CONSTRUCTION OF ONE-DIMENSIONAL SUBSETS OF THE REALS NOT CONTAINING SIMILAR COPIES OF GIVEN PATTERNS

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For any countable collection of sets of three points we construct a compact subset of the real line with Hausdorff dimension 1 that contains no similar copy of any of the given triplets.

1. Introduction

An old conjecture of Erdős [1974] (also known as the Erdős similarity problem) states that for any infinite set \(A \subset \mathbb{R}\) there exists a set \(E \subset \mathbb{R}\) of positive Lebesgue measure which does not contain any similar (that is, translated and rescaled) copy of \(A\). It is known that slowly decaying sequences are not counterexamples [Falconer 1984; Bourgain 1987; Kolountzakis 1997] (see for example [Humke and Laczkovich 1998; Komjáth 1983; Svetic 2000] for other related results) but nothing is known about any infinite sequence that converges to zero at least exponentially. On the other hand, it follows easily from Lebesgue’s density theorem that any set \(E \subset \mathbb{R}\) of positive Lebesgue measure contains similar copies of every finite set.

Bisbas and Kolountzakis [2006] gave an incomplete proof of a related statement: For every infinite set \(A \subset \mathbb{R}\) there exists a compact set \(E \subset \mathbb{R}\) of Hausdorff dimension 1 such that \(E\) contains no similar copy of \(A\). Kolountzakis asked whether the same holds for finite sets as well. Iosevich asked a similar question: if \(A \subset \mathbb{R}\) is a finite set and \(E \subset [0, 1]\) is a set of given Hausdorff dimension, must \(E\) contain a similar copy of \(A\)?

In this paper we answer these questions by showing that for any set \(A \subset \mathbb{R}\) of at least 3 elements there exists a 1-dimensional set that contains no similar copy of \(A\). In fact, we obtain a bit more by proving the following theorem, which immediately yields the two subsequent corollaries.

**Theorem 1.1.** For any countable set \(A \subset (1, \infty)\) there exists a compact set \(E \subset \mathbb{R}\) with Hausdorff dimension 1 such that if \(x < y < z\) and \(x, y, z \in E\), then

\[
\frac{z - x}{z - y} \notin A.
\]

**Corollary 1.2.** For any sequence \(B_1, B_2, \ldots \subset \mathbb{R}\) of sets of at least three elements there exists a compact set \(E \subset \mathbb{R}\) with Hausdorff dimension 1 that contains no similar copy of any of \(B_1, B_2, \ldots\).

**Corollary 1.3.** For any countable set \(B \subset \mathbb{R}\) there exists a compact set \(E \subset \mathbb{R}\) with Hausdorff dimension 1 that intersects any similar copy of \(B\) in at most two points.

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The method of the construction is similar to the method used in [Keleti 1998], where a compact set $A$ of Hausdorff dimension 1 is constructed such that $A$ does not contain any set of the form
\[
\{a, a + b, a + c, a + b + c\}
\]
for any $a, b, c \in \mathbb{R}$, $b, c \neq 0$, so in particular $A$ does not contain any nontrivial 3-term arithmetic progression.

Laba and Pramanik [2007] obtained a positive result by proving that if a compact set $E \subset \mathbb{R}$ has Hausdorff dimension sufficiently close to 1 and $E$ supports a probability measure whose Fourier transform has appropriate decay at infinity then $E$ must contain nontrivial 3-term arithmetic progressions. It would be interesting to know whether similar conditions could guarantee other finite patterns as well.

Perhaps one can even find conditions weaker than having positive measure that implies that a compact subset of $\mathbb{R}$ contains similar copies of all finite subsets. This is not impossible since Erdős and Kakutani [1957] constructed a compact set of measure zero with this property. The Erdős–Kakutani set has Hausdorff dimension 1 but, using the ideas from [Elekes and Steprāns 2004], Máté [≥ 2008] constructed such a set with Hausdorff dimension 0. However, the packing dimension of such a set must be 1, since the argument of the proof of [Darji and Keleti 2003, Theorem 2] gives that if a compact set $C \subset \mathbb{R}$ contains similar copies of all sets of $n$ points then $C$ has packing dimension at least $\frac{n-2}{n}$.

2. Proof of Theorem 1.1

Fix a sequence $\alpha_1, \alpha_2, \ldots \subset A$ so that each element of $A$ appears infinitely many times in the sequence $(\alpha_k)$. Let
\[
\beta_k = \max\left(6\alpha_k, \frac{6\alpha_k}{\alpha_k - 1}\right), \quad (k \in \mathbb{N}).
\]
Since $A \subset (1, \infty)$, the number $\beta_k$ is defined and $\beta_k > 6$ for every $k$. We can clearly choose a sequence $m_1, m_2, \ldots \subset \{3, 4, 5, \ldots\}$ so that
\[
\lim_{k \to \infty} \frac{\log(\beta_1 \cdots \beta_k)}{\log(m_1 \cdots m_{k-1})} = 0.
\]
Let
\[
\delta_k = \frac{1}{\beta_1 \cdots \beta_k \cdot m_1 \cdots m_k}.
\]
By induction we shall define sets
\[
E_0 \supset E_1 \supset E_2 \supset \ldots
\]
such that for each $k \in \mathbb{N}$

(*) $E_k$ consists of $m_1 \cdots m_k$ closed intervals of length $\delta_k$ which are separated by gaps of at least $\delta_k$ and each interval of $E_{k-1}$ contains $m_k$ intervals of $E_k$.

We will denote by
\[
I_{1}^{k}, I_{2}^{k}, \ldots, I_{m_1 \cdots m_k}^{k}
\]
the intervals of $E_k$ ordered from left to right, and by
\[
(J_{n}, K_{n}, L_{n})_{n \in \mathbb{Z}}
\]
an enumeration of the set

$$\Gamma = \{(I_a^k, I_b^k, I_c^k) : a, b, c, k \in \mathbb{N}, a < b < c \leq m_1 \cdots m_k\}$$

such that if $n > 1$ and $(J_n, K_n, L_n) = (I_a^k, I_b^k, I_c^k)$ then $n > k$. Since each element of $A$ appears infinitely many times in the sequence $(\alpha_k)$, by repeating each element of $\Gamma$ infinitely many times we can also guarantee that for all $a \in A$ and for all $(J, K, L) \in \Gamma$, there exists $n \in \mathbb{N}$ such that

$$\alpha_n = a, \quad \text{and} \quad (J_n, K_n, L_n) = (J, K, L). \quad (4)$$

Let $E_0 = [0, 1]$ and choose $E_1$ so that $(*)$ holds for $k = 1$. Suppose that $k \geq 2$ and $E_1, \ldots, E_{k-1}$ are already defined so that $(*)$ holds for $1, \ldots, k - 1$. Then $(J_k, K_k, L_k)$ is already defined and each interval of $E_{k-1}$ is either contained in exactly one of $J_k, K_k$ and $L_k$ or disjoint from them.

We shall define $E_k$ so that

$$x \in E_k \cap J_k, \quad y \in E_k \cap K_k \quad \text{and} \quad z \in E_k \cap L_k$$

will imply that

$$\frac{z - x}{z - y} \neq \alpha_k.$$

Let $I$ be an interval of $E_{k-1}$ which is contained in $J_k$. Since $I$ has length $\delta_{k-1}$ and using (3) and (1) we have

$$\frac{\delta_{k-1}}{3\alpha_k \delta_k} = \frac{m_k \beta_k}{3\alpha_k} \geq 2m_k > m_k + 1,$$

and $I$ contains more than $m_k$ points of the form $3\alpha_k \delta_k i$ for $i \in \mathbb{Z}$. Hence we can choose the $m_k$ intervals of $E_k$ in $I$ as segments of the form

$$\delta_k (3i \alpha_k + [0, 1]) \quad (i \in \mathbb{Z}).$$

If $I$ is an interval of $E_{k-1}$ which is contained in $K_k$, then similarly, since

$$\frac{\delta_{k-1}}{3\delta_k} = \frac{m_k \beta_k}{3} \geq 2m_k > m_k + 1,$$

we can choose the $m_k$ intervals of $E_k$ in $I$ as segments of the form

$$\delta_k (3j + [0, 1]) \quad (j \in \mathbb{Z}).$$

If $I$ is an interval of $E_{k-1}$ which is contained in $L_k$, then, since by (3) and (1) we have

$$\frac{\delta_{k-1}}{3\alpha_k \delta_k} = \frac{m_k \beta_k}{3\alpha_k} \geq 2m_k > m_k + 1,$$

we can choose the $m_k$ intervals of $E_k$ in $I$ as segments of the form

$$\delta_k \left(\frac{3\alpha_k}{\alpha_k - 1}(l + \frac{1}{2}) + [0, 1]\right) \quad (l \in \mathbb{Z}).$$

In each of the rest of the intervals of $E_{k-1}$ we define the $m_k$ intervals of length $\delta_k$ of $E_k$ arbitrarily so that they are separated by gaps of at least length $\delta_k$. 
This way we defined \( E_k \) so that (*) holds. Let

\[
E = \bigcap_{k=1}^{\infty} E_k.
\]

Then \( E \) is clearly a compact subset of \( \mathbb{R} \). Condition (**) implies that the Hausdorff dimension of \( E \) is at least

\[
\liminf_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_k)}
\]

(see [Falconer 1990, Example 4.6]). On the other hand, using (3) and (2) we get that

\[
\liminf_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_k)} = \liminf_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{\log(\beta_1 \cdots \beta_k) + \log(m_1 \cdots m_{k-1})} = 1,
\]

and therefore the Hausdorff dimension of \( E \) is 1.

Finally, to get a contradiction, suppose that

\[
x, y, z \in E, \quad x < y < z, \quad \text{and} \quad \frac{z - x}{z - y} \in A.
\]

Since \( \delta_k \to 0 \), there exists a \( k \in \mathbb{N} \) such that \( x, y \) and \( z \) are in distinct intervals of \( E_k \). Then, by (4) there exists an \( n \in \mathbb{N} \) so that

\[
x \in J_n, \quad y \in K_n, \quad z \in L_n \quad \text{and} \quad \frac{z - x}{z - y} = \alpha_n.
\]

By the construction of \( E_n \), there exists \( i, j, l \in \mathbb{Z} \) such that

\[
x \in \delta_n(3i\alpha_n + [0, 1]), \quad y \in \delta_n(3j + [0, 1]), \quad \text{and} \quad z \in \delta_n\left(\frac{3\alpha_n}{\alpha_n - 1}(l + \frac{1}{2}) + [0, 1]\right).
\]

Let

\[
X = 3i\alpha_n + [0, 1], \quad Y = 3j + [0, 1], \quad \text{and} \quad Z = \frac{3\alpha_n}{\alpha_n - 1}(l + \frac{1}{2}) + [0, 1].
\]

Then \( \frac{x}{\alpha_n} \in X \), \( \frac{y}{\alpha_n} \in Y \) and \( \frac{z}{\alpha_n} \in Z \). On the other hand, \( \frac{z - x}{z - y} = \alpha_n \) implies that \( \alpha_n y = x + (\alpha_n - 1)z \), so (by using the notation \( A + B = \{a + b : a \in A, b \in B\} \)) we must have

\[
\alpha_n Y \cap (X + (\alpha_n - 1)Z) \neq \emptyset. \tag{5}
\]

By definition (and using that \( \alpha_n > 1 \)),

\[
\alpha_n Y = \alpha_n(3j + [0, 1]) \tag{6}
\]

and

\[
X + (\alpha_n - 1)Z = 3i\alpha_n + [0, 1] + 3\alpha_n(l + \frac{1}{2}) + (\alpha_n - 1)[0, 1]
\]

\[
= 3(i + l)\alpha_n + \left[\frac{3}{2}\alpha_n, \frac{5}{2}\alpha_n\right]
\]

\[
= \alpha_n(3(i + l + \left[\frac{3}{2}, \frac{5}{2}\right]). \tag{7}
\]

Since \( i, j, l \in \mathbb{Z} \), (6) and (7) contradict (5).
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References


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