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SÖNKE HANSEN

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# RAYLEIGH-TYPE SURFACE QUASIMODES IN GENERAL LINEAR ELASTICITY

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Rayleigh-type surface waves correspond to the characteristic variety, in the elliptic boundary region, of the displacement-to-traction map. In this paper, surface quasimodes are constructed for the reduced elastic wave equation, anisotropic in general, with traction-free boundary. Assuming a global variant of a condition of Barnett and Lothe, the construction is reduced to an eigenvalue problem for a selfadjoint scalar first order pseudodifferential operator on the boundary. The principal and the subprincipal symbol of this operator are computed. The formula for the subprincipal symbol seems to be new even in the isotropic case.

## 1. Introduction

Rayleigh [1887] discovered the existence of surface waves which propagate along a traction-free flat boundary of an isotropic elastic body and which decay exponentially into the interior. The propagation speed of the surface wave is strictly less than that of body waves. Barnett and Lothe [1976] showed that Rayleigh-type surface waves can also exist at flat boundaries of anisotropic elastic media.

The goal of this paper is to construct, for elastic media which are not necessarily isotropic, Rayleigh-type surface quasimodes which are asymptotic to eigenvalues or resonances. We use a geometric version of semiclassical microlocal analysis.

The Rayleigh wave phenomenon of isotropic elastodynamics was explained by Taylor [1979] as propagation of singularities, over the elliptic boundary region, for the Neumann (displacement-to-traction) operator. Nakamura [1991] generalized this to anisotropic media, using the theory of Barnett and Lothe. Assuming isotropy of the elastic medium, Cardoso and Popov [1992] and Stefanov [2000] constructed Rayleigh quasimodes.

Let  $(M, g)$  be an oriented Riemannian manifold with nonempty compact smooth boundary  $X$ . The (infinitesimal) displacement of an elastic medium occupying  $M$  is a vector field  $u$  on  $M$ . The Lie derivative of the metric tensor is a symmetric tensor field,  $\text{Def } u = \mathcal{L}_u g / 2$ , called the deformation (strain) tensor caused by the displacement  $u$ . The elastic properties are defined by the elasticity (stiffness) tensor. This is a real fourth order tensor field  $C \in C^\infty(M; \text{End}(T^{0,2}M))$ ,  $e \mapsto Ce$ , which maps into symmetric tensors and vanishes on antisymmetric tensors. We assume positive definiteness of  $C$ , i.e.,  $(e | f)_C = (Ce | f)$  defines an inner product on the space of symmetric tensors  $e$  and  $f$ . Here  $(\cdot | \cdot)$  denotes the inner product on tensors induced from  $g$ . This assumption is often called the strong convexity condition. If coordinates are given, then the components of  $C$  satisfy symmetries,  $C^{ijkl} = C^{jikl} = C^{klij}$ , and  $C^{ijkl} e_{ij} e_{kl} > 0$  if  $e_{ij}$  is a nonzero symmetric tensor. (We use the summation convention.) Denote the

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Riemannian volume elements on  $M$  and  $X$  by  $dV_M$  and  $dV_X$ , respectively. The elasticity operator  $L$  and the traction  $T$  are defined, on compactly supported vector fields, by

$$\int_M (\text{Def } u \mid \text{Def } v)_C dV_M = \int_M (Lu \mid v) dV_M + \int_X (Tu \mid v) dV_X. \tag{1}$$

A positive mass density  $\rho \in C^\infty(M)$  and the elasticity tensor  $C$  define the material properties of the elastic medium. If the surface  $X$  is traction-free, then vibrations of the medium are solutions of the following eigenvalue problem:  $Lu = \lambda^2 \rho u$  in  $M$ ,  $Tu = 0$  at  $X$ . See [Marsden and Hughes 1983] for linear elasticity in the language of Riemannian geometry.

The principal symbol of  $L$ , and of the  $h$ -differential operator  $h^2L$ , equals the acoustic tensor,  $c(\xi) = c(\xi, \xi) \in \text{End}(\mathbb{C}T_x M)$ ,  $\xi \in T_x^* M$ ; see (31). Here the associated acoustical tensor  $c(\xi, \eta) \in \text{End}(\mathbb{C}T_x M)$ ,  $\xi, \eta \in T_x^* M$ , is defined as follows:

$$(c(\xi, \eta)v \mid w) = (v \otimes \eta \mid w \otimes \xi)_C. \tag{2}$$

(Using  $g$ , we identify vectors with covectors.) The  $ik$ -th covariant component of  $c(\xi, \eta)$  equals  $C^{ijkl} \xi_j \eta_l$ .

The existence of Rayleigh waves depends on the characteristic variety,  $\Sigma$ , of the surface impedance tensor,  $z$ . To define  $z$ , we first recall the definition of the elliptic boundary region,  $\mathcal{E} \subset T^*X$ . Let  $\nu$  denote the unit exterior conormal field of the boundary  $X$ . Identify  $T^*X = \nu^\perp \subset T_x^* M$ . By definition,  $\xi \in \mathcal{E}$  if and only if  $c(\xi + s\nu) - \rho \text{Id}$  is positive definite for real  $s$ . From the factorization theory of selfadjoint matrix polynomials one gets  $q(\xi) \in \text{End}(\mathbb{C}T_x M)$ ,  $\xi \in \mathcal{E} \cap T_x^* X$ , such that

$$c(\xi + s\nu) - \rho \text{Id} = (s \text{Id} - q^*(\xi))c(\nu)(s \text{Id} - q(\xi)), \tag{3}$$

$s \in \mathbb{C}$ . Moreover, the spectrum of  $q(\xi)$  lies in the lower half-plane,  $\text{spec } q(\xi) \subset \mathbb{C}_-$ , and these properties determine  $q(\xi)$  uniquely. The surface impedance tensor  $z$  is defined as follows:

$$z(\xi) = ic(\nu)q(\xi) + ic(\nu, \xi), \quad \xi \in \mathcal{E}. \tag{4}$$

The significance of  $z$  results from the fact, proved in Lemma 18, that  $z$  is the principal symbol of a parametrix of the displacement-to-traction operator. In physics, the meaning of  $z$  is that it relates the amplitudes of displacements to the amplitudes of tractions (forces) needed to sustain these.

The surface impedance tensor is Hermitian, and positive definite for large  $|\xi|$  [Barnett and Lothe 1985, Theorem 6]. If  $\dim M = 3$ , then

$$z(\xi), \xi \in \mathcal{E}, \text{ has at most one nonpositive eigenvalue.} \tag{U}$$

This property expresses the uniqueness of Rayleigh-type surface waves [Barnett and Lothe 1985, Theorem 8]. In case  $\dim X \neq 3$ , we shall assume (U) as a hypothesis. The characteristic variety of  $z$ ,

$$\Sigma = \{\xi \in \mathcal{E} ; \det z(\xi) = 0\},$$

is a smooth hypersurface, transversal to the radial directions of the fibers of  $T^*X$ . Compare [Barnett and Lothe 1985, Theorem 7]. Rayleigh waves exist only if  $\Sigma$  is not empty. We shall make the stronger assumption that  $\Sigma$  intersects every radial line:

$$\Sigma \cap \mathbb{R}_+ \xi \neq \emptyset \quad \text{if } \xi \in T^*X \setminus 0. \tag{E1}$$

Compare [Barnett and Lothe 1985, Theorem 12], [Nakamura 1991, Theorem 2.2], [Kawashita and Nakamura 2000, (ERW)]. Assuming (U) and (E1), there exists a unique  $p \in C^\infty(T^*X \setminus 0)$ ,  $p > 0$ , homogeneous of degree 1, such that

$$\Sigma = p^{-1}(1). \tag{5}$$

See Proposition 6. Furthermore, the kernel of  $z$  defines a line bundle,  $\ker z \rightarrow \Sigma$ , over the compact base  $\Sigma$ . We shall require that its first Chern class vanishes:

$$\ker z \rightarrow \Sigma \text{ is a trivial line bundle.} \tag{E2}$$

In particular, the bundle is assumed to possess a unit section. Property (E2) is stable with respect to homotopies in the material properties; see Corollary 7. In the case of isotropic elasticity with positive Lamé parameters, (U), (E1), and (E2) hold. Moreover,

$$\Sigma = \{c_r|\xi| = 1\} \subset \mathcal{C} = \{c_s|\xi| > 1\}, \quad p(\xi) = c_r|\xi|.$$

Here  $c_r$  is the propagation speed of the Rayleigh surface wave which is strictly less than the speeds of the body waves,  $0 < c_r < c_s < c_p$ . See Example 8.

Next we state the central result of this paper: The traction-free surface eigenvalue problem can be intertwined with a selfadjoint eigenvalue problem on the boundary. We employ a semiclassical pseudo-differential calculus, with distributions and operators depending on a small parameter,  $0 < h \leq 1$ . We write  $A_h \equiv B_h$  if and only if the Schwartz kernel of  $A_h - B_h$  belongs to  $C^\infty$  with seminorms satisfying  $\mathcal{O}_{C^\infty}(h^\infty)$ .

**Theorem 1.** *Assume  $\dim M = 3$ , or (U). Assume (E1), (E2). Given a unit section  $v$  of  $\ker z \rightarrow \Sigma$ , there exists a selfadjoint, elliptic operator  $P \in \Psi^1(X; \Omega^{1/2})$ , independent of  $h$ , and operators,*

$$B_h : L^2(X; \mathbb{C}T_X M) \rightarrow L^2(M; \mathbb{C}TM), \quad \|B_h\| = \mathcal{O}(h^{1/2}),$$

$$J_h, \tilde{J}_h \in \Psi^{0,0}(X; \Omega^{1/2}, \mathbb{C}T_X M), \quad J_h^* J_h \text{ elliptic at } \Sigma,$$

such that

$$(h^2 L - \rho)B_h \equiv 0, \quad TB_h J_h \equiv \tilde{J}_h(P - h^{-1}),$$

and  $B_h|_X = \text{Id}$  in a neighborhood of  $\Sigma$ . The principal symbol of  $P$  equals  $p$  of (5). Furthermore, there is a formula, (50), for the subprincipal symbol  $p_{\text{sub}}$  of  $P$ . If  $v$  is changed to another unit section,  $e^{i\varphi}v$ , then the subprincipal symbol changes to  $p_{\text{sub}} + \{p, \varphi\}$ , where  $\{p, \varphi\}$  denotes the Poisson bracket.

This result is known in the isotropic case [Cardoso and Popov 1992; Stefanov 2000], except for the assertions about the subprincipal symbol.

The operator  $B_h$  is a parametrix of the Dirichlet problem near  $\Sigma$ ; see Proposition 17. Its range consists of functions which are smooth in the interior of  $M$ , supported in a preassigned neighborhood of the boundary, and which decay like  $e^{-\delta \text{dist}_X/h}$  into the interior.

Ignoring finitely many eigenvalues the spectrum of  $P$  consists of a sequence of positive eigenvalues  $\mu_j \uparrow \infty$ . Applying Theorem 1 to an associated orthonormal system of eigenvectors we obtain, in Proposition 21, a sequence of quasimode states:  $Lu_j - \mu_j^2 \rho u_j = \mathcal{O}_{C^\infty}(h_j^\infty)$  with boundary tractions equal to zero. Moreover, the quasimode states are well-separated. The construction also works when starting with a sequence of almost orthogonal quasimode states of  $P$ .

Let  $D = \{u \in C_c^\infty; Tu = 0\}$ . The unbounded operator  $D \rightarrow L^2(M; \mathbb{C}TM; \rho dV_M)$ ,  $u \mapsto \rho^{-1}Lu$ , is symmetric and nonnegative. The associated quadratic form is given by the left-hand side of (1). Denote by  $L_T$  the Friedrichs extension of this operator. For a selfadjoint operator  $A$  with spectrum consisting of a sequence of eigenvalues accumulating at  $+\infty$ , denote by  $N_A(\lambda)$  the usual counting function for the eigenvalues of  $A$ . The following lower bound on  $N_{L_T}(\lambda)$  is an example application of our results.

**Corollary 2.** *Assume  $M$  compact,  $\dim M = 3$ , and (E1), (E2). Let  $P$  be the selfadjoint operator given in Theorem 1. For every  $m > 1$ ,  $N_{L_T}(\lambda) - N_P(\lambda - \lambda^{-m})$  is bounded from below.*

Rayleigh waves have been studied in several papers with the emphasis of getting information about resonances in scattering theory [Stefanov and Vodev 1994; 1995; 1996; Sjöstrand and Vodev 1997; Vodev 1997; Stefanov 2000], and, for anisotropic media, [Kawashita and Nakamura 2000]. Stefanov [2000] uses Rayleigh quasimodes to derive lower bounds on the number of resonances. See the remark at the end of Section 9 about going from quasimodes to resonances.

The subprincipal symbol  $p_{\text{sub}}$  affects the eigenvalue asymptotics of  $P$  [Duistermaat and Guillemin 1975], and it enters quasimode constructions [Cardoso and Popov 1992]. The subprincipal symbol occurs in the final formulas via integrals, such as  $\int_{S^*X} p_{\text{sub}}$  and  $\int_\gamma p_{\text{sub}}$ , where  $\gamma$  is a closed bicharacteristic. We point out that these integrals do not depend on the choice of the unit section  $v$  in Theorem 1, although  $p_{\text{sub}}$  itself does. An important aim of the present work is to give explicit formulas for the subprincipal symbol of  $P$ . These seem to be new even in the isotropic case which is dealt with in more detail in Proposition 25. The main difficulty comes from the fact that an invariant notion of subprincipal symbol has only been available for scalar operators. To overcome this obstacle we adapt and systematically use the geometric pseudodifferential calculus of Sharafutdinov [2004; 2005] which assumes given a differential geometric structure. The principal and subprincipal symbol levels are contained in the leading symbol of a (pseudo-)differential operator.

The paper is organized as follows. In Section 2 the surface impedance tensor is studied; in particular, a selfcontained treatment of Barnett–Lothe theory is given. The leading geometric symbols of some differential operators are computed in Section 3. In Section 4 we geometrically decompose the elasticity operator near the boundary into normal and tangential operators, keeping track of leading geometric symbols. Section 5 gives, microlocally at the elliptic region  $\mathcal{E}$ , a factorization of  $h^2L - \rho$  into a product of first order operators. Using the factorization, we construct in Section 6 a parametrix for the Dirichlet problem microlocally at  $\mathcal{E}$ . The displacement-to-traction operator  $Z$  is defined in Section 7, and its leading geometric symbol is determined. In Section 8 we derive a diagonalization of  $Z$ , and we prove Theorem 1. In Section 9 we construct localized traction-free surface quasimodes, and we prove Corollary 2. In Section 10 we calculate, for an isotropic elastic medium, the subprincipal symbol of  $P$ . The Appendix contains a detailed exposition of Sharafutdinov’s geometric pseudodifferential calculus in a semiclassical setting.

## 2. The surface impedance tensor

First we collect some well-known facts about spectral factorizations of selfadjoint matrix polynomials. Refer to [Gohberg et al. 1982, Chapter 11]. Let  $V$  be a finite-dimensional complex Hilbert space, and  $f(s) = as^2 + bs + c \in \text{End}(V)$  a quadratic polynomial in the complex variable  $s$ . The spectrum of  $f$  is the set of  $s \in \mathbb{C}$  such that  $\ker f(s) \neq 0$ . Assume that the leading coefficient of  $f$ ,  $a$ , is nonsingular. Then

the spectrum is finite. Assume that  $f$  is selfadjoint,  $f(s)^* = f(\bar{s})$ , and that, in addition  $f(s)$  is positive definite for real  $s$ . The spectrum of  $f$  is a disjoint union  $\sigma_+ \cup \sigma_-$ , where  $\sigma_+$  and  $\sigma_-$  are contained in the upper and lower half-planes, respectively. There is a unique  $q \in \text{End}(V)$  such that  $f(s) = (s - q^*)a(s - q)$ , and the spectrum of  $q$  equals  $\sigma_-$ . If  $\gamma$  is a closed Jordan curve which contains  $\sigma_-$  in its interior and  $\sigma_+$  in its exterior, then

$$q \oint_{\gamma} f(s)^{-1} ds = \oint_{\gamma} s f(s)^{-1} ds. \tag{6}$$

The integral on the left is nonsingular. Jordan–Keldysh chains are a means to compute  $q$ . In particular, one has  $qv = sv$  if  $f(s)v = 0$  and  $\text{Im } s < 0$ . Moreover, the solvency equation  $f(q) = 0$  holds.

The following representation of the factor  $q$  by integrals is important. We shall also apply it later to establish symbol properties. Denote by  $i = \sqrt{-1}$  the imaginary unit.

**Lemma 3.** *Let  $f$  and  $q$  be as above. Then*

$$a q f_0 = -\pi i \text{Id} + f_1, \tag{7}$$

where  $f_0 = \int_{-\infty}^{\infty} f(s)^{-1} ds$  is selfadjoint and positive definite, and

$$f_1 = \int_{|s| \leq 1} s a f(s)^{-1} ds + \int_{|s| > 1} s^{-1} (s^2 a - f(s)) f(s)^{-1} ds.$$

The integrals converge absolutely in  $\text{End}(V)$ .

*Proof.* Let  $\gamma_R$  be the negatively oriented closed contour composed of the semicircle  $\{|s| = R, \text{Im } s \leq 0\}$  and the interval  $[-R, R]$ . The integral representation (6) holds with  $\gamma = \gamma_R$  if  $R$  is sufficiently large. We have  $f(s)^{-1} = s^{-2} a^{-1} + \mathcal{O}(|s|^{-3})$  as  $|s| \rightarrow \infty$ . It follows that

$$\lim_{R \rightarrow \infty} \oint_{\gamma_R} f(s)^{-1} ds = \int_{-\infty}^{\infty} f(s)^{-1} ds$$

and

$$\lim_{R \rightarrow \infty} \oint_{\gamma_R} s a f(s)^{-1} ds = -\pi i \text{Id} + \lim_{R \rightarrow \infty} \int_{-R}^R s a f(s)^{-1} ds.$$

Using  $s^2 a f(s)^{-1} - \text{Id} = (s^2 a - f(s)) f(s)^{-1}$  we obtain

$$\int_{1 < |s| \leq R} s a f(s)^{-1} ds = \int_{1 < |s| \leq R} s^{-1} (s^2 a - f(s)) f(s)^{-1} ds.$$

This proves the formulas. The remaining assertions follow from these and the positive definiteness of  $f(s)$ . □

Let  $\xi \in T_x^* X$ , and denote by  $\nu \in T_x^* M$  the unit exterior normal. Set  $a = c(\nu)$ ,  $a_1(\xi) = c(\nu, \xi)$ , and  $a_2(\xi) = c(\xi)$ . Note that  $a_1(\xi)^* = c(\xi, \nu)$ . The polynomial

$$f(s) = c(\xi + s\nu) - \rho = a s^2 + (a_1 + a_1^*) s + a_2 - \rho, \tag{8}$$

$f(s) = f(s, \xi)$ , has values in  $\text{End}(CT_x M)$ . It is selfadjoint with real coefficients. By definition,  $\xi \in \mathcal{E}$  if and only if  $f(s)$  is positive definite for  $s \in \mathbb{R}$ .

**Lemma 4.** *The elliptic region  $\mathcal{E}$  is an open subset of  $T^* X$  with compact complement. Moreover,  $\mathcal{E}$  is symmetric and star shaped with respect to infinity, i.e.,  $t\xi \in \mathcal{E}$  whenever  $\xi \in \mathcal{E}$  and  $t$  real,  $|t| \geq 1$ .*

*Proof.* By positive definiteness of  $C$ , there exists  $\delta > 0$  such that  $g(v, c(\eta)v) \geq \delta|v \otimes \eta + \eta \otimes v|^2$  for (co-)vectors  $v, \eta$ . The symmetrization of a nonzero real elementary tensor is nonzero. Therefore, with a new  $\delta > 0$ , in the sense of selfadjoint maps,

$$c(\eta) \geq \delta|\eta|^2 \text{Id}. \tag{9}$$

Since  $|\xi + sv|^2 = |\xi|^2 + s^2$  the first assertions follow. The symmetry and the star-shapedness follow from  $c(t\eta) = t^2c(\eta)$ .  $\square$

If  $\xi \in \mathcal{E}$ , then (7) holds with  $q = q(\xi)$ ,  $f_j = f_j(\xi)$ . The spectral factor  $q$  solves (3); using current notation:

$$as^2 + (a_1 + a_1^*)s + a_2 - \rho = (s - q^*)a(s - q). \tag{10}$$

The spectrum of  $q$  lies in the lower half-plane, and  $q$  is uniquely determined by these properties. Notice that  $q$  is a smooth section of the bundle  $\pi^* \text{End}(\text{CT}_X M) \rightarrow \mathcal{E}$ , where  $\pi : \mathcal{E} \subset T^*X \rightarrow X$  denotes the canonical projection.

The surface impedance tensor, defined in (4), equals  $z = i(aq + a_1)$ . Lemma 3 implies

$$zf_0 = \pi \text{Id} + i(f_1 + a_1 f_0). \tag{11}$$

Since the  $f_j$ 's are real, this gives the decomposition of  $z$  into real and imaginary parts. Following [Mielke and Fu 2004], we shall use the Riccati-type equation

$$(z + ia_1^*)a^{-1}(z - ia_1) = a_2 - \rho \tag{12}$$

to deduce properties of  $z$ . Equation (12) follows upon insertion of  $q = -a^{-1}(iz + a_1)$  into the solvency equation associated with (10),

$$aq^2 + (a_1 + a_1^*)q + a_2 - \rho = 0. \tag{13}$$

A consequence of (12) is

$$(iq)^*z' + z'(iq) = a_1^{*'}q + q^*a_1' + (a_2 - \rho)' + q^*a'q, \tag{14}$$

where the prime denotes the derivative with respect to some chosen parameter. The spectra of  $q$  and  $q^*$  are disjoint. Therefore, the Sylvester equation  $(iq)^*x + x(iq) = i(xq - q^*x) = y$  has a unique solution  $x$  for given  $y$ . The solution is, in fact, given by an integral,  $x = \int_{-\infty}^0 \exp(irq)^* y \exp(irq) dr$ . It follows that  $x$  is positive definite if  $y$  is.

**Proposition 5.** *The impedance tensor  $z(\xi)$ ,  $\xi \in \mathcal{E}$ , has the following properties.*

- (i)  $z(\xi)$  is selfadjoint.
- (ii)  $z(\xi)$  is positive definite if  $|\xi|$  is sufficiently large.
- (iii)  $\text{Re } z(\xi)$  is positive definite.
- (iv)  $z(\xi)$  has at least two positive eigenvalues if  $\dim M \geq 3$ .
- (v)  $(d/dt)|_{t=1} t^{-1} z(t\xi) = \dot{z} - z$  is positive definite.
- (vi) The complex conjugate  $\overline{z(\xi)} = z(-\xi)$ .

We call  $\dot{z}(\xi) = (d/dt)|_{t=1} z(t\xi)$  the radial derivative of  $z$  at  $\xi$ . It follows from (v) that  $\dot{z}$  is positive definite on the kernel of  $z$ ,  $\ker z$ .

*Proof.* To prove (i) we follow the arguments in [Mielke and Fu 2004, Theorem 2.2]. First note that (12) remains true if  $z$  is replaced by  $z^*$ . Subtracting the two equations we get the Sylvester equation  $(iq)^*(z - z^*) + (z - z^*)(iq) = 0$ , implying  $z - z^* = 0$ .

It follows from (11) that  $\operatorname{Re} z = \pi f_0^{-1}$ . This proves (iii).

Suppose  $\dim M \geq 3$ . Aiming at an indirect proof of (iv), assume that  $z(\xi)$ ,  $\xi \in T_x^* X$ , has at most one positive eigenvalue. Then there exists  $w \in \mathbb{C}T_x M$  such that  $z(\xi)$  is negative semidefinite on the orthogonal complement  $w^\perp$ . Choose a real vector  $v \neq 0$  which is orthogonal to both  $\operatorname{Re} w$  and  $\operatorname{Im} w$ . Then  $v \in w^\perp$ , and  $(\operatorname{Re} z(\xi)v \mid v) = (z(\xi)v \mid v) \leq 0$ , contradicting the positive definiteness of  $\operatorname{Re} z$ .

Next we prove (v) following the method of [Mielke and Fu 2004, Theorem 2.3]. Since  $a_j(\xi)$  is homogeneous of degree  $j$  in  $\xi$ , (12) implies

$$(t^{-1}z(t\xi) + ia_1^*(\xi))a^{-1}(t^{-1}z(t\xi) - ia_1(\xi)) = a_2(\xi) - t^{-2}\rho.$$

Taking the derivative with respect to  $t$  at  $t = 1$ , we get

$$(iq)^*(\dot{z} - z) + (\dot{z} - z)(iq) = 2\rho.$$

By the remarks following (14), we see that  $\dot{z} - z$  is positive definite.

We now prove (vi). Note  $f(s, -\xi) = \overline{f(-s, \xi)}$ ,  $f_j(-\xi) = (-1)^j f_j(\xi)$ , and  $a_1(-\xi) = -a_1(\xi)$ . Using (11) we derive  $z(-\xi)f_0(\xi) = \overline{z(\xi)f_0(\xi)}$ . Since  $f_0$  is real and nonsingular the formula follows.

It remains to prove (ii). Let  $\eta \in T_x X$ ,  $|\eta| = 1$ . It suffices to show that  $z_\infty = \lim_{t \uparrow \infty} t^{-1}z(t\eta)$  exists and is positive definite. Set  $q_t = t^{-1}q(t\eta)$ ,  $t > 1$  large. From (10) deduce

$$as^2 + (a_1(\eta) + a_1^*(\eta))s + a_2(\eta) - t^{-2}\rho = (s - q_t^*)a(s - q_t), \quad s \in \mathbb{R}.$$

Using (7) and dominated convergence in the integrals giving  $f_j$  we infer that  $q_\infty = \lim_{t \uparrow \infty} q_t$  exists. In particular,  $t^{-1}z(t\eta)$  converges to  $z_\infty = i(aq_\infty + a_1(\eta))$  as  $t \uparrow \infty$ . Let  $y \in T_x M$  such that  $(z_\infty y \mid y) \leq 0$ . We must show  $y = 0$ . Set  $w(r) = \exp(irq_\infty)y$ ,  $r \leq 0$ . The solvency Equation (13) holds with  $q$  replaced by  $q_\infty$ ,  $\rho = 0$ . Therefore,  $aD_r^2 w + (a_1 + a_1^*)D_r w + a_2 w = 0$  holds, where we use the abbreviation  $a_j = a_j(\eta)$ . Take the inner product in  $\mathbb{C}T_x M$  with  $w$  and integrate. An integration by parts gives

$$\int_{-\infty}^0 (aD_r w + a_1 w \mid D_r w) + (D_r w \mid a_1 w) + (a_2 w \mid w) dr = i(aD_r w + a_1 w \mid w) \Big|_{-\infty}^0 = (z_\infty y \mid y) \leq 0.$$

Set  $W(r) = w(r) \otimes \eta + D_r w(r) \otimes v \in \operatorname{End}(\mathbb{C}T_x M)$ . Recall  $a = c(v)$ ,  $a_1 = c(v, \eta)$ ,  $a_2 = c(\eta)$ , and (2). We have shown:

$$\int_{-\infty}^0 (W(r) \mid W(r))_C dr \leq 0.$$

Recall that  $C$  is real, and that  $(\cdot \mid \cdot)_C$  is an inner product on symmetric 2-tensors. It follows that the symmetrization of  $W(r)$  vanishes for all  $r \leq 0$ . In particular,

$$(w(r) \otimes \eta + D_r w(r) \otimes v \mid \zeta \otimes v + v \otimes \zeta) = 0 \tag{15}$$

for  $\zeta \in \mathbb{C}T_x^* M$ ,  $r \leq 0$ . Recall  $(\eta \mid v) = 0$ . Setting  $\zeta = v$ , we derive  $D_r(w(r) \mid v) = (D_r w(r) \mid v) = 0$ . Since  $w(r) \rightarrow 0$  as  $r \rightarrow -\infty$ , we obtain  $(w(r) \mid v) = 0$ . Now, (15) simplifies to  $(D_r w(r) \mid \zeta) = 0$ . Since  $\zeta$  is arbitrary, this implies, successively,  $D_r w = 0$ ,  $w = 0$ ,  $y = 0$ .  $\square$



If  $\dim M = 3$  then (U) holds. This follows from (iv).

**Proposition 6.** *Assume (U). Then the characteristic variety of  $z$ ,  $\Sigma = \{\det z(\xi) = 0\}$ , is a smooth hypersurface in  $\mathcal{E}$ . Each radial line  $\mathbb{R}_+\xi \subset T^*X$  intersects  $\Sigma$  in at most one point, and the intersection is transversal. The kernel of  $z|_\Sigma$  defines a line bundle  $\ker z \rightarrow \Sigma$ . Assume, in addition, (E1). There is a unique  $p \in C^\infty(T^*X \setminus 0)$ , homogeneous of degree one, such that  $\Sigma = p^{-1}(1)$ . Moreover,  $p > 0$ , and  $p(-\xi) = p(\xi)$ .*

*Proof.* From the assumption and (v) of Proposition 5 it follows that  $(d/dt) \det z(t\xi) > 0$  if  $t\xi \in \Sigma$ ,  $t > 0$ . In particular, zero is a regular value of  $\det z$ . Hence  $\Sigma$  is a codimension one submanifold transversal to the radial field. Since  $\mathbb{R}_+\xi \cap \mathcal{E}$  is connected, a given radial line  $\mathbb{R}_+\xi$  intersects  $\Sigma$  in at most one point. Because of (U) and the selfadjointness of  $z$ , zero is simple eigenvalue of  $z$ . It follows that  $\ker z \rightarrow \Sigma$  is a line bundle. Now assume also (E1). Then each radial line intersects  $\Sigma$  in a unique point. Define  $p$  as follows. For  $0 \neq \xi \in T^*X$  set  $p(\xi) = 1/t$  if  $t\xi \in \Sigma$ ,  $t > 0$ . Smoothness of  $p$  follows from the implicit function theorem. The evenness of  $p$  is a consequence of (vi). The other properties of  $p$  are obvious. Clearly, the homogeneity and  $p|_\Sigma = 1$  determine  $p$  uniquely. □

**Corollary 7.** *Let  $\rho_t$  and  $C_t$  be homotopies of the mass densities and the elasticity tensors,  $0 \leq t \leq 1$ . Assume that the associated surface impedance tensors  $z_t$  and their characteristic varieties  $\Sigma_t$  satisfy (U) and (E1) for every  $t$ . The line bundles  $\ker z_0 \rightarrow \Sigma_0$  and  $\ker z_1 \rightarrow \Sigma_1$  are isomorphic.*

*Proof.* The factorization (3) and the definition of the impedance tensor imply that  $z_t$  depends continuously on the homotopy parameter  $t$ . It follows from Proposition 6 that the characteristic varieties are canonically diffeomorphic to the sphere bundle  $SX$ . We deduce that the Chern classes of the bundles  $\ker z_t \rightarrow SX$  do not depend on  $t$ . The assertion follows from this. □

**Example 8.** We consider, as special case, an isotropic elastic medium. We shall verify (U), (E1), and (E2). The elasticity tensor reads, in component notation,

$$C^{ijkl} = \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk}), \tag{16}$$

where  $\lambda, \mu$  denote the Lamé parameters. Equivalently,

$$c(\xi, \eta) = \lambda \xi \otimes \eta + \mu \eta \otimes \xi + \mu g(\xi, \eta) \text{Id}. \tag{17}$$

Positive definiteness of  $C$  is equivalent to  $\mu > 0, \lambda \dim M + 2\mu > 0$ . We make the stronger assumption  $\lambda, \mu > 0$ . Let  $\xi \in T_x^*X$ . We list the eigenvalues  $s \in \mathbb{C}$  and the eigenvectors  $v \in \mathbb{C}T_xM$  of the quadratic polynomial  $c(\xi + sv) - \rho$ :

- (a)  $(\lambda + 2\mu)(|\xi|^2 + s^2) - \rho = 0$  and  $v = \xi + sv$ ,
- (b)  $\mu(|\xi|^2 + s^2) - \rho = 0$  and  $v = s\xi - |\xi|^2 v$ ,
- (c)  $\mu(|\xi|^2 + s^2) - \rho = 0$  and  $v$  is orthogonal to  $\xi$  and  $v$ .

Introduce  $c_p = \sqrt{(\lambda + 2\mu)/\rho}$  and  $c_s = \sqrt{\mu/\rho}$ , the speeds of pressure and of shear waves, respectively. Assume that  $\xi \in \mathcal{E}$ . This is equivalent to  $c_s|\xi| > 1$ . The above eigenvalues and eigenvectors diagonalize  $q$ ,  $q(\xi)v = sv$  if  $\text{Im } s < 0$ . Denote by  $V$  the subbundle of  $\pi^*(\mathbb{C}T_XM) \rightarrow \mathcal{E}$  spanned by  $v$  and  $\xi$ , and  $V^\perp$  its orthogonal bundle. Fix the orthonormal frame  $v, \hat{\xi} = \xi/|\xi|$  of  $V$ , and choose an orthonormal frame of  $V^\perp$ . In block decompositions of matrices we let the indices 1 and 2 correspond to  $V$  and

$V^\perp$ , respectively. We denote by  $(e)_{ij}$  the block  $ij$  of the matrix which represents the endomorphism  $e$ . Observe that  $q$  leaves  $V$  and  $V^\perp$  invariant,  $(q)_{12} = 0 = (q)_{21}$ . A simple computation gives

$$(iq)_{11} = \frac{|\xi|}{b} \begin{bmatrix} ut\sqrt{1-t} & -i(b-t) \\ i(b-t) & t\sqrt{1-ut} \end{bmatrix}. \tag{18}$$

Here  $t = (c_s|\xi|)^{-2}$ ,  $u = (c_s/c_p)^2 = \mu/(\lambda + 2\mu)$ ,  $b = 1 - \sqrt{1-ut}\sqrt{1-t}$ . Moreover,  $(iq)_{22}$  equals  $|\xi|\sqrt{1-t}$  times the unit matrix. The maps  $a = c(v)$  and  $a_1 = c(v, \xi)$  also leave  $V$  and  $V^\perp$  invariant. We compute

$$(z)_{11} = \frac{\mu|\xi|}{b} \begin{bmatrix} t\sqrt{1-t} & -i(2b-t) \\ i(2b-t) & t\sqrt{1-ut} \end{bmatrix}, \tag{19}$$

and  $(z)_{22} = \mu(iq)_{22}$ . The determinant of  $z$  equals  $(\mu|\xi|\sqrt{1-t})^{\dim V^\perp}$  times

$$\det(z)_{11} = \mu^2|\xi|^2b^{-1}(4\sqrt{(1-t)(1-ut)} - (2-t)^2). \tag{20}$$

Given  $u \in ]0, 1/2[$ , the unique zero  $t \in ]0, 1[$  is found as the solution of Rayleigh’s cubic equation [1887, (24)], namely  $0 = ((t-2)^4 - 16(1-t)(1-ut))/t$ . Define the Rayleigh wave speed  $c_r = c_s\sqrt{t} \in C^\infty(X)$ . Set  $p(\xi) = c_r|\xi|$ . The characteristic variety  $\Sigma$  equals  $\{p(\xi) = 1\}$ . Thus (U) and (E1) hold. Obviously,  $i(2b-t)v + t\sqrt{1-t}\hat{\xi} \in \ker z(\xi)$ ,  $\xi \in \Sigma$ . Observe that

$$2(2b-t) = t(2-t) \quad \text{on } \Sigma. \tag{21}$$

Thus

$$i(2-t)v + 2\sqrt{1-t}\hat{\xi} \in \ker z, \quad t = (c_r/c_s)^2, \tag{22}$$

is a nowhere vanishing section of the kernel bundle. Hence also (E2) holds. This example is of course well-known.

**Remark.** The identity (11) goes back to Barnett and Lothe; compare [Lothe and Barnett 1976, (3.18)]. It is key to proving, in dimension three, the uniqueness of subsonic traction-free surface waves [Barnett and Lothe 1985, Theorem 8]. The second assumption in Proposition 6 is needed to prove the existence of Rayleigh surface waves. Compare with Theorem 12 of the same reference, where existence criteria are given in terms of the so-called limiting velocity which corresponds to the boundary of the elliptic region. See [Nakamura 1991, Theorem 2.2] for the Barnett–Lothe condition in a microlocal setting, and the real principal type property of the Lopatinski matrix it entails. See [Tanuma 2007] for a recent exposition of Barnett–Lothe theory, and for a treatment of isotropic and transversely isotropic media.

### 3. Connections and geometric symbols

The elasticity operator is defined in terms of the Levi-Civita connection and of the elasticity tensor. We use the geometric pseudodifferential calculus of the Appendix to define and compute the leading symbol of the elasticity operator. The leading symbol includes the principal and the subprincipal level. The calculus depends on the choice of connections.

Equip  $M$  with the Levi-Civita connection of  $g$ . Let  $\exp$  denote its exponential map. If  $x, y \in M$ , then denote by  $[y \leftarrow x]$  the shortest geodesic segment from  $x$  to  $y$ , assuming its interior does not intersect the boundary, and that it is unique.

Let  $E \rightarrow M$  be a (complex) vector bundle with connection  $\nabla^E$ . Denote by  $\tau_\gamma^E \in \text{End}(E_x, E_y)$  the parallel transport map along a given curve  $\gamma$  in  $M$  from  $x$  to  $y$ , e.g.,  $\tau_{[y \leftarrow x]}^E$ . The connection can be recovered from its parallel transport maps:

$$\nabla_v^E s(x) = \frac{d}{dt} \Big|_{t=0} \tau_{[x \leftarrow \exp_x tv]}^E s(\exp_x tv). \tag{23}$$

Denote by  $\pi^* E \rightarrow T^* M$  the pullback of  $E$  to the cotangent bundle  $\pi : T^* M \rightarrow M$ . Let  $a$  be a smooth section of  $\pi^* E \rightarrow T^* M$ . Following [Sharafutdinov 2004; 2005], we introduce the vertical and the horizontal covariant derivative of  $a$ . The vertical derivative

$${}^v \nabla a(x, \xi) \in E_x \otimes T_x M,$$

at  $\xi \in T_x^* M$ , is the derivative of the map  $T_x^* M \rightarrow E_x$ ,  $\xi \mapsto a(x, \xi)$ . The definition of the vertical derivative depends only on the linear structure of the fibers of  $T^* M$ . The horizontal derivative  ${}^h \nabla a(x, \xi) \in E_x \otimes T_x^* M$  is the derivative at  $v = 0$  of a map  $T_x M \rightarrow E_x$ ,

$${}^h \nabla a(x, \xi) = \frac{\partial}{\partial v} \Big|_{v=0} \tau_{[x \leftarrow \exp_x v]}^E a(\exp_x v, \tau_{[\exp_x v \leftarrow x]}^{T^* M} \xi). \tag{24}$$

The horizontal derivative depends on the Riemannian structure and on the connection  $\nabla^E$ . In the scalar case,  $E = \mathbb{C}$ , in local coordinates,

$${}^h \nabla a(x, \xi) = (\partial_{x_j} a(x, \xi) + \Gamma_{ij}^k(x) \xi_k \partial_{\xi_i} a(x, \xi)) dx^j,$$

where  $\Gamma_{ij}^k$  denote the Christoffel symbols of the Levi-Civita connection. Writing a local section of  $\pi^* E$  as a sum of products  $a_1(x, \xi) a_2(x)$  where  $a_1$  is scalar and  $a_2$  a section of  $E$  one readily derives local formulas for the horizontal derivative in terms of connection coefficients. The vertical and the horizontal derivative extend to first order differential operators,  ${}^v \nabla$  and  ${}^h \nabla$ , which map sections of  $\pi^*(E \otimes T^{r,s} M)$  to sections of  $\pi^*(E \otimes T^{r+1,s} M)$  and of  $\pi^*(E \otimes T^{r,s+1} M)$ , respectively. The operators  ${}^v \nabla$  and  ${}^h \nabla$  commute. It suffices to prove this when  $E$  is the trivial line bundle,  $E = \mathbb{C}$ . In this case the assertion is easily checked in normal coordinates.

Let  $F \rightarrow M$  be another vector bundle. Let  $A : C^\infty(M; E) \rightarrow C^\infty(M; F)$  be a differential operator of order  $m$ . We introduce a small parameter,  $0 < h \leq 1$ , and we replace  $A$  by the  $h$ -differential operator  $h^m A$ . Then  $A \in \Psi^{m,0}(M; E, F)$  as a semiclassical (pseudo-)differential operator. Refer to the [Appendix](#) for an exposition of Sharafutdinov’s geometric pseudodifferential calculus in a semiclassical setting. The formula (67) for the geometric symbol,  $\sigma_h(A) \in S^{m,0}$ , simplifies to

$$\sigma_h(A)(x, \xi) s = A_y(e^{i\langle \xi, \exp_x^{-1} y \rangle} / h \tau_{[y \leftarrow x]}^E s) \Big|_{y=x}, \tag{25}$$

where  $\xi \in T_x^* M$ ,  $s \in E_x$ , and  $i = \sqrt{-1}$ . The geometric symbol extends by continuity to the boundary of  $M$ . In symbol computations we track the leading geometric symbol, defined before [Proposition 27](#). In the following, the symbol of an operator is always its geometric symbol.

For the Laplace–Beltrami operator one has  $\sigma_h(-h^2 \Delta)(x, \xi) = |\xi|^2$ . This is readily checked using normal coordinates.

From (25) and (23) deduce

$$\sigma_h(-ih \nabla^E)(\xi) e = e \otimes \xi \in E_x \otimes T_x^* M. \tag{26}$$

As before, to ease notation, we usually do not write the base point  $x$  into the arguments of tensors and symbols.

If  $E \rightarrow M$  is a Hermitian vector bundle then we define, using the volume element  $dV_M$ , the Hilbert space  $L^2(E)$ . Assume  $E$  and  $F$  are Hermitian vector bundles having metric connections. The leading symbol of the formal adjoint  $A^*$  of  $A$  is given by

$$\sigma_h(A^*) \equiv \sigma_h(A)^* - ih \operatorname{tr} \left( {}^v\nabla^h \nabla \sigma_h(A)^* \right). \tag{27}$$

See [Proposition 27](#).

Equip the bundle  $E \otimes T^*M$  with the induced Hermitian structure and the induced connection. The connection is metric. Observe that the horizontal derivative of  $\sigma_h(-ih\nabla^E)^*$  vanishes. Therefore, (27) and (26) imply

$$\sigma_h((-ih\nabla^E)^*)(\xi)(e \otimes \eta) = g(\xi, \eta)e, \quad \xi, \eta \in T_x^*M, e \in E_x. \tag{28}$$

By [Proposition 28](#) the leading symbol of a composition is given as follows:

$$\sigma_h(AB) \equiv \sigma_h(A)\sigma_h(B) - ih \operatorname{tr} \left( {}^v\nabla \sigma_h(A) \cdot {}^h\nabla \sigma_h(B) \right). \tag{29}$$

The trace is the contraction of the  $TM \otimes T^*M$  factor which is produced by a pair of vertical and horizontal derivatives. *Note:* In (29) and below, the dot terminates a differentiated expression, serving as a closing bracket.

Let  $C \in C^\infty(M; \operatorname{End}(E \otimes T^*M))$ . View  $C$  as an operator acting by multiplication on sections of the bundle  $E \otimes T^*M \rightarrow M$ . Let  $\nabla$  denote the connection on the bundle  $\operatorname{End}(E \otimes T^*M) \rightarrow M$  induced from the Levi-Civita connection and from  $\nabla^E$ . Define sections  $c, \operatorname{div} c$  of  $\pi^* \operatorname{End}(E) \rightarrow T^*M$  as follows:

$$c(\xi)e = \langle \xi, \pi^*C(e \otimes \xi) \rangle, \quad (\operatorname{div} c)(\xi)e = \sum_j \langle \eta^j, (\pi^*\nabla_{v_j} C)(e \otimes \xi) \rangle,$$

where the angular brackets denote contractions on covectors, using  $g$ . Furthermore,  $(v_j)$  and  $(\eta^j)$  are any dual frames of  $TM$  and  $T^*M$ .

**Lemma 9.**  $\sigma_h(-h^2\nabla^{E^*} \circ C \circ \nabla^E) = c - ih \operatorname{div} c + \mathcal{O}(h^2)$ .

*Proof.* Observe that  $\sigma_h(C) = \pi^*C$ , and  ${}^h\nabla\pi^*C = \pi^*\nabla C$ . The symbol (28) is linear in  $\xi$ . Its vertical derivative is obvious. Using (29), the symbol  $a$  of  $-ih\nabla^{E^*} \circ C$  is found to be

$$a(\xi)(e \otimes \eta) = \langle \xi, \pi^*C(e \otimes \eta) \rangle - ih \sum_j \langle \eta^j, \pi^*\nabla_{v_j} C(e \otimes \eta) \rangle.$$

Here  $(v_j)$  and  $(\eta^j)$  are as in the definition of  $\operatorname{div} c$ . The horizontal derivative of the symbol of  $-ih\nabla^{E^*}$  vanishes. Therefore,

$$\sigma_h(-h^2\nabla^{E^*} \circ C \circ \nabla^E)(\xi)e = a(\xi)\sigma_h(-ih\nabla^E)(\xi)e = a(\xi)(e \otimes \xi),$$

where we used (29). □

Now assume  $E = \mathbb{C}TM$  and  $C$  the elasticity tensor. Identify

$$\operatorname{End}(\mathbb{C}T^{0,2}M) = \operatorname{End}(\mathbb{C}TM \otimes \mathbb{C}T^*M).$$

Let  $L$  the elasticity operator defined in (1). Recall from Riemannian geometry the following relation between the Levi-Civita connection and the Lie derivative:

$$(\mathcal{L}_u g)(v, w) = g(\nabla_v u, w) + g(v, \nabla_w u), \tag{30}$$

for (real) vector fields  $u, v, w$ . Using the symmetries of the elasticity tensor we get

$$L = \text{Def}^* \circ C \circ \text{Def} = (-i \nabla)^* \circ C \circ (-i \nabla).$$

We obtain the following corollary to [Lemma 9](#):

$$\sigma_h(h^2 L - \rho) = c - \rho \text{Id} - i h \text{div } c + \mathcal{O}(h^2). \tag{31}$$

If the  $C^{ijkl}$  represent  $C$  with respect to some given local coordinates, then (31) reads

$$\sigma_h(h^2 L - \rho)(\xi)^{ik} = C^{ijkl} \xi_j \xi_\ell - \rho \delta^{ik} - \sqrt{-1} h C^{ijkl} |_{|j} \xi_\ell + \mathcal{O}(h^2).$$

The vertical bar followed by  $j$  means covariant differentiation with respect to the  $j$ -th coordinate. If the elastic medium is isotropic, the leading symbol becomes

$$\begin{aligned} &\sigma_h(h^2 L - \rho)(\xi) \\ &\equiv \rho(c_p^2 |\xi|^2 - 1) P(\xi) + \rho(c_s^2 |\xi|^2 - 1) (\text{Id} - P(\xi)) - i h (\nabla \lambda \otimes \xi + (\nabla \mu \otimes \xi)^* + \langle \xi, \nabla \mu \rangle \text{Id}), \end{aligned} \tag{32}$$

where  $P(\xi) = \hat{\xi} \otimes \hat{\xi}$  denotes the orthogonal projection to the propagation direction  $\hat{\xi} = \xi/|\xi|$ .

#### 4. The elasticity operator in a boundary collar

In a boundary collar,  $]-\varepsilon, 0] \times X \subset M$ , we write the elasticity operator  $L$  in terms differential operators on  $X$  having coefficients which depend on  $r \in I$ , the negative distance to  $X$ .

Let  $N(x) \in T_x M$  denote the unit exterior normal at  $x \in X$ . There exists  $\varepsilon > 0$  such that, if we set  $I = ]-\varepsilon, 0]$ , the exponential map of the Levi-Civita connection defines a diffeomorphism onto a neighborhood of  $X$  in  $M$ :

$$I \times X \rightarrow M, \quad (r, x) \mapsto y = \exp(rN(x)).$$

Essentially without losing generality, we assume that this map is onto  $M$ . The inverse map is  $y \mapsto (r, x)$ , where  $-r = d(y, X)$  is the distance from  $y$  to  $X$ , and  $x = p(y)$  is the unique point in  $X$  closest to  $y$ . The distance function  $r$  satisfies the (eikonal) equation  $|\nabla r| = 1$  in  $M$ . Extend  $N$  to  $M$  by  $N = \nabla r$ . Also introduce the unit conormal field  $\nu = dr$ . The level hypersurfaces

$$M_r = \{y \in M ; r + d(y, X) = 0\}$$

are diffeomorphic to  $X = M_0$ . The shape operator  $S = \nabla N$  is a field of symmetric endomorphisms of  $TM$ ,  $g(Su, v) = g(u, Sv)$ . The second fundamental forms of the level hypersurfaces  $M_r$  assign  $(u, v) \mapsto -g(Su, v)$  (Weingarten equation). The dependence of the metric tensor on  $r$  is given by the formula  $(\mathcal{L}_N g)(v, w) = 2g(Sv, w)$ . This formula follows from (30). Introduce  $J \in C^\infty(I_r \times X)$ , the solution of  $\partial_r \log J = \text{tr } S$ ,  $J|_{r=0} = 1$ . Then we have the following formula for the volume form of  $M$ :

$$\int_M f(y) dV_M(y) = \int_I \int_X f(\exp(rN(x))) J(r, x) dV_X(x) dr, \tag{33}$$

$f \in C_c^\infty(M)$ . See [\[Petersen 1998, Chapter 2\]](#) for the geometry of hypersurfaces using distance functions.

Let  $E \rightarrow M$  be a vector bundle with connection  $\nabla^E$ . Denote by  $E_r \rightarrow M_r$  the bundles induced by the inclusions  $M_r \subset M, r \in I$ . Set  $E_X = E_0$ . Let  $u \in C^\infty(M; E)$  be a section of  $E$ . Using parallel transport in  $E$  along the geodesics which intersect the boundary orthogonally, define  $\tilde{u} : I \rightarrow C^\infty(X; E_X)$  by

$$\tilde{u}(r)(x) = \tilde{u}(r, x) = \tau_{[x \leftarrow y]}^E u(y) \quad \text{if } y = \exp(rN(x)).$$

The map

$$C^\infty(M; E) \rightarrow C^\infty(I, C^\infty(X; E_X)), \quad u \mapsto \tilde{u}, \tag{34}$$

is an isomorphism of Fréchet spaces. The isomorphism commutes with bundle operations such as tensor products and contractions.

The covariant derivative in normal direction is transformed into  $\partial_r$  under the above isomorphism:

$$\widetilde{\nabla_N^E u}(r) = \partial_r \tilde{u}(r), \quad r \in I. \tag{35}$$

To see this, consider the geodesic  $I \rightarrow M, r \mapsto y(r) = \exp(rN(x))$ . The tangent vectors are  $\dot{y}(r) = N(y(r))$ . Using (23), it follows that

$$(\nabla_{N(y(r))}^E u)(y(r)) = \frac{d}{ds} \Big|_{s=r} \tau_{[y(r) \leftarrow y(s)]}^E u(y(s)) = \tau_{[y(r) \leftarrow x]}^E \frac{d}{ds} \Big|_{s=r} \tilde{u}(s, x).$$

This implies (35). We have  $\nabla_N N = SN = 0$ . It follows that  $\partial_r \tilde{N} = 0$ , and  $\partial_r \tilde{v} = 0$ . Abusing notation, we write  $\partial_r$  to denote  $\nabla_N^E$ .

Define  $\epsilon_\nu e = e \otimes \nu$  and  $\iota_\nu(e \otimes \eta) = \langle \eta, \nu \rangle e$ . Notice that  $\epsilon_\nu$  and  $\iota_\nu$  commute with  $\partial_r$ .

Let  $F \rightarrow M$  be another vector bundle with a connection. Let  $B : C^\infty(M; E) \rightarrow C^\infty(M; F)$  be a differential operator. Assume that  $B$  is tangential. This means, by definition, that  $B$  commutes with the distance function  $r, [B, r] = 0$ . Then, for every  $r \in I, B$  restricts to an operator  $B_r : C^\infty(M_r; E_r) \rightarrow C^\infty(M_r; F_r), B_r U = (Bu)|_{M_r}$ , where  $u$  is a section of  $E \rightarrow M$  which extends a given section  $U$  of  $E_r \rightarrow M_r$ . The assumption  $[B, r] = 0$  implies that  $B_r$  is well-defined. Parallel transport along the geodesics orthogonal to  $X$  defines bundle isomorphisms  $E_r \cong E_X$  and  $F_r \cong F_X$ . Via these isomorphisms the  $B_r$ 's induce differential operators  $B(r) : C^\infty(X; E_X) \rightarrow C^\infty(X; F_X)$ , called associated with  $B$ , such that  $\widetilde{B}u(r) = B(r)\tilde{u}(r), r \in I$ . Each  $B(r)$  is a differential operator having coefficients which are  $C^\infty$  with respect to  $r$ . Conversely, an operator  $B$  is tangential if it is given in this way by a family of differential operators  $\{B(r); r \in I\}$  with coefficients depending smoothly on  $r$ .

**Lemma 10.** *Let  $E \rightarrow M$  be a real vector bundle with connection  $\nabla^E$ . Then*

$$\nabla^E = \epsilon_\nu \partial_r + B, \tag{36}$$

where  $B$  is tangential. Moreover,  $B(0) = \nabla^{E_X}$ .

Here  $E \otimes T^*M$  carries the induced connection. The lemma extends, by decomposition into real and imaginary parts, to complexifications of real bundles with connections. In particular, it holds for complexified tensor bundles with the Levi-Civita connection.

*Proof.* Let  $P^\perp, P^\parallel \in C^\infty(M; \text{End}(TM))$  denote the orthogonal projectors onto the span of  $N$  and onto its orthogonal complement,  $N^\perp$ , respectively. Identify  $E \otimes T^*M$  with  $\text{Hom}(TM, E)$ . Let  $u \in C^\infty(M; E)$ . We have the following decomposition in  $C^\infty(M; \text{Hom}(TM, E))$ :

$$\nabla^E u = (\nabla^E u)P^\perp + (\nabla^E u)P^\parallel = (\nabla_N^E u) \otimes \nu + Bu.$$

This defines  $B$ , and implies (36). Note that  $B$  is tangential. We have

$$B(0)\tilde{u}(0) = Bu|_X = ((\nabla^E u)P^\parallel)|_X = (\nabla^{E_X}(u|_X))(P^\parallel|_X).$$

This proves the asserted formula for  $B(0)$ . □

Assume  $E \rightarrow M$  a Hermitian bundle with a metric connection. Using (33), and the fact that parallel transport preserves inner products, we have

$$\int_M (u | v)_E dV_M = \int_I \int_X (\tilde{u} | \tilde{v})_{E_X} J dV_X dr, \tag{37}$$

if  $u, v \in C_c^\infty(M; E)$ . Formal adjoints of differential operators on  $M$  are taken with respect to these inner products. The inner product of sections  $u$  and  $v$  of  $E_X \rightarrow X$  is  $\int_X (u | v)_{E_X} dV_X$ . Formal adjoints of operators  $A(r)$  associated with a tangential operator  $A$  are defined with respect to this inner product.

Next we prove a formula expressing the elasticity operator  $L$  as a quadratic polynomial in  $D_r = -i \partial_r$  with tangential coefficients. Now assume  $E = \mathbb{C}TM$ , and let  $B$  as in (36). Define tangential operators

$$A_0 = \iota_\nu C \epsilon_\nu, \quad A_1 = -i \iota_\nu CB, \quad A_2 = B^* CB.$$

The order of  $A_j$  is  $j$ . Moreover,  $A_1^* = i B^* C \epsilon_\nu$ .

**Proposition 11.** *The elasticity and traction operators defined in (1) are as follows:*

$$L = (D_r - i \operatorname{tr} S)(A_0 D_r + A_1) + A_1^* D_r + A_2, \quad -iT = A_0(0)D_r + A_1(0).$$

Furthermore,  $A_1^*(0) = A_1(0)^*$ .

*Proof.* Let  $u, v \in C_c^\infty(M; \mathbb{C}TM)$ . It follows from (30) and the symmetry properties of the elasticity tensor that

$$\int_M (\operatorname{Def} u | \operatorname{Def} v)_C dV_M = \int_M (C \nabla u | \nabla v) dV_M.$$

Inserting (36) and using the definition of  $A_j$ , the right-hand side equals

$$\begin{aligned} \int_M (\iota_\nu C \nabla u | \partial_r v) dV_M + \int_M (B^* C \nabla u | v) dV_M \\ = \int_M (A_0 \partial_r u + i A_1 u | \partial_r v) dV_M + \int_M (-i A_1^* \partial_r u + A_2 u | v) dV_M. \end{aligned}$$

We integrate by parts, using (37), and get

$$\begin{aligned} \int_M (w | \partial_r v) dV_M \\ = \int_I \int_X (\tilde{w} | \partial_r \tilde{v}) J dV_X dr = \int_X (w(0) | v(0)) dV_X - \int_I \int_X ((\partial_r \log J) \tilde{w} + \partial_r \tilde{w} | \tilde{v}) J dV_X dr. \end{aligned}$$

Summing up we have

$$\begin{aligned} \int_M (\operatorname{Def} u | \operatorname{Def} v)_C dV_M = \int_M ((D_r - i \operatorname{tr} S)(A_0 D_r + A_1)u + A_1^* D_r u + A_2 u | v) dV_M \\ + \int_X (A_0(0)(\partial_r u)(0) + i A_1(0)u(0) | v(0)) dV_X. \end{aligned}$$

Comparing with (1) the formulas for  $L$  and  $T$  follow. The last assertion follows because  $J = 1$  at  $X$ . □

Next we compute the leading symbols of the operators (associated with)  $A_j$ . The symbols are  $r$ -dependent sections of  $\pi^* \text{End}(\text{CT}_X M) \rightarrow T^* X$ . Dropping tildes, the symbol of  $A_0$  equals

$$\sigma_h(A_0) = a = c(v) \in C^\infty(I, C^\infty(T^* X; \pi^* \text{End}(\text{CT}_X M))).$$

Introduce the divergence of the acoustic tensor restricted to  $X$  as follows:

$$(\text{div}_X c)(\zeta)v = \sum_\alpha \langle \eta^\alpha, (\pi^* \nabla_{v_\alpha} C)(v \otimes \zeta) \rangle,$$

if  $\zeta \in T_x^* M$ ,  $v \in T_x M$ ,  $x \in X$ . Here  $(v_\alpha)$  and  $(\eta^\alpha)$  are any dual frames of  $TX$  and  $T^* X$ . If local coordinates are chosen such that  $r$  is one coordinate and the other coordinates are constant along the geodesics orthogonal to  $X$ , then  $(\text{div}_X c)(\zeta)^{ik} = C^{i\alpha k\ell} |_\alpha \zeta_\ell$ . Here the summation convention is used with Latin indices referring to all coordinates, and Greek referring to all coordinates except  $r$ . We also need the contraction  $\langle C, S \rangle \in C^\infty(M; \text{End}(TM))$  of the elasticity tensor with the shape operator, in coordinates,

$$\langle C, S \rangle^{ik} = C^{ijk\ell} S_{j\ell}, \quad S_{j\ell} = v_{j|\ell}.$$

(Because  $\nabla_N S = 0$  one can also write Greek indices instead of  $j$  and  $\ell$ .)

**Lemma 12.** *Let  $a_1$  and  $a_2$  denote the principal symbols of the  $h$ -differential operators  $hA_1$  and  $h^2 A_2$ , respectively. At  $r = 0$ :  $a_1(\xi) = c(v, \xi)$ , and  $a_2(\xi) = c(\xi)$ . On the leading symbol level,  $\sigma_h(hA_1) = a_1$ ,  $\sigma_h(hA_1^*) = a_1^* - iha_{1-}$ , and  $\sigma_h(h^2 A_2) = a_2 - iha_{2-} + \mathcal{O}(h^2)$ , where, at  $r = 0$ ,*

$$a_{1-} = (\text{div}_X c)(v) + \pi^* \langle C, S \rangle, \quad a_{2-}(\xi) = (\text{div}_X c)(\xi).$$

*Proof.* By Lemma 10 we have

$$hA_1(0) = \iota_\nu C \circ (-ih\nabla), \quad h^2 A_2(0) = (-ih\nabla)^* \circ C \circ (-ih\nabla),$$

where  $\nabla = \nabla^{TX}$  is the Levi-Civita connection of the boundary. We compute the leading symbol of  $hA_1(0)$  using the composition formula (29). Recall (26). The vertical derivative of the symbol of  $\iota_\nu C$  vanishes, Hence

$$\sigma_h(hA_1)(0)(\xi) = \sigma_h(hA_1(0))(\xi) = c(v, \xi), \quad \xi \in T_x^* M.$$

The formula for  $\sigma_h(h^2 A_2(0))$  follows from Lemma 9. In view of (27),  $a_{1-} = \text{tr } {}^v \nabla^h \nabla a_1^*$ . Since  $a_1^*(\xi) = c(\xi, v) = \langle \xi, \pi^*(C\epsilon_\nu) \rangle$  is linear in  $\xi$ , its vertical derivative is immediate. Hence

$$\text{tr } {}^v \nabla^h \nabla a_1^* = \sum_\alpha \langle \eta^\alpha, \pi^* \nabla_{v_\alpha} (C\epsilon_\nu) \rangle.$$

Now,  $\nabla_\nu(C\epsilon_\nu)$  equals  $(\nabla_\nu C)\epsilon_\nu$  plus a contraction of  $C$  with  $\nabla_\nu v = Sv$ , proving the formula for  $a_{1-}$ .  $\square$

If the elastic medium is isotropic, (16), then a straightforward computation shows that, at  $r = 0$ ,

$$\begin{aligned} (\text{div}_X c)(\zeta) &= (\nabla\lambda \otimes \zeta) + (\nabla\mu \otimes \zeta)^* + \langle \zeta, \nabla\mu \rangle \text{Id}, \\ \langle C, S \rangle &= (\lambda + \mu)S + (\mu \text{tr } S) \text{Id}. \end{aligned}$$

Here  $\nabla\lambda, \nabla\mu \in TX \subset T_X M$  are the gradients of the Lamé parameters restricted to  $X$ .



### 5. Microlocal factorization

We factorize, microlocally in the elliptic region, the  $h$ -differential operator  $h^2L - \rho$  into a product with right factor  $hD_r - Q$ , where  $Q$  is a tangential  $h$ -pseudodifferential operator such that the spectrum of its principal symbol is contained in the lower half-plane,  $\mathbb{C}_-$ .

As in the previous section we identify  $M$  with a boundary collar  $I \times X$ , and sections of  $\mathbb{C}TM \rightarrow M$  with  $r$ -dependent sections of  $\mathbb{C}T_X M \rightarrow X$ . Operators are polynomials in  $D_r/h = hD_r$  with tangential  $h$ -(pseudo-)differential operators as coefficients. The latter are quantizations (64),  $B_h = \text{Op}_h(b_h) \in \Psi_{\text{tang}}^{m,k}$ , of tangential symbols,

$$b_h \in S_{\text{tang}}^{m,k} = C^\infty(I, S^{m,k}(T^*X; \pi^* \text{End}(\mathbb{C}T_X M))).$$

By Proposition 11 the principal symbol  $f(s, \xi) = c(\xi + s\nu) - \rho$  of  $h^2L - \rho$  at  $\xi + s\nu$  is a second order polynomial in  $s$ . View  $s$  as the symbol of  $D_r/h$ . The coefficients are  $h$ -independent tangential symbols. By (9), there exists a constant  $\delta > 0$  such that

$$f(s, \xi) \geq \delta(1 + |s|^2 + |\xi|^2) \text{Id}, \quad s \in \mathbb{R}, \tag{38}$$

holds if  $\xi$  is sufficiently large. If  $F \subset \mathcal{E}$  is closed and  $R > 0$ , then  $F \setminus \{|\xi| > R\}$  is compact. Hence there exist  $0 < \varepsilon', \delta$  such that (38) holds uniformly for  $(r, \xi) \in [-\varepsilon', 0] \times F$ . We say that a property holds at the elliptic region  $\mathcal{E}$  if it is true in every open subset of  $I \times \mathcal{E}$  where (38) holds uniformly.

Recall from Section 2 that we have a unique spectral factorization (10) at  $\mathcal{E}$ .

**Lemma 13.** *Let  $q = q(\xi)$ ,  $\xi \in \mathcal{E}$ , the unique solution of the spectral factorization  $f(s, \xi) = (s - q(\xi)^*)a(s - q(\xi))$ ,  $\text{spec } q(\xi) \subset \mathbb{C}_-$ . Then  $q \in S_{\text{tang}}^1$  at  $\mathcal{E}$ .*

*Proof.* By Lemma 3 we have  $aq = -\pi i f_0^{-1} + f_1 f_0^{-1}$  with integrals  $f_j = f_j(\xi)$  defined there. Using (38), we can estimate  $f_0(\xi) = \int f(s, \xi)^{-1} ds$  as follows:

$$|f_0(\xi)| \leq \int_{-\infty}^{\infty} \delta^{-1}(1 + |s|^2 + |\xi|^2)^{-1} ds = \pi/\delta \langle \xi \rangle,$$

$\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ . The integrand  $f(s, \xi)^{-1}$  remains integrable after applying  $\partial_r$ ,  ${}^h\nabla$ , and  ${}^v\nabla$  finitely many times. Therefore these derivatives can be interchanged with the integral. In view of the symbol properties of  $f$ , we deduce, using estimates as above,  $f_0 \in S_{\text{tang}}^{-1}$  at  $\mathcal{E}$ . Using an upper bound  $f(s, \xi) \leq \delta^{-1}(|s|^2 + \langle \xi \rangle^2) \text{Id}$ , we derive  $f_0(\xi) \geq \delta \langle \xi \rangle^{-1} \text{Id}$ , again in the sense of selfadjoint maps. Therefore  $f_0$  is an elliptic symbol, and  $f_0^{-1} \in S_{\text{tang}}^1$  at  $\mathcal{E}$ .

Write  $f_1 = f_{10} + f_{11}$ , where  $f_{10}(\xi) = \int_{|s| \leq 1} saf(s, \xi)^{-1} ds$ ,

$$f_{11}(\xi) = \int_{|s| > 1} s^{-1}(s^2 a - f(s, \xi)) f(s, \xi)^{-1} ds.$$

Recall that  $s^2 a - f(s) = -s(a_1 + a_1^*) - (a_2 - \rho)$ . Reasoning as in the proof of  $f_0 \in S_{\text{tang}}^{-1}$ , we see that the integrand of  $f_{11}$  and its derivatives are integrable. Moreover, we deduce  $f_{11} \in S_{\text{tang}}^0$ . It is easy to see that  $f_{10} \in S_{\text{tang}}^{-2}$ . Therefore, at  $\mathcal{E}$ ,  $f_1 \in S_{\text{tang}}^0$ . The lemma follows.  $\square$

For an  $h$ -tempered family  $(u_h) \in h^{-\infty} C^{-\infty}(X)$  one defines the semiclassical wavefront set  $\text{WF}_h(u_h) \subset T^*X \sqcup S^*X$  [Gérard 1988; Sjöstrand and Zworski 2002]. Below we deal with operators associated to

symbols which are not defined on all of  $T^*X$  but only at  $\mathcal{E}$ . These operators are defined microlocally in  $\mathcal{E}$  by letting them operate on the subspace of distributions  $(u_h)$  which satisfy  $\text{WF}_h(u_h) \subset \mathcal{E}$ , modulo the space  $h^\infty C^\infty$ .

**Lemma 14.** *Let  $q$  be as in Lemma 13. Microlocally at  $\mathcal{E}$ ,*

$$h^2 L - \rho = (D_{r/h} - Q^\sharp) A_0 (D_{r/h} - Q), \tag{39}$$

where  $Q, Q^\sharp \in \Psi_{\text{tang}}^{1,0}$ , such that  $Q - \text{Op}_h(q), Q^\sharp - \text{Op}_h(q^*) \in \Psi_{\text{tang}}^{0,-1}$ . Here  $A_0$  is as in Proposition 11.

*Proof.* Initially we set  $Q = \text{Op}_h(q)$  and  $Q^\sharp = \text{Op}_h(q^*)$ . At  $\mathcal{E}$ ,

$$h^2 L - \rho = (D_{r/h} - Q^\sharp) A_0 (D_{r/h} - Q) + R_1 + R_0 D_{r/h}, \tag{40}$$

where  $R_j \in \Psi_{\text{tang}}^{j,-1}$ . Here we used the formula for  $L$  given in Proposition 11. Observe that, if  $A \in \Psi_{\text{tang}}^{m,k}$ , then the commutator  $[D_{r/h}, A]$  belongs to  $\Psi_{\text{tang}}^{m,k-1}$ . Aiming at an inductive construction, we assume that (40) holds for some positive integer  $k$  such that  $R_j \in \Psi_{\text{tang}}^{j+1-k,-k}$ . The spectra of  $q$  and  $q^*$  are disjoint. It follows that the equation  $sq - q^*s = r$  has, at  $\mathcal{E}$ , for every symbol  $r \in S^m$  a unique solution  $s \in S^{m-1}$ . Applying this construction to the principal symbols of the  $R_j$ 's, we find operators  $S_j \in \Psi_{\text{tang}}^{j-k,-k}$  such that  $S_j Q - Q^\sharp S_j - R_j$  lies in  $\Psi_{\text{tang}}^{j-k,-k-1}$ . Set

$$Q_1 = Q - A_0^{-1}(S_0 Q + S_1), \quad Q_1^\sharp = Q^\sharp + (Q^\sharp S_0 + S_1) A_0^{-1}.$$

Then

$$(D_{r/h} - Q_1^\sharp) A_0 (D_{r/h} - Q_1) = (D_{r/h} - Q^\sharp) A_0 (D_{r/h} - Q) + (S_0 Q - Q^\sharp S_0) D_{r/h} + (S_1 Q - Q^\sharp S_1) + [D_{r/h}, S_0 Q + S_1] - (Q^\sharp S_0 + S_1) A_0^{-1} (S_0 Q + S_1).$$

Replace  $Q$  and  $Q^\sharp$  by  $Q_1$  and  $Q_1^\sharp$ , respectively. By the symbol calculus, (40) holds with smaller errors,  $R_j \in \Psi_{\text{tang}}^{j-k,-k-1}$ . The proof is completed using asymptotic summation.  $\square$

It follows from the foregoing construction that the symbol of  $Q$  is classical.

### 6. A Dirichlet parametrix

Microlocally at  $\mathcal{E}$ , we solve, constructing a parametrix,  $Bf = u$ , the Dirichlet problem  $h^2 Lu - \rho u = 0, u|_X = f$ . We adapt the method of [Taylor 1996, 7.12] to our setting.

Denote by

$$S_{\text{pois}}^m \subset C^\infty([-1, 0], C^\infty(T^*X; \pi^* \text{End}(\mathbb{C}T_X M)))$$

the space of symbols  $b(s, \eta), -1 \leq s \leq 0, \eta \in T^*X$ , that satisfy the estimates

$$|\partial_s^\tau (v \nabla)^j ({}^h \nabla)^\ell b(s, \eta)| \leq C_{\tau j \ell} \langle \eta \rangle^{m+\tau-j},$$

for all nonnegative integers  $\tau, j$ , and  $\ell$ . Let  $S_{\text{pois}}^{m,k}$  denote the corresponding space of  $h$ -dependent symbols  $b_h$ . Observe that  $g(s\langle \eta \rangle) \in S_{\text{pois}}^0$  if  $g(t) = |t|^j e^{\varepsilon t}, \varepsilon > 0, j$  a nonnegative integer.

We continue to work in a collar  $I \times X \subset M$ . Choose a cutoff  $\chi_0$  as in (64). Let  $\delta > 0$ . Given  $b_h \in S_{\text{pois}}^{m,k}$ , introduce the operator  $B_h = \text{Op}_{\delta,h}(b_h(r/h))$  by setting

$$B_h f(r, y) = (2\pi h)^{-n} \int_{T_y^*} \int_{T_y} e^{-i\langle \eta, v \rangle / h + \delta r \langle \eta \rangle / h} \chi_0(y, v) b_h(r/h, y, \eta) \tau_{[y \leftarrow \exp_y v]}^{\text{CT}_X M} f(\exp_y v) \, dv \, d\eta, \quad (41)$$

for  $r \in I$ ,  $n = \dim X$ . We call  $B_h$  a Poisson operator with symbol  $b_h$  and (exponential) decay  $\delta$ . The arguments in [Taylor 1996, Chapter 7, Proposition 12.4] apply to give  $B_h : L^2(X) \rightarrow H_h^{-m+1/2}(I \times X)$  with norm  $\mathcal{O}(h^{-k+1/2})$ . (The Sobolev spaces  $H_h^s$  are defined using  $hD$  instead of  $D$ .) If  $0 < \delta' < \delta$  and  $j \in \mathbb{N}$ , then

$$r^j B_h \in \text{Op}_{\delta',h} S_{\text{pois}}^{m-j,k-j}.$$

Moreover,  $B_h f \in C^\infty$  in  $r < 0$ , and  $B_h f(r)$  decays together with its derivatives as  $e^{\delta' r/h}$ , uniformly if  $f$  ranges in a bounded subset of  $L^2(X)$ . We call  $h$ -dependent operators negligible if they have Schwartz kernels which are smooth and  $\mathcal{O}_{C^\infty(M \times X)}(h^\infty)$ . We write  $A \equiv B$  if and only if  $A - B$  is negligible. Note that  $B_h$  in (41) is negligible if there exists  $\epsilon > 0$  such that  $b_h(s, \eta) = 0$  if  $-\epsilon < s \leq 0$ .

We need to handle the composition of a Poisson operator with a tangential operator. The following lemma deals with this when the symbols are classical, i.e., possess asymptotic expansions in powers of  $h$ .

**Lemma 15.** *Let  $0 < \delta' < \delta$ . Let  $A = \text{Op}_h a(r)$  and  $B = \text{Op}_{\delta,h} b(r/h)$ , where  $a = a(r, \eta) \in S_{\text{tang}}^1$  and  $b(s, \eta) \in S_{\text{pois}}^m$  are  $h$ -independent symbols. Then*

$$AB \equiv \text{Op}_{\delta',h} c(r/h),$$

where  $c = c_h \in S_{\text{pois}}^{m+1,0}$  has an asymptotic expansion  $c \sim \sum_{j \geq 0} h^j c_j$ ,  $c_j \in S_{\text{pois}}^{m+1-j}$ . The principal term equals

$$c_0(s, \eta) = a(0, \eta) b(s, \eta) e^{(\delta - \delta')s \langle \eta \rangle}.$$

*Proof.* Using Taylor expansions,  $a(r, \eta) = \sum_{j < N} r^j a_j(\eta) + r^N a'_N(r, \eta)$ , and the properties of  $r^j B$  noted above, we may assume without loss of generality that  $a$  does not depend on  $r$ . Arguing as in the proof of Proposition 28 we can write, at least formally,  $AB = \text{Op}_{0,h} \tilde{c}(r/h)$ , where

$$\tilde{c}(s, x, \xi) = (2\pi h)^{-2n} \int_{T_x \times T_x^* \times T_x \times T_x^*} e^{i\varphi/h} a(x, \eta) \tau_{[x \leftarrow y]}^{\pi^* \text{End}(\text{CT}_X M)} b(s, y, \zeta) e^{\delta s \langle \xi \rangle} M(x, w + v, v) \, d(v, \eta, w, \vartheta),$$

$\varphi$  as in (72). We use the standard arguments in handling compositions of symbols: dyadic decompositions and the method of (non-)stationary phase. We infer that there exist  $\epsilon > 0$  and  $d_j \in S_{\text{pois}}^{m+1-j}$ ,  $d_0(s, \eta) = a(\eta) b(s, \eta)$ , such that for every  $N$ ,

$$\tilde{c}(s, \eta) = \left( \sum_{j < 3N} h^j d_j(s, \eta) \right) e^{\delta s \langle \eta \rangle} + \tilde{d}_{Nh}(s, \eta) e^{\epsilon s \langle \eta \rangle},$$

where  $\tilde{d}_{Nh} \in S_{\text{pois}}^{m+1-N,-N}$ . Observe that  $\langle \xi \rangle / \langle \eta \rangle$  is uniformly bounded from below if  $\xi$  and  $\eta$  range in the same dyadic shell. Above we have chosen  $\epsilon$  less than  $\delta$  times this bound. Define  $c_h(s, \eta)$  as the product of an asymptotic sum  $\sum_{j \geq 0} h^j d_j(s, \eta)$  with the symbol  $e^{(\delta - \delta')s \langle \eta \rangle} \in S_{\text{pois}}^0$ . It follows that  $AB - \text{Op}_{\delta',h} c(r/h)$  belongs to  $\text{Op}_{\epsilon,h} S_{\text{pois}}^{m+1-N,-N}$  for every  $N$ . Thus  $AB \equiv \text{Op}_{\delta',h} c(r/h)$ .  $\square$

Let  $q$  and  $Q$  be as in Lemma 14. If  $\eta \in \mathcal{E}$  ranges in a set having a positive distance to the complement of the elliptic region, then there exist positive constants  $\delta_0$  and  $M$  such that

$$|e^{s iq(0,\eta)}| \leq M e^{s \delta_0 \langle \eta \rangle}, \quad s \leq 0. \tag{42}$$

This follows from the fact that the spectrum of  $q(0, \eta)/\langle \eta \rangle$  is contained in a compact subset of the lower half-plane then. We shall solve  $(D_{r/h} - Q)B \equiv 0$ ,  $B|_{r=0} = \text{Id}$ , microlocally at  $\mathcal{E}$ . On the symbol level we have to solve linear ordinary differential equations with constant coefficient matrices. The following assertions are true microlocally in  $\mathcal{E}$  where (42) holds.

**Lemma 16.** *Let  $0 < \delta < \delta_0$ . Let  $r \in S_{\text{pois}}^{1+m}$  and  $v \in S^m$ . Let  $b(s, \eta)$  be the solution of the initial value problem*

$$\partial_s b(s, \eta) = (iq(0, \eta) - \delta \langle \eta \rangle) b(s, \eta) + r(s, \eta), \quad -1 < s \leq 0, \tag{43}$$

and  $b(0, \eta) = v(\eta)$ . Then  $b(s, \eta) \in S_{\text{pois}}^m$ .

*Proof.* Note that the coefficient matrix of (43) does not depend on  $s$ . Representing  $b$  by Duhamel’s formula and using (42) we derive the estimate

$$|b(s, \eta)| \leq M |v(\eta)| + M \int_s^0 e^{(\delta_0 - \delta)s \langle \eta \rangle} |r(s, \eta)| ds \leq M |v(\eta)| + \frac{M}{\delta_0 - \delta} \sup_{s \leq 0} |r(s, \eta)| / \langle \eta \rangle.$$

Moreover, we can estimate  $\partial_s b(s, \eta)$  by estimating the right-hand side of (43). Differentiating (43) we derive linear ordinary differential equations for  $\partial_s^\tau ({}^v \nabla)^j ({}^h \nabla)^\ell b(s, \eta)$ . These equations are of the same structure as (43) with the same coefficient matrix. The asserted symbol estimates are obtained recursively. □

**Proposition 17.** *Let  $0 < \delta < \delta_0$ , and  $\epsilon > 0$ . There exists  $B \in \text{Op}_{\delta,h} S_{\text{pois}}^{0,0}$  with Schwartz kernel supported in  $-\epsilon < s \leq 0$ , such that, microlocally at  $\mathcal{E}$ ,  $(D_{r/h} - Q)B \equiv 0$  and  $B|_{r=0} = \text{Id}$ . Moreover,  $(h^2 L - \rho)B \equiv 0$ .*

*Proof.* It follows from Lemma 15 that, for a classical symbol  $b \in S_{\text{pois}}^{m,k}$ ,  $0 < \delta' < \delta$ , modulo negligible operators, the composition

$$(D_{r/h} - Q) \text{Op}_{\delta,h} b(r/h)$$

equals  $\text{Op}_{\delta',h} c(r/h)$ ,  $c \in S_{\text{pois}}^{m+1,k}$ . Moreover,  $c$  is classical, and, modulo  $S_{\text{pois}}^{m,k-1}$ ,

$$c(s, \eta) \equiv (-i \partial_s b(s, \eta) - i \delta \langle \eta \rangle b(s, \eta) - q(0, \eta) b(s, \eta)) e^{(\delta - \delta')s \langle \eta \rangle}.$$

Fix a sequence  $(\delta_j)$ ,  $\delta < \delta_{j+1} < \delta_j$ . Using Lemmas 15 and 16 we recursively find  $h$ -independent symbols  $b_j \in S_{\text{pois}}^{1-j}$ ,  $b_1|_{r=0} = \text{Id}$ ,  $b_j|_{r=0} = 0$  if  $j > 1$ , such that  $B_j = h^{j-1} \text{Op}_{\delta_j,h} b_j(r/h)$  satisfy

$$(D_{r/h} - Q)(B_1 + \dots + B_j) \in \text{Op}_{\delta',h} S_{\text{pois}}^{1-j,-j}, \quad \delta_{j+1} < \delta' < \delta_j.$$

Now  $B$  is constructed using asymptotic summation. The last assertion follows from the factorization (39). □

### 7. The displacement-to-traction operator

In this section we deal with operators on the boundary  $X$ . Therefore, in the following, operators and symbols are, as a rule, evaluated at  $r = 0$ .

Let  $B$  denote the Dirichlet parametrix given in Proposition 17 and  $T$  the traction defined in (1). The operator  $Z = hTB$  is called the semiclassical displacement-to-traction operator, or Neumann operator, at  $\mathcal{E}$ . By Propositions 11 and 17 we have, if  $\text{WF}_h(f) \subset \mathcal{E}$ ,

$$Zf = (iA_0D_{r/h}Bf + ihA_1Bf)|_X = iA_0(0)Q(0)f + ihA_1(0)f.$$

Therefore,  $Z = iA_0Q + ihA_1$ , and  $Z$  is, microlocally in  $\mathcal{E}$ , a pseudodifferential operator of class  $\Psi^{1,0}$ . The symbol of  $Z$  is classical since the symbols of  $A_j$  and  $Q$  are.

**Lemma 18.** *The displacement-to-traction operator  $Z$  is, in  $\mathcal{E}$ , up to a negligible operator, formally selfadjoint. The principal symbol of  $Z$  equals the surface impedance tensor*

$$z = i(aq + a_1) \in S^1(\mathcal{E}; \pi^* \text{End}(\mathbb{C}T_X M)). \tag{44}$$

The leading symbol of  $Z$  is  $z + hz_-$ , where  $z_- \in S^0$ ,

$$z_-q - q^*z_- = i \text{tr}(S)z + i \partial_r z - a_{2-} - a_{1-q} + \text{tr}({}^v\nabla q^* \cdot a {}^h\nabla q). \tag{45}$$

*Proof.* Let  $f_1, f_2 \in L^2(X; \mathbb{C}T_X M)$ ,  $\text{WF}_h(f_j) \subset \mathcal{E}$ , and set  $u_j = Bf_j$ . By (1),

$$\int_X (Zf_1 | f_2) dV_X - \int_X (f_1 | Zf_2) dV_X = h^{-1} \int_M (u_1 | h^2 Lu_2 - \rho u_2) - (h^2 Lu_1 - \rho u_1 | u_2) dV_M.$$

It follows from Proposition 17 that the right-hand side is  $\mathcal{O}(h^\infty)$ , uniformly if the  $f_j$ 's range in a bounded set and have  $h$ -wavefronts contained in a common closed subset of  $\mathcal{E}$ . Thus  $Z^* = Z$  in  $\mathcal{E}$ . Recalling  $Z = iA_0Q + ihA_1$ , we infer from the symbol calculus that  $z = i(aq + a_1)$  is the principal symbol.

It remains to prove the formula for  $z_-$ . Write the leading symbols of  $Q$  and  $Q^\sharp$  as  $q + hq_-$  and  $q^* + hq_-^\sharp$ , respectively. It is easy to see that  $z_- = iaq_-$ . Recall the formula for  $L$  in Proposition 11. The factorization (39) is equivalent to

$$(D_{r/h} - ih \text{tr}(S))hA_1 + (hA_1^* - ih \text{tr}(S)A_0)D_{r/h} + h^2 A_2 - \rho = -D_{r/h}A_0Q - Q^\sharp A_0D_{r/h} + Q^\sharp A_0Q.$$

This in turn is equivalent to the following two equations of tangential operators:

$$\begin{aligned} hA_1 + hA_1^* - ih \text{tr}(S)A_0 + A_0Q + Q^\sharp A_0 &= 0, \\ [D_{r/h}, A_0Q + hA_1] - ih \text{tr}(S)hA_1 + h^2 A_2 - \rho - Q^\sharp A_0Q &= 0. \end{aligned}$$

On the principal symbol level these equations become  $a_1 + a_1^* + aq + q^*a = 0$  and  $a_2 - \rho - q^*aq = 0$ . These equations agree with (10). On the leading symbol level the equations become, after division by  $h$ ,

$$\begin{aligned} -ia_{1-} - i \text{tr}(S)a + aq_- + q_-^\sharp a - i \text{tr}({}^v\nabla q^* \cdot {}^h\nabla a) &= 0, \\ -\partial_r z - i \text{tr}(S)a_1 - ia_{2-} - q^*aq_- - q_-^\sharp aq + i \text{tr}({}^v\nabla q^* \cdot {}^h\nabla aq) &= 0. \end{aligned}$$

Elimination of  $q_-^\sharp$  from these equations gives

$$(aq_-)q - q^*(aq_-) = ia_{1-q} + \text{tr}(S)z + \partial_r z + ia_{2-} - i \text{tr}({}^v\nabla q^* \cdot a {}^h\nabla q).$$

Formula (45) for  $z_- = iaq_-$  follows. □

In principle  $z_-$  is found as the unique solution of the linear Equation (45). The right-hand side of the equation consists of known quantities and their first order derivatives. Refer to Section 10 for an algorithm computing  $z_-$  if the elastic medium is isotropic.

### 8. Diagonalization of $Z$

Assume (U) and (E1). By Proposition 6 the kernel  $\ker z$  defines a line bundle over the characteristic variety  $\Sigma = p^{-1}(1)$  of the surface impedance tensor  $z$ . Since zero is a simple eigenvalue of  $z$  at  $\Sigma$ , there exist  $\epsilon > 0$  and an open neighborhood  $K \subset \mathcal{C}$  of  $\Sigma$  such that  $z(\xi)$ ,  $\xi \in K$ , has exactly one eigenvalue  $\lambda_0(\xi)$  of modulus  $< \epsilon$ . (In the following,  $K$  is to be replaced by a smaller neighborhood when necessary.) The line bundle  $E_0 = \ker(z - \lambda_0) \rightarrow K$  is a subbundle of  $\pi^* \mathbb{C}T_X M = \text{Hom}(\mathbb{C}, \pi^* \mathbb{C}T_X M)$ . The orthoprojector onto this bundle is given by a contour integral,  $u_0 = (2\pi i)^{-1} \oint_{|\lambda|=\epsilon} (\lambda - z)^{-1} d\lambda$ . Denote by  $u_1 = \text{Id} - u_0$  the orthoprojector onto the orthogonal bundle,  $E_1$ .

Assume also (E2). Choose a unit section  $v$  of  $\ker z \rightarrow \Sigma$ ,  $|v| = 1$ . Using  $u_0$ , extend  $v$  to a unit section of  $E_0 \rightarrow K$ . Call this section also  $v$ . Clearly,  $u_0 = v \otimes v^*$ . If  $R \in \Psi^{0,0}$  denotes the inverse of a square root of the scalar operator  $\text{Op}_h(v)^* \text{Op}_h(v)$ , then  $V = \text{Op}_h(v)R$  satisfies  $V^*V = \text{Id}$ , i.e.,  $V$  is an isometry.

**Lemma 19.** *Choose  $V \in \Psi^{0,0}(K; \mathbb{C}, \mathbb{C}T_X M)$ , with principal symbol  $v$ , such that  $V^*V = \text{Id}$ . Set  $U_0 = VV^*$ ,  $U_1 = \text{Id} - U_0$ . There exist  $B \in \Psi^{-1,-2}(X; \mathbb{C}T_X M)$ ,  $B^* = B$ , and  $R \in \Psi^{-1,-1}(X; \mathbb{C}T_X M)$ ,  $R^* + R = 0$ , such that, microlocally in  $K$ ,*

$$(\text{Id} - R^*)Z(\text{Id} - R) = U_0(Z + B)U_0 + U_1(Z + B)U_1. \tag{46}$$

In particular,

$$(\text{Id} - R^*)Z(\text{Id} - R)V = VV^*(Z + B)V. \tag{47}$$

The leading symbol of the scalar operator  $V^*(Z + B)V \in \Psi^{1,0}$  equals

$$\lambda_0 + h(z_-v | v) - ih \text{tr} (v^* v \nabla_z \cdot {}^h \nabla v + v \nabla v^* \cdot {}^h \nabla \lambda_0 \cdot v). \tag{48}$$

Here, as in Lemma 18,  $z + hz_-$  denotes the leading symbol of  $Z$ .

*Proof.* To prove (46) we adopt ideas from [Stefanov 2000]. The operators  $U_0$  and  $U_1$  are orthogonal projectors,  $U_j^* = U_j = U_j^2$ , and  $U_1U_0 = 0$ . Write

$$Z = U_0ZU_0 + U_1ZU_1 + B,$$

where  $B = U_0ZU_1 + U_1ZU_0$ . Since  $u_jz = zu_j$  and  $u_1u_0 = 0$  we have  $B \in \Psi^{0,-1}$ . Let  $hb$ , with  $b = b^* \in S^0$ , denote the principal symbol of  $B$ . Define the section  $z_j = z|_{E_j}$  of  $\text{End}(E_j)$ . The spectra of  $z_0$  and  $z_1$  are disjoint. Therefore the Sylvester equation  $sz_0 - z_1s = u_1bu_0$  has a unique solution  $s$  which is a section of  $\text{Hom}(E_0, E_1)$ . We extend  $s$  to a section of  $\text{End}(\pi^* \mathbb{C}T_X M)$  by  $s = u_1su_0$ . Then  $sz - zs = u_1bu_0$ , and  $s \in S^{-1}$ . Define  $S = \text{Op}_h(hs)$  and  $R = U_0S^*U_1 - U_1SU_0$ . Then,  $R^* = -R$  and  $B = U_0BU_1 + U_1BU_0 \equiv R^*Z + ZR$  modulo  $\Psi^{-1,-2}$ . Therefore, with a different  $B \in \Psi^{-N+1,-N}$ ,  $N = 2$ , and  $Z_0 = Z_1 = Z$ , we have

$$(\text{Id} - R^*)Z(\text{Id} - R) = U_0Z_0U_0 + U_1Z_1U_1 + B. \tag{49}$$

If  $N \geq 2$  then, using the same construction as before, we find  $R_1 \in \Psi^{-N,-N}$ ,  $R_1^* = -R_1$  such that  $U_0 B U_1 + U_1 B U_0 \equiv R_1^* Z + Z R_1$  modulo  $\Psi^{-N,-N-1}$ . Hence we get (49) with  $R$  and  $Z_j$  replaced by  $R + R_1$  and  $Z_j + B$ , respectively. The new error  $B$  belongs to  $\Psi^{-N,-N-1}$ . Iterating this construction and using asymptotic summation (46) follows. Since  $U_0 V = V$ , (46) implies (47).

Observe that the leading symbols of  $V^*(Z + B)V$  and  $V^*ZV$  are equal. The principal symbol equals  $(v | zv) = \lambda_0$  because  $|v| = 1$ . We write the leading symbol of  $V$  as  $(1 + h\gamma)v + hw$ , where  $v^*w = (w | v) = 0$ . Note that  $(v | zw) = 0$ . A straightforward symbol computation, using (68) and (71), gives

$$\sigma_h(V^*ZV) \equiv \lambda_0 + h(z-v | v) + h(\gamma + \bar{\gamma})\lambda_0 - ih \operatorname{tr}(v^* v \nabla_z \cdot {}^h \nabla v + v \nabla {}^h \nabla v^* \cdot {}_z v + v \nabla v^* \cdot {}^h \nabla_z v)$$

modulo  $\mathcal{O}(h^2)$ . From  $V^*V = 1$  it follows that the leading symbol of  $V^*V$  equals unity. Since  $|v|^2 = 1$  is the principal symbol, this implies

$$h(\gamma + \bar{\gamma}) - ih \operatorname{tr}(v \nabla {}^h \nabla v^* \cdot v + v \nabla v^* \cdot {}^h \nabla v) = 0.$$

Therefore the expression for the symbol of  $V^*ZV$  simplifies to

$$\sigma_h(V^*ZV) \equiv \lambda_0 + h(z-v | v) + ih \operatorname{tr}(\lambda_0 v \nabla v^* \cdot {}^h \nabla v - v^* v \nabla_z \cdot {}^h \nabla v - v \nabla v^* \cdot {}^h \nabla_z v)$$

modulo  $\mathcal{O}(h^2)$ . Using  ${}^h \nabla_z v = \lambda_0 {}^h \nabla v + {}^h \nabla \lambda_0 \cdot v$  we deduce (48). □

Denote by  $\Psi_{\text{phg}}^m$  the class of  $h$ -independent pseudodifferential operators  $A$  with polyhomogeneous symbols,  $a \sim \sum_{j \leq m} a_j$ ,  $a_j$  homogeneous of degree  $j$ . When regarded as an  $h$ -dependent operator,  $A \in \Psi^{m,m}$  has the classical symbol  $\sum_{j \leq m} h^{-j} a_j$ . In the next lemma, following [Popov and Vodev 1999] and [Stefanov 2000], we use this relation to conjugate the scalar operator constructed in Lemma 19 into  $hP - 1$ , where  $P$  is  $h$ -independent.

Recall that  $\Omega^{1/2} \rightarrow X$  denotes the bundle of half-densities.

**Lemma 20.** *There is a selfadjoint operator  $P \in \Psi_{\text{phg}}^1(X; \Omega^{1/2})$  with principal symbol  $p$ , and an operator  $A \in \Psi^{0,0}$  from half-density sections to scalar functions, elliptic near  $\Sigma$ , such that  $A^*V^*(Z + B)VA = hP - 1$  in a neighborhood of  $\Sigma$ . The subprincipal symbol of  $P$  equals, on  $\Sigma$ ,*

$$p_{\text{sub}} = (\dot{z}v | v)^{-1} (\operatorname{Re}(z-v | v) + \operatorname{Im} \operatorname{tr}(v^* v \nabla_z \cdot {}^h \nabla v)) + \operatorname{Im} \operatorname{tr}({}^h \nabla p \cdot v \nabla v^* \cdot v). \tag{50}$$

Here  $\dot{z}$  denotes the radial derivative of  $z$ . If instead of  $v$  another unit section  $\tilde{v} = e^{i\varphi}v$  of  $\ker z \rightarrow \Sigma$  is used to define  $V$ , and thus  $P$ , then the principal symbol of  $P$  remains unchanged, whereas the subprincipal changes to  $\tilde{p}_{\text{sub}} = p_{\text{sub}} + \{p, \varphi\}$  on  $\Sigma$ . Here  $\{p, \varphi\}$  denotes the Poisson bracket.

Obviously,  $P$  is elliptic and bounded from below.

*Proof.* The radial derivatives of  $p$  and of  $\lambda_0 = (zv | v)$  are, at  $\Sigma$ , equal to 1 and  $(\dot{z}v | v) > 0$ , respectively. Therefore, near  $\Sigma$ ,  $a_0^2 \lambda_0 = p - 1$  for some  $a_0 \in C^\infty$ ,  $a_0 > 0$ . Set  $\tilde{Z} = A_0^*V^*(Z + B)VA_0$ ,  $A_0 = \operatorname{Op}_h(a_0)$ . Choose  $\tilde{P}_1 \in \Psi^{1,0}$  (formally) selfadjoint with leading symbol  $p - ih \operatorname{tr}(v \nabla {}^h \nabla p)/2$ . The selfadjoint operators  $\tilde{Z}$  and  $\tilde{P}_1 - 1$  have the same principal symbol,  $p - 1$ . Therefore, the imaginary parts of their leading symbols are equal. It follows that the principal symbol  $q_0$  of  $\tilde{Q}_0 = \tilde{Z} - (\tilde{P}_1 - 1) \in \Psi^{0,0}$  equals, on  $\Sigma$ ,  $a_0^2 = (\dot{z}v | v)^{-1}$  times the real part of the coefficient of  $h$  in (48).

Define  $p_0 \in C^\infty(T^*X \setminus 0)$ , homogeneous of degree 0, and  $r_{-1} \in S^{-1}$  such that  $q_0 = p_0 + 2(p-1)r_{-1}$  holds in a neighborhood of  $\Sigma$ . Then

$$(1 - h \operatorname{Op}_h(r_{-1}))^* \tilde{Z}(1 - h \operatorname{Op}_h(r_{-1})) = \tilde{P}_1 + h\tilde{P}_0 - 1 + h\tilde{Q}_{-1},$$

where  $\tilde{P}_0$  is selfadjoint with principal symbol  $p_0$ . Proceeding inductively, we obtain selfadjoint operators  $\tilde{P}_j \in \Psi^{j,0}$  with classical symbols such that, for  $N < 1$ ,

$$(1 - hR_N^*) \tilde{Z}(1 - hR_N) = h \sum_{N < j \leq 1} h^{-j} \tilde{P}_j - 1 + h^{-N} \tilde{Q}_N,$$

where  $\tilde{Q}_N \in \Psi^{N,0}$ ,  $R_N \in \Psi^{-1,0}$ . Therefore, there is an  $h$ -independent operator  $P \in \Psi_{\text{phg}}^1$  such that  $(1 - hR^*) \tilde{Z}(1 - hR) = hP - 1$  near  $\Sigma$ . Moreover,  $P \equiv \tilde{P}_1 + h\tilde{P}_0$  modulo  $\Psi^{-1,-2}$ . The symbol of  $P$  equals  $p - i \operatorname{tr}(^v \nabla ^h \nabla p)/2 + p_0$  modulo  $S^{-1}$ . It follows from Corollary 29, or rather its analogue for  $h$ -independent operators, that  $p$  is the principal symbol of  $P$  and  $p_{\text{sub}} = p_0$  its subprincipal symbol. By construction  $p_0 = q_0$  on  $\Sigma$ . Formula (50) follows from the formula for  $q_0$  mentioned earlier.

Note that  $\{p, \varphi\} = \operatorname{tr}(^v \nabla p \cdot ^h \nabla \varphi - ^h \nabla p \cdot ^v \nabla \varphi)$ . The last assertion of the lemma follows from (50), using  $v^* ^v \nabla z \cdot v = ^v \nabla \lambda_0$ . □

*Proof of Theorem 1.* The following assertions hold microlocally in a neighborhood of  $\Sigma$ . It follows from Lemmas 19 and 20 that

$$(\operatorname{Id} - R^*)Z(\operatorname{Id} - R)VA = VA^{-*}(hP - 1),$$

where  $A^{-*}$  denotes a parametrix of  $A^*$ . Define  $J_h = (\operatorname{Id} - R)VA$  and  $\tilde{J}_h = (\operatorname{Id} - R^*)^{-1}VA^{-*}$ . We have  $J_h, \tilde{J}_h \in \Psi^{0,0}$ ,  $\tilde{J}_h - J_h \in \Psi^{-1,-1}$ . Moreover,  $J_h^* J_h$  is elliptic. By definition of  $Z$ ,  $TB_h J_h = \tilde{J}_h(P - h^{-1})$ , where  $B_h$  is the Dirichlet parametrix given in Proposition 17. Combining the results in Section 6 with Lemmas 19 and 20, the theorem follows. □

### 9. Construction of quasimodes

Given  $P$  of Theorem 1 we associate to the sequence of positive eigenvalues of  $P$  a sequence of quasimodes of  $L_T$ . We follow [Stefanov 2000, Section 4], differing in some details, however.

Let  $P$ ,  $B_h$ , and  $J_h$  as in Theorem 1. Assume given a sequence of quasimodes,  $(\mu_j)$ , with almost orthogonal quasimodes states:

$$Pf_j - \mu_j f_j = \mathcal{O}_{C^\infty}(h_j^\infty), \quad (f_j | f_k) - \delta_{jk} = \mathcal{O}((h_j + h_k)^\infty), \tag{51}$$

$$f_j \in C^\infty(X; \Omega^{1/2}), \quad 0 < \mu_j \leq \mu_{j+1} \rightarrow \infty, \quad h_j = \mu_j^{-1}.$$

We define quasimode states for the traction-free boundary problem. By Theorem 1 the traction  $t_j = TB_{h_j} J_{h_j} f_j = \mathcal{O}_{C^\infty}(h_j^\infty)$ . Choose  $u'_j \in \mathcal{O}_{C^\infty}(h_j^\infty)$  satisfying  $A_0(0)\partial_r u'_j|_X + t_j = 0$  and  $u'_j|_X = 0$ . Define  $u_j \in C_c^\infty(M; \mathbb{C}TM)$ ,

$$u_j = h_j^{-1/2} (B_{h_j} J_{h_j} f_j + u'_j). \tag{52}$$

By Theorem 1,

$$Lu_j - \mu_j^2 \rho u_j = \mathcal{O}_{C^\infty}(h_j^\infty), \quad Tu_j = 0, \tag{53}$$

and  $\|u_j\|_{L^2} = \mathcal{O}(1)$ . We can assume that the  $u_j$  are supported in a given neighborhood of  $X$ . Using the ellipticity of  $L$ , we deduce  $\|u_j\|_{H^2} = \mathcal{O}(h_j^{-2})$ .



To go from quasimodes to eigenvalues or, in scattering theory, to resonances, it is desirable to be able to decompose the quasimodes into well-separated clusters. In addition, the quasimode states of each cluster should be linearly independent, and remain so after applying small perturbations.

**Proposition 21.** *Let the assumptions of Theorem 1 hold. Assume given quasimodes  $\mu_j = h_j^{-1} > 0$  of  $P$  as in (51), and define  $u_j$  as in (52). Then (53) holds. Let  $m > \dim X$ . There exist  $\delta > 0$  and a covering of  $\{\mu_j\}$  by a sequence of intervals  $[a_k, b_k] \subset \mathbb{R}_+$ , such that*

$$b_k + 2\delta b_k^{-m - \dim X} < a_{k+1}, \quad b_k - a_k < b_k^{-m}.$$

Let  $w_j \in H^2(M; \mathbb{C}TM)$  be such that, for some  $N \geq 0$ ,

$$\|w_j\|_{H^2} = \mathcal{O}(h_j^{-2-N}), \quad w_j - u_j = \mathcal{O}_{L^2}(h_j^{2 \dim X + N}).$$

Then, for large  $k$ ,  $\{w_j\}_{a_k \leq \mu_j \leq b_k}$  is linearly independent.

*Proof.* Property (53) is clear by the arguments already given.

It is well-known that a quasimode sequence (51) is asymptotic to a subsequence of the sequence of eigenvalues of  $P$ . The latter satisfies the Weyl asymptotics. Hence we have a Weyl estimate  $j \leq C\mu_j^{\dim X}$ . It follows that every interval  $[a, b]$ ,  $1 \leq b$ , of length  $> L$  has a subinterval of length  $\geq Lb^{-\dim X}/C$  containing no quasimode  $\mu_j$ . The existence of intervals  $[a_k, b_k]$  having the stated properties follows from this observation. Compare [Stefanov 1999, Proof of Theorem 2]. Define the set of indices of the  $k$ -th cluster:  $I_k = \{j ; \mu_j \in [a_k, b_k]\}$ .

Choose a left inverse  $K_h \in \Psi^{0,0}(X; \mathbb{C}T_X M, \Omega^{1/2})$  of  $J_h$ ,  $K_h J_h = \text{Id}$  at  $\Sigma$ . Since  $J_h^* J_h$  is elliptic at  $\Sigma$ ,  $K_h$  is readily found.

Denote by  $\gamma : v \mapsto v|_X$  the trace map. By (52),  $h_j^{1/2} \gamma u_j = J_{h_j} f_j + \gamma u'_j$ . From (51) it follows that  $\text{WF}_{h_j} f_j \subset \Sigma$ . Therefore,

$$h_j^{1/2} K_{h_j} \gamma u_j = f_j + \mathcal{O}_{C^\infty}(h_j^\infty).$$

By the remark after Lemma 26 we can assume that there exists a constant  $C$  such that for all  $j, \ell \in I_k$ ,  $k \in \mathbb{N}$ ,

$$\|K_{h_\ell} - K_{h_j}\|_{L^2 \rightarrow L^2} \leq C b_k |h_\ell - h_j|.$$

Using  $b_k |h_\ell - h_j| \leq b_k a_k^{-2} |\mu_\ell - \mu_j| \leq a_k^{-2} b_k^{-m+1}$ , it follows that

$$h_j^{1/2} \|(K_{h_\ell} - K_{h_j}) \gamma u_j\|_{L^2} = \mathcal{O}(b_k^{-m}), \quad j, \ell \in I_k,$$

if  $k$  is sufficiently large. The assumptions on  $w_j$  imply  $\|w_j - u_j\|_{H^1} = \mathcal{O}(h_j^{1+\dim X})$ . Here we use the estimate  $\|v\|_{H^1}^2 \leq C \|v\|_{L^2} \|v\|_{H^2}$ . Applying the trace theorem,  $\|\gamma w_j - \gamma u_j\|_{L^2} = \mathcal{O}(h_j^{1+\dim X})$ . Summarizing the estimates, we have shown that, for some  $\varepsilon > 0$ ,

$$\|h_j^{1/2} K_{h_\ell} \gamma w_j - f_j\|_{L^2} = \mathcal{O}(h_\ell^{\varepsilon + \dim X}), \quad j, \ell \in I_k.$$

Because of almost orthogonality of the  $f_j$  and the Weyl estimate, we can apply [Stefanov 1999, Lemma 4]. We obtain, for every  $\ell \in I_k$ , the linear independence of  $\{K_{h_\ell} \gamma w_j\}_{j \in I_k}$  when  $k$  is sufficiently large. Since  $K_{h_\ell} \gamma$  is linear, also  $\{w_j\}_{j \in I_k}$  is linearly independent. □

*Proof of Corollary 2.* We apply Proposition 21 with  $\mu_j \uparrow \infty$  the sequence of positive eigenvalues of  $P$ , counted with multiplicities, and  $\{f_j\}$  a corresponding orthonormal system of eigenvectors. Fix  $m > \dim X$ . Let  $[a_k, b_k]$  be the intervals, clustering  $\{\mu_j\}$ , given in the proposition. The quasimode states defined in (52) belong to the domain of the selfadjoint operator  $L_T$ . Let  $\pi_k$  denote the spectral projector for  $L_T$  of the interval  $[a'_k, b'_k]$ , where  $a'_k = a_k - \delta b_k^{-m-\dim X}$ ,  $b'_k = b_k + \delta b_k^{-m-\dim X}$ . The intervals  $[a'_k, b'_k]$  are pairwise disjoint. Set  $w_j = \pi_k u_j$  if  $\mu_j \in [a_k, b_k]$ . A well-known argument, using the spectral theorem, gives

$$\delta^2 b_k^{-2m-2 \dim X} \|w_j - u_j\|_{L^2}^2 \leq \|(L_T - \mu_j^2)u_j\|_{L^2}^2 = \mathcal{O}(b_k^{-\infty})$$

if  $\mu_j \in [a_k, b_k]$ . Since  $L_T$  is elliptic, we have  $\|w_j\|_{H^2} = \mathcal{O}(\mu_j^2)$ . Now Proposition 21, with  $N = 0$ , implies that, for  $k$  sufficiently large, the rank of  $\pi_k$  equals  $\#\{j; \mu_j \in [a_k, b_k]\}$ . Hence an increase by  $n$  of  $N_P$  over  $[a_k, b_k]$  leads to an increase  $\geq n$  of  $N_{L_T}$  over  $[a'_k, b'_k]$ . Taking into account the widths of the intervals, the corollary follows.  $\square$

**Remark.** The foregoing arguments also apply to give lower bounds for the counting function of resonances. In this case,  $\pi_k$  is the projector onto the space of resonant states which correspond to resonances in rectangles  $[a_k, b_k] + i[0, s_k]$ . To satisfy the assumptions in Proposition 21 for  $w_j = \pi_k u_j$ , one establishes resolvent estimates. See [Stefanov and Vodev 1996; Tang and Zworski 1998; Stefanov 1999; Stefanov 2000] for ways from quasimodes to resonances. The clustering method was developed in this context [Stefanov 1999] to handle multiplicities appropriately. Resolvent estimates for anisotropic elastic systems are given in [Kawashita and Nakamura 2000].

### 10. The isotropic subprincipal symbol

In this section we assume that the elastic medium is isotropic. We evaluate the subprincipal symbol of  $P$ ,  $p_{\text{sub}}$ , starting from the general formula (50).

We continue with Example 8, referring to the notation introduced there. The kernel bundle  $\ker z$  is a line subbundle of  $V$ , the subbundle of  $\mathbb{C}T_X M$  spanned by  $v$ ,  $\hat{\xi} = \xi/|\xi|$ . Abbreviate (19) and (18) as follows:

$$(z)_{11} = \begin{bmatrix} \zeta_1 & -i\zeta_2 \\ i\zeta_2 & \zeta_3 \end{bmatrix}, \quad (iq)_{11} = \begin{bmatrix} \kappa_{11} & -i\kappa_{12} \\ i\kappa_{21} & \kappa_{22} \end{bmatrix}.$$

It will be convenient to use the velocities relative to the Rayleigh wave speed,  $\sigma_s = c_r/c_s$  and  $\sigma_p = c_r/c_p$ . Then  $t = \sigma_s^2$ ,  $ut = \sigma_p^2$  on  $\Sigma = \{c_r|\xi| = 1\}$ . Moreover, we set  $\tau_s = (1 - \sigma_s^2)^{1/2}$ ,  $\tau_p = (1 - \sigma_p^2)^{1/2}$ ,

We first show how to evaluate  $(z_- v | v)$ ,  $v \in \ker z$ ,  $z_-$  as in (45).

**Lemma 22.** Set  $K = (iq)_{11}$ . Define  $Y_j$  by (55), (56), and (58) below. Let  $X = (x_{jk})$  the selfadjoint  $2 \times 2$  matrix which is the unique solution of

$$XK + K^*X = -2Y_1 - Y_2 - Y_2^* + Y_3 + Y_3^*. \tag{54}$$

Let  $v = v_1 \hat{\xi} + v_2 \hat{\xi}^\perp \in \ker z$ . Then

$$2 \operatorname{Re}(z_- v | v) = x_{11}|v_1|^2 + x_{22}|v_2|^2 + 2 \operatorname{Re} x_{12} \bar{v}_1 v_2.$$

*Proof.* Set  $x = z_- + z_-^*$ . Then  $2 \operatorname{Re}(z_- v | v) = (xv | v)$ . By (45),  $x$  satisfies the uniquely solvable Sylvester equation  $x(iq) + (iq)^*x = iy + (iy)^*$ , where  $y$  equals the right-hand side of (45). Since  $q$

leaves  $V$  and  $V^\perp$  invariant,  $X = (x)_{11} = (x_{jk})$  is the unique solution of (54) provided the right-hand side of the equation equals  $(iy + (iy)^*)_{11}$ . The latter holds if

$$Y_1 = (\text{tr}(S)z + \partial_r z)_{11}, \quad Y_2 = (a_{1-}iq)_{11}, \quad Y_3 = (i \text{tr} \, {}^v\nabla(iq)^* \cdot a^h \nabla iq)_{11}.$$

Observe that the  $a_{2-}$  term of (45) drops out because of the skewness of  $(ia_{2-})_{11}$ . In the following we derive formulas for  $Y_j$ .

The basis vectors  $\nu$  and  $\hat{\xi}$  do not depend on  $r$ . Therefore,  $(\partial_r z)_{11} = \partial_r(z)_{11}$ . We obtain

$$Y_1 = \text{tr}(S) \begin{bmatrix} \zeta_1 & -i\zeta_2 \\ i\zeta_2 & \zeta_3 \end{bmatrix} + \begin{bmatrix} \partial_r \zeta_1 & -i\partial_r \zeta_2 \\ i\partial_r \zeta_2 & \partial_r \zeta_3 \end{bmatrix}. \tag{55}$$

Using Lemma 12 and the remark following it we obtain a formula for  $(a_{1-})_{11}$ . Clearly,  $(a_{1-}iq)_{11} = (a_{1-})_{11}(iq)_{11}$ . We derive

$$Y_2 = \begin{bmatrix} \mu \text{tr} S & \langle \hat{\xi}, \nabla \mu \rangle \\ \langle \hat{\xi}, \nabla \lambda \rangle & \mu \text{tr} S + (\lambda + \mu) \langle \hat{\xi}, S \hat{\xi} \rangle \end{bmatrix} \begin{bmatrix} \kappa_{11} & -i\kappa_{12} \\ i\kappa_{21} & \kappa_{22} \end{bmatrix}. \tag{56}$$

It remains to determine  $Y_3$ . Fix an orthonormal frame  $(\eta_j)$  of  $T_X^*M$ ,  $\eta_1 = \nu$ ,  $\eta_2 = \hat{\xi}$ . To compute the contraction we use the frame  $(\eta_j)_{j \geq 2}$  of  $T^*X$ , and the dual frame. We compute derivatives of

$$iq = |\xi| \sqrt{1-t} (\text{Id} - \nu \otimes \nu - \hat{\xi} \otimes \hat{\xi}) + \kappa_{11} \nu \otimes \nu - i\kappa_{12} \nu \otimes \hat{\xi} + i\kappa_{21} \hat{\xi} \otimes \nu + \kappa_{22} \hat{\xi} \otimes \hat{\xi}.$$

Set  $s_{jk} = \langle S\eta_j, \eta_k \rangle$ . A calculation using  ${}^h\nabla \nu = S$  and  ${}^h\nabla \hat{\xi} = 0$  gives

$$({}^h\nabla_j iq)_{11} = {}^h\nabla_j(iq)_{11} + s_{j2} |\xi| b^{-1} M, \quad j \geq 2,$$

where

$$M = \begin{bmatrix} 0 & (ut - b)\sqrt{1-t} \\ (ut - b)\sqrt{1-t} & i(ut - t) \end{bmatrix}.$$

Regard the coefficients  $\kappa_{jk}$  as functions of  $c_s, c_p, |\xi|$ . Then  ${}^h\nabla_j(iq)_{11} = \langle \eta_j, \nabla c_s \rangle K_s + \langle \eta_j, \nabla c_p \rangle K_p$ , where  $K_s$  and  $K_p$  denote the partial derivatives of  $(iq)_{11}$  with respect to  $c_s$  and  $c_p$ , respectively. In particular,

$$({}^h\nabla_2 iq)_{11} = \langle \hat{\xi}, \nabla c_s \rangle K_s + \langle \hat{\xi}, \nabla c_p \rangle K_p + s_{22} |\xi| b^{-1} M.$$

Define  $w_1 = [(ut - b)\sqrt{1-t}, -i(b - ut)]$ . The row  $k > 2$  in  $({}^h\nabla_j iq)_{21}$  equals  $s_{jk} b^{-1} |\xi| w_1$ .

The vertical derivative of a function  $\kappa$  which, when restricted to a fiber depends only on  $|\xi|$ , is given by its radial derivative:

$${}^v\nabla_\eta \kappa = |\xi|^{-1} \langle \hat{\xi}, \eta \rangle \dot{\kappa}. \tag{57}$$

A calculation using  ${}^v\nabla \nu = 0$  and  ${}^v\nabla \hat{\xi} = |\xi|^{-1} (\text{Id} - \hat{\xi} \otimes \hat{\xi})$  gives

$$({}^v\nabla_j iq)_{11} = {}^v\nabla_j(iq)_{11} = |\xi|^{-1} \delta_{2j} \dot{K}, \quad j \geq 2,$$

where we have set

$$\dot{K} = \begin{bmatrix} \kappa_{11} & -i\kappa_{12} \\ i\kappa_{21} & \kappa_{22} \end{bmatrix}.$$

Define  $w_2 = [i(b - t), \sqrt{1-ut} - \sqrt{1-t}]$ . The row  $k > 2$  in  $({}^v\nabla_j iq)_{21}$  equals  $\delta_{jk} b^{-1} w_2$ .

Set  $A = (a)_{11} = \text{diag}[\lambda + 2\mu, \mu]$ . Note that  $(a)_{22}$  equals  $\mu$  times the unit matrix. Summing over  $j \geq 2$  we derive

$$Y_3 = i \dot{K}^* A (|\hat{\xi}|^{-1} \langle \hat{\xi}, \nabla c_s \rangle K_s + |\hat{\xi}|^{-1} \langle \hat{\xi}, \nabla c_p \rangle K_p + s_{22} b^{-1} M) + i \mu b^{-2} |\xi| (\text{tr}(S) - s_{22}) w_2^* \otimes w_1, \quad (58)$$

evaluated at  $\Sigma$ . □

Denote by  $v$  the unique unit section of  $\ker(z - \lambda_0)$  satisfying  $\langle \hat{\xi} | v \rangle > 0$ , so

$$v = \gamma^{-1} (i \zeta_2 v + (\zeta_1 - \lambda_0) \hat{\xi}),$$

$\gamma > 0$  such that  $|v| = 1$ . We compute the  $v$ -dependent terms in the right-hand side of (50).

**Lemma 23.** *On  $\Sigma$  we have  $\text{Im tr } {}^h \nabla p \cdot {}^v \nabla v^* \cdot v = 0$ , and*

$$16\gamma^2 \text{Im tr}(v^* {}^v \nabla z \cdot {}^h \nabla v) = m^3 \mu^{-1} c_r^2 \sigma_s^6 (4 - \sigma_s^2)(2 - \sigma_s^2)(2\tau_s \dot{\zeta}_3 - (2 - \sigma_s^2) \dot{\zeta}_2) s_{22} + 2m^3 c_r \sigma_s^6 (2 - \sigma_s^2)(5\sigma_s^2 - 4 - \sigma_s^4) \text{tr}' S, \quad (59)$$

where  $\text{tr}' S = \text{tr } S - s_{22}$ ,  $s_{22} = \langle \hat{\xi}, S \hat{\xi} \rangle$ , and  $m = \mu |\xi| / b$ .

*Proof.* Set  $\gamma_1 = \zeta_2 / \gamma$  and  $\gamma_2 = (\zeta_1 - \lambda_0) / \gamma$ . We continue to use the frame  $(\eta_j)$ . For  $j \geq 2$  we have

$$\begin{aligned} {}^v \nabla_j v^* &= -i {}^v \nabla_j \gamma_1 \cdot v^* + {}^v \nabla_j \gamma_2 \cdot \hat{\xi}^* + |\xi|^{-1} (1 - \delta_{2j}) \gamma_2 \eta_j^*, \\ {}^h \nabla_j v &= i {}^h \nabla_j \gamma_1 \cdot v + {}^h \nabla_j \gamma_2 \cdot \hat{\xi} + i \gamma_1 S \eta_j. \end{aligned}$$

Note that  ${}^v \nabla_j v^* \cdot v$  is real. Hence  $\text{Im tr } {}^h \nabla p \cdot {}^v \nabla v^* \cdot v = 0$ . We need the vertical derivative of  $z$ . To compute it we proceed in the same way as we did when computing the derivatives of  $iq$ . Recall that  $z$  equals  $\zeta^\perp \text{Id}$  on  $V^\perp$ , where  $\zeta^\perp = \mu |\xi| \sqrt{1-t}$ . We obtain  $({}^v \nabla_j z)_{11} = {}^v \nabla_j (z)_{11}$ . Moreover, the column  $k > 2$  in  $({}^v \nabla_j z)_{12}$  equals  $\delta_{jk} |\xi|^{-1}$  times the transpose of the row vector  $[-i \zeta_2, \zeta_3 - \zeta^\perp]$ . We get

$$\begin{aligned} \text{Im } v^* {}^v \nabla_j z \cdot {}^h \nabla_j v &= \gamma_1 \text{Re } v^* {}^v \nabla_j z \cdot S \eta_j \\ &= \gamma_1 (\gamma_2 {}^v \nabla_j \zeta_3 - \gamma_1 {}^v \nabla_j \zeta_2) s_{2j} + \gamma_1 |\xi|^{-1} (\gamma_2 (\zeta_3 - \zeta^\perp) - \gamma_1 \zeta_2) s_{jj} (1 - \delta_{2j}). \end{aligned}$$

Summing over  $j \geq 2$  we obtain

$$\gamma^2 \text{Im tr}(v^* {}^v \nabla z \cdot {}^h \nabla v) = \zeta_1 \zeta_2 {}^v \nabla_{S \hat{\xi}} \zeta_3 - \zeta_2^2 {}^v \nabla_{S \hat{\xi}} \zeta_2 + c_r \zeta_2 (\zeta_1 (\zeta_3 - \zeta^\perp) - \zeta_2^2) \text{tr}'(S).$$

The first term on the right equals

$$m^2 |\xi|^{-1} s_{22} (2b - t) (t \sqrt{1-t} \dot{\zeta}_3 - (2b - t) \dot{\zeta}_2).$$

Moreover, using the definition of  $b$ , we calculate

$$\zeta_3 - \zeta^\perp = m (\sqrt{1-ut} - \sqrt{1-t}).$$

Using (21),  $4b = t(4-t)$ , we derive (59). □

The restriction to  $\Sigma$  of the radial derivative of the eigenvalue  $\lambda_0 = (zv | v) = a_0^{-2}(p-1)$  equals  $\dot{\lambda}_0 = (\dot{z}v | v) = a_0^{-2}$  because  $\dot{p} = 1$  on  $\Sigma$ .

**Lemma 24.** *Let  $m = \mu|\xi|/b$ . On  $\Sigma$ , we have*

$$\gamma^2 \dot{\lambda}_0 = m^3 \sigma_s^6 (4 - \sigma_s^2) \tau_s (\tau_p / \tau_s + c_s \tau_s / c_p \tau_p + \sigma_s^2 - 2),$$

*Proof.* The section  $w = i \zeta_2 v + \zeta_1 \hat{\xi}$  equals  $\gamma v$  on  $\Sigma$ . Therefore  $\gamma^2 \lambda_0 \equiv (zw | w) = \zeta_1 \det(z)_{11}$  to second order on  $\Sigma$ . Inserting (20),

$$\gamma^2 \lambda_0 \equiv m b t \sqrt{1-t} (4 \sqrt{(1-t)(1-ut)} - (2-t)^2).$$

Recall that  $p = c_r |\xi| = \sigma_s t^{-1/2}$ ,  $\Sigma = \{t = \sigma_s^2\}$ . The rule of de l'Hospital gives

$$\lim_{t \rightarrow \sigma_s^2} \frac{4 \sqrt{(1-t)(1-ut)} - (2-t)^2}{\sigma_s t^{-1/2} - 1} = 4 \sigma_s^2 (\tau_p / \tau_s + c_s \tau_s / c_p \tau_p + \sigma_s^2 - 2).$$

Summarizing, the formula for  $\gamma^2 \dot{\lambda}_0 = \gamma^2 \lambda_0 / (p - 1)$  follows. □

Inserting the formulas of the lemmas of this section into the general formula (50) for the subprincipal symbol of  $P$  we obtain a formula for the subprincipal symbol in the isotropic case.

**Proposition 25.** *Denote by  $X = (x_{jk})$  the  $2 \times 2$  matrix solving (54). Set*

$$N = \tau_s (\tau_p / \tau_s + c_s \tau_s / c_p \tau_p + \sigma_s^2 - 2).$$

*Let  $P$  be the operator of Lemma 20 determined by the unit section  $v$  of  $\ker z$  having positive  $\hat{\xi}$  component. The subprincipal symbol of  $P$  is given as follows.*

$$16 N p_{\text{sub}} = (c_r / 2\mu) (x_{11} (2 - \sigma_s^2)^2 + 4 x_{22} (1 - \sigma_s^2)^2 + 4 \operatorname{Im} x_{12} (2 - \sigma_s^2) \tau_s) + \mu^{-1} c_r^2 (2 - \sigma_s^2) (2 \tau_s \dot{\zeta}_3 - (2 - \sigma_s^2) \dot{\zeta}_2) \langle S \hat{\xi}, \hat{\xi} \rangle + 2 c_r (4 - \sigma_s^2)^{-1} (2 - \sigma_s^2) (5 \sigma_s^2 - 4 - \sigma_s^4) (\operatorname{tr}(S) - \langle S \hat{\xi}, \hat{\xi} \rangle).$$

*Proof.* On  $\Sigma$ ,  $w = \gamma v = (mt/2)(i(2-t)v + 2\sqrt{1-t}\hat{\xi})$ . Using Lemmas 22 and 24 we calculate  $16N \operatorname{Re}(z-w | w) / \gamma^2 \dot{\lambda}_0$ . The result is the first term on the right-hand side of the claimed formula. Similarly, we obtain the other terms combining the Lemmas 23 and 24. □

The constituents of the above formula for  $p_{\text{sub}}$  are curvature and velocities (Lamé parameters), assumed known. It seems difficult to analyze the formula further unless it is specialized to particular cases. However, it should be noted that the formula allows explicit numerical evaluation of  $p_{\text{sub}}$ . Therefore it can be used when solving transport equations for Rayleigh wave amplitudes numerically with a (seismic) ray tracing program, say. Formulas for the amplitudes of Rayleigh waves were given by in [Babich and Kirpichnikova 2004].

### Appendix: Geometric pseudodifferential calculus

Pseudodifferential operators on manifolds are usually introduced by reducing to the euclidean case via partitions of unity, [Hörmander 1985, 18.1; Zworski 2011]. The principal symbol of a pseudodifferential operator is invariantly defined. If the operator acts on sections of the line bundle of half-densities then there also is an invariantly defined subprincipal symbol [Hörmander 1985, Theorem 18.1.33; Sjöstrand and Zworski 2002, Appendix].

In the body of the paper we explicitly track, down to the subprincipal level, symbols of operators acting between vector bundles. To achieve this we use Sharafutdinov’s geometric pseudodifferential calculus [Sharafutdinov 2004; 2005]. The purpose of this appendix is to recall this calculus, presenting a semiclassical variant. Since we have to refer, in the main part of the present paper, to proofs of the calculus, we give a rather detailed presentation. The calculus depends on a symmetric connection of the manifold and on metric connections of the (Hermitian) bundles. We make the stronger assumption that the manifold is Riemannian and that the symmetric connection is the Levi-Civita connection. The important features of the calculus are a symbol isomorphism modulo order minus infinity, and complete symbol expansions for products and adjoints given solely in terms of geometric data. Using connections to develop a pseudodifferential calculus and to prove the existence of a complete symbol isomorphism was done earlier in [Widom 1980]. This was further developed by Pflaum [1998], who gave a convenient quantization map from symbols to operators. Sharafutdinov gave symbol expansions in terms of geometric data.

Let  $X$  a compact Riemannian manifold without boundary,  $\dim X = n$ . The exponential map,  $\exp$ , of the Levi-Civita connection defines a diffeomorphism,  $(x, v) \mapsto (x, y) = (x, \exp_x v)$ , between a neighborhood of the zero-section of the tangent bundle  $T = TX$  and a neighborhood of the diagonal in  $X^2$ . In the proofs of the propositions below we need the following properties of  $\exp$ . In local coordinates the exponential map satisfies

$$(\exp_x v)^i = x^i + v^i - \Gamma_{jk}^i(x)v^j v^k/2 + \mathcal{O}(|v|^3), \tag{60}$$

where  $\Gamma_{jk}^i$  denotes the Christoffel symbols. Normal coordinates centered at  $x$  satisfy  $(\exp_x v)^i = v^i$ . There exist  $0 < r < R < \text{inj}(x)$ , the injectivity radius of  $X$ , such that the equation

$$\exp_{\exp_x v} z = \exp_x w \tag{61}$$

defines, for every  $v \in T_x = T_x X$ ,  $|v| < R$ , a diffeomorphism  $w \mapsto z = z(x, v, w)$  from an open neighborhood of the origin, contained in  $\{|w| < R\} \subset T_x$ , onto the ball  $\{|z| < r\} \subset T_y$ ,  $y = \exp_x v$ . This map is used below to change variables of integration. Obviously,  $z(x, 0, w) = w$ . A computation in normal coordinates centered at  $x$  shows that

$$(z'_w)^{-1} z = w - v + \mathcal{O}((|v| + |w|)^3) \quad \text{as } v, w \rightarrow 0. \tag{62}$$

Recall, from Section 3, the notation for segments and for parallel transport maps. In local coordinates,

$$(\tau_{[\exp_x v \leftarrow x]}^{TX} w)^i = w^i - \Gamma_{jk}^i(x)w^j v^k + \mathcal{O}(|v|^2). \tag{63}$$

Let  $E \rightarrow X$  and  $F \rightarrow X$  be Hermitian vector bundles with metric connections. Recall from Section 3 the definition (24) of horizontal derivatives and the definition of vertical derivatives. A  $C^\infty$  section  $a$  of the bundle  $\pi^* \text{Hom}(E, F) \rightarrow T^* X$  is called a  $\text{Hom}(E, F)$ -valued symbol of order  $m \in \mathbb{R}$ ,  $a \in S^m = S^m(T^* X; \pi^* \text{Hom}(E, F))$ , if and only if for all nonnegative integers  $j$  and  $\ell$ ,

$$\sup_{x, \xi} (1 + |\xi|)^{j-m} |(v \nabla)^j (h \nabla)^\ell a(x, \xi)| < \infty.$$

These are the usual type  $(1, 0)$  symbol estimates. The symbol space  $S^m$  is a Fréchet space. The space  $S^{m,k} = S^{m,k}(T^* X; \pi^* \text{Hom}(E, F))$  of  $h$ -dependent symbols of order  $m$  and degree  $k$  is the Fréchet

space of families  $a_h \in S^m$  such that  $\{h^k a_h; 0 < h \leq 1\}$  is bounded in  $S^m$ . We call  $a \in S^{m,k}$  classical if there exists an asymptotic expansion  $a \sim \sum_j h^{j-k} a_j$  with  $h$ -independent symbols  $a_j \in S^{m-j}$ .

In the following lemma we define, in a semiclassical setting, the quantization of symbols according to Sharafutdinov’s geometric pseudodifferential calculus. We relate this definition of  $h$ -pseudodifferential operators to the definition in the euclidean situation. For semiclassical analysis, in particular, for the class  $\Psi^{m,k} = \text{Op}_h S^{m,k}$  of  $h$ -pseudodifferential operators, including mapping properties, and for frequency sets ( $h$ -wavefront sets), refer to [Gérard 1988; Ivrii 1998; Dimassi and Sjöstrand 1999; Sjöstrand and Zworski 2002; Zworski 2011]. The class of negligible operators,  $\Psi^{-\infty,-\infty}$ , consists of  $h$ -dependent operators whose Schwartz kernels are  $C^\infty$  with  $\mathcal{O}(h^\infty)$  seminorms.

Fix  $\chi_0 \in C^\infty(TX)$ , real-valued,  $|v| < r$  on the support of  $\chi_0(x, v)$ , such that  $\chi_0 = 1$  in a neighborhood of the zero-section in  $TX$ .

**Lemma 26.** *Let  $a_h \in S^{m,k}$  be a  $\text{Hom}(E, F)$ -valued symbol. Then*

$$A_h u_h(x) = (2\pi h)^{-n} \int_{T_x^*} \int_{T_x} e^{-i\langle \eta, v \rangle / h} \chi_0(x, v) \cdot a_h(x, \eta) \tau_{[x \leftarrow \exp_x v]}^E u_h(\exp_x v) \, dv \, d\eta, \tag{64}$$

defines an  $h$ -pseudodifferential operator  $A_h \in \Psi^{m,k}(X; E, F)$ . Given a point  $x$  there exists a geodesic ball  $U$  centered at  $x$ , and a symbol  $a_h^U \in S^{m,k}$  such that, for  $u_h$  compactly supported in  $U$ ,

$$A_h u_h(y) = (2\pi h)^{-n} \int_{T_x^*} \int_{T_x} e^{i\langle \theta, v-w \rangle / h} a_h^U(y, \theta) \tau_{[y \leftarrow y']}^E u_h(y') \, dw \, d\theta,$$

where  $y = \exp_x v$  and  $y' = \exp_x w$ . Moreover, at  $x$ ,  $a_h^U \equiv a_h$  modulo  $S^{m-2,k-2}$ . Every  $h$ -pseudodifferential operator is, modulo negligible operators, of the form (64).

The measures in (64) are the normalized Lebesgue measures of the euclidean spaces  $T_x$  and  $T_x^*$ .

*Proof.* We shall drop the subscript  $h$  from the notation. Fix  $x \in X$ . Let  $U$  denote a geodesic ball with center  $x$  and radius  $\leq R$ . In the following we assume that the support of  $u$  is a compact subset of  $U$ . In (64) we replace the variables  $x, v, \eta$  by  $y, z, \zeta$ . Next we change variables in the integral  $Au(y)$  such that the domain of integration does not depend on  $y$ . Set  $y = \exp_x v$ . Define  $z = z(x, v, w)$  by (61). Using the symplectic map  $(w, \vartheta) \mapsto (z, \zeta)$ ,  $\zeta = {}^t(z'_w)^{-1} \vartheta$ , we get

$$Au(y) = \int_{T_x} K(v, w) \tau_{[y \leftarrow \exp_x w]}^E u(\exp_x w) \, dw,$$

where the kernel  $K$  is given by

$$K(v, w) = (2\pi h)^{-n} \int_{T_x^*} e^{-i\varphi/h} \chi_0(y, z) a(y, \zeta) \, d\vartheta,$$

with  $\varphi = \langle \zeta, z \rangle = \langle \vartheta, (z'_w)^{-1} z \rangle$ . Since  $z = 0$  if and only if  $v = w$ , we have  $\varphi(v, w, \vartheta) = \langle \psi(v, w) \vartheta, w - v \rangle$ . Here  $\psi = \text{Id} + \mathcal{O}(|v|^2 + |w|^2)$  by (62). Decreasing the radius of  $U$  and making the linear change of variables  $\theta = \psi(v, w) \vartheta$ , we get

$$K(v, w) = (2\pi h)^{-n} \int_{T_x^*} e^{i\langle \theta, v-w \rangle / h} \chi_0(y, z) a(y, \zeta) J_1(v, w) \, d\theta,$$

with  $J_1(v, w) = 1 + \mathcal{O}(|v|^2 + |w|^2)$ . It follows that  $A$  restricted to  $U$  is an  $h$ -pseudodifferential operator of class  $\Psi^{m,k}$ . As it stands the symbol depends on  $v, \theta, w$ . Using the standard symbol reduction procedure we obtain  $a^U(\exp_x v, \theta)$ . Moreover, the asymptotic expansion implies that, as  $v \rightarrow 0$ ,  $a^U - a \in S^{m-2,k-2}$ .

Note that  $Au(y) = 0$  if the distance between  $y$  and  $\text{supp } u$  is  $> r$ . Using a partition of unity, we infer that the class of operators given by (64) equals the class of  $h$ -pseudodifferential operators with Schwartz kernels supported in small neighborhoods of the diagonal. □

Standard arguments show that up to a negligible operator  $A_h = \text{Op}_h(a_h)$  does not depend on the choice of the cutoff  $\chi_0$ . The space

$$\Psi^{m,k}(X; E, F) = \text{Op}_h S^{m,k} + \Psi^{-\infty, -\infty}.$$

is the space  $h$ -pseudodifferential order  $m$  and degree  $k$ . We denote the geometric symbol by  $\sigma_h(A_h) = a_h$ .

**Remark.** Let  $A_h = \text{Op}_h(a_h) \in \Psi^{0,0}$ . Then  $A_h$  is  $L^2$  bounded, uniformly in  $h$ . Assume, in addition, that  $a_h$  depends differentiably on  $h$  with  $\partial_h a_h \in S^{0,0}$ . Changing variables in (64) from  $\eta$  to  $\xi = \eta/h$ , we obtain  $A_{h_1} - A_{h_0} = \int_{h_0}^{h_1} h^{-1} \text{Op}_h(b_h) dh$ , where  $b_h \in S^{0,0}$ ,  $b_h(x, \eta) = h \partial_h a_h(x, \eta) + {}^v \nabla_\eta a_h(x, \eta)$ . This implies the following useful Lipschitz estimate:

$$\|A_{h_1} - A_{h_0}\|_{L^2 \rightarrow L^2} \leq Ch_0^{-1} |h_1 - h_0| \quad \text{if } h_0 < h_1,$$

where  $\|\text{Op}_h(b_h)\|_{L^2 \rightarrow L^2} \leq C < \infty$ . The assumption holds if  $a_h$  is classical and given as a Borel sum.

In the following, we often suppress from writing the  $h$ -dependence of symbols, operators and distributions. Moreover, when dealing with integrals like (64), we move, without explicitly writing this, the  $x$ -dependence from the domain of integration into the integrand using arguments as in the proof of the lemma.

Lebesgue measure  $dv$  on  $T_x X$  and Riemannian volume are related by

$$\int f(y) dV_X(y) = \int f(\exp_x v) J_0(x, v) dv,$$

where  $y = \exp_x v$  and  $J_0$  is the Jacobian, satisfying  $J_0 = 1 + \mathcal{O}(|v|^2)$  at  $v = 0$ . Let  $A = A_h$  be as in (64). The Schwartz kernel  $K_A$  of  $A$ ,

$$Au(x) = \int_X K_A(x, y) u(y) dV_X(y), \quad K_A(x, y) \in \text{Hom}(E_y, F_x).$$

equals in a neighborhood of the diagonal a partial Fourier transform of the symbol,

$$K_A(x, y) = (2\pi h)^{-n} \int_{T_x^*} e^{-i\langle \eta, \exp_x^{-1} y \rangle / h} a(x, \eta) d\eta \psi(x, y) \tau_{[x \leftarrow y]}^E. \tag{65}$$

Here  $\psi(x, y) = \chi_0(x, v) / J_0(x, v)$ ,  $y = \exp_x v$ . The symbol  $a$  is recovered via the inverse Fourier transform:

$$a(x, \xi) \equiv \int_{T_x} e^{i\langle \xi, v \rangle / h} (\chi_0 J_0)(x, v) K_A(x, \exp_x v) \tau_{[\exp_x v \leftarrow x]}^E dv \tag{66}$$



modulo  $S^{-\infty, -\infty}$ . The correspondence between an operator  $A = \text{Op}_h(a)$  and its full symbol  $a$ , named the geometric symbol of  $A$ , defines the complete symbol isomorphism

$$\Psi^{m,k}(X; E, F) / \Psi^{-\infty, -\infty} \cong S^{m,k}(T^*X; \text{Hom}(\pi^*E, \pi^*F)) / S^{-\infty, -\infty}.$$

The geometric symbol can also be computed by applying the operator to suitable testing functions:

$$a(x, \xi)s \equiv A_y(e^{i\langle \xi, \exp_x^{-1} y \rangle / h} \chi_0(x, \exp_x^{-1} y) \tau_{[y \leftarrow x]}^E s) |_{y=x}. \tag{67}$$

Here  $A_y$  means that  $A$  acts on functions of the variable  $y$ . In particular, in case  $E = \mathbb{C}$ , the geometric symbol is obtained at the center of normal coordinates  $x^j$  when  $A$  is applied to  $e^{i\xi_j x^j / h}$  and evaluated at  $x^j = 0$ .

We derive symbol properties and expansions using the method of stationary phase:

$$\begin{aligned} (\det(H/2\pi i h))^{1/2} \int e^{i\varphi(x)/h} a(x) dx &= \exp(2^{-1} i h \langle H^{-1} \partial, \partial \rangle) (e^{i\rho(x)/h} a(x)) |_{x=0} \\ &= \sum_{j < 3N} \frac{(i h)^j}{j! 2^j} \langle H^{-1} \partial, \partial \rangle^j (e^{i\rho(x)/h} a(x)) |_{x=0} + \mathcal{O}(h^N). \end{aligned}$$

Here  $\varphi \in C^\infty$  is real-valued,  $\varphi'(x) = 0$  if and only if  $x = 0$ ,  $H = \varphi''(0)$  is nonsingular, and  $\varphi(0) = 0$ . The remainder  $\rho(x) = \varphi(x) - \langle Hx, x \rangle / 2$  vanishes to third order at  $x = 0$ . The expansion has the advantage, when compared to that obtained using the Morse lemma, of giving an efficient algorithm for computing the asymptotic series.

See [Asada and Fujiwara 1978, Lemma 3.2] and [Hörmander 1990, Theorem 7.7.5], where the expansion is arranged in powers of  $\omega^{-1} = h$ .

We are mainly interested in the leading symbols of operators. We define the leading symbol of an operator  $\text{Op}_h(a) \in \Psi^{m,k}$  as the residue of  $a$  in  $S^{m,k} / S^{m-2, k-2}$ . The principal symbol is, of course, the residue in  $S^{m,k} / S^{m-1, k-1}$ .

**Proposition 27.** *Let  $A = \text{Op}_h(a)$  as in (64) with geometric symbol  $a \in S^{m,k}$ . The formal adjoint  $A^* \in \Psi^{m,k}(X; F, E)$  has the geometric symbol*

$$b \equiv a^* - i h \text{tr } {}^v \nabla^h \nabla a^* \pmod{S^{m-2, k-2}}. \tag{68}$$

If  $a$  is classical then so is  $b$ .

Notice that  ${}^v \nabla^h \nabla a^*$  is a section of  $\pi^*(\text{Hom}(F, E) \otimes T \otimes T^*)$ . The trace is taken of the  $T \otimes T^*$  part.

*Proof.* The formal adjoint of  $A$  is defined by

$$\int_X (u_1(x) | Au_2(x))_F dV_X(x) = \int_X (A^*u_1(y) | u_2(y))_E dV_X(y).$$

The Schwartz kernel satisfies  $K_{A^*}(x, y) = K_A(y, x)^*$ . Recall that parallel transport preserves inner products. It follows from (65) that

$$K_{A^*}(x, y) = (2\pi h)^{-n} \int_{T_y^*} e^{i\langle \eta, \exp_y^{-1} x \rangle / h} \tau_{[x \leftarrow y]}^E a(y, \eta)^* d\eta \psi(y, x),$$

and  $K_{A^*}(x, y) = 0$  if the distance between  $x$  and  $y$  is  $> r$ . Set  $y = \exp_x v$ . Define  $z \in T_y$  by  $\exp_y z = x$ . After a linear change variables from  $\eta \in T_y^*$  to  $\zeta = {}^t(\exp'_x(v))\eta \in T_x^*$  we have

$$K_{A^*}(x, y) = (2\pi h)^{-n} \int_{T_x^*} e^{i\langle \eta, z \rangle / h} \tau_{[x \leftarrow y]}^E a(y, \eta)^* d\zeta \psi(y, x) / J_1(x, v),$$

with Jacobian  $J_1(x, v) = 1 + \mathcal{O}(|v|^2)$ . Define

$$b(x, \xi) = \int_{T_x} e^{i\langle \xi, v \rangle / h} (\chi_0 J_0)(x, v) K_{A^*}(x, y) \tau_{[y \leftarrow x]}^F dv.$$

Inserting  $K_{A^*}$  we have

$$b(x, \xi) = (2\pi h)^{-n} \int_{T_x} \int_{T_x^*} e^{i\varphi/h} \tilde{a} J d\zeta dv, \tag{69}$$

where

$$\begin{aligned} \varphi &= \langle \xi, v \rangle + \langle \eta, z \rangle = -\langle \zeta - \xi, v \rangle + \langle \zeta, \Phi \rangle, \\ \tilde{a} &= \tau_{[x \leftarrow y]}^E a(y, \eta)^* \tau_{[y \leftarrow x]}^F = \tau_{[x \leftarrow y]}^{\text{Hom}(F, E)} a(y, \eta)^*, \\ J &= \chi_0(x, v) J_0(x, v) \psi(y, x) / J_1(x, v) = 1 + \mathcal{O}(|v|^2), \end{aligned}$$

and  $\Phi = \Phi(x, v) = \exp'_x(v)^{-1} z + v$ . A computation in normal coordinates centered at  $x$  shows that  $\Phi = \mathcal{O}(|v|^3)$  as  $v \rightarrow 0$ . If  $\varphi'_\xi = 0$  then  $z = 0$ , hence  $v = 0$ . It follows that the critical points of  $\varphi$  are defined by  $v = 0, \zeta = \xi$ .

Apply the method of stationary phase to (69) and deduce that  $b \in S^{m,k}$ . Moreover, the following asymptotic expansion holds:

$$b \sim \sum_j \frac{(ih)^j}{j!} \langle -\partial_\xi, \partial_v \rangle^j (e^{i\langle \zeta, \Phi \rangle / h} \tilde{a})|_{v=0, \zeta=\xi}. \tag{70}$$

Differentiation of the exponential factor brings out a nonzero factor only if it consumes at least three derivatives with respect to  $v$  and at most one derivative with respect to  $\zeta$ . It follows that the sum is asymptotic. Moreover,  $b$  is determined modulo  $S^{m-2, k-2}$  by the terms in the asymptotic sum with  $j < 2, b \equiv a^* - ih \langle \partial_\xi, \partial_v \rangle \tilde{a}$ . Observe that

$$\tau_{[x \leftarrow \exp_x v]}^T \circ \exp'_x(v) = \text{Id}_{T_x} + \mathcal{O}(|v|^2) \quad \text{as } v \rightarrow 0.$$

It follows that  $\partial_v \tilde{a}|_{v=0} = {}^h \nabla a^*(x, \xi)$ . Hence  $b \equiv a^* - ih \text{tr } {}^v \nabla {}^h \nabla a^*$ . The Schwartz kernels of  $\text{Op}_h(b)$  and  $A^*$  are equal in a neighborhood of the diagonal. Therefore  $A^* - B \in \Psi^{-\infty, -\infty}$ .  $\square$

**Proposition 28.** *Let  $A \in \Psi^{m_A, k_A}(X; F, G)$  and  $B \in \Psi^{m_B, k_B}(X; E, F)$  with geometric symbols  $a$  and  $b$ , respectively. Set  $k = k_A + k_B, m = m_A + m_B$ . Then  $AB \in \Psi^{m, k}(X; E, G)$  with geometric symbol*

$$c \equiv ab - ih \text{tr } ({}^v \nabla a \cdot {}^h \nabla b) \tag{71}$$

modulo  $S^{m-2, k-2}$ . If  $a$  and  $b$  are classical then so is  $c$ .

Again the trace is taken of the  $T \otimes T^*$  part, and the dot terminates differentiated expressions.

*Proof.* Setting  $y = \exp_x v$ , the operator  $C = AB$  is given by

$$Cu(x) = (2\pi h)^{-2n} \iiint_{T_x \times T_x^* \times T_y \times T_y^*} e^{-i(\langle \eta, v \rangle + \langle \xi, z \rangle)/h} a(x, \eta) \cdot \tau_{[x \leftarrow y]}^F(b(y, \zeta) \tau_{[y \leftarrow \exp_y z]}^E u(\exp_y z)) dz d\zeta dv d\eta.$$

Here and in the following we do not write the cutoff factors. Let  $z = z(x, v, w)$  be the solution of  $\exp_y z = \exp_x w$ . The symplectic change of variables  $(w, \vartheta) \mapsto (z, \zeta)$ ,  $\zeta = {}^t(z'_w)^{-1} \vartheta$ , preserves the volume form. We get  $Cu(x) = \int_{T_x} K_C(x, \exp_x w) u(\exp_x w) J_0(x, w) dw$ , with Schwartz kernel

$$K_C(x, \exp_x w) J_0(x, w) = (2\pi h)^{-2n} \int_{T_x^* \times T_x \times T_x^*} e^{-i(\langle \eta, v \rangle + \langle \xi, z \rangle)/h} c_0 d(\vartheta, v, \eta) \tau_{[x \leftarrow \exp_x w]}^E,$$

$c_0 = a(x, \eta) \tau_{[x \leftarrow y]}^{\text{Hom}(E, F)} b(y, \zeta) M(x, w, v)$ . Here  $M(x, w, v) \in GL(E_x)$  denotes the parallel transport in  $E$  along the geodesic triangle  $x \rightarrow \exp_x w \rightarrow \exp_x v \rightarrow x$ . It follows that the symbol of  $C$  equals

$$c(x, \xi) = (2\pi h)^{-2n} \int_{T_x \times T_x^* \times T_x \times T_x^*} e^{i\varphi/h} c_0 d(v, \eta, w, \vartheta), \tag{72}$$

$\varphi = \langle \xi, w \rangle - \langle \eta, v \rangle - \langle \zeta, z \rangle$ . We introduce  $w - v$  as a new variable,  $w$ . Then (72) holds with

$$\begin{aligned} \varphi &= -\langle \eta - \xi, v \rangle - \langle \vartheta - \xi, w \rangle + \langle \vartheta, \Phi \rangle, \\ c_0 &= a(x, \eta) \tau_{[x \leftarrow y]}^{\text{Hom}(E, F)} b(y, \zeta) M(x, w + v, v), \end{aligned}$$

Here  $\Phi = w - (z'_w(x, v, w + v))^{-1} z(x, v, w + v) \in T_x^*$ . By (62),  $\Phi$  vanishes to third order at  $v = w = 0$ . Clearly,  $v = 0 = z$  at a critical point of  $\varphi$ . It follows that  $v = w = 0$  and  $\eta = \vartheta = \xi$  define the critical points.

Now apply the method of stationary phase to (72) and deduce that  $c \in S^{m, k}$  is a symbol which, moreover, has an asymptotic expansion

$$c \sim \sum_j \frac{(-ih)^j}{j!} (\langle \partial_\vartheta, \partial_w \rangle + \langle \partial_\eta, \partial_v \rangle)^j (e^{i\langle \vartheta, \Phi \rangle/h} c_0) \Big|_{v=w=0, \eta=\vartheta=\xi}. \tag{73}$$

Using that  $\Phi$  does not depend on  $\eta$  and  $\vartheta$ , and vanishes to third order at  $v = w = 0$ , we infer that the summands with  $j > 1$  belong to  $S^{m-2, k-2}$ . It follows that

$$ab - ih \langle \partial_\eta a, \partial_v \tilde{b} M \rangle - iha \langle \partial_\vartheta, \partial_w \tilde{b} M \rangle,$$

evaluated at the critical point, is the leading symbol of  $C$ . Here  $\tilde{b} = \tau_{[x \leftarrow y]}^{\text{Hom}(E, F)} b(y, \zeta)$ . We have  $\partial_w \tilde{b} = 0$  at  $v = w = 0$ . This follows from  $\zeta'_w = 0$  which is a corollary of  $z = w$  at  $v = 0$ . The derivatives of  $M$  with respect to  $v$  and  $w$  vanish at  $v = w = 0$ . Using  $\tau_{[x \leftarrow \exp_x v]}^T \circ z'_w = \text{Id}_{T_x} + \mathcal{O}(|v|^2)$  at  $w = 0$ , we derive

$$\partial_v \tilde{b} = \partial_v \tau_{[x \leftarrow \exp_x v]}^{\text{Hom}(E, F)} b(\exp_x v, {}^t(z'_w)^{-1} \vartheta) = {}^h \nabla b(x, \vartheta),$$

at  $v = w = 0$ . Summarizing the computations, (71) follows. □

**Remark.** The proofs of Propositions 27 and 28 follow those in [Sharafutdinov 2004; 2005] closely with only minor modifications. Our derivation of the asymptotic expansions of the symbols of adjoints and products may be somewhat shorter, however. We differ in defining the adjoint with respect to the volume element rather than using half-densities. Notice that the symbol expansions (70) and (73) depend only on the given symbols and on the geometry. In the formulas (68) and (71), we extracted the leading symbols.

For the purposes of the present paper it suffices to assume  $X$  compact. A symbol calculus on general (complete) Riemannian manifolds needs to take the injectivity radius into account and handle mapping properties more explicitly.

It is well-known that a pseudodifferential operator acting on half-densities has an invariantly defined subprincipal symbol; see [Sjöstrand and Zworski 2002, Appendix] for a proof in the semiclassical case. We relate the subprincipal symbol to the leading geometric symbol. Equip the half-density bundle  $\Omega^{1/2} \rightarrow X$  with the inner product  $(u | v) = u \cdot \bar{v} / dV_X$ , where the operations on the right are in the sense of densities. The connection given by  $\nabla^{\Omega^{1/2}} dV_X^{1/2} = 0$  is metric with respect to the Hermitian structure of  $\Omega^{1/2}$ .

**Corollary 29.** *Let  $A \in \Psi^{m,k}(X; \Omega^{1/2})$ . The leading symbol of  $A$  equals that of the corresponding scalar operator  $\tilde{A} \in \Psi^{m,k}(X)$  which is given by  $\tilde{A}u = dV_X^{-1/2} A(u dV_X^{1/2})$ . If the geometric symbol  $a$  of  $A$  is classical,  $a \sim \sum_{j \geq 0} h^{j-k} a_j$ ,  $a_j \in S^{m-j}$ , then  $h^{-k} a_0$  is the principal symbol of  $A$ , and*

$$a_{\text{sub}} = h^{1-k} (a_1 + i \nabla a_0 \cdot \nabla a_0 / 2)$$

is its subprincipal symbol.

*Proof.* Consider the multiplication operator  $dV_X^{1/2} \in \Psi^{0,0}(X; \mathbb{C}, \Omega^{1/2})$ . The  $\text{Hom}(\mathbb{C}, \Omega^{1/2})$ -valued symbol  $\pi^* dV_X^{1/2}$  is the leading symbol of this operator. Note that its horizontal and vertical derivatives vanish. The equality of the leading symbols of  $A$  and  $\tilde{A}$  now follows from Proposition 28.

Let  $a^U$  denote the local symbol of  $A$  in a geodesic coordinate chart  $U$  centered at a given point  $x$ . We use normal coordinates centered at  $x$ . Assume  $a$  classical,  $h^k a = a_0 + h a_1 + \mathcal{O}(h^2)$ . Then  $a^U$  is classical, and  $h^k a^U = a_0 + \mathcal{O}(h)$ . Moreover, it follows from Lemma 26 that  $h^k a^U = a_0 + h a_1 + \mathcal{O}(h^2)$  at  $x$ . The subprincipal symbol equals, by definition,  $h^{1-k} (a_1 + 2^{-1} i \sum_j \partial^2 a_0 / \partial x_j \partial \xi_j)$ . The horizontal derivative in the  $j$ -th coordinate direction equals, at  $x$ , the partial derivative with respect to  $x_j$ . The formula for the subprincipal symbol follows. □

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SÖNKE HANSEN: [soenke@math.upb.de](mailto:soenke@math.upb.de)

*Institut für Mathematik, Universität Paderborn, 33095 Paderborn, Germany*

<http://www.math.upb.de/~soenke/>

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László Lempert	Purdue University, USA <a href="mailto:lempert@math.purdue.edu">lempert@math.purdue.edu</a>	Richard B. Melrose	Massachusetts Institute of Technology, USA <a href="mailto:rbm@math.mit.edu">rbm@math.mit.edu</a>
Frank Merle	Université de Cergy-Pontoise, France <a href="mailto:Frank.Merle@u-cergy.fr">Frank.Merle@u-cergy.fr</a>	William Minicozzi II	Johns Hopkins University, USA <a href="mailto:minicozz@math.jhu.edu">minicozz@math.jhu.edu</a>
Werner Müller	Universität Bonn, Germany <a href="mailto:mueller@math.uni-bonn.de">mueller@math.uni-bonn.de</a>	Yuval Peres	University of California, Berkeley, USA <a href="mailto:peres@stat.berkeley.edu">peres@stat.berkeley.edu</a>
Gilles Pisier	Texas A&M University, and Paris 6 <a href="mailto:pisier@math.tamu.edu">pisier@math.tamu.edu</a>	Tristan Rivière	ETH, Switzerland <a href="mailto:riviere@math.ethz.ch">riviere@math.ethz.ch</a>
Igor Rodnianski	Princeton University, USA <a href="mailto:irod@math.princeton.edu">irod@math.princeton.edu</a>	Wilhelm Schlag	University of Chicago, USA <a href="mailto:schlag@math.uchicago.edu">schlag@math.uchicago.edu</a>
Sylvia Serfaty	New York University, USA <a href="mailto:serfaty@cims.nyu.edu">serfaty@cims.nyu.edu</a>	Yum-Tong Siu	Harvard University, USA <a href="mailto:siu@math.harvard.edu">siu@math.harvard.edu</a>
Terence Tao	University of California, Los Angeles, USA <a href="mailto:tao@math.ucla.edu">tao@math.ucla.edu</a>	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA <a href="mailto:met@math.unc.edu">met@math.unc.edu</a>
Gunther Uhlmann	University of Washington, USA <a href="mailto:gunther@math.washington.edu">gunther@math.washington.edu</a>	András Vasy	Stanford University, USA <a href="mailto:andras@math.stanford.edu">andras@math.stanford.edu</a>
Dan Virgil Voiculescu	University of California, Berkeley, USA <a href="mailto:dvv@math.berkeley.edu">dvv@math.berkeley.edu</a>	Steven Zelditch	Northwestern University, USA <a href="mailto:zelditch@math.northwestern.edu">zelditch@math.northwestern.edu</a>

## PRODUCTION

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