



# GLOBAL EXISTENCE OF SMOOTH SOLUTIONS OF A 3D LOG-LOG ENERGY-SUPERCRITICAL WAVE EQUATION

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We prove global existence of smooth solutions of the 3D log-log energy-supercritical wave equation

$$\partial_{tt}u - \Delta u = -u^5 \log^c(\log(10 + u^2))$$

with  $0 < c < 8/225$  and smooth initial data  $(u(0) = u_0, \partial_t u(0) = u_1)$ . First we control the  $L_t^4 L_x^{12}$  norm of the solution on an arbitrary size time interval by an expression depending on the energy and an a priori upper bound of its  $L_t^\infty \tilde{H}^2(\mathbb{R}^3)$  norm, with  $\tilde{H}^2(\mathbb{R}^3) := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ . The proof of this long time estimate relies upon the use of some potential decay estimates and a modification of an argument by Tao. Then we find an a posteriori upper bound of the  $L_t^\infty \tilde{H}^2(\mathbb{R}^3)$  norm of the solution by combining the long time estimate with an induction on time of the Strichartz estimates.

## 1. Introduction

We shall consider the defocusing log-log energy-supercritical wave equation

$$\partial_{tt}u - \Delta u = -f(u) \tag{1-1}$$

where  $u : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is a real-valued scalar field and  $f(u) := u^5 g(u)$  with  $g(u) := \log^c(\log(10 + u^2))$ ,  $0 < c < 8/225$ . *Classical solutions* of (1-1) are solutions that are infinitely differentiable and compactly supported in space for each fixed time  $t$ . It is not difficult to see that classical solutions of (1-1) satisfy the energy conservation law

$$E := \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u(t, x))^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx + \int_{\mathbb{R}^3} F(u(t, x)) dx \tag{1-2}$$

where  $F(u) := \int_0^u f(v) dv$ . Classical solutions of (1-1) enjoy three symmetry properties that we use throughout this paper:

- *time translation invariance*: if  $u$  is a solution of (1-1) and  $t_0$  is a fixed time then  $\tilde{u}(t, x) := u(t - t_0, x)$  is also a solution of (1-1);
- *space translation invariance*: if  $u$  is a solution of (1-1) and  $x_0$  is a fixed point lying in  $\mathbb{R}^3$  then  $\tilde{u}(t, x) := u(t, x - x_0)$  is also a solution of (1-1);
- *time reversal invariance*: if  $u$  is a solution to (1-1) then  $\tilde{u}(t, x) := u(-t, x)$  is also a solution.

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The defocusing log-log energy-supercritical wave equation (1-1) is closely related to the power-type defocusing wave equations, namely,

$$\partial_{tt}u - \Delta u = -|u|^{p-1}u. \tag{1-3}$$

Solutions of (1-3) have an invariant scaling

$$u(t, x) \rightarrow u^\lambda(t, x) := \frac{1}{\lambda^{2/(p-1)}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \tag{1-4}$$

and (1-3) is  $s_c$ -critical, where  $s_c := \frac{3}{2} - \frac{2}{p-1}$ . Thus the  $\dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$  norm of  $(u(0), \partial_t u(0))$  is invariant under scaling, i.e.,

$$\begin{aligned} \|u^\lambda(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)} &= \|u(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)}, \\ \|\partial_t u^\lambda(0)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)} &= \|\partial_t u(0)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)}. \end{aligned}$$

If  $p = 5$ , then  $s_c = 1$  and this is why the quintic defocusing cubic wave equation

$$\partial_{tt}u - \Delta u = -u^5 \tag{1-5}$$

is called the energy-critical equation. If  $1 < p < 5$  then  $s_c < 1$  and (1-3) is energy-subcritical while if  $p > 5$  then  $s_c > 1$  and (1-3) is energy-supercritical. Notice that for every  $p > 5$  there exists two positive constant  $\lambda_1(p), \lambda_2(p)$  such that

$$\lambda_1(p)|u|^5 \leq |f(u)| \leq \lambda_2(p) \max(1, |u|^p). \tag{1-6}$$

This is why (1-1) is said to belong to the group of barely supercritical equations. There is another way to see that. Notice that a simple integration by part shows that

$$F(u) \sim \frac{u^6}{6} g(u), \tag{1-7}$$

and consequently the nonlinear potential term of the energy  $\int_{\mathbb{R}^3} F(u) dx \sim \int_{\mathbb{R}^3} u^6 g(u) dx$  just barely fails to be controlled by the linear component, in contrast to (1-5).

The energy-critical wave equation (1-5) has received a great deal of attention. Grillakis [1990; 1992] established global existence of smooth solutions (global regularity) of this equation with smooth initial data  $u(0) = u_0, \partial_t u(0) = u_1$ . His work followed that of Rauch [1981, part I] for small data and that of Struwe [1988] on the spherically symmetric case. Later Shatah and Struwe [1993] gave a simplified proof of this result. Kapitanski [1994] and, independently, Shatah and Struwe [1994] proved global existence of solutions with data  $(u_0, u_1)$  in the energy class.

We are interested in proving global regularity of (1-1) with smooth initial data  $(u_0, u_1)$ . By standard persistence of regularity results it suffices to prove global existence of solutions

$$u \in \mathcal{C}([0, T], \tilde{H}^2(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^1(\mathbb{R}^3)),$$

with data  $(u_0, u_1) \in \tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ . Here the following space

$$\tilde{H}^2(\mathbb{R}^3) := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3). \tag{1-8}$$

In view of the local well-posedness theory [Lindblad and Sogge 1995], standard limit arguments and the finite speed of propagation it suffices to find an a priori upper bound of the form

$$\| (u(T), \partial_t u(T)) \|_{\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq C_1 (\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)}, T) \tag{1-9}$$

for all times  $T > 0$  and for classical solutions  $u$  of (1-1) with smooth and compactly supported data  $(u_0, u_1)$ . Here  $C_1$  is a constant depending only on  $\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)}$  and the time  $T$ .

The global behavior of the solutions of the supercritical wave equations is poorly understood, mostly because of the lack of conservation laws in  $\tilde{H}^2(\mathbb{R}^3)$ . Nevertheless Tao [2007] was able to prove global regularity for another barely supercritical equation, namely

$$\partial_{tt}u - \Delta u = -u^5 \log(2 + u^2), \tag{1-10}$$

with radial data. The main result of this paper is:

**Theorem 1.** *The solution of (1-1) with smooth data  $(u_0, u_1)$  exists for all time. Moreover there exists a nonnegative constant  $M_0 = M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$  depending only on  $\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}$  and  $\|u_1\|_{H^1(\mathbb{R}^3)}$  such that*

$$\|u\|_{L_t^\infty \tilde{H}^2(\mathbb{R} \times \mathbb{R}^3)} + \|\partial_t u\|_{L_t^\infty H^1(\mathbb{R} \times \mathbb{R}^3)} \leq M_0. \tag{1-11}$$

We recall some basic properties and estimates. Let  $Q$  be a function, let  $J$  be an interval and let  $t_0 \in J$  be a fixed time. If  $u$  is a classical solution of the more general problem  $\partial_{tt}u - \Delta u = Q$  then  $u$  satisfies the Duhamel formula

$$u(t) = u_{l,t_0}(t) + u_{nl,t_0}(t), \quad t \in J, \tag{1-12}$$

with  $u_{l,t_0}, u_{nl,t_0}$  denoting the linear part and the nonlinear part respectively of the solution starting from  $t_0$ . Recall that

$$u_{l,t_0}(t) = \cos(t - t_0)Du(t_0) + \frac{\sin(t - t_0)D}{D}\partial_t u(t_0) \tag{1-13}$$

and

$$u_{nl,t_0}(t) = - \int_{t_0}^t \frac{\sin(t - t')D}{D}Q(t') dt', \tag{1-14}$$

with  $D$  the multiplier defined by  $\widehat{Df}(\xi) := |\xi|\widehat{f}(\xi)$ . An explicit formula for  $((\sin(t - t')D)/D)Q(t')$  and  $t \neq t'$  is

$$\left[ \frac{\sin(t - t')D}{D}Q(t') \right](x) = \frac{1}{4\pi|t - t'|} \int_{|x-x'|=|t-t'|} Q(t', x') dS(x'). \tag{1-15}$$

For a proof see [Sogge 1995]. We recall that  $u_{l,t_0}$  satisfies

$$\partial_{tt}u_{l,t_0} - \Delta u_{l,t_0} = 0, \quad u_{l,t_0}(t_0) = u(t_0), \quad \partial_t u_{l,t_0}(t_0) = \partial_t u(t_0),$$

while  $u_{nl,t_0}$  is the solution of

$$\partial_{tt}u_{nl,t_0} - \Delta u_{nl,t_0} = Q, \quad u_{nl,t_0}(t_0) = 0, \quad \partial_t u_{nl,t_0}(t_0) = 0.$$

We recall the Strichartz estimate [Ginibre and Velo 1995; Keel and Tao 1998; Lindblad and Sogge 1995; Sogge 1995]

$$\|u\|_{L_t^q L_x^r(J \times \mathbb{R}^3)} \lesssim \|\partial_t u(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|\nabla u(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(J \times \mathbb{R}^3)}, \tag{1-16}$$

if  $(q, r)$  is wave admissible, that is,  $(q, r) \in (2, \infty) \times [2, \infty]$  and  $1/q + 3/r = 1/2$ .

We set some notation that appears throughout the paper. We write  $C = C(a_1, \dots, a_n)$  if  $C$  only depends on the parameters  $a_1, \dots, a_n$ . We write  $A \lesssim B$  if there exists a universal nonnegative constant  $C' > 0$  such that  $A \leq C'B$ .  $A = O(B)$  means  $A \lesssim B$ . More generally we write  $A \lesssim_{a_1, \dots, a_n} B$  if there exists a nonnegative constant  $C' = C(a_1, \dots, a_n)$  such that  $A \leq C'B$ . We say that  $C''$  is the constant determined by  $\lesssim$  in  $A \lesssim_{a_1, \dots, a_n} B$  if  $C''$  is the smallest constant among the  $C'$ 's such that  $A \leq C'B$ . We write  $A \ll_{a_1, \dots, a_n} B$  if there exists a universal nonnegative small constant  $c = c(a_1, \dots, a_n)$  such that  $A \leq cB$ . Similar notions are defined for  $A \gtrsim B$ ,  $A \gtrsim_{a_1, \dots, a_n} B$  and  $A \gg B$ . In particular we say that  $C''$  is the constant determined by  $\gtrsim$  in  $A \gtrsim B$  if  $C''$  is the largest constant among the  $C'$ 's such that  $A \geq C'B$ . If  $x$  is number then  $x+$  and  $x-$  are slight variations of  $x$ :  $x+ := x + \alpha\epsilon$  and  $x- := x - \beta\epsilon$  for some  $\alpha > 0, \beta > 0$  and  $0 < \epsilon \ll 1$ .

Let  $\Gamma_+$  denote the forward light cone

$$\Gamma_+ = \{(t, x) : t > |x|\}, \tag{1-17}$$

and if  $J = [a, b]$  is an interval, let  $\Gamma_+(J)$  denote the light cone truncated to  $J$ , that is,

$$\Gamma_+(J) := \Gamma_+ \cap (J \times \mathbb{R}^3). \tag{1-18}$$

Let  $e(t)$  denote the local energy, that is,

$$e(t) := \frac{1}{2} \int_{|x| \leq t} (\partial_t u(t, x))^2 dx + \frac{1}{2} \int_{|x| \leq t} |\nabla u(t, x)|^2 dx + \int_{|x| \leq t} F(u(t, x)) dx. \tag{1-19}$$

If  $u$  is a solution of (1-1) then by using the finite speed of propagation and the Strichartz estimates we have

$$\|u\|_{L_t^q L_x^r(\Gamma_+(J))} \lesssim \|\nabla u(b)\|_{L_x^2(\mathbb{R}^3)} + \|\partial_t u(b)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(\Gamma_+(J))} \tag{1-20}$$

if  $(q, r)$  is wave admissible. If  $J_1 := [a_1, a_2]$  and  $J_2 := [a_2, a_3]$  then we also have

$$\|u\|_{L_t^q L_x^r(\Gamma_+(J_1))} \lesssim \|\nabla u(a_3)\|_{L_x^2(\mathbb{R}^3)} + \|\partial_t u(a_3)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(\Gamma_+(J_1 \cup J_2))}. \tag{1-21}$$

We recall also the well-known Sobolev embeddings. If  $h$  is a smooth function then

$$\|h\|_{L^\infty(\mathbb{R}^3)} \lesssim \|h\|_{\tilde{H}^2(\mathbb{R}^3)} \tag{1-22}$$

and

$$\|h\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla h\|_{L^2(\mathbb{R}^3)}. \tag{1-23}$$

If  $u$  is the solution of (1-1) with data  $(u_0, u_1) \in \tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ , then we get from (1-22)

$$E \lesssim \|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}^2 \max(1, \|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}^4 g(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)})). \tag{1-24}$$

We shall use the Paley–Littlewood technology. Let  $\phi(\xi)$  be a bump function adapted to  $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$  and equal to one on  $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$ . If  $(M, N) \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}}$  are dyadic numbers then the Paley–Littlewood projection operators  $P_M, P_{<N}$  and  $P_{\geq N}$  are defined in the Fourier domain by

$$\widehat{P_M f}(\xi) := \left( \phi\left(\frac{\xi}{M}\right) - \phi\left(\frac{\xi}{2M}\right) \right) \hat{f}(\xi), \quad \widehat{P_{<N} f}(\xi) := \sum_{M < N} \widehat{P_M f}(\xi), \quad \widehat{P_{\geq N} f}(\xi) := \sum_{M \geq N} \widehat{P_M f}(\xi).$$

The inverse Sobolev inequality can be stated as follows:

**Proposition 2** (Inverse Sobolev inequality [Tao 2006]). *Let  $g$  be a smooth function such that*

$$\|g\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim E^{1/2}l, \quad \|P_{\geq N}g\|_{L_x^6(\mathbb{R}^3)} \gtrsim \eta,$$

for some real number  $\eta > 0$  and for some dyadic number  $N > 0$ . Then there exists a ball  $B(x, r) \subset \mathbb{R}^3$  with  $r = O(1/N)$  such that we have the mass concentration estimate

$$\int_{B(x,r)} |g(y)|^2 dy \gtrsim \eta^3 E^{-1/2} r^2. \tag{1-25}$$

We also recall a result that shows that the mass of solutions of (1-1) can be locally in time controlled.

**Proposition 3** (Local mass is locally stable [Tao 2006]). *Let  $J$  be a time interval, let  $t, t' \in J$  and let  $B(x, r)$  be a ball. Let  $u$  be a solution of (1-1). Then*

$$\left( \int_{B(x,r)} |u(t', y)|^2 dy \right)^{1/2} = \left( \int_{B(x,r)} |u(t, y)|^2 dy \right)^{1/2} + O(E^{1/2}|t - t'|). \tag{1-26}$$

This result, proved for (1-5) in [Tao 2006], is also true for (1-1). Indeed the proof relied upon the fact that the  $L^2(\mathbb{R}^3)$  norm of the velocity of the solution of (1-5) at time  $t$  is bounded by the square root of its energy, which is also true for the solution of (1-1) (by (1-2) and (1-7)).

Now we make some comments with respect to Theorem 1. If the function  $g$  were a positive constant, it would be easy to prove that the solution of (1-1) with data  $(u_0, u_1)$  lies in  $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ , since we have a good global theory for (1-5). Therefore we can hope to prove global well-posedness for  $g$  slowly increasing to infinity, by extending the technology to prove global well-posedness for (1-5). Notice also that Tao [2006] found that the solution  $u$  of (1-5) satisfies

$$\|u\|_{L_t^4 L_x^{12}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \tilde{E}^{\tilde{E}^{O(1)}}, \tag{1-27}$$

with  $\tilde{E}$  the energy of  $u$ . The structure of  $g$  is a double log: it is, roughly speaking, the inverse function of the tower exponential bound in (1-27).

Now we explain the main ideas of this paper.

Tao [2006] was able to bound on arbitrary long time intervals the  $L_t^4 L_x^{12}$  norm of solutions of the energy-critical equation (1-5) by a quantity that depends exponentially on their energy. This estimate can be viewed as a long time estimate. Unfortunately we cannot expect to prove a similar result for (1-1) since we are not in the energy-critical regime. However we shall prove the following proposition:

**Proposition 4** (Long time estimate). *Let  $J = [t_1, t_2]$  be a time interval. Let  $u$  be a classical solution of (1-1). Assume that*

$$\|u\|_{L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)} \leq M \tag{1-28}$$

for some  $M \geq 0$ . Then there exist three constants  $C_{L,0} > 0$ ,  $C_{L,1} > 0$  and  $C_{L,2} > 0$  such that

- if  $E \ll \frac{1}{g^{1/2}(M)}$  (small energy regime) then

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq C_{L,0}; \tag{1-29}$$

- if  $E \gtrsim \frac{1}{g^{1/2}(M)}$  (large energy regime) then

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq (C_{L,1}(Eg(M)))^{C_{L,2}(E^{193/4+} g^{225/8+}(M))}. \tag{1-30}$$

This proposition shows that we can control the  $L_t^4 L_x^{12}(J \times \mathbb{R}^3)$  norm of solutions of (1-1) by their energy and an a priori bound of their  $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$  norm. We would like to control the pointwise-in-time  $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$  norm of  $u$  on an interval  $[0, T]$ , with  $T$  arbitrarily large. This is done by an induction on time. We assume that this norm is controlled on  $[0, T]$  by a number  $M_0$ . Then by continuity we can find a slightly larger interval  $[0, T']$  such that this norm is bounded by (say)  $2M_0$  on  $[0, T']$ . This is our a priori bound. We subdivide  $[0, T']$  into subintervals where the  $L_t^4 L_x^{12}$  norm of  $u$  is small and we control the pointwise-in-time  $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$  norm of  $u$  on each of these subintervals (see Lemma 6). Since  $g$  varies slowly we can estimate the number of intervals of this partition by using Proposition 4 and we can prove a posteriori that  $\|u(t)\|_{\tilde{H}^2(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\tilde{H}^1(\mathbb{R}^3)}$  is bounded on  $[0, T']$  by  $M_0$ , provided that  $M_0$  is large enough; see Section 2.

The proof of Proposition 4 is a modification of the argument used in [Tao 2006] to establish a tower-exponential bound of the  $L_t^4 L_x^{12}(J \times \mathbb{R}^3)$  norm of  $v$ , the solution of (1-5). We divide  $J$  into subintervals  $J_i$  where the  $L_t^4 L_x^{12}$  norm of  $u$ , the solution of (1-1), is “substantial”. Then by using the Strichartz estimates and the Sobolev embedding (1-22) we notice that the  $L_t^\infty L_x^6(J_i \times \mathbb{R}^3)$  norm of  $u$  is also substantial, more precisely, we find a lower bound that depends on the energy  $E$  and  $g(M)$ . Then by Proposition 2 we can localize a bubble where the mass concentrates and we prove that the size of these subintervals is also substantially large. Tao [2006] used the mass concentration to construct a solution  $\tilde{v}$  of (1-5) that has a smaller energy than  $v$  and that coincides with  $v$  outside a cone. The idea behind that is to use an induction on the levels of energy, due to Bourgain [1999], and the small energy theory following from the Strichartz estimates in order to control the  $L_t^4 L_x^{12}$  norm of  $v$  outside a cone. Unfortunately it seems almost impossible to apply this procedure to our problem. Indeed the energy of the constructed solution  $\tilde{u}$  is smaller than the energy  $E$  of  $u$  by an amount that depends on  $E$  but also on  $g(M)$  and therefore an induction on the levels of the energy is possible if the  $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$  norm of  $\tilde{u}$  can be controlled by  $M$ , which is far from being trivial. It turns out that we do not need to use the Bourgain induction method. Indeed since we know that the size of the subintervals  $J_i$  s is substantially large and since we have a good control of the  $L_t^4 L_x^{12}$  norm on these subintervals it suffices to find an upper bound of the size of their union in order to conclude. To this end we divide a cone containing the ball where the mass concentrates and the  $J_i$  s into truncated-in-time cones where the  $L_t^4 L_x^{12}$  norm of  $u$  is substantial. Let  $\tilde{J}_1, \tilde{J}_2, \dots$  be the sequence of time intervals resulting from this partition. The mass concentration helps us to control the size of the first time interval  $\tilde{J}_1$ . By using an asymptotic stability result we can prove, roughly speaking, that if we consider two successive subintervals  $\tilde{J}_j, \tilde{J}_{j+1}$  resulting from this partition of the cone then the size of  $\tilde{J}_{j+1}$  can be controlled by the size of  $\tilde{J}_j$ ; see (3-34). But a potential energy decay estimate shows that if the size of the union of the  $J_i$  s is too large then we can find a large subinterval  $[t'_1, t'_2]$  such that the  $L_t^4 L_x^{12}$  norm of  $u$  on the cone truncated to  $[t'_1, t'_2]$  is small. Therefore  $[t'_1, t'_2]$  cannot be covered by many  $\tilde{J}_j$  s and one of them is very large in comparison with its predecessor, which contradicts (3-34). At the end of the process we can find an upper bound of the size of the union of the subintervals  $J_i$  s and consequently we can control the  $L_t^4 L_x^{12}$  norm of  $u$  on the interval  $J$ .

**Remark 5.** We will frequently use the  $x+$  and  $x-$  notations. Indeed the point  $(2, \infty)$  is not wave admissible. Therefore we will work with the point  $(2+, \infty-)$ : see (5-6) and (7-9). This generates slight variations of many quantities throughout this paper. Sometimes we might deal with quantities like  $z := x+ / y-$ . We cannot conclude directly that  $z = (x/y)+$ . In this case we create a variation of  $y$  so

small (compared to that of  $x$ ) that we have  $z = (x/y)_+$ . These details have been omitted for the sake of readability. We strongly recommend that the reader ignores these slight variations at the first reading.

### 2. Proof of Theorem 1

The proof relies upon Proposition 4 and the following lemma, which we prove on page 268.

**Lemma 6** (Local boundedness). *Let  $J = [t_1, t_2]$  be an interval. Assume that  $u$  is a classical solution of (1-1). Let  $Z(t) := \|(u(t), \partial_t u(t))\|_{\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)}$ . There exists  $0 < \epsilon \ll \text{constant}$  such that if*

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq \frac{\epsilon}{g^{1/4}(Z(t_1))}, \tag{2-1}$$

then there exists  $C_l > 0$  such that

$$Z(t) \leq 2C_l Z(t_1) \quad \text{for } t \in J. \tag{2-2}$$

We claim that the set

$$\mathcal{F} := \{T \in [0, \infty) : \sup_{t \in [0, T]} \|(u(t), \partial_t u(t))\|_{\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq M_0\} \tag{2-3}$$

is equal to  $[0, \infty)$  for some constant  $M_0 := M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$  large enough. Indeed,  $0 \in \mathcal{F}$  (this is clear);  $\mathcal{F}$  is closed, by continuity; and  $\mathcal{F}$  is open. To see this last fact, let  $T \in \mathcal{F}$ . Then by continuity there exists  $\delta > 0$  such that

$$\sup_{t \in [0, T']} \|(u(t), \partial_t u(t))\|_{\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq 2M_0 \tag{2-4}$$

for every  $T' \in [0, T + \delta)$ . By (1-29) and (1-30) we have

$$\|u\|_{L_t^4 L_x^{12}([0, T'] \times \mathbb{R}^3)} \leq \max(C_{L,0}, (C_{L,1} E g(2M_0))^{C_{L,2}(E^{(193/4)+} g^{(225/8)+}(2M_0)})). \tag{2-5}$$

Let  $N \geq 1$  and let  $\underline{Z}(0) := \max(Z(0), 1)$ . Without loss of generality we can assume that  $C_l \gg 1$  so that  $2C_l \underline{Z}(0) \gg 1$  and  $\log^c(2C_l \underline{Z}(0)) \gg 1$ . We have, by the elementary rules of the logarithm and the inequality  $\log^c(2nx) \leq \log^c((2n)^x)$  for  $n \geq 1$  and  $x \gg 1$ :

$$\begin{aligned} \sum_{n=1}^N \frac{\epsilon^4}{g((2C_l)^n Z_0)} &\geq \sum_{n=1}^N \frac{\epsilon^4}{\log^c(\log((2C_l)^{2n} \underline{Z}^{2n}(0) + 10))} \gtrsim \sum_{n=1}^N \frac{1}{\log^c(2n \log(2C_l \underline{Z}(0)))} \\ &\gtrsim \frac{1}{\log^c(2C_l \underline{Z}(0))} \sum_{n=1}^N \frac{1}{\log^c(2n)} \gtrsim \frac{1}{\log^c(2C_l \underline{Z}(0))} \int_1^{N+1} \frac{1}{\log^c(2t)} dt \\ &\gtrsim \frac{1}{\log^c(2C_l \underline{Z}(0))} \int_1^{N+1} \frac{1}{t^{1/2}} dt \gtrsim \frac{N^{1/2}}{\log^c(2C_l \underline{Z}(0))}. \end{aligned} \tag{2-6}$$



By Lemma 6, (2-5) and (2-6) we can construct a partition  $(J_n)_{1 \leq n \leq N}$  of  $[0, T']$  such that

$$\begin{aligned} \|u\|_{L_t^4 L_x^{12}(J_n \times \mathbb{R}^3)} &= \frac{\epsilon}{g^{1/4} ((2C_l)^n Z_0)}, \quad 1 \leq n < N, \\ \|u\|_{L_t^4 L_x^{12}(J_N \times \mathbb{R}^3)} &\leq \frac{\epsilon}{g^{1/4} ((2C_l)^N Z_0)}, \quad Z(t) \leq (2C_l)^n Z(0), \end{aligned}$$

for  $t \in J_1 \cup \dots \cup J_n$  and

$$\frac{N^{1/2}}{\log^c(2C_l Z(0))} \leq \max(C_{L,0}, (C_{L,1} E g(2M_0))^{C_{L,2}} (E^{193/4+} g^{225/8+} (2M_0))). \tag{2-7}$$

Since  $c < 8/225$  we have by (1-24)

$$\begin{aligned} \log N &\lesssim \log^c(2C_l Z(0)) + \log(C_{L,0}) \\ &\quad + C_{L,2} E^{(193/4)+} \log^{(225c/8)+} \log(10+4M_0^2) \log(C_{L,1} E \log^c \log(10+4M_0^2)) \\ &\leq \log\left(\frac{\log(M_0/Z(0))}{\log(2C_l)}\right), \end{aligned} \tag{2-8}$$

if  $M_0 = M_0(\|u_0\|_{\dot{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$  is large enough. To prove the last inequality in (2-8) it is enough, by using (1-24), to notice that  $\lim_{M_0 \rightarrow \infty} f(M_0) = 0$  with

$$f(M_0) := \frac{\log^c(2C_l Z(0)) + \log(C_{L,0}) + C_{L,2} E^{(193/4)+} \log^{(225c/8)+} \log(10+4M_0^2) \log(C_{L,1} E \log^c \log(10+4M_0^2))}{\log\left(\frac{\log(M_0/Z(0))}{\log(2C_l)}\right)}. \tag{2-9}$$

Therefore we conclude that

$$\sup_{t \in [0, T']} \|(u(t), \partial_t u(t))\|_{H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)} \leq (2C_l)^N Z(0) \leq M_0. \tag{2-10}$$

*Proof of Lemma 6.* By the Strichartz estimates (1-16), the Sobolev embeddings (1-22) and (1-23) and the elementary estimate  $|u^5 \nabla(g(u))| \lesssim |u^4 \nabla u g(u)|$ , we have

$$\begin{aligned} Z(t) &\lesssim Z(t_1) + \|u^5 g(u)\|_{L_t^1 L_x^2([t_1, t] \times \mathbb{R}^3)} + \|u^4 \nabla u g(u)\|_{L_t^1 L_x^2([t_1, t] \times \mathbb{R}^3)} + \|u^5 \nabla(g(u))\|_{L_t^1 L_x^2([t_1, t] \times \mathbb{R}^3)} \\ &\lesssim Z(t_1) + \|u^5 g(u)\|_{L_t^1 L_x^2([t_1, t] \times \mathbb{R}^3)} + \|u^4 \nabla u g(u)\|_{L_t^1 L_x^2([t_1, t] \times \mathbb{R}^3)} \\ &\lesssim Z(t_1) + \|u\|_{L_t^4 L_x^{12}([t_1, t] \times \mathbb{R}^3)}^4 \|u\|_{L_t^\infty L_x^6([t_1, t] \times \mathbb{R}^3)} g(\|u\|_{L_t^\infty L_x^\infty([t_1, t] \times \mathbb{R}^3)}) \\ &\quad + \|u\|_{L_t^4 L_x^{12}([t_1, t] \times \mathbb{R}^3)}^4 \|\nabla u\|_{L_t^\infty L_x^6([t_1, t] \times \mathbb{R}^3)} g(\|u\|_{L_t^\infty L_x^\infty([t_1, t] \times \mathbb{R}^3)}) \\ &\lesssim Z(t_1) + \|u\|_{L_t^4 L_x^{12}([t_1, t] \times \mathbb{R}^3)}^4 Z(t) g(Z(t)). \end{aligned} \tag{2-11}$$

Let  $C_l$  be the constant determined by the last inequality in (2-11). From (2-1), (2-11) and a continuity argument, we have (2-2). □

### 3. Proof of Proposition 4

The proof relies upon five lemmas, which we state here and then prove in subsequent sections, after seeing how they imply the proposition.

**Lemma 7** (Long time estimate if energy small). *Let  $J = [t_1, t_2]$  be a time interval. Let  $u$  be a classical solution of (1-1). Assume that (1-28) holds. If*

$$E \ll \frac{1}{g^{1/2}(M)}, \tag{3-1}$$

then

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \lesssim 1. \tag{3-2}$$

**Lemma 8** (If  $\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)}$  is nonnegligible a mass concentration bubble exists and the size of  $J$  is bounded from below). *Let  $u$  be a classical solution of (1-1). Let  $J$  be a time interval. Assume that (1-28) holds. Let  $\eta$  be a positive number such that*

$$\eta \leq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-3}$$

If  $\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \geq \eta$ , then

$$\|u\|_{L^\infty L_x^6(J \times \mathbb{R}^3)} \gtrsim \eta^{2+} E^{-(1/2)+}. \tag{3-4}$$

Moreover, there exist a point  $x_0 \in \mathbb{R}^3$ , a time  $t_0 \in J$  and a positive number  $r$  such that we have the mass concentration estimate in the ball  $B(x_0, r)$

$$\int_{B(x_0, r)} |u(t_0, y)|^2 dy \gtrsim \eta^{6+} E^{-(2+)} r^2, \tag{3-5}$$

and the following lower bound on the size of  $J$ :

$$|J| \gtrsim \eta^4 E^{-2/3} r. \tag{3-6}$$

**Lemma 9** (Potential energy decay estimate). *Let  $u$  be a classical solution of (1-1). Let  $[a, b]$  be an interval. Then we have the potential energy decay estimate*

$$\int_{|x| \leq b} F(u(b, x)) dx \lesssim \frac{a}{b} (e(a) + e^{1/3}(a)) + e(b) - e(a) + (e(b) - e(a))^{1/3}. \tag{3-7}$$

**Lemma 10** ( $L_t^4 L_x^{12}$  norm of  $u$  is small on a large truncation of the forward light cone). *Let  $J = [t_1, t_2]$  be an interval. Let  $u$  be a classical solution of (1-1). Assume that (1-28) holds. Let  $\eta$  be a positive number such that*

$$\eta \ll \min\left(E^{1/4}, E^{5/18}, \frac{E^{1/12}}{g^{5/24}(M)}\right). \tag{3-8}$$

Assume also that there exists  $C_2 \gg 1$  such that

$$[t_1, (C_2 E^{10+} \eta^{-(36+)})^{4C_2} E^{10+} \eta^{-(36+)} t_1] \subset J. \tag{3-9}$$

Then there exists a subinterval  $J' = [t'_1, t'_2]$  such that  $|t'_2/t'_1| \sim E^{10+} \eta^{-(36+)}$  and

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J'))} \leq \eta. \tag{3-10}$$

**Lemma 11** (Asymptotic stability). *Let  $J = [t_1, t_2]$  be a time interval. Let  $J' = [t'_1, t'_2] \subset J$  and let  $t \in J/J'$ . Let  $u$  be a classical solution of (1-1). Assume that (1-28) holds. Then*

$$\|u_{l, t'_2}(t) - u_{l, t'_1}(t)\|_{L_x^\infty(\mathbb{R}^3)} \lesssim \frac{E^{5/6} g^{1/6}(M)}{\text{dist}^{1/2}(t, J')}. \tag{3-11}$$

We are ready to prove [Proposition 4](#). We assume that we have an a priori bound  $M$  of the  $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$  norm of the solution  $u$ . There are two steps:

- If  $E \ll 1/g^{1/2}(M)$ , then we know from [Lemma 7](#) that (1-29) holds.
- Therefore we assume that the energy is large, that is,

$$E \gtrsim \frac{1}{g^{1/2}(M)}. \tag{3-12}$$

We can assume without loss of generality that

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \geq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-13}$$

From (3-13) we can partition  $J$  into subintervals  $J_1, \dots, J_l$  such that for  $i = 1, \dots, l - 1$ ,

$$\|u\|_{L_t^4 L_x^{12}(J_i \times \mathbb{R}^3)} = \frac{E^{1/12}}{g^{5/24}(M)} \quad \text{and} \quad \|u\|_{L_t^4 L_x^{12}(J_l \times \mathbb{R}^3)} \leq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-14}$$

Before moving forward we say that an interval  $J_i$  is *exceptional* if

$$\|u_{l,t_1}\|_{L_t^4 L_x^{12}(J_i \times \mathbb{R}^3)} + \|u_{l,t_2}\|_{L_t^4 L_x^{12}(J_i \times \mathbb{R}^3)} \geq \frac{1}{(C_3 E g(M))^{C_4 (E^{193/4} + g^{(225/8)+}(M)})}, \tag{3-15}$$

for some  $C_3 \gg 1, C_4 \gg 1$  to be chosen later. (The numbers  $193/4$  and  $225/8$  will play an important role in (3-44).) Otherwise  $J_i$  is *unexceptional*. Let  $\mathcal{E}$  denote the set of  $J'_i$  s that are exceptional and let  $\overline{\mathcal{E}^c}$  denote the set of nonempty sequences of consecutive unexceptional intervals  $J_i$ . By (1-16), (3-12) and (3-15),

$$\text{card}(\mathcal{E}) \lesssim E^2 [O(Eg(M))]^{O(E^{193/4} + g^{(225/8)+}(M))} \lesssim [O(Eg(M))]^{O(E^{193/4} + g^{(225/8)+}(M))}. \tag{3-16}$$

Since  $\text{card}(\overline{\mathcal{E}^c}) \lesssim \text{card}(\mathcal{E})$  we have

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)}^4 \lesssim [O(Eg(M))]^{O(E^{193/4} + g^{(225/8)+}(M))} \left( \frac{E^{1/3}}{g^{5/6}(M)} + \sup_{K \in \overline{\mathcal{E}^c}} \|u\|_{L_t^4 L_x^{12}(K \times \mathbb{R}^3)}^4 \right). \tag{3-17}$$

Let  $K = J_{i_0} \cup \dots \cup J_{i_1}$  be a sequence of consecutive unexceptional intervals. If  $N(K)$  is the number of  $J_i$  s making  $K$  then by (3-12), (3-14) and (3-17) we have

$$\|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \lesssim \left( \sup_{K \in \overline{\mathcal{E}^c}} N(K) \right) [O(Eg(M))]^{O(E^{193/4} + g^{(225/8)+}(M))}. \tag{3-18}$$

Therefore it suffices to estimate  $N(K)$  for every  $K = J_{i_0} \cup \dots \cup J_{i_1}$ . We will do that by first determining a lower bound for the size of the elements  $J_i$ s and then by estimating the size of  $K$ . By (3-12), (3-14) and [Lemma 8](#), there exists for  $i \in [i_0, \dots, i_1]$  a  $(t_i, r_i, x_i) \in (J_i \times (0, \infty) \times \mathbb{R}^3)$  such that

$$\frac{1}{r_i^2} \int_{B(x_i, r_i)} |u(t_i, y)|^2 dy \gtrsim \frac{E^{-(3/2+)}}{g^{5/4+}(M)} \tag{3-19}$$

and

$$|J_i| \gtrsim \frac{E^{-1/3} r_i}{g^{5/6}(M)}. \tag{3-20}$$

Let  $k \in [i_0, \dots, i_1]$  be such that  $r_k = \min_{i \in [i_0, i_1]} r_i$ ; let  $f(t, r, x) := \frac{1}{r^2} \int_{B(x,r)} |u(t, y)|^2 dy$ ; let  $C_5$  be the constant determined by (3-19); and let  $r_0 = r_0(M)$  be defined by

$$r_0 M^2 = \frac{C_5 E^{-(3/2)+}}{4g^{(5/4)+(M)}}.$$

Since  $f(t, r, x) \leq rM^2$  we have

$$f(t, r_0, x) \leq \frac{C_5 E^{-(3/2)+}}{4g^{(5/4)+(M)}}.$$

The set  $A := \{(t, r, x) : t \in K, r_0 \leq r \leq r_k, x \in \mathbb{R}^3\}$  is connected. Therefore its image is connected by  $f$  and there exists  $(\tilde{t}, \tilde{r}, \tilde{x}) \in K \times [r_0, r_k] \times \mathbb{R}^3$  such that  $f(\tilde{t}, \tilde{r}, \tilde{x}) = (C_5 E^{-(3/2)+}) / (2g^{(5/4)+(M)})$ . In other words we have the following mass concentration

$$\frac{1}{\tilde{r}^2} \int_{B(\tilde{x}, \tilde{r})} u^2(\tilde{t}, x) dx = \frac{C_5 E^{-(3/2)+}}{2g^{(5/4)+(M)}}. \tag{3-21}$$

Moreover we have the useful lower bound for the size of  $J_i$ ,<sup>1</sup>  $i_0 \leq i \leq i_1$ :

$$|J_i| \gtrsim \tilde{r} \frac{E^{-1/3}}{g^{5/6}(M)}. \tag{3-22}$$

At this point we need to use the following lemma, which gives information about the size of  $K$ .

**Lemma 12.** *Let  $K$  be a sequence of unexceptional intervals. Assume there exist  $\bar{t} \in K, \bar{x} \in \mathbb{R}^3$  and  $\bar{r} \in (0, \infty)$  such that*

$$\frac{1}{\bar{r}^2} \int_{B(\bar{x}, \bar{r})} u^2(\bar{t}, y) dy \gtrsim E^{-(3/2)+} g^{(5/4)+(M)}. \tag{3-23}$$

Then there exist two constants  $C_6 \gg 1, C_7 \gg 1$  such that

$$|K| \leq (C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+(M)}} \bar{r}. \tag{3-24}$$

If we combine the lemma with (3-22) we can estimate  $N(K)$ . More precisely, by Lemma 12, (3-22) and (3-12) we have

$$N(K) \lesssim \frac{(C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+(M)}} \bar{r}}{\tilde{r} \frac{E^{-1/3}}{g^{5/6}(M)}} \lesssim (O(E g(M)))^{O(E^{(193/4)+} g^{(225/8)+(M)})}. \tag{3-25}$$

Plugging this upper bound for  $N(K)$  into (3-18) we get (1-30), completing the proof of the proposition (modulo the lemmas).

*Proof of Lemma 12.* By using the space translation invariance of (1-1) we can reduce to the case where  $\bar{x}$  vanishes.<sup>2</sup> By using the time reversal invariance and the time translation invariance<sup>3</sup> it suffices to estimate  $|K \cap [\bar{t}, \infty)|$ . By using the time translation invariance again<sup>4</sup> we can assume that  $\bar{t} = \bar{r}$  and

<sup>1</sup>Notice that we have the lower bound  $\tilde{r} \geq C_5 E^{-(3/2)+} / (4M^2 g^{(5/4)+(M)})$ . One might think that the presence of  $\tilde{r}$  in (3-22) is annoying since this lower bound is crude. However we will see that  $\tilde{r}$  disappears at the end of the process: see (3-25). Therefore a sharp lower bound is not required.

<sup>2</sup>We consider the function  $u_1(t, x) = u(t, x - \bar{x})$  and we abuse notation in the sequel by writing  $u_1$  for  $u$ .

<sup>3</sup>We consider the function  $u_2(t, x) := u(2\bar{t} - t, x)$  and we abuse notation in the sequel by writing  $u_2$  for  $u$ .

<sup>4</sup>We consider the function  $u_3(t, x) := u(t + (\bar{t} - \bar{r}), x)$  and we abuse notation in the sequel by writing  $u_3$  for  $u$ .

therefore  $\bar{r} \in K$ . Let  $K_+ := K \cap [\bar{r}, \infty)$ . We are interested in estimating  $|K_+|$ . We would like to use Lemma 10. Therefore, we consider the set  $\Gamma_+(K_+)$ . We have

$$\frac{1}{\bar{r}^2} \int_{B(0, \bar{r})} |u(\bar{r}, y)|^2 dy \gtrsim \frac{E^{-(3/2)+}}{g^{(5/4)+}(M)}. \tag{3-26}$$

Therefore by Proposition 3 and (3-26) we have

$$\int_{B(0, \bar{r})} |u(t, y)|^2 dy \gtrsim \frac{E^{-(3/2)+} \bar{r}^2}{g^{(5/4)+}(M)} \tag{3-27}$$

if  $(t - \bar{r})E^{1/2} \leq (c_0 E^{-(3/4)+} \bar{r} / g^{(5/8)+}(M))$  for some  $c_0 \ll 1$ . Therefore by Hölder there exists  $0 < c_1 \ll 1$  small enough such that

$$\|u\|_{L_t^4 L_x^{12} \left( \Gamma_+ \left( \left[ \bar{r}, \bar{r} + \frac{c_0 E^{-(5/4)+} \bar{r}}{g^{(5/8)+}(M)} \right] \right) \right)} \geq c_1 \frac{E^{-17/16}}{g^{25/32}(M)}. \tag{3-28}$$

Suppose first that  $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(K_+))} \leq c_1 \frac{E^{-(17/16)}}{g^{(25/32)}(M)}$ . In this case we get from (3-28)

$$K_+ \subset \left[ \bar{r}, \bar{r} + \frac{c_0 E^{-(5/4)+} \bar{r}}{g^{(5/8)+}(M)} \right], \tag{3-29}$$

and, using also (3-12), we get (3-24).

Now suppose instead that  $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(K_+))} \geq c_1 \frac{E^{-(17/16)+}}{g^{(25/32)+}(M)}$ . Define

$$\tilde{\eta} := \frac{c_1}{4} \frac{E^{-(17/16)+}}{g^{(25/32)+}(M)}, \tag{3-30}$$

and divide  $\Gamma_+(K_+)$  into consecutive cone truncations  $\Gamma_+(\tilde{J}_1), \dots, \Gamma_+(\tilde{J}_k)$  such that, for  $j = 1, \dots, k - 1$ ,

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} = \tilde{\eta} \tag{3-31}$$

and

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_k))} \leq \tilde{\eta}. \tag{3-32}$$

We get from (3-28)

$$\tilde{J}_1 \subset \left[ \bar{r}, \bar{r} + \frac{c_0 E^{-(5/4)+} \bar{r}}{g^{(5/8)+}(M)} \right]. \tag{3-33}$$

**Result 13.** *If  $j \in [1, \dots, k - 1]$  we either have*

$$|\tilde{J}_{j+1}| \lesssim |\tilde{J}_j| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M) \tag{3-34}$$

or

$$|\tilde{J}_j| \geq (C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \bar{r} \tag{3-35}$$

for some constants  $C_6 \gg 1, C_7 \gg 1$ .

*Proof.* We get from (1-21), (3-12) and (3-30)

$$\begin{aligned} \|u - u_{l,t_{j+1}}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} &\lesssim \|u^5 g(u)\|_{L_t^1 L_x^2(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \\ &\lesssim \|u^4\|_{L_t^1 L_x^3(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \|u g^{1/6}(u)\|_{L_t^\infty L_x^6(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} g^{5/6}(M) \\ &\lesssim \tilde{\eta}^4 E^{1/6} g^{5/6}(M) \\ &\ll \tilde{\eta}, \end{aligned} \quad (3-36)$$

with  $J_j = [t_{j-1}, t_j]$ . Therefore by (3-31) we have  $\|u_{l,t_{j+1}}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} \sim \tilde{\eta}$ . This implies that

$$\|u_{l,t_{j+1}} - u_{l,t_2}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta} \quad (3-37)$$

or

$$\|u_{l,t_2}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}. \quad (3-38)$$

**Case 1.**  $\|u_{l,t_{j+1}} - u_{l,t_2}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$ . By Lemma 11 and Hölder we have

$$\begin{aligned} \|u_{l,t_{j+1}} - u_{l,t_2}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} &\lesssim |\tilde{J}_j|^{1/4} \|u_{l,t_{j+1}} - u_{l,t_2}\|_{L_t^\infty L_x^{12}(\Gamma_+(\tilde{J}_j))} \\ &\lesssim |\tilde{J}_j|^{1/4} \|u_{l,t_{j+1}} - u_{l,t_2}\|_{L_t^\infty L_x^\infty(\Gamma_+(\tilde{J}_j))}^{1/2} \|u_{l,t_{j+1}} - u_{l,t_2}\|_{L_t^\infty L_x^6(\Gamma_+(\tilde{J}_j))}^{1/2} \\ &\lesssim \frac{|\tilde{J}_j|^{1/4} E^{2/3} g^{1/12}(M)}{|\tilde{J}_{j+1}|^{1/4}}. \end{aligned} \quad (3-39)$$

We get (3-34) from (3-37) and (3-39).

**Case 2.**  $\|u_{l,t_2}\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$ . In this case  $\|u_{l,t_2}\|_{L_t^4 L_x^{12}(\tilde{J}_j)} \gtrsim \tilde{\eta}$ . Recall that  $K_+$  is a subinterval of  $K = J_{i_0} \cup \dots \cup J_{i_1}$ , sequence of unexceptional intervals  $J_i$ ,  $i_0 \leq i \leq i_1$ . Consequently there are at least  $\sim \tilde{\eta}(C_3 E g(M))^{C_4 E^{(193/4)+g^{(225/8)+}(M)}}$  intervals  $J_j$  that cover  $\tilde{J}_i$ . Therefore we get (3-35) from (3-22) and (3-12).  $\square$

Using Result 13 and Lemma 10 we can get an upper bound on the size  $|K_+|$ :

**Result 14.** *We have*

$$|K_+| \leq (C_6 E g(M))^{C_7 (E^{(193/4)+g^{(225/8)+}(M)})} \bar{r}. \quad (3-40)$$

*Proof.* Let  $B := (C_6 E g(M))^{C_7 (E^{(193/4)+g^{(225/8)+}(M)})}$ . Assume that (3-40) fails. Let  $\tilde{J}_{j_1}$  be the first interval for which  $|\tilde{J}_1 \cup \dots \cup \tilde{J}_{j_1}|$  exceeds  $B\bar{r}$ . Then  $j_1 \neq 1$ ,  $|\tilde{J}_{j_1}| \lesssim |\tilde{J}_{j_1-1}| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M)$  and we have

$$\frac{c_1 E^{-5/4} \tilde{r}}{g^{(5/8)}(M)} + T_2 - T_1 + (T_2 - T_1) \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M) \gtrsim |\tilde{J}_1| + \dots + |\tilde{J}_{j_1}| \geq B\bar{r}, \quad (3-41)$$

if  $[T_1, T_2] := \tilde{J}_2 \cup \dots \cup \tilde{J}_{j_1-1}$ . Therefore by (3-12) and (3-41) we have

$$T_2 - T_1 \gtrsim \frac{\tilde{\eta}^4 E^{-(8/3)} B\bar{r}}{g^{1/3}(M)}. \quad (3-42)$$

Moreover  $T_1 \leq \bar{r} + (c_1 E^{-(5/4)+} \bar{r}) / (g^{(5/8)+}(M))$ . Therefore by (3-12) we have

$$T_1 = O(\bar{r}). \quad (3-43)$$

By (3-42) and (3-43) we have

$$\frac{T_2}{T_1} \geq \left( C_2 E^{10+} \left( \frac{\tilde{\eta}}{4} \right)^{-(36+)} \right)^{4C_2 E^{10+} (\tilde{\eta}/4)^{-(36+)}} \tag{3-44}$$

with  $C_2$  defined in Lemma 10, provided that  $C_6, C_7 \gg \max(c_1, C_2)$ . Therefore we can apply Lemma 10 and find a subinterval  $[t'_1, t'_2] \subset \tilde{J}_2 \cup \dots \cup \tilde{J}_{j_1-1}$  with  $|t'_2/t'_1| \sim E^{10+} \tilde{\eta}^{-(36+)}$  and  $\|u\|_{L_t^4 L_x^{12}([t'_1, t'_2])} \leq \tilde{\eta}/4$ . This means that  $[t'_1, t'_2] \subset [T_1, T_2]$  is covered by at most two consecutive intervals. It is convenient to introduce  $[t'_1, t'_2]_g$ , the geometric mean of  $t'_1$  and  $t'_2$ . We have  $[t'_1, t'_2]_g \sim \tilde{\eta}^{-18} E^5 t'_1$ . There are two cases.

**Case 1.**  $[t'_1, t'_2]$  is covered by one interval  $\tilde{J}_{\bar{j}} = [a_{\bar{j}}, b_{\bar{j}}]$ ,  $2 \leq \bar{j} \leq j_1 - 1$ . Then  $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(36+)} E^{10+} t'_1$  and  $|\tilde{J}_{\bar{j}-1}| \leq t'_1$ . Therefore  $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(36+)} E^{10+} |\tilde{J}_{\bar{j}-1}|$ . Contradiction with (3-12) and (3-34).

**Case 2.**  $[t'_1, t'_2]$  is covered by two intervals  $\tilde{J}_{\bar{j}} = [a_{\bar{j}}, b_{\bar{j}}]$  and  $\tilde{J}_{\bar{j}+1} = [a_{\bar{j}+1}, b_{\bar{j}+1}]$  for some  $2 \leq \bar{j} \leq j_1 - 2$ . Then there are two subcases.

*Case 2a.*  $b_{\bar{j}} \leq [t'_1, t'_2]_g$ . In this case  $|\tilde{J}_{\bar{j}+1}| \gtrsim \tilde{\eta}^{-(36+)} E^{10+} t'_1$  and  $|\tilde{J}_{\bar{j}}| \leq \tilde{\eta}^{-(18+)} E^5 t'_1$ . Therefore by (3-12) we have  $|\tilde{J}_{\bar{j}+1}| \gtrsim \tilde{\eta}^{-(18+)} E^5 |\tilde{J}_{\bar{j}}|$ . Contradiction with (3-12) and (3-34).

*Case 2b.*  $b_{\bar{j}} \geq [t'_1, t'_2]_g$ . In this case by (3-12)  $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(18+)} E^5 t'_1$  and  $|\tilde{J}_{\bar{j}-1}| \leq t'_1$ . Therefore  $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(18+)} E^5 |\tilde{J}_{\bar{j}-1}|$ . Contradiction with (3-12) and (3-34).

This exhausts all cases. Thus we have proved Result 14 and so also Lemma 12. □

**Remark 15.** It seems likely that we can find a better upper bound for  $|K_+|$  than (3-40) by exploiting Lemma 11 in a better way. For instance we can consider  $k$  successive time intervals  $\tilde{J}_{j+1}, \dots, \tilde{J}_{j+k}$ ,  $k > 1$  and prove an estimate like

$$|\tilde{J}_{j+1}| + \dots + |\tilde{J}_{j+k}| \lesssim |\tilde{J}_j| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M). \tag{3-45}$$

This estimate is stronger than (3-34). We can probably find a smaller  $B$  such that (3-44) holds with  $\tilde{\eta}$  substituted for something like  $k\tilde{\eta}$  and, by modifying the argument above, find a contradiction with (3-45). At the end of the process we can probably prove global existence of smooth solutions to (1-1) for  $0 < c < c_0$ , with  $c_0 > 8/225$  to be determined. We will not pursue these matters.

### 4. Proof of Lemma 7

Applying the Strichartz estimates and the Hölder inequality,

$$\begin{aligned} \|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} &\lesssim E^{1/2} + \|u^4\|_{L_t^1 L_x^2(J \times \mathbb{R}^3)} \|u g^{1/6}(u)\|_{L_t^\infty L_x^6(J \times \mathbb{R}^3)} \|g^{5/6}(u)\|_{L_t^\infty L_x^\infty(J \times \mathbb{R}^3)} \\ &\lesssim E^{1/2} + E^{1/6} g^{5/6}(M) \|u\|_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)}. \end{aligned} \tag{4-1}$$

Hence (3-2) by (3-1) and a continuity argument.

### 5. Proof of Lemma 8

Let  $J' = [t'_1, t'_2] \subset J$  be such that  $\|u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)} = \eta$ . Then by (1-22) and (3-3)

$$\begin{aligned} \|f(u)\|_{L_t^1 L_x^2(J' \times \mathbb{R}^3)} &\lesssim \|u g^{1/6}(u)\|_{L_t^\infty L_x^6(J' \times \mathbb{R}^3)} \|u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)}^4 \|g^{5/6}(u)\|_{L_t^\infty L_x^\infty(J' \times \mathbb{R}^3)} \\ &\lesssim E^{1/6} \eta^4 g^{5/6}(M) \lesssim E^{1/2}. \end{aligned} \quad (5-1)$$

It is slightly unfortunate that  $(2, \infty)$  is not wave admissible. Therefore we consider the admissible pair  $(2 + \epsilon, 6(2 + \epsilon)/\epsilon)$  with  $\epsilon \ll 1$ . By the Strichartz estimates and (5-1), we have

$$\|u\|_{L_t^{2+\epsilon} L_x^{6(2+\epsilon)/\epsilon}(J' \times \mathbb{R}^3)} \lesssim \|\nabla u(t'_1)\|_{L^2(\mathbb{R}^3)} + \|u(t'_1)\|_{L^2(\mathbb{R}^3)} + \|f(u)\|_{L_t^1 L_x^2(J' \times \mathbb{R}^3)} \lesssim E^{1/2}. \quad (5-2)$$

Let  $N$  be a frequency to be chosen later. By the Bernstein inequality and (1-7) we have

$$\|P_{<N} u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)} \lesssim N^{1/4} |J'|^{1/4} \|u\|_{L_t^\infty L_x^6(J' \times \mathbb{R}^3)} \lesssim N^{1/4} |J'|^{1/4} E^{1/6}. \quad (5-3)$$

Therefore

$$\|P_{<N} u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)} \lesssim |J'|^{1/4} N^{1/4} E^{1/6}. \quad (5-4)$$

Let  $c_2 \ll 1$ . Then if  $N = c_2^4 (\eta^4 / (|J'| E^{2/3}))$  we have

$$\|P_{\geq N} u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)} \gtrsim \eta \quad \text{and} \quad \|u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)} \sim \|P_{\geq N} u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)}. \quad (5-5)$$

By (5-2) and (5-5) we have

$$\begin{aligned} \eta &\sim \|P_{\geq N} u\|_{L_t^4 L_x^{12}(J' \times \mathbb{R}^3)} \\ &\lesssim \|P_{\geq N} u\|_{L_t^{2+\epsilon} L_x^{6(2+\epsilon)/\epsilon}(J' \times \mathbb{R}^3)}^{(2+\epsilon)/4} \|P_{\geq N} u\|_{L_t^\infty L_x^6(J' \times \mathbb{R}^3)}^{1-(2+\epsilon)/4} \\ &\lesssim E^{(2+\epsilon)/8} \|P_{\geq N} u\|_{L_t^\infty L_x^6(J' \times \mathbb{R}^3)}^{1-(2+\epsilon)/4}. \end{aligned} \quad (5-6)$$

Therefore we conclude that  $\|P_{\geq N} u\|_{L_t^\infty L_x^6(J' \times \mathbb{R}^3)} \gtrsim \eta^{2+} E^{-((1/2)+)}$ . Applying Proposition 2 we get (3-5).

### 6. Proof of Lemma 9

Bahouri and Gerard [1999, page 171] used arguments from Grillakis [1990; 1992] and Shatah–Struwe [1993] to derive an a priori estimate of the solution  $u$  to the 3D quintic defocusing wave equation, that is,  $\partial_{tt} u - \Delta u + u^5 = 0$ . More precisely they were able to prove

$$\int_{|x| \leq b} |u(b, x)|^6 dx \lesssim \frac{a}{b} (\tilde{e}(a) + \tilde{e}^{1/3}(a)) + \tilde{e}(b) - \tilde{e}(a) + (\tilde{e}(b) - \tilde{e}(a))^{1/3}, \quad (6-1)$$

with

$$\tilde{e}(t) := \frac{1}{2} \int_{|x| \leq t} (\partial_t u)^2 dx + \frac{1}{2} \int_{|x| \leq t} |\nabla u|^2 dx + \frac{1}{6} \int_{|x| \leq t} u^6 dx. \quad (6-2)$$

Since we apply their ideas to the potential  $f$  we just sketch the proof. Given the cone  $\Gamma_+([a, b])$  we denote by  $\partial\Gamma_+([a, b])$  the mantle of the cone  $\Gamma_+([a, b])$ , that is,

$$\partial\Gamma_+([a, b]) := \{(t', x) \in [a, b] \times \mathbb{R}^3, t = |x|\}. \quad (6-3)$$



The local energy identity

$$e(b) - e(a) = \frac{1}{2\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \left| \frac{x\partial_t u}{t} + \nabla u \right|^2 + \frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} F(u) \tag{6-4}$$

results from the integration of the identity  $\partial_t u (\partial_t u - \Delta u + f(u)) = 0$  on the cone  $\Gamma_+([a, b])$ . We have [Shatah and Struwe 1998]

$$\begin{aligned} &\partial_t \left( \frac{t}{2} (\partial_t u)^2 + \frac{t}{2} |\nabla u|^2 + (x \cdot \nabla u) \partial_t u + t F(u) + u \partial_t u \right) \\ &- \operatorname{div} \left( t \nabla u \partial_t u + (x \cdot \nabla u) \nabla u - \frac{|\nabla u|^2 x}{2} + \frac{(\partial_t u)^2 x}{2} - x F(u) + u \nabla u \right) + u f(u) - 4F(u) = 0. \end{aligned} \tag{6-5}$$

Integrating this identity on  $\Gamma_+([a, b])$ , we have

$$X(b) - X(a) + Y(a, b) = \int_{\Gamma_+([a,b])} 4F(u) - u f(u), \tag{6-6}$$

with

$$X(t) := \int_{|x| \leq t} \frac{t}{2} (\partial_t u)^2 + \frac{t}{2} |\nabla u|^2 + (x \cdot \nabla u) \partial_t u + t F(u) + u \partial_t u \tag{6-7}$$

and

$$\begin{aligned} Y(a, b) := &-\frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \left( \frac{t}{2} (\partial_t u)^2 + \frac{t}{2} |\nabla u|^2 + (x \cdot \nabla u) \partial_t u + t F(u) + u \partial_t u + t \frac{\nabla u \cdot x}{|x|} \partial_t u + \frac{|x \cdot \nabla u|^2}{|x|} \right. \\ &\left. - \frac{|\nabla u|^2}{2} |x| + \frac{(\partial_t u)^2 |x|}{2} - |x| F(u) + u \frac{\nabla u \cdot x}{|x|} \right). \end{aligned} \tag{6-8}$$

In fact we have [Shatah and Struwe 1993]

$$X(t) = \int_{|x| \leq t} t \left[ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} \left| \nabla u + \frac{ux}{|x|^2} \right|^2 \right] + \partial_t u (x \cdot \nabla u + u) + t F(u) - \int_{|x|=t} \frac{u^2}{2}. \tag{6-9}$$

Since  $t = |x|$  on  $\partial\Gamma_+([a, b])$  we have

$$Y(a, b) = -\frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} |x| (\partial_t u)^2 + 2(x \cdot \nabla u) \partial_t u + u \partial_t u + \frac{(x \cdot \nabla u)^2}{|x|} + u \frac{\nabla u \cdot x}{|x|}, \tag{6-10}$$

and after some computations [Shatah and Struwe 1993], we get

$$Y(a, b) = -\frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \frac{1}{t} (t \partial_t u + (\nabla u \cdot x) + u)^2 + \int_{|x|=b} \frac{u^2}{2} - \int_{|x|=a} \frac{u^2}{2}. \tag{6-11}$$

Therefore, if

$$H(t) := \int_{|x| \leq t} t \left[ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} \left| \nabla u + \frac{ux}{|x|^2} \right|^2 \right] + \partial_t u (x \cdot \nabla u + u) + t F(u), \tag{6-12}$$

then

$$H(b) - H(a) = \frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \frac{1}{t} (t \partial_t u + \nabla u \cdot x + u)^2 + \int_{\Gamma_+([a,b])} 4F(u) - u f(u). \tag{6-13}$$

We estimate  $H(t)$ , following [Bahouri and Gérard 1999]. We have

$$|\partial_t u(x \cdot \nabla u + u)| \leq \frac{t}{2} \left( (\partial_t u)^2 + \left| \nabla u + \frac{ux}{|x|^2} \right|^2 \right) \lesssim t \left( (\partial_t u)^2 + |\nabla u|^2 + \frac{u^2}{|x|^2} \right). \tag{6-14}$$

Therefore by (6-14), the Hölder inequality and (1-7), we have

$$H(t) \lesssim t \left( e(t) + \int_{|x| \leq t} \frac{u^2}{|x|^2} \right) \lesssim t \left( e(t) + \left( \int_{|x| \leq t} u^6 \right)^{1/3} \right) \lesssim t(e(t) + e^{1/3}(t)). \tag{6-15}$$

Moreover by (6-4), the Hölder inequality and (1-7), we have

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \frac{1}{t} (t\partial_t u + \nabla u \cdot x + u)^2 &\lesssim \frac{b}{2\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \left( \frac{\nabla u \cdot x}{t} + \partial_t u \right)^2 + \frac{1}{2\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \frac{u^2}{t^2} \\ &\lesssim b \int_{\partial\Gamma_+([a,b])} \left| \frac{x}{t} \partial_t u + \nabla u \right|^2 + \frac{1}{2\sqrt{2}} \left( \int_{\partial\Gamma_+([a,b])} u^6 \right)^{1/3} \\ &\lesssim b((e(b) - e(a)) + (e(b) - e(a))^{1/3}). \end{aligned} \tag{6-16}$$

We get from (1-7)

$$4F(u) - uf(u) \leq 0. \tag{6-17}$$

By (6-13), and (6-15)–(6-17), we have

$$\begin{aligned} \int_{|x| \leq b} F(u) &\lesssim \frac{H(b)}{b} \lesssim \frac{H(a) + \frac{1}{\sqrt{2}} \int_{\partial\Gamma_+([a,b])} \frac{1}{t} (t\partial_t u + \nabla u \cdot x + u)^2}{b} \\ &\lesssim \frac{a}{b} (e(a) + e^{1/3}(a)) + e(b) - e(a) + (e(b) - e(a))^{1/3}. \end{aligned} \tag{6-18}$$

### 7. Proof of Lemma 10

The proof relies upon two results that we prove in the subsections.

**Result 16.** *Let  $u$  be a classical solution of (1-1). Assume that (1-28) holds. Let  $\eta$  be a positive number such that (3-3) holds. If  $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J))} \geq \eta$  then*

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J))} \gtrsim \eta^{2+} E^{-((1/2)+)}. \tag{7-1}$$

**Result 17.** *Let  $u$  be a smooth solution to (1-1). Assume that (1-28) holds. Let  $\eta$  be a positive number such that*

$$\eta \leq \min(1, E^{1/18}). \tag{7-2}$$

*Let  $J = [t_1, t_2]$  be an interval such that  $[t_1, t_1(E\eta^{-18})^{4E\eta^{-18}}] \subset J$ . Then there exists a subinterval  $J' = [t'_1, t'_2]$  such that  $|t'_2/t'_1| = E\eta^{-18}$  and*

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \lesssim \eta. \tag{7-3}$$

Let  $C_9$  be the constant determined by  $\gtrsim$  in (7-1). Let  $C_{10}$  be the constant determined by  $\lesssim$  in (7-3). We get from (3-9):

$$\left[ t_1, t_1 \left( E \left( \frac{C_9 \eta^{2+} E^{-(1/2)+}}{2C_{10}} \right)^{-18} \right)^{4E} \left( \frac{C_9 \eta^{2+} E^{-(1/2)+}}{2C_{10}} \right)^{-18} \right] \subset [t_1, C_2 (E^{10+} \eta^{-(36+)})^{4C_2} E^{10+} \eta^{-(36+)} t_1] \subset J, \tag{7-4}$$

if  $C_2 \gg \max(C_9, C_{10})$ . Therefore, since  $(C_9 \eta^{2+} E^{-(1/2+)}) / (2C_{10})$  satisfies (7-2) by (3-8), we can use Result 17 and show that there exists a subinterval  $J' = [t'_1, t'_2]$  such that  $|t'_2/t'_1| \sim E^{10+} \eta^{-(36+)}$  and

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \leq \frac{C_9 \eta^{2+} E^{-(1/2+)} C_{10}}{2C_{10}} \leq C_9 \frac{\eta^{2+} E^{-(1/2+)}}{2}. \tag{7-5}$$

Now we claim that  $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J'))} \leq \eta$ . If not by (3-8) and Result 16 we have

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \geq C_9 \eta^{2+} E^{-(1/2+)}. \tag{7-6}$$

Contradiction with (7-5).

*Proof of Result 16.* We substitute  $J'$  for  $\Gamma_+(J')$  in (5-1) to get

$$\|f(u)\|_{L_t^1 L_x^2(\Gamma_+(J'))} \lesssim E^{1/2}. \tag{7-7}$$

By the Strichartz estimates (1-20) on the truncated cone  $\Gamma_+(J')$  we have

$$\|u\|_{L_t^{2+\epsilon} L_x^{(6(2+\epsilon))/\epsilon}(\Gamma_+(J'))} \lesssim E^{1/2}, \tag{7-8}$$

after following similar steps to prove (5-2). Therefore

$$\eta = \|u\|_{L_t^4 L_x^{12}(\Gamma_+(J))} \lesssim \|u\|_{L_t^{2+\epsilon} L_x^{(6(2+\epsilon))/\epsilon}(\Gamma_+(J'))}^{(2+\epsilon)/4} \|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))}^{1-((2+\epsilon)/4)} \lesssim E^{(2+\epsilon)/8} \|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))}^{1-((2+\epsilon)/4)}. \tag{7-9}$$

Therefore (7-1) holds. □

*Proof of Result 17.* By (7-2) we have  $E \eta^{-18} \geq 1$ . Let  $n$  be the largest integer such that  $2n \leq 4E \eta^{-18}$ . This implies that  $n \geq E \eta^{-18}$ . Let  $A := E \eta^{-18}$ . Now we consider the interval  $[t_1, A^{2n} t_1] \subset J$ . We write  $[t_1, A^{2n} t_1] = [t_1, A^2 t_1] \cup \dots \cup [A^{2(n-1)} t_1, A^{2n} t_1]$ . We have

$$\sum_{i=1}^n e(A^{2i} t_1) - e(A^{2(i-1)} t_1) \leq 2E, \tag{7-10}$$

and by the pigeonhole principle there exists  $i_0 \in [1, n]$  such that

$$e(A^{2i_0} t_1) - e(A^{2(i_0-1)} t_1) \lesssim \eta^{18}. \tag{7-11}$$

Now we choose  $a := A^{2(i_0-1)} t_1$  and  $b \in [A^{2i_0-1} t_1, A^{2i_0} t_1]$ . Let  $t'_1 := A^{2(i_0-1)} t_1$ ,  $t'_2 := A^{2i_0-1} t_1$  and  $J' := [t'_1, t'_2]$ . We apply (3-7) and (7-2) to get

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+([t'_1, t'_2]))} \lesssim \|F(u)\|_{L_t^\infty L_x^1(\Gamma_+([t'_1, t'_2]))} \lesssim (E^{-1} \eta^{18} (E + E^{1/3}) + \eta^{18} + \eta^6)^{1/6} \lesssim \eta. \quad \square$$

### Proof of Lemma 11

We have after computation of the derivative of  $e(t)$

$$\partial_t e(t) \geq \int_{|x|=t} F(u) dS, \quad (7-12)$$

and integrating with respect of time

$$\int_I \int_{|x| \leq t} g(u) u^6(t', x') dS dt' \lesssim E. \quad (7-13)$$

By using the space and time translation invariance

$$\int_J \int_{|x'-x|=|t'-t|} g(u) u^6(t', x') dS dt' \lesssim E. \quad (7-14)$$

Therefore (1-15), (1-22), (7-14) and the Hölder inequality give us

$$\begin{aligned} \left| -\int_{J'} \frac{\sin(t-t')D}{D} g(u) u^5 dt' \right| &= \left| \frac{1}{4\pi |t-t'|} \int_{|x'-x|=|t'-t|} g^{5/6}(u) u^5 g^{1/6}(u) dS dt' \right| \\ &\lesssim \int_{J'} \frac{1}{|t-t'|} \left( \int_{|x'-x|=|t'-t|} u^6 g(u) dS \right)^{5/6} \left( \int_{|x'-x|=|t'-t|} g(u) dS \right)^{1/6} dt' \\ &\lesssim g^{1/6}(M) \int_{J'} \frac{1}{|t-t'|^{2/3}} \left( \int_{|x'-x|=|t'-t|} u^6 g(u) dS \right)^{5/6} dt' \\ &\lesssim g^{1/6}(M) E^{5/6} \left( \int_{J'} \frac{1}{|t-t'|^4} \right)^{1/6} \lesssim g^{1/6}(M) \frac{E^{5/6}}{\text{dist}^{1/2}(t, J')}. \end{aligned} \quad (7-15)$$

Notice that

$$u(t) = u_{l, t_i}(t) - \int_{t_i}^t \frac{\sin(t-t')D}{D} u^5(t') g(u(t')) dt', \quad (7-16)$$

for  $i = 1, 2$ . We get (3-11) from (7-15) and (7-16).

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