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PERIODIC STOCHASTIC KORTEWEG–DE VRIES EQUATION WITH ADDITIVE SPACE-TIME WHITE NOISE

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We prove the local well-posedness of the periodic stochastic Korteweg–de Vries equation with the additive space-time white noise. To treat low regularity of the white noise in space, we consider the Cauchy problem in the Besov-type space $\hat{b}_{p,\infty}^s(\mathbb{T})$ for $s = -\frac{1}{2}+$, $p = 2+$ such that $sp < -1$. In establishing local well-posedness, we use a variant of the Bourgain space adapted to $\hat{b}_{p,\infty}^s(\mathbb{T})$ and establish a nonlinear estimate on the second iteration on the integral formulation. The deterministic part of the nonlinear estimate also yields the local well-posedness of the deterministic KdV in $M(\mathbb{T})$, the space of finite Borel measures on \mathbb{T} .

1. Introduction

In this paper, we prove the local well-posedness of the periodic stochastic Korteweg–de Vries (SKdV) equation with additive space-time white noise:

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = dW, \\ u(x, 0) = u_0(x), \end{cases} \quad (1)$$

where u is a real-valued function, $(x, t) \in \mathbb{T} \times \mathbb{R}^+$ with $\mathbb{T} = [0, 2\pi)$, and $W(t) = \partial B / \partial x$ is a cylindrical Wiener process on $L^2(\mathbb{T})$. With $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, we have $W(t) = \beta_0(t)e_0 + \sum_{n \neq 0} \frac{1}{\sqrt{2}} \beta_n(t)e_n(x)$, where $\{\beta_n\}_{n \geq 0}$ is a family of mutually independent complex-valued Brownian motions (here we take β_0 to be real-valued) in a fixed probability space (Ω, \mathcal{F}, P) associated with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and $\beta_{-n}(t) = \overline{\beta_n}(t)$ for $n \geq 1$. Note that $\text{Var}(\beta_n(1)) = 2$ for $n \geq 1$.

De Bouard et al. [2004] considered

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = \phi dW, \\ u(x, 0) = u_0(x), \end{cases} \quad (2)$$

where ϕ is a bounded linear operator in $L^2(\mathbb{T})$. They showed that (2) is locally well posed when ϕ is a Hilbert–Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. See the references in their paper for earlier work in the periodic and nonperiodic settings.

In this work, we consider the case when ϕ is the identity operator on $L^2(\mathbb{T})$, that is, we take the additive noise to be the space-time white noise $\partial^2 B / \partial t \partial x$, where $B(x, t)$ is a two parameter Brownian motion on $\mathbb{T} \times \mathbb{R}^+$. Note that ϕ is a Hilbert–Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$.

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Suppose that u is the solution to (1), or equivalently to (2) with $\phi = \text{Id}$, the identity operator on $L^2(\mathbb{T})$. Let $v_1(x, t) = u(x + \alpha_0 t, t) - \alpha_0$, where α_0 is the mean of u_0 . Then, v_1 satisfies (1) with the mean-zero initial condition $u_0 - \alpha_0$. Now, let \mathbb{P}_0 be the projection onto the spatial frequency 0, and $\mathbb{P}_{n \neq 0} = \text{Id} - \mathbb{P}_0$. Note that $\mathbb{P}_0 W(t) = \beta_0(t)e_0(x) = \frac{1}{\sqrt{2\pi}}\beta_0(t)$. By letting $v_2 = v_1 - \frac{1}{\sqrt{2\pi}}\beta_0(t)$, we see that u satisfies (1) if and only if v_2 satisfies

$$\begin{cases} dv_2 + (\partial_x^3 v_2 + (v_2 + \frac{1}{\sqrt{2\pi}}\beta_0(t))\partial_x v_2)dt = \mathbb{P}_{n \neq 0} dW, \\ v_2(x, 0) = u_0(x) - \alpha_0, \end{cases}$$

almost surely since $\beta_0(0) = 0$ a.s. By setting $v_3(x, t) = v_2(x + c_\omega(t), t)$ with $c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}}\beta_0(t')dt'$, it follows that v_3 satisfies

$$\begin{cases} dv_3 + (\partial_x^3 v_3 + v_3\partial_x v_3)dt = d\tilde{W}, \\ v_3(x, 0) = u_0(x) - \alpha_0, \end{cases}$$

where

$$\tilde{W}(x, t) = \sum_{n \neq 0} \frac{1}{\sqrt{2}}\beta_n(t)e_n(x + c_\omega(t)) = \sum_{n \neq 0} \frac{1}{\sqrt{2}}\beta_n(t)e^{inc_\omega(t)}e_n(x);$$

that is, v_3 solves (2), where

$$\phi = \text{diag}(\phi_n; n \neq 0) \quad \text{with} \quad \phi_n(t) = e^{inc_\omega(t)} \quad \text{and} \quad c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}}\beta_0(t')dt' \tag{3}$$

(with respect to the basis $\{e_n\}_{n \in \mathbb{Z}}$). Moreover, v_3 has spatial mean 0 (as long as it exists) since e_0 does not lie in the range of ϕ . Therefore, in the remainder of the paper, we concentrate on studying the local well-posedness of (2) with ϕ given by (3) and the mean-zero initial condition u_0 (which implies that u has spatial mean 0 as long as it exists).

Recall that u is called a (local-in-time) mild solution to (2) if u satisfies

$$u(t) = S(t)u_0 - \frac{1}{2} \int_0^t S(t-t')\partial_x u^2(t')dt' + \int_0^t S(t-t')\phi(t')dW(t') \tag{4}$$

at least for $t \in [0, T]$ for some $T > 0$, where $S(t) = e^{-t\partial_x^3}$.

Note that the first two terms in (4) also appear in the deterministic KdV theory. Thus, we briefly review recent well-posedness results of the periodic (deterministic) KdV.

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \tag{5}$$

Bourgain [1993] introduced a new weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n, \tau)\|_{L^2_{n,\tau}(\mathbb{Z} \times \mathbb{R})}, \tag{6}$$

where $\langle \cdot \rangle = 1 + |\cdot|$. He proved the local well-posedness of (5) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding the global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2 norm. Kenig et al. [1996] improved Bourgain's result and established the local well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the bilinear estimate

$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|v\|_{X^{s,\frac{1}{2}}} \tag{7}$$

for $s \geq -\frac{1}{2}$ under the mean-zero assumption on u and v . Colliander et al. [2003] proved the corresponding global well-posedness result via the I -method.

There are also results on (5) which exploit its complete integrability. Bourgain [1997] proved the global well-posedness of (5) in the class $M(\mathbb{T})$ of measures μ , assuming that its total variation $\|\mu\|$ is sufficiently small. His proof is based on the trilinear estimate on the second iteration of the integral formulation of (5), assuming an a priori uniform bound on the Fourier coefficients of the solution u of the form

$$\sup_{n \in \mathbb{Z}} |\hat{u}(n, t)| < C \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

He then established (8) using complete integrability. More recently, Kappeler and Topalov [2006] proved the global well-posedness of the KdV in $H^{-1}(\mathbb{T})$ via the inverse spectral method.

There are also results on the necessary conditions on the regularity with respect to smoothness or uniform continuity of the solution map $: u_0 \in H^s(\mathbb{T}) \rightarrow u(t) \in H^s(\mathbb{T})$. Bourgain [1997] showed that if the solution map is C^3 , then $s \geq -\frac{1}{2}$. Christ et al. [2003] proved that if the solution map is uniformly continuous, then $s \geq -\frac{1}{2}$. (See also [Kenig et al. 2001].) These results, in particular, imply that we cannot hope to have a local-in-time solution of KdV via the fixed point argument in H^s , $s < -\frac{1}{2}$. Recall that, for each fixed t , the space-time white noise $\partial^2 B / \partial t \partial x$ lies in $\bigcap_{s < -1/2} H^s \setminus H^{-\frac{1}{2}}$ almost surely. Hence, these results for KdV cannot be applied to study the local well-posedness of (1).

Now, let us discuss the spaces which capture the regularities of the spatial and space-time white noise. Recently, we proved the invariance of the (spatial) white noise for the (deterministic) KdV in [Oh 2009a] (also see [Oh 2009b]) by first establishing the local well-posedness in an appropriate Banach space containing the support of the (spatial) white noise. Define the Besov-type space via the norm

$$\|f\|_{\hat{b}_{p,\infty}^s} := \|\hat{f}\|_{b_{p,\infty}^s} = \sup_j \|\langle n \rangle^s \hat{f}(n)\|_{L^p_{|n| \sim 2^j}} = \sup_j \left(\sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\hat{f}(n)|^p \right)^{1/p}. \tag{9}$$

In [Oh 2009a], using the theory of abstract Wiener spaces, we showed that $\hat{b}_{p,\infty}^s$ contains the full support of the (spatial) white noise for $sp < -1$. (The statement also holds true for $sp = -1$.)

Let's consider the stochastic convolution $\Phi(t)$ given by

$$\Phi(t) = \int_0^t S(t-t') \phi(t') dW(t'), \tag{10}$$

where ϕ is given by (3). Define a variant of the $X^{s,b}$ space adjusted to $\hat{b}_{p,\infty}^s(\mathbb{T})$. Let $X_{p,q}^{s,b}$ be the completion of the Schwartz class $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm

$$\|u\|_{X_{p,q}^{s,b}} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n, \tau)\|_{b_{p,\infty}^0 L_t^q}. \tag{11}$$

Note that $X_{p,q}^{s,b}$ defined in (11) is the space of functions u such that $S(-t)u(\cdot, t) \in (\hat{b}_{p,\infty}^s)_x (\mathcal{FL}^{b,q})_t$, where $\mathcal{FL}^{b,q}$ is defined via the norm

$$\|f\|_{\mathcal{FL}^{b,q}} := \|\langle \tau \rangle^b \hat{f}(\tau)\|_{L^q}. \tag{12}$$

In the same paper we also showed that the local-in-time white noise is supported on $\mathcal{FL}^{c,q}$ for $cq < -1$. This implies that the Brownian motion belongs locally in time to $\mathcal{FL}^{b,q}$ for $(b-1)q < -1$. Hence, with

$b < \frac{1}{2}$ and $q = 2$, we see that the local-in-time stochastic convolution $\eta(t)\Phi(t)$ lies in $X_{p,q}^{s,b}$ almost surely, with $sp < -1$, $b < \frac{1}{2}$ and $q = 2$, where $\eta(t)$ is a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$.

The argument in [De Bouard et al. 2004] is based on the result by Roynette [1993] on the endpoint regularity of the Brownian motion, which states that the Brownian motion $\beta(t)$ belongs to the Besov space $B_{p,q}^{1/2}$ if and only if $q = \infty$ (with $1 \leq p < \infty$). The authors then proved a variant of the bilinear estimate (7) by Kenig, Ponce and Vega, adjusted to their Besov space setting, establishing the local well-posedness via the fixed point theorem. The use of a variant of the bilinear estimate (7) required a slight regularization of the noise in space via ϕ so that the smoothed noise has the spatial regularity $s > -\frac{1}{2}$. Thus, they could not treat the space-time white noise, that is, $\phi = \text{Id}$.

Our result is based on two observations. First, our l_n^p -based function spaces $\hat{b}_{p,\infty}^s$ in (9) and $X_{p,q}^{s,b}$ in (11) capture the regularity of the spatial and space-time white noise for $sp < -1$, $b < \frac{1}{2}$ and $q = 2$. The second is that we can indeed carry out the argument in [Bourgain 1997], a nonlinear estimate on the second iteration, *without* assuming the a priori bound (8), if we take the initial data $u_0 \in \hat{b}_{p,\infty}^s$ for $s > -\frac{1}{2}$ with $p > 2$. Then, we construct a solution u as a strong limit of the smooth solutions u^N (with smooth u_0^N and ϕ^N) of (2). Note that our nonlinear estimate on the second iteration in Section 5 depends on the stochastic term, whereas the bilinear estimate in [De Bouard et al. 2004] is entirely deterministic.

Finally, we present our main results.

Theorem 1. *Let ϕ be as in (3) and $p = 2+$. Let $s = -\frac{1}{2} + \delta$ with $(p - 2)/(4p) < \delta < (p - 2)/(2p)$, so $sp < -1$. Also, let u_0 be \mathcal{F}_0 -measurable such that it has mean 0 and belongs to $\hat{b}_{p,\infty}^s(\mathbb{T})$ almost surely. Then, there exists a stopping time $T_\omega > 0$ and a unique process $u \in C([0, T_\omega]; \hat{b}_{p,\infty}^s(\mathbb{T}))$ satisfying (2) on $[0, T_\omega]$ almost surely.*

As a corollary, we obtain:

Theorem 2. *The stochastic KdV (1) with the additive space-time white noise is locally well posed almost surely (with the prescribed mean on u_0).*

Remark 1.1. Our argument provides an answer to the question posed in [Bourgain 1997, remark on p. 120], at least in the local-in-time setting. The deterministic part of the nonlinear estimate in Section 5 can be used to establish the local well-posedness of (5) for a finite Borel measure $u_0 = \mu \in M(\mathbb{T})$ with $\|\mu\| < \infty$ *without* the complete integrability or the smallness assumption on μ . Note that $\mu \in \hat{b}_{p,\infty}^s$ for $sp \leq -1$ since $\sup_n |\hat{\mu}(n)| < \|\mu\| < \infty$. Hence, it can be used to study the Cauchy problem on $M(\mathbb{T})$ for nonintegrable KdV-variants. Also, see [Oh 2009b].

Remark 1.2. Let $\mathcal{FL}^{s,p}(\mathbb{T})$ be the space of functions on \mathbb{T} defined via the norm $\|f\|_{\mathcal{FL}^{s,p}} = \|\langle n \rangle^s \hat{f}(n)\|_{L_n^p}$. Recall from [Oh 2009a] that $\mathcal{FL}^{s,p}(\mathbb{T})$ contains the support of the (spatial) white noise when $sp < -1$. Then, Theorems 1 and 2 can also be established in $\mathcal{FL}^{s,p}(\mathbb{T})$ for $s = -\frac{1}{2}+$, $p = 2+$ with $sp < -1$. The modification is straightforward once we note that $\|f\|_{\mathcal{FL}^{s-\varepsilon,p}} \lesssim \|f\|_{\hat{b}_{p,\infty}^s}$ for any $\varepsilon > 0$, and thus we omit the details.

This paper is organized as follows: In Section 2, we introduce some notations. In Section 3, we introduce function spaces along with their embeddings and state deterministic linear estimates from [Bourgain 1993] and [Oh 2009a]. In Section 4, we study some basic properties of the stochastic convolution. In Section 5, we prove Theorem 1 by establishing the nonlinear estimate on the second iteration of the integral formulation (4).

2. Notation

In the periodic setting on \mathbb{T} , the spatial Fourier domain is \mathbb{Z} . Let dn be the normalized counting measure on \mathbb{Z} . We say that $f \in L^p(\mathbb{Z})$, where $1 \leq p < \infty$, if

$$\|f\|_{L^p(\mathbb{Z})} = \left(\int_{\mathbb{Z}} |f(n)|^p dn \right)^{1/p} := \left(\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |f(n)|^p \right)^{1/p} < \infty.$$

If $p = \infty$, we have the obvious definition involving the supremum. We often drop 2π for simplicity. If a function depends on both x and t , we use $\hat{\cdot}^x$ (and $\hat{\cdot}^t$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $\hat{\cdot}$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context.

For a Banach space $X \subset \mathcal{S}'(\mathbb{T} \times \mathbb{R})$, we use \hat{X} to denote the space of the Fourier transforms of the functions in X , which is a Banach space with the norm $\|f\|_{\hat{X}} = \|\mathcal{F}_{n,\tau}^{-1} f\|_X$, where $\mathcal{F}_{n,\tau}^{-1}$ denotes the inverse Fourier transform (in n and τ). Also, for a space Y of functions on \mathbb{Z} , we use \hat{Y} to denote the space of the inverse Fourier transforms of the functions in Y with the norm $\|f\|_{\hat{Y}} = \|\mathcal{F} f\|_Y$. Now, define $\hat{b}_{p,q}^s(\mathbb{T})$ by the norm

$$\|f\|_{\hat{b}_{p,q}^s(\mathbb{T})} = \|\hat{f}\|_{b_{p,q}^s(\mathbb{Z})} := \left\| \|\langle n \rangle^s \hat{f}(n)\|_{L^p_{|n| \sim 2^j}} \right\|_{t^q} = \left(\sum_{j=0}^{\infty} \left(\sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\hat{f}(n)|^p \right)^{q/p} \right)^{1/q} \tag{13}$$

for $q < \infty$ and by (9) when $q = \infty$.

Throughout the paper, $\eta(t)$ denotes a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$, and let $\eta_\tau(t) = \eta(T^{-1}t)$. We use c, C to denote various constants, usually depending only on s, p , and δ . If a constant depends on other quantities, we make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \leq CA$. We also use $a+$ and $a-$ to denote $a + \varepsilon$ and $a - \varepsilon$, respectively, for arbitrarily small $\varepsilon \ll 1$.

3. Function spaces and basic embeddings

Let $X^{s,b}$ denote the usual periodic Bourgain space defined in (6). We often use the shorthand notation $\|\cdot\|_{s,b}$ to denote the $X^{s,b}$ norm. Now, define $X_{p,q}^{s,b}$, the Bourgain space adapted to $\hat{b}_{p,\infty}^s$, to be the completion of the Schwartz functions on $\mathbb{T} \times \mathbb{R}$ with respect to the norm given by

$$\|u\|_{X_{p,q}^{s,b}} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n, \tau)\|_{b_{p,\infty}^0 L_\tau^q} = \sup_j \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n, \tau)\|_{L^p_{|n| \sim 2^j} L_\tau^q}. \tag{14}$$

In the following, we take $p = 2+$ and $s = -\frac{1}{2}+ = -\frac{1}{2} + \delta$ with $\delta < (p-2)/2p$ (and $\delta > (p-2)/4p$) such that $sp < -1$. Lastly, given $T > 0$, we define $X_{p,q}^{s,b,T}$ as a restriction of $X_{p,q}^{s,b}$ on $[0, T]$ by

$$\|u\|_{X_{p,q}^{s,b,T}} = \|u\|_{X_{p,q}^{s,b}[0,T]} = \inf\{\|\tilde{u}\|_{X_{p,q}^{s,b}} : \tilde{u}|_{[0,T]} = u\}.$$

We define the local-in-time versions of the other function spaces analogously.

Now, we discuss the basic embeddings. For $p \geq 2$, we have $\|a_n\|_{L_n^p} \leq \|a_n\|_{L_n^2}$. Thus, we have $\|f\|_{\hat{b}_{p,\infty}^s} \leq \|f\|_{H^s}$, and thus

$$\|u\|_{X_{p,2}^{s,b}} \leq \|u\|_{X^{s,b}}. \tag{15}$$

By the Hölder inequality, we have

$$\begin{aligned} \|f\|_{H^{-\frac{1}{2}-\delta}} &= \left(\sum_j (2^j)^{0-} \|\langle n \rangle^{-\frac{1}{2}-\delta+} \hat{f}(n)\|_{|n|\sim 2^j}^2 \right)^{1/2} \\ &\leq \sup_j \|\langle n \rangle^{-2\delta+}\|_{L^{2p/p-2}} \|\langle n \rangle^{-\frac{1}{2}+\delta} \hat{f}(n)\|_{L_n^p} \leq \|f\|_{\hat{b}_{p,\infty}^s} \end{aligned} \tag{16}$$

for $s = -\frac{1}{2} + \delta$ with $\delta > (p-2)/4p$. Hence, for $s = -\frac{1}{2} + \delta$ with $\delta > (p-2)/4p$, we have

$$\|u\|_{X^{-\frac{1}{2}-\delta,b}} \lesssim \|u\|_{X_{p,2}^{s,b}}. \tag{17}$$

Now, we briefly go over the linear estimates. Let $S(t) = e^{-t\partial_x^3}$ and $T \leq 1$ in the following. We first present the homogeneous and nonhomogeneous linear estimates. See [Bourgain 1993; Kenig et al. 1993; Oh 2009a] for details of the proofs.

Lemma 3.1. *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$, we have $\|S(t)u_0\|_{X_{p,2}^{s,b,T}} \lesssim T^{(1/2)-b} \|u_0\|_{\hat{b}_{p,\infty}^s}$.*

Lemma 3.2. *For any $s \in \mathbb{R}$ and $b \leq \frac{1}{2}$, we have*

$$\left\| \int_0^t S(t-t')F(x,t')dt' \right\|_{X_{p,2}^{s,b,T}} \lesssim \|F\|_{X_{p,2}^{s,b-1}} + \|F\|_{X_{p,1}^{s,-1}}.$$

Also, we have $\left\| \int_0^t S(t-t')F(x,t')dt' \right\|_{X_{p,2}^{s,b,T}} \lesssim \|F\|_{X_{p,2}^{s,b-1}}$ for $b > \frac{1}{2}$.

The next lemma is the periodic L^4 Strichartz estimate due to Bourgain [1993].

Lemma 3.3. *Let u be a function on $\mathbb{T} \times \mathbb{R}$. Then, we have $\|u\|_{L_{x,t}^4} \lesssim \|u\|_{X_{x,t}^{0, \frac{1}{3}}}$.*

Lastly, recall that by restricting the Bourgain spaces onto a small time interval $[0, T]$, we can gain a small power of T . See [Colliander and Oh 2009] for the proof.

Lemma 3.4. *For $0 \leq b' < b \leq \frac{1}{2}$, we have*

$$\|u\|_{X^{s,b',T}} = \|\eta_T u\|_{X^{s,b',T}} \lesssim T^{b-b'} \|u\|_{X^{s,b}}.$$

4. Stochastic convolution

In this section, we study basic properties of the stochastic convolution $\Phi(t)$ defined in (10). In particular, we prove that $\eta\Phi$ belongs to $X_{p,2}^{s,b,T}$ and is continuous from $[0, T]$ into $\hat{b}_{p,\infty}^s$ for $T \leq 1$ almost surely for $sp < -1$ and $(b-1) \cdot 2 < -1$, where $\eta(t)$ is a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$.

Before stating the main results, we point out the following. Let ϕ be the identity operator on $L^2(\mathbb{T})$ or be as in (3). Then, we know that such ϕ is Hilbert–Schmidt from $L^2(\mathbb{T})$ into $H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. In other words, with a slight abuse of notation, define

$$\phi := \sum_{n \in \mathbb{Z}} \phi e_n = \sum_{n \in \mathbb{Z}} \phi_n e_n \tag{18}$$

in view of $\phi = \text{diag}(\phi_n; n \neq 0)$. Then, we have $\phi \in H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. Moreover, we have $\|\phi\|_{HS(L^2; H^s)} = \|\phi\|_{H^s}$, where $\|\cdot\|_{HS(L^2; H^s)}$ denotes the Hilbert–Schmidt norm from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$. For such ϕ , we also have $\phi \in \hat{b}_{p,\infty}^s(\mathbb{T})$ if and only if $sp \leq -1$, and we can use $\|\phi\|_{\hat{b}_{p,\infty}^s}$ to discuss the regularity of ϕ in place of the Hilbert–Schmidt norm. This is one of the reasons for using this space. (We need only $sp < -1$ for our purpose since the nonlinear estimate in Section 5 holds for $s = -\frac{1}{2}$ and $p = 2+$ with $sp < -1$.)

Proposition 4.1. *Let $0 < T \leq 1$ and $p = 2+$. Let $s = -\frac{1}{2} + \delta$ and $b = \frac{1}{2} - \delta$, with $(p-2)/4p < \delta < (p-2)/2p$ such that $sp < -1$ and $(b-1) \cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in (10) with ϕ as in (3), we have*

$$\mathbb{E}[\|\eta\Phi\|_{X_{p,2}^{s,b,T}}] \leq C(\eta, s, p) < \infty. \tag{19}$$

In particular, $\Phi \in X_{p,2}^{-\frac{1}{2}+\delta, \frac{1}{2}-\delta, T}$ almost surely.

Before going into the proof of Proposition 4.1, recall the following. Let β_1 and β_2 be independent real-valued Brownian motions on (Ω, \mathcal{F}, P) , and $f_1(t, \omega)$ and $f_2(t, \omega)$ be real-valued stochastic processes independent of β_1 and β_2 . Then, we can regard β_j and f_j as $\beta_j(t, \omega) = \beta_j(t, \omega_1)$ and $f_j(t, \omega) = f_j(t, \omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Thus, in taking an expectation, we can first integrate over $\omega_1 \in \Omega_1$. Then, for $m \in \mathbb{N}$, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \int_a^b f_1(t) d\beta_1(t) + \int_a^b f_2(t) d\beta_2(t) \right|^{2m} \right] \\ &= \mathbb{E} \left[\sum_{k=0}^{2m} \binom{2m}{k} \left(\int_a^b f_1(t) d\beta_1(t) \right)^k \left(\int_a^b f_2(t) d\beta_2(t) \right)^{2m-k} \right] \\ &= \mathbb{E}_{\Omega_2} \left[\sum_{n=0}^m \binom{2m}{2n} \frac{(2n)!}{2^n n!} \|f_1(\cdot, \omega_2)\|_{L^2(a,b)}^{2n} \frac{(2(m-n))!}{2^{m-n} (m-n)!} \|f_2(\cdot, \omega_2)\|_{L^2(a,b)}^{2(m-n)} \right]. \tag{20} \end{aligned}$$

In the computation above, we used the fact that, for each fixed ω_2 , $\int_a^b f_j(t, \omega_2) d\beta_j(t, \omega_1)$ is a Gaussian random variable on Ω_1 with variance $\|f_j(\cdot, \omega_2)\|_{L^2(a,b)}^2$.

Proof. By the Hölder inequality, we have

$$\|\langle \tau - n^3 \rangle^{(1/2)-\delta} \hat{u}(n, \tau)\|_{L^2_\tau} \leq \|\langle \tau - n^3 \rangle^{-2\delta}\|_{L^2_\tau} \| \langle \tau - n^3 \rangle^{(1/2)+\delta} \hat{u}(n, \tau) \|_{L^p_\tau},$$

that is, we have $\|\eta\Phi\|_{X_{p,2}^{s,1/2-\delta}} \lesssim \|\eta\Phi\|_{X_{p,p}^{s,1/2+\delta}}$ as long as $\delta > (p-2)/4p$. Thus, we will work in $X_{p,p}^{s,1/2+\delta}$ in the following.

Define $g(t) = \eta(t) \int_0^t S(-r)\phi(r)dW(r)$ such that $\eta(t)\Phi(\cdot, t) = S(t)g(\cdot, t)$. Assume that each β_n is extended to a Brownian motion on \mathbb{R} in such a way that the family $\{\beta_n\}_{n \geq 0}$ is still independent. Note that for $t \in [0, T]$, we have

$$\hat{g}(n, t) = \eta(t) \int_0^t \eta(r) e^{-irn^3} \phi_n(r) \chi_{[0,T]}(r) \frac{1}{\sqrt{2}} d\beta_n(r). \tag{21}$$

We have inserted $\eta(r)$ and $\chi_{[0,T]}(r)$ in the integrand since $\eta(r)\chi_{[0,T]}(r) \equiv 1$ for $r \in [0, t] \subset [0, T]$. For notational simplicity, we use $\phi_n(r)$ to denote $\phi_n(r)\chi_{[0,T]}(r)$ in the following, that is, we assume that ϕ_n is supported on $[0, T]$. By (3), we have $|\phi_n(r)| \leq 1$ for $r \in \mathbb{R}$.

Now, we write the left-hand side of (19) as

$$\begin{aligned} \mathbb{E}\left[\|\eta\Phi\|_{X_{p,p}^{s,\frac{1}{2}+\delta,T}}\right] &\lesssim \mathbb{E}\left[\sup_j 2^{js} \left(\sum_{|n|\sim 2^j} \sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} |\hat{g}(n,\tau)|^p d\tau\right)^{1/p}\right] \\ &\quad + \mathbb{E}\left[\sup_j 2^{js} \left(\sum_{|n|\sim 2^j} \int_{|\tau|\leq 2} |\hat{g}(n,\tau)|^p d\tau\right)^{1/p}\right]. \end{aligned} \tag{22}$$

Part 1. First, we estimate the second term in (22). Let

$$G_n(r,\tau) = \eta(r)e^{-irn^3} \phi_n(r) \int_r^\infty \eta(t)e^{-it\tau} dt. \tag{23}$$

Also write $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$ where $\beta_n^{(r)} = \text{Re } \beta_n$ and $\beta_n^{(i)} = \text{Im } \beta_n$. Then, by the stochastic Fubini Theorem, we have, for $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}[|\hat{g}(n,\tau)|^{2m}] &= \mathbb{E}\left[\left|\int_{\mathbb{R}} \eta(t)e^{-it\tau} \int_{-\infty}^t \eta(r)e^{-irn^3} \phi_n(r) \frac{1}{\sqrt{2}} d\beta_n(r) dt\right|^{2m}\right] \\ &= 2^{-m} \mathbb{E}\left[\left|\int_{-1}^2 G_n(r,\tau) d\beta_n(r)\right|^{2m}\right] \\ &\lesssim \mathbb{E}\left[\left|\int_{-1}^2 \text{Re } G_n(r,\tau) d\beta_n^{(r)}(r) - \int_{-1}^2 \text{Im } G_n(r,\tau) d\beta_n^{(i)}(r)\right|^{2m}\right] \\ &\quad + \mathbb{E}\left[\left|\int_{-1}^2 \text{Im } G_n(r,\tau) d\beta_n^{(r)}(r) + \int_{-1}^2 \text{Re } G_n(r,\tau) d\beta_n^{(i)}(r)\right|^{2m}\right]. \end{aligned} \tag{24}$$

Note that $|\text{Re } G_n(r,\tau)|, |\text{Im } G_n(r,\tau)| \leq |G_n(r,\tau)| \leq \|\eta\|_{L^1} |\phi_n(r)| \lesssim \|\eta\|_{L^1} \chi_{[0,T]}(r)$. Thus, we have

$$\|\text{Re } G_n(r,\tau)\|_{L_r^2}^{2k} \|\text{Im } G_n(r,\tau)\|_{L_r^2}^{2(m-k)} \lesssim \|\eta\|_{L^1}^{2m}$$

for $k = 0, \dots, m$. Then, by (20) along with the independence of $\phi_n, \beta_n^{(r)}$, and $\beta_n^{(i)}$, we have

$$\|\hat{g}(n,\tau)\|_{L^{2m}(\Omega)} \leq C = C(\eta, m)$$

independent of n and τ . Hence, for $p \in (2, 4)$, we have

$$(\mathbb{E}[|\hat{g}(n,\tau)|^p])^{1/p} \leq \|\hat{g}(n,\tau)\|_{L^2(\Omega)}^\theta \|\hat{g}(n,\tau)\|_{L^4(\Omega)}^{1-\theta} \lesssim 1 \tag{25}$$

by interpolation, where $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{4}$. Then, the second term in (22) is estimated by

$$\begin{aligned} (22) &\leq \left(\sum_{j=0}^\infty 2^{jsp} \sum_{|n|\sim 2^j} \int_{|\tau|\leq 2} \mathbb{E}[|\hat{g}(n,\tau)|^p] d\tau\right)^{1/p} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{jsp} \sum_{|n|\sim 2^j} 1\right)^{1/p} \sim \left(\sum_{j=0}^\infty 2^{(sp+1)j}\right)^{1/p} \leq C < \infty, \end{aligned} \tag{26}$$

since $sp < -1$.

Part 2. Next, we estimate the first term in (22). Let

$$\begin{cases} G_n^{(1)}(r, \tau) = \eta(r)e^{-irn^3} \phi_n(r) \int_r^\infty \eta'(t)(e^{-it\tau}/i\tau)dt, \\ G_n^{(2)}(r, \tau) = \eta^2(r)e^{-irn^3} \phi_n(r)(e^{-ir\tau}/i\tau). \end{cases} \tag{27}$$

Then, by the stochastic Fubini theorem and integration by parts, we have

$$\begin{aligned} \sqrt{2}\hat{g}(n, \tau) &= \int_{-1}^2 G_n(r, \tau)d\beta_n(r) = \int_{-1}^2 G_n^{(1)}(r, \tau)d\beta_n(r) + \int_{-1}^2 G_n^{(2)}(r, \tau)d\beta_n(r) \\ &=: I_n^{(1)}(\tau) + I_n^{(2)}(\tau). \end{aligned} \tag{28}$$

Thus $|\hat{g}(n, \tau)|^p \lesssim |I_n^{(1)}(\tau)|^p + |I_n^{(2)}(\tau)|^p$.

First, we estimate the contribution from $G_n^{(1)}$. For $|\tau| \sim 2^k$, we have

$$\left| \int_r^\infty \eta'(t) \frac{e^{-it\tau}}{i\tau} dt \right| \leq |\tau^{-2}\eta'(r)| + \left| \int_r^\infty \eta''(t) \frac{e^{-it\tau}}{\tau^2} dt \right| \leq C\eta 2^{-2k} \tag{29}$$

by partial integration. Thus, we have $|G_n^{(1)}(r, \tau)| \lesssim 2^{-2k}$. Then, repeating a similar computation as in Part 1, we obtain

$$\left(\mathbb{E}[|I_n^{(1)}(\tau)|^p]\right)^{1/p} \leq \|I_n^{(1)}(\tau)\|_{L^2(\Omega)}^\theta \|I_n^{(1)}(\tau)\|_{L^4(\Omega)}^{1-\theta} \lesssim 2^{-2k}, \tag{30}$$

by (20) and interpolation. Hence, the contribution to (22) is estimated by

$$\begin{aligned} (22) &\leq \left(\sum_{j=0}^\infty 2^{jsp} \sum_{|n|\sim 2^j} \sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} \mathbb{E}[|I_n^{(1)}(\tau)|^p]d\tau \right)^{1/p} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{j(sp+1)} \sum_{k=1}^\infty 2^{k(-3p/2+\delta p+1)} \right)^{1/p} \leq C < \infty, \end{aligned} \tag{31}$$

since $sp < -1$ and $-\frac{3}{2}p + \delta p + 1 < 0$.

Now, we consider the contribution from $I_n^{(2)}(\tau)$. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have $|I_n^{(2)}(\tau)|^2 \lesssim \left| \int_{-1}^2 G_n^{(2)}(r, \tau)d\beta_n^{(r)}(r) \right|^2 + \left| \int_{-1}^2 G_n^{(2)}(r, \tau)d\beta_n^{(i)}(r) \right|^2$. We only estimate the first term since the second term is estimated in the same way. By the Ito formula (see [De Bouard et al. 2004]), we have

$$\begin{aligned} \left| \int_{-1}^2 G_n^{(2)}(r, \tau)d\beta_n^{(r)}(r) \right|^2 &= \int_{-1}^2 \eta^4(t) \frac{|\phi_n(t)|^2}{\tau^2} dt + 2 \operatorname{Re} \int_{-1}^2 \int_{-\infty}^t G_n^{(2)}(r, \tau)d\beta_n^{(r)}(r) \overline{G_n^{(2)}(t, \tau)d\beta_n^{(r)}(t)} \\ &=: I_n'(\tau) + I_n''(\tau). \end{aligned}$$

The contribution from $I_n'(\tau)$ is at most

$$\begin{aligned} (22) &\lesssim \left(\sum_{j=0}^\infty 2^{jsp} \sum_{|n|\sim 2^j} \sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} |\tau|^{-p} d\tau \left(\int_{-1}^2 \eta^4(t)dt \right)^{p/2} \right)^{1/p} \\ &\lesssim \|\eta\|_{L^4}^2 \left(\sum_{j=0}^\infty 2^{j(sp+1)} \sum_{k=1}^\infty 2^{k(-\frac{p}{2}+\delta p+1)} \right)^{1/p} \leq C < \infty, \end{aligned} \tag{32}$$

since $sp < -1$ and $\delta < (p-2)/2p$.

We finally estimate the contribution from $I_n''(\tau)$. Write

$$I_n''(\tau) = \int_{-1}^2 H_n(t) d\beta_n^{(r)}(t),$$

where $H_n(t) = \int_{-\infty}^t \tilde{H}_n(r, t) d\beta_n^{(r)}(r)$ with

$$\tilde{H}_n(r, t) = 2\tau^{-2} \operatorname{Re}(\eta^2(r)\eta^2(t)e^{i(t-r)n^3} \phi_n(r)\overline{\phi_n(t)}e^{i(t-r)\tau}). \tag{33}$$

Then, by the Ito isometry and $|\phi_n(\omega, t)| \leq 1$ for all $(\omega, t) \in \Omega \times \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}[|I_n''(\tau)|^2] &= \mathbb{E}\left[\left(\int_{-1}^2 H_n(t) d\beta_n^{(r)}(t)\right)^2\right] \sim \int_{-1}^2 \mathbb{E}[H_n^2(t)] dt \\ &= \int_{-1}^2 \mathbb{E}\left[\left(\int_{-\infty}^t \tilde{H}_n(r, t) d\beta_n^{(r)}(r)\right)^2\right] dt = \int_{-1}^2 \int_{-1}^t \mathbb{E}[|\tilde{H}_n(r, t)|^2] dr dt \\ &\lesssim \tau^{-4} \int_{-1}^2 \int_{-1}^t \eta^4(r)\eta^4(t) dr dt \lesssim \tau^{-4}. \end{aligned} \tag{34}$$

Hence, the contribution from $I_n''(\tau)$ is at most

$$\begin{aligned} (22) &\lesssim \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n|\sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} \mathbb{E}[|I_n''(\tau)|^{p/2}] d\tau\right)^{1/p} \\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n|\sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} (\mathbb{E}[|I_n''(\tau)|^2])^{p/4} d\tau\right)^{1/p} \\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{j(sp+1)} \sum_{k=1}^{\infty} 2^{k(-\frac{p}{2}+\delta p+1)}\right)^{1/p} \leq C < \infty, \end{aligned} \tag{35}$$

for $p \leq 4$, $sp < -1$, and $\delta < (p-2)/2p$. □

We state a corollary to the proof of [Proposition 4.1](#) for a general diagonal covariance operator $\phi(t, \omega) = \operatorname{diag}(\phi_n(t, \omega); n \in \mathbb{Z})$, which is independent of $\{\beta_n\}_{n \geq 1}$.

Corollary 4.2. *Let $0 < T \leq 1$, $p = 2+$, and $s, s' \in \mathbb{R}$ with $s < s'$. Moreover, let $b = \frac{1}{2} - \delta$ with $(p-2)/4p < \delta < (p-2)/2p$, so $(b-1) \cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in [\(10\)](#) with $\phi \in L^p([0, T] \times \Omega; \hat{b}_{p,\infty}^{s',\infty})$, independent of $\{\beta_n\}_{n \geq 1}$, we have*

$$\mathbb{E}[\|\eta\Phi\|_{X_{p,2}^{s,b,T}}] \leq C(\eta, s, s', p) \|\phi\|_{L^p([0,T] \times \Omega; \hat{b}_{p,\infty}^{s',\infty})}. \tag{36}$$

In particular, $\Phi \in X_{p,2}^{s, \frac{1}{2}-\delta, T}$ almost surely.

Proof. In the proof of [Proposition 4.1](#), we used $|\phi_n(t)| \leq 1$ whenever $\phi_n(t)$ appeared. Now, we briefly go through that proof, keeping track of $\phi_n(t)$. Since ϕ is independent of $\{\beta_n\}_{n \geq 1}$, we regard β_n and ϕ_n as $\beta_n(t, \omega) = \beta_n(t, \omega_1)$ and $\phi_n(t, \omega) = \phi_n(t, \omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$.

In (25), we have $\mathbb{E}[|\hat{g}(n, \tau)|^p] \lesssim \mathbb{E}_{\Omega_2} \|\phi_n(\cdot, \omega_2)\|_{L^2[0, T]}^p$. Then, in (26), we have

$$(22) \leq \left(\sum_{j=0}^{\infty} 2^{j s p} \sum_{|n| \sim 2^j} \int_{|\tau| \leq 2} \mathbb{E}_{\Omega_2} \|\phi_n(\cdot, \omega_2)\|_{L^2[0, T]}^p d\tau \right)^{1/p} \\ \leq \left(\sum_{j=0}^{\infty} 2^{j(s-s')p} 2^{j s' p} \sum_{|n| \sim 2^j} \|\phi_n(\cdot, \omega_2)\|_{L^p([0, T] \times \Omega_2)}^p \right)^{1/p} \lesssim \|\phi\|_{L^p([0, T] \times \Omega; \hat{b}_{p, \infty}^{s'})},$$

since $s - s' < 0$. A similar modification in (30) and (31) (alternatively, (32)) takes care of the contribution from $I_n^{(1)}(\tau)$ (alternatively, $I_n'(\tau)$). Now, as for $I_n''(\tau)$, we first integrate only over Ω_1 in (34) and obtain

$$\mathbb{E}_{\Omega_1} [|I_n''(\tau)|^2] \lesssim \tau^{-4} \int_{-1}^2 \int_{-1}^t \eta^4(r) \eta^4(t) |\phi_n(r)|^2 |\phi_n(t)|^2 dr dt \lesssim \tau^{-4} \|\phi_n\|_{L^2[0, T]}^4.$$

Then, in (35), we have

$$\mathbb{E}[|I_n''(\tau)|^{p/2}] = \mathbb{E}_{\Omega_2} [\|I_n''(\tau)\|_{L^{p/2}(\Omega_1)}^{p/2}] \leq \mathbb{E}_{\Omega_2} [\|I_n''(\tau)\|_{L^2(\Omega_1)}^{p/2}] \lesssim \tau^{-p} \mathbb{E}_{\Omega_2} \|\phi_n(\cdot, \omega_2)\|_{L^2[0, T]}^p$$

for $p \in [2, 4]$. The rest follows as before. □

Now, we discuss the continuity of the stochastic convolution. In the remaining of this section, we show that the stochastic convolution $\Phi(t)$ defined in (10) belongs to $C([0, T]; \hat{b}_{p, \infty}^s(\mathbb{T}))$ almost surely. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have

$$\Phi(t) = \frac{1}{\sqrt{2}} \sum_{n \neq 0} \int_0^t S(t-r) \phi_n(r) e_n d\beta_n^{(r)}(r) + i \frac{1}{\sqrt{2}} \sum_{n \neq 0} \int_0^t S(t-r) \phi_n(r) e_n d\beta_n^{(i)}(r), \quad (37)$$

since $\phi e_0 = 0$ and $\phi e_n = \phi_n e_n$, $n \neq 0$. In the following, we only show the continuity of the first stochastic convolution in (37), which we denote by $\Phi^{(r)}(t)$. Also, let $W^{(r)}(t) = \frac{1}{\sqrt{2}} \sum_n \beta_n^{(r)}(t) e_n$. As in [Da Prato 2004], we use the factorization method based on the elementary identity

$$\int_r^t (t-t')^{\alpha-1} (t'-r)^{-\alpha} dt' = \frac{\pi}{\sin \pi \alpha} \quad (38)$$

with $\alpha \in (0, 1)$ for $0 \leq r \leq t' \leq t$. Using (38), we can write the first term in (37) as

$$\Phi^{(r)}(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t S(t-t')(t-t')^{\alpha-1} Y(t') dt', \quad (39)$$

where

$$Y(t') = \int_0^{t'} S(t'-r)(t'-r)^{-\alpha} \phi(r) dW^{(r)}(r). \quad (40)$$

First, we present a lemma that provides a criterion for the continuity of (39) in terms of the L^{2m} -integrability of $Y(t')$.

Lemma 4.3 [Da Prato 2004, Lemma 2.7]. *Let $T > 0$, $\alpha \in (0, 1)$, and $m > \frac{1}{2\alpha}$. For $f \in L^{2m}([0, T]; \hat{b}_{p, \infty}^s(\mathbb{T}))$, let*

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t') dt', \quad 0 \leq t \leq T.$$

Then, $F \in C([0, T]; \hat{b}_{p,\infty}^s(\mathbb{T}))$. Moreover, there exists $C = C(m, T)$ such that

$$\|F(t)\|_{\hat{b}_{p,\infty}^s} \leq C \|f\|_{L^{2m}([0, T]; \hat{b}_{p,\infty}^s)}, \quad 0 \leq t \leq T.$$

Remark 4.4. Although Da Prato states his Lemma 2.7 for a Hilbert space H , his proof makes no use of the Hilbert space structure of H . Thus the same result holds for $\hat{b}_{p,\infty}^s(\mathbb{T})$ as well.

In view of Lemma 4.3, it suffices to show that $Y(t') \in L^{2m}([0, T]; \hat{b}_{p,\infty}^s(\mathbb{T}))$ a.s.

Proposition 4.5. Let $T > 0$, $m \geq 2$, $s = -\frac{1}{2}+$, and $p = 2+$ such that $sp < -1$. Let ϕ be as in (3). Then, the stochastic convolution $\Phi^{(r)}(t)$ is continuous from $[0, T]$ into $\hat{b}_{p,\infty}^s$ almost surely. Moreover, there exists

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|\Phi^{(r)}(t)\|_{\hat{b}_{p,\infty}^s}^{2m} \right) \leq C(m, T, s, p) < \infty.$$

Proof. Let $\alpha \in (\frac{1}{2m}, \frac{1}{2})$ and Y as in (40). First, note that Y is real-valued since $\phi_{-n}(s)e_{-n} = \overline{\phi_n(s)e_n}$ and $\beta_{-n}^{(r)} = \beta_n^{(r)}$. Note that $\{\beta_n^{(r)}\}_{n \neq 0}$ and ϕ are independent since ϕ depends only on β_0 . Thus, we can regard $\beta_n^{(r)}$ and ϕ as $\beta_n^{(r)}(\omega) = \beta_n^{(r)}(\omega_1)$ and $\phi(\omega) = \phi(\omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Then, for each fixed ω_2 and $t' \in [0, t]$, $\widehat{Y(t')}(n)$ is a Gaussian random variable on Ω_1 with $\text{Var}_{\Omega_1}(\widehat{Y(t')}(n)) = \mathbb{E}_{\Omega_1} [|\widehat{Y(t')}(n)|^2]$.

Let $G_n(r, \omega_2) = (t' - r)^{-\alpha} e^{i(t'-r)n^3} \phi_n(r, \omega_2)$. Note that $|G_n(r, \omega_2)| = (t' - r)^{-\alpha}$ for $0 < r < t'$ and $n \neq 0$. By the Ito isometry, we have

$$\mathbb{E}_{\Omega_1} [|\widehat{Y(t')}(n)|^2] = \frac{1}{2} \mathbb{E}_{\Omega_1} \left[\left| \int_0^{t'} G_n(r, \omega_2) d\beta(r, \omega_1) \right|^2 \right] = \frac{1}{2} \int_0^{t'} |G_n(r, \omega_2)|^2 dr \sim \int_0^{t'} (t' - r)^{-2\alpha} dr.$$

By the Minkowski integral inequality (with $p = 2+ < 2m$) after replacing \sup_j by \sum_j , we have

$$\begin{aligned} \mathbb{E}_{\Omega_1} (\|Y(t', \cdot, \omega_2)\|_{\hat{b}_{p,\infty}^s}^{2m}) &= \mathbb{E}_{\Omega_1} \left[\left(\sup_j \sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\widehat{Y(t')}(n)|^p \right)^{2m/p} \right] \\ &\lesssim \left(\sum_{j=0}^{\infty} \sum_{|n| \sim 2^j} 2^{jsp} \left(\mathbb{E}_{\Omega_1} [|\widehat{Y(t')}(n)|^{2m}] \right)^{p/2m} \right)^{2m/p} \\ &\sim \left(\sum_{j=0}^{\infty} 2^{j(sp+1)} \right)^{2m/p} \left(\int_0^{t'} (t' - r)^{-2\alpha} dr \right)^m \lesssim \left(\frac{(t')^{1-2\alpha}}{1-2\alpha} \right)^m, \end{aligned}$$

since $sp < -1$. Therefore

$$\begin{aligned} \int_0^T \mathbb{E} (\|Y(t')\|_{\hat{b}_{p,\infty}^s}^{2m}) dt' &= \int_0^T \mathbb{E}_{\Omega_2} \mathbb{E}_{\Omega_1} (\|Y(t')\|_{\hat{b}_{p,\infty}^s}^{2m}) dt' \\ &\lesssim \int_0^T \left(\frac{(t')^{1-2\alpha}}{1-2\alpha} \right)^m dt' \lesssim T^{(1-2\alpha)m+1} < C(m, T, s, p) < \infty. \end{aligned}$$

In particular, it follows that $Y(\cdot, \omega) \in L^{2m}([0, T]; \hat{b}_{p,\infty}^s)$ almost surely. Then, the desired result follows from Lemma 4.3. □

5. Nonlinear estimate on the second iteration

Now, we present the crucial nonlinear analysis. First, we briefly go over Bourgain’s argument [1997]. By writing the integral equation, the deterministic KdV (5) is equivalent to

$$u(t) = S(t)u_0 - \frac{1}{2}\mathcal{N}(u, u)(t), \tag{41}$$

where $\mathcal{N}(\cdot, \cdot)$ is given by

$$\mathcal{N}(u_1, u_2)(t) := \int_0^t S(t-t')\partial_x(u_1u_2)(t')dt'. \tag{42}$$

In the following, we assume that the initial condition u_0 has mean 0, which implies that $u(t)$ has spatial mean 0 for each $t \in \mathbb{R}$. We use (n, τ) , (n_1, τ_1) , and (n_2, τ_2) to denote the Fourier variables for uu , the first factor, and the second factor u of uu in $\mathcal{N}(u, u)$, respectively, thus we have $n = n_1 + n_2$ and $\tau = \tau_1 + \tau_2$. By the mean-zero assumption on u and since we have $\partial_x(uu)$ in the definition of $\mathcal{N}(u, u)$, we assume $n, n_1, n_2 \neq 0$. We also use the following notation:

$$\sigma_0 := \langle \tau - n^3 \rangle \text{ and } \sigma_j := \langle \tau_j - n_j^3 \rangle.$$

One of the main ingredients is the observation due to Bourgain [1993]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2 \quad \text{for } n = n_1 + n_2, \tag{43}$$

which in turn implies that

$$\text{MAX} := \max(\sigma_0, \sigma_1, \sigma_2) \gtrsim \langle nn_1n_2 \rangle. \tag{44}$$

Now, define

$$A_j = \{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}^3 \times \mathbb{R}^3 : \sigma_j = \text{MAX}\}, \tag{45}$$

and let $\mathcal{N}_j(u, u)$ denote the contribution of $\mathcal{N}(u, u)$ on A_j . By the standard bilinear estimate as in [Bourgain 1993; Kenig et al. 1996], we have

$$\|\mathcal{N}_0(u, u)\|_{-\frac{1}{2}+\delta, \frac{1}{2}-\delta} \leq o(1)\|u\|_{-\frac{1}{2}-\delta, \frac{1}{2}-\delta}^2, \tag{46}$$

where $o(1) = T^\theta$ with some $\theta > 0$ by considering the estimate on a short time interval $[-T, T]$ (for example, Lemma 3.4). See (2.17), (2.26), and (2.68) in [Bourgain 1997]. Here, we abuse the notation and use $\|\cdot\|_{s,b} = \|\cdot\|_{X^{s,b}}$ to denote the local-in-time version as well. Note that the temporal regularity b equals $\frac{1}{2} - \delta < \frac{1}{2}$. This allows us to improve the spatial regularity by 2δ . Clearly, we cannot expect to do the same for $\mathcal{N}_1(u, u)$. (By symmetry, we do not consider $\mathcal{N}_2(u, u)$ in the following.) The bilinear estimate (7) is known to fail for any $s \in \mathbb{R}$ if $b < \frac{1}{2}$ due to the contribution from $\mathcal{N}_1(u, u)$ [Kenig et al. 1996]. Following the notation in [Bourgain 1997], let

$$I_{s,b} = \|\mathcal{N}_1(u, u)\|_{X^{s,b}} \quad \text{and} \quad \alpha := \frac{1}{2} - \delta < \frac{1}{2}. \tag{47}$$

Then, by Lemma 3.2 and duality with $\|d(n, \tau)\|_{L_{n,\tau}^2} \leq 1$, we have

$$I_{-\alpha, 1-\alpha} = \|\mathcal{N}_1(u, u)\|_{-\alpha, 1-\alpha} \lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \hat{u}(n_1, \tau_1) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha}, \tag{48}$$

where

$$c(n_2, \tau_2) = \langle n_2 \rangle^{-(1-\alpha)} \sigma_2^\alpha \hat{u}(n_2, \tau_2) \quad \text{so that } \|c\|_{L_{n,\tau}^2} = \|u\|_{-(1-\alpha),\alpha} = \|u\|_{-\frac{1}{2}-\delta, \frac{1}{2}-\delta}. \quad (49)$$

The main idea here is to consider the second iteration, that is, substitute (41) for $\hat{u}(n_1, \tau_1)$ in (48), thus leading to a trilinear expression. Since $\sigma_1 = \text{MAX} \gtrsim \langle nn_1n_2 \rangle \gg 1$ on A_1 , we can assume that

$$\hat{u}(n_1, \tau_1) = (\mathcal{N}(u, u))^\wedge(n_1, \tau_1) \sim \frac{|n_1|}{\sigma_1} \sum_{\substack{n_1=n_3+n_4 \\ \tau_1=\tau_3+\tau_4}} \int \hat{u}(n_3, \tau_3) \hat{u}(n_4, \tau_4) d\tau_4. \quad (50)$$

Note that $\hat{u}(n_1, \tau_1)$ cannot come from $S(t)u_0$ of (41) since we have $\sigma_1 \sim 1$ for the linear part. Moreover, by the standard computation [Bourgain 1993], we have

$$\begin{aligned} \mathcal{N}(u, u)(x, t) &= -i \sum_{k=1}^\infty \frac{i^k k^k}{k!} \sum_{n \neq 0} e^{i(nx+n^3t)} \int \eta(\lambda - n^3) \widehat{\partial_x u^2}(n, \lambda) d\lambda \\ &\quad + i \sum_{n \neq 0} e^{inx} \int \frac{(1-\eta)(\tau - n^3)}{\tau - n^3} \widehat{\partial_x u^2}(n, \tau) e^{i\tau t} d\tau \\ &\quad + i \sum_{n \neq 0} e^{i(nx+n^3t)} \int \frac{(1-\eta)(\lambda - n^3)}{\lambda - n^3} \widehat{\partial_x u^2}(n, \lambda) d\lambda \\ &=: \mathcal{M}_1(u, u)(x, t) + \mathcal{M}_2(u, u)(x, t) + \mathcal{M}_3(u, u)(x, t). \end{aligned} \quad (51)$$

Note that $(\mathcal{M}_1(u, u))^\wedge(n_1, \tau_1)$ and $(\mathcal{M}_3(u, u))^\wedge(n_1, \tau_1)$ are distributions supported on $\{\tau_1 - n_1^3 = 0\}$, so $\sigma_1 \sim 1$. Hence, the only contribution for the second iteration on A_1 comes from $\mathcal{M}_2(u, u)$ whose Fourier transform is given in (50). This shows the validity of the assumption (50).

The σ_1 appearing in the denominator allows us to cancel $\langle n \rangle^{1-\alpha}$ and $\langle n_2 \rangle^{1-\alpha}$ in the numerator in (48). Then, $I_{-\alpha, 1-\alpha}$ can be estimated by

$$\lesssim \sum_{\substack{n=n_1+n_2 \\ n_1=n_3+n_4}} \int_{\substack{\tau=\tau_1+\tau_2 \\ \tau_1=\tau_3+\tau_4}} \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{|n_1|}{\sigma_1} \hat{u}(n_3, \tau_3) \hat{u}(n_4, \tau_4) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha}. \quad (52)$$

Bourgain then divided the argument into several cases, depending on the sizes of $\sigma_0, \dots, \sigma_4$. Here, the key algebraic relation is

$$n^3 - n_2^3 - n_3^3 - n_4^3 = 3(n_2 + n_3)(n_3 + n_4)(n_4 + n_2) \quad \text{with } n = n_2 + n_3 + n_4. \quad (53)$$

Then, Bourgain proved [1997, (2.69)]

$$I_{-\alpha, 1-\alpha} \leq o(1) \|u\|_{-(1-\alpha), \alpha} I_{-\alpha, 1-\alpha} + o(1) \|u\|_{-(1-\alpha), \alpha}^3 + o(1) \|u\|_{-(1-\alpha), \alpha}, \quad (54)$$

assuming the a priori estimate (8): $|\hat{u}(n, t)| < C$ for all $n \in \mathbb{Z}, t \in \mathbb{R}$. Indeed, the estimates involving the first two terms on the right-hand side of (54) were obtained without (8), and *only* the last term in (54) required (8) [Bourgain 1997, “Estimation of (2.62)”], which was then used to deduce

$$\|\hat{u}(n, \cdot)\|_{L_t^2} < C. \quad (55)$$

The a priori estimate (8) is derived via the isospectral property of the KdV flow and is false for a general function in $X^{-(1-\alpha),\alpha}$. (It is here that the smallness of the total variation $\|\mu\|$ is used.)

Our goal is to carry out a similar analysis for SKdV (2) on the second iteration *without* the a priori estimates (8) and (55) coming from the complete integrability of KdV. We achieve this goal by considering the estimate in

$$X_{p,2}^{-\alpha,\alpha} = X_{p,2}^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta},$$

where $p = 2+$ and $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. By (15) and (17) (recall $-\alpha = -\frac{1}{2} + \delta$ and $-(1-\alpha) = -\frac{1}{2} - \delta$), we have

$$\|u\|_{X_{p,2}^{-\alpha,\alpha}} \leq \|u\|_{X^{-\alpha,\alpha}} \quad \text{and} \quad \|u\|_{X^{-(1-\alpha),\alpha}} \lesssim \|u\|_{X_{p,2}^{-\alpha,\alpha}}. \tag{56}$$

Then, it follows from (46) and (56) that

$$\|\mathcal{N}_0(u, u)\|_{X_{p,2}^{-\alpha,\alpha}} \leq o(1)\|u\|_{X_{p,2}^{-\alpha,\alpha}}^2. \tag{57}$$

Now, we consider the estimate on $\|\mathcal{N}_1(u, u)\|_{X_{p,2}^{-\alpha,\alpha}}$. From (56) and $\alpha < 1 - \alpha$, it suffices to control $I_{-\alpha,1-\alpha}$. As in the deterministic case, we consider the second iteration, and substitute (4) for $\hat{u}(n_1, \tau_1)$ in (48). As before, there is no contribution from $S(t)u_0$, or $\mathcal{M}_1(u, u)$, $\mathcal{M}_3(u, u)$ defined in (51). There are two contributions:

- (i) $\mathcal{N}_1(\mathcal{M}_2(u, u), u)$ from the deterministic nonlinear part: In this case, we can use the estimates from [Bourgain 1997] *except* when the a priori bound (8) was assumed; that is, we need to estimate the contribution from [Bourgain 1997, (2.62)]:

$$R_\alpha := \sum_n \int_{\tau=\tau_2+\tau_3+\tau_4} \chi_B \frac{d(n, \tau)}{\langle n \rangle^{1+\alpha} \sigma_0^\alpha} \hat{u}(-n, \tau_2) \hat{u}(n, \tau_3) \hat{u}(n, \tau_4) d\tau_2 d\tau_3 d\tau_4, \tag{58}$$

where $\|d(n, \tau)\|_{L_{n,\tau}^2} \leq 1$ and $B = \{\sigma_0, \sigma_2, \sigma_3, \sigma_4 < |n|^\gamma\}$ with some small parameter $\gamma > 0$. Note that this corresponds to the case $n_2 = -n$ and $n_3 = n_4 = n$ in (52) after some reduction. In our analysis, we directly estimate R_α in terms of $\|u\|_{X_{p,2}^{-\alpha,\alpha}}$. The key observation is that we can take the spatial regularity $s = -\alpha$ to be greater than $-\frac{1}{2}$ by choosing $p > 2$.

- (ii) $\mathcal{N}_1(\Phi, u)$ from the stochastic convolution Φ in (10): In view of (56), we estimate

$$\mathbb{E}[\|\mathcal{N}_1(\eta\Phi, u)\|_{X^{-\alpha,1-\alpha}}] \tag{59}$$

via the stochastic analysis from Section 4.

Remark 5.1. In fact, we do not need to take an expectation in (59) since we establish local well-posedness pathwise in ω , that is, for almost every *fixed* ω . Nonetheless, we estimate (59) with the expectation since it shows how F_1^N and F_2^N defined in (70) arise along with their estimates.

Estimate on (i). In [Bourgain 1997], the parameter $\gamma = \gamma(\alpha)$, subject to the conditions (2.43) and (2.60) therein, played a certain role in estimating R_α along with the a priori bound (8). However, it plays no role in our analysis. By the Cauchy-Schwarz and Young’s inequalities, we have

$$(58) \leq \sum_n \|d(n, \cdot)\|_{L_\tau^2} \langle n \rangle^{-1-\alpha} \|\hat{u}(-n, \tau_2)\|_{L_{\tau_2}^{6/5}} \|\hat{u}(n, \tau_3)\|_{L_{\tau_3}^{6/5}} \|\hat{u}(n, \tau_4)\|_{L_{\tau_4}^{6/5}}.$$

By the Hölder inequality (with appropriate \pm signs) and the fact that $-1 - \alpha < -3\alpha$,

$$(58) \leq \sum_n \|d(n, \cdot)\|_{L^2_{\tau}} \prod_{j=2}^4 \langle n \rangle^{-\alpha} \|\sigma_j^{-\alpha}\|_{L^3_{\tau_j}} \|\sigma_j^{\alpha} \hat{u}(\pm n, \tau_j)\|_{L^2_{\tau_j}} \leq \|d(\cdot, \cdot)\|_{L^2_{n,\tau}} \|u\|_{X_{6,2}^{-\alpha,\alpha}}^3 \leq \|u\|_{X_{p,2}^{-\alpha,\alpha}}^3,$$

where the last two inequalities follow by choosing $\alpha > \frac{1}{3}$ and $p = 2+ < 6$.

Estimate on (ii). We use the notation from the proof of Proposition 4.1. It follows from (28) and $\eta(t)\Phi(\cdot, t) = S(t)g(\cdot, t)$ that

$$(\eta\Phi)^\wedge(n_1, \tau_1) = \hat{g}(n_1, \tau_1 - n_1^3) = \frac{1}{\sqrt{2}} I_{n_1}^{(1)}(\tau_1 - n_1^3) + \frac{1}{\sqrt{2}} I_{n_1}^{(2)}(\tau_1 - n_1^3).$$

Recall that $\sigma_1 = \langle \tau_1 - n_1^3 \rangle \gtrsim \langle nn_1n_2 \rangle$. Also, recall from the proof of Proposition 4.1 that $|\phi_{n_1}(r)| = \chi_{[0,T]}(r)$ is independent of ω .

- Contribution from $I_{n_1}^{(1)}(\tau_1 - n_1^3)$: From (48) with (27), (28), and (29), we estimate (59) by

$$(59) \lesssim \mathbb{E} \left[\sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1^2} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha} \right]. \quad (60)$$

By the Cauchy–Schwarz inequality in ω and the Ito isometry,

$$(59) \lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n, \tau)}{\sigma_0^\alpha} \frac{\|\phi_{n_1}\|_{L^2[0,T]} \|c(n_2, \tau_2)\|_{L^2(\Omega)}}{\sigma_1^{(3/2)-\delta} \langle n_1 \rangle^{(1/2)+\delta} \sigma_2^\alpha}. \quad (61)$$

By the $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$ -Hölder inequality along with Lemma 3.3, (16), (18), (49), and (56), this leads to

$$(59) \lesssim T^\theta \|d\|_{L^2_{n,\tau}} \|\phi\|_{L^2([0,T]; H^{-\frac{1}{2}-\delta})} \|c\|_{L^2(\Omega; L^2_{n,\tau})} \leq T^\theta \|\phi\|_{L^p([0,T]; \hat{b}_{p,\infty}^{-\alpha})} \|u\|_{L^2(\Omega; X^{-(1-\alpha),\alpha})} \\ \lesssim T^\theta \|\phi\|_{L^p([0,T]; \hat{b}_{p,\infty}^{-\alpha})} \|u\|_{L^2(\Omega; X_{p,2}^{-\alpha,\alpha})}.$$

Remark 5.2. Strictly speaking, we need to take the supremum over $\{\|d\|_{L^2_{n,\tau}} = 1\}$ inside the expectation in (60). However, we do not worry about this issue to simplify the presentation, since we have

$$(59) \leq \|\mathcal{N}_1(\eta\Phi, u)\|_{L^2(\Omega; X^{-\alpha,1-\alpha})} \\ \leq \left(\sum_n \int \frac{\langle n \rangle^{2-2\alpha}}{\sigma_0^{2\alpha}} \mathbb{E} \left[\left| \int_0^T |\phi_{n_1}(r)| \sum_{\substack{n=n_1+n_2 \\ \tau=\tau_1+\tau_2}} \int \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_1^2 \sigma_2^\alpha} d\tau_1 d\beta_{n_1}(r) \right|^2 d\tau \right] \right)^{1/2} \\ = \sup_{\|d\|_{L^2_{n,\tau}}=1} (61)$$

by the Ito isometry. Also, recall that we have $I_{n_1}^{(1)}(\tau_1 - n_1^3) = \int_0^T G_{n_1}^{(1)}(r, \tau_1 - n_1^3) d\beta_{n_1}(r)$ where $G_n^{(1)}(r, \tau)$ is defined in (27). Hence, strictly speaking, we should replace $G_{n_1}^{(1)}(r, \tau_1 - n_1^3)$ by $\sigma_1^{-2} |\phi_{n_1}(r)|$ in (60) only after the application of the Ito isometry. Once again, we do not worry about this issue to simplify the presentation. The same remark applies to the following as well.

- Contribution from $I_{n_1}^{(2)}(\tau_1 - n_1^3)$: Suppose $\max(\sigma_0, \sigma_2) \gtrsim \langle nn_1n_2 \rangle^{1/100}$; say $\sigma_0 \geq \langle nn_1n_2 \rangle^{1/100}$. Then

$$\begin{aligned}
 (59) &\lesssim \mathbb{E} \left[\sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha} \right] \\
 &\lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n, \tau)}{\sigma_0^{\alpha-200\delta}} \frac{\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{(1/2)+\delta} \langle n_1 \rangle^{(1/2)+\delta}} \frac{\|c(n_2, \tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha}. \tag{62}
 \end{aligned}$$

Then we can conclude this case as before by the $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$ -Hölder inequality as long as $\alpha - 200\delta > \frac{1}{3}$, which can be guaranteed by taking $\delta > 0$ sufficiently small, or equivalently, taking $p > 2$ sufficiently close to 2.

Now assume instead $\max(\sigma_0, \sigma_2) \ll \langle nn_1n_2 \rangle^{1/100}$. We invoke a result contained in [Colliander et al. 2003, (7.50) and Lemma 7.4]. The conclusion there is stated with -1 as the exponent of $\langle \tau - n^3 \rangle$, instead of $-\frac{3}{4}$; but by examining the proof, one sees that it will work with any exponent more negative than $-(\frac{2}{3} + \frac{1}{100})$.

Lemma 5.3. For $\Omega(n) = \{\eta \in \mathbb{R} : \eta = -3nn_1n_2 + o(\langle nn_1n_2 \rangle^{1/100}) \text{ for some } n_1 \in \mathbb{Z} \text{ with } n = n_1 + n_2\}$,

$$\int \langle \tau - n^3 \rangle^{-3/4} \chi_{\Omega(n)}(\tau - n^3) d\tau \lesssim 1. \tag{63}$$

We have

$$(59) \lesssim \mathbb{E} \left[\sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3)}{\sigma_1} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha} \right].$$

By the Cauchy–Schwarz inequality and the Ito isometry, this yields

$$(59) \lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n, \tau)}{\sigma_0^\alpha} \frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3) \|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{(1/2)-\delta} \langle n_1 \rangle^{(1/2)+\delta}} \frac{\|c(n_2, \tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha}. \tag{64}$$

By the $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$ -Hölder inequality along with Lemma 3.3, Lemma 5.3, and Equations (16), (18), (49), and (56), we get

$$\begin{aligned}
 (59) &\lesssim T^\theta \|d\|_{L^2_{n,\tau}} \|\langle n_1 \rangle^{-\frac{1}{2}-\delta} \|\phi_{n_1}\|_{L^2[0,T]} \|\chi_{\Omega(n_1)}(\tau_1 - n_1^3) \sigma_1^{-\frac{1}{2}+\delta}\|_{L^2_t} \|c\|_{L^2(\Omega; L^2_{n,\tau})} \\
 &\leq T^\theta \|\phi\|_{L^2([0,T]; H^{-\frac{1}{2}-\delta})} \|u\|_{L^2(\Omega; X^{-(1-\alpha), \alpha})} \lesssim T^\theta \|\phi\|_{L^p([0,T]; \hat{b}_{p,\infty}^{-\alpha})} \|u\|_{L^2(\Omega; X_{p,2}^{-\alpha,\alpha})}.
 \end{aligned}$$

Proof of Theorem 1. Fix a mean-zero $u_0 \in \hat{b}_{p,\infty}^{-\alpha'}(\mathbb{T})$ and ϕ as in (3), where $\alpha' = \frac{1}{2} - \delta -$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$ such that $(-\alpha')p < -1$. Consider sequences of initial data $u_0^N \in L^2(\mathbb{T})$ and diagonal covariance operator $\phi^N \in HS(L^2; L^2)$, given by

$$u_0^N = \mathbb{P}_{\leq N} u_0 = \sum_{|n| \leq N} \hat{u}_0(n) e^{inx} \quad \text{and} \quad \phi^N(t, \omega) := \text{diag}(\phi_n(t, \omega); 0 < |n| \leq N), \tag{65}$$

where ϕ_n is given in (3). Now, fix $\alpha = \frac{1}{2} - \delta > \alpha'$ as in (47). Note that such u_0^N converges to u_0 in $\mathcal{F}L^{-\alpha,p}(\mathbb{T})$, and thus in $\hat{b}_{p,\infty}^{-\alpha}(\mathbb{T})$. Also, ϕ^N converges to ϕ in $\mathcal{F}L^{-\frac{1}{2}-\cdot,p}(\mathbb{T})$ for each t and ω , and thus

in $\widehat{b}_{p,\infty}^{-\frac{1}{2}-}(\mathbb{T})$. Then, by the monotone convergence theorem, ϕ^N converges to ϕ in $L^p([0, 1] \times \Omega; \widehat{b}_{p,\infty}^{-\frac{1}{2}-})$. (Indeed, the convergence is in $L^\infty([0, 1] \times \Omega; \widehat{b}_{p,\infty}^{-\frac{1}{2}-})$, since we have $|\phi_n(t, \omega)| = 1$ for all n , independent of $t \in \mathbb{R}$ and $\omega \in \Omega$.) Note that a slight loss of the regularity $-\alpha < -\alpha'$ was necessary since u_0^N defined in (65) does not necessarily converge to u_0 in $\widehat{b}_{p,\infty}^{-\alpha'}(\mathbb{T})$ due to the L^∞ nature of the norm over the dyadic blocks. We can avoid such a loss of the regularity if we start with $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$.

Now, let $\Gamma^N = \Gamma_{u_0^N}^N$ be the map defined by

$$\Gamma^N v = \Gamma_{u_0^N}^N v := S(t)u_0^N - \frac{1}{2}\mathcal{N}(v, v) + \eta\Phi^N, \tag{66}$$

where Φ^N is the stochastic convolution defined in (10) with the covariance operator ϕ^N . By the well-posedness result in [De Bouard et al. 2004], there exists a unique global solution $u^N \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T})) \cap C(\mathbb{R}^+; B_{2,1}^{0-}(\mathbb{T}))$ a.s. to (66) for each N since $\phi^N \in HS(L^2; L^2)$.

Now, we put all the estimates together. Note that all the implicit constants are independent of N . Also, when there is no superscript N , it means that $N = \infty$. From Lemma 3.1, we have

$$\|S(t)u_0^N\|_{X_{p,2}^{s,b,T}} \leq C_1 \|u_0^N\|_{\widehat{b}_{p,\infty}^s} \tag{67}$$

for any $s, b \in \mathbb{R}$ with $C_1 = C_1(b)$. In particular, by taking $b > \frac{1}{2}$, we see that $S(t)u_0$ is continuous on $[0, T]$ with values in $\widehat{b}_{p,\infty}^s$. Also, by taking $b < \frac{1}{2}$, we gain a power of T . From the definition of $\mathcal{N}_j(\cdot, \cdot)$ and (57), we have

$$\|\mathcal{N}(u^N, u^N)\|_{X_{p,2}^{-a,a,T}} \leq C_2 T^{\theta_1} \|u^N\|_{X_{p,2}^{-a,a,T}}^2 + 2\|\mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-a,a,T}}. \tag{68}$$

Also, from (47) and (56), we have

$$\|\mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-a,1-a,T}} \leq I_{-\alpha,1-\alpha}^N. \tag{69}$$

Recall that $\eta\Phi \in X_{p,2}^{-a,\alpha}$ a.s. from Proposition 4.1. Moreover, by defining F_1^N and F_2^N on $\mathbb{T} \times \mathbb{R} \times \Omega$ via their Fourier transforms

$$\begin{aligned} \widehat{F_1^N}(n, \tau) &= \langle n \rangle^{-\frac{1}{2}-\delta} (\sigma_0^{-\frac{3}{2}+\delta} + \sigma_0^{-\frac{1}{2}-\delta}) \int_0^T |\phi_n(r)| d\beta_n(r), \\ \widehat{F_2^N}(n, \tau) &= \langle n \rangle^{-\frac{1}{2}-\delta} \chi_{\Omega(n)}(\tau - n^3) \sigma_0^{-\frac{1}{2}+\delta} \int_0^T |\phi_n(r)| d\beta_n(r) \end{aligned} \tag{70}$$

for $|n| \leq N$, we have $F_1^N, F_2^N \in L^2(\Omega; L_{x,t}^2)$ by the Ito isometry and Lemma 5.3, which is basically shown in the estimate on (ii). See (61) and (64). Then, from (54) and the estimates on (i) and (ii), we have

$$I_{-\alpha,1-\alpha}^N \leq C_3 (T^{\theta_2} \|u^N\|_{X_{p,2}^{-a,a,T}} I_{-\alpha,1-\alpha}^N + T^{\theta_3} \|u^N\|_{X_{p,2}^{-a,a,T}}^3 + T^{\theta_4} L_\omega^N \|u^N\|_{X_{p,2}^{-a,a,T}}), \tag{71}$$

where $L_\omega^N = L^N(F_1^N, F_2^N)(\omega) := \|F_1^N(\omega)\|_{L_{x,t}^2} + \|F_2^N(\omega)\|_{L_{x,t}^2} < \infty$ a.s. Moreover, L_ω^N is nondecreasing in N .

For fixed $R > 0$, choose $T > 0$ small such that $C_3 T^{\theta_2} R \leq \frac{1}{2}$. Then, from (71), we have

$$I_{-a,1-a}^N \leq 2C_3 (T^{\theta_3} \|u^N\|_{X_{p,2}^{-a,a,T}}^3 + T^{\theta_4} L_\omega^N \|u^N\|_{X_{p,2}^{-a,a,T}}) \tag{72}$$

for $\|u^N\|_{X_{p,2}^{-a,a,T}} \leq R$. From (66)–(72), we have

$$\begin{aligned} \|u^N\|_{X_{p,2}^{-a,a,T}} &= \|\Gamma^N u^N\|_{X_{p,2}^{-a,a,T}} \leq C_1 \|u_0^N\|_{\hat{b}_{p,\infty}^{-a}} + \frac{1}{2} C_2 T^{\theta_1} \|u^N\|_{X_{p,2}^{-a,a,T}}^2 \\ &\quad + 2C_3 (T^{\theta_3} \|u^N\|_{X_{p,2}^{-a,a,T}}^3 + T^{\theta_4} L_\omega^N \|u^N\|_{X_{p,2}^{-a,a,T}}) + C_4 \|\eta \Phi^N(\omega)\|_{X_{p,2}^{-a,a}}, \end{aligned} \tag{73}$$

and

$$\begin{aligned} \|u^N - u^M\|_{X_{p,2}^{-a,a,T}} &= \|\Gamma^N u^N - \Gamma^M u^M\|_{X_{p,2}^{-a,a,T}} \\ &\leq C_1 \|u_0^N - u_0^M\|_{\hat{b}_{p,\infty}^{-a}} + \frac{1}{2} C_2 T^{\theta_1} (\|u^N\|_{X_{p,2}^{-a,a,T}} + \|u^M\|_{X_{p,2}^{-a,a,T}}) \|u^N - u^M\|_{X_{p,2}^{-a,a,T}} \\ &\quad + C_5 T^{\theta_3} (\|u^N\|_{X_{p,2}^{-a,a,T}}^2 + \|u^M\|_{X_{p,2}^{-a,a,T}}^2) \|u^N - u^M\|_{X_{p,2}^{-a,a,T}} \\ &\quad + 2C_3 T^{\theta_4} L_\omega^N \|u^N - u^M\|_{X_{p,2}^{-a,a,T}} + 2C_3 T^{\theta_4} \tilde{L}_\omega^{N,M} \|u^M\|_{X_{p,2}^{-a,a,T}} \\ &\quad + C_4 \|\eta(\Phi^N - \Phi^M)\|_{X_{p,2}^{-a,a}}, \end{aligned} \tag{74}$$

where

$$\tilde{L}_\omega^{N,M} := \|F_1^N - F_1^M\|_{L_{x,t}^2} + \|F_2^N - F_2^M\|_{L_{x,t}^2}. \tag{75}$$

Note that in estimating the difference $\Gamma^N u^N - \Gamma^M u^M$ on A_1 , one needs to consider

$$\tilde{I}_{-a,1-a} := \|\mathcal{N}_1(u^N, u^N) - \mathcal{N}_1(u^M, u^M)\|_{-a,1-a} \tag{76}$$

as in [Bourgain 1997]. We can follow the argument on pages 135–136 in that reference, except for R_α defined in (58), which yields the third term on the right-hand side of (74). As for R_α , we can write

$$\mathcal{N}(\mathcal{N}(u, u), u) - \mathcal{N}(\mathcal{N}(v, v), v) = \mathcal{N}(\mathcal{N}(u + v, u - v), u) + \mathcal{N}(\mathcal{N}(v, v), u - v) \tag{77}$$

as in [Bourgain 1997, (3.4)], and then we can repeat the computation done for R_α in the estimate on (i), also yielding the third term on the right-hand side of (74).

By the definition of u_0^N , we have $2C_1 \|u_0^N\|_{\hat{b}_{p,\infty}^{-a}} \leq 2C_1 \|u_0\|_{\hat{b}_{p,\infty}^{-a}} + \frac{1}{2}$ for N sufficiently large. And since ϕ^N converges to ϕ in $L^p([0, 1] \times \Omega; \hat{b}_{p,\infty}^{-a+})$, it follows from Corollary 4.2 and the estimate on (ii), see (61), (62), and (64), that $\mathbb{E}[\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-a,a}}]$ and $\mathbb{E}[\tilde{L}_\omega^{N,\infty}]$ defined in (75) converge to 0. Hence, $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-a,a}} + \tilde{L}_\omega^{N,\infty} \rightarrow 0$ a.s. after selecting a subsequence (which we still denote with the index N .) Then, by Egoroff’s theorem, given $\varepsilon > 0$, there exists a set Ω_ε with $\mathbb{P}(\Omega_\varepsilon^c) < 2^{-1}\varepsilon$ such that $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-a,a}} + \tilde{L}_\omega^{N,\infty} \rightarrow 0$ uniformly in Ω_ε . In particular, $2C_4 \|\eta \Phi^N\|_{X_{p,2}^{-a,a}} \leq 2C_4 \|\eta \Phi\|_{X_{p,2}^{-a,a}} + \frac{1}{2}$ for large N uniformly on Ω_ε . In the following, we will work on Ω_ε .

Now, let $R_\omega = 2(C_1 \|u_0\|_{\hat{b}_{p,\infty}^{-a}} + C_4 \|\eta \Phi(\omega)\|_{X_{p,2}^{-a,a}}) + 1$, and define the stopping time T_ω by

$$T_\omega = \inf\{T > 0 : \max(C_3 T^{\theta_2} R_\omega, P_1(T, R_\omega, \omega), P_2(T, R_\omega, \omega)) \geq \frac{1}{2}\}, \tag{78}$$

where

$$\begin{aligned} P_1(T, R_\omega, \omega) &= \frac{1}{2} C_2 T^{\theta_1} R_\omega + 2C_3 T^{\theta_3} (R_\omega)^2 + 2C_3 T^{\theta_4} L_\omega \quad \text{from (73),} \\ P_2(T, R_\omega, \omega) &= C_2 T^{\theta_1} R_\omega + 2C_5 T^{\theta_3} (R_\omega)^2 + 2C_3 T^{\theta_4} L_\omega \quad \text{from (74).} \end{aligned}$$

The first condition in the definition of T_ω guarantees (72), and hence (73) and (74), for $\|u^N\|_{X_{p,2}^{-a,a,T}} \leq R_\omega$. The second condition along with (73) indeed guarantees that

$$\|u^N\|_{X_{p,2}^{-a,a,T}} \leq R_\omega \tag{79}$$

for $T \leq T_\omega$, for the following reason. Because of the temporal regularity $b = \alpha < \frac{1}{2}$, we have $\|u^N\|_{X_{p,2}^{-a,a,T}} = \|\chi_{[0,T]} u^N\|_{X_{p,2}^{-a,a}}$, where $\chi_{[0,T]}$ denotes the characteristic function of the time interval $[0, T]$ [Bourgain 1999]. Hence, $\|u^N\|_{X_{p,2}^{-a,a,T}}$ is continuous in T since

$$\left| \|u^N\|_{X_{p,2}^{-a,a,T+\delta}} - \|u^N\|_{X_{p,2}^{-a,a,T}} \right| \leq \|u^N\|_{X_{p,2}^{-a,a}[T,T+\delta]} \lesssim \delta^\theta \|u^N\|_{X^{0-\frac{1}{2}}[T,T+\delta]} \tag{80}$$

for sufficiently small $\delta > 0$. Note that the last term in (80) is finite for small δ since the local-in-time solutions constructed in [De Bouard et al. 2004] are controlled in this norm (indeed in a stronger norm adapted to the Besov space $B_{2,1}^{0-}$.) Then, (79) follows from (73), the second condition in (78), and the continuity of the norm in T since (79) clearly holds at $T = 0$.

From (74) along with the third condition in (78), we have

$$\|u^N - u^M\|_{X_{p,2}^{-a,a,T_\omega}} \leq 2C_1 \|u_0^N - u_0^M\|_{\hat{b}_{p,\infty}^{-a}} + 4C_3 T^{\theta_4} R_\omega \tilde{L}_\omega^{N,M} + 2C_4 \|\eta(\Phi^N - \Phi^M)\|_{X_{p,2}^{-a,a}}. \tag{81}$$

The right-hand side of (81) goes to 0 as $N, M \rightarrow \infty$ since u_0^N is Cauchy in $\hat{b}_{p,\infty}^{-a}$ and

$$\|\eta(\Phi^N - \Phi^M)\|_{X_{p,2}^{-a,a}} + \tilde{L}_\omega^{N,M} \rightarrow 0$$

on Ω_ε uniformly in N, M . Let u denote the limit in $X_{p,2}^{-a,a,T_\omega}$.

In the following, we give a brief discussion to show that the limit u is a solution to (4). Clearly, $S(t)u_0^N$ and $\eta\Phi^N$ converge to $S(t)u_0$ and $\eta\Phi$ in $X_{p,2}^{-a,a,T_\omega}$. It follows from (57) that $\mathcal{N}_0(u^N, u^N)$ converges to $\mathcal{N}_0(u, u)$ in $X_{p,2}^{-a,a,T_\omega}$. In view of (72), (74), and (76), we see that $\mathcal{N}_j(u^N, u^N)$ is Cauchy in a slightly stronger space $X_{p,2}^{-a,1-a,T_\omega}$, $j = 1, 2$. Let v_j denote the corresponding limit. Thus, from (66), we have

$$u = S(t)u_0 - \frac{1}{2}\mathcal{N}_0(u, u) - \frac{1}{2}(v_1 + v_2) + \eta\Phi. \tag{82}$$

Now, we need to show that $\mathcal{N}_j(u^N, u^N)$ indeed converges to $\mathcal{N}_j(u, u)$, $j = 1, 2$. By symmetry, we only consider $\mathcal{N}_1(u, u) - \mathcal{N}_1(u^N, u^N)$. As before, we substitute (82) (and (66)) in the first factor u (and u^N) of $\mathcal{N}_1(\cdot, \cdot)$, respectively. There are three contributions to consider.

(A) Contribution from the stochastic terms: We have

$$\mathcal{N}_1(\eta\Phi, u) - \mathcal{N}_1(\eta\Phi^N, u^N) = \mathcal{N}_1(\eta(\Phi - \Phi^N), u) + \mathcal{N}_1(\eta\Phi^N, u - u^N). \tag{83}$$

From the estimate on (ii), we have

$$\|(83)\|_{X_{p,2}^{-a,a,T_\omega}} \lesssim \tilde{L}_\omega^{N,\infty} \|u\|_{X_{p,2}^{-a,a,T_\omega}} + L_\omega^N \|u^N - u\|_{X_{p,2}^{-a,a,T_\omega}} \rightarrow 0$$

as $N \rightarrow \infty$, since $\|u\|_{X_{p,2}^{-a,a,T}} \leq R_\omega$ and $\tilde{L}_\omega^{N,\infty} \rightarrow 0$ uniformly on Ω_ε .

(B) Contribution from $\mathcal{N}_0(\cdot, \cdot)$: In this case, we consider

$$\mathcal{N}_1(\mathcal{N}_0(u, u), u) - \mathcal{N}_1(\mathcal{N}_0(u^N, u^N), u^N). \tag{84}$$

Note that we have $\sigma_1 \geq \sigma_0, \sigma_2, \sigma_3, \sigma_4$ from the definition of $\mathcal{N}_1(\cdot, \cdot)$ and $\mathcal{N}_0(\cdot, \cdot)$, see (50) and (52). Indeed, we have $\sigma_1 \geq \sigma_0, \sigma_2$ since we are on A_1 defined in (45), and also $\sigma_1 \geq \sigma_3, \sigma_4$ since we are on the support of $\mathcal{N}_0(\cdot, \cdot)$ in the first factor of $\mathcal{N}_1(\cdot, \cdot)$. Once again, one can easily follow the argument in [Bourgain 1997, page 136] and show

$$\|(84)\|_{X_{p,2}^{-a,a,T_\omega}} \lesssim (\|u^N\|_{X_{p,2}^{-a,a,T_\omega}}^2 + \|u\|_{X_{p,2}^{-a,a,T_\omega}}^2) \|u^N - u\|_{X_{p,2}^{-a,a,T_\omega}} \rightarrow 0.$$

In treating $R_\alpha - R_\alpha^N$ defined in (58), one needs to proceed as before, using (77) and the estimate on (i).

(C) Contribution from v_j and $\mathcal{N}_j(u^N, u^N)$, $j = 1$ or 2 : By symmetry, assume $j = 1$. In this case, we have $\sigma_1 \geq \sigma_0, \sigma_2$ but $\sigma_3 \geq \sigma_1, \sigma_4$, thus we control (54) by the first term on the right-hand side [Bourgain 1997, (II.1) on page 126]. Now, we need to estimate

$$\mathcal{N}_1(v_1, u) - \mathcal{N}_1(\mathcal{N}_1(u^N, u^N), u^N) = \mathcal{N}_1(v_1 - \mathcal{N}_1(u^N, u^N), u) + \mathcal{N}_1(\mathcal{N}_1(u^N, u^N), u - u^N) =: \text{I} + \text{II}. \quad (85)$$

Then, by proceeding as in [Bourgain 1997] with (56) and (72), we have

$$\|\text{II}\|_{X_{p,2}^{-a,1-a,T_\omega}} \lesssim I_{-a,1-a}^N \|u - u^N\|_{X^{-(1-a),a,T_\omega}} \lesssim \|u - u^N\|_{X_{p,2}^{-a,a,T_\omega}} \rightarrow 0.$$

By proceeding as in [Bourgain 1997, (II.1)] with $|n_1|^\alpha$ replaced by $|n_1|^{1-\alpha}$, followed by (56), we have

$$\begin{aligned} \|\text{I}\|_{X_{p,2}^{-a,1-a,T_\omega}} &\lesssim \|v_1 - \mathcal{N}_1(u^N, u^N)\|_{-(1-a),1-a} \|u\|_{-(1-a),a} \\ &\lesssim \|v_1 - \mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-a,1-a,T_\omega}} \|u\|_{X_{p,2}^{-a,a,T_\omega}} \rightarrow 0, \end{aligned}$$

since $v_1 = \lim_{N \rightarrow \infty} \mathcal{N}_1(u^N, u^N)$ in $X_{p,2}^{-a,1-a,T_\omega}$ by definition.

Hence, we have $u = \Gamma_{u_0} u$ for each $\omega \in \Omega_\varepsilon$, so u is a mild solution to (2) on $[0, T_\omega]$. Let $\Omega^{(1)} = \Omega_\varepsilon$. Now, we can recursively construct

$$\Omega^{(j+1)} \subset \Omega \setminus \bigcup_{k=1}^j \Omega^{(k)}$$

for $j = 1, 2, \dots$ with $\mathbb{P}(\Omega \setminus \bigcup_{k=1}^j \Omega^{(k)}) < 2^{-j} \varepsilon$ such that $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-a,a}}$ and $\tilde{L}_\omega^{N,\infty}$ converge to 0 uniformly in each $\Omega^{(j)}$. Then, by repeating the argument, we can construct a solution u on $\bigcup_{j=1}^\infty \Omega^{(j)}$. Note that $\mathbb{P}(\Omega \setminus \bigcup_{j=1}^\infty \Omega^{(j)}) = 0$.

We have constructed a solution u to (2) in $X_{p,2}^{-a,a,T_\omega}$ with $u_0 \in \hat{b}_{p,\infty}^{-a'}$. Since u is a solution, the a priori estimate (73) holds with the regularity $(s, b) = (-a', a')$ in place of $(-a, a)$. Then, we easily see that $u \in X_{p,2}^{-a',a',T_\omega}$ by redefining R_ω and T_ω with this regularity. In the remaining of the paper, we work only with the spatial regularity $s = -a'$, that is, there is no approximating sequences any more. Hence, for notational simplicity, we will use $-a$ in place of $-a'$ to denote the spatial regularity of the solution in the following.

We still need to take care of several issues. Note that the temporal regularity $b = \alpha = \frac{1}{2} - \delta$ of the solution u is less than $\frac{1}{2}$. In particular, we need to show that the solution u is continuous from $[0, T_\omega]$ into $\hat{b}_{p,\infty}^{-a}$. We also need to show its uniqueness and continuous dependence on the initial data.

From Proposition 4.5, $\eta\Phi \in C([0, T_\omega]; \hat{b}_{p,\infty}^{-\alpha})$ a.s. Also, it follows from (67) with $b = \frac{1}{2} + \delta$, (69), (72), and symmetry on σ_1 and σ_2 , that

$$S(t)u_0 + \mathcal{N}_1(u, u) + \mathcal{N}_2(u, u) \in X_{p,2}^{-\alpha, \frac{1}{2} + \delta, T_\omega} \subset C([0, T_\omega]; \hat{b}_{p,\infty}^{-\alpha})$$

almost surely. Now, we consider $\mathcal{N}_0(u, u)$, that is, when $\sigma_0 = \text{MAX}$. Note that the contribution comes only from $\mathcal{M}_2(u, u)$ defined in (51). Define

$$\begin{aligned} \mathcal{N}_3(u, u) &= \text{the contribution of } \mathcal{N}_0(u, u) \text{ on } \{\max(\sigma_1, \sigma_2) \gtrsim \langle nn_1n_2 \rangle^{1/100}\}, \\ \mathcal{N}_4(u, u) &= \mathcal{N}_0(u, u) - \mathcal{N}_3(u, u). \end{aligned}$$

(a) First, we consider $\mathcal{N}_3(u, u)$, so $\max(\sigma_1, \sigma_2) \gtrsim \langle nn_1n_2 \rangle^{1/100}$; say $\sigma_1 \gtrsim \langle nn_1n_2 \rangle^{1/100}$. Then, by (15) and Lemma 3.2, we have

$$\|\mathcal{N}_3(u, u)\|_{X_{p,2}^{-\alpha, \frac{1}{2} + \delta, T_\omega}} \lesssim \|\partial_x(u^2)\|_{X_{p,2}^{-\alpha, -\frac{1}{2} + \delta, T_\omega}} \lesssim \|\partial_x(u^2)\|_{X^{-\alpha, -\frac{1}{2} + \delta, T_\omega}}.$$

By duality and (44), the right-hand side equals

$$\begin{aligned} \sup_{\|d\|_{L_{n,\tau}^2} = 1} \sum_{\substack{n, n_1 \\ n = n_1 + n_2}} \int_{\tau = \tau_1 + \tau_2} \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^{(1/2) - \delta}} \prod_{j=1}^2 \frac{\langle n_j \rangle^{1-\alpha} c(n_j, \tau_j)}{\sigma_j^\alpha} d\tau d\tau_1 \\ \lesssim \sup_{\|d\|_{L_{n,\tau}^2} = 1} \sum_{\substack{n, n_1 \\ n = n_1 + n_2}} \int_{\tau = \tau_1 + \tau_2} d(n, \tau) \frac{c(n_1, \tau_1)}{\sigma_1^{\alpha - 200\delta}} \frac{c(n_2, \tau_2)}{\sigma_2^\alpha} d\tau d\tau_1, \end{aligned}$$

where $c(n, \tau)$ is defined in (49). Thus, by the $L_{x,t}^2, L_{x,t}^4, L_{x,t}^4$ -Hölder inequality along with Lemma 3.3, (49), and (56), we conclude that

$$\|\mathcal{N}_3(u, u)\|_{X_{p,2}^{-\alpha, \frac{1}{2} + \delta, T_\omega}} \lesssim \|c\|_{L_{n,\tau}^2}^2 \leq \|u\|_{X_{p,2}^{-\alpha, -1, a}}^2 \lesssim \|u\|_{X_{p,2}^{-\alpha, a}}^2 < \infty.$$

(b) Now, consider $\mathcal{N}_4(u, u)$, so $\max(\sigma_1, \sigma_2) \ll \langle nn_1n_2 \rangle^{1/100}$. It suffices to show that $\mathcal{N}_0(u, u) \in X_{p,1}^{-\alpha, 0, T_\omega}$, since $X_{p,1}^{-\alpha, 0, T_\omega} \subset C([0, T_\omega]; \hat{b}_{p,\infty}^{-\alpha})$. Then, by Cauchy–Schwarz inequality, Lemma 5.3 and duality, we have

$$\begin{aligned} \|\mathcal{N}_4(u, u)\|_{X_{p,1}^{-\alpha, 0, T_\omega}} &\leq \|\partial_x(u^2)\|_{X_{2,1}^{-\alpha, -1, T_\omega}} \leq \left\| \|\langle n \rangle^{-\alpha} \langle \tau - n^3 \rangle^{-1} \chi_{\Omega(n)}(\tau - n^3) \widehat{\partial_x(u^2)}(n, \tau) \|_{L_\tau^1} \right\|_{L_n^2} \\ &\leq \left\| \langle \tau - n^3 \rangle^{-\frac{1}{2} + \delta} \chi_{\Omega(n)}(\tau - n^3) \right\|_{L_\tau^2} \|\partial_x(u^2)\|_{-\alpha, -\frac{1}{2} - \delta} \\ &\lesssim \sup_{\|d\|_{L_{n,\tau}^2} = 1} \sum_{\substack{n, n_1 \\ n = n_1 + n_2}} \int_{\tau = \tau_1 + \tau_2} \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^{(1/2) + \delta}} \prod_{j=1}^2 \frac{\langle n_j \rangle^{1-\alpha} c(n_j, \tau_j)}{\sigma_j^\alpha} d\tau d\tau_1 \\ &\lesssim \sup_{\|d\|_{L_{n,\tau}^2} = 1} \sum_{\substack{n, n_1 \\ n = n_1 + n_2}} \int_{\tau = \tau_1 + \tau_2} d(n, \tau) \frac{c(n_1, \tau_1)}{\sigma_1^\alpha} \frac{c(n_2, \tau_2)}{\sigma_2^\alpha} d\tau d\tau_1. \end{aligned}$$

The rest follows as before. Hence, the solution u is continuous from $[0, T_\omega]$ to $\hat{b}_{p,\infty}^{-\alpha}$.

Lastly, we show the uniqueness and the continuous dependence of the solutions on the initial data. Let u and v be the mild solutions of (2) on $[0, T_\omega]$ with initial data u_0 and v_0 ; then

$$u - v = \Gamma_{u_0}u - \Gamma_{v_0}v = S(t)(u_0 - v_0) - \frac{1}{2}(\mathcal{N}(u, u) - \mathcal{N}(v, v)), \tag{86}$$

where Γ is defined in (66). Moreover, assume that

$$\|u_0\|_{\hat{b}_{p,\infty}^{-a}} \leq R, \quad \|v_0\|_{\hat{b}_{p,\infty}^{-a}} \leq R, \quad \|u\|_{X_{p,2}^{-a,a,T_\omega}} \leq R, \quad \|v\|_{X_{p,2}^{-a,a,T_\omega}} \leq R. \tag{87}$$

Let $\tilde{\mathcal{N}}_j(u, v) := -\frac{1}{2}(\mathcal{N}_j(u, u) - \mathcal{N}_j(v, v))$ for $j = 1, \dots, 4$. First, note that $\|\tilde{\mathcal{N}}_4(u, v)\|_{X_{p,1}^{-a,\varepsilon,T_\omega}} \lesssim R^2 < \infty$ from (a slight variation of) Case (b), and we have

$$\|(u - v) - \tilde{\mathcal{N}}_4(u, v)\|_{X_{p,1}^{-a,\varepsilon,T_\omega}} \leq \left\| S(t)(u_0 - v_0) + \sum_{j=1}^3 \tilde{\mathcal{N}}_j(u, v) \right\|_{X_{p,2}^{-a,\frac{1}{2}+\delta,T_\omega}} \lesssim C_1(R) < \infty$$

by Cauchy–Schwarz inequality with $\varepsilon < \delta$, followed by (67), (69), (72), Case (a), and (87). Then, by interpolation and Cauchy–Schwarz inequality, we have

$$\|u - v\|_{C([0,T_\omega];\hat{b}_{p,\infty}^{-a})} \lesssim \|u - v\|_{X_{p,1}^{-a,0,T_\omega}} \lesssim \|u - v\|_{X_{p,1}^{-a,-\delta,T_\omega}}^\beta \|u - v\|_{X_{p,1}^{-a,\varepsilon,T_\omega}}^{1-\beta} \lesssim C_2(R) \|u - v\|_{X_{p,2}^{-a,\frac{1}{2}-\delta,T_\omega}}^\beta \tag{88}$$

with $\beta = \frac{\varepsilon}{\varepsilon + \delta} \in (0, 1)$. From (67) and the nonlinear estimates (see (68), (72), (74), (76)), we have

$$\|u - v\|_{X_{p,2}^{-a,\frac{1}{2}-\delta,T_\omega}} \lesssim \|u_0 - v_0\|_{\hat{b}_{p,\infty}^{-a}} + C_3(R) T_\omega^\theta \|u - v\|_{X_{p,2}^{-a,\frac{1}{2}-\delta,T_\omega}}.$$

Hence, for sufficiently small $T > 0$, we have

$$\|u - v\|_{X_{p,2}^{-a,\frac{1}{2}-\delta,T_\omega}} \lesssim \|u_0 - v_0\|_{\hat{b}_{p,\infty}^{-a}}. \tag{89}$$

Therefore, it follows from (88) and (89) that the solution map is Hölder continuous with the bound

$$\|u - v\|_{C([0,T_\omega];\hat{b}_{p,\infty}^{-a})} \leq C_4(R) \|u_0 - v_0\|_{\hat{b}_{p,\infty}^{-a}}^\beta.$$

In particular, the solution is unique. This completes the proof of Theorem 1. □

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