

THE LINEAR PROFILE DECOMPOSITION FOR THE AIRY EQUATION AND THE EXISTENCE OF MAXIMIZERS FOR THE AIRY STRICHARTZ INEQUALITY

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We establish the linear profile decomposition for the Airy equation with complex or real initial data in L^2 . As an application, we obtain a dichotomy result on the existence of maximizers for the symmetric Airy Strichartz inequality.

1. Introduction

In this paper, we consider the problem of the linear profile decomposition for the Airy equation with the L^2 initial data

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0, x) = u_0(x) \in L^2, \end{cases} \quad (1)$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ or \mathbb{C} . Roughly speaking, the profile decomposition is to investigate the general structure of a sequence of solutions to the Airy equation with bounded initial data in L^2 . We expect that it can be expressed, up to a subsequence, as a sum of a superposition of concentrating waves—profiles—and a reminder term. The profiles are “almost orthogonal” in the Strichartz space and in L^2 while the remainder term is small in the same Strichartz norm and can be negligible in practice. The profile decomposition is also referred to as the *bubble decomposition* in the literature; see [Killip and Visan 2008b, p.35] for an interesting historical discussion.

The same problem in the context of the wave or Schrödinger equations has been intensively studied recently. For the wave equations, Bahouri and Gérard [1999] established a linear profile decomposition for the energy critical wave equation in \mathbb{R}^3 (their argument can be generalized to higher dimensions). Following [Bahouri and Gérard 1999], Keraani [2001] obtained a linear profile decomposition for energy critical Schrödinger equations; see also [Shao 2009]. For the mass critical Schrödinger equations, when $d = 2$, Merle and Vega [1998] established a linear profile decomposition, similar in spirit to that in [Bourgain 1998]; Carles and Keraani [2007] treated the $d = 1$ case, while the higher-dimensional analogue was obtained by Bégout and Vargas [2007]. In general, a nonlinear profile decomposition can be achieved from the linear case via a perturbation argument. The first ingredient of the proof of linear profile decompositions is to start with some refined inequality: the refined Sobolev embedding or the refined Strichartz inequality. Usually establishing such refinements needs some nontrivial work. For instance, in the Schrödinger case, the two-dimensional improvement is due to Moyua et al. [1999] involving the X_p^q spaces; the one-dimensional improvement due to Carles and Keraani [2007] using the

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Hausdorff–Young inequality and the weighted Fefferman–Phong inequality [Fefferman 1983], which Kenig et al. [2000] first introduced to prove their refined Strichartz inequality (5) for the Airy equation; the higher-dimensional refinement is due to Bégout and Vargas [2007] based on a new bilinear restriction estimate for paraboloids by Tao [2003]. Another important ingredient of the arguments is the idea of the concentration-compactness principle which aims to compensate for the defect of compactness of the Strichartz inequality, and was exploited in [Bahouri and Gérard 1999; Merle and Vega 1998; Carles and Keraani 2007; Bégout and Vargas 2007]; also see [Schindler and Tintarev 2001] for an abstract version of this principle in the Hilbert space. The profile decompositions turn out to be quite useful in nonlinear dispersive equations. For instance, they can be used to analyze the mass concentration phenomena near the blow up time for the mass critical Schrödinger equation; see [Merle and Vega 1998; Carles and Keraani 2007; Bégout and Vargas 2007]. They were also used to show the existence of minimal mass or energy blow-up solutions for the Schrödinger or wave equations at critical regularity, which is an important step in establishing the global well-posedness and scattering results for such equations; see [Kenig and Merle 2006; 2007; Killip et al. 2007; Tao et al. 2007; Killip and Visan 2008a. Shao [2009] used it to establish the existence of maximizers for the nonendpoint Strichartz and Sobolev–Strichartz inequalities for the Schrödinger equation.

The discussion above motivates the question of profile decompositions for the Airy equation, which is the free form of the mass critical generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u \pm u^4 \partial_x u = 0, & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0, x) = u_0(x). \end{cases} \quad (2)$$

This is one of the (generalized) KdV equations [Tao 2006b] and is the natural analogy to the mass critical nonlinear Schrödinger equation in one spatial dimension. The KdV equations arise from describing the waves on shallow water surfaces, and turn out to have connections to many other physical problems. As is well known, the class of solutions to (1) enjoys a number of symmetries that preserve the mass $\int |u|^2 dx$. We will employ the notations from [Killip et al. 2007] and first discuss the symmetries at the initial time $t = 0$.

Definition 1.1 (Mass-preserving symmetry group). For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}$ and scaling parameter $h_0 > 0$, we define the unitary transform $g_{\theta, x_0, h_0} : L^2 \rightarrow L^2$ by the formula

$$[g_{\theta, x_0, h_0} f](x) := \frac{1}{h_0^{1/2}} e^{i\theta} f\left(\frac{x - x_0}{h_0}\right).$$

We let G be the collection of such transformations. It is easy to see that G is a group.

Unlike the free Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0, & t \in \mathbb{R}, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), \end{cases} \quad (3)$$

two important symmetries are missing for (1), namely, the Galilean symmetry

$$u(t, x) \mapsto e^{ix\xi_0 + it|\xi_0|^2} u(t, x + 2t\xi_0),$$

and the pseudo-conformal symmetry

$$u(t, x) \mapsto |t|^{-d/2} e^{-i|x|^2/(4t)} u(-1/t, x/t).$$

This lack of symmetries causes difficulties if we try to mimic the existing argument of profile decompositions for the Schrödinger equations. In this paper, we will show how to compensate for the lack of the Galilean symmetry when developing the analogous version of linear profile decompositions for the Airy Equation (1).

Like Schrödinger equations, an important family of inequalities, the Airy Strichartz inequality [Kenig et al. 1991, Theorem 2.1], is associated with the Airy equation (1). It is invariant under the symmetry group and asserts that

$$\|D^a e^{-t\partial_x^3} u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2}, \quad (4)$$

if and only if $-a + 3/q + 1/r = 1/2$ and $-1/2 < a \leq 1/q$, where $e^{-t\partial_x^3} u_0$ and D^a are defined in Section 2. When $q = r = 6$ and $a = 1/6$, we also have the following refined Strichartz estimate due to Kenig–Ponce–Vega, which is the key to establishing the profile decomposition results for the Airy equation in this paper.

Lemma 1.2 (KPV’s refined Strichartz [Kenig et al. 2000]). *Let $p > 1$. Then*

$$\|D^{1/6} e^{-t\partial_x^3} u_0\|_{L_{t,x}^6} \leq C \left(\sup_{\tau} |\tau|^{\frac{1}{2} - \frac{1}{p}} \|\widehat{u_0}\|_{L^p(\tau)} \right)^{\frac{1}{3}} \|u_0\|_{L^2}^{\frac{2}{3}}, \quad (5)$$

where τ denotes an interval of the real line with length $|\tau|$.

In Section 3, we will present a new proof suggested by Terence Tao by using the Whitney decomposition.

As in the Schrödinger case, the Airy Strichartz inequality (4) cannot guarantee the solution map from the L^2 space to the Strichartz space to be compact, namely, every L^2 -bounded sequence will produce a convergent subsequence of solutions in the Strichartz space. The particular Strichartz space we are interested in is equipped with the norm $\|D^{1/6} u\|_{L_{t,x}^6}$. The failure of compactness can be seen explicitly from creating counter-examples by considering the symmetries in L^2 such as the space and time translations, or scaling symmetry or frequency modulation. Indeed, given $x_0 \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $h_0 \in (0, \infty)$, we denote by τ_{x_0} , S_{h_0} and R_{t_0} the operators defined by

$$\tau_{x_0} \phi(x) := \phi(x - x_0), \quad S_{h_0} \phi(x) := \frac{1}{h_0^{1/2}} \phi\left(\frac{x}{h_0}\right), \quad R_{t_0} \phi(x) := e^{-t_0 \partial_x^3} \phi(x).$$

Let $(x_n)_{n \geq 1}$, $(t_n)_{n \geq 1}$ be sequences both going to infinity, and $(h_n)_{n \geq 1}$ be a sequence going to zero as n goes to infinity. Then for any nontrivial $\phi \in \mathcal{S}$, $(\tau_{x_n} \phi)_{n \geq 1}$, $(S_{h_n} \phi)_{n \geq 1}$ and $(R_{t_n} \phi)_{n \geq 1}$ weakly converge to zero in L^2 . However, their Strichartz norms are all equal to $\|D^{1/6} e^{-t \partial_x^3} \phi\|_{L_{t,x}^6}$, which is nonzero. Hence these sequences are not relatively compact in the Strichartz spaces. Moreover, the frequency modulation also exhibits the defect of compactness: for $\xi_0 \in \mathbb{R}$, we define M_{ξ_0} via

$$M_{\xi_0} \phi(x) := e^{ix \xi_0} \phi(x).$$

Choosing $(\xi_n)_{n \geq 1}$ to be a sequence going to infinity as n goes to infinity, we see that $(M_{\xi_n} \phi)_{n \geq 1}$ converges weakly to zero. However, from Remark 1.7, $\|D^{1/6} e^{-it \partial_x^3} (e^{i(\cdot) \xi_n} \phi)\|_{L_{t,x}^6}$ converges to $3^{-1/6} \|e^{-it \partial_x^3} \phi\|_{L_{t,x}^6}$, which is not zero. This shows that the modulation operator M_{ξ_0} is not compact either.

It will be clear from the statements of [Theorem 1.5](#) and [Theorem 1.6](#) that these four symmetries in L^2 above are the only obstructions to the compactness of the solution map. Hence the parameter (h_0, ζ_0, x_0, t_0) plays a special role in characterizing this defect of compactness; moreover, a sequence of such parameters needs to satisfy some *orthogonality* constraint (the term is used in the sense of [Lemma 5.2](#)).

Definition 1.3 (Orthogonality). For $j \neq k$, two sequences

$$\Gamma_n^j := (h_n^j, \zeta_n^j, x_n^j, t_n^j)_{n \geq 1} \quad \text{and} \quad \Gamma_n^k := (h_n^k, \zeta_n^k, x_n^k, t_n^k)_{n \geq 1}$$

in $(0, \infty) \times \mathbb{R}^3$ are orthogonal if one of the following holds:

- $\lim_{n \rightarrow \infty} \left(\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + h_n^j |\zeta_n^j - \zeta_n^k| \right) = \infty$,
- $(h_n^j, \zeta_n^j) = (h_n^k, \zeta_n^k)$ and

$$\lim_{n \rightarrow \infty} \left(\frac{|t_n^k - t_n^j|}{(h_n^j)^3} + \frac{3|(t_n^k - t_n^j)\zeta_n^j|}{(h_n^j)^2} + \frac{|x_n^j - x_n^k + 3(t_n^j - t_n^k)(\zeta_n^j)^2|}{h_n^j} \right) = \infty.$$

Remark 1.4. For any $\Gamma_n^j = (h_n^j, \zeta_n^j, x_n^j, t_n^j)_{n \geq 1}$, it is clear that, up to a subsequence, $\lim_{n \rightarrow \infty} |h_n^j \zeta_n^j|$ is either finite or infinite. For the former, we can reduce to $\zeta_n^j \equiv 0$ for all n by changing profiles; see [Remark 3.6](#). For the latter, the corresponding profiles exhibit a Schrödinger behavior in some sense; see [Remark 1.7](#). In view of this, we will group the decompositions accordingly in the statements of our main theorems below.

Now we are able to state the main theorems. When the initial data to [Equation \(1\)](#) is complex, the following theorem on the linear Airy profile decomposition is proven in [Section 5](#).

Theorem 1.5 (Complex version). *Let $(u_n)_{n \geq 1}$ be a sequence of complex-valued functions satisfying $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence, there exists a sequence of L^2 functions $(\phi^j)_{j \geq 1} : \mathbb{R} \rightarrow \mathbb{C}$ and a family of pair-wise orthogonal sequences $\Gamma_n^j = (h_n^j, \zeta_n^j, x_n^j, t_n^j) \in (0, \infty) \times \mathbb{R}^3$ such that, for any $l \geq 1$, there exists an L^2 function $w_n^l : \mathbb{R} \rightarrow \mathbb{C}$ satisfying*

$$u_n = \sum_{\substack{1 \leq j \leq l, \zeta_n^j \equiv 0 \\ \text{or } |h_n^j \zeta_n^j| \rightarrow \infty}} e^{t_n^j \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] + w_n^l, \quad (6)$$

where $g_n^j := g_{0, x_n^j, h_n^j} \in G$ and

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \|D^{1/6} e^{-t \partial_x^3} w_n^l\|_{L_{t,x}^6} = 0. \quad (7)$$

Moreover, for every $l \geq 1$,

$$\lim_{n \rightarrow \infty} \left(\|u_n\|_{L^2}^2 - \left(\sum_{j=1}^l \|\phi^j\|_{L^2}^2 + \|w_n^l\|_{L^2}^2 \right) \right) = 0. \quad (8)$$

When the initial sequence is of real-value, we analogously obtain the following real-version profile decomposition. Note that we can restrict the frequency parameter ζ_n^j to be nonnegative.

Theorem 1.6 (Real version). *Let $(u_n)_{n \geq 1}$ be a sequence of real-valued functions satisfying $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence there exists a sequence of L^2 functions, $(\phi^j)_{j \geq 1}: \mathbb{R} \rightarrow \mathbb{C}$, and a family of orthogonal sequences $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j) \in (0, \infty) \times [0, \infty) \times \mathbb{R}^2$ such that, for any $l \geq 1$, there exists an L^2 function $w_n^l: \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

$$u_n = \sum_{\substack{1 \leq j \leq l, \xi_n^j \equiv 0 \\ \text{or } |h_n^j \xi_n^j| \rightarrow \infty}} e^{t_n^j \partial_x^3} g_n^j [\operatorname{Re}(e^{i(\cdot) h_n^j \xi_n^j} \phi^j)] + w_n^l, \quad (9)$$

where $g_n^j := g_{0, x_n^j, h_n^j} \in G$ and

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \|D^{1/6} e^{-t \partial_x^3} w_n^l(x)\|_{L_{t,x}^6} = 0. \quad (10)$$

Moreover, for every $l \geq 1$,

$$\lim_{n \rightarrow \infty} \left(\|u_n\|_{L^2}^2 - \left(\sum_{\substack{1 \leq j \leq l, \xi_n^j \equiv 0 \\ \text{or } |h_n^j \xi_n^j| \rightarrow \infty}} \|\operatorname{Re}(e^{i(\cdot) h_n^j \xi_n^j} \phi^j)\|_{L^2}^2 + \|w_n^l\|_{L^2}^2 \right) \right) = 0. \quad (11)$$

When $\lim_{n \rightarrow \infty} |h_n^j \xi_n^j| = \infty$ for some $1 \leq j \leq l$, the profile will exhibit asymptotic ‘‘Schrödinger’’ behavior. For simplicity, we just look at the complex case.

Remark 1.7 (Asymptotic Schrödinger behavior). Without loss of generality, we assume $\phi^j \in \mathcal{S}$ with the compact Fourier support $[-1, 1]$. Then

$$\begin{aligned} D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \xi_n^j} \phi^j](x) &= \int e^{i(x-x_n^j)\xi + i(t-t_n^j)\xi^3} |\xi|^{1/6} (h_n^j)^{1/2} \widehat{\phi^j}(h_n^j(\xi - \xi_n^j)) d\xi \\ &= (h_n^j)^{-1/2} |\xi_n^j|^{1/6} e^{i(x-x_n^j)\xi_n^j + i(t-t_n^j)(\xi_n^j)^3} \\ &\quad \times \int e^{i \left[\frac{\eta(x-x_n^j + 3(t-t_n^j)(\xi_n^j)^2)}{h_n^j} + \frac{\eta^3(t-t_n^j)}{(h_n^j)^3} + \frac{3\eta^2(t-t_n^j)\xi_n^j}{(h_n^j)^2} \right]} \left| 1 + \frac{\eta}{h_n^j \xi_n^j} \right|^{1/6} \widehat{\phi^j}(\eta) d\eta. \end{aligned}$$

Set $x' := \frac{x - x_n^j + 3(t - t_n^j)(\xi_n^j)^2}{h_n^j}$ and $t' := \frac{3(t - t_n^j)\xi_n^j}{(h_n^j)^2}$. Then the dominated convergence theorem yields

$$\begin{aligned} \|D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \xi_n^j} \phi^j]\|_{L_{t,x}^6} &= 3^{-1/6} \left\| \int e^{ix'\eta + it'\eta^2} e^{it' \frac{\eta^3}{3h_n^j \xi_n^j}} \left| 1 + \frac{\eta}{h_n^j \xi_n^j} \right|^{1/6} \widehat{\phi^j} d\eta \right\|_{L_{t',x'}^6} \\ &\rightarrow_{n \rightarrow \infty} 3^{-1/6} \|e^{-it' \partial_x^2} \phi^j\|_{L_{t',x'}^6}, \end{aligned}$$

where $e^{-it \partial_x^2}$ denotes the Schrödinger evolution operator defined via

$$e^{-it \partial_x^2} f(x) := \int_{\mathbb{R}} e^{ix\xi + it|\xi|^2} \widehat{f}(\xi) d\xi.$$

Indeed,

$$\int e^{ix'\eta + it'\eta^2} e^{it' \frac{\eta^3}{3h_n^j \xi_n^j}} \left| 1 + \frac{\eta}{h_n^j \xi_n^j} \right|^{1/6} \widehat{\phi^j} d\eta \rightarrow e^{-it' \partial_x^2} \phi^j(x') \quad \text{a.e.},$$

and by using [Stein 1993, Corollary, p. 334] or integration by parts,

$$\left| \int e^{ix'\eta + it'\eta^2} e^{it' \frac{\eta^3}{3h_n^j \xi_n^j}} \left| 1 + \frac{\eta}{h_n^j \xi_n^j} \right|^{1/6} \widehat{\phi}^j d\eta \right| \leq C_{\phi^j} B(t', x')$$

for n large enough but still uniform in n . Here

$$B(t', x') = \begin{cases} (1 + |t'|)^{-1/2} \leq C[(1 + |x'|)(1 + |t'|)]^{-1/4} & \text{for } |x'| \leq 6|t'|, \\ (1 + |x'|)^{-1} \leq C[(1 + |x'|)(1 + |t'|)]^{-1/2} & \text{for } |x'| > 6|t'|. \end{cases}$$

It is easy to observe that $B \in L_{t', x'}^6$.

In the next three paragraphs, we outline the proof of [Theorem 1.5](#) in three steps; [Theorem 1.6](#) follows similarly. Given an L^2 -bounded sequence $(u_n)_{n \geq 1}$, at the first step, we use the refined Strichartz inequality (5) and an iteration argument to obtain a preliminary decomposition for $(u_n)_{n \geq 1}$: up to a subsequence

$$u_n = \sum_{j=1}^N f_n^j + q_n^N,$$

where $\widehat{f_n^j}$ is supported on an interval $(\xi_n^j - \rho_n^j, \xi_n^j + \rho_n^j)$ and $|\widehat{f_n^j}| \leq C(\rho_n^j)^{-1/2}$, and $e^{-t\partial_x^3} q_n^N$ is small in the Strichartz norm. Then we impose the orthogonality condition on (ρ_n^j, ξ_n^j) : for $j \neq k$,

$$\lim_{n \rightarrow \infty} \left(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} \right) = \infty,$$

to regroup the decomposition.

At the second step, for each $j \in [1, N]$, we will perform a further decomposition to f_n^j to extract the space and time parameters. For simplicity, we suppress all the superscripts j and rescale $(f_n)_{n \geq 1}$ to obtain $P = (P_n)_{n \geq 1}$ by setting

$$\widehat{P}_n(\cdot) := \rho_n^{1/2} \widehat{f}_n(\rho_n(\cdot + \rho_n^{-1} \xi_n)),$$

from which we can infer that each \widehat{P}_n is bounded and supported on a finite interval centered at the origin. We apply the concentration-compactness argument to $(P_n)_{n \geq 1}$ to extract (y_n^α, s_n^α) : for any $A \geq 1$, up to a subsequence,

$$P_n(x) = \sum_{\alpha=1}^A e^{-ix\rho_n^{-1}\xi_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot)](x - y_n^\alpha) + P_n^A(x). \quad (12)$$

More precisely, we will investigate the set of weak limits,

$$\mathcal{W}(P) := \left\{ w\text{-}\lim_{n \rightarrow \infty} e^{-ix\rho_n^{-1}\xi_n} e^{-s_n \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\xi_n} P_n(\cdot)](x + y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2 \right\},$$

where the notion $w\text{-}\lim_{n \rightarrow \infty} f_n$ denotes, up to a subsequence, the weak limit of $(f_n)_{n \geq 1}$ in L^2 . Note that due to the lack of Galilean transform and the additional multiplier weight in the current Strichartz norm,

it is a slight but necessary modification to the Schrödinger case [Carles and Keraani 2007], where $\mathring{W}(P)$ is the set

$$\{w\text{-}\lim_{n \rightarrow \infty} e^{is_n \partial_x^2} P_n(x + y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2\}.$$

In (12), we impose the orthogonality condition on (y_n^α, s_n^α) : for $\alpha \neq \beta$,

$$\lim_{n \rightarrow \infty} \left(\left| y_n^\beta - y_n^\alpha + \frac{3(s_n^\beta - s_n^\alpha)(\zeta_n)^2}{(\rho_n)^2} \right| + \left| \frac{3(s_n^\beta - s_n^\alpha)\zeta_n}{\rho_n} \right| + |s_n^\beta - s_n^\alpha| \right) = \infty. \quad (13)$$

The error term $P^A := (P_n^A)_{n \geq 1}$ is small in the weak sense that

$$\lim_{A \rightarrow \infty} \mu(P^A) := \lim_{A \rightarrow \infty} \sup\{\|\phi\|_{L^2} : \phi \in \mathring{W}(P^A)\} = 0. \quad (14)$$

Since $f_n(x) = \sqrt{\rho_n} e^{ix\zeta_n} P_n(\rho_n x)$,

$$f_n(x) = \sum_{\alpha=1}^A \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\zeta_n} \phi^\alpha(\cdot)](\rho_n x - y_n^\alpha) + \sqrt{\rho_n} e^{ix\zeta_n} P_n^A(\rho_n x).$$

Let $e_n^A := \sqrt{\rho_n} e^{ix\zeta_n} P_n^A(\rho_n x)$. Now the major task is to upgrading this weak convergence in (14) to

$$\lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} e_n^A\|_{L_{t,x}^6} = 0.$$

To achieve this, we will interpolate $L_{t,x}^6$ between $L_{t,x}^q$ and $L_{t,x}^\infty$ for some $4 \leq q < 6$. The $L_{t,x}^q$ norm is controlled by some localized restriction estimates and the $L_{t,x}^\infty$ norm is expected to be controlled by $\mu(P^A)$. Unlike the Schrödinger case, we will distinguish the case $\lim_{n \rightarrow \infty} |\rho_n^{-1}\zeta_n| = +\infty$ from $\lim_{n \rightarrow \infty} |\rho_n^{-1}\zeta_n| < +\infty$ due to the additional multiplier weight in the current Strichartz norm.

The final decomposition is obtained by setting

$$(h_n^j, \zeta_n^j, x_n^j, t_n^j) := ((\rho_n^j)^{-1}, \zeta_n^j, (\rho_n^j)^{-1} y_n^j, (\rho_n^j)^{-3} s_n^j),$$

and showing two orthogonality results for the profiles.

1.8. The second part of this paper is devoted to applying the linear profile decomposition result to the problem of the existence of maximizers for the Airy Strichartz inequality. As a corollary of Theorems 1.5 and 1.6, we will establish a dichotomy result. Denote

$$S_{\text{airy}}^{\mathbb{C}} := \sup\{\|D^{1/6} e^{-t\partial_x^3} u_0\|_{L_{t,x}^6} : \|u_0\|_{L^2} = 1\}, \quad (15)$$

when u_0 is complex-valued; similarly we define $S_{\text{airy}}^{\mathbb{R}}$ for real-valued initial data. We are interested in determining whether there exists a maximizing function u_0 with $\|u_0\|_{L^2} = 1$ for which

$$\|D^{1/6} e^{-t\partial_x^3} u_0\|_{L_{t,x}^6} = S_{\text{airy}} \|u_0\|_{L^2},$$

where S_{airy} represents either $S_{\text{airy}}^{\mathbb{C}}$ or $S_{\text{airy}}^{\mathbb{R}}$. The analogous question to the Schrödinger Strichartz inequalities was studied by Kunze [2003], Foschi [2007], Hundertmark and Zharnitsky [2006], Carneiro [2008],

Bennett et al. [2008] and Shao [2009]. We set

$$S_{\text{schr}}^{\mathbb{C}} := \sup \left\{ \|e^{-it\Delta} u_0\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{R}^d)} : \|u_0\|_{L^2(\mathbb{R}^d)} = 1 \right\}. \quad (16)$$

The fact $S_{\text{schr}}^{\mathbb{C}} < \infty$ is due to Strichartz [1977] which in turn had precursors in [Tomas 1975]. For the problem of existence of such optimal $S_{\text{schr}}^{\mathbb{C}}$ and explicitly characterizing the maximizers, Kunze [2003] treated the $d = 1$ case and showed that maximizers exist by an elaborate concentration-compactness method. Foschi [2007] explicitly determined the best constants when $d = 1, 2$, and showed that the only maximizers are Gaussians up to the natural symmetries associated to the Strichartz inequality by using the sharp Cauchy–Schwarz inequality and the space-time Fourier transform. Hundertmark and Zharnitsky [2006] independently obtained this result by an interesting representation formula of the Strichartz inequalities in lower dimensions. Recently, Carneiro [2008] proved a sharp Strichartz-type inequality by following the arguments in [Hundertmark and Zharnitsky 2006] and found its maximizers, which derives the same results in [Hundertmark and Zharnitsky 2006] as a corollary when $d = 1, 2$. Very recently, Bennett et al. [2008] offered a new proof to determine the best constants by using the method of heat-flow. Shao [2009] showed that a maximizer exists for all nonendpoint Strichartz inequalities and in all dimensions by relying on the recent linear profile decomposition results for the Schrödinger equations. We will continue this approach for (15). Additionally, we will use a simple but beautiful idea of asymptotic embedding of a NLS solution to an approximate gKdV solution, which was previously exploited in [Christ et al. 2003; Tao 2007]. This gives that in the complex case, $S_{\text{schr}}^{\mathbb{C}} \leq 3^{1/6} S_{\text{airy}}^{\mathbb{C}}$ while in the real case, $S_{\text{schr}}^{\mathbb{C}} \leq 2^{1/2} 3^{1/6} S_{\text{airy}}^{\mathbb{R}}$.

Theorem 1.9. *We have the following dichotomy on the existence of maximizers for (15) with the complex- or real-valued initial data, respectively:*

- *In the complex case, either a maximizer is attained for (15), or there exists ϕ of complex value satisfying*

$$\|\phi\|_{L^2} = 1 \quad \text{and} \quad S_{\text{schr}}^{\mathbb{C}} = \|e^{-it\partial_x^2} \phi\|_{L_{t,x}^6},$$

and a sequence $(a_n)_{n \geq 1}$ satisfying $\lim_{n \rightarrow \infty} |a_n| = \infty$ such that

$$\lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} \phi]\|_{L_{t,x}^6} = S_{\text{airy}}^{\mathbb{C}}, \quad S_{\text{schr}}^{\mathbb{C}} = 3^{1/6} S_{\text{airy}}^{\mathbb{C}}.$$

- *In the real case, a similar statement holds; more precisely, either a maximizer is attained for (15), or there exists ϕ of complex value satisfying*

$$S_{\text{schr}}^{\mathbb{C}} = \frac{\|e^{-it\partial_x^2} \phi\|_{L_{t,x}^6}}{\|\phi\|_{L^2}},$$

and a positive sequence $(a_n)_{n \geq 1}$ satisfying $\lim_{n \rightarrow \infty} a_n = \infty$ and $\lim_{n \rightarrow \infty} \|\text{Re}(e^{i(\cdot)a_n} \phi)\|_{L^2} = 1$ such that

$$\lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} \text{Re}(e^{i(\cdot)a_n} \phi)\|_{L_{t,x}^6} = S_{\text{airy}}^{\mathbb{R}}, \quad S_{\text{schr}}^{\mathbb{C}} = 2^{1/2} 3^{1/6} S_{\text{airy}}^{\mathbb{R}}.$$

Remark 1.10. Note that when $S_{\text{schr}}^{\mathbb{C}} = 3^{1/6} S_{\text{airy}}^{\mathbb{C}}$ or $S_{\text{schr}}^{\mathbb{C}} = 2^{1/2} 3^{1/6} S_{\text{airy}}^{\mathbb{R}}$, the explicit ϕ had been uniquely determined by Foschi [2007] and Hundertmark and Zharnitsky [2006] independently: they are Gaussians up to the natural symmetries enjoyed by the Strichartz inequality for the Schrödinger equation.

This paper is organized as follows: in [Section 2](#) we establish some notation. In [Section 3](#), we make a preliminary decomposition for an L^2 -bounded sequence $(u_n)_{n \geq 1}$ of complex value. In [Section 4](#), we obtain similar results for a real sequence. In [Section 5](#), we prove [Theorems 1.5](#) and [1.6](#). In [Section 6](#), we prove [Theorem 1.9](#).

2. Notation

We use $X \lesssim Y$, $Y \gtrsim X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some constant $0 < C < \infty$, which might depend on the dimension but not on the functions. If $X \lesssim Y$ and $Y \lesssim X$ we will write $X \sim Y$. If the constant C depends on a special parameter, we shall denote it explicitly by subscripts.

We define the space-time norm $L_t^q L_x^r$ of f on $\mathbb{R} \times \mathbb{R}$ by

$$\|f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R})} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t, x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}$ is replaced by a small space-time region. When $q = r$, we abbreviate it by $L_{t,x}^q$. Unless specified, all the space-time integrations are taken over $\mathbb{R} \times \mathbb{R}$, and all the spatial integrations over \mathbb{R} .

We fix the notation that $\lim_{n \rightarrow \infty}$ should be understood as $\limsup_{n \rightarrow \infty}$ throughout this paper.

The spatial Fourier transform is defined via

$$\widehat{u}_0(\xi) := \int_{\mathbb{R}} e^{-ix\xi} u_0(x) dx;$$

the space-time Fourier transform is defined analogously.

The Airy evolution operator $e^{-t\partial_x^3}$ is defined via

$$e^{-t\partial_x^3} u_0(x) := \int_{\mathbb{R}} e^{ix\xi + it\xi^3} \widehat{u}_0(\xi) d\xi.$$

The spatial derivative ∂_x^k , for k a positive integer, is defined via the Fourier transform

$$\widehat{\partial_x^k}(\xi) = (i\xi)^k.$$

The fractional differentiation operator D^α , $\alpha \in \mathbb{R}$, is defined via

$$D^\alpha f(x) := \int_{\mathbb{R}} e^{ix\xi} |\xi|^\alpha \widehat{f}(\xi) d\xi.$$

The inner product $\langle \cdot, \cdot \rangle_{L^2}$ in the Hilbert space L^2 is defined via

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}} f(x) \bar{g}(x) dx,$$

where \bar{g} denotes the usual complex conjugate of g in the complex plane \mathbb{C} .

3. Preliminary decomposition: complex version

To begin proving [Theorems 1.5](#) and [1.6](#), we present a new proof of the refined Strichartz inequality [\(5\)](#) based on the Whitney decomposition. The following notation is taken from [\[Killip and Visan 2008b\]](#).

Definition 3.1. Given $j \in \mathbb{Z}$, we denote by \mathcal{D}_j the set of all dyadic intervals in \mathbb{R} of length 2^j :

$$\mathcal{D}_j := \{2^j[k, k+1) : k \in \mathbb{Z}\}.$$

We also write $\mathcal{D} := \cup_{j \in \mathbb{Z}} \mathcal{D}_j$. Given $I \in \mathcal{D}$, we define f_I by $\widehat{f}_I := \widehat{f}1_I$ where 1_I denotes the indicator function of I .

Then the Whitney decomposition we need is as follows: Given two distinct $\xi, \xi' \in \mathbb{R}$, there is a unique maximal pair of dyadic intervals $I \in \mathcal{D}$ and $I' \in \mathcal{D}$ such that

$$|I| = |I'|, \text{dist}(I, I') \geq 4|I|, \quad (17)$$

where $\text{dist}(I, I')$ denotes the distance between I and I' , and $|I|$ denotes the length of the dyadic interval I . Let \mathcal{F} denote all such pairs as $\xi \neq \xi'$ varies over $\mathbb{R} \times \mathbb{R}$. Then we have

$$\sum_{(I, I') \in \mathcal{F}} 1_I(\xi)1_{I'}(\xi') = 1, \text{ for a.e. } (\xi, \xi') \in \mathbb{R} \times \mathbb{R}. \quad (18)$$

Since I and I' are maximal, $\text{dist}(I, I') \leq 10|I|$. This shows that for a given $I \in \mathcal{D}$, there exists a bounded number of I' so that $(I, I') \in \mathcal{F}$, that is,

$$\#\{I' : (I, I') \in \mathcal{F}\} \lesssim 1 \quad \text{for all } I \in \mathcal{D}. \quad (19)$$

Proof of Lemma 1.2. Given $p > 1$, we normalize $\sup_{\tau \in \mathbb{R}} |\tau|^{1/2-1/p} \|\widehat{f}\|_{L^p(\tau)} = 1$. Then for all dyadic intervals $I \in \mathcal{D}$,

$$\int_I |\widehat{f}|^p d\xi \leq |I|^{1-p/2}. \quad (20)$$

We square the left side of (5) and reduce to proving

$$\left\| \iint e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{1/6} \widehat{f}(\xi) \overline{\widehat{f}}(\eta) d\xi d\eta \right\|_{L^3_{t,x}} \lesssim \|\widehat{f}\|_{L^2}^{4/3}. \quad (21)$$

We change variables $a := \xi - \eta$ and $b := \xi^3 - \eta^3$ and use the Hausdorff–Young inequality in both t and x , we need to show

$$\iint \frac{|\xi\eta|^{1/4} |\widehat{f}(\xi)\widehat{f}(\eta)|^{3/2}}{|\xi + \eta|^{1/2} |\xi - \eta|^{1/2}} d\xi d\eta \lesssim \int |\widehat{f}|^2 d\xi. \quad (22)$$

By symmetries of this expression, it is sufficient to work in the region $\{(\xi, \eta) : \xi \geq 0, \eta \geq 0\}$. In this case, $|\xi\eta|^{1/4} \lesssim |\xi + \eta|^{1/2}$; so we reduce to proving

$$\iint \frac{|\widehat{f}(\xi)\widehat{f}(\eta)|^{3/2}}{|\xi - \eta|^{1/2}} d\xi d\eta \lesssim \int |\widehat{f}|^2 d\xi. \quad (23)$$

In view of (23), we assume $\widehat{f} \geq 0$ from now on. Then we apply the Whitney decomposition to obtain

$$\widehat{f}(\xi)\widehat{f}(\eta) = \sum_{(I, I') \in \mathcal{F}} \widehat{f}_I(\xi)\widehat{f}_{I'}(\eta), \text{ for a. e. } (\xi, \eta) \in \mathbb{R} \times \mathbb{R}, \quad (24)$$

and

$$\text{for all } (\xi, \eta) \in I \times I' \text{ with } (I, I') \in \mathcal{F}, |\xi - \eta| \sim |I|. \quad (25)$$

If we choose a slightly larger dyadic interval containing both I and I' but still of length comparable to I , still denoted by I , we reduce to proving

$$\sum_{I \in \mathcal{D}} \frac{\left(\int \widehat{f}_I^{3/2} d\xi \right)^2}{|I|^{1/2}} \lesssim \int \widehat{f}^2 d\xi. \quad (26)$$

To prove (26) we will make a further decomposition to $f_I = \sum_{n \in \mathbb{Z}} f_{n,I}$: for any $n \in \mathbb{Z}$, define $f_{n,I}$ via

$$\widehat{f_{n,I}} := \widehat{f} \mathbf{1}_{\{\xi: 2^n |I|^{-1/2} \leq \widehat{f}(\xi) \leq 2^{n+1} |I|^{-1/2}\}}.$$

By the Cauchy–Schwarz inequality, for any $\varepsilon > 0$,

$$\left(\int \widehat{f}_I^{3/2} d\xi \right)^2 = \left(\sum_{n \in \mathbb{Z}} \int \widehat{f_{n,I}}^{3/2} d\xi \right)^2 \lesssim_\varepsilon \sum_{n \in \mathbb{Z}} 2^{|n|\varepsilon} \left(\int \widehat{f_{n,I}}^{3/2} d\xi \right)^2. \quad (27)$$

Now (26) is an easy consequence of the following claim:

$$\sum_{I \in \mathcal{D}} \frac{\left(\int \widehat{f_{n,I}}^{3/2} d\xi \right)^2}{|I|^{1/2}} \lesssim 2^{-|n|\varepsilon} \int \widehat{f}^2 d\xi, \text{ for some } \varepsilon > 0. \quad (28)$$

By the Cauchy–Schwarz inequality,

$$\left(\int \widehat{f_{n,I}}^{3/2} d\xi \right)^2 \lesssim \int \widehat{f_{n,I}}^2 d\xi \int \widehat{f_{n,I}} d\xi. \quad (29)$$

On the one hand, when $n \geq 0$, by the Chebyshev's inequality and (20),

$$\begin{aligned} \int \widehat{f_{n,I}} d\xi &\lesssim 2^n |I|^{-1/2} |\{\xi \in I : \widehat{f}(\xi) \geq 2^n |I|^{-1/2}\}| \\ &\lesssim 2^n |I|^{-1/2} \frac{\int_I \widehat{f}^p d\xi}{2^{np} |I|^{-p/2}} \\ &\lesssim 2^{n(1-p)} |I|^{-1/2} |I|^{p/2} |I|^{1-p/2} \\ &= 2^{-|n|(p-1)} |I|^{1/2}, \end{aligned}$$

for any $p > 1$. On the other hand, when $n < 0$,

$$\int \widehat{f_{n,I}} d\xi \lesssim 2^n |I|^{-1/2} |I| = 2^{-|n|} |I|^{1/2}.$$

Combining these estimates, there exists an $\varepsilon > 0$ such that

$$\sum_{I \in \mathcal{D}} \frac{\left(\int \widehat{f_{n,I}}^{3/2} d\xi \right)^2}{|I|^{1/2}} \lesssim 2^{-|n|\varepsilon} \sum_{I \in \mathcal{D}} \int \widehat{f_{n,I}}^2 d\xi. \quad (30)$$

Interchanging the summation order, we have

$$\sum_{I \in \mathcal{D}} \int \widehat{f_{n,I}}^2 d\xi = \sum_{j \in \mathbb{Z}} \sum_{I \in \mathcal{D}_j} \int \widehat{f}^2 1_{\{\xi \in I: \widehat{f} \sim 2^{n-j/2}\}} d\xi = \int_{\mathbb{R}} \sum_{j: \widehat{f} \sim 2^{n-j/2}} \widehat{f}^2 d\xi \lesssim \int \widehat{f}^2 d\xi. \quad (31)$$

Then the claim (28) follows from (30) and (31). Hence the proof of Lemma 1.2 is complete. \square

By using this refined Airy Strichartz inequality (5), we extract the scaling and frequency parameters ρ_n^j and ξ_n^j following the approach in [Carles and Keraani 2007].

Lemma 3.2 (Complex version: extraction of ρ_n^j and ξ_n^j). *Let $(u_n)_{n \geq 1}$ be a sequence of complex valued functions with $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence, for any $\delta > 0$, there exists $N := N(\delta)$, a family $(\rho_n^j, \xi_n^j)_{\substack{1 \leq j \leq N \\ n \geq 1}} \in (0, \infty) \times \mathbb{R}$ and a family $(f_n^j)_{\substack{1 \leq j \leq N \\ n \geq 1}}$ of L^2 -bounded sequences such that, if $j \neq k$,*

$$\lim_{n \rightarrow \infty} \left(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} \right) = \infty, \quad (32)$$

for every $1 \leq j \leq N$, there exists a compact K in \mathbb{R} such that

$$\sqrt{\rho_n^j} |\widehat{f_n^j}(\rho_n^j \xi + \xi_n^j)| \leq C_\delta 1_K(\xi), \quad (33)$$

and

$$u_n = \sum_{j=1}^N f_n^j + q_n^N, \quad (34)$$

which satisfies

$$\|D^{\frac{1}{6}} e^{-t\partial_x^3} q_n^N\|_{L_{t,x}^6} \leq \delta, \quad (35)$$

and

$$\lim_{n \rightarrow \infty} \left(\|u_n\|_{L^2}^2 - \left(\sum_{j=1}^N \|f_n^j\|_{L^2}^2 + \|q_n^N\|_{L^2}^2 \right) \right) = 0. \quad (36)$$

Proof. For $\gamma_n = (\rho_n, \xi_n) \in (0, \infty) \times \mathbb{R}$, we define $G_n : L^2 \rightarrow L^2$ by setting

$$G_n(f)(\xi) := \sqrt{\rho_n} f(\rho_n \xi + \xi_n).$$

We will induct on the Strichartz norm. If $\|D^{\frac{1}{6}} e^{-t\partial_x^3} u_n\|_{L_{t,x}^6} \leq \delta$, then there is nothing to prove. Otherwise, up to a subsequence, we have

$$\|D^{\frac{1}{6}} e^{-t\partial_x^3} u_n\|_{L_{t,x}^6} > \delta.$$

On the one hand, applying Lemma 1.2 with $p = 4/3$, we see that there exists a family of intervals $I_n^1 := [\xi_n^1 - \rho_n^1, \xi_n^1 + \rho_n^1]$ such that

$$\int_{I_n^1} |\widehat{u_n}|^{4/3} d\xi \geq C_1 \delta^4 (\rho_n^1)^{\frac{1}{3}},$$

where C_1 only depends on C , the constant in [Lemma 1.2](#); note that we used $\|u_n\|_{L^2} \leq 1$ here. On the other hand, for any $A > 0$,

$$\int_{I_n^1 \cap \{|\widehat{u}_n| > A\}} |\widehat{u}_n|^{4/3} d\xi \leq A^{-2/3} \|\widehat{u}_n\|_{L^2}^2 \leq A^{-2/3}.$$

Let $C_\delta := (C_1/2)^{-3/2} \delta^{-6}$. Then from the two considerations above, we have

$$\int_{I_n^1 \cap \{|\widehat{u}_n| \leq C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u}_n|^{4/3} d\xi \geq \frac{C_1}{2} \delta^4 (\rho_n^1)^{1/3}.$$

From the Hölder inequality, we have

$$\int_{I_n^1 \cap \{|\widehat{u}_n| \leq C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u}_n|^{4/3} d\xi \leq C_2 \left(\int_{I_n^1 \cap \{|\widehat{u}_n| \leq C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u}_n|^2 d\xi \right)^{2/3} (|I_n^1|)^{1/3}.$$

This yields

$$\int_{I_n^1 \cap \{|\widehat{u}_n| \leq C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u}_n|^2 d\xi \geq C' \delta^6,$$

where $C' > 0$ is some constant depending only on C_1 and C_2 . Define v_n^1 and γ_n^1 by

$$\widehat{v}_n^1 := \widehat{u}_n \mathbf{1}_{I_n^1 \cap \{|\widehat{u}_n| \leq C_\delta (\rho_n^1)^{-1/2}\}}, \quad \gamma_n^1 := (\rho_n^1, \xi_n^1).$$

Then $\|v_n^1\|_{L^2} \geq (C')^{1/2} \delta^3$. Also by the definition of G , we have

$$|G_n^1(\widehat{v}_n^1)(\xi)| = |(\rho_n^1)^{1/2} \widehat{v}_n^1(\rho_n^1 \xi + \xi_n^1)| \leq C_\delta \mathbf{1}_{[-1,1]}(\xi).$$

Moreover, since the supports are disjoint on the Fourier side, we have

$$\|u_n\|_{L^2}^2 = \|u_n - v_n^1\|_{L^2}^2 + \|v_n^1\|_{L^2}^2.$$

We repeat the same argument with $u_n - v_n^1$ in place of u_n . At each step, the L^2 -norm decreases by at least $(C')^{1/2} \delta^3$. Hence after $N := N(\delta)$ steps, we obtain $(v_n^j)_{1 \leq j \leq N}$ and $(\gamma_n^j)_{1 \leq j \leq N}$, so

$$u_n = \sum_{j=1}^N v_n^j + q_n^N, \quad \|u_n\|_{L^2}^2 = \sum_{j=1}^N \|v_n^j\|_{L^2}^2 + \|q_n^N\|_{L^2}^2,$$

where the latter equality is due to the disjoint Fourier supports. We have the error term estimate

$$\|D^{1/6} e^{-t\partial_x^3} q_n^N\|_{L_{t,x}^6} \leq \delta,$$

which gives [\(35\)](#). The properties we obtain now are almost the case except for the first point of this lemma [\(32\)](#). To obtain it, we will reorganize the decomposition. We impose the following condition on $\gamma_n^j := (\rho_n^j, \xi_n^j)$: γ_n^j and γ_n^k are orthogonal if

$$\lim_{n \rightarrow \infty} \left(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} \right) = \infty.$$

Then we define f_n^1 to be a sum of those v_n^j whose γ_n^j are not orthogonal to γ_n^1 . Then taking the least $j_0 \in [2, N]$ such that $\gamma_n^{j_0}$ is orthogonal to γ_n^1 , we can define f_n^2 to be a sum of those v_n^j whose γ_n^j are

orthogonal to γ_n^1 but not to $\gamma_n^{j_0}$. Repeating this argument a finite number of times, we obtain (34). This decomposition automatically gives (32). Since the supports of the functions are disjoint on the Fourier side, we also have (36). Finally we want to make sure that, up to a subsequence, (33) holds.

By construction, those v_n^j kept in the definition of f_n^j are such that the γ_n^j are not orthogonal to γ_n^1 , that is, for those j , we have

$$\lim_{n \rightarrow \infty} \frac{\rho_n^j}{\rho_n^1} + \frac{\rho_n^1}{\rho_n^j} < \infty, \quad \lim_{n \rightarrow \infty} \frac{|\zeta_n^j - \zeta_n^1|}{\rho_n^j} < \infty. \quad (37)$$

To show (33), it is sufficient to show that, up to a subsequence, $G_n^1(\widehat{v}_n^j)$ is bounded by a compactly supported and bounded function, which will imply (33) with $j = 1$. On the one hand, by construction,

$$|G_n^j(\widehat{v}_n^j)| \leq C_\delta 1_{[-1,1]}.$$

On the other hand, we observe that

$$G_n^1(\widehat{v}_n^j) = G_n^1(G_n^j)^{-1} G_n^j(\widehat{v}_n^j), \quad G_n^1(G_n^j)^{-1} f(\zeta) = \sqrt{\frac{\rho_n^1}{\rho_n^j}} f\left(\frac{\rho_n^1}{\rho_n^j} \zeta + \frac{\zeta_n^1 - \zeta_n^j}{\rho_n^j}\right),$$

which yields the desired estimate for $G_n^1(\widehat{v}_n^j)$ by (37). Inductively we obtain (32). Hence the proof of Lemma 3.2 is complete. \square

The following lemma is useful in upgrading the weak convergence of error terms to the strong convergence in the Strichartz norm in Lemma 3.5.

Lemma 3.3. *We have the following two localized restriction estimates: for $9/2 < q < 6$ and $\widehat{G} \in L^\infty(B(0, R))$ for some $R > 0$,*

$$\|D^{1/6} e^{-t\partial_x^3} G\|_{L_{t,x}^q} \leq C_{q,R} \|\widehat{G}\|_{L^\infty}. \quad (38)$$

For the same G , $4 \leq q < 6$ and $|\zeta_0| \geq 10R$,

$$\|e^{-t\partial_x^3} (e^{i(\cdot)\zeta_0} G)\|_{L_{t,x}^q} \leq C_{q,R} |\zeta_0|^{-1/q} \|\widehat{G}\|_{L^\infty}. \quad (39)$$

Proof. Let us start with the proof of (38). Let $q = 2r$ with $9/4 < r < 3$. After squaring, we are reduced to proving

$$\left\| \int_{B(0,R)} \int_{B(0,R)} e^{ix(\xi_1 - \xi_2) + it(\xi_1^3 - \xi_2^3)} |\xi_1 \xi_2|^{1/6} \widehat{G}(\xi_1) \overline{\widehat{G}(\xi_2)} d\xi_1 d\xi_2 \right\|_{L_{t,x}^r} \leq C_{q,R} \|\widehat{G}\|_{L^\infty(B(0,R))}^2.$$

Let $s_1 := \xi_1 - \xi_2$ and $s_2 := \xi_1^3 - \xi_2^3$ and denote the resulting image of $B(0, R) \times B(0, R)$ by Ω under this change of variables. Then by using the Hausdorff–Young inequality since $r > 2$, we see that the left side of the inequality above is bounded by

$$C \left(\int_{\Omega} \left| |\xi_1 \xi_2|^{1/6} \frac{\widehat{G}(\xi_1) \widehat{G}(\xi_2)}{|\xi_1^2 - \xi_2^2|} \right|^{r'} ds_1 ds_2 \right)^{\frac{1}{r'}}.$$

Then if we change variables back, we obtain

$$C \left(\int_{B(0,R) \times B(0,R)} \frac{|\xi_1 \xi_2|^{r'/6}}{|\xi_1 + \xi_2|^{r'-1} |\xi_1 - \xi_2|^{r'-1}} |\widehat{G}(\xi_1) \widehat{G}(\xi_2)|^{r'} d\xi_1 d\xi_2 \right)^{\frac{1}{r'}}.$$

As in the proof of [Lemma 1.2](#), we may assume that $\xi_1, \xi_2 \geq 0$. So we have $|\xi_1 \xi_2|^{\frac{1}{2}} \lesssim |\xi_1 + \xi_2|$, which leads to $(\xi_1 \xi_2)^{r'/6} \lesssim (\xi_1 + \xi_2)^{r'/3}$ and thus

$$\frac{|\xi_1 \xi_2|^{r'/6}}{|\xi_1 + \xi_2|^{r'-1} |\xi_1 - \xi_2|^{r'-1}} \lesssim \frac{1}{|\xi_1 - \xi_2|^{\frac{5}{3}r'-2}} + \frac{1}{|\xi_1 + \xi_2|^{\frac{5}{3}r'-2}}.$$

Then since $|\zeta|^{-\frac{5}{3}r'+2}$ is locally integrable when $3/2 < r' < 9/5$ and $\widehat{G} \in L^\infty$, we obtain [\(38\)](#).

The proof of [\(39\)](#) is similar. Setting $q = 2r$ with $2 \leq r < 3$ and following the same procedure as above, we have

$$\begin{aligned} \|e^{-t\partial_x^3}(e^{i(\cdot)\xi_0} G)\|_{L_{t,x}^q}^2 &= \|e^{-t\partial_x^3}(e^{i(\cdot)\xi_0} G) \overline{e^{-t\partial_x^3}(e^{i(\cdot)\xi_0} G)}\|_{L_{t,x}^r} \\ &= \left\| \int e^{ix(\xi-\eta) + it[(\xi+\xi_0)^3 - (\eta+\xi_0)^3]} \widehat{G}(\xi) \overline{\widehat{G}(\eta)} d\xi d\eta \right\|_{L_{t,x}^r} \\ &\lesssim \left(\int \frac{|\widehat{G}(\xi)|^{r'} |\widehat{G}(\eta)|^{r'}}{|\xi - \eta|^{r'-1} |\xi + \eta + 2\xi_0|^{r'-1}} d\xi d\eta \right)^{1/r'} \\ &\lesssim \left(\int \frac{|\widehat{G}(\xi)|^{r'} |\widehat{G}(\eta)|^{r'}}{|\xi - \eta|^{r'-1} |\xi_0|^{r'-1}} d\xi d\eta \right)^{1/r'} \\ &\leq C_{q,R} |\xi_0|^{-1+1/r'} \|\widehat{G}\|_{L^\infty}^2 \leq C_{q,R} |\xi_0|^{-2/q} \|\widehat{G}\|_{L^\infty}^2, \end{aligned}$$

where we have used $|\xi + \eta + 2\xi_0| \sim |\xi_0|$ since $\xi, \eta \in B(0, R)$ and $|\xi_0| \geq 10R$. \square

In [Lemma 3.2](#), we have determined the scaling and frequency parameters. Recall that from [Section 1](#), we are left with extracting the space and time translation parameters. For this purpose, we will apply the concentration–compactness argument. For simplicity, we present the following lemma of this kind adapted to Airy evolution but not involving the frequency and scaling parameters. The general case is similar and will be done in the next lemma.

Lemma 3.4 (Concentration–compactness). *Suppose $P := (P_n)_{n \geq 1}$ with $\|P_n\|_{L^2} \leq 1$. Then up to a subsequence, there exists a sequence $(\phi^\alpha)_{\alpha \geq 1} \in L^2$ and a family $(y_n^\alpha, s_n^\alpha) \in \mathbb{R}^2$ such that they satisfy the following constraints: for $\alpha \neq \beta$,*

$$\lim_{n \rightarrow \infty} (|y_n^\alpha - y_n^\beta| + |s_n^\alpha - s_n^\beta|) = \infty, \quad (40)$$

and for $A \geq 1$, there exists $P_n^A \in L^2$ so that

$$P_n(x) = \sum_{\alpha=1}^A e^{s_n^\alpha \partial_x^3} \phi^\alpha(x - y_n^\alpha) + P_n^A(x), \quad (41)$$

and

$$\lim_{A \rightarrow \infty} \mu(P^A) = 0,$$

where $\mu(P^A)$ is defined in the argument below; moreover we have the following almost orthogonality identity: for any $A \geq 1$,

$$\lim_{n \rightarrow \infty} \left(\|P_n\|_{L^2}^2 - \left(\sum_{\alpha=1}^A \|\phi^\alpha\|_{L^2}^2 + \|P_n^A\|_{L^2}^2 \right) \right) = 0. \quad (42)$$

Proof. Let $\mathcal{W}(P)$ be the set of weak limits of subsequences of P in L^2 after the space and time translations:

$$\mathcal{W}(P) := \{w\text{-}\lim_{n \rightarrow \infty} e^{-s_n \partial_x^3} P_n(x + y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2\}.$$

We set $\mu(P) := \sup\{\|\phi\|_{L^2} : \phi \in \mathcal{W}(P)\}$. Clearly we have

$$\mu(P) \leq \liminf_{n \rightarrow \infty} \|P_n\|_{L^2}.$$

If $\mu(P) = 0$, then there is nothing to prove. Otherwise $\mu(P) > 0$, then up to a subsequence, there exists a $\phi^1 \in L^2$ and a sequence $(y_n^1, s_n^1)_{n \geq 1} \in \mathbb{R}^2$ such that

$$\phi^1(x) = w\text{-}\lim_{n \rightarrow \infty} e^{-s_n^1 \partial_x^3} P_n(x + y_n^1) \text{ in } L^2, \quad (43)$$

and $\|\phi^1\|_{L^2} \geq \frac{1}{2}\mu(P)$. We set $P_n^1 := P_n - e^{s_n^1 \partial_x^3} \phi^1(x - y_n^1)$. Then since $e^{-t \partial_x^3}$ is an unitary operator on L^2 , we have

$$\begin{aligned} \|P_n^1\|_{L^2}^2 &= \langle P_n^1, P_n^1 \rangle_{L^2} \\ &= \langle P_n - e^{s_n^1 \partial_x^3} \phi^1(x - y_n^1), P_n - e^{s_n^1 \partial_x^3} \phi^1(x - y_n^1) \rangle_{L^2} \\ &= \langle e^{-s_n^1 \partial_x^3} (P_n - e^{s_n^1 \partial_x^3} \phi^1(x - y_n^1)), e^{-s_n^1 \partial_x^3} (P_n - e^{s_n^1 \partial_x^3} \phi^1(x - y_n^1)) \rangle_{L^2} \\ &= \langle e^{-s_n^1 \partial_x^3} P_n - \phi^1(x - y_n^1), e^{-s_n^1 \partial_x^3} P_n - \phi^1(x - y_n^1) \rangle_{L^2} \\ &= \langle e^{-s_n^1 \partial_x^3} P_n(x + y_n^1) - \phi^1(x), e^{-s_n^1 \partial_x^3} P_n(x + y_n^1) - \phi^1(x) \rangle_{L^2} \\ &= \langle P_n, P_n \rangle_{L^2} + \langle \phi^1, \phi^1 \rangle_{L^2} - \langle e^{-s_n^1 \partial_x^3} P_n(x + y_n^1), \phi^1 \rangle_{L^2} - \langle \phi^1, e^{-s_n^1 \partial_x^3} P_n(x + y_n^1) \rangle_{L^2}. \end{aligned}$$

Taking $n \rightarrow \infty$ and using (43), we see that

$$\lim_{n \rightarrow \infty} (\|P_n\|_{L^2}^2 - (\|\phi^1\|_{L^2}^2 + \|P_n^1\|_{L^2}^2)) = 0, \quad e^{-s_n^1 \partial_x^3} P_n^1(x + y_n^1) \rightarrow 0, \text{ weakly in } L^2.$$

We replace P_n with P_n^1 and repeat the same process: if $\mu(P^1) > 0$, we obtain ϕ^2 and $(y_n^2, s_n^2)_{n \geq 1}$ so that $\|\phi^2\|_{L^2} \geq \frac{1}{2}\mu(P^1)$ and

$$\phi^2(x) = w\text{-}\lim_{n \rightarrow \infty} e^{-s_n^2 \partial_x^3} P_n^1(x + y_n^2) \text{ in } L^2.$$

Moreover, $(y_n^1, s_n^1)_{n \geq 1}$ and $(y_n^2, s_n^2)_{n \geq 1}$ satisfy (40). Otherwise, up to a subsequence, we may assume that

$$\lim_{n \rightarrow \infty} s_n^2 - s_n^1 = s_0, \quad \lim_{n \rightarrow \infty} y_n^2 - y_n^1 = y_0,$$

where $(s_0, y_0) \in \mathbb{R}^2$. Then for any $\phi \in \mathcal{F}$,

$$\lim_{n \rightarrow \infty} \left\| e^{-(s_n^2 - s_n^1) \partial_x^3} \phi(x + (y_n^2 - y_n^1)) - e^{-s_0 \partial_x^3} \phi(x + y_0) \right\|_{L^2} = 0.$$

That is to say,

$$\left(e^{-(s_n^2 - s_n^1) \partial_x^3} \phi(x + (y_n^2 - y_n^1)) \right)_{n \geq 1}$$

converges strongly in L^2 . On the other hand, we rewrite,

$$e^{-s_n^2 \partial_x^3} P_n^1(x + y_n^2) = e^{-(s_n^2 - s_n^1) \partial_x^3} (e^{-s_n^1 \partial_x^3} P_n^1(x + y_n^1))(x + (y_n^2 - y_n^1)).$$

Now the strong convergence and weak convergence together yield $\phi^2 = 0$, hence $\mu(P^1) = 0$, a contradiction. Hence (40) holds.

Iterating this argument, a diagonal process produces a family of pairwise orthogonal sequences

$$(y_n^\alpha, s_n^\alpha)_{\alpha \geq 1} \quad \text{and} \quad (\phi^\alpha)_{\alpha \geq 1}$$

satisfying (41) and (42). From (42), $\sum_\alpha \|\phi^\alpha\|_{L^2}^2$ is convergent and hence $\lim_{\alpha \rightarrow \infty} \|\phi^\alpha\|_{L^2} = 0$. This gives

$$\lim_{A \rightarrow \infty} \mu(P^A) = 0,$$

since $\mu(P^A) \leq 2\|\phi^A\|_{L^2}$ by construction. □

We are ready to extract the space and time parameters of the profiles.

Lemma 3.5 (Complex version: extraction of $x_n^{j,\alpha}$ and $s_n^{j,\alpha}$). *Suppose an L^2 -bounded sequence $(f_n)_{n \geq 1}$ satisfies*

$$\sqrt{\rho_n} |\widehat{f_n}(\rho_n(\xi + (\rho_n)^{-1} \xi_n))| \leq F(\xi),$$

with $F \in L^\infty(K)$ for some compact set K in \mathbb{R} independent of n . Then up to a subsequence, there exists a family $(y_n^\alpha, s_n^\alpha) \in \mathbb{R} \times \mathbb{R}$ and a sequence $(\phi^\alpha)_{\alpha \geq 1}$ of L^2 functions such that, if $\alpha \neq \beta$,

$$\lim_{n \rightarrow \infty} \left(\left| y_n^\beta - y_n^\alpha + \frac{3(s_n^\beta - s_n^\alpha)(\xi_n)^2}{(\rho_n)^2} \right| + \left| \frac{3(s_n^\beta - s_n^\alpha)\xi_n}{\rho_n} \right| + |s_n^\beta - s_n^\alpha| \right) = \infty, \quad (44)$$

and for every $A \geq 1$, there exists $e_n^A \in L^2$,

$$f_n(x) = \sum_{\alpha=1}^A \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot)](\rho_n x - y_n^\alpha) + e_n^A(x), \quad (45)$$

and

$$\lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \|D^{\frac{1}{6}} e^{-t \partial_x^3} e_n^A\|_{L^6_{t,x}} = 0, \quad (46)$$

and for any $A \geq 1$,

$$\lim_{n \rightarrow \infty} \left(\|f_n\|_{L^2}^2 - \left(\sum_{\alpha=1}^A \|\phi^\alpha\|_{L^2}^2 + \|e_n^A\|_{L^2}^2 \right) \right) = 0. \quad (47)$$

Proof. Setting $P := (P_n)_{n \geq 1}$ with $\widehat{P}_n(\xi) := \sqrt{\rho_n} \widehat{f}_n(\rho_n(\xi + (\rho_n)^{-1} \xi_n))$. Then

$$\widehat{P}_n \in L^\infty(K).$$

Let $\mathcal{W}(P)$ be the set of weak limits in L^2 defined by

$$\mathcal{W}(P) := \left\{ w\text{-}\lim_{n \rightarrow \infty} e^{-ix\rho_n^{-1}\xi_n} e^{-s_n \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\xi_n} P_n(\cdot)](x + y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2 \right\},$$

and $\mu(P)$ as in the previous lemma. Then a similar concentration-compactness argument shows that, up to a subsequence, there exists a family $(y_n^\alpha, s_n^\alpha)_{\substack{\alpha \geq 1 \\ n \geq 1}}$ and $(\phi^\alpha)_{\alpha \geq 1} \in L^2$ such that (44) holds, and

$$P_n(x) = \sum_{\alpha=1}^A e^{-ix\rho_n^{-1}\xi_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot)](x - y_n^\alpha) + P_n^A(x).$$

As weak limits, each $\widehat{\phi}^\alpha$ has the same support as \widehat{P}_n , so does \widehat{P}_n^A . Furthermore, we may assume that $\widehat{\phi}^\alpha, \widehat{P}_n^A \in L^\infty(K)$. Setting $P^A := (P_n^A)_{n \geq 1}$. Then the sequence $(P^A)_{A \geq 1}$ satisfies

$$\lim_{A \rightarrow \infty} \mu(P^A) = 0. \quad (48)$$

For any $A \geq 1$, we also have

$$\lim_{n \rightarrow \infty} \left(\|P_n\|_{L^2}^2 - \left(\sum_{\alpha=1}^A \|\phi^\alpha\|_{L^2}^2 + \|P_n^A\|_{L^2}^2 \right) \right) = 0.$$

Since $f_n(x) = \sqrt{\rho_n} e^{ix\xi_n} P_n(\rho_n x)$, the decomposition (45) of f_n follows after setting

$$e_n^A(x) := \sqrt{\rho_n} e^{ix\xi_n} P_n^A(\rho_n x).$$

What remains to show is that

$$\lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \left\| D^{\frac{1}{6}} e^{-t\partial_x^3} [\sqrt{\rho_n} e^{iy\xi_n} P_n^A(\rho_n y)] \right\|_{L_{t,x}^6} = 0,$$

which will follow from (48) and the restriction estimates in Lemma 3.3 by an interpolation argument. Indeed, by scaling, it is equivalent to showing that

$$\lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \left\| D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A] \right\|_{L_{t,x}^6} = 0, \quad (49)$$

where $a_n := (\rho_n)^{-1} \xi_n$. Up to a subsequence, we split into two cases according to whether $\lim_{n \rightarrow \infty} |a_n| = \infty$ or not.

Case 1. $\lim_{n \rightarrow \infty} |a_n| = \infty$. By using the Hörmander–Mikhlin multiplier theorem [Tao 2006a, Theorem 4.4], for sufficiently large n , we have

$$\left\| D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A] \right\|_{L_{t,x}^6} \lesssim |a_n|^{1/6} \left\| e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A] \right\|_{L_{t,x}^6}.$$

We will show that, after taking limits in n , the right hand side is bounded by $C_q \mu(P^A)^{1-q/6}$ for some $4 \leq q < 6$. Then $\lim_{A \rightarrow \infty} \mu(P^A) = 0$ yields the result. We choose a cut-off $\chi_n(t, x) := \chi_{n,1}(t) \chi_{n,2}(x)$ satisfying

$$\chi_{n,2}(x) := \chi_2(x) e^{ix a_n}, \quad \chi_2 \in \mathcal{S},$$

where $\widehat{\chi}_2$ is compactly supported and $\widehat{\chi}_2(\xi) := 1$ on the common support K of \widehat{P}_n , and

$$\widehat{\chi}_{n,1}((\xi + a_n)^3) := \widehat{\chi}_1(\xi^3), \quad \chi_1 \in \mathcal{S},$$

where $\widehat{\chi}_1(\xi^3) := 1$ on $\text{Supp } \widehat{\chi}_2$. Let $*$ denote space-time convolution; then

$$\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)] = e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A). \quad (50)$$

Indeed, the space-time Fourier transform of χ_n is equal to

$$\widehat{\chi}_n(\tau, \xi) := \int e^{-it\tau - ix\xi} \chi_n(t, x) dt dx = \widehat{\chi}_2(\xi - a_n) \widehat{\chi}_{n,1}(\tau).$$

On the support of the space-time Fourier transform of $e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)$, we see that

$$\widehat{\chi}_n(\tau, \xi) \equiv 1.$$

This gives (50). Then by the Hölder inequality and the restriction estimate (39) in Lemma 3.3, for sufficiently large n ,

$$\begin{aligned} \|e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)\|_{L_{t,x}^6} &= \|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)]\|_{L_{t,x}^6} \\ &\lesssim \|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)]\|_{L_{t,x}^q}^{q/6} \|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)]\|_{L_{t,x}^\infty}^{1-q/6} \\ &\lesssim |a_n|^{-1/6} \|F\|_{L^\infty}^{q/6} \|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)]\|_{L_{t,x}^\infty}^{1-q/6}, \end{aligned}$$

for some $4 < q < 6$. There exists $(t_n, y_n)_{n \geq 1}$ such that

$$\|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)]\|_{L_{t,x}^\infty} \sim \left| \chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)](t_n, y_n) \right|.$$

We expand the right side out,

$$\left| \iint \chi_{n,1}(-t) \chi_{n,2}(-x) e^{-t\partial_x^3} [e^{-t_n\partial_x^3}(e^{i(\cdot)a_n} P_n^A)(\cdot + y_n)](x) dx dt \right|.$$

Setting $p_n(x) = e^{-t_n\partial_x^3}(e^{i(\cdot)a_n} P_n^A)(x + y_n)$, then it equals

$$\left| \iint \widehat{\chi}_1(\eta^3) \widehat{\chi}_2(\eta) e^{-ix\eta} d\eta e^{-ix a_n} p_n(x) dx \right| = \left| \int \chi_2(-x) e^{-ix a_n} p_n(x) dx \right|.$$

Taking $n \rightarrow \infty$, and using the definition of $\mathcal{W}(P^A)$ followed by the Cauchy–Schwarz inequality, we obtain

$$\lim_{n \rightarrow \infty} \|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n} P_n^A)]\|_{L_{t,x}^\infty} \lesssim \|\chi_2\|_{L^2} \mu(P^A) \lesssim \chi_2 \mu(P^A).$$

Hence the claim (49) follows.

Case 2. $\lim_{n \rightarrow \infty} |a_n| < \infty$. From the Hölder inequality, we have the $L_{t,x}^6$ norm in (49) is bounded by

$$\|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A]\|_{L_{t,x}^q}^{q/6} \|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A]\|_{L_{t,x}^\infty}^{1-q/6}$$

for some $4 < q < 6$. On the one hand, since $\lim_{n \rightarrow \infty} |a_n|$ is finite and $\widehat{P_n^A} \in L^\infty(K)$, there exists a large $R > 0$ so that

$$\text{Supp } \mathcal{F}[e^{i(\cdot)a_n} P_n^A] \subset B(0, R),$$

where $\mathcal{F}(f)$ denotes the spatial Fourier transform of f . Then from (38) in Lemma 3.3, we see that

$$\|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A]\|_{L_{t,x}^q} \leq C_{q,R} \|F\|_{L^\infty},$$

which is independent of n . On the other hand, from the Bernstein inequality, we have

$$\|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A]\|_{L_{t,x}^\infty} \leq C_{q,R} \|e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A]\|_{L_{t,x}^\infty}.$$

Then a similar argument as in Case 1 shows that $\|e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A]\|_{L_{t,x}^\infty}$ is bounded by $\mu(P^A)^c$ for some $c > 0$. Hence (49) follows and the proof of Lemma 3.5 is complete. \square

Remark 3.6. In view of the previous lemma, we will make a very useful reduction when $\lim_{n \rightarrow \infty} \rho_n^{-1} \zeta_n = a$ is finite: we will take $\zeta_n \equiv 0$. Indeed, we first replace $e^{i(\cdot)\rho_n^{-1}\zeta_n} \phi^\alpha$ with $e^{i(\cdot)a} \phi^\alpha$ by putting the difference into the error term; then we can reduce it further by regarding $e^{i(\cdot)a} \phi^\alpha$ as a new ϕ^α .

Next we will show that the profiles obtained in (45) are strongly decoupled under the orthogonality condition (44); more general version is in Lemma 5.2. Abusing notation, we define

$$\widetilde{g}_n^\alpha(\phi^\alpha)(x) := \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\zeta_n} \phi^\alpha(\cdot)](\rho_n x - y_n^\alpha),$$

where $\zeta_n \equiv 0$ when $\lim_{n \rightarrow \infty} \rho_n^{-1} \zeta_n$ is finite.

Corollary 3.7. Under (44), for any $\alpha \neq \beta$, we have

$$\lim_{n \rightarrow \infty} |(\widetilde{g}_n^\alpha(\phi^\alpha), \widetilde{g}_n^\beta(\phi^\beta))_{L^2}| = 0 \quad (51)$$

and for any $1 \leq \alpha \leq A$,

$$\lim_{n \rightarrow \infty} |(\widetilde{g}_n^\alpha(\phi^\alpha), e_n^A)_{L^2}| = 0. \quad (52)$$

Proof. Without loss of generality, we assume that ϕ^α and ϕ^β are Schwartz functions with compact Fourier support. We first prove (51). By changing variables, we have

$$\begin{aligned} |(\widetilde{g}_n^\alpha(\phi^\alpha), \widetilde{g}_n^\beta(\phi^\beta))_{L^2}| &= \left| \left\langle \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\zeta_n} \phi^\alpha(\cdot)](\rho_n x - y_n^\alpha), \sqrt{\rho_n} e^{s_n^\beta \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\zeta_n} \phi^\beta(\cdot)](\rho_n x - y_n^\beta) \right\rangle_{L^2} \right| \\ &= \left| \left\langle e^{-(s_n^\beta - s_n^\alpha) \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\zeta_n} \phi^\alpha(\cdot)](x + y_n^\beta - y_n^\alpha), e^{ix\rho_n^{-1}\zeta_n} \phi^\beta(x) \right\rangle_{L^2} \right| \\ &\leq \left(\left| \int e^{i\xi(x + y_n^\beta - y_n^\alpha + 3\frac{(s_n^\beta - s_n^\alpha)\zeta_n^2}{\rho_n^2}) + i\xi^3(s_n^\beta - s_n^\alpha) + 3i\xi^2\frac{(s_n^\beta - s_n^\alpha)\zeta_n}{\rho_n}} \widehat{\phi^\alpha}(\xi) d\xi \right|, |\phi^\beta| \right)_{L^2}. \end{aligned}$$

Hence if (44) holds, by using [Stein 1993, Corollary, p. 334] or integration by parts combined with the dominated convergence theorem, we conclude that this expression goes to zero as n goes to infinity.

To prove (52), we write

$$e_n^A = \sum_{\beta=A+1}^B \widetilde{g}_n^\beta(\phi^\beta) + e_n^B,$$

for any $B > A$. Recall

$$e_n^B = \sqrt{\rho_n} (e^{i(\cdot)\rho_n^{-1}\xi_n} P_n^B)(\rho_n x).$$

Then

$$|\langle \widetilde{g}_n^\alpha(\phi^\alpha), e_n^A \rangle_{L^2}| \leq \sum_{\beta=A+1}^B \left| \langle \widetilde{g}_n^\alpha(\phi^\alpha), \widetilde{g}_n^\beta(\phi^\beta) \rangle_{L^2} \right| + \left| \langle \phi^\alpha, e^{-ix\rho_n^{-1}\xi_n} e^{-s_n^\alpha \partial_x^3} (e^{i(\cdot)\rho_n^{-1}\xi_n} P_n^B)(x + y_n^\alpha) \rangle_{L^2} \right|.$$

When n goes to infinity, the first term goes to zero because of (51). The second term is less than $\|\phi^\alpha\|_{L^2} \mu(P^B)$ by the definitions of $\mathcal{W}(P^B)$ and $\mu(P^B)$, and the Cauchy–Schwarz inequality; so it can be made arbitrarily small if taking B large enough. Hence (52) is obtained by taking $B \rightarrow \infty$. \square

4. Preliminary decomposition: real version

To prove Theorem 1.6, we need the corresponding real version of lemmas in the previous section, especially of Lemmas 3.2 and 3.5. To develop the real analogue of Lemma 3.2, we recall the following lemma due to Kenig et al. [2000].

Lemma 4.1. *Let $u_0 \in L^2$ be a real-valued function with $\|u_0\|_{L^2} = 1$. Then for any $\delta > 0$, there exists a positive integer $N = N(\delta)$, real-valued functions f^1, \dots, f^N and e^N , intervals τ_1, \dots, τ_N , and a positive constant C_δ such that*

$$\widehat{f}^j(\xi) = \widehat{f}^j(-\xi), \quad \text{Supp } \widehat{f}^j \subset \tau_j \cup (-\tau_j), \quad |\tau_j| = \rho_j, \quad |\widehat{f}^j| \leq C_\delta \rho_j^{-1/2},$$

and

$$u_0 = \sum_{j=1}^N f^j + e^N,$$

with

$$\|u_0\|_{L^2}^2 = \sum_{j=1}^N \|f^j\|_{L^2}^2 + \|e^N\|_{L^2}^2, \quad \|D^{1/6} e^{-t\partial_x^3} e^N\|_{L_{t,x}^6} < \delta.$$

The proof of this lemma is similar to that of the previous Lemma 3.2 with the help that, for real function f , $\widehat{f} = \widehat{f}(-\xi)$. For our purpose, we will do a little more on the decomposition above. Indeed, from the proof in [Kenig et al. 2000] we know that $\widehat{f}^j(\xi) = 1_{\{\xi \in \tau_j \cup (-\tau_j): |\widehat{u}_0| \leq C_\delta \rho_j^{-1/2}\}} \widehat{u}_0(\xi)$ and $\tau_j \subset (0, \infty)$. We can decompose f^j further by setting

$$\begin{aligned} f^j &:= f^{j,+} + f^{j,-}, \\ \widehat{f}^{j,+} &:= 1_{\{\xi \in \tau_j: |\widehat{u}_0| \leq C_\delta \rho_j^{-1/2}\}} \widehat{u}_0, \\ \widehat{f}^{j,-} &:= 1_{\{\xi \in -\tau_j: |\widehat{u}_0| \leq C_\delta \rho_j^{-1/2}\}} \widehat{u}_0. \end{aligned}$$

Since u_0 is real, we have $\widehat{u}_0(\xi) = \widehat{u}_0(-\xi)$, which yields

$$\overline{f^{j,+}(\xi)} = \widehat{f}^{j,-}(-\xi), \quad \text{and} \quad f^{j,-} = \overline{f^{j,+}}.$$

Hence

$$f^j = 2 \operatorname{Re} f^{j,+}.$$

Now we return to prove [Theorem 1.6](#). We repeat the process above for each real-valued u_n to obtain v_n^1, \dots, v_n^N and real-valued e_n^N such that

$$u_n = \sum_{j=1}^N 2 \operatorname{Re}(v_n^j) + e_n^N, \quad (53)$$

with

$$\sqrt{\rho_n^j} |\widehat{v_n^j}(\rho_n^j \zeta + \zeta_n^j)| \leq C_\delta 1_K(\zeta), \quad \text{with } \zeta_n^j > 0, \text{ for some compact } K, \quad (54)$$

and

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^N 4 \|\operatorname{Re}(v_n^j)\|_{L^2}^2 + \|e_n^N\|_{L^2}^2. \quad (55)$$

Still we define the real version of the orthogonality condition on the sequence $(\rho_n^j, \zeta_n^j)_{n \geq 1} \in (0, +\infty)^2$ as before: for $j \neq k$,

$$\lim_{n \rightarrow \infty} \left(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\zeta_n^j - \zeta_n^k|}{\rho_n^j} \right) = \infty. \quad (56)$$

Based on [\(53\)](#) and [\(54\)](#), the basic idea of obtaining the real version is to apply the procedure in the previous section to v_n^j , and then take the real part. The only issue here is to show that the error term is still small in the Strichartz norm, and the almost orthogonality in L^2 norm still holds. We omit the details.

Lemma 4.2 (Real version: extraction of ρ_n^j and ζ_n^j). *Let $(u_n)_{n \geq 1}$ be a sequence of real-valued functions with $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence, for any $\delta > 0$, there exists $N = N(\delta)$, an orthogonal family $(\rho_n^j, \zeta_n^j)_{\substack{1 \leq j \leq N \\ n \geq 1}} \in (0, \infty)^2$ satisfying [\(56\)](#) and a sequence $(f_n^j)_{\substack{1 \leq j \leq N \\ n \geq 1}} \in L^2$ such that, for every $1 \leq j \leq N$, there is a compact set K in \mathbb{R} such that*

$$\sqrt{\rho_n^j} |\widehat{f_n^j}(\rho_n^j \zeta + \zeta_n^j)| \leq C_\delta 1_K(\zeta), \quad (57)$$

and for any $N \geq 1$, there exists a real-valued $q_n^N \in L^2$ such that

$$u_n = 2 \sum_{j=1}^N \operatorname{Re}(f_n^j) + q_n^N, \quad (58)$$

with

$$\|D^{\frac{1}{6}} e^{-t\partial_x^3} q_n^N\|_{L_{t,x}^6} \leq \delta, \quad (59)$$

and for any $N \geq 1$,

$$\lim_{n \rightarrow \infty} \left(\|u_n\|_{L^2}^2 - \left(\sum_{j=1}^N 4 \|\operatorname{Re}(f_n^j)\|_{L^2}^2 + \|q_n^N\|_{L^2}^2 \right) \right) = 0. \quad (60)$$

Then we focus on decomposing f_n^j further as in [Lemma 3.5](#). Taking real parts automatically produces a decomposition for $\operatorname{Re}(f_n^j)$. We will be sketchy on how to resolve issues of the convergence of the error term and the almost L^2 orthogonality.

Lemma 4.3 (Real version: extraction of $x_n^{j,\alpha}$ and $s_n^{j,\alpha}$). *Let $(f_n)_{n \geq 1} \in L^2$ be a sequence of real-valued functions satisfying $\|f_n\|_{L^2} \leq 1$ and*

$$\sqrt{\rho_n} \left| \widehat{f_n}(\rho_n(\xi + (\rho_n)^{-1}\xi_n)) \right| \leq F(\xi),$$

with $F \in L^\infty(K)$ for some compact set K and $\xi_n > 0$. Then up to a subsequence, there exists a family $(y_n^\alpha, s_n^\alpha) \in \mathbb{R} \times \mathbb{R}$ and a sequence of complex-valued functions $(\phi^\alpha)_{\alpha \geq 1} \in L^2$ such that, if $\alpha \neq \beta$,

$$\lim_{n \rightarrow \infty} \left(\left| y_n^\beta - y_n^\alpha + \frac{3(s_n^\beta - s_n^\alpha)(\xi_n)^2}{(\rho_n)^2} \right| + \left| \frac{3(s_n^\beta - s_n^\alpha)\xi_n}{\rho_n} \right| + |s_n^\beta - s_n^\alpha| \right) = \infty, \quad (61)$$

and for each $A \geq 1$, there exists $e_n^A \in L^2$ of complex-value such that

$$f_n(x) = \sum_{\alpha=1}^A \widetilde{g}_n^\alpha(\phi^\alpha)(x) + \text{Re}(e_n^A)(x), \quad (62)$$

where

$$\widetilde{g}_n^\alpha(\phi^\alpha)(x) = \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} [\text{Re}(e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha)](\rho_n x - y_n^\alpha),$$

with $\xi_n \equiv 0$ when $\rho_n^{-1}\xi_n$ converges to some finite limit, and

$$\lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \|D^{\frac{1}{6}} e^{-t \partial_x^3} \text{Re}(e_n^A)\|_{L_{t,x}^6} = 0, \quad (63)$$

and for any $A \geq 1$,

$$\lim_{n \rightarrow \infty} \left(\|f_n\|_{L^2}^2 - \left(\sum_{\alpha=1}^A \|\text{Re}(e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha)\|_{L^2}^2 + \|\text{Re}(e_n^A)\|_{L^2}^2 \right) \right) = 0. \quad (64)$$

Moreover, for any $\alpha \neq \beta$,

$$\lim_{n \rightarrow \infty} \left| \langle \widetilde{g}_n^\alpha(\phi^\alpha), \widetilde{g}_n^\beta(\phi^\beta) \rangle_{L^2} \right| = 0, \quad (65)$$

and for any $1 \leq \alpha \leq A$,

$$\lim_{n \rightarrow \infty} \left| \langle \widetilde{g}_n^\alpha(\phi^\alpha), \text{Re}(e_n^A) \rangle_{L^2} \right| = 0. \quad (66)$$

Proof. We briefly describe how to obtain these identities. Equations (61), (62) follow along similar lines as in Lemma 3.5. Equation (63) follows from (46) and the pointwise inequality

$$\left| D^{\frac{1}{6}} e^{-t \partial_x^3} \text{Re}(e_n^A)(x) \right| = \left| \text{Re}(D^{\frac{1}{6}} e^{-t \partial_x^3} e_n^A)(x) \right| \leq \left| D^{\frac{1}{6}} e^{-t \partial_x^3} e_n^A(x) \right|.$$

Equation (64) follows from (65) and (66), which are proven similarly as in Corollary 3.7. \square

5. Final decomposition: proof of Theorems 1.5 and 1.6

In this section, we will only prove the complex version Theorem 1.5 by following the approach in [Keraani 2001]; the real version Theorem 1.6 can be obtained similarly. We go back to the decompositions (34), (45) and set

$$(h_n^j, \xi_n^j, x_n^{j,\alpha}, t_n^{j,\alpha}) := ((\rho_n^j)^{-1}, \xi_n^j, (\rho_n^j)^{-1} y_n^{j,\alpha}, (\rho_n^j)^{-3} s_n^{j,\alpha}).$$

Then we use [Remark 3.6](#) and put all the error terms together,

$$u_n = \sum_{\substack{1 \leq j \leq N, \zeta_n^j \neq 0 \\ \text{or } |h_n^j \zeta_n^j| \rightarrow \infty}} \sum_{\alpha=1}^{A_j} e^{t_n^{j,\alpha} \partial_x^3} g_n^{j,\alpha} [e^{i(\cdot)h_n^j \zeta_n^j} \phi^{j,\alpha}] + w_n^{N, A_1, \dots, A_N}, \quad (67)$$

where $g_n^{j,\alpha} = g_{0, x_n^{j,\alpha}, t_n^j} \in G$ and

$$w_n^{N, A_1, \dots, A_N} = \sum_{j=1}^N e_n^{j, A_j} + q_n^N. \quad (68)$$

We enumerate the pairs (j, α) by ω satisfying

$$\omega(j, \alpha) < \omega(k, \beta) \text{ if } j + \alpha < k + \beta \text{ or } j + \alpha = k + \beta \text{ and } j < k. \quad (69)$$

After relabeling, [Equation \(67\)](#) can be further rewritten as

$$u_n = \sum_{\substack{1 \leq j \leq l, \zeta_n^j \neq 0 \\ \text{or } |h_n^j \zeta_n^j| \rightarrow \infty}} e^{t_n^j \partial_x^3} g_n^j [e^{i(\cdot)h_n^j \zeta_n^j} \phi^j] + w_n^l, \quad (70)$$

where $w_n^l = w_n^{N, A_1, \dots, A_N}$ with $l = \sum_{j=1}^N A_j$. To establish [Theorem 1.5](#), we are thus left with three points to investigate.

(i) The family $\Gamma_n^j = (h_n^j, \zeta_n^j, t_n^j, x_n^j)$ is pairwise orthogonal, that is, it satisfies [Definition 1.3](#). In fact, we have two possibilities:

(a) The two pairs are in the form $\Gamma_n^j = (h_n^i, \zeta_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^m, \zeta_n^m, t_n^{m,\beta}, x_n^{m,\beta})$ with $i \neq m$. In this case, the orthogonality follows from

$$\lim_{n \rightarrow \infty} \left(\frac{h_n^i}{h_n^m} + \frac{h_n^m}{h_n^i} + h_n^i |\zeta_n^i - \zeta_n^m| \right) = \infty,$$

which is [\(32\)](#) in [Lemma 3.2](#).

(b) The two pairs are in form $\Gamma_n^j = (h_n^i, \zeta_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^i, \zeta_n^i, t_n^{i,\beta}, x_n^{i,\beta})$ with $\alpha \neq \beta$. In this case, the orthogonality follows from

$$\lim_{n \rightarrow \infty} \left(\frac{|t_n^{i,\beta} - t_n^{i,\alpha}|}{(h_n^i)^3} + \frac{3|t_n^{i,\beta} - t_n^{i,\alpha}| |\zeta_n^i|}{(h_n^i)^2} + \left| \frac{x_n^{i,\beta} - x_n^{i,\alpha} + 3(t_n^{i,\beta} - t_n^{i,\alpha})(\zeta_n^i)^2}{h_n^i} \right| \right) = \infty,$$

which is [\(44\)](#) in [Lemma 3.5](#).

(ii) The almost orthogonality identity (8) is satisfied. In fact, combining (36) and (47), we obtain that for any $N \geq 1$,

$$\begin{aligned} \|u_n\|_{L^2}^2 &= \sum_{j=1}^N \left(\sum_{\alpha=1}^{A_j} \|\phi^{j,\alpha}\|_{L^2}^2 + \|e_n^{j,A_j}\|_{L^2}^2 \right) + \|q_n^N\|_{L^2}^2 + o_n(1) \\ &= \sum_{j=1}^N \left(\sum_{\alpha=1}^{A_j} \|\phi^{j,\alpha}\|_{L^2}^2 \right) + \|w_n^{N,A_1,\dots,A_N}\|_{L^2}^2 + o_n(1) = \sum_{j=1}^l \|\phi^j\|_{L^2}^2 + \|w_n^l\|_{L^2}^2 + o_n(1), \end{aligned}$$

where $\lim_{n \rightarrow \infty} o_n(1) = 0$. Note that we have used the fact that

$$\|w_n^l\|_{L^2}^2 = \|w_n^{N,A_1,\dots,A_N}\|_{L^2}^2 = \sum_{j=1}^N \|e_n^{j,A_j}\|_{L^2}^2 + \|q_n^N\|_{L^2}^2,$$

which is due to the disjoint supports on the Fourier side.

(iii) The remainder $e^{-t\partial_x^3} \omega_n^{N,A_1,\dots,A_N}$ converges to zero in the Strichartz norm. In view of the adapted enumeration, we have to prove that

$$\lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} \omega_n^{N,A_1,\dots,A_N}\|_{L_{t,x}^6} \rightarrow 0, \text{ as } \inf_{1 \leq j \leq N} \{N, j + A_j\} \rightarrow \infty. \quad (71)$$

Let $\delta > 0$ be an arbitrarily small number. Take N_0 such that, for every $N \geq N_0$,

$$\lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} q_n^N\|_{L_{t,x}^6} \leq \delta/3. \quad (72)$$

For every $N \geq N_0$, there exists B_N such that, whenever $A_j \geq B_N$,

$$\lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} e_n^{j,A_j}\|_{L_{t,x}^6} \leq \delta/3N. \quad (73)$$

The remainder w_n^{N,A_1,\dots,A_N} can be rewritten in the form

$$w_n^{N,A_1,\dots,A_N} = q_n^N + \sum_{1 \leq j \leq N} w_n^{j,A_j \vee B_N} + S_n^{N,A_1,\dots,A_N},$$

where $A_j \vee B_N := \max\{A_j, B_N\}$ and

$$S_n^{N,A_1,\dots,A_N} = \sum_{\substack{1 \leq j \leq N \\ A_j < B_N}} (w_n^{j,A_j} - w_n^{j,B_N}),$$

that is,

$$S_n^{N,A_1,\dots,A_N} = \sum_{\substack{1 \leq j \leq N \\ A_j < B_N}} \sum_{A_j < \alpha \leq B_N} e^{i_n^\alpha \partial_x^3} g_n^{j,\alpha} [e^{i(\cdot)h_n^j \xi_n^j} \phi^{j,\alpha}],$$

with $\xi_n^j \equiv 0$ when $\lim_{n \rightarrow \infty} |h_n^j \xi_n^j| < \infty$. From (72) and (73), it follows that

$$\lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} w_n^{N,A_1,\dots,A_N}\|_{L_{t,x}^6} \leq 2\delta/3 + \lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} S_n^{N,A_1,\dots,A_N}\|_{L_{t,x}^6}. \quad (74)$$

Now we need the following almost-orthogonality result:

Lemma 5.1. *Let $\Gamma_n^j = (h_n^j, \zeta_n^j, x_n^j, t_n^j)$ be a family of orthogonal sequences. Then for every $l \geq 1$,*

$$\lim_{n \rightarrow \infty} \left(\left\| \sum_{j=1}^l D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j [e^{i(\cdot)h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6}^6 - \sum_{j=1}^l \left\| D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j [e^{i(\cdot)h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6}^6 \right) = 0, \quad (75)$$

with $\zeta_n^j \equiv 0$ when $\lim_{n \rightarrow \infty} |h_n^j \zeta_n^j| < \infty$.

Suppose this lemma were proven, we show how to conclude the proof of (71). From Lemma 5.1, it follows that

$$\lim_{n \rightarrow \infty} \left\| D^{1/6} e^{-t\partial_x^3} S_n^{N, A_1, \dots, A_N} \right\|_{L_{t,x}^6}^6 = \sum_{\substack{1 \leq j \leq N \\ A_j < B_N}} \sum_{A_j < \alpha \leq B_N} \lim_{n \rightarrow \infty} \left\| D^{1/6} e^{-(t-t_n^{j,\alpha})\partial_x^3} g_n^{j,\alpha} [e^{i(\cdot)h_n^j \zeta_n^j} \phi^{j,\alpha}] \right\|_{L_{t,x}^6}^6. \quad (76)$$

The Strichartz inequality gives

$$\begin{aligned} \sum_{\substack{1 \leq j \leq N \\ A_j < B_N}} \sum_{A_j < \alpha \leq B_N} \left\| D^{1/6} e^{-(t-t_n^{j,\alpha})\partial_x^3} g_n^{j,\alpha} [e^{i(\cdot)h_n^j \zeta_n^j} \phi^{j,\alpha}] \right\|_{L_{t,x}^6}^6 &\lesssim \sum_{\substack{1 \leq j \leq N \\ A_j < B_N}} \sum_{A_j < \alpha \leq B_N} \|\phi^{j,\alpha}\|_{L^2}^6 \\ &\leq \sum_{j,\alpha} \|\phi^{j,\alpha}\|_{L^2}^6. \end{aligned} \quad (77)$$

On the other hand, $\sum_{j,\alpha} \|\phi^{j,\alpha}\|_{L^2}^2$ is convergent; hence the right side of (77) is finite. This shows that

$$\left(\sum_{\substack{j,\alpha \\ \alpha > A_j}} \left\| D^{1/6} e^{-(t-t_n^{j,\alpha})\partial_x^3} g_n^{j,\alpha} [e^{i(\cdot)h_n^j \zeta_n^j} \phi^{j,\alpha}] \right\|_{L_{t,x}^6}^6 \right)^{1/6} \leq \delta/3 \quad (78)$$

provided that $\inf_{1 \leq j \leq N} \{N, j + A_j\}$ is large enough. Combining (74), (76) and (78), we obtain

$$\lim_{n \rightarrow \infty} \left\| D^{1/6} e^{-t\partial_x^3} w_n^{N, A_1, \dots, A_N} \right\|_{L_{t,x}^6} = 0 \quad (79)$$

provided that $\inf_{1 \leq j \leq N} \{N, j + A_j\}$ is large enough. Hence the proof of (71) is complete.

Proof of Lemma 5.1. By using the Hölder inequality, we need to show that for $j \neq k$, as n goes to infinity,

$$\left\| D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j [e^{i(\cdot)h_n^j \zeta_n^j} \phi^j] D^{1/6} e^{-(t-t_n^k)\partial_x^3} g_n^k [e^{i(\cdot)h_n^k \zeta_n^k} \phi^k] \right\|_{L_{t,x}^3} \rightarrow 0. \quad (80)$$

By the pigeonhole principle, we can assume that ζ_n^j and ζ_n^k are of the same sign if they are not zero; moreover by a density argument, we also assume that ϕ^j and ϕ^k are Schwartz functions with compact Fourier supports. Evidence in favor of (80) is that, if $\lim_{n \rightarrow \infty} |h_n \zeta_n| = \infty$, $D^{1/6} e^{-(t-t_n)\partial_x^3} g_n [e^{i(\cdot)h_n \zeta_n} \phi]$ is somehow a Schrödinger wave in the sense of Remark 1.7. For the pairwise orthogonal Schrödinger waves, however, the analogous result to (80) is true; see [Merle and Vega 1998; Carles and Keraani 2007; Bégout and Vargas 2007].

To prove (80) we will have two possibilities. First, the two pairs are in the form $\Gamma_n^j = (h_n^i, \zeta_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^m, \zeta_n^m, t_n^{m,\beta}, x_n^{m,\beta})$ with $i \neq m$. In this case, the orthogonality is given by

$$\lim_{n \rightarrow \infty} \left(\frac{h_n^i}{h_n^m} + \frac{h_n^m}{h_n^i} + h_n^i |\zeta_n^i - \zeta_n^m| \right) = \infty.$$

So we have two subcases. We begin with the case where $\lim_{n \rightarrow \infty} h_n^i |\zeta_n^i - \zeta_n^m| = \infty$; moreover, we may assume that $h_n^i = h_n^m$ for all n (when both limits are infinity, the reasoning is similar, using the argument below). By changing variables, we see that the left side of (80) equals

$$\left\| D^{1/6} e^{-t \partial_x^3} (e^{i(\cdot) h_n^i \zeta_n^i} \phi^{i,\alpha}) D^{1/6} e^{-\left(t + \frac{t_n^{i,\alpha} - t_n^{m,\beta}}{(h_n^i)^3}\right) \partial_x^3} (e^{i(\cdot) h_n^m \zeta_n^m} \phi^{m,\beta}) \left(x + \frac{x_n^{m,\alpha} - x_n^{i,\beta}}{h_n^i}\right) \right\|_{L_{t,x}^3}. \quad (81)$$

The integrand above equals

$$\iint e^{ix[(\zeta + h_n^i \zeta_n^i) + (\eta + h_n^i \zeta_n^m)] + it[(\zeta + h_n^i \zeta_n^i)^3 + (\eta + h_n^i \zeta_n^m)^3]} |\zeta + h_n^i \zeta_n^i|^{1/6} |\eta + h_n^i \zeta_n^m|^{1/6} \\ \times e^{i(\eta + h_n^i \zeta_n^m)(x_n^{i,\alpha} - x_n^{m,\beta})/h_n^i + i(\eta + h_n^i \zeta_n^m)^3 (t_n^{i,\alpha} - t_n^{m,\beta})/(h_n^i)^3} \widehat{\phi^{i,\alpha}}(\zeta) \widehat{\phi^{m,\beta}}(\eta) d\zeta d\eta.$$

Applying the change of variables $a := (\zeta + h_n^i \zeta_n^i) + (\eta + h_n^i \zeta_n^m)$ and $b := (\zeta + h_n^i \zeta_n^i)^3 + (\eta + h_n^i \zeta_n^m)^3$, followed by the Hausdorff–Young inequality, we see that (81) is bounded by

$$C \left(\iint \frac{|\zeta + h_n^i \zeta_n^i|^{1/4} |\eta + h_n^i \zeta_n^m|^{1/4} |\widehat{\phi^{i,\alpha}}(\zeta) \widehat{\phi^{m,\beta}}(\eta)|^{3/2}}{|\zeta + h_n^i \zeta_n^i + \eta + h_n^i \zeta_n^m|^{1/2} |\zeta - \eta + h_n^i (\zeta_n^i - \zeta_n^m)|^{1/2}} d\zeta d\eta \right)^{2/3}.$$

We consider two subcases according to the limits of $|h_n^i \zeta_n^i|$ and $|h_n^m \zeta_n^m|$. Note that $\lim_{n \rightarrow \infty} h_n^i |\zeta_n^i - \zeta_n^m| = \infty$, then either both are infinity or only one is.

- In the former case, since ζ_n^i and ζ_n^m are of the same sign, we have

$$\frac{|\zeta + h_n^i \zeta_n^i|^{1/4} |\eta + h_n^i \zeta_n^m|^{1/4}}{|\zeta + \eta + h_n^i (\zeta_n^i + \zeta_n^m)|^{1/2}} \sim \frac{|\zeta_n^i \zeta_n^m|^{1/4}}{|\zeta_n^i + \zeta_n^m|^{1/2}} \lesssim 1.$$

Then (81) is further bounded by $C_{\phi^{i,\alpha}, \phi^{m,\beta}} (h_n^i |\zeta_n^i - \zeta_n^m|)^{-1/3}$, which goes to zero as n goes to infinity.

- In the latter case, say $\lim_{n \rightarrow \infty} |h_n^i \zeta_n^i| = \infty$, we will have $\zeta_n^m = 0$. Then

$$\frac{|\zeta + h_n^i \zeta_n^i|^{1/4} |\eta + h_n^i \zeta_n^m|^{1/4}}{|\zeta + \eta + h_n^i (\zeta_n^i + \zeta_n^m)|^{1/2}} \lesssim |h_n^i \zeta_n^i|^{-1/4}.$$

Then (81) is further bounded by $C_{\phi^{i,\alpha}, \phi^{m,\beta}} |h_n^i \zeta_n^i|^{-1/2}$, which goes to zero as n goes to infinity.

Under the first possibility, we still need to consider the case when

$$\lim_{n \rightarrow \infty} \left(\frac{h_n^i}{h_n^m} + \frac{h_n^m}{h_n^i} \right) = \infty.$$

We may assume that $\lim_{n \rightarrow \infty} |h_n^i \zeta_n^i - h_n^m \zeta_n^m| < \infty$. It follows that $\lim_{n \rightarrow \infty} |h_n^i \zeta_n^i|$ and $\lim_{n \rightarrow \infty} |h_n^m \zeta_n^m|$ are finite or infinite simultaneously. We will consider the case where they are both infinite since the other

follows similarly. Under this consideration, we deduce that

$$\left| \frac{h_n^m \xi_n^m}{h_n^i \xi_n^i} \right| \sim 1$$

for sufficiently large n . To prove (80), we will use the idea of regarding the profile term as a Schrödinger wave as in Remark 1.7. We recall that

$$\begin{aligned} D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] &= (h_n^i)^{-1/2} |\xi_n^i|^{1/6} e^{i\xi_n^i(x-x_n^{i,\alpha})+i(\xi_n^i)^3(t-t_n^{i,\alpha})} \\ &\quad \times \int e^{i\xi \left[\frac{x-x_n^{i,\alpha}}{h_n^i} + 3(\xi_n^i)^2 \frac{t-t_n^{i,\alpha}}{h_n^i} \right] + i\xi^3 \frac{t-t_n^{i,\alpha}}{(h_n^i)^3} + 3i\xi^2 \xi_n^i \frac{t-t_n^{i,\alpha}}{(h_n^i)^2}} \left| 1 + \frac{\xi}{h_n^i \xi_n^i} \right|^{1/6} \widehat{\phi^{i,\alpha}} d\xi, \end{aligned}$$

Similarly for $D^{1/6} e^{-(t-t_n^k)\partial_x^3} g_n^k [e^{i(\cdot)h_n^k \xi_n^k} \phi^k]$. For any $R > 0$, we set

$$\begin{aligned} A_R^i &:= \left\{ (t, x) \in \mathbb{R} \times \mathbb{R} : \left| 3\xi_n^i \frac{t-t_n^{i,\alpha}}{(h_n^i)^2} \right| + \left| \frac{x-x_n^{i,\alpha}}{h_n^i} + 3(\xi_n^i)^2 \frac{t-t_n^{i,\alpha}}{h_n^i} \right| \leq R \right\}, \\ A_R^m &:= \left\{ (t, x) \in \mathbb{R} \times \mathbb{R} : \left| 3\xi_n^m \frac{t-t_n^{m,\beta}}{(h_n^m)^2} \right| + \left| \frac{x-x_n^{m,\beta}}{h_n^m} + 3(\xi_n^m)^2 \frac{t-t_n^{m,\beta}}{h_n^m} \right| \leq R \right\}. \end{aligned}$$

By the Hölder inequality, the Strichartz inequality and Remark 1.7, we only need to show, for a large $R > 0$,

$$\lim_{n \rightarrow \infty} \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m \xi_n^m} \phi^k] \right\|_{L_{t,x}^3(A_R^i \cap A_R^m)} = 0. \quad (82)$$

Indeed, $\mathbb{R}^2 \setminus (A_R^i \cap A_R^m) \subset (\mathbb{R}^2 \setminus A_R^i) \cup (\mathbb{R}^2 \setminus A_R^m)$; here we only consider the integration over the region $\mathbb{R}^2 \setminus A_R^i$ since the other case is similar. By the Hölder inequality and the Strichartz inequality,

$$\begin{aligned} &\left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m \xi_n^m} \phi^k] \right\|_{L_{t,x}^3(\mathbb{R}^2 \setminus A_R^i)} \\ &\lesssim \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L_{t,x}^6(\mathbb{R}^2 \setminus A_R^i)} \left\| D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m \xi_n^m} \phi^k] \right\|_{L_{t,x}^6} \\ &\lesssim \|\phi^k\|_{L^2} \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L_{t,x}^6(\mathbb{R}^2 \setminus A_R^i)}. \end{aligned}$$

Let

$$x' := \frac{x - x_n^{i,\alpha} + 3(\xi_n^i)^2(t - t_n^{i,\alpha})}{h_n^i} \quad \text{and} \quad t' := \frac{3\xi_n^i(t - t_n^{i,\alpha})}{(h_n^i)^2}.$$

Then a change of variables and similar computations as in Remark 1.7 show that

$$\begin{aligned} \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L_{t,x}^6(\mathbb{R}^2 \setminus A_R^i)} &\lesssim \left\| \int e^{i(x'\xi + t'\xi^2) + i\frac{\xi^3 t'}{3h_n^i \xi_n^i}} \left| 1 + \frac{\xi}{h_n^i \xi_n^i} \right|^{1/6} \widehat{\phi^{i,\alpha}}(\xi) d\xi \right\|_{L_{t',x'}^6(|t'| + |x'| \geq R)} \\ &\rightarrow \|e^{-it'\Delta} \phi^{i,\alpha}\|_{L_{t',x'}^6(|t'| + |x'| \geq R)} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$ followed by $R \rightarrow \infty$. Returning to (82), using L^∞ -bounds for the integrands, we see that it is bounded by

$$\begin{aligned}
C \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L^\infty} &\left\| D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m \xi_n^m} \phi^k] \right\|_{L^\infty} \min\{|A_R^i|^{1/3}, |A_R^m|^{1/3}\} \\
&\leq C_{R,\phi^j,\phi^k} (h_n^i h_n^m)^{-1/2} |\xi_n^i \xi_n^m|^{1/6} \min\{[(h_n^i)^3 |\xi_n^i|^{-1}]^{1/3}, [(h_n^m)^3 |\xi_n^m|^{-1}]^{1/3}\} \\
&\leq C_{R,\phi^j,\phi^k} \min\left\{ \left(\frac{h_n^i}{h_n^m}\right)^{2/3} \left|\frac{h_n^m \xi_n^m}{h_n^i \xi_n^i}\right|^{1/6}, \left(\frac{h_n^m}{h_n^i}\right)^{2/3} \left|\frac{h_n^i \xi_n^i}{h_n^m \xi_n^m}\right|^{1/6} \right\}.
\end{aligned}$$

Hence (80) holds when $\lim_{n \rightarrow \infty} (h_n^i/h_n^m + h_n^m/h_n^i) = \infty$.

Secondly, the two pairs are in form $\Gamma_n^j = (h_n^i, \xi_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^i, \xi_n^i, t_n^{i,\beta}, x_n^{i,\beta})$, with $\alpha \neq \beta$. In this case, the orthogonality is given by

$$\lim_{n \rightarrow \infty} \left(\frac{|t_n^{i,\beta} - t_n^{i,\alpha}|}{(h_n^i)^3} + \frac{3|t_n^{i,\beta} - t_n^{i,\alpha}| |\xi_n^i|}{(h_n^i)^2} + \frac{|x_n^{i,\beta} - x_n^{i,\alpha}| + 3(t_n^{i,\beta} - t_n^{i,\alpha})(\xi_n^i)^2}{h_n^i} \right) = \infty.$$

We assume $\lim_{n \rightarrow \infty} |h_n^i \xi_n^i| = \infty$ since the other case is similar. We expand the left-hand side of (80) out, which is equal to

$$\begin{aligned}
(h_n^i)^{-\frac{4}{3}} &\left\| D^{1/6} e^{-\frac{t-t_n^i}{(h_n^i)^3} \partial_x^3} [e^{i(\cdot)h_n^i \xi_n^i} \phi^{i,\alpha}] \left(\frac{x - x_n^{i,\alpha}}{h_n^i}\right) D^{1/6} e^{-\frac{t-t_n^m}{(h_n^i)^3} \partial_x^3} [e^{i(\cdot)h_n^i \xi_n^i} \phi^{m,\beta}] \left(\frac{x - x_n^{m,\beta}}{h_n^i}\right) \right\|_{L_{i,x}^3} \\
&= \frac{|\xi_n^i|^{1/3}}{h_n^i} \left\| \int e^{i \left[\frac{\eta(x-x_n^{i,\alpha} + 3(t-t_n^{i,\alpha})(\xi_n^i)^2)}{h_n^i} + \frac{\eta^3(t-t_n^{i,\alpha})}{(h_n^i)^3} + \frac{3\eta^2(t-t_n^{i,\alpha})\xi_n^i}{(h_n^i)^2} \right]} \left| 1 + \frac{\eta}{h_n^i \xi_n^i} \right|^{1/6} \widehat{\phi}^{i,\alpha}(\eta) d\eta \right. \\
&\quad \times \left. \int e^{i \left[\frac{\eta(x-x_n^{i,\beta} + 3(t-t_n^{i,\beta})(\xi_n^i)^2)}{h_n^i} + \frac{\eta^3(t-t_n^{i,\beta})}{(h_n^i)^3} + \frac{3\eta^2(t-t_n^{i,\beta})\xi_n^i}{(h_n^i)^2} \right]} \left| 1 + \frac{\eta}{h_n^i \xi_n^i} \right|^{1/6} \widehat{\phi}^{i,\beta}(\eta) d\eta \right\|_{L_{i,x}^3}.
\end{aligned}$$

Through the change of variables $t' = \frac{3(t - t_n^{i,\beta})\xi_n^i}{(h_n^i)^2}$, $x' = \frac{x - x_n^{i,\beta} + 3(t - t_n^{i,\beta})(\xi_n^i)^2}{h_n^i}$, this reduces to

$$\begin{aligned}
C \left\| \int e^{i\eta \left[x' + \frac{i,\beta}{x_n^{i,\beta} - x_n^{i,\alpha} + 3(t-t_n^{i,\beta})(\xi_n^i)^2} \right]} + i\eta^3 \left[\frac{i,\beta}{(h_n^i)^3} - \frac{i,\alpha}{3h_n^i \xi_n^i} \right] + i\eta^2 \left[t' + \frac{3(t-t_n^{i,\beta})\xi_n^i}{(h_n^i)^2} \right] \right. \\
\left. \times \left| 1 + \frac{\eta}{h_n^i \xi_n^i} \right|^{1/6} \widehat{\phi}^{i,\alpha}(\eta) d\eta \int e^{ix'\eta + it'\eta^2} e^{i\eta^3 \frac{t'}{3h_n^i \xi_n^i}} \left| 1 + \frac{\eta}{h_n^i \xi_n^i} \right|^{1/6} \widehat{\phi}^{i,\beta}(\eta) d\eta \right\|_{L_{t',x'}^3}.
\end{aligned}$$

Using the Hölder inequality followed by the principle of the stationary phase or integration by parts, we see that (80) holds. \square

Similarly, we can obtain the following generalization of Corollary 3.7 about the orthogonality of profiles in L^2 space. Its proof will be omitted.

Lemma 5.2. Assume $\Gamma_n^j = (h_n^j, \xi_n^j, t_n^j, x_n^j)$ and $\Gamma_n^k = (h_n^k, \xi_n^k, t_n^k, x_n^k)$ are pairwise orthogonal. Then

$$\lim_{n \rightarrow \infty} \langle e^{t_n^j \partial_x^3} g_n^j [e^{i(\cdot)h_n^j \xi_n^j} \phi^j], e^{t_n^k \partial_x^3} g_n^k [e^{i(\cdot)h_n^k \xi_n^k} \phi^k] \rangle_{L^2} = 0, \quad (83)$$

and for $1 \leq j \leq l$,

$$\lim_{n \rightarrow \infty} \langle e^{t_n^j \partial_x^3} g_n^j [e^{i(\cdot)h_n^j \xi_n^j} \phi^j], w_n^l \rangle_{L^2} = 0, \quad (84)$$

with $\xi_n^j \equiv 0$ when $\lim_{n \rightarrow \infty} |h_n^j \xi_n^j| < \infty$.

6. The existence of maximizers for the symmetric Airy Strichartz inequality

This section is devoted to establishing [Theorem 1.9](#), a dichotomy result on the existence of maximizers for the symmetric Airy Strichartz inequality. First, we will exploit the idea of asymptotically embedding a Schrödinger solution into an approximate Airy solution. We will show that the best constant for the Airy Schrödinger Strichartz bounds that for the symmetric Schrödinger Strichartz inequality up to a constant. We will follow the approach in [\[Tao 2007\]](#), in which Tao shows that any qualitative scattering result on the mass critical gKdV equation $\partial_t u + \partial_x^3 u \pm |u|^4 \partial_x u = 0$ automatically implies an analogous scattering result for the mass critical nonlinear Schrödinger equation $i\partial_t u + \partial_x^2 u \pm |u|^4 u = 0$.

Lemma 6.1 (Asymptotic embedding of Schrödinger into Airy). *Corresponding to [Theorems 1.5](#) and [1.6](#), we have, respectively,*

$$S_{\text{schr}}^{\mathbb{C}} \leq 3^{1/6} S_{\text{airy}}^{\mathbb{C}}, \quad (85)$$

$$S_{\text{schr}}^{\mathbb{C}} \leq 2^{1/2} 3^{1/6} S_{\text{airy}}^{\mathbb{R}}. \quad (86)$$

Proof. We first prove [\(86\)](#). Let u_0 a maximizer to [\(16\)](#). Since $d=1$, from the work in [\[Foschi 2007\]](#), we can assume that u_0 is a standard Gaussian; hence it is even and its Fourier transform is another Gaussian. Denote

$$u_N(0, x) := \frac{1}{(3N)^{1/4}} \operatorname{Re} \left(e^{ixN} u_0 \left(\frac{x}{\sqrt{3N}} \right) \right).$$

Let $u_N(t, x)$ solve the Airy [Equation \(1\)](#) with initial data $u_N(0, x)$. From the Airy Strichartz inequality,

$$\|D^{1/6} u_N\|_{L_{t,x}^6} \leq S_{\text{airy}}^{\mathbb{R}} \|u_N(0, x)\|_{L^2}. \quad (87)$$

On the one hand, a computation shows that

$$\|u_N(0, x)\|_{L^2}^2 = \frac{1}{2} \int |u_0(x)|^2 + \operatorname{Re}(e^{2\sqrt{3}iN^{3/2}x} u_0^2(x)) dx. \quad (88)$$

From the Riemann–Lebesgue lemma, we know the second term above rapidly goes to zero as $N \rightarrow \infty$. On the other hand,

$$\widehat{u}_N(0, \xi) = \frac{(3N)^{1/4}}{2} (\widehat{u}_0(\sqrt{3N}(\xi - N)) + \widehat{u}_0(\sqrt{3N}(\xi + N))),$$

which yields

$$\begin{aligned} D^{1/6} u_N(t, x) &= \int e^{ix\xi + it\xi^3} |\xi|^{1/6} \widehat{u}_N(0, \xi) d\xi \\ &= \frac{(3N)^{1/4}}{2} \int e^{ix\xi + it\xi^3} |\xi|^{1/6} (\widehat{u}_0(\sqrt{3N}(\xi - N)) + \widehat{u}_0(\sqrt{3N}(\xi + N))) d\xi \\ &= 2^{-1} 3^{-1/4} N^{-1/12} e^{ixN + itN^3} \int e^{i[\eta((3N)^{-1/2}x + \sqrt{3}N^{3/2}t) + t\eta^2 + t(3N)^{-3/2}\eta^3]} \\ &\quad \times \left| 1 + \frac{\eta}{N\sqrt{3N}} \right|^{1/6} (\widehat{u}_0(\eta) + \widehat{u}_0(\eta + 2N\sqrt{3N})) d\eta. \end{aligned}$$

Changing variables $x' = (3N)^{-1/2}x + \sqrt{3}N^{3/2}t$ and $t' = t$, we obtain

$$\begin{aligned} & \|D^{1/6}u_N(t, x)\|_{L_{t,x}^6} \\ &= 2^{-1}3^{-1/6} \left\| \int e^{i[x'\eta + t'\eta^2 + t'(3N)^{-3/2}\eta^3]} \times \left| 1 + \frac{\eta}{N\sqrt{3N}} \right|^{1/6} (\widehat{u}_0(\eta) + \widehat{u}_0(\eta + 2N\sqrt{3N})) d\eta \right\|_{L_{t',x'}^6} \end{aligned} \quad (89)$$

Comparing (87), (88), (89) and letting $N \rightarrow \infty$, as in Remark 1.7, we obtain,

$$2^{-1}3^{-1/6} \left\| \int e^{ix'\eta + it'\eta^2} \widehat{u}_0(\eta) d\eta \right\|_{L_{t',x'}^6} \leq 2^{-1/2} S_{\text{airy}}^{\mathbb{R}} \|u_0\|_{L^2}. \quad (90)$$

By the choice of u_0 , we have

$$2^{-1}3^{-1/6} S_{\text{schr}}^{\mathbb{C}} \leq 2^{-1/2} S_{\text{airy}}^{\mathbb{R}},$$

that is, $S_{\text{schr}}^{\mathbb{C}} \leq 2^{1/2}3^{1/6} S_{\text{airy}}^{\mathbb{R}}$. Hence (86) follows. To show (85), we choose

$$\phi_N(x) := \frac{1}{(3N)^{1/4}} e^{ixN} u_0\left(\frac{x}{\sqrt{3N}}\right).$$

Then

$$\|\phi_N\|_{L^2} = \|u_0\|_{L^2}, \quad \|e^{-it\partial_x^2} \phi_N\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{R})} = S_{\text{schr}}^{\mathbb{C}} \|u_0\|_{L^2}.$$

Also an easy computation shows that

$$\|D^{1/6} e^{-t\partial_x^3} \phi_N\|_{L_{t,x}^6} \rightarrow 3^{-1/6} \|e^{-it\partial_x^2} u_0\|_{L_{t,x}^6}, \quad \text{as } N \rightarrow \infty.$$

From the Airy Strichartz inequality,

$$\|D^{1/6} e^{-t\partial_x^3} \phi_N\|_{L_{t,x}^6} \leq S_{\text{airy}}^{\mathbb{C}} \|\phi_N\|_{L^2} = S_{\text{airy}}^{\mathbb{C}} \|u_0\|_{L^2},$$

we conclude that (85) follows. \square

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. We only prove the complex version by using Theorem 1.5. For the real version, we use Theorem 1.6 instead but its proof is similar.

We choose a maximizing sequence $(u_n)_{n \geq 1}$ with $\|u_n\|_{L^2} = 1$, and decompose it into the linear profiles as in Theorem 1.5 to obtain

$$u_n = \sum_{\substack{1 \leq j \leq l, \zeta_n^j \equiv 0 \\ \text{or } |h_n^j \zeta_n^j| \rightarrow \infty}} e^{t_n^j \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] + w_n^l. \quad (91)$$

Then from the asymptotically vanishing Strichartz norm (7) and the triangle inequality, we obtain that, up to a subsequence, for any given $\varepsilon > 0$, there exists n_0 , for all $l \geq n_0$ and $n \geq n_0$,

$$\left\| \sum_{j=1}^l D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6} \geq S_{\text{airy}}^{\mathbb{C}} - \varepsilon,$$

with $\zeta_n^j \equiv 0$ when $\lim_{n \rightarrow \infty} |h_n^j \zeta_n^j| < \infty$. On the other hand, [Lemma 5.1](#) yields,

$$\left\| \sum_{j=1}^l D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6}^6 \leq \sum_{j=1}^l \left\| D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6}^6 + o_n(1). \quad (92)$$

Then up to a subsequence, there exists n_1 such that, for large $n \geq n_1$ and $l \geq n_1$,

$$\sum_{j=1}^l \left\| D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6}^6 \geq (S_{\text{airy}}^{\mathbb{C}})^6 - 2\varepsilon. \quad (93)$$

Choosing j_0 such that

$$D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} [e^{i(\cdot) h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}]$$

has the biggest Strichartz norm among $1 \leq j \leq l$, we see that, by Strichartz and the almost orthogonal identity [\(8\)](#),

$$\begin{aligned} (S_{\text{airy}}^{\mathbb{C}})^6 - 2\varepsilon &\leq \left\| D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} [e^{i(\cdot) h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}] \right\|_{L_{t,x}^6}^4 \sum_{j=1}^l \left\| D^{1/6} e^{-(t-t_n^j) \partial_x^3} g_n^j [e^{i(\cdot) h_n^j \zeta_n^j} \phi^j] \right\|_{L_{t,x}^6}^2 \\ &\leq \left\| D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} [e^{i(\cdot) h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}] \right\|_{L_{t,x}^6}^4 \sum_{j=1}^l (S_{\text{airy}}^{\mathbb{C}} \|\phi^j\|_{L^2})^2 \\ &\leq (S_{\text{airy}}^{\mathbb{C}})^2 \left\| D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} [e^{i(\cdot) h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}] \right\|_{L_{t,x}^6}^4. \end{aligned}$$

This yields,

$$\left\| D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} [e^{i(\cdot) h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}] \right\|_{L_{t,x}^6} \geq \left((S_{\text{airy}}^{\mathbb{C}})^{-2} [(S_{\text{airy}}^{\mathbb{C}})^6 - 2\varepsilon] \right)^{1/4} \geq S_{\text{airy}}^{\mathbb{C}} - \varepsilon. \quad (94)$$

Moreover, [\(8\)](#) implies that there exists $J > 0$ such that

$$\|\phi^j\|_{L^2} \leq 1/100 \quad \text{for all } j > J.$$

This, together with [\(94\)](#) and the Strichartz inequality

$$\left\| D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} [e^{i(\cdot) h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}] \right\|_{L_{t,x}^6} \leq S_{\text{airy}}^{\mathbb{C}} \|\phi^{j_0}\|_{L^2},$$

shows that, for ε small enough, j_0 is between 1 and J ; otherwise $S_{\text{airy}}^{\mathbb{C}}/2 \leq S_{\text{airy}}^{\mathbb{C}}/100$, a contradiction. Hence j_0 does not depend on l , n and ε . So we can freely take ε to zero without changing j_0 . Now we consider two cases:

Case I. When $h_n^{j_0} \zeta_n^{j_0} \rightarrow \zeta^{j_0} \in \mathbb{R}$, we can take $\zeta_n^{j_0} \equiv 0$. Then

$$\left\| D^{1/6} e^{-(t-t_n^{j_0}) \partial_x^3} g_n^{j_0} (\phi^{j_0}) \right\|_{L_{t,x}^6} = \left\| D^{1/6} e^{-t \partial_x^3} \phi^{j_0} \right\|_{L_{t,x}^6}.$$

Then we take $\varepsilon \rightarrow 0$ in (94) to obtain

$$\|\phi^{j_0}\|_{L^2} = 1, \quad S_{\text{airy}}^{\mathbb{C}} = \|D^{1/6} e^{-t\partial_x^3} \phi^{j_0}\|_{L_{t,x}^6}.$$

This shows that ϕ^{j_0} is a maximizer for (15).

Case II. When $|h_n^{j_0} \zeta_n^{j_0}| \rightarrow \infty$, we take $n \rightarrow \infty$ in (94) and use Remark 1.7,

$$\begin{aligned} S_{\text{airy}}^{\mathbb{C}} - \varepsilon &\leq \lim_{n \rightarrow \infty} \|D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} [e^{i(\cdot)h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}]\|_{L_{t,x}^6} \\ &= \lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}]\|_{L_{t,x}^6} \\ &= 3^{-1/6} \|e^{-it\partial_x^2} \phi^{j_0}\|_{L_{t,x}^6} \leq 3^{-1/6} S_{\text{schr}}^{\mathbb{C}} \|\phi^{j_0}\|_{L^2} \\ &\leq S_{\text{airy}}^{\mathbb{C}} \|\phi^{j_0}\|_{L^2}. \end{aligned}$$

Taking $\varepsilon \rightarrow 0$ forces all the inequality signs to be equal. Hence we obtain

$$\|\phi^{j_0}\|_{L^2} = 1, \quad S_{\text{airy}}^{\mathbb{C}} = 3^{-1/6} S_{\text{schr}}^{\mathbb{C}}$$

and

$$S_{\text{airy}}^{\mathbb{C}} = \lim_{n \rightarrow \infty} \|D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)h_n^{j_0} \zeta_n^{j_0}} \phi^{j_0}]\|_{L_{t,x}^6} = 3^{-1/6} \|e^{-it\partial_x^2} \phi^{j_0}\|_{L_{t,x}^6}.$$

This shows that $S_{\text{schr}}^{\mathbb{C}} = \|e^{-it\partial_x^2} \phi^{j_0}\|_{L_{t,x}^6}$; hence ϕ^{j_0} is a maximizer for (16). Set $a_n := h_n^{j_0} \zeta_n^{j_0}$. Then the proof of Theorem 1.9 is complete. \square

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References

- [Bahouri and Gérard 1999] H. Bahouri and P. Gérard, “High frequency approximation of solutions to critical nonlinear wave equations”, *Amer. J. Math.* **121**:1 (1999), 131–175. [MR 2000i:35123](#) [Zbl 0919.35089](#)
- [Bégout and Vargas 2007] P. Bégout and A. Vargas, “Mass concentration phenomena for the L^2 -critical nonlinear Schrödinger equation”, *Trans. Amer. Math. Soc.* **359**:11 (2007), 5257–5282. [MR 2008g:35190](#)
- [Bennett et al. 2008] J. Bennett, N. Bez, A. Carbery, and D. Hundertmark, “Heat-flow monotonicity of Strichartz norms”, preprint, 2008, Available at [arXiv:0809.4783](#).
- [Bourgain 1998] J. Bourgain, “Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity”, *Internat. Math. Res. Notices* **1998**:5 (1998), 253–283. [MR 99f:35184](#) [Zbl 0917.35126](#)
- [Carles and Keraani 2007] R. Carles and S. Keraani, “On the role of quadratic oscillations in nonlinear Schrödinger equations, II: The L^2 -critical case”, *Trans. Amer. Math. Soc.* **359**:1 (2007), 33–62. [MR 2008a:35260](#) [Zbl 1115.35119](#)
- [Carneiro 2008] E. Carneiro, “A sharp inequality for the Strichartz norm”, preprint, 2008, Available at [arXiv:0809.4054](#).
- [Christ et al. 2003] M. Christ, J. Colliander, and T. Tao, “Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations”, *Amer. J. Math.* **125**:6 (2003), 1235–1293. [MR 2005d:35223](#)

- [Fefferman 1983] C. L. Fefferman, “The uncertainty principle”, *Bull. Amer. Math. Soc. (N.S.)* **9**:2 (1983), 129–206. [MR 85f:35001](#) [Zbl 0526.35080](#)
- [Foschi 2007] D. Foschi, “Maximizers for the Strichartz inequality”, *J. Eur. Math. Soc. (JEMS)* **9**:4 (2007), 739–774. [MR 2008k:35389](#)
- [Hundertmark and Zharnitsky 2006] D. Hundertmark and V. Zharnitsky, “On sharp Strichartz inequalities in low dimensions”, *Int. Math. Res. Not.* **2006** (2006), Art. ID 34080, 18. [MR 2007b:35277](#) [Zbl 1131.35308](#)
- [Kenig and Merle 2006] C. E. Kenig and F. Merle, “Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case”, *Invent. Math.* **166**:3 (2006), 645–675. [MR 2007g:35232](#) [Zbl 1115.35125](#)
- [Kenig and Merle 2007] C. E. Kenig and F. Merle, “On the energy critical focusing non-linear wave equation”, pp. Exp. No. V, 14 in *Séminaire: Équations aux Dérivées Partielles. 2006–2007*, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2007. [MR 2385192](#)
- [Kenig et al. 1991] C. E. Kenig, G. Ponce, and L. Vega, “Oscillatory integrals and regularity of dispersive equations”, *Indiana Univ. Math. J.* **40**:1 (1991), 33–69. [MR 92d:35081](#) [Zbl 0738.35022](#)
- [Kenig et al. 2000] C. E. Kenig, G. Ponce, and L. Vega, “On the concentration of blow up solutions for the generalized KdV equation critical in L^2 ”, pp. 131–156 in *Nonlinear wave equations (Providence, RI, 1998)*, Contemp. Math. **263**, Amer. Math. Soc., Providence, RI, 2000. [MR 2001g:35224](#) [Zbl 0970.35125](#)
- [Keraani 2001] S. Keraani, “On the defect of compactness for the Strichartz estimates of the Schrödinger equations”, *J. Differential Equations* **175**:2 (2001), 353–392. [MR 2002j:35281](#)
- [Killip and Visan 2008a] R. Killip and M. Visan, “The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher”, preprint, 2008, Available at arXiv:0804.1018.
- [Killip and Visan 2008b] R. Killip and M. Visan, “Nonlinear Schrödinger equations at critical regularity”, Lecture notes, 2008. Summer School of Clay Mathematics Institute.
- [Killip et al. 2007] R. Killip, T. Tao, and M. Visan, “The cubic nonlinear Schrödinger equations in two dimension with radial data”, preprint, 2007, Available at arxiv:math.AP/0708.0849.
- [Kunze 2003] M. Kunze, “On the existence of a maximizer for the Strichartz inequality”, *Comm. Math. Phys.* **243**:1 (2003), 137–162. [MR 2004i:35006](#) [Zbl 1060.35133](#)
- [Merle and Vega 1998] F. Merle and L. Vega, “Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D”, *Internat. Math. Res. Notices* **1998**:8 (1998), 399–425. [MR 99d:35156](#) [Zbl 0913.35126](#)
- [Moyua et al. 1999] A. Moyua, A. Vargas, and L. Vega, “Restriction theorems and maximal operators related to oscillatory integrals in \mathbf{R}^3 ”, *Duke Math. J.* **96**:3 (1999), 547–574. [MR 2000b:42017](#) [Zbl 0946.42011](#)
- [Schindler and Tintarev 2001] I. Schindler and K. Tintarev, “An abstract version of the concentration compactness principle”, pp. 3531–3536 in *Proceedings of the Third World Congress of Nonlinear Analysts, Part 5 (Catania, 2000)*, vol. 47, 2001. [MR 2004b:35056](#) [Zbl 1042.35525](#)
- [Shao 2009] S. Shao, “Maximizers for the Strichartz inequalities and the Sobolev-Strichartz inequalities for the Schrödinger equation”, *Electronic Journal of Differential Equations* **2009**:3 (2009), 1–13.
- [Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, Princeton, NJ, 1993. [MR 95c:42002](#) [Zbl 0821.42001](#)
- [Strichartz 1977] R. S. Strichartz, “Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations”, *Duke Math. J.* **44**:3 (1977), 705–714. [MR 58 #23577](#) [Zbl 0372.35001](#)
- [Tao 2003] T. Tao, “A sharp bilinear restrictions estimate for paraboloids”, *Geom. Funct. Anal.* **13**:6 (2003), 1359–1384. [MR 2004m:47111](#) [Zbl 1068.42011](#)
- [Tao 2006a] T. Tao, “Course description for Math 247A: Fourier analysis”, 2006, Available at <http://www.math.ucla.edu/~tao/247a.1.06f/>.
- [Tao 2006b] T. Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics **106**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis. [MR 2008i:35211](#) [Zbl 1106.35001](#)

- [Tao 2007] T. Tao, “Two remarks on the generalised Korteweg-de Vries equation”, *Discrete Contin. Dyn. Syst.* **18**:1 (2007), 1–14. [MR 2007k:35428](#) [Zbl 1123.35059](#)
- [Tao et al. 2007] T. Tao, M. Visan, and X. Zhang, “Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions”, *Duke Math. J.* **140**:1 (2007), 165–202. [MR 2355070](#)
- [Tomas 1975] P. A. Tomas, “A restriction theorem for the Fourier transform”, *Bull. Amer. Math. Soc.* **81** (1975), 477–478. [MR 50 #10681](#) [Zbl 0298.42011](#)

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