ANALYSIS & PDE

Volume 4 No. 2 2011

mathematical sciences publishers

Analysis & PDE

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in LaTeX

ANALYSIS AND PDE
Vol. 4, No. 2, 2011

ON THE AREA OF THE SYMMETRY ORBITS OF COSMOLOGICAL SPACETIMES WITH TOROIDAL OR HYPERBOLIC SYMMETRY

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We prove several global existence theorems for spacetimes with toroidal or hyperbolic symmetry with respect to a geometrically defined time. More specifically, we prove that generically, the maximal Cauchy development of T^2-symmetric initial data with positive cosmological constant $\Lambda > 0$, in the vacuum or with Vlasov matter, may be covered by a global areal foliation with the area of the symmetry orbits tending to zero in the contracting direction. We then prove the same result for surface symmetric spacetimes in the hyperbolic case with Vlasov matter and $\Lambda \geq 0$. In all cases, there is no restriction on the size of initial data.

1. Introduction

The study of the global Cauchy problem constitutes one of the main areas of research in mathematical relativity and is one of the most natural problems to investigate in view of the hyperbolicity of the Einstein equations and of the theorems concerning the local Cauchy problem [Fourès-Bruhat 1952; Choquet-Bruhat and Geroch 1969]. These theorems assert that given an appropriate initial data set, there exists a maximal solution of the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{1}$$

MSC2000: 83C05.
Keywords: Einstein equations, singularities, hyperbolic partial differential equations.

coupled if necessary to appropriate matter equations,[1] which is unique up to diffeomorphism in the class of globally hyperbolic spacetimes. We call this solution the *maximal Cauchy development* of the initial data. The global hyperbolicity assumption guarantees the domain of dependence property and is essential to the uniqueness statement.

The global Cauchy problem consists in understanding the global geometry of the maximal Cauchy development. A fundamental conjecture, known as *strong cosmic censorship*,[2] states that *the maximal Cauchy development of generic compact or asymptotically flat initial data is inextendible as a regular*[3] *Lorentzian manifold*. This can be thought of as a statement of uniqueness in a class of spacetimes not assumed a priori to be globally hyperbolic.

The expression "generic initial data" in the statement of the conjecture reflects the fact that there exist particular initial data for which the maximal Cauchy development fails to be inextendible. However, the extendibility property of the maximal Cauchy development for these particular initial data is expected to be dynamically unstable and, as we shall see below, this expectation has been verified in several cases. From the point of view of physics, uniqueness means predictability and thus, strong cosmic censorship asserts that, generically, general relativity is a deterministic theory in the same sense that Newtonian mechanics is deterministic.

1A. ***Areal foliations of T^2-symmetric and $k \leq 0$ surface-symmetric spacetimes.*** In full generality, the questions tied to the global Cauchy problem are not accessible with the current set of mathematical techniques. In order to make progress, one may try to look at simpler but connected problems, such as the study of the global Cauchy problem within certain classes of symmetries.

Following this approach, two classes of solutions arising from compact initial data with symmetry have been given much attention recently, the so-called *T^2-symmetric* and *surface-symmetric* spacetimes. The T^2-symmetric spacetimes constitute a class of solutions arising from initial data with spatial topology T^3 and admitting a torus action. They contain as special subcases the T^3-Gowdy spacetimes and the polarized T^2-symmetric spacetimes.[4] The surface-symmetric spacetimes constitute a class of solutions arising from initial data where the initial Riemannian 3-manifold is given by a doubly warped product $S^1 \times \mathcal{S}$, where $\mathcal{S}$ is a compact 2-surface of constant curvature k and such that the rest of the initial data is invariant under the local isometries of $\mathcal{S}$. By rescaling, k may be taken as being -1, 0 or $+1$ and the different cases are known as hyperbolic, plane[5] or spherical symmetry.

In the case of T^2-symmetric or $k \leq 0$ surface-symmetric spacetimes, the local geometry of the solution possesses the particular property that, unless the spacetime is flat, the symmetry orbits are either trapped or antitrapped, a feature which is shared by the spheres of symmetry in the black or white hole regions of a Schwarzschild solution with $m > 0$. If we denote by t the area of the symmetry orbits, this means that

[1] See [Choquet-Bruhat 1970; 1971] for the case of Vlasov matter.

[2] The conjecture was originally developed by R. Penrose [1979] and first formulated as a statement about the global geometry of the maximal Cauchy development in [Moncrief and Eardley 1981]. See also the presentation in [Christodoulou 1999].

[3] The regularity concerns here the degree of differentiability of the possible extensions and gives rise to different versions of the conjecture. For instance, the C^2 formulation of the conjecture is obtained by replacing "regular" with C^2 in the above statement of the conjecture.

[4] They also contain the even more special case of polarized T^3-Gowdy spacetimes.

[5] Note that the plane symmetric case is a special case of T^3-Gowdy polarized solutions.

the gradient of t is everywhere timelike and that t may be used as a time coordinate.[6] For the vacuum T^2-symmetric case with zero cosmological constant ($\Lambda = 0$), the existence of a global areal foliation where t takes value in (t_0, ∞) with $t_0 \geq 0$ was proven in [Berger et al. 1997]. The proof was then extended to the Vlasov case [Andréasson 1999; Andréasson et al. 2004] and to the case with $\Lambda > 0$ [Clausen and Isenberg 2007]. Similarly, the existence of a global areal foliation for the surface-symmetric case with $k = -1$, $\Lambda = 0$ and Vlasov matter[7] was proven in [Andréasson et al. 2003] and extended to the case with $\Lambda > 0$ in [Tchapnda and Rendall 2003; Tchapnda and Noutchegueme 2005].

It was soon realized that in the vacuum T^3-Gowdy case with $\Lambda = 0$, one has $t_0 = 0$ unless the spacetime is flat [Moncrief 1981; Chruściel 1990]. The natural question arose: *Is $t_0 = 0$ generically for all the possible cases?* The proofs that $t_0 = 0$ generically for T^2-symmetric spacetimes with $\Lambda = 0$, in the vacuum or with Vlasov matter, were given in [Isenberg and Weaver 2003] and [Weaver 2004]. It has also been proven that $t_0 = 0$ in the case of plane symmetric initial data with $\Lambda = 0$ and Vlasov matter as well as in the case of plane or hyperbolic symmetric initial data with $\Lambda \geq 0$ and Vlasov matter under an extra small data assumption [Rein 1996; Tchapnda 2004]. Moreover, the results for $k \leq 0$ surface-symmetric initial data have been extended to the Einstein–Vlasov-scalar field system [Tegankong and Rendall 2006].

1B. ***Strong cosmic censorship for T^2-symmetric or surface-symmetric spacetimes.*** One motivation for the study of the value of t_0 was the expectation that, in the cases were $t_0 = 0$, the curvature should in general blow up as t goes to 0, thus providing a proof of inextendibility (and thus of strong cosmic censorship) for these cases. Indeed, for vacuum T^3-Gowdy spacetimes with $\Lambda = 0$, first in the polarized case[8] and then for the full class, detailed asymptotic expansions were obtained and used in this sense to establish a proof of the C^2 formulation of the strong cosmic censorship conjecture [Chruściel et al. 1990; Ringström 2006; 2009].

While it seemed difficult to extend the analysis of the vacuum T^3-Gowdy spacetimes to the more general case of T^2-symmetric spacetimes, strong cosmic censorship was nonetheless proven for T^2-symmetric spacetimes with $\Lambda = 0$ in the presence of Vlasov matter [Dafermos and Rendall 2006]. The analysis starts with the remark that for T^2-symmetric or $k \leq 0$ surface-symmetric spacetimes, with or without Vlasov matter and with $\Lambda \geq 0$, the fact that t is unbounded implies inextendibility in the expanding direction because of the continuous extension of the Killing fields to possible Cauchy horizons [Dafermos and Rendall 2005]. Thus it is sufficient to study the contracting direction in order to complete the proof of strong cosmic censorship for these classes of spacetimes. The proof given in [Dafermos and Rendall 2006] relied on a rigidity of the possible Cauchy horizon, linked with the fact that $t_0 = 0$, and on the particular properties of the Vlasov equation. The assumption that $\Lambda = 0$ was necessary only as to ensure that $t_0 = 0$. Therefore the proof remained valid in the case where $\Lambda > 0$, if one added

[6]In particular, any nonflat T^2-symmetric or $k \leq 0$ surface-symmetric spacetime can be oriented by ∇t. With this choice of orientation, the future corresponds to the direction where t increases (expanding direction) and the past to the direction where t decreases (contracting direction).

[7]Note that, in the surface-symmetric case, a result analogous to Birkhoff's theorem applies, by which we mean that these spacetimes have no dynamical degree of freedom in the vacuum.

[8]Note that, in [Chruściel et al. 1990], strong cosmic censorship was also proved for polarized Gowdy spacetimes arising from initial data given on $S^2 \times S^1$, S^3 and $L(p, q)$.

the extra assumption that $t_0 = 0$. In [Smulevici 2008], we studied the remaining cases, namely the T^2-symmetric spacetimes with $\Lambda > 0$ and Vlasov matter for which $t_0 > 0$, and proved their inextendibility, thus completing a proof of strong cosmic censorship for T^2-symmetric spacetimes with $\Lambda \geq 0$ and Vlasov matter. In the same article, we proved that vacuum T^2-symmetric spacetimes with $\Lambda > 0$ and $t_0 > 0$ were also generically inextendible. Finally, in the surface-symmetric case with Vlasov matter, strong cosmic censorship was resolved in the affirmative for $k \leq 0$ and $\Lambda \geq 0$ and for $k = 1$ and $\Lambda = 0$, some obstructions remaining in the spherical case with $\Lambda > 0$, in particular the possible formation of Schwarzschild–de-Sitter or, even worse, extremal Schwarzschild–de-Sitter black holes [Dafermos and Rendall 2007].

1C. ***The past asymptotic value of*** t ***and the main theorems.*** The results of [Smulevici 2008], as well as the proof of inextendibility for the $k \leq 0$ surface-symmetric cases where $t_0 > 0$ contained in [Dafermos and Rendall 2007], gave satisfactory answers to the strong cosmic censorship conjecture. However, they did not address the question of the value of t_0. It is the subject of this article to resolve this question.

First, in Theorem 1 (see Section 5), we will extend the work of M. Weaver [2004] proving that *the maximal Cauchy development of* T^2*-symmetric initial data with* $\Lambda \geq 0$ *and nonvanishing Vlasov matter can be covered by global areal foliations with* t *going to zero in the contracting direction.* Thus $t_0 = 0$ for these spacetimes.

As often happens in these types of problems, the vacuum case is more difficult than the Vlasov case. This is already reflected in the fact that for vacuum T^2-symmetric spacetimes with $\Lambda > 0$, one can find special families of (nonflat) solutions for which $t_0 > 0$. That these solutions are indeed special is the content of Theorem 2 which states that *vacuum* T^2*-symmetric spacetimes with* $\Lambda \geq 0$ *for which* $t_0 > 0$ *are necessarily polarized*. Thus, generically, $t_0 = 0$ for vacuum T^2-symmetric spacetimes with $\Lambda \geq 0$.

Finally, we will show that the proof given for the T^2 case with Vlasov matter may be adapted to the hyperbolic case. We will obtain Theorem 3 which asserts that $t_0 = 0$ *for* $k \leq 0$ *surface-symmetric spacetimes with* $\Lambda \geq 0$ *and nonvanishing Vlasov matter*. Thus Theorem 3 asserts that the results of [Rein 1996; Tchapnda 2004] are true in general and do not require any smallness assumption. To summarize, we provide in Tables 1 and 2 a picture of the current status of the analysis of singularities for the T^2-symmetric and surface-symmetric spacetimes in the vacuum or with Vlasov matter.

1D. ***Outline.*** The outline of this article is as follows. We start in Section 2 with an introduction to the different classes of symmetry and present the classes of initial data that we will consider in the rest of the paper. In Section 3, we recall the existence and uniqueness of the maximal Cauchy development and in Section 4, we present the previous results concerning the global foliations of T^2-symmetric and $k \leq 0$ surface-symmetric spacetimes that we shall use as a starting point for our analysis. The statements of the theorems proved in this article then follow in Section 5. Before giving the proofs of the three theorems in sections 7, 8 and 9, it will be useful to describe the approach that we will take, especially for the proof of Theorem 2, and this is done in Section 6. We end this paper with some comments and open questions in Section 10. In Appendix A, we provide some information on the initial data sets of the Einstein and Einstein–Vlasov systems for the reader not familiar with this. In Appendix B, we very briefly describe a coordinate transformation for $k = -1$ surface symmetric spacetimes and finally in Appendix C, we recall the classical results that symmetric initial data lead to symmetric spacetimes.

T^3-Gowdy	vacuum	$\Lambda = 0$	$t_0 = 0$	[Moncrief 1981; Chruściel 1990]
		$\Lambda > 0$	$t_0 = 0$	Theorem 2
	Vlasov	$\Lambda = 0$	$t_0 = 0$	[Weaver 2004]
		$\Lambda > 0$	$t_0 = 0$	Theorem 1
T^2-symmetric	vacuum	$\Lambda = 0$	$t_0 = 0$	[Isenberg and Weaver 2003]
		$\Lambda > 0$	$t_0 = 0$	Theorem 2
	Vlasov	$\Lambda = 0$	$t_0 = 0$	[Weaver 2004]
		$\Lambda > 0$	$t_0 = 0$	Theorem 1
$k = 0$	Vlasov	$\Lambda = 0$	$t_0 = 0$	[Weaver 2004]
		$\Lambda > 0$	$t_0 = 0$	Theorem 3; see also [Tchapnda 2004] with small data
$k = -1$	Vlasov	$\Lambda = 0$	$t_0 = 0$	Theorem 3; see also [Rein 1996] with small data
		$\Lambda > 0$	$t_0 = 0$	Theorem 3; see also [Tchapnda 2004] with small data

Table 1. Value of t_0 for generic T^2-symmetric and $k \leq 0$ surface-symmetric spacetimes.

T^3-Gowdy	vacuum	$\Lambda = 0$	Holds [Chruściel et al. 1990; Ringström 2006; 2009]
		$\Lambda > 0$	Holds for cases with $t_0 > 0$ [Smulevici 2008]; open otherwise
	Vlasov	$\Lambda = 0$	Holds [Dafermos and Rendall 2006]
		$\Lambda > 0$	Holds [Dafermos and Rendall 2006; Smulevici 2008]
T^2-symmetric	vacuum	$\Lambda = 0$	Open
		$\Lambda > 0$	Holds for cases with $t_0 > 0$ [Smulevici 2008]; open otherwise
	Vlasov	$\Lambda = 0$	Holds [Dafermos and Rendall 2006]
		$\Lambda > 0$	Holds [Dafermos and Rendall 2006; Smulevici 2008]
$k = 0$	Vlasov	$\Lambda = 0$	Holds [Dafermos and Rendall 2007]
		$\Lambda > 0$	Holds [Dafermos and Rendall 2007]
$k = 1$	Vlasov	$\Lambda = 0$	Holds [Dafermos and Rendall 2007]
		$\Lambda > 0$	Holds under some conditions [Dafermos and Rendall 2007]; open otherwise

Table 2. Status of strong cosmic censorship for T^2-symmetric and surface-symmetric spacetimes.

2. Preliminaries

2A. ***T^2-symmetric spacetimes with spatial topology T^3.*** A spacetime $(\mathcal{M}, g)$ is said to be T^2-symmetric if the metric is invariant under an effective action of the Lie group T^2 and the group orbits are spatial. The Lie algebra of T^2 is spanned by two commuting Killing fields X and Y everywhere nonvanishing and we may normalize them so that the area element t of the group orbits is given by

$$g(X, X)g(Y, Y) - g(X, Y)^2 = t^2.$$

In previous analysis of these spacetimes [Chruściel 1990; Berger et al. 1997; Andréasson et al. 2004; Clausen and Isenberg 2007], it has been shown that any nonflat globally hyperbolic T^2-symmetric spacetime with spatial topology T^3 that satisfies the Einstein equations in the vacuum or with Vlasov matter and with $\Lambda \geq 0$ admits a metric in areal coordinates of the form

$$ds^2 = e^{2(\nu-U)}(-\alpha dt^2 + d\theta^2) + e^{2U}\big(dx + A dy + (G + AH) d\theta\big)^2 + e^{-2U} t^2 (dy + H d\theta)^2, \tag{2}$$

where all functions depend only on t and θ and are periodic in the latter. Note that the form (2) of the metric is unchanged under an SL(2, $\mathbb{R}$) transformation of the Killing vectors $X = \partial/\partial x$ and $Y = \partial/\partial y$.

As T^2-symmetric spacetimes contain several dynamical degrees of freedom, certain special cases have been introduced. A first simplification appears in the case where the Killing fields X and Y may be chosen such that their inner product, and thus the function A, vanishes. Such cases are called *polarized* T^2-symmetric spacetimes.

Associated with T^2-symmetric spacetimes are certain quantities called the twist quantities, defined by

$$J = \epsilon_{abcd} X^a Y^b \nabla^c X^d, \tag{3}$$

$$K = \epsilon_{abcd} X^a Y^b \nabla^c Y^d. \tag{4}$$

These are related to the metric functions by

$$J = -\frac{t e^{-2\nu+4U}}{\sqrt{\alpha}} (G_t + A H_t), \tag{5}$$

$$K = AJ - \frac{t^3 e^{-2\nu}}{\sqrt{\alpha}} H_t. \tag{6}$$

For any pair of commuting Killing vectors on a spacetime satisfying the vacuum Einstein equations, the associated twists quantities are constant [Geroch 1972]. Thus, for vacuum T^2-symmetric spacetimes, by an SL(2, $\mathbb{R}$) transformation of the Killing fields X and Y, we may ensure that the form of the metric (2) is unchanged while one of twist quantities vanishes. Therefore, in the vacuum, we shall always assume that $J = 0$.

The cases where both $J = 0$ and $K = 0$ are called T^3-*Gowdy* spacetimes. By Frobenius's theorem, the conditions $J = K = 0$ are equivalent to the integrability of the planes orthogonal to dx, dy.

2B. ***Spacetimes with a hyperbolic surface of symmetry.*** A spacetime $(\mathcal{M}, g)$ is said to be $k = -1$ *surface-symmetric* if it can be foliated by spacelike surfaces Σ_t such that, for all t, Σ_t is isometric to a doubly warped product $(S^1 \times \mathcal{S}, h_t)$ where S is a fixed compact surface of constant curvature -1.

It follows from previous analysis [Rendall 1995] that any $k = -1$ surface symmetric spacetime which is globally hyperbolic and satisfies the Einstein equations with $\Lambda \geq 0$, in the vacuum or with Vlasov matter, admits a metric in areal coordinates of the form[9]

$$ds^2 = -\frac{e^{2\nu}}{t}(\alpha dt^2 - d\theta^2) + t\gamma_{ab} dx^a dx^b, \tag{7}$$

where the functions ν and α depend only on t and θ and are periodic in the latter with period 1 and γ induces a metric of constant curvature -1 on the orbits of symmetry.

[9]Compared to the usual metric for these spacetimes, we use the square of the radius function $t = r^2$ as the time coordinate rather than the radius function r itself. Moreover, we have introduced the functions α and ν by analogy with the T^2 case, so as to ease the application of the method of the T^2 case to this class of spacetimes. See Appendix B for a description of the change of coordinates from the usual parametrization to this one.

2C. ***The Einstein–Vlasov system.*** Apart from the vacuum case, where we will set $T_{\mu\nu} = 0$ in (1), we will couple the Einstein equations to the Vlasov matter model, which we present in this section.

Let $\mathcal{P} \subset \mathcal{T}\mathcal{M}$ denote the set of all future-directed timelike vectors of length -1. $\mathcal{P}$ is classically called the *mass shell*. Let f denote a nonnegative function on the mass shell. The Vlasov equation for f is derived from the condition that f be preserved along geodesics. In coordinates, we therefore have

$$p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0, \tag{8}$$

where p^α denotes the momentum coordinates on the tangent bundle conjugate to x^α.

The energy-momentum tensor is defined by

$$T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f, \tag{9}$$

where $\pi : \mathcal{P} \to \mathcal{M}$ is the natural projection from the mass shell to the spacetime and the integral is with respect to the natural volume form on $\pi^{-1}(x)$.

2D. ***The classes of initial data.*** After this introduction to the symmetry classes and the matter fields, we are ready to present the initial data sets that will be studied in this article. For convenience, we will require that the initial data be smooth and, in the nonvacuum case, that the support of the Vlasov field be compact. These assumptions may clearly be relaxed if necessary.[10]

Definition 1. A *vacuum T^2-symmetric initial data set* is a triplet (Σ, h, K) such that

(1) Σ is a smooth differential 3-manifold with topology T^3 (in particular, Σ admits an effective action of T^2),

(2) h is a smooth Riemannian metric on Σ that is invariant under an effective action of the Lie group T^2,

(3) K is a smooth symmetric 2-tensor also invariant under the same T^2 action,

(4) (Σ, h, K_{ab}) satisfies the vacuum constraint equations of general relativity.

We describe in Appendix A the constraint equations in the vacuum or in the presence of Vlasov matter for the reader not familiar with them.

Definition 2. A *T^2-symmetric initial data set with Vlasov matter* is a quadruplet $(\Sigma, h, K, \hat{f})$ such that

(1) conditions (1), (2) and (3) of the preceding definition hold,

(2) $\hat{f}$ is a smooth, nonnegative function of compact support defined on $T\Sigma$ which is invariant under the natural lift to $T\Sigma$ of the T^2 action,

(3) $(\Sigma, h, K_{ab}, \hat{f})$ satisfies the constraint equations of the Einstein–Vlasov system.

Let us also define the notion of polarized T^2-symmetric initial data and of Gowdy initial data:

Definition 3. A vacuum T^2-symmetric initial data set (Σ, h, K) (respectively a T^2-symmetric initial data set with Vlasov matter $(\Sigma, h, K, \hat{f})$) is said to be *polarized* if there exist two Killing fields (X, Y) which generate the T^2 action such that $h(X, Y) = 0$ and $K(X, Y) = 0$ on Σ.

[10]For instance, we could have chosen the initial data to be compatible with the statement of Theorem 4.1 of [Dafermos and Rendall 2006]. However, we decided to give preference to clarity and will therefore stick with compact data for the Vlasov field.

Definition 4. A vacuum T^2-symmetric initial data set (Σ, h, K) (respectively a T^2-symmetric initial data set with Vlasov matter) is said to be a *Gowdy initial data set* if there exist linearly independent, commuting vector fields Z, X, Y on Σ such that X, Y are Killing fields which generate the T^2 action and such that $h(Z, X) = h(Z, Y) = K(Z, Y) = K(Z, X) = 0$.

We define $k = -1$ surface-symmetric initial data with Vlasov matter as follows:

Definition 5. A $k=-1$ *surface-symmetric initial data set* with Vlasov matter is a quadruplet $(\Sigma, h, K, \hat{f})$ such that

(1) $\Sigma = S^1 \times \mathcal{S}$ where $\mathcal{S}$ is a smooth compact surface,

(2) h is a smooth Riemannian doubly-warped product metric on Σ of the form $a(\theta)\, d\theta^2 + b(\theta)\gamma_{\mathcal{S}}$, where $\gamma_{\mathcal{S}}$ is a metric of constant curvature -1,

(3) $\hat{f}$ is a smooth, nonnegative function of compact support defined on $T\Sigma$ and invariant under the natural lift of the local isometries of $\mathcal{S}$ to $T\Sigma$,

(4) $(\Sigma, h, K_{ab}, \hat{f})$ satisfies the constraint equations of the Einstein–Vlasov system.

3. The maximal Cauchy development

We will recall in this section the classical results concerning the existence and uniqueness of the maximal Cauchy development to which we will refer often in the rest of this article. We will state the theorem in the case of Vlasov matter. For the vacuum case, it suffices to replace all matter terms by zero.

Theorem. *Let* $(\Sigma, h, K, \hat{f})$ *be an initial data set for the Einstein–Vlasov system. There exists a triplet* $(\mathcal{M}, g, f)$, *called the maximal Cauchy development of* $(\Sigma, h, K, \hat{f})$, *satisfying these conditions*:

(1) $(\mathcal{M}, g)$ *is a smooth globally hyperbolic spacetime and* f *is a smooth, nonnegative function of compact support defined on the mass shell* $\mathcal{P}$.

(2) $(\mathcal{M}, g, f)$ *satisfies the Einstein–Vlasov system* (1), (9), (8).

(3) *There exists a smooth embedding* $\phi : \Sigma \to \mathcal{M}$ *such that* $\phi(\Sigma)$ *is a Cauchy surface for* $\mathcal{M}$ *and if* h', K', f' *denotes respectively the first and second fundamental form of* $\phi(\Sigma)$ *and the restriction of* f *to the tangent bundle of* $\phi(\Sigma)$ *then* $\phi^*(h') = h$, $\phi^*(K') = K$, $\phi^*(f') = \hat{f}$.

(4) *If* $(\bar{\mathcal{M}}, \bar{g}, \bar{f})$ *is another triplet satisfying (1), (2) and (3) and if* $\bar{\phi}$ *denotes the corresponding embedding of* Σ *in* $\bar{\mathcal{M}}$ *then there exists an smooth isometry* ψ *from* $(\bar{\mathcal{M}}, \bar{g})$ *onto a subset of* $(\mathcal{M}, g)$ *such that* $\psi^* \bar{f} = f$ *and* $\psi(\bar{\phi}(\Sigma)) = \phi(\Sigma)$.

See [Fourès-Bruhat 1952; Choquet-Bruhat and Geroch 1969; Choquet-Bruhat 1970; 1971] for the original proofs of these results.

4. Global areal foliations of T^2-symmetric or $k = -1$ surface-symmetric spacetimes

We present in this section certain previous results concerning areal foliations of T^2-symmetric or $k = -1$ surface-symmetric spacetimes. Let us first recall that symmetries of the initial data are transmitted to the maximal Cauchy development. For the reader not familiar with these results, they are presented in Appendix C. Thus, T^2-symmetric (respectively surface-symmetric) initial data lead to T^2-symmetric (respectively surface-symmetric) spacetimes.

From [Chruściel 1990; Berger et al. 1997; Andréasson et al. 2004; Clausen and Isenberg 2007] for the T^2 case and from [Andréasson et al. 2003] for the hyperbolic case, we have:

Proposition 1. *Let $(\mathcal{M}, g, f)$ be the maximal Cauchy development of nonflat T^2-symmetric initial data (respectively $k = -1$ surface-symmetric initial data) with $\Lambda \geq 0$, either in the vacuum or with Vlasov matter.*

(1) *$(\mathcal{M}, g)$ is T^2-symmetric (respectively $k = -1$ surface-symmetric) and f is invariant under the natural lift to $T\mathcal{M}$ of the T^2 action (respectively under the natural lift to $T\mathcal{M}$ of the local isometries of $\mathcal{S}$, with $\mathcal{S}$ as in Definition 5).*

(2) *$(\mathcal{M}, g)$ can be covered by areal coordinates (t, θ, x, y) where the metric takes the form (2) (respectively (7)) and t ranges from $t_0 \geq 0$ to $+\infty$.*

(3) *In the T^2 case, $(\mathcal{M}, g)$ is a polarized T^2-symmetric spacetime (respectively a T^3-Gowdy spacetime) if and only if the initial data are polarized (respectively Gowdy).*

We also have a continuation criterion, which follows from the standard local well-posedness theory for the Einstein–Vlasov system as found in [Choquet-Bruhat 1970; 1971]:

Proposition 2. *Let $(\mathcal{M}, g, f)$ be a past development[11] of T^2-symmetric initial data (respectively $k = -1$ surface-symmetric initial data) with $\Lambda \geq 0$, either in the vacuum or with Vlasov matter and assume that $(\mathcal{M}, g)$ can be covered by areal coordinates (t, θ, x, y), where t ranges from $t_f > 0$ to t_i, $t_f < t_i$ and the metric takes the form (2) (respectively (7)) . Assume that*

(1) *all metric functions and their derivatives admit a continuous extension to $t = t_f$, and*

(2) *in the Vlasov case, f and all its derivatives admit a continuous extension to $t = t_f$.*

Then there exists a past development $(\tilde{\mathcal{M}}, \tilde{g}, \tilde{f})$ of the initial data and an isometric embedding i of $\mathcal{M}$ into $\tilde{\mathcal{M}}$ satisfying $i^*(\tilde{f}) = f$ and such that $i(\mathcal{M}) \neq \tilde{\mathcal{M}}$.

5. The theorems

Theorem 1. *Let $(\mathcal{M}, g, f)$ be the maximal development of T^2-symmetric initial data with Vlasov matter and $\Lambda \geq 0$. Suppose that the Vlasov field f does not vanish identically. Then $(\mathcal{M}, g)$ admits a global foliation by areal coordinates with the time coordinate t taking all values in $(0, \infty)$; that is, $t_0 = 0$ in the notation of Proposition 1.*

Thus the presence of Vlasov matter forbids $t_0 > 0$. In the vacuum case, we know that nonflat solutions with $t_0 > 0$ exist in [Smulevici 2008, Appendix E], an indication that this case is more difficult.

Theorem 2. *Let $(\mathcal{M}, g)$ be the maximal Cauchy development of vacuum T^2-symmetric initial data with $\Lambda > 0$ and suppose that the spacetime is not polarized. Then $(\mathcal{M}, g)$ admits a global foliation by areal coordinates with the time coordinate t taking all values in $(0, \infty)$; that is, $t_0 = 0$ in the notation of Proposition 1.*

The last theorem is the analogous of Theorem 1 in the hyperbolic symmetric case:

[11] Here and everywhere else in the paper, we will consider that, by definition, a development of an initial data set for the Einstein equations is a globally hyperbolic spacetime that satisfies the Einstein equations and agrees with the given data initially in the usual sense of general relativity.

Theorem 3. *Let $(\mathcal{M}, g, f)$ be the maximal development of $k = -1$ surface-symmetric initial data with Vlasov matter and $\Lambda \geq 0$. Suppose that the Vlasov field f does not vanish identically. Then $(\mathcal{M}, g)$ admits a global foliation by areal coordinates with the time coordinate t taking all values in $(0, \infty)$; that is, $t_0 = 0$ in the notation of Proposition 1.*

Note that in the vacuum case, there exist solutions of the Einstein equations with hyperbolic symmetry such that $t_0 > 0$ [Rendall 1995]. Thus, the assumption on the Vlasov field is necessary.

6. Remarks on the strategy of the proofs

We will present here the main ideas of the proofs of the theorems. We will place particular emphasis on the proof of Theorem 2 as it is the most difficult one. The reader might want to return to this section while reading the proof of Theorem 2 in order to better follow the arguments.

The proofs of Theorems 1 and 3 are based on the strategies developed in [Isenberg and Weaver 2003; Weaver 2004]. However, some crucial arguments of these previous works fail in the case of Theorem 2 and we have thus been forced to introduce a different approach, which we will present below.

In order to explain these differences and before presenting this new approach, let us first briefly revisit some of the ideas of the proofs contained in [Isenberg and Weaver 2003] and [Weaver 2004] for T^2-symmetric spacetimes with $\Lambda = 0$, respectively in the vacuum and in the Vlasov case.

6A. ***Previous work.*** Let us thus assume that $(\mathcal{M}, g, f)$ is a past development of T^2-symmetric initial data, with $\Lambda = 0$, in the vacuum or with Vlasov matter. Suppose that $(\mathcal{M}, g)$ is covered by areal coordinates with $t \in (t_f, t_i]$, where $t_f > 0$. In view of Proposition 2, in order to obtain a statement analogue to that of Theorem 1, it is sufficient to prove that for all such $(\mathcal{M}, g)$, all metric functions, the Vlasov field f and all their derivatives admit continuous extensions to $t = t_f$.

The conformal coordinate system and the function α. Let us first recall from [Berger et al. 1997] that another coordinate system may be introduced in $(\mathcal{M}, g)$, the so-called conformal coordinate system. In this coordinate system, the metric takes the form

$$ds^2 = e^{2(\nu-U)}(-d\tau^2 + d\chi^2) + e^{2U}\big(dx + A\,dy + (G + AH)\,d\chi\big)^2 + e^{-2U}t^2(dy + H\,d\chi)^2. \tag{10}$$

In the coordinate system (τ, χ, x, y), if one assumes that the area of the symmetry orbits t is uniformly bounded from below by a strictly positive constant, one may obtain[12] continuous extensions of all metric functions, the Vlasov field and their derivatives [Berger et al. 1997]. Thus, it is clear that in order to obtain the same statement in areal coordinates, the key point is to control the function α appearing in (2), as well as its derivatives, since it is this function which dictates the change of coordinates from conformal to areal coordinates. Moreover, it turns out that the function α is necessarily nondecreasing in the past, and in fact increasing if $K > 0$ (i.e., the spacetime is not of T^3-Gowdy type). This implies that, in essence, one only need prove that α is bounded above.

[12]The proof (in the vacuum case) is essentially based on energy and null cone estimates where the energies considered arose naturally from the wave map background structure of the equations. On the other hand, these estimates and the results obtained in conformal coordinates do not provide any information concerning the behavior of the function t, apart from what is already contained in the statement of Proposition 1, see [Berger et al. 1997].

The energy estimates. For this purpose, one introduces the energy density[13]

$$g = U_t^2 + \alpha U_\theta^2 + \frac{e^{2U}}{4t^2}\left(A_t^2 + \alpha A_\theta^2\right) \tag{11}$$

and the energy integral

$$E_g = \int_{\theta \in [0,1]} \frac{g}{\sqrt{\alpha}}. \tag{12}$$

This energy can easily be shown to be bounded from above.

The estimate on α. Moreover, one can obtain an estimate of the type:

$$\alpha e^{2\nu}(t,\theta) \le C \frac{E_g(t_i)}{E_g(t)}, \tag{13}$$

for some positive constant C which depends only on the initial data and the value of $t_f > 0$. Thus, in order to obtain an upper bound on $\alpha e^{2\nu}$, it is sufficient to have a lower bound on E_g. In the vacuum case with $\Lambda = 0$, this lower bound follows easily from the Einstein equations, as E_g is necessarily nondecreasing in the past direction. From the bound on $\alpha e^{2\nu}$, the upper bound on α follows easily by integration of the evolution equation for α; see (106) with $\Lambda = 0$. The key points are thus the estimate (13) and the monotonicity of E_g.

In the Vlasov case, the monotonicity of E_g is actually broken and thus one loses the easy upper bound on $\alpha e^{2\nu}$. In order to obtain a bound on α, one introduces another energy integral, which we shall call here $E_{g,f}$, which can also be proven to be bounded from above. It turns out that $E_{g,f}$ controls ρ, an energy density associated with the energy-momentum tensor, and using the fact that f does not vanish identically, Weaver [2004] proved that one can extract enough information from ρ to obtain the estimate

$$\min_{\theta \in [0,1]} \alpha(t,\cdot) < M, \tag{14}$$

for some constant $M > 0$. Thus, using the fact that f does not vanish, one obtains an estimate on the function α. This estimate is not as strong as in the vacuum case but it turns out that, together with the upper bound on E_g, this control on α is sufficient to derive pointwise estimates on g and bounds on the support of f, from which it is easy to derive all the remaining estimates.

6B. *The proofs of Theorem 1 and Theorem 3.* Assume now that we are in the setting of Theorem 1, where we focus on the T^2-symmetric case with Vlasov matter and $\Lambda \ge 0$. In this case, as in the case where $\Lambda = 0$, $f \ne 0$ discussed in Section 6A, we do not have monotonicity of E_g. However, all other important monotonicity properties hold and the estimate concerning $\min_{\theta \in [0,1]} \alpha(t,\cdot)$ still holds. This implies that the proof in the Vlasov case with $\Lambda = 0$ can be extended without too much difficulty to the case where $\Lambda > 0$. This is treated in detail in Section 7.

Remark 6B.1. In particular, we note that the assumption of the nonvanishing of the Vlasov field is necessary only so as to establish the estimate (14). In other words, we have the following proposition, which will be useful in the course of the proof of Theorem 2.

[13]In the Gowdy case, this energy quantity arises naturally from the wave map structure of the equations. For the T^2 case, the vacuum Einstein equations may be regarded as the equations of a wave map problem with source, for which the natural associated energy density is g.

Proposition 3. *Let $(\mathcal{M}, g, f)$ be a development of T^2-symmetric initial data in the vacuum or with Vlasov matter and with $\Lambda \geq 0$. Assume that $(\mathcal{M}, g)$ admits a global areal foliation (t, θ, x, y) where t ranges from $t_f > 0$ to t_i, $t_f < t_i$. Assume moreover that the estimate* (14) *holds. Then, all the metric functions, the Vlasov field f and all their derivatives admits continuous extensions to $t = t_f$, i.e the assumptions of Proposition 2 are verified.*

Let us also note that the important monotonicity properties used in the proof of Theorem 1 remain valid in the case of hyperbolic symmetry. We will prove Theorem 3 by adapting the strategy of the proof of Theorem 1 to the hyperbolic symmetric case. This is treated in detail in Section 9.

6C. ***The proof of Theorem 2.*** In the vacuum case with $\Lambda > 0$, we lose again the monotonicity property of E_g. Thus, one does not have a priori the lower bound on E_g required to apply (13). Moreover, we cannot obtain an a priori estimate on $\min_{\theta\in[0,1]} \alpha(t, \cdot)$ as in the Vlasov case as this required that certain matter terms do not vanish. However, estimates similar to (13) hold and thus, we easily obtain that the statement that α is bounded above is equivalent to the statement that E_g is bounded from below by a strictly positive constant.

Different parametrizations for the orbits of symmetry and explicit solutions of the equations. The monotonicity of E_g is linked with the homogeneity or inhomogeneity of the wave equation for the metric function U defined in (2). When $\Lambda > 0$, an extra term arises in the time derivative of E_g which has the wrong sign; see (124). In fact, when both twists quantities vanish, i.e., in the T^3-Gowdy case ($K = 0$), there is a way to recover the monotonicity argument. Indeed, one may apply a simple transformation to the function U such that the wave equation for the resulting metric function P is homogeneous; see (113) with $K = 0$. Using (U, A) or (P, A) corresponds to a different choice of parametrization for the extrinsic geometry of the orbits of symmetry. The system of wave equations satisfied by (P, A) has a similar structure to that of (U, A) and one may introduce an energy E_h associated with it, which plays a role similar to that of E_g.

The interpretation of the transformation is as follows. In the case $(K = 0, \Lambda = 0)$, all flat Kasner spacetimes corresponding to $U = k \ln t$, $A =$ constant are possible solutions of the equations. In the case $(K > 0, \Lambda = 0)$, the only Kasner spacetimes of the form $U = k \ln t$, $A =$ constant satisfying the Einstein equations are those for which $U = 0$ and $A =$ constant. Another characterization of these solutions is that they correspond to $E_g = 0$. In the case $K = 0$, $\Lambda > 0$, there can be naturally no flat Kasner solutions, but there are plane symmetric solutions which are characterized by $E_h = 0$. We also remark that in both cases, $(K > 0, \Lambda = 0)$ and $(K = 0, \Lambda > 0)$, there are solutions, with respectively $E_g = 0$ or $E_h = 0$, for which $t_0 > 0$; see [Isenberg and Weaver 2003; Smulevici 2008, Appendix E].

The easy case $(K = 0, \Lambda > 0)$. In this case, as mentioned above, the system of wave equations for (P, A) is homogeneous. Moreover, one can easily prove that E_h is nondecreasing in the past direction (see Remark 8E.1). An estimate similar to (13) can be derived, from which we obtain the desired upper bound on α under the assumption that $E_h \neq 0$ initially. This case can thus be treated separately and we present it in Proposition 4 (see Section 8E).

The general case. The contradiction setting. When both $K > 0$ and $\Lambda > 0$, there is no easy way to recover a monotonicity property on E_h or E_g and thus there are no a priori lower bounds on E_g or E_h. We will prove Theorem 2 by recovering such a lower bound via other methods. The aim will therefore be to bound away from 0 the energy integrals E_h and E_g associated with the nonlinear system of wave equations describing the motion of the orbits of symmetry. For this, we will proceed by contradiction, assuming that $t_0 > 0$ for the maximal Cauchy development.

This will allow us to obtain two important facts: α is uniformly blowing up (Section 8C) and the energy integrals E_g and E_h tend to 0 as $t \to t_0$ (Section 8H). (The uniform blow up of α is in fact an immediate consequence of the Remark 6B.1.)

Control on the spatial differences of some metric functions. We will use the uniform blow up of α and the vanishing limit of E_h and E_g to obtain successively more and more control on the solutions and improve our understanding of the nonlinear terms in the equations. First, the vanishing of E_h and E_g in the limit $t \to t_0$ will imply a strong control on the spatial differences of some the metric functions (Section 8I). In particular, control on $\max_{\theta\in[0,1]} \alpha e^{\nu} - \min_{\theta\in[0,1]} \alpha e^{\nu}$ and similar quantities will be used extensively in the null cone estimates and the analysis of the characteristics which we will pursue later.

Some tools for the null cone estimates. In Sections 8O and 8P, we will derive null cone estimates. In order to do so, it will be necessary to have at hand the following tools:

- an estimate on $(\partial/\partial\theta)(\ln\alpha)$ (Section 8J),
- estimates for the integrals of small powers of α (Section 8M), which will essentially be used to control some error terms in the null cone estimates,
- a parametrization of the null rays in areal coordinates (Section 8K).

Moreover, to exploit these null cone estimates in the last step of the proof, we will need to control a change of coordinates from the coordinates adapted to the null rays to the areal system of coordinates. The required estimate is proved in Section 8L.

Finally, in order to prove the pointwise estimates from below of Section 8P, we will need to start with large data. The analysis of the polarization energy, which we describe below, will enable us to exhibit such large data.

The polarization energy E_A. In Section 8L, we will focus our attention on the polarization energy E_A of the spacetime associated with the wave equation satisfied by the polarization function A defined in (2). Since by definition, $E_A \le E_h$, a lower bound on E_A is sufficient to obtain a lower bound on E_h and close the estimates. (Motivation for considering this energy comes from the fact that the evolution equation for A stays homogeneous even with $\Lambda > 0$ and the simple remark that one of the common features of all known cases with $t_0 > 0$ is that all such spacetimes are polarized and thus have $E_A = 0$.) From the contradiction setting, it follows that $E_A \to 0$ as $t \to t_0$. Using the assumption $E_A > 0$ and the vanishing limit of E_A, we will exhibit a sequence of points in the spacetime where the energy density h is of the order of α.

The null cone estimates. These points will be used in Section 8P as large initial data for some null cone estimates along the characterisitics of the spacetime. The aim of these null cone estimates will be to prove that not only is h of order α at some points, but it is in fact blowing up at least like $\alpha^{1-\epsilon}$ along certain characteristics. However, in order to control the spatial derivatives and the nonlinearity of the equations, we will also need an estimate from above for h. Thus, we will first prove that h is blowing up at most like $\alpha^{1+\epsilon}$. To derive these pointwise estimates on h, we will use the tools developed in the previous sections and apply null cone estimates similar to those we introduced in [Smulevici 2008].

By a continuity argument, it will actually follow that h necessarily blows up along a whole family of characteristics.

The contradiction. In the previous step, we have obtained the blow up of h as $\alpha^{1-\epsilon}$ along a strip of characteristics. This can be integrated in space but if we want to relate the resulting integral to E_h, we need to control the difference between the integral of h over the spacelike foliation associated with the conformal coordinate system and its integral over the spacelike foliations associated with the areal coordinate system. Using the results of Section 8L, we will prove in Section 8Q that the two integrals differ at most by a factor of α^{ϵ}. It follows that

$$E_h = \int_{[0,1]} \frac{h}{\sqrt{\alpha}} d\theta$$

is bounded from below by $\delta \min_{\theta \in [0,1]} \alpha^{1/2-2\epsilon}$ for some $\delta > 0$ and thus, in particular, does not vanish as t goes to t_0. This is a contradiction, which concludes the proof of Theorem 2.

7. Proof of Theorem 1

We will prove Theorem 1 in this section. As discussed above, the method will follow [Weaver 2004]. It would be sufficient to check that the extra terms arising from the introduction of $\Lambda > 0$ do not spoil any of the monotonicity arguments and may be controlled when required, but in order to be self-contained, we will provide a full proof. Moreover, some of the estimates given here will be useful later in order to prove Theorem 2 in Section 8, in particular, to obtain the uniform blow up of α of Lemma 8.1. We start by recalling the Einstein–Vlasov system for T^2-symmetric spacetimes in areal coordinates.

7A. ***Vlasov matter in T^2-symmetric spacetimes.*** Let $(\mathcal{M}, g, f)$ be a past development of T^2-symmetric initial data with Vlasov matter as described in Section 2D and assume that (t, θ, x, y) is a system of areal coordinates such that the metric takes the form (2). Let v_i, for $i = 0, 1, 2, 3$, denote the components of the velocity vector in the untwisted set of 1-forms:

$$\{dt, d\theta, dx + G d\theta, dy + H d\theta\}. \tag{15}$$

The dual basis is

$$\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} - G\frac{\partial}{\partial x} - H\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \tag{16}$$

In this new frame, the metric (2) and its inverse are given by

$$\tilde{g}_{ij} = \begin{pmatrix} -\alpha e^{2(\nu-U)} & 0 & 0 & 0 \\ 0 & e^{2(\nu-U)} & 0 & 0 \\ 0 & 0 & e^{2U} & e^{2U}A \\ 0 & 0 & e^{2U}A & e^{-2U}t^2 + e^{2U}A^2 \end{pmatrix}, \tag{17}$$

$$\tilde{g}^{ij} = \begin{pmatrix} -\alpha^{-1} e^{-2(\nu-U)} & 0 & 0 & 0 \\ 0 & e^{-2(\nu-U)} & 0 & 0 \\ 0 & 0 & e^{-2U} + e^{2U}A^2 t^{-2} & -e^{2U}At^{-2} \\ 0 & 0 & -e^{2U}At^{-2} & e^{2U}t^{-2} \end{pmatrix}. \tag{18}$$

Note that, along a geodesic, the components v_2 and v_3 of the velocity vector are constant, since if we denote the tangent vector to a geodesic by V, the geodesic and the Killing equations give

$$v_2 = g\Big(V, \frac{\partial}{\partial x}\Big), \tag{19}$$

$$\nabla_V g\Big(V, \frac{\partial}{\partial x}\Big) = g\Big(\nabla_V V, \frac{\partial}{\partial x}\Big) + g\Big(V, \nabla_V \frac{\partial}{\partial x}\Big) = 0. \tag{20}$$

We will parametrize the mass shell $\mathcal{P}$ by the coordinates $(t, \theta, x, y, v_1, v_2, v_3)$, where by an abuse of notation we denote the lift to $\mathcal{P}$ of the coordinates on $\mathcal{M}$ by the same symbols. The Vlasov field f can then be identified with a function of $(t, \theta, x, y, v_1, v_2, v_3)$ or, using the symmetry, with a function of $(t, \theta, v_1, v_2, v_3)$ only. We shall, abusing notation, use both definitions and always denote it by f.

With these definitions, the mass shell relation $v_\mu v^\mu = -1$, which holds on the support of the Vlasov field, is given by[14]

$$v_0 = -\sqrt{\alpha e^{2(\nu-U)} + \alpha v_1^2 + \alpha e^{2(\nu-2U)} v_2^2 + \alpha t^{-2} e^{2\nu}(v_3 - A v_2)^2}, \tag{21}$$

and the Vlasov equation reads as

$$\frac{\partial f}{\partial t} = \frac{\partial v_0}{\partial v_1}\frac{\partial f}{\partial \theta} - \bigg(\frac{\partial v_0}{\partial \theta} + \frac{\sqrt{\alpha} e^{\nu}}{t^3}(K - AJ)(v_3 - A v_2) + \frac{\sqrt{\alpha} e^{2\nu-4U}}{t} J v_2\bigg)\frac{\partial f}{\partial v_1}. \tag{22}$$

7B. ***The Einstein equations in areal coordinates.*** The Einstein equations (1) give rise to the following system of equations in areal coordinates:

Constraint equations:

$$\frac{\nu_t}{t} = U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2) + \frac{\alpha e^{2\nu-4U}}{4t^2} J^2 + \frac{\alpha e^{2\nu}(K - AJ)^2}{4t^4} + \alpha e^{2(\nu-U)}\Lambda + 8\pi \frac{\sqrt{\alpha}}{t}\int_{\mathbb{R}^3} f|v_0|\, dv_1\, dv_2\, dv_3, \tag{23}$$

$$\frac{\alpha_t}{\alpha} = -\frac{\alpha e^{2\nu-4U} J^2}{t} - \frac{\alpha e^{2\nu}(K - AJ)^2}{t^3} - 4t\alpha e^{2(\nu-U)}\Lambda - 16\pi \alpha^{3/2} e^{2(\nu-U)} \int_{\mathbb{R}^3} \frac{f(1 + e^{-2U} v_2^2 + e^{2U} t^{-2}(v_3 - A v_2)^2)}{|v_0|}\, dv_1\, dv_2\, dv_3, \tag{24}$$

$$\frac{\nu_\theta}{t} = 2U_t U_\theta + \frac{e^{4U}}{2t^2} A_t A_\theta - \frac{\alpha_\theta}{2t\alpha} - 8\pi \frac{\sqrt{\alpha}}{t}\int_{\mathbb{R}^3} f v_1\, dv_1\, dv_2\, dv_3. \tag{25}$$

Evolution equations:

$$\nu_{tt} - \alpha \nu_{\theta\theta} = \frac{\alpha_\theta \nu_\theta}{2} + \frac{\alpha_t \nu_t}{2\alpha} - \frac{\alpha_\theta^2}{4\alpha} + \frac{\alpha_{\theta\theta}}{2} - U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 - \alpha A_\theta^2) - \frac{\alpha e^{2\nu-4U} J^2}{4t^2} - \frac{3\alpha e^{2\nu}(K - AJ)^2}{4t^4} + \alpha \Lambda e^{2(\nu-U)} - 8\pi \frac{\alpha^{3/2} e^{2\nu}}{t^3}\int_{\mathbb{R}^3} \frac{f(v_3 - A v_2)^2}{|v_0|}\, dv_1\, dv_2\, dv_3, \tag{26}$$

[14] Note that $v_0 < 0$ since $v^0 > 0$.

$$U_{tt}-\alpha U_{\theta\theta}=-\frac{U_t}{t}+\frac{\alpha_\theta U_\theta}{2}+\frac{\alpha_t U_t}{2\alpha}+\frac{e^{4U}}{2t^2}(A_t^2-\alpha A_\theta^2)$$
$$+\alpha\Lambda e^{2(\nu-U)}+\frac{\alpha e^{2\nu-4U}J^2}{2t^2}+8\pi\frac{\alpha^{3/2}e^{2(\nu-U)}}{2t}\int_{\mathbb{R}^3}\frac{f(1+2e^{-2U}v_2^2)}{|v_0|}\,dv_1\,dv_2\,dv_3, \quad (27)$$

$$A_{tt}-\alpha A_{\theta\theta}=\frac{A_t}{t}+\frac{\alpha_\theta A_\theta}{2}+\frac{\alpha_t A_t}{2\alpha}-4(A_tU_t-\alpha A_\theta U_\theta)$$
$$+\frac{\alpha e^{2\nu-4U}J(K-AJ)}{t^2}+16\pi\frac{\alpha^{3/2}e^{2\nu-4U}}{t}\int_{\mathbb{R}^3}\frac{fv_2(v_3-Av_2)}{|v_0|}\,dv_1\,dv_2\,dv_3. \quad (28)$$

Auxiliary equations:

$$J_t=-16\pi\alpha\int_{\mathbb{R}^3}\frac{fv_1v_2}{|v_0|}\,dv_1\,dv_2\,dv_3, \quad (29)$$

$$J_\theta=\ \ 16\pi\int_{\mathbb{R}^3}f\,v_2\,dv_1\,dv_2\,dv_3, \quad (30)$$

$$K_t=-16\pi\alpha\int_{\mathbb{R}^3}\frac{fv_1v_3}{|v_0|}\,dv_1\,dv_2\,dv_3, \quad (31)$$

$$K_\theta=\ \ 16\pi\int_{\mathbb{R}^3}f\,v_3\,dv_1\,dv_2\,dv_3. \quad (32)$$

We will now proceed to the proof of Theorem 1.

In the rest of this section, $(\mathcal{M}, g, f)$ will be a past development of T^2-symmetric initial data with Vlasov matter and $\Lambda \geq 0$. We will cover $(\mathcal{M}, g)$ by areal coordinates (t, θ, x, y), where the range of the coordinates is $(t_f, t_i] \times [0, 1]^3$ with $0 < t_f < t_i$. The metric is then given by (2) where all functions depend on t and θ and are periodic in θ with period 1. The Einstein–Vlasov system implies that the system (23)–(32) completed by (22) holds for all $(t, \theta) \in (t_f, t_i] \times [0, 1]$. Moreover, we will assume that f does not vanish identically. From what has been said in Section 6, we will prove that for all such $(\mathcal{M}, g, f)$, the hypotheses of Proposition 2 are satisfied, from which Theorem 1 follows immediately.

First we recall some standard facts about the Vlasov field in such spacetimes.

7C. ***Conservation laws.*** From the conservation of the Vlasov field f along geodesics, if follows immediately that f is bounded above by some constant $F > 0$:

$$f \leq F. \quad (33)$$

Since v_2 and v_3 are constant along geodesics, it follows that the support of f in v_2 and v_3 is conserved. By compactness of the initial Cauchy surface, we therefore have an upper bound on the support of f in v_2 and v_3 in $(\mathcal{M}, g)$. Let X be such an upper bound:

$$X=\sup\big\{\max(|v_2|, |v_3|) \,:\, \exists(t, \theta, v_1) \text{ such that } f(t, \theta, v_1, v_2, v_3) > 0\big\} < \infty. \quad (34)$$

The particle current is given by

$$N^\mu=\frac{\sqrt{\alpha}}{t}\int_{\mathbb{R}^3}\frac{f}{|v_0|}v^\mu\,dv_1\,dv_2\,dv_3. \quad (35)$$

From the Vlasov equation it follows that N^μ is divergence free: $\nabla_\mu N^\mu = 0$. We therefore have the conservation law, for all t,

$$\int_{[0,1]} N^0 t\sqrt{\alpha} e^{2(\nu-U)} d\theta = \int_{[0,1]} \left(\int_{\mathbb{R}^3} f \, dv_1 \, dv_2 \, dv_3 \right) d\theta = Q, \tag{36}$$

for some constant Q. And since, by assumption, the Vlasov field does not vanish identically, we have

$$Q > 0. \tag{37}$$

7D. ***Lower bound on the mean value of*** $|v_1|$**.** We now prove a lower bound on the mean value of $|v_1|$ for the measure $f \, dv \, d\theta$. This lower bound is the important estimate that takes advantage of the assumption that $f \neq 0$. Coupled to the energy estimates derived in the next section, this estimate will give us uniform control of $\min_{[0,1]} \alpha(t, \cdot)$; see Section 7H.

Lemma 7.1. *There exists* $\delta > 0$ *such that*

$$\int_{[0,1]} \int_{\mathbb{R}^3} f |v_1| \, dv_1 \, dv_2 \, dv_3 d\theta > \delta \tag{38}$$

for all $t \in (t_f, t_i]$.

Proof. Let

$$\epsilon = \frac{Q}{16X^2F},$$

so that $Q - \epsilon 8X^2 F = Q/2 > 0$. We have

$$\begin{aligned} \int_{[0,1]} \int_{\mathbb{R}^3} f|v_1| \, dv_1 \, dv_2 \, dv_3 &= \int_{[0,1]} \int_{\mathbb{R}^2} \left(\int_{-\epsilon}^{\epsilon} f|v_1| \, dv_1 \right) dv_2 \, dv_3 + \int_{[0,1]} \int_{\mathbb{R}^2} \left(\int_{|v_1|>\epsilon} f|v_1| \, dv_1 \right) dv_2 \, dv_3 \\ &\geq \epsilon \int_{[0,1]} \int_{\mathbb{R}^2} \left(\int_{|v_1|>\epsilon} f \, dv_1 \right) dv_2 \, dv_3 \geq \epsilon Q/2. \qquad \square \end{aligned}$$

7E. ***Energy estimates.*** The following energy estimates take their origins in the underlying wave map structure of the equations, visible in the vacuum case [Berger et al. 1997] and easily modifiable to suit the Vlasov case.

Define the energy integral $E_{g,K,\Lambda,f}(t)$ by[15]

$$E_{g,K,\Lambda,f} = \int_{[0,1]} \frac{\nu_t}{\sqrt{\alpha} t} d\theta. \tag{39}$$

From the constraint equation (23), it follows that

$$\begin{aligned} E_{g,K,\Lambda,f} = \int_{[0,1]} \frac{1}{\sqrt{\alpha}} \bigg(& U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2) + \frac{\alpha e^{2\nu-4U} J^2}{4t^2} \\ & + \frac{\alpha e^{2\nu}(K - AJ)^2}{4t^4} + \alpha e^{2(\nu-U)} \Lambda + 8\pi \frac{\sqrt{\alpha}}{t} \int_{\mathbb{R}^3} f|v_0| \, dv_1 \, dv_2 \, dv_3 \bigg) d\theta. \end{aligned} \tag{40}$$

[15]The motivation for the notation $E_{g,K,\Lambda,f}$ is that this energy may be decomposed into four terms, containing respectively g, K, Λ and f. Later, we will introduce several other energy integrals and the notation will follow the same pattern.

Using the Einstein equations, we may compute the time derivative of $E_{g,K,\Lambda,f}$:

$$\frac{dE_{g,K,\Lambda,f}}{dt} = -\int_{[0,1]}\Bigg[\frac{2}{t}\left(\frac{U_t^2}{\sqrt{\alpha}}+\frac{e^{4U}}{4t^2}\sqrt{\alpha}A_\theta^2\right)+\frac{\sqrt{\alpha}e^{2\nu-4U}J^2}{2t^3}+\frac{\sqrt{\alpha}e^{2\nu}(K-AJ)^2}{t^5} + 8\pi\int_{\mathbb{R}^3}\left(\frac{f|v_0|}{t^2}+\frac{\alpha e^{2\nu}f(v_3-Av_2)^2}{t^4|v_0|}\right)dv_1\,dv_2\,dv_3\Bigg]d\theta. \quad (41)$$

Since the right-hand side is nonpositive, $E_{g,K,\Lambda,f}$ is nondecreasing when t is decreasing.[16]

Lemma 7.2. *$E_{g,K,\Lambda,f}$ is bounded on $(t_f, t_i]$ and admits a continuous extension at t_f.*

Proof. From (39), (41) and the mass shell relation (21), we obtain

$$\frac{dE_{g,K,\Lambda,f}}{dt} \geq -\frac{4}{t}E_{g,K,\Lambda,f}, \quad (42)$$

where the factor of 4 arises because of the terms containing $(K-AJ)^2$. Applying Gronwall's lemma and using the lower bound $t \geq t_f > 0$, we then obtain a uniform bound on $E_{g,K,f}$. □

7F. *Estimate for $\sqrt{\alpha}e^{2\nu+bU}$.* Here we exploit the monotonicity properties of the constraint equations.

Lemma 7.3. *For any real number b,*

$$\sqrt{\alpha}e^{2\nu+bU} \quad (43)$$

is uniformly bounded on $(t_f, t_i] \times [0, 1]$.

Proof. Using equations (23) and (24), we see that $t^{b^2/8}\sqrt{\alpha}e^{2\nu+bU}$ is decreasing with decreasing t:

$$\begin{aligned}\partial_t(t^{b^2/8}\sqrt{\alpha}e^{2\nu+bU}) &= b^2/8t^{b^2/8-1}\sqrt{\alpha}e^{2\nu+bU}+t^{b^2/8}\frac{\alpha_t}{2\sqrt{\alpha}}e^{2\nu+bU}+t^{b^2/8}\sqrt{\alpha}e^{2\nu+bU}(2\nu_t+bU_t)\\ &= t^{b^2/8}\sqrt{\alpha}e^{2\nu+bU}\Bigg(2t\left[\left(U_t+\frac{b}{4t}\right)^2+\alpha U_\theta^2+\frac{e^{4U}}{4t^2}(A_t^2+\alpha A_\theta^2)\right]\\ &\qquad + 8\pi\sqrt{\alpha}\int_{\mathbb{R}^3} f\left(|v_0|+\frac{\alpha v_1^2}{|v_0|}\right)dv_1\,dv_2\,dv_3\Bigg) \geq 0. \quad \square\end{aligned}$$

Thanks to the freedom in the choice of the Killing fields, we also have:

Lemma 7.4. *For any positive real number r and any real number λ,*

$$\alpha^{r/2}e^{2r\nu+\lambda U}A^2 \quad (44)$$

is bounded on $(t_f, t_i] \times [0, 1]$.

Proof. Consider inverting the role of X and Y in the metric:

$$\tilde{X} = Y, \quad (45)$$

$$\tilde{Y} = -X. \quad (46)$$

[16]In contrast, $E_g = \int_{[0,1]}\frac{1}{\sqrt{\alpha}}\left(U_t^2+\alpha U_\theta^2+\frac{e^{4U}}{4t^2}(A_t^2+\alpha A_\theta^2)\right)d\theta$ is not necessarily monotonic.

This is an SL(2, ℝ) transformation and therefore (see Section 2A), the form of the metric is unchanged if we relabel the metric functions as follows, using tilde notations for the new metric functions:

$$e^{2\tilde{U}} = e^{2U} A^2 + t^2 e^{-2U}, \tag{47}$$

$$e^{2\tilde{U}} \tilde{A} = -A e^{2U}, \tag{48}$$

$$\tilde{\alpha} = \alpha, \tag{49}$$

$$\tilde{\alpha} e^{2(\tilde{\nu}-\tilde{U})} = \alpha e^{2(\nu-U)}. \tag{50}$$

Let $q < r$, using the previous equations, it follows that

$$\tilde{\alpha}^{q/2} e^{2q\tilde{\nu}+2(1-q)\tilde{U}} = \alpha^{q/2} e^{2q\nu+2(1-q)U} A^2 + \alpha^{q/2} t^2 e^{2q\nu-2(1+q)U} \tag{51}$$

Since the tilde metric functions satisfy the same equations with respect to the same t, the left-hand side of (51) is bounded on $(t_f, t_i] \times [0, 1]$ from Lemma 7.3. Since the second term on the right hand side is positive, the first term is bounded. By Lemma 7.3,

$$\alpha^{(r-q)/2} e^{2(r-q)\nu+(\lambda-2(1-q))U} \tag{52}$$

is bounded, and multiplying this by the first term on the right in (51), we obtain the desired estimate. □

The quantity $\sqrt{\alpha} e^{\nu}$ will play an important role in the analysis. To simplify some of the computations, let us define β by

$$e^{\beta} = \sqrt{\alpha} e^{\nu}. \tag{53}$$

7G. ***Estimates for spatial derivatives integrals.*** From (25), we derive

$$\beta_\theta = 2t \left(U_t U_\theta + \frac{e^{4U}}{4t^2} A_t A_\theta \right) - 8\pi \sqrt{\alpha} \int_{\mathbb{R}^3} f v_1 \, dv_1 \, dv_2 \, dv_3. \tag{54}$$

It follows from this equation and the energy estimates obtained in Lemma 7.2 that we can uniformly control the variation in θ of the metric functions. In other words:

Lemma 7.5. *The integrals*

$$\int_{[0,1]} |\beta_\theta| \, d\theta, \quad \int_{[0,1]} |U_\theta| \, d\theta, \quad \int_{[0,1]} e^{2U} |A_\theta| \, d\theta, \quad \int_{[0,1]} |J_\theta| \, d\theta, \quad \int_{[0,1]} |K_\theta| \, d\theta$$

are uniformly bounded on $(t_f, t_i]$.

Proof. From (54), we obtain

$$\frac{|\beta_\theta|}{t} \leq \frac{\nu_t}{\sqrt{\alpha} t} \tag{55}$$

and by integration we obtain a bound on $\int_{[0,1]} |\beta_\theta| \, d\theta$ in view of (39) and the bound on $E_{g,K,\Lambda,f}$. The bounds on $\int_{[0,1]} |J_\theta| \, d\theta$ and $\int_{[0,1]} |K_\theta| \, d\theta$ follow from the auxiliary equations (30), (32), and the conservation of the flux (36) together with (34). The bounds on the remaining quantities follow from the definition of $E_{g,K,f}$ and the monotonicity in t of α. □

7H. ***Control of α along special curves.*** In this section, we obtain a bound on $\min_{\theta\in[0,1]}\alpha(t,\cdot)$, using the lower bound on the mean value of $|v_1|$.

Lemma 7.6. $\min_{\theta\in[0,1]}\alpha(t,\cdot)$ *is uniformly bounded on* $(t_f, t_i]$.

Proof. From the definition of $E_{g,K,\Lambda,f}$, we have

$$8\pi\int_{[0,1]}\int_{\mathbb{R}^3} f|v_0|\,dv_1\,dv_2\,dv_3\,d\theta \le tE_{g,K,\Lambda,f}, \tag{56}$$

and from the mass shell relation (21), we obtain

$$\int_{[0,1]}\int_{\mathbb{R}^3} f\sqrt{\alpha}|v_1|\,dv_1\,dv_2\,dv_3\,d\theta \le \frac{tE_{g,K,\Lambda,f}}{8\pi}, \tag{57}$$

$$\sqrt{\min_{\theta\in[0,1]}\alpha(t,\cdot)}\int_{[0,1]}\int_{\mathbb{R}^3} f|v_1|\,dv_1\,dv_2\,dv_3\,d\theta \le \frac{tE_{g,K,\Lambda,f}}{8\pi}, \tag{58}$$

$$\sqrt{\min_{\theta\in[0,1]}\alpha(t,\cdot)} \le \frac{tE_{g,K,\Lambda,f}}{8\pi\delta}, \tag{59}$$

where we have used the lower bound of Lemma 7.1 to obtain the last inequality. □

Remark 7H.1. This is the only step in the proof of Theorem 1 where we need the assumption that f does not vanish. In particular, in the proof by contradiction of Theorem 2 given in Section 8, we will be able to assume that the above lemma does not hold (see Section 8D).

Corollary 1. *There exists* $\bar\theta\in[0,1]$ *such that* $\alpha(t,\bar\theta)$ *is bounded on* $(t_f,t_i]$.

Proof. Let M be a bound for $\min_{[0,1]}\alpha$. Suppose that for every $\theta\in[0,1]$, $\alpha(t,\theta)$ is unbounded. By assumption, for every θ, there exists a $t^*(\theta)$, for which $\alpha(t^*(\theta),\theta)>2M$ and by continuity, there exists an open interval $I_\theta=(\theta-\delta_\theta,\theta+\delta_\theta)$ such that

$$\alpha(t^*(\theta),\theta')>M \quad \text{for all } \theta'\in I_\theta. \tag{60}$$

Consider $\bigcup_{\theta\in[0,1]} I_\theta$. This is an open cover of $[0,1]$; by compactness, it has a finite subcover. Let $\theta_0,\theta_1,\dots,\theta_n$ be such that $[0,1]=\bigcup_{0\le k\le n} I_{\theta_k}$ and let $T=\min_{0\le k\le n} t^*(\theta_k)$. Since α is increasing with decreasing time, it follows that $\alpha(T,\theta)>M$ for every $\theta\in[0,1]$ which contradicts the definition of M. □

7I. ***Estimate for*** $\sqrt{\alpha}e^{\nu+bU}$**.**

Lemma 7.7. *For any real number* b, $\sqrt{\alpha}e^{\nu+bU}=e^{\beta+bU}$ *is uniformly bounded on* $(t_f,t_i]\times[0,1]$.

Proof. By Lemma 7.3 and Corollary 1, we have

$$e^{2\beta(t,\bar\theta)+bU(t,\bar\theta)}=\sqrt{\alpha(t,\bar\theta)}\sqrt{\alpha(t,\bar\theta)}\,e^{2\nu(t,\bar\theta)+bU(t,\bar\theta)}\le B, \tag{61}$$

for some constant $B>0$. The uniform bound on $e^{\beta+bU}$ then follows from Lemma 7.5, since we have

$$|\beta(t,\bar\theta)-\beta(t,\theta)|\le B', \tag{62}$$

$$|U(t,\bar\theta)-U(t,\theta)|\le B', \tag{63}$$

for all $(t,\theta)\in(t_f,t_i]\times[0,1]$ and a fixed constant $B'>0$, which implies

$$e^{\beta(t,\theta)+bU(t,\theta)}\le Be^{B'+|b|B'}.$$ □

7J. ***Control of the polarization.*** Corollary 1 also implies a sharper estimate on the inner product of the Killing fields:

Lemma 7.8. *For any real numbers r and b, $e^{r\beta+bU}A$ is uniformly bounded on $(t_f, t_i] \times [0, 1]$.*

Proof. It follows from Corollary 1 and Lemma 7.4 that $e^{r\beta+bU}A$ is bounded on $(t_f, t_I] \times \{\bar{\theta}\}$. Furthermore,

$$e^{r\beta+bU}A(t,\theta) \le e^{r\beta+bU}A(t,\bar{\theta}) + \int_{\bar{\theta}}^{\theta} \left(e^{r\beta+bU}A(r\beta_\theta + bU_\theta) + e^{r\beta+(b-2)U}e^{2U}A_\theta \right) d\theta'. \tag{64}$$

Using the bound on $e^{r\beta+bU}A(t,\bar{\theta})$, we therefore obtain

$$\left|e^{r\beta+bU}A(t,\theta)\right| \le B + \left| \int_{\bar{\theta}}^{\theta} e^{r\beta+bU}|A|(|r\beta_\theta| + |bU_\theta|)\, d\theta \right|. \tag{65}$$

for some constant $B > 0$ and we can conclude using Gronwall's inequality and Lemma 7.5. □

7K. ***Estimates for the time integrals of the twist quantities.*** To estimate the first derivatives of U and A in the next section, we will need the following estimates for the time integrals of the twist quantities:

Lemma 7.9. *The quantity*

$$\int_t^{t_i} \max_{\theta\in[0,1]} [e^{2\beta-4U}J^2](t',\theta)\, dt'$$

is uniformly bounded on $(t_f, t_i]$.

Proof. From Lemma 7.5, there exists a constant M such that

$$|J(t',\theta)| \le M + |J(t',\bar{\theta})| \quad \text{and} \quad e^{2\beta-4U}(t',\theta) \le e^{6M}e^{2\beta-4U}(t',\bar{\theta}). \tag{66}$$

Thus,

$$\int_t^{t_i} \max_{\theta\in[0,1]} [e^{2\beta-4U}J^2](t',\theta)\, dt' \le \int_t^{t_i} e^{6M}\left(e^{2\beta-4U}(J^2 + 2M|J| + M^2)\right)(t',\bar{\theta})\, dt' \tag{67}$$

and using $2|J| \le J^2 + 1$ as well as Lemma 7.7, we obtain

$$\int_t^{t_i} \max_{\theta\in[0,1]} [e^{2\beta-4U}J^2](t',\theta)dt' \le B + B' \int_t^{t_i} [e^{2\beta-4U}J^2](t',\bar{\theta})\, dt', \tag{68}$$

for some constants B and B'.

Since by integration of (24) we have

$$\int_t^{t_i} [e^{2\beta-4U}J^2](t',\bar{\theta})\, dt' \le t_i \ln \frac{\alpha(t,\bar{\theta})}{\alpha(t_i,\bar{\theta})}, \tag{69}$$

which is bounded from Corollary 1, the right-hand side of (68) is uniformly bounded. □

Similarly:

Lemma 7.10. *The quantity*

$$\int_t^{t_i} \max_{\theta\in[0,1]} [e^{2\beta}(K - AJ)^2](t',\theta)\, dt'$$

is uniformly bounded on $(t_f, t_i]$.

Proof. We first integrate the θ derivative of $e^{\beta}(K-AJ)$, using the auxiliary equations (32) and (30) to replace the derivatives of K_θ and J_θ by matter terms:

$$\begin{aligned} &e^{\beta}|K-AJ|(t,\theta) \\ &\quad\le e^{\beta}|K-AJ|(t,\bar\theta)+\left|\int_{\bar\theta}^{\theta}\left(e^{\beta}|K-AJ||\beta_\theta|+e^{\beta-2U}|J|e^{2U}|A_\theta|\right.\right. \\ &\qquad\qquad \left.\left.+16\pi e^{\beta}\int_{\mathbb{R}^3} f|v_3|\,dv_1\,dv_2\,dv_3+16\pi e^{\beta}A\int_{\mathbb{R}^3} f|v_2|\,dv_1\,dv_2\,dv_3\right)d\theta'\right|. \end{aligned} \tag{70}$$

Using Lemmas 7.5, 7.7 and 7.8, as well as the conservation law (36) and the uniform boundedness (34) of the support of f in v_3 and v_2, we obtain

$$e^{\beta}|K-AJ|(t,\theta)\le e^{\beta}|K-AJ|(t,\bar\theta)+B\left|\int_{\bar\theta}^{\theta}e^{\beta}|K-AJ||\beta_\theta|\,d\theta'\right|+C\max_{\theta\in[0,1]}[e^{\beta-2U}|J|(t,\cdot)]+D, \tag{71}$$

for some constants B, C and D. Applying Gronwall's lemma, we obtain

$$e^{\beta}|K-AJ|(t,\theta)\le\big(e^{\beta}|K-AJ|(t,\bar\theta)+B\max_{\theta\in[0,1]}[e^{\beta-2U}|J|(t,\cdot)]+C\big)\big(1+e^{|\int_{\bar\theta}^{\theta}|\beta_\theta|\,d\theta'|}\big) \tag{72}$$

and therefore, using Lemma 7.5 again, we have

$$\max_{\theta\in[0,1]}e^{\beta}|K-AJ|(t,\cdot)\le D\big(e^{\beta}|K-AJ|(t,\bar\theta)+B\max_{\theta\in[0,1]}[e^{\beta-2U}|J|(t,\cdot)]+C\big). \tag{73}$$

We now conclude by integrating (24) and applying Lemma 7.9 to bound the term containing J. □

7L. *Null cone estimates for the first derivatives of U and A coupled to an estimate for the support of f.* We will perform null cone energy estimates to bound the first derivatives of U and A. However, to close the estimates we will also need to estimate the support of f.

Recall the definition of the energy density:

$$g=U_t^2+\alpha U_\theta^2+\frac{e^{4U}}{4t^2}\left(A_t^2+\alpha A_\theta^2\right).$$

Lemma 7.11. *The function g is uniformly bounded on $(t_f, t_i]\times[0,1]$ and the support of f is uniformly bounded on $(t_f, t_i]\times[0,1]\times\mathbb{R}^3$.*

Proof. We define $g^{\times}$ by

$$g^{\times}=2\sqrt{\alpha}\left(U_tU_\theta+\frac{e^{4U}}{4t^2}A_tA_\theta\right). \tag{74}$$

We have $g\pm g^{\times}\ge 0$. Let $\partial_u=\partial_t-\sqrt{\alpha}\partial_\theta$ and $\partial_v=\partial_t+\sqrt{\alpha}\partial_\theta$.

Using the Einstein equations, we can compute the null derivatives of $g+g^{\times}$ and $g-g^{\times}$:

$$\begin{aligned} \partial_u(g+g^{\times})&=-\frac{2}{t}\left(U_t^2+\frac{e^{4U}}{4t^2}\alpha A_\theta^2\right)+\frac{\alpha_t}{\alpha}(g+g^{\times})-\frac{g^{\times}}{t} \\ &\quad+2(U_t+\sqrt{\alpha}U_\theta)\left(\frac{e^{2\beta-4U}}{2t^2}J^2+8\pi\frac{\sqrt{\alpha}e^{2\beta-2U}}{2t}\int_{\mathbb{R}^3}\frac{f(1+2e^{-2U}v_2^2)}{|v_0|}\,dv_1\,dv_2\,dv_3+e^{2\beta-2U}\Lambda\right) \\ &\quad+\frac{e^{4U}}{2t^2}(A_t+\sqrt{\alpha}A_\theta)\left(\frac{e^{2\beta-4U}}{t^2}J(K-AJ)+16\pi\sqrt{\alpha}\frac{e^{2\beta-4U}}{t}\int_{\mathbb{R}^3}\frac{fv_2(v_3-Av_2)}{|v_0|}\,dv_1\,dv_2\,dv_3\right), \end{aligned} \tag{75}$$

$$
\begin{aligned}
\partial_u(g-g^\times) = &-\frac{2}{t}\left(U_t^2+\frac{e^{4U}}{4t^2}\alpha A_\theta^2\right)+\frac{\alpha_t}{\alpha}(g-g^\times)+\frac{g^\times}{t} \\
&+2(U_t-\sqrt{\alpha}U_\theta)\left(\frac{e^{2\beta-4U}}{2t^2}J^2+8\pi\frac{\sqrt{\alpha}e^{2\beta-2U}}{2t}\int_{\mathbb{R}^3}\frac{f(1+2e^{-2U}v_2^2)}{|v_0|}\,dv_1\,dv_2\,dv_3+e^{2\beta-2U}\Lambda\right) \\
&+\frac{e^{4U}}{2t^2}(A_t-\sqrt{\alpha}A_\theta)\left(\frac{e^{2\beta-4U}}{t^2}J(K-AJ)+16\pi\sqrt{\alpha}\frac{e^{2\beta-4U}}{t}\int_{\mathbb{R}^3}\frac{f\,v_2(v_3-Av_2)}{|v_0|}\,dv_1\,dv_2\,dv_3\right).
\end{aligned}
\tag{76}
$$

Define T_1 and T_2 by

$$
T_1=\frac{e^{2\beta-4U}}{2t^2}J^2+8\pi\frac{\sqrt{\alpha}e^{2\beta-2U}}{2t}\int_{\mathbb{R}^3}\frac{f(1+2e^{-2U}v_2^2)}{|v_0|}\,dv_1\,dv_2\,dv_3+e^{2\beta-2U}\Lambda, \tag{77}
$$

$$
T_2=\frac{e^{2\beta-2U}}{t^3}J(K-AJ)+16\pi\sqrt{\alpha}\frac{e^{2\beta-2U}}{t^2}\int_{\mathbb{R}^3}\frac{f\,v_2(v_3-Av_2)}{|v_0|}\,dv_1\,dv_2\,dv_3. \tag{78}
$$

T_1 and T_2 can be estimated using (24):

$$
|T_1|\le\left|\frac{\alpha_t}{2t\alpha}\right|, \tag{79}
$$

$$
|T_2|\le\left|\frac{\alpha_t}{2\alpha}\right|. \tag{80}
$$

We therefore obtain

$$
|\partial_u(g+g^\times)|\le\left|\frac{\alpha_t}{\alpha}\right|\left(\frac{g}{t}+\frac{1}{2t}+2g\right)+\frac{3g}{t}, \tag{81}
$$

$$
|\partial_v(g-g^\times)|\le\left|\frac{\alpha_t}{\alpha}\right|\left(\frac{g}{t}+\frac{1}{2t}+2g\right)+\frac{3g}{t}. \tag{82}
$$

To perform null cone estimates, in view of the last two inequalities, we need to control the time integral of α_t/α, that is to say, we need to control $\ln\alpha$. Consider the right-hand side of (24). The time integral of the two terms containing the twist quantities are bounded from Lemmas 7.9, 7.10 and the term containing the cosmological constant is bounded from Lemma 7.7. Therefore to control $|\alpha_t/\alpha|$, we only need to control the last term, which is the term containing the Vlasov field. While we already have a bound on the support of the Vlasov field in v_2 and v_3, we still cannot estimate the support of f in v_1. Therefore, the best we can obtain from (24) is an estimate for $|\alpha_t/\alpha|$ which depends on the support of f in v_1 and quantities which have been shown to be bounded. On the other hand, using the characteristic equations associated with the Vlasov equation, i.e., using the geodesic equations, we can obtain a bound on the support of v_1 in terms of g and quantities which have been shown to be bounded. The strategy, which was originally developed by Andréasson [1999], is therefore to combine the two. For this, we define the functions

$$
u_1=\sqrt{\alpha}v_1, \tag{83}
$$

$$
\bar{u}_1(t)=\sup\left\{\sqrt{\alpha}|v_1|\;:\;\exists(t',\theta,v_2,v_3)\in[t,t_i]\times[0,1]\times\mathbb{R}^2\text{ such that }f(t',\theta,v_1,v_2,v_3)\neq 0\right\}, \tag{84}
$$

$$
\psi(t)=\max\Big(\sup_{\theta\in[0,1]}g(t,\cdot)+\bar{u}_1^2(t),2\Big). \tag{85}
$$

We start by estimating $\left|\frac{\alpha_t}{\alpha}\right| = -\frac{\alpha_t}{\alpha}$ in terms of $\bar{u}_1$:

$$-\frac{\alpha_t}{\alpha}(t,\theta) \le C(t) + B(t,\theta), \tag{86}$$

for some nonnegative function $C(t)$ whose integral in time is bounded and where $B(t,\theta)$ is given by

$$B(t,\theta) = 16\pi e^{2\beta-2U} \int_{\mathbb{R}^3} \frac{f(1+e^{-2U}v_2^2+t^{-2}e^{2U}(v_3-Av_2)^2)}{|v_0|}\, du_1\, dv_2\, dv_3.$$

We have

$$\begin{aligned}
&B(t,\theta)\\
&\le 16\pi e^{2\beta-2U}\int_{\mathbb{R}^3} \frac{f(1+e^{-2U}v_2^2+t^{-2}e^{2U}(v_3-Av_2)^2)}{e^{\beta-U}\sqrt{1+e^{-2\beta+2U}u_1^2}}\, du_1\, dv_2\, dv_3\\
&\le 16\pi e^{\beta-U} F\Big(1+e^{-2U}X^2+\frac{e^{2U}}{t^2}(X+|A|X)^2\Big)4X^2\int_{-\bar{u}_1}^{\bar{u}_1}\frac{du_1}{\sqrt{1+e^{-2\beta+2U}u_1^2}}\\
&\le 16\pi e^{\beta-U} F\Big(1+e^{-2U}X^2+\frac{e^{2U}}{t^2}(X+|A|X)^2\Big)4X^2\cdot 2\left(e^{\beta-U}\ln\Big(\bar{u}_1+\sqrt{e^{2\beta-2U}+\bar{u_1}^2}\Big)+e^{-1}\right).
\end{aligned} \tag{87}$$

Therefore, using Lemmas 7.7 and 7.8, it follows from (86) that there exist a nonnegative function $C(t)$ whose integral in time is bounded and a constant $D>0$ such that we have the estimate

$$-\frac{\alpha_t}{\alpha}(t,\theta) \le C(t) + D\ln(1+\bar{u}_1^2). \tag{88}$$

On the other hand, from the characteristic equation of the Vlasov equation (22), it follows that

$$\begin{aligned}
\frac{du_1^2}{ds} &= \frac{\alpha_t}{\alpha}u_1^2\\
&\quad + \frac{2\sqrt{\alpha}u_1}{v_0}\Big(e^{2\beta-2U}(\beta_\theta-U_\theta)+e^{2\beta-4U}(\beta_\theta-2U_\theta)v_2^2+\frac{e^{2\beta}}{t^2}(v_3-Av_2)\big((v_3-Av_2)\beta_\theta-A_\theta v_2\big)\Big)\\
&\quad + \frac{2e^{2\beta}u_1}{t}\left(\frac{(K-AJ)(v_3-Av_2)}{t^2}+e^{-4U}Jv_2\right)
\end{aligned} \tag{89}$$

and therefore we have, by integration,

$$\begin{aligned}
|u_1^2(s)-u_1^2(t_i)| = \bigg|\int_{t_i}^{s}\bigg(&\frac{\alpha_t}{\alpha}u_1^2+\frac{2\sqrt{\alpha}u_1}{v_0}\Big(e^{2\beta-2U}(\beta_\theta-U_\theta)+e^{2\beta-4U}(\beta_\theta-2U_\theta)v_2^2\\
&+\frac{e^{2\beta}}{t^2}(v_3-Av_2)\big((v_3-Av_2)\beta_\theta-A_\theta v_2\big)\Big)\\
&+\frac{2e^{2\beta}u_1}{t}\left(\frac{(K-AJ)(v_3-Av_2)}{t^2}+e^{-4U}Jv_2\right)\bigg)\,ds'\bigg|.
\end{aligned} \tag{90}$$

Let us estimate one by one the terms on the right-hand side.

The first term can be estimated using (88) as follows.[17] For $s < t_i$, we have

$$\left|\int_{t_i}^{s} \frac{\alpha_t}{\alpha} u_1^2 ds'\right| \leq \left|\int_{t_i}^{s} \left|\frac{\alpha_t}{\alpha}\bar{u}_1^2(s')\right| ds'\right| \leq \left|\int_{t_i}^{s} \big(C(s) + D\ln(1+\bar{u}_1^2(s')\big)\,\bar{u}_1^2(s')\,ds'\right|, \tag{91}$$

where $C(s)$ is a nonnegative function whose integral is uniformly bounded and D is a nonnegative constant.

To estimate the second term on the right-hand side of (90), we use (54) to obtain

$$\sqrt{\alpha}|\beta_\theta| \leq Bg + D\bar{u}_1^2, \tag{92}$$

for some constants B and D which depend on the bounds on t, f and the support of f in v_2 and v_3. Moreover, from the definition of g, we have

$$\sqrt{\alpha}|U_\theta| \leq \frac{g}{2} + \frac{1}{2}, \tag{93}$$

$$\sqrt{\alpha}\frac{e^{2U}|A_\theta|}{t} \leq 2g + \frac{1}{2}. \tag{94}$$

From the uniform bounds on $e^{2\beta-2U}$, $e^{2\beta}A$, and the support of f in v_2 and v_3, and from the estimate for β_θ, we have, using $|u_1| \leq |v_0|$, that along a characteristic for which f does not uniformly vanish:

$$\left|\int_{t_i}^{s} \frac{2\sqrt{\alpha}u_1}{v_0}\left(e^{2\beta-2U}(\beta_\theta - U_\theta) + e^{2\beta-4U}(\beta_\theta - 2U_\theta)v_2^2 + \frac{e^{2\beta}}{t^2}(v_3 - Av_2)\big((v_3-Av_2)\beta_\theta - A_\theta v_2\big)\right) ds'\right|$$
$$\leq B + \left|\int_{t_i}^{s} \big(Dg + E\bar{u}_1^2\big)\,ds'\right|, \tag{95}$$

for some constants B, D and E.

Consider the last term on the right-hand side of (90). We have

$$\left|\int_{t_i}^{s} \frac{2e^{2\beta}u_1}{t^3}\big((K-AJ)(v_3-Av_2) + e^{-4U}Jv_2\big)\,ds'\right|$$
$$\leq \left|\int_{t_i}^{s} \frac{2\bar{u}_1}{t_f^3}\Big(e^{\beta}\max_{\theta\in[0,1]}(e^{\beta}|K-AJ|)(t,\cdot)\,\big|X + |A|X\big| + e^{\beta-2U}X\max_{\theta\in[0,1]}(e^{\beta-2U}|J|)(t,\cdot)\Big)\,ds'\right|. \tag{96}$$

Using Lemmas 7.7, 7.8, 7.9, 7.10 and the inequality $2a \leq a^2+1$ to replace $\bar{u}_1$, $\max_{\theta\in[0,1]}(e^{\beta}|K-AJ|)(t,\cdot)$ and $\max_{\theta\in[0,1]}(e^{\beta-2U}|J|)(t,\cdot)$ by their respective squares, we obtain

$$\left|\int_{t_i}^{s} \frac{2e^{2\beta}u_1}{t^3}\big((K-AJ)(v_3-Av_2) + e^{-4U}Jv_2\big)\,ds'\right| \leq B + \left|\int_{t_i}^{s} \bar{u}_1^2 F(s)\,ds'\right|, \tag{97}$$

where B is a constant and $F(s)$ is a nonnegative function whose integral is uniformly bounded. Using (91), (95) and (97), we therefore obtain for $\bar{u}_1$ the estimate

$$\bar{u}_1^2(t) \leq B + \int_{t_i}^{s} \big(C(s) + B\ln(1+\bar{u}_1^2(s'))\big)\,\bar{u}_1^2 ds' + \int_{t_i}^{s} \big(Bg + B\bar{u}_1^2\big)\,ds' + \int_{t_i}^{s} \bar{u}_1^2 F(s)\,ds'. \tag{98}$$

[17] Note the importance of the independence in θ of the right-hand side of (88) to perform the estimate along the characteristics.

where B is a nonnegative constant and $C(s)$, $F(s)$ are nonnegative function whose integrals are uniformly bounded.

These estimates are sufficient to obtain an upper bound on ψ. We first use equations (81) and (82) to do a null cone estimate for $g(t,\theta)$. For this let (t,θ) be in $(t_f, t_i] \times [0,1]$ and integrate (81) and (82) along the integral curves of ∂_u, ∂_v ending at (t,θ). Adding the equations obtained, we have

$$2g(t,\theta) \le B + \int_u \left|\frac{\alpha_t}{\alpha}\right| \left(\left(\frac{2g}{t} + 1 + 2g \right) + \frac{3g}{t} \right) du' + \int_v \left|\frac{\alpha_t}{\alpha}\right| \left(\left(\frac{2g}{t} + 1 + 2g \right) + \frac{3g}{t} \right) dv'. \tag{99}$$

where B is a constant which depends on the maximum of g on the initial hypersurface and is finite by compactness. Using the estimate (88) and taking the maximum for θ in $[0,1]$, we obtain, for $t \in (t_f, t_i]$,

$$\max_{\theta\in[0,1]} g(t,\cdot) \le B + \int_t^{t_i} \big(C(t') + B \ln(1+\bar{u}_1^2(t')) \big) \max_{\theta\in[0,1]} g(t',\cdot)\, dt', \tag{100}$$

where B is a nonnegative constant and $C(t)$ is a nonnegative function whose integral is uniformly bounded. Combining this with (98), we derive for ψ the estimate

$$\psi(t) \le B + \int_t^{t_i} F(s) \ln(\psi)(s) \psi(s)\, ds, \tag{101}$$

where B is nonnegative constant and $F(s)$ is a nonnegative function whose integral is uniformly bounded. From the last line it follows that

$$F\psi \ln\psi \left(B + \int_t^{t_i} F(s)\ln(\psi)(s)\psi(s)\, ds \right)^{-1} \left(\ln\left(B + \int_t^{t_i} F(s)\ln(\psi)(s)\psi(s)\, ds \right) \right)^{-1} \le F(s), \tag{102}$$

and by integration we obtain

$$\psi(t) \le B^{\exp \int_t^{t_i} F(s)\, ds}. \tag{103}$$

Since the integral is uniformly bounded, it follows that ψ is uniformly bounded. ☐

7M. ***Continuous extension of the metric functions.*** Now that g and the support of f have been proven to be uniformly bounded, it follows easily that:

Lemma 7.12. *The first derivatives of U, A, J, K, together with ν_t, α_t are uniformly bounded on $(t_f, t_i] \times [0,1]$ and U, A, ν, α, J, K admit continuous extension to $t = t_f$.*

7N. ***Estimates for the derivatives of f, ν_θ, α_θ and higher order estimates.*** Such estimates follow by standard methods, which can be found for instance in [Weaver 2004].

7O. ***The conclusion.*** Since all metric functions, the Vlasov field and all their derivatives have been shown to be uniformly bounded, the assumptions of Proposition 2 have been retrieved. In particular, the maximal Cauchy development cannot have $t_f > 0$ which concludes the proof of Theorem 1.

8. Proof of Theorem 2

8A. ***The Einstein equations in areal coordinates for vacuum T^2-symmetric spacetimes.*** The Einstein equations (1) for vacuum T^2-symmetric solutions reduce in areal coordinates to the following system of equations:

Constraint equations:

$$\frac{\nu_t}{t} = U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2) + \frac{\alpha e^{2\nu} K^2}{4t^4} + \alpha e^{2(\nu-U)}\Lambda, \tag{104}$$

$$\frac{\nu_\theta}{t} = 2U_t U_\theta + \frac{e^{4U}}{2t^2} A_t A_\theta - \frac{\alpha_\theta}{2t\alpha}, \tag{105}$$

$$\frac{\alpha_t}{\alpha} = -4t\alpha e^{2(\nu-U)}\Lambda - \frac{\alpha e^{2\nu} K^2}{t^3}. \tag{106}$$

Evolution equations:

$$\nu_{tt} - \alpha\nu_{\theta\theta} = \frac{\alpha_\theta \nu_\theta}{2} + \frac{\alpha_t \nu_t}{2\alpha} - \frac{\alpha_\theta^2}{4\alpha} + \frac{\alpha_{\theta\theta}}{2} - U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 - A_\theta^2) - \frac{3\alpha e^{2\nu} K^2}{4t^4} + \alpha\Lambda e^{2(\nu-U)}, \tag{107}$$

$$U_{tt} - \alpha U_{\theta\theta} = -\frac{U_t}{t} + \frac{\alpha_\theta U_\theta}{2} + \frac{\alpha_t U_t}{2\alpha} + \frac{e^{4U}}{2t^2}(A_t^2 - \alpha A_\theta^2) + \alpha\Lambda e^{2(\nu-U)}, \tag{108}$$

$$A_{tt} - \alpha A_{\theta\theta} = \frac{A_t}{t} + \frac{\alpha_\theta A_\theta}{2} + \frac{\alpha_t A_t}{2\alpha} - 4(A_t U_t - \alpha A_\theta U_\theta). \tag{109}$$

Auxiliary equations:

$$0 = G_t + A H_t, \tag{110}$$

$$0 = H_t - \frac{\sqrt{\alpha} e^{2\nu} K}{t^3}. \tag{111}$$

Note that the Killing fields have been chosen such that the twist quantity J vanishes and note that K is a nonnegative constant (see Section 2A).

Let us define the following replacement for the function U:

$$P = 2U - \ln t. \tag{112}$$

We refer to the discussion on page 202 for the motivation for the introduction of the quantity P.

The evolution equation for U leads to the following equation for P:

$$P_{tt} - \alpha P_{\theta\theta} = \left(-\frac{1}{t} + \frac{1}{2}\frac{\alpha_t}{\alpha}\right) P_t + \frac{\alpha_\theta P_\theta}{2} + e^{2P}(A_t^2 - \alpha A_\theta^2) - \frac{1}{2t^4}\alpha e^{2\nu} K^2. \tag{113}$$

As mentioned on page 202, this equation is homogeneous in the Gowdy case $K = 0$, since there are no terms containing Λ compared to (108). In the following, it will be useful to work both with P and U and to use two energy densities, one associated with the system of wave equations for (U, A) and one associated with the system of wave equations for (P, A).

8B. ***The universal cover of $\mathcal{M}/T^2$.*** In Section 8K, we will study the characteristic equation which defines null rays in areal coordinates. It will be easier to address this problem in the universal cover of the quotient of the spacetime. For any T^2-symmetric spacetimes $(\mathcal{M}, g)$, we introduce $\mathcal{Q} = \mathcal{M}/T^2$, the quotient of the spacetime by the orbits of symmetry, and then define $\tilde{\mathcal{Q}}$ as the universal cover of $\mathcal{Q}$. Let $\pi_1 : \mathcal{M} \to \mathcal{Q}$ be the natural projection from $\mathcal{M}$ to $\mathcal{Q}$.

Suppose $(\mathcal{M}, g)$ is foliated by areal coordinates with the metric taking the form (2). Let α_Q be such that α is the pull-back of α_Q by π_1^*. We then define $\tilde{\alpha}$ to be the lift to $\tilde{\mathcal{Q}}$ of α_Q. We may define similarly tilde functions for all metric functions, such as $\tilde{\nu}$, $\tilde{U}$, etc. Note that $\tilde{\mathcal{Q}}$ has topology $\mathbb{R}\times\mathbb{R}$ and admits areal coordinates $(\tilde{t},\tilde{\theta})\in(t_f.t_i]\times\mathbb{R}$ and Lorentzian metric:

$$ds^2=-e^{2(\tilde{\nu}-\tilde{U})}(\tilde{\alpha}d\tilde{t}^2-d\tilde{\theta}^2). \tag{114}$$

Note also that all tilde functions $\tilde{\nu}$, $\tilde{U}$, etc. are periodic in θ with period 1 and that they satisfy the system of equations (104)–(111) on $(t_f.t_i]\times\mathbb{R}$.

In the following, we will often,[18] by an abuse of notation,[19] drop the tildes on functions defined on $\tilde{\mathcal{Q}}$.

8C. ***The contradiction setting.*** As explained in Section 6, the proof will follow by contradiction. Let us thus assume that $(\mathcal{M}, g)$ is the past maximal development of vacuum T^2-symmetric spacetimes with $\Lambda>0$ such that $t_0>0$. By Proposition 1, there exist a global areal foliation where the metric takes the form (2) and such that t lies in $(t_0,t_i]$. Thus, there exists functions α, ν, U, A defined on $(t_0,t_i]\times[0,1]$ which are periodic in θ with period 1, and a constant K such that α, ν, U, A and K satisfy the system of equations (104), (109). Moreover, since the cases where $\Lambda=0$ have already been treated, and since the cases where $K=0$, $\Lambda>0$ may be treated by similar methods as we explained in the previous section, we will suppose that we are in the case where $K>0$ and $\Lambda>0$. Finally, let us assume that the assumptions of Theorem 1 hold, i.e., the spacetime is not polarized.

8D. ***Uniform blow up of α.*** The contradiction setting immediately implies the following:

Lemma 8.1. *Under the assumptions of Section 8C, for all $\theta\in[0,1]$, we have $\alpha(t,\theta)\to\infty$ as $t\to t_0$ and $\min_{\theta\in[0,1]}\alpha(t,\theta)\to\infty$ as $t\to t_0$.*

Proof. Suppose the lemma does not hold. Because α is monotonic, it follows that $\min_{\theta\in[0,1]}\alpha(t,\theta)$ is uniformly bounded, i.e., results similar to those of Section 7H hold. We may then apply similar estimates as the estimates of sections 7I to 7N, replacing f by 0 everywhere. Indeed, the presence of the Vlasov matter was necessary only so as to ensure that the content of Section 7H is valid. Proposition 2 then applies, and thus $(\mathcal{M}, g)$ is not maximal, a contradiction. □

Remark 8D.1. Since the rest of the proof of Theorem 2 will rely on the assumptions of Section 8C, it will be from now on assumed that they hold.

8E. ***The basic energy estimates.*** We will need to work with several energy densities and several energy integrals. Let us thus define

$$g=U_t^2+\alpha U_\theta^2+\frac{e^{4U}}{4t^2}\left(A_t^2+\alpha A_\theta^2\right), \tag{115}$$

$$h=P_t^2+\alpha P_\theta^2+e^{2P}\left(A_t^2+\alpha A_\theta^2\right). \tag{116}$$

$$E_g(t)=\int_{[0,1]}\frac{g}{\sqrt{\alpha}}d\theta, \tag{117}$$

$$E_h(t)=\int_{[0,1]}\frac{h}{\sqrt{\alpha}}d\theta, \tag{118}$$

[18] That is to say, we shall use the same symbol for a function defined on $\mathcal{M}$ and for its associated tilde function.

[19] Note that strictly speaking, in the analysis of Section 7, all metric functions were also defined on $\mathcal{Q}$ rather than $\mathcal{M}$ since we had considered them to be function of (t,θ). The same remark applies for the analysis carried in Section 9.

$$E_{h,K}(t) = E_h(t) + \int_{[0,1]} \frac{\sqrt{\alpha}e^{2\nu}K^2}{t^4} d\theta, \tag{119}$$

$$E_{h,K,\Lambda}(t) = E_h(t) + \int_{[0,1]} \left(\frac{\sqrt{\alpha}e^{2\nu}K^2}{t^4} + 4\Lambda \frac{\sqrt{\alpha}e^{2\nu-P}}{t} \right) d\theta. \tag{120}$$

Several computations will also be useful for the rest of the analysis. First, using the constraint equations (104) and (106), we have the identities

$$\frac{\partial}{\partial t}\left(\frac{\sqrt{\alpha}e^{2\nu-P}}{t} \right) = \frac{1}{2}\sqrt{\alpha}e^{2\nu-P}\left(h - \frac{1}{t^2} \right), \tag{121}$$

$$\frac{\partial}{\partial t}\left(\sqrt{\alpha}e^{2\nu} \right) = 2t\sqrt{\alpha}e^{2\nu} g. \tag{122}$$

Taking the time derivative of E_h and using the Einstein equations, we obtain

$$\frac{dE_h}{dt} = -\frac{2}{t}\int_{[0,1]} \frac{P_t^2}{\sqrt{\alpha}} + e^{2P}\sqrt{\alpha}A_t^2 - 2\Lambda \int_{[0,1]} \sqrt{\alpha}e^{2\nu-P}h - \frac{2}{t^3}\int_{[0,1]} \sqrt{\alpha}e^{2\nu}K^2 g + \frac{1}{2t^5}\int_{[0,1]} \sqrt{\alpha}e^{2\nu}K^2 - \frac{2}{t}\int_{[0,1]} \sqrt{\alpha}P_{\theta\theta} + \frac{\alpha_\theta}{2\sqrt{\alpha}}P_\theta. \tag{123}$$

The terms on the last line vanish thanks to the θ periodicity, so we obtain[20]

$$\frac{dE_h}{dt} = -\frac{2}{t}\int_{[0,1]} \frac{P_t^2}{\sqrt{\alpha}} + \frac{e^{2P}}{\sqrt{\alpha}}A_t^2 - 2\Lambda \int_{[0,1]} \sqrt{\alpha}e^{2\nu-P}h - \frac{2}{t^3}\int_{[0,1]} \sqrt{\alpha}e^{2\nu}K^2 g + \frac{1}{2t^5}\int_{[0,1]} \sqrt{\alpha}e^{2\nu}K^2. \tag{124}$$

or, written only in terms of h and P_t,

$$\frac{dE_h}{dt} = -\frac{2}{t}\int_{[0,1]} \frac{P_t^2}{\sqrt{\alpha}} + \frac{e^{2P}}{\sqrt{\alpha}}A_t^2 - \int_{[0,1]} \frac{P_t}{\sqrt{\alpha}t^4}\alpha e^{2\nu}K^2 + \int_{[0,1]} \frac{1}{2}\frac{\alpha_t}{\alpha^{3/2}}h. \tag{125}$$

We see that the last term on the right-hand side of (124) is competing against the others.

Remark 8E.1. In the case where $K = 0$, the last term vanishes, thus, we obtain the desired monotonicity[21] on E_h and we could conclude as in [Isenberg and Weaver 2003]. Thus, we obtain:

Proposition 4. *Let $(\mathcal{M}, g)$ be the maximal development of T^2-symmetric initial data in the vacuum with $\Lambda \geq 0$ and $K = 0$. Suppose that E_h does not vanish identically. Then $(\mathcal{M}, g)$ admits a global foliation by areal coordinates with the time coordinate t taking all values in $(0, \infty)$, i.e., $t_0 = 0$ in the notation of Proposition 1.*

Unfortunately, in the general case, we lose this monotonicity and the analysis is, as we will see, more complex.

[20] That the terms involving derivatives in θ add up to an exact derivative is due to the wave map background structure of the equations. See [Berger et al. 1997].

[21] Note that the parallelism between the cases ($K > 0$, $\Lambda = 0$) and ($K = 0$, $\Lambda > 0$) does not extend beyond the issue of the value of t_0. Indeed, once we know that $t_0 = 0$, the different powers of t for the terms containing Λ and K in (106) are likely to yield different asymptotics for the solutions.

We may also compute the time derivative of $E_{h,K}$ and $E_{h,K,\Lambda}$:

$$\frac{dE_{h,K}}{dt} = -\frac{2}{t}\int_{[0,1]} \frac{P_t^2}{\sqrt{\alpha}} + e^{2P}\sqrt{\alpha}A_t^2 - 2\Lambda\int_{[0,1]} \sqrt{\alpha}e^{2\nu-P}h - \frac{7}{2t^5}\int_{[0,1]} \sqrt{\alpha}e^{2\nu}K^2, \tag{126}$$

$$\frac{dE_{h,K,\Lambda}}{dt} = -\frac{2}{t}\int_{S^1[0,1]} \frac{P_t^2}{\sqrt{\alpha}} + e^{2P}\sqrt{\alpha}A_t^2 - \frac{7}{2t^5}\int_{[0,1]} \sqrt{\alpha}e^{2\nu}K^2 - 2\Lambda\int_{[0,1]} \frac{\sqrt{\alpha}e^{2\nu-P}}{t^2}. \tag{127}$$

We see in particular that $E_{h,K}$ and $E_{h,K,\Lambda}$ are nondecreasing with decreasing time.[22]

Lemma 8.2. *E_g, E_h, $E_{h,K}$ and $E_{h,K,\Lambda}$ are uniformly bounded on $(t_0, t_i]$ and the last two quantities can be continuously extended to t_0.*

Proof. From (127), we have

$$\frac{dE_{h,K,\Lambda}}{dt} \geq -\frac{7}{2t}E_{h,K,\Lambda}. \tag{128}$$

By application of Gronwall's lemma, therefore, $E_{h,K,\Lambda}$ is bounded uniformly if $t_0 > 0$. However, since

$$E_h \leq E_{h,K} \leq E_{h,K,\Lambda}, \tag{129}$$

we also obtain a uniform bound on E_h and $E_{h,K}$. Since $E_{h,K}$ and $E_{h,K,\Lambda}$ are monotonically increasing they admit strictly positive limits at $t = t_0$. A similar analysis implies the uniform bound on E_g. □

8F. *Continuous extensions of the twist and cosmological energies.* In order to extract some information from the continuous extensions of $E_{h,K}$ and $E_{h,K,\Lambda}$, we will need the following:

Lemma 8.3. *The functions $\sqrt{\alpha}e^{2\nu}$ and $\sqrt{\alpha}e^{2\nu-P}$, and therefore also $\sqrt{\alpha}e^{2\nu}K^2/t^4$ and $\Lambda\sqrt{\alpha}e^{2\nu-P}/t$, admit continuous extensions to $t = t_0$ and are uniformly bounded in $(t_0, t_i] \times [0, 1]$.*

Proof. The derivatives with respect to t of $\sqrt{\alpha}e^{2\nu}$ and $t^{-1/2}\sqrt{\alpha}e^{2\nu-P}$ are positive, as can be verified by direction computation. Therefore, they are monotonically decreasing in the past direction and admit continuous extensions to $t = t_0$. Moreover, they are bounded by the maximum of their values on the initial data surface, which is finite by compactness. □

Since $\sqrt{\alpha}e^{2\nu}$ and $t^{-1/2}\sqrt{\alpha}e^{2\nu-P}$ are pointwise decreasing with t in the past direction and are positive, their integrals over θ at fix t are positive functions which are decreasing in the past direction and therefore, they admit a limit as t goes to t_0. Thus:

Lemma 8.4. *The integrals*

$$\int_{[0,1]} \frac{\sqrt{\alpha}e^{2\nu}K^2}{t^4}d\theta \quad \text{and} \quad \int_{[0,1]} \Lambda\frac{\sqrt{\alpha}e^{2\nu-P}}{t}d\theta$$

admit continuous extensions to $t = t_0$.

8G. *Estimate for the spatial derivatives of β and $\beta - P/2$.* We define β as in the Vlasov case by

$$e^{2\beta} = \alpha e^{2\nu}. \tag{130}$$

[22] Note that this monotonicity cannot be used as a replacement of the monotonicity of E_g or E_h, since no estimate similar to (13) can hold when E_g is replaced by $E_{h,K}$ or $E_{h,K,\Lambda}$, as can be seen by studying homogeneous plane symmetric solutions.

It follows as in Lemma 7.5 that β_θ is bounded by $g/\sqrt{\alpha}$:

$$|\beta_\theta| \le t\frac{g}{\sqrt{\alpha}}. \tag{131}$$

By integration, we obtain:

Lemma 8.5. *For all $t \in (t_0, t_i]$,*

$$\max_{[0,1]} \beta(t,\cdot) - \min_{[0,1]} \beta(t,\cdot) \le t_i E_g. \tag{132}$$

In particular, $\max_{[0,1]} \beta(t,\cdot) - \min_{[0,1]} \beta(t,\cdot)$ *is uniformly bounded.*

We may do the same analysis using h and P. First, we rewrite (105) as

$$\beta_\theta - \frac{P_\theta}{2} = \frac{t}{2}\left(P_t P_\theta + e^{2P} A_t A_\theta\right), \tag{133}$$

from which we obtain that

$$\left|\beta_\theta - \frac{P_\theta}{2}\right| \le \frac{t}{4}\frac{h}{\sqrt{\alpha}}. \tag{134}$$

Therefore, using the bounds on E_h, we have:

Lemma 8.6. *For all $t \in (t_0, t_i]$,*

$$\max_{[0,1]}(2\beta - P)(t,\cdot) - \min_{[0,1]}(2\beta - P)(t,\cdot) \le \frac{t_i}{2}E_h. \tag{135}$$

In particular, $\max_{[0,1]}(2\beta - P)(t,\cdot) - \min_{[0,1]}(2\beta - P)(t,\cdot)$ *is uniformly bounded.*

8H. *Limit of the gravitational energy of the orbits of symmetry.*

Lemma 8.7. *For all $\epsilon > 0$ there exists $t_\epsilon > t_0$ such that either $E_h(t_\epsilon) \le \epsilon$ or $E_g(t_\epsilon) \le \epsilon$.*

Proof. Suppose the lemma does not hold. Then there exists an $\epsilon > 0$ such that $\min(E_h, E_g) > \epsilon$ for all $t > t_0$.

By integration of (124), we have, for all $t \in (t_0, t_i]$,

$$\int_t^{t_i} 2\Lambda \int_{[0,1]} \sqrt{\alpha} e^{2\nu - P} h\, d\theta\, dt' + \int_t^{t_i} \frac{2K^2}{t^3}\sqrt{\alpha} e^{2\nu} g\, d\theta\, dt' \le E_h(t) - E_h(t_i) + \int_t^{t_i} \frac{1}{2t^5}\int_{[0,1]} \sqrt{\alpha} e^{2\nu} K^2. \tag{136}$$

Since all terms on the right-hand side are bounded by Lemmas 8.2 and 8.4, we have in particular, that, there exits some constant $D > 0$ such that

$$\int_t^{t_i} 2\Lambda \int_{[0,1]} \sqrt{\alpha} e^{2\nu - P} h\, d\theta\, dt \le D. \tag{137}$$

Using the control on the spatial derivatives of $2\beta - P$ obtained in Lemma 8.6, we obtain, for all (t, θ) in $(t_0, t_i] \times [0, 1]$,

$$\int_t^{t_i} \int_{[0,1]} e^{2\beta - P} \frac{h}{\sqrt{\alpha}}\, d\theta\, ds \le B, \tag{138}$$

$$\int_t^{t_i} \min_{\theta' \in [0,1]} e^{2\beta - P}(s,\cdot) E_h(s)\, ds \le B, \tag{139}$$

$$\int_t^{t_i} \min_{\theta' \in [0,1]} e^{2\beta - P}(s, \cdot)\, ds \leq \frac{B}{\epsilon}, \tag{140}$$

$$\int_t^{t_i} e^{2\beta - P}(s, \theta)\, ds \leq \frac{B}{\epsilon} + B'(t_i - t), \tag{141}$$

$$\int_t^{t_i} e^{2\beta - P}(s, \theta)\, ds \leq B'', \tag{142}$$

for some constants $B > 0$, $B' > 0$ and $B'' > 0$.

Similarly, one obtains from inequality (136) and Lemma 8.5 the existence of a constant $B''' > 0$ such that, for all $(t, \theta) \in (t_0, t_i]$,

$$\int_t^{t_i} e^{2\beta}(s, \theta)\, ds \leq B'''. \tag{143}$$

It follows from (142) and (143) that the right-hand side of (106) is bounded and by integration, $\ln \alpha$ and therefore α are uniformly bounded above, which contradicts Lemma 8.1. □

We may now prove a stronger version of Lemma 8.7:

Lemma 8.8. *$E_h \to 0$ as $t \to 0$ and $E_g \to 0$ as $t \to 0$.*

Proof. We have $E_h = E_{h,K} - \int_{[0,1]} (\sqrt{\alpha} e^{2\nu} K^2 / t^4)\, d\theta$. In view of Lemmas 8.2 and 8.4, both terms on the right-hand side have a limit, thus E_h has a limit. Similarly, E_g has a limit. In view of the last lemma, both limits cannot be strictly positive and therefore at least one of them has to be zero. Suppose for instance, that E_h tends to 0 as t tends to t_0. From the definition of h, g, P and U it follows that

$$g = \frac{h}{4} + \frac{P_t}{2t} + \frac{1}{4t^2},$$

and therefore

$$g \leq \frac{h}{2} + \frac{1}{2t^2}. \tag{144}$$

Since on the other hand, $\sqrt{\alpha}$ tends to infinity uniformly in θ by Lemma 8.1, it follows from the last inequality that E_g also tends to 0 as t tends to t_0. The case where we know a priori that E_g tends to 0 and we need to deduce that E_h tends to 0 may be treated similarly. □

8I. *Strong control on the spatial derivative of β.* An immediate application of these limits allows an improvement to Lemmas 8.5 and 8.6:

Lemma 8.9.

$$\lim_{t \to t_0} \bigl(\max_{[0,1]} \beta(t, \cdot) - \min_{[0,1]} \beta(t, \cdot) \bigr) = 0,$$

$$\lim_{t \to t_0} \bigl(\max_{[0,1]} (2\beta - P)(t, \cdot) - \min_{[0,1]} (2\beta - P)(t, \cdot) \bigr) = 0. \tag{145}$$

From this it follows that:

Lemma 8.10. *For all $\epsilon > 0$, there exists $t' > t_0$ such that, for all $t \in (t_0, t']$,*

$$\max_{\theta \in [0,1]} e^{2\beta}(t, \cdot) \leq e^{\epsilon} \min_{[0,1]} e^{2\beta}(t, \cdot), \tag{146}$$

$$\max_{\theta \in [0,1]} e^{2\beta - P}(t, \cdot) \leq e^{\epsilon} \min_{[0,1]} e^{2\beta - P}(t, \cdot), \tag{147}$$

$$\max_{\theta \in [0,1]} \Bigl(-\frac{\alpha_t}{\alpha}(t, \cdot) \Bigr) \leq e^{\epsilon} \min_{\theta \in [0,1]} \Bigl(-\frac{\alpha_t}{\alpha}(t, \cdot) \Bigr). \tag{148}$$

Proof. The first two inequalities follow directly from the last lemma. Next, write the Einstein equation for α, (106), in terms of β and P:

$$\frac{\alpha_t}{\alpha} = -4e^{2\beta-P}\Lambda - \frac{e^{2\beta}K^2}{t^3}. \tag{149}$$

Now the last inequality of the lemma follows from the first two. □

Note that by integration, we could easily obtain from the last line that for all $\epsilon > 0$, there exists $t' > t_0$ and a constant $C > 0$ such that

$$\max_{\theta\in[0,1]} \alpha(t,\cdot) \le C \min_{\theta\in[0,1]} \alpha(t,\cdot)^{1+\epsilon} \quad \text{for all } t \in (t_0, t']. \tag{150}$$

Unfortunately, the exponent of the right-hand side is not 1 and this will not be sufficient for our analysis. Thus, we need a stronger estimate than this one, which we provide in the next section.

8J. ***An estimate for*** $(\partial/\partial\theta)(\ln\alpha)$**.** The estimates on β_θ and $2\beta_\theta - P_\theta$ coming from the inequalities (131) and (134) were based on previously known estimates for T^2-symmetric spacetimes written in areal coordinates. Here, we will derive a stronger estimate from these inequalities, using the identities (122) and (121) and Equation (106). The estimate that we obtain is the following:

Lemma 8.11. *There exists a constant $C > 0$ such that, for all $(t,\theta) \in (t_0, t_i] \times [0,1]$,*

$$\left|\frac{\partial}{\partial\theta}(\ln(\alpha))(t,\theta)\right| \le C. \tag{151}$$

Proof. Multiplying (131) and (134) by $e^{2\beta}$ and $e^{2\beta-P}$, we obtain

$$|\beta_\theta| e^{2\beta} \le t\frac{g}{\sqrt{\alpha}} e^{2\beta} = \frac{1}{2}\partial_t\big(\sqrt{\alpha}e^{2\nu}\big), \tag{152}$$

$$\left|\beta_\theta - \frac{P_\theta}{2}\right| e^{2\beta-P} \le \frac{t}{4}\frac{h}{\sqrt{\alpha}}e^{2\beta-P} = \frac{t^{1/2}}{2}\partial_t\big(t^{-1/2}\sqrt{\alpha}e^{2\nu-P}\big), \tag{153}$$

where we have used the identities (122) and (121) arising from the constraints to rewrite the right-hand sides of the equations.

On the other end, from (106), we have

$$-\frac{\partial}{\partial t}\ln\alpha = 4\Lambda e^{2\beta-P} + \frac{K^2 e^{2\beta}}{t^3}. \tag{154}$$

Thus, taking the θ derivative of the last equation, we obtain

$$-\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial t}\ln\alpha\right) = 4\Lambda(2\beta_\theta - P_\theta)e^{2\beta-P} + 2\beta_\theta\frac{K^2 e^{2\beta}}{t^3}. \tag{155}$$

We now integrate the last line and commute the θ and t partial derivatives in the integrand of the left-hand side to obtain, for all $(t,\theta) \in (t_0, t_i] \times [0,1]$,

$$\partial_\theta \ln\alpha(t,\theta) = \partial_\theta \ln\alpha(t_i,\theta) + \int_t^{t_i} 4\Lambda(2\beta_\theta - P_\theta)e^{2\beta-P}(s,\theta)\,ds + \int_t^{t_i} 2\beta_\theta\frac{K^2 e^{2\beta}}{t^3}(s,\theta)\,ds. \tag{156}$$

Using (152) and (153), we have

$$|\partial_\theta \ln \alpha(t,\theta)| \le \sup_{\theta\in[0,1]} |\partial_\theta \ln \alpha(t_i,\cdot)| + \int_t^{t_i} 4\Lambda \left|(2\beta_\theta - P_\theta)e^{2\beta-P}\right|(s,\theta)\,ds + \int_t^{t_i} \left|2\beta_\theta \frac{K^2 e^{2\beta}}{t^3}\right|(s,\theta)\,ds,$$
$$\le \sup_{\theta\in[0,1]} |\partial_\theta \ln \alpha(t_i,\cdot)| + 4\Lambda t_i^{1/2} \int_t^{t_i} \partial_t\big(t^{-1/2}\sqrt{\alpha}e^{2\nu-P}\big) + \frac{K^2}{t_0^3}\int_t^{t_i} \partial_t(\sqrt{\alpha}e^{2\nu}), \tag{157}$$

and the lemma follows from the uniform bounds on $\sqrt{\alpha}e^{2\nu}$ and $\sqrt{\alpha}e^{2\nu-P}$. □

By integration, we immediately obtain:

Corollary 2. *There exists a constant $C > 0$ such that, for all $t \in (t_0, t_i]$, we have*

$$\max_{\theta\in[0,1]} \alpha(t,\theta) \le C \min_{\theta\in[0,1]} \alpha(t,\theta). \tag{158}$$

Combining this with Lemma 8.5, we may obtain:

Corollary 3. *There exist constants M_1 and M_2 such that for all $(t,\theta) \in (t_0, t_i]$, we have*

$$M_1\sqrt{\alpha}(t,\theta) \ge e^{2\beta}(t,\theta) \ge M_2\sqrt{\alpha}(t,\theta). \tag{159}$$

Similarly, there exist constants M_1' and M_2' such that for all for all $(t,\theta) \in (t_0, t_i]$, we have

$$M_1'\sqrt{\alpha}(t,\theta) \ge e^{2\beta-P}(t,\theta) \ge M_2'\sqrt{\alpha}(t,\theta). \tag{160}$$

Proof. Given that $E_{h,K}$ is nondecreasing in the past direction, that E_h tends to zero as t tends to t_0 and that $K > 0$, it follows that the limit of $\int_{[0,1]} \sqrt{\alpha}e^{2\nu} d\theta$ is nonzero. This implies, using the monotonicity of $\sqrt{\alpha}e^{2\nu}$ as a function of t and the monotone convergence theorem, that there exists a θ_0 and a constant $M > 0$ such that $\sqrt{\alpha}e^{2\nu}(t,\theta_0) \ge M$ for all $t \in (t_0, t_i]$. Let M' be an upper bound for $\sqrt{\alpha}e^{2\nu}(t,\theta_0)$. By Lemma 8.5, there exits a constant M'' such that, for all $(t,\theta) \in (t_0, t_i] \times [0,1]$,

$$e^{M''}e^{2\beta}(t,\theta_0) \ge e^{2\beta}(t,\theta) \ge e^{-M''}e^{2\beta}(t,\theta_0) \tag{161}$$

and thus

$$M'e^{M''}\sqrt{\alpha}(t,\theta_0) \ge e^{2\beta}(t,\theta) \ge Me^{-M''}\sqrt{\alpha}(t,\theta_0) \tag{162}$$

Let M''' be such that, for all $(t,\theta) \in t \in (t_i, t_0] \times [0,1]$,

$$e^{M'''}\sqrt{\alpha}(t,\theta) \ge \sqrt{\alpha}(t,\theta_0) \ge e^{-M'''}\sqrt{\alpha}(t,\theta). \tag{163}$$

Then we have

$$M'e^{M'''}e^{M''}\sqrt{\alpha}(t,\theta) \ge e^{2\beta}(t,\theta) \ge Me^{-M'''}e^{-M''}\sqrt{\alpha}(t,\theta) \tag{164}$$

This proves the inequalities (159). The second set of inequalities can be treated similarly, using E_g and another energy integral

$$E_{g,\Lambda} = \int_{[0,1]} \left(\frac{g}{\sqrt{\alpha}} + \alpha e^{2(\nu-U)}\Lambda\right) d\theta, \tag{165}$$

which may be easily proven to be nondecreasing in the past direction and uniformly bounded. □

The aim of the next two sections will be to describe the characteristics curves and to establish several estimates about their behavior for t close to t_0. We will actually not need to analyze all null curves, but only null curves orthogonal to the orbits of symmetry. Note that in the next sections, we will often, by an abuse of notation, denote by the same name functions defined on $\mathcal{M}$ or $\mathcal{Q}$ together with their lifts to $\tilde{\mathcal{Q}}$, the universal cover of $\mathcal{Q}$.

8K. ***An analysis of the characteristics in areal coordinates.*** Consider a null curve γ in $\mathcal{M}$ which is orthogonal to the orbits of symmetry and let $\tilde{\gamma}$ be the lift to $\tilde{\mathcal{Q}}$ of the projection to $\mathcal{Q}$ of γ. In null coordinates as those used in [Smulevici 2008], γ is given by $u =$ constant or $v =$ constant. In areal coordinates, we obtain γ by solving the characteristic equation

$$\Theta'(s) = \pm\sqrt{\alpha(s, \Theta(s))}, \tag{166}$$

with appropriate initial conditions. If $\Theta(t)$ is a solution to the above equation, then γ is given in areal coordinates by $(t, \Theta(t))$.

By standard arguments, solutions of (166) exist and are smooth and unique on $(t_0, t]$ for any $t \in (t_0, t_i]$ once initial conditions have been fixed.

Now let us consider the characteristics parallel to the constant v lines. They are parametrized by $(s, \Theta(s, \theta, t))$, where $\Theta(s, \theta, t)$ satisfies

$$\Theta(s, \theta, t) = \theta - \int_t^s \sqrt{\alpha(s', \Theta(s', \theta, t))}ds'. \tag{167}$$

Taking the θ derivative,

$$\Theta_\theta(s, \theta, t) = 1 - \int_t^s \frac{1}{2}\left(\frac{\alpha_\theta}{\sqrt{\alpha}}\right)\big(s', \Theta(s', \theta, t)\big)\,\Theta_\theta(s', \theta, t)\,ds'. \tag{168}$$

Solving this equation implicitly, we see that

$$\Theta_\theta(s, \theta, t) = \exp\int_s^t \frac{1}{2}\left(\frac{\alpha_\theta}{\sqrt{\alpha}}\right)(s', \Theta(s, \theta, t))\,ds'. \tag{169}$$

We are naturally lead to estimate the integral on the right. This is the subject of the next section.

8L. ***Estimates for the integral along the characteristics of $\alpha_\theta/\sqrt{\alpha}$.***

Lemma 8.12. *For all $\epsilon > 0$, there exists a $\bar{t} > t_0$, such that for all $t' \in (t_0, \bar{t}]$ there exists a negative constant M_1 and a positive constant M_2 such that, for all $(t, \theta) \in (t_0, t'] \times [0, 1]$,*

$$M_1 - \epsilon \ln\alpha(t, \Theta(t, \theta, t')) \le \int_t^{t'} -\frac{\alpha_\theta}{\sqrt{\alpha}}(s, \Theta(s, \theta, t'))\,ds \le M_2 + \epsilon \ln\alpha(t, \Theta(t, \theta, t')). \tag{170}$$

Proof. Let $\epsilon > 0$ and let $\bar{t} \in (t_0, t_i]$ be such that Lemma 8.10 holds in the following way: for all $(t, \theta, \theta') \in (t_0, \bar{t}] \times [0, 1]^2$,

$$-(1-\epsilon)\frac{\alpha_t}{\alpha}(t, \theta') < -\frac{\alpha_t}{\alpha}(t, \theta) < -(1+\epsilon)\frac{\alpha_t}{\alpha}(t, \theta'). \tag{171}$$

Fix $t' \in (t_0, \bar{t}]$ and let $\Theta(t, \theta, t')$ be a characteristic such that

$$\Theta(t, \theta, t') = \theta - \int_{t'}^{t} \sqrt{\alpha}(s, \Theta(s, \theta, t'))\, ds. \tag{172}$$

We have, for all $(t, \theta) \in (t_0, t'] \times [0, 1]$,

$$\int_t^{t'} -\frac{\alpha_\theta}{\sqrt{\alpha}}(s, \Theta(s, \theta, t'))\, ds = \int_t^{t'} \left(\frac{\alpha_t}{\alpha} - \frac{\alpha_\theta}{\sqrt{\alpha}} - \frac{\alpha_t}{\alpha}\right)(s, \Theta(s, \theta, t'))\, ds \tag{173}$$

$$= \int_t^{t'} \frac{d}{ds}\left(\ln \alpha(s, \Theta(s, \theta, t')\right) ds - \int_t^{t'} \frac{\alpha_t}{\alpha}(s, \Theta(s, \theta, t'))\, ds. \tag{174}$$

We now use (171) to estimate the second integral on the right-hand side. Let θ_0 be in $[0, 1]$. Then

$$-\int_t^{t'} \frac{\alpha_t}{\alpha}(s, \Theta(s, \theta, t'))\, ds \geq -(1-\epsilon) \int_t^{t'} \frac{\alpha_t}{\alpha}(s, \theta_0)\, ds, \tag{175}$$

$$-\int_t^{t'} \frac{\alpha_t}{\alpha}(s, \Theta(s, \theta, t'))\, ds \geq -(1-\epsilon)\left(\ln \alpha(t', \theta_0) - \ln(\alpha(t, \theta_0))\right). \tag{176}$$

Using Corollary 2, there exists a constant $M > 0$ such that

$$-\int_t^{t'} \frac{\alpha_t}{\alpha}(s, \Theta(s, \theta, t'))\, ds \geq -(1-\epsilon)\big(\ln \alpha(t', \Theta(t', \theta, t')) - \ln \alpha(t, \Theta(t, \theta, t'))\big) - M. \tag{177}$$

Similarly, we obtain

$$-\int_t^{t'} \frac{\alpha_t}{\alpha}(s, \Theta(s, \theta, t'))\, ds \leq -(1+\epsilon)\big(\ln \alpha(t', \Theta(t', \theta, t')) - \ln \alpha(t, \Theta(t, \theta, t'))\big) + M. \tag{178}$$

Thus we have, from (174) and (177):

$$\ln \alpha(t', \Theta(t', \theta, t')) - \ln \alpha(t, \Theta(t, \theta, t')) - (1-\epsilon)\big(\ln \alpha(t', \Theta(t', \theta, t')) - \ln \alpha(t, \Theta(t, \theta, t'))\big) - M \leq \int_t^{t'} -\frac{\alpha_\theta}{\sqrt{\alpha}}(s, \Theta(s, \theta, t'))\, ds \tag{179}$$

and similarly

$$\int_t^{t'} -\frac{\alpha_\theta}{\sqrt{\alpha}}(s, \Theta(s, \theta, t'))\, ds \leq \ln \alpha(t', \Theta(t', \theta, t')) - \ln \alpha(t, \Theta(t, \theta, t')) - (1+\epsilon)\big(\ln \alpha(t', \Theta(t', \theta, t')) - \ln \alpha(t, \Theta(t, \theta, t'))\big) + M. \tag{180}$$

The lemma follows by simplifying the terms containing $\alpha(t, \Theta(t, \theta))$ in (179) and (180). □

8M. ***Estimates for the integrals of small powers of α.*** It will be useful for the derivation of pointwise energy estimates to have some control over the integral of α^p for small enough p. We first need the following result:

Lemma 8.13. *There exists $\theta \in [0, 1]$, such that*

$$\lim_{t \to t_0} \sqrt{\alpha} e^{2\nu}(t, \theta) > 0. \tag{181}$$

Proof. Suppose that the lemma does not hold. Since $\sqrt{\alpha}e^{2\nu}(t,\theta)$ is a decreasing function of t, it must then tend to 0 as t tends to t_0 for any θ. From the compactness of [0, 1] and using again the fact that $\sqrt{\alpha}e^{2\nu}$ is decreasing in t, it follows that $\int_{[0,1]}\sqrt{\alpha}e^{2\nu}d\theta$ tends to 0 as t tends to t_0. This contradicts the facts that $E_{h,K}$ tends to a strictly positive value by monotonicity and E_h has limit 0. □

Lemma 8.14. *For all $p<\frac{1}{2}$, there exists a function $B(t')$ such that $B(t')\to 0$ as $t'\to t_0$ and such that for all $(t,\theta)\in(t_0,t']\times[0,1]$, with $t'>t_0$, we have*

$$\int_t^{t'}\alpha^p(s,\theta)\,ds\le B(t'). \tag{182}$$

Proof. Let $\theta_0\in[0,1]$ be such that the previous lemma holds, and thus such that $\sqrt{\alpha}e^{2\nu}(\,\cdot\,,\theta_0)$ is bounded from below by a strictly positive constant on $(t_0,t']$.

We then rewrite (106) as

$$-\big(\tfrac{1}{2}-p\big)\frac{\alpha_t}{\alpha^{3/2-p}}=\big(\tfrac{1}{2}-p\big)\alpha^p f(t,\theta_0), \tag{183}$$

where

$$f(t,\theta_0)=4\Lambda\sqrt{\alpha}e^{2\nu-P}+\frac{\sqrt{\alpha}e^{2\nu}K^2}{t^3}$$

is a function bounded from below by a strictly positive constant. Integrating (183), we obtain

$$\int_t^{t'}\frac{d}{dt}(\alpha^{p-1/2})\,ds=\int_t^{t'}\big(\tfrac{1}{2}-p\big)\alpha^p f(s,\theta_0)\,ds. \tag{184}$$

Using the lower bound on $f(s,\theta)$, we therefore obtain

$$\int_t^{t'}\alpha^p(s,\theta_0)\,ds\le\frac{C}{\alpha^{1/2-p}(t',\theta_0)}, \tag{185}$$

for some constant $C>0$. The lemma then follows by application of Corollary 2 of Section 8J and the fact that $\lim_{t'\to t_0}\alpha(t',\theta_0)=+\infty$. □

From (124), we have seen that E_h is a priori not monotonic. In the next section, we will analyze an energy integral associated with the polarization function A. The advantage of this energy integral over E_h is that, as the wave equation for A is homogeneous, we will be able to extract useful information from the sign of dE_A/dt.

8N. ***Analysis of the polarization energy.*** Define the energy associated with the wave equation for A as

$$E_A=\int_{[0,1]}\frac{e^{2P}}{\sqrt{\alpha}}\left(A_t^2+\alpha A_\theta^2\right)\,d\theta. \tag{186}$$

Since by definition $E_A\le E_h$, we immediately obtain that $E_A\to 0$, when $t\to t_0$. The aim of this section is to extract some information from this remark. Note that the wave equation for A, (109), may also be written as

$$\partial_t\left(\frac{te^{2P}A_t}{\sqrt{\alpha}}\right)-\partial_\theta\left(te^{2P}\sqrt{\alpha}A_\theta\right)=0. \tag{187}$$

We first compute the time derivative of E_A:

$$\begin{aligned}
\frac{dE_A}{dt} &= \int_{[0,1]} \frac{\partial}{\partial t}\left(\frac{e^{2P}}{\sqrt{\alpha}}A_t^2\right) + \frac{\partial}{\partial t}\left(e^{2P}\sqrt{\alpha}A_\theta^2\right) \\
&= \int_{[0,1]} A_t\partial_t\left(\frac{e^{2P}}{\sqrt{\alpha}}A_t\right) + A_{tt}\frac{e^{2P}}{\sqrt{\alpha}}A_t + A_\theta^2\partial_t(e^{2P}\sqrt{\alpha}) + 2A_{\theta t}A_\theta e^{2P}\sqrt{\alpha} \\
&= \int_{[0,1]} A_t\left(\partial_\theta(e^{2P}\sqrt{\alpha}A_\theta) - \frac{e^{2P}A_t}{t\sqrt{\alpha}}\right) \\
&\quad + \left(\alpha A_{\theta\theta} + \left(-\frac{1}{t} + \frac{\alpha_t}{2\alpha}\right)A_t + \tfrac{1}{2}\alpha_\theta A_\theta - 2(A_tP_t - \alpha A_\theta P_\theta)\right)\frac{e^{2P}}{\sqrt{\alpha}}A_t \\
&\quad + 2P_te^{2P}\sqrt{\alpha}A_\theta^2 + \frac{\alpha_t}{2\sqrt{\alpha}}e^{2P}A_\theta^2 + 2A_{\theta t}A_\theta e^{2P}\sqrt{\alpha} \\
&= \int_{[0,1]} -\frac{1}{t}\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + A_t\partial_\theta\left(e^{2P}\sqrt{\alpha}A_\theta\right) + \frac{e^{2P}}{\sqrt{\alpha}}A_t\alpha A_{\theta\theta} - \frac{1}{t}\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + \frac{\alpha_t}{2\alpha}\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + \tfrac{1}{2}\alpha_\theta A_\theta\frac{e^{2P}A_t}{\sqrt{\alpha}} \\
&\quad - 2A_t^2P_t\frac{e^{2P}}{\sqrt{\alpha}} + 2\alpha P_\theta A_\theta\frac{e^{2P}A_t}{\sqrt{\alpha}} + 2P_te^{2P}\sqrt{\alpha}A_\theta^2 + \frac{\alpha_t}{2\sqrt{\alpha}}e^{2P}A_\theta^2 + 2A_{\theta t}A_\theta e^{2P}\sqrt{\alpha}, \\
&= \int_{[0,1]} -2\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + 2\partial_\theta(A_te^{2P}\sqrt{\alpha}A_\theta) + \left(\frac{\alpha_t}{2\alpha} - 2P_t\right)\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + \left(\frac{\alpha_t}{2\alpha} + 2P_t\right)e^{2P}\sqrt{\alpha}A_\theta^2.
\end{aligned} \tag{188}$$

Since the second term vanishes due to the periodicity, we obtain

$$\frac{dE_A}{dt} = \int_{[0,1]} -2\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + \left(\frac{\alpha_t}{2\alpha} - 2P_t\right)\frac{e^{2P}A_t^2}{\sqrt{\alpha}} + \left(\frac{\alpha_t}{2\alpha} + 2P_t\right)e^{2P}\sqrt{\alpha}A_\theta^2. \tag{189}$$

Note that by assumption, the spacetime is not polarized and thus E_A cannot identically vanish on any Cauchy surface, in particular, on any surface of constant t. Now, if there exists $t' \in (t_0, t_i]$ such that, for all $(t, \theta) \in (t_0, t'] \times [0, 1]$, both $(\alpha_t/2\alpha) + P_t$ and $(\alpha_t/2\alpha) - P_t$ tend to 0, it follows that E_A is increasing in the past direction, which contradicts the fact that $E_A \to 0$ as $t \to t_0$. We are led to the following:

Lemma 8.15. *There exists a constant $C > 0$ and a sequence of points (t_n, θ_n) in $(t_0, t_i] \times [0, 1]$, with $t_n \to t_0$, as $n \to +\infty$ such that $\frac{|P_t|}{\sqrt{\alpha}}(t_n, \theta_n) \geq C$.*

Proof. As explained above, we have a sequence of points (t_n, θ_n) such that $|P_t| + \alpha_t/(2\alpha) \geq 0$; otherwise E_A is increasing for t close to t_0. From Corollary 3 of Section 8J and (149), there exists a constant $M > 0$ such that, for all $(t, \theta) \in (t_0, t_i] \times [0, 1]$,

$$\frac{\alpha_t}{\alpha}(t, \theta) \leq -M\sqrt{\alpha}(t, \theta), \tag{190}$$

from which we obtain

$$\left(|P_t| - \frac{M}{2}\sqrt{\alpha}\right)(t_n, \theta_n) \geq 0, \tag{191}$$

which proves the lemma. □

The set of points we have just obtained will be used as initial data for some null cone estimates, where the aim will be to estimate from below the energy density h. However, we will need to treat some of the nonlinear terms as error terms, and for this, it will be necessary to first control h from above, which is the subject of the next section.

8O. ***Pointwise null cone energy estimates: control from above.*** We introduce the energy density:

$$h^\times = 2\sqrt{\alpha}P_t P_\theta + 2e^{2P}\sqrt{\alpha}A_t A_\theta. \tag{192}$$

Let us compute the sum and the difference of h and $h^\times$:

$$h + h^\times = (P_t + \sqrt{\alpha}P_\theta)^2 + e^{2P}(A_t + \sqrt{\alpha}A_\theta)^2, \tag{193}$$
$$h - h^\times = (P_t - \sqrt{\alpha}P_\theta)^2 + e^{2P}(A_t - \sqrt{\alpha}A_\theta)^2. \tag{194}$$

Define

$$D_u = \partial_t - \sqrt{\alpha}\partial_\theta, \tag{195}$$
$$D_v = \partial_t + \sqrt{\alpha}\partial_\theta, \tag{196}$$
$$P_u = D_u P, \quad P_v = D_v P, \tag{197}$$
$$A_u = D_u A, \quad A_v = D_v A. \tag{198}$$

With this notation, we have

$$h + h^\times = P_v^2 + e^{2P}A_v^2, \tag{199}$$
$$h - h^\times = P_u^2 + e^{2P}A_u^2. \tag{200}$$

We may also rewrite the wave equations (113) and (109) for P and A as[23]

$$D_u D_v P = \frac{\alpha_t}{2\alpha}P_v - \frac{1}{2t}(P_u + P_v) + e^{2P}A_u A_v - \frac{1}{2t^4}\alpha e^{2\nu}K^2, \tag{201}$$
$$D_v D_u P = \frac{\alpha_t}{2\alpha}P_u - \frac{1}{2t}(P_u + P_v) + e^{2P}A_u A_v - \frac{1}{2t^4}\alpha e^{2\nu}K^2, \tag{202}$$
$$D_u D_v A = \frac{\alpha_t}{2\alpha}A_v - \frac{1}{2t}(A_u + A_v) - A_u P_v - A_v P_u, \tag{203}$$
$$D_v D_u A = \frac{\alpha_t}{2\alpha}A_u - \frac{1}{2t}(A_u + A_v) - A_u P_v - A_v P_u. \tag{204}$$

We have

$$D_u(h + h^\times) = \Big(-\frac{1}{t} + \frac{\alpha_t}{\alpha}\Big)(P_v^2 + e^{2P}A_v^2) - \frac{1}{t}(P_u P_v + e^{2P}A_v A_u) - \frac{P_v}{t^4}\alpha e^{2\nu}K^2, \tag{205}$$
$$D_v(h - h^\times) = \Big(-\frac{1}{t} + \frac{\alpha_t}{\alpha}\Big)(P_u^2 + e^{2P}A_u^2) - \frac{1}{t}(P_u P_v + e^{2P}A_v A_u) - \frac{P_u}{t^4}\alpha e^{2\nu}K^2, \tag{206}$$

or, equivalently,

$$D_u(h + h^\times) = \Big(-\frac{1}{t} + \frac{\alpha_t}{\alpha}\Big)(h + h^\times) - \frac{1}{t}(P_u P_v + e^{2P}A_v A_u) - \frac{P_v}{t^4}\alpha e^{2\nu}K^2, \tag{207}$$
$$D_v(h - h^\times) = \Big(-\frac{1}{t} + \frac{\alpha_t}{\alpha}\Big)(h - h^\times) - \frac{1}{t}(P_u P_v + e^{2P}A_v A_u) - \frac{P_u}{t^4}\alpha e^{2\nu}K^2. \tag{208}$$

[23] Note that $D_u D_v = D_v D_u + \frac{\alpha_t}{\sqrt{\alpha}}\partial_\theta$.

We will prove the next result using null cone estimates:

Lemma 8.16. *For all $\epsilon > 0$, there exists a constant $B > 0$, a $t' > t_0$ and a $\theta_0 \in [0, 1]$ such that, for all $t' \geq t > t_0$,*

$$\sup_{\theta \in [0,1]} h(t, \cdot) \leq B \alpha^{1+\epsilon}(t, \theta_0). \tag{209}$$

Proof. Let $t \in (t_0, t_i]$ and let $\Theta(s, \theta, t)$ denote a solution of the characteristic equation with initial conditions $\Theta(t, \theta, t) = \theta$ such that $(s, \Theta(s, \theta, t))$ corresponds to a constant v line in null coordinates, as introduced in Section 8K.

We have

$$\frac{\partial}{\partial s}\big((h + h^\times)(s, \Theta(s, \theta, t))\big) = \frac{\partial (h + h^\times)}{\partial t} - \sqrt{\alpha}\frac{\partial (h + h^\times)}{\partial \theta} = D_u(h + h^\times)(s, \Theta(s, \theta, t)) \tag{210}$$

and therefore (207) can be rewritten as follows, for any $t' > t_0$:

$$\begin{aligned} &\frac{\partial}{\partial s}\left((h + h^\times)(s, \Theta(s, \theta, t)) \exp \int_s^{t'} \left(-\frac{1}{s'} + \frac{\alpha_t}{\alpha}\right)(s', \Theta(s', \theta, t))\, ds'\right) \\ &\qquad = \left(\exp \int_s^{t'} \left(-\frac{1}{s'} + \frac{\alpha_t}{\alpha}\right)(s', \Theta(s', \theta, t))\, ds'\right) \phi(s, \Theta(s, \theta, t)), \end{aligned} \tag{211}$$

where

$$\phi = -\frac{1}{s}(P_u P_v + e^{2P} A_v A_u) - \frac{P_v}{s^4} \alpha e^{2\nu} K^2.$$

Let $t' \geq t > t_0$ and integrate the last line between t' and t to obtain

$$\begin{aligned} &(h + h^\times)(t', \Theta(t', \theta, t)) - (h + h^\times)(t, \theta) \exp \int_t^{t'} \Big(-\frac{1}{s'} + \frac{\alpha_t}{\alpha}\Big)(s', \Theta(s', \theta, t))\, ds' \\ &\qquad = \int_t^{t'} \left[\left(\exp \int_s^{t'} \Big(-\frac{1}{s'} + \frac{\alpha_t}{\alpha}\Big)(s', \Theta(s', \theta, t))\, ds'\right) \phi(s, \Theta(s, \theta, t))\right] ds. \end{aligned} \tag{212}$$

Let $\epsilon > 0$ and fix a θ_0 in $[0, 1]$. Assume t' is such that Lemma 8.10 holds in the following sense: for all $(t, \theta) \in (t_0, t'] \times [0, 1]$,

$$(1 + \epsilon)\frac{\alpha_t}{\alpha}(t, \theta_0) \leq \frac{\alpha_t}{\alpha}(t, \theta) \leq (1 - \epsilon)\frac{\alpha_t}{\alpha}(t, \theta_0), \tag{213}$$

which implies the estimates

$$\frac{t}{t'}\left(\frac{\alpha(t', \theta_0)}{\alpha(t, \theta_0)}\right)^{1+\epsilon} \leq \exp \int_t^{t'} \left(-\frac{1}{s'} + \frac{\alpha_t}{\alpha}\right)(s', \Theta(s', \theta, t))\, ds' \tag{214}$$

and

$$\exp \int_t^{t'} \left(-\frac{1}{s'} + \frac{\alpha_t}{\alpha}\right)(s', \Theta(s', \theta, t))\, ds' \leq \frac{t}{t'}\left(\frac{\alpha(t', \theta_0)}{\alpha(t, \theta_0)}\right)^{1-\epsilon}. \tag{215}$$

Define $F(s, \theta, t)$ by

$$F(s, \theta, t) = (h + h^\times)(s, \Theta(s, \theta, t))\frac{s}{t'}\left(\frac{\alpha(t', \theta_0)}{\alpha(s, \theta_0)}\right)^{1+\epsilon}. \tag{216}$$

Note that it follows from the definition of $F(s,\theta,t)$ that, for all $t, t' \in (t_0, t_i]$,

$$\sup_{\theta\in[0,1]} F(s,\theta,t) = \sup_{\theta\in[0,1]} F(s,\theta,t')$$

Thus, we may define $Z(s)$ by

$$Z(s) = \sup_{\theta\in[0,1]} F(s,\theta,t).$$

From (212), (214) and (215), we have

$$F(t,\theta,t) \le F(t',\theta,t) + \int_t^{t'} \frac{s}{t'} \left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} |\phi(s,\Theta(s,\theta,t))|\,ds. \tag{217}$$

We will now estimate the second term on the right-hand side of the last inequality. First note that

$$|\phi(s,\Theta(s,\theta,t))| = \left| -\frac{1}{s}(P_u P_v + e^{2P} A_v A_u) - \frac{P_v}{s^4}\alpha e^{2\nu} K^2 \right| \tag{218}$$

$$\le \frac{h}{s} + \sqrt{h+h^\times}\frac{e^{2\beta}K^2}{s^4}. \tag{219}$$

Thus

$$\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} |\phi(s,\Theta(s,\theta))| \le \left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} \tfrac{1}{2}h + \left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} \sqrt{h+h^\times}\,\frac{e^{2\beta}K^2}{s^4}. \tag{220}$$

The second term on the right-hand side of this last line may then be rewritten in terms of $F(s,\theta,t)$:

$$\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} \sqrt{h+h^\times}\frac{e^{2\beta}K^2}{s^4} = \left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1/2-3\epsilon/2} \frac{e^{2\beta}K^2}{s^4}\left(\frac{t'}{s}\right)^{1/2}\sqrt{F(s,\theta,t)}. \tag{221}$$

Moreover, from Lemmas 8.5 and 8.14 and Corollary 3, for ϵ small enough we have

$$\int_t^{t'} \left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1/2-3\epsilon/2} \frac{e^{2\beta}K^2}{s^4}\left(\frac{t'}{s}\right)^{1/2} ds \le \int_t^{t'} C\alpha^{(3\epsilon)/2}(s,\theta_0)\,ds \le M, \tag{222}$$

for some constant $M > 0$. We now use estimates of the type found in [Smulevici 2008]. Let t_m be such that $Z(t_m)$ is a maximum of Z on $[t,t']$. Note the trivial fact that $\sup_{[t,t']} Z = Z(t_m) = \sup_{[t_m,t']} Z$.

It follows from (217), (220) and (222) that

$$F(t_m,\theta,t_m) \le F(t',\theta,t_m) + \sqrt{Z(t_m)}M + \int_{t_m}^{t'} \frac{s}{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} \frac{h(s,\Theta(s,\theta,t_m))}{s}\,ds, \tag{223}$$

for some constant $M > 0$. Note that $F(t',\theta,t_m)$ is uniformly bounded since, by definition,

$$F(t',\theta,t_m) = (h+h^\times)(t',\Theta(t',\theta,t_m)) \le \sup_{\theta\in[0,1]} (h+h^\times)(t',\cdot) \le C \tag{224}$$

for some constant $C > 0$. Thus we have, from (223),

$$F(t_m,\theta,t_m) \le C + \sqrt{Z(t_m)}M + \int_{t_m}^{t'} \frac{s}{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon} \frac{h(s,\Theta(s,\theta,t_m))}{s}\,ds, \tag{225}$$

and taking the supremum over θ, we obtain

$$Z(t_m) \le C + \sqrt{Z(t_m)}M + D\int_{t_m}^{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon}\sup_{\theta\in[0,1]}(h(s,\cdot))\,ds. \tag{226}$$

We interpret the last line as an inequality for a second-order polynomial equation in $\sqrt{Z(t_m)}$. Thus $\sqrt{Z(t_m)}$ must lie between the roots of this polynomial, and we obtain easily that

$$Z(t_m) \le C' + C'\int_{t_m}^{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon}\sup_{\theta\in[0,1]}(h(s,\cdot))\,ds, \tag{227}$$

for some constant $C' > 0$. Since $Z(t) \le Z(t_m)$ and since $t \le t_m$, we have

$$Z(t) \le C' + C'\int_{t}^{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon}\sup_{\theta\in[0,1]}(h(s,\cdot))\,ds, \tag{228}$$

Thus, we have established that

$$\Big(\sup_{\theta\in[0,1]}(h+h^{\times})(t,\cdot)\Big)\frac{t}{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(t,\theta_0)}\right)^{1+\epsilon} \le B + C\int_{t}^{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(s,\theta_0)}\right)^{1-\epsilon}\sup_{\theta\in[0,1]}(h(s,\cdot))\,ds, \tag{229}$$

for some constants $B, C > 0$. A similar estimate may be obtained using $h - h^{\times}$ and (208). Adding the estimate for $h - h^{\times}$ to (229), we obtain easily that

$$\Big(\sup_{\theta\in[0,1]} h(t,\cdot)\Big)\frac{t}{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(t,\theta_0)}\right)^{1+\epsilon} \le B + C\int_{t}^{t'}\left(\frac{\alpha(t',\theta_0)}{\alpha(t,\theta_0)}\right)^{1-\epsilon}\sup_{\theta\in[0,1]}(h(s,\cdot))\,ds.$$

Applying Gronwall's lemma, together with Lemma 8.14, completes the proof. □

8P. ***Pointwise null cone energy estimates: control from below.*** With the control from above for h that we have just obtained, we may now prove an estimate from below for h if we have appropriate initial data:

Lemma 8.17. *Suppose that there exists a constant $B > 0$ and a sequence of points (t_n, θ_n) with $t_n \to t_0$ as $n \to +\infty$, such that*

$$\frac{|P_v|}{\sqrt{\alpha}}(t_n,\theta_n) > B \quad \textit{for all } n.$$

For all $\epsilon > 0$, there exists $C > 0$, $t' > t_0$, $\theta' \in [0, 1]$ and an interval $[\theta'-\delta, \theta'+\delta]$ with $\delta > 0$ such that, for all $(t,\theta) \in (t_0, t'] \times [\theta'-\delta, \theta'+\delta]$,

$$h(t,\Theta(t,\theta,t')) \ge C\alpha^{1-\epsilon}(t,\Theta(t,\theta,t')), \tag{230}$$

where $(s,\Theta(s,\theta,t'))$ denote the parametrizations of the null lines parallel to the constant v lines starting at (t',θ) which were introduced in Section 8K.

Proof. Let $\epsilon > 0$ and $n_0 \in \mathbb{N}$ be such that Lemma 8.16 holds and Lemma 8.10 holds as in (213), with t' replaced by t_{n_0} in both lemmas. Let $n \ge n_0$. We will integrate (207) in a way similar to the proof of the last lemma. Let us denote by $\Theta_n(t,\theta)$ the null lines parallel to the constant v lines starting at (t_n,θ), i.e., $\Theta_n(t,\theta) = \Theta(t,\theta,t_n)$. Equation (207) can then be integrated as

$$(h+h^\times)(t_n,\theta)-(h+h^\times)(t,\Theta_n(t,\theta))\exp\int_t^{t_n}\Big(-\frac{1}{s'}+\frac{\alpha_t}{\alpha}\Big)(s',\Theta_n(s',\theta))\,ds'$$
$$=\int_t^{t_n}\Big(\Big(\exp\int_s^{t_n}\Big(-\frac{1}{s'}+\frac{\alpha_t}{\alpha}\Big)(s',\Theta_n(s',\theta))\,ds'\Big)\phi(s,\Theta_n(s,\theta))\Big)\,ds, \quad (231)$$

where

$$\phi=-\frac{1}{s}(P_uP_v+e^{2P}A_vA_u)-\frac{P_v}{s^4}\alpha e^{2\nu}K^2.$$

Fix $\theta_0\in[0,1]$. Since Lemma 8.10 holds in the sense of (213) for $t\in(t_0,t_n]$, we have again the estimates

$$\frac{t}{t_n}\Big(\frac{\alpha(t_n,\theta_0)}{\alpha(t,\theta_0)}\Big)^{1+\epsilon}\le\exp\int_t^{t_n}\Big(\frac{-1}{s}+\frac{\alpha_t}{\alpha}\Big)(s,\Theta_n(s,\theta))\,ds, \quad (232)$$

$$\exp\int_t^{t_n}\Big(\frac{-1}{s}+\frac{\alpha_t}{\alpha}\Big)(s,\Theta_n(s,\theta))\,ds\le\frac{t}{t_n}\Big(\frac{\alpha(t_n,\theta_0)}{\alpha(t,\theta_0)}\Big)^{1-\epsilon}. \quad (233)$$

Using this, we may estimate the last term on the right-hand side of (231):

$$\begin{aligned}\int_t^{t_n}&\Big(\Big(\exp\int_s^{t_n}\Big(-\frac{1}{s'}+\frac{\alpha_t}{\alpha}\Big)(s',\Theta_n(s',\theta))\,ds'\Big)\phi(s,\Theta_n(s,\theta))\Big)\,ds\\ &\le\int_t^{t_n}\frac{t}{t_n}\Big(\frac{\alpha(t_n,\theta_0)}{\alpha(t,\theta_0)}\Big)^{1-\epsilon}|\phi|(s,\Theta_n(s,\theta))\,ds\\ &\le\int_t^{t_n}\frac{t}{t_n}\Big(\frac{\alpha(t_n,\theta_0)}{\alpha(t,\theta_0)}\Big)^{1-\epsilon}\Big(\frac{h}{s}+\sqrt{h+h^\times}\frac{e^{2\beta}K^2}{s^4}\Big)\,ds.\end{aligned} \quad (234)$$

We now use Lemma 8.14 and the estimates $h+h^\times\le 2h$, $h\le C\alpha^{1+\epsilon}$, $\sqrt{h}\le\sqrt{C}\sqrt{\alpha^{1+\epsilon}}$, and $e^{2\beta}\le C\sqrt{\alpha}$, for some constant $C>0$ independent of n, to obtain

$$\begin{aligned}\int_t^{t_n}\Big(\Big(\exp\int_s^{t_n}\Big(-\frac{1}{s'}+\frac{\alpha_t}{\alpha}\Big)(s',\Theta_n(s',\theta))\,ds'\Big)\phi(s,\Theta_n(s,\theta))\Big)\,ds&\le\alpha(t_n,\theta_0)^{1-\epsilon}\int_t^{t_n}C\alpha^{2\epsilon}\\ &\le C_n'\alpha(t_n,\theta_0),\end{aligned} \quad (235)$$

where $C_n'\to 0$ as $n\to\infty$.

We obtain from (231) that

$$(h+h^\times)(t,\Theta_n(t,\theta))\ge\big((h+h^\times)(t_n,\theta)-C_n'(t)\alpha(t_n,\theta_0)\big)\exp\int_t^{t_n}\Big(\frac{1}{s'}-\frac{\alpha_t}{\alpha}\Big)(s',\Theta_n(s',\theta))\,ds', \quad (236)$$

$$(h+h^\times)(t,\Theta_n(t,\theta))\ge\big((h+h^\times)(t_n,\theta)-C_n'(t)\alpha(t_n,\theta_0)\big)\frac{t_n}{t}\Big(\frac{\alpha(t,\theta_0)}{\alpha(t_n,\theta_0)}\Big)^{1-\epsilon}. \quad (237)$$

By assumption, for all $n\in\mathbb{N}$, we have

$$\frac{|P_v|}{\sqrt{\alpha}}(t_n,\theta_n)>B,$$

and thus, from (199), $(h+h^\times)(t_n,\theta_n)/\alpha(t_n,\theta_n)\ge A$, for some $A>0$. By application of Corollary 2, we obtain $(h+h^\times)(t_n,\theta_n)/\alpha(t_n,\theta_0)\ge A'$, for some constant $A'>0$. Thus, for all $n\in\mathbb{N}$, there exists an interval around θ_n, say $[\theta_n-\delta_n,\theta_n+\delta_n]$ with $\delta_n>0$, such that, for all $\theta\in[\theta_n-\delta_n,\theta_n+\delta_n]$,

$$\frac{(h+h^{\times})(t_n,\theta)}{\alpha(t_n,\theta_0)}\geq\frac{A'}{2}. \tag{238}$$

Let n_1 be such that for all $n\geq n_1$ and all $t\in(t_0,t_n]$, $C'_n\leq A'/4$. Let $n_2=\max(n_0,n_1)$. Then we have, from (237) and (238), for all $(t,\theta)\in(t_0,t_{n_2}]\times[\theta_{n_2}-\delta_{n_2},\theta_{n_2}+\delta_{n_2}]$,

$$(h+h^{\times})(t,\Theta_{n_2}(t,\theta))\geq\frac{A'}{4}\alpha(t_{n_2},\theta_0)\frac{t_{n_2}}{t}\left(\frac{\alpha(t,\theta_0)}{\alpha(t_{n_2},\theta_0)}\right)^{1-\epsilon}. \tag{239}$$

Moreover, we have $\alpha(t,\Theta_{n_2}(t,\theta))\leq M\alpha(t,\theta_0)$ for some constant $M>0$; thus we obtain

$$(h+h^{\times})(t,\Theta_{n_2}(t,\theta))\geq\frac{A'}{4M^{1-\epsilon}}\alpha(t_{n_2},\theta_0)\frac{t_{n_2}}{t_0}\left(\frac{\alpha(t,\Theta_{n_2}(t,\theta))}{\alpha(t_{n_2},\theta_0)}\right)^{1-\epsilon}, \tag{240}$$

which proves the lemma. □

Remark 8P.1. With the notation of Lemma 8.17, it is possible to choose t' so that $t'\in(t_0,\bar{t}\,]$, where $\bar{t}$ is such that Lemma 8.12 holds. To see this, just replace in the above proof n_0 by $n'_0\geq n_0$ such that $t_{n'_0}\in(t_0,\bar{t}]$.

8Q. ***The contradiction.*** From Lemma 8.15, there exists a sequence of points (t_n,θ_n) and a constant $A>0$ such that $\frac{|P_t|}{\sqrt{\alpha}}(t_n,\theta_n)\geq A$. Thus, without of generality, we may assume that there exists a sequence of points (t'_n,θ'_n) and a constant $A>0$ such $\frac{|P_v|}{\sqrt{\alpha}}(t'_n,\theta'_n)\geq\frac{A}{2}$, exchanging the role of u and v if necessary. Therefore, Lemma 8.17 applies, and for all $\epsilon>0$, there exists a $C>0$, a $t'>t_0$, a $\theta'\in[0,1]$ and an interval $[\theta'-\delta,\theta'+\delta]$ with $\delta>0$ such that, for all $(t,\theta)\in(t_0,t']\times[\theta'-\delta,\theta'+\delta]$,

$$h(t,\Theta(t,\theta,t'))\geq C\alpha^{1-\epsilon}(t,\Theta(t,\theta,t')), \tag{241}$$

where $(s,\Theta(s,\theta,t'))$ denote the parametrizations of the null lines parallel to the constant v lines, starting at (t',θ). Moreover, let us choose t' so that $t'\in(t_0,\bar{t}]$, where $\bar{t}$ is such that Lemma 8.12 holds, as in Remark 8P.1.

Consider the integral in θ of $h(t,\Theta(t,\theta,t'))$ and fix a $\theta_0\in[0,1]$. We have

$$\int_{[0,1]}h(t,\Theta(t,\theta,t'))\,d\theta\geq 2\delta C\alpha^{1-\epsilon}(t,\theta_0), \tag{242}$$

using Corollary 2. On the other hand, we have, by the change of variable $\theta'=\Theta(t,\theta,t')$,

$$\int_{[0,1]}h(t,\Theta(t,\theta,t'))\,d\theta=\int_{[0,1]}h(t,\theta')\Theta_\theta^{-1}d\theta'. \tag{243}$$

From (169), we therefore have

$$\int_{[0,1]}h(t,\Theta(t,\theta,t'))\,d\theta=\int_{[0,1]}h(t,\theta')\left(\exp\int_{t'}^{t}\frac{1}{2}\left(\frac{\alpha_\theta}{\sqrt{\alpha}}\right)ds\right)d\theta', \tag{244}$$

where the integral in the exponential is taken along the characteristics.

Since Lemma 8.12 holds, we have

$$\exp \int_{t'}^{t} \frac{1}{2}\left(\frac{\alpha_\theta}{\sqrt{\alpha}}\right) ds \leq M\alpha^{\epsilon}. \tag{245}$$

Thus, we obtain

$$\int_{[0,1]} h(t, \Theta(t,\theta,t'))\, d\theta \leq \int_{[0,1]} hM\alpha^{\epsilon}(t,\theta')\, d\theta'. \tag{246}$$

Using again Corollary 2 and the boundedness of $E_h = \int_{[0,1]}(h/\sqrt{\alpha})d\theta$, we see that the right-hand of the last inequality is bounded by $M'\alpha^{1/2+\epsilon}(t,\theta_0)$ for some constant M'. Choosing ϵ small enough, this contradicts (242) since $\alpha \to \infty$ as $t \to t_0$. Thus Theorem 2 is proved.[24]

9. Proof of Theorem 3

We will prove Theorem 3 in this section. For this, we will adapt the proof found by Isenberg and Weaver [2003] to the case of $k = -1$ surface-symmetric spacetimes. To exploit their methods, we have rewritten the metric in a form similar to the T^2 case (see (7) in Section 2B). In particular, the coordinate t used in (7) denotes the square of the usual areal time used for these spacetimes, as found for instance in [Tchapnda 2004].

To start, we recall the Einstein–Vlasov system for spacetimes with a hyperbolic surface of symmetry.

9A. ***Vlasov matter in k = −1 surface-symmetric spacetimes.*** Let $(\mathcal{M}, g, f)$ be a past development of $k = -1$ surface-symmetric initial data with Vlasov matter as described in Section 2D and assume that (t, θ, x, y) is a system of areal coordinates such that the metric in $\mathcal{M}$ takes the form (7). Let v_i, $i = 0, 1, 2, 3$ denotes the components of the velocity vector in the canonical basis of 1-forms associated with the coordinate system (t, θ, x, y). We will parametrize the mass shell $\mathcal{P}$ by the coordinates $(t, \theta, x, y, v_1, v_2, v_3)$, where by an abuse of notation, we denote the lift to $\mathcal{P}$ of the coordinates on $\mathcal{M}$ by the same symbols. The Vlasov field f can be seen as a function of $(t, \theta, x, y, v_1, v_2, v_3)$ or, using the symmetry, as a function depending only on t, θ, $w = (\sqrt{t}/e^{\nu})v_1$ and $L = \gamma^{ab}v_a v_b$, and we will, by an abuse of notation, use both definitions and denote it by f in either case.[25]

With these definitions, the mass shell relation $v_\mu v^\mu = -1$ is given by

$$v_0 = -\sqrt{\frac{\alpha}{t}e^{2\nu} + \alpha v_1^2 + \frac{\alpha e^{2\nu}}{t^2}\gamma^{ab}v_a v_b} = -\frac{\sqrt{\alpha}e^{\nu}}{\sqrt{t}}\sqrt{1+w^2+L/t} \tag{247}$$

and the Vlasov equation for $f(t, \theta, w)$ reads as

$$2\sqrt{t}\partial_t f + \frac{2\sqrt{t\alpha}w}{\sqrt{1+w^2+L/t}}\partial_\theta f - \left(\sqrt{t}(2\nu_t - 1/t)w + \left(\nu_\theta + \frac{\alpha_\theta}{2\alpha}\right)2\sqrt{t\alpha}\sqrt{1+w^2+L/t}\right)\partial_w f = 0. \tag{248}$$

[24] We see that the margin of error is, up to ϵ, $\alpha^{1/2}$. This margin follows from our estimates because, up to α^{ϵ}, we have $h \sim \alpha$ along certain characteristics. On the other hand, if we did not have this margin, i.e., if we had $h \sim \alpha^{1/2}$, then it would follow that for t' close enough to t_0, $(\alpha_t/2\alpha) \pm P_t \leq 0$ for all $\theta \in [0, 1]$, and (189) would the imply that that E_A is increasing the past. This would contradict the fact that $E_A \to 0$ as $t \to t_0$.

[25] The indices on the velocities v_i are raised or lowered using the metric (7), not using γ_{ab}. This implies that if p^a denotes the canonical momentum associated with the coordinates system $(t.\theta.x.y)$, then $L = t^2\gamma_{ab}p^a p^b$.

9B. ***The Einstein equations.*** The Einstein equations (1) reduce to the following system of equations:

Constraint equations:

$$\nu_t = \frac{1}{4t} + \alpha e^{2\nu}\Lambda - \frac{k\alpha e^{2\nu}}{t} + 8\pi\sqrt{\alpha}\int_{\mathbb{R}^3} f|v_0|\gamma^{-1/2}\,dv_1\,dv_2\,dv_3, \tag{249}$$

$$\frac{\alpha_t}{\alpha} = -4\Lambda\alpha e^{2\nu} + \frac{4k\alpha e^{2\nu}}{t} - 16\pi\alpha^{3/2}e^{2\nu}\int_{\mathbb{R}^3}\frac{f(t^{-1}+t^{-2})}{|v_0|}\gamma^{-1/2}\,dv_1\,dv_2\,dv_3, \tag{250}$$

$$\nu_\theta + \frac{\alpha_\theta}{2\alpha} = -8\pi\sqrt{\alpha}\int_{\mathbb{R}^3} f v_1\gamma^{-1/2}\,dv_1\,dv_2\,dv_3. \tag{251}$$

Evolution equation:

$$\nu_{tt} - \alpha\nu_{\theta\theta} = \frac{1}{2}\alpha_{\theta\theta} - \frac{\alpha_\theta^2}{4\alpha} + \frac{\nu_\theta\alpha_\theta}{2} - \frac{1}{4t^2} + \frac{\alpha_t\nu_t}{2\alpha} + \frac{\alpha e^{2\nu}\Lambda}{t} - 4\pi\frac{\alpha^{3/2}e^{2\nu}}{t^3}\int_{\mathbb{R}^3}\frac{fL}{|v_0|}\gamma^{-1/2}\,dv_1\,dv_2\,dv_3. \tag{252}$$

Here k denotes the curvature of the surface of symmetry and will therefore be -1 in the case of hyperbolic symmetry. γ denotes the determinant of the metric γ_{ab}.

In the rest of this section, $(\mathcal{M}, g, f)$ will be a past development of $k = -1$ surface-symmetric initial data with Vlasov matter and $\Lambda \geq 0$. We will cover $(\mathcal{M}, g)$ by areal coordinates (t, θ, x, y), where the range of the coordinates (t, θ) is $(t_f, t_i] \times [0, 1]$ with $0 < t_f < t_i$. The metric will be given by (7) with the functions α and ν depending only on (t, θ) and being periodic in θ with period 1. The Einstein–Vlasov system implies that the system (249)–(252) completed with (248) holds for all $(t, \theta) \in (t_f, t_i] \times [0, 1]$. Moreover, we will assume that f does not vanish identically. Following what has been said in Section 6, we will prove that for all such $(\mathcal{M}, g, f)$, the hypotheses of Proposition 2 are satisfied, from which Theorem 3 follows immediately.

First, we recall some properties of the Vlasov field for such spacetimes.

9C. ***Conservation laws.*** As in Section 7C, since f is conserved along geodesics, we have an immediate upper bound on f:

$$f \leq F, \tag{253}$$

for some $F > 0$. Since by assumption, f has compact support, conservation of angular momentum along geodesics implies an upper bound on the support of f in L; that is, we have

$$X = \sup_{L\in\mathrm{supp}(f)} L < \infty. \tag{254}$$

The particle current is given by

$$N^\mu = \frac{\sqrt{\alpha}}{t}\int_{\mathbb{R}^3}\frac{f}{|v_0|}v^\mu\gamma^{-1/2}\,dv_1\,dv_2\,dv_3. \tag{255}$$

From the Vlasov equation it follows that N^μ is divergence free $\nabla_\mu N^\mu = 0$ and therefore, we have the conservation law,

$$\int_{[0,1]} N^0\sqrt{\alpha}e^{2\nu}d\theta = \int_{[0,1]}\left(\int_{\mathbb{R}^3} f\gamma^{-1/2}\,dv_1\,dv_2\,dv_3\right)d\theta = Q \quad \text{for all } t, \tag{256}$$

for some nonnegative constant Q. Moreover, since by assumption, the Vlasov field does not vanish identically, we have

$$Q > 0. \tag{257}$$

9D. *Lower bound on the mean value of* $|v_1|$. The proof of Lemma 7.1 is easily adapted to the present setting to give:

Lemma 9.1. *There exists* $\delta > 0$ *such that, for all* t,

$$\int_{[0,1]} \left(\int_{\mathbb{R}^3} f \gamma^{-1/2} |v_1| \, dv_1 \, dv_2 \, dv_3 \right) d\theta > \delta. \tag{258}$$

9E. *Energy estimates.* We define $E(t)$ as the energy integral

$$E(t) = \int_{[0,1]} \frac{\nu_t}{t\sqrt{\alpha}} d\theta. \tag{259}$$

Lemma 9.2. *E admits a continuous extension to t_f. In particular E is uniformly bounded on $(t_f, t_i]$.*

Proof. As usual, we take the time derivative of E and use the Einstein equations and the periodicity to simplify the resulting equations. It follows that

$$\frac{dE}{dt} = -\int_{[0,1]} \left(\frac{1}{2t^3\sqrt{\alpha}} - \frac{k\sqrt{\alpha}e^{2\nu}}{t^3} + 8\pi \int_{\mathbb{R}^3} \left(\frac{f|v_0|}{t^2} + \frac{\alpha e^{2\nu} f L}{2t^4 |v_0|} \right) \gamma^{-1/2} \, dv_1 \, dv_2 \, dv_3 \right) d\theta. \tag{260}$$

Since $k = -1$, we see that E is increasing with decreasing t. Moreover from the last equation, the definition of E, and (249), it follows that

$$\frac{dE}{dt} \geq -\frac{4E}{t} \tag{261}$$

and by integration of the last line, we obtain an upper bound for E on $(t_f, t_i]$. □

9F. *Estimate for* $\sqrt{\alpha}e^{2\nu}$.

Lemma 9.3. *$\sqrt{\alpha}e^{2\nu}$ is uniformly bounded on $(t_f, t_i]$.*

Proof. It follows from equations (249) and (250) that

$$\partial_t(\sqrt{\alpha}e^{2\nu}) \geq 0. \tag{262}$$

We will use this bound in order to estimate the terms containing $\alpha e^{2\nu}$ in the right-hand side of (250). This will follow from the next two lemmas.

9G. *Estimate for* $\int_{[0,1]} |(\sqrt{\alpha}e^{\nu})_\theta| \, d\theta$. Let $e^\beta = \sqrt{\alpha}e^\nu$. Equation (251) can now be written as

$$\beta_\theta = -8\pi\sqrt{\alpha} \int_{\mathbb{R}^3} f v_1 \gamma^{-1/2} \, dv_1 \, dv_2 \, dv_3. \tag{263}$$

Lemma 9.4. *$\int_{[0,1]} |\beta_\theta| \, d\theta$ is bounded on $(t_f, t_i]$. In particular, there exists a bound independent of $t \in (t_f, t_i]$ on the difference between the maximum and the minimum of $\beta(t, \cdot)$.*

Proof. From (263), we have

$$|\beta_\theta| \leq 8\pi\sqrt{\alpha}\int_{\mathbb{R}^3} f v_1 \gamma^{-1/2}\, dv_1\, dv_2\, dv_3 \leq 8\pi\int_{\mathbb{R}^3} f v_0 \gamma^{-1/2}\, dv_1\, dv_2\, dv_3 \leq \frac{\nu_t}{\sqrt{\alpha}}, \tag{264}$$

where we have used the fact that $\sqrt{\alpha}|v_1| \leq v_0$ from the mass shell relation to obtain the second inequality and (249) to obtain the third.

Dividing (264) by t and integrating $\nu_t/\sqrt{\alpha}$ over [0, 1], we obtain a bound on $\int_{[0,1]} |\beta_\theta|\, d\theta$ from the bounds on t and E. □

9H. ***Control of α along special curves.*** Similar to Section 7H, we now prove:

Lemma 9.5. $\min_{[0,1]} \alpha(t, \cdot)$ *is bounded on* $(t_f, t_i]$.

Proof. From the definition of E and from (249),

$$8\pi \int_{[0,1]}\int_{\mathbb{R}^3} f\gamma^{-1/2}|v_0|\, dv_1\, dv_2\, dv_3\, d\theta \leq tE(t). \tag{265}$$

Since $\sqrt{\alpha}|v_1| \leq v_0$, we obtain

$$\min_{[0,1]}(\sqrt{\alpha}) \int_{[0,1]}\int_{\mathbb{R}^3} f|v_1|\gamma^{-1/2}\, dv_1\, dv_2\, dv_3\, d\theta \leq \frac{tE(t)}{8\pi} \leq A, \tag{266}$$

for some constant A depending on the bound on E. However from Lemma 9.1, we have

$$\delta \leq \int_{[0,1]}\int_{\mathbb{R}^3} f|v_1|\sqrt{\gamma}^{-1}\, dv_1\, dv_2\, dv_3, \tag{267}$$

for some $\delta > 0$. Therefore $\min_{[0,1]}(\sqrt{\alpha}) \leq A/\delta$. □

This concludes the proof of Lemma 9.3. □

As in Corollary 1 of Section 7H, we obtain:

Corollary 4. *There exists* $\bar{\theta}$ *such that* $\alpha(t, \bar{\theta})$ *is bounded on* $(t_p, t_i]$.

9I. ***Estimate for*** $e^{2\beta}$**.**

Lemma 9.6. $e^{2\beta} = \alpha e^{2\nu}$ *is uniformly bounded on* $(t_f, t_i] \times [0, 1]$.

Proof. This follows from Corollary 4 and Lemmas 9.3 and 9.4 by an argument similar to the one given for the proof of Lemma 7.7. □

9J. ***Estimates for the support of*** f**.** Let

$$u_1 = \sqrt{\alpha}v_1 = \frac{\sqrt{\alpha}e^{\nu}}{\sqrt{t}}w \tag{268}$$

and define $\bar{u}_1$ by

$$\bar{u}_1(t) = \sup\left\{|u_1| \,:\, \exists(\theta, L) \text{ such that } f\left(t, \theta, \frac{u_1}{\sqrt{\alpha}}, L\right) \neq 0\right\} \tag{269}$$

Lemma 9.7. $\bar{u}_1$ *is uniformly bounded on* $(t_f, t_i]$.

Proof. The characteristic equation for u_1 associated with the Vlasov equation written (248) in terms of the coordinates (t, θ, u_1, L) gives

$$\frac{d(u_1^2)}{ds} = \frac{\alpha_t}{\alpha} u_1^2 + \frac{2\sqrt{\alpha} u_1}{v_0} \frac{e^{2\beta}}{t} \beta_\theta (1 + L/t). \tag{270}$$

The transformation (268) from w to u_1 will avoid the difficulty arising from the term containing β_θ in (248). Indeed, this term contains the factor $\sqrt{1 + w^2 + L/t}$, which depends in w in a not completely trivial way. On the other hand, having v_0 at the denominator of the last term in the right-hand side of (270) will enable us to easily estimate this term.

Let us first estimate the factor α_t/α appearing in the first term of the right-hand side of (270). From (250) and the bounds on e^β obtained previously, we have, for appropriate constants C and A',

$$\begin{aligned}
\left|\frac{\alpha_t}{\alpha}\right| &\le C + 16\pi\alpha^{3/2} e^{2\nu} \int_{\mathbb{R}^3} \frac{f\left(t^{-1} + Lt^{-2}\right)}{|v_0|} \gamma^{-1/2}\, dv_1\, dv_2\, dv_3 \\
&\le C + C'\sqrt{\alpha} \int_{-\bar{u}_1}^{\bar{u}_1} \int_{-X}^{X} \frac{f\left(t^{-1} + Lt^{-2}\right)}{|v_0|} \frac{du_1}{\sqrt{\alpha}} \pi\, dL \\
&\le C + C'' F \left(\frac{1}{t_f} + \frac{X}{t_f^2}\right) \int_{-\bar{u}_1}^{\bar{u}_1} \frac{du_1}{|v_0|} \\
&\le C + A \int_{-\bar{u}_1}^{\bar{u}_1} \frac{du_1}{\sqrt{1 + te^{-2\beta} u_1^2}} \\
&\le C + A \left[e^\beta t^{-1/2} \ln\left(u_1 + \sqrt{e^{2\beta}/t + u_1^2}\right)\right]_{-\bar{u}_1}^{\bar{u}_1} \\
&\le C + A'\left(\ln\left(\bar{u}_1 + \sqrt{e^{2\beta}/t + \bar{u}_1^2}\right) + e^{-1}\right).
\end{aligned} \tag{271}$$

We now estimate the second term on the right-hand side of (270). First note that the mass shell relation written in terms of u_1 reads as

$$v_0 = -\sqrt{\frac{\alpha}{t} e^{2\nu} + u_1^2 + \frac{\alpha e^{2\nu}}{t^2} \gamma^{ab} v_a v_b} \tag{272}$$

and thus, we have $|u_1|/|v_0| < 1$. Moreover, from (263), we have

$$\sqrt{\alpha}\beta_\theta \le 8\pi^2 F X \bar{u}_1^2. \tag{273}$$

Integrating (270) and using the estimates (273) and (271), we obtain an inequality of the form

$$u_1^2(t) \le A + B \int_t^{t_i} u_1^2(s) \ln(1 + \bar{u}_1^2)(s)\, ds + C \int_t^{t_i} \bar{u}_1^2(s)\, ds, \tag{274}$$

for some positive constants A, B and C. It follows from the last line, as in (101)–(103), that $\bar{u}_1$ is uniformly bounded. ☐

9K. ***Estimates for α, β, ν and β_θ.***

Lemma 9.8. *α, β, ν and β_θ are uniformly bounded on $(t_f, t_i] \times [0, 1]$.*

Proof. This follows easily from the Einstein equations since the right-hand sides of equations (249), (250) and (251) contain only quantities that have been shown to be bounded. □

9L. ***Estimates for the derivatives of f, α_θ, ν_θ and higher-order estimates.*** This follows by standard arguments, which can be found for instance in [Weaver 2004].

9M. ***The conclusion.*** Since all metric functions, the Vlasov field and all their derivatives have been shown to be uniformly bounded, the assumptions of Proposition 2 have been retrieved. In particular, the maximal Cauchy development cannot have $t_0 > 0$, which concludes the proof of Theorem 3.

10. Comments and open questions

10A. ***Weaver's estimate for Vlasov matter.*** The result of Theorem 3 was obtained in [Rein 1996; Tchapnda 2004] under a small data assumption. The main difference in our analysis which enables us to remove this smallness assumption, is to use, following [Weaver 2004], the presence of the Vlasov field to obtain a lower bound on one of the matter terms (see Lemma 9.1). It would be interesting to see if this estimate could be applied in other geometries and what would be the consequences.

Let us also note that if we couple the Einstein–Vlasov system to extra matter fields, a statement analogous to Lemma 9.1 would certainly be true if the extra matter fields satisfy the strong energy condition. For instance, the results of Theorems 1 and 3 can certainly be extended to include a massless scalar field.

10B. ***Theorem 2 and the hierarchization of the equations.*** The proof of Theorem 2 is based on the recovery of the lower bound on the energy quantities E_h and E_g. In the vacuum case, this lower bound is obtained directly from the monotonicity of E_g. However, this monotonicity is unstable to any perturbation in the setting of the problem, such as the introduction of matter or of a positive cosmological constant.

Our strategy has been to prove that, while E_h is not necessarily monotone, one can recover a monotonicity for another energy, namely E_A, which controls E_h from below and thus is sufficient to obtain the required lower bound on E_h. Since E_A is the energy associated with the wave equation for A only, while E_h is associated for the system of equations for (U, A), this shows that, in the contradiction setting that we have deployed, a certain hierarchy in the evolution equations appears, in the sense that one may first focus on the evolution equation for A and extract information from it, which we then reintroduce in the whole system.

Let us also note that not all estimates derived during the proof of Theorem 2 require the contradiction setting of Section 8Q. In particular, in Section 8J, we have proven a new estimate for T^2-symmetric spacetimes which might be useful in a further study of these solutions.

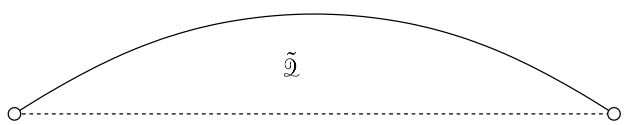
10C. ***Antitrapped initial data.*** One of the common features of T^2-symmetric and $k = -1$ surface-symmetric spacetimes is the antitrapping of the orbits of symmetry. This property arises from the positivity of the Hawking mass (excluding the flat case) and the fact that the orbits of symmetry have nonpositive curvature. The positivity of the Hawking mass is itself a consequence of the topology of Cauchy surfaces and of the Einstein equations, especially the Raychaudhuri equations. The proofs of the positivity of the Hawking mass and of antitrapping for vacuum T^2 symmetry and for $k \leq 0$ surface-symmetric spacetimes with Vlasov matter or with a massless scalar field were first obtained in [Chruściel 1990] and [Rendall 1995]. In [Rendall 1997], the results on T^2-symmetry were extended to the nonvacuum cases where local T^2-symmetry only is assumed. In order to improve our understanding of the structure of cosmological singularities, it would be interesting to try to generalize these results. One might ask for instance the following question. Assume that Σ is a compact Cauchy surface of a given spacetime satisfying the vacuum Einstein equations such that there exist a diffeomorphism ϕ between Σ and $S^1 \times \mathcal{R}$ where $\mathcal{R}$ is a compact surface. Assume moreover that for every point $\theta \in S^1$, $\phi^{-1}(\{\theta\} \times \mathcal{R})$ has nonpositive curvature. Is it then true that $\phi^{-1}(\{\theta\} \times \mathcal{R})$ is necessarily trapped or antitrapped?

10D. ***Strong cosmic censorship in polarized T^2-symmetric spacetimes.*** Theorems 1, 2 and 3 complete our understanding of the value of t_0 for T^2-symmetric and surface-symmetric spacetimes, as can be observed in Table 2, and we should therefore focus our attention to the remaining, very difficult, open problems presented in Table 1. One of the first questions to consider is that of strong cosmic censorship for vacuum polarized T^2-symmetric spacetimes with $\Lambda = 0$. While it is likely that the dynamics of these spacetimes are very different from those of general vacuum T^2-symmetric spacetimes, they are the simplest examples of vacuum inhomogeneous cosmological models where, writing the Einstein equations in areal coordinates, the constraint equations do not decouple from the evolution equations, as can be seen by removing the terms involving A in (104)–(110).

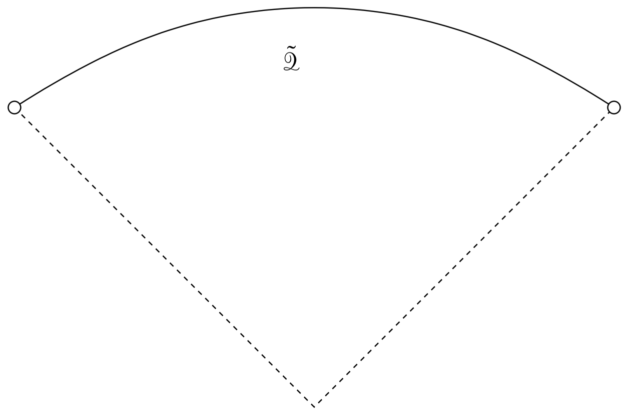
10E. ***Future causal geodesic completeness of T^2-symmetric and $k = -1$ surface-symmetric spacetimes.*** By the arguments of [Dafermos and Rendall 2005], (nonflat) T^2-symmetric and $k = -1$ surface-symmetric spacetimes are future inextendible. In the Gowdy case, where a complete understanding of the asymptotics has been obtained [Ringström 2004], and in the $k = -1$ surface-symmetric case with either small data [Rein 2004] or $\Lambda > 0$ [Tchapnda and Rendall 2003], future geodesic completeness has also been proven. More generally, we have the following conjecture:

Conjecture 1. *Let $(\mathcal{M}, g)$ be the maximal Cauchy development of T^2-symmetric or $k = -1$ surface-symmetric initial data in the vacuum or with Vlasov matter and with $\Lambda \geq 0$. Assume $(\mathcal{M}, g)$ is nonflat. Denote by t the area of the orbits of symmetry and orient $(\mathcal{M}, g)$ by ∇t. Then $(\mathcal{M}, g)$ is future causally complete.*

10F. ***The past boundary of $\tilde{\mathcal{Q}}$.*** One might also consider the following question about the structure of singularities in T^2-symmetric or $k = -1$ surface-symmetric spacetimes. Let $\tilde{\mathcal{Q}}$ be the universal cover of the quotient by the group orbits of the maximal Cauchy development. It is possible to draw a Penrose diagram of $\tilde{\mathcal{Q}}$, by introducing bounded double null coordinates on $\tilde{\mathcal{Q}}$ and then regarding $\tilde{\mathcal{Q}}$ as a bounded subset of $\mathbb{R}^{1+1}$. In the case of vacuum nonflat T^3-Gowdy initial data with $\Lambda = 0$, it is then a well known fact that its past boundary is spacelike with respect to the causality of $\mathbb{R}^{1+1}$ and thus the Penrose diagram takes the following form:



On the other hand, for nongeneric vacuum T^2-symmetric spacetimes[26] with $t_0 > 0$, the past boundary is null:



It is natural to ask where the general case stands compared to these two particular cases, whether the past boundary is spacelike, null or neither.

Appendix A. Initial data and constraint equations for the Einstein and Einstein–Vlasov systems

We present below the constraint equations of the Einstein–Vlasov system. To obtain the constraint equations in the vacuum case, it suffices to replace all matter terms (i.e., all terms containing $\hat{f}$) by zero.

Recall that a smooth initial data set for the Einstein–Vlasov system is a quadruplet $(\Sigma, h, K, \hat{f})$ such that:

(1) Σ is a smooth 3-dimensional manifold,

(2) h is a smooth Riemannian metric on Σ,

(3) K is a smooth symmetric 2-tensor on Σ,

(4) $\hat{f}$ is a smooth function defined on the tangent bundle of Σ,

(5) $(\Sigma, h, K, \hat{f})$ satisfies the constraint equations

$$R^{(3)} - K_{ab}K^{ab} + (tr K)^2 = 16\pi\rho + 2\Lambda, \tag{275}$$

$$\nabla^{(3)}_a K_b{}^a - \nabla^{(3)}_b (tr K) = 8\pi j_b, \tag{276}$$

where $\nabla^{(3)}$ and $R^{(3)}$ denote the Levi-Civita and the Ricci curvature scalar of h and ρ and j_b are given by

$$\rho = \int_{\mathbb{R}^3} \hat{f}(1 + p^a p_a)^{1/2}\sqrt{h}\, dp^1\, dp^2\, dp^3, \tag{277}$$

$$j_a = \int_{\mathbb{R}^3} \hat{f} p_a \sqrt{h}\, dp^1\, dp^2\, dp^3, \tag{278}$$

where it has been assumed in the above definitions that, if π_Σ denotes the natural projection from $T\Sigma$ to Σ, then (p^1, p^2, p^3) are global coordinates on $\pi_\Sigma^{-1}(x)$ for any $x \in \Sigma$.

[26] See the appendix of [Smulevici 2008], for instance.

Appendix B. Surface-symmetric spacetimes in areal coordinates

We present in this appendix a change of coordinates and parametrization of the metric which brings the metric (7) from the usual parametrization:

$$ds^2 = -e^{2\mu(r,\theta)}dr^2 + e^{2\lambda(r,\theta)}d\theta^2 + r^2\gamma_{ab}dx^a dx^b. \tag{279}$$

We define the new time coordinate by $t = r^2$. The metric now takes the form

$$ds^2 = -\frac{e^{2\mu}}{4t}dt^2 + e^{2\lambda}d\theta^2 + t\gamma_{ab}\,dx^a\,dx^b. \tag{280}$$

We can then define the functions α and ν by

$$e^{2\lambda} = \frac{e^{2\nu}}{t}, \tag{281}$$

$$e^{2\mu} = 4\alpha e^{2\nu}. \tag{282}$$

in order to obtain the metric in the form (7).

Appendix C. From symmetric initial data to symmetric spacetimes

We recall in this section that the symmetries of initial data are transmitted to the maximal Cauchy development. For the proofs in the vacuum case and a more exhaustive treatment of these questions, we refer the reader to the classical work of Chruściel [Chruściel 1991]. We will write the theorems in the vacuum case for simplicity.

First, we recall that Killing data leads to Killing vector fields:

Proposition 5. *Let (Σ, h, K) be a vacuum initial data set for the Einstein equations. Assume that there exists a smooth vector field Y such that*

$$\mathcal{L}_Y h = \mathcal{L}_Y K = 0 \tag{283}$$

Let $(\mathcal{M}, g)$ denote the maximal Cauchy development of (Σ, h, K) as in the statement of the theorem of Section 3 and let $\phi : \Sigma \to \mathcal{M}$ be the corresponding embedding. Then there exists a smooth vector field X on $\mathcal{M}$ such that

$$\mathcal{L}_X g = 0, \quad X_{|\phi(\Sigma)} = \phi_*(Y). \tag{284}$$

Proposition 6. *Let (Σ, h, K) be a vacuum initial data set for the Einstein equations. Assume that there exists a topological group G and a smooth action of G by isometries on (Σ, h, K), that is, a map*

$$\phi : G \times \Sigma \to \Sigma, \quad (q, p) \mapsto \phi_q(p) \qquad \textit{such that} \quad \phi_g^* h = h \quad \textit{and} \quad \phi_g^* K = K. \tag{285}$$

Let $(\mathcal{M}, g)$ denote the maximal Cauchy development of (Σ, h, K) as in the statement of the theorem of Section 3 and let i be the corresponding embedding of Σ in $\mathcal{M}$. There exists an action ψ of G on $\mathcal{M}$,

$$\psi : G \times \mathcal{M} \to \mathcal{M}, \quad (q, p) \mapsto \psi_q(p), \tag{286}$$

such that, for all $q \in G$,

$$\psi_q^* g = g, \quad \psi_q \circ i = i \circ \phi_g. \tag{287}$$

Acknowledgements

I would like to thank Mihalis Dafermos for his consistent support, his encouragements, his precious help and for introducing me to the problems addressed in this article. I would also like to thank John Stewart for much useful advice and Gustav Holzegel for many interesting discussions. Finally, I would like to thank Hans Ringström for his comments on a previous version of this text contained in my thesis. The work necessary to prove Theorems 1 and 3 was conducted during my stay in MIT from October to December 2007 and I wish to thank the Department of Mathematics of MIT and the NSF for the help and funding I received during my stay. Finally, I wish to gratefully acknowledge funding from EPSRC.

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Received 1 Sep 2009. Revised 24 Dec 2009. Accepted 31 Jul 2010.

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ANALYSIS AND PDE
Vol. 4, No. 2, 2011

ON A MAXIMUM PRINCIPLE AND ITS APPLICATION TO THE LOGARITHMICALLY CRITICAL BOUSSINESQ SYSTEM

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In this paper we study a transport-diffusion model with some logarithmic dissipations. We look for two kinds of estimates. The first is a maximum principle whose proof is based on Askey theorem concerning characteristic functions and some tools from the theory of C_0-semigroups. The second is a smoothing effect based on some results from harmonic analysis and submarkovian operators. As an application we prove the global well-posedness for the two-dimensional Euler–Boussinesq system where the dissipation occurs only on the temperature equation and has the form $|\mathrm{D}|/\log^\alpha(e^4+\mathrm{D})$, with $\alpha \in [0, \frac{1}{2}]$. This result improves on an earlier critical dissipation condition ($\alpha = 0$) needed for global well-posedness.

1. Introduction

The first goal of this paper is to study some mathematical problems related to the following transport-diffusion model with logarithmic dissipations:

$$\begin{cases} \partial_t\theta + v\cdot\nabla\theta + \kappa\dfrac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, \\ \operatorname{div} v = 0, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{1}$$

Here, the unknown is the scalar function θ, the velocity v is a time-dependent vector field with zero divergence and θ_0 is the initial datum. The parameters are $\kappa \geq 0$, $\lambda > 1$ and $\alpha, \beta \in \mathbb{R}$. The operator $|\mathrm{D}|^\beta/\log^\alpha(\lambda+|\mathrm{D}|)$ is defined through its Fourier transform:

$$\mathscr{F}\left(\frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}f\right)(\xi) = \frac{|\xi|^\beta}{\log^\alpha(\lambda+|\xi|)}(\mathscr{F}f)(\xi).$$

We will discuss along this paper some quantitative properties for this model; especially two kinds of information will be established: maximum principle and some smoothing effects. We notice that the

MSC2000: primary 35Q35; secondary 76D03.
Keywords: Boussinesq system, logarithmic dissipation, global existence.

special case of (1) corresponding to $\alpha = 0$ and $\beta \in [0, 2]$ appears naturally in some fluid models like quasigeostrophic equations or Boussinesq systems. In this context A. Córdoba and D. Córdoba [2004] established *a priori* L^p estimates: for $p \in [1, \infty]$ and $t \geq 0$,

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}. \tag{2}$$

We remark also that the proof in the case $p = +\infty$ can be obtained from the following representation of the fractional Laplacian $|\mathrm{D}|^\beta$:

$$|\mathrm{D}|^\beta f(x) = c_d \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\beta}}\, dy.$$

Indeed, one can check that if a continuous function reaches its maximum at a point x_0, then $|\mathrm{D}|^\beta f(x_0) \geq 0$ and hence we conclude as for the heat equation. Our first main result is a generalization of the result of [Córdoba and Córdoba 2004] to (1).

Theorem 1.1. *Let* $\kappa \geq 0$, $d \in \{2, 3\}$, $\beta \in]0, 1]$, $\alpha \geq 0$, $\lambda \geq e^{(3+2\alpha)/\beta}$ *and* $p \in [1, \infty]$. *Then any smooth solution of* (1) *satisfies*

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

Remark 1.2. The restriction on the parameter β is technical and we believe that the above theorem remains true for $\beta \in]1, 2[$ and $\alpha > 0$.

We discuss the proof in the special case $v \equiv 0$. Equation (1) is reduced to the fractional heat equation

$$\partial_t \theta + \kappa\, \mathscr{L} \theta = 0 \quad \text{with} \quad \mathscr{L} := \frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda + |\mathrm{D}|)}.$$

The solution is explicitly given by the convolution formula

$$\theta(t, x) = K_t \star \theta_0(x) \quad \text{with} \quad \widehat{K_t}(\xi) = e^{-t|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}.$$

We will show that the family $(K_t)_{t\geq 0}$ is a convolution semigroup of probabilities, which means that $\mathscr{L}$ is the generator of a Lévy semigroup. Consequently, this family is a C_0-semigroup of contractions on L^p for every $p \in [1, \infty[$. The important step in the proof is to get the positivity of the kernel K_t. For this purpose we use Askey's criterion for characteristic functions; see Theorem 3.4. The restrictions on the dimension d and the values of β are due to the use of this criterion. Now to deal with the full transport-diffusion equation (1) we use some results from the theory of C_0-semigroups of contractions.

The second estimate that we intend to establish is a generalized Bernstein inequality. Before stating the result we recall that for $q \in \mathbb{N}$ the operator Δ_q is the frequency localization around an annulus of size 2^q; see next section for more details.

Theorem 1.3. *Let* $d \in \{1, 2, 3\}$, $\beta \in]0, 1]$, $\alpha \geq 0$, $\lambda \geq e^{(3+2\alpha)/\beta}$ *and* $p \in]1, \infty[$. *For* $q \in \mathbb{N}$ *and* $f \in \mathscr{S}(\mathbb{R}^d)$,

$$2^{q\beta}(q+1)^{-\alpha} \|\Delta_q f\|_{L^p}^p \leq C \int_{\mathbb{R}^d} \left(\frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda + |\mathrm{D}|)} \Delta_q f \right) |\Delta_q f|^{p-2} \Delta_q f \, dx,$$

where C *is a constant depending on* p, α, β *and* λ.

The proof relies on some tools from the theory of Lévy operators or more generally submarkovians operators combined with some results from harmonic analysis.

Remarks 1.4. (1) When $\alpha = 0$ then the inequality above is valid for all $\beta \in [0, 2]$. The case $\beta = 2$ was discussed in [Danchin 2001; Planchon 2000]. The remaining case $\beta \in [0, 2[$ was treated in [2007], but only for $p \geq 2$.

(2) The proof for the case $p = 2$ is an easy consequence of the Plancherel identity and does not require any assumption on the parameters α, β and λ.

The second part of this paper is concerned with an application of Theorems 1.1 and 1.3 to the following Boussinesq model with general dissipation

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla \pi = \theta e_2, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \theta + v \cdot \nabla \theta + \kappa \mathscr{L} \theta = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0, \theta_{|t=0} = \theta_0. \end{cases} \tag{3}$$

Here, the velocity field v is given by $v = (v^1, v^2)$, while the pressure π and the temperature θ are scalar functions. The force term θe_2 in the velocity equation, with e_2 the vector (0, 1), models the effect of gravity on the fluid motion. The operator $\mathscr{L}$, whose form may vary, is used to take into account anomalous diffusion in the fluid motion.

From a mathematical point of view, the question of global well-posedness for the inviscid model, corresponding to $\kappa = 0$, is extremely hard to deal with. We point out that the classical theory of symmetric hyperbolic quasilinear systems can be applied to this system and thus we can get local well-posedness for smooth initial data. The significant quantity that one needs to bound in order to get global existence is the L^∞-norm of the vorticity, defined by $\omega = \operatorname{curl} v = \partial_1 v^2 - \partial_2 v^1$. Now we observe from the first equation of (3) that ω solves the equation

$$\partial_t \omega + v \cdot \nabla \omega = \partial_1 \theta. \tag{4}$$

The main difficulty encountered for global existence is due to the lack of strong dissipation in the temperature equation: we don't see how to estimate in a suitable way the quantity $\int_0^T \|\partial_1 \theta\|_{L^\infty}$. However, the situation in the viscous case, $\kappa > 0$ and $\mathscr{L} = -\Delta$, is well understood, and the question of global existence was solved recently in a series of papers. Chae [2006] proved global existence and uniqueness for initial data $(v_0, \theta_0) \in H^s \times H^s$, with $s > 2$; see also [Hou and Li 2005]. This result was improved in [Hmidi and Keraani 2009] to initial data $v_0 \in B_{p,1}^{(2/p)+1}$ and $\theta_0 \in B_{p,1}^{-1+(2/p)} \cap L^r$, $r > 2$. The global existence of Yudovich solutions for this system was treated in [Danchin and Paicu 2009]. The same authors [2008] constructed global strong solutions for a dissipative term of the form $\partial_{11}\theta$ instead of $\Delta\theta$. In [Hmidi and Zerguine 2010; Hmidi et al. 2011] we try to understand the lower dissipation $\mathscr{L} = |\mathrm{D}|^\alpha$ needed for global existence. In the first of these papers we proved global well-posedness when $\alpha \in]1, 2[$. The proof relies on the fact that the dissipation is sufficiently strong to counterbalance the possible amplification of the vorticity due to $\partial_1\theta$. However the case $\alpha = 1$ is not reached by this method, and this value of α is called critical, in the sense that the dissipation and the amplification of the vorticity due to $\partial_1\theta$ have the same order.

In [Hmidi et al. 2011] we proved there is some hidden structure leading to global existence in the critical case. More precisely, we introduced the mixed quantity

$$\Gamma = \omega + \mathscr{R}\theta, \quad \text{with} \quad \mathscr{R} := \frac{\partial_1}{|\mathrm{D}|};$$

it satisfies the equation

$$\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla]\theta.$$

As a matter of fact, the problem in the framework of Lebesgue spaces is reduced to estimating the commutator between the advection $v \cdot \nabla$ and Riesz transform $\mathcal{R}$, which is homogenous of degree zero. Since Riesz transform is a Calderón–Zygmund operator, we can, using the smoothing effects for θ in a suitable way, get a global estimate of $\|\omega(t)\|_{L^p}$. We can then use this information to control more strong norms of the vorticity, like $\|\omega(t)\|_{L^\infty}$ or $\|\omega(t)\|_{B^0_{\infty,1}}$. For more discussion about the global well-posedness problem concerning other classes of Boussinesq systems we refer to [Hmidi et al. 2010; Miao and Xue 2009].

Our goal here is to relax the critical dissipation needed for global well-posedness by some logarithmic factor. More precisely, we will study the logarithmically critical Boussinesq model

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla \pi = \theta e_2, \\ \partial_t \theta + v \cdot \nabla \theta + \dfrac{|\mathrm{D}|}{\log^\alpha(\lambda + |\mathrm{D}|)}\theta = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases} \tag{5}$$

Before stating our result we need some new definitions. We define the logarithmic Riesz transform $\mathcal{R}_\alpha$ by $\mathcal{R}_\alpha = (\partial_1/|\mathrm{D}|)\log^\alpha(\lambda + |\mathrm{D}|)$. Next, for given $\alpha \in \mathbb{R}$ we define the function spaces $\{\mathcal{X}_p\}_{1 \le p \le \infty}$ by

$$u \in \mathcal{X}_p \iff \|u\|_{\mathcal{X}_p} := \|u\|_{B^0_{\infty,1} \cap L^p} + \|\mathcal{R}_\alpha u\|_{B^0_{\infty,1} \cap L^p} < \infty.$$

Our result reads as follows (see Section 2 for the definitions and the basic properties of Besov spaces).

Theorem 1.5. *Let $\alpha \in [0, \frac{1}{2}]$, $\lambda \ge e^4$ and $p \in]2, \infty[$. Let $v_0 \in B^1_{\infty,1} \cap \dot{W}^{1,p}$ be a divergence-free vector field of $\mathbb{R}^2$ and $\theta_0 \in \mathcal{X}_p$. Then there exists a unique global solution (v, θ) to the system* (5) *with*

$$v \in L^\infty_{\text{loc}}(\mathbb{R}_+; B^1_{\infty,1} \cap \dot{W}^{1,p}), \qquad \theta \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{X}_p) \cap \widetilde{L}^1_{\text{loc}}(\mathbb{R}_+; B^{1,-\alpha}_{p,\infty}).$$

The proof shares the same ideas as the case $\alpha = 0$ treated in [Hmidi et al. 2011] but with more technical difficulties. We define

$$\mathcal{R}_\alpha = \frac{\partial_1}{|\mathrm{D}|}\log^\alpha(\lambda + |\mathrm{D}|) \quad \text{and} \quad \Gamma = \omega + \mathcal{R}_\alpha \theta.$$

Then we get

$$\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}_\alpha, v \cdot \nabla]\theta.$$

To estimate the commutator in the framework of Lebesgue spaces we use the paradifferential calculus combined with Theorems 1.1 and 1.3.

Remarks 1.6. (1) For global well-posedness for the generalized Navier–Stokes system in dimension three, Tao [2009] proved that we can improve the dissipation $|\mathrm{D}|^{5/2}$ to

$$\frac{|\mathrm{D}|^{5/2}}{\log^{1/2}(2 + |\mathrm{D}|)}.$$

(2) The space $\mathscr{X}_p$ is less regular than the space $B^{\varepsilon}_{\infty,1} \cap B^{\varepsilon}_{p,1}$, for all $\varepsilon > 0$. More precisely, we will see in Corollary 4.3 that $B^{\varepsilon}_{\infty,1} \cap B^{\varepsilon}_{p,1} \hookrightarrow \mathscr{X}_p$.

(3) If we take $\theta = 0$ then the system (5) is reduced to the two-dimensional Euler system. It is well-known that this system is globally well-posed in H^s for $s > 2$. The main tool for global existence is the BKM criterion [Beale et al. 1984] ensuring that the development of finite-time singularities for Kato's solutions is related to the blowup of the L^∞ norm of the vorticity near maximal time existence. Vishik [1998] extended the global existence of strong solutions to initial data belonging to Besov spaces $B^{1+2/p}_{p,1}$. These spaces have the same scale as Lipschitz functions and in this sense they are called critical and it is not at all clear whether BKM criterion can be used in this context.

(4) Since $B^{1+2/r}_{r,1} \hookrightarrow B^1_{\infty,1} \cap \dot{W}^{1,p}$ for all $r \in [1, +\infty[$ and $p > \max\{r, 2\}$, the space of initial velocity in our theorem contains all the critical spaces $B^{1+2/p}_{p,1}$ except the biggest one, $B^1_{\infty,1}$. For the limiting case we have been able to prove the global existence only under the extra assumption that $\nabla v_0 \in L^p$ for some $p \in]2, \infty[$. The reason behind this extra assumption is that to obtain a global L^∞ bound for the vorticity we need first to establish an L^p estimate for some $p \in]2, \infty[$ and it is not clear how to get rid of this condition.

(5) Since $\nabla v \in L^1_{\text{loc}}(\mathbb{R}_+; L^\infty)$, we can propagate all the higher regularities, both critical (for example $v_0 \in B^{1+2/p}_{p,1}$ with $p < \infty$) and subcritical (for example $v_0 \in H^s$, with $s > 2$).

2. Notation and preliminaries

Throughout this paper we will use the following notation.

- For any positive A and B the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.
- For any tempered distribution u, both $\hat{u}$ and $\mathscr{F}u$ denote the Fourier transform of u.
- For every $p \in [1, \infty]$, we denote by $\|\cdot\|_{L^p}$ the norm in the Lebesgue space L^p.
- The norm in the mixed space time Lebesgue space $L^p([0, T], L^r(\mathbb{R}^d))$ is denoted by $\|\cdot\|_{L^p_T L^r}$ (with the obvious generalization to $\|\cdot\|_{L^p_T \mathscr{X}}$ for any normed space $\mathscr{X}$).
- For any pair of operators P and Q on some Banach space $\mathscr{X}$, the commutator $[P, Q]$ is given by $PQ - QP$.
- For $p \in [1, \infty]$, we denote by $\dot{W}^{1,p}$ the space of distributions u such that $\nabla u \in L^p$.

Functional spaces. We introduce the so-called Littlewood–Paley decomposition and the corresponding cut-off operators. There exists radial positive functions $\chi \in \mathscr{D}(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d \backslash \{0\})$ such that:

(i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1$ and $\operatorname{supp} \chi \cap \operatorname{supp} \varphi(2^{-q}) = \varnothing$ for all $q \geq 1$.

(ii) $\operatorname{supp} \varphi(2^{-j} \cdot) \cap \operatorname{supp} \varphi(2^{-k} \cdot) = \varnothing$ if $|j - k| \geq 2$.

For every $v \in \mathscr{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1} v = \chi(D) v, \quad \Delta_q v = \varphi(2^{-q} D) v \ \text{ for } q \in \mathbb{N}, \quad S_q = \sum_{j=-1}^{q-1} \Delta_j \ \text{ for } q \in \mathbb{N}.$$

The homogeneous operators are defined by

$$\dot{\Delta}_q v = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad \dot{S}_q = \sum_{j\leq q-1} \dot{\Delta}_j \quad \text{for } q \in \mathbb{Z}.$$

From [Bony 1981] we split the product uv into three parts:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v \quad \text{and} \quad R(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v, \quad \text{where } \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

For $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := (2^{qs} \|\Delta_q u\|_{L^p})_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := (2^{qs} \|\dot{\Delta}_q u\|_{L^p})_{\ell^r(\mathbb{Z})} < +\infty.$$

For $s, s' \in \mathbb{R}$ and $p, r \in [1, \infty]$ we define the generalized Besov space $B^{s,s'}_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^{s,s'}_{p,r}} := (2^{qs}(|q|+1)^{s'} \|\Delta_q u\|_{L^p})_{\ell^r} < \infty.$$

Let $T > 0$ and $\rho \geq 1$. We denote by $L^\rho_T B^{s,s'}_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^{s,s'}_{p,r}} := \left\| (2^{qs}(|q|+1)^{s'} \|\Delta_q u\|_{L^p})_{\ell^r} \right\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T B^{s,s'}_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T B^s_{p,r}} := (2^{qs}(|q|+1)^{s'} \|\Delta_q u\|_{L^\rho_T L^p})_{\ell^r} < +\infty.$$

By a direct application of the Minkowski inequality, we have the following links between these spaces, for $\varepsilon > 0$:

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\varepsilon}_{p,r}, \quad \text{if } r \geq \rho,$$
$$L^\rho_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \quad \text{if } \rho \geq r.$$

We will make frequent use of Bernstein inequalities (see [Chemin 1998], for instance).

Lemma 2.1. *There exists a constant C such that, for $q, k \in \mathbb{N}$, $1 \leq a \leq b$ and $f \in L^a(\mathbb{R}^d)$, we have*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} \leq C^k\, 2^{q(k+d(1/a-1/b))} \|S_q f\|_{L^a}$$

and

$$C^{-k} 2^{qk} \|\Delta_q f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q f\|_{L^a} \leq C^k 2^{qk} \|\Delta_q f\|_{L^a}.$$

3. Maximum principle

Our task is to establish some useful estimates for the following equation generalizing (1):

$$\begin{cases} \partial_t\theta + v\cdot\nabla\theta + \dfrac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}\,\theta = f, \\ \operatorname{div} v = 0, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{6}$$

Two special problems will be studied. One deals with L^p estimates that give in particular Theorem 1.1. The second consists in establishing some logarithmic estimates in Besov spaces with index regularity 0.

The first main result of this section generalizes Theorem 1.1:

Theorem 3.1. *Let $p\in[1,\infty]$, $\beta\in\,]0,1]$, $\alpha\geq 0$ and $\lambda\geq e^{(3+2\alpha)/\beta}$. Any smooth solution of* (6) *satisfies*

$$\|\theta(t)\|_{L^p}\leq\|\theta_0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}\,d\tau.$$

The proof is in two steps. The first is to check the result for the free fractional heat equation. More precisely, we will establish that the semigroup $e^{t\mathscr{L}}$, with

$$\mathscr{L}:=\frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)},$$

is a contraction in Lebesgue spaces L^p, for every $p\in[1,\infty[$ of course under suitable conditions on the parameters α,β,λ. This problem is reduced to showing that $\|K_t\|_{L^1}\leq 1$. This is equivalent to $K_t\in L^1$ and $K_t\geq 0$. As we will see, to get the integrability of the kernel we do not need any restriction on the value of our parameters. Nevertheless, the positivity of K_t requires some restrictions, which are detailed in Theorem 3.1. The second step is to establish the L^p estimate for the system (6) and for this purpose we use some results about Lévy operators or, more generally, submarkovian operators.

Positive definite functions. As we will see, there is a strong connection between the positivity of the kernel K_t introduced above and the notion of positive definite functions. We will first gather some well-known properties about positive definite functions and recall some useful criteria for characteristic functions. Then, as an application, we will show that the kernel K_t is positive under suitable conditions on the parameters involved.

Definition 3.2. Let $f:\mathbb{R}^d\to\mathbb{C}$ be a complex-valued function. We say that f is positive definite if for every integer $n\in\mathbb{N}^*$ and every set of points $\{x_j,\ j=1,\dots,n\}$ of $\mathbb{R}^d$ the matrix $(f(x_j-x_k))_{1\leq j,k\leq n}$ is positive Hermitian, that is,

$$\sum_{j,k=1}^n f(x_j-x_k)\xi_j\bar{\xi}_k\geq 0\quad\text{for all }\ \xi_1,\dots,\xi_n\in\mathbb{C}.$$

We will give some results about positive definite functions.

(1) From the definition, every positive definite function f satisfies

$$f(0)\geq 0,\qquad f(-x)=\overline{f(x)},\qquad |f(x)|\leq f(0).$$

(2) Continuity of a positive definite function f at zero implies continuity everywhere. More precisely,

$$|f(x)-f(y)| \leq 2f(0)\big(f(0)-f(x-y)\big).$$

(3) The sum of two positive definite functions is also positive definite and according to Schur's lemma the product of two positive definite functions is also positive definite and therefore the class of positive definite functions is a convex cone closed under multiplication.

(4) Let μ be a finite positive measure. Its Fourier–Stieltjes transform is given by

$$\hat{\mu}(\xi)=\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\,d\mu(x).$$

It is easy to see that $\hat{\mu}$ is a positive definite function. Indeed,

$$\sum_{j,k=1}^{n}\hat{\mu}(x_j-x_k)\xi_j\bar{\xi}_k=\int_{\mathbb{R}^d}\Big(\sum_{j,k=1}^{n}e^{-ix\cdot x_j}\xi_j e^{ix\cdot x_k}\bar{\xi}_k\Big)d\mu(x)=\int_{\mathbb{R}^d}\Big|\sum_{j=1}^{n}e^{-ix\cdot x_j}\xi_j\Big|^2 d\mu(x)\geq 0.$$

The converse of (4) is due to Bochner; see for instance [Bochner 1959, Theorem 19].

Theorem 3.3 (Bochner's theorem). *Let $f:\mathbb{R}^d\to\mathbb{C}$ be a continuous positive definite function. Then f is the Fourier transform of a finite positive Borel measure.*

There are some criteria for radial functions to be positive definite. For example in dimension one the celebrated criterion of Pólya [1949] states that if $F:[0,+\infty[\,\to\mathbb{R}$ is continuous and convex with $F(0)=1$ and $\lim_{r\to+\infty}F(r)=0$ then $f(x)=F(|x|)$ is positive definite. This criterion was extended to higher dimensions by numerous authors [Askey 1973; Gneiting 2001; Trigub 1989]. We mention only one extension:

Theorem 3.4 (Askey). *Let $d\in\mathbb{N}$ and let $F:[0,+\infty[\,\to\mathbb{R}$ be a continuous function such that*

(1) $F(0)=1$,

(2) *the function $r\mapsto(-1)^dF^{(d)}(r)$ exists and is convex on* $]0,+\infty[$, *and*

(3) $\lim_{r\to+\infty}F(r)=\lim_{r\to+\infty}F^{(d)}(r)=0$.

Then for every $k\in\{1,2,\dots,2d+1\}$ the function $x\mapsto F(|x|)$ is the Fourier transform of a probability measure on $\mathbb{R}^k$.

Remark 3.5. As an application of Askey's theorem we have that $x\mapsto e^{-t|x|^\beta}$ is positive definite for all $t>0$, $\beta\in\,]0,1]$ and $d\in\mathbb{N}$. Indeed, the function $F(r)=e^{-tr^\beta}$ is completely monotone, that is, $(-1)^kF^{(k)}(r)\geq 0$, for all $r>0,\ k\in\mathbb{N}$. Although the case $\beta\in\,]1,2]$ cannot be reached by this criterion, the result is still true.

The perturbation of the function above by a logarithmic damping is also positive definite:

Proposition 3.6. *Let $\alpha,t\in[0,+\infty[\,\times\,]0,+\infty[$, $\beta\in\,]0,1]$, $\lambda\geq e^{(3+2\alpha)/\beta}$, and define $f:\mathbb{R}^d\to\mathbb{R}$ by*

$$f(x)=e^{-t|x|^\beta/\log^\alpha(\lambda+|x|)}.$$

Then f is a positive definite function for $d\in\{1,2,3\}$.

Remarks 3.7. (1) It is possible that this result remains true for higher dimension $d \geq 4$ but we will avoid dealing with this more computational case. We also think that the radial function associated to f is completely monotone.

(2) The lower bound of λ is not optimal by our method. In fact we can obtain more precise bounds, but this seems to be irrelevant.

Proof. We write $f(x) = F(|x|)$ with

$$F(r) = e^{-t\phi(r)} \quad \text{and} \quad \phi(r) = \frac{r^\beta}{\log^\alpha(\lambda+r)}\cdot$$

The function F is smooth on $]0, \infty[$ and assumptions (1) and (3) of Theorem 3.4 are satisfied. It follows that the function f is positive definite for $d \in \{1, 2, 3\}$ if

$$F^{(3)}(r) \leq 0.$$

Easy computations give for $r > 0$,

$$F^{(3)}(r) = \big[-t\,\phi^{(3)}(r) + 3t^2\,\phi'(r)\phi^{(2)}(r) - t^3\,(\phi'(r))^3\big]F(r).$$

We will prove that

$$\phi'(r) \geq 0,\ \phi^{(2)}(r) \leq 0 \quad \text{and} \quad \phi^{(3)}(r) \geq 0.$$

This is sufficient to get $F^{(3)}(r) \leq 0$, for all $r > 0$. The first derivative of ϕ is given by

$$\begin{aligned}\phi'(r) &= \frac{\beta\, r^{\beta-1}}{\log^\alpha(\lambda+r)} - \frac{\alpha\, r^\beta}{(\lambda+r)\log^{\alpha+1}(\lambda+r)}\\ &= \frac{r^{\beta-1}}{(\lambda+r)\log^{\alpha+1}(\lambda+r)}\big(\beta\lambda\log(\lambda+r) + r(\beta\log(\lambda+r) - \alpha)\big).\end{aligned}$$

We see that if λ satisfies

$$\lambda \geq e^{\alpha/\beta} \tag{7}$$

then $\phi'(r) \geq 0$. For the second derivative of ϕ we obtain

$$\begin{aligned}\phi^{(2)}(r) &= -\frac{\beta(1-\beta)r^{\beta-2}}{\log^\alpha(\lambda+r)} - \frac{2\alpha\beta r^{\beta-1}}{(\lambda+r)\log^{1+\alpha}(\lambda+r)} + \frac{\alpha r^\beta}{(\lambda+r)^2\log^{\alpha+1}(\lambda+r)} + \frac{\alpha(\alpha+1)r^\beta}{(\lambda+r)^2\log^{\alpha+2}(\lambda+r)}\\ &= \frac{r^{\beta-2}}{\log^\alpha(\lambda+r)}\Big[-\beta(1-\beta) - \frac{2\alpha\beta r}{(\lambda+r)\log(\lambda+r)} + \frac{\alpha r^2}{(\lambda+r)^2\log(\lambda+r)} + \frac{\alpha(\alpha+1)r^2}{(\lambda+r)^2\log^2(\lambda+r)}\Big].\end{aligned}$$

Since $(r^2)/(\lambda+r)^2 \leq r/(\lambda+r) \leq 1$, we have

$$\begin{aligned}\phi^{(2)}(r) &\leq \frac{r^{\beta-2}}{\log^\alpha(\lambda+r)}\Big[(1-\beta)\Big(-\beta + \frac{2\alpha}{\log(\lambda+r)}\Big) - \frac{\alpha r}{(\lambda+r)\log(\lambda+r)}\Big(1 - \frac{\alpha+1}{\log(\lambda+r)}\Big)\Big]\\ &\leq \frac{r^{\beta-2}}{\log^\alpha(\lambda+r)}\Big[(1-\beta)\Big(-\beta + \frac{2\alpha}{\log\lambda}\Big) - \frac{\alpha r}{(\lambda+r)\log(\lambda+r)}\Big(1 - \frac{\alpha+1}{\log\lambda}\Big)\Big].\end{aligned}$$

Now we choose λ such that

$$-\beta + \frac{2\alpha}{\log\lambda} \leq 0 \quad \text{and} \quad 1 - \frac{\alpha+1}{\log\lambda} \geq 0,$$

which is true whenever

$$\lambda \geq \max(e^{2\alpha/\beta}, e^{\alpha+1}). \tag{8}$$

Under this assumption we get

$$\phi^{(2)}(r) \leq 0 \quad \text{for all } r > 0.$$

Similarly we have

$$\phi^{(3)}(r) = I_1 + I_2 + I_3 + I_4,$$

with

$$\begin{aligned}
I_1 &= \alpha(\alpha+1)r^{\beta-1}\frac{\log^{-\alpha-3}(\lambda+r)}{(\lambda+r)^2}\big(3\lambda\beta\log(\lambda+r) + r(3\beta\log(\lambda+r) - (2+\alpha))\big),\\
I_2 &= \alpha r^{\beta}\frac{\log^{-2-\alpha}(\lambda+r)}{(\lambda+r)^3}\big(-3(1+\alpha) + (-3\beta^2+6\beta-2)\log(\lambda+r)\big),\\
I_3 &= \alpha r^{\beta-2}\,\frac{\log^{-1-\alpha}(\lambda+r)}{(\lambda+r)^3}\big(\lambda\beta(9-6\beta)r + 3\lambda^2\beta(1-\beta)\big),\\
I_4 &= (2-\beta)(1-\beta)\beta r^{\beta-3}\log^{-\alpha}(\lambda+r).
\end{aligned}$$

It is easy to see that I_3 and I_4 are nonnegative. On the other hand, $I_1 + I_2$ equals

$$3\lambda\beta\alpha(\alpha+1)r^{\beta-1}\frac{\log^{-\alpha-2}(\lambda+r)}{(\lambda+r)^2} + \alpha r^{\beta}\frac{\log^{-2-\alpha}(\lambda+r)}{(\lambda+r)^3}\Big[-3(1+\alpha) + (\alpha+1)(\lambda+r)\Big(3\beta - \frac{2+\alpha}{\log(\lambda+r)}\Big) + (-3\beta^2+6\beta-2)\log(\lambda+r)\Big].$$

Since $-3\beta^2+6\beta-2 \geq -2$ for $\beta \in [0, 1]$, and since $-\dfrac{\log x}{x} \geq -\dfrac{\log\lambda}{\lambda}$ for all $x \geq \lambda \geq e$, we have

$$\begin{aligned}
I_1 + I_2 &\geq \alpha(\alpha+1)r^{\beta}\frac{\log^{-2-\alpha}(\lambda+r)}{(\lambda+r)^3}\Big[-3 + (\lambda+r)\Big(3\beta - \frac{2+\alpha}{\log\lambda} - 2\frac{\log(\lambda+r)}{(\alpha+1)(\lambda+r)}\Big)\Big]\\
&\geq \alpha(\alpha+1)r^{\beta}\frac{\log^{-2-\alpha}(\lambda+r)}{(\lambda+r)^2}\Big[3\beta - \frac{3}{\lambda} - \frac{2+\alpha}{\log\lambda} - \frac{2\log\lambda}{(\alpha+1)\lambda}\Big].
\end{aligned}$$

We can check that $\log\lambda \leq \lambda$ and $\log^2\lambda \leq \lambda$, for all $\lambda \geq e$. Thus

$$\begin{aligned}
I_1 + I_2 &\geq \alpha(\alpha+1)r^{\beta}\frac{\log^{-2-\alpha}(\lambda+r)}{(\lambda+r)^2}\Big[3\beta - \frac{1}{\log\lambda}\Big(5+\alpha+\frac{2}{\alpha+1}\Big)\Big]\\
&\geq \alpha(\alpha+1)r^{\beta}\frac{\log^{-2-\alpha}(\lambda+r)}{(\lambda+r)^2}\Big[3\beta - \frac{7+\alpha}{\log\lambda}\Big].
\end{aligned}$$

We choose λ such that

$$3\beta - \frac{7+\alpha}{\log\lambda} \geq 0.$$

It follows that $I_1 + I_2$ is nonnegative if

$$\lambda \geq e^{(7+\alpha)/(3\beta)}.$$

A condition that implies this inequality and also (7) and (8) is

$$\lambda \geq e^{(3+2\alpha)/\beta}.$$

Finally, we get: for all $\alpha \geq 0$, $\beta \in]0, 1]$, and $\lambda \geq e^{(3+2\alpha)/\beta}$,

$$\phi^{(3)}(r) \geq 0 \quad \text{ for all } r > 0.$$

This achieves the proof. □

More precise information about the kernel K_t is listed now:

Lemma 3.8. *Let $\lambda \geq 2$ and denote by K_t the element of $\mathcal{S}'(\mathbb{R}^d)$ such that*

$$\widehat{K}_t(\xi) = e^{-t|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}.$$

(1) *For $(t, \alpha, \beta) \in]0, \infty[\times \mathbb{R} \times]0, \infty[$ the function K_t belongs to $\in L^1 \cap C_0$.*

(2) *For $d \in \{1, 2, 3\}$, $(t, \alpha, \beta) \in]0, +\infty[\times [0, \infty[\times]0, 1]$ and $\lambda \geq e^{(3+2\alpha)/\beta}$, we have*

$$K_t(x) \geq 0 \text{ for all } x \in \mathbb{R}_+ \quad \text{and} \quad \|K_t\|_{L^1} = 1.$$

Proof. (1) By definition we have

$$K_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t|\xi|^\beta/\log^\alpha(\lambda+|\xi|)} e^{i x\cdot\xi}\, d\xi.$$

Let $\mu \geq 0$. Integrating by parts we get

$$|x|^\mu x_j^d K_t(x) = (-2i\pi)^{-d} \int_{\mathbb{R}^d} \partial_{\xi_j}^d \big(e^{-t((|\xi|^\beta)/\log^\alpha(\lambda+|\xi|))}\big)|x|^\mu e^{i x\cdot\xi} d\xi.$$

On the other hand we have

$$|x|^\mu e^{i x\cdot\xi} = |\mathrm{D}|^\mu e^{i x\cdot\xi},$$

where $|\mathrm{D}|$ is a fractional derivative on the variable ξ. Thus we get

$$|x|^\mu x_j^d K_t(x) = (-2i\pi)^{-d} \int_{\mathbb{R}^d} |\mathrm{D}|^\mu \partial_{\xi_j}^d \big(e^{-t|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}\big) e^{i x\cdot\xi}\, d\xi.$$

Now we use the following representation for $|\mathrm{D}|^\mu$ when $\mu \in]0, 2]$:

$$|\mathrm{D}|^\mu f(x) = C_{\mu,d} \int_{\mathbb{R}^d} \frac{f(x)-f(x-y)}{|y|^{d+\mu}}\, dy.$$

It follows that

$$|x|^\mu |x_j^d K_t(x)| \leq C_{\mu,d} \int_{\mathbb{R}^{2d}} \frac{|\mathcal{K}_j(\xi)-\mathcal{K}_j(\xi-y)|}{|y|^{d+\mu}}\, dy\, d\xi$$

with

$$\mathcal{K}_j(\xi) := \partial_{\xi_j}^d \big(e^{-t|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}\big).$$

Now we decompose the integral into two parts:

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \frac{|\mathcal{K}_j(\xi)-\mathcal{K}_j(\xi-y)|}{|y|^{d+\mu}}\, dy\, d\xi &= \int_{|y|\geq\frac{|\xi|}{2}} \frac{|\mathcal{K}_j(\xi)-\mathcal{K}_j(\xi-y)|}{|y|^{d+\mu}}\, dy\, d\xi + \int_{|y|\leq\frac{|\xi|}{2}} \frac{|\mathcal{K}_j(\xi)-\mathcal{K}_j(\xi-y)|}{|y|^{d+\mu}}\, dy\, d\xi \\ &= I_1 + I_2.\end{aligned}$$

To estimate the first term we use the following estimate, which can be obtained by straightforward computations:

$$|\mathcal{K}_j(\xi)| \le C_{t,\alpha,\beta}\frac{|\xi|^{\beta-d}}{\log^\alpha(\lambda+|\xi|)}e^{-t|\xi|^\beta/\log^\alpha(\lambda+|\xi|)} \le C_{t,\alpha,\beta}|\xi|^{\beta-d}e^{-(t/2)|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}.$$

Hence we get, under the assumption $\mu \in]0, \beta[$,

$$\begin{aligned} I_1 &\le C_{t,\alpha,\beta}\int_{|\xi|\le 2|y|}\frac{1}{|y|^{d+\mu}}\big(|\xi|^{\beta-d}e^{-(t/2)|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}+|\xi-y|^{\beta-d}e^{-(t/2)|\xi-y|^\beta/\log^\alpha(\lambda+|\xi-y|)}\big)\,d\xi\,dy\\ &\le C_{t,\alpha,\beta}\int_{|\xi|\le 3|y|}\frac{1}{|y|^{d+\mu}}|\xi|^{\beta-d}e^{-(t/2)|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}d\xi\,dy \le C_{t,\alpha,\beta}\int_{\mathbb{R}^d}\frac{1}{|\xi|^{d+\mu-\beta}}e^{-(t/2)|\xi|^\beta/\log^\alpha(\lambda+|\xi|)}d\xi\\ &\le C_{t,\alpha,\beta}. \end{aligned}$$

To estimate the second term we use the mean-value theorem to write

$$|\mathcal{K}_j(\xi)-\mathcal{K}_j(\xi-y)| \;\le\; |y|\sup_{\eta\in[\xi-y,\xi]}|\nabla\mathcal{K}_j(\eta)|;$$

we also have

$$|\nabla\mathcal{K}_j(\eta)| \le C_{t,\alpha,\beta}|\eta|^{\beta-d-1}e^{-(t/2)|\eta|^\beta/\log^\alpha(\lambda+|\eta|)}.$$

Since $|y| \le \frac{1}{2}|\xi|$, for $\eta \in [\xi - y, \xi]$ we have

$$\tfrac{1}{2}|\xi| \le |\eta| \le \tfrac{5}{2}|\xi|.$$

This yields

$$|\mathcal{K}_j(\xi)-\mathcal{K}_j(\xi-y)| \le C_t|y||\xi|^{\beta-d-1}e^{-Ct|\xi|^{\beta/2}}.$$

Therefore we find, for $\mu \in]0, \beta[\,\cap\,]0, 1[$,

$$\begin{aligned} I_2 &\le C_{t,\alpha,\beta}\int_{|y|\le(1/2)|\xi|}\frac{1}{|y|^{d+\mu-1}}|\xi|^{\beta-d-1}e^{-Ct|\xi|^{\beta/2}}\,dy\,d\xi \le C_{t,\alpha,\beta}\int_{\mathbb{R}^2}(1/|\xi|^{d+\mu-\beta})e^{-Ct|\xi|^{\beta/2}}d\xi\\ &\le C_{t,\alpha,\beta}. \end{aligned}$$

Finally we get

$$|x|^\mu|x_j|^d|K_t(x)| \le C_{t,\alpha,\beta} \qquad \text{for } j = 1, \dots, d.$$

Since $K_t \in C_0$, we have

$$(1+|x|^{d+\mu})|K_t(x)| \le C_t.$$

This proves that $K_t \in L^1(\mathbb{R}^d)$.

(2) Using Theorem 3.4 and Proposition 3.6 we get $K_t \ge 0$. Since $K_t \in L^1$, this implies $\|K_t\|_{L^1} = \widehat{K}_t(0) = 1$. □

Now set

$$\mathcal{L} := \frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}. \tag{9}$$

We define the propagator $e^{-t\mathcal{L}}$ by convolution:

$$e^{-t\mathcal{L}}f = K_t \star f.$$

Corollary 3.9. *Let $\alpha \geq 0$, $\beta \in]0, 1]$, $\lambda \geq e^{(3+2\alpha)/\beta}$ and $p \in [1, \infty]$. Then*

$$\|e^{-t\mathscr{L}} f\|_{L^p} \leq \|f\|_{L^p} \quad \text{for all } f \in L^p.$$

Proof. From the classical convolution inequalities combined with Lemma 3.8 we get

$$\|e^{-t\mathscr{L}} f\|_{L^p} \leq \|K_t\|_{L^1} \|f\|_{L^p} \leq \|f\|_{L^p}. \qquad \square$$

Structure of the semigroup $(e^{-t\mathscr{L}})_{t\geq 0}$. We first recall the notions of C_0-semigroup and submarkovian generators.

Definition 3.10. Let X be a Banach space and $(T_t)_{t\geq 0}$ a family of bounded operators from X into X. This family is called a *strongly continuous semigroup* on X or a *C_0-semigroup* if

(1) $T_0 = \mathrm{Id}$,

(2) $T_{t+s} = T_t T_s$ for every $t, s \geq 0$, and

(3) $\lim_{t\to 0^+} \|T_t x - x\| = 0$ for every $x \in X$.

If in addition the semigroup satisfies the estimate $\|T_t\|_{\mathscr{L}(X)} \leq 1$, then it is called a *$C_0$-semigroup of contractions*.

For a given C_0-semigroup of contractions $(T_t)_{t\geq 0}$ we define its domain $\mathscr{D}(A)$ by

$$\mathscr{D}(A) := \left\{ f \in X : \lim_{t\to 0^+} \frac{T_t f - f}{t} \text{ exists in } X \right\}$$

and we set

$$Af = \lim_{t\to 0^+} \frac{T_t f - f}{t} \quad \text{for } f \in \mathscr{D}(A).$$

It is well-known that the operator A is densely defined: its domain $\mathscr{D}(A)$ is dense in X.

Definition 3.11. Let $X = L^p(\mathbb{R}^d)$, with $p \in [1, \infty[$ and $d \in \mathbb{N}^*$. A C_0-semigroup $(T_t)_{t\geq 0}$ of contractions on X is said a *submarkovian semigroup* if it satisfies these two conditions:

(1) If $f \in X$ satisfies $f(x) \geq 0$ a.e., then $T_t f(x) \geq 0$ a.e. for every $t \geq 0$.

(2) If $f \in X$ satisfies $|f| \leq 1$, then $|T_t f| \leq 1$ for every $t \geq 0$.

Define $L^p_+ := \{f \in L^p;\ f(x) \geq 0,\ \text{a.e.}\}$. The next result is classical.

Theorem 3.12 (Beurling–Deny theorem). *Let A be a nonnegative self-adjoint operator of L^2. Then we have equivalence between*:

(1) $f \in L^2_+ \Rightarrow e^{-tA} f \in L^2_+$ *for all* $t > 0$.

(2) $f \in \mathscr{D}(A^{1/2}) \Rightarrow |f| \in \mathscr{D}(A^{1/2})$ *and* $\|A^{1/2}|f|\|_{L^2} \leq \|A^{1/2} f\|_{L^2}$.

Proposition 3.13. *Let $d \in \{1, 2, 3\}$, $p \in [1, \infty[$, $\alpha \geq 0$, $\beta \in]0, 1]$ and $\lambda \geq e^{(3+2\alpha)/\beta}$. With $\mathscr{L}$ as in* (9), *we have*:

(1) *The family $(e^{-t\mathscr{L}})_{t\geq 0}$ defines a C_0-semigroup of contractions in $L^p(\mathbb{R}^d)$.*

(2) *The family $(e^{-t\mathscr{L}})_{t\geq 0}$ defines a submarkovian semigroup in $L^p(\mathbb{R}^d)$.*

(3) *The operator $(e^{-t\mathscr{L}})_{t\geq 0}$ satisfies the Beurling–Deny theorem described in Theorem 3.12.*

Proof. (1) For $f \in L^p$ we define the action of the semigroup on f

$$e^{-t\mathscr{L}} f(x) = K_t \star f(x),$$

where $\widehat{K}_t(\xi) = e^{-t|\xi|/\log^\alpha(\lambda+|\xi|)}$. From Corollary 3.9, the semigroup maps L^p to itself if $p \in [1, \infty]$, and

$$\|K_t \star f\|_{L^p} \le \|f\|_{L^p}.$$

Conditions (1) and (2) of Definition 3.10 are easy to check. It remains to prove condition (3), concerning the strong continuity of the semigroup. Since $\|K_t\|_{L^1} = 1$ and $K_t \ge 0$, we have, for $\eta > 0$,

$$\begin{aligned} K_t \star f(x) - f(x) &= \int_{\mathbb{R}^d} K_t(y)(f(x-y) - f(x))\, dy \\ &= \int_{|y|\le\eta} K_t(y)(f(x-y) - f(x))\, dy + \int_{|y|\ge\eta} K_t(y)(f(x-y) - f(x))\, dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

The first term is estimated as follows:

$$\|I_1\|_{L^p} \le \int_{|y|\le\eta} K_t(y)\|f(\cdot-y) - f(\cdot)\|_{L^p}\, dy \le \sup_{|y|\le\eta} \|f(\cdot-y) - f(\cdot)\|_{L^p}.$$

For the second term we write $\|I_2\|_{L^p} \le 2\|f\|_{L^p} \int_{|y|\ge\eta} K_t(y)\, dy$. Combining these estimates we get

$$\|K_t \star f - f\|_{L^p} \le \sup_{|y|\le\eta} \|f(\cdot-y) - f(\cdot)\|_{L^p} + 2\|f\|_{L^p} \int_{|y|\ge\eta} K_t(y)\, dy.$$

It is well known that for every $p \in [1, \infty[$ we have

$$\lim_{\eta\to 0^+} \sup_{|y|\le\eta} \|f(\cdot-y) - f(\cdot)\|_{L^p} = 0.$$

Thus for a given $\varepsilon > 0$ we can find $\eta > 0$ small enough that

$$\sup_{|y|\le\eta} \|f(\cdot-y) - f(\cdot)\|_{L^p} \le \varepsilon.$$

Now to conclude the proof it suffices to prove that

$$\lim_{t\to 0^+} \int_{|y|\ge\eta} K_t(y)\, dy = 0.$$

This will follow from

$$K_t \overset{t\to 0^+}{\rightharpoonup} \delta_0.$$

To prove this last statement we write, for $\phi \in \mathscr{S}$,

$$\langle K_t, \phi\rangle = \frac{1}{(2\pi)^d} \langle \widehat{K}_t, \hat{\phi}\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} \hat{\phi}(\xi) d\xi.$$

We can use now Lebesgue theorem and the inversion Fourier transform leading to

$$\lim_{t\to 0^+} \langle K_t, \phi\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) d\xi = \phi(0).$$

Finally we get that $(K_t\star)_{t\geq 0}$ defines a C_0-semigroup of contractions for every $p\in[1,\infty[$.

(2) From Definition 3.11 and the first part of Proposition 3.13 it remains to show that

- for $f\in L^p$ with $f(x)\geq 0$ a.e. we have $e^{-t\mathscr{L}}f(x)\geq 0$;
- for $f\in L^p$ with $|f(x)|\leq 1$ a.e. we have $|e^{-t\mathscr{L}}f(x)|\leq 1$.

This is a direct consequence of the explicit formula

$$e^{-t\mathscr{L}}f(x)=K_t\star f(x),$$

where according to Lemma 3.8 we have $K_t\geq 0$ and $\|K_t\|_{L^1}=1$.

(3) It is not hard to see that the operator $|\mathrm{D}|^\beta/\log^\alpha(\lambda+|\mathrm{D}|)$ is a nonnegative self-adjoint operator of L^2. This operator satisfies the first condition of Theorem 3.12 since the kernel K_t is positive. □

The following result gives in particular Theorem 3.1.

Proposition 3.14. *Let A be a generator of a C_0-semigroup of contractions.*

(1) *Let $p\in[1,\infty[$ and $u\in\mathscr{D}(A)$. Then*

$$\int_{\mathbb{R}^2}Au\,|u|^{p-1}\mathrm{sign}\,u\,dx\leq 0.$$

(2) *Let θ be a smooth solution of the equation*

$$\partial_t\theta+v\cdot\nabla\theta-A\theta=f,$$

where v is a smooth vector field with zero divergence and f is a smooth function. Then

$$\|\theta(t)\|_{L^p}\leq\|\theta_0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}d\tau\quad\textit{for every } p\in[1,\infty].$$

Proof. (1) We introduce the operation $[h,g]$ between two functions by

$$[h,g]=\|g\|_{L^p}^{2-p}\int_{\mathbb{R}^2}h(x)|g(x)|^{p-1}\,\mathrm{sign}\,g(x)\,dx.$$

Define the function $\psi:[0,\infty[\,\to\mathbb{R}$ by

$$\psi(t)=[e^{tA}u,u].$$

We have $\psi(0)=\|u\|_{L^p}^2$. From the Hölder inequality and the fact that the operator e^{tA} is a contraction on L^p we get

$$\psi(t)\leq\|e^{tA}u\|_{L^p}\|u\|_{L^p}\leq\|u\|_{L^p}^2.$$

Thus we find $\psi(t)\leq\psi(0)$, for all $t\geq 0$. Therefore we get $\lim_{t\to 0^+}\dfrac{\psi(t)-\psi(0)}{t}\leq 0$. This gives

$$\int_{\mathbb{R}^2}Au(x)|u(x)|^{p-1}\,\mathrm{sign}\,u(x)\,dx\leq 0.$$

(2) Let $p \in [1, \infty[$. Multiplying the first equation in (6) by $|\theta|^{p-1} \operatorname{sign} \theta$, integrating by parts and using $\operatorname{div} v = 0$ we get

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + \int_{\mathbb{R}^2} |A\theta(x)\theta(x)|^{p-1} \operatorname{sign}\theta(x)dx \le \|f(t)\|_{L^p}\|\theta(t)\|_{L^p}^{p-1}.$$

Using Proposition 3.14 we find

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p \le \|f(t)\|_{L^p}\|\theta(t)\|_{L^p}^{p-1}.$$

Simplifying, we get $\frac{d}{dt}\|\theta(t)\|_{L^p} \le \|f(t)\|_{L^p}$. Integrating in time we get for $p \in [1, \infty[$

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}.$$

Since the estimates are uniform on the parameter p, we can get the limit case $p = +\infty$. □

Logarithmic estimate. In the last part of this section we show some logarithmic estimates generalizing results in [Vishik 1998; Hmidi and Keraani 2009]. We recall the following result on the propagation of Besov regularities.

Proposition 3.15. *Let $\kappa \ge 0$ and let A be a C_0 semigroup of contractions on $L^m(\mathbb{R}^d)$ for every $m \in [1, \infty[$. We assume that for every $q \in \mathbb{N} \cup \{-1\}$, the operator Δ_q commutes with A on a dense subset of L^p. Let $(p, r) \in [1, \infty]^2$, $s \in]-1, 1[$, and let θ be a smooth solution of*

$$\partial_t\theta + v\cdot\nabla\theta - \kappa A\theta = f.$$

Then

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,r}^s} \lesssim e^{CV(t)}\left(\|\theta_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B_{p,r}^s}\, d\tau\right),$$

where $V(t) = \|\nabla v\|_{L_t^1 L^\infty}$ and C is a constant depending only on s and d.

Proof. We set $\theta_q := \Delta_q\theta$. By localizing in frequency the equation of θ we get

$$\partial_t\theta_q + v\cdot\nabla\theta_q - \kappa A\theta_q = -[\Delta_q, v\cdot\nabla]\theta + f_q.$$

Using Proposition 3.14 we get

$$\|\theta_q(t)\|_{L^p} \le \|\theta_q(0)\|_{L^p} + \int_0^t \left\|[\Delta_q, v\cdot\nabla]\theta(\tau)\right\|_{L^p} d\tau + \int_0^t \|f_q(\tau)\|_{L^p} d\tau.$$

At the same time, we have the classical commutator estimate [Chemin 1998]:

$$\left\|[\Delta_q, v\cdot\nabla]\theta\right\|_{L^p} \le C2^{-qs}c_q\|\nabla v\|_{L^\infty}\|\theta\|_{B_{p,r}^s}, \quad \|(c_q)\|_{\ell^r} = 1.$$

Thus

$$\|\theta(t)\|_{B_{p,r}^s} \le \|\theta_0\|_{B_{p,r}^s} + C\int_0^t \|\nabla v\|_{L^\infty}\|\theta\|_{B_{p,r}^s} + \int_0^t \|f(\tau)\|_{B_{p,r}^s} d\tau.$$

It suffices now to use the Gronwall's inequality. □

Now we will show that for the index regularity $s = 0$ we can obtain a better estimate with a linear growth on the norm of the velocity.

Proposition 3.16. *Let v be a smooth divergence-free vector field on $\mathbb{R}^d$. Let $\kappa \geq 0$ and let A be a generator of C_0-semigroup of contractions on $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty[$. We assume that for every $q \in \mathbb{N}$, the operators Δ_q and A commute on a dense subset of L^p. Let θ be a smooth solution of*

$$\partial_t \theta + v \cdot \nabla\theta - \kappa A\theta = f.$$

Then, for every $p \in [1, \infty]$,

$$\|\theta\|_{\widetilde{L}_t^\infty B^0_{p,1}} \leq C(\|\theta_0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}})\left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right),$$

where the constant C does not depend on p or κ.

This was first proved in [Vishik 1998] for the case $\kappa = 0$ by using the special structure of the transport equation. In [Hmidi and Keraani 2008] we generalized Vishik's result for a transport-diffusion equation where the dissipation term has the form $-\kappa\Delta\theta$. The method described in there can be easily adapted here for our model.

Proof. Let $q \in \mathbb{N} \cup \{-1\}$ and denote by $\overline{\theta}_q$ the unique global solution of the initial value problem

$$\begin{cases} \partial_t \overline{\theta}_q + v \cdot \nabla\overline{\theta}_q - \kappa A\overline{\theta}_q = \Delta_q f, \\ \overline{\theta}_{q\,|t=0} = \Delta_q \theta^0. \end{cases} \tag{10}$$

Using Proposition 3.15 with $s = \pm\frac{1}{2}$ we get

$$\|\overline{\theta}_q\|_{\widetilde{L}_t^\infty B^{\pm(1/2)}_{p,\infty}} \lesssim \big(\|\Delta_q\theta_0\|_{B^{\pm(1/2)}_{p,\infty}} + \|\Delta_q f\|_{L^1_t B^{\pm(1/2)}_{p,\infty}}\big)e^{CV(t)},$$

where $V(t) = \|\nabla v\|_{L^1_t L^\infty}$. Combining this with the definition of Besov spaces this yields, for $j, q \geq -1$,

$$\|\Delta_j \overline{\theta}_q\|_{L_t^\infty L^p} \lesssim 2^{-\frac{1}{2}|j-q|}\big(\|\Delta_q\theta_0\|_{L^p} + \|\Delta_q f\|_{L^1_t L^p}\big)e^{CV(t)}. \tag{11}$$

By linearity and again the definition of Besov spaces we have

$$\|\theta\|_{\widetilde{L}_t^\infty B^0_{p,1}} \leq \sum_{|j-q|\geq N} \|\Delta_j \overline{\theta}_q\|_{L_t^\infty L^p} + \sum_{|j-q|<N} \|\Delta_j \overline{\theta}_q\|_{L_t^\infty L^p}, \tag{12}$$

where $N \in \mathbb{N}$ is to be chosen later. To deal with the first sum we use (11):

$$\sum_{|j-q|\geq N} \|\Delta_j \overline{\theta}_q\|_{L_t^\infty L^p} \lesssim 2^{-N/2} \sum_{q\geq -1} \big(\|\Delta_q\theta_0\|_{L^p} + \|\Delta_q f\|_{L^1_t L^p}\big)e^{CV(t)} \lesssim 2^{-N/2}\big(\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}}\big)e^{CV(t)}.$$

We now turn to the second sum in the right-hand side of (12). It is clear that

$$\sum_{|j-q|<N} \|\Delta_j \overline{\theta}_q\|_{L_t^\infty L^p} \lesssim \sum_{|j-q|<N} \|\overline{\theta}_q\|_{L_t^\infty L^p}.$$

Applying Proposition 3.14 to the system (10) yields $\|\overline{\theta}_q\|_{L_t^\infty L^p} \leq \|\Delta_q\theta_0\|_{L^p} + \|\Delta_q f\|_{L^1_t L^p}$. It follows that

$$\sum_{|j-q|<N} \|\Delta_j \overline{\theta}_q\|_{L_t^\infty L^p} \lesssim N\big(\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}}\big).$$

The outcome is that

$$\|\theta\|_{\widetilde{L}_t^\infty B^0_{p,1}} \lesssim \big(\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}}\big)(2^{-N/2}e^{CV(t)} + N).$$

Choosing

$$N=\left[\frac{2CV(t)}{\log 2}\right]+1,$$

we get the desired result. □

Combining Propositions 3.16 and 3.13 we get:

Corollary 3.17. *Let v be a smooth divergence-free vector field on $\mathbb{R}^d$, with $d\in\{2,3\}$. Let $\kappa,\alpha\geq 0$, $\beta\in\,]0,1]$, $\lambda\geq e^{(3+2\alpha)/\beta}$, and $p\in[1,\infty]$. Let θ be a smooth solution of*

$$\partial_t\theta+v\cdot\nabla\theta+\kappa|\mathrm{D}|^\beta\log^{-\alpha}(\lambda+|\mathrm{D}|)\theta=f.$$

Then

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^0}\leq C\big(\|\theta_0\|_{B_{p,1}^0}+\|f\|_{L_t^1B_{p,1}^0}\big)\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big),$$

where the constant C depends only on λ and α.

4. Proof of the generalized Bernstein inequality (Theorem 1.3)

We first extend the classical Bernstein inequality of Lemma 2.1 to more general operators:

Proposition 4.1. *Let $\alpha\in\mathbb{R}$, $\beta>0$ and $\lambda\geq 2$. Then there exists a constant C such that for every $f\in\mathcal{S}(\mathbb{R}^d)$ and for every $q\geq -1$ and $p\in[1,\infty]$ we have*

$$\big\|\Delta_q(\mathcal{L}f)\big\|_{L^p}\leq C2^{q\beta}(|q|+1)^{-\alpha}\|\Delta_qf\|_{L^p},$$

where $\mathcal{L}=\dfrac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}$ (as in (9)). *Moreover,*

$$\big\|S_q(\mathcal{L}f)\big\|_{L^p}\leq C2^{q\beta}(|q|+1)^{-\alpha}\|S_qf\|_{L^p}.$$

Remark 4.2. The first result of Proposition 4.1 remains true for more general situation where $q\in\mathbb{N}$ and the operator $|\mathrm{D}|^\beta$ is replaced by $a(\mathrm{D})$, where $a\in C^\infty(\mathbb{R}\backslash\{0\})$ is a homogeneous distribution of order $\beta\in\mathbb{R}$ satisfying

$$|\partial_\xi^\gamma a(\xi)|\leq C|\xi|^{\beta-|\gamma|}\quad\text{for every }\gamma\in\mathbb{N}^d.$$

Proof of Proposition 4.1. Case $q\in\mathbb{N}$. It is easy to see that

$$\Delta_q\Big(\frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}f\Big)=K_q\star\Delta_qf,$$

with

$$\widehat{K}_q(\xi)=\frac{\tilde{\phi}(2^{-q}\xi)|\xi|^\beta}{\log^\alpha(\lambda+|\xi|)},$$

for $\tilde{\phi}$ a smooth function supported in the annulus $\left\{\frac14\leq|x|\leq 3\right\}$ and taking the value 1 on the support of the function ϕ introduced in Section 2. By Fourier inversion and change of variables we get

$$K_q(x)=c_d\int_{\mathbb{R}^d}e^{ix\cdot\xi}\frac{\tilde{\phi}(2^{-q}\xi)|\xi|^\beta}{\log^\alpha(\lambda+|\xi|)}d\xi=c_d2^{q\beta}2^{qd}\int_{\mathbb{R}^d}e^{i2^qx\cdot\xi}\frac{\tilde{\phi}(\xi)|\xi|^\beta}{\log^\alpha(\lambda+2^q|\xi|)}d\xi=c_d2^{q\beta}2^{qd}\tilde{K}_q(2^qx),$$

with

$$\tilde{K}_q(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\tilde{\phi}(\xi)|\xi|^\beta}{\log^\alpha(\lambda + 2^q|\xi|)} d\xi.$$

Obviously we have

$$\|K_q\|_{L^1} = c_d 2^{q\beta} \|\tilde{K}_q\|_{L^1}.$$

Hence to prove Proposition 4.1 it suffices to establish

$$\|\tilde{K}_q\|_{L^1} \leq C(q+1)^{-\alpha}. \tag{13}$$

From the definition of $\tilde{K}_q$ we see that

$$\tilde{K}_q(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\tilde{\psi}(\xi)}{\log^\alpha(\lambda + 2^q|\xi|)} d\xi,$$

where $\tilde{\psi}$ belongs to the Schwartz class and is supported in $\left\{\frac{1}{4} \leq |x| \leq 3\right\}$. Integrating by parts we get, for $j \in \{1, 2, \ldots, d\}$,

$$x_j^{d+1} \tilde{K}_q(x) = (-i)^{d+1} \int_{\frac{1}{4}\leq|\xi|\leq 3} e^{ix\cdot\xi}\, \partial_{\xi_j}^{d+1} \left(\frac{\tilde{\psi}(\xi)}{\log^\alpha(\lambda + 2^q|\xi|)} \right) d\xi.$$

Now we claim that

$$\left| \partial_{\xi_j}^{d+1} \left(\frac{\tilde{\psi}(\xi)}{\log^\alpha(\lambda+2^q|\xi|)} \right) \right| \leq C_{\lambda,\alpha,d} \frac{g(\xi)}{\log^\alpha(\lambda+2^q)},$$

where $g \in \mathcal{S}(\mathbb{R}^d)$. This is an easy consequence of Leibniz formula and the fact that

$$\left| \partial_{\xi_j}^{n} \left(\frac{1}{\log^\alpha(\lambda+2^q|\xi|)} \right) \right| \leq \sum_{l,k=1}^{n} c_{l,k} \left(\frac{2^q}{\lambda+2^q|\xi|} \right)^l \frac{1}{\log^{\alpha+k}(\lambda+2^q|\xi|)} \leq \frac{C_{\lambda,\alpha,n}}{\log^\alpha(\lambda+2^q)} \quad \text{for } \tfrac{1}{4} \leq |\xi| \leq 2.$$

Thus we get for $j \in \{1, \ldots, d\}$

$$|x_j|^{d+1} |\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q) \quad \text{for } x \in \mathbb{R}^d.$$

It follows that

$$|x|^{d+1} |\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q) \quad \text{for } x \in \mathbb{R}^d.$$

It is easy to see that $\tilde{K}_q$ is continuous and

$$|\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q)$$

Consequently,

$$|\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q)(1 + |x|)^{-d-1} \quad \text{for } x \in \mathbb{R}^d.$$

This yields $\|\tilde{K}_q\|_{L^1} \leq C\log^{-\alpha}(\lambda + 2^q) \leq C(q+1)^{-\alpha}$, which concludes the proof when $q \in \mathbb{N}$.

Case $q = -1$. Here we can write the kernel K_{-1} as

$$K_{-1}(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\tilde{\chi}(\xi)|\xi|^\beta}{\log^\alpha(\lambda + |\xi|)} d\xi = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \chi_1(\xi) d\xi,$$

where $\tilde{\chi}$ is a smooth compactly supported function taking the value 1 on the support of the function χ introduced in Section 2. The function χ_1 is given by

$$\chi_1(\xi) = \frac{\tilde{\chi}(\xi)|\xi|^\beta}{\log^\alpha(\lambda + |\xi|)}.$$

We can see by means of easy computations that $\tilde{\chi}$ is smooth outside zero and satisfies, for every $\gamma \in \mathbb{N}^d$,

$$|\partial_\xi^\gamma \tilde{\chi}(\xi)| \le C_\gamma |\xi|^{\beta - |\gamma|} \quad \text{for all } \xi \neq 0.$$

Using the Mikhlin–Hörmander theorem we get

$$|K_{-1}(x)| \le C|x|^{-d-\beta}.$$

Since K_{-1} is continuous at zero we have

$$|K_{-1}(x)| \le C(1 + |x|)^{-d-\beta}.$$

This proves that $K_{-1} \in L^1$.

To prove the second estimate we use the first result combined with the identity $S_{q+2}S_q = S_q$:

$$\left\| S_q(\mathscr{L} f) \right\|_{L^p} \le \sum_{j=-1}^{q+1} \left\| \Delta_j(\mathscr{L} S_q f) \right\|_{L^p} \le C \| S_q f \|_{L^p} \sum_{j=-1}^{q+1} 2^{j\beta}(|j| + 1)^{-\alpha}.$$

Since $\beta > 0$, this last series diverges and

$$\sum_{j=-1}^{q+1} 2^{j\beta}(|j| + 1)^{-\alpha} \le C 2^{q\beta}(|q| + 1)^{-\alpha}.$$

This can be deduced from the asymptotic behavior

$$\int_1^x e^{\beta t} t^{-\alpha} dt \approx \frac{1}{\beta} e^{\beta x} x^{-\alpha} \quad \text{as } x \to +\infty.$$ □

As a consequence of Proposition 4.1 we get the following result, which describes the action of the logarithmic Riesz transform

$$\mathscr{R}_\alpha = \frac{\partial_1 \log^\alpha(\lambda + |\mathrm{D}|)}{|\mathrm{D}|}$$

on Besov spaces.

Corollary 4.3. *Let $\alpha \in \mathbb{R}$, $\lambda > 1$ and $p \in [1, \infty]$. The map*

$$(\mathrm{Id} - \Delta_{-1})\mathscr{R}_\alpha : B_{p,r}^{s,\alpha} \to B_{p,r}^s$$

is continuous.

The generalized Bernstein inequality. In this section we prove Theorem 1.3, which we restate here for convenience:

Theorem 1.3. *Let $d \in \{1, 2, 3\}$, $\beta \in]0, 1]$, $\alpha \geq 0$, $\lambda \geq e^{(3+2\alpha)/\beta}$ and $p > 1$. For $q \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$2^{q\beta}(q+1)^{-\alpha}\|\Delta_q f\|_{L^p}^p \leq C \int_{\mathbb{R}^d} \left(\frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}\Delta_q f\right)|\Delta_q f|^{p-1}\operatorname{sign}\Delta_q f\,dx.$$

where C depends on p, α, β and λ.

Some preliminary lemmas will be needed. The first is a Stroock–Varopoulos inequality for submarkovian operators. For the proof see [Liskevich et al. 1996; Liskevich and Semenov 1996].

Theorem 4.4. *If $p > 1$ and A is a submarkovian generator, we have*

$$4\frac{p-1}{p^2}\left\|A^{1/2}(|f|^{p/2}\operatorname{sign} f)\right\|_{L^2}^2 \leq \int_{\mathbb{R}^d}(Af)\,|f|^{p-1}\operatorname{sign}\, f\,dx \leq C_p\left\|A^{1/2}(|f|^{p/2}\operatorname{sign} f)\right\|_{L^2}^2.$$

The generator A satisfies the first Beurling–Deny condition

$$4\frac{p-1}{p^2}\|A^{1/2}(|f|^{p/2})\|_{L^2}^2 \leq \int_{\mathbb{R}^d}(Af)\,|f|^{p-1}\operatorname{sign} f\,dx.$$

Combining this result with Proposition 3.13 we get:

Corollary 4.5. *Let $p > 1$, $\beta \in]0, 1]$, $\alpha \geq 0$ and $\lambda \geq e^{(3+2\alpha)/\beta}$. Then*

$$4\frac{p-1}{p^2}\left\|\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|f|^{p/2})\right\|_{L^2}^2 \leq \int_{\mathbb{R}^d}\left(\frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}f\right)|f|^{p-1}\operatorname{sign} f\,dx.$$

We will make use of the following composition results:

Lemma 4.6. (1) *Let $\mu \geq 1$ and $s \in [0, \mu[\cap[0, 2[$. Then*

$$\left\||f|^\mu\right\|_{B^s_{2,2}} \leq C\|f\|_{B^s_{2\mu,2}}\|f\|^{\mu-1}_{B^0_{2\mu,2}}$$

(2) *Let $\mu \in]0, 1]$, $p, q \in [1, \infty]$ and $0 < s < 1 + \dfrac{1}{p}$. Then*

$$\left\||f|^\mu\right\|_{B^{s\mu}_{(p/\mu),(q/\mu)}} \leq C\|f\|^\mu_{B^s_{p,q}}.$$

The first estimate is a particular case of a general result in [Chen et al. 2007]. The second was established in [Sickel 1999]; see also [Kateb 2003, Theorem 1.4].

Next we recall the following result, proved in [Chen et al. 2007; Danchin 2001; Planchon 2000].

Proposition 4.7. *Let $d \geq 1$, $\beta \in]0, 2]$ and $p \geq 2$. Then we have for $q \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$2^{q\beta}\|\Delta_q f\|_{L^p}^p \leq C\int_{\mathbb{R}^d}(|\mathrm{D}|^\beta\Delta_q f)\,|\Delta_q f|^{p-1}\operatorname{sign}\Delta_q f\,dx.$$

where C depends on p and β. For $\beta = 2$ we can extend the inequality above to $p \in]1, \infty[$.

Proof of Theorem 1.3. Using Corollary 4.5 it suffices to prove

$$C^{-1}2^{q\beta}(q+1)^{-\alpha}\|\Delta_q f\|_{L^p}^p \leq \left\|\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|\Delta_q f|^{p/2})\right\|_{L^2}^2.$$

We will use an idea from [Chen et al. 2007]. Let $N \in \mathbb{N}$ then we have

$$\big\| |\mathrm{D}|(|f_q|^{p/2})\big\|_{L^2} \le \big\| S_N|\mathrm{D}|(|f_q|^{p/2})\big\|_{L^2} + \big\|(\mathrm{Id}-S_N)|\mathrm{D}|(|f_q|^{p/2})\big\|_{L^2}.$$

It is clear that for $s \ge 0$

$$\big\|(\mathrm{Id}-S_N)|\mathrm{D}|(|f_q|^{p/2})\big\|_{L^2} \le C2^{-Ns}\||f_q|^{p/2}\|_{B^{1+s}_{2,2}}. \tag{14}$$

We have now to deal with fraction powers in Besov spaces. We will treat differently the cases $p > 2$ and $p \le 2$.

Case $p > 2$. Combining Lemma 4.6(1) with the Bernstein inequality we get, under the assumption that $0 < s < \min(p/2-1, 2)$,

$$\||f_q|^{p/2}\|_{B^{1+s}_{2,2}} \le C\|f_q\|^{(p/2)-1}_{B^0_{p,2}}\|f_q\|_{B^{1+s}_{p,2}} \le C2^{q(1+s)}\|f_q\|^{p/2}_{L^p}.$$

Case $1 < p \le 2$. Using Lemma 4.6(2) and the Bernstein inequality we get, for $0 < s < (p-1)/2$,

$$\||f_q|^{p/2}\|_{B^{1+s}_{2,2}} \le C\|f_q\|^{p/2}_{B^{(2+2s)/p}_{p,p}} \le C2^{q(1+s)}\|f_q\|^{p/2}_{L^p}.$$

It follows from (14) and the previous inequalities that there exists $s_p > 0$ such that for $0 < s < s_p$

$$\big\|(\mathrm{Id}-S_N)|\mathrm{D}|(|f_q|^{p/2})\big\|_{L^2} \le C2^{-Ns}2^{q(1+s)}\|f_q\|^{p/2}_{L^p}.$$

On the other hand Proposition 4.1 gives

$$\begin{aligned}\big\| S_N|\mathrm{D}|(|f_q|^{p/2})\big\|_{L^2} &\le \left\| S_N|\mathrm{D}|^{1-\beta/2}\log^{\alpha/2}(\lambda+|\mathrm{D}|)\left(\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|f_q|^{p/2})\right)\right\|_{L^2}\\ &\le C2^{N(1-\beta/2)}N^{\alpha/2}\left\|\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|f_q|^{p/2})\right\|_{L^2}.\end{aligned}$$

Therefore we get

$$\||\mathrm{D}|(|f_q|^{p/2})\|_{L^2} \le C2^{-Ns}2^{q(1+s)}\|f_q\|^{p/2}_{L^p} + C2^{N(1-\beta/2)}N^{\alpha/2}\left\|\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|f_q|^{p/2})\right\|_{L^2}.$$

According to Proposition 4.7 we have for $p \in \,]1, \infty[$,

$$C_p 2^q\|f_q\|^{p/2}_{L^p} \le \||\mathrm{D}|(|f_q|^{p/2})\|_{L^2}.$$

Combining the last two estimates we get

$$2^q\|f_q\|^{p/2}_{L^p} \le C2^{s(q-N)}\,2^q\|f_q\|^{p/2}_{L^p} + C2^{N(1-\beta/2)}N^{\alpha/2}\left\|\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|f_q|^{p/2})\right\|_{L^2}.$$

We take $N = q + N_0$ such that $C2^{-N_0 s} \le \frac{1}{2}$. Then we get

$$\|f_q\|^{p/2}_{L^p} \le C2^{-q\beta/2}(q+1)^{\alpha/2}\left\|\frac{|\mathrm{D}|^{\beta/2}}{\log^{\alpha/2}(\lambda+|\mathrm{D}|)}(|f_q|^{p/2})\right\|_{L^2}.$$

This gives the desired result. □

5. Commutator estimates

We will establish in this section some commutator estimates. The following result was proved in [Hmidi et al. 2011].

Lemma 5.1. *Given $(p, m) \in [1, \infty]^2$ such that $p \geq m'$ with m' the conjugate exponent of m. Let f, g, h be functions such that $\nabla f \in L^p$, $g \in L^m$ and $xh \in L^{m'}$. Then,*

$$\|h \star (fg) - f(h \star g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m}.$$

Lemma 5.2. *Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of strictly nonnegative real numbers such that*

$$M := \max\Big(\sup_{n\in\mathbb{Z}} a_n^{-1} \sum_{j\leq n} a_j, \sup_{n\in\mathbb{Z}} a_n \sum_{j\geq n} a_j^{-1}\Big) < \infty.$$

For every $p \in [1, \infty]$, the linear operator $T : \ell^p \to \ell^p$ defined by

$$T((b_n)_{n\in\mathbb{Z}}) = \Big(\sum_{j\leq n} a_j a_n^{-1} b_j\Big)_{n\in\mathbb{Z}}$$

is continuous and $\|T\|_{\mathscr{L}(\ell^p)} \leq M$.

Proof. By interpolation it suffices to prove the cases $p = 1$ and $p = +\infty$. Let's start with $p = 1$ and set $\boldsymbol{b} = (b_n)_{n\in\mathbb{Z}}$. From the Fubini lemma and the hypothesis we have

$$\|T\boldsymbol{b}\|_{\ell^1} \leq \sum_{n\in\mathbb{Z}} \sum_{j\leq n} a_j a_n^{-1} |b_j| \leq \sum_{j\in\mathbb{Z}} |b_j| a_j \sum_{n\geq j} a_n^{-1} \leq M\|\boldsymbol{b}\|_{\ell^1}.$$

For the case $p = +\infty$, we write

$$\|T\boldsymbol{b}\|_{\ell^\infty} \leq \sup_{n\in\mathbb{Z}} \sum_{j\leq n} a_j a_n^{-1} |b_j| \leq \|\boldsymbol{b}\|_{\ell^\infty} \sup_{n\in\mathbb{Z}} a_n^{-1} \sum_{j\leq n} a_j \leq M\|\boldsymbol{b}\|_{\ell^\infty}.$$

This completes the proof. ☐

The goal now is to study the commutation between the operators

$$\mathscr{R}_\alpha = \frac{\partial_1}{|\mathrm{D}|} \log^\alpha(\lambda + |\mathrm{D}|) \quad \text{and} \quad v \cdot \nabla.$$

Recall that $B^{s,s'}_{\infty,2}$ is the space given by the set of tempered distributions u such that

$$\|u\|_{B^{s,s'}_{\infty,r}} = \Big\| \big(2^{qs}(|q|+1)^{s'} \|\Delta_q u\|_{L^\infty}\big)_q \Big\|_{\ell^r}.$$

The main result of this section reads as follows.

Proposition 5.3. *Let $\alpha \in \mathbb{R}$, $\lambda > 1$, and let v be a smooth divergence-free vector field and θ a smooth scalar function.*

(1) *For every $(p, r) \in [2, \infty[\times [1, \infty]$ there exists a constant $C = C(p, r)$ such that*

$$\|[\mathscr{R}_\alpha, v \cdot \nabla]\theta\|_{B^0_{p,r}} \leq C\|\nabla v\|_{L^p}\big(\|\theta\|_{B^{0,\alpha}_{\infty,r}} + \|\theta\|_{L^p}\big).$$

(2) *For every $(r, \rho) \in [1, \infty] \times]1, \infty[$ and $\epsilon > 0$ there exists a constant $C = C(r, \rho, \epsilon)$ such that*

$$\|[\mathscr{R}_\alpha, v \cdot \nabla]\theta\|_{B^0_{\infty,r}} \leq C\big(\|\omega\|_{L^\infty} + \|\omega\|_{L^\rho}\big)\big(\|\theta\|_{B^\epsilon_{\infty,r}} + \|\theta\|_{L^\rho}\big).$$

Proof. (1) We split the commutator into three parts according to Bony's decomposition [1981]:

$$
\begin{aligned}
[\mathcal{R}_\alpha, v\cdot\nabla]\theta &= \sum_{q\in\mathbb{N}}[\mathcal{R}_\alpha, S_{q-1}v\cdot\nabla]\Delta_q\theta + \sum_{q\in\mathbb{N}}[\mathcal{R}_\alpha, \Delta_q v\cdot\nabla]S_{q-1}\theta + \sum_{q\geq -1}[\mathcal{R}_\alpha, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta \\
&= \sum_{q\in\mathbb{N}}\mathrm{I}_q + \sum_{q\in\mathbb{N}}\mathrm{II}_q + \sum_{q\geq -1}\mathrm{III}_q = \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
$$

We start with the estimate of the term I. It is easy to see that there exists $\tilde{\varphi}\in\mathcal{S}$ whose spectrum does not meet the origin such that

$$\mathrm{I}_q(x) = h_q\star(S_{q-1}v\cdot\nabla\Delta_q\theta) - S_{q-1}v\cdot(h_q\star\nabla\Delta_q\theta),$$

where

$$\hat{h}_q(\xi) = i\tilde{\varphi}(2^{-q}\xi)\frac{\xi_1}{|\xi|}\log^\alpha(\lambda+|\xi|).$$

Applying Lemma 5.1 with $m=\infty$ we get

$$\|\mathrm{I}_q\|_{L^p} \lesssim \|xh_q\|_{L^1}\|\nabla S_{q-1}v\|_{L^p}\|\Delta_q\nabla\theta\|_{L^\infty} \lesssim 2^q\|xh_q\|_{L^1}\|\Delta_q\theta\|_{L^\infty}\|\nabla v\|_{L^p}. \tag{15}$$

We can easily check that

$$\|xh_q\|_{L^1} = 2^{-q}\|x\tilde{h}_q\|_{L^1} \quad \text{with} \quad \widehat{\tilde{h}_q}(\xi) = i\tilde{\varphi}(\xi)\frac{\xi_1}{|\xi|}\log^\alpha(\lambda+2^q|\xi|).$$

We can get, in a way similar to the proof of Proposition 4.1,

$$\|\tilde{h}_q\|_{L^1} \leq C(1+|q|)^\alpha.$$

Thus estimate (15) becomes

$$\|\mathrm{I}_q\|_{L^p} \leq C(1+|q|)^\alpha\|\Delta_q\theta\|_{L^\infty}\|\nabla v\|_{L^p}.$$

Combined with the trivial fact

$$\Delta_j\sum_q \mathrm{I}_q = \sum_{|j-q|\leq 4}\mathrm{I}_q,$$

this yields

$$\|\mathrm{I}\|_{B^0_{p,r}} \lesssim \Big(\sum_{q\geq -1}\|\mathrm{I}_q\|^r_{L^p}\Big)^{1/r} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^{0,\alpha}_{\infty,r}}.$$

Let us move to the term II. As before we write

$$\mathrm{II}_q(x) = h_q\star(\Delta_q v\cdot\nabla S_{q-1}\theta) - \Delta_q v\cdot(h_q\star\nabla S_{q-1}\theta),$$

and then we obtain the estimate

$$\|\mathrm{II}_q\|_{L^p} \lesssim 2^{-q}(1+|q|)^\alpha\|\Delta_q\nabla v\|_{L^p}\|S_{q-1}\nabla\theta\|_{L^\infty} \lesssim \|\nabla v\|_{L^p}\sum_{j\leq q-2}\frac{2^j(1+|j|)^{-\alpha}}{2^q(1+|q|)^{-\alpha}}((1+|j|)^\alpha\|\Delta_j\theta\|_{L^\infty}).$$

Combined with Lemma 5.2 this yields

$$\|\mathrm{II}\|_{B^0_{p,r}} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^{0,\alpha}_{\infty,r}}.$$

To deal with III, we use the fact that the divergence of $\Delta_q v$ vanishes to write

$$\mathrm{III}=\sum_{q\geq 2}\mathcal{R}_\alpha\,\mathrm{div}(\Delta_q v\,\widetilde{\Delta}_q\theta)-\sum_{q\geq 2}\mathrm{div}(\Delta_q v\,\mathcal{R}_\alpha\widetilde{\Delta}_q\theta)+\sum_{q\leq 1}[\mathcal{R}_\alpha,\,\Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta=J_1+J_2+J_3.$$

Using Remark 4.2 we get

$$\big\|\Delta_j\mathcal{R}_\alpha\,\mathrm{div}(\Delta_q v\,\widetilde{\Delta}_q\theta)\big\|_{L^p}\lesssim 2^j(1+|j|)^\alpha\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty};$$

and since $q\geq 2$,

$$\big\|\Delta_j\,\mathrm{div}(\Delta_q v\,\mathcal{R}_\alpha\widetilde{\Delta}_q\theta)\big\|_{L^p}\lesssim 2^j\|\Delta_q v\|_{L^p}\|\mathcal{R}_\alpha\widetilde{\Delta}_q\theta\|_{L^\infty}\lesssim 2^j(1+|q|)^\alpha\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}.$$

Therefore we get

$$\|\Delta_j(J_1+J_2)\|_{L^p}\lesssim\sum_{\substack{q\in\mathbb{N}\\ q\geq j-4}}2^j(1+|q|)^\alpha\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}\lesssim\|\nabla v\|_{L^p}\sum_{\substack{q\in\mathbb{N}\\ q\geq j-4}}2^{j-q}(1+|q|)^\alpha\|\Delta_q\theta\|_{L^\infty},$$

where we have again used Bernstein inequality to get the last inequality. It suffices now to use Lemma 5.2:

$$\|J_1+J_2\|_{B^0_{p,r}}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{B^{0,\alpha}_{\infty,r}}.$$

For the term J_3 we can write

$$\sum_{-1\leq q\leq 1}[\mathcal{R}_\alpha,\,\Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta(x)=\sum_{q\leq 1}[\mathrm{div}\,\widetilde{\chi}(\mathrm{D})\mathcal{R}_\alpha,\,\Delta_q v]\widetilde{\Delta}_q\theta(x),$$

where $\widetilde{\chi}$ belongs to $\mathcal{D}(\mathbb{R}^d)$. From the proof of Proposition 4.1 we know that $\mathrm{div}\,\widetilde{\chi}(\mathrm{D})\mathcal{R}_\alpha$ is a convolution operator with a kernel $\tilde{h}$ satisfying

$$|\tilde{h}(x)|\lesssim(1+|x|)^{-d-1}.$$

Thus

$$J_3=\sum_{q\leq 1}\tilde{h}\star(\Delta_q v\cdot\tilde{\Delta}_q\theta)-\Delta_q v\cdot(\tilde{h}\star\tilde{\Delta}_q\theta).$$

Note that $\Delta_j J_3=0$ for $j\geq 6$; thus we just need to estimate the low frequencies of J_3. Since $x\tilde{h}$ belongs to $L^{p'}$ for $p'>1$, we can use Lemma 5.1 with $m=p\geq 2$ to obtain

$$\|\Delta_j J_3\|_{L^p}\lesssim\sum_{q\leq 1}\|x\tilde{h}\|_{L^{p'}}\|\Delta_q\nabla v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^p}\lesssim\|\nabla v\|_{L^p}\sum_{-1\leq q\leq 1}\|\Delta_q\theta\|_{L^p}.$$

This yields

$$\|J_3\|_{B^0_{p,r}}\lesssim\|\nabla v\|_{L^p}\|\theta\|_{L^p},$$

completing the proof of the first part of Proposition 5.3.

(2) The second part can be done similarly, so we give a shorter proof. To estimate the terms I and II we use two estimates: $\|\Delta_q\nabla u\|_{L^\infty}\approx\|\Delta_q\omega\|_{L^\infty}$ for all $q\in\mathbb{N}$, and

$$\|\nabla S_{q-1}v\|_{L^\infty}\lesssim\|\nabla\Delta_{-1}v\|_{L^\infty}+\sum_{j=0}^{q-2}\|\Delta_j\nabla v\|_{L^\infty}\lesssim\|\omega\|_{L^\rho}+q\|\omega\|_{L^\infty}.$$

Thus (15) becomes

$$\|\mathrm{I}_q\|_{L^\infty}\leq\|\omega\|_{L^\infty}(1+|q|)^{1+\alpha}\|\Delta_q\theta\|_{L^\infty}$$

and by Corollary 4.3

$$\|\mathrm{I}\|_{B^0_{\infty,r}} \le \|\omega\|_{L^\infty}\|\theta\|_{B^{0,1+\alpha}_{\infty,r}} \le \|\omega\|_{L^\infty}\|\theta\|_{B^\epsilon_{\infty,\infty}}.$$

The second term II is estimated as

$$\|\mathrm{II}\|_{B^0_{\infty,r}} \le \|\omega\|_{L^\infty}\|\theta\|_{B^{0,\alpha}_{\infty,r}} \le \|\omega\|_{L^\infty}\|\theta\|_{B^\epsilon_{\infty,\infty}}.$$

For the remaining term the analysis is the same as before, except for J_3, where we apply Lemma 5.1 with $p=\infty$ and $m=\rho$, leading to

$$\|\Delta_j J_3\|_{L^\rho} \lesssim \sum_{q\le 1}\|x\tilde{h}\|_{L^{\rho'}}\|\Delta_q\nabla v\|_{L^\infty}\|\widetilde{\Delta}_q\theta\|_{L^\rho} \lesssim \|\nabla v\|_{L^\rho}\sum_{-1\le q\le 1}\|\Delta_q\theta\|_{L^\rho} \lesssim \|\omega\|_{L^\rho}\|\theta\|_{L^\rho}.$$

This ends the proof of the theorem. □

6. Smoothing effects

In this section we describe smoothing effects for the model (6), with zero source term f:

$$\begin{cases}\partial_t\theta + v\cdot\nabla\theta + \mathscr{L}\theta = 0,\\ \theta_{|t=0} = \theta_0.\end{cases} \tag{TD}$$

with

$$\mathscr{L} := \frac{|\mathrm{D}|^\beta}{\log^\alpha(\lambda+|\mathrm{D}|)}$$

and $\operatorname{div} v = 0$.

Theorem 6.1. *Let $\alpha\ge 0,\ \lambda\ge e^{3+2\alpha},\ d\in\{2,3\},\ \beta\in\,]0,1]$ and let v be a smooth divergence-free vector field of $\mathbb{R}^d$ with vorticity ω. Then, for every $p\in\,]1,\infty[$, there exists a constant C such that*

$$\sup_{q\in\mathbb{N}} 2^{q\beta}(1+q)^{-\alpha}\|\Delta_q\theta\|_{L^1_tL^p} \le C\|\theta_0\|_{L^p} + C\|\theta_0\|_{L^\infty}\|\omega\|_{L^1_tL^p},$$

for every smooth solution θ of (TD).

Remark 6.2. We give the proof in the case $\beta=1$ for simplicity, but the result remains true for $\beta\in\,]0,1[$.

Proof of Theorem 6.1 in the case $\beta=1$. We start by localizing the equation in frequencies. for $q\ge -1$ we set $\theta_q := \Delta_q\theta$. Then

$$\partial_t\theta_q + v\cdot\nabla\theta_q + \frac{|\mathrm{D}|}{\log^\alpha(\lambda+|\mathrm{D}|)}\theta_q = -[\Delta_q, v\cdot\nabla]\theta.$$

Recall that θ_q is real function since the functions involved in the dyadic partition of the unity are radial. Then multiplying the above equation by $|\theta_q|^{p-2}\theta_q$, integrating by parts and using Hölder inequalities we get

$$\frac{1}{p}\frac{d}{dt}\|\theta_q\|^p_{L^p} + \int_{\mathbb{R}^2}\Big(\frac{|\mathrm{D}|}{\log^\alpha(\lambda+|\mathrm{D}|)}\theta_q\Big)|\theta_q|^{p-2}\theta_q\,dx \le \|\theta_q\|^{p-1}_{L^p}\|[\Delta_q, v\cdot\nabla]\theta\|_{L^p}.$$

Using Theorem 1.3 we get for $q\ge 0$

$$c2^q(1+q)^{-\alpha}\|\theta_q\|^p_{L^p} \le \int_{\mathbb{R}^2}\Big(\frac{|\mathrm{D}|}{\log^\alpha(\lambda+|\mathrm{D}|)}\theta_q\Big)|\theta_q|^{p-2}\theta_q dx,$$

where c depends on p. Inserting this estimate in the previous one we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta_q\|_{L^p}^p + c2^q(1+q)^{-\alpha}\|\theta_q\|_{L^p}^p \lesssim \|\theta_q\|_{L^p}^{p-1}\|[\Delta_q, v\cdot\nabla]\theta\|_{L^p}.$$

Thus we find

$$\frac{d}{dt}\|\theta_q\|_{L^p} + c2^q(1+q)^{-\alpha}\|\theta_q\|_{L^p} \lesssim \|[\Delta_q, v\cdot\nabla]\theta\|_{L^p}. \tag{16}$$

To estimate the right side we will use the following result; see [Hmidi et al. 2011, Proposition 3.3].

$$\|[\Delta_q, v\cdot\nabla]\theta\|_{L^p} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}}.$$

Combined with (16) this lemma yields

$$\begin{aligned}\frac{d}{dt}(e^{ct2^q(1+q)^{-\alpha}}\|\theta_q(t)\|_{L^p}) &\lesssim e^{ct2^q(1+q)^{-\alpha}}\|\nabla v(t)\|_{L^p}\|\theta(t)\|_{B^0_{\infty,\infty}}\\ &\lesssim e^{ct2^q(1+q)^{-\alpha}}\|\omega(t)\|_{L^p}\|\theta_0\|_{L^\infty}.\end{aligned}$$

To get the last line, we have used the conservation of the L^∞ norm of θ and the classical fact that

$$\|\nabla v\|_{L^p} \lesssim \|\omega\|_{L^p} \quad \text{for } p \in]1, +\infty[.$$

Integrating the differential inequality we get for $q \in \mathbb{N}$

$$\|\theta_q(t)\|_{L^p} \lesssim \|\theta_q^0\|_{L^p}e^{-ct2^q(1+q)^{-\alpha}} + \|\theta_0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^q(1+q)^{-\alpha}}\|\omega(\tau)\|_{L^p}\,d\tau.$$

Integrating in time yields

$$2^q(1+q)^{-\alpha}\|\theta_q\|_{L^1_tL^p} \lesssim \|\theta_q^0\|_{L^p} + \|\theta_0\|_{L^\infty}\int_0^t\|\omega(\tau)\|_{L^p}\,d\tau \lesssim \|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty}\int_0^t\|\omega(\tau)\|_{L^p}\,d\tau,$$

which is the desired result. □

7. Proof of Theorem 1.5

Throughout this section we use the notation Φ_k to denote any function of the form

$$\Phi_k(t) = C_0\underbrace{\exp(\ldots\exp}_{k\text{ times}}(C_0t)\ldots),$$

where C_0 depends on the relevant norms of the initial data and its value may vary from line to line up to some absolute constants. We will make frequent and tacit use of the trivial inequalities

$$\int_0^t\Phi_k(\tau)\,d\tau \le \Phi_k(t) \quad\text{and}\quad \exp\biggl(\int_0^t\Phi_k(\tau)\,d\tau\biggr) \le \Phi_{k+1}(t).$$

The proof of Theorem 1.5 is done in several steps. We first give some a priori estimates for the equations (5). Next we prove uniqueness. Finally, we discuss the construction of the solutions.

A priori estimates. Theorem 1.5 deals with critical regularities and one needs to bound the Lipschitz norm of the velocity in order to get the global persistence of the initial regularities. For this purpose we will proceed in several steps: one of the main steps is to give an L^∞-bound of the vorticity, but due to technical difficulties related to Riesz transforms, this is not done directly. First we prove an L^p-estimate for the vorticity with $2 < p < \infty$.

Proposition 7.1. *Let $\alpha \in [0, \frac{1}{2}]$, $\lambda \geq e^{3+2\alpha}$ and $p \in]2, \infty[$. Let (v, θ) be a solution of* (5) *with $\omega^0 \in L^p$, $\theta_0 \in L^p \cap L^\infty$ and $\mathcal{R}_\alpha\theta_0 \in L^p$. Then, for every $\epsilon > 0$,*

$$\|\omega(t)\|_{L^p} + \|\theta\|_{L^1_t B^{1-\epsilon}_{p,1}} \leq \Phi_2(t).$$

Proof. Applying the transform $\mathcal{R}_\alpha$ to the temperature equation we get

$$\partial_t \mathcal{R}_\alpha\theta + v\cdot\nabla\mathcal{R}_\alpha\theta + \frac{|\mathrm{D}|}{\log^\alpha(\lambda+|\mathrm{D}|)}\mathcal{R}_\alpha\theta = -[\mathcal{R}_\alpha, v\cdot\nabla]\theta. \tag{17}$$

Since $\dfrac{|\mathrm{D}|}{\log^\alpha(\lambda+|\mathrm{D}|)}\mathcal{R}_\alpha = \partial_1$, the function $\Gamma := \omega + \mathcal{R}_\alpha\theta$ satisfies

$$\partial_t\Gamma + v\cdot\nabla\Gamma = -[\mathcal{R}_\alpha, v\cdot\nabla]\theta. \tag{18}$$

According to Proposition 5.3(1), applied with $r = 2$,

$$\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{B^0_{p,2}} \lesssim \|\nabla v\|_{L^p}(\|\theta\|_{B^{0,\alpha}_{\infty,2}} + \|\theta\|_{L^p}).$$

Using the classical embedding $B^0_{p,2} \hookrightarrow L^p$ which is true only for $p \in [2, \infty)$

$$\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{L^p} \leq \|\nabla v\|_{L^p}(\|\theta\|_{B^{0,\alpha}_{\infty,2}} + \|\theta\|_{L^p}).$$

Since $\operatorname{div} v = 0$, the L^p estimate applied to the transport equation (18) gives

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^p}\, d\tau.$$

Applying Theorem 3.1 to (17) yields

$$\|\mathcal{R}_\alpha\theta(t)\|_{L^p} \leq \|\mathcal{R}_\alpha\theta_0\|_{L^p} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^p}\, d\tau.$$

Set $f(t) := \|\omega(t)\|_{L^p} + \|\mathcal{R}_\alpha\theta(t)\|_{L^p}$. From the previous estimates we get

$$\begin{aligned} f(t) &\lesssim \|\Gamma_0\|_{L^p} + \|\mathcal{R}_\alpha\theta_0\|_{L^p} + \int_0^t \|\nabla v(\tau)\|_{L^p}(\|\theta(\tau)\|_{B^{0,\alpha}_{\infty,2}} + \|\theta\|_{L^p})\, d\tau \\ &\lesssim f(0) + \int_0^t f(\tau)(\|\theta(\tau)\|_{B^{0,\alpha}_{\infty,2}} + \|\theta_0\|_{L^p})\, d\tau. \end{aligned}$$

Here we have used the Calderón–Zygmund estimate, to the effect that $\|\nabla v\|_{L^p} \leq C\|\omega\|_{L^p}$ for $p \in (1, \infty)$, and also the estimate $\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}$ from Theorem 3.1.

According to Gronwall's lemma we get

$$f(t) \lesssim f(0) e^{C\|\theta_0\|_{L^p} t} e^{C\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}}}. \tag{19}$$

For $N \in \mathbb{N}$, the Bernstein inequalities and Theorem 3.1 give

$$\begin{aligned}
\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} &\le t \Big\| \Big(\sum_{q<N} (1+|q|)^{2\alpha} \|\Delta_q \theta\|^2_{L^\infty} \Big)^{1/2} \Big\|_{L^\infty_t} + \|(\mathrm{Id} - S_N)\theta\|_{L^1_t B^{0,\alpha}_{\infty,1}} \\
&\lesssim t \|\theta\|_{L^\infty_{t,x}} N^{1/2+\alpha} + \sum_{q\ge N} (1+|q|)^\alpha \|\Delta_q\theta\|_{L^1_t L^\infty} \\
&\lesssim t \|\theta_0\|_{L^\infty} N^{1/2+\alpha} + \sum_{q\ge N} (1+|q|)^\alpha \|\Delta_q\theta\|_{L^1_t L^\infty} \\
&\lesssim N^{1/2+\alpha} \|\theta_0\|_{L^\infty} t + \sum_{q\ge N} 2^{q(2/p)} (1+|q|)^\alpha \|\Delta_q\theta\|_{L^1_t L^p}.
\end{aligned}$$

Using Theorem 6.1, we obtain for $p > 2$ and $0 < \varepsilon < 1 - 2/p$

$$\begin{aligned}
\sum_{q\ge N} (1+|q|)^\alpha 2^{q(2/p)} \|\Delta_q\theta\|_{L^1_t L^p} &\lesssim \sum_{q\ge N} (1+|q|)^{2\alpha} 2^{q((2/p)-1)} \big(\|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \|\omega\|_{L^1_t L^p} \big) \\
&\lesssim \sum_{q\ge N} 2^{q((2/p)+\varepsilon-1)} \big(\|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \|\omega\|_{L^1_t L^p} \big) \\
&\lesssim \|\theta_0\|_{L^p} + 2^{N(-1+\varepsilon+2/p)} \|\theta_0\|_{L^\infty} \|\omega\|_{L^1_t L^p}.
\end{aligned}$$

Consequently,

$$\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} \lesssim N^{(1/2)+\alpha} \|\theta_0\|_{L^\infty} t + \|\theta_0\|_{L^p} + 2^{N(-1+\varepsilon+2/p)} \|\theta_0\|_{L^\infty} \|\omega\|_{L^1_t L^p}.$$

We choose

$$N = \left[\frac{\log(e + \|\omega\|_{L^1_t L^p})}{(1-\varepsilon-2/p)\log 2} \right].$$

This yields

$$\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} \lesssim \|\theta_0\|_{L^\infty \cap L^p} + \|\theta_0\|_{L^\infty} t \log^{(1/2)+\alpha} \Big(e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau \Big).$$

Combining this estimate with (19) we get

$$\begin{aligned}
\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} &\lesssim \|\theta_0\|_{L^\infty\cap L^p} + \|\theta_0\|_{L^\infty} t \, \log^{(1/2)+\alpha} \big(e + C f(0) e^{C\|\theta_0\|_{L^p} t} e^{C\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}}} \big) \\
&\le C_0 \log^{(1/2)+\alpha}(e + f(0)) \, (1 + t^{(3/2)+\alpha}) + C \|\theta_0\|_{L^\infty} t \|\theta\|^{(1/2)+\alpha}_{L^1_t B^{0,\alpha}_{\infty,2}},
\end{aligned} \tag{20}$$

where C_0 is a constant depending on $\|\theta_0\|_{L^p \cap L^\infty}$.

Case 1: $\alpha < \frac{1}{2}$.

Lemma 7.2. *There exists a number C, depending only on $\alpha \in [0, 1[$, such that if $a, b > 0$ and if $x \in \mathbb{R}_+$ is a solution of the inequality*

$$x \le a + b x^\alpha, \tag{21}$$

then

$$x \le C(a + b^{1/(1-\alpha)}).$$

Proof. Set $y = a^{-1}x$. Then (21) becomes

$$y \le 1 + ba^{\alpha-1}y^{\alpha}.$$

We will look for a number $\mu > 0$ such that $y \le e^{\mu}$. It suffices to find μ such that

$$1 + ba^{\alpha-1}e^{\mu\alpha} \le e^{\mu}.$$

In particular (since $e^{\mu\alpha} > 1$) we can take for μ the solution of

$$(1 + ba^{\alpha-1})e^{\mu\alpha} = e^{\mu}.$$

This gives $e^{\mu} = (1 + ba^{\alpha-1})^{1/(1-\alpha)}$. Now recall that there is a constant $C = C_{\alpha}$ such that, for every $t, s \ge 0$,

$$(t+s)^{1/(1-\alpha)} \le C(t^{1/(1-\alpha)} + s^{1/(1-\alpha)}).$$

With this constant we have $y \le C(1 + b^{1/(1-\alpha)}a^{-1})$, or equivalently $x \le C(a + b^{1/(1-\alpha)})$, as required. □

Applying this lemma to (20) we get, for every $t \in \mathbb{R}_+$,

$$\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} \le C_0(t^{3/2} + t^{2/(1-2\alpha)}) \le C_0(1 + t^{2/(1-2\alpha)}) \le \Phi_1(t). \tag{22}$$

It follows from (19) that

$$f(t) \le C_0 e^{C_0 t^{2/(1-2\alpha)}} \le \Phi_2(t) \tag{23}$$

Applying Theorem 6.1 and (23) we get, for every $\epsilon > 0$ and $q \in \mathbb{N}$,

$$2^q(1+|q|)^{-\alpha}\|\Delta_q\theta\|_{L^1_t L^p} \le C_0 e^{C_0 t^{2/(1-2\alpha)}} \le \Phi_2(t).$$

Case 2: $\alpha = \frac{1}{2}$. The estimate (20) becomes

$$\|\theta\|_{L^1_t B^{0,1/2}_{\infty,2}} \le C_0 \log(e + f(0))(1+t^2) + C\|\theta_0\|_{L^\infty} t \|\theta\|_{L^1_t B^{0,1/2}_{\infty,2}},$$

with C_0 depending on $\|\theta_0\|_{L^p \cap L^\infty}$. Hence if we choose t small enough that

$$C\|\theta_0\|_{L^\infty} t = \tfrac{1}{2}, \tag{24}$$

then

$$\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} \le C_0 \log(e + f(0)).$$

From (19) we get that

$$f(t) \le C_0(e + f(0))^{C_0}.$$

Now let t be a given positive time and choose a partition $(t_i)_{i=1}^N$ of $[0, t]$ such that

$$C\|\theta_0\|_{L^\infty}(t_{i+1} - t_i) \approx \tfrac{1}{2}. \tag{25}$$

Set $a_i := \int_{t_i}^{t_{i+1}} \|\theta(\tau)\|_{B^{0,1/2}_{\infty,2}}\, d\tau$ and $b_i = f(t_i)$. Computations similar to (20) yield

$$a_i \le C_0 \log(e + b_i)(1 + (t_{i+1} - t_i)^2) + C\|\theta_0\|_{L^\infty}(t_{i+1} - t_i)a_i.$$

Hence we get

$$a_i \le C_0 \log(e + b_i). \tag{26}$$

The analogous estimate to (19) is

$$b_{i+1} \lesssim b_i e^{C(t_{i+1}-t_i)\|\theta_0\|_{L^p}} e^{Ca_i} \le C_0 b_i e^{Ca_i}. \tag{27}$$

Combining (26) and (27) yields

$$b_{i+1} \le C_0(e+b_i)^{C_0}.$$

By induction we can prove that for every $i \in \{1, \dots, N\}$ we have $b_i \le C_0 e^{\exp C_0 i}$, and consequently, from (26), $a_i \le C_0 e^{C_0 i}$. It follows that

$$\|\theta\|_{L^1_t B^{0,1/2}_{\infty,2}} = \sum_{i=1}^{N} a_i \le C_0 e^{C_0 N} \le C_0 e^{C_0 t}.$$

We have used in the last inequality the fact that $N \approx C_0 t$, a consequence of (25). We have also obtained

$$f(t) \le C_0 e^{\exp C_0 t}.$$

It is not hard to see that (24) implies

$$\|\theta\|_{L^1_t B^s_{p,1}} \le \|\theta\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}} \le \Phi_2(t) \quad \text{for every } s < 1. \tag{28}$$

This ends the proof of Proposition 7.1. □

Remark 7.3. Combining (28) with the Bernstein inequalities and the fact that $p > 2$ yields

$$\|\theta\|_{L^1_t B^\epsilon_{\infty,1}} \le \Phi_2(t) \quad \text{for every } \epsilon < 1 - \frac{2}{p}. \tag{29}$$

We are now ready to prove an L^∞-bound on the vorticity.

Proposition 7.4. *Let $\alpha \in [0, \frac{1}{2}]$, $\lambda \ge e^{3+2\alpha}$, $p \in]2, \infty[$, and let (v, θ) be a smooth solution of the system* (5) *such that $\omega^0, \theta_0, \mathcal{R}_\alpha\theta_0 \in L^p \cap L^\infty$. Then we have*

$$\|\omega(t)\|_{L^\infty} + \|\mathcal{R}_\alpha\theta(t)\|_{L^\infty} \le \Phi_3(t) \tag{30}$$

and

$$\|v(t)\|_{L^\infty} \le \Phi_4(t). \tag{31}$$

Proof of (30). By using the maximum principle for the transport equation (18), we get

$$\|\Gamma(t)\|_{L^\infty} \le \|\Gamma^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^\infty}\, d\tau.$$

Since the function $\mathcal{R}_\alpha\theta$ satisfies the equation

$$\big(\partial_t + v\cdot\nabla + |\mathrm{D}| \log^{-\alpha}(\lambda + |\mathrm{D}|)\big)\mathcal{R}_\alpha\theta = -[\mathcal{R}_\alpha, v\cdot\nabla]\theta,$$

we get, using Theorem 3.1,

$$\|\mathcal{R}_\alpha\theta(t)\|_{L^\infty} \le \|\mathcal{R}_\alpha\theta(t)\|_{L^\infty} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^\infty}\, d\tau.$$

Thus we obtain

$$\begin{aligned}\|\Gamma(t)\|_{L^\infty}+\|\mathcal{R}_\alpha\theta(t)\|_{L^\infty} &\le \|\Gamma^0\|_{L^\infty}+\|\mathcal{R}_\alpha\theta_0\|_{L^\infty}+2\int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^\infty}d\tau\\ &\le C_0+\int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{B^0_{\infty,1}}d\tau.\end{aligned}$$

It follows from Theorem 3.1, Proposition 5.3(2) and Proposition 7.1 that

$$\begin{aligned}&\|\omega(t)\|_{L^\infty}+\|\mathcal{R}_\alpha\theta(t)\|_{L^\infty}\\ &\qquad\lesssim C_0+\int_0^t \|\omega(\tau)\|_{L^\infty\cap L^p}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}}+\|\theta(\tau)\|_{L^p}\big)\,d\tau\\ &\qquad\lesssim C_0+\|\omega\|_{L^\infty_t L^p}\big(\|\theta\|_{L^1_t B^\epsilon_{\infty,1}}+t\|\theta_0\|_{L^p}\big)+\int_0^t \|\omega(\tau)\|_{L^\infty}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}}+\|\theta_0\|_{L^p}\big)\,d\tau.\end{aligned}$$

Let $0<\epsilon<1-\dfrac{2}{p}$. Using (29) we get

$$\|\omega(t)\|_{L^\infty}+\|\mathcal{R}_\alpha\theta(t)\|_{L^\infty}\lesssim \Phi_2(t)+\int_0^t \|\omega(\tau)\|_{L^\infty}(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}}+\|\theta_0\|_{L^p})\,d\tau.$$

Therefore we obtain by the Gronwall lemma and (29)

$$\|\omega(t)\|_{L^\infty}+\|\mathcal{R}_\alpha\theta(t)\|_{L^\infty}\le \Phi_3(t).$$ □

Proof of (31). Let $N\in\mathbb{N}$ to be chosen later. Using the fact that $\|\dot\Delta_q v\|_{L^\infty}\approx 2^{-q}\|\dot\Delta_q\omega\|_{L^\infty}$, we get

$$\begin{aligned}\|v(t)\|_{L^\infty}&\le \|\chi(2^N|\mathrm{D}|)v(t)\|_{L^\infty}+\sum_{q\ge -N}2^{-q}\|\dot\Delta_q\omega(t)\|_{L^\infty}\\ &\le \|\chi(2^N|\mathrm{D}|)v(t)\|_{L^\infty}+2^N\|\omega(t)\|_{L^\infty}.\end{aligned}$$

Applying the frequency localizing operator to the velocity equation we get

$$\chi(2^N|\mathrm{D}|)v=\chi(2^N|\mathrm{D}|)v_0+\int_0^t \mathcal{P}\chi(2^N|\mathrm{D}|)\theta(\tau)\,d\tau+\int_0^t \mathcal{P}\chi(2^N|\mathrm{D}|)\,\mathrm{div}(v\otimes v)(\tau)\,d\tau,$$

where $\mathcal{P}$ stands for Leray projector. From Lemma 2.1, a Calderón–Zygmund estimate and the uniform boundedness of $\chi(2^N|\mathrm{D}|)$ we get

$$\int_0^t \|\chi(2^N|\mathrm{D}|)\mathcal{P}\theta(\tau)\|_{L^\infty}d\tau\lesssim 2^{-N(2/p)}\int_0^t \|\theta(\tau)\|_{L^p}d\tau\lesssim t\|\theta_0\|_{L^p}.$$

Using Corollary 3.9(2) we find

$$\int_0^t \|\mathcal{P}\chi(2^N|\mathrm{D}|)\,\mathrm{div}(v\otimes v)(\tau)\|_{L^\infty}d\tau\lesssim 2^N\int_0^t \|v(\tau)\|^2_{L^\infty}d\tau.$$

The outcome is

$$\|v(t)\|_{L^\infty}\lesssim \|v_0\|_{L^\infty}+t\|\theta_0\|_{L^p}+2^{-N}\int_0^t \|v(\tau)\|^2_{L^\infty}d\tau+2^N\|\omega(t)\|_{L^\infty}\lesssim 2^{-N}\int_0^t \|v(\tau)\|^2_{L^\infty}d\tau+2^N\Phi_3(t).$$

Choosing judiciously N we find

$$\|v(t)\|_{L^\infty} \le \Phi_3(t)\left(1+\left(\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau\right)^{1/2}\right).$$

From the Gronwall lemma we get $\|v(t)\|_{L^\infty} \le \Phi_4(t)$, as desired. □

Finally, we turn to a Lipschitz bound of the velocity.

Proposition 7.5. *Let* $\alpha \in [0, \frac{1}{2}]$, $\lambda \ge e^{3+2\alpha}$, $p \in]2, \infty[$, *and let* (v, θ) *be a smooth solution of the system* (5) *with* $\omega^0, \theta_0, \mathcal{R}_\alpha\theta_0 \in B^0_{\infty,1} \cap L^p$. *Then*

$$\|\mathcal{R}_\alpha\theta(t)\|_{B^0_{\infty,1}} + \|\omega(t)\|_{B^0_{\infty,1}} + \|v(t)\|_{B^1_{\infty,1}} \le \Phi_4(t).$$

Proof. Applying Corollary 3.17 to the equations (17) and (18), we obtain

$$\|\Gamma(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}_\alpha\theta(t)\|_{B^0_{\infty,1}} \lesssim \big(C_0 + \big\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\big\|_{L^1_t B^0_{\infty,1}}\big)\big(1+\|\nabla v\|_{L^1_t L^\infty}\big). \tag{32}$$

Thanks to Propositions 5.3, 7.4, 7.1 and Equation (29) we get

$$\big\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\big\|_{L^1_t B^0_{\infty,1}} \lesssim \int_0^t \big(\|\omega(\tau)\|_{L^\infty} + \|\omega(\tau)\|_{L^p}\big)\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}} + \|\theta(\tau)\|_{L^p}\big)\, d\tau \lesssim \Phi_3(t).$$

By easy computations we get

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\le \|\nabla\Delta_{-1}v\|_{L^\infty} + \sum_{q\in\mathbb{N}} \|\Delta_q\nabla v\|_{L^\infty} \lesssim \|\omega\|_{L^p} + \sum_{q\in\mathbb{N}} \|\Delta_q\omega\|_{L^\infty}\\ &\lesssim \Phi_2(t) + \|\omega(t)\|_{B^0_{\infty,1}}.\end{aligned} \tag{33}$$

Putting together (32) and (33) leads to

$$\|\omega(t)\|_{B^0_{\infty,1}} \le \|\Gamma(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}_\alpha\theta(t)\|_{B^0_{\infty,1}} \le \Phi_3(t)\left(1+\int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}} d\tau\right).$$

Thus we obtain from the Gronwall inequality

$$\|\omega(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}_\alpha\theta(t)\|_{B^0_{\infty,1}} \le \Phi_4(t). \tag{34}$$

Coming back to (33) we get

$$\|\nabla v(t)\|_{L^\infty} \le \Phi_4(t).$$

Let us move to the estimate of v in the space $B^1_{\infty,1}$. By definition we have

$$\|v(t)\|_{B^1_{\infty,1}} \lesssim \|v(t)\|_{L^\infty} + \|\omega(t)\|_{B^0_{\infty,1}}.$$

Combined with (31) and (34) this yields

$$\|v(t)\|_{B^1_{\infty,1}} \le \Phi_4(t).$$

The proof of Proposition 7.5 is now achieved, and with it the first step in the proof of Theorem 1.5, according to outline on page 274. □

Uniqueness. We will show that the Boussinesq system (5) has a unique solution in the function space

$$\mathscr{E}_T = (L^\infty_T B^0_{\infty,1} \cap L^1_T B^1_{\infty,1}) \times (L^\infty_T L^p \cap \widetilde{L}^1_T B^{1,-\alpha}_{p,\infty}), \quad 2 < p < \infty.$$

Let (v^1, θ^1) and (v^2, θ^2) be two solutions of (5) belonging to the space $\mathscr{E}_T$, and set

$$v = v^2 - v^1, \quad \theta = \theta^2 - \theta^1.$$

Then we get

$$\begin{cases} \partial_t v + v^2 \cdot \nabla v = -\nabla \pi - v \cdot \nabla v^1 + \theta e_2, \\ \partial_t \theta + v^2 \cdot \nabla \theta + \dfrac{|\mathrm{D}|}{\log^\alpha(\lambda + |\mathrm{D}|)} \theta = -v \cdot \nabla \theta^1, \\ v_{|t=0} = v_0, \quad \theta_{|t=0} = \theta_0. \end{cases}$$

According to Proposition 3.15 we have

$$\|v(t)\|_{B^0_{\infty,1}} \le C e^{CV_1(t)} \big(\|v_0\|_{B^0_{\infty,1}} + \|\nabla \pi\|_{L^1_t B^0_{\infty,1}} + \|v \cdot \nabla v^1\|_{L^1_t B^0_{\infty,1}} + \|\theta\|_{L^1_t B^0_{\infty,1}} \big),$$

with $V_1(t) = \|\nabla v^1\|_{L^1_t L^\infty}$. Straightforward computations using the incompressibility of the flows gives

$$\begin{aligned} \nabla \pi &= -\nabla \Delta^{-1} \operatorname{div}(v \cdot \nabla (v^1 + v^2)) + \nabla \Delta^{-1} \partial_2 \theta \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

To estimate the term I we use the definition

$$\|\mathrm{I}\|_{B^0_{\infty,1}} \lesssim \|(\nabla \Delta^{-1} \operatorname{div}) \operatorname{div} \Delta_{-1}(v \otimes (v^1 + v^2))\|_{L^\infty} + \|v \cdot \nabla (v^1 + v^2)\|_{B^1_{\infty,1}}.$$

From Proposition 3.1(2) of [Hmidi et al. 2011] and Besov embeddings we have

$$\|(\nabla \Delta^{-1} \operatorname{div}) \operatorname{div} \Delta_{-1}(v \otimes (v^1 + v^2))\|_{L^\infty} \lesssim \|v \otimes (v^1 + v^2)\|_{L^\infty} \lesssim \|v\|_{B^0_{\infty,1}} \|v^1 + v^2\|_{B^0_{\infty,1}}.$$

Using the incompressibility of v and Bony's decomposition one can easily obtain

$$\|v \cdot \nabla (v^1 + v^2)\|_{B^0_{\infty,1}} \lesssim \|v\|_{B^0_{\infty,1}} \|v^1 + v^2\|_{B^1_{\infty,1}}.$$

Putting together these estimates yields

$$\|\mathrm{I}\|_{B^0_{\infty,1}} \lesssim \|v\|_{B^0_{\infty,1}} \|v^1 + v^2\|_{B^1_{\infty,1}}. \tag{35}$$

We now turn to the term II. By using Besov embeddings and a Calderón–Zygmund estimate we get

$$\|\mathrm{II}\|_{B^0_{\infty,1}} \lesssim \|\nabla \Delta^{-1} \partial_2 \theta\|_{B^{2/p}_{p,1}} \lesssim \|\theta\|_{B^{2/p}_{p,1}}.$$

Combining this estimate with (35) yields

$$\|v(t)\|_{B^0_{\infty,1}} \lesssim e^{CV(t)} \bigg(\|v_0\|_{B^0_{\infty,1}} + \int_0^t \|v(\tau)\|_{B^0_{\infty,1}} \big(1 + \|(v^1, v^2)(\tau)\|_{B^1_{\infty,1}}\big)\, d\tau \bigg) + e^{CV(t)} \|\theta\|_{L^1_t B^{2/p}_{p,1}},$$

where $V(t) := \|(v^1, v^2)\|_{L^1_t B^1_{\infty,1}}$.

Now we have to estimate $\|\theta\|_{L^1_t B^{2/p}_{p,1}}$. By applying Δ_q to the equation of θ and arguing similarly to the proof of Theorem 6.1 we obtain for $q \in \mathbb{N}$

$$\|\theta_q(t)\|_{L^p} \lesssim e^{-ct2^q(1+q)^{-\alpha}}\|\theta_q^0\|_{L^p} + \int_0^t e^{-c2^q(1+q)^{-\alpha}(t-\tau)}\big\|\Delta_q(v\cdot\nabla\theta^1)(\tau)\big\|_{L^p}\,d\tau + \int_0^t e^{-c2^q(1+q)^{-\alpha}(t-\tau)}\big\|\big[v^2\cdot\nabla,\,\Delta_q\big]\theta(\tau)\big\|_{L^p}\,d\tau.$$

Remark, first, that an obvious Hölder inequality yields that for every $\varepsilon \in [0, 1]$ there exists an absolute constant C such that

$$\int_0^t e^{-c\tau 2^q(1+q)^{-\alpha}}\,d\tau \le Ct^{1-\varepsilon}2^{-q\varepsilon}(1+q)^{\alpha\varepsilon} \quad \text{for all } t \ge 0.$$

Using this fact and integrating in time we obtain

$$2^{q2/p}\|\theta_q\|_{L^1_t L^p} \lesssim (q+1)^{\alpha}2^{q(-1+2/p)}\|\theta_q^0\|_{L^p} + \mathrm{I}_q(t) + \mathrm{II}_q(t), \tag{36}$$

where

$$\mathrm{I}_q(t) = t^{1-\varepsilon}(q+1)^{\alpha\varepsilon}2^{q(-\varepsilon+2/p)}\int_0^t \|\Delta_q(v\cdot\nabla\theta^1)(\tau)\|_{L^p}\,d\tau,$$
$$\mathrm{II}_q(t) = t^{1-\varepsilon}(q+1)^{\alpha\varepsilon}2^{q(-\varepsilon+2/p)}\int_0^t \|[v^2\cdot\nabla,\,\Delta_q]\theta(\tau)\|_{L^p}\,d\tau.$$

Using Bony's decomposition we get easily

$$\begin{aligned}\|\Delta_q(v\cdot\nabla\theta^1)(t)\|_{L^p} &\lesssim \|v(t)\|_{L^\infty}\sum_{j\le q+2}2^j\|\Delta_j\theta^1(t)\|_{L^p} + 2^q\|v(t)\|_{L^\infty}\sum_{j\ge q-4}\|\Delta_j\theta^1(t)\|_{L^p}\\ &\lesssim \|v(t)\|_{L^\infty}\sum_{j\le q+2}(1+|j|)^{\alpha}\big(2^j(1+|j|)^{-\alpha}\|\Delta_j\theta^1(t)\|_{L^p}\big)\\ &\quad + \|v(t)\|_{L^\infty}\sum_{j\ge q-4}2^{q-j}(1+|j|)^{\alpha}\big(2^j(1+|j|)^{-\alpha}\|\Delta_j\theta^1(t)\|_{L^p}\big).\end{aligned}$$

Integrating in time we get

$$\begin{aligned}\mathrm{I}_q(t) &\lesssim t^{1-\varepsilon}\|v\|_{L^\infty_t L^\infty}2^{q((2/p)-\varepsilon)}(q+1)^{1+\alpha(1+\varepsilon)}\|\theta^1\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}}\\ &\quad + t^{1-\varepsilon}\|v\|_{L^\infty_t L^\infty}\|\theta^1\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}}2^{q((2/p)+1-\varepsilon)}(q+1)^{\alpha(1+\varepsilon)}\sum_{j\ge q-4}2^{-j}(1+|j|)^{\alpha}\\ &\lesssim t^{1-\varepsilon}\|v\|_{L^\infty_t L^\infty}2^{q((2/p)-\varepsilon)}(q+1)^{1+\alpha(1+\varepsilon)}\|\theta^1\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}}.\end{aligned} \tag{37}$$

To estimate the term II_q we use the following classical commutator (since $\frac{2}{p} < 1$) [Chemin 1998]:

$$\|[v^2\cdot\nabla,\,\Delta_q]\theta\|_{L^p} \lesssim 2^{-q(2/p)}\|\nabla v^2\|_{L^\infty}\|\theta\|_{B^{2/p}_{p,1}}.$$

Thus we obtain

$$II_q(t) \lesssim t^{1-\varepsilon}(q+1)^{\alpha\varepsilon}2^{-q\varepsilon}\|\nabla v^2\|_{L^\infty_t L^\infty}\|\theta\|_{L^1_t B^{2/p}_{p,1}}. \tag{38}$$

We choose $\varepsilon \in]0,1[$ such that $\frac{2}{p} - \varepsilon < 0$, which is possible since $p > 2$. Combining (36), (37) and (38) we get

$$\|\theta\|_{L^1_t B^{2/p}_{p,1}} \lesssim \|\theta_0\|_{L^p} + t^{1-\varepsilon}\|v\|_{L^\infty_t L^\infty}\|\theta^1\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}} + t^{1-\varepsilon}\|\nabla v^2\|_{L^\infty_t L^\infty}\|\theta\|_{L^1_t B^{2/p}_{p,1}}.$$

It follows that there exists a small $\delta > 0$ such that for $t \in [0,\delta]$

$$\|\theta\|_{L^1_t B^{2/p}_{p,1}} \lesssim \|\theta_0\|_{L^p} + t^{1-\varepsilon}\|v\|_{L^\infty_t L^\infty}\|\theta^1\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}}.$$

Plugging this estimate into (36) we find

$$\|v\|_{L^\infty_t B^0_{\infty,1}} \lesssim e^{CV(t)}\big(\|v_0\|_{B^0_{\infty,1}} + \|\theta_0\|_{L^p} + t\|v\|_{L^\infty_t B^0_{\infty,1}} + t^\varepsilon\|v\|_{L^\infty_t L^\infty}\|\theta^1\|_{\widetilde{L}^1_t B^{1,-\alpha}_{p,\infty}}\big).$$

If δ is sufficiently small then we get for $t \in [0,\delta]$

$$\|v\|_{L^\infty_t B^0_{\infty,1}} \lesssim \|v_0\|_{B^0_{\infty,1}} + \|\theta_0\|_{L^p}. \tag{39}$$

This gives in turn

$$\|\theta\|_{L^1_t B^{2/p}_{p,1}} \lesssim \|v_0\|_{B^0_{\infty,1}} + \|\theta_0\|_{L^p}. \tag{40}$$

This gives in particular the uniqueness on $[0,\delta]$. Iterating this argument yields the uniqueness in $[0,T]$.

Existence. We consider the system

$$\begin{cases} \partial_t v_n + v_n\cdot\nabla v_n + \nabla\pi_n = \theta_n e_2, \\ \partial_t\theta_n + v_n\cdot\nabla\theta_n + \dfrac{|\mathrm{D}|}{\log^\alpha(\lambda+|\mathrm{D}|)}\theta_n = 0, \\ \operatorname{div} v_n = 0, \\ v_{n|t=0} = S_n v^0, \quad \theta_{n|t=0} = S_n\theta^0. \end{cases} \tag{B$_n$}$$

By using the same method as [Hmidi and Keraani 2009] we can prove that this system has a unique local smooth solution (v_n,θ_n). The global existence of these solutions is governed by the following criterion: we can push the construction beyond the time T if the quantity $\|\nabla v_n\|_{L^1_T L^\infty}$ is finite. Now from the *a priori* estimates the Lipschitz norm cannot blow up in finite time and then the solution (v_n,θ_n) is globally defined. Once again from the *a priori* estimates we have for $2 < p < \infty$:

$$\|v_n\|_{L^\infty_T B^1_{\infty,1}} + \|\omega_n\|_{L^\infty_T L^p} + \|\theta_n\|_{L^\infty_T \mathscr{X}_p} \le \Phi_4(T).$$

The space $\mathscr{X}_p$ was introduced before the statement of Theorem 1.5. It follows that up to an extraction the sequence (v_n,θ_n) is weakly convergent to (v,θ) belonging to $L^\infty_T B^1_{\infty,1} \times L^\infty_T \mathscr{X}_p$, with $\omega \in L^\infty_T L^p$. For $(n,m) \in \mathbb{N}^2$ we set $v_{n,m} = v_n - v_m$ and $\theta_{n,m} = \theta_n - \theta_m$ then according to the estimate (39) and (40) we get for $T = \delta$,

$$\|v_{n,m}\|_{L^\infty_T B^0_{\infty,1}} + \|\theta_{n,m}\|_{L^1_T B^{2/p}_{p,1}} \lesssim \|S_n v_0 - S_m v_0\|_{B^0_{\infty,1}} + \|S_n\theta_0 - S_m\theta_0\|_{L^p}.$$

This shows that (v_n,θ_n) is a Cauchy sequence in the Banach space $L^\infty_T B^0_{\infty,1} \times L^1_T B^{2/p}_{p,1}$ and then it converges strongly to (v,θ). This allows to pass to the limit in the system (B_n) and then we get that (v,θ) is a solution of the Boussinesq system (5).

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Received 13 Nov 2009. Accepted 18 Mar 2010.

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ANALYSIS AND PDE
Vol. 4, No. 2, 2011

DEFECTS IN SEMILINEAR WAVE EQUATIONS AND TIMELIKE MINIMAL SURFACES IN MINKOWSKI SPACE

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We study semilinear wave equations with Ginzburg–Landau-type nonlinearities, multiplied by a factor of ε^{-2}, where $\varepsilon > 0$ is a small parameter. We prove that for suitable initial data, the solutions exhibit energy-concentration sets that evolve approximately via the equation for timelike Minkowski minimal surfaces, as long as the minimal surface remains smooth. This gives a proof of the predictions made (on the basis of formal asymptotics and other heuristic arguments) by cosmologists studying cosmic strings and domain walls, as well as by applied mathematicians.

1. Introduction

In this paper we prove that, if Γ is a timelike minimal surface of codimension $k = 1$ or 2 in Minkowski space $\mathbb{R}^{1+N}$, smooth in a time interval $(-T, T)$, then, for suitable initial data and $N > k$, the solutions $u : \mathbb{R}^{1+N} \to \mathbb{R}^k$ of the equation

$$\Box u + \frac{1}{\varepsilon^2} f(u) = 0, \qquad 0 < \varepsilon \ll 1, \tag{1-1}$$

exhibit an energy-concentration set that approximately follows Γ, at least up to time T. Here, the model nonlinearity is $f(u) = (|u|^2 - 1)\,u$ in low dimensions; in higher dimensions, we take f to be a qualitatively similar nonlinearity, satisfying growth conditions that leave equation (1-1) globally well-posed; see (1-9) and (1-19) for the precise assumptions.

Our main motivation for this work comes from the very rich mathematical literature on corresponding questions about elliptic and parabolic analogues of (1-1), which have been studied in great detail for the past 30 years or so. In the elliptic case, these past results establish deep connections between energy-concentration sets for the solutions $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^k$ of the equation

$$-\Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \qquad 0 < \varepsilon \ll 1, \tag{1-2}$$

and (Euclidean) minimal surfaces of codimension k in Ω. Similarly, the parabolic equation

$$u_t - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \qquad 0 < \varepsilon \ll 1, \qquad u : (0, T) \times \mathbb{R}^N \to \mathbb{R}^k, \tag{1-3}$$

The author was partially supported by the National Science and Engineering Research Council of Canada under operating Grant 261955.
MSC2000: primary 35B40, 35L70, 53C44; secondary 85A40.
Keywords: Minkowski minimal surface, semilinear wave equation, topological defects, defect dynamics.

is related to the geometric evolution problem of codimension-k motion by mean curvature. Our results address the natural question of whether any parallel results hold, relating the semilinear wave equation (1-1) to the timelike Minkowski minimal surface problem, which is a geometric wave equation.

It turns out that this question is also relevant to the description of cosmological domain walls ($k = 1$) and strings ($k = 2$); see [Kibble 1976] for a seminal early paper, and [Vilenkin and Shellard 1994] for an in-depth survey of a large body of work on related questions. The problems we study have also been addressed in the applied math literature by [Neu 1990], with some generalizations considered by [Rotstein and Nepomnyashchy 2000]. We will not say any more about any of these applications in this paper, except to note that our main results can be described as giving a rigorous derivation, in the relatively simple and physically unrealistic setting of a scalar particle described by equation (1-1), of the laws of motion for cosmic strings and domain walls, deduced formally by cosmologists over 30 years ago.

1.1. ***Mathematical background.*** We first review results about the elliptic and parabolic equations (1-2) and (1-3). Throughout this discussion, we consider the model nonlinearity $f(u) = (|u|^2 - 1)u$.

In the elliptic case, and when $k = 1$ (so that equation (1-2) is scalar), the general heuristic principle (underlying essentially every work we know of) is that

$$u \approx q\Big(\frac{d}{\varepsilon}\Big), \tag{1-4}$$

where $q : \mathbb{R} \to \mathbb{R}$ solves

$$-q'' + f(q) = 0, \qquad q(0) = 0 \quad \text{and} \quad q(x) \to \pm 1 \text{ as } x \to \pm\infty, \tag{1-5}$$

and $d : \Omega \to \mathbb{R}$ is the signed-distance function to a *minimal* hypersurface $\Gamma \subset \Omega$, so that d is characterized near Γ by the properties

$$d = 0 \text{ on } \Gamma, \qquad |\nabla d|^2 = 1 \text{ near } \Gamma, \tag{1-6}$$

and Γ satisfies

$$\text{(Euclidean) mean curvature} = 0. \tag{1-7}$$

There are a vast number of results establishing various forms of these assertions. Roughly speaking, these fall into two families. The first (see for example [Modica 1987] or [Hutchinson and Tonegawa 2000]) employ variational and measure-theoretic methods, together with elliptic estimates, to characterize the limiting behavior of sequences of solutions as $\varepsilon \to 0$. These proofs generally establish some form of what is called "equipartition of energy", which can be viewed as a weak form of the description (1-4). The second family of proofs (see for example [Pacard and Ritoré 2003]) employ the Liapunov–Schmidt reduction and related arguments, relying ultimately on the implicit function theorem and control of the spectrum of some linearized operator. These arguments yield existence results that give very precise descriptions, in the spirit of (1-4), of the solutions that are constructed.

In the $k = 1$ scalar case of the parabolic equation (1-3), more or less the same heuristic (1-4), (1-5) holds, except that now d is a function of t and x, and, for every t, $d(t, \cdot)$ is the signed-distance function from a hypersurface Γ_t, so that

$$d(t, \cdot) = 0 \text{ on } \Gamma_t \quad \text{and} \quad |\nabla_x d(t, \cdot)|^2 = 1 \text{ near } \Gamma_t,$$

with $\Gamma := \bigcup_{t>0} \{t\} \times \Gamma_t$ inside $(0, T) \times \mathbb{R}^N$, satisfying

$$\text{velocity} = \text{mean curvature}. \tag{1-8}$$

Different versions of this result have been established by a variety of proofs, including linearization techniques (see [de Mottoni and Schatzman 1995]), which establish a strong form of (1-4) but are valid only locally in t; maximum principle arguments, which ultimately rely on an ansatz based on (1-4) to build sub- and super-solutions (see [Chen 1992; Evans et al. 1992]), or which employ a change of variables motivated by (1-4) and techniques for weak passage to limits [Barles et al. 1993]; and measure-theoretic methods combined with parabolic estimates (as in [Ilmanen 1993]), in which (1-4) appears in the weak form of assertions about equipartition of energy. The maximum principle and measure-theoretic arguments give weaker descriptions that are, however, valid globally in t, with (1-8) understood in a weak sense.

In the vector-valued $k = 2$ case, for both the elliptic (1-2) and parabolic (1-3) systems, we do not know of any characterization as precise as (1-4); obstacles to such results include the difficulty of describing rotational degrees of freedom, and the related poor behavior of the spectrum of certain linearized operators. However, there are a number of results showing, in various degrees of generality for solutions of equation (1-2) (including, among others, [Lin and Rivière 1999; Bethuel et al. 2001; Alberti et al. 2005]) and of equation (1-3) (see [Ambrosio and Soner 1997; Lin and Rivière 2001; Bethuel et al. 2006], for example) with suitable energy bounds, that energy concentrates around a codimension-2 submanifold Γ satisfying (1-7) and (1-8), respectively. These results generally employ elliptic or parabolic estimates, some of which are extremely delicate, in combination with measure-theoretic arguments, and they provide information, customarily phrased in the language of varifold convergence, about the precise way in which energy concentrates around the codimension-2 surface Γ.

All results about (1-2) and (1-3) rely very heavily on tools that are not available for hyperbolic equations, such as maximum principles (in the scalar case) and elliptic or parabolic regularity. Thus, they do not give much indication of how to proceed for the nonlinear wave equation (1-1). We know of only two partial exceptions to this rule. First, there is no abstract reason that linearization arguments should be impossible in the hyperbolic setting; they appear however to be hard to carry through. Second, a number of papers, starting with [Bronsard and Kohn 1991], study (1-3) using weighted energy estimates. In particular, we mention an argument presented by Soner in a 1995 lecture series [1998] for the scalar parabolic equation (1-3), and developed in [Jerrard and Soner 1999; Lin 1998] for parabolic systems. This argument relies on a rather straightforward but remarkable computation of $\frac{d}{dt}\int_{\mathbb{R}^N} \zeta\, e_\varepsilon(u)\, dx$, where $e_\varepsilon(u)$ is a natural energy density associated with a solution u of (1-3), and ζ is a smooth function such that $\zeta(t, x) = \frac{1}{2}\operatorname{dist}(x, \Gamma_t)^2$ near Γ_t, where the latter solves (1-8). This calculation certainly uses the parabolic character of (1-3), but it is not clear if it uses it in an essential way. Indeed, our main proofs originated as an attempt to develop an analogue of this argument in the hyperbolic setting.

Much less work has been done on the hyperbolic equation (1-1) than on its elliptic and parabolic counterparts. The few papers of which we are aware mostly study situations rather different from those we consider here, including

- works [Jerrard 1999; Lin 1999] that characterize the behavior of solutions of (1-1) in the limit $\varepsilon \to 0$ in the case $N = k = 2$, for the model nonlinearity $f(u) = (|u|^2 - 1)u$.
- [Gustafson and Sigal 2006], on the Maxwell–Higgs model, in which (1-1), with the model nonlinearity $f(u) = (|u|^2 - 1)u$, is coupled to an electromagnetic field, when $N = k = 2$ and $0 < \varepsilon \ll 1$.

- [Stuart 2004a], studying an equation of the form (1-1) on a Lorentzian manifold and with a focusing nonlinearity, for $0 < \varepsilon \ll 1$; see also [Stuart 2004b].

In all these papers, energy concentrates around points (known as "vortices" or "quasiparticles", depending on the situation), and these points evolve according to an ODE. These results are valid only as long as the points remain separated from each other. The fact that points are geometrically very simple objects makes the analysis easier, in some ways, than in the problems we consider here, where the same role is now played by submanifolds of dimension $n \geq 1$. An additional significant, simplifying factor in all the papers cited above (except those of Stuart) is that they study a scaling in which vortices move at subrelativistic velocities, that is, velocities that tend to 0 as $\varepsilon \to 0$.

It is also worth mentioning the work of Cuccagna [2008] that studies (1-1) in $\mathbb{R}^{1+3}$ with $\varepsilon = 1$, and establishes scattering for initial data $(u, u_t)|_{t=0}$, which is a small, very smooth perturbation of $(q(x^3), 0)$. This can be seen as an analogue for (1-1) of results [Lindblad 2004; Brendle 2002] that establish scattering for solutions of the timelike Minkowski minimal surface problem, with initial data that is a small perturbation of a motionless hyperplane.

As far as we know, the only work of rigorous mathematics that addresses exactly the questions we consider here is a recent preprint of Bellettini, Novaga, and Orlandi [2008]. Its main result identifies some conditions that, if they could be verified, would suffice to imply that a varifold, obtained from a sequence of solutions (u_ε) of (1-1) satisfying natural energy bounds, is stationary with respect to the Minkowski inner product structure. These conditions include lower density bounds, as well as, roughly speaking, some quite strong constraints on the limiting tangent space. As discussed in Remark 1.6, the results we obtain here are stronger than those projected in [Bellettini et al. 2008].

1.2. ***New results.*** In many ways, our results follow the pattern described previously. In the case $k = 1$ of a scalar equation, as in the earlier work on the elliptic and parabolic problems, we obtain, for suitable initial data, a description of solutions of (1-1) parallel to (1-4), (1-5), (1-6), (1-7) with, in the last two identities, the Euclidean metric replaced by the Minkowski metric. And in the case $k = 2$, we prove that, for solutions of (1-1) with suitable initial data, energy concentrates around a codimension-2 surface Γ that satisfies (1-7), again with the Euclidean metric replaced by the Minkowski metric. We also give a precise description of the way in which this concentration occurs; in fact, we obtain this description for the case $k = 1$ as well.

The strongest results (for example [Bethuel et al. 2006]) on the parabolic equation (1-3) hold globally for $t > 0$, and they assume only natural energy bounds on the initial data. Our results, by contrast, are valid only locally in t — that is, as long as the surface Γ remains smooth — and require rather special initial data. We note, however, that results like those we obtain are almost certainly *not true* globally in t or for general initial data.

In all our results, we take the timelike minimal surface Γ to have the topology of $(-T, T) \times \mathbb{T}^n$, where $n = N - k$. When $k = 2$, this covers the important example of a closed string in $\mathbb{R}^3$. In fact, we view the global topology of Γ as relatively unimportant, since our results are in some sense local, and since both the semilinear wave equation (1-1) and the timelike minimal surface equation enjoy finite propagation speed. In any case, our methods should extend to $\Gamma \cong (-T, T) \times M$ for more general M.

The quite general results in [Milbredt 2008] imply, in particular, the local existence of smooth timelike minimal surfaces Γ, given smooth data at $t = 0$.

In the scalar case, we assume that the nonlinearity f in (1-1) has the form $f = F'$, where $F : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$F(\pm 1) = 0 \quad \text{and} \quad c(1-|s|)^2 \le F(s). \tag{1-9}$$

We also assume that f grows sufficiently slowly so that (1-1) is globally well posed in $\dot{H}^1 \times L^2$. If $N \le 4$, we may take $f(u) = (u^2-1)u$.

In the statement of our results, we use the notation

$$e_\varepsilon(u) := \frac{1}{2}\big(u_t^2 + |\nabla u|^2\big) + \frac{1}{\varepsilon^2} F(u) \tag{1-10}$$

and

$$\kappa_1 := \int_{-1}^{1} \sqrt{2F(s)}\, ds. \tag{1-11}$$

One can think of κ_1 as a constant, related to the surface tension of an interface. In the scalar case, our main results can be summarized as:

Theorem 1. *Let $\Gamma \subset (-T, T) \times \mathbb{R}^N$ be a smooth timelike minimal hypersurface. Let $\Gamma_t := \Gamma \cap (\{t\} \times \mathbb{R}^N)$ and assume that, for every $t \in (-T, T)$, Γ_t is diffeomorphic to the torus $\mathbb{T}^n$, for $n = N-1$.*

Given $T_0 < T$, there exists a neighborhood $\mathcal{N}$ of Γ in $(-T_0, T_0) \times \mathbb{R}^N$ in which there exists a smooth solution $d : \mathcal{N} \to \mathbb{R}$ of the problem

$$d = 0 \ \text{ on } \Gamma, \qquad -d_t^2 + |\nabla d|^2 = 1 \ \text{ near } \Gamma. \tag{1-12}$$

(*In other words*, *d is the signed* Minkowski *distance to Γ*; *compare with* (1-6).) *Moreover*, *there exists a solution u of* (1-1) (*with f as described above*) *such that*, *for any $T_0 < T$*,

$$\Big\| u - q\Big(\frac{d}{\varepsilon}\Big) \Big\|_{L^2(\mathcal{N})} \le C\sqrt{\varepsilon}, \tag{1-13}$$

where q solves (1-5) *and*

$$\int_{\mathcal{N}} d^2 e_\varepsilon(u)\, dt\, dx \; + \int_{[(-T_0,T_0)\times\mathbb{R}^N]\setminus\mathcal{N}} e_\varepsilon(u)\, dt\, dx \; \le C\varepsilon. \tag{1-14}$$

In addition, *if $\mathcal{T}_{\varepsilon(u)} = (\mathcal{T}^\alpha_{\varepsilon,\beta}(u))^N_{\alpha,\beta=0}$ and $\mathcal{T}(\Gamma) = (\mathcal{T}^\alpha_\beta(\Gamma))^N_{\alpha,\beta=0}$ denote the energy-momentum tensors for u and Γ* (*defined in* (2-8) *and* (2-9), *respectively*), *then*

$$\Big\| \frac{\varepsilon}{\kappa_1}\mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma) \Big\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C\varepsilon. \tag{1-15}$$

In all these conclusions, *$C = C(T_0, \Gamma)$ is independent of ε.*

Remark 1.1. The definitions imply that $\mathcal{T}^0_{\varepsilon,0}(u) = e_\varepsilon(u)$, and that $\mathcal{T}^0_0(\Gamma)$ is a measure supported on Γ and defined by

$$\int f(t,x)\, d\mathcal{T}^0_0 = \int_{-T}^{T} \int_{\Gamma_t} f(t,x)(1-V^2)^{-1/2} \mathcal{H}^n(dx)\, dt,$$

where $V(t,x)$ denotes the (Euclidean) normal velocity of Γ at a point $(t,x) \in \Gamma$. We can denote this measure by $(1-V^2)^{-1/2}(\mathcal{H}^n \llcorner \Gamma_t) \otimes dt$. The conclusion (1-15) thus implies, in particular, that

$$\Big\| \frac{\varepsilon}{\kappa_1} e_\varepsilon(u) \; - \; (1-V^2)^{-1/2}(\mathcal{H}^n \llcorner \Gamma_t) \otimes dt \Big\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C\varepsilon. \tag{1-16}$$

A parallel remark holds for conclusion (1-21) of Theorem 2 below.

Remark 1.2. Our arguments show that a solution u of (1-1) satisfies (1-13), (1-14) and (1-15) if, for example,

$$u(0,x) = q\Big(\frac{d(0,x)}{\varepsilon}\Big) \quad \text{and} \quad u_t(0,x) = \frac{1}{\varepsilon} q'\Big(\frac{d(0,x)}{\varepsilon}\Big)\, d_t(0,x) \tag{1-17}$$

in a neighborhood $\mathcal{N}_0$ of Γ_0, and if

$$\int_{\{0\}\times(\mathbb{R}^N\setminus\mathcal{N}_0)} e_\varepsilon(u)\,dx \le \varepsilon. \tag{1-18}$$

For details, see Lemma 9 and Theorem 22.

In the vector case, we can again take $f(u) = (|u|^2-1)u$ if $N \le 4$, or, in other words, $f = \nabla_u F$, for $F(u) = \frac{1}{4}(|u|^2-1)^2$. More generally, we require from f only that the equation (1-1) be globally well posed in $\dot H^1 \times L^2$, and that $f = \nabla_u F$ where

$$c(1-|u|)^2 \le F(u) \le C(1-|u|)^2 \ \text{ for } |u| \le 2 \quad \text{and} \quad F(u) \ge c > 0 \ \text{ for } |u| \ge 2. \tag{1-19}$$

We summarize our results in the $k = 2$ vector case in:

Theorem 2. *Let $\Gamma \subset (-T,T)\times\mathbb{R}^N$ be a smooth timelike minimal surface of codimension $k = 2$. Let $\Gamma_t := \Gamma \cap (\{t\}\times\mathbb{R}^N)$ and assume that, for every $t \in (-T,T)$, Γ_t is diffeomorphic to the torus $\mathbb{T}^n$, for $n = N-2 \ge 1$.*

When $k = 2$, there exists a solution for (1-1) such that, for any $T_0 < T$, there is a constant C such that

$$\int_{(-T_0,T_0)\times\mathbb{R}^N} \tilde d^2 e_\varepsilon(u)\,dt\,dx \le C, \tag{1-20}$$

where $\tilde d(t,x) = \min\{1, \operatorname{dist}((t,x),\Gamma)\}$ and

$$\Big\| \frac{1}{\pi|\ln\varepsilon|}\mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\Big\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C|\ln\varepsilon|^{-1/2} \tag{1-21}$$

where $\mathcal{T}_\varepsilon(u)$ and $\mathcal{T}(\Gamma)$ denote the energy-momentum tensors for u and Γ defined in (2-8) and (2-9), respectively. In all these conclusions, $C = C(T_0,\Gamma)$ is independent of ε.

Remark 1.3. In Lemma 9 we give an explicit construction of initial data for which the conclusions of the theorem hold.

Remark 1.4. The proof shows that the solutions u from Theorem 2 have a defect near Γ; see (6-5) for a precise, if opaque, version of this assertion.

Remark 1.5. In both the above theorems, the constants C in the conclusions are at least exponential in T_0. That is, our proofs yield constants of the form $C = ae^{bT_0}$, where a, b themselves depend on Γ and T_0, and may blow up as $T_0 \nearrow T$.

Remark 1.6. Our results imply in particular that, if we fix Γ as in either of the theorems above, there exists a sequence (u_ε) of solutions of (1-1) such that the energy-momentum tensors $\delta_\varepsilon\mathcal{T}_\varepsilon(u_\varepsilon)$ converge weakly, as measures in $(-T,T)\times\mathbb{R}^N$, to $\mathcal{T}(\Gamma)$, if the scaling factor $\delta_\varepsilon = \delta_\varepsilon(k)$ is chosen correctly. This fact can be seen as a form of varifold convergence, analogous to results proved in [Ilmanen 1993;

Ambrosio and Soner 1997; Bethuel et al. 2001; 2006] for elliptic and parabolic equations, and discussed in the hyperbolic case in [Bellettini et al. 2008].

However, by providing quantitative estimates of $\|\delta_\varepsilon \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\|_{W^{-1,1}}$, our results are sharper than simple convergence results. This sharpening is significant, because convergence results, strictly analogous to known results in the elliptic or parabolic cases, *can fail* in the hyperbolic setting. That is, in our setting (but *not* for elliptic or parabolic problems) there exist sequences of solutions (u_ε) such that $\delta_\varepsilon \mathcal{T}_\varepsilon(u_\varepsilon)$ converges to a measure-valued tensor $\mathcal{T}$ supported on a codimension-k set, but such that $\mathcal{T}$ is not the energy-momentum tensor for any timelike minimal surface Γ; in other words, $\mathcal{T}$ is not weakly stationary; see Section 1.4 for explicit examples.

Remark 1.7. If we fix Γ and consider an associated sequence (u_ε) of solutions as found in Theorem 2, with $\varepsilon \to 0$, the uniform energy bounds (1-20) away from Γ, combined with a classical argument of [Shatah 1988], imply that, after passing to a subsequence, u_ε converges weakly in $H^1_{loc}([(-T,T) \times \mathbb{R}^N] \setminus \Gamma)$ to a wave map into S^1.

Remark 1.8. In both theorems, we ultimately rely on energy estimates in a frame that moves with Γ. These estimates (summarized in Theorem 22) assert more or less that energy remains concentrated around Γ on the same scale for $0 < t < T$, as it is at $t = 0$. The hypotheses for Theorem 22 are:

- small energy away from Γ_0 — see (2-31);
- a defect near Γ_0 — see (2-36);
- small energy, *given the presence of the defect*, near Γ_0, *in a frame that moves with* Γ — see (2-34) and (2-35).

Theorems 1 and 2 follow from the special case of Theorem 22 in which the energy is, roughly speaking, as concentrated as possible around Γ_0. The fact that our results for $k = 1$ are considerably stronger than for $k = 2$ stems ultimately from the fact that, when $k = 1$, for initial data that is nearly energetically optimal (essentially, (1-17) and (1-18) or suitably small perturbations thereof) the energy is very sharply concentrated around Γ_0, whereas when $k = 2$, for the model initial data, energy is quite spread out. A more precise expression of this fact appears in (1-33).

1.3. ***About the proofs.*** A main issue in the analysis of (1-1) is to establish some kind of stability property of the moving defect; that is, the interface ($k = 1$) or "string" ($k = 2$). The relativistic invariance of the equation suggests that a defect should acquire extra energy when it accelerates (and this is confirmed by our results; see, for example, (1-16)), so we must rule out this extra energy as a potential source of instability. Our analysis starts from the observation that, for a solution that behaves as predicted in the formal arguments of [Vilenkin and Shellard 1994; Neu 1990; Rotstein and Nepomnyashchy 2000] and others, a moving defect will always appear to be energetically optimal in the frame of reference of an observer who is moving with the defect.

1.3.1. *Change of variables.* Motivated by this, we begin by rewriting the equation in a frame that follows the timelike minimal surface Γ, where the defect is expected to remain. In these variables, our task is to show that the solution is approximately constant, and we expect the defect to have some optimality property that we can exploit.

To define the change of variables, we start with a map H defined on $(-T,T)\times\mathbb{T}^n$ and parametrizing $\Gamma\subset(-T,T)\times\mathbb{R}^N$, and we extend H to a diffeomorphism ψ between, essentially, a neighborhood in $(-T,T)\times\mathbb{T}^n\times\mathbb{R}^k$ of $(-T,T)\times\mathbb{T}^n$ and a neighborhood of Γ in $\mathbb{R}^{1+N}$. We write ψ as a function of variables $y=(y^0,\dots,y^N)=(y^\tau,y^\nu)$, where $y^\tau=(y^0,\dots,y^n)$ are variables tangent to Γ, and $y^\nu=(y^{n+1},\dots,y^N)$ correspond to directions normal to Γ. We always arrange that y^0 is a timelike coordinate, and that all other coordinates are spacelike.

We will also write, for example, $D_\tau=(\partial_{y^0},\dots,\partial_{y^n})$ and $\nabla_\nu=(\partial_{y^{n+1}},\dots,\partial_{y^N})$. We generally write D for a space-time gradient, and ∇ for a gradient involving space-like variables only.

We then define $v=u\circ\psi$ on the domain of ψ. We find it convenient to write the equation satisfied by v (that is, equation (1-1) expressed in terms of the y variables) in the form

$$-\partial_{y^\alpha}(g^{\alpha\beta}\partial_{y^\beta}v)-b\cdot Dv+\frac{1}{\varepsilon^2}f(v)=0 \quad\text{and}\quad b^\beta:=\frac{\partial_{y^\alpha}\sqrt{-g}}{\sqrt{-g}}g^{\alpha\beta}. \tag{1-22}$$

Here, $G=(g_{\alpha\beta})$ is the expression in the y coordinates of the Minkowski metric, $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$, $g=\det(g_{\alpha\beta})$, and we implicitly sum over repeated indices. Equation (1-22) enjoys certain useful properties, which are summarized in Proposition 4. Some of these follow from the specific form we chose for the map ψ. The fact that Γ is a timelike minimal surface implies a key property of the coefficient b of the first-order term:

$$|b^\nu|\le C|y^\nu| \quad\text{at } y=(y^\tau,y^\nu), \quad\text{for } b^\nu:=(b^{n+1},\dots,b^N). \tag{1-23}$$

We emphasize that the verification of (1-23) is *the only place* in our analysis where we explicitly invoke the fact that Γ is a minimal surface.

1.3.2. *Energy estimates.* We now focus on v solving (1-22) on, say, $(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$ for some $T_1<T$ and $\rho_0>0$, where $B_\nu(\rho_0):=\{y^\nu\in\mathbb{R}^k_\nu : |y^\nu|<\rho_0\}$. We will use the notation

$$e_{\varepsilon,\nu}(v):=\frac{1}{2}|\nabla_\nu v|^2+\frac{1}{\varepsilon^2}F(v). \tag{1-24}$$

We introduce a scaling factor $\delta_\varepsilon=\delta_\varepsilon(k)$ (see (2-1)), chosen so that, heuristically,

$$\delta_\varepsilon\int_{\{y^\nu\in\mathbb{R}^k_\nu : |y^\nu|\le\rho_1\}} e_{\varepsilon,\nu}(v)(y^\tau,\cdot)\,dy^\nu\ \ge\ 1-o_\varepsilon(1) \quad\text{if } v(y^\tau,\cdot)\text{ has a defect near } y^\nu=0, \tag{1-25}$$

for every fixed ρ_1; this is made precise later. One of our goals is to show that, if

$$\zeta_3(s):=\delta_\varepsilon\int_{\mathbb{T}^n\times W_\nu(s)}|D_\tau v|^2+|y^\nu|^2\,e_{\varepsilon,\nu}(v)\,dy^1\cdots dy^N\Big|_{y^0=s}$$

is small when $s=0$, say, then it remains small for a range of positive s. Here, $W_\nu(s)$ is a neighborhood of the origin in $\mathbb{R}^k_\nu$ that may depend on the parameter s, but will always contain a ball of fixed radius ρ. The smallness of ζ_3 is consistent with v having a large amount of energy, as long as it involves mostly the normal energy $e_{\varepsilon,\nu}(v)$ and is concentrated very near the codimension-k surface $\{y^\nu=0\}$.

Our strategy is to define some quantity $\zeta_1(s)$ such that

$$\zeta_1'(s)\le C\zeta_3(s) \tag{1-26}$$

and such that, under suitable additional assumptions,

$$\zeta_1(s) \geq c\zeta_3(s) - o_\varepsilon(1). \tag{1-27}$$

A main task will then be to show that these additional assumptions are preserved by equation (1-22). If we can do this, we can easily use Grönwall's inequality to control the growth of ζ_3.

For the verification of (1-26), we define the approximately[1] conserved energy density

$$e_\varepsilon(v) = \frac{1}{2}a^{\alpha\beta} v_{y^\alpha} v_{y^\beta} + \frac{1}{\varepsilon^2}F(v), \tag{1-28}$$

where $a^{\alpha\beta}$ is a positive-definite matrix related to $g^{\alpha\beta}$; see (2-16). (When we want to avoid any possibility of confusion, we will write $e_\varepsilon(v; G)$ for the above quantity, and $e_\varepsilon(u; \eta)$ for the energy defined in (1-10), with η denoting the expression in the original coordinates of the Minkowski metric.) We further define

$$\zeta_1(s) := \delta_\varepsilon \int_{\mathbb{T}^n \times W_\nu(s)} \big(1 + \kappa_2 |y^\nu|^2\big) e_\varepsilon(v)\, dy^1 \cdots dy^N \Big|_{y_0 = s} - 1,$$

where κ_2 is a constant to be selected in a moment. (It will turn out later that we can take $\kappa_2 = 1$ in the scalar case.) We hope to show that ζ_1 satisfies the properties (1-26) and (1-27) above.

Indeed, as long as the sets $W_\nu(s)$ are chosen to shrink rapidly enough, we will show in Section 3 that the verification of (1-26) follows quite easily from the differential inequality

$$\frac{\partial}{\partial y^0} e_\varepsilon(v) \leq \sum_{i=1}^{N} \frac{\partial}{\partial y^i}\varphi^i + C\big(|D_\tau v|^2 + |y^\nu|^2 |\nabla_\nu v|^2\big) \tag{1-29}$$

for some vector $\varphi = (\varphi^1, \ldots, \varphi^N)$. The differential inequality (1-29), in turn, follows easily from (1-22); see Lemma 6. The key point in (1-29) is the factor $|y^\nu|^2$, which follows from (1-23) and, hence, from the fact that Γ is a minimal surface.

To check (1-27), we first note that some of the good properties of (1-22) alluded to above imply that if κ_2 is chosen in a suitable way (see (2-23)), then

$$\big(1 + \kappa_2 |y^\nu|^2\big) e_\varepsilon(v) \geq \; c|D_\tau v|^2 + \big(1 + |y^\nu|^2\big) e_{\varepsilon,\nu}(v).$$

With this choice of κ_2,

$$\zeta_1(s) \geq c\zeta_3(s) + \int_{\mathbb{T}^n} \left(\delta_\varepsilon \int_{W_\nu(s)} e_{\varepsilon,\nu}(v)\, dy^\nu - 1 \right) dy^1 \cdots dy^n \Big|_{y_0 = s}.$$

Thus, in view of the choice (1-25) of δ_ε, we can deduce (1-27) as long as we can check that $v(s, \cdot)$ has a defect confined near $\{(y^1, \ldots, y^N) \in \mathbb{T}^n \times \mathbb{R}^k_\nu : y^\nu = 0\}$. (This is the additional assumption mentioned before equation (1-27).)

[1] The *exact* law expressing conservation of energy for (1-1) can, of course, be transposed to the y coordinates. As far as we know, this is not useful for our problem, since it does not distinguish any good property of equation (1-22) resulting from the fact that the change of variables is built around a parametrization of a minimal surface.

1.3.3. *A certain stability property.* We therefore introduce a "defect confinement functional"

$$\mathfrak{D} : H^1(\mathbb{T}^n \times B_\nu(\rho_0)) \to \mathbb{R}$$

that is designed to have two properties. (This functional takes quite different forms in the two cases $k = 1$ and $k = 2$ that we consider; see (3-1) and (5-1).) First, we require that

$$\mathfrak{D}(v(s, \cdot)) \text{ small} \quad \Rightarrow \quad \text{“defect is confined”} \quad \Rightarrow \quad \text{lower energy bounds} \quad \Rightarrow \quad \text{(1-27) holds.} \tag{1-30}$$

This sort of argument will eventually lead to an inequality of the simple form

$$\zeta_3(s) \le C[\zeta_1(s) + \zeta_2(s)] + o_\varepsilon(1), \tag{1-31}$$

where

$$\zeta_2(s) = \mathfrak{D}(v(s)).$$

Second, we need $\mathfrak{D}$ to be such that

$$\text{changes in } \zeta_2(s) \text{ can be controlled by } \zeta_3(s). \tag{1-32}$$

Concrete versions of (1-30) and (1-32) are established in Section 3 for $k = 1$, and Section 5 for $k = 2$. Heuristically, (1-32) should hold because, if the defect strays away from $y^\nu = 0$, then it should carry with it concentrations of energy that can be detected by ζ_3. In the case $k = 1$, (1-32) will take the simple form $\zeta_2(s) \le 2\zeta_2(0) + C\int_0^s \zeta_3(\sigma)\, d\sigma$. The corresponding estimate for $k = 2$ is similar but slightly more complicated. In both cases, however, by combining (1-31) and a specific concrete version of (1-32) with (1-26), we obtain control over $\zeta_i(s)$ for $i = 1, 2, 3$. This gives us a good deal of information about the behavior of v, from which all of our main conclusions are ultimately deduced.

One can view (1-31) and (1-32) as a weak stability property of states w for which $\mathfrak{D}(w)$ is small and for which the inequality in (1-31) is almost saturated.

The difference in the strength of our conclusions in the cases $k = 1$ and $k = 2$, discussed in Remark 1.8, stems from the fact that, for optimal initial data,

$$\text{for } i = 1, 2, 3, \qquad \zeta_i(0) \approx \begin{cases} \varepsilon^2 & \text{when } k = 1, \\ |\ln \varepsilon|^{-1} & \text{for } k = 2; \end{cases} \tag{1-33}$$

see Lemma 9. This reflects sharper energy concentration around $\{y^\nu = 0\}$ in the case $k = 1$.

1.3.4. *Some other issues.* The change of variables that we employ is defined only in a neighborhood of Γ. We must therefore combine estimates of v near Γ with estimates of u away from Γ, and then iterate. We verify in Section 6 that this can be done in such a way as to genuinely yield estimates valid up to $(-T_0, T_0) \times \mathbb{R}^N$ for arbitrary $T_0 < T$.

Spacelike hypersurfaces of the form $\{y^0 = \text{constant}\}$ play a distinguished role in our argument, as it is along these surfaces that the defect structure is nearly energetically optimal for the solutions v that we consider. This near-optimality is manifested, for example, in the fact that inequality (1-31) is nearly saturated. In general, our change of variables ψ^{-1} maps the hypersurface $\{(t, x) \in \mathbb{R}^{1+N} : t = 0\}$ (on which we assume the data for the solution u of (1-1) is given) onto a hypersurface that is smooth and spacelike, but otherwise can be quite arbitrary. So, a certain amount of work is needed to obtain control of v on a suitable portion of some hypersurface $\{y^0 = \text{constant}\}$. This is done in Sections 4 and 5.3, and involves mainly technical adjustments to our basic energy estimates as outlined above. This means that we carry out our main energy estimates twice, once in a simpler form that can easily be iterated, and once to deal with complications caused by the geometry of the initial hypersurface in the transformed variables. This, and the similarity between the cases $k = 1$ and $k = 2$, leads to a certain amount of redundancy which, however, enables us to present out argument first in a relatively simple setting, in Section 3; we believe this makes the main ideas easier to grasp.

The technical work of Section 4 could be avoided if we insisted on prescribing data only on spacelike hypersurfaces that have the form $\{y^0 = \text{constant}\}$ near Γ_0, but we feel that this would be unnecessarily restrictive.

Finally, we extract all the conclusions of the main theorems from control over quantities such as ζ_1, ζ_2 and ζ_3 above; this is done in Section 6. In the vector case, these arguments require a useful recent estimate of [Kurzke and Spirn 2009], without which we would not be able to establish the full energy-momentum tensor estimate (1-21).

1.4. ***Some examples.*** It is well known that the timelike minimal surface equation for $(1+1)$-dimensional surfaces in $\mathbb{R}^{1+N}$ is explicitly solvable for every $N \geq 2$. In particular, if $a : \mathbb{R} \to \mathbb{R}^N$ and $b : \mathbb{R} \to \mathbb{R}^N$ are smooth maps such that $|a'| = |b'| = 1$, then the function

$$X(s,t) := (t, x(s,t)) \qquad \text{with} \qquad x(s,t) := \tfrac{1}{2}(a(s+t) + b(s-t))$$

parametrizes a surface that satisfies the timelike minimal surface equation wherever it is smooth. (See, for example, the exposition in [Vilenkin and Shellard 1994, Chapter 6].) From this one can deduce,[2] in particular, that, if $g : \mathbb{R} \to \mathbb{R}^k$ is any smooth function (where $k = N - 1$), then

$$\Gamma := \{(t, s, g(s-t)) : t, s \in \mathbb{R}\} \tag{1-34}$$

is a $(1+1)$-dimensional minimal surface in $\mathbb{R}^{1+N}$. For a timelike minimal surface Γ of this very simple form, it turns out that there are corresponding solutions of the nonlinear wave equation (1-1) that *exactly* follow Γ. Indeed, if $q : \mathbb{R}^k \to \mathbb{R}^k$ is any smooth solution of

$$-\Delta q + (q^2 - 1)q = 0,$$

then, after writing $x \in \mathbb{R}^N = \mathbb{R}^{1+k}$ as $(x^1, x^\nu) \in \mathbb{R} \times \mathbb{R}^k$,

$$u(t,x) := q\Big(\frac{x^\nu - g(x^1 - t)}{\varepsilon}\Big) \tag{1-35}$$

solves (1-1) in all of $\mathbb{R}^{1+N}$.

In particular, consider a family of surfaces $(\Gamma^\varepsilon)_{\varepsilon\in(0,1]}$ of the form (1-34) associated with a sequence of smooth rapidly oscillating functions (g_ε), converging weakly in H^1 to a limiting function g_0. Although Γ^ε converges in Hausdorff distance to the minimal surface Γ_0 associated via (1-34) with the function g_0, one can arrange the oscillation in such a way that $\mathcal{T}(\Gamma^\varepsilon)$ converges weakly to a limiting measure that is *not* equal to $\mathcal{T}(\Gamma_0)$. (This is a simple special case of the phenomenon known in the cosmology literature as “wiggly strings”; see again [Vilenkin and Shellard 1994, chapter 6]. Related issues are also discussed in [Neu 1990].)

[2]Take $a(s) = (s, 0)$ and $b(s)$ of the form $b = (\sigma(s), h(\sigma(s))$, for $h : \mathbb{R} \to \mathbb{R}^{k-1}$ smooth, and σ strictly increasing and adjusted so that $|b'| \equiv 1$. Then, a change of variables shows that the surface parametrized by $x(s,t)$ can be written in the form (1-34), if g is defined by requiring that $\frac{1}{2}h(\sigma(r)) = g(\frac{1}{2}(\sigma(r)+r))$ for all r. One can check that any smooth g can be realized in this way.

To illustrate this in detail, let us for simplicity assume that $k = 1$ and $g_0 = 0$. One can check that, if u_ε is the solution of the form (1-35) associated with g_ε, then (using notation defined in Section 2.3)

$$\mathcal{T}_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon^2} q'^2 \begin{pmatrix} 1+g_\varepsilon'^2 & -g_\varepsilon'^2 & g_\varepsilon' \\ g_\varepsilon'^2 & 1-g_\varepsilon'^2 & g_\varepsilon' \\ -g_\varepsilon' & g_\varepsilon' & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{T}(\Gamma_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{H}^{1+1} \llcorner \Gamma_0.$$

From these it is easy to see that, unless $g_\varepsilon \to g_0 = 0$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R})$, $(\varepsilon/\kappa_1)\mathcal{T}_\varepsilon(u_\varepsilon)$ converges to a limit that does not equal $\mathcal{T}(\Gamma_0)$. One can further check that this limit in general is not the energy-momentum tensor for any smooth string.

2. Notation and assumptions

2.1. ***General notation.*** We will write $B(\rho)$ to denote an open ball of radius ρ centered at the origin.

In order to emphasize the parallels between the two cases we consider, we will use the same notation for $k = 1$ and $k = 2$, normally without indicating the dependence on k. For example, we will write

$$\delta_\varepsilon := \begin{cases} \varepsilon/\kappa_1 & \text{when } k = 1, \text{ for } \kappa_1 \text{ defined in (1-11);} \\ (\pi\,|\ln \varepsilon|)^{-1} & \text{for } k = 2. \end{cases} \tag{2-1}$$

Similarly, $\mathcal{D}$ and $\mathcal{D}_\nu$ will have different meanings in the cases $k = 1$ and $k = 2$; see (3-1)–(3-3) and (5-1)–(5-2).

Throughout this work, we consider $(1+n)$-dimensional submanifolds in $(1+N)$-dimensional Minkowski space. We will always write $k = N - n$ for the codimension of the manifold. The same number k is also the dimension of the target space for the semilinear wave equation (1-1).

A parametric $(1+n)$-dimensional submanifold Γ of $\mathbb{R}^{1+N}$ is a submanifold described as the image of a smooth map $H : U \to \mathbb{R}^{1+N}$, where U is an open subset of $\mathbb{R}^{1+n}$. We will generally assume that this map H is injective. Given a map H parametrizing a surface Γ, we will often define a map $\psi : U \times (\text{small ball in } \mathbb{R}^k) \to \mathbb{R}^{1+N}$ that parametrizes a neighborhood of Γ and agrees with H on $U \times \{0\}$. In this situation, we will typically write points in $U \times \mathbb{R}^k \subset \mathbb{R}^{1+N}$ in the form

$$y = (y^\tau, y^\nu) \quad \text{with } y^\tau = (y^0, \dots, y^n) \in U \text{ and } y^\nu = (y^{n+1}, \dots, y^N) \in \mathbb{R}^k. \tag{2-2}$$

The superscripts stand for "tangential" and "normal", respectively. We will also sometimes use the alternative notation

$$y^\nu = (y^{\nu,1}, \dots, y^{\nu,k}) \tag{2-3}$$

for y^ν. We will always arrange that y^0 is a timelike coordinate, and we will often write $y^{\tau\prime} = (y^1, \dots y^n)$ and $y' := (y^{\tau\prime}, y^\nu)$, so that a "prime" denotes spatial variables only.

For notational consistency, we may sometimes write y^τ to denote a point $(y^0, \dots, y^n) \in U \subset \mathbb{R}^{1+n}$ even when there are no normal y^ν variables present. We may also write, for example, $\mathbb{R}^k_\nu$ to denote a copy of $\mathbb{R}^k$ that should be thought of as being in the normal y^ν variables, and we will write $B_\nu(\rho) := \{y^\nu \in \mathbb{R}^k_\nu : |y^\nu| < \rho\}$, where k should be clear from the context. We will generally write ∇ to denote the gradient in spatial directions only, and D to denote the spacetime gradient, so that $D = (\partial_t, \nabla)$. When using the notation (2-2), we will similarly write $D = (D_\tau, \nabla_\nu) = (\partial_{y^0}, \nabla_\tau, \nabla_\nu)$, where for example $\nabla_\nu = (\partial_{y^{n+1}}, \dots, \partial_{y^N})$.

We write $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta})$ to denote the diagonal matrix $\mathrm{diag}(-1, 1, \dots, 1)$.

We normally follow the convention that Latin indices i, j, k run from 1 to N, while Greek indices α, β, γ run from 0 to N; we sum over repeated upper and lower indices. When summing implicitly over the (t, x) variables, we will identify x^0 with t.

2.2. ***Assumptions and notation related to timelike minimal surfaces.*** A parametric submanifold is said to be *timelike* if $\gamma(DH) := \det(DH^T \eta\, DH) < 0$ at every point of U. The Minkowski area of a timelike parametric submanifold is defined to be

$$\mathcal{L}(H) := \int_U \sqrt{-\gamma} \tag{2-4}$$

A timelike submanifold $\Gamma = \mathrm{Image}(H)$ is said to be a *timelike minimal surface* if H is a critical point of $\mathcal{L}$. (The terminology, although standard, is misleading, as a minimal surface Γ is in general not a minimizer or local minimizer of $\mathcal{L}$.)

Our main results all involve a timelike minimal surface Γ that is the image of a smooth, injective map $H : (-T, T) \times \mathbb{T}^n \to (-T, T) \times \mathbb{R}^N$ of the form

$$H(y^0, \dots, y^n) = \big(y^0, h(y^0, \dots, y^n)\big) \quad \text{for some smooth } h : (-T, T) \times \mathbb{T}^n \to \mathbb{R}^N, \tag{2-5}$$

where $\mathbb{T}^n$ denotes the n-dimensional torus, thought of as the periodic unit cube (so that $\mathcal{H}^n(\mathbb{T}^n) = 1$). We will require that our parametrization satisfies[3]

$$H^T_{y_0} \eta\, H_{y_i} = h_{y_0} \cdot h_{y_i} = 0 \quad \text{for } i > 0, \tag{2-6}$$

where, here and throughout, we view H and h as column vectors. One can easily check that, if Γ is a timelike parametric submanifold given as the image of a map H satisfying (2-5) and (2-6), then, for any $T_1 < T$, there exists some $\alpha > 0$ such that

$$H^T_{y_0} \eta\, H_{y_0} = -1 + |h_{y_0}|^2 \le -\alpha \quad \text{and} \quad \nabla H^T \nabla H \ge \alpha I_n \quad \text{for all } y^\tau \in (-T_1, T_1) \times \mathbb{T}^n. \tag{2-7}$$

2.3. ***Energy-momentum tensors.*** Among other results, we establish a relationship between the energy-momentum tensors for a codimension-k timelike Minkowski minimal surface in $\mathbb{R}^{1+N}$ and its counterpart for the semilinear wave equation (1-1) for a function $\mathbb{R}^{1+N} \to \mathbb{R}^k$ with $0 < \varepsilon \ll 1$. We recall the definitions:

If u solves (1-1), then $\mathcal{T}_\varepsilon(u)$ is defined to be the tensor whose components are

$$\mathcal{T}^\alpha_{\varepsilon,\beta}(u) := \delta^\alpha_\beta \Big(\frac{1}{2} \eta^{\gamma\delta} u_{x^\gamma} \cdot u_{x^\delta} + \frac{1}{\varepsilon^2} F(u) \Big) - \eta^{\alpha\gamma} u_{x^\gamma} \cdot u_{x^\beta}. \tag{2-8}$$

Here, $(\eta^{\alpha\beta}) = \mathrm{diag}(-1, 1, \dots, 1)$ as usual. (We deviate from convention in taking $\mathcal{T}_\varepsilon(u)$ and $\mathcal{T}(\Gamma)$ to be tensors of type $(1, 1)$ rather than of type $(0, 2)$; to recover the standard definition, one must lower an index.)

And, if Γ is a timelike minimal surface, we define $\mathcal{T}(\Gamma)$ to be the tensor whose components are the

[3] Assumption (2-6) does not entail any loss of generality. Indeed, for H of the form (2-5), we can always achieve (2-6) by replacing h by a function $\tilde{h}$ of the form $\tilde{h}(y_0, \dots, y_n) = h(y_0, \Psi(y_0, \dots, y_n))$ for a suitable $\Psi : (-T, T) \times \mathbb{T}^n \to (-T, T) \times \mathbb{T}^n$. The suitable Ψ can be found by making the ansatz $\tilde{H}(y) = (y_0, \tilde{h}(y))$ for $\tilde{h}$, and substituting into (2-6). This yields an ordinary differential equation for Ψ that we can supplement with the initial conditions $\Psi(0, y') = y'$ and then solve by appealing to standard theory.

signed measures

$$\mathcal{T}^\alpha_\beta(\Gamma)(A) := \int_A P^\alpha_\beta(t,x)\,d\lambda_\Gamma, \tag{2-9}$$

where λ_Γ denotes the Minkowski area density of Γ, and where $P(t,x) = (P^\alpha_\beta(t,x))$ is the matrix corresponding to the Minkowski orthogonal projection onto $T_{(t,x)}\Gamma$, for λ_Γ-a.e. $(t,x) \in \Gamma$. That is, if $H : U \subset \mathbb{R}^{1+n} \to \mathcal{U} \subset \mathbb{R}^{1+N}$ is a smooth injective map such that $\Gamma = H(U)$, then λ_Γ denotes the measure on $\mathcal{U}$ defined by

$$\int_{\mathbb{R}^{1+N}} f(x)\,d\lambda_\Gamma \;:=\; \int_U f(H(y^\tau))\sqrt{-\gamma(y^\tau)}\,dy^\tau.$$

where as before $\gamma = \det(DH^T\eta\,DH)$. (It is easy to check that λ_Γ depends only on Γ.) $P = P(t,x)$ is characterized by

$$P^\alpha_\beta v^\beta = v^\alpha \quad \text{for } v \in T_{(t,x)}\Gamma \quad \text{and} \quad P^\alpha_\beta w^\beta = 0 \quad \text{if } w^T\eta v = 0 \text{ for all } v \in T_{(t,x)}\Gamma.$$

For both models, the energy-momentum tensor may be obtained by considering variations of the relevant action functional with respect to suitable one-parameter families of diffeomorphisms. We recall this in some detail for $\mathcal{T}(\Gamma)$, as we will need to refer to this later:

Lemma 3. *Suppose that $H : U \subset \mathbb{R}^{1+n} \to \mathcal{U} \subset \mathbb{R}^{1+N}$ is a smooth injective map whose image $\Gamma := H(U)$ is a timelike surface. Given $\tau \in C^\infty_c(\mathcal{U};\mathbb{R}^{1+N})$, define $\Phi_\sigma(x) := x + \sigma\tau(x)$. We have*

$$\frac{d}{d\sigma}\mathcal{L}(\Phi_\sigma\circ H)\Big|_{\sigma=0} = \int_{\mathcal{U}} \tau^\beta_{x^\alpha}(x)\,P^\alpha_\beta\,d\lambda_\Gamma = \int_{\mathcal{U}} \tau^\beta_{x^\alpha}(x)\,d\mathcal{T}^\alpha_\beta(\Gamma). \tag{2-10}$$

Note that (2-10) exactly parallels the well-known first variation formula in the Euclidean case, in which λ_Γ is replaced by the restriction to Γ of the Hausdorff measure of the suitable dimension, and P^α_β is replaced by the orthogonal projection with respect to the Euclidean inner product.

Exactly parallel to (2-10), $\mathcal{T}_\varepsilon(u)$ arises from domain variations of the action functional, say $\mathcal{A}_\varepsilon$, whose Euler–Lagrange equation is (1-1); see for example [Shatah and Struwe 1998] for the proof. Thus, the results (1-15) and (1-21) assert that the first variation of $\mathcal{A}_\varepsilon$ (with respect to domain variations) at the critical point u is close (in a weak topology, and after suitable rescaling) to the first variation of $\mathcal{L}$ at the associated timelike minimal surface Γ.

We present the standard calculation that leads to (2-10), since we will need it later:

Proof of Lemma 3. We will write $H_\sigma := \Phi_\sigma\circ H$,

$$\begin{aligned}\gamma_{\sigma,ab} &= H^T_{\sigma,y^a}\eta H_{\sigma,y^b} = H^\alpha_{\sigma,y^a}\eta_{\alpha\beta}H^\beta_{\sigma,y^b},\\ (\gamma^{ab}_\sigma) &= (\gamma_{\sigma,ab})^{-1}, \quad \text{and}\\ \gamma_\sigma &= \det(\gamma_{\sigma,ab}),\end{aligned}$$

where the indices a, b run from 0 to n and α, β, as usual, run from 0 to N. Using the fact that

$$\frac{d}{d\sigma}\gamma_\sigma = \gamma_\sigma\gamma^{ab}_\sigma\frac{d}{d\sigma}\gamma_{\sigma,ab},$$

we find that

$$\frac{d}{d\sigma}\mathcal{L}(H_\sigma)\Big|_{\sigma=0} = \frac{d}{d\sigma}\int_U \sqrt{-\gamma_\sigma}\Big|_{\sigma=0} = \int_U (\tau^\beta \circ H)_{y^a}\,\eta_{\beta\delta}\,H^\delta_{y^b}\gamma^{ab}\sqrt{-\gamma}\,dy^\tau$$
$$= \int_U (\tau^\beta_{x^\alpha} \circ H)\,H^\alpha_{y^a}\,\eta_{\beta\delta}\,H^\delta_{y^b}\,\gamma^{ab}\sqrt{-\gamma}\,dy^\tau$$
$$= \int_{\mathcal{U}} \tau^\beta_{x^\alpha}(t,x)\,P^\alpha_\beta(t,x)\,d\lambda_\Gamma$$

where

$$P^\alpha_\beta(H(y^\tau)) := H^\alpha_{y^a}(y^\tau)\gamma^{ab}(y^\tau)\,H^\delta_{y^b}(y^\tau)\,\eta_{\delta\beta}.$$

Note that P^α_β is defined for λ_Γ-a.e. (t,x), so the above integral makes sense. In order to complete the proof, we must check that $P^\alpha_\beta(t,x)$ is the orthogonal projection onto $T_{(t,x)}\Gamma$. To see this, first note that, at any $y^\tau \in \mathbb{R}^{1+n}$,

$$(PH_{y^c})^\alpha = P^\alpha_\beta H^\beta_{y^c} = H^\alpha_{y^a}\,\gamma^{ab}H^\delta_{y^b}\,\eta_{\delta\beta}\,H^\beta_{y^c} = H^\alpha_{y^a}\,\gamma^{ab}\gamma_{bc} = H^\alpha_{y^a}\,\delta^a_c = H^\alpha_{y^c}.$$

Thus, $PH_{y^c} = H_{y^c}$. And, if v is orthogonal to H_{y^b} for all b, then

$$(Pv)^\alpha = P^\alpha_\beta v^\beta = H^\alpha_{y^a}\,\gamma^{ab}H^\delta_{y^b}\,\eta_{\delta\beta}\,v^\beta = 0,$$

since the orthogonality of v means exactly that $H^\delta_{y^b}\,\eta_{\delta\beta}\,v^\beta = 0$ for every b. Since $T_{(t,x)}\Gamma$ at $(t,x) = H(y^\tau)$ is spanned by $\{H_{y^b}(y^\tau)\}_{b=0}^n$, the above calculations state exactly that $P(t,x)$ is the matrix corresponding to orthogonal projection onto $T_{(t,x)}\Gamma$. □

2.4. ***Change of variables.*** We next define the change of variables that, as mentioned earlier, is the starting point of our argument. We will use the notation (2-2).

We assume, as always, that Γ is a smooth timelike minimal surface, given as the image[4] of a smooth injective map $H : (-T,T)\times\mathbb{T}^n \to \mathbb{R}^{1+N}$ satisfying (2-5) and (2-7). For this section, we allow $k = N-n$ to be an arbitrary positive integer, since all the proofs for $k=2$ apply without change to $k \geq 3$. (The case $k=1$ is simpler.) Although we do not use them in this paper, the results for $k \geq 3$ may be useful for problems such as the dynamics of defects in certain nonabelian gauge theories.

First, we fix smooth maps $\overline{\nu}_i : (-T,T)\times\mathbb{T}^n \to \mathbb{R}^{1+N}$ for $i = 1,\dots,k$, such that

$$\overline{\nu}_i^T\eta\,\overline{\nu}_j = \delta_{ij} \quad \text{and} \quad H^T_{y^\alpha}\eta\,\overline{\nu}_i = 0 \quad \text{in } (-T,T)\times\mathbb{T}^n \to \mathbb{R}^{1+N}, \tag{2-11}$$

for all $i, j \in \{1,\dots,k\}$ and $\alpha \in \{0,\dots,n\}$. (Here and throughout the paper, we are thinking of $\overline{\nu}_i$ as a column vector.) This states that $\{\overline{\nu}_1(y^\tau),\dots,\overline{\nu}_k(y^\tau)\}$ form an orthonormal basis for the normal space to Γ at $H(y^\tau)$, where words like "normal" and "orthonormal" are understood with respect to the Minkowski inner product, and y^τ denotes a generic point in $(-T,T)\times\mathbb{T}^n$. Note that, when $k=1$, (2-11) determines $\overline{\nu}_1$ up to a sign, whereas for $k \geq 2$ there are rotational degrees of freedom that we have not specified (and will not specify).

[4] All the results of this section are local, so the topology of Γ (that is, the fact that H is defined on $(-T,T)\times\mathbb{T}^n$) is irrelevant here. However, it is convenient to keep the same set-up as in the rest of the paper.

Next, we define, using the notation (2-2),

$$\psi(y) := H(y^\tau) + \sum_{i=1}^{k} \overline{\nu}_i(y^\tau)\, y^{n+i}. \tag{2-12}$$

It is clear that $\psi(y^\tau, 0) = H(y^\tau)$ for all $y^\tau \in (-T, T) \times \mathbb{T}^n$.

Recall that the statement of Theorems 1 and 2 involve a number $T_0 < T$. We henceforth fix $T_1 \in (T_0, T)$, and we let $\rho_0 > 0$ be so small that

$$\psi\big(\{-T_1\} \times \mathbb{T}^n \times B_\nu(\rho_0)\big) \Subset (-T, -T_0) \times \mathbb{R}^N, \quad \psi\big(\{T_1\} \times \mathbb{T}^n \times B_\nu(\rho_0)\big) \subset (T_0, T) \times \mathbb{R}^N, \tag{2-13}$$

and

$$\psi \text{ is injective, with smooth inverse } \varphi, \text{ on } (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0). \tag{2-14}$$

The latter condition can be satisfied due to the inverse function theorem; indeed, we will check below that $D\psi(y^\tau, 0)$ is invertible for $y^\tau \in (-T_1, T_1) \times \mathbb{T}^n$. We next define

$$(g_{\alpha\beta})_{\alpha,\beta=0}^N = G := D\psi^T \eta\, D\psi, \tag{2-15}$$

so that G represents the Minkowski metric in the y coordinates. We further define $g := \det G$ and $(g^{\alpha\beta})_{\alpha,\beta=0}^N := G^{-1}$, and we finally define $(a^{\alpha\beta})_{\alpha,\beta=0}^N$ by

$$a^{ij} = g^{ij} \text{ if } i, j \geq 1, \quad a^{00} = -g^{00}, \quad \text{and} \quad a^{i0} = a^{0j} = 0 \text{ for } i, j = 1, \dots, N. \tag{2-16}$$

When we write (1-1) in terms of the y coordinates as in (1-22), $(g^{\alpha\beta})$ and g appear in the coefficients and $(a^{\alpha\beta})$ appears in a natural associated energy density $e_\varepsilon(v) = e_\varepsilon(v; G)$, defined in (1-28). We summarize the properties of g and $(g^{\alpha\beta})$ that we will use:

Proposition 4. *Let ψ, g, $(g^{\alpha\beta})$ be the functions on $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ defined above. After taking ρ_0 smaller if necessary, there exist positive constants $c \leq C$ such that*

$$\|g^{\alpha\beta}\|_{W^{1,\infty}} \leq C, \qquad g^{\alpha\beta}_{y^0}\xi_\alpha\xi_\beta \;\leq\; C\big(|\xi_\tau|^2 + |y^\nu|^2\,|\xi_\nu|^2\big), \tag{2-17}$$

$$\frac{\partial_{y^\alpha}\sqrt{-g}}{\sqrt{-g}}\, g^{\alpha\beta}\xi_\beta\xi_0 \;\leq\; C\big(|\xi_\tau|^2 + |y^\nu|^2\,|\xi_\nu|^2\big), \tag{2-18}$$

$$|g^{\alpha\beta}\xi_\beta| \;\leq\; C\big(|\xi_\tau| + |y^\nu|\,|\xi_\nu|\big) \quad \textit{if } \alpha \leq n, \quad \textit{and} \tag{2-19}$$

$$c|\xi_\tau|^2 + \big(1 - C|y^\nu|^2\big)|\xi_\nu|^2 \;\leq\; a^{\alpha\beta}(y)\xi_\alpha\xi_\beta \;\leq\; C|\xi_\tau|^2 + \big(1 + C|y^\nu|^2\big)|\xi_\nu|^2 \tag{2-20}$$

for all $y = (y^\tau, y^\nu) \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ and $\xi = (\xi_\tau, \xi_\nu) \in \mathbb{R}^{1+N} \cong \mathbb{R}^{1+n} \times \mathbb{R}^k$. In addition,

$$\psi^0_{y^0} \geq c \quad \textit{in } (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0). \tag{2-21}$$

We emphasize that the main point in the proof of (2-18) is that $\nabla_\nu\sqrt{-g} = 0$ when $y^\nu = 0$. This is equivalent to Γ having zero mean curvature.

We will use the notation

$$\mathcal{N} := \psi\,\big((-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)\big) \cap [(-T_0, T_0) \times \mathbb{R}^N] \tag{2-22}$$

For future use, it is convenient to fix a constant $\kappa_2 \geq 1$ such that

$$\big(1+\kappa_2|y^\nu|^2\big)e_\varepsilon(v) \geq \frac{\lambda}{2}|D_\tau v|^2 + \big(1+|y^\nu|^2\big)e_{\varepsilon,\nu}(v) \tag{2-23}$$

everywhere in $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ and for all $v \in H^1$, where $\varepsilon_{\varepsilon,\nu}$ was defined in (1-24). This is possible due to (2-20).

When Γ is a hypersurface, we have a slightly better behavior:

Proposition 5. *Suppose that $k = 1$, and let ψ, g, and $(g^{\alpha\beta})$ be as defined above. After taking ρ_0 smaller if necessary*,

$$g^{\alpha N} = g^{N\alpha} = \begin{cases} 1 & \text{if } \alpha = N, \\ 0 & \text{otherwise}, \end{cases} \tag{2-24}$$

and, *in addition*, *there exist positive constants $\lambda < \Lambda$ such that*

$$\lambda|\xi_\tau|^2 + |\xi_\nu|^2 \;\leq\; a^{\alpha\beta}(y)\xi_\alpha\xi_\beta \;\leq\; \Lambda|\xi_\tau|^2 + |\xi_\nu|^2 \tag{2-25}$$

everywhere in $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$.

Conclusion (2-25) is not essential, but will allow us to simplify our notation, for example by taking $\kappa_2 = 1$ in (2-23) and everywhere else that this constant occurs (for $k = 1$).

We defer the proofs of Propositions 4 and 5 to an Appendix.

For a solution $u : \mathbb{R}^{1+N} \to \mathbb{R}^k$ of (1-1), we will define $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^k$ by $v = u \circ \psi$. Then v satisfies

$$\Box_G v + \frac{1}{\varepsilon^2} f(v) = 0 \tag{2-26}$$

on its domain. Here,

$$\Box_G v = -\frac{1}{\sqrt{-g}}\,\partial_{y^\alpha}\big(\sqrt{-g}\,g^{\alpha\beta}\partial_{y^\beta} v\big).$$

As noted earlier, we find it convenient to write (2-26) in the form (1-22). We now derive a key differential inequality for the energy density $e_\varepsilon(v)$ from (1-28):

Lemma 6. *If $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^k$ is a smooth solution of* (2-26), *with coefficients satisfying* (2-17), *then*

$$\frac{\partial}{\partial y^0} e_\varepsilon(v) \leq C\big(|D_\tau v|^2 + |y^\nu|^2|\nabla_\nu v|^2\big) + \nabla\cdot\varphi, \quad \textit{with} \tag{2-27}$$

$$\varphi := (\varphi^1, \dots, \varphi^N), \quad \varphi^i := g^{i\alpha} v_{y^\alpha}\cdot v_{y^0}. \tag{2-28}$$

Proof. Multiply (1-22) by v_{y^0}, and rewrite to find that

$$-\partial_{y^\alpha}(g^{\alpha\beta} v_{y^\beta}\cdot v_{y^0}) + g^{\alpha\beta} v_{y^\beta}\cdot v_{y^0 y^\alpha} + \frac{1}{\varepsilon^2}F(v)_{y^0} = -(b\cdot Dv)\cdot v_{y^0}.$$

We rewrite $g^{\alpha\beta} v_{y^\beta}\cdot v_{y^0 y^\alpha}$ as

$$\tfrac{1}{2}\partial_{y^0}(g^{\alpha\beta} v_{y^\beta}\cdot v_{y^\alpha}) - \tfrac{1}{2} g^{\alpha\beta}_{y^0} v_{y^\beta}\cdot v_{y^\alpha}.$$

Gathering all the terms of the form $\partial_{y^0}(\cdots)$ on the left-hand side, we find that

$$\partial_{y^0}\Big(-g^{0\beta} v_{y^\beta}\cdot v_{y^0} + \frac{1}{2} g^{\alpha\beta} v_{y^\alpha}\cdot v_{y^\beta} + \frac{1}{\varepsilon^2}F(v)\Big) = \partial_{y^i}(g^{i\beta} v_{y^\beta}\cdot v_{y^0}) - (b\cdot Dv)\,v_{y^0} + \tfrac{1}{2} g^{\alpha\beta}_{y^0} v_{y^\beta}\cdot v_{y^\alpha}.$$

The definition (2-16) of $a^{\alpha\beta}$ implies that the left-hand side is just $\partial_{y^0} e_\varepsilon(v)$. To complete the proof, we use (2-17) and (2-18) to check that the non-divergence terms on the left-hand side are bounded by $C(|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2)$. □

As an easy consequence of Proposition 5, we obtain a quite explicit description of the signed Minkowski distance function, defined by the eikonal equation (1-12) in the case $k = 1$.

Corollary 7. *If* $k = 1$, ψ *is defined as above, and* $\varphi = (\varphi^0, \dots, \varphi^N)$ *denotes the inverse of* ψ, *then* φ^N *solves the eikonal equation* (1-12) *on* $\mathrm{Image}(\psi)$.

In particular, Corollary 7 shows that it makes sense to speak of the signed-distance function in the set $\mathcal{N}$ defined in (2-22).

Proof. Fix a point in the image of ψ, say, $(t, x) = \psi(y)$. Then, since $\eta = \eta^{-1}$,

$$(g^{\alpha\beta})(y) = [D\psi^T(y)\,\eta\, D\psi(y)]^{-1} = (D\psi)^{-1}(y)\,\eta\,(D\psi)^{-T}(y) = D\varphi(t, x)\,\eta\, D\varphi^T(t, x).$$

Thus, according to (2-24),

$$1 = g^{NN}(y) = -(\varphi^N_t)^2 + |\nabla\varphi^N|^2,$$

so that (1-12) holds. And, it is clear that $\varphi^N(t, x) = 0$ for $(t, x) \in \Gamma$. □

In fact the curves $s \mapsto H(y^\tau) + s\nu(y^\tau) = \psi(y^\tau, s)$ are exactly the characteristic curves for the eikonal equation (1-12).

The eikonal equation (1-12) determines the distance function d only up to a sign; we will always choose to identify d with φ^N (so that our choice of a sign is ultimately determined by our choice of the sign for the unit normal ν.) Then, it follows that

$$d(\psi(y)) = y^N \quad \text{for } y \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0), \tag{2-29}$$

2.5. ***Initial data.*** In this section, we describe our general assumptions on the initial data.

We will eventually combine estimates for $v = u \circ \psi$ on $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ (which we use to control the behavior of u near Γ) with standard energy estimates for (1-1) away from Γ. We start by making a number of smallness assumptions, in all of which a parameter ζ_0 appears. We will prove below that one can find data for which $\zeta_0 \approx \varepsilon^2$ when $k = 1$, and $\zeta_0 \approx |\ln\varepsilon|^{-1}$ when $k = 2$. Although we omit the proof, it is in fact true that one cannot find data satisfying our assumptions with $\zeta_0 \ll \varepsilon^2$ (for $k = 1$) or $\zeta_0 \ll |\ln\varepsilon|^{-1}$. We, therefore, will assume that

$$\zeta_0 \ge \varepsilon^2 \quad \text{if } k = 1, \qquad \zeta_0 \ge |\ln\varepsilon|^{-1} \quad \text{if } k = 2. \tag{2-30}$$

This is convenient, as it will enable us to absorb small error terms into expressions of the form $C\zeta_0$.

Our first assumption is that the energy is small away from Γ_0:

$$\delta_\varepsilon \int_{\{x\in\mathbb{R}^N : (0,x)\notin \mathrm{image}(\psi)\}} e_\varepsilon(u)\,dx\Big|_{t=0} \le \zeta_0 \tag{2-31}$$

where $e_\varepsilon(u) = e_\varepsilon(u; \eta)$ is defined in (1-10), and $\delta_\varepsilon = \delta_\varepsilon(k)$ is defined in (2-1).

Near Γ_0 it is convenient to state our assumptions in terms of $v = u \circ \psi$. Note that the initial data for u at $t = 0$ corresponds to data for v on a hypersurface that does *not*, in general, have the form $\{y_0 = \text{constant}\}$. This hypersurface is described next:

Lemma 8. *There exists a Lipschitz function* $b : \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}$ *such that, for arbitrary* $y = (y_0, y')$ *in* $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$,

$$\psi(y_0, y') \in \{0\} \times \mathbb{R}^N \quad \textit{if and only if} \quad y_0 = b(y'). \tag{2-32}$$

Moreover, $\|\nabla b\|_\infty \le C$.

Proof. Fix $y' \in \mathbb{T}^n \times B_\nu(\rho_0)$ and, for $s \in (-T_1, T_1)$, let $y(s) := (s, y')$ and let $X(s) = \psi(y(s)) \in \mathbb{R}^{1+N}$. To prove that $\psi^{-1}(\{0\} \times \mathbb{R}^N)$ is the graph of a function, we need to show that $y(s)$ intersects $\psi^{-1}(\{0\} \times \mathbb{R}^N)$ exactly once or, equivalently, that $X(s)$ intersects $\{0\} \times \mathbb{R}^N$ for exactly one value of s. To prove this, note that the definition of G and (2-7) imply that, after taking ρ_0 smaller if necessary,

$$X'(s)^T \eta X'(s) = y'(s)^T G(y(s)) y'(s) = g_{00}(y(s)) < 0$$

for every s. Thus, $s \mapsto X(s) = (X^0(s), X'(s))$ is a timelike curve, from which the claim is obvious. It follows that there exists a function b satisfying (2-32). Then, by differentiating the identity $\psi^0(b(y'), y') = 0$, we find that $\psi^0_{y^0}(b(y'), y')\, \nabla b(y') + \nabla \psi^0(b(y'), y') = 0$. We know from (2-21) that $\psi^0_{y^0}$ is bounded away from 0, and this, together with the smoothness of ψ^0, implies that $\|\nabla b\|_\infty \le C$. □

Using the lemma, we define

$$v_0(y') := v(b(y'), y') \quad \text{for } y' \in \mathbb{T}^n \times B_\nu(\rho_0). \tag{2-33}$$

Our next assumptions specify that the energy near Γ_0 is small, in the frame that moves with Γ:

$$\delta_\varepsilon \int_{\mathbb{T}^n \times B_\nu(\rho_0)} \big(1 + \kappa_2 |y^\nu|^2\big) e_\varepsilon(v_0; G)\, dy' - 1 \le \zeta_0, \tag{2-34}$$

$$\delta_\varepsilon \int_{\mathbb{T}^n \times B_\nu(\rho_0)} \big(|v_{y^0}|^2 + |v_{y^0}|\, |\nabla_\nu v_0|\big)\, (b(y'), y')\, dy' \le \zeta_0. \tag{2-35}$$

Finally, using notation discussed in the Introduction and defined in (3-1) for $k = 1$, and in (5-1) and (5-3) for $k = 2$, we require that

$$\mathfrak{D}(v_0; \rho_0) \le \zeta_0. \tag{2-36}$$

This specifies that the initial profile possesses a defect — that is, an interface or vortex — near Γ_0.

Note that conditions (2-31) and (2-34)–(2-36) are always satisfied if we define ζ_0 to be the maximum of the left-hand sides of these inequalities. The smallest possible values of ζ_0 depend on k and, as mentioned earlier, account for the fact that our conclusions for $k = 1$ are stronger than for $k = 2$.

Lemma 9. *In the scalar case* ($k = 1$), *there exist initial data* $(u, u_t)|_{t=0} \in \dot{H}^1 \times L^2(\mathbb{R}^N)$ *for* (1-1), *satisfying conditions* (2-31)–(2-36) *with* $\zeta_0 = C\varepsilon^2$, *and such that*

$$\int_{\mathcal{N}_0} \Big(u(0, x) - q\Big(\frac{d(0, x)}{\varepsilon}\Big)\Big)^2 \le C\varepsilon, \quad \textit{where } \mathcal{N}_0 = \{x \in \mathbb{R}^N : (0, x) \in \mathcal{N}\}. \tag{2-37}$$

In the vector ($k = 2$) *case, there exist initial data* $(u, u_t)|_{t=0} \in \dot{H}^1 \times L^2(\mathbb{R}^N; \mathbb{R}^2)$ *for* (1-1), *satisfying conditions* (2-31)–(2-36) *with* $\zeta_0 = C|\ln \varepsilon|^{-1}$.

Although we do not prove it, these scalings for ζ_0 are, in fact, optimal.

Proof. In both cases, $k=1$ and $k=2$, we define a function U in Image(ψ) such that

$$U \circ \psi = \tilde{q}\Big(\frac{y^\nu}{\varepsilon}\Big), \tag{2-38}$$

where $\tilde{q} : \mathbb{R}^k \to \mathbb{R}^k$ is a nearly optimal profile. We then require that

$$u(0,x) = U(0,x) \quad \text{and} \quad u_t(0,x) = U_t(0,x) \quad \text{in } \mathcal{N}_0, \tag{2-39}$$

and we verify (2-34)–(2-36). (Note that (2-29) then implies that $u(x,0) = \tilde{q}(d/\varepsilon)$ when $k=1$, which will make (2-37) obvious.) Finally, we argue that $u(0,\cdot)$ can be extended to $\mathbb{R}^N \setminus \mathcal{N}_0$ such that (2-31) holds.

<u>Case $k=1$</u>: By integrating the equation (1-5) solved by q, and using the boundary conditions at $\pm\infty$, one finds that $q' = \sqrt{2F(q)}$ and, hence, that

$$\int_{\mathbb{R}} \tfrac{1}{2} q'^2 + F(q)\,dx = \int_{\mathbb{R}} \sqrt{2F(q)}\, q'(s)\,ds = \int_{-1}^{1} \sqrt{2F(s)}\,ds = \kappa_1. \tag{2-40}$$

Using (1-5) and (1-9), standard ODE arguments show that, for suitable constants,

$$|q'(s)| + |q(s) - \operatorname{sign}(s)| \le Ce^{-c|s|} \quad \text{for all } s.$$

It follows that, given $\varepsilon > 0$, we can find a function $\tilde{q}$ such that $\tilde{q}(s/\varepsilon) = q(s/\varepsilon)$ if $|s| < \frac{1}{2}\rho_0$, and

$$\tilde{q}\Big(\frac{s}{\varepsilon}\Big) = q\Big(\frac{s}{\varepsilon}\Big) \quad \text{if } |s| < \tfrac{1}{3}\rho_0, \qquad \tilde{q}\Big(\frac{s}{\varepsilon}\Big) = \operatorname{sign}(s) \quad \text{if } |s| > \tfrac{2}{3}\rho_0, \qquad \|\tilde{q} - q\|_{W^{1,\infty}} \le Ce^{-c/\varepsilon},$$

and

$$\kappa_1 < \int_{-\rho_0/\varepsilon}^{\rho_0/\varepsilon} t\frac{1}{2}\tilde{q}'^2 + F(\tilde{q})\,dx \;\le\; \kappa_1 + Ce^{-c/\varepsilon}. \tag{2-41}$$

Now, define U as in (2-38) and define $u|_{t=0}$ near Γ_0 by (2-39). Then, by construction, v_0 as defined in (2-33) is given by $v_0(y) = \tilde{q}(y^N/\varepsilon)$, and $v_{y_0} = 0$. The latter fact immediately implies that (2-35) holds, and (2-31) and (2-35) are easily verified. For example, the explicit form of v_0 and (2-25) imply that $e_\varepsilon(v_0; G) = \frac{1}{2}\tilde{q}'^2(y^\nu/\varepsilon) + \varepsilon^{-2}F(\tilde{q}(y^\nu/\varepsilon))$. Then, recalling that $\delta_\varepsilon = \varepsilon/\kappa_1$, we infer from (2-41) and the change of variables $y^N/\varepsilon \mapsto y^N$ that

$$\delta_\varepsilon \int_{\mathbb{T}^n \times B_\nu(\rho_0)} \big(1 + \kappa_2 |y^\nu|^2\big) e_\varepsilon(v_0; G)\,dy' - 1 \;\le\; C\varepsilon^2 \int_{-\rho_0/\varepsilon}^{\rho_0/\varepsilon} \Big(\frac{\tilde{q}'^2}{2} + F(\tilde{q})\Big)(y^N)^2\,dy^N + Ce^{-c/\varepsilon}.$$

The exponential decay of q implies that $\int_R \big(\frac{1}{2}\tilde{q}'^2 + F(\tilde{q})\big)(y^N)^2\,dy^N \le C$ independently of ε, and (2-34) follows, with $\zeta_0 = C\varepsilon^2$. The verifications of (2-35) and (2-36) are similar and a little easier.

Finally, on $\mathbb{R}^N \setminus \mathcal{N}_0$, we set $u_t(0,\cdot) \equiv 0$, and we require that $u(0,\cdot) = \pm 1$ and that u be continuous (hence, smooth) across $\partial\mathcal{N}_0$. This can be done, since $\mathbb{R}^N \setminus \Gamma_0$ consists of two components, one of which meets $\mathcal{N}_0$ where $d = \rho_0$ (and, hence, $u = 1$), and the other where $d = -\rho_0$. (Here, we have used the fact that ρ_0 is sufficiently small; see (2-13).)

Case $k=2$: In this case, we may define $\tilde q(s) = s\min\{1, 1/|s|\}$ for $s \in \mathbb{R}^2$, and go on to make the definitions (2-38) and (2-39) as above, so that $v_0(y) = \tilde q(y^\nu/\varepsilon)$. Then, an easy calculation shows that

$$\frac{1}{\pi|\ln\varepsilon|}\int_{B_\nu(\rho_0/\varepsilon)} \frac{1}{2}|\nabla\tilde q|^2 + F(\tilde q)\,ds \le 1 + C|\ln\varepsilon|^{-1}.$$

This plays a role analogous to (2-41) above, and allows us to verify along the previous lines (but using (2-20) in place of (2-25)) that (2-34) holds with $\zeta_0 = C|\ln\varepsilon|^{-1}$. As before, (2-35) follows from the fact that $v_{y^0}(b(y'), y') = 0$ in $\mathbb{T}^n \times B_\nu(\rho_0)$. One can check (2-36) directly from the definitions (see Section 5), noting that

$$J_\nu v_0(y^\tau, y^\nu) = \begin{cases} \varepsilon^{-2} & \text{if } |y^\nu| < \varepsilon, \\ 0 & \text{if } |y^\nu| > \varepsilon. \end{cases}$$

It remains to show that $u_0 = U(0,\cdot)$, as defined in $\mathcal{N}_0$ by (2-39), can be extended to a function in $H^1(\mathbb{R}^N)$ satisfying (2-31). It is clear that we can extend u_0 by a finite-energy map in a neighborhood $\mathcal{V}$ of $\mathcal{N}_0$. Next, we point out that, since Γ_0 is a smooth, compact, oriented codimension-2 submanifold without boundary of $\mathbb{R}^N$, results in [Alberti et al. 2003] imply that we may find a function $w \in H^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \Gamma; \mathbb{C})$ with $\int_{\mathbb{R}^N\setminus\mathcal{N}_0} |\nabla w|^2 < \infty$, such that $|w| = 1$ a.e. and, in addition, such that $Jw = J(u_0/|u_0|)$ in Γ_0, where $J(\cdots)$ denotes the distributional Jacobian of $(\cdots)$. This implies that there exists a real-valued function $\theta \in H^1_{\mathrm{loc}}(\mathcal{V}\setminus\Gamma_0; \mathbb{R})$ such that $u_0 = |u_0|\,w\,e^{i\theta}$ in $\mathcal{V}$. Thus, we define $u(0,\cdot)$ globally in $\mathbb{R}^N$ by setting

$$u(0,\cdot) = \begin{cases} |u_0|\,w\,e^{i\chi\theta} & \text{in } \mathcal{V}, \\ w & \text{in } \mathbb{R}^N\setminus\mathcal{V}, \end{cases}$$

where $\chi \in C^\infty_c(\mathcal{V})$ and $\chi \equiv 1$ in $\mathcal{N}_0$; we may set $u_t(0,x) = 0$ outside of $\mathcal{N}_0$. □

3. Basic energy estimates, $k = 1$

The main result of this section — Proposition 10 below — contains the simplest case of our main estimate.

In this section and the next, we restrict our attention to the case $k = 1$, so that[5] $N = n+1$, $y^\nu = y^N \in \mathbb{R}$, and $\nabla_\nu = \partial_N$. Thus, in this section, $B_\nu(\rho)$ denotes the interval $(-\rho, \rho)$ along the y^N axis. We also follow other conventions for $k = 1$, so that, for example, $\delta_\varepsilon = \varepsilon/\kappa_1$; see (2-1).

Throughout this section, we let ψ denote the change of variables from Section 2.4, in the case $k = 1$. We also use the notation g, $g_{\alpha\beta}$, $g^{\alpha\beta}$, etc. from the previous section.

In the Introduction we discussed a "defect confinement" functional $\mathcal{D}$. In the case $k = 1$, we define it to be

$$\mathcal{D}(v;\rho) := \int_{\mathbb{T}^n\times B_\nu(\rho)} |y^\nu|\,|v - \mathrm{sign}(y^\nu)|^2\,dy' \tag{3-1}$$

for $v : \mathbb{T}^n \times B_\nu(\rho) \to \mathbb{R}$. We will also write

$$\mathcal{D}(v;\rho) = \int_{\mathbb{T}^n} \mathcal{D}_\nu(v(y^{\tau\prime});\rho)\,dy^{\tau\prime}. \tag{3-2}$$

[5]Although here there is not much point in writing y^ν and ∇_ν instead of y^N and ∂_N, this notation will prove useful when we consider the vector case, and we use it here to emphasize the parallels.

where $v(y^{\tau\prime})(y^\nu) = v(y^{\tau\prime}, y^\nu)$, and

$$\mathcal{D}_\nu(w;\rho) := \int_{B_\nu(\rho)} |y^\nu|\,|w - \operatorname{sign}(y^\nu)|^2\,dy^\nu \quad \text{for } w : B_\nu(\rho) \to \mathbb{R}. \tag{3-3}$$

Let c_* be a constant such that

$$|g^{N\alpha}(y)\xi_\alpha\xi_0| = |a^{N\alpha}(y)\xi_\alpha\xi_0| \le \tfrac{1}{2}c_*\, a^{\alpha\beta}\xi_\alpha\xi_\beta \tag{3-4}$$

for all $\xi \in \mathbb{R}^{1+N}$ and $y \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$.

Proposition 10. *Let $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}$ satisfy* (2-26), *where $f = F'$ and F satisfies* (1-9). *Recalling that $\delta_\varepsilon = \varepsilon/\kappa_1$, where κ_1 is defined in* (1-11), *assume that there exist some $s_1 \in (-T_1, T_1)$, $\rho_1 \in (0, \rho_0)$, and $\zeta_0 \ge \varepsilon^2$ such that*

$$\delta_\varepsilon \int_{\{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1)} \big(1 + (y^\nu)^2\big)\, e_\varepsilon(v)\,dy' - 1 \le \zeta_0, \tag{3-5}$$

$$\textit{and} \quad \mathcal{D}(v(s_1), \rho_1/2) \le \zeta_0. \tag{3-6}$$

There exists a constant C, independent of v and of $\varepsilon \in (0,1]$, such that

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1 - c_* s)} |D_\tau v|^2 + (y^\nu)^2\Big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\Big)\,dy' \le C\zeta_0,$$

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1 - c_* s)} \big(1 + (y^\nu)^2\big)\, e_\varepsilon(v)\,dy' - 1 \le C\zeta_0,$$

$$\textit{and} \quad \mathcal{D}(s_1 + s; \rho_1/2) \le C\zeta_0,$$

for all $s \in [0, \rho_1/2c_*]$ such that $s_1 + s < T_1$.

Our first lemma will be needed to establish requirement (1-30), as discussed in the Introduction. In the statement and proof, we take all the y^τ variables to be frozen and consider a function v of a single real variable y^ν.

Lemma 11. *Let $B_\nu(\rho) := (-\rho, \rho) \subset \mathbb{R}_\nu$ be an interval as above. There exists a constant $\kappa_3 = \kappa_3(\rho)$ such that, if $v \in H^1(B_\nu(\rho))$ and*

$$\mathcal{D}_\nu(v;\rho) \le \kappa_3, \tag{3-7}$$

then

$$\delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)\,dy^\nu \ge 1 - Ce^{-C/\varepsilon}. \tag{3-8}$$

Moreover, there exists a constant $\kappa_4 = \kappa_4(\rho)$ such that, if (3-7) *holds and*

$$\delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)\,dy^\nu \le 1 + \zeta_0 \quad \textit{for some } \zeta_0 \in (0, \kappa_4), \tag{3-9}$$

then

$$\int_{B_\nu(\rho)} \Big|\tfrac{1}{2}\varepsilon v_{y^\nu}^2 - \frac{1}{\varepsilon}F(v)\Big|\,dy^\nu \le C\big(\sqrt{\zeta_0} + e^{-c/\varepsilon}\big). \tag{3-10}$$

The proof of Proposition 10 uses only the first conclusion (3-8) of this lemma. The other conclusion (3-10) is used in the proof of Theorem 22, when we deduce control over the full energy-momentum tensor from simpler energy estimates, like those of Proposition 10.

Proof of Lemma 11. Step 1: Note that (3-8) is obvious if (3-9) fails, so it suffices to show that, if (3-7) and (3-9) hold, then both conclusions, (3-8) and (3-10), follow.

First, we define $Q(s) := \int_0^s \sqrt{2F(\sigma)}\,d\sigma$ and, for any function $w \in H^1(B_\nu(\rho))$, we estimate

$$\varepsilon e_{\varepsilon,\nu}(w) = \tfrac{1}{2}\varepsilon w_{y^\nu}^2 + \frac{1}{\varepsilon}F(w) \geq \sqrt{2F(w)}\,|w_{y^\nu}| = |\partial_{y^\nu}(Q \circ w)|.$$

Thus, since $\delta_\varepsilon = \varepsilon/\kappa_1$,

$$\delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(w) \;\geq \frac{1}{\kappa_1}\int_{B_\nu(\rho)} |\partial_{y^\nu}(Q \circ w)| \tag{3-11}$$

and, for any w, to obtain lower bounds for the left-hand side, it suffices to show that $y^\nu \mapsto Q(w(y^\nu))$ has large total variation on $B_\nu(\rho) = (-\rho, \rho)$.

Step 2: Next, fix $\alpha > 0$ so that $F' = f$ is decreasing on $(1-\alpha, 1)$; this is possible as F is C^2 and attains its minimum at 1 with $F''(1) > 0$.

Let $v^+ := \sup_{y^\nu \in (\rho/4,\, 3\rho/4)} v(y^\nu)$. If $v^+ \leq 1$, then (3-7) implies that

$$\kappa_3 \geq \int_{\rho/4}^{3\rho/4} y^\nu |1 - v(y^\nu)|^2 dy^\nu \geq C\rho^2(1 - v^+)^2.$$

Thus, by choosing κ_3 small enough, we can arrange that $v^+ \geq 1 - \theta\alpha$ for some $\theta \in (0, 1/2)$, to be chosen below. It then follows by the same argument that $v^- := \inf_{y^\nu \in (-3\rho/4,\, -\rho/4)} v(y^\nu) \leq -1 + \theta\alpha$.

Step 3: We next claim that, once κ_3 and κ_4 are fixed in a suitable way, our hypotheses imply that

$$v \geq 1 - \alpha \;\text{ in } (3\rho/4, \rho) \quad \text{and} \quad v \leq -1 + \alpha \;\text{ in } (-\rho, -3\rho/4). \tag{3-12}$$

This follows from (3-11) and Step 2 — the latter implies lower bounds on the total variation of $Q \circ w$ if (3-12) fails, and these lower bounds can be made to contradict (3-11) and (3-9).

In more detail, let us suppose (toward a contradiction) that the first inequality in (3-12) fails. Then, using Step 2, there exist points $y^{\nu,1} < y^{\nu,2} < y^{\nu,3}$ such that $v(y^{\nu,1}) < -1 + \theta\alpha$, $v(y^{\nu,2}) > 1 - \theta\alpha$, and $v(y^{\nu,3}) < 1 - \alpha$. Hence, using the fact that Q is nondecreasing (as the antiderivative of the positive function $\sqrt{2F}$), we have (the first inequality following from (3-9) and (3-11))

$$\begin{aligned}
\kappa_1(1+\kappa_4) \geq \int_{B_\nu(\rho)} |\partial_{y^\nu}(Q \circ v)| &\geq \left|\int_{y^{\nu,1}}^{y^{\nu,2}} \partial_{y^\nu}(Q \circ v)\,dy^\nu\right| + \left|\int_{y^{\nu,2}}^{y^{\nu,3}} \partial_{y^\nu}(Q \circ v)\,dy^\nu\right| \\
&\geq \big|Q(1-\theta\alpha) - Q(-1+\theta\alpha)\big| + \big|Q(1-\alpha) - Q(1-\theta\alpha)\big| \\
&\geq \big|Q(1-\theta\alpha) - Q(-1+\theta\alpha)\big| + 2\kappa_1\kappa_4,
\end{aligned}$$

where for the last step we chose $\kappa_4 := (2\kappa_1)^{-1}\,\big|Q(1-\alpha) - Q(1-\alpha/2)\big|$ (recall that $\theta \leq \frac{1}{2}$). This inequality is false when $\theta = 0$, since $\kappa_1 = Q(1) - Q(-1)$, and so it also fails for sufficiently small $\theta \in (0, \frac{1}{2})$. Hence, we can choose κ_3 small enough to obtain a contradiction.

Step 4: We now replace v on the interval $(3\rho/4, \rho)$ by the minimizer of the functional

$$w \mapsto \int_{3\rho/4}^{\rho} e_{\varepsilon,\nu}(w)\,dy^\nu$$

subject to the boundary conditions $w(3\rho/4) = v(3\rho/4)$ and $w(\rho) = v(\rho)$. Let v_1 denote the function obtained in this way. Standard maximum principle arguments[6] imply that $v_1(\frac{7}{8}\rho) \geq 1 - Ce^{-c/\varepsilon}$. In a similar way, we can modify v_1 on $(-\rho, -3\rho/4)$ to obtain a function v_2 with less energy than that of v_1, and such that $v_2(-\rho) = v(-\rho)$, and $v_2(-7\rho/8) \leq -1 + Ce^{-c/\varepsilon}$.

Thus, $\big|Q(v_2(\frac{7}{8}\rho)) - Q(1)\big| \leq Ce^{-c/\varepsilon}$ and similarly $\big|Q(v_2(-\frac{7}{8}\rho)) - Q(-1)\big| \leq Ce^{-c/\varepsilon}$. As a result, using (3-9) and (3-11) as in Step 3, and recalling that $\kappa_1 = Q(1) - Q(-1)$, we obtain

$$\begin{aligned}\kappa_1(1+\zeta_0) &\geq \int_{B_\nu(\rho)} \varepsilon\, e_{\varepsilon,\nu}(v)\, dy^\nu \geq \int_{B_\nu(\rho)} \varepsilon\, e_{\varepsilon,\nu}(v_2)\, dy^\nu \\ &\geq \big|Q(v_2(-\rho)) - Q(v_2(-\tfrac{7}{8}\rho))\big| + \big|Q(v_2(-\tfrac{7}{8}\rho)) - Q(v_2(\tfrac{7}{8}\rho))\big| + \big|Q(v_2(\tfrac{7}{8}\rho)) - Q(v_2(\rho))\big| \\ &\geq \big|Q(v_2(-\rho)) - Q(-1)\big| + \kappa_1 + \big|Q(1) - Q(v_2(\rho))\big| - Ce^{-c/\varepsilon}.\end{aligned}$$

This implies (3-8). Also, since $v_2 = v$ at $\pm\rho$, the above implies that

$$\begin{aligned}Q(v(\rho)) - Q(v(-\rho)) &= \kappa_1 + Q(v(\rho)) - Q(1) - \big(Q(v(-\rho)) - Q(-1)\big) \\ &\geq \kappa_1 - \big|Q(v(\rho)) - Q(1)\big| - \big(Q(v(-\rho)) - Q(-1)\big) \\ &\geq \kappa_1(1-\zeta_0) - Ce^{-c/\varepsilon}. \qquad (3\text{-}13)\end{aligned}$$

<u>Step 5</u> We now use (3-13) to prove (3-10). First note that

$$\begin{aligned}\int_{B_\nu(\rho)} \Big|\tfrac{1}{2}\varepsilon v_{y^\nu}^2 - \frac{1}{\varepsilon}F(v)\Big|\, dy^\nu &\leq \int_{B_\nu(\rho)} \Big|\sqrt{\varepsilon} v_{y^\nu} - \frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\Big|\, \Big|\sqrt{\varepsilon} v_{y^\nu} + \frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\Big|\, dy^\nu \\ &\leq C\bigg(\int_{B_\nu(\rho)} \Big|\sqrt{\varepsilon}\, v_{y^\nu} - \frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\Big|^2 dy^\nu\bigg)^{1/2} \bigg(\int_{B_\nu(\rho)} \varepsilon\, e_{\varepsilon,\nu}(v)\, dy^\nu\bigg)^{1/2}.\end{aligned}$$

Expanding the square and recalling that $\sqrt{2F} = Q'$, we see that

$$\begin{aligned}\int_{B_\nu(\rho)} \frac{1}{2}\Big|\sqrt{\varepsilon} v_{y^\nu} - \frac{\sqrt{2F(v)}}{\sqrt{\varepsilon}}\Big|^2 dy^\nu &= \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)\, dy^\nu - \int_{B_\nu(\rho)} Q'(v) v_{y^\nu}\, dy^\nu \\ &= \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)\, dy^\nu - \big(Q(v(\rho)) - Q(v(-\rho))\big) \\ &\leq \kappa_1(1+\zeta_0) - \big(\kappa_1(1-\zeta_0) - Ce^{-c/\varepsilon}\big) \quad \text{(using (3-9) and (3-13))} \\ &\leq C\zeta_0 + Ce^{-c/\varepsilon}.\end{aligned}$$

Combining these inequalities and again appealing to (3-9), we arrive at (3-10). □

The next lemma is used to establish requirement (1-32), as discussed in the Introduction. In this lemma we write v as a function of two variables, y^0 and y^ν.

[6] The point is that one can easily check that

$$w(y^\nu) := 1 - \alpha \frac{\cosh(b(y^\nu - 7\rho/8)/\varepsilon)}{\cosh(b\rho/8\varepsilon)}$$

satisfies $-w'' + \varepsilon^{-2} f(w) \leq 0$ in $(3\rho/4, \rho)$, if b is fixed small enough (depending on F). Then, in view of (3-12) and the fact that f is decreasing on $(1-\alpha, 1)$, one can use the maximum principle to find that $v_1 > w$ in $(3\rho/4, \rho)$.

Lemma 12. *Let $B_\nu(\rho) \subset \mathbb{R}$ be an interval as above, and let $v \in H^1((0,\tau)\times B_\nu(\rho))$ for some $\tau > 0$. There exists a constant C, depending on ρ but independent of τ and $\varepsilon \in (0,1]$, such that*

$$\int_{B_\nu(\rho)} |y^\nu|\,|v(0,y^\nu) - v(\tau,y^\nu)|^2\,dy^\nu \le C\int_{(0,\tau)\times B_\nu(\rho)} \tfrac{1}{2}\varepsilon v_{y^0}^2 + \frac{(y^\nu)^2}{\varepsilon}F(v)\,dy^\nu\,dy^0.$$

Proof. For $Q:\mathbb{R}\to\mathbb{R}$ as above, such that $Q'(s) = \sqrt{2F(s)}$,

$$\tfrac{1}{2}\varepsilon v_{y^0}^2 + \frac{(y^\nu)^2}{\varepsilon}F(v) \;\ge\; |y^\nu|\sqrt{2F(v)}\,|v_{y^0}| \;=\; |y^\nu|\,|Q(v)_{y^0}|.$$

By integrating this inequality, we find that

$$\begin{aligned}\int_{(0,\tau)\times B_\nu(\rho)} \tfrac{1}{2}\varepsilon v_{y^0}^2 + \frac{y^2}{\varepsilon}F(v)\,dy^\nu\,dy^0 &\ge \int_{B_\nu(\rho)}\int_0^\tau |y^\nu|\,|Q(v)_{y^0}|\,dy^0\,dy^\nu\\ &\ge \int_{B_\nu(\rho)} |y^\nu|\,\big|Q(v(\tau,y^\nu)) - Q(v(0,y^\nu))\big|\,dy^\nu.\end{aligned}$$

Finally, our assumption (1-9) that $F(s) \ge (1-|s|)^2$ and elementary calculus imply that

$$|Q(b) - Q(a)| \ge c(b-a)^2,$$

and the lemma follows. □

Proof of Proposition 10. Since the equation is well posed in $H^1\times L^2$, and since all the quantities in the statement are continuous in $H^1\times L^2$, we may prove the proposition for v smooth.

In the proof we will write simply $\mathcal{D}(\,\cdot\,)$ instead of $\mathcal{D}(\,\cdot\,;\rho_1/2)$.

<u>Step 1</u>: We may assume that $s_1 = 0$. We will use the notation $s_{\max} := \min\{\rho_1/2c_*, T_1\}$ and

$$W_\nu(s) := B_\nu(\rho_1 - c_* s), \qquad W(s) := \mathbb{T}^n \times W_\nu(s).$$

We define

$$\begin{aligned}\zeta_1(s) &:= \delta_\varepsilon \int_{\{s\}\times W(s)} (1+(y^\nu)^2)\,e_\varepsilon(v)\,dy' - 1,\\ \zeta_2(s) &:= \mathcal{D}(v(s)),\\ \zeta_3(s) &:= \delta_\varepsilon \int_{\{s\}\times W(s)} |D_\tau v|^2 + (y^\nu)^2\left(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\right)dy'.\end{aligned}$$

We first claim that

$$\zeta_1(s) \le C\zeta_0 + C\int_0^s \zeta_3(\sigma)\,d\sigma \quad \text{for } s\in(0,s_{\max}]. \tag{3-14}$$

Towards this end, we compute

$$\zeta_1'(s) = I_1 - c_* I_2, \quad \text{where } \begin{aligned}[t] I_1 &:= \delta_\varepsilon \int_{\{s\}\times W(s)} \big(1+(y^\nu)^2\big)\,\frac{\partial}{\partial y^0} e_\varepsilon(v)\,dy',\\ I_2 &:= \delta_\varepsilon \int_{\{s\}\times\mathbb{T}^n\times\partial W_\nu(s)} \big(1+(y^\nu)^2\big)\,e_\varepsilon(v)\,dy^{\tau\prime}.\end{aligned}$$

To estimate I_1, we use Lemma 6 and integrate by parts in the spatial variables. From (2-28), we easily see that $|y^\nu|\,|\varphi^\nu| \le C\big(|D_\tau v|^2 + (y^\nu)^2|\nabla_\nu v|^2\big)$. Thus, we arrive at

$$I_1 \le C\delta_\varepsilon \int_{\{s\}\times W(s)} \big(|D_\tau v|^2 + (y^\nu)^2|\nabla_\nu v|^2\big)\, dy' + \delta_\varepsilon \int_{\{s\}\times \mathbb{T}^n \times \partial W_\nu(s)} \big(1+(y^\nu)^2\big)\, |\varphi^N|\, dy^{\tau\prime}.$$

Our choice (3-4) of c_* exactly guarantees that $|\varphi_N| \le c_* e_\varepsilon(v)$, so that the boundary term above is dominated by $-c_* I_2$. It follows that $\zeta_1' \le C\zeta_3$. Since it is clear from (3-5) that $\zeta_1(0) \le \zeta_0$, we conclude that (3-14) holds.

Step 2: Next, we estimate ζ_2. Using the hypotheses and Lemma 12, we find that

$$\begin{aligned} \zeta_2(s) &\le 2\mathfrak{D}(v(0)) + 2\int_{\mathbb{T}^n\times B_\nu(\rho_1/2)} |y^\nu|\, |v(s,y') - v(0,y')|^2\, dy' \\ &\le 2\zeta_0 + C\int_{\mathbb{T}^n} \Big(\int_0^s \int_{B_\nu(\rho_1/2)} \tfrac12 \varepsilon\, |v_{y^0}|^2 + \frac{(y^\nu)^2}{\varepsilon} F(v)\, dy^\nu\, dy^0 \Big)\, dy^{\tau\prime} \\ &\le 2\zeta_0 + C\int_0^s \zeta_3(\sigma)\, d\sigma \end{aligned} \tag{3-15}$$

for $s \le s_{\max}$. We have changed the order of integration and used the fact that

$$\mathbb{T}^n \times B_\nu(\rho_1/2) = W_\nu(\rho_1/2c_*) \subset W_\nu(s) \quad \text{for } s \le s_{\max} \le \rho_1/2c_*.$$

Step 3: Finally, we claim that

$$\zeta_3(s) \le C\big(\zeta_1(s) + \zeta_2(s) + e^{-C/\varepsilon}\big) \tag{3-16}$$

for every $s \in (0, s_{\max}]$. We fix such an s, and we often write $v(\,\cdot\,)$ instead of $v(s,\,\cdot\,)$. Note that (2-25) implies that

$$\big(1+(y^\nu)^2\big)\, e_\varepsilon(v) \ge \tfrac12 \lambda |D_\tau v|^2 + \big(1+(y^\nu)^2\big)\, e_{\varepsilon,\nu}(v).$$

It follows from this and the definitions of ζ_1 and ζ_3 that

$$\zeta_1(s) \ge c\,\zeta_3(s) + \delta_\varepsilon \int_{\{s\}\times W(s)} e_{\varepsilon,\nu}(v)\, dy' - 1.$$

Thus, it suffices to show that

$$1 - \delta_\varepsilon \int_{\{s\}\times W(s)} e_{\varepsilon,\nu}(v)\, dy' \le C\,\zeta_2(s) \;+\; Ce^{-c/\varepsilon}. \tag{3-17}$$

To do this, we say that a point $y^{\tau\prime} \in \mathbb{T}^n$ is *good* if

$$\mathfrak{D}_\nu(v(y^{\tau\prime})) \le \kappa_3,$$

and bad otherwise, for $v(y^{\tau\prime})(y^\nu) := v(y^{\tau\prime}, y^\nu)$. Then, Chebyshev's inequality implies that

$$\big|\{y^{\tau\prime} \in \mathbb{T}^n : y^{\tau\prime} \text{is bad}\}\big| \le \frac{1}{\kappa_3} \int_{\{s\}\times\mathbb{T}^n} \mathfrak{D}_\nu(v(y^{\tau\prime}))\, dy^{\tau\prime} = C\mathfrak{D}(v(s)) = C\,\zeta_2(s). \tag{3-18}$$

Thus $\big|\{y^{\tau\prime} \in \mathbb{T}^n : y^{\tau\prime}\text{is good}\}\big| \geq 1 - C\zeta_2(s)$ and, so, Lemma 11 implies that

$$\delta_\varepsilon \int_{\{s\}\times W(s)} e_{\varepsilon,\nu}(v)\, dy' \geq \int_{\{(s,y^{\tau\prime}) : y^{\tau\prime}\in\mathbb{T}^n\text{is good}\}} \Big(\delta_\varepsilon \int_{W_\nu(s)} e_{\varepsilon,\nu}(v)\, dy^\nu\Big)\, dy^{\tau\prime}$$
$$\geq (1 - C\zeta_2(s))(1 - Ce^{-c/\varepsilon}) \quad \text{by using (3-8).} \tag{3-19}$$

This proves (3-17), and hence (3-16).

Step 4: By combining the previous few steps and recalling that $\zeta_0 \geq \varepsilon^2$, we see that

$$\zeta_3(s) \leq C\zeta_0 + C\int_0^s \zeta_3(\sigma)d\sigma,$$

so Gronwall's inequality implies that there exists some C such that $\zeta_3(s) \leq C\zeta_0$ for all $s \in (0, \rho_1/2c_*)$. Then, (3-14) and (3-15) imply that $\zeta_1(s), \zeta_2(s) \leq C\zeta_0$. These estimates imply all the conclusions of the proposition. □

4. Initial energy estimates, $k = 1$

In this section, we indicate how to modify the above arguments to obtain control over v on a portion of a hypersurface of the form $\{y^0 = \text{constant}\}$, starting from our assumptions (2-31)–(2-36) about u at $t = 0$, which translate to information about v on a hypersurface of the form $\{(b(y'), y') : y' \in \mathbb{T}^n \times B_\nu(\rho_0)\}$, with b in general a non-constant function. (Recall that the function b was found in Lemma 8.) This is in general needed before we can start to iterate Proposition 10.

We note that if we assume that the minimal surface Γ has velocity 0 at time $t = 0$, then it is easy to check that $b(y') \equiv 0$. As a result, the hypotheses (3-5) and (3-6) of Proposition 10 follow immediately in this case from our general assumptions (2-31) and (2-34)–(2-36) on the initial data. So, the reader who is willing to accept this restriction on Γ can skip this section (and Section 5.3) without any loss.

We continue to follow the notational conventions for the case $k = 1$, summarized at the beginning of Section 3. We will prove:

Proposition 13. *Assume that $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}$ is a solution of* (2-26) *with data that satisfies* (2-34)–(2-36) *on the hypersurface* $\{(b(y'), y') : y' \in \mathbb{T}^n \times B_\nu(\rho_0)\}$.

There exist $s_1 > 0$ and $\rho_1 > 0$ for which v satisfies the hypotheses (3-5) *and* (3-6) *of Proposition 10, with ζ_0 replaced by $C\zeta_0$, and such that, in addition,*

$$\delta_\varepsilon \int_{\{y\in(-T_1,s_1)\times\mathbb{T}^n\times B_\nu(\rho_1) : \psi^0(y)>0\}} \Big(|D_\tau v|^2 + |y^\nu|^2\big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\big)\Big)\, dy \leq C\zeta_0.$$

If we simply tried to repeat our earlier arguments, we would have to worry about the way in which a cone with slope c_* intersects the initial hypersurface, and these considerations would force us to impose unnatural restrictions on the initial velocity of the surface Γ. We, therefore, exploit finite propagation speed in a different and sharper way than in our earlier arguments. (We could have done this earlier, but we wanted to present our basic estimate in a relatively simple setting.) This, and other considerations, forces us to introduce a certain amount of notation.

We start by defining

$$\mathscr{C} := \big\{(t, x) \in \mathbb{R}^{1+N} : \operatorname{dist}(x, \Gamma_0) < \tau - t \text{ and } t > 0\big\}. \tag{4-1}$$

where dist denotes the Euclidean distance function, $\Gamma_0 = \{H(0, y^{\tau\prime}) : y^{\tau\prime} \in \mathbb{T}^n\}$, and $\tau > 0$ is chosen so small that

$$\mathscr{C} \Subset \mathrm{Image}(\psi). \tag{4-2}$$

Note that $\mathscr{C}$ consists of the set of points for which the solution of the semilinear wave equation (1-1) depends solely on the data in the set $\mathscr{C}_0 := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Gamma_0) < \tau\}$. We continue by defining

$$\begin{aligned}
&V := \psi^{-1}(\mathscr{C}),\\
&s_0 := \inf\big\{y^0 \in (-T_1, T_1) \,:\, \big(\{y^0\}\times\mathbb{T}^n\times B_\nu(\rho_0)\big)\cap V \neq \varnothing\big\}, \quad \text{and}\\
&V^* := \big\{y = (y^0, y') \in (s_0, T_1)\times\mathbb{T}^n\times B_\nu(\rho_0) \,:\, (s, y') \in V \text{ for some } s \geq y^0\big\}
\end{aligned}$$

Thus V^* is just V "extended downward" in the timelike y^0 variable, to s_0. For $s \in R$, we define

$$V(s) := \{y \in V : y^0 < s\} \quad \text{and} \quad V^*(s) := \{y \in V^* : y^0 < s\}.$$

We further define

$$\begin{aligned}
&\partial_0 V(s) := \{y \in \partial V(s) : \psi^0(y) = 0\},\\
&\partial_1 V(s) := \{y = (y^0, y') \in \partial V(s) : y^0 = s\},\\
&\partial_2 V(s) := \partial V(s) \setminus \big(\partial_0 V(s) \cup \partial_1 V(s)\big).
\end{aligned}$$

We will also write

$$\begin{aligned}
&\partial_1 V^*(s) := \{y = (y^0, y') \in \partial V^*(s) : y^0 = s\}\\
&\partial_0 V := \{y \in \partial V : \psi^0(y) = 0\}\\
&W_0 := \{y' \in \mathbb{T}^n\times B_\nu(\rho_0) : (y^0, y') \in \partial_0 V \text{ for some } y^0\}.
\end{aligned}$$

Finally, for $i = 0, 1, 2$, we define

$$W_i(s) := \{y' \in \mathbb{T}^n\times B_\nu(\rho_0) : (y^0, y') \in \partial_i V(s) \text{ for some } y^0\}$$

and similarly $W_i^*(s)$.

The next lemma collects some geometric facts that we will need about the sets defined above.

Lemma 14. *We have*

$$(W_0(s)\setminus W_1(s)) \cap W_1^*(s) = \varnothing \quad \textit{for all } s. \tag{4-3}$$

In addition, there exist $s_1 > 0$ *and* $\rho_1 > 0$ *such that*

$$(s_0, s_1)\times\mathbb{T}^n\times B_\nu(\rho_1) \subset V^* \quad \textit{and} \quad \{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1) \subset V. \tag{4-4}$$

Proof. To prove (4-3), fix $y' \in W_0(s) \setminus W_1(s)$. The definitions imply that the line $\{(y^0, y') : y^0 \in \mathbb{R}\}$ intersects $\partial_0 V(s)$ and does not meet $\partial_1 V(s)$, so it must leave $\overline{V}$ at a point (σ, y') with $\sigma < s$. Arguments like those of Lemma 8 show that once the line has left $\overline{V}$, it cannot re-enter, since, if it did, the timelike curve $s \mapsto X(s) := \psi(s, y')$ (see Lemma 8) would intersect $\partial^+\mathscr{C} := \{(t, x) \in \partial\mathscr{C} : t > 0\}$ more than once, which is impossible. Thus, the line does not intersect $\overline{V}^*$ at any point (y^0, y') with $y' > \sigma$ and, so, it cannot intersect $\partial_1 V^*(s) \subset \{(y^0, s) \in \overline{V}^* : y^0 = s\}$. Thus, $y' \notin W_1^*(s)$, proving (4-3).

Next, the existence of $s_1, \rho_1 > 0$ satisfying (4-4) follows from the fact that the (Euclidean) distance from $\{0\} \times \mathbb{T}^n \times \{0\} = \psi^{-1}(\Gamma_0)$ to $\partial^+ V := \partial V \setminus \partial_0 V = \psi^{-1}(\partial^+ \mathscr{C})$ is positive. This last fact, in turn, is clear from the fact that the distance from Γ_0 to $\partial^+\mathscr{C}$ is positive, together with the smoothness of ψ. □

Recall that $v_0 : \mathbb{T}^n \times B_\nu(\rho_0) \cong \{0\} \times \mathbb{T}^n \times B_\nu(\rho_0)$ was defined in (2-33). We extend v_0 to $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho)$ in such a way that it is independent of y^0; this extended function is still denoted by v_0.

The remainder of this section contains the proof of Proposition 13. In the proof, when we want to distinguish between row vectors and column vectors (which one can think as vectors and covectors, respectively), we will write $\overline{\xi}$ to denote a column vector, with components ξ^α, and $\underline{\xi}$ for a row vector, with components ξ_α.

Proof of Proposition 13. As in Proposition 10, it suffices to prove the statement for v smooth in $\overline{V}$.

Step 1: We define $v^* : V^* \to \mathbb{R}$ by

$$v^*(y) = \begin{cases} v(y) & \text{if } y \in V \\ v_0(y) & \text{if } y \in V^* \setminus V. \end{cases} \tag{4-5}$$

Since $v = v_0$ on $\overline{V} \cap (V^* \setminus V) = \partial_0 V$, it is easy to see that v^* is Lipschitz in V^*. Note, however, that the derivatives of v^* are in general discontinuous across $\partial_0 V$.

We define

$$\zeta_1(s) := \delta_\varepsilon \int_{\partial_1 V^*(s)} \bigl(1 + (y^\nu)^2\bigr)\, e_\varepsilon(v^*)\, dy' - 1,$$
$$\zeta_2(s) := \mathscr{D}(v^*(s); \rho_1/2),$$
$$\zeta_3(s) := \delta_\varepsilon \int_{\partial_1 V^*(s)} \Bigl(|D_\tau v^*|^2 + (y^\nu)^2 e_{\varepsilon,\nu}(v^*)\Bigr)\, dy'.$$

In view of (4-4), we can repeat word for word the arguments from the proof of Proposition 10, to find

$$\zeta_3(s) \le C\bigl(\zeta_1(s) + \zeta_2(s) + e^{-c/\varepsilon}\bigr), \tag{4-6}$$

$$\zeta_2(s) \le 2\zeta_2(s_0) + C \int_{s_0}^{s} \zeta_3(\sigma)\, d\sigma, \tag{4-7}$$

for every $s \in [s_0, s_1]$. Also, the definition of s_0 implies that $v^* = v_0$ on $\partial_1 V^*(s_0) := \{s_0\} \times W_0$, so that $\zeta_2(s_0) \le \zeta_0$ by (2-36). Thus,

$$\zeta_2(s) \le C\zeta_0 + C \int_{s_0}^{s} \zeta_3(\sigma)\, d\sigma \tag{4-8}$$

for every $s \in [s_0, s_1]$.

The remainder of the proof is devoted to the estimate of ζ_1. Since v^* is smooth away from $\partial_0 V$ and (by Fubini's Theorem) $\partial_1 V^*(s) \cap \partial_0 V$ has $\mathscr{H}^N$ measure 0 for $\mathscr{L}^1$-a.e. s, the definition of v^* implies that

$$e_\varepsilon(v^*) = \begin{cases} e_\varepsilon(v) & \mathscr{H}^N\text{-a.e. in } \partial_1 V(s) \\ e_\varepsilon(v_0) & \mathscr{H}^N\text{-a.e. in } \partial_1 V^*(s) \setminus \partial_1 V(s) \end{cases} \tag{4-9}$$

for a.e. s. Also, if $[\cdots]$ denotes an integrand that does not depend on the y^0 variable, then clearly $\int_{\partial_1^* V(s) \setminus \partial_1 V(s)} [\cdots]\, dy' = \int_{W_1^*(s) \setminus W(s)} [\cdots]\, dy'$. Thus, for a.e. s,

$$\int_{\partial_1 V^*(s)} \bigl(1 + (y^\nu)^2\bigr)\, e_\varepsilon(v^*)\, dy' = \int_{\partial_1 V(s)} \bigl(1 + (y^\nu)^2\bigr)\, e_\varepsilon(v)\, dy' + \int_{W_1^*(s) \setminus W_1(s)} \bigl(1 + (y^\nu)^2\bigr)\, e_\varepsilon(v_0)\, dy' \tag{4-10}$$

Step 2: We claim that, for a.e. s,

$$\delta_\varepsilon \int_{\partial_1 V(s)} \big(1+(y^\nu)^2\big)\, e_\varepsilon(v)\, dy' \\ \leq \delta_\varepsilon \int_{\partial_0 V(s)} \big(1+(y^\nu)^2\big)(-n_0 e_\varepsilon(v) + n_i \varphi^i)\, d\mathcal{H}^N(dy) + C\int_{s_0}^s \zeta_3(\sigma)\, d\sigma, \quad (4\text{-}11)$$

where $\underline{n}(y)$ denotes the (Euclidean) outer unit normal at a point $y \in \partial V(s)$, thought of as a row vector with components n_α, and φ^i is defined in (2-28) and appears in the local energy estimate of Lemma 6.

Step 2.1: To prove (4-11), we will first integrate by parts and show that some of the boundary terms have a sign, and hence can be discarded. (In this, we basically follow the proof of Proposition 10.) For this, it is useful to define $\widetilde{\mathcal{T}}_\varepsilon = \widetilde{\mathcal{T}}_\varepsilon(v)$ by

$$\widetilde{\mathcal{T}}^\alpha_{\varepsilon,\beta} := \delta^\alpha_\beta\Big(\frac{1}{2} g^{\gamma\delta} v_{y^\gamma} v_{y^\delta} + \frac{1}{\varepsilon^2} F(v)\Big) - g^{\alpha\gamma} v_{y^\gamma} v_{y^\beta}. \quad (4\text{-}12)$$

Observe, from the definitions, that[7]

$$\widetilde{\mathcal{T}}^0_{\varepsilon,0}(v) = e_\varepsilon(v) \quad \text{and} \quad \widetilde{\mathcal{T}}^i_{\varepsilon,0}(v) = -\varphi^i, \quad (4\text{-}13)$$

so that the conclusion of Lemma 6 can be written $\partial_{y^\alpha} \widetilde{\mathcal{T}}^\alpha_{\varepsilon,0} \leq C\big(|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2\big)$.

We now compute

$$\begin{aligned} \delta_\varepsilon \int_{V(s)} \partial_{y^\alpha}\Big(\big(1+(y^\nu)^2\big)\, \widetilde{\mathcal{T}}^\alpha_{\varepsilon,0}\Big)\, dy &\leq C\,\delta_\varepsilon \int_{V(s)} \Big(\big(|D_\tau v|^2 + (y^\nu)^2 |\nabla_\nu v|^2\big) + y^N \widetilde{\mathcal{T}}^N_{\varepsilon,0}\Big)\, dy \\ &\leq C\,\delta_\varepsilon \int_{V(s)} \big(|D_\tau v|^2 + (y^\nu)^2|\nabla_\nu v|^2\big)\, dy \\ &\leq C\int_{s_0}^s \zeta_3(\sigma)\, d\sigma. \end{aligned} \quad (4\text{-}14)$$

On the other hand, we can integrate by parts to rewrite the left-hand side as an integral over $\partial V(s)$. Then, noting that $\underline{n}(y) = (1, 0, \dots, 0)$ for $y \in \partial_1 V(s)$, we find that

$$\delta_\varepsilon \int_{V(s)} \partial_{y^\alpha}\Big(\big(1+(y^\nu)^2\big)\, \widetilde{\mathcal{T}}^\alpha_{\varepsilon,0}\Big) = \delta_\varepsilon \int_{\partial_1 V(s)} \big(1+(y^\nu)^2\big)\, e_\varepsilon(v)\, dy' + \delta_\varepsilon \int_{\partial_0 V(s)} \Big(\big(1+(y^\nu)^2\big)\, n_\alpha \widetilde{\mathcal{T}}^\alpha_{\varepsilon,0}\Big)\, d\mathcal{H}^N(y) \\ + \delta_\varepsilon \int_{\partial_2 V(s)} \Big(\big(1+(y^\nu)^2\big)\, n_\alpha \widetilde{\mathcal{T}}^\alpha_{\varepsilon,0}\Big)\, d\mathcal{H}^N(y).$$

By combining this with (4-14) and recalling (4-13), we see that our claim (4-11) will follow if we can show that the last integral on the right-hand side is positive.

Step 2.2: To do this, we will show that

$$n_\alpha(y)\, \widetilde{\mathcal{T}}^\alpha_{\varepsilon,0}(y) \geq 0 \quad \text{for a.e. } y \in \partial_2 V(s). \quad (4\text{-}15)$$

We first check that

$$g^{\alpha\beta} n_\alpha n_\beta = 0 \quad \text{for a.e. } y \in \partial_2 V. \quad (4\text{-}16)$$

[7] In fact, $\widetilde{T}_\varepsilon$ is just the energy-momentum tensor for u, expressed in terms of the y coordinates. The fact that, when written in the y coordinates, the energy-momentum tensor is divergence-free, takes the form $\partial_{y^\alpha}\big(\widetilde{\mathcal{T}}^\alpha_{\varepsilon,\beta}(v)\sqrt{-g}\big) = 0$ for all β.

In fact, we will show that this holds at every $y \in \partial_2 V$ such that ∂C has a tangent plane at $x = \psi(y)$; this is a set of full measure. Fix such a y and let $\overline{w} = (w^\alpha)$ be any (column) vector tangent to $\partial\mathscr{C}$ at x. Also, let $\underline{m}(x)$ denote the (Euclidean) outer unit normal to $\mathscr{C}$ at $x \in \partial\mathscr{C}$, again thought of as a row vector with components m_α. Writing $\varphi = \psi^{-1}$ as usual, since φ maps $\partial\mathscr{C}$ to ∂V, it is clear that $D\varphi(x)\,\overline{w}$ is tangent to ∂V at $\varphi(x) = y$, which implies that $\underline{n}(y)\, D\varphi(x)\,\overline{w} = 0$. Since this holds for all tangent vectors $\overline{w}$ at x, it follows that $\underline{n}(y)\, D\varphi(x)$ is parallel to the Euclidean unit normal $\underline{m}$ to ∂C at x; that is, $\underline{n}(y)\, D\varphi(x) = \lambda\underline{m}(x)$ for some $\lambda \in \mathbb{R}$. The form of $\mathscr{C}$ implies that $\underline{m}$ is a null vector, so that

$$0 = \lambda^2\eta^{\alpha\beta} m_\alpha m_\beta = \lambda^2\eta^{\alpha\beta} n_\gamma\varphi^\gamma_\alpha m_\delta\varphi^\delta_\beta = g^{\gamma\delta} n_\gamma n_\delta,$$

proving (4-16). Note also that $n_0(y) > 0$ for $y \in \partial_2 V$, and recall further that $F(u) \geq 0$. Thus,

$$\begin{aligned} n_\alpha\widetilde{\mathscr{T}}^\alpha_{\varepsilon,0} &= \frac{n_0}{\varepsilon^2}F(u) + \frac{n_0}{2}g^{\alpha\beta}v_{y^\alpha}v_{y^\beta} - v_{y^0}g^{\alpha\beta}n_\alpha v_{y^\beta} \\ &\geq \frac{n_0}{2}g^{\alpha\beta}v_{y^\alpha}v_{y^\beta} - v_{y^0}g^{\alpha\beta}n_\alpha v_{y^\beta} = \frac{n_0}{2}g^{\alpha\beta}\Big(Dv - \frac{v_{y^0}}{n_0}n\Big)_\alpha\Big(Dv - \frac{v_{y^0}}{n_0}n\Big)_\beta, \end{aligned}$$

using (4-16). If we write $\xi := Dv - \dfrac{v_{y^0}}{n_0}n$, then clearly $\xi_0 = 0$, which implies that

$$g^{\alpha\beta}\xi_\alpha\xi_\beta = g^{ij}\xi_i\xi_j = a^{\alpha\beta}\xi_\alpha\xi_\beta \geq 0.$$

Thus, we have proved (4-15).

Step 3: Next, we note that

$$-\int_{\partial_0 V(s)} \big(1 + (y^\nu)^2\big)\, n_0(y)\, e_\varepsilon(v)(y)\, d\mathscr{H}^N(dy) = \int_{W_0(s)} \big(1 + (y^\nu)^2\big)\, e_\varepsilon(v)(b(y'), y')\, dy, \tag{4-17}$$

where we recall that $\partial_0 V = \{(b(y'), y') : y' \in W_0\}$, and hence that $\partial_0 V(s) = \{(b(y'), y') : y' \in W_0(s)\}$. This is obvious, because the Euclidean outer unit normal to $V(s)$ is given by $\underline{n} = (-1, \nabla b)/(1+|\nabla b|^2)^{1/2}$, with the minus sign appearing because V sits above the graph. Thus, $-n_0(b(y'), y') = \big(1 + |\nabla b(y')|^2\big)^{-1/2}$, and then (4-17) follows from a change of variables using the area formula.

Step 4: Now, we combine (4-17) with (4-10) and (4-11), to find that

$$\zeta_1(s) \leq C\int_{s_0}^{s} \zeta_3(\sigma)\, d\sigma + A + B,$$

for a.e. $s \in [s_0, s_1]$, where

$$\begin{aligned} A &:= \delta_\varepsilon\int_{W_0(s)} \big(1 + (y^\nu)^2\big)\big(e_\varepsilon(v) - e_\varepsilon(v_0)\big)(b(y'), y')\, dy' + \delta_\varepsilon\int_{\partial_0 V(s)} \big(1 + (y^\nu)^2\big)\, n_i\varphi^i\, d\mathscr{H}^N, \\ B &:= \delta_\varepsilon\int_{W_1^*(s)\setminus W_1(s)} \big(1 + (y^\nu)^2\big)\, e_\varepsilon(v_0)\, dy' + \delta_\varepsilon\int_{W_0(s)} \big(1 + (y^\nu)^2\big)\, e_\varepsilon(v_0)\, dy' - 1. \end{aligned}$$

We have checked in Lemma 14 that $(W_1^*(s)\setminus W_1(s)) \cap W_0(s) = \varnothing$; this is equivalent to (4-3). Thus,

$$B \leq \delta_\varepsilon\int_{W_0} \big(1 + (y^\nu)^2\big)\, e_\varepsilon(v_0)\, dy' - 1 \leq \zeta_0,$$

by (2-34). To estimate A, we differentiate the identity $v(b(y'), y') = v_0(y')$ to find that $v_{y^0}\nabla b + \nabla v = \nabla v_0$. Thus, $|D(v - v_0)| = |v_{y^0}(1, -\nabla b)| \leq C|v_{y^0}|$ at points $(b(y'), y') \in \partial_0 V$, using the control over $\|\nabla b\|_\infty$

obtained in Lemma 8. It follows that, at such points,

$$e_\varepsilon(v) - e_\varepsilon(v_0) = \tfrac{1}{2} a^{\alpha\beta}(v - v_0)_{y^\alpha}(v + v_0)_{y^\beta} \le C\big(v_{y^0}^2 + |D_\tau v_0|^2 + |v_{y^0}|\,|\nabla_\nu v_0|\big).$$

Similarly, using (2-19), we see that $|\varphi^i| \le C\big(v_{y^0}^2 + |D_\tau v_0|^2 + (y^\nu)^2 |\nabla_\nu v_0|^2\big)$, so

$$A \le C\delta_\varepsilon \int_{\partial_0 V} \big(v_{y^0}^2 + |v_{y^0}|\,|\nabla_\nu v_0|\big)\, d\mathcal{H}^N + C\delta_\varepsilon \int_{W_0} \big(|D_\tau v_0|^2 + (y^\nu)^2 |\nabla_\nu v_0|^2\big)\, dy'.$$

Also, since $v_0(y') = v^*(s_0, y')$, we have

$$\int_{W_0} \varepsilon\big(|D_\tau v_0|^2 + (y^\nu)^2\, |\nabla_\nu v_0|^2\big)\, dy' \le \zeta_3(s_0) \le C\big(\zeta_1(s_0) + \zeta_2(s_0) + e^{-c/\varepsilon}\big) \le C\zeta_0;$$

here we used (4-7) for the second inequality, and (2-34) and (2-35) for the last. Using this and (2-35), we conclude that $A \le C\zeta_0$ and, hence, that

$$\zeta_1(s) \le C \int_{s_0}^{s} \zeta_3(\sigma)\, d\sigma + C\zeta_0.$$

Step 5: The rest of the proof follows exactly that of Proposition 10. In the end, we find that $\zeta_i(s_1) \le C\zeta_0$ for $i = 1, 2, 3$ and, in view of (4-4), these estimates immediately imply the conclusion. □

5. Energy estimates, $k = 2$

In this section, we prove energy estimates like those from Sections 3 and 4, but now in the case $k = 2$, so that we consider a vector-valued function $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho) \to \mathbb{R}^2$ solving (2-26), where $B_\nu(\rho) \subset \mathbb{R}^2_\nu$ now denotes a 2-dimensional ball, κ_2 is the constant chosen in (2-23), $\delta_\varepsilon = (\pi |\ln \varepsilon|)^{-1}$, and the nonlinearity in (1-1) is $f = \nabla F$, with $F : \mathbb{R}^2 \to [0, \infty)$ satisfying (1-19).

The main results and proofs in this section are strictly analogous to Propositions 10 and 13. The chief difference is that the "defect-confinement functional" $\mathfrak{D}$ (discussed in the Introduction) has quite a different form than in the case $k = 1$. Thus, the arguments we need in order to verify that the desired properties (1-30) and (1-32) hold, are quite different from (and more delicate than) their counterparts in the scalar case. Once suitable forms of these facts are established, we follow our earlier proofs with only cosmetic changes.

We will use machinery that relates the Jacobian and the Ginzburg–Landau energy. We will give precise statements of the facts we need from the literature, in the hope of rendering our arguments somewhat accessible to people who are not familiar with these results; see also the book [Sandier and Serfaty 2007] for a general reference on these topics. The results we use (see Lemmas 18, 19 and 21) are proved for $F_{\mathrm{model}}(u) = \frac{1}{4}(|u|^2 - 1)^2$ in the sources we cite, but it is evident[8] from the proofs that they still apply to functions F satisfying the assumptions (1-19) that we impose here.

[8] In all the proofs we will cite, easy truncation arguments are used to reduce to, for example, the case of u with $|u| \le M$ a.e. for $M = 2$; then, (1-19) implies that $\big(1/(C\varepsilon)^2\big)\, F_{\mathrm{model}}(u) \le (1/\varepsilon^2)\, F(u) \le \big(1/(\varepsilon/C)^2\big)\, F_{\mathrm{model}}(u)$. It is then clear that results established for F_{model} carry over to energy functionals that instead contain F, since everything we use is essentially unaffected if ε is replaced by $C\varepsilon$ or ε/C.

For $v \in H^1(\mathbb{T}^n \times B_\nu(\rho); \mathbb{R}^2)$ we take $\mathfrak{D}$ to have the form (as when $k = 1$)

$$\mathfrak{D}(v; \rho) := \int_{\mathbb{T}^n} \mathfrak{D}_\nu(v(y^{\tau\prime}); \rho)\, dy^{\tau\prime}, \tag{5-1}$$

where $v(y^{\tau\prime})(y^\nu) = v(y^{\tau\prime}, y^\nu)$. For $w = (w^1, w^2) \in H^1(B_\nu(\rho); \mathbb{R}^2)$, we define

$$\mathfrak{D}_\nu(w; \rho) := |||J_\nu w - \pi\delta_0|||_\rho \tag{5-2}$$

where, for a measure μ on $B_\nu(\rho)$,

$$|||\mu|||_\rho := \sup\Big\{ \int \omega(y^\nu) f(y^\nu)\, dy^\nu \,:\, \omega \in C^2_c(B_\rho),\ |\nabla\omega(y^\nu)| \le |y^\nu|^2,\ \|\omega\|_{W^{2,\infty}} \le 1 \Big\}. \tag{5-3}$$

(Clearly, $|||\cdot|||_\rho$ also makes sense for some distributions that are less regular than measures, but we will not need that here.) Here, we are using the notation $J_\nu w = \det \nabla_\nu w$. We will also write $\boldsymbol{J}_\nu w$ for the 2-form $\boldsymbol{J}_\nu w = J_\nu w\, dy^\nu$, where $dy^\nu := dy^{\nu,1} \wedge dy^{\nu,2}$. Note that

$$\boldsymbol{J}_\nu w := d_\nu w^1 \wedge d_\nu w^2, \quad \text{where } d_\nu w^i = \frac{\partial w^i}{\partial y^{\nu,1}} dy^{\nu,1} + \frac{\partial w^i}{\partial y^{\nu,2}} dy^{\nu,2}.$$

(Recall that $y^{\nu,i} = y^{n+i}$.)

General results and heuristics about Jacobians and vortices (see, for example, [Sandier and Serfaty 2007]), together with the definition of the $|||\cdot|||_\rho$ norm, suggest that, if $w : B_\nu(\rho) \to \mathbb{R}^2$ is a function possessing a single "vortex of degree 1" localized near some point in $B_\nu(\rho/2)$, then, roughly speaking,

$$|||J_\nu w - \pi\delta_0|||_\rho \approx (\text{the distance from the origin to the vortex})^3$$

(The cubic scaling on the right-hand side is related to the condition $|\nabla\omega(y^\nu)| \le |y^\nu|^2$ imposed on test functions, in the definition of $|||\cdot|||_\rho$.) Thus, the right-hand side of (5-1) is the average of the above quantity over the tangential y^τ variables.

The first main result of this section parallels Proposition 10 above:

Proposition 15. *Let $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^2$ satisfy* (2-26), *where $B_\nu(\rho) \subset \mathbb{R}^2_\nu$ and $f = \nabla F$, with $F : \mathbb{R}^2 \to \mathbb{R}$ satisfying* (1-19). *Recalling that $\delta_\varepsilon = (\pi \ln \varepsilon|)^{-1}$, assume that there exist $s_1 \in (-T_1, T_1)$, $\rho_1 \in (0, \rho_0)$, and $\zeta_0 \ge \delta_\varepsilon$ such that*

$$\delta_\varepsilon \int_{\{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1)} \big(1 + \kappa_2|y^\nu|^2\big)\, e_\varepsilon(v)\, dy' - 1 \le \zeta_0 \quad \textit{and} \tag{5-4}$$

$$\mathfrak{D}(v(0); \rho_1/2) \le \zeta_0. \tag{5-5}$$

There exists a constant C such that

$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1 - c_* s)} \Big(|D_\tau v|^2 + |y^\nu|^2 \Big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v)\Big)\Big)\, dy' \le C\zeta_0,$$
$$\delta_\varepsilon \int_{\{s_1+s\}\times\mathbb{T}^n\times B_\nu(\rho_1 - c_* s)} e_\varepsilon(v)\, \big(1 + \kappa_2|y^\nu|^2\big)\, dy' - 1 \le C\zeta_0,$$
$$\textit{and} \quad \mathfrak{D}(v(s); \rho_1/2) \le C\zeta_0,$$

for all $s \in [0, \rho_1/2c_*]$ such that $s_1 + s < T_1$. Here, c_* is as defined in (3-4).

As remarked earlier, there does not exist any initial data satisfying (5-4) and (5-5) with $\zeta_0 \ll \delta_\varepsilon$ when $k = 2$, so the condition $\zeta_0 \geq \delta_\varepsilon$ is not restrictive.

The second main result of this section parallels Proposition 13:

Proposition 16. *Assume that $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^2$ is a solution of* (2-26), *with data that satisfies* (2-34)–(2-36) *on the hypersurface $\{(b(y'), y') : y' \in \mathbb{T}^n \times B_\nu(\rho_0)\}$, with $\zeta_0 \geq \delta_\varepsilon$ and $\mathfrak{D}$ as defined in* (5-1).

There exist some $s_1 > 0$ and $\rho_1 > 0$ for which v satisfies the hypotheses (5-4), (5-5) *of* Proposition 15, *with ζ_0 replaced by $C\zeta_0$, and such that, in addition,*

$$\delta_\varepsilon \int_{\{y \in (-T_1, s_1) \times \mathbb{T}^n \times B_\nu(\rho_1) : \psi^0(y) > 0\}} \Big(|D_\tau v|^2 + |y^\nu|^2 \Big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v) \Big) \Big) \, dy \leq C\zeta_0.$$

5.1. *Variational-stability estimates.* We start by establishing some properties relating the $|||\cdot|||_\rho$ norm of the Jacobian Jv and the Ginzburg–Landau energy $e_{\varepsilon,\nu}(v)$. These will be used to show that $\mathfrak{D}(\cdot)$ satisfies the requirements (1-30) and (1-32) from the Introduction.

Our first result is analogous to Lemma 11, and establishes a form of (1-30). It is a straightforward consequence of the Jacobian machinery mentioned above.

Proposition 17. *For $\rho > 0$, there exist constants κ_3 and C, both depending on ρ, such that, if $w \in H^1(B_\nu(\rho); \mathbb{R}^2)$ and*

$$\mathfrak{D}_\nu(w; \rho) = |||J_\nu w - \pi\delta_0|||_\rho \leq \kappa_3, \tag{5-6}$$

then

$$|\ln\varepsilon|^{-1} \int_B e_{\varepsilon,\nu}(w) \, dy^\nu \geq \pi - |\ln\varepsilon|^{-1} C. \tag{5-7}$$

The proof of Proposition 17 uses the following facts:

Lemma 18. *If $\varepsilon \in (0, 1]$, $w \in H^1(B_\nu(\rho); \mathbb{R}^2)$, and*

$$\|J_\nu w - \pi\delta_0\|_{W^{-1,1}(B_\nu(\rho))} \leq \frac{\rho}{10},$$

then

$$\frac{1}{|\ln\varepsilon|} \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(w) \, dy^\nu \geq \pi - \frac{C}{|\ln\varepsilon|}.$$

This follows, for example, from a much sharper estimate proved in [Jerrard and Spirn 2007, Theorem 1.3]. A slightly different norm is used there in place of the $W^{-1,1}$ norm, but that result is easily seen to imply the one stated here.

Lemma 19. *Suppose that $\varepsilon \in (0, 1]$ and that $w \in H^1(B_\nu(\rho); \mathbb{R}^2)$ satisfies*

$$\frac{1}{|\ln\varepsilon|} \int_B e_{\varepsilon,\nu}(w) \, dy^\nu \leq 3\pi/2.$$

There exists an integer $\ell \in \{0, \pm 1\}$ and a point $\xi \in B$ such that

$$\|J_\nu w - \pi\ell\delta_\xi\|_{W^{-1,1}(B_\nu(\rho))} \leq C |\ln\varepsilon| \, \varepsilon^{1/4}.$$

This follows from [Jerrard and Spirn 2007, Theorem 1.1].

Proof of Proposition 17. Fix $w \in H^1(B_\nu(\rho); \mathbb{R}^2)$. We may assume that

$$\frac{1}{|\ln\varepsilon|}\int_{B_\nu(\rho)} e_{\varepsilon,\nu}(w)\,dy^\nu \le 3\pi/2, \tag{5-8}$$

since otherwise (5-7) is immediate. So, in view of Lemma 18, it suffices to show that there exists a constant $\kappa_3(\rho)$ such that, if (5-8) holds and $|||J_\nu w - \pi\delta_0|||_\rho < \kappa_3$, then

$$\|J_\nu w - \pi\delta_0\|_{W^{-1,1}(B_\nu(\rho))} \le \frac{\rho}{10}. \tag{5-9}$$

In fact, it suffices to show that there exists some $\varepsilon_0 > 0$ such that the above conclusion holds if $\varepsilon \in (0, \varepsilon_0)$ in (5-8), since we can arrange that (5-7) holds for $\varepsilon > \varepsilon_0$ by choosing C large enough.

Now, (5-8) and Lemma 19 imply that there exist an integer ℓ with $|\ell| \le 1$, and a point $\xi \in B_\nu(\rho)$ such that $\|J_\nu w - \pi\ell\delta_\xi\|_{W^{-1,1}(B_\nu(\rho))} \le C|\ln\varepsilon|\,\varepsilon^{1/4}$. Fix a function $\omega_* \in C^2_c(B)$, with $|\nabla\omega_*(y)| \le |y|^2$ and $\|\omega_*\|_{W^{2,\infty}} \le 1$, and such that $\omega_*(y) < \omega_*(0)$ if $y \ne 0$. Then, (5-6) and the definition of the $|||\cdot|||_\rho$ norm imply that

$$\int \omega_*\; J_\nu w\,dy^\nu - \pi\omega_*(0) \ge -\kappa_3.$$

On the other hand, the estimate $\|J_\nu w - \pi\ell\delta_\xi\|_{W^{-1,1}(B)} \le C|\ln\varepsilon|\,\varepsilon^{1/4}$ implies that

$$\int \omega_*\; J_\nu w\,dy^\nu - \pi\ell\,\omega_*(\xi) \le C\|\omega_*\|_{W^{1,\infty}}|\ln\varepsilon|\,\varepsilon^{1/4} \le C|\ln\varepsilon|\,\varepsilon^{1/4}.$$

Thus,

$$\ell\,\omega_*(\xi) \ge \omega_*(0) - \frac{\kappa_3}{\pi} - C|\ln\varepsilon|\,\varepsilon^{1/4}. \tag{5-10}$$

Since $\omega_*(0) > 0$, this implies that $\ell = 1$ for all sufficiently small $\varepsilon > 0$, if κ_3 is fixed small enough. Then, $\|Jw(\tau) - \pi\delta_\xi\|_{W^{-1,1}(B_\nu(\rho))} \le C|\ln\varepsilon|\,\varepsilon^{1/4}$ and, as a result,

$$\begin{aligned}\|J(w(\tau)) - \pi\delta_0\|_{W^{-1,1}(B_\nu(\rho))} &\le C|\ln\varepsilon|\,\varepsilon^{1/4} + \pi\,\|\delta_\xi - \delta_0\|_{W^{-1,1}(B_\nu(\rho))}\\ &\le C|\ln\varepsilon|\,\varepsilon^{1/4} + \pi\,|\xi|,\end{aligned}$$

where the last inequality follows immediately from the definition of the $W^{-1,1}$ norm. Since ω_* is continuous and achieves its maximum exactly at the origin, (5-10) implies that, fixing κ_3 still smaller if necessary, $\pi|\xi| < \rho/20$ and, as a result, (5-9) holds for all small ε. □

The second result about the $|||\cdot|||_\rho$ norm is analogous to Lemma 11, and establishes a form of the requirement (1-32); in fact, the norm is designed exactly so that an estimate of the form (5-11) holds. In the lemma, we write v as a function of $(y^0, y^\nu) \in \mathbb{R}\times\mathbb{R}^2_\nu$.

Proposition 20. *Let $v \in H^1\big((0,\tau)\times B_\nu(\rho); \mathbb{R}^2\big)$ for some $\rho, \tau > 0$. There exist positive constants C and α, depending on ρ but independent of τ and $\varepsilon \in (0, 1]$, such that*

$$\begin{aligned}|||J_\nu v(\tau,\cdot) - J_\nu v(0,\cdot)|||_\rho \le C\delta_\varepsilon &\int_{(0,\tau)\times B_\nu(\rho)} \big(|y^\nu|^2 + \varepsilon^\alpha\big)\Big(\tfrac12|Dv|^2 + \frac{1}{\varepsilon^2}F(v)\Big)\,dy^\nu\,dy^0\\ &+ C\varepsilon^\alpha\bigg(1 + \int_{\{0\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v)\,dy^\nu + \int_{\{\tau\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v)\,dy^\nu\bigg).\end{aligned} \tag{5-11}$$

We believe that the ε^α in the first integral on the right-hand side of (5-11) could be removed with some work, but the estimate is false without the boundary terms in the second line of (5-11). In any case, all these terms will be negligible in our later arguments.

The proof of Proposition 20 requires the following:

Lemma 21. *There exist universal constants $C, \alpha > 0$ such that, given any $U \subset \mathbb{R}^3 = \mathbb{R}_{y^0} \times \mathbb{R}^2_\nu$ and $w \in H^1(U; \mathbb{R}^2)$,*

$$\left|\int_U \omega \wedge \boldsymbol{J} w\right| \le \frac{C}{|\ln \varepsilon|} \int_U |\omega| \Big(\tfrac{1}{2}|Dw|^2 + \frac{1}{\varepsilon^2} F(w)\Big) + C\varepsilon^\alpha (1 + \|D\omega\|_\infty) \left(1 + \|\omega\|_\infty + \int_U (1+|\omega|)\Big(\tfrac{1}{2}|Dw|^2 + \frac{1}{\varepsilon^2}F(w)\Big)\right) \tag{5-12}$$

for every compactly supported Lipschitz continuous 1-form ω in U, and every $\varepsilon \in (0, 1]$. Here, $\boldsymbol{J}w$ denotes the 2-form $dw^1 \wedge dw^2 = (w^1_{y^0}\, dy^0 + d_\nu w^1) \wedge (w^2_{y^0}\, dy^0 + d_\nu w^2)$.

This is [Jerrard 2007, Lemma 9], with notation adapted to our setting. In (5-12), Dw denotes, as usual, the gradient in all three variables.

Proof of Proposition 20.
Step 1: Fix $v \in H^1((0, \tau) \times B_\nu(\rho); \mathbb{R}^2)$. In order to prove (5-11), we must estimate

$$\int_{B_\nu(\rho)} \omega\big(J_\nu v(\tau, y^\nu) - J_\nu v(0, y^\nu)\big)\, dy^\nu$$

for an arbitrary $\omega \in C^\infty_c(B_\nu(\rho))$ such that $|\nabla\omega(y)| \le |y|^2$ and $\|\omega\|_{W^{2,\infty}} \le 1$. We fix such a test function ω, and we start by rewriting the previous expression. For this, let δ denote a positive number to be fixed later (not to be confused with δ_ε), and define $V : (-\delta, \tau + \delta) \times B_\nu(\rho) \to \mathbb{R}^2$ by

$$V(y^0, y^\nu) = \begin{cases} v(0, y^\nu) & \text{if } -\delta < y^0 \le 0, \\ v(y^0, y^\nu) & \text{if } 0 \le y^0 \le \tau, \\ v(\tau, y^\nu) & \text{if } \tau \le y^0 \le \tau + \delta. \end{cases}$$

Let $\chi \in C^\infty_c(-\delta, \tau+\delta)$ be a function such that

$$\chi(y^0) \equiv 1 \ \text{ for } y^0 \in [0, \tau] \quad \text{ and } \quad \|\chi'\|_\infty \le C(1 + \delta^{-1}).$$

Since $J_\nu V(y^0) = J_\nu v(0)$ for $y^0 \in (-\delta, 0]$, and $J_\nu V(y^0) = J_\nu v(\tau)$ for $y^0 \in [\tau, \tau+\delta)$,

$$\begin{aligned}\int_{B_\nu(\rho)} \omega\big(J_\nu v(\tau, y^\nu) - J_\nu v(0, y^\nu)\big)\, dy^\nu &= -\int_{-\delta}^{\tau+\delta} \chi'(y^0) \Big(\int_{B_\nu(\rho)} \omega(y^\nu)\, J_\nu V\, dy^\nu\Big)\, dy^0 \\ &= -\int_{(-\delta, \tau+\delta)\times B_\nu(\rho)} \big(\omega(y^\nu)\, \chi'(y^0)\, dy^0\big) \wedge \boldsymbol{J} V.\end{aligned} \tag{5-13}$$

We continue by observing that

$$\omega(y^\nu)\, \chi'(y^0)\, dy^0 = \omega(y^\nu)\, d\chi(y^0) = d\big(\omega(y^\nu)\, \chi(y^0)\big) - \chi(y^0)\, d\omega(y^\nu).$$

Also, since $\boldsymbol{J}V = d(V^1 \wedge dV^2)$, it is clear that $d(\boldsymbol{J}V) = 0$, so that $d(\chi\omega) \wedge \boldsymbol{J}V = d(\chi\omega \wedge \boldsymbol{J}V)$ and, thus, the right-hand side of (5-13) can be rewritten

$$-\int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} (\omega\,\chi'\,dy^0) \wedge \boldsymbol{J}V = \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\,d\omega \wedge \boldsymbol{J}V - \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} d(\chi\omega) \wedge \boldsymbol{J}V$$
$$= \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\,d\omega \wedge \boldsymbol{J}V. \tag{5-14}$$

Step 2: The properties of ω and the choice of χ imply that

$$|\chi\,d\omega(y)| \le |y^\nu|^2 \quad \text{and} \quad \|D(\chi d\omega)\|_\infty \le C\delta^{-1}.$$

It thus follows from Lemma 21 that

$$\left|\int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\,d\omega \wedge \boldsymbol{J}V\right| \le C\,|\ln\varepsilon|^{-1} \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} |y^\nu|^2 \Big(\tfrac{1}{2}|DV|^2 + \frac{1}{\varepsilon^2}F(V)\Big)\,dy^\nu\,dy^0$$
$$+ C\varepsilon^\alpha(1+\delta^{-1})\bigg(1 + \int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \Big(\tfrac{1}{2}|DV|^2 + \frac{1}{\varepsilon^2}F(V)\Big)\,dy^\nu\,dy^0\bigg).$$

We now fix $\delta := \varepsilon^{\alpha/2}$ and recall the definition of V, to find that

$$\left|\int_{(-\delta,\tau+\delta)\times B_\nu(\rho)} \chi\,d\omega \wedge \boldsymbol{J}V\right| \le C\,|\ln\varepsilon|^{-1} \int_{(0,\tau)\times B_\nu(\rho)} \big(|y^\nu|^2 + \varepsilon^{\alpha/2}\big)\Big(\tfrac{1}{2}|v_{y^0}|^2 + e_{\varepsilon,\nu}(v)\Big)\,dy^\nu\,dy^0$$
$$+ C\varepsilon^{\alpha/2}\bigg(1 + \int_{\{0\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v)\,dy^\nu + \int_{\{\tau\}\times B_\nu(\rho)} e_{\varepsilon,\nu}(v)\,dy^\nu\bigg).$$

The conclusion now follows by recalling (5-13) and (5-14), and renaming α. □

5.2. *Proof of Proposition 15.* As in Proposition 10 it suffices to consider smooth solutions v.

To simplify we will write $\mathcal{D}(\,\cdot\,)$ and $|||\cdot|||$ instead of $\mathcal{D}(\,\cdot\,;\rho_1/2)$ and $|||\cdot|||_{\rho_1/2}$.

Step 1: For simplicity we assume that $s_1 = 0$. We will use the notation $s_{\max} := \min\{\rho_1/2c_*, T_1\}$,

$$W_\nu(s) := B_\nu(\rho_1 - c_* s) \quad \text{and} \quad W(s) := \mathbb{T}^n \times W_\nu(s).$$

We define

$$\zeta_1(s) := \delta_\varepsilon \int_{\{s\}\times W(s)} \big(1 + \kappa_2|y^\nu|^2\big)\,e_\varepsilon(v)\,dy' - 1,$$
$$\zeta_2(s) := \mathcal{D}(v(s)),$$
$$\zeta_3(s) := \delta_\varepsilon \int_{\{s\}\times W(s)} \big(|D_\tau v|^2 + |y^\nu|^2 e_{\varepsilon,\nu}(v)\big)\,dy'.$$

(Recall that κ_2 was fixed in (2-23), and that we took $\kappa_2 = 1$ for $k = 1$.) We first claim that

$$\zeta_1(s) \le \zeta_0 + C\int_0^s \zeta_3(\sigma)\,d\sigma \quad \text{for } 0 < s \le s_{\max}. \tag{5-15}$$

Indeed, exactly as before, we compute that $\zeta_1'(s) = I_1 - c_* I_2$, where

$$I_1 := \delta_\varepsilon \int_{\{s\}\times W(s)} \big(1+\kappa_2|y^\nu|^2\big) \frac{\partial}{\partial y^0} e_\varepsilon(v)\, dy'$$

$$I_2 = \delta_\varepsilon \int_{\{s\}\times \mathbb{T}^n \times \partial W_\nu(s)} \big(1+\kappa_2|y^\nu|^2\big)\, e_\varepsilon(v)\, d\mathcal{H}^{N-1}(y').$$

And, exactly as before, in I_1 we use the differential inequality (2-27) satisfied by the energy, and integrate by parts in the spatial variables. As before, our choice (3-4) of c_* guarantees that the boundary term that arises, involving an integral over $\{s\} \times \mathbb{T}^n \times \partial W_\nu(s)$, is dominated by $-c_* I_2$. This leads, as before, to the differential inequality

$$\zeta_1' \le C\zeta_3.$$

Since our assumption (5-4) states exactly that $\zeta_1(0) \le \zeta_0$, we conclude that (5-15) holds.

Step 2: We estimate ζ_2. It is clear that $|||\cdot|||$ is a norm, so

$$\mathcal{D}_\nu(v(s, y^{\tau\prime})) \le \mathcal{D}_\nu(v(0, y^{\tau\prime})) + |||J_\nu v(s, y^{\tau\prime}) - J_\nu v(0, y^{\tau\prime})|||$$

for every $(s, y^{\tau\prime})$, by the triangle inequality. It follows that

$$\begin{aligned}\zeta_2(s) &\le \mathcal{D}(v(0)) + \int_{\mathbb{T}^n} |||J_\nu v(0, y^{\tau\prime}) - J_\nu v(s, y^{\tau\prime})|||\, dy^{\tau\prime} \\ &\overset{(5\text{-}5),\,(5\text{-}11)}{\le} \zeta_0 + C\delta_\varepsilon \int_{\mathbb{T}^n}\int_{(0,s)\times B_\nu(\rho_1/2)} |D_\tau v|^2 + \big(|y^\nu|^2+\varepsilon^\alpha\big)\, e_{\varepsilon,\nu}(v)\, dy^\nu\, dy^0\, dy^{\tau\prime} \\ &\qquad + C\varepsilon^\alpha + C\varepsilon^\alpha \int_{\mathbb{T}^n}\Big(\int_{\{0\}\times B_\nu(\rho_1/2)} e_{\varepsilon,\nu}(v)\, dy^\nu + \int_{\{s\}\times B_\nu(\rho_1/2)} e_{\varepsilon,\nu}(v)\, dy^\nu\Big)\, dy^{\tau\prime}.\end{aligned}$$

Also, since $B_\nu(\rho_1/2) \subset W_\nu(s)$ for every $s \le \rho_0/2c_*$, the definitions yield

$$\int_{\mathbb{T}^n}\int_{\{s\}\times B_\nu(\rho_1/2)} e_\varepsilon(v)\, dy^\nu\, dy^{\tau\prime} \le C{\delta_\varepsilon}^{-1}(\zeta_1(s)+1) \le C|\ln\varepsilon|(\zeta_1(s)+1),$$

and similarly for $s = 0$. By combining these and rearranging, we find that if $0 \le s \le s_{\max}$, then

$$\zeta_2(s) \le \zeta_0 + C\int_0^s \big(\zeta_3(\sigma) + \varepsilon^\alpha(\zeta_1(\sigma)+C)\big)\, d\sigma + C\varepsilon^\alpha + C\varepsilon^{\alpha/2}\big(\zeta_0+\zeta_1(s)+C\big). \tag{5-16}$$

Step 3: Finally, we show (by *exactly* the same arguments as in the corresponding step of the proof of Proposition 10) that

$$\zeta_3(s) \le C\big(\zeta_1(s) + \zeta_2(s) + |\ln\varepsilon|^{-1}\big) \tag{5-17}$$

for every $s \in [0, s_{\max}]$. We fix such an s, and we write $v(\cdot)$ instead of $v(s, \cdot)$. It follows, from the definitions of ζ_1, ζ_3 and the choice (2-23) of κ_2, that

$$\zeta_1(s) \ge c\,\zeta_3(s) + \delta_\varepsilon \int_{\{s\}\times W(\rho_1/2c_*)} e_{\varepsilon,\nu}(v)\, dy' - 1. \tag{5-18}$$

We say that a point $y^{\tau\prime} \in \mathbb{T}^n$ is *good* if $\mathcal{D}_\nu(v(y^{\tau\prime})) \le \kappa_3$, and bad otherwise. Then, Chebyshev's inequality and (5-1) imply that $|\{y^{\tau\prime} \in \mathbb{T}^n : y^{\tau\prime} \text{ is good }\}| \ge 1 - C\zeta_2(s)$ and, exactly as in (3-19), but appealing to

Proposition 17 instead of Lemma 11, we infer that

$$\delta_\varepsilon \int_{\{s\}\times W(\rho_1/2c_*)} e_{\varepsilon,\nu}(v)\,dy' \geq (1-C\zeta_2(s))(1-C|\ln\varepsilon|^{-1}).$$

Combining this inequality with (5-18), we obtain (5-17).

Step 4: By combining the previous few steps, we see that

$$\zeta_3(s) \leq C\zeta_0 + C|\ln\varepsilon|^{-1} + C\int_0^s \zeta_3(\sigma)\,d\sigma + C\varepsilon^\alpha \int_0^s\int_0^\sigma \zeta_3(t)\,dt\,d\sigma.$$

If we define $\zeta_4(s) := \zeta_3(s) + \zeta_0 + |\ln\varepsilon|^{-1} + \varepsilon^\alpha \int_0^s \zeta_3(\sigma)\,d\sigma$, it follows (since $\zeta_0 \geq \delta_\varepsilon$) that

$$\zeta_4(s) \leq C\int_0^s \zeta_4(\sigma)\,d\sigma \quad \forall s\in[0,s_{\max}], \qquad \zeta_4(0)\leq C\zeta_0.$$

Gronwall's inequality then implies that $\zeta_4(s) \leq C\zeta_0$ for all $s\in[0,s_{\max}]$. The conclusions of the proposition follow from this, together with (5-15) and (5-16). □

5.3. *Proof of Proposition 16.* Finally, we present the proof of Proposition 16. We use notation from Section 4, such as $V^*(s)$, $\partial_i V^*(s)$ and so on.

As usual, we may assume by an approximation argument, relying on standard well-posedness theory for (2-26), that v is smooth on $\overline{V}$. Define v^* as in (4-5), and set

$$\begin{aligned}
\zeta_1(s) &= \delta_\varepsilon \int_{\partial_1 V^*(s)} \bigl(1+\kappa_2|y^\nu|^2 bigr) e_\varepsilon(v^*)\,dy' - 1,\\
\zeta_2(s) &= \mathcal{D}(v^*(s);\rho_1/2),\\
\zeta_3(s) &= \delta_\varepsilon \int_{\partial_1 V^*(s)} \bigl(|D_\tau v^*|^2 + |y^\nu|^2 e_{\varepsilon,\nu}(v^*)\bigr)\,dy'.
\end{aligned}$$

We repeat exactly the arguments of Proposition 15, to find that

$$\zeta_2(s) \leq C\int_{s_0}^s \zeta_3(\sigma) + \varepsilon^\alpha(\zeta_1(\sigma)+C)\,d\sigma + C\varepsilon^\alpha + C\varepsilon^{\alpha/2}(\zeta_0+\zeta_1(s)+C)$$

and

$$\zeta_3(s) \leq C(\zeta_1(s)+\zeta_2(s)+|\ln\varepsilon|^{-1}).$$

To estimate ζ_1, we argue as in the proof of Proposition 13; that is, we apply the divergence theorem to

$$\int_{V(s)} \partial_{y^\alpha}\bigl((1+\kappa_2|y^\nu|^2)\widetilde{T}^\alpha_{\varepsilon,0}\bigr),$$

where

$$\widetilde{T}^\alpha_{\varepsilon,\beta}(y) := \delta^\alpha_\beta\Bigl(\tfrac{1}{2}\varepsilon g^{\gamma\delta} v_{y^\gamma}\cdot v_{y^\delta} + \frac{1}{\varepsilon}F(v)\Bigr) - \varepsilon\, g^{\alpha\gamma} v_{y^\gamma}\cdot v_{y^\beta},$$

and we rewrite, noting that $n_\alpha(y)\,\widetilde{T}^\alpha_{\varepsilon,0}(y) \geq 0$ for a.e. $y\in\partial_2 V(s)$, exactly as before. This eventually yields, for a.e. $s\in[s_0,s_1]$,

$$\zeta_1(s) \leq C\int_{s_0}^s \zeta_3(\sigma)\,d\sigma + A + B,$$

where

$$A := \delta_\varepsilon \int_{W_0(s)} \big(1+\kappa_2|y^\nu|^2\big)\big(e_\varepsilon(v)-e_\varepsilon(v_0)\big)(b(y'),y')\,dy' + \delta_\varepsilon \int_{\partial_0 V(s)} \big(1+\kappa_2|y^\nu|^2\big) n_i \varphi^i \, d\mathcal{H}^N,$$

$$B := \delta_\varepsilon \int_{(W_1^*(s)\setminus W_1(s))\cup W_0(s)} \big(1+\kappa_2|y^\nu|^2\big) e_\varepsilon(v_0)\,dy' - 1.$$

We proceed exactly as in the proof of Proposition 13, using Lemmas 8 and 14, the hypotheses (2-34)–(2-36), and elementary arguments, to show that $A \le C\zeta_0$ and $B \le C\zeta_0$ for a.e. $s \in [s_0, s_1]$ and, hence, that $\zeta_1 \le C\int_{s_0}^s \zeta_3(\sigma)d\sigma + C\zeta_0$.

The proof is now finished, exactly as in Step 4 of the proof of Proposition 15. □

6. Proof of Theorems 1 and 2

We combine the estimates proved in the previous sections with standard energy estimates in the original (t, x) variables, iterate, and harvest consequences, to complete the proofs of our main results. We mostly give a unified treatment of the cases $k = 1$ and $k = 2$. To distinguish between the relevant energy densities in the (t, x) and the y variables, in this section we will often use the notation $e_\varepsilon(u; \eta)$ and $e_\varepsilon(v; G)$; see (1-28) and the following discussion.

The next theorem assembles most of our main estimates, and will easily imply Theorems 1 and 2; it can be seen as the main result of this paper. In it, and throughout this section, when we write $C(\Gamma, T_0)$, it will denote a constant that may depend upon various choices made in the construction (2-12) of the map ψ that we use to change variables; these choices, however, are constrained only by Γ and T_0.

Theorem 22. *Let $k = 1$ or 2, $n \ge 1$, and $N = n + k$.*

Let $\Gamma \subset (-T, T) \times \mathbb{R}^N$ be a smooth timelike Minkowski minimal surface of codimension k, satisfying our standing assumptions (2-5)–(2-7). *Let $u : (-T, T) \times \mathbb{R}^N \to \mathbb{R}^k$ solve* (1-1) *with initial data satisfying assumptions* (2-31) *and* (2-34)–(2-36), *for some ζ_0 verifying* (2-30).

Given $T_0 < T$, fix $T_1 \in (T_0, T)$ and $\rho_0 > 0$ so small that (2-13), (2-14), *and the conclusions of* Proposition 4 *hold on $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$.*

There exists a constant $C(\Gamma, T_0)$ such that

$$\delta_\varepsilon \int_{((-T_0,T_0)\times\mathbb{R}^N)\setminus\mathcal{N}} e_\varepsilon(u;\eta)\,dx\,dt \le C\zeta_0, \tag{6-1}$$

for $\mathcal{N} = \mathrm{image}(\psi) \cap ((-T_0, T_0) \times \mathbb{R}^N)$, and such that $v = u \circ \psi$ satisfies

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |D_\tau v|^2 + |y^\nu|^2\Big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\Big)\,dy \le C\zeta_0, \tag{6-2}$$

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \big(1+\kappa_2|y^\nu|^2\big) e_\varepsilon(v;G)\,dy - \mathcal{H}^{1+n}\big((-T_1,T_1)\times\mathbb{T}^n\big) \le C\zeta_0, \tag{6-3}$$

(*for κ_2 as in* (2-23), *with $\kappa_2 = 1$ when $k = 1$*), *and*

$$\int_{(-T_1,T_1)\times\mathbb{T}^n} \mathcal{D}_\nu(v(y^\tau);\rho_1/2)\,dy^\tau \le C\zeta_0, \tag{6-4}$$

where $\mathscr{D}_\nu$ was defined in (3-3) *for $k=1$, and in* (5-2) *for $k=2$, while ρ_1 was found in Lemma 14. Finally,*

$$\left\| \delta_\varepsilon \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma) \right\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \le C\sqrt{\zeta_0}. \tag{6-5}$$

The following lemma will be used repeatedly.

Lemma 23. *There exists a constant $C>0$, depending only on Γ, T_1, ρ_0, such that*

$$C^{-1} e_\varepsilon(u;\eta)(\psi(y)) \le \; e_\varepsilon(v;G)(y) \; \le \; C e_\varepsilon(u;\eta)(\psi(y))$$

and

$$C^{-1} \le |\det D\psi(y)| = \sqrt{-g(y)} \le C$$

for all $y \in (-T_1, T_1)\times \mathbb{T}^n \times B_\nu(\rho_0)$.

Proof. This is clear from the construction of the diffeomorphism ψ, see in particular (2-20). □

Next, we show that Theorems 1 and 2 follow directly from Theorem 22 and the above Lemma. The rest of this section will then be devoted to the proof of Theorem 22.

Proofs of Theorems 1 and 2. First we consider the scalar case, i.e., that of Theorem 1. We define $\mathcal{N}$ as in (2-22). Then, as noted in Corollary 7, the function d defined by (2-29) satisfies the eikonal equation (1-12) in $\mathcal{N}$, as required.

Let u solve (1-1) with the initial data given by Lemma 9, in the case $k=1$, so that it satisfies the assumptions of Theorem 22 with $\zeta_0 = C\varepsilon^2$ and, in addition,

$$\int_{\mathbb{T}^n\times B_\nu(\rho_0)} \Big(v_0 - q\Big(\frac{y^N}{\varepsilon}\Big)\Big)^2 dy' \le C\varepsilon \tag{6-6}$$

for v_0 as defined in (2-33).

Then, conclusion (1-15) of Theorem 1 is exactly (6-5).

To prove (1-14), we recall that $\mathcal{N} \subset \psi((-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0))$ and use (2-29) and Lemma 23 to estimate

$$\begin{aligned} \delta_\varepsilon \int_{\mathcal{N}} d^2 e_\varepsilon(u;\eta)\,dx\,dt \; &\le \; C\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} (y^\nu)^2 e_\varepsilon(v;G)\,dx\,dt \\ &\le C\zeta_0 - \bigg(\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} e_\varepsilon(v;G)\,dy - \mathcal{H}^{1+n}((-T_1,T_1)\times\mathbb{T}^n)\bigg), \end{aligned}$$

where the latter inequality is due to (6-3). Next, by using Lemma 11 and arguing exactly as in the proof of (3-17), we see that

$$2T_1 - \delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} e_\varepsilon(v;G)\,dy \le C\int_{(-T_1,T_1)\times\mathbb{T}^n} \mathscr{D}_\nu(v(y^\tau))\,dy^\tau + Ce^{-c/\varepsilon}. \tag{6-7}$$

The above inequalities and (6-4) imply that $\delta_\varepsilon \int_{\mathcal{N}} d^2 e_\varepsilon(u;\eta)\,dx\,dt \le C\varepsilon^2$. By combining this with (6-1), we obtain (1-14).

Finally, to prove (1-13), note that for every $y' \in \mathbb{T}^n\times B_\nu(\rho_1)$ and $y^0 \in (-T_1,T_1)$,

$$\left|v(y^0,y') - v_0(y')\right| = \left|v(y^0,y') - v(b(y'),y')\right| \le \left|y^0 - b(y')\right|^{1/2} \bigg(\int_{-T_1}^{T_1} \left|\partial_{y^0} v(s,y')\right|^2 ds\bigg)^{1/2}.$$

Since $|\partial_{y^0} v| \le |D_\tau v|$, we find by integrating that

$$\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \bigl|v(y^0,y') - v_0(y')\bigr|^2 dy \;\le\; C\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |D_\tau v|^2\,dy \le C\varepsilon,$$

using (6-2) (for the last inequality) and the fact that $\zeta_0 \le C\varepsilon^2$. Then, (6-6) implies that

$$\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \bigl|v(y) - q(y^N/\varepsilon)\bigr|^2 dy \le C\varepsilon.$$

By changing variables, using Lemma 23, and recalling (2-29), we obtain (1-13).

The proof of Theorem 2 is much the same, except that we make no claim about $\int |v(y^0,y') - v_0(y')|^2$, as the estimate $\int |\partial_{y^0} v|^2 dy \le C$ is too weak to provide good control over this quantity.

Otherwise, we follow the above proof; that is, we let u solve (1-1) with the initial data given by Lemma 9, in the case $k=2$, so that it satisfies the assumptions of Theorem 22 with $\zeta_0 = C|\ln\varepsilon|^{-1}$. Then, conclusion (1-21) of Theorem 2 is exactly (6-5). To prove (1-20), it suffices, in view of (6-1), to prove that

$$\int_{\mathcal{N}} \operatorname{dist}(\,\cdot\,,\Gamma)^2 e_\varepsilon(u;\eta)\,dx\,dt \le C.$$

Using Lemma 23 to change variables, and noting that $\operatorname{dist}(\psi(y),\Gamma)^2 \le C|y^\nu|^2$ (since the left-hand side is a smooth function of y that vanishes when $y^\nu = 0$), it suffices to prove that

$$\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |y^\nu|^2 e_\varepsilon(v;G)\,dy \;\le\; C.$$

This follows exactly the proof of (1-14) in the case $k=1$ above. The estimate corresponding to (6-7) has *exactly* the same form, except that the last term on the right-hand side is now $C|\ln\varepsilon|^{-1}$; this is proved by arguing as before, but using Proposition 17 in place of Lemma 11. ☐

The proof of Theorem 22 will use some standard energy estimates that we now recall.

Lemma 24. *Let $u:\mathbb{R}^{1+N}\to\mathbb{R}^k$ be a smooth, finite-energy solution of the semilinear wave equation* (1-1). *For any $a<b$ and bounded Lipschitz function $\chi:\mathbb{R}^{1+N}\to\mathbb{R}$,*

$$\Bigl|\int_{\{b\}\times\mathbb{R}^N} e_\varepsilon(u;\eta)\,\chi\,dx - \int_{\{a\}\times\mathbb{R}^N} e_\varepsilon(u;\eta)\,\chi\,dx\Bigr| \le \int_{(a,b)\times\mathbb{R}^N} e_\varepsilon(u;\eta)\,|D\chi|\,dx\,dt. \tag{6-8}$$

Also, for any pair a,b of real numbers and open $A\subset\mathbb{R}^N$,

$$\int_{\{b\}\times A_{|b-a|}} e_\varepsilon(u;\eta)\,dx \le \int_{\{a\}\times A} e_\varepsilon(u;\eta)\,dx, \tag{6-9}$$

where $A_s := \{x\in A : \operatorname{dist}(x,\partial A) > s\}$.

Proof. Both conclusions are standard, and follow from the identity $\partial_t e_\varepsilon(u;\eta) = \nabla\cdot(u_t\nabla u)$ satisfied by solutions of (1-1), integration by parts, and the elementary inequality $|u_t\nabla u| \le e_\varepsilon(u;\eta)$. For the second inequality, assuming for concreteness that $a<b$, it is easy to see that the set $\{(t,x): a<t<b,\ x\in A_{t-a}\}$ is a set of finite perimeter, so that the divergence theorem holds and there is no problem in justifying the standard argument. ☐

Lemma 25. *Let $v : (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) \to \mathbb{R}^k$ be a smooth solution of* (2-26). *For any $-T_1 \le a < b \le T_1$ and $\chi \in W_0^{1,\infty}\big((-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)\big)$,*

$$\left|\int_{\{b\}\times\mathbb{R}^N} e_\varepsilon(v; G)\,\chi\,dy' - \int_{\{a\}\times\mathbb{R}^N} e_\varepsilon(v; G)\,\chi\,dy'\right| \le C\int_{(a,b)\times\mathbb{R}^N} e_\varepsilon(v; G)\,(|\chi| + |D\chi|)\,dy.$$

Proof. Lemma 6 implies that $\partial_{y^0} e_\varepsilon(v; G) \le C e_\varepsilon(v; G) + \nabla\cdot\varphi$, and the positivity (2-16) of the matrix $(a^{\alpha\beta})$ together with the definition (2-28) of φ imply that $|\varphi| \le C e_\varepsilon(v; G)$. The conclusion follows from these facts, together with integration by parts, exactly as in the previous lemma. □

Now, we present:

Proof of Theorem 22. We treat both cases $k = 1$ and $k = 2$ simultaneously. We may assume, as usual, that u and $v = u \circ \psi$ are smooth.

It is convenient to define $\rho : (-T_0, T_0) \times \mathbb{R}^N \to [0, +\infty]$ by

$$\rho(t, x) = \begin{cases} |y^\nu| & \text{if } (t, x) = \psi(y) \\ +\infty & \text{if } (t, x) \notin \mathcal{N} = \text{image}(\psi). \end{cases}$$

Note that, when $k = 1$, $\rho(t, x) = |d(t, x)|$ for $(t, x) \in \mathcal{N}$.

Step 1: Given $u : (-T, T) \times \mathbb{R}^N \to \mathbb{R}^k$ solving (1-1), $k = 1$ or $k = 2$, we will say that u is *controlled* on a set $W \subset (-T_0, T_0) \times \mathbb{R}^N$ if there exists a constant C, depending on W, Γ, and ψ, such that ,for any function u satisfying the hypotheses of Theorem 22,

$$\int_W e_\varepsilon(u; \eta) \le C\zeta_0.$$

(We will only say this about sets that are bounded away from Γ.) If W is an open set, the integral is understood as $\int \cdots\, dx\,dt$ and, if W is a subset of some $\{t\} \times \mathbb{R}^N$, it is understood as $\int \cdots\, dx$.

Similarly, for a set $W \subset (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$, we say that $v = u \circ \psi$ is *controlled* on W if there exists a constant $C = C(W, \Gamma, T_0)$ such that

$$\delta_\varepsilon \int_W \Big(|D_\tau v|^2 + |y^\nu|^2\Big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\Big)\Big) \le C\zeta_0,$$

Again, W may be either an open set or a subset of $\{s\} \times \mathbb{T}^n \times B_\nu(\rho_0)$ for some s, with the integral understood accordingly.

We make some easy remarks. First, if v is controlled on a set W, then, since

$$e_\varepsilon(v; G) \le C(\hat\rho)\Big(|D_\tau v|^2 + |y^\nu|^2\Big(|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2}F(v)\Big)\Big) \quad \text{whenever } |y^\nu| \ge \hat\rho,$$

it follows that, for any $\hat\rho \in (0, \rho_0)$,

$$\int_{\{y=(y^\tau, y^\nu)\in W : |y^\nu| \ge \hat\rho\}} e_\varepsilon(v; G) \le C\zeta_0.$$

As a result, Lemma 23 implies that, if $A \subset \text{Image}(\psi)$ is bounded away from Γ, then u is controlled on A if and only if v is controlled on $\psi^{-1}(A)$, Finally, we remark that, for any $\hat\rho > 0$, the assumptions (2-31), (2-34), (2-35), and a change of variables imply that

$$u \text{ is controlled on } \{(0,x) \in \mathbb{R}^{1+N} : \rho(0,x) \ge \hat\rho\}, \tag{6-10}$$

with the implicit constants depending on $\hat\rho$ and ψ or, more precisely, on the behavior of ψ on $\{y \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0) : \psi^0(y) = 0\}$.

Step 2: We next claim that for $0 < s'_+ < s_+ \le T_1$ and $\rho' < \rho_1/2$,

$$\begin{aligned} &\text{if } v \text{ is controlled on } \{y \in (-T_1, s_+) \times \mathbb{T}^n \times B_\nu(\rho_1/2) : \psi^0(y) > 0\},\\ &\text{then, for every } t \in [0, s'_+],\ u \text{ is controlled on } \{(t,x) : \rho(t,x) > \rho'\}, \end{aligned} \tag{6-11}$$

and this control is uniform for $t \in [0, s'_+]$. To prove this, we fix $\hat\rho > 0$ so small that $\hat\rho \le \rho'$ and

$$\begin{aligned} \{(t,x) : 0 < t < s'_+,\ \rho(t,x) < \hat\rho\} &= \{\psi(y) : y \in (-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\hat\rho),\ 0 < \psi^0(y) < s'_+\}\\ &\subset \{\psi(y) : y \in (-T_1, s_+) \times \mathbb{T}^n \times B_\nu(\hat\rho) : \psi^0(y) > 0\}. \end{aligned}$$

The point is that, if $|y^\nu|$ is small enough and $\psi^0(y) < s'_+$, then $y^0 < s_+$. Such a number $\hat\rho$ exists, because $\left|\psi^0(y^0, y^{\tau'}, y^\nu) - y^0\right| \le C|y^\nu|$; this is an easy consequence of the definition of ψ.

Then, it follows from the control of v and Step 1 that

$$u \text{ is controlled on } \{(t,x) : 0 < t < s'_+,\ \hat\rho/2 < \rho(t,x) < \hat\rho\}, \tag{6-12}$$

since this set is bounded away from Γ and is contained, by the choice of $\hat\rho$, in the image via ψ of a set on which we have assumed that v is controlled.

Now, we fix a function $\chi \in C^\infty([0, s'_+] \times \mathbb{R}^N)$ such that $\chi = 1$ wherever $\rho(t,x) > \hat\rho$, and $\chi = 0$ where $\rho(t,x) \le \hat\rho/2$. Then, we apply (6-8) with this choice of χ and with $a = 0$ and $b \in (0, s'_+)$ and, using (6-10) and (6-12), we find that u is controlled on $\{(b,x) : \rho(b,x) > \hat\rho\}$, with implicit constants that are uniform for $b \in (0, s'_+]$. Thus, we have proved (6-11).

Step 3: We next claim that, for $0 < s_+ \le T_1$ as above,

$$\begin{aligned} &\text{if } v \text{ is controlled on } \{y \in (-T_1, s_+) \times \mathbb{T}^n \times B_\nu(\rho_1/2) : \psi^0(y) > 0\},\\ &\text{then } v \text{ is controlled on } \{s_+\} \times \mathbb{T}^n \times \big(B_\nu(\rho_1) \setminus B_\nu(\rho_1/2)\big). \end{aligned} \tag{6-13}$$

We first apply (6-11), with parameters $s'_+ < s_+$ and $\rho' < \frac{1}{2}\rho_1$ to be fixed later. We then apply (6-9) with $a = s'_+$ and $|b| \le T$, to conclude that u is controlled on

$$S(s'_+, \rho') := \{(t,x) : |t| \le T,\ \operatorname{dist}(x, A(s'_+, \rho')) > |t - s'_+|\}$$

where

$$A(s'_+, \rho') := \{x : \rho(s'_+, x) > \rho'\}.$$

Then, Step 1 implies that v is controlled on $\psi^{-1}(S(s'_+, \rho'))$.

We will show that we can fix $s'_+ < s_+$ and $\rho' > 0$ such that

$$\psi\big(\{s_+\} \times \mathbb{T}^n \times (B_\nu(\rho_0) \setminus B_\nu(\rho_1/4))\big) \Subset S(s'_+, \rho'), \tag{6-14}$$

by which we mean that some open neighborhood of $\psi(\cdots)$ is contained in $S(s'_+, \rho')$. For now, we assume that we have selected s'_+ and ρ' so that (6-14) holds, and we complete the proof of (6-13).

Indeed, if (6-14) holds, then clearly

$$\{s_+\} \times \mathbb{T}^n \times \big(B_\nu(\rho_0) \setminus B_\nu(\rho_1/4)\big) \Subset \psi^{-1}(S(s'_+, \rho')).$$

and, so, there exists some $a < s_+$ such that $(a, s_+) \times \mathbb{T}^n \times (B_\nu(\rho_0) \setminus B_\nu(\rho_1/4)) \Subset \psi^{-1}(S(s'_+, \rho'))$. Now, we can find some smooth nonnegative function χ such that

$$\chi = 1 \text{ on } \{s_+\} \times \mathbb{T}^n \times \big(\overline{B}_\nu(\rho_1) \setminus B_\nu(\rho_1/2)\big) \quad \text{and} \quad \operatorname{spt}(\chi) \subset (a, T_1) \times \mathbb{T}^n \times \big(B_\nu(\rho_0) \setminus B_\nu(\rho_1/4)\big).$$

Since v is controlled in $\psi^{-1}(S(s'_+, \rho'))$, (6-13) follows from applying Lemma 25 with this choice of χ and a, and with $b = s_+$.

Step 4: We next verify (6-14). Since the sets $S(s, \rho)$ depend continuously on s and ρ in an obvious way, (6-14) will follow (for suitable $s'_+ < s_+$ and $\rho' > 0$) if we can show that

$$\psi\big(\{s_+\} \times \mathbb{T}^n \times (B_\nu(\rho_0) \setminus B_\nu(\rho_1/4))\big) \Subset S(s_+, 0). \tag{6-15}$$

We will deduce this as a consequence of the following fact: If Σ is a connected spacelike hypersurface in $(1+N)$-dimensional Minkowski space, and we define the solid light cone with vertex (t, x) to be

$$LC(t, x) := \{(t', x') : |x - x'| \geq |t - t'|\},$$

then $LC(t, x) \cap \Sigma = \{(t, x)\}$ for every $(t, x) \in \Sigma$.

To reduce (6-15) to this geometric fact, we define

$$\Sigma := \psi\big(\{s_+\} \times \mathbb{T}^n \times B_\nu(\rho_0)\big).$$

Clearly, Σ is a connected hypersurface. We claim that it is also spacelike. To see this, recall (see (2-20)) that $(g_{ij})_{i,j=1}^N$ is positive definite; this implies that $\psi^{-1}(\Sigma) = \{s_+\} \times \mathbb{T}^n \times B_\nu(\rho_0)$ is spacelike with respect to the $(g_{\alpha\beta})$ metric. The claim then follows, since ψ is an isometry between $(-T_1, T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)$ with the $(g_{\alpha\beta})$ metric and image$(\psi) \subset \mathbb{R}^{1+N}$ with the Minkowski metric in standard form $ds^2 = -dt^2 + (dx^1)^2 + \ldots + (dx^N)^2$.

Next, note that the definition of ψ and the choice (2-13) of ρ_0 imply that

$$\psi\big(\{s_+\} \times \mathbb{T}^n \times (B_\nu(\rho_0) \setminus B_\nu(\rho_1/4))\big) \Subset (-T, T) \times \mathbb{R}^N.$$

Thus, in order to prove (6-15), it suffices to show that the closure of $\psi\big(\{s_+\} \times \mathbb{T}^n \times (B_\nu(\rho_0) \setminus B_\nu(\rho_1/4))\big)$ does not intersect $((-T, T) \times \mathbb{R}^N) \setminus S(s_+, 0)$. However, by inspection of the definition of $S(s, \rho)$, one sees that

$$((-T, T) \times \mathbb{R}^N) \setminus S(s_+, 0) \;\subset \bigcup_{\{(x \in \mathbb{R}^N : \rho(s_+, x) = 0\}} LC(s_+, x) = \bigcup_{\{x \in \mathbb{R}^N : (s_+, x) \in \Gamma\}} LC(s_+, x).$$

In addition, since $\Gamma \cap (\{s_+\} \times \mathbb{R}^N) = \psi(\{s_+\} \times \mathbb{T}^n \times \{0\})$, it is clear that

$$\Sigma \;\supset\; \Gamma \cap (\{s_+\} \times \mathbb{R}^N).$$

Then, the geometric fact mentioned above implies that

$$\Sigma \cap \big(((-T, T) \times \mathbb{R}^N) \setminus S(s_+, 0)\big) \;\subset \bigcup_{\{x : (s_+, x) \in \Gamma\}} \Sigma \cap LC(s_+, x) = \Gamma \cap (\{s_+\} \times \mathbb{R}^N) = \psi(\{s_+\} \times \mathbb{T}^n \times \{0\}).$$

Since ψ is injective, this implies that $\psi\big(\{s_+\}\times\mathbb{T}^n\times(B_\nu(\rho_0)\backslash\{0\})\big)$ does not intersect $((-T,T)\times\mathbb{R}^N)\setminus S(s_+,0)$, completing the proof of (6-15).

Step 5: We introduce more terminology. For a set $W\subset(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$, if there exists $W_\tau\subset(-T_1,T_1)\times\mathbb{T}^n$ such that

$$W_\tau\times B_\nu(\rho_1/2)\subset W\subset W_\tau\times B_\nu(\rho_0),$$

then we say that v is *completely controlled* on W if v is controlled on W and, in addition, there exists a constant $C(W,\psi)$ such that

$$\delta_\varepsilon\int_W\big((1+\kappa_2|y^\nu|^2)\,e_\varepsilon(v;G)\big)-\mathcal{H}^{\dim W_\tau}(W_\tau)\le C\zeta_0\quad\text{and}\quad\int_{W_\tau}\mathcal{D}_\nu(v(y^\tau))\ \le C\zeta_0.$$

And, as above, we allow W to be either an open set or a subset of some $\{y^0=\text{constant}\}$ slice, with the integral understood accordingly, and with $\dim W_\tau=1+n$ in the first case and $\dim W_\tau=n$ in the second.

Now, we establish estimates (6-1)–(6-4). First, by assumption, $v=u\circ\psi$ satisfies the hypotheses of Proposition 13 (if $k=1$) and Proposition 16 (if $k=2$), and these imply that

$$v\text{ is controlled on }\{y\in(-T_1,s_1)\times\mathbb{T}^n\times B_\nu(\rho_1):\psi^0(y)>0\},\quad\text{and}\tag{6-16}$$

$$v\text{ is completely controlled on }\{s_1\}\times\mathbb{T}^n\times B_\nu(\rho_1).\tag{6-17}$$

In particular, (6-17) implies that v satisfies the hypotheses of Proposition 10 ($k=1$) or Proposition 15 ($k=2$), with ζ_0 replaced by $C\zeta_0$. These propositions assert that

$$v\text{ is completely controlled on }\{s\}\times\mathbb{T}^n\times B_\nu(\rho_1/2)\text{ for }s_1\le s\le s_2,\tag{6-18}$$

where $s_2:=\min\{T_1,s_1+(\rho_1/2c_*)\}$. Then, (6-16) and (6-18) imply that v is controlled on

$$\{y\in(-T_1,s_2)\times\mathbb{T}^n\times B_\nu(\rho_1/2):\psi^0(y)>0\}.$$

Next, we invoke (6-13) to find that v is controlled on $\{s_2\}\times\mathbb{T}^n\times(B_\nu(\rho_1)\setminus B_\nu(\rho_1/2))$. Hence, appealing again to (6-18), we see that v is completely controlled on $\{s_2\}\times\mathbb{T}^n\times B_\nu(\rho_1)$.

Thus, we can apply Proposition 10 or 15, with s_1 replaced by s_2 and ζ_0 multiplied by a suitable constant, but with the same fixed valued of ρ_1 already used. We can repeat this argument as necessary to find, after a finite number of iterations, that v is completely controlled on $\{s\}\times\mathbb{T}^n\times B_\nu(\rho_1/2)$ for $s_1\le s\le T_1$. Since all our energy estimates are clearly valid backwards in the timelike variables, we can also iterate Proposition 10 or 15 backwards, starting from s_1 and arguing as above, to conclude that

$$v\text{ is completely controlled on }\{s\}\times\mathbb{T}^n\times B_\nu(\rho_1/2)\text{ for }-T_1\le s\le T_1.\tag{6-19}$$

Since $T_1>T_0$, we deduce, by applying (6-11) in both directions in the t variable, that

$$u\text{ is controlled on }\{(t,x)\in(-T_0,T_0)\times\mathbb{R}^N:\rho(t,x)\ge\rho_1/4\}.\tag{6-20}$$

Using these and Lemma 24, we can deduce (arguing as in the proof of (6-11) and (6-13)) that in fact v is completely controlled on $(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)$. These estimates imply (6-1)–(6-4).

Step 6: It remains to prove (6-5). The point is that it essentially suffices to prove the same estimate in the y variables, in which (6-2)–(6-4) imply a great deal of information about the way in which energy concentrates around Γ, which in these variables is $(-T_1, T_1)\times\mathbb{T}^n\times\{0\}$. We will extract this information using Lemma 11 for the case $k=1$, and an estimate of [Kurzke and Spirn 2009] for $k=2$.

If $m=(m^\beta_\alpha)$ and $\mathcal{T}=(\mathcal{T}^\alpha_\beta)$, let us write

$$\langle m, \mathcal{T}\rangle := \int m^\beta_\alpha \, d\mathcal{T}^\alpha_\beta.$$

Then, we must estimate $\langle m, \delta_\varepsilon \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\rangle$ for $(m^\beta_\alpha)\in W^{1,\infty}((-T_0, T_0)\times\mathbb{R}^N)$, with compact support and with $\|m\|_{W^{1,\infty}}\le 1$. To do this, let χ be a smooth function with support in image(ψ) and such that $\chi=1$ on $\{(t,x) : |t|<T_0,\ \rho(t,x)<\rho_0/2\}$. Then,

$$\bigl\langle m,\ \mathcal{T}_\varepsilon(u)-\mathcal{T}(\Gamma)\bigr\rangle = \bigl\langle (1-\chi)m,\ \mathcal{T}_\varepsilon(u)\bigr\rangle + \bigl\langle \chi m,\ \mathcal{T}_\varepsilon(u)-\mathcal{T}(\Gamma)\bigr\rangle.$$

It is clear from the definition (2-8) of $\mathcal{T}_\varepsilon$ that $|\mathcal{T}^\alpha_{\varepsilon,\beta}(u)|\le Ce_\varepsilon(u;\eta)$, so that

$$\bigl|\bigl\langle (1-\chi)m,\ \mathcal{T}_\varepsilon(u)\bigr\rangle\bigr| \le \sum_{\alpha,\beta}\|m^\beta_\alpha\|_\infty \int_{\{(t,x)\in(-T_0,T_0)\times\mathbb{R}^N\,:\,\rho(t,x)\ge\rho_0/2\}} e_\varepsilon(u;\eta)\,dt\,dx \tag{6-21}$$
$$\le C\zeta_0, \tag{6-22}$$

using (6-1) and (6-2) together with Lemma 23.

<u>Step 7</u>: Let us write $\overline{m} := \chi m$. Note that $\overline{m}$ is supported in image(ψ), and $\|\overline{m}\|_{W^{1,\infty}}\le C$. We will write

$$\check{m}^\gamma_\delta(y) = \overline{m}^\beta_\alpha\circ\psi(y)\ \psi^\alpha_{y_\delta}(y)\ \varphi^\gamma_{x^\beta}\circ\psi(y)\sqrt{-g(y)} \quad\text{ and }\quad \varphi := \psi^{-1} \text{ as usual.} \tag{6-23}$$

Note that $\|\check{m}\|_{W^{1,\infty}}\le C$. We claim that

$$\langle\overline{m}, \mathcal{T}_\varepsilon(u)\rangle = \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\gamma_\delta(y)\,\widetilde{\mathcal{T}}^\delta_{\varepsilon,\gamma}(v)(y)\,dy \tag{6-24}$$

and

$$\langle\overline{m}, \mathcal{T}(\Gamma)\rangle = \int_{(-T_1,T_1)\times\mathbb{T}^n} \check{m}^\gamma_\delta(y^\tau,0)\,\tilde{P}^\delta_\gamma\,dy^\tau, \tag{6-25}$$

where $\widetilde{\mathcal{T}}_\varepsilon(v)$ was defined[9] in (4-12), $\tilde{P}^\delta_\gamma = 1$ if $\delta=\gamma\in\{0,\dots,n\}$ and $\tilde{P}^\delta_\gamma=0$ otherwise. These are arguably obvious from the tensorial nature of the quantities involved. However, for the convenience of the reader, we note that the definitions (2-8), (4-12) and (6-23) imply that

$$\overline{m}^\beta_\alpha(t,x)\,\mathcal{T}^\alpha_{\varepsilon,\beta}(u)(t,x) = \check{m}^\beta_\alpha(y)\,\widetilde{\mathcal{T}}^\alpha_{\varepsilon,\beta}(v)(y)(-g(y))^{-1/2} \ \text{ for } (t,x)=\psi(y).$$

Then, (6-24) follows from a change of variables, noting that $|\det D\psi| = \sqrt{-g}$, so $dt\,dx = \sqrt{-g(y)}\,dy$. To rewrite $\langle\overline{m}, \mathcal{T}(\Gamma)\rangle$, note that our proof of Lemma 3 (to which we refer for notation) showed that

$$\langle\overline{m}, \mathcal{T}(\Gamma)\rangle = \int_{(-T,T)\times\mathbb{T}^n} (\overline{m}^\beta_\alpha\circ H)\,H^\alpha_{y^a}\,\eta_{\beta\delta}\,H^\delta_{y^b}\,\gamma^{ab}\sqrt{-\gamma}\,dy^\tau,$$

[9] As remarked earlier, $\widetilde{\mathcal{T}}_\varepsilon(v)$ is just the energy-momentum tensor for u, expressed in terms of the y variables.

where $H : (-T, T) \times \mathbb{T}^n \to (-T, T) \times \mathbb{R}^N$ is the given map parametrizing Γ (see (2-5)), and with a, b summed implicitly from 0 to n. Since $\psi(y^\tau, 0) = H(y^\tau)$, we can rewrite the integrand above in terms of ψ and g, and this leads to (6-25). For this, it is useful to note that

$$g_{\alpha\beta}(y^\tau, 0) = \begin{cases} \gamma_{\alpha\beta}(y^\tau) & \text{if } \alpha, \beta \le n, \\ \delta_{\alpha\beta} & \text{if } \alpha, \beta > n, \\ 0 & \text{otherwise,} \end{cases}$$

and that

$$g^{ab}\, \psi^\delta_{y_b}\, \eta_{\delta\beta} = \varphi^a_\alpha \circ \psi\; \eta^{\alpha\gamma}\; \varphi^b_\gamma \circ \psi\; \psi^\delta_{y_b}\, \eta_{\delta\beta} = \varphi^a_{x^\beta} \circ \psi$$

for $a \in \{0, \dots, n\}$.

Step 8: We now apply our earlier estimates to control various terms in $\langle \overline{m}, \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\rangle$ represented, as in (6-24) and (6-25), in terms of the y coordinates. In these calculations, we do *not* sum over indices γ and δ when they are repeated.

Case 1: $\gamma \ne \delta$, $\delta \le n$: When this holds, we have, using successively (4-12) and (2-19),

$$\begin{aligned} |\widetilde{T}^\delta_{\varepsilon,\gamma}| &= |g^{\delta\alpha} v_{y^\alpha} v_{y^\gamma}| \\ &\le C(|D_\tau v|^2 + |y^\nu|^2\, |\nabla_\nu v|^2). \end{aligned}$$

Thus, in this case,

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\gamma_\delta\, \widetilde{\mathcal{T}}^\delta_{\varepsilon,\gamma}\, dy \;\le\; C\|\check{m}\|_\infty \zeta_0,$$

where we have used (6-2).

Case 2: $\gamma \ne \delta$, $\gamma \le n$: In this case, we have the weaker estimate

$$\begin{aligned} |\widetilde{T}^\delta_{\varepsilon,\gamma}| &= |g^{\delta\alpha} v_{y^\alpha} v_{y^\gamma}| \\ &\le C(|D_\tau v|^2 + |D_\tau v|\, |\nabla_\nu v|), \end{aligned}$$

again using (4-12) and (2-19). So, we have

$$\begin{aligned} \delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\gamma_\delta\, \widetilde{\mathcal{T}}^\delta_{\varepsilon,\gamma}\, dy &\le C\delta_\varepsilon \|\check{m}\|_\infty \big(\|D_\tau v\|_2^2 + \|D_\tau v\|_2\, \|\nabla_\nu v\|_2\big) \\ &\le C\|\check{m}\|_\infty \big(\zeta_0 + \sqrt{\zeta_0}\sqrt{\delta_\varepsilon}\|\nabla_\nu v\|_2\big) \quad \text{(using (6-2))} \\ &\le C\|\check{m}\|_\infty \big(\zeta_0 + \sqrt{\zeta_0}\sqrt{(\zeta_0 + 2T_1)}\big) \quad \text{(using (6-3)).} \end{aligned}$$

Case 3: $\gamma = \delta \le n$: This is the only case in which $\langle m, \mathcal{T}(\Gamma)\rangle$ makes a nonzero contribution. Indeed, by (2-19), $|g^{\delta\alpha} v_{y^\alpha} v_{y^\delta}| \le C(|D_\tau v|^2 + |D_\tau v|\, |\nabla_\nu v|)$, so that

$$\begin{aligned} \widetilde{T}^\delta_{\varepsilon,\delta} &= \tfrac{1}{2} g^{\alpha\beta} v_{y_\alpha} v_{y^\beta} + \tfrac{1}{\varepsilon^2} F(v) + O(|D_\tau v|^2 + |D_\tau v|\, |\nabla_\nu v|) \\ &= \frac{1}{2}|\nabla_\nu v|^2 + \frac{1}{\varepsilon^2} F(v) + O(|D_\tau v|^2 + |D_\tau v|\, |\nabla_\nu v|), \end{aligned}$$

using (2-19) and (2-20). Thus,

$$\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\delta\, \widetilde{\mathcal{T}}^\delta_{\varepsilon,\delta}\, dy = \delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\delta\, e_{\varepsilon,\nu}(v)\, dy + O(\|\check{m}\|_\infty \sqrt{\zeta_0}\,).$$

The contribution to $\langle \overline{m}, \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\rangle$ from a summand with $\delta = \gamma \le n$ is, thus,

$$\begin{aligned}\int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\delta\, \delta_\varepsilon\, e_{\varepsilon,\nu}(v)\, dy - \int_{(-T_1,T_1)\times\mathbb{T}^n} m^\delta_\delta(y^\tau, 0)\, dy^\tau + O(\|\check{m}\|_\infty \sqrt{\zeta_0}\,)\\ = \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \big(\check{m}^\delta_\delta(y^\tau, y^\nu) - \check{m}^\delta_\delta(y^\tau, 0)\big)\, \delta_\varepsilon\, e_{\varepsilon,\nu}(v)\, dy\\ - \int_{(-T_1,T_1)\times\mathbb{T}^n} m^\delta_\delta(y^\tau, 0)\left(1 - \delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)(y^\tau, y^\nu)\, dy^\nu\right) dy^\tau + O(\|\check{m}\|_\infty \sqrt{\zeta_0}\,)\\ =: A + B + O(\|\check{m}\|_\infty \sqrt{\zeta_0}\,).\end{aligned}$$

To estimate A, note that $\big|\check{m}^\delta_\delta(y^\tau, y^\nu) - \check{m}^\delta_\delta(y^\tau, 0)\big| \le \|\check{m}\|_{W^{1,\infty}}\, |y^\nu| \le C|y^\nu|$, so that

$$|A| \le \left(\delta_\varepsilon \int |y^\nu|^2\, e_{\varepsilon,\nu}(v)\, dy\right)^{1/2} \left(\delta_\varepsilon \int e_{\varepsilon,\nu}(v)\, dy\right)^{1/2} \le C\sqrt{\zeta_0}$$

after arguing, as in Case 2 above, to estimate $\int e_{\varepsilon,\nu}(v) \le C$.

As for the other term, since $\|\check{m}\|_\infty \le C$,

$$|B| \le C \int_{(-T_1,T_1)\times\mathbb{T}^n} |\Theta_1(y^\tau)|\, dy^\tau \quad \text{for } \Theta_1(y^\tau) := \delta_\varepsilon \int_{B_\nu(\rho)} e_{\varepsilon,\nu}(v)(y^\tau, y^\nu)\, dy^\nu - 1.$$

We say that y^τ is *good* if $\mathcal{D}_\nu(v(y^\tau)) \le \kappa_3$, where κ_3 is the constant from Lemma 11 and Proposition 17, for $k = 1$ and $k = 2$, respectively. A point will be called *bad* if it is not good. In particular, these results show that if y^τ is good, then

$$\Theta_1(y^\tau) \ge \begin{Bmatrix} -Ce^{-c/\varepsilon} & \text{if } k = 1\\ -C|\ln\varepsilon|^{-1} & \text{if } k = 2\end{Bmatrix} \ge -C\zeta_0 \text{ in both cases,}$$

since $\delta_\varepsilon \le \zeta_0$. Thus, since clearly $\Theta_1(y^\tau) \ge -1$ everywhere, we see that

$$|\Theta_1(y^\tau)| \le \begin{cases}\Theta_1(y^\tau) + C\zeta_0 & \text{if } y^\tau \text{ is good,}\\ \Theta_1(y^\tau) + 2 & \text{if } y^\tau \text{ is bad.}\end{cases}$$

Thus, we compute

$$\begin{aligned}|B| &\le C \int_{\{\text{good points}\}} (\Theta_1(y^\tau) + C\zeta_0)\, dy^\tau + C \int_{\{\text{bad points}\}} (\Theta_1(y^\tau) + 2)\, dy^\tau\\ &\le C \int_{(-T_0,T_0)\times\mathbb{T}^n} \Theta_1(y^\tau)\, dy^\tau + C\zeta_0 + 2\,\mathcal{H}^{1+n}\big(\{y^\tau \in (-T_0, T_0)\times\mathbb{T}^n : y^\tau \text{ is bad}\}\big).\end{aligned}$$

To conclude the estimate, we note that (6-3) implies that $\int_{(-T_0,T_0)\times\mathbb{T}^n} \Theta_1(y^\tau)\, dy^\tau \le C\zeta_0$, and (6-4), together with Chebyshev's inequality, implies that

$$\mathcal{H}^{1+n}\big(\{y^\tau \in (-T_0, T_0)\times\mathbb{T}^n : y^\tau \text{ is bad}\}\big) \le C \int_{(-T_0,T_0)\times\mathbb{T}^n} \mathcal{D}_\nu(v(y^\tau, \cdot))\, dy^\tau \le C\zeta_0.$$

Thus, $|B| \le C\zeta_0$.

Case 4: $\gamma, \delta > n$: Here, we consider the cases $k = 1$ and $k = 2$ separately.

<u>$k=1$</u>: The assumption of Case 4 reduces to $\gamma=\delta=N$ and, using (2-24), we see that

$$\widetilde{\mathcal{T}}^N_{\varepsilon,N} \;=\; \frac{1}{2}\sum_{a,b=0}^{n} g^{ab} v_{y^a} v_{y^b} - \frac{1}{2}(v_{y^N})^2 + \frac{1}{\varepsilon^2}F(v) = -\frac{1}{2}(v_{y^N})^2 + \frac{1}{\varepsilon^2}F(v) + O(|D_\tau v|^2).$$

We will write

$$\Theta_2(y^\tau) := \delta_\varepsilon \int_{B_\nu(\rho_0)} \left| v^2_{y^N} - \frac{1}{\varepsilon^2}F(v)\right| dy^\nu,$$

and we will now say that $y^\tau \in (-T_1, T_1)\times \mathbb{T}^n$ is *good* if

$$\mathcal{D}_\nu(v(y^\tau)) \le \kappa_3 \quad \text{and, in addition,} \quad \Theta_1(y^\tau) \le \kappa_4 \tag{6-26}$$

for κ_3, κ_4 found in Lemma 11. Then, Lemma 11 implies that, if y^τ is good, then

$$\Theta_2(y^\tau) \le C\sqrt{|\Theta_1(y^\tau)| + \zeta_0}.$$

Thus, using Hölder's inequality

$$\int_{\{\text{good points}\}} \Theta_2(y^\tau)\, dy^\tau \le \sqrt{C\textstyle\int_{\{\text{good points}\}}\big(|\Theta_1(y^\tau)| + C\zeta_0\big)\, dy^\tau} \le C\sqrt{\zeta_0},$$

using estimates from Case 3 above. And, if y^τ is bad, then

$$\Theta_2(y^\tau) \le C\left(1 + \Theta_1(y^\tau)\right)$$

so that

$$\begin{aligned}\int_{\{\text{bad points}\}} \Theta_2(y^\tau)\, dy^\tau &\le \sqrt{C\textstyle\int_{\{\text{bad points}\}}\big(\Theta_1(y^\tau) + 1\big)\, dy^\tau} \\ &\le C\sqrt{\zeta_0} + C\big(\mathcal{H}^{1+n}\big(\{y^\tau \in (-T_0, T_0)\times\mathbb{T}^n : y^\tau \text{ is bad}\}\big)\big)^{1/2} \le C\sqrt{\zeta_0},\end{aligned}$$

where at the end we used Chebyshev's inequality with (6-3) and (6-4). Hence,

$$\left|\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^N_N\, \widetilde{\mathcal{T}}^N_{\varepsilon,N}\, dy\right| \le C\int_{(-T_0,T_0)\times\mathbb{T}^n} \Theta_2(y^\tau)\, dy^\tau + O(\zeta_0) \le C\sqrt{\zeta_0}.$$

<u>$k=2$</u>: We claim that, when $k=2$,

$$\left|\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} \check{m}^\delta_\gamma\, \widetilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\, dy\right| \le C\sqrt{\zeta_0} \tag{6-27}$$

if $\delta, \gamma \in \{N-1, N\}$. This will complete the proof of (6-5). To prove (6-27), we first note that Proposition 4 implies that

$$\begin{pmatrix} \widetilde{\mathcal{T}}^{N-1}_{\varepsilon,N-1} & \widetilde{\mathcal{T}}^{N-1}_{\varepsilon,N} \\ \widetilde{\mathcal{T}}^{N}_{\varepsilon,N-1} & \widetilde{\mathcal{T}}^{N}_{\varepsilon,N}\end{pmatrix} = \begin{pmatrix} \frac{1}{2}\big(|v_{y^N}|^2 - |v_{y^{N-1}}|^2\big) + \varepsilon^{-2}F(v) & -v_{y^{N-1}}\cdot v_{y^N} \\ -v_{y^{N-1}}\cdot v_{y^N} & \frac{1}{2}\big(-|v_{y^N}|^2 + |v_{y^{N-1}}|^2\big) + \varepsilon^{-2}F(v)\end{pmatrix} + O\big(|D_\tau v|^2 + |y^\nu|^2\,|\nabla_\nu v|^2\big). \tag{6-28}$$

At this point, we need [Kurzke and Spirn 2009, Theorem 1], which implies that, if $w \in H^1(B_\nu(\rho_0), \mathbb{R}^2)$,

$$\mathcal{D}_\nu(w) \le \kappa_3 \quad \text{and} \quad \Theta_1(y^\tau) \le \tfrac{3}{2}, \tag{6-29}$$

then

$$\left|\delta_\varepsilon \int_{B_\nu(\rho_0)} \begin{pmatrix} |w_{y^{N-1}}|^2 & w_{y^{N-1}} \cdot w_{y^N} \\ w_{y^{N-1}} \cdot v_{y^N} & |w_{y^N}|^2 \end{pmatrix} dy^\nu - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| \le C\Big(\delta_\varepsilon \int_{B_\nu(\rho_0)} e_{\varepsilon,\nu}(w)\, dy^\nu - 1 + C\delta_\varepsilon\Big)^{1/2}.$$

(The main hypothesis of the Kurzke–Spirn estimate is (5-9), and we have shown in the proof of Proposition 17 that this follows from (6-29).) Accordingly, we will continue to say (exactly parallel to the case $k = 1$; see (6-26)) that $y^\tau \in (-T_1, T_1) \times \mathbb{T}^n$ is *good* if $v(y^\tau, \cdot) \in H^1(B_\nu(\rho_0); \mathbb{R}^2)$ satisfies (6-29). A point that is not good is said to be *bad*. It follows, as usual, from Chebyshev's inequality and (6-3), (6-4) that

$$\mathcal{H}^{1+n}\big(\{y^\tau \in (-T_0, T_0) \times \mathbb{T}^n : y^\tau \text{ is bad}\}\big) \le C\zeta_0.$$

The Kurzke–Spirn inequality implies that, if y^τ is good, then

$$\frac{\delta_\varepsilon}{2} \int_{B_\nu(\rho)} |\nabla_\nu v(y^\tau, y^\nu)|^2\, dy^\nu \ge 1 - C(\Theta_1(y^\tau) + C\delta_\varepsilon)^{1/2}.$$

Thus, for a good point y^τ,

$$\begin{aligned} \delta_\varepsilon \int_{B_\nu(\rho_0)} \frac{1}{\varepsilon^2} F(v)(y^\tau, y^\nu)\, dy^\nu &= \Theta_1(y^\tau) + \bigg(1 - \frac{\delta_\varepsilon}{2} \int_{B_\nu(\rho)} |\nabla_\nu v(y^\tau, y^\nu)|^2\, dy^\nu\bigg) \\ &\le \Theta_1(y^\tau) + C(\Theta_1(y^\tau) + C\delta_\varepsilon)^{1/2}. \end{aligned}$$

Similarly, the Kurzke–Spirn estimate also implies that, if y^τ is good, then

$$\delta_\varepsilon \left|\int_{B_\nu(\rho_0)} (|v_{y^N}|^2 - |v_{y^{N-1}}|^2)\, dy^\nu\right| + \delta_\varepsilon \left|\int_{B_\nu(\rho_0)} v_{y^N} \cdot v_{y^{N-1}}\, dy^\nu\right| \le C(\Theta_1(y^\tau) + C\delta_\varepsilon)^{1/2}.$$

Combining these and recalling (6-28), we see that, if y^τ is good, then, for $\gamma, \delta \in \{N-1, N\}$,

$$\left|\delta_\varepsilon \int_{B_\nu(\rho_0)} \mathcal{T}^\delta_{\varepsilon,\gamma}(y^\tau, y^\nu)\, dy^\nu\right| \le C(\Theta_1(y^\tau) + C\delta_\varepsilon)^{1/2} + \Theta_3(y^\tau), \tag{6-30}$$

where $\Theta_3(y^\tau) := \delta_\varepsilon \int_{B_\nu(\rho_0)} (|D_\tau v|^2 + |y^\nu|^2\, |\nabla_\nu v|^2)\, dy^\nu$. Also, if y^τ is bad, then (6-28) implies that

$$\delta_\varepsilon \int_{B_\nu(\rho_0)} \big|\mathcal{T}^\delta_{\varepsilon,\gamma}(y^\tau, y^\nu)\big|\, dy^\nu \le C(\Theta_1(y^\tau) + 1).$$

This last fact, together with the estimate of the bad set's size and (6-3), implies that

$$\left|\delta_\varepsilon \int_{(-T_1,T_1) \times \mathbb{T}^n \times B_\nu(\rho_0)} \check{m}^\delta_\gamma\, \widetilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\, dy\right| \le |A| + |B| + C\zeta_0,$$

where

$$A := \delta_\varepsilon \int_{\{\text{good points} \in (-T_1,T_1)\times\mathbb{T}^n\}} \int_{B_\nu(\rho)} \check{m}^\delta_\gamma(y^\tau, 0)\, \widetilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\, dy^\nu\, dy^\tau,$$

$$B := \delta_\varepsilon \int_{\{\text{good points} \in (-T_1,T_1)\times\mathbb{T}^n\}} \int_{B_\nu(\rho)} \big(\check{m}^\delta_\gamma(y^\tau, y^\nu) - \check{m}^\delta_\gamma(y^\tau, 0)\big)\, \widetilde{\mathcal{T}}^\gamma_{\varepsilon,\delta}\, dy^\nu\, dy^\tau.$$

From (6-30), Hölder's inequality, (6-2), and (6-3), we see that

$$|A| \le \|\check{m}\|_\infty \int_{\{\text{good points} \in (-T_1,T_1)\times\mathbb{T}^n\}} C\big((\Theta_1(y^\tau) + C\delta_\varepsilon)^{1/2} + \Theta_3(y^\tau)\big)\, dy^\tau \le C\sqrt{\zeta_0}.$$

And, since $\|\hat{m}\|_{W^{1,\infty}} \le C$,

$$|B| \le C\delta_\varepsilon \int_{(-T_1,T_1)\times\mathbb{T}^n\times B_\nu(\rho_0)} |y^\nu|\, e_{\varepsilon,\nu}(v)\, dy.$$

We have shown, in Case 3 above, that the right-hand side is bounded by $C\sqrt{\zeta_0}$, so we find that $|A|+|B| \le C\sqrt{\zeta_0}$. Therefore, we have proved (6-27), and (6-5) follows. □

Appendix

In this appendix, we give the proof of Propositions 4 and 5. Recall that we have defined $G = D\psi^T \eta\, D\psi$, where $\psi(y^\tau, y^\nu) := H(y^\tau) + \sum_{i=1}^k \bar{\nu}_i(y^\tau)\, y^{n+i}$. Here, H is the given parametrization of the minimal surface Γ, and the vectors $\{\bar{\nu}_i\}$ form an orthonormal frame for the normal bundle of Γ; see (2-11). The proposition asserts certain properties of $g := \det G$ and $(g^{ij}) = G^{-1}$. We will use the following lemma to read off properties of G^{-1} from those of G.

Lemma 26. *Let M be a matrix written in block form as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(*where the blocks need not be of equal size, i.e.*, $A \in M^{n\times n}$, $B \in M^{n\times m}$, $C \in M^{m\times n}$ *and* $D \in M^{m\times m}$, *for some m and n*). *If*

$$N = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{-31}$$

is well defined, then $N = M^{-1}$.

This is proved by simply verifying that $MN = I$.

Proof of Propositions 4 and 5. We will think of $\nu(y^\tau)$ as a $(1+N)\times k$ matrix, with columns $\bar{\nu}_i$ for $i = 1, \ldots, k$, and of y^ν as a $k \times 1$ vector, so that $\nu y^\nu := \sum_{i=1}^k \bar{\nu}_i(y^\tau)\, y^{n+i}$.

<u>Step 1</u>: To start, note that $\nabla_\nu \psi(y) = \nu(y^\tau)$, so the choice (2-11) of ν implies that G can be written in block form as

$$G = \begin{pmatrix} G_{\tau\tau} & G_{\tau\nu} \\ G_{\nu\tau} & I_k \end{pmatrix}$$

where

$$G_{\tau\tau} := D_\tau\psi^T \eta\, D_\tau\psi \in M^{(1+n)\times(1+n)}, \qquad G_{\tau\nu} = G_{\nu\tau}^T := D_\tau(\nu y^\nu)^T \eta\, \nu \in M^{(1+n)\times k},$$

and I_k denotes the $k \times k$ identity matrix. Observe that

$$|G_{\tau\nu}| \le C|y^\nu| \text{ when } k \ge 2 \quad \text{and} \quad G_{\tau\nu} \equiv 0 \text{ for } k = 1. \tag{-32}$$

The second assertion above follows from differentiating the identity $(\nu^T \eta\, \nu)(y^\tau) \equiv 1$. Since $\psi(y^\tau, 0) = H(y^\tau)$, we can write $G_{\tau\tau}(y^\tau, 0)$ in block form as

$$G_{\tau\tau}(y^\tau, 0) = \begin{pmatrix} H_{y_0}^T \eta\, H_{y_0} & H_{y_0}^T \eta\, \nabla H \\ \nabla H^T \eta\, H_{y_0} & \nabla H^T \eta\, \nabla H \end{pmatrix} = \begin{pmatrix} -1 + |h_{y^0}|^2 & 0 \\ 0 & \nabla h^T \nabla h \end{pmatrix},$$

where we have used (2-5) and (2-6). It then follows, from (2-7) and the smoothness of H, that $G_{\tau\tau}(y^\tau, 0)$ is invertible, with uniformly bounded inverse, for $y^\tau \in [-T_1, T_1] \times \mathbb{T}^n$. If ρ_0 is chosen small enough, it follows by continuity that $G_{\tau\tau}(y)$ is invertible, with uniformly bounded inverse, for $y \in [-T_1, T_1] \times \mathbb{T}^n \times B_\nu(\rho_0)$.

Step 2: Next, we note that $G_{\tau\tau}(y) = G_{\tau\tau}(y^\tau, 0) + O(|y^\tau|)$; we use (-31) and (-32) to find that, taking ρ_0 smaller if necessary, $G(y)$ is invertible for $y \in [-T_1, T_1] \times \mathbb{T}^n \times B_\nu(\rho_0)$, with

$$\begin{aligned} G^{-1}(y) &= \begin{pmatrix} (G_{\tau\tau}(y^\tau, 0) + O(|y^\nu|))^{-1} & O(|y^\nu|) \\ O(|y^\nu|) & (I_k - O(|y^\nu|^2))^{-1} \end{pmatrix} \\ &= \begin{pmatrix} G_{\tau\tau}^{-1}(y^\tau, 0) & 0 \\ 0 & I_k \end{pmatrix} + \begin{pmatrix} O(|y^\nu|) & O(|y^\nu|) \\ O(|y^\nu|) & O(|y^\nu|^2) \end{pmatrix}. \end{aligned} \tag{-33}$$

We have used (more than once) the fact that $G_{\tau\tau}^{-1}(y^\tau, 0)$ is uniformly bounded, which implies that $|(G_{\tau\tau} + A)^{-1} - G_{\tau\tau}^{-1}| \le C|A|$ for A sufficiently small, with a uniform constant C.

From (-33) and (2-7), we easily conclude that (2-20), (2-19), and the first estimate of (2-17) hold. Moreover, if $k = 1$, then, in view of (-32),

$$G^{-1}(y) = \begin{pmatrix} G_{\tau\tau}(y)^{-1} & 0 \\ 0 & I_k \end{pmatrix} = G^{-1}(y) = \begin{pmatrix} G_{\tau\tau}(y^\tau, 0)^{-1} + O(|y^\nu|) & 0 \\ 0 & I_k \end{pmatrix},$$

from which we infer (2-24) and (2-25).

To establish the second conclusion of (2-17), we differentiate the identity $G^{-1}G = I$ to find that

$$G_{y_0}^{-1} = -G^{-1}\, G_{y_0}\, G^{-1}.$$

Our earlier expression for G implies that

$$G_{y_0} = \begin{pmatrix} G_{\tau\tau, y_0} & O(|y^\nu|) \\ O(|y^\nu|) & 0 \end{pmatrix},$$

and one can readily check that this implies that $|g_{y^0}^{\alpha\beta}\, \xi_\alpha\, \xi_\beta| \le C(|\xi_\tau|^2 + |y^\nu|^2\, |\xi_\nu|^2)$, which completes the proof of (2-17).

Step 3: It remains to establish (2-18). To do this, fix $\zeta \in C_0^\infty([-T_1, T_1] \times \mathbb{T}^n; \mathbb{R}^k)$ and, for $\sigma \in \mathbb{R}$, define

$$f(\sigma) = \int_V \bigl(-\det(D_\tau H_\sigma^T\, \eta\, DH_\sigma)\bigr)^{1/2} dy^\tau, \tag{-34}$$

where

$$H_\sigma(y^\tau) = H(y^\tau) + \sigma\, \nu(y^\tau)\, \zeta(y^\tau) = \psi(y^\tau, \sigma\, \zeta(y^\tau)).$$

Note that, for σ small, H_σ parametrizes a surface Γ_σ that is a small variation of the original surface Γ. Because Γ is a Minkowski minimal surface, it follows that $f'(0) = 0$. We will show that this yields the conclusion of the lemma.

Thinking of $D\zeta$ as a $k \times (1{+}n)$ matrix, a direct computation yields

$$DH_\sigma(y^\tau) = D_\tau\psi(y^\tau, \sigma\,\zeta(y^\tau)) + \sigma\,\nu(y^\tau)\,D\zeta(y^\tau)$$

It then follows from (2-11) that $DH_\sigma^T\eta\,DH_\sigma$ has the form

$$\big(DH_\sigma^T\eta\,DH_\sigma\big)(y^\tau) = \big(D_\tau\psi^T\eta\,D_\tau\psi\big)(y^\tau, \sigma\,\zeta(y^\tau)) + \sigma^2 B(y^\tau).$$

for some matrix $B(y^\tau)$ that depends smoothly on y^τ. Since, if $A(\sigma)$ are square matrices depending smoothly on a real parameter σ, then

$$\frac{d}{d\sigma}\det(A(\sigma) + \sigma^2 B)\big|_{\sigma=0} = \frac{d}{d\sigma}\det A(\sigma)\big|_{\sigma=0},$$

it follows that, at $\sigma = 0$,

$$\frac{d}{d\sigma}\det\big(DH_\sigma^T\eta\,DH_\sigma\big)(y^\tau) = \frac{d}{d\sigma}\det\big(D_\tau\psi^T\eta\,D_\tau\psi\big)(y^\tau, \sigma\,\zeta(y^\tau)) \tag{-35}$$

Step 4: We next note that

$$\det\big(D_\tau\psi^T\eta\,D_\tau\psi\big)(y^\tau, \sigma\,\zeta(y^\tau)) = \det\big(D\psi^T\eta\,D\psi\big)(y^\tau, \sigma\,\zeta(y^\tau)) + O(\sigma^2). \tag{-36}$$

Indeed, this follows (by rather easy linear algebra considerations) from the fact that

$$D\psi^T\eta\,D\psi\;(y^\tau, \sigma\,\zeta(y^\tau)) = \begin{pmatrix} D_\tau\psi^T\eta\,D_\tau\psi & O(\sigma) \\ O(\sigma) & I_k + O(\sigma^2) \end{pmatrix}. \tag{-37}$$

By combining (-35) and (-36),

$$\frac{d}{d\sigma}\det\big(DH_\sigma^T\eta\,DH_\sigma\big)(y^\tau)\big|_{\sigma=0} = \frac{d}{d\sigma}g(y^\tau, \sigma\,\zeta(y^\tau))\big|_{\sigma=0} = \nabla_\nu g(y^\tau, 0)\cdot\zeta$$

at $\sigma = 0$. Also, it follows from (2-7) and continuity that $\det(DH_\sigma^T\eta\,DH_\sigma)(y^\tau)$ and $g(y^\tau, \sigma\,\zeta(y^\tau))$ are bounded away from 0 for ζ small enough, and hence

$$\frac{d}{d\sigma}\big(-\det(DH_\sigma^T\;\eta\;DH_\sigma)(y^\tau)\big)^{1/2}\big|_{\sigma=0} = \nabla_\nu\sqrt{-g}\cdot\zeta$$

Thus, the identity $f'(0) = 0$ reduces to

$$0 = \int \nabla_\nu\sqrt{-g}\cdot\zeta\;dy^\tau$$

Since ζ is arbitrary, we conclude that $\nabla_\nu\sqrt{-g} = 0$. This and (-33) imply the required estimate (2-18). □

Acknowledgments

The author is very grateful to Alberto Montero for numerous useful discussions, including conversations which provided the initial impetus for this project, and for carefully reading and commenting on drafts of large parts of this paper.

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Received 1 Dec 2009. Accepted 15 Apr 2010.

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ANALYSIS AND PDE
Vol. 4, No. 2, 2011

WELL- AND ILL-POSEDNESS ISSUES FOR ENERGY SUPERCRITICAL WAVES

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We investigate the initial value problem for some energy supercritical semilinear wave equations. We establish local existence in suitable spaces with continuous flow. The proof uses the finite speed of propagation and a quantitative study of the associated ODE. It does not require any scaling invariance of the equation. We also obtain some ill-posedness and weak ill-posedness results.

1. Introduction

In this work, we discuss some well-posedness issues of the Cauchy problem associated to the semilinear wave equation

$$\partial_t^2 u - \Delta u + F'(u) = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^d, \tag{1}$$

where $d \geq 2$ and $F : \mathbb{R} \to \mathbb{R}$ is an *even* regular function satisfying

$$F(0) = F'(0) = 0 \quad \text{and} \quad uF'(u) \geq 0. \tag{2}$$

These assumptions on F include the massive case, that is, the Klein–Gordon equation. With hypothesis (2), one can construct a global weak solution with finite energy data using a standard compactness argument; see, for example [Strauss 1989]. However, the construction of (even local) strong solutions requires some control on the growth at infinity and more tools. As regards the growth of the nonlinearity F, we distinguish two cases. For dimensions $d \geq 3$ we shall assume that our Cauchy problem is H^1-supercritical in the sense that

$$\frac{F(u)}{|u|^{2d/(d-2)}} \nearrow +\infty, \quad u \to \infty\,. \tag{3}$$

In two space dimensions and thanks to Sobolev embedding, any Cauchy problem with polynomially growing nonlinearities is locally well-posed regardless of the sign of the nonlinearity and the growth of F at infinity. This is a limit case of (3). Square exponential nonlinearities were investigated first in [Nakamura and Ozawa 1999b], where global existence and scattering for small Cauchy data were proved, then in [Atallah-Baraket 2004], where local existence was obtained under restrictive conditions, and finally in [Ibrahim et al. 2007a], where a new notion of criticality based on the size of the energy

Majdoub is grateful to the Laboratory of PDE and Applications at the Faculty of Sciences of Tunis. Ibrahim is partially supported by NSERC# 371637-2009 grant and start-up fund from the University of Victoria. Masmoudi is partially supported by NSF Grant DMS-0703145.
MSC2000: 34C25, 35L05, 49K40, 65F22.
Keywords: nonlinear wave equation, well-posedness, ill-posedness, finite speed of propagation, oscillating second order ODE.

appears. In this paper, we examine the situation of other growths of exponential nonlinearities (not necessarily square). More precisely, when $d = 2$, we assume either

$$\frac{\log(F(u))}{|u|^2} \nearrow +\infty \quad \text{as } u \to \infty, \tag{4}$$

or

$$\text{for some } q \text{ with } 0 < q \leq 2, \qquad \frac{\log(F(u))}{|u|^q} = O(1) \quad \text{as } u \to \infty. \tag{5}$$

The model example that we are going to work with when $d = 3$ is given by

$$\partial_t^2 u - \Delta u + u^7 = 0. \tag{6}$$

It is a good prototype for all higher dimensions $d \geq 3$ illustrating assumption (3). In two dimensions, we take

$$\partial_t^2 u - \Delta u + u\,(1+u^2)^{((q-2)/2)} \mathrm{e}^{4\pi((1+u^2)^{(q/2)}-1)} = 0, \tag{7}$$

with $q > 0$, illustrating either the cases (4) or (5), depending on whether $q > 2$ or $q \leq 2$.

Define the total energy of u by

$$E(u(t)) \stackrel{\text{def}}{=} \|\nabla_{t,x} u(t)\|_{L_x^2}^2 + \int_{\mathbb{R}^d} 2F(u(t))\,dx.$$

The energy of data $(\varphi, \psi) \in \dot{H}^1 \times L^2$ is given by

$$E(\varphi, \psi) \stackrel{\text{def}}{=} \|\nabla\varphi\|_{L_x^2}^2 + \|\psi\|_{L_x^2}^2 + \int_{\mathbb{R}^d} 2F(\varphi)\,dx.$$

When $\psi = 0$, we abbreviate $E(\varphi, 0)$ to simply $E(\varphi)$.

In the sequel, we adopt the following definitions of weak solution and local/global well-posedness of the Cauchy problem associated to (1).

Definition 1.1. Let $X := X_1 \times X_0$ be a Banach space.[1] A weak solution of (1) is a function $u : \mathbb{R} \to X_1$ with $(\partial_t u, \nabla_x u) \in L^\infty(\mathbb{R}, X_0)$ satisfying (1) in the distributional sense and having finite propagation speed. When $X = H^1 \times L^2$ is the energy space, we have in addition $F(u) \in L^\infty(\mathbb{R}, L^1)$ and $E(u(t)) \leq E(u(0))$ for all t.

The existence of such solutions will one of our results.

Definition 1.2.

- The Cauchy problem associated to (1) is *locally well-posed* in X, abbreviated as LWP, if for every data $(u_0, u_1) \in X$, there exists a time $T > 0$ and a unique[2] (distributional) solution

$$u : [-T, T] \times \mathbb{R}^d \to \mathbb{R}$$

  to (1) such that $(u, \partial_t u) \in \mathscr{C}([-T, T]; X)$, $(u, \partial_t u)(t = 0) = (u_0, u_1)$, and such that the solution map $(u_0, u_1) \mapsto (u, \partial_t u)$ is continuous from X to $\mathscr{C}([-T, T]; X)$.
- The Cauchy problem is *globally well-posed* (GWP) if the time T can be taken arbitrary.

[1] Typically, $X = B^s_{p,q} \times B^{s-1}_{p,q}$, for some suitable choice of s, p and q.

[2] In some cases the uniqueness holds in more restrictive space.

- The Cauchy problem is *strongly well-posed* (SWP) if the solution map is uniformly continuous.
- The Cauchy problem is *ill-posed* (IP) if the solution map is not continuous.
- The Cauchy problem is *weakly ill-posed* on a set $\boldsymbol{Y} \subset \boldsymbol{X}$ (WIP) if the solution map

$$(u_0, u_1) \in \boldsymbol{Y} \mapsto (u, \partial_t u)$$

is not uniformly continuous from $\boldsymbol{Y}$ to $\mathscr{C}([-T, T]; \boldsymbol{X})$.

We recall a few historic facts about this problem. First, in space dimensions $d \geq 3$, the defocusing semilinear wave equation with power p reads

$$\partial_t^2 u - \Delta u + |u|^{p-1} u = 0, \tag{8}$$

where $p > 1$. This problem has been widely investigated and there is a large literature dealing with the well-posedness theory of (8) in the scale of the Sobolev spaces H^s. Second, for the global solvability in the energy space $\dot{H}^1 \times L^2$, there are mainly three cases. In the subcritical case

$$p < p^* \stackrel{\text{def}}{=} \frac{d+2}{d-2},$$

Ginibre and Velo [1985] finally settled global well-posedness in the energy space, by using the Strichartz estimate, nonlinear estimates in Besov space, and energy conservation.

The critical case $p = p^*$ is more delicate, due to possibility of energy concentration. Struwe [1988] proved global existence of radially symmetric regular solutions. Then Grillakis [1990; 1992] extended this result to nonradial data. In the energy space, Ginibre, Soffer and Velo [Ginibre et al. 1992] proved global well-posedness in the radial case, where the Morawetz estimate effectively precludes concentration. The case of general data was solved by Shatah and Struwe [1994], and Kapitanski [1994]. See also [Ibrahim and Majdoub 2003] for variable metrics. Note that uniqueness in the energy space is not yet fully solved. We refer to [Planchon 2003] for $d \geq 4$, to [Struwe 1999; Masmoudi and Planchon 2006] for partial results in $d = 3$, and to [Struwe 2006] for the case of classical solutions.

The supercritical case $p > p^*$ is even harder, and the global well-posedness problem for general data remains open, except for the existence of global weak solutions [Strauss 1989], local well-posedness in higher Sobolev spaces (H^s with $s \geq d/2 - 2/p > 1$) as well as global well-posedness with scattering for small data [Lindblad and Sogge 1995; Wang 1998], and some negative results concerning nonuniform continuity of the solution map [Burq et al. 2007; Christ et al. 2003; Lebeau 2001]. See also [Lebeau 2005] for a result concerning a loss of regularity and [Tao 2007] for a result about global regularity for a logarithmically energy-supercritical wave equation in the radial case.

It is worth noticing that the nonlinearities considered in [Burq et al. 2007; Christ et al. 2003; Lebeau 2001; 2005] are homogeneous, and thus at first glance, the proofs cannot be adapted to the case of inhomogeneous nonlinearities. But as suggested in [Alazard and Carles 2009], it might be that homogeneity is used only to guess a suitable ansatz. We also mention the NLS analogues of [Lebeau 2005] (see for example [Alazard and Carles 2009; Carles 2007; Thomann 2008]). Several different techniques are used there, to get some results which seem out of reach with an ODE approach (in [Alazard and Carles 2009], the case $d = 1$ is allowed, and the trick used in [Lebeau 2005] and [Burq et al. 2007] cannot be adapted, apparently). See also [Burq and Tzvetkov 2008] about random data Cauchy theory for supercritical wave equations.

In dimension two, H^1-critical nonlinearities seem to be of exponential type[3], since every power is H^1-subcritical. On the one hand, in a recent work [Ibrahim et al. 2006], the case $F(u) = 1/8\pi\,(\mathrm{e}^{4\pi u^2} - 1)$ was investigated and an energy threshold was proposed. Local strong well-posedness was shown under the size restriction $\|\nabla u_0\|_{L^2} < 1$ and the global well-posedness was obtained in both the sub and critical cases (when the energy is below or equal to the energy threshold). Very recently, Struwe [2009] has constructed global smooth solutions with radially symmetric data of arbitrary size. On the other hand, the ill posedness results of [Lebeau 2005; Christ et al. 2003; Burq et al. 2002] show the nonuniform continuity of the solution map (or sometimes its noncontinuity at the zero data). In the two-dimensional exponential case and since small data are in the subcritical regime, we prove only the nonuniform continuity of the solution map. It is worth to note that the results of [Christ et al. 2003] are based on the scaling invariances of the wave and Schrödinger equations with homogeneous nonlinearities. The idea developed there [Christ et al. 2003] is to approximate the solution by its corresponding ODE (at the zero dispersion limit). Since solutions of the ODE are periodic in time, then a decoherence phenomena occurs for small time since the ODE solutions oscillate fast. Note that the original result in this field appears in [Lebeau 2001].

Hence, in this paper our main aim is to investigate the local well and ill posedness regardless of the size of the initial data. Our idea to overcome the absence of scaling invariance is to choose regularized step functions as initial data (i.e., functions constant near zero). The presence of the step immediately guarantees the equality between the PDE and the ODE solutions in a backward light cone, thanks to the finite speed of propagation. The length of the step can be adjusted (in the supercritical regime) so that ill-posedness/weak ill-posedness occurs inside the light cone.

This paper is organized as follows. In Section 2, we state our main results. In Section 3, we recall some basic definitions and auxiliary lemmas. In Section 4, we investigate the energy regularity regime. Section 5 is devoted to the low regularity data.

Finally, we mention that, C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Main results

Energy regularity data. First we show that if the general assumptions (2)+(3) or (2)+(4) are satisfied, the nonlinearity is too strong to ensure the local well-posedness in the energy space:

Theorem 2.1. *Assume that $d \geq 3$ and (2)+(3), or $d = 2$ and (2)+(4).*

(1) *There exist a sequence (φ_k) in $\dot{H}^1$ and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\nabla\varphi_k\|_{L^2_x} \to 0, \quad t_k \to 0, \quad \sup_k E(\varphi_k) < \infty,$$

and such that any weak solution u_k of (6) with initial data $(\varphi_k, 0)$ satisfies

$$\liminf_{k\to+\infty} \|\partial_t u_k(t_k)\|_{L^2_x} \gtrsim 1.$$

In particular the Cauchy problem is ill-posed in $H^1 \times L^2$.

[3] In fact, the critical nonlinearity is of exponential type in any dimension d with respect to $H^{d/2}$ norm.

(2) *If we relax the condition* $\sup_k E(\varphi_k) < \infty$ *by taking* $\lim_{k\to+\infty} \int F(\varphi_k) = +\infty$, *we can even get*

$$\lim_{k\to+\infty} \|\partial_t u_k(t_k)\|_{L^2_x} = \infty\,.$$

Remark 2.2. Lebeau [2001] proved a loss of regularity result for energy supercritical homogeneous wave equation; see also [Christ et al. 2003]. Recently, Tao [2007] has shown the global well-posedness in the radial case of a logarithmic energy supercritical wave equation in $H^{1+\varepsilon} \times H^{\varepsilon}$ for any $\varepsilon > 0$. The above Theorem shows that ε cannot be taken zero.

The above theorem covers model (7) in two space dimensions with $q > 2$. When $q < 2$, recall that the global well-posedness in the energy space can easily be obtained through the sharp Trudinger–Moser inequality combined with the simple observation that for $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|(1+u^2)^{(q-2)/2} e^{4\pi(1+u^2)^{q/2}} - e^{4\pi}\right| \le C_\varepsilon\,(e^{\varepsilon u^2}-1) \quad \text{for all } u \in \mathbb{R}.$$

In the case $q = 2$, the local well-posedness for the Cauchy problem associated to (7) in the energy space was first established in [Nakamura and Ozawa 1999a; 1999b] for small Cauchy data. Later on, optimal smallness for well-posedness was investigated, first in [Atallah-Baraket 2004] for radially symmetric initial data $(0, u_1)$, and then in [Ibrahim et al. 2006; 2007b] for general data. The following result generalizes the previous results to any data in the energy space regardless of its size.

Theorem 2.3. *Let* $(u_0, u_1) \in H^1 \times L^2$. *There exists a time* $T > 0$ *and a unique solution* u *of* (7) *with* $q = 2$ *in the space* $C_T(H^1) \cap C^1_T(L^2)$ *satisfying* $u(0, x) = u_0(x)$ *and* $\dot u(0, x) = u_1(x)$. *Moreover, the solution map is continuous on* $H^1 \times L^2$.

In [Ibrahim et al. 2007b] it is shown that the local solutions of (7) (with $q = 2$) are global whenever the total energy $E \le 1$, where

$$E(u(t)) \overset{\text{def}}{=} \|\nabla_{t,x} u(t)\|^2_{L^2_x} + \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{4\pi u^2} - 1\, dx.$$

Indeed, in that case, the Cauchy problem is strongly well-posed. The following result shows the weak ill-posedness on the set $\{ E < 1+\delta \}$ for any $\delta > 0$. More precisely

Theorem 2.4. *Let* $\nu > 0$. *There exist a sequence of positive real numbers* (t_k) *tending to zero and two sequences* (u_k) *and* (v_k) *of solutions of the nonlinear Klein–Gordon equation*

$$\Box u + u e^{4\pi u^2} = 0, \tag{9}$$

satisfying

$$\|(u_k - v_k)(t=0,\cdot)\|^2_{H^1} + \|\partial_t(u_k - v_k)(t=0,\cdot)\|^2_{L^2} = o(1) \quad \text{as } k \to +\infty,$$
$$0 < E(u^k, 0) - 1 \le e^3 \nu^2, \quad 0 < E(v^k, 0) - 1 \le \nu^2,$$
$$\liminf_{k\to\infty} \|\partial_t(u_k - v_k)(t_k,\cdot)\|^2_{L^2} \ge \frac{\pi}{4}(e^2 + e^{3-8\pi})\nu^2.$$

Notice that Theorem 2.3 yields the continuity with respect to the initial data and Theorem 2.4 yields that there is no uniform continuity if the energy is larger than 1 (supercritical regime).

Remark 2.5. Struwe [2009] has constructed global smooth solutions for the two-dimensional energy critical wave equation with radially symmetric data. Although the techniques are different, this result might be seen as an analogue of Tao's result [2007] for the three-dimensional energy supercritical wave equation. Our Theorem 2.4 shows just the weak ill-posedness in the supercritical case. This is weaker than the result in higher dimensions where the flow fails to be continuous at zero as shown in [Christ et al. 2003]. The reason behind this is that small data are always subcritical in the exponential case.

Low regularity data for the model (7)**.** Now that the local well/ill-posedness is clarified in the energy space for dimension $d \geq 2$, our next task in this paper is to seek for the "largest possible spaces" in which we have local well-posedness for the Cauchy problem associated to the model (7). Recall that we have the embeddings

$$H^1(\mathbb{R}^2) \hookrightarrow B^1_{2,\infty}(\mathbb{R}^2) \hookrightarrow H^s(\mathbb{R}^2), \quad s < 1\,. \tag{10}$$

The next theorem show the failure of the well-posedness in spaces slightly bigger than the energy space in the case $q = 2$. This means that the Cauchy problem posed either in $B^1_{2,\infty}$ or H^s with $s < 1$ becomes supercritical. More specifically:

Theorem 2.6. *Assume $q = 2$. Let $\mathcal{W} := \{ u \in L^2 : \nabla u \in L^{2,\infty}\}$, where $L^{2,\infty}$ is the classical Lorentz space.*[4]

(1) *There exists a sequence (φ_k) in $\mathcal{W}$ and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\varphi_k\|_{\mathcal{W}} \to 0 \quad \textit{as } t_k \to 0,$$

and such that any weak solution u_k of (7) *with initial data $(\varphi_k, 0)$ satisfies*

$$\lim_{k\to\infty} \|\partial_t u_k(t_k)\|_{L^{2,\infty}} = \infty.$$

(2) *There exists a sequence (φ_k) in $\mathcal{B}^1_{2,\infty}$ and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\varphi_k\|_{\mathcal{B}^1_{2,\infty}} \to 0 \quad \textit{as } t_k \to 0,$$

and such that any weak solution u_k of (7) *with initial data $(\varphi_k, 0)$ satisfies*

$$\lim_{k\to\infty} \|\partial_t u_k(t_k)\|_{\mathcal{B}^0_{2,\infty}} = \infty.$$

In particular, the flow fails to be continuous at 0 *in the $\mathcal{W} \times L^{2,\infty}$ topology or $\mathcal{B}^1_{2,\infty} \times \mathcal{B}^0_{2,\infty}$ topology.*

(3) *Let $s < 1$. There exists a sequence (φ_k) in H^s and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\varphi_k\|_{H^s} \to 0 \quad \textit{as } t_k \to 0,$$

and such that any weak solution u_k of (7) *with initial data $(\varphi_k, 0)$ satisfies*

$$\lim_{k\to\infty} \|\partial_t u_k(t_k)\|_{H^{s-1}} = \infty.$$

In particular, the flow fails to be continuous at 0 *in the $H^s \times H^{s-1}$ topology.*

This theorem can be seen as a consequence of the following general result about arbitrary $1 \leq q < \infty$. Indeed, Equation (7) is subcritical at the regularity of the Besov space $\mathcal{B}^1_{2,q'}$ but supercritical at the H^s regularity level with $s < 1$, where, as usual, q' denotes the Lebesgue conjugate exponent of q. More precisely:

[4]It is defined by its norm $\|u\|_{L^{2,\infty}} := \sup_{\sigma>0}(\sigma \operatorname{meas}^{1/2}\{\,|u(x)| > \sigma\})$.

Theorem 2.7. *Assume that* $1 \le q < \infty$.

(1) *Let* $(u_0, u_1) \in \mathcal{B}^1_{2,q'} \times \mathcal{B}^0_{2,q'}$.[5] *There exists a time* $T > 0$ *and a unique solution* u *of* (7) *with initial data* (u_0, u_1) *in the space* $C_T(\mathcal{B}^1_{2,q'}) \cap C^1_T(\mathcal{B}^0_{2,q'})$.

(2) *Let* $s < 1$. *There exists a sequence* (φ_k) *in* H^s *and a sequence* (t_k) *in* $(0, 1)$ *satisfying*

$$\|\varphi_k\|_{H^s} \to 0 \quad \text{as } t_k \to 0,$$

and such that any weak solution u_k *of* (7) *with initial data* $(\varphi_k, 0)$ *satisfies*

$$\lim_{k\to+\infty} \|\partial_t u_k(t_k)\|_{H^{s-1}} = \infty.$$

In particular, the flow fails to be continuous at 0 *in the* $H^s \times H^{s-1}$ *topology.*

Remark 2.8. The same well-posedness results can be derived for the corresponding two dimensional nonlinear Schrödinger equations.

We end this section with a table summarizing the picture of well/ill-posedness.

Setting	Data regularity		
	H^1	$\mathcal{B}^1_{2,\infty}$	H^s with $s < 1$
$d \ge 3$ and (3)	WIP	IP	IP
$d = 2$ and (4)	IP	IP	IP
$d = 2$ and $q < 2$	GWP & SWP	LWP	IP
$d = q = 2$ and $E > 1$	LWP & WIP	IP	IP
$d = q = 2$ and $E \le 1$	GWP & SWP	IP	IP

3. Background

Besov spaces. For the convenience of the reader, we recall the definition and some properties of Besov spaces.

Definition 3.1. Let χ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| > 2$. Define the function $\psi(\xi) = \chi(\xi/2) - \chi(\xi)$. The (homogeneous) frequency localization operators are defined by

$$\widehat{\dot\triangle_j u}(\xi) = \psi(2^{-j}\xi)\,\hat{u}(\xi) \quad \text{for all } j \in \mathbb{Z}.$$

If $s < d/p$, then u belongs to the homogeneous Besov space $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^d)$ if and only if the partial sum $\sum_{-m}^{m} \dot\triangle_j u$ converges to u as a tempered distribution and the sequence $(2^{sj}\,\|\triangle_j u\|_{L^p})$ belongs to $\ell^q(\mathbb{Z})$.

To define the inhomogeneous Besov spaces, we need an inhomogeneous frequency localization.

Definition 3.2. The inhomogeneous frequency localization operators are defined by

$$\widehat{\triangle_j u}(\xi) = \begin{cases} 0 & \text{if } j \le -2, \\ \chi(\xi)\hat{u}(\xi) & \text{if } j = -1, \\ \psi(2^{-j}\xi)\,\hat{u}(\xi) & \text{if } j \ge 0. \end{cases}$$

[5] As we will see in the proof, when $q' = \infty$ the appropriate space is $\tilde{\mathcal{B}}^1_{2,\infty}$, the closure of smooth compactly supported function in the usual Besov space $\mathcal{B}^1_{2,\infty}$.

For $N \in \mathbb{N}$, set

$$S_N = \sum_{j \le N-1} \triangle_j .$$

We say that u belongs to the inhomogeneous Besov space $\mathcal{B}^s_{p,q}(\mathbb{R}^d)$ if $u \in \mathcal{S}'$ and $\|u\|_{\mathcal{B}^s_{p,q}} < \infty$, where

$$\|u\|_{\mathcal{B}^s_{p,q}} = \begin{cases} \|\triangle_{-1} u\|_{L^p} + \Big(\sum_{j=0}^{\infty} 2^{jqs} \, \|\triangle_j u\|^q_{L^p} \Big)^{1/q} & \text{if } q < \infty, \\ \|\triangle_{-1} u\|_{L^p} + \sup_{j \ge 0} 2^{js} \, \|\triangle_j u\|_{L^p} & \text{if } q = \infty . \end{cases}$$

We recall without proof the following properties of the operators $\triangle_j$ and Besov spaces [Runst and Sickel 1996; Triebel 1983; 1992; 1978].

- Bernstein's inequality: For all $1 \le p \le q \le \infty$ we have

$$\|\triangle_j u\|_{L^q(\mathbb{R}^d)} \le C \, 2^{jd(1/p-1/q)} \, \|\triangle_j u\|_{L^p(\mathbb{R}^d)} .$$

- Embeddings:

$$\mathcal{B}^s_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d), \tag{11}$$

  whenever

$$s - \frac{d}{p} \ge s_1 - \frac{d}{p_1}, \quad 1 \le p \le p_1 \le \infty, \quad 1 \le q \le q_1 \le \infty, \quad s, s_1 \in \mathbb{R} .$$

- Equivalent norm: For $s > 0$ we have

$$\|u\|_{\mathcal{B}^s_{p,q}} \approx \|u\|_{L^p} + \|\nabla u\|_{\dot{\mathcal{B}}^{s-1}_{p,q}} . \tag{12}$$

Sobolev spaces and Hölder spaces are special cases of Besov spaces: $H^s = \mathcal{B}^s_{2,2}$ and $C^\sigma = \mathcal{B}^\sigma_{\infty,\infty}$, for noninteger $\sigma > 0$.

We shall also use a result about functions that operate by pointwise multiplication in Besov spaces:

Theorem 3.3 [Runst and Sickel 1996, Theorem 4.6.2]. *Let $|s| < d/2$. Any function in $\dot{\mathcal{B}}^{d/2}_{2,\infty} \cap L^\infty(\mathbb{R}^d)$ is a pointwise multiplier in the Besov space $\dot{\mathcal{B}}^s_{2,q}(\mathbb{R}^d)$.*

An important application of this theorem[6] which will be used in the sequel is the fact that the function $f(x) := x/r$ operates on $\dot{\mathcal{B}}^0_{2,\infty}(\mathbb{R}^2)$ via pointwise multiplication. Indeed, according to Theorem 3.3 it suffices to show that f belongs to $\dot{\mathcal{B}}^1_{2,\infty}(\mathbb{R}^2)$. For this, note that $\hat{f}$ is an homogeneous distribution of degree -2, belonging to the C^∞ class outside the origin. We can then define $g \in \mathcal{S}$ by $\hat{g} = \psi \hat{f}$. Hence $\triangle_j f(x) = g(2^j x)$ and $\|\triangle_j f\|_{L^2} = 2^{-j} \|g\|_{L^2}$.

Two-dimensional Strichartz estimate and logarithmic inequality.

Proposition 3.4 [Miao et al. 2004; Nakamura and Ozawa 2001].

$$\|u\|_{L^4((0,T);B^{1/4}_{\infty,2})} \lesssim \|\partial_t^2 u - \Delta u + u\|_{L^1((0,T);L^2)} + \|u(0)\|_{H^1} + \|\partial_t u(0)\|_{L^2}. \tag{13}$$

Using the embedding (11), we can replace $B^{1/4}_{\infty,2}$ with the Hölder space $C^{1/4}$.

[6] We are grateful to Gérard Bourdaud for providing us this reference and a proof of the application.

The following lemma shows that we can estimate the L^∞ norm by a stronger norm but with a weaker growth (namely logarithmic).

Lemma 3.5. *Let* $0<\alpha<1$ *and* $1\le q\le\infty$. *There exists a constant* C *such that*

$$\|u\|_{L^\infty}\le C\|u\|_{\mathcal{B}^1_{2,q'}}\log^{1/q}\left(\mathrm{e}+\frac{\|u\|_{\mathcal{C}^\alpha}}{\|u\|_{\mathcal{B}^1_{2,q'}}}\right). \tag{14}$$

Similar inequalities appeared in [Brézis and Gallouet 1980]; they have been improved (with respect to the best constant) as follows:

Lemma 3.6 [Ibrahim et al. 2007a, Theorem 1.3]. *Let* $0<\alpha<1$. *For any* $\lambda>1/(2\pi\alpha)$ *and any* $0<\mu\le 1$, *a constant* $C_\lambda>0$ *exists such that, for any function* $u\in H^1(\mathbb{R}^2)\cap\mathcal{C}^\alpha(\mathbb{R}^2)$

$$\|u\|^2_{L^\infty}\le\lambda\|u\|^2_{H_\mu}\log\left(C_\lambda+\frac{8^\alpha\mu^{-\alpha}\|u\|_{\mathcal{C}^\alpha}}{\|u\|_{H_\mu}}\right), \tag{15}$$

where H_μ *is defined by the norm* $\|u\|^2_{H_\mu}:=\|\nabla u\|^2_{L^2}+\mu^2\|u\|^2_{L^2}$.

Proof of Lemma 3.5. Write $u=\sum_{j=-1}^{N-1}\triangle_j u+\sum_{j=N}^{\infty}\triangle_j u$, with $N\ge 0$ an integer to be chosen later. Using Bernstein's inequality, we get

$$\|u\|_{L^\infty}\le C\sum_{j=-1}^{N-1}2^j\|\triangle_j u\|_{L^2}+\sum_{j=N}^{\infty}2^{-j\alpha}\left(2^{j\alpha}\|\triangle_j u\|_{L^\infty}\right)\le C\Big(N^{1/q}\|u\|_{\mathcal{B}^1_{2,q'}}+\frac{2^{-N\alpha}}{1-2^{-\alpha}}\|u\|_{\mathcal{C}^\alpha}\Big).$$

Choosing $N\sim\dfrac{1}{\alpha\log 2}\log\left(\mathrm{e}+\dfrac{\|u\|_{\mathcal{C}^\alpha}}{\|u\|_{\mathcal{B}^1_{2,q'}}}\right)$, we obtain (14) as desired. □

Oscillating second order ODE. Here we recall a classical result about ordinary differential equations.

Lemma 3.7 [Arnaudiès and Lelong-Ferrand 1997, Section III.5]. *Let* $F:\mathbb{R}\to\mathbb{R}$ *be a smooth function. The ODE*

$$\ddot{x}(t)+F'(x(t))=0, \tag{16}$$

with initial conditions $x(0)=x_0>0$ *and* $\dot{x}(0)=0$, *has a nonconstant periodic solution if and only if the function* $G:y\mapsto 2(F(x_0)-F(y))$ *has two distinct simple zeros* α *and* β *with* $\alpha\le x_0\le\beta$ *and* G *has no zero in the interval* $]\alpha,\beta[$. *The period is then given by*

$$T=2\int_\alpha^\beta\frac{dy}{\sqrt{G(y)}}=\sqrt{2}\int_\alpha^\beta\frac{dy}{\sqrt{F(x_0)-F(y)}}.$$

In addition, x *is decreasing on* $[0,T/4]$ *and* $x(T/4)=0$.

Trudinger–Moser inequalities. It is known that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces L^p for $2\le p<\infty$ but not in L^∞. Moreover, H^1 functions are in the so-called Orlicz space, that is, their exponentials are integrable for every growth less than e^{u^2}. Precisely, we have the following Trudinger–Moser inequality (see [Adachi and Tanaka 2000; Ruf 2005] and references therein).

Proposition 3.8. *Let $\alpha \in (0, 4\pi)$. A constant c_α exists such that*

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u(x)|^2} - 1\right) dx \le c_\alpha \|u\|_{L^2}^2 \tag{17}$$

for all u in $H^1(\mathbb{R}^2)$ such that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$. Moreover, if $\alpha \ge 4\pi$, then (17) *is false.*

We point out that $\alpha = 4\pi$ becomes admissible in (17) if we require $\|u\|_{H^1(\mathbb{R}^2)} \le 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$. Precisely, we have

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)} \le 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2} - 1\right) dx < \infty$$

and this is false for $\alpha > 4\pi$. See [Ruf 2005] for more details.

The estimates above obviously control any exponential power with smaller growth ($q < 2$). However, no estimate holds if the growth is higher ($q > 2$). Hence, the value $q = 2$ is also another criticality threshold for problems involving such nonlinearities.

Some technical lemmas.

Lemma 3.9. *For any* $0 < a < 1$,

$$\int_a^1 r\mathrm{e}^{4a^2 \log^2 r}\, dr \le 2. \tag{18}$$

Proof. Let $I(a)$ be the integral in (18). The change of variable $s = -2a \log r$ yields

$$I(a) = \frac{1}{2a}\mathrm{e}^{-1/(4a^2)} \int_0^{-2a\log a} \mathrm{e}^{(s-1/2a)^2} ds = \frac{1}{2a}\mathrm{e}^{-1/(4a^2)} \int_{-1/(2a)}^{-2a\log a - 1/(2a)} \mathrm{e}^{y^2} dy.$$

But $-2a\log a - \dfrac{1}{2a} \le \dfrac{1}{2a}$ for $0 < a < 1$; thus

$$I(a) \le 2A\mathrm{e}^{-A^2} \int_0^A \mathrm{e}^{y^2} dy,$$

where $A = \dfrac{1}{2a}$. It remains to prove that for all nonnegative A

$$\int_0^A \mathrm{e}^{y^2}\, dy \le \frac{\mathrm{e}^{A^2}}{A}. \tag{19}$$

Estimate (19) is obvious when $A \le 1$. If $A \ge 1$, we write

$$\int_0^A \mathrm{e}^{y^2} dy = \int_0^1 \mathrm{e}^{y^2} dy + \int_1^A 2y\mathrm{e}^{y^2} \frac{dy}{2y},$$

and an integration by parts gives

$$\int_0^A \mathrm{e}^{y^2} dy \le \frac{\mathrm{e}}{2} + \frac{\mathrm{e}^{A^2}}{2A} + \int_1^A \frac{\mathrm{e}^{y^2}}{2y^2} dy.$$

Using the monotonicity of the function $y \mapsto \mathrm{e}^{y^2}/(2y^2)$, the estimate (19) follows. □

Lemma 3.10. *For any $a \geq 1$ and $k \geq 1$,*

$$\int_{e^{-k/2}}^{1} r e^{(4a^2/k)\log^2 r}\, dr \leq 2e^{(a^2-1)k}. \tag{20}$$

Proof. Let $I(a, k)$ be the integral in (20). The change of variable $u = -\frac{2a}{\sqrt{k}} \log r$ yields

$$I(a, k) = \frac{\sqrt{k}}{2a}\, e^{-k/(4a^2)} \int_0^{a\sqrt{k}} e^{(u-\sqrt{k}/(2a))^2}\, du.$$

Changing once more the variable to $v = u - \frac{\sqrt{k}}{2a}$ yields

$$I(a, k) = \frac{\sqrt{k}}{2a}\, e^{-k/(4a^2)} \int_{-\sqrt{k}/(2a)}^{(2a^2-1)\sqrt{k}/(2a)} e^{v^2} dv.$$

Hence, for any $a \geq 1$ we have

$$I(a, k) \leq \frac{\sqrt{k}}{a}\, e^{-k/(4a^2)} \int_0^{(2a^2-1)\sqrt{k}/(2a)} e^{v^2} dv.$$

Now, using the estimate $\int_0^A e^{u^2} du \leq \frac{e^{A^2} - 1}{A} \leq \frac{e^{A^2}}{A}$, true for all nonnegative A, we obtain (20). □

Lemma 3.11. *For any $\lambda > 0$ and $A > \lambda$,*

$$\int_{A-\lambda^2/A}^{A} \frac{du}{\sqrt{e^{A^2} - e^{u^2}}} \leq \frac{A\, e^{2\lambda^2}}{A^2 - \lambda^2}\, e^{-A^2/2}. \tag{21}$$

Proof. Choosing $h(u) = \frac{-1}{u e^{u^2}}$ and $g'(u) = \frac{u e^{u^2}}{\sqrt{e^{A^2} - e^{u^2}}}$, and integrating by parts, we deduce (21). □

Lemma 3.12. *For any $A > 1$,*

$$\int_0^A \frac{du}{\sqrt{e^{A^2} - e^{u^2}}} \approx A\, e^{-A^2/2}. \tag{22}$$

Proof. Let $I(A)$ be the integrating in (22). In one hand, it is clear that

$$I(A) \geq A\, e^{-A^2/2}.$$

In the other hand, write

$$I(A) = \int_0^{A-1/(4A)} \frac{du}{\sqrt{e^{A^2} - e^{u^2}}} + J\big(A, \tfrac{1}{2}\big). \tag{23}$$

By Lemma 3.11, we get

$$J\big(A, \tfrac{1}{2}\big) \leq \frac{A\, e^{1/2}}{A^2 - \frac{1}{4}}\, e^{-A^2/2} \lesssim A\, e^{-A^2/2}.$$

For any $0 \leq u \leq A - \frac{1}{4A}$, we have

$$\frac{1}{\sqrt{e^{A^2} - e^{u^2}}} \leq \frac{1}{\sqrt{e^{A^2} - e^{(A-1/4A)^2}}} \lesssim e^{-\frac{A^2}{2}}.$$

Hence, the first integral in (23) can be estimated by $\displaystyle\int_0^{A-1/(4A)} \frac{du}{\sqrt{e^{A^2}-e^{u^2}}} \lesssim A\,e^{-A^2/2}$, and (22) follows. □

4. Energy regularity data

This section is devoted to the well-posedness issues in energy space stated in Section 2. Some of these results were announced in [Ibrahim et al. 2007b]. We begin with Theorem 2.1.

Proof of Theorem 2.1. First, consider the case $d \geq 3$. We prove statement (1) of the theorem.

- Construction of φ_k. For $k \geq 1$ and $\varepsilon > 0$ (depending on k as we will see later) define φ_k by

$$\varphi_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ a(k,\varepsilon)(|x|^{2-d}-1) & \text{if } \varepsilon/k \leq |x| \leq 1, \\ k^{(d-2)/2} & \text{if } |x| \leq \varepsilon/k, \end{cases}$$

where $a(k,\varepsilon) = \dfrac{\varepsilon^{d-2}k^{(d-2)/2}}{k^{d-2}-\varepsilon^{d-2}}$ is chosen such that φ_k is continuous. An easy computation yields

$$\|\nabla\varphi_k\|_{L^2}^2 \lesssim \frac{\varepsilon^{d-2}k^{d-2}}{k^{d-2}-\varepsilon^{d-2}} \lesssim \varepsilon^{d-2}.$$

Using assumption (3), we get

$$\begin{aligned}\int_{\mathbb{R}^d} F(\varphi_k(x))dx &\lesssim F(k^{(d-2)/2})\Big(\frac{\varepsilon}{k}\Big)^d + \int_{\varepsilon/k}^1 F\big(a(k,\varepsilon)(r^{2-d}-1)\big)r^{d-1}dr \\ &\lesssim F(k^{(d-2)/2})\Big(\frac{\varepsilon}{k}\Big)^d\bigg(1+\frac{1-(\varepsilon/k)^d}{\big(1-(\varepsilon/k)^{d-2}\big)^{2d/(d-2)}}\bigg).\end{aligned}$$

Since $k\big(F(k^{(d-2)/2})\big)^{-1/d} \to 0$ we will choose

$$\varepsilon = \varepsilon_k \stackrel{\text{def}}{=} k\big(F(k^{(d-2)/2})\big)^{-1/d}.$$

With this choice, we can see that $\|\nabla\varphi_k\|_{L^2} \to 0$ and $\sup_k E(\varphi_k) < \infty$.

- Construction of t_k. Consider the ordinary differential equation associated to (1):

$$\ddot{\Phi} + F'(\Phi) = 0, \quad (\Phi(0), \dot{\Phi}(0)) = (k^{(d-2)/2}, 0). \tag{24}$$

Using Lemma 3.7 and the assumptions on F, we can see that (24) has a unique global periodic solution Φ_k with period

$$T_k = 2\sqrt{2}\int_0^{k^{(d-2)/2}} \frac{d\Phi}{\sqrt{F(k^{(d-2)/2})-F(\Phi)}} = 2\sqrt{2}\frac{k^{(d-2)/2}}{\sqrt{F(k^{(d-2)/2})}}\int_0^1\bigg(1-\frac{F(vk^{(d-2)/2})}{F(k^{(d-2)/2})}\bigg)^{-1/2}dv.$$

By assumption (3), we get

$$T_k \leq 2\sqrt{2}\frac{k^{(d-2)/2}}{\sqrt{F(k^{(d-2)/2})}}\int_0^1\big(1-v^{2d/(d-2)}\big)^{-1/2}dv \lesssim k^{(d-2)/2}\big(F(k^{(d-2)/2})\big)^{-1/2}.$$

It follows that (for k large enough)

$$T_k \ll \frac{\varepsilon_k}{k}.$$

Now we are in a position to construct the sequence (t_k). Recall that by finite speed of propagation, any weak solution u_k of (1) with data $(\varphi_k, 0)$ satisfies

$$u_k(t,x) = \Phi_k(t) \qquad \text{if } \ 0 < t < \frac{\varepsilon_k}{k} \text{ and } |x| < \frac{\varepsilon_k}{k} - t.$$

Hence

$$|\partial_t u_k(t,x)| = |\dot{\Phi}_k(t)| = \sqrt{2}\sqrt{F(k^{(d-2)/2}) - F(\Phi_k(t))}.$$

Let us choose $t_k = \dfrac{T_k}{4}$; then $\Phi_k(t_k) = 0$, $t_k \ll \dfrac{\varepsilon_k}{k}$ and, for $|x| < \dfrac{\varepsilon_k}{k} - t_k$,

$$|\partial_t u_k(t_k,x)| = \sqrt{2}\sqrt{F(k^{(d-2)/2}) - F(\Phi_k(t_k))} \gtrsim \sqrt{F(k^{(d-2)/2})}.$$

So

$$\|\partial_t u_k(t_k)\|_{L^2}^2 \gtrsim F(k^{(d-2)/2})\Big(\frac{\varepsilon_k}{k} - t_k\Big)^d = \Big(\frac{\varepsilon_k}{k}\Big)^d F(k^{(d-2)/2})\Big(1 - t_k\frac{k}{\varepsilon_k}\Big)^d,$$

and the conclusion follows.

Now we turn to the proof of the second claim of Theorem 2.1. For clarity, we restrict ourselves to the model example (6). For any real $a > 0$, we denote by Φ_a the unique global solution of

$$\ddot{\Phi}(t) + \Phi^7(t) = 0, \quad (\Phi(0), \dot{\Phi}(0)) = (a, 0)\ . \tag{25}$$

By Lemma 3.7, Φ_a is periodic with period $T(a)$. By a scaling argument, we have $T(a) = a^{-3}T(1)$, and therefore

$$T(a) = C\,a^{-3}, \tag{26}$$

for some absolute positive constant C.

- Construction of t_k. Let (M_k) be a sequence of integers tending to infinity and such that

$$M_k = o(k^{1/6}) \quad \text{as } \ k \to \infty\,. \tag{27}$$

We denote by (η_k) the unique sequence in $(0, \infty)$ satisfying

$$4M_k = \frac{1}{1-(1-\eta_k)^3}\,. \tag{28}$$

As a consequence of these choices, we obtain the crucial identity

$$M_k\,T(\sqrt{k}) = \big(M_k - \tfrac{1}{4}\big)\,T(\sqrt{k}(1-\eta_k))\,. \tag{29}$$

A good choice for the sequence (t_k) is then

$$t_k = M_k\,T(\sqrt{k})\,. \tag{30}$$

Taking advantage of (26) and (27), we get $t_k \ll k^{-4/3}$.

- Construction of φ_k. The idea is to take a function φ_k oscillating between $\sqrt{k}$ and $\sqrt{k}(1-\eta_k)$ a certain number of times. Choose a sequence (N_k) of even integers tending to infinity and such that

$$N_k \sim C\, k^{1/6}\, M_k^2, \tag{31}$$

and set $\alpha_k := 10\, t_k\, N_k\, k^{4/3} \sim C\, M_k^3$. Divide the radial interval $k^{-4/3} \le r \le (\alpha_k+1)k^{-4/3}$ into N_k subintervals each of them has a length $10\, t_k$ and write

$$[k^{-4/3}, (\alpha_k+1)k^{-4/3}] = \bigcup_{j=0}^{N_k-1} [a_k^{(j)}, a_k^{(j+1)}],$$

where $a_k^{(j)} = k^{-4/3} + 10\, j\, t_k$. Now consider a φ_k that is continuous and oscillates between $\sqrt{k}$ and $\sqrt{k}(1-\eta_k)$ as follows:

$$\begin{aligned}
\varphi_k(r) &= \sqrt{k} && \text{if } r \le k^{-4/3},\\
\varphi_k(r) &= \sqrt{k}(1-\eta_k) && \text{if } k^{-4/3}+t_k \le r \le k^{-4/3}+9t_k,\\
\varphi_k(r) &= \sqrt{k} && \text{if } k^{-4/3}+11t_k \le r \le k^{-4/3}+19t_k,\\
\varphi_k(r) &= \cdots,\\
\varphi_k(r) &= \sqrt{k} && \text{if } k^{-4/3}+(10N_k-9)t_k \le r \le k^{-4/3}+(10N_k-1)t_k,\\
\varphi_k(r) &= \sqrt{k} && \text{if } r \ge k^{-4/3}+10N_k t_k;
\end{aligned}$$

in the remaining intervals, φ_k is affine. An easy computation shows that

$$\|\nabla\varphi_k\|_{L^2}^2 \lesssim N_k \Big(\frac{\sqrt{k}\eta_k}{t_k}\Big)^2 (k^{-4/3})^3\, t_k\, k^{4/3} \lesssim \frac{1}{M_k}\,. \tag{32}$$

Moreover, using the finite speed of propagation and the fact that

$$\Phi_{\sqrt{k}}(t_k) = \sqrt{k}, \quad \Phi_{\sqrt{k}(1-\eta_k)}(t_k) = 0,$$

we conclude that any weak solution u_k to (6) with data $(\varphi_k, 0)$ satisfies

$$\|\partial_t\, u_k(t_k)\|_{L^2}^2 \gtrsim N_k k^4 (k^{-4/3})^4 t_k k^{4/3} \gtrsim M_k^3\,. \tag{33}$$

This finishes the proof for $d \ge 3$. The case $d = 2$ can be handled in a similar way. We have just to make a suitable choice of the initial data.

- Construction of φ_k. For $k \ge 1$, we define φ_k by

$$\varphi_k(x) = \begin{cases} 0 & \text{if } |x| \ge 1,\\ \dfrac{-2\sqrt{k}}{\log F(\sqrt{k})}\log|x| & \text{if } \varepsilon_k \mathrm{e}^{-k/2} \le |x| \le 1,\\ \sqrt{k} & \text{if } |x| \le \varepsilon_k \mathrm{e}^{-k/2}, \end{cases}$$

where $\varepsilon_k = e^{k/2}(F(\sqrt{k}))^{-1/2}$. Remark that, by (4), we have $\varepsilon_k \to 0$. An easy computation using (4) yields

$$\|\nabla\varphi_k\|_{L^2}^2 \lesssim \frac{-1}{\log \varepsilon_k},$$

and

$$\int_{\mathbb{R}^2} F(\varphi_k(x))\,dx \lesssim \varepsilon_k^2 e^{-k} F(\sqrt{k}) + \int_{\varepsilon_k e^{-k/2}}^1 r \exp\biggl(4\frac{\log^2 r}{(\log F(\sqrt{k}))^2}\biggr)\,dr.$$

The choice of ε_k implies that the first summand on the right side is $\lesssim 1$. For the second summand, we use Lemma 3.9.

- Construction of t_k. As in higher dimensions, we consider the associated ordinary differential equation with data $(\sqrt{k}, 0)$. This equation has a unique global periodic solution with period

$$T_k = 2\sqrt{2}\int_0^{\sqrt{k}} \frac{d\Phi}{\sqrt{F(\sqrt{k}) - F(\Phi)}}.$$

By assumption (4), we get

$$T_k \lesssim \sqrt{k}\frac{1}{A}\int_0^A \frac{du}{\sqrt{e^{A^2} - e^{u^2}}},$$

where $A = \sqrt{\log F(\sqrt{k})}$. It follows from Lemma 3.12 that $T_k \ll \varepsilon_k e^{-k/2}$. Now, arguing exactly in the same manner as in higher dimensions, we finish the proof for $d = 2$. □

Proof of Theorem 2.3. The idea here is to split the initial data into a small part in $H^1 \times L^2$ and a smooth one. First we solve the IVP with smooth initial data to obtain a local and bounded solution v. Then we consider the perturbed equation satisfied by $w := u - v$ and with small initial data. (A similar idea was used in [Gallagher and Planchon 2003; Germain 2008; Kenig et al. 2000; Planchon 2000].) Now we come to the details.

Existence. Given initial data (u_0, u_1) in the energy space $H^1 \times L^2$, we decompose it as

$$(u_0, u_1) = S_n(u_0, u_1) + (I - S_n)(u_0, u_1),$$

where the first term is defined as $(u_0, u_1)_{<n}$ and the second as $(u_0, u_1)_{>n}$, for n a (large) integer to be chosen later. Note that

$$(u_0, u_1)_{>n} \to 0 \quad \text{in } H^1 \times L^2 \quad \text{as } n \to \infty,$$

and that, for every n, $(u_0, u_1)_{<n} \in H^2 \times H^1$. First we consider the IVP with regular data

$$\Box v + v + f(v) = 0, \quad (v(0, x), \partial_t v(0, x)) = (u_0, u_1)_{<n}, \quad f(v) = v(e^{4\pi v^2} - 1). \tag{34}$$

It is known that (34) is well-posed. More precisely, there exist a time $T_n = T(\|(u_0, u_1)_{<n}\|_{H^2\times H^1}) > 0$ and a unique solution v to (34) in $C_{T_n}(H^2) \cap C^1_{T_n}(H^1)$. Moreover, we can choose T_n such that $\|v\|_{L^\infty_{T_n}(H^2)} \le (\|((u_0)_{<n}, (u_1)_{<n})\|_{H^2\times H^1} + 1)$.

Next we consider the perturbed IVP with small data

$$\Box w + w + f(w + v) - f(v) = 0, \quad (w(0, x), \partial_t w(0, x)) = (u_0, u_1)_{>n}. \tag{35}$$

We shall prove that (35) has a local in time solution in the space $\mathscr{E}_T := C_T(H^1)\cap C_T^1(L^2)\cap L_T^4(C^{1/4})$ for suitable time $T>0$. This will be achieved by a standard fixed point argument. We denote by w_ℓ the solution of the linear Klein–Gordon equation with data $(u_0,u_1)_{>n}$,

$$\Box w_\ell + w_\ell = 0, \quad (w_\ell(0,x), \partial_t w_\ell(0,x)) = (u_0,u_1)_{>n}.$$

For a positive time $T\le T_n$ and a positive real number δ, we denote by $\mathscr{E}_T(\delta)$ the closed ball in $\mathscr{E}_T$ of radius δ and center at the origin. On the ball $\mathscr{E}_T(\delta)$, we define the map Φ by

$$\Phi : w\in\mathscr{E}_T(\delta)\mapsto \tilde{w}$$

where

$$\Box\tilde{w}+\tilde{w}+f(w+w_\ell+v)-f(v)=0, \qquad (\tilde{w}(0,x),\partial_t\tilde{w}(0,x))=(0,0)\,.$$

By energy and Strichartz estimates, we get

$$\|\Phi(w)\|_{\mathscr{E}_T}\lesssim \|f(w+w_\ell+v)-f(v)\|_{L_T^1(L^2)}\lesssim \|w+w_\ell\|_{L_T^\infty(L^2)}\left\|\mathrm{e}^{C\|w+w_\ell+v\|_\infty^2}+\mathrm{e}^{C\|v\|_\infty^2}\right\|_{L_T^1}$$

It is clear that

$$\left\|\mathrm{e}^{C\|v\|_\infty^2}\right\|_{L_T^1}\lesssim T\mathrm{e}^{C(\|(u_0)_{<n}\|_{H^2}+1)^2}.$$

On the other hand, using the logarithmic inequality we infer

$$\mathrm{e}^{C\|w+w_\ell+v\|_\infty^2}\lesssim \mathrm{e}^{C\|(u_0)_{<n}\|_{H^2}^2}\Big(C+\frac{\|w+w_\ell\|_{C^{1/4}}}{\delta+\varepsilon}\Big)^{C(\delta+\varepsilon)^2},$$

where $\varepsilon^2=\|w_0\|_{H^1}^2+\|w_1\|_{L^2}^2$. By the Hölder inequality in time we deduce

$$\left\|\mathrm{e}^{C\|w+w_\ell+v\|_\infty^2}\right\|_{L_T^1}\lesssim \mathrm{e}^{C\|(u_0)_{<n}\|_{H^2}^2}T^{1-\beta/4}(T^{1/4}+\delta+\varepsilon)^\beta,$$

where $\beta:=C(\delta+\varepsilon)^2<4$ for δ and ε small enough. Finally, we get

$$\|\Phi(w)\|_{\mathscr{E}_T}\lesssim(\delta+\varepsilon)\mathrm{e}^{C\|(u_0)_{<n}\|_{H^2}^2}\big(T+T^{1-\beta/4}(T^{1/4}+\delta+\varepsilon)^\beta\big).$$

From this inequality it follows immediately that Φ maps $\mathscr{E}_T(\delta)$ into itself if T is small enough. To prove that Φ is a contraction (at least for T small), we consider two elements w_1 and w_2 in $\mathscr{E}_T(\delta)$ and define

$$w=w_1-w_2,\quad \tilde{w}=\tilde{w_1}-\tilde{w_2},\quad \bar{w}=(1-\theta)(w_\ell+w_1)+\theta(w_\ell+w_2)+v\quad \text{with } 0\le\theta\le 1.$$

We can write

$$f(w_\ell+w_1)-f(w_\ell+w_2)=w[(1+8\pi\bar{w}^2)\mathrm{e}^{4\pi\bar{w}^2}-1]$$

for some choice of $0\le\theta(t,x)\le 1$. By the energy estimate and the Strichartz inequality we have

$$\left\|\Phi(w_1)-\Phi(w_2)\right\|_{\mathscr{E}_T}\lesssim\left\|w\mathrm{e}^{C|\bar{w}|^2}\right\|_{L_T^1(L_x^2)}.$$

By convexity, we obtain

$$\|\Phi(w_1)-\Phi(w_2)\|_{\mathscr{E}_T}\lesssim\|w(\mathrm{e}^{C|w_\ell+w_1|^2}+\mathrm{e}^{C|w_\ell+w_2|^2})\|_{L_T^1(L_x^2)}.$$

So arguing as before, we get

$$\begin{aligned}\|\Phi(w_1)-\Phi(w_2)\|_{\mathscr{E}_T} &\lesssim \|w\|_{L^\infty_T(L^2)}\big(\|\mathrm{e}^{C\|w_\ell+w_1\|_\infty^2}\|_{L^1_T}+\|\mathrm{e}^{C\|w_\ell+w_2\|_\infty^2}\|_{L^1_T}\big),\\ &\lesssim T^{1-\beta/4}(T^{1/4}+\delta+\varepsilon)^\beta\|w_1-w_2\|_T,\end{aligned}$$

for some $\beta<4$. If the parameters $\varepsilon>0$, $\delta>0$ and $T>0$ are suitably chosen, then Φ is a contraction map on $\mathscr{E}_T(\delta)$ and thus a local in time solution is constructed.

Uniqueness. We shall prove the uniqueness in the space

$$\mathscr{F}_\eta := C_T(H^2)\cap C^1_T(H^1)+\{w\in\mathscr{E}_T : \|w\|_T\le\eta\},$$

for any $\eta<1/\sqrt{2}$. Let $u:=v+w$ and $U:=V+W$ be two solutions of (9) in $\mathscr{F}_\eta$ with the same initial data. Since v, $V\in C_t(H^2)$ and H^2 is embedded in L^∞, we can choose a time $T>0$ such that (for some constant C)

$$\|v\|_{L^\infty([0,T],L^\infty)}\le C \quad\text{and}\quad \|V\|_{L^\infty([0,T],L^\infty)}\le C\,. \tag{36}$$

The difference $U-u$ satisfies

$$\Box(U-u)+U-u=f(v+w)-f(V+W),\quad ((U-u),\partial_t(U-u))(t=0)=(0,0).$$

Using the energy estimate and Strichartz inequality, we get

$$\begin{aligned}\|U-u\|_{\mathscr{E}_T} &\lesssim \big\|f(v+w)-f(V+W)\big\|_{L^1_T(L^2)}\\ &\lesssim \big\|(U-u)\big(U^2(\mathrm{e}^{4\pi U^2}-1)+u^2(\mathrm{e}^{4\pi u^2}-1)\big)\big\|_{L^1_T(L^2)}\\ &\lesssim \|U-u\|_{L^\infty_T(L^{2/\varepsilon})}\big\|U^2(\mathrm{e}^{4\pi U^2}-1)+u^2(\mathrm{e}^{4\pi u^2}-1)\big\|_{L^1_T(L^{2/(1-\varepsilon)})},\end{aligned}$$

where $\varepsilon>0$ is to be chosen small enough. To conclude the proof of the uniqueness, we have to estimate the term

$$\big\|u^2(\mathrm{e}^{4\pi u^2}-1)\big\|_{L^1_T(L^{2/(1-\varepsilon)})},$$

for example. Observe that, for any $\beta>0$ and $a>1$,

$$x^2(\mathrm{e}^{4\pi x^2}-1)\le C_\beta(\mathrm{e}^{4\pi(1+\beta)x^2}-1), \tag{37}$$

and

$$(x+y)^2\le\frac{a}{a-1}\,x^2+a\,y^2\,. \tag{38}$$

Hence

$$\big\|u^2(\mathrm{e}^{4\pi u^2}-1)\big\|_{L^1_T(L^{2/(1-\varepsilon)})}\lesssim\int_0^T\bigg(\int_{\mathbb{R}^2}\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}u^2}-1\Big)\,dx\bigg)^{(1-\varepsilon)/2}dt\,.$$

Moreover, using (38), we can write

$$\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}u^2}-1\le\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}\frac{a}{a-1}v^2}-1\Big)+\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}aw^2}-1\Big)+\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}\frac{a}{a-1}v^2}-1\Big)\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}aw^2}-1\Big). \tag{39}$$

To estimate the first term on the right-hand side of (39), we use (36). For the second term, observe that

$$\sqrt{2}\,\eta\sqrt{\frac{(1+\beta)a}{1-\varepsilon}}\to\eta\sqrt{2}<1\quad\text{as } a\to1 \text{ and } \varepsilon,\ \beta\to0\,.$$

This enables us to use the Trudinger–Moser inequality. We do the same for the last term. This concludes the proof of the uniqueness in the space $\mathcal{F}_\eta$. Note that we can weaken the hypothesis $\eta < \frac{1}{\sqrt{2}}$ to $\eta < 1$ if we use the sharp logarithmic inequality (15). □

Remark 4.1. In higher dimensions $d \geq 3$, we obtain a similar result in $H^{d/2} \times H^{d/2-1}$ for (1) by using a decomposition in $H^{d/2+1} \times H^{d/2}$ and small in $H^{d/2} \times H^{d/2-1}$.

Proof of Theorem 2.4. For any $k \geq 1$ define f_k by

$$f_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ -\dfrac{\log|x|}{\sqrt{k\pi}} & \text{if } \mathrm{e}^{-k/2} \leq |x| \leq 1, \\ \sqrt{k/4\pi} & \text{if } |x| \leq \mathrm{e}^{-k/2}. \end{cases}$$

These functions were introduced in [Moser 1971] to show the optimality of the exponent 4π in Trudinger–Moser inequality. An easy computation shows that $\|\nabla f_k\|_{L^2(\mathbb{R}^2)} = 1$ and $\|f_k\|_{L^2(\mathbb{R}^2)} \lesssim 1/\sqrt{k}$. Denote by u_k and v_k any weak solutions of (9) with initial data $((1+\frac{1}{k}) f_k(\frac{\cdot}{\nu}), 0)$ and $(f_k(\frac{\cdot}{\nu}), 0)$, respectively. By construction,

$$\big\|(u_k - v_k)(0)\big\|_{H^1}^2 + \big\|\partial_t(u_k - v_k)0)\big\|_{L^2}^2 = \frac{1}{k^2}\|f_k\Big(\frac{\cdot}{\nu}\Big)\|_{H^1}^2 = o(1) \quad \text{as } k \to \infty.$$

Also, using estimate (20), it is clear that

$$0 < E\Big(\Big(1+\frac{1}{k}\Big) f_k\Big(\frac{\cdot}{\nu}\Big)\Big) - 1 \leq \mathrm{e}^3\nu^2 \quad \text{and} \quad 0 < E\Big(f_k\Big(\frac{\cdot}{\nu}\Big)\Big) - 1 \leq \nu^2.$$

Now, we shall construct the sequence of time t_k. A good approximation of u_k and v_k is provided by the corresponding ordinary differential equation,

$$\ddot{\Phi} + \Phi \mathrm{e}^{4\pi\Phi^2} = 0. \tag{40}$$

More precisely, let Φ_k and Ψ_k be the solutions of (40) with initial data

$$\Phi_k(0) = \Big(1+\frac{1}{k}\Big)\sqrt{\frac{k}{4\pi}}, \quad \dot{\Phi}_k(0) = 0,$$

and

$$\Psi_k(0) = \sqrt{\frac{k}{4\pi}}, \quad \dot{\Psi}_k(0) = 0,$$

respectively. Note that by finite speed of propagation, we have $\Phi_k = u_k$ and $\Psi_k = v_k$ in the backward light cone $|x| < \nu\mathrm{e}^{-k/2} - t,\ t < \nu\mathrm{e}^{-k/2}$.

On the other hand, recall that the period T_k of Φ_k is given by

$$T_k = 2\int_0^{(1+1/k)\sqrt{k}} \frac{du}{\sqrt{\mathrm{e}^{(1+1/k)^2 k} - \mathrm{e}^{u^2}}};$$

hence, using Lemma 3.12 we can prove that $T_k \approx \sqrt{k}\,\mathrm{e}^{-(1+1/k)^2 k/2}$. Therefore, one need to choose time $t_k << \mathrm{e}^{-(1+1/k)^2 k/2}$ and check that the decoherence of Φ_k and Ψ_k occurs at time t_k. Choose $t_k \in]0, T_k/4[$

such that

$$\Phi_k(t_k) = \Big(1+\frac{1}{k}\Big)\sqrt{\frac{k}{4\pi}} - \bigg(\Big(1+\frac{1}{k}\Big)\sqrt{\frac{k}{4\pi}}\bigg)^{-1}.$$

It follows that

$$t_k = \int_{\sqrt{k}+1/\sqrt{k}-4\pi\sqrt{k}/(k+1)}^{\sqrt{k}+1/\sqrt{k}} \frac{du}{\sqrt{e^{k(1+1/k)^2} - e^{u^2}}}.$$

Using (21), we obtain $t_k \lesssim (1/\sqrt{k})e^{-k/2}$. In particular, if k is large enough then $t_k \lesssim (\nu/2)e^{-k/2}$. Now we show that this time t_k is sufficient to let instability occurs. Since Ψ_k is decreasing on the interval $[0, (T_k/4)]$, we have

$$e^{4\pi\psi_k(0)^2} - e^{4\pi\psi_k(t_k)^2} = |e^k - e^{4\pi\psi_k(t_k)^2}| \lesssim e^k,$$

Therefore,

$$\big|(\dot\Phi_k(t_k))^2 - (\dot\Psi_k(t_k))^2\big| = \frac{1}{4\pi}\big|(e^{4\pi\Phi_k(0)^2} - e^{4\pi\Phi_k(t_k)^2}) - (e^{4\pi\Psi_k(0)^2} - e^{4\pi\Psi_k(t_k)^2})\big| \gtrsim e^k.$$

Finally, we deduce that

$$\int_{\mathbb{R}^2} \big|\partial_t(u_k - v_k)(t_k)\big|^2\,dx \gtrsim \int_{|x|<(\nu/2)e^{-k/2}} \big|\partial_t(u_k - v_k)(t_k)|^2\,dx \gtrsim \nu^2 e^{-k}|\dot\Phi_k(t_k)) - \dot\Psi_k(t_k)\big|^2$$

and the conclusion follows. □

5. Low regularity data

Proof of Theorem 2.6. (1) For $k \ge 1$ and $\gamma > 1$, let $\varphi_k = \gamma f_k$. An easy computation shows that

$$\|\nabla\varphi_k\|_{L^{2,\infty}} \lesssim \frac{\gamma}{\sqrt{k}}.$$

Next we consider the solution Φ_k of the associated ODE with Cauchy data $(\gamma\sqrt{k/4\pi}, 0)$. The period T_k of Φ_k satisfies

$$T_k \approx \gamma\sqrt{k}\,e^{-(\gamma^2/2)k} \ll e^{-k/2}.$$

Arguing as in the previous section, we construct a sequence (t_k) going to zero such that any weak solution u_k with Cauchy data $(\varphi_k, 0)$ satisfies

$$\|\partial_t u_k(t_k)\|^2_{L^{2,\infty}} \gtrsim e^{(\gamma^2-1)k},$$

and we are done.

(2) Now we will prove the ill-posedness in $\mathcal{B}^1_{2,\infty}$. The main difficulty is the construction of the initial data. For this end, consider a radial smooth function $h \in C_0^\infty(\mathbb{R}^2)$ satisfying $h(r)=0$ if $r \ge 2$ and $h(r)=1$ if $r < 1$. For $a > 0$, set $h_a(r) = h(r/a)$. Since $\hat h_a(\xi) = a^2\hat h(a\xi)$, we get

$$|\hat h_a(\xi)| \le \frac{C}{|\xi|^2} \quad \text{uniformly in } a. \tag{41}$$

Now we define the function g_a via

$$g_a(r) = \frac{1-h_a(r)}{r}.$$

Proposition 5.1. $$|\hat{g}_a(\xi)| \le \frac{C}{|\xi|} \quad \textit{uniformly in } a.$$

Proof. Write
$$\hat{g}_a(\xi) = \frac{C}{|\xi|} - C\Big(\frac{1}{|\xi|} \star \hat{h}_a(\xi)\Big),$$
using the fact that $\widehat{r^{-1}} = C|\xi|^{-1}$. (The convolution here is well defined.) Thus, we have to prove that, for fixed ξ,
$$\Big| \int \frac{\hat{h}_a(\eta)}{|\xi - \eta|}\, d\eta \Big| \lesssim \frac{1}{|\xi|} \quad \text{uniformly in } a.$$
The idea now is the following: fix ξ such that $|\xi| \sim 2^j$ for some $j \in \mathbb{Z}$ and write
$$\int \frac{\hat{h}_a(\eta)}{|\xi - \eta|}\, d\eta = \int_{|\eta| \le c2^j} \frac{\hat{h}_a(\eta)}{|\xi - \eta|}\, d\eta + \int_{|\eta| \sim 2^j} \frac{\hat{h}_a(\eta)}{|\xi - \eta|}\, d\eta + \int_{|\eta| \ge C2^j} \frac{\hat{h}_a(\eta)}{|\xi - \eta|}\, d\eta. \tag{42}$$
Using (41), we can easily estimate the second and third terms on the right-hand side. To estimate the first term, we use the fact that $\hat{h}_a$ is uniformly in L^1. □

Corollary 5.2. $$\sup_{a>0} \|g_a\|_{\dot{\mathcal{B}}^0_{2,\infty}} < \infty.$$

Proof. Write $\|g_a\|_{\dot{\mathcal{B}}^0_{2,\infty}} \approx \sup_{j\in\mathbb{Z}} \int_{2^{j-1}<|\xi|<2^{j+1}} |\hat{g}_a(\xi)|^2\, d\xi \lesssim \sup_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^{j+1}} \frac{dr}{r} \lesssim 1$, uniformly in a. □

Now we are ready to construct the sequence of initial data (φ_k). Let $\theta \in C_0^\infty(\mathbb{R}^2)$ be a radial function such that $\theta(r) = 1$ if $r \le 1$ and $\theta(r) = 0$ if $r \ge 2$. For $k \ge 1$, set
$$\tilde{g}_k(r) = \frac{1}{\sqrt{k}} g_{\mathrm{e}^{-k/2}}(r)\theta(r). \tag{43}$$
It follows from Corollary 5.2 that $\|\tilde{g}_k\|_{\dot{\mathcal{B}}^0_{2,\infty}} \lesssim \dfrac{1}{\sqrt{k}}$. Moreover, one can see easily that
$$\frac{1}{C}\sqrt{k} \le \int_0^2 \tilde{g}_k(r)\, dr\, C\sqrt{k}\,.$$
To finish the construction set
$$\varphi_k(r) = \gamma\sqrt{\frac{k}{4\pi}} - c_k \int_0^r \tilde{g}_k(\tau)\, d\tau,$$
where $\gamma > 1$ and c_k is chosen so that $\varphi_k(2) = 0$. We now summarize some crucial properties of φ_k.

Proposition 5.3. (a) $\varphi_k(r) = \gamma\sqrt{k/4\pi}$ *if* $r \le \mathrm{e}^{-k/2}$. (b) $\varphi_k \to 0$ *in* $\mathcal{B}^1_{2,\infty}(\mathbb{R}^2)$.

Proof. Part (a) follows directly from the definition of the function $\tilde{g}_k$. To prove (b), recall that
$$\|\varphi_k\|_{\mathcal{B}^1_{2,\infty}} \approx \|\varphi_k\|_{L^2} + \|\nabla\varphi_k\|_{\dot{\mathcal{B}}^0_{2,\infty}}\,.$$
Since $\|\varphi_k\|_{L^2} \lesssim 1/\sqrt{k}$ we have just to prove that $\|\nabla\varphi_k\|_{\dot{\mathcal{B}}^0_{2,\infty}}$ goes to zero. As $\nabla\varphi_k = (x/r)\tilde{g}_k(r)$, it suffices to apply Theorem 3.3 together with the fact that $x/r \in \dot{\mathcal{B}}^1_{2,\infty} \cap L^\infty$. □

We resume the proof of Theorem 2.6, considering the associated ODE with Cauchy data $(\gamma\sqrt{k/4\pi}, 0)$ and denoting by Φ_k the (global periodic) solution with period

$$T_k \lesssim \int_0^{\gamma\sqrt{k}} \frac{du}{\sqrt{e^{\gamma^2 k} - e^{u^2}}} \lesssim \gamma\sqrt{k}\, e^{-(\gamma^2/2)k} \ll e^{-k/2} \quad (\gamma > 1).$$

Set $t_k = T_k/4$ so that $\Phi_k(t_k) = 0$. Note that by finite speed of propagation any weak solution u_k of (7) with Cauchy data $(\varphi_k, 0)$ satisfies

$$u_k(t, x) = \Phi_k(t) \quad \text{for } 0 < t < e^{-k/2} \text{ and } |x| < e^{-k/2} - t.$$

Hence

$$-\partial_t u_k(t_k, x) \gtrsim e^{(\gamma^2/2)k} \quad \text{for } |x| < e^{-k/2} - t_k. \tag{44}$$

It remains to estimate from below the norm $\|\partial_t u_k(t_k)\|_{\dot{\mathcal{B}}^0_{2,\infty}}$. To get the desired estimate we proceed in the following way. First recall that

$$\|\partial_t u_k(t_k)\|_{\dot{\mathcal{B}}^0_{2,\infty}} = \sup_{\|v\|_{\dot{\mathcal{B}}^0_{2,1}}=1} \int_{\mathbb{R}^2} v(x)\, \partial_t u_k(t_k, x)\, dx.$$

Then we have to make a suitable choice of v. Let v be a smooth compactly supported function such that

$$v(x) = 1 \quad \text{for } |x| \le \tfrac{1}{4}, \qquad v(x) = 0 \quad \text{for } |x| \ge \tfrac{1}{2}.$$

For $k \ge 1$ let $v_k(x) = e^{k/2}\, v(e^{k/2} x)$. We remark that $\|v_k\|_{\dot{\mathcal{B}}^0_{2,\infty}} = \|v\|_{\dot{\mathcal{B}}^0_{2,\infty}}$ is a constant. Using (44), we get

$$\begin{aligned}\|\partial_t u_k(t_k)\|_{\dot{\mathcal{B}}^0_{2,\infty}} &\ge \int -\partial_t u_k(t_k, x)\, v_k(x)\, dx \ge e^{k/2} \int_{|x|\le \frac{1}{4}e^{-k/2}} -\partial_t u_k(t_k, x)\, dx \\ &\gtrsim e^{k/2}(e^{-k/2})^2\, e^{(\gamma^2/2)k} = e^{(\gamma^2-1)/2k}.\end{aligned}$$

This finishes the proof of the part (2) of the theorem, since $\gamma > 1$.

(3) Without loss of generality, we may assume that $0 \le s < 1$. Let $0 < \gamma < \frac{1}{2}(1-s)$ and consider $\varphi_k = k^\gamma f_k$. It is clear that

$$\|\varphi_k\|_{H^s} \lesssim k^\gamma\, k^{-(1-s)/2} \to 0$$

Denote by u_k any weak solution of (9) with initial data $(\varphi_k, 0)$ and Φ_k the solution of the associated ODE with Cauchy data $(k^\gamma\sqrt{k/4\pi}, 0)$. The period T_k of Φ_k satisfies

$$T_k \lesssim k^{\gamma+1/2}\, e^{-(k^{2\gamma+1})/2} \ll e^{-k/2}.$$

Choose $t_k = \dfrac{T_k}{4}$, so that $\Phi_k(t_k) = 0$. By finite speed of propagation, we have

$$u_k(t, x) = \Phi_k(t) \quad \text{for } |x| < e^{-k/2} - t, \;\; 0 < t < e^{-k/2}.$$

Hence $|x| < e^{-k/2} - t_k$,

$$-\partial_t u_k(t_k, x) \;=\; -\dot{\Phi}_k(t_k) = \frac{1}{2\sqrt{\pi}}\sqrt{e^{k^{2\gamma+1}} - e^{4\pi\Phi_k^2(t_k)}} = \frac{1}{2\sqrt{\pi}} e^{k^{2\gamma+1}/2}. \tag{45}$$

To conclude the proof we need to estimate from below $\|\partial_t u_k(t_k)\|_{H^{s-1}}$. Write

$$\|\partial_t u_k(t_k)\|_{H^{s-1}} = \sup_{\|v\|_{H^{1-s}}=1} \int_{\mathbb{R}^2} v(x)\, \partial_t u_k(t_k, x)\, dx.$$

Set $v_k(x) = \mathrm{e}^{sk/2}\, v(\mathrm{e}^{k/2}\, x)$, where v is as above. It follows that

$$\|\partial_t u_k(t_k)\|_{H^{s-1}} \geq \int -\partial_t u_k(t_k, x)\, v_k(x)\, dx \geq \mathrm{e}^{sk/2} \int_{|x|\leq \frac{1}{4}\mathrm{e}^{-k/2}} -\partial_t u_k(t_k, x)\, dx \gtrsim \mathrm{e}^{sk/2}\, (\mathrm{e}^{-k/2})^2\, \mathrm{e}^{\frac{1}{2}k^{2\gamma+1}} = \mathrm{e}^{(s/2-1)k+\frac{1}{2}k^{2\gamma+1}},$$

which goes to infinity when $k \to \infty$. □

Proof of Theorem 2.7. (1) Our aim here is to prove the local well-posedness of (7) in $\mathcal{B}^1_{2,q'} \times \mathcal{B}^0_{2,q'}$ for any $1 \leq q < \infty$. The strategy is the same as in the proof of Theorem 2.3. We decompose the initial data (u_0, u_1) into a small part[7] in $\mathcal{B}^1_{2,q'} \times \mathcal{B}^0_{2,q'}$ and a regular one:

$$(u_0, u_1) = (u_0, u_1)_{>N} + (u_0, u_1)_{<N}\,.$$

First we solve the IVP with regular data to obtain a local regular solution v, and then we solve the perturbed IVP with small data using a fixed point argument to obtain finally the expected solution u. Let us start by studying the free equation. For a given $(u^0_\ell, u^1_\ell) \in \mathcal{B}^1_{2,q'} \times \mathcal{B}^0_{2,q'}$ we denote by u_ℓ the free solution with data (u^0_ℓ, u^1_ℓ), that is

$$\Box u_\ell + u_\ell = 0, \quad (u_\ell, \partial_t\, u_\ell)(t=0) = (u^0_\ell, u^1_\ell)\,. \tag{46}$$

Using a localization in frequency, an energy estimate and te Strichartz inequality (13), we derive the following result.

Proposition 5.4. *Let $T > 0$. For any $1 < q' \leq \infty$, there exists $0 \leq \varepsilon(q') < \frac{1}{4}$ such that*

$$\|u_\ell\|_{L^\infty_T(\mathcal{B}^1_{2,q'})} + \|u_\ell\|_{L^4_T(\mathcal{C}^{\frac{1}{4}-\varepsilon})} \lesssim \|u^0_\ell\|_{\mathcal{B}^1_{2,q'}} + \|u^1_\ell\|_{\mathcal{B}^0_{2,q'}}. \tag{47}$$

(*In fact, when $q' \leq 4$, we have a zero loss of derivatives, meaning $\varepsilon(q') = 0$, and when $q' > 4$, one can choose an arbitrary $0 < \varepsilon < \frac{1}{4}$.*)

Proof. From the energy and Strichartz estimates applied to $\Delta_j u_\ell$, we have

$$2^j\, \|\Delta_j u_\ell\|_{L^\infty_T(L^2)} + 2^{j/4} \|\Delta_j u_\ell\|_{L^4_T(L^\infty)} \lesssim 2^j\, \|\Delta_j u^0_\ell\|_{L^2} + \|\Delta_j u^1_\ell\|_{L^2}\,. \tag{48}$$

Summing this estimate in $\ell^{q'}$ we have $\big\|2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)}\big\|_{\ell^{q'}} \leq \|u^0_\ell\|_{\mathcal{B}^1_{2,q'}} + \|u^1_\ell\|_{\mathcal{B}^0_{2,q'}}$. In the case $q' \leq 4$, the proposition follows from the observation

$$\big\|u_\ell\|_{L^4(\mathcal{B}^{1/4}_{\infty,q'})} \leq \|2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)}\big\|_{\ell^{q'}},$$

together with the Sobolev embedding $\mathcal{B}^{1/4}_{\infty,q'} \to \mathcal{C}^{1/4}$. When $q' > 4$, notice that for any $0 < \varepsilon < \frac{1}{4}$,

$$\|u_\ell\|_{L^4_T(\mathcal{B}^{1/4-\varepsilon}_{\infty,4})} = \big\|(2^{j/4-j\varepsilon}\|\Delta_j u_\ell\|_{L^\infty})_{\ell^4}\big\|_{L^4_T} = \big\|2^{-j\varepsilon}(2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)})\big\|_{\ell^4}.$$

[7] To do so in the case $q' = \infty$ we have to work with $\tilde{\mathcal{B}}^1_{2,\infty} := \overline{\mathcal{D}}^{\mathcal{B}^1_{2,\infty}}$ and $\tilde{\mathcal{B}}^0_{2,\infty} := \overline{\mathcal{D}}^{\mathcal{B}^0_{2,\infty}}$.

Using (48) and Hölder's inequality in j — writing $\frac{1}{4} = \frac{1}{q'} + \frac{1}{r}$, with $r = \frac{4q'}{q'-4}$ — we get

$$\|u_\ell\|_{L^4_T(\mathscr{B}^{1/4-\varepsilon}_{\infty,4})} \leq \|(2^{-j\varepsilon})\|_{\ell^r} \, \big\|(2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)})\big\|_{\ell^{q'}} \lesssim \|u^0_\ell\|_{\mathscr{B}^1_{2,q'}} + \|u^1_\ell\|_{\mathscr{B}^0_{2,q'}} .$$

Again, Sobolev embedding enables us to finish the proof. □

Define $g_q(u) := u\big((1+u^2)^{(q-2)/2}\mathrm{e}^{4\pi((1+u^2)^{q/2}-1)} - 1\big)$, so that (7) reads

$$\Box u + u + g_q(u) = 0. \tag{49}$$

An easy computation shows that

$$|g_q(u) - g_q(v)| \leq \begin{cases} C|u-v|(\mathrm{e}^{C|u|^q} - 1 + \mathrm{e}^{C|v|^q} - 1) & \text{if } 1 \leq q \leq 2, \\ C|u-v|(u^2 + \mathrm{e}^{C|u|^q} - 1 + v^2 + \mathrm{e}^{C|v|^q} - 1) & \text{if } 2 < q < \infty. \end{cases} \tag{50}$$

According to (50) and the Sobolev embeddings

$$H^1 \hookrightarrow \mathscr{B}^1_{2,q'} \quad \text{if } q \leq 2, \qquad H^2 \hookrightarrow \mathscr{B}^1_{2,q'} \hookrightarrow H^1 \quad \text{if } q > 2,$$

we will distinguish two cases.

Case $1 \leq q < 2$. We solve $\Box v + v + g_q(v) = 0$ with Cauchy data $(u_0, u_1)_{<N} \in H^1 \times L^2$ to obtain a global solution $v \in \mathscr{C}(\mathbb{R}, H^1)$. Next we have to solve

$$\Box w + w + g_q(v+w) - g_q(v) = 0, \quad (w, \partial_t w)(t=0) = (u_0, u_1)_{>N} . \tag{51}$$

We seek w in the form

$$w = u_\ell + \boldsymbol{w},$$

where u_ℓ is the free solution with Cauchy data $(u_0, u_1)_{>N}$. Hence $\boldsymbol{w}$ solves

$$\Box \boldsymbol{w} + \boldsymbol{w} + g_q(v + u_\ell + \boldsymbol{w}) - g_q(v) = 0, \quad (\boldsymbol{w}, \partial_t \boldsymbol{w})(t=0) = (0,0) . \tag{52}$$

We rely on estimates for the linear part u_ℓ given by Proposition 5.4 in order to choose appropriate functional spaces for which a fixed point argument can be performed. We introduce, for any nonnegative time T and some $0 \leq \varepsilon < \frac{1}{4}$, the complete metric space

$$\mathscr{E}_T = \mathscr{C}\big([0,T], H^1(\mathbb{R}^2)\big) \cap \mathscr{C}^1\big([0,T], L^2(\mathbb{R}^2)\big) \cap L^4_T\big(\mathscr{C}^{\frac{1}{4}-\varepsilon}(\mathbb{R}^2)\big),$$

endowed with the norm

$$\|u\|_{\mathscr{E}_T} := \sup_{0\leq t\leq T} \big[\|u(t,\cdot)\|_{H^1} + \|\partial_t u(t,\cdot)\|_{L^2}\big] + \|u\|_{L^4_T(\mathscr{C}^{\frac{1}{4}-\varepsilon})}.$$

For a positive real number δ, we denote by $\mathscr{E}_T(\delta)$ the ball in $\mathscr{E}_T$ of radius δ and centered at the origin. On the ball $\mathscr{E}_T(\delta)$, we define the map Φ by

$$\boldsymbol{w} \mapsto \Phi(\boldsymbol{w}) := \tilde{\boldsymbol{w}}, \tag{53}$$

where

$$\Box \tilde{\boldsymbol{w}} + \tilde{\boldsymbol{w}} = g_q(v) - g_q(v + u_\ell + \boldsymbol{w}), \quad (\tilde{\boldsymbol{w}}, \partial_t \tilde{\boldsymbol{w}})(t=0) = (0,0). \tag{54}$$

To show that, for small T and δ, Φ maps $\mathscr{E}_T(\delta)$ into itself and it is a contraction, we use Proposition 5.4 together with Lemma 3.5 and (50). We skip the details here and refer to [Ibrahim et al. 2006] for similar arguments.

Case $2 < q < \infty$. The method is almost the same as above, except for the choice of the functional spaces. First we solve $\Box v + v + g_q(v) = 0$ with Cauchy data $(u_0, u_1)_{<N} \in H^2 \times H^1$ to obtain a local solution $v \in \mathscr{C}((-T, T), H^2)$. Remember that in this case, the nonlinearity is too strong to solve the Cauchy problem in $H^1 \times L^2$ (see Theorem 2.1). Next we have to solve

$$\Box w + w + g_q(v + w) - g_q(v) = 0, \quad (w, \partial_t w)(t = 0) = (u_0, u_1)_{>N}. \tag{55}$$

We seek w in the form

$$w = u_\ell + \boldsymbol{w},$$

where u_ℓ is the free solution with Cauchy data $(u_0, u_1)_{>N}$. Hence $\boldsymbol{w}$ solves

$$\Box \boldsymbol{w} + \boldsymbol{w} + g_q(v + u_\ell + \boldsymbol{w}) - g_q(v) = 0, \quad (\boldsymbol{w}, \partial_t \boldsymbol{w})(t = 0) = (0, 0). \tag{56}$$

We introduce, for any nonnegative time T, the complete metric space

$$\mathscr{E}_T = \mathscr{C}\big([0, T], H^2(\mathbb{R}^2)\big) \cap \mathscr{C}^1\big([0, T], H^1(\mathbb{R}^2)\big) \cap L^4_T\big(\mathscr{C}^{1/4}(\mathbb{R}^2)\big),$$

endowed with the norm

$$\|u\|_{\mathscr{E}_T} := \sup_{0 \le t \le T} \big[\|u(t, \cdot)\|_{H^2} + \|\partial_t u(t, \cdot)\|_{H^1}\big] + \|u\|_{L^4_T(\mathscr{C}^{1/4})}.$$

We denote by $\mathscr{E}_T(\delta)$ the ball in $\mathscr{E}_T$ of radius δ and centered at the origin. On the ball $\mathscr{E}_T(\delta)$, we define the map Φ by

$$\boldsymbol{w} \mapsto \Phi(\boldsymbol{w}) := \tilde{\boldsymbol{w}}, \tag{57}$$

where

$$\Box \tilde{\boldsymbol{w}} + \tilde{\boldsymbol{w}} = g_q(v) - g_q(v + u_\ell + \boldsymbol{w}), \quad (\tilde{\boldsymbol{w}}, \partial_t \tilde{\boldsymbol{w}})(t = 0) = (0, 0). \tag{58}$$

Having in hand Proposition 5.4, Lemma 3.5, and (50), we proceed in a similar way as in the previous case (see also [Ibrahim et al. 2006]) but now we need to be more careful since the source term has to be estimated in $L^1_T(H^1)$ instead of $L^1_T(L^2)$. We refer also to [Colliander et al. 2009] for similar computation in the context of nonlinear Schrödinger equation.

(2) We turn to the second part of the theorem. Without loss of generality, we may assume that $0 \le s < 1$. Also, for the sake of simplicity, we take $q = 1$. Let $\gamma > \frac{1}{2}$ and, for $k \ge 1$, consider the function g_k defined by

$$g_k(x) = \begin{cases} \sqrt{k} & \text{if } |x| \le \mathrm{e}^{-k/2}, \\ -\dfrac{\sqrt{k}}{\log 2} \log|x| + \sqrt{k} - \dfrac{k^{3/2}}{2\log 2} & \text{if } \mathrm{e}^{-k/2} \le |x| \le 2\mathrm{e}^{-k/2}, \\ 0 & \text{if} |x| \ge 2\mathrm{e}^{-k/2}. \end{cases}$$

We remark that

$$\|k^\gamma g_k\|_{H^s} \lesssim k^{\gamma - s + 3/2}\, \mathrm{e}^{-(1-s)k/2} \to 0 \quad \text{as } k \to \infty.$$

Denote by Φ_k the solution of the associated ODE with Cauchy data $(k^{\gamma+\frac{1}{2}}, 0)$. The period T_k of Φ_k satisfies

$$T_k \lesssim k^{\gamma+1/2}\, \mathrm{e}^{-1/2k^{\gamma+1/2}} \ll \mathrm{e}^{-k/2}.$$

Choose $t_k = \dfrac{T_k}{4}$ so that $\Phi_k(t_k) = 0$. By finite speed of propagation, any weak solution u_k of (7) satisfies

$$-\partial_t u_k(t_k, x) \;=\; \dot{\Phi}_k(t_k) = \frac{\mathrm{e}^{-2\pi}}{2\sqrt{\pi}} \sqrt{\mathrm{e}^{\sqrt{k^{2\gamma+1}+1}} - \mathrm{e}^{\sqrt{\Phi_k^2(t_k)+1}}} \gtrsim \mathrm{e}^{\frac{1}{2}k^{\gamma+\frac{1}{2}}} \qquad \text{for } |x| < \mathrm{e}^{-k/2} - t_k.$$

So arguing exactly as before, we get

$$\|\partial_t u_k(t_k)\|_{H^{s-1}} \gtrsim (\mathrm{e}^{-k/2})^2\, \mathrm{e}^{sk/2}\, \mathrm{e}^{\frac{1}{2}k^{\gamma+\frac{1}{2}}} = \mathrm{e}^{(s/2-1)k+\frac{1}{2}k^{\gamma+\frac{1}{2}}}.$$

This concludes the proof once $\gamma > \frac{1}{2}$. □

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Received 6 Dec 2009. Revised 31 May 2010. Accepted 29 Jun 2010.

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ANALYSIS & PDE

Volume 4 No. 2 2011
