

ANALYSIS & PDE

Volume 3

No. 2

2010

MAGDALENA CZUBAK

**LOCAL WELLPOSEDNESS FOR
THE 2+1-DIMENSIONAL MONOPOLE EQUATION**



 mathematical sciences publishers

LOCAL WELLPOSEDNESS FOR THE 2+1-DIMENSIONAL MONOPOLE EQUATION

MAGDALENA CZUBAK

The space-time monopole equation on \mathbb{R}^{2+1} can be derived by a dimensional reduction of the antiselfdual Yang–Mills equations on \mathbb{R}^{2+2} . It can be also viewed as the hyperbolic analog of Bogomolny equations. We uncover null forms in the nonlinearities and employ optimal bilinear estimates in the framework of wave–Sobolev spaces. As a result, we show the equation is locally wellposed in the Coulomb gauge for initial data sufficiently small in H^s for $s > \frac{1}{4}$.

1. Introduction

In this paper we study local wellposedness of the Cauchy problem for the monopole equation on \mathbb{R}^{2+1} Minkowski space in the Coulomb gauge. The space-time monopole equation can be derived by a dimensional reduction from the antiselfdual Yang–Mills equations on \mathbb{R}^{2+2} , and is given by

$$F_A = *D_A\phi, \tag{ME}$$

where F_A is the curvature of a one-form connection A on \mathbb{R}^{2+1} , $D_A\phi$ is a covariant derivative of the Higgs field ϕ , and $*$ is the Hodge star operator with respect to the Minkowski \mathbb{R}^{2+1} metric. (ME) is a hyperbolic analog of Bogomolny equations, and was first introduced by Ward [1989] and discussed from the point of view of twistors. Ward [1999] also studied its soliton solutions. Recently, Dai, Terng and Uhlenbeck [2006] gave a broad survey on the space-time monopole equation. In particular, using the inverse scattering transform they have shown global existence and uniqueness up to a gauge transformation for small initial data in $W^{2,1}$. However, L^2 based wellposedness theory for this equation has not been investigated. The objective of this paper is to fill this gap by specifically treating the Cauchy problem for rough initial data in H^s .

Written in coordinates, (ME) is a system of first order hyperbolic partial differential equations. The unknowns are a pair (A, ϕ) . If (A, ϕ) solve the equation, then so do

$$\lambda A(\lambda t, \lambda x) \quad \text{and} \quad \lambda\phi(\lambda t, \lambda x), \quad \text{for any } \lambda > 0.$$

This results in the critical exponent $s_c = 0$. Since in general one expects local wellposedness for $s > s_c$ the goal would be to show (ME) is wellposed for $s > 0$. Nevertheless, the two dimensions create an obstacle, which so far only allows $s > \frac{1}{4}$. We explain this now. In Section 4 we choose a Coulomb gauge, and

MSC2000: 35L70, 70S15.

Keywords: monopole, null form, Coulomb gauge, wellposedness.

reformulate (ME) as a system of semilinear wave equations coupled with an elliptic equation, to which we refer as auxiliary monopole equations (aME). Schematically it looks as follows:

$$\begin{aligned}\square u &= \mathfrak{B}_+(\partial u, \partial v, A_0), \\ \square v &= \mathfrak{B}_-(\partial u, \partial v, A_0), \\ \Delta A_0 &= \mathcal{C}(\partial u, \partial v, A_0),\end{aligned}\tag{aME}$$

where $\mathfrak{B}_\pm, \mathcal{C}$ are bilinear forms,¹ A_0 is the temporal part of the connection A , $\partial u, \partial v$ denote space-time derivatives of u and v respectively, and are given in terms of ϕ and the spatial part of A . As a result, showing wellposedness of (ME) for $s > 0$ can follow from showing (aME) is wellposed for $s > 1$ (see Theorem 4.1). Also, the most difficult nonlinearity that we have to handle is contained in $\mathfrak{B}_\pm(\partial u, \partial v, A_0)$. Luckily, it exhibits a structure of a null form. There are two standard null forms:

$$Q_0(u, v) = -\partial_t u \partial_t v + \nabla u \cdot \nabla v, \quad Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v.$$

The null condition was introduced by Klainerman [1984], and it was first applied to produce better local wellposedness results for wave equations with a null form by Klainerman and Machedon [1993]. Indeed, in low dimensions, for these kind of nonlinearities one can assume much less regularity of the initial data than for the general products. Counterexamples for general products were shown by Lindblad [1996]. We uncover the null form $Q_{\alpha\beta}$ in our system of wave equations as well as a new type of a null form which is related to $Q_{\alpha\beta}$. Unfortunately, the results in two spatial dimensions for $Q_{\alpha\beta}$ are not as optimal as they are in higher dimensions or as they are for Q_0 . In fact, the best result in literature so far for $Q_{\alpha\beta}$ in two dimensions is due to Zhou [1997]. He establishes local wellposedness for initial data in $H^s \times H^{s-1}$ for $s > \frac{5}{4}$. In addition, by examining the first iterate Zhou shows that this is as close as one can get to the critical level using iteration methods.² On the other hand, for dimensions $n \geq 3$ Klainerman and Machedon [1996] showed almost optimal local wellposedness in $H^s \times H^{s-1}$ for $s > n/2$. The articles [Klainerman and Machedon 1995; Klainerman and Selberg 2002] give equally satisfying results for Q_0 , and in all dimensions $n \geq 2$.

Now, one of the nonlinearities in the system (aME) is $Q_{\alpha\beta}$, so showing (aME) is locally wellposed for $s > \frac{5}{4}$ would be sharp by iteration methods. This is what we do, and as a result we obtain local wellposedness of (ME) in the Coulomb gauge for $s > \frac{1}{4}$ (see the Main Theorem below). However, (aME) is not exactly (ME), so we hope to treat (ME) directly in the near future and improve the results. What should be mentioned here is that we have considered other traditional gauges such as Lorentz and Temporal, but they have not been as nearly useful as the Coulomb gauge. Perhaps other, less traditional gauges could be used. Moreover, we note that even the estimates involving the temporal variable A_0 seem to require $s > \frac{1}{4}$.

We include a brief discussion about global wellposedness. As already mentioned, in [Dai et al. 2006] the inverse scattering transform is used to show global existence and uniqueness up to a gauge transformation for small data in $W^{2,1}$. To extend it to global wellposedness in L^2 based theory, we would like to benefit from the local result in this article. It is not immediately clear how this can be accomplished

¹ See Section 4 for the precise formula for \mathfrak{B}_\pm and \mathcal{C} .

² The discussion of the first iterate can be also found in the appendix of [Klainerman and Selberg 2002], and it can be deduced from the estimates and counterexamples found within [Foschi and Klainerman 2000].

since, for example, the energy functional is not positive definite and, in fact, it vanishes for the solutions of (ME) [Ward 1989]. Global wellposedness is an interesting question, and we would like to investigate it in the future.

The main result of this paper is contained in the following theorem.

Main Theorem. *Let $\frac{1}{4} < s < \frac{1}{2}$ and $r \in (0, 2s]$ and consider the space-time monopole equation*

$$F_A = *D_A\phi, \quad (\text{ME})$$

with initial data

$$(A_1, A_2, \phi)|_{t=0} = (a_1, a_2, \phi_0),$$

then (ME) in the Coulomb gauge is locally wellposed for initial data sufficiently small in $H^s(\mathbb{R}^2)$ in the following sense.

- **Local existence:** *For all $a_1, a_2, \phi_0 \in H^s(\mathbb{R}^2)$ sufficiently small there exist $T > 0$ depending continuously on the norm of the initial data, and functions*

$$A_0 \in C_b([0, T], \dot{H}^r), \quad A_1, A_2, \phi \in C_b([0, T], H^s),$$

which solve (ME) in the Coulomb gauge on $[0, T] \times \mathbb{R}^2$ in the sense of distributions and such that the initial conditions are satisfied.

- **Uniqueness:** *If $T > 0$ and (A, ϕ) and (A', ϕ') are two solutions of (ME) in the Coulomb gauge on $(0, T) \times \mathbb{R}^2$ belonging to*

$$C_b([0, T], \dot{H}^r) \times (H_T^{s,\theta})^3$$

with the same initial data, then $(A, \phi) = (A', \phi')$ on $(0, T) \times \mathbb{R}^2$.

- **Continuous dependence on the initial data:** *For any $a_1, a_2, \phi_0 \in H^s(\mathbb{R}^2)$ there is a neighborhood U of a_1, a_2, ϕ_0 in $(H^s(\mathbb{R}^2))^3$ such that the solution map $(a, \phi_0) \rightarrow (A, \phi)$ is continuous from U into $C_b([0, T], \dot{H}^r) \times (C_b([0, T], H^s))^3$.*

Remark 1.1. The spaces $H_T^{s,\theta}$ are defined in Section 2B.

Remark 1.2. The initial data does not have to be given in the Coulomb gauge. See Theorem 3.3.

Remark 1.3. There are two reasons for the requirement of the small initial data. First, the construction of the global Coulomb gauge requires an assumption on the size of the data (see Section 3B). The second obstacle comes from the elliptic equation for A_0 in (aME), and including A_0 in the Picard iteration. See Remark 4.2 for further discussion.

Remark 1.4. We do not prescribe initial data for A_0 , because when A is in the Coulomb gauge, $A_0(t)$ can be determined at any time by solving the elliptic equation. See Section 4 for more details.

Remark 1.5. To simplify the exposition, in this paper we assume $\frac{1}{4} < s < \frac{1}{2}$. See [Czubak 2008] for all $s > \frac{1}{4}$. In general, the higher the value of s , the less delicate the estimates have to be. We have a uniform way to handle all $s > \frac{1}{4}$ for the estimates involving the null forms (see Section 5B1 for a discussion). Therefore the reason for restricting the range of s is rather due to the technicalities of the estimates for A_0 (Theorems 5.3 and 5.5 and Corollaries 5.4 and 5.6) and the regularity of the gauge transformations (Lemma 3.1). The technicalities are not very interesting and are handled in [Czubak 2008] with similar arguments as those presented here.

The outline of the paper is as follows. [Section 2](#) sets notation, introduces spaces, and estimates used throughout the paper. In [Section 3](#) we take a closer look at the equations and discuss gauge transformations. In [Section 4](#) we rewrite (ME) as a system of wave equations coupled with an elliptic equation. We also show local wellposedness of the new system implies local wellposedness of (ME) in the Coulomb gauge. [Section 5](#) is devoted to the proof of the [Main Theorem](#), which is reduced to establishing estimates (5-4)–(5-8).

2. Preliminaries

First we establish notation, then we introduce function spaces as well as estimates used.

2A. Notation. The expression $a \lesssim b$ means $a \leq Cb$ for some positive constant C . A point in the 2+1-dimensional Minkowski space is written as $(t, x) = (x^\alpha)_{0 \leq \alpha \leq 2}$. Greek indices range from 0 to 2, and Roman indices range from 1 to 2. We raise and lower indices with the Minkowski metric $\text{diag}(-1, 1, 1)$. We write $\partial_\alpha = \partial_{x^\alpha}$ and $\partial_t = \partial_0$, and we also use the Einstein notation. Therefore, $\partial^i \partial_i = \Delta$, and $\partial^\alpha \partial_\alpha = -\partial_t^2 + \Delta = \square$. When we refer to spatial and time derivatives of a function f , we write ∂f , and when we consider only spatial derivatives of f , we write ∇f . Finally, d denotes the exterior differentiation operator and d^* its dual given by $d^* = (-1)^k ***d*$, where $*$ is the Hodge $*$ operator (see, for example, [Roe 1998]) and k comes from d^* acting on some given k -form. It will be clear from the context, when $*$ and d^* operators act with respect to the Minkowski metric and when with respect to the Euclidean metric. For the convenience of the reader we include the following: with respect to the Euclidean metric on \mathbb{R}^2 we have

$$*dx = dy, \quad *dy = -dx, \quad *1 = dx \wedge dy,$$

and with respect to the $\text{diag}(-1, 1, 1)$ metric on \mathbb{R}^{2+1} ,

$$*dt = dx \wedge dy, \quad *dx = dt \wedge dy, \quad *dy = -dt \wedge dx.$$

2B. Function spaces. We use Picard iteration. Here we introduce the spaces, in which we are going to perform the iteration³. First we define the following Fourier multiplier operators

$$\begin{aligned} \widehat{\Lambda^\alpha f}(\xi) &= (1 + |\xi|^2)^{\alpha/2} \widehat{f}(\xi), & \widehat{\Lambda_+^\alpha u}(\tau, \xi) &= (1 + \tau^2 + |\xi|^2)^{\alpha/2} \widehat{u}(\tau, \xi), \\ \widehat{\Lambda_-^\alpha u}(\tau, \xi) &= \left(1 + \frac{(\tau^2 - |\xi|^2)^2}{1 + \tau^2 + |\xi|^2}\right)^{\alpha/2} \widehat{u}(\tau, \xi), \end{aligned}$$

where the symbol of Λ_-^α is comparable to $(1 + ||\tau| - |\xi||)^\alpha$. The corresponding homogeneous operators are denoted by D^α , D_+^α , D_-^α , respectively.

Now, the spaces of interest are the wave-Sobolev spaces, $H^{s,\theta}$ and $\mathcal{H}^{s,\theta}$, given by⁴

$$\|u\|_{H^{s,\theta}} = \|\Lambda^s \Lambda_-^\theta u\|_{L^2(\mathbb{R}^{2+1})}, \quad \|u\|_{\mathcal{H}^{s,\theta}} = \|u\|_{H^{s,\theta}} + \|\partial_t u\|_{H^{s-1,\theta}}.$$

³We are also going to employ a combination of the standard $L_t^p W_x^{s,q}$ spaces for A_0 . See [Section 5C](#).

⁴These spaces, together with results in [Selberg 2002b], allowed Klainerman and Selberg to present a unified approach to local wellposedness for wave maps, Yang–Mills and Maxwell–Klein–Gordon types of equations in [Klainerman and Selberg 2002], and are now the natural choice for low regularity subcritical local wellposedness for wave equations. See also [Tao 2006].

An equivalent norm for $\mathcal{H}^{s,\theta}$ is $\|u\|_{\mathcal{H}^{s,\theta}} = \|\Lambda^{s-1}\Lambda_+\Lambda_-u\|_{L^2(\mathbb{R}^{2+1})}$. By results in [Selberg 1999] if $\theta > 1/2$, we have

$$H^{s,\theta} \hookrightarrow C_b(\mathbb{R}, H^s), \quad (2-1)$$

$$\mathcal{H}^{s,\theta} \hookrightarrow C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1}). \quad (2-2)$$

This is a crucial fact needed to localize our solutions in time. We denote the restrictions to the time interval $[0, T]$ by

$$H_T^{s,\theta} \quad \text{and} \quad \mathcal{H}_T^{s,\theta},$$

respectively.

2C. Estimates used. Throughout the paper we use the following estimates:

$$\|D^{-\sigma}(uv)\|_{L_t^p L_x^q} \lesssim \|u\|_{H^{s,\theta}} \|v\|_{H^{s,\theta}} \quad (2-3)$$

is a theorem established by Klainerman and Tataru [1999] for the space-time operator D_+ . The proof for the spatial operator D is in [Selberg 1999]. There are several conditions that σ, p, q have to satisfy, and they are listed in Section 5D, where we discuss the application of the estimate. Further,

$$\|u\|_{L_t^p L_x^2} \lesssim \|u\|_{H^{0,\theta}} \leq \infty \quad \text{if } 2 \leq p \leq \infty, \theta > \frac{1}{2}, \quad (2-4)$$

$$\|u\|_{L_t^p L_x^q} \lesssim \|u\|_{H^{1-2/q-1/p,\theta}} \quad \text{if } 2 \leq p \leq \infty, 2 \leq q < \infty, 2/p \leq \frac{1}{2} - 1/q, \theta > \frac{1}{2}, \quad (2-5)$$

$$\|uv\|_{L_{t,x}^2} \lesssim \|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}} \quad \text{if } a, b, \alpha, \beta \geq 0, a+b > 1, \alpha+\beta > \frac{1}{2}. \quad (2-6)$$

Estimate (2-4) can be proved by interpolation between $H^{0,\theta} \hookrightarrow L_{t,x}^2$ and (2-1) with $s=0$. Estimate (2-5) is a two-dimensional case of Theorem D in [Klainerman and Selberg 2002]. Finally, (2-6) is a special case of the proposition in [Klainerman and Selberg 2002, Appendix A.2].

3. A closer look at the monopole equations

3A. Derivation and background. Electric charge is quantized, which means that it appears in integer multiples of an electron. This is called the principle of quantization and has been observed in nature. The only theoretical proof so far was presented by Paul Dirac [1931]. In the proof Dirac introduced the concept of a magnetic monopole, of an isolated point-source of a magnetic charge. Despite extensive research, magnetic monopoles have not been (yet) found in nature. We refer to magnetic monopoles as Euclidean monopoles. The Euclidean monopole equation has exactly the same form as our space-time monopole equation (ME),

$$F_A = *D_A\phi,$$

with the exception that $*$ acts here with respect to the Euclidean metric and the base manifold is \mathbb{R}^3 instead of \mathbb{R}^{2+1} . The Euclidean monopole equations are also referred to as Bogomolny equations. For more on Euclidean monopoles we refer the reader to [Jaffe and Taubes 1980] and [Atiyah and Hitchin 1988]. In this paper we study the space-time monopole equation, which was first introduced by Ward [1989]. Both the Euclidean and the space-time monopole equations are examples of integrable systems

and have an equivalent formulation as a Lax pair. This and much more can be found in [Dai et al. 2006].

Given a space-time monopole equation

$$F_A = *D_A\phi, \quad (\text{ME})$$

the unknowns are a pair (A, ϕ) . A is a connection 1-form given by

$$A = A_0 dt + A_1 dx + A_2 dy, \quad \text{where } A_\alpha : \mathbb{R}^{2+1} \rightarrow \mathfrak{g}. \quad (3-1)$$

Here \mathfrak{g} is the Lie algebra of a Lie group G , which is typically taken to be a matrix group $SU(n)$ or $U(n)$. In this paper we consider $G = SU(n)$, but everything we say here should generalize to any compact Lie group.

To be more general we could say A is a connection on a principal G -bundle. Then observe that the G -bundle we deal here with is a trivial bundle $\mathbb{R}^{2+1} \times G$.

Next, ϕ is a section of a vector bundle associated to the G -bundle by a representation. We use the adjoint representation. Since we have a trivial bundle, we can just think of the Higgs field ϕ as a map from $\mathbb{R}^{2,1} \rightarrow \mathfrak{g}$.

F_A is the curvature of A . It is a Lie algebra valued 2-form on \mathbb{R}^{2+1}

$$F_A = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \quad \text{where } F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta], \quad (3-2)$$

where $[\cdot, \cdot]$ denotes the Lie bracket, which for matrices can be thought of simply as $[X, Y] = XY - YX$.

When we write $[\phi, B]$, where B is a 1-form, we mean

$$[\phi, B] = [\phi, B_i] dx^i \quad \text{and} \quad [B, C] = \frac{1}{2} [B_i, C_j] dx^i \wedge dx^j, \quad \text{for two 1-forms } B, C. \quad (3-3)$$

In the physics language, frequently adopted by the mathematicians, A is called a gauge potential, ϕ a scalar field and F_A is called an electromagnetic field.

Next, D_A is the covariant exterior derivative associated to A , and $D_A\phi$ is given by

$$D_A\phi = D_\alpha\phi dx^\alpha, \quad \text{where } D_\alpha\phi = \partial_\alpha\phi + [A_\alpha, \phi]. \quad (3-4)$$

The space-time monopole equation (ME) is obtained by a dimensional reduction of the antiselfdual Yang–Mills equations on \mathbb{R}^{2+2} , given by

$$F_A = -*F_A. \quad (\text{ASDYM})$$

If the curvature of a connection A satisfies (ASDYM), then A is called an antiselfdual connection. (The corresponding selfdual Yang–Mills equation is

$$F_A = *F_A; \quad (\text{SDYM})$$

in either case F_A satisfies the Yang–Mills equation $D_A *F = 0$, since $F_A = \pm *F_A$ implies

$$D_A *F = \pm D_A F = 0,$$

as can be seen from the second Bianchi identity, or by direct computation from (3-2).)

Both (ASDYM) and (SDYM) are known to give rise to many different integrable equations (see [Ward 1985; Ablowitz et al. 2003] and references therein). In particular, the 2+2 signature is used to derive both KDV and NLS in [Mason and Sparling 1989], and harmonic maps from \mathbb{R}^2 and \mathbb{R}^{1+1} into a Lie

group in [Uhlenbeck 1992]. Also see [Ward 1989], where Einstein's vacuum equation for cylindrically symmetric space-times, and the sine-Gordon equation are derived from (ME).

We now present the details of the derivation of (ME) from (ASDYM) outlined in [Dai et al. 2006]. Let

$$dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$$

be a metric on \mathbb{R}^{2+2} , then in coordinates (ASDYM) is

$$F_{12} = -F_{34}, \quad F_{13} = -F_{24}, \quad F_{23} = F_{14}. \quad (3-5)$$

The next step is the dimensional reduction, where we assume the connection A is independent of x_3 , and we let $A_3 = \phi$. Then (3-5) becomes

$$D_0\phi = F_{12}, \quad D_1\phi = F_{02}, \quad D_2\phi = F_{10}, \quad (3-6)$$

where we use index 0 instead of 4. This is exactly (ME) written out in components.

Remark 3.1. Equivalently we could write (ME) as

$$F_{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma} D^\gamma \phi, \quad (3-7)$$

where $\varepsilon_{\alpha\beta\gamma}$ is a completely antisymmetric tensor with $\varepsilon_{012} = 1$, and where we raise the index γ using the Minkowski metric. We choose to work with the Hodge operator $*$ as it simplifies our task in Section 4.

Following [Dai et al. 2006], there is another way to write (ME), which is very useful for computations. (ME) is an equation involving 2-forms on both sides. By taking the parts corresponding to $dt \wedge dx$ and $dt \wedge dy$ on the one hand, and the parts corresponding to $dx \wedge dy$ on the other, we obtain, respectively,

$$\partial_t A + [A_0, A] - dA_0 = *d\phi + [*A, \phi], \quad (3-8)$$

$$dA + [A, A] = *(\partial_t \phi + [A_0, \phi]). \quad (3-9)$$

Observe that now operators d and $*$ act only with respect to the spatial variables. Similarly, A now denotes only the spatial part of the connection, i.e., $A = (A_1, A_2)$. Moreover, (3-8) is an equation involving 1-forms, and (3-9) involves 2-forms.

3B. Gauge transformations. (ME) is invariant under gauge transformations. Indeed, if we have a smooth map g , with compact support such that $g : \mathbb{R}^{2+1} \rightarrow G$, and

$$A \rightarrow A_g = gAg^{-1} + gdg^{-1}, \quad \phi \rightarrow \phi_g = g\phi g^{-1}, \quad (3-10)$$

then a computation shows $F_A \rightarrow gF_A g^{-1}$ and $D_A \phi \rightarrow gD_A \phi g^{-1}$. Therefore if a pair (A, ϕ) solves (ME), so does (A_g, ϕ_g) .

We would like to discuss the regularity of the gauge transformations. If $A \in X$, $\phi \in Y$ where X, Y are some Banach spaces, the smoothness and compact support assumption on g can be lowered just enough so the gauge transformation defined above is a continuous map from X back into X , and from Y back into Y . First note that since we are mapping into a compact Lie group, we can assume $g \in L_{t,x}^\infty$ and $\|g\|_{L_{t,x}^\infty} = \|g^{-1}\|_{L_{t,x}^\infty}$. Next, note that the Main Theorem produces a solution so that ϕ and the spatial parts of the connection $A_1, A_2 \in C_b(I, H^s)$, $\frac{1}{4} < s < \frac{1}{2}$, and $A_0 \in C_b(I, \dot{H}^r)$, $r \in (0, 2s]$.

Lemma 3.1. *Let $0 < \alpha < 1$, and $Y = C_b(I, \dot{H}^1 \cap \dot{H}^{\alpha+1}) \cap L^\infty$, then the gauge action is a continuous map from*

$$C_b(I, H^\alpha) \times Y \rightarrow C_b(I, H^\alpha), \quad (h, g) \mapsto ghg^{-1} + gdg^{-1}, \quad (3-11)$$

and the following estimate holds:

$$\|h_g\|_{C_b(I, H^\alpha)} \lesssim (\|h\|_{C_b(I, H^\alpha)} + 1)\|g\|_Y^2. \quad (3-12)$$

Proof. The continuity of the map easily follows from the inequalities we obtain below. Next, for fixed t we have

$$\|g(t)h(t)g^{-1}(t) + g(t)dg^{-1}(t)\|_{H^\alpha} \lesssim \|ghg^{-1}\|_{L^2} + \|D^\alpha(ghg^{-1})\|_{L^2} + \|gdg^{-1}\|_{H^\alpha},$$

where for the ease of notation we eliminated writing of the variable t on the right side of the inequality. The first term is bounded by $\|h(t)\|_{H^\alpha}\|g\|_{L^\infty}^2$. For the second one we have

$$\|D^\alpha(ghg^{-1})\|_{L^2} \lesssim \|D^\alpha gh\|_{L^2}\|g\|_{L^\infty} + \|hD^\alpha g^{-1}\|_{L^2}\|g\|_{L^\infty} + \|h\|_{\dot{H}^\alpha}\|g\|_{L^\infty}^2.$$

It is enough to only look at the first term since g and g^{-1} have the same regularity. By Hölder's inequality and Sobolev embedding

$$\|D^\alpha gh\|_{L^2} \leq \|D^\alpha g\|_{L^{2/\alpha}}\|h\|_{L^{(1/2-\alpha/2)^{-1}}} \lesssim \|g\|_{\dot{H}^1}\|h\|_{\dot{H}^\alpha}, \quad (3-13)$$

where we use that $\frac{\alpha}{2} = \frac{1}{2} - \frac{1-\alpha}{2}$. Finally for the last term we have

$$\|gdg^{-1}\|_{H^\alpha} \lesssim \|g\|_{\dot{H}^1}\|g\|_{L^\infty} + \|D^\alpha gdg^{-1}\|_{L^2} + \|g\|_{\dot{H}^{\alpha+1}}\|g\|_{L^\infty}, \quad (3-14)$$

and we are done if we observe that the second term can be handled exactly as in (3-13). \square

Remark 3.2. We assume $0 < \alpha < 1$ since this is the case we need. However it is not difficult to see the lemma still holds with $\alpha = 0$ or $\alpha \geq 1$ [Czubak 2008].

From the lemma, we trivially obtain the following corollary.

Corollary 3.2. *Let $0 < r, s < 1$,*

$$X = C_b(I, \dot{H}^r) \times C_b(I, H^s) \times C_b(I, H^s), \quad \text{and} \quad Y = C_b(I, \dot{H}^1 \cap \dot{H}^{s+1} \cap \dot{H}^{r+1}) \cap L^\infty.$$

Then the gauge action is a continuous map from

$$X \times Y \rightarrow X, \quad (A_0, A_1, A_2) \mapsto A_g, \quad (3-15)$$

as well as from

$$C_b(I, H^s) \times Y \rightarrow C_b(I, H^s), \quad \phi \mapsto \phi_g = g\phi g^{-1}, \quad (3-16)$$

and the following estimates hold:

$$\begin{aligned} \|A_g\|_X &\lesssim (1 + \|A\|_X)\|g\|_Y^2, \\ \|\phi_g\|_{C_b(I, H^s)} &\lesssim \|\phi\|_{C_b(I, H^s)}\|g\|_Y^2. \end{aligned} \quad (3-17)$$

Since in this paper we work in the Coulomb gauge, we ask: given any initial data $a_1, a_2, \phi_0 \in H^s(\mathbb{R}^2)$, can we find a gauge transformation so that the initial data is placed in the Coulomb gauge? Dell'Antonio and Zwanziger [1991] produce a global \dot{H}^1 Coulomb gauge using variational methods. Here, we also require $g \in \dot{H}^{s+1}$, and two dimensions are tricky. Fortunately, if the initial data is small, we can obtain a global gauge with the additional regularity as needed. This has been studied by the author and Uhlenbeck for two dimensions and higher; the result in two dimensions is the following:

Theorem 3.3 [Czubak and Uhlenbeck \geq 2010]. *Let $s > 0$. Given $A(0) = a$ sufficiently small in*

$$H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2),$$

there exists a gauge transformation $g \in \dot{H}^{s+1}(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2) \cap L^\infty$ such that $\partial^i(ga_i g^{-1} + g\partial_i g^{-1}) = 0$.

4. The monopole equation in the Coulomb gauge as a system of wave and elliptic equations

We begin by rewriting the monopole equation in the Coulomb gauge as a system of wave equations coupled with an elliptic equation. We refer to the new system as the auxiliary monopole equations (aME). Then we establish that local wellposedness (LWP) for (ME) in the Coulomb gauge can be obtained from LWP of (aME).

4A. Derivation of (aME) from (ME). Suppose we have initial data

$$A_i|_{t=0} = a_i \quad \text{for } i = 1, 2 \quad \text{and} \quad \phi|_{t=0} = \phi_0, \quad (4-1)$$

where $\partial^i a_i = 0$. Recall equations (3-8) and (3-9):

$$\partial_t A + [A_0, A] - dA_0 = *d\phi + [*A, \phi], \quad (4-2)$$

$$dA + [A, A] = *(\partial_t \phi + [A_0, \phi]), \quad (4-3)$$

where d and $*$ act only with respect to the spatial variables, and A denotes only the spatial part of the connection. If we impose the Coulomb gauge condition, then

$$d^*A = 0. \quad (4-4)$$

By equivalence of closed and exact forms on \mathbb{R}^n , we can further suppose that

$$A = *df, \quad (4-5)$$

for some $f : \mathbb{R}^{2+1} \rightarrow \mathfrak{g}$. Observe that

$$d * df = \Delta f dx \wedge dy, \quad [*df, *df] = [df, df] = 1/2[\partial_i f, \partial_j f] dx^i \wedge dx^j, \quad (4-6)$$

and $**\omega = -\omega$ for a one-form on \mathbb{R}^2 . Hence (4-2) and (4-3) become

$$\partial_t * df + [A_0, *df] - dA_0 = *d\phi - [df, \phi], \quad (4-7)$$

$$\Delta f + [\partial_1 f, \partial_2 f] = \partial_t \phi + [A_0, \phi]. \quad (4-8)$$

Take d^* of (4-7) to obtain

$$\Delta A_0 = -d^*[A_0, *df] - d^*[df, \phi].$$

This is the elliptic equation in (aME). To uncover the wave equations we proceed as follows. Take d of (4-7)

$$\partial_t \Delta f + \partial^j [A_0, \partial_j f] = \Delta \phi + \partial_2 [\partial_1 f, \phi] - \partial_1 [\partial_2 f, \phi]. \quad (4-9)$$

Consider (4-9) and (4-8) on the spatial Fourier transform side:

$$-\partial_t |\xi|^2 \hat{f} + |\xi|^2 \hat{\phi} = i(\xi_2 [\widehat{\partial_1 f, \phi}] - \xi_1 [\widehat{\partial_2 f, \phi}] - \xi_j [\widehat{A_0, \partial_j f}]), \quad (4-10)$$

$$-|\xi|^2 \hat{f} - \partial_t \hat{\phi} = -[\widehat{\partial_1 f, \partial_2 f}] + [\widehat{A_0, \phi}]. \quad (4-11)$$

This allows us to write (4-10) and (4-11) as a system for ϕ and df :

$$(\partial_t - i|\xi|)(\hat{\phi} + i|\xi|\hat{f}) = -\hat{\mathcal{B}}_+(\phi, df, A_0), \quad (4-12)$$

$$(\partial_t + i|\xi|)(\hat{\phi} - i|\xi|\hat{f}) = -\hat{\mathcal{B}}_-(\phi, df, A_0), \quad (4-13)$$

where

$$\hat{\mathcal{B}}_{\pm} = -[\widehat{\partial_1 f, \partial_2 f}] + [\widehat{A_0, \phi}] \pm \left(\frac{\xi_1}{|\xi|} [\widehat{\partial_2 f, \phi}] - \frac{\xi_2}{|\xi|} [\widehat{\partial_1 f, \phi}] + \frac{\xi_j}{|\xi|} [\widehat{A_0, \partial_j f}] \right). \quad (4-14)$$

Indeed, multiply (4-10) by $i/|\xi|$, and first add the resulting equation to (4-11) to obtain (4-12), and then subtract it from (4-11) to obtain (4-13). Now we let

$$\hat{\phi} + i|\xi|\hat{f} = (\partial_t + i|\xi|)\hat{u} \quad \text{and} \quad \hat{\phi} - i|\xi|\hat{f} = (\partial_t - i|\xi|)\hat{v}, \quad (4-15)$$

where $u, v : \mathbb{R}^{2+1} \rightarrow \mathfrak{g}$. This gives

$$\square u = \mathcal{B}_+(\phi, df, A_0), \quad \square v = \mathcal{B}_-(\phi, df, A_0).$$

See Remark 4.1 below.

Now we discuss initial data. From (4-15), we have

$$\partial_t \widehat{u}(0) = \hat{\phi}_0 + i|\xi|\widehat{f}(0) - i|\xi|\widehat{u}(0), \quad (4-16)$$

$$\partial_t \widehat{v}(0) = \hat{\phi}_0 - i|\xi|\widehat{f}(0) + i|\xi|\widehat{v}(0). \quad (4-17)$$

We are free to choose any data for u and v , as long as in the end we can recover the original data for ϕ and A . Hence we just let $u(0) = v(0) = 0$. We still need to say what $|\xi|\widehat{f}(0)$ is in terms of the initial data (a, ϕ_0) . Let $\hat{h} = |\xi|\widehat{f}(0)$. By (4-1) and (4-5) we have $a_1 = A_1(0) = -\partial_2 f(0)$ and $a_2 = A_2(0) = \partial_1 f(0)$. Therefore

$$R_1 h = a_2, \quad R_2 h = -a_1,$$

where R_j denotes the Riesz transform, $(-\Delta)^{-1/2} \partial_j$. Differentiate the first equation with respect to x , the second with respect to y , and add them together to obtain

$$\Delta D^{-1} h = \partial_1 a_2 - \partial_2 a_1.$$

So

$$h = R_2 a_1 - R_1 a_2, \quad (4-18)$$

It follows that the initial data for u and v are

$$u(0) = v(0) = 0, \quad \partial_t u(0) = \phi_0 + ih, \quad \partial_t v(0) = \phi_0 - ih, \quad (4-19)$$

with h defined by (4-18).

In summary, the monopole equation in the Coulomb gauge

$$F_A = *D_A\phi, \quad d^*A = 0,$$

with initial data (4-1) can be rewritten as the system

$$\begin{aligned} \square u &= \mathcal{B}_+(\phi, \nabla f, A_0), \\ \square v &= \mathcal{B}_-(\phi, \nabla f, A_0), \\ \Delta A_0 &= \mathcal{C}(\phi, \nabla f, A_0), \end{aligned} \tag{aME}$$

where

$$\mathcal{C} = -\partial_1[A_0, \partial_2 f] + \partial_2[A_0, \partial_1 f] + \partial_j[\partial_j f, \phi], \tag{4-20}$$

$$\mathcal{B}_\pm = -\mathcal{B}_1 \mp i\mathcal{B}_2 + \mathcal{B}_3 \mp i\mathcal{B}_4, \tag{4-21}$$

and

$$\mathcal{B}_1 = [\partial_1 f, \partial_2 f], \quad \mathcal{B}_2 = R_1[\partial_2 f, \phi] - R_2[\partial_1 f, \phi], \quad \mathcal{B}_3 = [A_0, \phi], \quad \mathcal{B}_4 = R_j[A_0, \partial_j f]. \tag{4-22}$$

The initial data for (aME) is given by (4-19).

Remark 4.1. u and v are our new unknowns, but we are really interested in ϕ and df . Therefore, we observe that once we know what u and v are, we can determine ϕ and df by using

$$\hat{\phi} = \frac{(\partial_t + i|\xi|)\hat{u} + (\partial_t - i|\xi|)\hat{v}}{2}, \quad i|\xi|\hat{f} = \frac{(\partial_t + i|\xi|)\hat{u} - (\partial_t - i|\xi|)\hat{v}}{2}, \tag{4-23}$$

or equivalently

$$\phi = \frac{(\partial_t + iD)u + (\partial_t - iD)v}{2}, \quad \partial_j f = -iR_j \left(\frac{(\partial_t + iD)u - (\partial_t - iD)v}{2} \right). \tag{4-24}$$

From df we get A by letting $A = *df$. Finally, for simplicity we usually keep the nonlinearities in terms of ϕ and df . However, since ϕ and df can be written in terms of derivatives of u and v we sometimes write $\mathcal{B}_\pm(\phi, df, A_0)$ as $\mathcal{B}_\pm(\partial u, \partial v, A_0)$.

Remark 4.2. (aME) has some resemblance to a system considered by Selberg [2002a] for the Maxwell–Klein–Gordon (MKG) equations, where he successfully obtains almost optimal local wellposedness in dimensions $1 + 4$. Besides the dimension considered, there are two fundamental technical differences applicable to our problem. First comes from the fact that the monopole equation we consider here is an example of a system in the nonabelian gauge theory whereas MKG is an example of a system in the abelian gauge theory. The existence of a global Coulomb gauge requires smallness of initial data in the nonabelian gauge theories, but is not needed in the abelian theories. Another technical difference arises from Selberg being able to solve the elliptic equation for his temporal variable A_0 using the Riesz representation theorem, where he does not require smallness of the initial data. The elliptic equation in (aME) is more difficult, so we include A_0 in the Picard iteration. As a result we are not able to allow large data by taking a small time interval, which we could do if we only had the two wave equations. Finally, we point out that the proof of our estimates involving A_0 is modeled after Selberg’s proof in [Selberg 2002a] (see Remark 5.1 and Section 5C).

4B. Return to the monopole equation. Now we have a theorem, where we show how LWP for (aME) implies LWP for (ME) in the Coulomb gauge. For completeness, we first state exactly what we mean by LWP of (aME).

Let $r \in (0, \min(2s, 1 + s)]$, $s > 0$. Consider the system (aME) with initial data

$$(u, u_t)|_{t=0} = (u_0, u_1) \quad \text{and} \quad (v, v_t)|_{t=0} = (v_0, v_1)$$

in $H^{s+1} \times H^s$, then (aME) is LWP if the following conditions are satisfied:

Local existence. There exist $T > 0$ depending continuously on the norm of the initial data, and functions

$$A_0 \in C_b([0, T], \dot{H}^r), \quad u, v \in \mathcal{H}_T^{s+1, \theta} \hookrightarrow C_b([0, T], H^{s+1}) \cap C_b^1([0, T], H^s),$$

which solve (aME) on $[0, T] \times \mathbb{R}^2$ in the sense of distributions and such that the initial conditions are satisfied.

Uniqueness. If $T > 0$ and (A_0, u, v) and (A'_0, u', v') are two solutions of (aME) on $(0, T) \times \mathbb{R}^2$ belonging to

$$C_b([0, T], \dot{H}^r) \times \mathcal{H}_T^{s+1, \theta} \times \mathcal{H}_T^{s+1, \theta},$$

with the same initial data, then $(A_0, u, v) = (A'_0, u', v')$ on $(0, T) \times \mathbb{R}^2$.

Continuous dependence on initial data. For any $(u_0, u_1), (v_0, v_1) \in H^{s+1} \times H^s$ there is a neighborhood U of the initial data such that the solution map $(u_0, u_1), (v_0, v_1) \rightarrow (A_0, u, v)$ is continuous from U into $C_b([0, T], \dot{H}^r) \times (C_b([0, T], H^{s+1}) \cap C_b^1([0, T], H^s))^2$.

In fact, by the results in [Selberg 2002b] combined with estimates for the elliptic equation, we can show the stronger estimates

$$\begin{aligned} \|u - u'\|_{\mathcal{H}_T^{s+1, \theta}} + \|v - v'\|_{\mathcal{H}_T^{s+1, \theta}} + \|A_0 - A'_0\|_{C_b([0, T], \dot{H}^r)} \\ \lesssim \|u_0 - u'_0\|_{H^{s+1}} + \|u_1 - u'_1\|_{H^s} + \|v_0 - v'_0\|_{H^{s+1}} + \|v_1 - v'_1\|_{H^s}, \end{aligned} \quad (4-25)$$

where $(u'_0, u'_1), (v'_0, v'_1)$ are sufficiently close to $(u_0, u_1), (v_0, v_1)$.

Remark 4.3. Note that below we have no restriction on s , that is, if we could show (aME) is LWP in $H^{s+1} \times H^s$, $s > 0$, we would get LWP of (ME) in the Coulomb gauge in H^s for $s > 0$ as well.

Theorem 4.1. Consider (ME) in the Coulomb gauge with the following initial data in H^s for $s > 0$:

$$A_i|_{t=0} = a_i, \quad i = 1, 2, \quad \phi|_{t=0} = \phi_0, \quad \text{with } \partial^i a_i = 0. \quad (4-26)$$

Then local wellposedness of (aME) with initial data as in (4-19) implies local wellposedness of (ME) in the Coulomb gauge with initial data given by (4-26).

Proof. First, in view of Section 4A it is clear that if u, v satisfy (aME) with initial data as in (4-19), then solutions of (ME) in the Coulomb gauge satisfy the initial data as given in (4-26).

Local existence. From (4-24), if

$$u, v \in \mathcal{H}_T^{s+1, \theta}, \quad \text{then} \quad \phi, A = *df \in H_T^{s, \theta},$$

as needed. We now verify that if (u, v, A_0) solve (aME), then (ϕ, df, A_0) solve (ME) in the Coulomb gauge. Since $A = *df$, A is in the Coulomb gauge: $d*A = -*d(*df) = 0$. Next note that (ME) in the

Coulomb gauge is equivalent to (4-7) and (4-8). Suppose u, v, A_0 solve (aME). It follows (df, ϕ) solve (4-12) and (4-13). Add (4-12) to (4-13) to recover (4-11), which is equivalent to (4-8).

Next given (aME) we need to show (4-7) holds. Write (4-7) in coordinates,

$$\partial_1 A_0 - \partial_2 \phi + \partial_t \partial_2 f = [\partial_1 f, \phi] - [A_0, \partial_2 f], \quad (4-27)$$

$$\partial_2 A_0 + \partial_1 \phi - \partial_t \partial_1 f = [\partial_2 f, \phi] + [A_0, \partial_1 f]. \quad (4-28)$$

From the elliptic equation in (aME) we have:

$$A_0 = \Delta^{-1}(-\partial_1[A_0, \partial_2 f] + \partial_2[A_0, \partial_1 f] + \partial_1[\partial_1 f, \phi] + \partial_2[\partial_2 f, \phi]). \quad (4-29)$$

Also subtract (4-12) from (4-13) and multiply by $|\zeta|$ on both sides to obtain (4-9), which implies

$$\phi - \partial_t f = \Delta^{-1}(\partial_j[A_0, \partial_j f] - \partial_2[\partial_1 f, \phi] + \partial_1[\partial_2 f, \phi]). \quad (4-30)$$

In order to recover (4-27), first use (4-29) to get

$$\partial_1 A_0 = \Delta^{-1}(-\partial_1^2[A_0, \partial_2 f] + \partial_1 \partial_2[A_0, \partial_1 f] + \partial_1^2[\partial_1 f, \phi] + \partial_1 \partial_2[\partial_2 f, \phi]). \quad (4-31)$$

Next use (4-30) to get

$$\partial_2(\phi - \partial_t f) = \Delta^{-1}(\partial_2 \partial_1[A_0, \partial_1 f] + \partial_2^2[A_0, \partial_2 f] - \partial_2^2[\partial_1 f, \phi] + \partial_2 \partial_1[\partial_2 f, \phi]), \quad (4-32)$$

and subtract this from (4-31) to get (4-27) as needed. We recover (4-28) in the exactly same way.

Continuous dependence on initial data. We would like to show that

$$\begin{aligned} \|A_0 - A'_0\|_{C_b([0, T], \dot{H}^r)} + \|A_1 - A'_1\|_{H_T^{s, \theta}} + \|A_2 - A'_2\|_{H_T^{s, \theta}} + \|\phi - \phi'\|_{H_T^{s, \theta}} \\ \lesssim \|a_1 - a'_1\|_{H^s} + \|a_2 - a'_2\|_{H^s} + \|\phi_0 - \phi'_0\|_{H^s} \end{aligned} \quad (4-33)$$

for any a'_1, a'_2, ϕ'_0 sufficiently close to a_1, a_2, ϕ_0 . In view of LWP for (aME) with data given by

$$u(0) = v(0) = 0, \quad \partial_t u(0) = \phi_0 + ih, \quad \partial_t v(0) = \phi_0 - ih, \quad h = R_2 a_1 - R_1 a_2,$$

and by (4-25) we have

$$\begin{aligned} \|u - u'\|_{\mathcal{Y}_T^{s+1, \theta}} + \|v - v'\|_{\mathcal{Y}_T^{s+1, \theta}} + \|A_0 - A'_0\|_{C_b([0, T], \dot{H}^r)} \\ \lesssim \|u'_0\|_{H^{s+1}} + \|\phi_0 + ih - u'_1\|_{H^s} + \|v'_0\|_{H^{s+1}} + \|\phi_0 - ih - v'_1\|_{H^s}, \end{aligned} \quad (4-34)$$

for all u'_0, v'_0, u'_1, v'_1 satisfying

$$\|u'_0\|_{H^{s+1}} + \|\phi_0 + ih - u'_1\|_{H^s} + \|v'_0\|_{H^{s+1}} + \|\phi_0 - ih - v'_1\|_{H^s} \leq \delta, \quad \text{for some } \delta > 0. \quad (4-35)$$

In particular choose

$$u'_0 = v'_0 = 0, \quad u'_1 = \phi'_0 + ih', \quad v'_1 = \phi'_0 - ih', \quad h' = R_2 a'_1 - R_1 a'_2, \quad (4-36)$$

such that

$$\begin{aligned} \|\phi_0 + ih - \phi'_0 - ih'\|_{H^s} + \|\phi_0 - ih - \phi'_0 + ih'\|_{H^s} &\lesssim \|\phi_0 - \phi'_0\|_{H^s} + \|R_1(a_2 - a'_2)\|_{H^s} + \|R_2(a_1 - a'_1)\|_{H^s} \\ &\leq \|\phi_0 - \phi'_0\|_{H^s} + \|a_1 - a'_1\|_{H^s} + \|a_2 - a'_2\|_{H^s} \\ &\leq \delta. \end{aligned} \quad (4-37)$$

Then, by (4-34)–(4-37), $\|A_0 - A'_0\|_{C_b([0,T],\dot{H}^r)}$ is bounded by the right side of (4-33). Next observe that

$$\begin{aligned} \|A_1 - A'_1\|_{H_T^{s,\theta}} &\lesssim \|R_2(\partial_t + iD)(u - u')\|_{H_T^{s,\theta}} + \|R_2(\partial_t - iD)(v - v')\|_{H_T^{s,\theta}} \\ &\leq \|u - u'\|_{\mathcal{H}_T^{s+1,\theta}} + \|v - v'\|_{\mathcal{H}_T^{s+1,\theta}}. \end{aligned}$$

So again by (4-34)–(4-37) $\|A_1 - A'_1\|_{H_T^{s,\theta}}$ is bounded by the right side of (4-33). We bound the difference for A_2 and ϕ in a similar fashion.

Uniqueness. By LWP of (aME), A_0 is unique in the required class. We need to show A and ϕ are unique in $H_T^{s,\theta}$. However, by (4-33) this is obvious. \square

5. Proof of the Main Theorem

By Theorem 4.1 it is enough to show LWP for (aME). We start by explaining how we are going to perform our iteration.

5A. Set up of the iteration. Equations (aME) are written for functions u and v . Nevertheless, functions u and v are only our auxiliary functions, and we are really interested in solving for df and ϕ . In addition, the nonlinearities \mathcal{B}_\pm are a linear combination of terms \mathcal{B}_i , $i = 1, 2, 3, 4$, given by (4-22), and the \mathcal{B}_i are written in terms of ϕ , df and A_0 . Also, when we do our estimates, it is easier to keep the \mathcal{B}_i in terms of ϕ and df with the exception of \mathcal{B}_2 , which we rewrite in terms of ∂u and ∂v (see Section 5B2 for the details). These comments motivate the following procedure for our iteration. Start with $\phi_{-1} = df_{-1} = 0$. Then $\mathcal{B}_\pm \equiv 0$. Solve the homogeneous wave equations for u_0, v_0 with the initial data given by (4-19). Then to solve for df_0, ϕ_0 , use (4-24). Then feed ϕ_0 and df_0 into the elliptic equation,

$$\Delta A_{0,0} = -d^*([A_{0,0}, *df_0] + [df_0, \phi_0]), \quad (5-1)$$

and solve for $A_{0,0}$. Next we take df_0, ϕ_0 and $A_{0,0}$ plug them into $\mathcal{B}_1, \mathcal{B}_3, \mathcal{B}_4$, but rewrite \mathcal{B}_2 in terms of $\partial u_0, \partial v_0$. We continue in this manner, so at the j th step of the iteration, $j \geq 1$, we solve

$$\begin{aligned} \square u_j &= -\mathcal{B}_1(\nabla f_{j-1}) - i\mathcal{B}_2(\partial u_{j-1}, \partial v_{j-1}) + \mathcal{B}_3(A_{0,j-1}, \phi_{j-1}) - i\mathcal{B}_4(A_{0,j-1}, \nabla f_{j-1}), \\ \square v_j &= -\mathcal{B}_1(\nabla f_{j-1}) + i\mathcal{B}_2(\partial u_{j-1}, \partial v_{j-1}) + \mathcal{B}_3(A_{0,j-1}, \phi_{j-1}) + i\mathcal{B}_4(A_{0,j-1}, \nabla f_{j-1}), \\ \Delta A_{0,j} &= -d^*([A_{0,j}, *df_j] + [df_j, \phi_j]). \end{aligned}$$

5B. Estimates needed. The elliptic equation is discussed in Section 5C. Therefore we begin by discussing the inversion of the wave operator in $\mathcal{H}^{s+1,\theta}$ spaces. The main idea is that for the purposes of local in time estimates \square^{-1} can be replaced with $\Lambda_\pm^{-1} \Lambda_\mp^{-1}$. The first estimates, leading to wellposedness for small initial data, were proved by Klainerman and Machedon [1995]. The small data assumption was removed by Selberg [2002b], who showed that by introducing ε small enough in the invertible version

of the wave operator, that is, $\Lambda_+^{-1}\Lambda_-^{-1+\varepsilon}$, we can use initial data as large as we wish.⁵ Selberg [2002b] also gave a very useful, general framework for local wellposedness of wave equations, which reduces the proof of the [Main Theorem](#) to establishing the estimates below, for the nonlinearities \mathcal{B}_\pm , and to combining them with appropriate elliptic estimates from [Section 5C](#). The needed estimates for \mathcal{B}_\pm are

$$\|\Lambda_+^{-1}\Lambda_-^{-1+\varepsilon}\mathcal{B}_\pm(\partial u, \partial v, A_0)\|_{\mathcal{H}^{s+1,\theta}} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} + \|v\|_{\mathcal{H}^{s+1,\theta}}, \quad (5-2)$$

$$\|\Lambda_+^{-1}\Lambda_-^{-1+\varepsilon}(\mathcal{B}_\pm(\partial u, \partial v, A_0) - \mathcal{B}_\pm(\partial u', \partial v', A'_0))\|_{\mathcal{H}^{s+1,\theta}} \lesssim \|u - u'\|_{\mathcal{H}^{s+1,\theta}} + \|v - v'\|_{\mathcal{H}^{s+1,\theta}}, \quad (5-3)$$

where the suppressed constants depend continuously on the $\mathcal{H}^{s+1,\theta}$ norms of u, u', v, v' . Since \mathcal{B}_\pm are bilinear, (5-3) can follow from (5-2). In this paper small initial data is necessary (see [Theorem 3.3](#) and [Section 5C](#)), so we do not need ε , but we keep it to make the estimates general. Let $\frac{1}{4} < s < \frac{1}{2}$ and set θ, ε as follows:

$$\frac{3}{4} - \frac{\varepsilon}{2} < \theta \leq s + \frac{1}{2} - \varepsilon, \quad \text{and} \quad \theta < 1 - \varepsilon, \quad 0 \leq \varepsilon < \min\left(2s - \frac{1}{2}, \frac{1}{2}\right).$$

Next observe $\Lambda_+\Lambda_-^{-1-\varepsilon}\mathcal{H}^{s+1,\theta} = H^{s,\theta-1+\varepsilon}$, as well as that

$$\|\nabla f\|_{H^{s,\theta}}, \|\phi\|_{H^{s,\theta}} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} + \|v\|_{\mathcal{H}^{s+1,\theta}}.$$

Therefore, using (4-21) and (4-22), it is enough to prove the following:

$$\|\mathcal{B}_1\|_{H^{s,\theta-1+\varepsilon}} = \|[\partial_1 f, \partial_2 f]\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|\nabla f\|_{H^{s,\theta}}^2, \quad (5-4)$$

$$\|\mathcal{B}_2\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|[\partial_j f, \phi]\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|\partial_j f\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \quad \text{for } j = 1, 2, \quad (5-5)$$

$$\|\mathcal{B}_3\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|A_0\phi\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|A_0\| \|\phi\|_{H^{s,\theta}}, \quad (5-6)$$

$$\|\mathcal{B}_4\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|A_0\partial_j f\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|A_0\| \|\partial_j f\|_{H^{s,\theta}} \quad \text{for } j = 1, 2, \quad (5-7)$$

where the norm used for A_0 is immaterial, mainly because in [Section 5C](#) we show that

$$\|A_0\| \lesssim \|\nabla f\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}. \quad (5-8)$$

A few remarks are in order. Estimate (5-4) corresponds to estimates for the null form Q_{ij} , and estimate (5-5) gives rise to a new null form Q (this is discussed in the next two sections). A_0 in estimates (5-6) and (5-7) solves the elliptic equation in (aME), which results in a quite good regularity for A_0 . As a result, we do not have to look for any special structures to get (5-6) and (5-7) to hold, so we can drop the brackets, and also treat these estimates as equivalent since ϕ and df exhibit the same regularity. Finally, since Riesz transforms are clearly bounded on L^2 , we ignore them in the estimates needed in (5-5) and (5-7). The estimates (5-4) and (5-5) for the null forms are the most interesting. Hence we discuss them first, and then we consider the elliptic terms.

5B1. *Null forms: proof of estimate (5-4).* $[\partial_1 f, \partial_2 f]$ has a structure of a null form Q_{ij} :

$$[\partial_1 f, \partial_2 f] = \partial_1 f \partial_2 f - \partial_2 f \partial_1 f = Q_{12}(f, f).$$

⁵See also [Klainerman and Selberg 2002, Section 5] for an excellent discussion and motivation of the issues involved in the Picard iteration.

It follows that (5-4) is equivalent to

$$\|Q_{12}(f, f)\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|\nabla f\|_{H^{s,\theta}}^2.$$

Fortunately the hard work for null forms of type $Q_{\alpha,\beta}$ in two dimensions is already carried out by Zhou [1997]. His proof is done using spaces $N^{s+1,\theta}$ with the norm given by⁶

$$\|u\|_{N^{s+1,\theta}} = \|\Lambda_+^{s+1} \Lambda_-^\theta u\|_{L^2}. \tag{5-9}$$

In his work $\theta = s + \frac{1}{2}$. We state Zhou’s result.

Theorem [Zhou 1997]. *Consider in \mathbb{R}^{2+1} the space-time norms (5-9) and functions φ, ψ defined on \mathbb{R}^{2+1} . The estimates*

$$\|Q_{\alpha\beta}(\varphi, \psi)\|_{N^{s,s-1/2}} \lesssim \|\varphi\|_{N^{s+1,s+1/2}} \|\psi\|_{N^{s+1,s+1/2}}$$

hold for any $\frac{1}{4} < s < \frac{1}{2}$.

Our iteration is done using spaces $\mathcal{H}^{s+1,\theta}$. Inspection of Zhou’s proof shows that it could be easily modified to be placed in the context of $\mathcal{H}^{s+1,\theta}$ spaces. However, even though our auxiliary functions’ iterates u_j and v_j belong to $\mathcal{H}^{s+1,\theta}$, from (4-24) we only have

$$df \in H^{s,\theta} \Rightarrow \|\Lambda^s \Lambda_-^\theta Df\|_{L^2(\mathbb{R}^{2+1})} < \infty, \tag{5-10}$$

but again inspection of Zhou’s proof shows we can still handle $Q_{12}(f, f)$ given only that (5-10) holds. Zhou’s proof works for $\frac{1}{4} < s < \frac{1}{2}$. However, it motivates an alternate proof that uses $\mathcal{H}^{s+1,\theta}$ and works for all values of $s > \frac{1}{4}$. The proof is closely related to the original proof in [Zhou 1997], but on the surface it seems more concise. The reason for this is that we use Theorem F from [Klainerman and Selberg 2002], which involves all the technicalities. See [Czubak 2008] for the details.

5B2. Null forms: proof of estimate (5-5). We need

$$\|[\partial_j f, \phi]\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|\partial_j f\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}, \quad j = 1, 2.$$

However, analysis of the first iterate shows that for this estimate to hold we need $s > \frac{3}{4}$, so we need to work a little bit harder, and use (4-24)⁷

$$[\partial_j f, \phi] = -\frac{i}{4}[R_j(\partial_t u + iDu - \partial_t v + iDv), \partial_t u + iDu + \partial_t v - iDv]. \tag{5-11}$$

If we use the bilinearity of the bracket, we can group (5-11) by terms involving brackets of u with itself, v with itself, and then also by the terms that are mixed, that is, involve both u and v . So we have

$$\begin{aligned} 4i[\partial_j f, \phi] = & [R_j(\partial_t + iD)u, (\partial_t + iD)u] - [R_j(\partial_t - iD)v, (\partial_t - iD)v] \\ & + [R_j(\partial_t + iD)u, (\partial_t - iD)v] - [R_j(\partial_t - iD)v, (\partial_t + iD)u]. \end{aligned}$$

Since u and v are matrix-valued and do not commute we need to combine the last two brackets to take advantage of a null form structure. This corresponds to (5-13) below (note the plus sign in the formula).

The needed estimates are contained in the following theorem.

⁶See [Selberg 1999, Section 3.5] for a comparison with $\mathcal{H}^{s+1,\theta}$ spaces.

⁷The obvious way is to just substitute for ϕ and leave $\partial_j f$ the same, but it is an exercise to see that this does not work — for several reasons!

Theorem 5.1. *Suppose $s > \frac{1}{4}$ and $\frac{3}{4} - \frac{\varepsilon}{2} < \theta \leq s + \frac{1}{2}$, with $\theta < 1 - \varepsilon$ and $0 \leq \varepsilon < \min\left(2s - \frac{1}{2}, \frac{1}{2}\right)$. Let $Q(\varphi, \psi)$ be given by*

$$Q(\varphi, \psi) = (\partial_t \pm iD)R_j\varphi(\partial_t \pm iD)\psi - (\partial_t \pm iD)\varphi(\partial_t \pm iD)R_j\psi, \quad \text{or} \quad (5-12)$$

$$Q(\varphi, \psi) = (\partial_t \pm iD)R_j\varphi(\partial_t \mp iD)\psi + (\partial_t \pm iD)\varphi(\partial_t \mp iD)R_j\psi. \quad (5-13)$$

Then

$$Q(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\varepsilon} \quad (5-14)$$

or, equivalently,

$$\|Q(\varphi, \psi)\|_{H^{s,\theta-1+\varepsilon}} \lesssim \|\varphi\|_{\mathcal{H}^{s+1,\theta}} \|\psi\|_{\mathcal{H}^{s+1,\theta}}. \quad (5-15)$$

Proof. We show the details only for

$$(\partial_t + iD)R_j\varphi(\partial_t - iD)\psi + (\partial_t + iD)\varphi(\partial_t - iD)R_j\psi,$$

as the rest follows similarly. Observe that the symbol of Q is then

$$q(\tau, \zeta, \lambda, \eta) = \left(\frac{\zeta_j}{|\zeta|} + \frac{\eta_j}{|\eta|}\right)(\tau + |\zeta|)(\lambda - |\eta|).$$

Suppose $\tau\lambda \geq 0$, then

$$q \leq 2|(\tau + |\zeta|)(\lambda - |\eta|)| \leq \begin{cases} 2|\tau + |\zeta||\lambda - |\eta|| & \text{if } \tau, \lambda \geq 0, \\ 2|\tau - |\zeta||\eta| + |\lambda| & \text{if } \tau, \lambda \leq 0. \end{cases}$$

It follows

$$\iint_{\tau\lambda \geq 0} |\Lambda^s \Lambda_-^{\theta-1+\varepsilon} Q(\varphi, \psi)|^2 d\tau d\xi \lesssim \|D_+\varphi D_-\psi\|_{H^{s,\theta-1+\varepsilon}}^2 + \|D_-\varphi D_+\psi\|_{H^{s,\theta-1+\varepsilon}}^2, \quad (5-16)$$

and the estimate follows by [Theorem 5.2](#) below.

Suppose $\tau\lambda < 0$. If we break down the computations into the region

$$\{(\tau, \zeta), (\lambda, \eta) : |\tau| \geq 2|\zeta| \text{ or } |\lambda| \geq 2|\eta|\} \quad (5-17)$$

and its complement, then in the region [\(5-17\)](#), we bound q by

$$q \leq 2(|\tau| + |\zeta|)(|\lambda| + |\eta|),$$

since there we do not need any special structure. (This is a simple exercise in this region; see [\[Czubak 2008, Appendix B\]](#).)

In the complementary region, we have

$$q \leq 4|\zeta||\eta| \left| \frac{\zeta_i}{|\zeta|} + \frac{\eta_i}{|\eta|} \right|,$$

which is the absolute value of the symbol of the null form Q_{ij} in the first iterate. It has received a lot of attention, but we have not seen a reference where it was discussed in a context other than that of the initial data in $H^{s+1} \times H^s$. This may be because it has not come up as a nonlinearity before, and/or because it can be handled in the same way as the null form Q_{ij} . The details are in [\[Czubak 2008\]](#). \square

Now we prove an estimate needed to show (5-16) is bounded by the square of the right side of (5-15).

Theorem 5.2. *Let $s > 0$, $\max(\frac{1}{2}, 1 - s) < \theta < 1$, and $0 \leq \varepsilon \leq 1 - \theta$. Then*

$$\|D_+ \varphi D_- \psi\|_{H^{s, \theta-1+\varepsilon}} \lesssim \|\varphi\|_{\mathcal{H}^{s+1, \theta}} \|\psi\|_{\mathcal{H}^{s+1, \theta}}.$$

Proof. We would like to show that $\|\Lambda^s \Lambda_-^{\theta-1+\varepsilon} (D_+ \varphi D_- \psi)\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|\varphi\|_{\mathcal{H}^{s+1, \theta}} \|\psi\|_{\mathcal{H}^{s+1, \theta}}$. This follows from showing that

$$H^{s, \theta} \cdot \mathcal{H}^{s+1, \theta-1} \hookrightarrow H^{s, \theta-1+\varepsilon},$$

which by the product rule⁸ for the operator Λ^s in turn follows from

$$H^{0, \theta} \cdot \mathcal{H}^{s+1, \theta-1} \hookrightarrow H^{0, \theta-1+\varepsilon}, \quad H^{s, \theta} \cdot \mathcal{H}^{1, \theta-1} \hookrightarrow H^{0, \theta-1+\varepsilon}.$$

It is easy to check that $\mathcal{H}^{s+1, \theta-1} \hookrightarrow H^{s+1+\theta-1, 0}$ and $\mathcal{H}^{1, \theta-1} \hookrightarrow H^{\theta, 0}$, so we just need to show

$$H^{0, \theta} \cdot H^{s+\theta, 0} \hookrightarrow H^{0, \theta-1+\varepsilon}, \quad H^{s, \theta} \cdot H^{\theta, 0} \hookrightarrow H^{0, \theta-1+\varepsilon},$$

which are weaker than

$$H^{0, \theta} \cdot H^{s+\theta, 0} \hookrightarrow L^2, \quad H^{s, \theta} \cdot H^{\theta, 0} \hookrightarrow L^2,$$

but those follow from the Klainerman–Selberg estimate (2-6) as long as $s + \theta > 1$, which holds by the conditions we impose on s and θ . \square

An alternate approach could be to follow the set up used by [Klainerman and Machedon 1995] and estimate the integral directly.

5B3. Elliptic piece: proof of estimate (5-6). Recall we wish to show

$$\|A_0 w\|_{H^{s, \theta-1+\varepsilon}} \lesssim \|A_0\| \|w\|_{H^{s, \theta}}. \tag{5-18}$$

We need this estimate during our iteration, so we really mean $A_{0, j}$, but for simplicity we omit writing of the index j . Now we choose a norm for A_0 to be anything that makes (5-18) possible to establish. This results in

$$\|A_0\| = \|A_0\|_{L_t^{\tilde{p}} L_x^\infty} + \|D^s A_0\|_{L_t^p L_x^q},$$

where

$$\tilde{p} \in \left(1 - 2s, \frac{1}{2}\right), \quad \frac{2}{p} = 1 - \frac{1}{q}, \quad \max\left(\frac{1}{3}(1 - 2s), \frac{s}{2}\right) < \frac{1}{q} < \frac{2}{3}s. \tag{5-19}$$

For now we assume we can show $A_0 \in L_t^{\tilde{p}} L_x^\infty \cap L_t^p \dot{W}_x^{s, q}$ and delay the proof to Section 5C, where the reasons for our choices of \tilde{p} , p , q should become clear. We start by using $\theta - 1 + \varepsilon < 0$:

$$\|A_0 w\|_{H^{s, \theta-1+\varepsilon}} \leq \|\Lambda^s (A_0 w)\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|A_0 w\|_{L^2(\mathbb{R}^{2+1})} + \|D^s (A_0 w)\|_{L^2(\mathbb{R}^{2+1})}. \tag{5-20}$$

For the first term, by Hölder’s inequality,

$$\begin{aligned} \|A_0 w\|_{L^2(\mathbb{R}^{2+1})} &\leq \|A_0\|_{L_t^{\tilde{p}} L_x^\infty} \|w\|_{L_t^{\tilde{p}'} L_x^2}, \quad \text{for } \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = \frac{1}{2}, \quad p \text{ as in (5-19)} \\ &\lesssim \|A_0\| \|w\|_{H^{0, \theta}}, \quad \text{by (2-4)} \\ &\leq \|A_0\| \|w\|_{H^{s, \theta}}. \end{aligned} \tag{5-21}$$

⁸On L^2 this is very easy to establish using triangle inequality. See [Klainerman and Selberg 2002].

We bound the second term in (5-20) by

$$\|D^s(A_0 w)\|_{L^2(\mathbb{R}^{2+1})} \lesssim \underbrace{\|A_0\|_{L_t^{\tilde{p}} L_x^\infty} \|D^s w\|_{L_t^{\tilde{p}'} L_x^2}}_I + \underbrace{\|D^s A_0\|_{L_t^p L_x^q} \|w\|_{L_t^{p'} L_x^{q'}}}_{II},$$

where

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{2} = \frac{1}{q} + \frac{1}{q'}$$

and p, q are as in (5-19) and \tilde{p} as in (5-21). I is handled similarly to (5-21) as follows. Apply (2-4) with $u = D^s w$ to obtain

$$I \lesssim \|A_0\| \|D^s w\|_{H^{0,\theta}} \leq \|A_0\| \|w\|_{H^{s,\theta}}. \tag{5-22}$$

We now consider II . By the choices of p, q , the Klainerman–Selberg estimate (2-5) applies (see the discussion in Section 5D for an explanation) and gives

$$II \leq \|A_0\| \|w\|_{L_t^{p'} L_x^{q'}} \lesssim \|A_0\| \|w\|_{H^{1-2/q'-1/p',\theta}}. \tag{5-23}$$

From (5-19) we also have

$$II \lesssim \|A_0\| \|w\|_{H^{1-2/q'-1/p',\theta}} \lesssim \|A_0\| \|w\|_{H^{s,\theta}}. \tag{5-24}$$

and (5-18) follows now from (5-21), (5-22) and (5-24).

Remark 5.1. The above proof illustrates other difficulties due to working in two dimensions. Initially, we wanted to follow the proof of estimate (38) in [Selberg 2002a], and just use the $\|\Lambda^s A_0\|_{L_t^p L_x^q}$ norm. Unfortunately in two dimensions, the condition $sq > 2$ needed to show that $A_0 \in L_t^p L_x^\infty$ is disjoint from the conditions needed to use Klainerman–Tataru estimate (2-3) and establish that $\Lambda^s A_0 \in L_t^p L_x^q$ in the first place. This resulted in the $L_t^{\tilde{p}} L_x^\infty \cap L_t^p \dot{W}_x^{s,q}$ space above and also having to employ the Klainerman–Selberg estimate (2-5), which was not needed for the proof of [Selberg 2002a, estimate (38)].

5C. Elliptic regularity: estimates for A_0 . Here we present a variety of a priori estimates for the non-dynamical variable A_0 . At each point we could add the index j to A_0, df and ϕ . Therefore the presentation also applies to the iterates $A_{0,j}$. It is an exercise to show that the estimates we obtain here are enough to solve for $A_{0,j}$ at each step as well as to close the iteration for A_0 . Let A_0 solve

$$\Delta A_0 = -d^*[A_0, *df] - d^*[df, \phi] = -\partial_1[A_0, \partial_2 f] + \partial_2[A_0, \partial_1 f] + \partial_j[\partial_j f, \phi].$$

There is a wide range of estimates A_0 satisfies. Nevertheless, the two spatial dimensions limit our “range of motion.” For example, it does not seem possible to place $A_0(t)$ in L^2 . We state the general results and only show the cases we need to prove $A_0 \in L_t^{\tilde{p}} L_x^\infty \cap L_t^p \dot{W}_x^{s,q}$ as required in the last section. The rest of the cases can be found in [Czubak 2008]. We add that the proofs of both of the following theorems were originally inspired by the proof of estimate (45) in [Selberg 2002a]. We start with the homogeneous estimates.

Theorem 5.3. *Let $s > 0$, and let $0 \leq a \leq s + 1$ be given, and suppose $1 \leq p \leq \infty$ and $1 < q < \infty$ satisfy*

$$\max\left(\frac{1}{3}(1 + 2a - 4s), \frac{1}{2}(1 + a - 4s), \frac{1}{2} \min(a, 1)\right) < \frac{1}{q} < \frac{1+a}{2}, \tag{5-25}$$

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} \leq \frac{1}{2}\left(1 - \frac{1}{q}\right), \quad \frac{1}{p} < \left(1 - \frac{2}{q} + a\right). \tag{5-26}$$

(i) If $0 \leq a \leq 1$ and the $H^{s,\theta}$ norm of ∇f is sufficiently small, then $A_0 \in L_t^p \dot{W}_x^{a,q}$ and we have the estimate

$$\|A_0\|_{L_t^p \dot{W}_x^{a,q}} \lesssim \|\phi\|_{H^{s,\theta}} \|\nabla f\|_{H^{s,\theta}}. \quad (5-27)$$

(ii) If $1 < a \leq s+1$ and $A_0 \in L_t^p L_x^{(1/q-1/2)^{-1}}$, then $A_0 \in L_t^p \dot{W}_x^{a,q}$ and we have

$$\|A_0\|_{L_t^p \dot{W}_x^{a,q}} \lesssim (\|A_0\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} + \|\phi\|_{H^{s,\theta}}) \|\nabla f\|_{H^{s,\theta}}. \quad (5-28)$$

Corollary 5.4. Let $s > 0$, then $A_0 \in C_b(I : \dot{H}_x^a)$, where

$$0 < a \leq \begin{cases} 2s & \text{if } 0 < s \leq 1, \\ 1+s & \text{if } 1 < s. \end{cases}$$

Proof of Corollary 5.4. Suppose $0 < s < \frac{1}{2}$. Then use part (i) of the theorem with $q = 2$ and $p = \infty$ to obtain $A_0 \in L_t^\infty \dot{H}_x^a$ for $a \leq 2s$. A_0 continuous as a function of time easily follows from a contraction argument in $C_b(I : \dot{H}_x^a)$ using $L_t^\infty \dot{H}_x^a$ estimates. The case $s \geq \frac{1}{2}$ is considered in [Czubak 2008]. \square

So far we just need $s > 0$ in order to make the estimates work. The requirement for $s > \frac{1}{4}$ does not come in until we start looking at the nonhomogeneous spaces, where also the range of p and q is smaller. However, we can distinguish two cases, $aq < 2$ and $aq > 2$.

Theorem 5.5. Let $s > 0$, and suppose the $H^{s,\theta}$ norm of ∇f is sufficiently small.

(i) If $aq < 2$ for $0 < a < \min(2s, 1)$ and if p and q satisfy

$$\max\left(\frac{1}{2} + a - 2s, \frac{a}{2}\right) < \frac{1}{q} < \frac{1}{2}, \quad (5-29)$$

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} < \frac{1}{2} - \frac{1}{q}, \quad (5-30)$$

then $A_0 \in L_t^p W_x^{a,q}$ and we have the estimate

$$\|A_0\|_{L_t^p W_x^{a,q}} \lesssim \|\phi\|_{H^{s,\theta}} \|\nabla f\|_{H^{s,\theta}}. \quad (5-31)$$

(ii) If $aq > 2$, we need $s > \frac{1}{4}$ and $0 < a < \min(4s-1, 1+s, 2s)$. Suppose p and q also satisfy

$$\max\left(\frac{a-s}{2}, \frac{1}{2} + a - 2s\right) < \frac{1}{q} < \frac{1}{2} \min(a, 1), \quad (5-32)$$

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} < \frac{1}{2} - \frac{1}{q}; \quad (5-33)$$

then $A_0 \in L_t^p W_x^{a,q}$ and we have the estimate

$$\|A_0\|_{L_t^p W_x^{a,q}} \lesssim \|\phi\|_{H^{s,\theta}} \|\nabla f\|_{H^{s,\theta}}. \quad (5-34)$$

Corollary 5.6. *If $s > \frac{1}{4}$ and the $H^{s,\theta}$ norm of ∇f is sufficiently small, we have in particular $A_0 \in L_t^p L_x^\infty$ for p satisfying*

$$1 - 2s < \frac{1}{p} < \frac{1}{2}, \quad (5-35)$$

and we have the estimate

$$\|A_0\|_{L_t^p L_x^\infty} \lesssim \|\phi\|_{H^{s,\theta}} \|\nabla f\|_{H^{s,\theta}}. \quad (5-36)$$

Proof of Corollary 5.6. For each $p \in (1 - 2s, \frac{1}{2})$ we can find some a and q , which satisfy the conditions of [Theorem 5.5](#), part (ii). The corollary then follows from the Sobolev embedding: $W^{a,q}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for $aq > 2$. \square

Remark 5.2. Estimate (5-34) is where $s > \frac{1}{4}$ is needed. Conditions on $\frac{1}{p}$ in (5-33) are needed so we can use (below) the Klainerman–Tataru estimate (2-3). In order to be able to choose such $\frac{1}{p}$, obviously $1 - \frac{2}{q} + a - 2s$ must be strictly less than $\frac{1}{2} - \frac{1}{q}$. This forces $\frac{1}{q}$ to be strictly greater than $\frac{1}{2} + a - 2s$. We also need $aq > 2$ to use the Sobolev embedding in [Corollary 5.6](#), so if we want to be able to find q between $\frac{1}{2} + a - 2s$ and $\frac{a}{2}$, a is forced to be strictly less than $4s - 1$. Therefore s must be greater than $\frac{1}{4}$. See below for another instance of requiring $s > \frac{1}{4}$.

5D. Proof of estimates needed in 5B3. Recall we would like to show $A_0 \in L_t^{\tilde{p}} L_x^\infty \cap L_t^p \dot{W}_x^{s,q}$. Therefore, we are interested in part (i) of [Theorem 5.3](#) and part ii) in [Theorem 5.5](#), so we can conclude [Corollary 5.6](#). Moreover, we need a specific case of part (i) in [Theorem 5.3](#), because we need $A_0 \in L_t^p \dot{W}_x^{s,q}$, where p, q in addition satisfy

$$1 - \frac{2}{p} \leq \frac{1}{q} < \frac{1}{2} \quad \text{and} \quad \frac{2}{q} - \frac{1}{2} + \frac{1}{p} \leq s, \quad (5-37)$$

so we can use the embedding

$$H^{s,\theta} \hookrightarrow H^{1-(1-2/q)-(1/2-1/p),\theta} \hookrightarrow L_t^{(1/2-1/p)^{-1}} L_x^{(1/2-1/q)^{-1}} \quad (5-38)$$

in (5-23) and (5-24). When we put (5-37) together with (5-25) and (5-26) with $a = s$, we obtain the second line of (5-19), namely

$$\frac{2}{p} = 1 - \frac{1}{q}, \quad \max\left(\frac{1}{3}(1 - 2s), \frac{s}{2}\right) < \frac{1}{q} < \frac{2}{3}s. \quad (5-39)$$

Remark 5.3. Observe that in order to be able to find such q we must have $s > \frac{1}{4}$.

Consider

$$\begin{aligned} \|A_0\|_{L_t^p \dot{W}_x^{s,q}} &= \|\Delta^{-1}(d^*[A_0, *df] + d^*[df, \phi])\|_{L_t^p \dot{W}_x^{s,q}} \\ &\lesssim \|D^{-1}(A_0 \nabla f)\|_{L_t^p \dot{W}_x^{s,q}} + \|D^{-1}(\nabla f \phi)\|_{L_t^p \dot{W}_x^{s,q}} \\ &\lesssim \|D^{s-1}(A_0 \nabla f)\|_{L_t^p L_x^q} + \|D^{s-1}(\nabla f \phi)\|_{L_t^p L_x^q} \\ &\lesssim \|A_0 \nabla f\|_{L_t^p L_x^r} + \|D^{s-1}(\nabla f \phi)\|_{L_t^p L_x^q}, \end{aligned} \quad (5-40)$$

where we use the Sobolev embedding with $\frac{1}{q} = \frac{1}{r} - \frac{1-s}{2}$. The latter term is bounded by $\|\nabla f\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}$ using the Klainerman–Tataru estimate (2-3), whose application we discuss in the section below. For the former we use $\frac{1}{r} = \frac{1}{q} + \frac{1-s}{2} = (\frac{1}{q} - \frac{s}{2}) + \frac{1}{2}$:

$$\|A_0 \nabla f\|_{L_t^p L_x^r} \leq \|A_0\|_{L_t^p L_x^{(1/q-s/2)^{-1}}} \|\nabla f\|_{L_t^\infty L_x^2} \lesssim \|A_0\|_{L_t^p \dot{W}_x^{s,q}} \|\nabla f\|_{H^{s,\theta}}. \quad (5-41)$$

Then if the $H^{s,\theta}$ norm of ∇f is sufficiently small, we obtain

$$\|A_0\|_{L_t^p \dot{W}_x^{s,q}} \lesssim \|\nabla f\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}, \tag{5-42}$$

as needed.

For the nonhomogeneous estimate, since here $\frac{1}{4} < s < \frac{1}{2}$ the upper bound for a is simply $4s - 1$. In addition, for our purposes right now it suffices to show the estimate for one particular a . Therefore we set $0 < a < \min(s, 4s - 1)$ for $\frac{1}{4} < s < \frac{1}{2}$, and we let p, q satisfy (5-32) and (5-33). We have

$$\begin{aligned} \|A_0\|_{L_t^p W_x^{a,q}} &\lesssim \|D^{-1}(A_0 \nabla f)\|_{L_t^p W_x^{a,q}} + \|D^{-1}(\nabla f \phi)\|_{L_t^p W_x^{a,q}} \\ &\lesssim \|D^{-1}(A_0 \nabla f)\|_{L_t^p L_x^q} + \|D^{-1}(\nabla f \phi)\|_{L_t^p L_x^q} + \|D^{a-1}(A_0 \nabla f)\|_{L_t^p L_x^q} + \|D^{a-1}(\nabla f \phi)\|_{L_t^p L_x^q}. \end{aligned} \tag{5-43}$$

The Klainerman–Tataru estimate (2-3) handles the second and the last term (see below). Consider the first term:

$$\begin{aligned} \|D^{-1}(A_0 \nabla f)\|_{L_t^p L_x^q} &\lesssim \|A_0 \nabla f\|_{L_t^p L_x^r}, && \text{for } \frac{1}{q} = \frac{1}{r} - \frac{1}{2}, \\ &\leq \|A_0\|_{L_t^p L_x^q} \|\nabla f\|_{L_t^\infty L_x^2} \\ &\leq \|A_0\|_{L_t^p W_x^{a,q}} \|\nabla f\|_{H^{s,\theta}}. \end{aligned} \tag{5-44}$$

For the third term we have

$$\begin{aligned} \|D^{a-1}(A_0 \nabla f)\|_{L_t^p L_x^q} &\lesssim \|A_0 \nabla f\|_{L_t^p L_x^r}, && \text{for } \frac{1}{q} = \frac{1}{r} - \frac{1-a}{2} \\ &\lesssim \|A_0\|_{L_t^p L_x^q} \|D^a \nabla f\|_{L_t^\infty L_x^2}, && \text{for } \frac{1}{r} = \frac{1}{q} + \left(\frac{1}{2} - \frac{a}{2}\right) \\ &\lesssim \|A_0\|_{L_t^p W_x^{a,q}} \|\nabla f\|_{H^{s,\theta}}, \end{aligned}$$

Then as before, this completes the proof if the $H^{s,\theta}$ norm of ∇f is sufficiently small.

Applying the Klainerman–Tataru theorem. We said that several of the above estimates follow from the Klainerman–Tataru estimate (2-3). We need to check that this is in fact the case. We begin by stating the theorem. We state it for two dimensions only, and as it is given in [Klainerman and Selberg 2002] (the original result holds for $n \geq 2$).

Theorem [Klainerman and Tataru 1999]. *Let $1 \leq p \leq \infty, 1 \leq q < \infty$. Assume that*

$$\frac{1}{p} \leq \frac{1}{2} \left(1 - \frac{1}{q}\right), \tag{5-45}$$

$$0 < \sigma < 2 \left(1 - \frac{1}{q} - \frac{1}{p}\right), \tag{5-46}$$

$$s_1, s_2 < 1 - \frac{1}{q} - \frac{1}{2p}, \tag{5-47}$$

$$s_1 + s_2 + \sigma = 2 \left(1 - \frac{1}{q} - \frac{1}{2p}\right). \tag{5-48}$$

Then

$$\|D^{-\sigma}(uv)\|_{L_t^p L_x^q(\mathbb{R}^2)} \lesssim \|u\|_{H^{s_1,\theta}} \|v\|_{H^{s_2,\theta}},$$

provided $\theta > \frac{1}{2}$.

The first time we use the theorem is in (5-40) for the term $\|D^{s-1}(\nabla f\phi)\|_{L_t^p L_x^q}$. Note that $\sigma = 1 - s$. Clearly $1 \leq p \leq \infty$, $1 \leq q < \infty$. Next by (5-39) $\frac{2}{p} = 1 - \frac{1}{q}$, so (5-45) holds. Since $s < \frac{1}{2}$, $\sigma > 0$, and we can see (5-46) holds when we substitute $\frac{1}{2} - \frac{1}{2q}$ for $\frac{1}{p}$ in the right side and use $\frac{1}{q} < \frac{2}{3}s$. Next we let $s_1 = s_2$ and with $\sigma = 1 - s > 0$, (5-48) implies (5-47), so we only check (5-48). To that end we must be able to choose s_1 so that

$$2s_1 = 1 - \frac{2}{q} - \frac{1}{p} + s \leq 2s,$$

which is equivalent to our condition on p and one of the lower bounds on $\frac{1}{q}$.

The next place we use the theorem is in (5-43) for $\|D^{-1}(\nabla f\phi)\|_{L_t^p L_x^q}$, $\|D^{a-1}(\nabla f\phi)\|_{L_t^p L_x^q}$, where p and q are as in (5-32) and (5-33) with $0 < a < \min(s, 4s - 1) < 1$. Then for $\sigma = 1$, by the right side of (5-33), (5-46) holds and implies (5-45). Note, since (5-46) is true with $\sigma = 1$, it is true with $\sigma = 1 - a$. Next, for $\sigma = 1$ (5-48) gives (5-47) and also for $\sigma = 1 - a$ as long as $0 < a < 1$. So again it is sufficient to see we can have s_1 defined by (5-48) such that $s_1 \leq s$, but for $\sigma = 1 - a$ that follows from the left side of (5-33), and shows we can find it for $\sigma = 1$ as well.

Acknowledgment

The author expresses deep gratitude to her thesis advisor Karen Uhlenbeck for her time, many helpful discussions, and in particular for suggesting the problem with the reformulation (4-12)–(4-13). The author also thanks the referees for many insightful comments and their careful reading of the manuscript.

References

- [Ablowitz et al. 2003] M. J. Ablowitz, S. Chakravarty, and R. G. Halburd, “Integrable systems and reductions of the self-dual Yang–Mills equations”, *J. Math. Phys.* **44**:8 (2003), 3147–3173. [MR 2004h:70034](#) [Zbl 1062.70050](#)
- [Atiyah and Hitchin 1988] M. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, 1988. [MR 89k:53067](#) [Zbl 0671.53001](#)
- [Czubak 2008] M. Czubak, *Well-posedness for the space-time monopole equation and Ward wave map*, Ph.D. thesis, University of Texas, Austin, 2008.
- [Czubak and Uhlenbeck \geq 2010] M. Czubak and K. Uhlenbeck, “On the existence of Coulomb gauges”, In preparation.
- [Dai et al. 2006] B. Dai, C.-L. Terng, and K. Uhlenbeck, “On the space-time monopole equation”, pp. 1–30 in *Surveys in differential geometry*, edited by S.-T. Yau, Surv. Differ. Geom. **10**, Int. Press, Somerville, MA, 2006. [MR 2009f:53033](#) [Zbl 1157.53016](#)
- [Dell’Antonio and Zwanziger 1991] G. Dell’Antonio and D. Zwanziger, “Every gauge orbit passes inside the Gribov horizon”, *Comm. Math. Phys.* **138**:2 (1991), 291–299. [MR 92i:58029](#) [Zbl 0726.53067](#)
- [Dirac 1931] P. A. M. Dirac, “Quantised singularities in the electromagnetic field”, *Proceedings of the Royal Society of London, Series A* **133**:821 (1931), 60–72.
- [Foschi and Klainerman 2000] D. Foschi and S. Klainerman, “Bilinear space-time estimates for homogeneous wave equations”, *Ann. Sci. École Norm. Sup. (4)* **33**:2 (2000), 211–274. [MR 2001g:35145](#) [Zbl 0959.35107](#)
- [Jaffe and Taubes 1980] A. Jaffe and C. Taubes, *Vortices and monopoles: Structure of static gauge theories*, Progress in Physics **2**, Birkhäuser, Mass., 1980. [MR 82m:81051](#) [Zbl 0457.53034](#)
- [Klainerman 1984] S. Klainerman, “Long time behaviour of solutions to nonlinear wave equations”, pp. 1209–1215 in *Proceedings of the International Congress of Mathematicians* (Warsaw, 1983), vol. 2, PWN, 1984. [MR 804771](#) [Zbl 0581.35052](#)
- [Klainerman and Machedon 1993] S. Klainerman and M. Machedon, “Space-time estimates for null forms and the local existence theorem”, *Comm. Pure Appl. Math.* **46**:9 (1993), 1221–1268. [MR 94h:35137](#) [Zbl 0803.35095](#)

- [Klainerman and Machedon 1995] S. Klainerman and M. Machedon, “Smoothing estimates for null forms and applications”, *Duke Math. J.* **81**:1 (1995), 99–133. [MR 97h:35022](#) [Zbl 0909.35094](#)
- [Klainerman and Machedon 1996] S. Klainerman and M. Machedon, “Estimates for null forms and the spaces $H_{s,\delta}$ ”, *Internat. Math. Res. Notices* **17** (1996), 853–865. [MR 98j:46028](#) [Zbl 0909.35095](#)
- [Klainerman and Selberg 2002] S. Klainerman and S. Selberg, “Bilinear estimates and applications to nonlinear wave equations”, *Commun. Contemp. Math.* **4**:2 (2002), 223–295. [MR 2003d:35182](#) [Zbl 1146.35389](#)
- [Klainerman and Tataru 1999] S. Klainerman and D. Tataru, “On the optimal local regularity for Yang–Mills equations in \mathbf{R}^{4+1} ”, *J. Amer. Math. Soc.* **12**:1 (1999), 93–116. [MR 2000c:58052](#) [Zbl 0924.58010](#)
- [Lindblad 1996] H. Lindblad, “Counterexamples to local existence for semi-linear wave equations”, *Amer. J. Math.* **118**:1 (1996), 1–16. [MR 97b:35124](#) [Zbl 0855.35080](#)
- [Mason and Sparling 1989] L. J. Mason and G. A. J. Sparling, “Nonlinear Schrödinger and Korteweg–de Vries are reductions of self-dual Yang–Mills”, *Phys. Lett. A* **137**:1-2 (1989), 29–33. [MR 90d:58169](#)
- [Roe 1998] J. Roe, *Elliptic operators, topology and asymptotic methods*, 2nd ed., Pitman Research Notes in Mathematics Series **395**, Longman, Harlow, 1998. [MR 99m:58182](#) [Zbl 0919.58060](#)
- [Selberg 1999] S. Selberg, *Multilinear spacetime estimates and applications to local existence theory for nonlinear wave equations*, Ph.D. thesis, Princeton University, 1999.
- [Selberg 2002a] S. Selberg, “Almost optimal local well-posedness of the Maxwell–Klein–Gordon equations in 1 + 4 dimensions”, *Comm. Partial Differential Equations* **27**:5-6 (2002), 1183–1227. [MR 2003f:35247](#) [Zbl 1013.35077](#)
- [Selberg 2002b] S. Selberg, “On an estimate for the wave equation and applications to nonlinear problems”, *Differential Integral Equations* **15**:2 (2002), 213–236. [MR 2002h:35204](#) [Zbl 1032.35121](#)
- [Tao 2006] T. Tao, *Nonlinear dispersive equations: Local and global analysis*, CBMS Regional Conference Series in Mathematics **106**, Amer. Math. Soc., Providence, 2006. [MR 2008i:35211](#) [Zbl 1106.35001](#)
- [Uhlenbeck 1992] K. Uhlenbeck, “On the connection between harmonic maps and the self-dual Yang–Mills and the sine-Gordon equations”, *J. Geom. Phys.* **8**:1-4 (1992), 283–316. [MR 93f:58050](#) [Zbl 0747.58025](#)
- [Ward 1985] R. S. Ward, “Integrable and solvable systems, and relations among them”, *Philos. Trans. Roy. Soc. London Ser. A* **315**:1533 (1985), 451–457. [MR 87e:58105](#) [Zbl 0579.35078](#)
- [Ward 1989] R. S. Ward, “Twistors in 2 + 1 dimensions”, *J. Math. Phys.* **30**:10 (1989), 2246–2251. [MR 90k:32089](#) [Zbl 0699.58065](#)
- [Ward 1999] R. S. Ward, “Two integrable systems related to hyperbolic monopoles”, *Asian J. Math.* **3**:1 (1999), 325–332. [MR 2000j:37098](#) [Zbl 0986.37061](#)
- [Zhou 1997] Y. Zhou, “Local existence with minimal regularity for nonlinear wave equations”, *Amer. J. Math.* **119**:3 (1997), 671–703. [MR 98e:35119](#) [Zbl 0881.35077](#)

Received 10 Feb 2009. Revised 15 Sep 2009. Accepted 21 Jan 2010.

MAGDALENA CZUBAK: czubak@math.toronto.edu

Department of Mathematics, University of Toronto, 40 Saint George Street, Toronto, Ontario M5S 2E4, Canada

<http://www.math.toronto.edu/czubak/>

Analysis & PDE

pjm.math.berkeley.edu/apde

EDITORS

EDITOR-IN-CHIEF

Maciej Zworski
University of California
Berkeley, USA

BOARD OF EDITORS

Michael Aizenman	Princeton University, USA aizenman@math.princeton.edu	Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr
Luis A. Caffarelli	University of Texas, USA caffarel@math.utexas.edu	Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Charles Fefferman	Princeton University, USA cf@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Nigel Higson	Pennsylvania State University, USA higson@math.psu.edu
Vaughan Jones	University of California, Berkeley, USA vfr@math.berkeley.edu	Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr
László Lempert	Purdue University, USA lempert@math.purdue.edu	Richard B. Melrose	Massachusetts Institute of Technology, USA rbm@math.mit.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu
Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de	Yuval Peres	University of California, Berkeley, USA peres@stat.berkeley.edu
Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Igor Rodnianski	Princeton University, USA irod@math.princeton.edu	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu	András Vasy	Stanford University, USA andras@math.stanford.edu
Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu	Steven Zelditch	Johns Hopkins University, USA szelditch@math.jhu.edu

PRODUCTION

apde@mathscipub.org

Paulo Ney de Souza, Production Manager

Sheila Newbery, Production Editor

Silvio Levy, Senior Production Editor


See inside back cover or pjm.math.berkeley.edu/apde for submission instructions.

The subscription price for 2010 is US \$120/year for the electronic version, and \$180/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer-review and production is managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
<http://www.mathscipub.org>

A NON-PROFIT CORPORATION

Typeset in L^AT_EX

Copyright ©2010 by Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 3 No. 2 2010

Polynomials with no zeros on the bidisk	109
GREG KNESE	
Local wellposedness for the 2+1-dimensional monopole equation	151
MAGDALENA CZUBAK	
Regularity of almost periodic modulo scaling solutions for mass-critical NLS and applications	175
DONG LI and XIAOYI ZHANG	
Estimées des noyaux de Green et de la chaleur sur les espaces symétriques	197
GILLES CARRON	
Lower bounds for resonances of infinite-area Riemann surfaces	207
DMITRY JAKOBSON and FRÉDÉRIC NAUD	