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# GLOBAL EXISTENCE OF SMOOTH SOLUTIONS OF A 3D LOG-LOG ENERGY-SUPERCRITICAL WAVE EQUATION

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We prove global existence of smooth solutions of the 3D log-log energy-supercritical wave equation

$$\partial_{tt} u - \triangle u = -u^5 \log^c(\log(10+u^2))$$

with $0 < c < 8/225$ and smooth initial data $(u(0) = u_0,\ \partial_t u(0) = u_1)$. First we control the $L_t^4 L_x^{12}$ norm of the solution on an arbitrary size time interval by an expression depending on the energy and an a priori upper bound of its $L_t^\infty \tilde{H}^2(\mathbb{R}^3)$ norm, with $\tilde{H}^2(\mathbb{R}^3) := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$. The proof of this long time estimate relies upon the use of some potential decay estimates and a modification of an argument by Tao. Then we find an a posteriori upper bound of the $L_t^\infty \tilde{H}^2(\mathbb{R}^3)$ norm of the solution by combining the long time estimate with an induction on time of the Strichartz estimates.

## 1. Introduction

We shall consider the defocusing log-log energy-supercritical wave equation

$$\partial_{tt} u - \triangle u = -f(u) \tag{1-1}$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a real-valued scalar field and $f(u) := u^5 g(u)$ with $g(u) := \log^c(\log(10+u^2))$, $0 < c < 8/225$. *Classical solutions* of (1-1) are solutions that are infinitely differentiable and compactly supported in space for each fixed time $t$. It is not difficult to see that classical solutions of (1-1) satisfy the energy conservation law

$$E := \frac{1}{2}\int_{\mathbb{R}^3} (\partial_t u(t,x))^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 \, dx + \int_{\mathbb{R}^3} F(u(t,x))\, dx \tag{1-2}$$

where $F(u) := \int_0^u f(v)\, dv$. Classical solutions of (1-1) enjoy three symmetry properties that we use throughout this paper:

- *time translation invariance*: if $u$ is a solution of (1-1) and $t_0$ is a fixed time then $\tilde{u}(t,x) := u(t-t_0, x)$ is also a solution of (1-1);
- *space translation invariance*: if $u$ is a solution of (1-1) and $x_0$ is a fixed point lying in $\mathbb{R}^3$ then $\tilde{u}(t,x) := u(t, x-x_0)$ is also a solution of (1-1);
- *time reversal invariance*: if $u$ is a solution to (1-1) then $\tilde{u}(t,x) := u(-t,x)$ is also a solution.

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The defocusing log-log energy-supercritical wave equation (1-1) is closely related to the power-type defocusing wave equations, namely,

$$\partial_{tt} u - \triangle u = -|u|^{p-1} u. \tag{1-3}$$

Solutions of (1-3) have an invariant scaling

$$u(t,x) \to u^{\lambda}(t,x) := \frac{1}{\lambda^{2/(p-1)}} u\Big(\frac{t}{\lambda}, \frac{x}{\lambda}\Big) \tag{1-4}$$

and (1-3) is $s_c$-critical, where $s_c := \frac{3}{2} - \frac{2}{p-1}$. Thus the $\dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$ norm of $(u(0), \partial_t u(0))$ is invariant under scaling, i.e.,

$$\begin{aligned} \|u^{\lambda}(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)} &= \|u(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)}, \\ \|\partial_t u^{\lambda}(0)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)} &= \|\partial_t u(0)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)}. \end{aligned}$$

If $p = 5$, then $s_c = 1$ and this is why the quintic defocusing cubic wave equation

$$\partial_{tt} u - \triangle u = -u^5 \tag{1-5}$$

is called the energy-critical equation. If $1 < p < 5$ then $s_c < 1$ and (1-3) is energy-subcritical while if $p > 5$ then $s_c > 1$ and (1-3) is energy-supercritical. Notice that for every $p > 5$ there exists two positive constant $\lambda_1(p)$, $\lambda_2(p)$ such that

$$\lambda_1(p)|u|^5 \le |f(u)| \le \lambda_2(p) \max(1, |u|^p). \tag{1-6}$$

This is why (1-1) is said to belong to the group of barely supercritical equations. There is another way to see that. Notice that a simple integration by part shows that

$$F(u) \sim \frac{u^6}{6} g(u), \tag{1-7}$$

and consequently the nonlinear potential term of the energy $\int_{\mathbb{R}^3} F(u)\, dx \sim \int_{\mathbb{R}^3} u^6 g(u)\, dx$ just barely fails to be controlled by the linear component, in contrast to (1-5).

The energy-critical wave equation (1-5) has received a great deal of attention. Grillakis [1990; 1992] established global existence of smooth solutions (global regularity) of this equation with smooth initial data $u(0) = u_0$, $\partial_t u(0) = u_1$. His work followed that of Rauch [1981, part I] for small data and that of Struwe [1988] on the spherically symmetric case. Later Shatah and Struwe [1993] gave a simplified proof of this result. Kapitanski [1994] and, independently, Shatah and Struwe [1994] proved global existence of solutions with data $(u_0, u_1)$ in the energy class.

We are interested in proving global regularity of (1-1) with smooth initial data $(u_0, u_1)$. By standard persistence of regularity results it suffices to prove global existence of solutions

$$u \in \mathscr{C}\big([0,\ T],\ \tilde{H}^2(\mathbb{R}^3)\big) \cap \mathscr{C}^1\big([0,\ T], H^1(\mathbb{R}^3)\big),$$

with data $(u_0, u_1) \in \tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Here the following space

$$\tilde{H}^2(\mathbb{R}^3) := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3). \tag{1-8}$$

In view of the local well-posedness theory [Lindblad and Sogge 1995], standard limit arguments and the finite speed of propagation it suffices to find an a priori upper bound of the form

$$\left\|(u(T), \partial_t u(T))\right\|_{\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq C_1\left(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)},\ \|u_1\|_{H^1(\mathbb{R}^3)},\ T\right) \tag{1-9}$$

for all times $T > 0$ and for classical solutions $u$ of (1-1) with smooth and compactly supported data $(u_0, u_1)$. Here $C_1$ is a constant depending only on $\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}$, $\|u_1\|_{H^1(\mathbb{R}^3)}$ and the time $T$.

The global behavior of the solutions of the supercritical wave equations is poorly understood, mostly because of the lack of conservation laws in $\tilde{H}^2(\mathbb{R}^3)$. Nevertheless Tao [2007] was able to prove global regularity for another barely supercritical equation, namely

$$\partial_{tt} u - \triangle u = -u^5 \log(2+u^2), \tag{1-10}$$

with radial data. The main result of this paper is:

**Theorem 1.** *The solution of* (1-1) *with smooth data* $(u_0, u_1)$ *exists for all time. Moreover there exists a nonnegative constant* $M_0 = M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$ *depending only on* $\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}$ *and* $\|u_1\|_{H^1(\mathbb{R}^3)}$ *such that*

$$\|u\|_{L_t^\infty \tilde{H}^2(\mathbb{R}\times\mathbb{R}^3)} + \|\partial_t u\|_{L_t^\infty H^1(\mathbb{R}\times\mathbb{R}^3)} \leq M_0. \tag{1-11}$$

We recall some basic properties and estimates. Let $Q$ be a function, let $J$ be an interval and let $t_0 \in J$ be a fixed time. If $u$ is a *classical solution* of the more general problem $\partial_{tt} u - \triangle u = Q$ then $u$ satisfies the Duhamel formula

$$u(t) = u_{l,t_0}(t) + u_{nl,t_0}(t), \quad t \in J, \tag{1-12}$$

with $u_{l,t_0}$, $u_{nl,t_0}$ denoting the linear part and the nonlinear part respectively of the solution starting from $t_0$. Recall that

$$u_{l,t_0}(t) = \cos(t-t_0)D u(t_0) + \frac{\sin(t-t_0)D}{D}\partial_t u(t_0) \tag{1-13}$$

and

$$u_{nl,t_0}(t) = -\int_{t_0}^{t} \frac{\sin(t-t')D}{D} Q(t')\,dt', \tag{1-14}$$

with $D$ the multiplier defined by $\widehat{Df}(\xi) := |\xi|\widehat{f}(\xi)$. An explicit formula for $((\sin(t-t')D)/D)Q(t')$ and $t \neq t'$ is

$$\left[\frac{\sin(t-t')D}{D} Q(t')\right](x) = \frac{1}{4\pi|t-t'|}\int_{|x-x'|=|t-t'|} Q(t', x')\,dS(x'). \tag{1-15}$$

For a proof see [Sogge 1995]. We recall that $u_{l,t_0}$ satisfies

$$\partial_{tt} u_{l,t_0} - \triangle u_{l,t_0} = 0, \quad u_{l,t_0}(t_0) = u(t_0), \quad \partial_t u_{l,t_0}(t_0) = \partial_t u(t_0),$$

while $u_{nl,t_0}$ is the solution of

$$\partial_{tt} u_{nl,t_0} - \triangle u_{nl,t_0} = Q, \quad u_{nl,t_0}(t_0) = 0, \quad \partial_t u_{nl,t_0}(t_0) = 0.$$

We recall the Strichartz estimate [Ginibre and Velo 1995; Keel and Tao 1998; Lindblad and Sogge 1995; Sogge 1995]

$$\|u\|_{L_t^q L_x^r(J\times\mathbb{R}^3)} \lesssim \|\partial_t u(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|\nabla u(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(J\times\mathbb{R}^3)}, \tag{1-16}$$

if $(q,r)$ is wave admissible, that is, $(q,r)\in(2,\ \infty]\times[2,\ \infty]$ and $1/q+3/r=1/2$.

We set some notation that appears throughout the paper. We write $C=C(a_1,\dots,a_n)$ if $C$ only depends on the parameters $a_1,\dots,a_n$. We write $A\lesssim B$ if there exists a universal nonnegative constant $C'>0$ such that $A\leq C'B$. $A=O(B)$ means $A\lesssim B$. More generally we write $A\lesssim_{a_1,\dots,a_n} B$ if there exists a nonnegative constant $C'=C(a_1,\dots,a_n)$ such that $A\leq C'B$. We say that $C''$ is the constant determined by $\lesssim$ in $A\lesssim_{a_1,\dots,a_n} B$ if $C''$ is the smallest constant among the $C'$s such that $A\leq C'B$. We write $A\ll_{a_1,\dots,a_n} B$ if there exists a universal nonnegative small constant $c=c(a_1,\dots,a_n)$ such that $A\leq cB$. Similar notions are defined for $A\gtrsim B$, $A\gtrsim_{a_1,\dots,a_n} B$ and $A\gg B$. In particular we say that $C''$ is the constant determined by $\gtrsim$ in $A\gtrsim B$ if $C''$ is the largest constant among the $C'$s such that $A\geq C'B$. If $x$ is number then $x+$ and $x-$ are slight variations of $x$: $x+:=x+\alpha\epsilon$ and $x-:=x-\beta\epsilon$ for some $\alpha>0$, $\beta>0$ and $0<\epsilon\ll 1$.

Let $\Gamma_+$ denote the forward light cone

$$\Gamma_+=\{(t,x):t>|x|\}, \tag{1-17}$$

and if $J=[a,b]$ is an interval, let $\Gamma_+(J)$ denote the light cone truncated to $J$, that is,

$$\Gamma_+(J):=\Gamma_+\cap(J\times\mathbb{R}^3). \tag{1-18}$$

Let $e(t)$ denote the local energy, that is,

$$e(t):=\frac{1}{2}\int_{|x|\leq t}(\partial_t u(t,x))^2\,dx+\frac{1}{2}\int_{|x|\leq t}|\nabla u(t,x)|^2\,dx+\int_{|x|\leq t}F(u(t,x))\,dx. \tag{1-19}$$

If $u$ is a solution of (1-1) then by using the finite speed of propagation and the Strichartz estimates we have

$$\|u\|_{L_t^q L_x^r(\Gamma_+(J))}\lesssim\|\nabla u(b)\|_{L_x^2(\mathbb{R}^3)}+\|\partial_t u(b)\|_{L_x^2(\mathbb{R}^3)}+\|Q\|_{L_t^1 L_x^2(\Gamma_+(J))} \tag{1-20}$$

if $(q,r)$ is wave admissible. If $J_1:=[a_1,a_2]$ and $J_2:=[a_2,a_3]$ then we also have

$$\|u\|_{L_t^q L_x^r(\Gamma_+(J_1))}\lesssim\|\nabla u(a_3)\|_{L_x^2(\mathbb{R}^3)}+\|\partial_t u(a_3)\|_{L_x^2(\mathbb{R}^3)}+\|Q\|_{L_t^1 L_x^2(\Gamma_+(J_1\cup J_2))}. \tag{1-21}$$

We recall also the well-known Sobolev embeddings. If $h$ is a smooth function then

$$\|h\|_{L^\infty(\mathbb{R}^3)}\lesssim\|h\|_{\tilde{H}^2(\mathbb{R}^3)} \tag{1-22}$$

and

$$\|h\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla h\|_{L^2(\mathbb{R}^3)}. \tag{1-23}$$

If $u$ is the solution of (1-1) with data $(u_0,u_1)\in\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, then we get from (1-22)

$$E\lesssim\|u_0\|^2_{\tilde{H}^2(\mathbb{R}^3)}\max\bigl(1,\|u_0\|^4_{\tilde{H}^2(\mathbb{R}^3)}g(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)})\bigr). \tag{1-24}$$

We shall use the Paley–Littlewood technology. Let $\phi(\xi)$ be a bump function adapted to $\{\xi\in\mathbb{R}^3:|\xi|\leq 2\}$ and equal to one on $\{\xi\in\mathbb{R}^3:|\xi|\leq 1\}$. If $(M,N)\in 2^{\mathbb{Z}}\times 2^{\mathbb{Z}}$ are dyadic numbers then the Paley–Littlewood projection operators $P_M$, $P_{<N}$ and $P_{\geq N}$ are defined in the Fourier domain by

$$\widehat{P_M f}(\xi):=\biggl(\phi\Bigl(\frac{\xi}{M}\Bigr)-\phi\Bigl(\frac{\xi}{2M}\Bigr)\biggr)\hat{f}(\xi),\quad \widehat{P_{<N}f}(\xi):=\sum_{M<N}\widehat{P_M f}(\xi),\quad \widehat{P_{\geq N}f}(\xi):=\sum_{M\geq N}\widehat{P_M f}(\xi).$$

The inverse Sobolev inequality can be stated as follows:

**Proposition 2** (Inverse Sobolev inequality [Tao 2006]). *Let $g$ be a smooth function such that*

$$\|g\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim E^{1/2} l, \qquad \|P_{\geq N} g\|_{L_x^6(\mathbb{R}^3)} \gtrsim \eta,$$

*for some real number $\eta > 0$ and for some dyadic number $N > 0$. Then there exists a ball $B(x,r) \subset \mathbb{R}^3$ with $r = O(1/N)$ such that we have the mass concentration estimate*

$$\int_{B(x,r)} |g(y)|^2 \, dy \gtrsim \eta^3 E^{-1/2} r^2. \tag{1-25}$$

We also recall a result that shows that the mass of solutions of (1-1) can be locally in time controlled.

**Proposition 3** (Local mass is locally stable [Tao 2006]). *Let $J$ be a time interval, let $t$, $t' \in J$ and let $B(x,r)$ be a ball. Let $u$ be a solution of* (1-1). *Then*

$$\left( \int_{B(x,r)} |u(t',y)|^2 \, dy \right)^{1/2} = \left( \int_{B(x,r)} |u(t,y)|^2 \, dy \right)^{1/2} + O\big(E^{1/2} |t - t'|\big). \tag{1-26}$$

This result, proved for (1-5) in [Tao 2006], is also true for (1-1). Indeed the proof relied upon the fact that the $L^2(\mathbb{R}^3)$ norm of the velocity of the solution of (1-5) at time $t$ is bounded by the square root of its energy, which is also true for the solution of (1-1) (by (1-2) and (1-7)).

Now we make some comments with respect to Theorem 1. If the function $g$ were a positive constant, it would be easy to prove that the solution of (1-1) with data $(u_0, u_1)$ lies in $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, since we have a good global theory for (1-5). Therefore we can hope to prove global well-posedness for $g$ slowly increasing to infinity, by extending the technology to prove global well-posedness for (1-5). Notice also that Tao [2006] found that the solution $u$ of (1-5) satisfies

$$\|u\|_{L_t^4 L_x^{12}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \tilde{E}^{\tilde{E}^{O(1)}}, \tag{1-27}$$

with $\tilde{E}$ the energy of $u$. The structure of $g$ is a double log: it is, roughly speaking, the inverse function of the towel exponential bound in (1-27).

Now we explain the main ideas of this paper.

Tao [2006] was able to bound on arbitrary long time intervals the $L_t^4 L_x^{12}$ norm of solutions of the energy-critical equation (1-5) by a quantity that depends exponentially on their energy. This estimate can be viewed as a long time estimate. Unfortunately we cannot expect to prove a similar result for (1-1) since we are not in the energy-critical regime. However we shall prove the following proposition:

**Proposition 4** (Long time estimate). *Let $J = [t_1, t_2]$ be a time interval. Let $u$ be a classical solution of* (1-1). *Assume that*

$$\|u\|_{L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)} \leq M \tag{1-28}$$

*for some $M \geq 0$. Then there exist three constants $C_{L,0} > 0$, $C_{L,1} > 0$ and $C_{L,2} > 0$ such that*

- *if $E \ll \dfrac{1}{g^{1/2}(M)}$ (small energy regime) then*

$$\|u\|^4_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq C_{L,0}; \tag{1-29}$$

- *if $E \gtrsim \dfrac{1}{g^{1/2}(M)}$ (large energy regime) then*

$$\|u\|^4_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq \big(C_{L,1}(E g(M))\big)^{C_{L,2}(E^{193/4+} g^{225/8+}(M))}. \tag{1-30}$$

This proposition shows that we can control the $L_t^4 L_x^{12}(J \times \mathbb{R}^3)$ norm of solutions of (1-1) by their energy and an a priori bound of their $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$ norm. We would like to control the pointwise-in-time $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ norm of $u$ on an interval $[0, T]$, with $T$ arbitrarily large. This is done by an induction on time. We assume that this norm is controlled on $[0, T]$ by a number $M_0$. Then by continuity we can find a slightly larger interval $[0, T']$ such that this norm is bounded by (say) $2M_0$ on $[0, T']$. This is our a priori bound. We subdivide $[0, T']$ into subintervals where the $L_t^4 L_x^{12}$ norm of $u$ is small and we control the pointwise-in-time $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ norm of $u$ on each of these subintervals (see Lemma 6). Since $g$ varies slowly we can estimate the number of intervals of this partition by using Proposition 4 and we can prove a posteriori that $\|u(t)\|_{\tilde{H}^2(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\tilde{H}^1(\mathbb{R}^3)}$ is bounded on $[0, T']$ by $M_0$, provided that $M_0$ is large enough; see Section 2.

The proof of Proposition 4 is a modification of the argument used in [Tao 2006] to establish a tower-exponential bound of the $L_t^4 L_x^{12}(J \times \mathbb{R}^3)$ norm of $v$, the solution of (1-5). We divide $J$ into subintervals $J_i$ where the $L_t^4 L_x^{12}$ norm of $u$, the solution of (1-1), is "substantial". Then by using the Strichartz estimates and the Sobolev embedding (1-22) we notice that the $L_t^\infty L_x^6(J_i \times \mathbb{R}^3)$ norm of $u$ is also substantial, more precisely, we find a lower bound that depends on the energy $E$ and $g(M)$. Then by Proposition 2 we can localize a bubble where the mass concentrates and we prove that the size of these subintervals is also substantially large. Tao [2006] used the mass concentration to construct a solution $\tilde{v}$ of (1-5) that has a smaller energy than $v$ and that coincides with $v$ outside a cone. The idea behind that is to use an induction on the levels of energy, due to Bourgain [1999], and the small energy theory following from the Strichartz estimates in order to control the $L_t^4 L_x^{12}$ norm of $v$ outside a cone. Unfortunately it seems almost impossible to apply this procedure to our problem. Indeed the energy of the constructed solution $\tilde{u}$ is smaller than the energy $E$ of $u$ by an amount that depends on $E$ but also on $g(M)$ and therefore an induction on the levels of the energy is possible if the $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$ norm of $\tilde{u}$ can be controlled by $M$, which is far from being trivial. It turns out that we do not need to use the Bourgain induction method. Indeed since we know that the size of the subintervals $J_i$ s is substantially large and since we have a good control of the $L_t^4 L_x^{12}$ norm on these subintervals it suffices to find an upper bound of the size of their union in order to conclude. To this end we divide a cone containing the ball where the mass concentrates and the $J_i$s into truncated-in-time cones where the $L_t^4 L_x^{12}$ norm of $u$ is substantial. Let $\tilde{J}_1, \tilde{J}_2, \dots$ be the sequence of time intervals resulting from this partition. The mass concentration helps us to control the size of the first time interval $\tilde{J}_1$. By using an asymptotic stability result we can prove, roughly speaking, that if we consider two successive subintervals $\tilde{J}_j$, $\tilde{J}_{j+1}$ resulting from this partition of the cone then the size of $\tilde{J}_{j+1}$ can be controlled by the size of $\tilde{J}_j$; see (3-34). But a potential energy decay estimate shows that if the size of the union of the $J_i$ s is too large then we can find a large subinterval $[t_1', t_2']$ such that the $L_t^4 L_x^{12}$ norm of $u$ on the cone truncated to $[t_1', t_2']$ is small. Therefore $[t_1', t_2']$ cannot be covered by many $\tilde{J}_j$s and one of them is very large in comparison with its predecessor, which contradicts (3-34). At the end of the process we can find an upper bound of the size of the union of the subintervals $J_i$ s and consequently we can control the $L_t^4 L_x^{12}$ norm of $u$ on the interval $J$.

**Remark 5.** We will frequently use the $x+$ and $x-$ notations. Indeed the point $(2, \infty)$ is not wave admissible. Therefore we will work with the point $(2+, \infty-)$: see (5-6) and (7-9). This generates slight variations of many quantities throughout this paper. Sometimes we might deal with quantities like $z := x+/y-$. We cannot conclude directly that $z = (x/y)+$. In this case we create a variation of $y$ so small (compared to that of $x$) that we have $z = (x/y)+$. These details have been omitted for the sake of readability. We strongly recommend that the reader ignores these slight variations at the first reading.

## 2. Proof of Theorem 1

The proof relies upon Proposition 4 and the following lemma, which we prove on page 268.

**Lemma 6** (Local boundedness). *Let $J = [t_1, t_2]$ be an interval. Assume that $u$ is a classical solution of* (1-1). *Let $Z(t) := \|(u(t), \partial_t u(t))\|_{\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)}$. There exists $0 < \epsilon \ll$ constant such that if*

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \leq \frac{\epsilon}{g^{1/4}(Z(t_1))}, \tag{2-1}$$

*then there exists $C_l > 0$ such that*

$$Z(t) \leq 2C_l Z(t_1) \quad \textit{for } t \in J. \tag{2-2}$$

We claim that the set

$$\mathscr{F} := \Big\{T \in [0,\ \infty) : \sup_{t\in[0,T]} \big\|(u(t), \partial_t u(t))\big\|_{\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq M_0\Big\} \tag{2-3}$$

is equal to $[0,\ \infty)$ for some constant $M_0 := M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$ large enough. Indeed, $0 \in \mathscr{F}$ (this is clear); $\mathscr{F}$ is closed, by continuity; and $\mathscr{F}$ is open. To see this last fact, let $T \in \mathscr{F}$. Then by continuity there exists $\delta > 0$ such that

$$\sup_{t\in[0,T']} \|\,(u(t), \partial_t u(t))\,\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq 2M_0 \tag{2-4}$$

for every $T' \in [0, T+\delta)$. By (1-29) and (1-30) we have

$$\|u\|^4_{L_t^4 L_x^{12}([0,T']\times\mathbb{R}^3)} \leq \max\big(C_{L,0}, (C_{L,1} E\, g(2M_0))^{C_{L,2}(E^{(193/4)+} g^{(225/8)+}(2M_0))}\big). \tag{2-5}$$

Let $N \geq 1$ and let $\underline{Z}(0) := \max(Z(0),\ 1)$. Without loss of generality we can assume that $C_l \gg 1$ so that $2C_l\underline{Z}(0) \gg 1$ and $\log^c\big(2C_l\underline{Z}(0)\big) \gg 1$. We have, by the elementary rules of the logarithm and the inequality $\log^c(2nx) \leq \log^c((2n)^x)$ for $n \geq 1$ and $x \gg 1$:

$$\begin{aligned}\sum_{n=1}^{N} \frac{\epsilon^4}{g\,((2C_l)^n Z_0)} &\geq \sum_{n=1}^{N} \frac{\epsilon^4}{\log^c\big(\log((2C_l)^{2n}\underline{Z}^{2n}(0)+10)\big)} \gtrsim \sum_{n=1}^{N} \frac{1}{\log^c\big(2n\log(2C_l\underline{Z}(0))\big)} \\ &\gtrsim \frac{1}{\log^c\big(2C_l\underline{Z}(0)\big)} \sum_{n=1}^{N} \frac{1}{\log^c(2n)} \gtrsim \frac{1}{\log^c\big(2C_l\underline{Z}(0)\big)} \int_1^{N+1} \frac{1}{\log^c(2t)}\,dt \\ &\gtrsim \frac{1}{\log^c\big(2C_l\underline{Z}(0)\big)} \int_1^{N+1} \frac{1}{t^{1/2}}\,dt \gtrsim \frac{N^{1/2}}{\log^c\big(2C_l\underline{Z}(0)\big)}.\end{aligned} \tag{2-6}$$

By Lemma 6, (2-5) and (2-6) we can construct a partition $(J_n)_{1 \leq n \leq N}$ of $[0, T']$ such that

$$\|u\|_{L_t^4 L_x^{12}(J_n \times \mathbb{R}^3)} = \frac{\epsilon}{g^{1/4}\left(\left(2C_l\right)^n Z_0\right)}, \quad 1 \leq n < N,$$
$$\|u\|_{L_t^4 L_x^{12}(J_N \times \mathbb{R}^3)} \leq \frac{\epsilon}{g^{1/4}\left(\left(2C_l\right)^N Z_0\right)}, \quad Z(t) \leq (2C_l)^n Z(0),$$

for $t \in J_1 \cup \cdots \cup J_n$ and

$$\frac{N^{1/2}}{\log^c(2C_l \underline{Z}(0))} \leq \max\left(C_{L,0}, (C_{L,1} E\, g(2M_0))^{C_{L,2}(E^{193/4+} g^{225/8+}(2M_0))}\right). \tag{2-7}$$

Since $c < 8/225$ we have by (1-24)

$$\begin{aligned} \log N &\lesssim \log^c(2C_l \underline{Z}(0)) + \log(C_{L,0}) \\ &\qquad + C_{L,2} E^{(193/4)+} \log^{(225c/8)+} \log(10+4M_0^2) \log\left(C_{L,1} E \log^c \log(10+4M_0^2)\right) \\ &\leq \log\Big(\frac{\log(M_0/Z(0))}{\log(2C_l)}\Big), \end{aligned} \tag{2-8}$$

if $M_0 = M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R})})$ is large enough. To prove the last inequality in (2-8) it is enough, by using (1-24), to notice that $\lim_{M_0 \to \infty} f(M_0) = 0$ with

$$f(M_0) := \frac{\log^c(2C_l \underline{Z}(0)) + \log(C_{L,0}) + C_{L,2} E^{(193/4)+} \log^{(225c/8)+} \log(10+4M_0^2) \log(C_{L,1} E \log^c \log(10+4M_0^2))}{\log\left(\frac{\log(M_0/Z(0))}{\log(2C_l)}\right)}. \tag{2-9}$$

Therefore we conclude that

$$\sup_{t \in [0,T']} \|(u(t), \partial_t u(t))\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq (2C_l)^N Z(0) \leq M_0. \tag{2-10}$$

*Proof of Lemma 6.* By the Strichartz estimates (1-16), the Sobolev embeddings (1-22) and (1-23) and the elementary estimate $|u^5 \nabla(g(u))| \lesssim |u^4 \nabla u g(u)|$, we have

$$\begin{aligned} Z(t) &\lesssim Z(t_1) + \|u^5 g(u)\|_{L_t^1 L_x^2([t_1,t] \times \mathbb{R}^3)} + \|u^4 \nabla u g(u)\|_{L_t^1 L_x^2([t_1,t] \times \mathbb{R}^3)} + \|u^5 \nabla(g(u))\|_{L_t^1 L_x^2([t_1,t] \times \mathbb{R}^3)} \\ &\lesssim Z(t_1) + \|u^5 g(u)\|_{L_t^1 L_x^2([t_1,t] \times \mathbb{R}^3)} + \|u^4 \nabla u g(u)\|_{L_t^1 L_x^2([t_1,t] \times \mathbb{R}^3)} \\ &\lesssim Z(t_1) + \|u\|^4_{L_t^4 L_x^{12}([t_1,t] \times \mathbb{R}^3)} \|u\|_{L_t^\infty L_x^6([t_1,t] \times \mathbb{R}^3)} g\left(\|u\|_{L_t^\infty L_x^\infty([t_1,t] \times \mathbb{R}^3)}\right) \\ &\qquad + \|u\|^4_{L_t^4 L_x^{12}([t_1,t] \times \mathbb{R}^3)} \|\nabla u\|_{L_t^\infty L_x^6([t_1,t] \times \mathbb{R}^3)} g\left(\|u\|_{L_t^\infty L_x^\infty([t_1,t] \times \mathbb{R}^3)}\right) \\ &\lesssim Z(t_1) + \|u\|^4_{L_t^4 L_x^{12}([t_1,t] \times \mathbb{R}^3)} Z(t) g(Z(t)). \end{aligned} \tag{2-11}$$

Let $C_l$ be the constant determined by the last inequality in (2-11). From (2-1), (2-11) and a continuity argument, we have (2-2). □

## 3. Proof of Proposition 4

The proof relies upon five lemmas, which we state here and then prove in subsequent sections, after seeing how they imply the proposition.

**Lemma 7** (Long time estimate if energy small). *Let $J=[t_1,t_2]$ be a time interval. Let $u$ be a classical solution of* (1-1). *Assume that* (1-28) *holds. If*

$$E \ll \frac{1}{g^{1/2}(M)}, \tag{3-1}$$

*then*

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \lesssim 1. \tag{3-2}$$

**Lemma 8** (If $\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)}$ is nonnegligible a mass concentration bubble exists and the size of $J$ is bounded from below). *Let $u$ be a classical solution of* (1-1). *Let $J$ be a time interval. Assume that* (1-28) *holds. Let $\eta$ be a positive number such that*

$$\eta \le \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-3}$$

*If $\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \ge \eta$, then*

$$\|u\|_{L_t^\infty L_x^6(J\times\mathbb{R}^3)} \gtrsim \eta^{2+} E^{-((1/2)+)}. \tag{3-4}$$

*Moreover, there exist a point $x_0\in\mathbb{R}^3$, a time $t_0\in J$ and a positive number $r$ such that we have the mass concentration estimate in the ball $B(x_0,r)$*

$$\int_{B(x_0,r)} |u(t_0,y)|^2\,dy \gtrsim \eta^{6+} E^{-(2+)} r^2, \tag{3-5}$$

*and the following lower bound on the size of $J$:*

$$|J| \gtrsim \eta^4 E^{-2/3} r. \tag{3-6}$$

**Lemma 9** (Potential energy decay estimate). *Let $u$ be a classical solution of* (1-1). *Let $[a,b]$ be an interval. Then we have the potential energy decay estimate*

$$\int_{|x|\le b} F(u(b,x))\,dx \lesssim \frac{a}{b}\left(e(a)+e^{1/3}(a)\right) + e(b)-e(a) + (e(b)-e(a))^{1/3}. \tag{3-7}$$

**Lemma 10** ($L_t^4 L_x^{12}$ norm of $u$ is small on a large truncation of the forward light cone). *Let $J=[t_1,\ t_2]$ be an interval. Let $u$ be a classical solution of* (1-1). *Assume that* (1-28) *holds. Let $\eta$ be a positive number such that*

$$\eta \ll \min\left(E^{1/4}, E^{5/18}, \frac{E^{1/12}}{g^{5/24}(M)}\right). \tag{3-8}$$

*Assume also that there exists $C_2 \gg 1$ such that*

$$\left[t_1,\ (C_2 E^{10+}\eta^{-(36+)})^{4C_2 E^{10+}\eta^{-(36+)}} t_1\right] \subset J. \tag{3-9}$$

*Then there exists a subinterval $J'=[t_1',t_2']$ such that $\left|t_2'/t_1'\right| \sim E^{10+}\eta^{-(36+)}$ and*

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J'))} \le \eta. \tag{3-10}$$

**Lemma 11** (Asymptotic stability). *Let $J=[t_1,t_2]$ be a time interval. Let $J'=[t_1',\ t_2']\subset J$ and let $t\in J/J'$. Let $u$ be a classical solution of* (1-1). *Assume that* (1-28) *holds. Then*

$$\|u_{l,t_2'}(t) - u_{l,t_1'}(t)\|_{L_x^\infty(\mathbb{R}^3)} \lesssim \frac{E^{5/6} g^{1/6}(M)}{\operatorname{dist}^{1/2}(t,\ J')}. \tag{3-11}$$

We are ready to prove Proposition 4. We assume that we have an a priori bound $M$ of the $L_t^\infty \tilde{H}^2(J\times\mathbb{R}^3)$ norm of the solution $u$. There are two steps:

- If $E \ll 1/g^{1/2}(M)$, then we know from Lemma 7 that (1-29) holds.
- Therefore we assume that the energy is large, that is,

$$E \gtrsim \frac{1}{g^{1/2}(M)}. \tag{3-12}$$

We can assume without loss of generality that

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \geq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-13}$$

From (3-13) we can partition $J$ into subintervals $J_1, \dots, J_l$ such that for $i = 1, \dots, l-1$,

$$\|u\|_{L_t^4 L_x^{12}(J_i\times\mathbb{R}^3)} = \frac{E^{1/12}}{g^{5/24}(M)} \quad \text{and} \quad \|u\|_{L_t^4 L_x^{12}(J_l\times\mathbb{R}^3)} \leq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-14}$$

Before moving forward we say that an interval $J_i$ is *exceptional* if

$$\|u_{l,t_1}\|_{L_t^4 L_x^{12}(J_i\times\mathbb{R}^3)} + \|u_{l,t_2}\|_{L_t^4 L_x^{12}(J_i\times\mathbb{R}^3)} \geq \frac{1}{(C_3 E g(M))^{C_4(E^{(193/4)+} g^{(225/8)+}(M))}}, \tag{3-15}$$

for some $C_3 \gg 1$, $C_4 \gg 1$ to be chosen later. (The numbers 193/4 and 225/8 will play an important role in (3-44).) Otherwise $J_i$ is *unexceptional*. Let $\mathscr{E}$ denote the set of $J_i'$ s that are exceptional and let $\overline{\mathscr{E}^c}$ denote the set of nonempty sequences of consecutive unexceptional intervals $J_i$. By (1-16), (3-12) and (3-15),

$$\operatorname{card}(\mathscr{E}) \lesssim E^2 \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))} \lesssim \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))}. \tag{3-16}$$

Since $\operatorname{card}(\overline{\mathscr{E}^c}) \lesssim \operatorname{card}(\mathscr{E})$ we have

$$\|u\|^4_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \lesssim \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))} \left(\frac{E^{1/3}}{g^{5/6}(M)} + \sup_{K\in\overline{\mathscr{E}^c}} \|u\|^4_{L_t^4 L_x^{12}(K\times\mathbb{R}^3)}\right). \tag{3-17}$$

Let $K = J_{i_0} \cup \dots \cup J_{i_1}$ be a sequence of consecutive unexceptional intervals. If $N(K)$ is the number of $J_i$ s making $K$ then by (3-12), (3-14) and (3-17) we have

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \lesssim \left(\sup_{K\in\overline{\mathscr{E}^c}} N(K)\right) \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))}. \tag{3-18}$$

Therefore it suffices to estimate $N(K)$ for every $K = J_{i_0} \cup \dots \cup J_{i_1}$. We will do that by first determining a lower bound for the size of the elements $J_i$s and then by estimating the size of $K$. By (3-12), (3-14) and Lemma 8, there exists for $i \in [i_0, \dots i_1]$ a $(t_i, r_i, x_i) \in \left(J_i \times (0,\infty) \times \mathbb{R}^3\right)$ such that

$$\frac{1}{r_i^2} \int_{B(x_i,r_i)} |u(t_i, y)|^2 \, dy \gtrsim \frac{E^{-(3/2+)}}{g^{5/4+}(M)} \tag{3-19}$$

and

$$|J_i| \gtrsim \frac{E^{-1/3} r_i}{g^{5/6}(M)}. \tag{3-20}$$

Let $k \in [i_0, \dots, i_1]$ be such that $r_k = \min_{i \in [i_0, i_1]} r_i$; let $f(t, r, x) := \frac{1}{r^2} \int_{B(x,r)} |u(t, y)|^2 \, dy$; let $C_5$ be the constant determined by (3-19); and let $r_0 = r_0(M)$ be defined by

$$r_0 M^2 = \frac{C_5 E^{-((3/2)+)}}{4 g^{(5/4)+}(M)}.$$

Since $f(t, r, x) \leq r M^2$ we have

$$f(t, r_0, x) \leq \frac{C_5 E^{-((3/2)+)}}{4 g^{(5/4)+}(M)}.$$

The set $A := \{(t, r, x) : t \in K, \ r_0 \leq r \leq r_k, \ x \in \mathbb{R}^3\}$ is connected. Therefore its image is connected by $f$ and there exists $(\tilde{t}, \tilde{r}, \tilde{x}) \in K \times [r_0, r_k] \times \mathbb{R}^3$ such that $f(\tilde{t}, \tilde{r}, \tilde{x}) = (C_5 E^{-((3/2)+)})/(2 g^{(5/4)+}(M))$ . In other words we have the following mass concentration

$$\frac{1}{\tilde{r}^2} \int_{B(\tilde{x}, \tilde{r})} u^2(\tilde{t}, x) \, dx = \frac{C_5 E^{-(3/2+)}}{2 g^{(5/4)+}(M)}. \tag{3-21}$$

Moreover we have the useful lower bound for the size of $J_i$,[1] $i_0 \leq i \leq i_1$:

$$|J_i| \gtrsim \tilde{r} \frac{E^{-1/3}}{g^{5/6}(M)}. \tag{3-22}$$

At this point we need to use the following lemma, which gives information about the size of $K$.

**Lemma 12.** *Let $K$ be a sequence of unexceptional intervals. Assume there exist $\bar{t} \in K$, $\bar{x} \in \mathbb{R}^3$ and $\bar{r} \in (0, \infty)$ such that*

$$\frac{1}{\bar{r}^2} \int_{B(\bar{x}, \bar{r})} u^2(\bar{t}, y) \, dy \gtrsim E^{-((3/2)+)} g^{(5/4)+}(M). \tag{3-23}$$

*Then there exist two constants $C_6 \gg 1$, $C_7 \gg 1$ such that*

$$|K| \leq (C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \bar{r}. \tag{3-24}$$

If we combine the lemma with (3-22) we can estimate $N(K)$. More precisely, by Lemma 12, (3-22) and (3-12) we have

$$N(K) \lesssim \frac{(C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \tilde{r}}{\tilde{r} \frac{E^{-(1/3)}}{g^{5/6}(M)}} \lesssim \big(O(E g(M))\big)^{O(E^{(193/4)+} g^{(225/8)+}(M))}. \tag{3-25}$$

Plugging this upper bound for $N(K)$ into (3-18) we get (1-30), completing the proof of the proposition (modulo the lemmas).

*Proof of Lemma 12.* By using the space translation invariance of (1-1) we can reduce to the case where $\bar{x}$ vanishes.[2] By using the time reversal invariance and the time translation invariance[3] it suffices to estimate $|K \cap [\bar{t}, \infty)|$. By using the time translation invariance again[4] we can assume that $\bar{t} = \bar{r}$ and

[1] Notice that we have the lower bound $\tilde{r} \geq C_5 E^{-((3/2)+)}/(4 M^2 g^{(5/4)+}(M))$. One might think that the presence of $\tilde{r}$ in (3-22) is annoying since this lower bound is crude. However we will see that $\tilde{r}$ disappears at the end of the process: see (3-25). Therefore a sharp lower bound is not required.

[2] We consider the function $u_1(t, x) = u(t, x - \bar{x})$ and we abuse notation in the sequel by writing $u_1$ for $u$.

[3] We consider the function $u_2(t, x) := u(2\bar{t} - t, x)$ and we abuse notation in the sequel by writing $u_2$ for $u$.

[4] We consider the function $u_3(t, x) := u(t + (\bar{t} - \bar{r}), x)$ and we abuse notation in the sequel by writing $u_3$ for $u$.

therefore $\bar{r} \in K$. Let $K_+ := K \cap [\bar{r}, \infty)$. We are interested in estimating $|K_+|$. We would like to use Lemma 10. Therefore, we consider the set $\Gamma_+(K_+)$. We have

$$\frac{1}{\bar{r}^2} \int_{B(0,\bar{r})} |u(\bar{r}, y)|^2 \, dy \gtrsim \frac{E^{-((3/2)+)}}{g^{(5/4)+}(M)}. \tag{3-26}$$

Therefore by Proposition 3 and (3-26) we have

$$\int_{B(0,\bar{r})} |u(t, y)|^2 \, dy \gtrsim \frac{E^{-((3/2)+)} \bar{r}^2}{g^{(5/4)+}(M)} \tag{3-27}$$

if $(t-\bar{r})E^{1/2} \le (c_0 E^{-((3/4)+)} \bar{r} / g^{(5/8)+}(M))$ for some $c_0 \ll 1$. Therefore by Hölder there exists $0 < c_1 \ll 1$ small enough such that

$$\|u\|_{L_t^4 L_x^{12}\left(\Gamma_+\left(\left[\bar{r}, \bar{r} + \frac{c_0 E^{-((5/4)+)} \bar{r}}{g^{(5/8)+}(M)}\right]\right)\right)} \ge c_1 \frac{E^{-17/16}}{g^{25/32}(M)}. \tag{3-28}$$

Suppose first that $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(K_+))} \le c_1 \dfrac{E^{-(17/16)}}{g^{(25/32)}(M)}$. In this case we get from (3-28)

$$K_+ \subset \left[\bar{r}, \, \bar{r} + \frac{c_0 E^{-((5/4)+)} \bar{r}}{g^{(5/8)+}(M)}\right], \tag{3-29}$$

and, using also (3-12), we get (3-24).

Now suppose instead that $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(K_+))} \ge c_1 \dfrac{E^{-((17/16)+)}}{g^{(25/32)+}(M)}$. Define

$$\tilde{\eta} := \frac{c_1}{4} \frac{E^{-((17/16))+}}{g^{(25/32)+}(M)}, \tag{3-30}$$

and divide $\Gamma_+(K_+)$ into consecutive cone truncations $\Gamma_+(\tilde{J}_1), \dots, \Gamma_+(\tilde{J}_k)$ such that, for $j = 1, \dots, k-1$,

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} = \tilde{\eta} \tag{3-31}$$

and

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_k))} \le \tilde{\eta}. \tag{3-32}$$

We get from (3-28)

$$\tilde{J}_1 \subset \left[\bar{r}, \bar{r} + \frac{c_0 E^{-((5/4)+)} \bar{r}}{g^{(5/8)+}(M)}\right]. \tag{3-33}$$

**Result 13.** *If $j \in [1, \dots, k-1]$ we either have*

$$|\tilde{J}_{j+1}| \lesssim |\tilde{J}_j| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M) \tag{3-34}$$

*or*

$$|\tilde{J}_j| \ge (C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \bar{r} \tag{3-35}$$

*for some constants $C_6 \gg 1$, $C_7 \gg 1$.*

*Proof.* We get from (1-21), (3-12) and (3-30)

$$\begin{aligned}\|u-u_{l,t_{j+1}}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))} &\lesssim \|u^5 g(u)\|_{L_t^1L_x^2(\Gamma_+(\tilde{J}_j\cup \tilde{J}_{j+1}))}\\ &\lesssim \|u^4\|_{L_t^1L_x^3(\Gamma_+(\tilde{J}_j\cup \tilde{J}_{j+1}))}\|ug^{1/6}(u)\|_{L_t^\infty L_x^6(\Gamma_+(\tilde{J}_j\cup \tilde{J}_{j+1}))}g^{5/6}(M)\\ &\lesssim \tilde{\eta}^4E^{1/6}g^{5/6}(M)\\ &\ll \tilde{\eta},\end{aligned} \tag{3-36}$$

with $J_j=[t_{j-1},t_j]$. Therefore by (3-31) we have $\|u_{l,t_{j+1}}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\sim\tilde{\eta}$. This implies that

$$\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\gtrsim\tilde{\eta} \tag{3-37}$$

or

$$\|u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\gtrsim\tilde{\eta}. \tag{3-38}$$

***Case 1.*** $\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_i))}\gtrsim\tilde{\eta}$. By Lemma 11 and Hölder we have

$$\begin{aligned}\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))} &\lesssim |\tilde{J}_j|^{1/4}\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^\infty L_x^{12}(\Gamma_+(\tilde{J}_j))}\\ &\lesssim |\tilde{J}_j|^{1/4}\|u_{l,t_{j+1}}-u_{l,t_2}\|^{1/2}_{L_t^\infty L_x^\infty(\Gamma_+(\tilde{J}_j))}\|u_{l,t_{j+1}}-u_{l,t_2}\|^{1/2}_{L_t^\infty L_x^6(\Gamma_+(\tilde{J}_j))}\\ &\lesssim \frac{|\tilde{J}_j|^{1/4}E^{2/3}g^{1/12}(M)}{|\tilde{J}_{j+1}|^{1/4}}.\end{aligned} \tag{3-39}$$

We get (3-34) from (3-37) and (3-39).

***Case 2.*** $\|u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\gtrsim\tilde{\eta}$. In this case $\|u_{l,t_2}\|_{L_t^4L_x^{12}(\tilde{J}_j)}\gtrsim\tilde{\eta}$. Recall that $K_+$ is a subinterval of $K=J_{i_0}\cup\cdots\cup J_{i_1}$, sequence of unexceptional intervals $J_i$, $i_0\le i\le i_1$. Consequently there are at least $\sim\tilde{\eta}(C_3Eg(M))^{C_4E^{(193/4)+}g^{(225/8)+}(M)}$ intervals $J_j$ that cover $\tilde{J}_i$. Therefore we get (3-35) from (3-22) and (3-12). □

Using Result 13 and Lemma 10 we can get an upper bound on the size $|K_+|$:

**Result 14.** *We have*

$$|K_+|\le(C_6Eg(M))^{C_7(E^{(193/4)+}g^{(225/8)+}(M))}\bar{r}. \tag{3-40}$$

*Proof.* Let $B:=(C_6Eg(M))^{C_7(E^{(193/4)+}g^{(225/8)+}(M))}$. Assume that (3-40) fails. Let $\tilde{J}_{j_1}$ be the first interval for which $|\tilde{J}_1\cup\cdots\cup\tilde{J}_{j_1}|$ exceeds $B\bar{r}$. Then $j_1\neq 1$, $|\tilde{J}_{j_1}|\lesssim|\tilde{J}_{j_1-1}|\tilde{\eta}^{-4}E^{8/3}g^{1/3}(M)$ and we have

$$\frac{c_1E^{-5/4}\bar{r}}{g^{(5/8)}(M)}+T_2-T_1+(T_2-T_1)\tilde{\eta}^{-4}E^{8/3}g^{1/3}(M)\gtrsim|\tilde{J}_1|+\cdots+|\tilde{J}_{j_1}|\ge B\bar{r}, \tag{3-41}$$

if $[T_1,T_2]:=\tilde{J}_2\cup\cdots\cup\tilde{J}_{j_1-1}$. Therefore by (3-12) and (3-41) we have

$$T_2-T_1\gtrsim\frac{\tilde{\eta}^4E^{-(8/3)}B\bar{r}}{g^{1/3}(M)}. \tag{3-42}$$

Moreover $T_1\le\bar{r}+(c_1E^{-((5/4)+)}\bar{r})/(g^{(5/8)+}(M))$. Therefore by (3-12) we have

$$T_1=O(\bar{r}). \tag{3-43}$$

By (3-42) and (3-43) we have

$$\frac{T_2}{T_1} \geq \left(C_2 E^{10+} \left(\frac{\tilde{\eta}}{4}\right)^{-(36+)}\right)^{4C_2 E^{10+} (\tilde{\eta}/4)^{-(36+)}}, \tag{3-44}$$

with $C_2$ defined in Lemma 10, provided that $C_6,\ C_7 \gg \max(c_1, C_2)$. Therefore we can apply Lemma 10 and find a subinterval $[t'_1, t'_2] \subset \tilde{J}_2 \cup \cdots \cup \tilde{J}_{j_1-1}$ with $\left|t'_2/t'_1\right| \sim E^{10+}\tilde{\eta}^{-(36+)}$ and $\|u\|_{L_t^4 L_x^{12}([t'_1, t'_2])} \leq \tilde{\eta}/4$. This means that $[t'_1, t'_2] \subset [T_1, T_2]$ is covered by at most two consecutive intervals. It is convenient to introduce $[t'_1, t'_2]_g$, the geometric mean of $t'_1$ and $t'_2$. We have $[t'_1, t'_2]_g \sim \tilde{\eta}^{-18} E^5 t'_1$. There are two cases.

***Case 1.*** $[t'_1, t'_2]$ is covered by one interval $\tilde{J}_{\bar{j}} = [a_{\bar{j}}, b_{\bar{j}}]$, $2 \leq \bar{j} \leq j_1 - 1$. Then $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(36+)} E^{10+} t'_1$ and $|\tilde{J}_{\bar{j}-1}| \leq t'_1$. Therefore $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(36+)} E^{10+} |\tilde{J}_{\bar{j}-1}|$. Contradiction with (3-12) and (3-34).

***Case 2.*** $[t'_1, t'_2]$ is covered by two intervals $\tilde{J}_{\bar{j}} = [a_{\bar{j}}, b_{\bar{j}}]$ and $\tilde{J}_{\bar{j}+1} = [a_{\bar{j}+1}, b_{\bar{j}+1}]$ for some $2 \leq \bar{j} \leq j_1 - 2$. Then there are two subcases.

*Case 2a.* $b_{\bar{j}} \leq [t'_1, t'_2]_g$. In this case $|\tilde{J}_{\bar{j}+1}| \gtrsim \tilde{\eta}^{-(36+)} E^{10+} t'_1$ and $|\tilde{J}_{\bar{j}}| \leq \tilde{\eta}^{-(18+)} E^{5+} t'_1$. Therefore by (3-12) we have $|\tilde{J}_{\bar{j}+1}| \gtrsim \tilde{\eta}^{-(18+)} E^{5+} |\tilde{J}_{\bar{j}}|$. Contradiction with (3-12) and (3-34).

*Case 2b.* $b_{\bar{j}} \geq [t'_1, t'_2]_g$. In this case by (3-12) $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(18+)} E^{5+} t'_1$ and $|\tilde{J}_{\bar{j}-1}| \leq t'_1$. Therefore $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(18+)} E^{5+} |\tilde{J}_{\bar{j}-1}|$. Contradiction with (3-12) and (3-34).

This exhausts all cases. Thus we have proved Result 14 and so also Lemma 12. □

**Remark 15.** It seems likely that we can find a better upper bound for $|K_+|$ than (3-40) by exploiting Lemma 11 in a better way. For instance we can consider $k$ successive time intervals $\tilde{J}_{j+1}, \ldots, \tilde{J}_{j+k}$, $k > 1$ and prove an estimate like

$$|\tilde{J}_{j+1}| + \cdots |\tilde{J}_{j+k}| \lesssim |\tilde{J}_j| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M). \tag{3-45}$$

This estimate is stronger than (3-34). We can probably find a smaller $B$ such that (3-44) holds with $\tilde{\eta}$ substituted for something like $k\tilde{\eta}$ and, by modifying the argument above, find a contradiction with (3-45). At the end of the process we can probably prove global existence of smooth solutions to (1-1) for $0 < c < c_0$, with $c_0 > 8/225$ to be determined. We will not pursue these matters.

## 4. Proof of Lemma 7

Applying the Strichartz estimates and the Hölder inequality,

$$\begin{aligned}\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} &\lesssim E^{1/2} + \|u^4\|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} \|ug^{1/6}(u)\|_{L_t^\infty L_x^6(J\times\mathbb{R}^3)} \|g^{5/6}(u)\|_{L_t^\infty L_x^\infty(J\times\mathbb{R}^3)} \\ &\lesssim E^{1/2} + E^{1/6} g^{5/6}(M) \|u\|^4_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)}. \end{aligned} \tag{4-1}$$

Hence (3-2) by (3-1) and a continuity argument.

## 5. Proof of Lemma 8

Let $J' = [t_1', t_2'] \subset J$ be such that $\|u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} = \eta$. Then by (1-22) and (3-3)

$$\begin{aligned}\|f(u)\|_{L_t^1 L_x^2(J'\times\mathbb{R}^3)} &\lesssim \|ug^{1/6}(u)\|_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)}\|u\|^4_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)}\|g^{5/6}(u)\|_{L_t^\infty L_x^\infty(J'\times\mathbb{R}^3)}\\ &\lesssim E^{1/6}\eta^4 g^{5/6}(M) \lesssim E^{1/2}.\end{aligned} \tag{5-1}$$

It is slightly unfortunate that $(2,\infty)$ is not wave admissible. Therefore we consider the admissible pair $(2+\epsilon,\ 6(2+\epsilon)/\epsilon)$ with $\epsilon \ll 1$. By the Strichartz estimates and (5-1), we have

$$\|u\|_{L_t^{2+\epsilon} L_x^{(6(2+\epsilon))/\epsilon}(J'\times\mathbb{R}^3)} \lesssim \|\nabla u(t_1')\|_{L^2(\mathbb{R}^3)} + \|u(t_1')\|_{L^2(\mathbb{R}^3)} + \|f(u)\|_{L_t^1 L_x^2(J'\times\mathbb{R}^3)} \lesssim E^{1/2}. \tag{5-2}$$

Let $N$ be a frequency to be chosen later. By the Bernstein inequality and (1-7) we have

$$\|P_{<N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \lesssim N^{1/4}|J'|^{1/4}\|u\|_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)} \lesssim N^{1/4}|J'|^{1/4}E^{1/6}. \tag{5-3}$$

Therefore

$$\|P_{<N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \lesssim |J'|^{1/4}N^{1/4}E^{1/6}. \tag{5-4}$$

Let $c_2 \ll 1$. Then if $N = c_2^4(\eta^4/(|J'|E^{2/3}))$ we have

$$\|P_{\geq N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \gtrsim \eta \quad \text{and} \quad \|u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \sim \|P_{\geq N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)}. \tag{5-5}$$

By (5-2) and (5-5) we have

$$\begin{aligned}\eta &\sim \|P_{\geq N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)}\\ &\lesssim \|P_{\geq N}u\|^{(2+\epsilon)/4}_{L_t^{2+\epsilon} L_x^{(6(2+\epsilon))/\epsilon}(J'\times\mathbb{R}^3)}\|P_{\geq N}u\|^{1-(2+\epsilon)/4}_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)}\\ &\lesssim E^{(2+\epsilon)/8}\|P_{\geq N}u\|^{1-((2+\epsilon)/4)}_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)}.\end{aligned} \tag{5-6}$$

Therefore we conclude that $\|P_{\geq N}\|_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)} \gtrsim \eta^{2+}E^{-((1/2)+)}$. Applying Proposition 2 we get (3-5).

## 6. Proof of Lemma 9

Bahouri and Gerard [1999, page 171] used arguments from Grillakis [1990; 1992] and Shatah–Struwe [1993] to derive an a priori estimate of the solution $u$ to the 3D quintic defocusing wave equation, that is, $\partial_{tt}u - \Delta u + u^5 = 0$. More precisely they were able to prove

$$\int_{|x|\leq b}|u(b,x)|^6\,dx \lesssim \frac{a}{b}(\tilde{e}(a) + \tilde{e}^{1/3}(a)) + \tilde{e}(b) - \tilde{e}(a) + (\tilde{e}(b) - \tilde{e}(a))^{1/3}, \tag{6-1}$$

with

$$\tilde{e}(t) := \frac{1}{2}\int_{|x|\leq t}(\partial_t u)^2\,dx + \frac{1}{2}\int_{|x|\leq t}|\nabla u|^2\,dx + \frac{1}{6}\int_{|x|\leq t}u^6\,dx. \tag{6-2}$$

Since we apply their ideas to the potential $f$ we just sketch the proof. Given the cone $\Gamma_+([a,b])$ we denote by $\partial\Gamma_+([a,b])$ the mantle of the cone $\Gamma_+([a,b])$, that is,

$$\partial\Gamma_+([a,b]) := \big\{(t',x) \in [a,b]\times\mathbb{R}^3,\ t = |x|\big\}. \tag{6-3}$$

The local energy identity

$$e(b)-e(a)=\frac{1}{2\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\left|\frac{x\partial_t u}{t}+\nabla u\right|^2+\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}F(u) \tag{6-4}$$

results from the integration of the identity $\partial_t u\,(\partial_{tt}u-\triangle u+f(u))=0$ on the cone $\Gamma_{+}([a,b])$. We have [Shatah and Struwe 1998]

$$\partial_t\Big(\frac{t}{2}(\partial_t u)^2+\frac{t}{2}|\nabla u|^2+(x.\nabla u)\partial_t u+tF(u)+u\partial_t u\Big)$$
$$-\operatorname{div}\Big(t\nabla u\partial_t u+(x.\nabla u)\nabla u-\frac{|\nabla u|^2x}{2}+\frac{(\partial_t u)^2x}{2}-xF(u)+u\nabla u\Big)+uf(u)-4F(u)=0. \tag{6-5}$$

Integrating this identity on $\Gamma_{+}([a,b])$, we have

$$X(b)-X(a)+Y(a,b)=\int_{\Gamma_{+}([a,b])}4F(u)-uf(u), \tag{6-6}$$

with

$$X(t):=\int_{|x|\leq t}\frac{t}{2}(\partial_t u)^2+\frac{t}{2}|\nabla u|^2+(x.\nabla u)\partial_t u+tF(u)+u\partial_t u \tag{6-7}$$

and

$$Y(a,b):=$$
$$-\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\Big(\frac{t}{2}(\partial_t u)^2+\frac{t}{2}|\nabla u|^2+(x.\nabla u)\partial_t u+tF(u)+u\partial_t u+t\frac{\nabla u.x}{|x|}\partial_t u+\frac{|x.\nabla u|^2}{|x|}$$
$$-\frac{|\nabla u|^2}{2}|x|+\frac{(\partial_t u)^2|x|}{2}-|x|F(u)+u\frac{\nabla u.x}{|x|}\Big). \tag{6-8}$$

In fact we have [Shatah and Struwe 1993]

$$X(t)=\int_{|x|\leq t}t\Big[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}\Big|\nabla u+\frac{ux}{|x|^2}\Big|^2\Big]+\partial_t u(x.\nabla u+u)+tF(u)-\int_{|x|=t}\frac{u^2}{2}. \tag{6-9}$$

Since $t=|x|$ on $\partial\Gamma_{+}([a,b])$ we have

$$Y(a,b)=-\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}|x|(\partial_t u)^2+2(x.\nabla u)\partial_t u+u\partial_t u+\frac{(x.\nabla u)^2}{|x|}+u\frac{\nabla u.x}{|x|}, \tag{6-10}$$

and after some computations [Shatah and Struwe 1993], we get

$$Y(a,b)=-\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\frac{1}{t}(t\partial_t u+(\nabla u.x)+u)^2+\int_{|x|=b}\frac{u^2}{2}-\int_{|x|=a}\frac{u^2}{2}. \tag{6-11}$$

Therefore, if

$$H(t):=\int_{|x|\leq t}t\Big[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}\Big|\nabla u+\frac{ux}{|x|^2}\Big|^2\Big]+\partial_t u(x.\nabla u+u)+tF(u), \tag{6-12}$$

then

$$H(b)-H(a)=\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\frac{1}{t}(t\partial_t u+\nabla u.x+u)^2+\int_{\Gamma_{+}([a,b])}4F(u)-uf(u). \tag{6-13}$$

We estimate $H(t)$, following [Bahouri and Gérard 1999]. We have

$$|\partial_t u(x.\nabla u+u)| \leq \frac{t}{2}\Big((\partial_t u)^2+\Big|\nabla u+\frac{ux}{|x|^2}\Big|^2\Big) \lesssim t\Big((\partial_t u)^2+|\nabla u|^2+\frac{u^2}{|x|^2}\Big). \tag{6-14}$$

Therefore by (6-14), the Hölder inequality and (1-7), we have

$$H(t) \lesssim t\Big(e(t)+\int_{|x|\leq t}\frac{u^2}{|x|^2}\Big) \lesssim t\Big(e(t)+\Big(\int_{|x|\leq t}u^6\Big)^{1/3}\Big) \lesssim t\big(e(t)+e^{1/3}(t)\big). \tag{6-15}$$

Moreover by (6-4), the Hölder inequality and (1-7), we have

$$\begin{aligned}\frac{1}{\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\frac{1}{t}\big(t\partial_t u+\nabla u.x+u\big)^2 &\lesssim \frac{b}{2\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\Big(\frac{\nabla u\cdot x}{t}+\partial_t u\Big)^2+\frac{1}{2\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\frac{u^2}{t^2}\\ &\lesssim b\int_{\partial\Gamma_+([a,b])}\Big|\frac{x}{t}\partial_t u+\nabla u\Big|^2+\frac{1}{2\sqrt{2}}\Big(\int_{\partial\Gamma_+([a,b])}u^6\Big)^{1/3}\\ &\lesssim b\big((e(b)-e(a))+(e(b)-e(a))^{1/3}\big).\end{aligned} \tag{6-16}$$

We get from (1-7)

$$4F(u)-uf(u)\leq 0. \tag{6-17}$$

By (6-13), and (6-15)–(6-17), we have

$$\begin{aligned}\int_{|x|\leq b}F(u) \lesssim \frac{H(b)}{b} &\lesssim \frac{H(a)+\frac{1}{\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\frac{1}{t}\,(t\partial_t u+\nabla u.x+u)^2}{b}\\ &\lesssim \frac{a}{b}\,\big(e(a)+e^{1/3}(a)\big)+e(b)-e(a)+(e(b)-e(a))^{1/3}.\end{aligned} \tag{6-18}$$

## 7. Proof of Lemma 10

The proof relies upon two results that we prove in the subsections.

**Result 16.** *Let $u$ be a classical solution of* (1-1). *Assume that* (1-28) *holds. Let $\eta$ be a positive number such that* (3-3) *holds. If $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J))} \geq \eta$ then*

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J))} \gtrsim \eta^{2+}E^{-((1/2)+)}. \tag{7-1}$$

**Result 17.** *Let $u$ be a smooth solution to* (1-1). *Assume that* (1-28) *holds. Let $\eta$ be a positive number such that*

$$\eta \leq \min(1, E^{1/18}). \tag{7-2}$$

*Let $J=[t_1,t_2]$ be an interval such that $[t_1, t_1(E\eta^{-18})^{4E\eta^{-18}}] \subset J$. Then there exists a subinterval $J'=[t_1',t_2']$ such that $|t_2'/t_1'| = E\eta^{-18}$ and*

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \lesssim \eta. \tag{7-3}$$

Let $C_9$ be the constant determined by $\gtrsim$ in (7-1). Let $C_{10}$ be the constant determined by $\lesssim$ in (7-3). We get from (3-9):

$$\left[t_1,\ t_1\left(E\left(\frac{C_9\eta^{2+}E^{-(1/2)+}}{2C_{10}}\right)^{-18}\right)^{4E\left(\frac{C_9\eta^{2+}E^{-(1/2)+}}{2C_{10}}\right)^{-18}}\right] \subset \left[t_1,\ C_2(E^{10+}\eta^{-(36+)})^{4C_2E^{10+}\eta^{-(36+)}}t_1\right] \subset J, \tag{7-4}$$

if $C_2 \gg \max(C_9, C_{10})$. Therefore, since $(C_9\eta^{2+}E^{-(1/2+)})/(2C_{10})$ satisfies (7-2) by (3-8), we can use Result 17 and show that there exists a subinterval $J' = [t_1',\ t_2']$ such that $|t_2'/t_1'| \sim E^{10+}\eta^{-(36+)}$ and

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \le \frac{C_9\eta^{2+}E^{-(1/2+)}C_{10}}{2C_{10}} \le C_9\frac{\eta^{2+}E^{-(1/2+)}}{2}. \tag{7-5}$$

Now we claim that $\|u\|_{L_t^4L_x^{12}(\Gamma_+(J'))} \le \eta$. If not by (3-8) and Result 16 we have

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \ge C_9\eta^{2+}E^{-(1/2+)}. \tag{7-6}$$

Contradiction with (7-5).

*Proof of Result 16.* We substitute $J'$ for $\Gamma_+(J')$ in (5-1) to get

$$\|f(u)\|_{L_t^1L_x^2(\Gamma_+(J'))} \lesssim E^{1/2}. \tag{7-7}$$

By the Strichartz estimates (1-20) on the truncated cone $\Gamma_+(J')$ we have

$$\|u\|_{L_t^{2+\epsilon}L_x^{(6(2+\epsilon))/\epsilon}(\Gamma_+(J'))} \lesssim E^{1/2}, \tag{7-8}$$

after following similar steps to prove (5-2). Therefore

$$\eta = \|u\|_{L_t^4L_x^{12}(\Gamma_+(J))} \lesssim \|u\|^{(2+\epsilon)/4}_{L_t^{2+\epsilon}L_x^{(6(2+\epsilon))/\epsilon}(\Gamma_+(J'))}\|u\|^{1-((2+\epsilon)/4)}_{L_t^\infty L_x^6(\Gamma_+(J'))} \lesssim E^{(2+\epsilon)/8}\|u\|^{1-((2+\epsilon)/4)}_{L_t^\infty L_x^6(\Gamma_+(J'))}. \tag{7-9}$$

Therefore (7-1) holds. □

*Proof of Result 17.* By (7-2) we have $E\eta^{-18} \ge 1$. Let $n$ be the largest integer such that $2n \le 4E\eta^{-18}$. This implies that $n \ge E\eta^{-18}$. Let $A := E\eta^{-18}$. Now we consider the interval $[t_1, A^{2n}t_1] \subset J$. We write $[t_1, A^{2n}t_1] = [t_1,\ A^2t_1] \cup \cdots \cup [A^{2(n-1)}t_1,\ A^{2n}t_1]$. We have

$$\sum_{i=1}^{n} e(A^{2i}t_1) - e(A^{2(i-1)}t_1) \le 2E, \tag{7-10}$$

and by the pigeonhole principle there exists $i_0 \in [1, n]$ such that

$$e(A^{2i_0}t_1) - e(A^{2(i_0-1)}t_1) \lesssim \eta^{18}. \tag{7-11}$$

Now we choose $a := A^{2(i_0-1)}t_1$ and $b \in [A^{2i_0-1}t_1,\ A^{2i_0}t_1]$. Let $t_1' := A^{2(i_0-1)}t_1,\ t_2' := A^{2i_0-1}t_1$ and $J' := [t_1', t_2']$. We apply (3-7) and (7-2) to get

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+([t_1', t_2']))} \lesssim \|F(u)\|_{L_t^\infty L_x^1(\Gamma_+([t_1', t_2']))} \lesssim (E^{-1}\eta^{18}(E+E^{1/3}) + \eta^{18} + \eta^6)^{1/6} \lesssim \eta. \qquad □$$

**Proof of Lemma 11**

We have after computation of the derivative of $e(t)$

$$\partial_t e(t) \geq \int_{|x|=t} F(u)\, dS, \tag{7-12}$$

and integrating with respect of time

$$\int_I \int_{|x|\leq t} g(u)u^6(t',x')\, dS\, dt' \lesssim E. \tag{7-13}$$

By using the space and time translation invariance

$$\int_J \int_{|x'-x|=|t'-t|} g(u)u^6(t',x')\, dS\, dt' \lesssim E. \tag{7-14}$$

Therefore (1-15), (1-22), (7-14) and the Hölder inequality give us

$$\begin{aligned}\left| -\int_{J'} \frac{\sin(t-t')D}{D} g(u)u^5\, dt' \right| &= \left| \frac{1}{4\pi |t-t'|} \int_{|x'-x|=|t'-t|} g^{5/6}(u)u^5 g^{1/6}(u)\, dS dt' \right| \\ &\lesssim \int_{J'} \frac{1}{|t-t'|} \Big( \int_{|x'-x|=|t'-t|} u^6 g(u) dS \Big)^{5/6} \Big( \int_{|x'-x|=|t'-t|} g(u)\, dS \Big)^{1/6} dt' \\ &\lesssim g^{1/6}(M) \int_{J'} \frac{1}{|t-t'|^{2/3}} \Big( \int_{|x'-x|=|t'-t|} u^6 g(u)\, dS \Big)^{5/6} dt' \\ &\lesssim g^{1/6}(M) E^{5/6} \Big( \int_{J'} \frac{1}{|t-t'|^4} \Big)^{1/6} \lesssim g^{1/6}(M) \frac{E^{5/6}}{\operatorname{dist}^{1/2}(t,J')}. \end{aligned} \tag{7-15}$$

Notice that

$$u(t) = u_{l,t_i}(t) - \int_{t_i}^t \frac{\sin(t-t')D}{D} u^5(t') g(u(t'))\, dt', \tag{7-16}$$

for $i = 1, 2$. We get (3-11) from (7-15) and (7-16).

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