ is a composition of a closed immersion followed by a finite étale surjection and one may therefore apply various natural pushforward constructions in this setting (see, e.g., [Graham and Shah 2023, Proposition A.5]). In all cases, the natural map

$$\iota_\infty : \mathrm{Sh}_{\boldsymbol{H}}(\mathbb{C}) \to \mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C}) \tag{2.6}$$

at the infinite level is injective, again by [Deligne 1971, Proposition 1.15]. The following generalization of [Loeffler et al. 2022, Proposition 5.3.1] gives a criterion for immersion at finite levels.

Lemma 2.7. *Assume that* $(\boldsymbol{G}, X_{\boldsymbol{G}})$ *satisfies* (SV5) *and that there exists a* $w \in \boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q})$ *such that* $\boldsymbol{H}$ *is the centralizer of* w *in* $\boldsymbol{G}$. *Let* K *be a compact–open subgroup of* G *such that there exists a neat compact–open subgroup of* G *that contains both* K *and* wKw^{-1}. *Then* $\iota_{H\cap K,K}$ *is a closed immersion.*

Proof. Since $\iota_{H\cap K,K}$ is a finite unramified morphism, it suffices by [Stacks 2005–, Lemma 04XV] to check that $\iota_{H\cap K,K}$ is universally injective. By [ibid., Lemma 01S4], this amounts to the surjectivity of the diagonal map

$$\Delta : \mathrm{Sh}_{\boldsymbol{H}}(H \cap K) \to \mathrm{Sh}_{\boldsymbol{H}}(H \cap K) \times_{\mathrm{Sh}_{\boldsymbol{G}}(K)} \mathrm{Sh}_{\boldsymbol{H}}(H \cap K).$$

By [Demazure and Gabriel 1980, I, Section 3, 6.11],[4] one can check surjectivity on closed points and, therefore, also on $\mathbb{C}$-points. But it is easily seen that $\Delta(\mathbb{C})$ is surjective if and only if $\iota_{H\cap K,K}(\mathbb{C})$ is injective. So it suffices to check the injectivity of $\iota_{H\cap K,K}$ on $\mathbb{C}$-points.

[4] See also the notion of *ultraschemes* in [EGA I 1971, Appendice].

To this end, say there exists a $P \in \mathrm{Sh}_{\boldsymbol{H}}(\mathbb{C})$ and $\kappa \in K$ such that $P\kappa$ lies in $\mathrm{Sh}_{\boldsymbol{H}}(\mathbb{C})$, where we are viewing $\mathrm{Sh}_{\boldsymbol{H}}(\mathbb{C})$ as a subset of $\mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C})$ via (2.6). Since $w \in \boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q})$ fixes $X_{\boldsymbol{H}}$ pointwise and commutes with all elements of $\boldsymbol{H}(\mathbb{A}_f)$, the (right) action of w on $\mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C})$ fixes $\mathrm{Sh}_{\boldsymbol{H}}(\mathbb{C})$ pointwise. Therefore,

$$P w\kappa = P\kappa = P\kappa w.$$

This implies that $\lambda := \kappa w\kappa^{-1}w^{-1}$ stabilizes P. If L is a compact–open subgroup of G that contains both K and wKw^{-1}, then $\lambda = \kappa(w\kappa^{-1}w^{-1}) \in L$. Recall that, if $(\boldsymbol{G}, X_{\boldsymbol{G}})$ satisfies (SV5), then the right action of any neat compact–open subgroup of G on $\mathrm{Sh}_{\boldsymbol{G}}(\mathbb{C})$ is free. Thus, if L is neat, the only element of L that can stabilize P is the identity. So our assumptions imply that λ must be the identity, i.e., $w = \kappa w\kappa^{-1}$. Since the centralizer of w in $\boldsymbol{G}$ is $\boldsymbol{H}$, we have $\kappa \in H$. Therefore, $\kappa \in H \cap K$ and the classes of P and $P\kappa$ in $\mathrm{Sh}_{\boldsymbol{H}}(H \cap K)(\mathbb{C})$ are forced to be equal. This implies that $\iota_{H\cap K,K}$ is injective on $\mathbb{C}$-points and is therefore a closed immersion. □

Remark 2.8. While we exclusively work in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$, many of our results hold true over canonical models over the reflex field of $\boldsymbol{H}$. For instance, (2.5) is defined over this field and the criterion of Lemma 2.7 applies if $\iota_{H\cap K,K}$ is viewed over this number field.

2.3. ***Connected components.*** Recall that we are viewing all Shimura varieties in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$. To simplify the description of the connected components of the Shimura varieties of $\boldsymbol{H}$, we assume for the rest of this note that the derived group $\boldsymbol{H}^{\mathrm{der}}$ is simply connected. Let $\boldsymbol{T} = \boldsymbol{H}/\boldsymbol{H}^{\mathrm{der}}$ and $\nu : \boldsymbol{H} \to \boldsymbol{T}$ denote the natural map. Let $\boldsymbol{T}(\mathbb{R})^{\dagger} \subset \boldsymbol{T}(\mathbb{R})$ denote the image of the real points of the center of $\boldsymbol{H}$ in the real points of $\boldsymbol{T}$ and let $\boldsymbol{T}(\mathbb{Q})^{\dagger} := \boldsymbol{T}(\mathbb{Q}) \cap \boldsymbol{T}(\mathbb{R})^{\dagger}$. Let U be a neat compact–open subgroup of H. Then $\nu(U)$ is a compact–open subgroup of $\boldsymbol{T}(\mathbb{A}_f)$ by [Milne 2005, Lemma 5.21] and neat by [Borel 2019, Corollary 17.3]. The set of connected components of $\mathrm{Sh}_{\boldsymbol{H}}(U)$ can be described via an explicit bijection (of sets)

$$\pi_0(\mathrm{Sh}_{\boldsymbol{H}}(U))(\mathbb{C}) \xrightarrow{\sim} \boldsymbol{T}(\mathbb{Q})^{\dagger}\backslash\boldsymbol{T}(\mathbb{A}_f)/\nu(U) \tag{2.9}$$

once a connected component $X_{\boldsymbol{H}}^{+} \subset X_{\boldsymbol{H}}$ in the analytic topology of $X_{\boldsymbol{H}}$ is fixed [Milne 2005, Theorem 5.17], which we do in what follows. Since $\pi_0(\mathrm{Sh}_{\boldsymbol{H}}(U))$ is a finite étale scheme and we are working over $\overline{\mathbb{Q}}$, there is no harm in identifying it with

$$\pi_0(\mathrm{Sh}_{\boldsymbol{H}}(U))(\overline{\mathbb{Q}}) \xrightarrow{\sim} \pi_0(\mathrm{Sh}_{\boldsymbol{H}}(U))(\mathbb{C}),$$

and we will merely view these as sets without a scheme structure. We however do view $\pi_0(\mathrm{Sh}_{\boldsymbol{H}}(U))$ as an abelian group by transport of structure via (2.9), and will denote this abelian group by $\pi_{0,U}$ for the rest of this note for simplicity.

Definition 2.10. By Y_U°, we denote the connected component of $\mathrm{Sh}_{\boldsymbol{H}}(U)$ over the identity element $[1] \in \pi_{0,U}$, which we consider as an object in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$. If V is a compact–open subgroup of U, we denote by $\varphi_{V,U} : \pi_{0,V} \to \pi_{0,U}$ the quotient homomorphism.

For convenience, we will write Y° for Y_U° when the level is apparent from context. We observe that the connected components of $\mathrm{Sh}_{\boldsymbol{H}}(U)$ are equidimensional of dimension $\dim \mathrm{Sh}_{\boldsymbol{H}}(U)$. It is easy to see

that the twisting isomorphism $[h]:\mathrm{Sh}_{\boldsymbol{H}}(U)\to\mathrm{Sh}_{\boldsymbol{H}}(h^{-1}Uh)$ sends Y_U° to the connected component of $\mathrm{Sh}_{\boldsymbol{H}}(h^{-1}Uh)$ indexed by the class of $\nu(h)$ in $\pi_{0,h^{-1}Uh}=\pi_{0,U}$.

Remark 2.11. Each component in $\pi_{0,U}$ is defined over a finite abelian extension of the reflex field, which is determined by an explicit Shimura–Deligne reciprocity law, and the eventual expression we derive for the Hecke action in Question 1.4 is defined over the compositum of the fields of definition of the cycles involved. This is, however, not relevant to the issues considered in this note.

3. Abstract pushforwards

In this section, we carefully define the cycles $z_K(-)$ from the introduction and consider their analogs in the more general setting of abstract pushforwards, which model the behavior of any suitable cohomology theory for Shimura varieties.

3.1. ***Special cycles.*** We maintain the notation and assumptions of Section 2. For each integer $m\geq 0$ and neat compact–open subgroup K of G, we denote by $\mathscr{C}^m(K)$ the free abelian group on the set of codimension m closed integral $\overline{\mathbb{Q}}$-subschemes of $\mathrm{Sh}_{\boldsymbol{G}}(K)$. We refer to elements of $\mathscr{C}^m(K)$ as *algebraic cycles of codimension m on* $\mathrm{Sh}_{\boldsymbol{G}}(K)$. Recall from (2.2) that n denotes the codimension of Shimura varieties of $\boldsymbol{H}$ in those of $\boldsymbol{G}$. Recall also that $\iota_{H\cap K,K}$ of (2.5) is finite and therefore proper. In particular, $\iota_{H\cap K,K}$ is a closed map.

Definition 3.1. For $g\in G$, we denote by $z_K(g)\in\mathscr{C}^n(K)$ the algebraic cycle on $\mathrm{Sh}_{\boldsymbol{G}}(K)$ given by the fundamental cycle of the reduced closed subscheme of $\mathrm{Sh}_{\boldsymbol{G}}(K)$ whose underlying topological space is given by the (necessarily irreducible and closed) image of $Y^\circ_{H\cap gKg^{-1}}$ under the morphism

$$\mathrm{Sh}_{\boldsymbol{H}}(H\cap gKg^{-1})\xrightarrow{\iota}\mathrm{Sh}_{\boldsymbol{G}}(gKg^{-1})\xrightarrow{[g]}\mathrm{Sh}_{\boldsymbol{G}}(K). \tag{3.2}$$

We refer to these as *irreducible special cycles*. We denote by $\mathscr{Z}_K$ the $\mathbb{Z}$-submodule of $\mathscr{C}^n(K)$ spanned by $z_K(g)$ for $g\in G$.

One may equivalently describe $z_K(g)$ as the fundamental cycle associated with the integral $\overline{\mathbb{Q}}$-subscheme of $\mathrm{Sh}_{\boldsymbol{G}}(K)$ whose $\mathbb{C}$-points are the image of $X^+_{\boldsymbol{H}}\times g$ in $\mathrm{Sh}_{\boldsymbol{G}}(K)(\mathbb{C})$. Then it is easily seen that the twisting isomorphism $[h]_{K,h^{-1}Kh}$ for any $h\in G$ sends $z_K(g)$ to $z_{h^{-1}Kh}(gh)$.

It is clear that the $z_K(g)$ are not uniquely labeled by $g\in G$. For instance, $z_K(g)=z_K(g\kappa)$ for any $\kappa\in K$. Let $\boldsymbol{H}(\mathbb{R})_+$ denote the stabilizer of $X^+_{\boldsymbol{H}}$ in $\boldsymbol{H}(\mathbb{R})$ and set

$$\boldsymbol{H}(\mathbb{Q})_+:=\boldsymbol{H}(\mathbb{R})_+\cap\boldsymbol{H}(\mathbb{Q}).$$

From the explicit description of $\mathbb{C}$-points underlying the scheme of $z_K(g)$, it is also clear that $z_K(g)=z_K(hg)$ for any $h\in\boldsymbol{H}(\mathbb{Q})_+$. Thus, the cycles $z_K(g)$ can be indexed by $\boldsymbol{H}(\mathbb{Q})_+\backslash G/K$ and we obtain a $\mathbb{Z}$-linear surjection

$$\Phi:\mathscr{S}_{\mathbb{Z}}(\boldsymbol{H}(\mathbb{Q})_+\backslash G/K)\to\mathscr{Z}_K, \tag{3.3}$$

where $\mathcal{S}_{\mathbb{Z}}(X)$ for a set X denotes the free $\mathbb{Z}$-module on the elements of X. Since G/K is discrete, we are free to replace $\boldsymbol{H}(\mathbb{Q})_+$ by its topological closure inside G in (3.3). If $\boldsymbol{H}^{\mathrm{der}}(\mathbb{R})$ is noncompact and contained in $\boldsymbol{H}(\mathbb{R})_+$, then this closure contains $\boldsymbol{H}^{\mathrm{der}}(\mathbb{A}_f)$ by strong approximation. In this case, one obtains a large set of relations among the labels g for $z_K(g)$, even locally at each prime.

Remark 3.4. A detailed study of such cycles for certain orthogonal Shimura varieties is done in [Cornut 2018, Section 5.14]. In [ibid., Proposition 5.1], it is shown using the Baire category theorem that a map closely related to Φ is in fact a bijection. A similar fact is established in [Jetchev 2014, Section 2.3]. See Remark 6.6 for an alternative approach to Question 1.4 when such a description is available. In general, it seems unclear what the full set of relations among the labels of these cycles is.

3.2. ***RIC functors.*** To address Question 1.4 at a greater level of generality and to streamline certain arguments, we will require some notions of functors on compact–open subgroups of locally profinite groups from [Shah 2024, Section 2]. These notions roughly coincide with those introduced in [Graham and Shah 2023, Section 2], except that the monoid "Σ" is taken to be the full group and axiom "(T2)" of [loc. cit.] is relaxed slightly. Note also that the terminology in the former has been slightly updated to acknowledge the prior widespread use of these functors (and their variations) [Lewis et al. 1981; Bley and Boltje 2004; Barwick 2017]. We briefly recall these notions below and explain their relevance in our context.

Fix any nonempty collection Υ_G of neat compact–open subgroups of $G = \boldsymbol{G}(\mathbb{A}_f)$ which is closed under intersections and conjugations by G, where any two subgroups have the same intersection with $\boldsymbol{Z}_{\boldsymbol{G}}(\mathbb{Q})$ and, for any $K, L \in \Upsilon_G$, there exists a third subgroup in Υ_G that is contained in $K \cap L$ and normal in K. It is elementary to see that such collections always exist [Shah 2024, Lemma 2.1.1]. We let $\mathcal{P}(G, \Upsilon_G)$ denote the small category whose set of objects is Υ_G and whose morphisms are given by $\mathrm{Hom}_{\mathcal{P}(G,\Upsilon_G)}(L, K) = \{g \in G \mid g^{-1}Lg \subset K\}$ for all $L, K \in \Upsilon_G$. A morphism will usually be written as either $(L \xrightarrow{g} K)$ or $[g]_{L,K}$, where, in the latter notation, we omit the subscripts if they are understood from context.[5] Composition in $\mathcal{P}(G, \Upsilon_G)$ is given by $[g] \circ [h] = [hg]$. If $1 = 1_G$ denotes the identity of G, the morphism $[1]_{L,K}$ is also denoted by $\mathrm{pr}_{L,K}$. Fix a commutative ring with identity R.

Definition 3.5. A *RIC functor M on (G, Υ_G) valued in R-*Mod is a pair of covariant functors $M^* : \mathcal{P}(G, \Upsilon_G)^{\mathrm{op}} \to R\text{-Mod}$ and $M_* : \mathcal{P}(G, \Upsilon_G) \to R\text{-Mod}$ satisfying the following three conditions:

(C1) $M^*(K) = M_*(K)$ for all $K \in \Upsilon_G$. This common R-module is denoted by $M(K)$.

(C2) For all $K \in \Upsilon_G$ and $g \in G$,

$$(gKg^{-1} \xrightarrow{g} K)^* = (K \xrightarrow{g^{-1}} gKg^{-1})_* \in \mathrm{Hom}_{R\text{-Mod}}(M(K), M(gKg^{-1})).$$

Here, for a morphism $\phi \in \mathcal{P}(G, \Upsilon_G)$, we let $\phi_* := M_*(\phi)$ and $\phi^* := M^*(\phi)$.

(C3) $[\gamma]_{K,K,*} : M(K) \to M(K)$ is the identity map for all $K \in \Upsilon_G$ and $\gamma \in K$.

[5] A priori, the notation $[g]$ here conflicts with (2.3). However, $\mathcal{P}(G, \Upsilon_G)$ can be canonically identified with the corresponding system of Shimura varieties and their degeneracy maps. This justifies our abuse of notation.

The pair of functors above will be denoted simply as $M : \mathcal{P}(G, \Upsilon_G) \to R\text{-Mod}$. We say that M is

(G) *Galois* if $\mathrm{pr}^*_{L,K}$ injects $M(K)$ onto $M(L)^{K/L}$ for all $L, K \in \Upsilon_G$ such that $L \triangleleft K$.

(Co) *cohomological* if $\mathrm{pr}_{L,K,*} \circ \mathrm{pr}^*_{L,K} = [K : L] \cdot \mathrm{id}$ for all $L, K \in \Upsilon_G$ with $L \subset K$.

(M) *Mackey* if, for all $K, L, L' \in \Upsilon_G$ with $L, L' \subset K$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_\delta M(L'_\delta) & \xrightarrow{\sum [\delta]_*} & M(L) \\ {\scriptstyle \oplus \mathrm{pr}^*}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ M(L') & \xrightarrow[\mathrm{pr}_*]{} & M(K) \end{array} \tag{3.6}$$

where the direct sum is over $\delta \in L' \backslash K / L$, and $L'_\delta := L' \cap \delta L \delta^{-1} \in \Upsilon_G$. This condition is independent of the choice of coset representatives.

When (Co) and (M) are both satisfied, we say that M is *CoMack* for brevity.

Remark 3.7. The acronym "RIC" stands for *restriction*, *induction*, *conjugation* and may be pronounced "Ric". To any smooth left G-representation π, one can attach a RIC functor M_π by setting $M_\pi(K) = \pi^K$ (the K-invariants of π) for all $K \in \Upsilon_G$. For $L \subset K$, the map $\mathrm{pr}^* : M_\pi(K) \to M_\pi(L)$ is the inclusion $\pi^K \hookrightarrow \pi^L$, the map $\mathrm{pr}_* : M_\pi(L) \to M_\pi(K)$ is the trace $\sum_{\gamma \in K/L} \gamma \cdot (-)$, and $[g]^* : M_\pi(K) \to M_\pi(gKg^{-1})$ is given by the action of g on π. Then M_π is Galois, cohomological and Mackey. However, not all RIC functors arise in this manner, e.g., the cohomology of Shimura varieties with $\mathbb{Z}_p$-coefficients generally fails to be Galois.

Remark 3.8. The role of Υ_G in the above is primarily to restrict any given cohomology theory for $\mathrm{Sh}_{\boldsymbol{G}}$ to those levels where pushforwards and pullbacks behave in the expected way (i.e., the cohomology over varying levels constitutes a CoMack functor). This is especially relevant in the case of Hilbert modular varieties, for which axiom (SV5) of [Milne 2005] fails. We observe that some claims in [Grossi 2020, Section 6.1] do not hold for arbitrary neat levels for precisely this reason, but this can be easily remedied by restricting to levels in a collection Υ_G as above and appealing to [Shah 2024, Lemma 2.7.1].

We can similarly define these notions for $H = \boldsymbol{H}(\mathbb{A}_f)$. To speak of maps between functors on H and G, we assume that the collection

$$\Upsilon_H := \{H \cap K \mid K \in \Upsilon_{\boldsymbol{G}}\} \tag{3.9}$$

is such that any two elements in Υ_H have the same intersection with $\boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q})$. This condition is automatic if either $\boldsymbol{Z}_{\boldsymbol{H}} \subset \boldsymbol{Z}_{\boldsymbol{G}}$ or $(\boldsymbol{H}, X_{\boldsymbol{H}})$ satisfies (SV5), but is otherwise a running assumption in this note. The other analogous conditions for Υ_H however always hold.

For convenience, we will call a pair $(U, K) \in \Upsilon_H \times \Upsilon_G$ *compatible* if U is contained in K. We fix for the rest of this subsection RIC functors $N : \mathcal{P}(H, \Upsilon_H) \to R\text{-Mod}$ and $M : \mathcal{P}(G, \Upsilon_G) \to R\text{-Mod}$.

Definition 3.10. A *pushforward* $\iota_* : N \to M$ is a collection of R-module homomorphisms $\iota_{U,K,*} : N(U) \to M(K)$ indexed by compatible pairs (U, K) such that, if $(V, L) \in \Upsilon_H \times \Upsilon_G$ is another such

pair and $h \in H$ labels morphisms $\phi = [h]_{V,U} \in \mathcal{P}(H, \Upsilon_H)$ and $\psi = [h]_{L,K} \in \mathcal{P}(G, \Upsilon_G)$, we have $\psi_* \circ \iota_{V,L,*} = \iota_{U,K,*} \circ \phi_*$.

Definition 3.11. Suppose $\iota_* : N \to M$ is a pushforward. For any $U \in \Upsilon_H$, $K \in \Upsilon_G$ and $\sigma \in G$, the *mixed Hecke correspondence* $[U\sigma K]_* : N(U) \to M(K)$ is defined as the composition

$$N(U) \xrightarrow{\mathrm{pr}^*} N(U \cap \sigma K \sigma^{-1}) \xrightarrow{\iota_*} M(\sigma K \sigma^{-1}) \xrightarrow{[\sigma]_*} M(K).$$

The *degree* of $[U\sigma K]_*$ is defined to be the index $[H \cap \sigma K \sigma^{-1} : U \cap \sigma K \sigma^{-1}]$ and denoted by $\deg[U\sigma K]_*$. We say that the pushforward ι_* is *Mackey* if, for all $V \in \Upsilon_H$ and $L, K \in \Upsilon_G$ satisfying $V, L \subset K$, we have $\mathrm{pr}^*_{L,K} \circ \iota_{V,K,*} = \sum_\gamma [V\gamma L]_*$, where γ runs over $V\backslash K/L$.

Note 3.12. Suppose $\boldsymbol{H} = \boldsymbol{G}$ in the above situation. Then there is a natural pushforward $M \to M$ given by ϕ_* for various $\phi \in \mathcal{P}(G, \Upsilon_G)$, and it is easy to see that M is Mackey if and only if this natural pushforward is. Given $K, K' \in \Upsilon_G$ and $\sigma \in G$, we refer to $[K\sigma K']_* : M(K) \to M(K')$ defined with respect to this pushforward as a *covariant Hecke correspondence* and $[K'\sigma K] := [K\sigma^{-1}K']_* : M(K) \to M(K')$ as a *contravariant Hecke correspondence*. Then our notion of degree applied to these recovers the usual notion of degrees of Hecke correspondences.

3.3. ***Pushforwards of cycles.*** We now specialize the notions of Section 3.2 to the situation of interest. Recall that, for $L \in \Upsilon_G$, $\mathscr{C}^n(L)$ denotes the abelian group of algebraic cycles of codimension n on $\mathrm{Sh}_{\boldsymbol{G}}(L) \in \mathbf{Sch}_{\overline{\mathbb{Q}}}$. Since the maps in the inverse system of Shimura varieties for $\boldsymbol{G}$ are all finite étale, the functoriality results established in [Stacks 2005–, Lemmas 02RD and 02R5] apply and we see that the groups $\mathscr{C}^n(L)$ for varying L assemble into a RIC functor

$$\mathscr{C}^n : \mathcal{P}(G, \Upsilon_G) \to \mathbb{Z}\text{-Mod}. \tag{3.13}$$

For $V \in \Upsilon_H$, we let $N(V)$ denote the free abelian group on the fundamental cycles indexed by $\pi_{0,V}$ from Section 2.3. For the same reasons, $N(V)$ for varying V assemble into a RIC functor

$$N : \mathcal{P}(H, \Upsilon_H) \to \mathbb{Z}\text{-Mod}. \tag{3.14}$$

Moreover, there exists a pushforward $\mathrm{cyc}_* : N \to \mathscr{C}^n$ in the sense of Definition 3.10 given by proper pushforward of algebraic cycles, again by [Stacks 2005–, Lemma 02R5].

Lemma 3.15. *The functors N and $\mathscr{C}^n$ are CoMack and Galois, and the pushforward cyc_* is Mackey.*

Proof. That N and $\mathscr{C}^n$ are Mackey follows by applying [Stacks 2005–, Lemma 02RG] to the diagram in [Shah 2024, Corollary 2.7.3], which is Cartesian in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$ by [ibid., Lemma 2.7.4]. That cyc_* is Mackey follows by similar considerations and our assumptions on Υ_H (which guarantee that the analog of the second commutative diagram in the proof of [Graham and Shah 2023, Proposition 4.12] in our context is also Cartesian). That N and $\mathscr{C}^n$ are cohomological follows by [Stacks 2005–, Lemma 02RH]. So it only remains to see that these functors are Galois. Since $\mathscr{C}^n$ specializes to N when $\boldsymbol{H} = \boldsymbol{G}$, it suffices to focus

on $\mathscr{C}^n$. So suppose that $L, K \in \Upsilon_G$ with $L \lhd K$. Since $\mathscr{C}^n$ is cohomological,

$$\mathrm{pr}^*_{L,K} : \mathscr{C}^n(K) \to \mathscr{C}^n(L)$$

is necessarily injective (as its postcomposition with $\mathrm{pr}_{L,K,*}$ is multiplication by $[K : L]$). It is also clear that the image of $\mathrm{pr}^*_{L,K}$ lands in the K/L-invariants of $\mathscr{C}^n(L)$. Finally, the surjectivity of $\mathrm{pr}^*_{L,K}$ onto $\mathscr{C}^n(L)^{K/L}$ follows by Galois descent for algebraic cycles with integral coefficients. Since we were unable to find a satisfactory reference for this fact, we provide a full proof below.

Let us momentarily write $X := \mathrm{Sh}_{\boldsymbol{G}}(L)$, $Y := \mathrm{Sh}_{\boldsymbol{G}}(K)$, $f : X \to Y$ for the degeneracy map induced by the inclusion $L \hookrightarrow K$, and $\Gamma := K/L$. Then f is a Galois covering with Galois group Γ in the sense of [Bosch et al. 1990, Section 6.2, Example B], i.e., the induced map $X \times \Gamma \to X \times_Y X$ given by $(x, \gamma) \mapsto (x, x\gamma)$ is an isomorphism (where Γ is viewed as a constant étale group scheme). Let Z_0 be a closed integral subscheme of X of codimension n, and Z be the scheme-theoretic union of the distinct Γ-conjugates of Z_0. Then Z is a closed and reduced subscheme of X and its ideal sheaf $\mathscr{I}$ is Γ-invariant under the induced action $\Gamma \times \mathbb{O}_X \to \mathbb{O}_X$. It therefore descends to a quasicoherent sheaf of ideals $\mathscr{J}$ for $\mathbb{O}_Y$ by [Bosch et al. 1990, Section 6.1, Theorem 4]. If W denotes the closed subscheme of Y corresponding to $\mathscr{J}$, we have $f^{-1}(W) = Z$ scheme-theoretically by construction. Then f^* sends the fundamental cycle $[W]$ to the fundamental cycle $[Z]$ by [Fulton 1998, Lemma 1.7.1].[6] Since cycles of the form $[Z]$ span $\mathscr{C}^n(L)^\Gamma$, the surjectivity of $f^* = \mathrm{pr}^*_{L,K}$ onto $\mathscr{C}^n(L)^\Gamma$ follows. □

Remark 3.16. The Galois descent for cycles with integral coefficients presumably follows from [Anschütz 2017]. See also [Poonen 2017, Section 7.6.2] for an argument similar to ours in the context of fields. For rational coefficients, Galois descent follows by [Shah 2024, Corollary 2.1.12].

One may also make similar considerations when $\mathscr{C}^n$ is replaced by Chow groups, or with the p-adic étale cohomology with coefficients as in [Graham and Shah 2023, Section 5.1]. The conclusions of Lemma 3.15 remain valid, except that the target functor is no longer necessarily Galois. However, it still makes sense to pose the analog of Question 1.4 in these settings. To make our results applicable to such situations, we will work with an arbitrary CoMack functor on (G, Υ_G) and an arbitrary Mackey pushforward to this CoMack functor. The situation of Question 1.4 can be recovered by specializing to $\mathscr{C}^n$ and cyc_*.

To this end, let R be any commutative ring with identity and $N_R := N \otimes_{\mathbb{Z}} R$ be the RIC functor on (H, Υ_H) obtained by base change, i.e., $N_R(V) := N(V) \otimes_{\mathbb{Z}} R$ for all $V \in \Upsilon_H$. For any $V \in \Upsilon_H$, denote by $[Y^\circ] \in N_R(V)$ the fundamental cycle associated with $Y^\circ = Y^\circ_V$ introduced in Definition 2.10.

Definition 3.17. Let $M_R : \mathcal{P}(G, \Upsilon_G) \to R\text{-Mod}$ be any RIC functor and $\iota_* : N_R \to M_R$ be any Mackey pushforward. For $g \in G$ and $K \in \Upsilon_G$, we define $y_K(g) \in M_R(K)$ to be the image of $[Y^\circ] \in N_R(H \cap gKg^{-1})$ under the composition

$$N_R(H \cap gKg^{-1}) \xrightarrow{\iota_*} M_R(gKg^{-1}) \xrightarrow{[g]_*} M_R(K). \tag{3.18}$$

[6] The scheme W can be shown to be reduced and irreducible, but we do not need this.

Similarly, we define $x_K(g) \in M_R(K)$ to be the image of the fundamental cycle $[\mathrm{Sh}_{\boldsymbol{H}}(H \cap gKg^{-1})] \in N_R(H \cap gKg^{-1})$ under (3.18).

Remark 3.19. Since the fundamental cycle of $\mathrm{Sh}_{\boldsymbol{H}}(V)$ is a formal sum of cycles indexed by $\pi_{0,V}$ for any $V \in \Upsilon_H$, it is not hard to see that each $x_K(g)$ is a formal sum of various $y_K(-)$. Indeed, say Z is a (geometric) connected component of $\mathrm{Sh}_{\boldsymbol{H}}(H \cap gKg^{-1})$ and $h \in H$ is such that Z is indexed by $[\nu(h)] \in \pi_{0,H\cap gKg^{-1}}$. Then the image of $[Z] \in N_R(H \cap gKg^{-1})$ under (3.18) is equal to $y_K(hg)$. Thus, $x_K(g)$ equals the sum of $y_K(hg)$ as h varies over representatives for $\pi_{0,H\cap gKg^{-1}}$.

As already noted, we consider a general ring R and a CoMack functor M_R in order to capture the various pushforward constructions and cohomology theories one may consider. For instance, one may take $R = \mathbb{Z}_p$ and

$$M_{\mathbb{Z}_p}(L) := \mathrm{H}^{2n}_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(L), \mathbb{Z}_p(n)),$$

where the right-hand side denotes Jannsen's continuous étale cohomology [1988]. Indeed, $N_{\mathbb{Z}_p}(V)$ can be identified with $\mathrm{H}^0_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{H}}(V), \mathbb{Z}_p)$ and the pushforward is obtained via the Gysin triangle in Ekedahl's "derived" category of constructible $\mathbb{Z}_p$-sheaves as in [Graham and Shah 2023, Appendix A]. The relevant properties of this construction can be established as in [ibid., Section 4], and the failure of (SV5) can be handled by [Shah 2024, Corollary 2.7.3]. Note however that $M_{\mathbb{Z}_p}$ in this case is *not* necessarily Galois.

Remark 3.20. The inductive limit $\varinjlim_L M_R(L)$ for $L \in \Upsilon_G$ over restriction maps is naturally a smooth G-representation, where an element $g \in G$ acts by $[g]^*$. The Mackey axiom for M_R implies that

$$[K\sigma K]_* \cdot x = \sum_{\delta \in K \backslash K\sigma K} \delta^{-1} \cdot x$$

for any $x \in M_R(K)$, where the equality is viewed in the inductive limit. This is the analog of (1.2) from the introduction. See [Shah 2024, Corollary 2.4.3] or [Graham and Shah 2023, Lemma 2.7(a)] for a justification.

3.4. ***A comparison.*** Assume for this subsection only that $R = \mathbb{Z}$, $M_R = \mathscr{C}^n$ and $\iota_* = \mathrm{cyc}_*$. Then $y_K(g) \in \mathscr{C}^n(K)$ (Definition 3.17) is not necessarily equal to $z_K(g)$ (Definition 3.1), but the two are very closely related.

Lemma 3.21. *There exists a unique positive integer $d_{g,K}$ such that $y_K(g) = d_{g,K} \cdot z_K(g)$. Moreover, if the morphism $\iota_{H\cap gKg^{-1}, gKg^{-1}}$ is a closed immersion, $d_{hg,K} = 1$ for all $h \in H$.*

Proof. The first part follows by [Fulton 1998, Section 1.4]. More precisely, $d_{g,K}$ is the degree of the field extension of the function fields that corresponds to the dominant morphism of integral schemes

$$Y^\circ_{H\cap gKg^{-1}} \to \iota_{g,K}(Y^\circ_{H\cap gKg^{-1}})^{\mathrm{red}}, \tag{3.22}$$

where $\iota_{g,K}$ denotes $\iota_{H\cap gKg^{-1}, gKg^{-1}}$ for simplicity and the right-hand side of (3.22) denotes the reduced induced closed subscheme of $\mathrm{Sh}_{\boldsymbol{G}}(gKg^{-1})$ on the image of Y° under $\iota_{g,K}$. It follows that $d_{g,K} = 1$ if

$\iota_{g,K}$ is a closed immersion. In that case, $\iota_{hg,K}$ is a closed immersion for any $h \in H$ as well, since we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{\boldsymbol{H}}(H \cap hgK(hg)^{-1}) & \xrightarrow{\iota} & \mathrm{Sh}_{\boldsymbol{G}}(hgK(hg)^{-1}) \\ {\scriptstyle [h]}\downarrow & & \downarrow{\scriptstyle [h]} \\ \mathrm{Sh}_{\boldsymbol{H}}(H \cap gKg^{-1}) & \xrightarrow{\iota} & \mathrm{Sh}_{\boldsymbol{G}}(gKg^{-1}) \end{array}$$

whose vertical arrows are isomorphisms. □

Remark 3.23. Since the isomorphism class of the map (3.22) depends only on the class of g in $\boldsymbol{H}(\mathbb{Q})_+\backslash G/K$, we have $d_{g,K} = d_{hg\kappa,K}$ for all $h \in \boldsymbol{H}(\mathbb{Q})_+$ and $\kappa \in K$. We also observe that the cycle $x_K(g)$ from Definition 3.17 in this case is closely related to, but not exactly the same as, the "natural cycle" defined in [Kudla 1997, Section 2], since the former is a sum of various $y_K(-)$ (see Remark 3.19), whereas the latter is a sum of various $z_K(-)$.

4. The formula

In this section, we derive our formula for Hecke action on the classes introduced in Definition 3.17 and the special cycles of Definition 3.1. We also highlight two scenarios where the resulting expression simplifies and agrees with (1.5).

4.1. ***The computation.*** The notation, conventions and assumptions introduced in Sections 2 and 3 are maintained. In particular, the derived group of $\boldsymbol{H}$ is assumed to be simply connected and the existence of a collection Υ_H in (3.9) whose elements have the same intersection with $\boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q})$ is also assumed. Recall also that all our Shimura varieties are viewed in $\mathbf{Sch}_{\overline{\mathbb{Q}}}$.

As in Section 3.3, we fix for all of this section a commutative ring R with identity, a CoMack functor $M_R : \mathcal{P}(G, \Upsilon_G) \to R\text{-Mod}$, and a Mackey pushforward $\iota_* : N_R \to M_R$, where N_R denotes the base change of N (see (3.14)) to R. We also fix a compact–open subgroup $K \in \Upsilon_G$ and two elements $g, \sigma \in G$. Our main goal in this subsection is to compute an expression for

$$[K\sigma K]_* \cdot y_K(g) \in M_R(K)$$

in terms of $y_K(-)$ from Definition 3.17. Here, $[K\sigma K]_*$ is as in Note 3.12.

Lemma 4.1. *Let $K' \in \Upsilon_G$ denote gKg^{-1}. Then $[K\sigma K]_* \cdot y_K(g) = [K'g\sigma K]_* \cdot y_{K'}(1)$.*

Proof. This follows by unraveling the definitions. □

Lemma 4.1 reduces our problem to computing $[K'\varsigma K]_* \cdot y_{K'}(1) \in M_R(K)$ for arbitrary $K' \in \Upsilon_G$ and $\varsigma \in G$, which we also fix for all of this subsection.[7] For this purpose, we introduce the following notation.

Notation 4.2. We let U denote the intersection $H \cap K' \in \Upsilon_H$ and I denote the finite double coset space $U\backslash K'\varsigma K/K$. For each $i \in I$, we let $\varsigma_i \in G$ denote a representative element for i.

[7] Later on, we will specialize to $K' = gKg^{-1}$ and $\varsigma = g\sigma$.

Lemma 4.3. *We have* $[K'\varsigma K]_* \cdot y_{K'}(1) = \sum_{i\in I}[U\varsigma_i K]_*([Y_U^\circ])$, *where* $[U\varsigma_i K]_*$ *denotes the mixed Hecke correspondence.*

Proof. It is elementary to deduce from the Mackey axiom for ι_* that $[K'\varsigma K]_* \circ \iota_{U,K',*} = \sum_{i\in I}[U\varsigma_i K]_*$ as R-linear maps $N_R(U) \to M_R(K)$ (see [Shah 2024, Lemma 2.5.7]). The claim now follows since $y_{K'}(1) = \iota_{U,K',*}([Y_U^\circ])$ by definition. □

Lemma 4.3 in turn reduces our problem to computing the effect of certain mixed Hecke correspondences on $[Y_U^\circ]$. It will be convenient to make the following notational convention.

Notation 4.4. For any subgroup X of H and any element $\tau \in G$, we denote by $X_{\tau,K}$ the intersection $X \cap \tau K \tau^{-1}$. We will write X_τ for $X_{\tau,K}$ if K is fixed in context.

Unraveling the definition of $[U\varsigma_i K]_*$, we obtain the following commutative diagram for each $i \in I$:

$$\begin{array}{ccccccc} & & & & N_R(U_{\varsigma_i}) & & \\ & & & \overset{\mathrm{pr}^*}{\nearrow} & \downarrow{\scriptstyle \mathrm{pr}_*} & \searrow^{\iota_*} & \\ N_R(U) & & & & N_R(H_{\varsigma_i}) & \xrightarrow{\iota_*} & M_R(\varsigma_i K \varsigma_i^{-1}) \xrightarrow{[\varsigma_i]_*} M_R(K) \\ & & & \searrow & & & \nearrow \\ & & & & [U\varsigma_i K]_* & & \end{array} \tag{4.5}$$

We wish to compute the effect of the individual maps in diagram (4.5). To this end, recall (Section 2.3) that, for $V \in \Upsilon_H$, $\pi_{0,V}$ is an abelian group that parametrizes the connected components of $\mathrm{Sh}_{\boldsymbol{H}}(V)$, and, for $V' \in \Upsilon_H$ contained in V, $\varphi_{V',V} : \pi_{0,V'} \to \pi_{0,V}$ denotes the quotient morphism.

Notation 4.6. For any $\tau \in G$, we let A_τ denote a set of representatives in H for the kernel of the homomorphism $\varphi_{U_\tau,U} : \pi_{0,U_\tau} \to \pi_{0,U}$.

Here the representatives are picked under the composition $H \xrightarrow{\nu} \boldsymbol{T}(\mathbb{A}_f) \to \pi_{0,U_\tau}$, which is surjective by [Milne 2005, Lemma 5.21] and simple connectedness of $\boldsymbol{H}^{\mathrm{der}}$.

Lemma 4.7. *For any* $\tau \in G$,

$$\mathrm{pr}^*_{U_\tau,U}([Y_U^\circ]) = \sum_{h\in A_\tau}[h]_*([Y^\circ_{hU_\tau h^{-1}}]) \in N_R(U_\tau),$$

where $[h]$ *on the right-hand side above denotes the morphism* $(hU_\tau h^{-1} \xrightarrow{h} U_\tau) \in \mathcal{P}(H, \Upsilon_H)$.

Proof. Let $\eta \in H$. Recall (Section 2.3) that the twisting isomorphism $[h] : \mathrm{Sh}_{\boldsymbol{H}}(hU_\tau h^{-1}) \xrightarrow{\sim} \mathrm{Sh}_{\boldsymbol{H}}(U_\tau)$ sends the component indexed by the class of η in $\pi_{0,hU_\tau h^{-1}} = \pi_{0,U_\tau}$ to the component of $\mathrm{Sh}_{\boldsymbol{H}}(U_\tau)$ indexed by the class of $h\eta$ (or that of ηh) in π_{0,U_τ}. So the right-hand side of the equality above is just the formal sum of the fundamental cycles corresponding to the connected components of $\mathrm{Sh}_{\boldsymbol{H}}(U_\tau)$ indexed by A_τ. We argue that this also equals the left-hand side. Since the morphism $\mathrm{pr}_{U_\tau,U} : \mathrm{Sh}_{\boldsymbol{H}}(U_\tau) \to \mathrm{Sh}_{\boldsymbol{H}}(U)$ is étale, so is its pullback $\mathrm{pr}^{-1}_{U_\tau,U}(Y_U^\circ) \to Y_U^\circ$ along $Y_U^\circ \to \mathrm{Sh}_{\boldsymbol{H}}(U)$. This implies that $\mathrm{pr}^{-1}_{U_\tau,U}(Y_U^\circ)$ is reduced [Stacks 2005–, Tag 03PC]. So the scheme $\mathrm{pr}^{-1}_{U_\tau,U}(Y_U^\circ)$ is equal to the disjoint union of the components of $\mathrm{Sh}_{\boldsymbol{H}}(U_\tau)$ indexed by A_τ. The claim now follows by definition of flat pullback [Fulton 1998, Sections 1.7 and 1.5]. □

Next we need a result for degrees of maps between connected components of $\mathrm{Sh}_{\boldsymbol{H}}$.

Lemma 4.8. *Let $V_1, V_2 \in \Upsilon_H$ with $V_2 \subset V_1$ and $\eta \in \boldsymbol{T}(\mathbb{A}_f)$. For $j = 1, 2$, let Y_{η,V_j} denote the component of $\mathrm{Sh}_{\boldsymbol{H}}(V_j)$ indexed by the class of η in π_{0,V_j} and let e denote the cardinality of the kernel of $\varphi_{V_2,V_1} : \pi_{0,V_2} \to \pi_{0,V_1}$. Then the natural map $Y_{\eta,V_2} \to Y_{\eta,V_1}$ is finite étale of degree $[V_1 : V_2]/e$.*

Proof. Let W be a compact–open subgroup of V_2 such that $W \triangleleft V_1$. To ease notation, we let V denote an element of $\{V_1, V_2\}$. By enlarging W if necessary, we may assume that $W \cap \boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q}) = V \cap \boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q})$, so that the map $\mathrm{Sh}_{\boldsymbol{H}}(W) \to \mathrm{Sh}_{\boldsymbol{H}}(V)$ is a Galois cover with Galois group V/W. Since $Y_{\eta,V} \hookrightarrow \mathrm{Sh}_{\boldsymbol{H}}(V)$ is an open immersion,

$$Z := \mathrm{pr}_{W,V}^{-1}(Y_{\eta,V}) \to Y_{\eta,V}$$

is a Galois cover of degree $[V:W]$ as well. Let e_V denote the cardinality of $\ker(\varphi_{W,V})$ and let $\upsilon_1, \ldots \upsilon_{e_V} \in V$ be representatives of $\ker(\varphi_{W,V})$. Then Z is the union of components of $\mathrm{Sh}_{\boldsymbol{H}}(W)$ indexed by the classes $[\eta\nu(\upsilon_k)] \in \pi_{0,W}$ for $k = 1, \ldots, e_V$. Since $[\upsilon_k] : \mathrm{Sh}_{\boldsymbol{H}}(W) \to \mathrm{Sh}_{\boldsymbol{H}}(W)$ are automorphisms that act transitively on the connected components contained in Z, we see that the degree of $Y_{\eta\nu(\upsilon_k),W} \to Y_{\eta,V}$ is independent of k and therefore equal to $[V : W]/e_V$. Since $e_{V_1} = e_{V_2} \cdot e$ and $[V_1 : W] = [V_1 : V_2] \cdot [V_2 : W]$, we conclude that $Y_{\eta,V_2} \to Y_{\eta,V_1}$ is finite étale with degree as claimed. □

Remark 4.9. In the proof, we do not require $W \in \Upsilon_H$.

Corollary 4.10. *For any $\tau \in G$ and $\eta \in \pi_{0,U_\tau}$,*

$$\mathrm{pr}_{U_\tau,H_\tau,*}([Y_{\eta,U_\tau}]) = e_\tau^{-1} \deg[U\tau K]_* \cdot [Y_{\eta,H_\tau}],$$

where Y_{η,U_τ} denotes the component of $\mathrm{Sh}_{\boldsymbol{H}}(U_\tau)$ indexed by η, Y_{η,H_τ} denotes the component of $\mathrm{Sh}_{\boldsymbol{H}}(H_\tau)$ indexed by $\varphi_{U_\tau,H_\tau}(\eta) \in \pi_{0,H_\tau}$ and e_τ denotes the cardinality of $\ker(\varphi_{U_\tau,H_\tau})$.

Proof. This follows by Lemma 4.8 and the definition of proper pushforward [Fulton 1998, Section 1.4]. □

We now apply these results to the maps in diagram (4.5) by specializing to $\tau = \varsigma_i$. Let us first fix some additional notation.

Notation 4.11. For each $i \in I$, we will let U_i, H_i, A_i denote U_{ς_i}, H_{ς_i}, A_{ς_i}, respectively. We let $B_i \subset A_i$ denote a set of representatives for the image of A_i under the quotient morphism $\varphi_{U_i,H_i} : \pi_{0,U_i} \to \pi_{0,H_i}$. Here we are identifying $A_i = A_{\varsigma_i}$ with the kernel of $\varphi_{U_i,U} : \pi_{0,U_i} \to \pi_{0,U}$. We let $e_i = e_{\varsigma_i}$ denote the cardinality of $\ker(\varphi_{U_i,H_i})$ and c_i denote the cardinality of $\ker(\varphi_{U_i,H_i}) \cap \ker(\varphi_{U_i,U})$.

If we identify A_i with $\ker(\varphi_{U_i,U})$ and B_i with $\varphi_{U_i,H_i}(A_i)$, then c_i is the cardinality of the kernel of the surjective homomorphism $A_i \to B_i$, and so $c_i = |A_i|/|B_i|$. Let us also emphasize that $H_i = H_{\varsigma_i} = H \cap \varsigma_i K \varsigma_i^{-1}$ is a *compact–open* subgroup of H (despite the notational similarity with H). To make the statement of our main theorem more self-contained, we recall most of the necessary notation.

Theorem 4.12 (explicit descent). *For $K', K \in \Upsilon_G$ and $\varsigma \in G$, let $U = H \cap K'$ and $I = U\backslash K'\varsigma K/K$. For each $i \in I$, let $\varsigma_i \in G$ denote a representative for the class i and write $U_i = U \cap \varsigma_i K \varsigma_i^{-1}$, $H_i = H \cap \varsigma_i K \varsigma_i^{-1}$ and $\deg[U\varsigma_i K]_* = [H_i : U_i]$. For each i, let $A_i \subset H$ denote a set of representatives for $\ker(\varphi_{U_i,U})$, let*

$B_i \subset A_i$ denote a set of representatives for $\varphi_{U_i,H_i}(\ker(\varphi_{U_i,U}))$, and set $c_i = |A_i|/|B_i|$, $e_i = |\ker(\varphi_{U_i,H_i})|$. Then

$$[K'\varsigma K]_* \cdot y_{K'}(1) = \sum_{i\in I}\sum_{h\in B_i} c_i e_i^{-1} \deg[U\varsigma_i K]_* \cdot y_K(h\varsigma_i)$$

as elements of $M_R(K)$.

Proof. By Lemma 4.3, it suffices to compute $[U\varsigma_i K]_*([Y_U^\circ])$ for each $i \in I$. Note that the integer c_i is the number of connected components of $\mathrm{Sh}_{\boldsymbol{H}}(U_i)$ contained in $\mathrm{pr}_{U_i,U}^{-1}(Y_U^\circ)$ that collapse into a single component of $\mathrm{Sh}_{\boldsymbol{H}}(H_i)$ under pr_{U_i,H_i}. Invoking Lemma 4.7 and Corollary 4.10, we see that

$$\mathrm{pr}_{U_i,H_i,*} \circ \mathrm{pr}_{U_i,U}^*([Y_U^\circ]) = \sum_{h\in B_i} (c_i e_i^{-1} \deg[U\varsigma_i K]_*) \cdot [h]_*([Y^\circ_{H_{h\varsigma_i}}]), \tag{4.13}$$

where $[h]$ above is the morphism $(H_{h\varsigma_i} \xrightarrow{h} H_{\varsigma_i}) \in \mathcal{P}(H, \Upsilon_H)$. Now, for each $i \in I$ and $h \in B_i$, we have a commutative diagram

$$\begin{array}{ccccc} N_R(H_{h\varsigma_i}) & \xrightarrow{\iota_*} & M_R(h\varsigma_i K \varsigma_i^{-1} h^{-1}) & & \\ \wr\downarrow{\scriptstyle [h]_*} & & \wr\downarrow{\scriptstyle [h]_*} & \searrow^{[h\varsigma_i]_*} & \\ N_R(H_{\varsigma_i}) & \xrightarrow{\iota_*} & M_R(\varsigma_i K \varsigma_i^{-1}) & \xrightarrow{[\varsigma_i]_*} & M_R(K) \end{array} \tag{4.14}$$

Let us momentarily write $K_i = \varsigma_i K \varsigma_i^{-1}$ to simplify notation. Using (4.13) and the commutativity of diagrams (4.5) and (4.14), we see that

$$\begin{aligned} [U\varsigma_i K]_*([Y_U^\circ]) &= [\varsigma_i]_{K_i,K,*} \circ \iota_{U_i,K_i,*} \circ \mathrm{pr}_{U_i,U}^*([Y_U^\circ]) \\ &= [\varsigma_i]_{K_i,K,*} \circ \iota_{H_i,K_i,*} \circ \mathrm{pr}_{U_i,H_i,*} \circ \mathrm{pr}_{U_i,U}^*([Y_U^\circ]) \\ &= [\varsigma_i]_{K_i,K,*} \circ \iota_{H_i,K_i,*} \Big(\sum_{h\in B_i} (c_i e_i^{-1} \deg[U\varsigma_i K]_*) \cdot [h]_{H_{h\varsigma_i},H_i,*}([Y^\circ_{H_{h\varsigma_i}}]) \Big) \\ &= \sum_{h\in B_i} (c_i e_i^{-1} \deg[U\varsigma_i K]_*) \cdot [h\varsigma_i]_{hK_ih^{-1},K,*} \circ \iota_{H_{h\varsigma_i},hK_ih^{-1},*}([Y^\circ_{H_{h\varsigma_i}}]) \\ &= \sum_{h\in B_i} c_i e_i^{-1} \deg[U\varsigma_i K]_* \cdot y_K(h\varsigma_i). \end{aligned}$$

□

In the formula above, we may take the inner sum to be over A_i (instead of B_i) after removing c_i from the expression, since $y_K(h\varsigma_i) = y_K(h'\varsigma_i)$ for $h, h' \in A_i$ if the classes of h, h' are equal in π_{0,H_i}.

Theorem 4.12 bis. *With notation as above,*

$$[K'\varsigma K]_* \cdot y_{K'}(1) = \sum_{i\in I}\sum_{h\in A_i} e_i^{-1} \deg[U\varsigma_i K]_* \cdot y_K(h\varsigma_i).$$

One reason for preferring the first version is that a simplification occurs when $\nu(U)$ contains $\nu(H_i)$ for all $i \in I$. We record it as a lemma for ease of reference in Section 5.

Lemma 4.15. *If $\nu(U)$ contains $\nu(H_i)$ for some $i \in I$, we have $c_i = e_i$. If, moreover, $\nu(U)$ equals $\nu(H_i)$, then B_i is a singleton.*

Remark 4.16. In the setting of Corollary 4.19, suppose that $K = K_\ell K^\ell$ and $\sigma \in \boldsymbol{G}(\mathbb{Q}_\ell)$ for a rational prime ℓ at which $\boldsymbol{T}$ is unramified. Write $U = U_\ell U^\ell$, and suppose that $\nu(U_\ell)$ is the maximal compact–open subgroup of $\boldsymbol{T}(\mathbb{Q}_\ell)$. Then $\nu(U) \supseteq \nu(H_i)$ for all $i \in I$. So, in this case, the coefficients in the expression of Theorem 4.12 only involve mixed degrees. See [Shah 2024, Section 5], where several techniques were developed to aid their computation and [ibid., Part II] for several concrete examples.

If we replace $y_{K'}(1)$ with $x_{K'}(1)$, the formula is much simpler and does not require as much work.

Proposition 4.17.
$$[K'\varsigma K]_* \cdot x_{K'}(1) = \sum_{i\in I} \deg[U\varsigma_i K]_* \cdot x_K(\varsigma_i).$$

Proof. Let $N_{\mathrm{triv},R} : \mathcal{P}(H, \Upsilon_H) \to R\text{-Mod}$ denote the functor associated with the trivial representation of H (see Remark 3.7), meaning $N_{\mathrm{triv},R}(V) = R$ for all V. Since fundamental cycles of $\mathrm{Sh}_{\boldsymbol{H}}(V)$ for varying $V \in \Upsilon_H$ map to themselves under pullbacks and to multiples by degree under pushforwards along degeneracy maps, they realize the trivial functor $N_{\mathrm{triv},R}$. The class $x_K(g)$ can then be defined as the image of $1_R \in R = N_{\mathrm{triv},R}(H_g)$ under the analogous twisting map in Definition 3.17. The claim easily follows by the obvious analog of Lemma 4.3 and the diagram (4.5) with N_R replaced by $N_{\mathrm{triv},R}$. □

Remark 4.18. Proposition 4.17 holds without the assumption that $\boldsymbol{H}^{\mathrm{der}}$ be simply connected, since the definition of $x_K(-)$ etc. does not rely on a description of the connected components of $\mathrm{Sh}_{\boldsymbol{H}}$. One may also drop (SV3) for $(\boldsymbol{H}, X_{\boldsymbol{H}})$ in light of [Graham and Shah 2023, Appendix A], which extends the formalism of Shimura varieties in the absence of (SV3), *assuming* that $(\boldsymbol{H}, X_{\boldsymbol{H}})$ embeds into a datum which does satisfy (SV3). In our case, this latter datum is $(\boldsymbol{G}, X_{\boldsymbol{G}})$.

If we specialize to $K' = gKg^{-1}$ and $\varsigma = g\sigma$ in Theorem 4.12 and invoke Lemma 4.1, we obtain the following.

Corollary 4.19. *For $K \in \Upsilon_G$ and $g, \sigma \in G$, let $U = H \cap gKg^{-1}$ and $I = U\backslash gK\sigma K/K$. For each $i \in I$, let $\varsigma_i \in G$ denote a representative for i and let $U_i = U \cap \varsigma_i K \varsigma_i^{-1}$, $H_i = H \cap \varsigma_i K \varsigma_i^{-1}$ and $\deg[U\varsigma_i K]_* = [H_i : U_i]$. For each i, let $A_i \subset H$ denote a set of representatives for $\ker(\varphi_{U_i,U})$, $B_i \subset A_i$ denote a set of representatives for $\varphi_{U_i,H_i}(\ker(\varphi_{U_i,U}))$, and set $c_i = |A_i|/|B_i|$, $e_i = |\ker(\varphi_{U_i,H_i})|$. Then*
$$[K\sigma K]_* \cdot y_K(g) = \sum_{i\in I} \sum_{h\in B_i} c_i e_i^{-1} \deg[U\varsigma_i K]_* \cdot y_K(h\varsigma_i).$$

We can finally answer Question 1.4 now. Recall from Lemma 3.21 that, when $M_R = \mathscr{C}^n$ (with $R = \mathbb{Z}$ and $\iota_* = \mathrm{cyc}_*$), there exists for each $\tau \in G$ a unique positive integer $d_{\tau,K}$ such that $y_K(\tau) = d_{\tau,K} \cdot z_K(\tau)$ as elements of $\mathscr{C}^n(K)$.

Corollary 4.20. *With notation as in Corollary 4.19, we have*
$$[K\sigma K]_* \cdot z_K(g) = \frac{1}{d_{g,K}} \sum_{i\in I} \sum_{h\in B_i} c_i d_{h\varsigma_i,K} e_i^{-1} \deg[U\varsigma_i K]_* \cdot z_K(h\varsigma_i). \tag{4.21}$$

In particular, $[K\sigma K]_* \cdot z_K(g)$ lies in the submodule of $\mathscr{C}^n(K)$ spanned by irreducible special cycles.

Proof. The expression follows from Corollary 4.19 after specializing to $R = \mathbb{Z}$, $M_R = \mathscr{C}^n$ and $\iota_* = \mathrm{cyc}_*$ (which we can do by Lemma 3.15). Since $[K\sigma K]_* \cdot z_K(g)$ belongs to both $\mathscr{C}^n(K)$ (by definition) and $\mathscr{Z}_K \otimes_{\mathbb{Z}} \mathbb{Q}$ by (4.21), it must lie inside $\mathscr{Z}_K$ (where $\mathscr{Z}_K$ is as in Definition 3.1). □

Remark 4.22. The coefficients of individual summands in (4.21) are not a priori integers, since we do not know if $d_{g,K}$ divides the product $c_i d_{h\varsigma_i,K} e_i^{-1} \deg[U\varsigma_i K]_*$ for all $i \in I$, $h \in B_i$. The point is that the coefficients in our expression can be made integral (if not already) by collecting together the coefficients of all $z_K(h\varsigma_i)$ that represent the same irreducible cycle in $\mathrm{Sh}_{\boldsymbol{G}}(K)$.

4.2. ***Examples.*** Below, we record two simple instances in which the right-hand side of (4.21) matches that of (1.5). The notation above is maintained.

Example 4.23. Suppose $\boldsymbol{H}$ is a torus. In this case, we are asking for a Hecke action on special points on $\mathrm{Sh}_{\boldsymbol{G}}(K)$. We have $\boldsymbol{H} = \boldsymbol{T}$ and ν is the identity map. Our assumption on Υ_H implies that A_i identifies with U/U_i and $e_i = [H_i : U_i] = \deg[U\varsigma_i K]_*$. Choose the representatives in A_i to lie in U. Moreover, $d_{\tau,K} = 1$ for all $\tau \in G$, since $\mathrm{Sh}_{\boldsymbol{H}}(H_\tau)$ is a finite set of reduced points over $\overline{\mathbb{Q}}$. Thus,

$$[K\sigma K]_* \cdot z_K(g) = \sum_{i\in I} \sum_{h\in B_i} c_i \cdot z_K(h\varsigma_i) = \sum_{i\in I} \sum_{h\in A_i} z_K(h\varsigma_i).$$

Now, for each $i \in I$, we have $U\varsigma_i K/K = \bigsqcup_{h\in A_i} h\varsigma_i K$ and therefore $gK\sigma K/K = \bigsqcup_{i\in I, h\in A_i} h\varsigma_i K$. So

$$[K\sigma K]_* \cdot z_K(g) = \sum_{\gamma \in K\sigma K/K} z_K(g\gamma),$$

which agrees with (1.5). This is of course what one gets by directly computing the result of a correspondence on a general zero-cycle on $\mathrm{Sh}_{\boldsymbol{G}}(K)$ as in (1.1).

Example 4.24. Suppose that $\boldsymbol{H} = \boldsymbol{G}$. In this case, we are asking for a Hecke action on connected components of $\mathrm{Sh}_{\boldsymbol{G}}(K)$ itself. We have $U = gKg^{-1}$, so $I = \{1\}$ is a singleton and we may take $\varsigma_1 = g\sigma$. By definition, we have $\deg[U\varsigma_1 K]_* = [\varsigma_1 K \varsigma_1^{-1} : U_{\varsigma_1}]$, which equals $[K : \varsigma_1^{-1} U_{\varsigma_1} \varsigma_1]$. Since $\varsigma_1^{-1} U_{\varsigma_1} \varsigma_1 = \sigma^{-1} K\sigma \cap K$, we see that

$$\deg[U\varsigma_1 K]_* = |K\sigma^{-1}K/K|.$$

Since $\nu(U) = \nu(K) = \nu(H_{\varsigma_1})$, we have $c_1 = e_1$ and $B_1 = \{1\}$ is a singleton by Lemma 4.15. Clearly, $d_{\tau,K} = 1$ for all $\tau \in G$. Therefore,

$$[K\sigma K]_* \cdot z_K(g) = |K\sigma^{-1}K/K| \cdot z_K(g\sigma).$$

Now $|K\sigma^{-1}K/K| = |K\sigma K/K|$ by unimodularity of G [Renard 2010, p. 58] and $z_K(g\gamma\sigma) = z_K(g\sigma)$ for any $\gamma \in K$ since $\pi_{0,K} = \pi_0(\mathrm{Sh}_{\boldsymbol{G}}(K))$ is abelian. So

$$[K\sigma K]_* \cdot z_K(g) = \sum_{\gamma \in K\sigma K/K} z_K(g\gamma),$$

which again agrees with (1.5).

5. Counterexamples

In this section, we furnish two (families of) examples where (1.5) fails to hold.

5.1. ***Counting cycles.*** Since irreducible special cycles form a $\mathbb{Z}$-module basis of $\mathscr{Z}_K$ by definition, we see that (1.5) holds only if

$$|K\sigma K/K| \stackrel{?}{=} \frac{1}{d_{g,K}} \sum_{i\in I} \sum_{h\in B_i} c_i d_{h\varsigma_i,K} e_i^{-1} \deg[U\varsigma_i K]_*.$$

Indeed, the right-hand side above is the sum of the coefficients in the right-hand side of (4.21). The strategy is therefore to compute both these integers explicitly and show they are not equal. We will pick K and σ so that the various $d_{-,K}$ appearing above are forced to be 1. For the computation of mixed degrees, we rely on the calculations done in [Shah 2024], though alternatives are also provided for the reader and the computations are mostly self-contained. In both our examples, we will have $g = 1_G$ (so $U = H \cap K$) and the letter g will be used for other purposes. For this reason, we denote the representatives ς_i for $i \in U\backslash K\sigma K/K$ by σ_i. In this notation, we are interested in checking if

$$|K\sigma K/K| \stackrel{?}{=} \frac{1}{d_{1_G,K}} \sum_{i\in I} \sum_{h\in B_i} c_i d_{h\sigma_i,K} e_i^{-1} \deg[U\sigma_i K]_*. \tag{5.1}$$

Throughout Sections 5.2 and 5.3, ℓ denotes a rational prime and $\mathbb{A}_f^\ell = \mathbb{A}_f/\mathbb{Q}_\ell$ denotes the group of finite rational adeles away from ℓ.

Remark 5.2. The counterexamples below also work if $z_K(1)$ is replaced by $x_K(1)$.

5.2. ***Symplectic groups.*** We let

$$\boldsymbol{H} = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2, \quad \boldsymbol{G} = \mathrm{GSp}_4,$$

which we consider as reductive group schemes over $\mathbb{Z}$. Here, we define $\boldsymbol{G}$ with respect to the standard symplectic matrix which has the identity matrix in the top right 2×2 block, negative identity in the bottom left 2×2 block, and zeros elsewhere. The embedding of $\boldsymbol{H}$ in $\boldsymbol{G}$ is as in [Shah 2024, Section 9.3], which gives a morphism of Shimura data (see [Milne 2005, Section 6]). Moreover, both data satisfy (SV1)–(SV6) of [Milne 2005] by the discussion in [ibid., Section 6]. In addition, the derived group of $\boldsymbol{H}$ is $\mathrm{SL}_2 \times \mathrm{SL}_2$, which is simply connected. Finally, our assumption on the existence of Υ_H is also satisfied, since the datum for $\boldsymbol{H}$ satisfies (SV5). Thus, Corollary 4.20 applies to this embedding of Shimura data. Let

$$w := \mathrm{diag}(1, -1, 1, -1) \in \boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q}).$$

Over $\mathbb{Q}$, the centralizer of w in $\boldsymbol{G}$ equals $\boldsymbol{H}$. We have $\boldsymbol{T} = \boldsymbol{H}/\boldsymbol{H}^{\mathrm{der}} = \mathbb{G}_m$, and $\nu : \boldsymbol{H} \to \boldsymbol{T}$ is the map given by taking the common determinant of the two components. As above, we write $G = \boldsymbol{G}(\mathbb{A}_f)$ and $H = \boldsymbol{H}(\mathbb{A}_f)$. For $N \geq 1$, we let $K(N)$ denote the principal congruence subgroup of G at level N. Then

$K(N)$ is neat if $N \geq 3$. Now say K is a compact–open subgroup contained in $K(N)$ for some $N \geq 3$. Since $w \in K(1) = \boldsymbol{G}(\widehat{\mathbb{Z}})$ and $K(N) \trianglelefteq K(1)$, we have $wKw^{-1} \subset wK(N)w^{-1} = K(N)$. Thus,

$$\iota_{U,K} : \mathrm{Sh}_{\boldsymbol{H}}(U) \to \mathrm{Sh}_{\boldsymbol{G}}(K)$$

is a closed immersion by Lemma 2.7. We fix such a K from now on and write as above $U := H \cap K$, etc. We let ℓ be a rational prime such that K is unramified at ℓ, i.e., $K = K_\ell K^\ell$, where $K_\ell = \mathrm{GSp}_4(\mathbb{Z}_\ell)$ and $K^\ell \subset \mathrm{GSp}_4(\mathbb{A}_f^\ell)$ is a compact–open subgroup. Then $U = U_\ell U^\ell$ with $U_\ell = \boldsymbol{H}(\mathbb{Z}_\ell)$. Set

$$\tau_\ell := \begin{pmatrix} 1 & & & 1/\ell \\ & 1 & 1/\ell & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \boldsymbol{G}(\mathbb{Q}_\ell)$$

and denote by τ the image of τ_ℓ under the embedding $\boldsymbol{G}(\mathbb{Q}_\ell) \hookrightarrow \boldsymbol{G}(\mathbb{A}_f)$, so $\tau = \tau^\ell \tau_\ell$, where $\tau^\ell \in \boldsymbol{G}(\mathbb{A}_f^\ell)$ is the identity. The convention introduced in Notation 4.4 is maintained.

Lemma 5.3. *If $\ell = 2$, the morphism $\iota : \mathrm{Sh}_{\boldsymbol{H}}(H_\tau) \to \mathrm{Sh}_{\boldsymbol{G}}(\tau K \tau^{-1})$ is a closed immersion.*

Proof. Let $L = L_\ell L^\ell$ be a neat compact–open subgroup of G that contains both K and wKw (for example, take $L = K(N)$). Clearly, $L_\ell = K_\ell$ by maximality of K_ℓ. Write $w = w_\ell w^\ell$ and set $L' := \tau L \tau^{-1}$. Then L' is neat and contains $\tau K \tau^{-1}$. By Lemma 2.7, $\iota = \iota_{H_\tau, \tau K \tau^{-1}}$ is a closed immersion whenever L' contains $w\tau K \tau^{-1} w$. Note that

$$L' = L^\ell \cdot L'_\ell, \quad \text{where } L'_\ell = \tau_\ell K_\ell \tau_\ell^{-1}.$$

Now L^ℓ contains $w^\ell \tau^\ell K^\ell \tau^\ell w^\ell = w^\ell K^\ell w^\ell$ by our choice. So L' contains $w\tau K\tau^{-1}w$ if and only if $\tau_\ell K_\ell \tau_\ell^{-1}$ contains (and hence equals) $w_\ell \tau_\ell K_\ell \tau_\ell^{-1} w_\ell$, i.e., when

$$\gamma_\ell := \tau_\ell^{-1} w_\ell \tau_\ell = \begin{pmatrix} 1 & & & 2/\ell \\ & -1 & -2/\ell & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

normalizes K_ℓ. This is true for $\ell = 2$, where we even have $\gamma_\ell \in K_\ell$. □

Lemma 5.4. *For all $\eta \in \{h, h\tau \mid h \in H\}$, we have $\nu(H_\eta) = \nu(U)$.*

Proof. For any $h \in H$ and $\eta \in G$, we have $\nu(H_{h\eta}) = \nu(H_\eta)$ as $H_{h\eta} = hH_\eta h^{-1}$. So it suffices to restrict our attention to $\eta \in \{1_G, \tau\}$. The case $\eta = 1_G$ is trivial and the argument for $\eta = \tau$ is as follows. Let $\boldsymbol{A} \subset \boldsymbol{G}_{\mathbb{Q}_\ell}$ denote the maximal diagonal torus and let $A_\ell := \boldsymbol{A}(\mathbb{Q}_\ell)$, $A_{\tau_\ell}^\circ = A_\ell \cap \tau_\ell K_\ell \tau_\ell^{-1}$. Then $\mathrm{diag}(a, b, c, d) \in A_{\tau_\ell}^\circ$ if and only if

$$a, b, c, d \in \mathbb{Z}_\ell^\times, \quad ac = bd, \quad a - d, b - c \in \ell\mathbb{Z}_\ell.$$

It is then easily seen that $\nu(A_{\tau_\ell}^\circ) = \mathbb{Z}_\ell^\times$. Since H_τ contains $A_{\tau_\ell}^\circ U^\ell$, we have $\nu(H_\tau) = \mathbb{Z}_\ell^\times \nu(U^\ell) = \nu(U)$. □

Define $\sigma = \sigma^\ell \sigma_\ell \in \boldsymbol{G}(\mathbb{A}_f)$ by setting $\sigma_\ell = \mathrm{diag}(\ell, \ell, 1, 1) \in \boldsymbol{G}(\mathbb{Q}_\ell)$ and $\sigma^\ell = 1$.

Lemma 5.5. *We have* $K\sigma K = U\sigma K \sqcup U\sigma\tau K$. *Moreover*:

(a) $|K\sigma K/K| = 1+\ell+\ell^2+\ell^3$.

(b) $\deg[U\sigma K]_* = (1+\ell)^2$.

(c) $\deg[U\sigma\tau K]_* = 1$.

Proof. The first statement is [Shah 2024, Proposition 9.3.3]. Alternatively, note that, by [Taylor 1988, p. 38] or [Roberts and Schmidt 2007, p. 189], a set of representatives for $K\sigma K/K$ (in our convention) is given by

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ell \\ & & & \ell \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & \ell & & b \\ & & \ell & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} \ell & a & b & \\ & 1 & & \\ & & 1 & \\ & & -a & \ell \end{pmatrix}, \quad \begin{pmatrix} \ell & & b & a \\ & \ell & a & c \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where the entries a, b, c in each of the displayed matrices run over $0, 1, \ldots, \ell-1$. One now shows by applying elementary row and column operations that the classes of these matrices in $U\backslash G/K$ are represented by σ and $\sigma\tau$. That σ and $\sigma\tau$ represent distinct classes in $U\backslash G/K$ follows by noticing that $HK \neq H\tau K$ (which one can see by showing that $h\tau \notin K$ for any $h \in H$).

Part (a) follows from the decomposition above. Part (b) follows from [Shah 2024, Lemma 9.4.1(a)] by evaluating the function computed there at the zero matrix, and part (c) by [ibid., Lemma 9.3.2]. Alternatively, (b) can also be computed by relating $\deg[U\sigma K]_*$ to the degree of the T_ℓ Hecke operator for GL_2, and part (c) by observing that, for $h \in H$, the conditions on its matrix entries imposed by $(\sigma\tau)^{-1}h(\sigma\tau) \in K$ already force $h \in U$. □

Let $\sigma_1 := \sigma$ and $\sigma_2 := \sigma\tau$, so $I = \{1, 2\}$ is our indexing set. By our choice of K and Lemma 3.21, we have $d_{\sigma_1,K} = d_{1_G,K} = 1$. Lemmas 5.4 and 4.15 imply that we can take $B_i = \{1_H\}$ and that $c_i = e_i$ for $i = 1, 2$. Invoking Lemma 5.5, equation (5.1) reads

$$1+\ell+\ell^2+\ell^3 \stackrel{?}{=} (\ell+1)^2 + d_{\sigma_2,K}. \tag{5.6}$$

If $\ell = 2$ (e.g., take $K = K(N)$ for $N \geq 3$ odd), we have $d_{\tau,K} = 1$ by Lemma 5.3 and so $d_{\sigma_2,K} = 1$ by Lemma 3.21. But, in that case, the left-hand side of (5.6) is 15 while the right-hand side is 10.

5.3. ***Unitary groups.*** In Section 5.2, our eventual counterexample only worked for $\ell = 2$, which might seem a little unsatisfactory in terms of scope. In this section, we consider certain unitary Shimura varieties for which there is an abundance of elements w satisfying Lemma 2.7. This allows us to furnish a set of counterexamples for all primes ℓ that are split in an imaginary quadratic extension used to define the Shimura variety. Although the ideas are the same as before, a little more work is involved.

Let $E = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let $\gamma \in \mathrm{Gal}(E/\mathbb{Q})$ denote the nontrivial element. For a positive integer p, let $\mathrm{GU}(p, p)$ be the unitary similitude group over $\mathbb{Q}$ of signature (p, p), defined with respect to the Hermitian pairing over E given by the Hermitian matrix $J = \mathrm{diag}(1, -1, 1, -1, \ldots, 1, -1)$. That is, J is the $2p \times 2p$ diagonal matrix with 1 in the odd-numbered

entries and -1 in the even-numbered entries (cf. [Graham and Shah 2023, Section 3.1]). Let $c : \mathrm{GU}(p, p) \to \mathbb{G}_m$ denote the similitude map and $\det : \mathrm{GU}(p, p) \to \mathrm{GU}(1)$ the determinant. Set

$$\boldsymbol{H} = \mathrm{GU}(1,1) \times_c \mathrm{GU}(1,1), \quad \boldsymbol{G} = \mathrm{GU}(2,2)$$

and let $\iota : \boldsymbol{H} \to \boldsymbol{G}$ be the embedding $(h_1, h_2) \mapsto \mathrm{diag}(h_1, h_2)$. Then both $\boldsymbol{H}$ and $\boldsymbol{G}$ admit standard Shimura data, given by sending $z \in \mathbb{C}^\times$ to alternating copies of z and $\bar{z}$ along the diagonal. Moreover, both of these data satisfy (SV5), and ι constitutes an embedding of Shimura data. We also observe that the derived group of $\boldsymbol{H}$ is simply connected, since $(\boldsymbol{H}^{\mathrm{der}})_{\overline{\mathbb{Q}}} \simeq \mathrm{SL}_2 \times \mathrm{SL}_2$. Therefore, Corollary 4.20 applies in this context as well.

Let U_1 denote the torus of norm one elements in $E^\times$, as in [Graham and Shah 2023, Section 3.1]. Then $\boldsymbol{T} = \boldsymbol{H}/\boldsymbol{H}^{\mathrm{der}}$ is isomorphic to $\mathrm{U}_1 \times \mathbb{G}_m \times \mathrm{U}_1$ in such a way that $\nu : \boldsymbol{H} \to \boldsymbol{T}$ is identified with the map

$$\nu : \boldsymbol{H} \to \mathrm{U}_1 \times \mathbb{G}_m \times \mathrm{U}_1, \quad h = (h_1, h_2) \mapsto \left(\frac{c(h)}{\det h_1}, c(h), \frac{c(h)}{\det h_2} \right),$$

where $c(h) := c(h_1) = c(h_2)$ is the common similitude. It is easily seen that $\boldsymbol{T}(\mathbb{Q})$ is discrete in $\boldsymbol{T}(\mathbb{A}_f)$. Let $\mathscr{N} : E^\times \to \mathbb{Q}^\times$ denote the norm map and let $S \subset E^\times \setminus \{1\}$ denote the set of elements such that $\mathscr{N}(\xi) = \xi \cdot \gamma(\xi) = 1$. For any $\xi \in S$, define

$$w_\xi := \mathrm{diag}(1, 1, \xi, \xi) \in \boldsymbol{Z}_{\boldsymbol{H}}(\mathbb{Q}).$$

For any $\mathbb{Q}$-algebra R and $g \in \boldsymbol{G}(R)$, the condition $g w_\xi = w_\xi g$ forces g to be block diagonal. Thus, the centralizer of w_ξ in $\boldsymbol{G}$ equals $\boldsymbol{H}$. As before, we will consider $w_\xi \in \boldsymbol{G}(\mathbb{Q})$ as an element of G via the diagonal embedding $\boldsymbol{G}(\mathbb{Q}) \hookrightarrow \boldsymbol{G}(\mathbb{A}_f) = G$.

Let $\ell \nmid 2d$ be a fixed rational prime that is split in E. If $\jmath : E \to \mathbb{Q}_\ell$ is an embedding and $\xi \in S$ is such that $\jmath(\xi) \in 1 + \ell\mathbb{Z}_\ell$, then $\jmath \circ \gamma(\xi) \in 1 + \ell\mathbb{Z}_\ell$ as well, since $\gamma(\xi)$ is the inverse of ξ. We will refer to such $\xi \in S$ as *ℓ-invertible*. For positive integers m,

$$\xi_m := \left(\frac{1 - dm^2\ell^2}{1 + dm^2\ell^2} \right) + \left(\frac{2m\ell}{1 + dm^2\ell^2} \right) \sqrt{-d} \in S$$

are examples of such elements.

Fix now a neat compact–open subgroup K and let ℓ be a prime split in E such that K is hyperspecial at ℓ, i.e., $K = K^\ell K_\ell$ with $K_\ell = \boldsymbol{G}(\mathbb{Z}_\ell)$. For any $\xi \in S$, define $K_\xi := K \cap w_\xi^{-1} K w_\xi$. Then K_ξ is a neat compact–open subgroup of G, and both K_ξ and $w_\xi K_\xi w_\xi^{-1}$ are contained in K. If ξ is also ℓ-invertible, then $\xi \in E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$ lies in $\mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times$ and the ℓ-component $w_{\xi,\ell} \in \boldsymbol{G}(\mathbb{Q}_\ell)$ of w_ξ lies in K_ℓ. Thus, for an ℓ-invertible element $\xi \in S$, which we fix in what follows, K_ξ is hyperspecial at ℓ. The upshot of this discussion is that, by replacing K with K_ξ, we may assume that:

- K and $w_\xi K w_\xi^{-1}$ are hyperspecial (and equal) at ℓ.
- K and $w_\xi K w_\xi^{-1}$ are contained in a common neat compact–open subgroup.

Set $U := K \cap \boldsymbol{H}(\mathbb{A}_f)$. Then $\iota_{U,K}$ is a closed immersion by Lemma 2.7 and our assumptions on K.

Since ℓ is split in E, we can fix isomorphisms $\boldsymbol{H}_{\mathbb{Q}_\ell} \simeq \mathbb{G}_m \times \mathrm{GL}_2 \times \mathrm{GL}_2$ and $\boldsymbol{G}_{\mathbb{Q}_\ell} \simeq \mathbb{G}_m \times \mathrm{GL}_4$ such that the local embedding $\iota : \boldsymbol{H}_{\mathbb{Q}_\ell} \hookrightarrow \boldsymbol{G}_{\mathbb{Q}_\ell}$ is identified with the embedding

$$\mathbb{G}_m \times \mathrm{GL}_2 \times \mathrm{GL}_2 \hookrightarrow \mathbb{G}_m \times \mathrm{GL}_4, \quad (c, h_1, h_2) \mapsto (c, \mathrm{diag}(h_1, h_2)),$$

using the embedding $j : E \hookrightarrow \mathbb{Q}_\ell$. In the 4×4 matrices below, we suppress the similitude coordinate, which is 1, and write ξ for $j(\xi)$.

Let

$$\tau_\ell := \begin{pmatrix} 1 & & 1/\ell & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \boldsymbol{G}(\mathbb{Q}_\ell)$$

and let $\tau = \tau_\ell \tau^\ell \in \boldsymbol{G}(\mathbb{A}_f)$, where τ^ℓ is the identity.

Lemma 5.7. *The morphism $\iota : \mathrm{Sh}_{\boldsymbol{H}}(H_\tau) \to \mathrm{Sh}_{\boldsymbol{G}}(\tau K \tau^{-1})$ is a closed immersion.*

Proof. As in the proof of Lemma 5.3, it suffices to show that

$$\tau_\ell^{-1} w_{\xi,\ell} \tau_\ell = \begin{pmatrix} 1 & & (1-\xi)/\ell & \\ & 1 & & \\ & & \xi & \\ & & & \xi \end{pmatrix} \in \boldsymbol{G}(\mathbb{Q}_\ell)$$

lies in K_ℓ, which it does by ℓ-invertibility of ξ. □

Lemma 5.8. *For all $\eta \in \{h, h\tau \mid h \in H\}$, $\nu(H_\eta) = \nu(U)$.*

Proof. As in Lemma 5.4, it suffices to restrict to $\eta \in \{1_H, \tau\}$, and the claim for $\eta = 1_H$ is again trivial. At a split prime, the choice of an $\alpha \in \mathbb{Q}_\ell$ such that $\alpha^2 = -d$ determines compatible isomorphisms $\boldsymbol{T}_{\mathbb{Q}_\ell} \simeq \mathbb{G}_m^3$ and $\boldsymbol{H}_{\mathbb{Q}_\ell} \simeq \mathbb{G}_m \times \mathrm{GL}_2 \times \mathrm{GL}_2$ such that $\nu : \boldsymbol{H}_{\mathbb{Q}_\ell} \to \boldsymbol{T}_{\mathbb{Q}_\ell}$ is given by $(c, h_1, h_2) \mapsto (c \det h_1^{-1}, c, c \det h_2^{-1})$. Then one can show that $H_{\tau_\ell} := \boldsymbol{H}(\mathbb{Q}_\ell) \cap \tau_\ell K_\ell \tau_\ell^{-1}$ contains the subgroup

$$\{(u, \mathrm{diag}(a, b), \mathrm{diag}(a, c)) \in \mathbb{G}_m \times \mathrm{GL}_2 \times \mathrm{GL}_2 \mid a, b, c, u \in \mathbb{Z}_\ell^\times\}.$$

From this, one deduces that $\nu(H_{\tau_\ell}) = \mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times = \nu(U_\ell)$. □

Let $\sigma, \sigma' \in G$ be defined so that their components away from ℓ are the identity and at ℓ are given by $\sigma_\ell := (1, \mathrm{diag}(\ell, 1, 1, 1))$, $\sigma'_\ell = (1, \mathrm{diag}(1, 1, \ell, 1))$.

Lemma 5.9. *We have $K\sigma K = U\sigma K \sqcup U\sigma' K \sqcup U\sigma\tau K$. Moreover*:

(a) $|K\sigma K/K| = 1 + \ell + \ell^2 + \ell^3$.

(b) $\deg[U\sigma K]_* = \deg[U\sigma' K]_* = 1 + \ell$.

(c) $\deg[U\sigma\tau K]_* = 1$.

Proof. The first claim is a special case of [Shah 2024, Proposition 7.4.5]. Alternatively, note that the stratification of the projective space $\mathbb{P}^3$ over the field $\mathbb{Z}/\ell\mathbb{Z}$ implies that a set of representatives for

$K\sigma K/K$ is given by

$$\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&\ell\end{pmatrix},\quad \begin{pmatrix}1&&&\\&1&&\\&&\ell&a\\&&&1\end{pmatrix},\quad \begin{pmatrix}1&&&\\&\ell&a&b\\&&1&\\&&&1\end{pmatrix},\quad \begin{pmatrix}\ell&a&b&c\\&1&&\\&&1&\\&&&1\end{pmatrix},$$

where the $\mathbb{Q}_\ell^\times$-component (in $\boldsymbol{G}(\mathbb{Q}_\ell)\simeq\mathbb{Q}_\ell^\times\times\mathrm{GL}_4(\mathbb{Q}_\ell)$) is taken to be 1, and where the entries a, b, c in each of the displayed matrices run over $0,1,\dots,\ell-1$. One then reduces each of these matrices by appropriate row and column operations to show that the classes of these matrices in $U\backslash G/K$ are represented by σ, σ' and $\sigma\tau$. It is easily checked that $HK\neq H\tau K$, which distinguishes $U\sigma\tau K$ from $U\sigma K$ and $U\sigma' K$. To distinguish $U\sigma K$ from $U\sigma' K$, we use the Cartan decomposition for the double quotient $\boldsymbol{H}(\mathbb{Z}_\ell)\backslash\boldsymbol{H}(\mathbb{Q}_\ell)/\boldsymbol{H}(\mathbb{Z}_\ell)$ and an elementary trick established in [ibid., Lemma 5.9.2].

Part (a) follows from the decomposition above. Parts (b) and (c) can be deduced along the lines outlined in the proof of Lemma 5.5. □

Set $\sigma_1:=\sigma$, $\sigma_2:=\sigma'$, $\sigma_3:=\sigma\tau$, so $I=\{1,2,3\}$ is our indexing set. We have $d_{1_G,K}=d_{\sigma_i,K}=1$ for $i=1,2,3$ by our choice of K and Lemmas 5.7 and 3.21. Lemma 5.8 (in conjunction with Lemma 4.15) implies that we may take $B_i=\{1_H\}$ and that $c_i=e_i$ for $i=1,2,3$. Equation (5.1) then reads

$$1+\ell+\ell^2+\ell^3\overset{?}{=}(1+\ell)+(1+\ell)+1,$$

which is false for all ℓ.

6. Schwartz spaces

To have another — and a technically simpler — perspective on the failure of (1.5) in general, we investigate an auxiliary question in the setting of Schwartz spaces motivated by the discussion in Section 3.1. This section can be read independently of the rest of this note.

Let G be a locally profinite group and J be a closed subgroup of G. Then $X:=J\backslash G$ is a locally compact Hausdorff and totally disconnected topological space with a continuous right action $X\times G\to X$ given by $(J\gamma,g)\mapsto J\gamma g$. Let $\mathcal{S}_{\mathbb{Z}}(X)$ denote the $\mathbb{Z}$-module of all $\mathbb{Z}$-valued locally constant compactly supported functions on X. We have an induced smooth left action

$$G\times\mathcal{S}_{\mathbb{Z}}(X)\to\mathcal{S}_{\mathbb{Z}}(X),\quad (g,\xi)\mapsto\xi((-)g). \tag{6.1}$$

For each compact–open subgroup K of G, we let $M(K):=\mathcal{S}_{\mathbb{Z}}(X/K)=\mathcal{S}_{\mathbb{Z}}(X)^K$ denote the $\mathbb{Z}$-module of all functions in $\mathcal{S}_{\mathbb{Z}}(X)$ that are K-invariant under the natural G-action. For $\sigma\in G$, we define $\mathbb{Z}$-linear maps

$$\mathcal{T}(\sigma):M(K)\to M(K),\qquad \mathrm{ch}(JgK)\mapsto\sum_{\gamma\in K\sigma K/K}\mathrm{ch}(Jg\gamma K),$$

$$[K\sigma K]_*:M(K)\to M(K),\qquad \mathrm{ch}(JgK)\mapsto\sum_{\delta\in K\backslash K\sigma K}\mathrm{ch}(JgK\delta),$$

where ch(−) denotes the characteristic function. Then $\mathcal{T}(\sigma)$ and $[K\sigma K]_*$ are respectively the analogs of (1.1) and (1.2) in this setting.

Question 6.2. *Is $\mathcal{T}(\sigma) = [K\sigma K]_*$ for all $\sigma \in G$?*

Of course, this is the case when J is the trivial subgroup, since both $\mathcal{T}(\sigma)$ and $[K\sigma K]_*$ send $\mathrm{ch}(gK)$ to the characteristic function of $gK\sigma K$. More generally, we have the following.

Proposition 6.3. *Suppose that, for any $\gamma \in G$, $J \cap \gamma K \gamma^{-1}$ equals a fixed subgroup of J. Then, for all $\sigma \in G$, $\mathcal{T}(\sigma) = [K\sigma K]_*$.*

Proof. Let Σ be the collection of all compact–open subgroups of G that are equal to a finite intersection of conjugates of K. Our assumption on J implies that, for any $g \in G$ and $L \in \Sigma$ satisfying $L \subset K$, $JgK = \bigsqcup_{\mu \in K/L} Jg\mu L$. If we take L to be the intersection of K and $\bigcap_{\delta \in K\backslash K\sigma K} \delta^{-1} K \delta$, then L is normal in K and its conjugates by $\delta \in K\sigma K$ are contained in K. These properties imply that, for any $\gamma, \delta \in K\sigma K$ and $g \in G$,

$$\mathrm{ch}(Jg\gamma K) = \sum_{\mu \in K/L} \mathrm{ch}(Jg\gamma\mu L), \quad \mathrm{ch}(JgK\delta) = \sum_{\nu \in \delta^{-1}K\delta/L} \mathrm{ch}(Jg\delta\nu L).$$

We wish to show that

$$\sum_{\gamma \in K\sigma K/K} \sum_{\mu \in K/L} \mathrm{ch}(Jg\gamma\mu L) = \sum_{\delta \in K\backslash K\sigma K} \sum_{\nu \in \delta^{-1}K\delta/L} \mathrm{ch}(Jg\delta\nu L).$$

Both sides are sums of characteristic functions on cosets $J\backslash G/L$, possibly with repetitions. But both the lists

- $\gamma\mu L$ for $\gamma \in K\sigma K/K$ and $\mu \in K/L$,
- $\delta\nu L$ for $\delta \in K\backslash K\sigma K$ and $\nu \in \delta^{-1}K\delta/L$

of cosets in G/L enumerate each element of $K\sigma K/L$ exactly once. □

An example where the condition of Proposition 6.3 is satisfied is when $G = \boldsymbol{G}(\mathbb{A}_f)$ for some reductive group $\boldsymbol{G}$ over $\mathbb{Q}$, K is a neat compact–open subgroup of G, and $J = \boldsymbol{T}(\mathbb{Q})$, where $\boldsymbol{T} \hookrightarrow \boldsymbol{G}$ is a torus in $\boldsymbol{G}$ such that $\boldsymbol{T}(\mathbb{Q})$ is discrete in $\boldsymbol{T}(\mathbb{A}_f)$. Indeed, $\boldsymbol{T}(\mathbb{A}_f) \cap \gamma K \gamma^{-1}$ is a compact–open subgroup of $\boldsymbol{T}(\mathbb{A}_f)$ for any $\gamma \in G$. So the discreteness of $J = \boldsymbol{T}(\mathbb{Q})$ in $\boldsymbol{T}(\mathbb{A}_f)$ implies that $J \cap \gamma K \gamma^{-1}$ is finite and the neatness of K forces $J \cap \gamma K \gamma^{-1}$ to be torsion free (see [Graham and Shah 2023, Definition B.6]). See Example 4.23. It is however also easy to find situations where $\mathcal{T}(\sigma) \neq [K\sigma K]_*$.

Example 6.4. Suppose $J = K$. Then $M(K) = \mathbb{Z}[K\backslash G/K]$ is the free $\mathbb{Z}$-module on the characteristic functions of double cosets in $K\backslash G/K$. Now, for any $g \in G$, $\mathcal{T}(\sigma) \cdot \mathrm{ch}(KgK) = \sum_{\gamma \in K\sigma K/K} \mathrm{ch}(Kg\gamma K)$ by definition, whereas

$$[K\sigma K]_* \cdot \mathrm{ch}(KgK) = \mathrm{ch}(KgK) * \mathrm{ch}(K\sigma K),$$

where $*$ denotes the convolution product on $M(K)$ with respect to a Haar measure μ on G such that $\mu(K) = 1$. Note that the map $\mathrm{ind} : M(K) \to \mathbb{Z}$ given by $\mathrm{ch}(K\gamma K) \mapsto |K\gamma K/K|$ is a $\mathbb{Z}$-algebra

homomorphism (see, e.g., [Shah 2024, Section 2.3]), so $\mathcal{T}(\sigma)\cdot \mathrm{ch}(KgK) = [K\sigma K]_* \cdot \mathrm{ch}(KgK)$ would imply that

$$\sum_{\gamma \in K\sigma K/K} |Kg\gamma K/K| = |KgK/K| \cdot |K\sigma K/K|. \tag{6.5}$$

This is clearly false in general. For instance, if $g = 1_G$, the left-hand side of (6.5) is $|K\sigma K/K|^2$ while the right-hand side is $|K\sigma K/K|$, and these are not equal as soon as $|K\sigma K/K| \neq 1$.

Remark 6.6. We may consider Question 6.2 as a problem of "explicit descent" in the following sense. We know that $\sum_{\delta \in K\backslash K\sigma K} \mathrm{ch}(JgK\delta)$ is an element of $M(K)$ as it is K-invariant (a descent phenomenon) and we even know that its support is $JgK\sigma K$. We however want an *explicit* linear combination of the basis $\{\mathrm{ch}(J\gamma K) \mid \gamma \in G\}$ of $M(K)$ that equals this element. This amounts to computing certain volumes of subsets of J. If one establishes that (3.3) or an appropriate variant of it is a bijection and the maps are equivariant for varying K, then one has an alternative strategy for deriving Corollary 4.20. This approach, however, does not work in the generality of Theorem 4.12.

Acknowledgements

The author is grateful to Christophe Cornut for several discussions and to Barry Mazur for valuable feedback on an earlier draft. The author also thanks the referees for their diligent reading of the manuscript and for numerous helpful suggestions that significantly improved the exposition of this note.

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Communicated by Shou-Wu Zhang
Received 2024-10-22 Revised 2025-04-05 Accepted 2025-06-27

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 2000 Allston Way #59, Berkeley, CA 94701-4004, is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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