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ENERGY IDENTITY FOR INTRINSICALLY BIHARMONIC MAPS IN FOUR DIMENSIONS

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Let u be a mapping from a bounded domain $S \subset \mathbb{R}^4$ into a compact Riemannian manifold N. Its intrinsic biharmonic energy $E_2(u)$ is given by the squared L^2-norm of the intrinsic Hessian of u. We consider weakly converging sequences of critical points of E_2. Our main result is that the energy dissipation along such a sequence is fully due to energy concentration on a finite set and that the dissipated energy equals a sum over the energies of finitely many entire critical points of E_2.

1. Introduction and main result

Let $S \subset \mathbb{R}^4$ be a bounded Lipschitz domain and let N be a compact Riemannian manifold without boundary. For convenience we assume that N is embedded in $\mathbb{R}^n$ for some $n \geq 2$. We denote the second fundamental form of this embedding by A and we denote the Riemannian curvature tensor of N by R. For $u \in C^\infty(S, N)$ define the pull-back vector bundle $u^{-1}TN$ in the usual way and denote the norm on it and on related bundles by $|\cdot|$. Together with the Levi-Cività connection on the tangent bundle TN, the mapping u induces a covariant derivative ∇^u on $u^{-1}TN$. We extend this covariant derivative to tensor fields in the usual way. Denote by π_N the nearest point projection from a neighborhood of N onto N and set $P_u(x) = D\pi_N(u(x))$. Then $P_u(x)$ is the orthogonal projection from $\mathbb{R}^4$ onto the tangent space $T_{u(x)}N$ to N at $u(x)$. Let $X \in L^2(S, \mathbb{R}^n)$ be a section of $u^{-1}TN$. Following [Moser 2008] we define

$$\nabla^u X = (P_u \partial_\alpha X) \otimes dx^\alpha$$

Denote the derivative of u by $Du = (\partial_\alpha u) \otimes dx^\alpha$. The intrinsic Hessian $\nabla^u Du$ is a section of $(TS)^* \otimes (TS)^* \otimes u^{-1}TN$. By a standard fact about $D\pi_N$, it is given by

$$\begin{aligned}\nabla^u Du &= (P_u \partial_\alpha \partial_\beta u) \otimes dx^\alpha \otimes dx^\beta \\ &= \big(\partial_\alpha \partial_\beta u + A(u)(\partial_\alpha u, \partial_\beta u)\big) \otimes dx^\alpha \otimes dx^\beta.\end{aligned}$$

We define the Sobolev spaces

$$W^{k,p}(S, N) = \{u \in W^{k,p}(S, \mathbb{R}^n) : u(x) \in N \text{ for almost all } x \in S\}$$

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and we introduce the energy functional $E_2 : W^{2,2}(S, N) \to \mathbb{R}_+$ given by

$$E_2(u) = \frac{1}{4}\int_S |\nabla^u Du|^2.$$

Critical points of E_2 are called intrinsically biharmonic mappings. There are also other kinds of second order functionals whose critical points are called "biharmonic" mappings. The functional E_2 is defined intrinsically, that is, it does not depend on the embedding of N into $\mathbb{R}^n$. Another intrinsically defined second order functional that is naturally associated with u is $F_2(u) = \frac{1}{4}\int_S |\tau(u)|^2$, where $\tau(u) := \operatorname{trace} \nabla^u Du$ denotes the tension field of u. Critical points of F_2 are usually called intrinsically biharmonic mappings. Another functional that can be associated with u is the energy $\tilde{E}_2(u) = \frac{1}{4}\int_S |D^2 u|^2$. Its critical points are usually called extrinsically biharmonic mappings. The functional $\tilde{E}_2$ enjoys better analytical properties than E_2 and F_2, but it has the drawback of depending on the particular embedding of N into $\mathbb{R}^n$.

Biharmonic mappings, being the next higher order equivalent of harmonic mappings, have attracted a lot of attention in the differential geometry literature; see [Montaldo and Oniciuc 2006] for an overview. Analytic aspects of the problem are less well understood, and on questions other than regularity (see [Chang et al. 1999; Wang 2004b; Wang 2004a; Wang 2004c; Lamm and Rivière 2008; Struwe 2008]) not much work has been done. This is the case in particular for intrinsic biharmonic mappings, because the problem is difficult due to a lack of coercivity of the corresponding functions in the Sobolev spaces traditionally used. Thus despite the fact that the intrinsic case is geometrically more interesting, the problem has not widely been studied from the analysis point of view.

Recent progress has been made, however, based on the observation that the lack of coercivity can be removed for one type of intrinsic biharmonic mappings (the type studied in the present paper), provided that one works in a geometrically motivated variant of Sobolev spaces [Moser 2008; Scheven 2009]. This approach permits methods analogous to what has been used for harmonic mappings. But since we have a fourth order equation for biharmonic mappings (in contrast to second order for harmonic mappings), and since we have to work in different spaces, such an approach still requires additional ideas and arguments. In this paper, we develop the theory a step further.

The existence of minimizers of E_2 under given boundary conditions on the mapping itself and on its first derivatives was established in [Moser 2008] using the direct method of the calculus of variations. For simplicity, from now on we will omit the adverb "intrinsically":

In the present paper, a mapping $u \in W^{2,2}(S, N)$ will be called *biharmonic* if it is critical for E_2 under outer variations, that is,

$$\frac{d}{dt}\Big|_{t=0} E_2(\pi_N(u + t\phi)) = 0 \quad \text{for all } \phi \in C_0^\infty(S, \mathbb{R}^n);$$

see [Scheven 2009; Moser 2008]. In [Scheven 2009] it is shown that a mapping $u \in W^{2,2}(S, N)$ is biharmonic precisely if it satisfies

$$\int_S \nabla_\alpha \partial_\beta u \cdot \big(\nabla_\alpha \nabla_\beta \phi + R(u)(\phi, \partial_\alpha u)\partial_\beta u\big) = 0 \tag{1}$$

for every section $\phi \in W_0^{2,2}(S, \mathbb{R}^n) \cap L^\infty(S, \mathbb{R}^n)$ of $u^{-1}TN$.

We will study sequences of biharmonic mappings $(u_k) \subset W^{2,2}(S, N)$ with uniformly bounded energy, that is, $\limsup_{k\to\infty} E_2(u_k) < \infty$. Since our results are analogous to known facts about harmonic mappings, we describe the situation encountered in that context: Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain in $\mathbb{R}^2$. A mapping $u \in W^{1,2}(\Omega, N)$ is said to be (weakly) harmonic if it is a critical point for the Dirichlet energy

$$E_1(u) = \frac{1}{2}\int_\Omega |Du|^2.$$

A given sequence $(u_k) \subset W^{1,2}(\Omega, N)$ of harmonic mappings with uniformly bounded Dirichlet energy has a subsequence that converges weakly in $W^{1,2}$ to some mapping $u \in W^{1,2}(\Omega, N)$. This convergence in general fails to be strong, that is, in general $\liminf_{k\to\infty} E_1(u_k) > E_1(u)$. The only reason for this loss is that the energy can concentrate on a lower dimensional subset $\Sigma_0 \subset \Omega$. In particular, $u_k \to u$ in $C^1_{\mathrm{loc}}(\Omega \setminus \Sigma_0, \mathbb{R}^n)$. By the results in [Hélein 1991; Hélein 1990], the mappings u_k and u are smooth. In addition, the set Σ_0 is finite. Moreover, for each point $x \in \Sigma_0$ there exist $M_x \in \mathbb{N}$ and entire harmonic mappings $v^x_1, \dots, v^x_{M_x} \in C^\infty(\mathbb{R}^2, N)$ such that, after passing to a subsequence,

$$\lim_{k\to\infty}\int_\Omega |Du_k|^2 \geq \int_\Omega |Du|^2 + \sum_{x\in\Sigma_0}\sum_{j=1}^{M_x}\int_{\mathbb{R}^2} |v^x_j|^2.$$

Later the converse inequality was shown to hold as well [Jost 1991; Parker 1996; Ding and Tian 1995]. Our main result is the analogue of these facts for critical points of the functional E_2. It is summarized in the following theorem:

Theorem 1.1. *Let $S \subset \mathbb{R}^4$ be a bounded Lipschitz domain and let N be a smooth compact manifold without boundary embedded in $\mathbb{R}^n$. Let $(u_k) \subset W^{2,2}(S, N)$ be a sequence of biharmonic mappings and assume that*

$$\limsup_{k\to\infty}\int_S |\nabla^{u_k} Du_k|^2 + |Du_k|^4 < \infty. \tag{2}$$

Then $u_k \in C^\infty(S, N)$ and we may pass to a subsequence in k (again called (u_k)) and find a biharmonic map $u \in C^\infty(S, N)$ and a finite set $\Sigma_0 \subset S$ such that

(i) $u_k \rightharpoonup u$ *weakly in* $(W^{2,2} \cap W^{1,4})(S, \mathbb{R}^n)$,

(ii) $u_k \to u$ *in* $C^2_{\mathrm{loc}}(S \setminus \Sigma_0, \mathbb{R}^n)$.

Moreover, for each $x \in \Sigma_0$ there exist $M_x \in \mathbb{N}$ and biharmonic mappings $v^x_1, \dots, v^x_{M_x} \in C^\infty(\mathbb{R}^4, N)$ such that

$$\begin{aligned}\lim_{k\to\infty}\int_S |\nabla^{u_k} Du_k|^2 &= \int_S |\nabla^u Du|^2 + \sum_{x\in\Sigma_0}\sum_{j=1}^{M_x}\int_{\mathbb{R}^4} |\nabla^{v^x_j} Dv^x_j|^2,\\ \lim_{k\to\infty}\int_S |Du_k|^4 &= \int_S |Du|^4 + \sum_{x\in\Sigma_0}\sum_{j=1}^{M_x}\int_{\mathbb{R}^4} |Dv^x_j|^4.\end{aligned} \tag{3}$$

Remarks. (i) By [Moser 2008, Theorem 2.1] the hypothesis (2) is equivalent to the seemingly weaker hypothesis $\limsup_{k\to\infty}\int_S |\nabla^{u_k} Du_k|^2 + |Du_k|^2 < \infty$ and also to the seemingly stronger hypothesis

$$\limsup_{k\to\infty} \|u_k\|_{W^{2,2}(S,N)} < \infty.$$

(ii) Moser [2008] showed that every biharmonic mapping $v \in W^{2,2}(S, N)$ in fact satisfies $v \in C^\infty(S, N)$.

(iii) To obtain smoothness of the limiting mapping u as well, one needs a removability result for isolated singularities of biharmonic mappings. This is derived in Lemma 2.3 below. Another auxiliary result is the existence of a uniform lower bound on the energy of entire nonconstant biharmonic mappings, given in Lemma 2.6 below. Analogues of these facts are well known for harmonic mappings and also for critical points of other higher order functionals; see for example [Wang 2004b].

(iv) The main contribution of Theorem 1.1 are the energy identities of (3). To obtain an equality (and not just a lower bound for the left hand sides), one has to show that no energy concentrates in a "neck" region around a concentration point $x \in \Sigma_0$. This is proven in Section 3 below. Similar results are known in the context of harmonic mappings; see for example [Jost 1991; Parker 1996; Ding and Tian 1995; Lin and Rivière 2002]. They are also known for other kinds of biharmonic mappings, but only if the target manifold is a round sphere, since then the Euler–Lagrange equations enjoy a special structure [Wang 2004b]. Under the general hypotheses of Theorem 1.1 no such structure seems available, so a different approach is needed.

Notation. By $e_1, \dots, e_4$ we denote the standard basis of $\mathbb{R}^4$. We also set $e_r(x) = x/|x|$ for all $x \in \mathbb{R}^4$. By $B_r(x)$ we denote the open ball in $\mathbb{R}^4$ with center x and radius r. We set $B_r = B_r(0)$. If A and B are tensors of the same type, then $A \cdot B$ denotes their scalar product. We will often write ∇Du instead of $\nabla^u Du$, and we identify $\mathbb{R}^k$ with its dual $(\mathbb{R}^k)^*$, writing, for example, e_α instead of dx^α.

2. Proof of Theorem 1.1

We define the energy densities

$$e_1(u) = |Du|^4 \quad \text{and} \quad e_2(u) = |\nabla Du|^2.$$

(These should not be confused with the unit vectors in $\mathbb{R}^4$.) We also set $e(u) = e_1(u) + e_2(u)$. For $U \subset S$ we define $\mathscr{E}_i(u; U) = \int_U e_i(u)$, where $i = 1, 2$, and we define $\mathscr{E}(u; U) = \mathscr{E}_1(u; U) + \mathscr{E}_2(u; U)$.

Theorem 1.1 is a consequence of the following two propositions.

Proposition 2.1. *There exists an $\varepsilon_1 > 0$ such that the following holds: Let $(u_k) \subset W^{2,2}(S, N)$ be a sequence of biharmonic mappings (so $u_k \in C^\infty(S, N)$) and assume that $u \in W^{2,2}(S, N)$ is such that*

$$u_k \rightharpoonup u \quad \textit{weakly in } (W^{2,2} \cap W^{1,4})(S, \mathbb{R}^n). \tag{4}$$

Define

$$\Sigma_0 = \{x \in S : \liminf_{k\to\infty} \mathscr{E}(u_k; B_r(x)) \geq \varepsilon_1/2 \textit{ for all } r > 0\}.$$

Then $u \in C^\infty(S, N)$ *is biharmonic and* $u_k \to u$ *in* $C^2_{\mathrm{loc}}(S \setminus \Sigma_0, N)$. *Moreover, there exist functions* θ_1, $\theta_2 : \Sigma_0 \to (0, \infty)$ *such that* $\theta_1(x) \geq \varepsilon_1$ *for all* $x \in \Sigma_0$ *and*

$$\mathscr{L}^4 \lfloor e_i(u_k) \overset{*}{\rightharpoonup} \mathscr{L}^4 \lfloor e_i(u) + \sum_{x\in\Sigma_0} \theta_i(x)\delta_{\{x\}} \quad \text{for } i = 1, 2 \tag{5}$$

weakly-∗ in the dual space of $C^0_0(S)$.

Remarks. (i) By Remark (i) following Theorem 1.1, the hypothesis (2) implies (4) for a subsequence.

(ii) The measures $\sum_{x\in\Sigma_0} \theta_i(x)\delta_{\{x\}}$ are called defect measures. Their common support Σ_0 is empty if and only if the convergence (4) is strong. In that case the last sum in (5) is defined to be zero.

Proposition 2.2. *Let* u_k, u, Σ_0 *and* θ_i *be as in Proposition 2.1. Then, for each* $x \in \Sigma_0$, *there exists* $M_x \in \mathbb{N}$ *and biharmonic mappings* $v^x_1, \dots, v^x_{M_x} \in C^\infty(\mathbb{R}^4, N)$ *such that* $\theta_i(x) = \sum_{j=1}^{M_x} \mathscr{E}_i(v^x_j; \mathbb{R}^4)$. *In particular*,

$$\lim_{k\to\infty} \mathscr{E}_i(u_k; S) = \mathscr{E}_i(u; S) + \sum_{x\in\Sigma_0}\sum_{j=1}^{M_x} \mathscr{E}_i(v^x_j; \mathbb{R}^4) \quad \text{for } i = 1, 2.$$

For the proof of Proposition 2.1 we need three auxiliary results. The following lemma is a simple consequence of [Moser 2008, Theorem 2.1]:

Lemma 2.1. *There exists a universal constant* C *such that the following holds*: *Let* $r > 0$, *let* $u \in W^{2,2}(B_r, N)$ *and let* $X \in L^2(B_r, \mathbb{R}^n)$ *be a section of* $u^{-1}TN$. *If* $\nabla^u X \in L^2(B_r)$ *then* $X \in L^4(B_r)$, *and*

$$\|X\|_{L^4(B_r)} \leq C(\|\nabla^u X\|_{L^2(B_r)} + r^{-1}\|X\|_{L^2(B_r)}).$$

For $u \in C^k$ we introduce the notation $[u]_{C^k}(x) = \sum_{j=1}^k |D^j u(x)|^{1/j}$. An obvious consequence of [Scheven 2009, Lemma 5.3] is the following:

Lemma 2.2. *There exists* $\varepsilon_1 > 0$ *such that, for all* $r > 0$ *and for all biharmonic* $u \in C^\infty(B_r, N)$ *satisfying*

$$\int_{B_r} |Du|^4 \leq \varepsilon_1 \quad \text{we have} \quad \sup_{x\in B_{r/2}} |x|[u]_{C^3}(x) \leq 1.$$

The following lemma shows that isolated singularities of biharmonic mappings are removable.

Lemma 2.3. *Let* $\Sigma \subset S$ *be finite and let* $u \in W^{2,2}(S, N)$ *be biharmonic on* $S \setminus \Sigma$. *Then* u *is biharmonic on* S. *In particular*, $u \in C^\infty(S, N)$.

Proof. This proof closely follows that of [Jost 2005, Lemma 8.5.3]. We assume without loss of generality that $S = B_1$ and that $\Sigma = \{0\}$. Then (1) is equivalent to

$$\int_{B_1} \nabla_\alpha\partial_\beta u \cdot \nabla_\alpha\nabla_\beta\phi = \int_{B_1} f(u, Du\otimes Du\otimes D^2u)\cdot\phi \tag{6}$$

for some $\mathbb{R}^n$-valued mapping f that is smooth in the first argument and linear in the second argument. Since u is biharmonic on $B_1 \setminus \{0\}$, Equation (6) is satisfied for all $\phi \in (L^\infty \cap W^{2,2}_0)(B_1 \setminus \{0\}, \mathbb{R}^n)$ that are sections of $u^{-1}TN$. From the properties of f we deduce that

$$|f(u, Du\otimes Du\otimes D^2u)| \leq C(|D^2u|^2 + |Du|^4). \tag{7}$$

Hence $f(u, Du \otimes Du \otimes D^2u) \in L^1(B_1, \mathbb{R}^n)$. For small $R \in (0, 1)$ we set

$$\tau_R(t) = \begin{cases} 0 & \text{for } t \in [0, R^2], \\ 1 - \log(t/R)/|\log R| & \text{for } t \in [R^2, R], \\ 1 & \text{for } t \in [R, 1). \end{cases}$$

One readily checks that

$$\lim_{R\to 0} \int_{B_1} |D^2\tau_R(|x|)|^2 + |D\tau_R(|x|)|^4 \, dx = 0. \tag{8}$$

Now let $\phi \in (L^\infty \cap W^{2,2})(B, \mathbb{R}^n)$ be a section of $u^{-1}TN$. Then, for all $R \in (0, 1)$,

$$\phi_R(x) = \tau(|x|)\phi(x)$$

is still a section of $u^{-1}TN$, and $\phi_R \in (L^\infty \cap W_0^{2,2})(B_1 \setminus \{0\}, \mathbb{R}^n)$. Hence it is an admissible test function for (6). Using (7) and (8) it is easy to check that (6) holds for all ϕ as above, that is, u is biharmonic. Since $u \in W^{2,2}(S, N)$, Remark (ii) to Theorem 1.1 implies that $u \in C^\infty(S, N)$. $\square$

Proof of Proposition 2.1. Clearly (4) implies $\limsup_{k\to\infty} \mathscr{E}(u_k; S) < \infty$. Hence Σ_0 is finite whatever the choice of ε_1. We choose ε_1 as in the statement of Lemma 2.2. Then the Arzèla–Ascoli theorem implies that $u_k \to u$ in $C^2_{\text{loc}}(S \setminus \Sigma_0, N)$. Hence u is biharmonic on $S \setminus \Sigma_0$. Lemma 2.3 therefore implies that $u \in C^\infty(S, N)$ and that u is biharmonic on S.

Weak lower semicontinuity of the L^2-norm and (4) imply the existence of (positive) Radon measures μ_1 and μ_2 on S such that

$$\mathscr{L}^4 \lfloor e_i(u_k) \overset{*}{\rightharpoonup} \mathscr{L}^4 \lfloor e_i(u) + \mu_i \quad \text{for } i = 1, 2. \tag{9}$$

We claim that

$$\mu_1(\{x\}) \geq \varepsilon_1 \quad \text{for all } x \in \operatorname{spt} \mu_1. \tag{10}$$

In fact, let $x \in S$ be such that $\mu_1(\{x\}) < \varepsilon_1$. Then by (9) there exists $r > 0$ such that

$$\limsup_{k\to\infty} \int_{B_r(x)} e_1(u_k) \leq \int_{B_r(x)} e_1(u) + \mu_1(\overline{B}_r(x)) < \varepsilon_1.$$

Thus $u_k \to u$ in $C^2(B_{r/2}(x))$ by Lemma 2.2 and the Arzèla–Ascoli theorem. (First only for a subsequence, but all subsequences must converge to the same limit u because $u_k \rightharpoonup u$ in $W^{2,2}(S, \mathbb{R}^n)$.) Thus $\mu_1(B_r/2(x)) = 0$, so $x \notin \operatorname{spt} \mu_1$. This proves (10), which in turn implies that $\operatorname{spt} \mu_1$ is finite and that $\mu_1 = \sum_{x \in \operatorname{spt} \mu_1} \theta_1(x)\delta_{\{x\}}$ for a function $\theta_1 : \operatorname{spt} \mu_1 \to [\varepsilon_1, \infty)$.

If $x \notin \operatorname{spt} \mu_1$, then (9) implies that

$$\inf_{r>0} \lim_{k\to\infty} \int_{B_r(x)} e_1(u_k) = \inf_{r>0} \int_{B_r(x)} e_1(u) = 0. \tag{11}$$

On the other hand, if $x \in \operatorname{spt} \mu_1$ then there exists $r > 0$ such that $B_{2r}(x) \cap \operatorname{spt} \mu_1 = \{x\}$ because $\operatorname{spt} \mu_1$ is finite. Thus $\mu(\partial B_r(x)) = 0$, and so (9) implies

$$\lim_{k\to\infty} \int_{B_r(x)} e_1(u_k) = \int_{B_r(x)} e_1(u) + \mu_1(\{x\}).$$

We conclude that

$$\inf_{r>0}\lim_{k\to\infty}\int_{B_r(x)} e_1(u_k)=\mu_1(\{x\}) \quad \text{for all } x\in S. \tag{12}$$

Now (12) together with (10) imply that $\operatorname{spt}\mu_1\subset\Sigma_0$. On the other hand, if $x\notin\operatorname{spt}\mu_1$ then (11) and Lemma 2.2 imply that there is $r>0$ such that $u_k\to u$ on $C^2(B_r(x),N)$; hence $x\notin\operatorname{spt}\mu_2$ and $x\notin\Sigma_0$. Thus $\operatorname{spt}\mu_2\subset\operatorname{spt}\mu_1=\Sigma_0$. It remains to check that $\operatorname{spt}\mu_1\subset\operatorname{spt}\mu_2$. But (9) implies that, for $r\in(0,\operatorname{dist}_{\partial S}(x))$,

$$\limsup_{k\to\infty}\int_{B_r(x)}\Big(\frac{|Du_k|^2}{r^2}+e_2(u_k)\Big)\le\int_{B_r(x)}\Big(\frac{|Du|^2}{r^2}+e_2(u)\Big)+\mu_2(\overline{B}_r(x)), \tag{13}$$

because by Sobolev embedding we have $Du_k\to Du$ strongly in L^2. If $x\notin\operatorname{spt}\mu_2$, then the infimum over $r>0$ of the right side of (13) is zero, since $Du\in L^4$. Hence Lemma 2.1 implies that $x\notin\Sigma_0$. □

For the proof of Proposition 2.2 we will need the following three lemmas:

Lemma 2.4. *There exists a modulus of continuity ω (that is, $\omega\in C^0([0,\infty))$ is nondecreasing and $\omega(0)=0$) such that, whenever $r>0$ and $u\in W^{2,2}(B_r,N)$ is biharmonic, then*

$$\operatorname{dist}_{\partial B_r}(x)[u]_{C^3}(x)\le\omega\Big(\int_{B_r}|Du|^4\Big) \quad \textit{for all } x\in B_r.$$

Proof. Notice that $u\in C^\infty(B_r,N)$ by Remark (ii) to Theorem 1.1. The claim follows from a scaled version of [Scheven 2009, Lemma 5.3] and from the fact that, by Jensen's inequality,

$$\Big(\rho^{-2}\int_{B_\rho(a)}|Du|^2\Big)^2\le\int_{B_\rho(a)}|Du|^4. \qquad \square$$

We will also need the following crucial estimate.

Lemma 2.5. *There exists a constant C_3 such that the following holds: For all $R\in(0,3/8)$ and for all biharmonic $u\in C^\infty(B_1,N)$ satisfying*

$$\varepsilon:=\sup_{\rho\in(R,1/2)}\mathscr{E}(u;B_{2\rho}\setminus B_\rho)\le C_3^{-1}$$

we have

$$\mathscr{E}(u;B_1\setminus B_R)\le C_3\omega(\varepsilon)+2\varepsilon. \tag{14}$$

Here, ω is as in the conclusion of Lemma 2.4.

The proof of Lemma 2.5 will be given in Section 3.

Finally, we will need the existence of a uniform lower bound on the energy of nonconstant entire biharmonic mappings. An analogous fact is well known for harmonic mappings and also for other kinds of biharmonic mappings; see for example [Wang 2004b].

Lemma 2.6. *There exists a constant $\alpha>0$ such that $\mathscr{E}(u;\mathbb{R}^4)\ge\alpha$ for every nonconstant biharmonic mapping $u\in C^\infty(\mathbb{R}^4,N)$.*

Proof. If the claim were false then there would exist nonconstant biharmonic $u_m \in C^\infty(\mathbb{R}^4, N)$ such that $\lim_{m\to\infty} \mathscr{E}(u_m; \mathbb{R}^4) = 0$. After passing to a subsequence we have $Du_m \to 0$ pointwise almost everywhere. Therefore, since u_m is nonconstant and since Du_m is continuous, there exist $x_m \in \mathbb{R}^4$ such that $r_m := |Du_m(x_m)|$ are nonzero but $\lim_{m\to\infty} r_m = 0$. Define $\tilde{u}_m(x) = u_m(x_m + x/r_m)$. Then $\mathscr{E}(\tilde{u}_m; \mathbb{R}^4) = \mathscr{E}(u_m; \mathbb{R}^4)$ converges to zero as $m \to \infty$. By Lemma 2.2 this implies the existence of a constant mapping u such that $\tilde{u}_m \to u$ in $C^2_{\text{loc}}(\mathbb{R}^4, N)$. But on the other hand, $|D\tilde{u}_m(0)| = 1$ for all m, so $|Du(0)| = 1$. This contradiction finishes the proof. $\square$

Proof of Proposition 2.2. By Proposition 2.1 we have u_k, $u \in C^\infty(S, N)$. Since the case $\Sigma_0 = \varnothing$ is trivial, we assume that Σ_0 is nonempty. After translating, rescaling (the energy $\mathscr{E}$ is scaling invariant) and restricting, we may assume that $\Sigma_0 = \{0\}$ and that $S = B_1$. By Proposition 2.1 we have $u_k \rightharpoonup u$ weakly in $(W^{2,2} \cap W^{1,4})(B_1, \mathbb{R}^n)$ and $u_k \to u$ in $C^2_{\text{loc}}(B_1 \setminus \{0\}, N)$. Moreover, there is some

$$\theta \geq \varepsilon_1 \tag{15}$$

such that

$$\mathscr{L}^4 \lfloor e(u_k) \overset{*}{\rightharpoonup} \mathscr{L}^4 \lfloor e(u) + \theta\delta_{\{0\}}. \tag{16}$$

Let $\varepsilon \in (0, 1)$ be such that $C_3\omega(\varepsilon) + 3\varepsilon \leq \min\{\alpha/4, \varepsilon_1/4\}$, where ω is as in Lemma 2.4, C_3 is as in Lemma 2.5 and ε_1 is as in Lemma 2.2. Since $u \in W^{2,2}(B_1, \mathbb{R}^n)$, there exists $Q \in (0, 1)$ such that

$$\int_{B_Q} e(u) \leq \varepsilon/2. \tag{17}$$

We claim that there exists a sequence $R_k \to 0$ such that, for all k large enough,

$$\mathscr{E}(u_k; B_{2\rho} \setminus B_\rho) \leq \varepsilon \quad \text{for all } \rho \in [R_k, Q/2], \tag{18}$$

$$\mathscr{E}(u_k; B_{2R_k} \setminus B_{R_k}) = \varepsilon. \tag{19}$$

In fact, set

$$\mathscr{R}_k = \{r \in (0, Q/2) : \mathscr{E}(u_k; B_{2r} \setminus B_r) > \varepsilon\}.$$

If infinitely many of the $\mathscr{R}_k$ were empty, Lemma 2.5 would imply that there exists $k_i \to \infty$ such that $\mathscr{E}(u_{k_i}; B_Q \setminus B_{r_i}) \leq C_3\omega(\varepsilon) + 2\varepsilon$ for any sequence $r_i \to 0$. Choosing this sequence in such a way that $\mathscr{E}(u_{k_i}; B_{r_i}) \leq \varepsilon$ for all i, we would conclude that $\mathscr{E}(u_{k_i}; B_Q) \leq C_3\omega(\varepsilon) + 3\varepsilon \leq \varepsilon_1/4$, contradicting (15).

Thus, for k large, $\mathscr{R}_k \neq \varnothing$ and we can define $R_k = \sup \mathscr{R}_k$. Clearly $R_k > 0$ because $\int_{B_{2r}\setminus B_r} e(u_k) \leq \int_{B_{2r}} e(u_k) \to 0$ as $r \to 0$. On the other hand, $R_k \to 0$, since otherwise $\rho = \frac{1}{2} \liminf_{k\to\infty} R_k$ is positive, so

$$\limsup_{k\to 0} \int_{B_{2R_k}\setminus B_{R_k}} e(u_k) \leq \lim_{k\to 0} \int_{B_Q\setminus B_\rho} e(u_k) = \int_{B_Q\setminus B_\rho} e(u) \leq \varepsilon/2$$

by (17). This contradicts the fact that R_k is contained in the closure of $\mathscr{R}_k$, which by continuity of $r \mapsto \int_{B_{2r}\setminus B_r} e(u_k)$ implies that $\int_{B_{2R_k}\setminus B_{R_k}} e(u_k) \geq \varepsilon$. This also proves (19). Then (18) follows from the definition of R_k.

Combining (18) with (a scaled version of) Lemma 2.5, we conclude that

$$\mathscr{E}(u_k;\, B_Q \setminus B_{R_k}) \le C_3\omega(\varepsilon) + 2\varepsilon \le \alpha/4. \tag{20}$$

Set $v_k(x) = u_k(R_k x)$. Then by (16)

$$\limsup_{k\to\infty} \mathscr{E}(v_k;\, B_R) = \limsup_{k\to\infty} \mathscr{E}(u_k;\, B_{RR_k}) \le \inf_{\rho>0} \limsup_{k\to\infty} \mathscr{E}(u_k;\, B_\rho) = \theta \tag{21}$$

for all $R > 0$. Set

$$\Sigma^{(1)} = \{x \in \mathbb{R}^4 : \liminf_{k\to\infty} \mathscr{E}(v_k;\, B_r(x)) \ge \varepsilon_1/2 \ \text{ for all } r > 0\}.$$

By (21) we can apply Proposition 2.1 to each B_R. We conclude that $\Sigma^{(1)}$ is locally finite and that there exists a biharmonic mapping $v \in C^\infty(\mathbb{R}^4, N)$ such that, after passing to a subsequence, $v_k \rightharpoonup v$ weakly in $(W^{1,4}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}})(\mathbb{R}^4, \mathbb{R}^n)$ and

$$v_k \to v \quad \text{in } \ C^2_{\mathrm{loc}}(\mathbb{R}^4 \setminus \Sigma^{(1)}, \mathbb{R}^n), \tag{22}$$

and we find that there are a functions $\theta_1^{(1)}, \theta_2^{(1)} : \Sigma^{(1)} \to (0, \infty)$ such that

$$\mathscr{L}^4 \lfloor e_i(v_k) \overset{*}{\rightharpoonup} \mathscr{L}^4 \lfloor e_i(v) + \sum_{x\in\Sigma^{(1)}} \theta_i^{(1)}(x)\delta_{\{x\}} \quad \text{for } \ i = 1, 2. \tag{23}$$

On the other hand, the bound (20) implies that

$$\limsup_{k\to\infty} \mathscr{E}(v_k;\, B_R \setminus \overline{B}_1) \le C_3\omega(\varepsilon) + 2\varepsilon \quad \text{for all } \ R > 1.$$

Thus $\Sigma^{(1)} \subset \overline{B}_1$ (so $\Sigma^{(1)}$ is finite) and therefore

$$v_k \to v \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^4 \setminus \overline{B}_1, \mathbb{R}^n)$$

by (22). From this and since $\mathscr{E}(v_k;\, B_2 \setminus \overline{B}_1) = \mathscr{E}(u_k;\, B_{2R_k} \setminus \overline{B}_{R_k}) = \varepsilon$ for all k by (19), we conclude that $\mathscr{E}(v;\, \mathbb{R}^4) \ge \varepsilon$. Hence Lemma 2.6 implies that $\mathscr{E}(v;\, \mathbb{R}^4) \ge \alpha$.

Claim #1. *For all $\eta > 0$, there exist $R > 1$ and $\rho \in (0, 1)$ such that*

$$\liminf_{k\to\infty} \mathscr{E}(u_k;\, B_\rho \setminus B_{RR_k}) \le \eta.$$

To prove this claim, let us first show that for all $\delta > 0$ there exist R and ρ and a sequence $k_i \to \infty$ such that

$$\mathscr{E}(u_{k_i};\, B_{2r} \setminus B_r) \le \delta \quad \text{for all } i \in \mathbb{N} \text{ and all } r \in [RR_{k_i}, \rho/2]. \tag{24}$$

In fact, assume that this were not the case. Then there would exist $\delta \in (0, \varepsilon)$ such that for all R and ρ, the set

$$\hat{\mathscr{R}}_k = \{r \in [RR_k, \rho/2] : \mathscr{E}(u_k;\, B_{2r} \setminus B_r) > \delta\}$$

is nonempty for all k large enough. We choose $R > 2$ so large and $\rho \in (0, Q)$ so small that

$$\mathscr{E}(v;\, B_{4\hat{R}} \setminus B_{\hat{R}/2}) \le \delta/4 \quad \text{for all } \hat{R} \ge R, \text{ and} \tag{25}$$

$$\mathscr{E}(u;\, B_\rho) \le \delta/4. \tag{26}$$

This is clearly possible because $e(v) \in L^1(\mathbb{R}^4)$. Let $\hat{R}_k = \sup \hat{\mathscr{R}}_k$, hence $\hat{R}_k \in [RR_k, \rho/2]$. Arguing as above for R_k, using (26) one readily checks that $\hat{R}_k \to 0$. We claim that

$$\hat{R}_k / R_k \to \infty. \tag{27}$$

Indeed, if this were not the case then (after passing to a subsequence) there would exist $\hat{R} \in [R, \infty)$ such that $\hat{R}_k/R_k \in [\hat{R}/2, 2\hat{R}]$ for k large enough. Thus by the definition of $\hat{R}_k$ and since $\hat{R} \ge R > 2$ and $\Sigma^{(1)} \subset \overline{B}_1$,

$$\delta \le \limsup_{k\to\infty} \mathscr{E}(u_k;\, B_{2\hat{R}_k} \setminus B_{\hat{R}_k}) \le \limsup_{k\to\infty} \mathscr{E}(v_k;\, B_{4\hat{R}} \setminus B_{\hat{R}/2}).$$

This contradiction to (25) shows that (27) must be true.

Now define $\hat{v}_k(x) = u_k(\hat{R}_k x)$. As done above for R_k and v_k, using the fact that $\delta \le \varepsilon$, one shows that there exists a nontrivial biharmonic mapping $\hat{v} \in C^\infty(\mathbb{R}^4, N)$ such that, after passing to a subsequence, $\hat{v}_k \rightharpoonup v$ in $(W^{2,2}_{\mathrm{loc}} \cap W^{1,4}_{\mathrm{loc}})(\mathbb{R}^4, \mathbb{R}^n)$. Since $\hat{v}$ is nontrivial, Lemma 2.6 implies that $\mathscr{E}(\hat{v}; \mathbb{R}^4) \ge \alpha$. Hence by (27) and since $\hat{R}_k \to 0$, for all $\hat{R} > 1$ we have

$$\begin{aligned}
\liminf_{k\to\infty} \mathscr{E}(u_k;\, B_\rho \setminus B_{RR_k}) &\ge \liminf_{k\to\infty} \mathscr{E}(u_k;\, B_{\hat{R}\hat{R}_k} \setminus B_{RR_k}) \\
&= \liminf_{k\to\infty} \mathscr{E}(\hat{v}_k;\, B_{\hat{R}} \setminus B_{R(R_k/\hat{R}_k)}) \\
&\ge \sup_{r>0} \liminf_{k\to\infty} \mathscr{E}(\hat{v}_k;\, B_{\hat{R}} \setminus B_r) \ge \mathscr{E}(\hat{v};\, B_{\hat{R}})
\end{aligned}$$

because $\hat{v}_k \rightharpoonup \hat{v}$ on $B_{\hat{R}}$. Taking the supremum over all $\hat{R} > 1$ and recalling that $\mathscr{E}(\hat{v}; \mathbb{R}^4) \ge \alpha$, we conclude that $\liminf_{k\to\infty} \mathscr{E}(u_k; B_\rho \setminus B_{RR_k}) \ge \alpha$. This contradiction to (20) concludes the proof of (24).

Combining Lemma 2.5 with (24) and choosing δ small enough shows that Claim #1 is true.

The results obtained so far apply to any $\theta > 0$. Now we argue by induction: Assume that $m \in \mathbb{N}$ is such that $\theta \in ((m-1)\alpha, m\alpha]$. If $m \ge 2$ then assume, in addition, that Proposition 2.2 is true for all $\theta \in (0, (m-1)\alpha]$. On one hand, for $i = 1, 2$, for all $R \in (1, \infty)$ and for all $\rho \in (0, 1)$ we have

$$\begin{aligned}
\theta_i + \mathscr{E}_i(u;\, B_\rho) = \lim_{k\to\infty} \big(\mathscr{E}_i(u_k;\, B_\rho \setminus B_{RR_k}) + \mathscr{E}_i(u_k;\, B_{RR_k})\big) &\ge \lim_{k\to\infty} \mathscr{E}_i(v_k;\, B_R) \\
&= \mathscr{E}_i(v;\, B_R) + \sum_{x\in\Sigma^{(1)}} \theta_i^{(1)}(x).
\end{aligned}$$

(First we used (5) and that $\mu_i(\partial B_\rho) = 0$ for all $\rho \in (0, 1)$, and then we used (23) together with the fact that $\Sigma^{(1)} \subset \overline{B}_1$.) Taking $\rho \to 0$ and $R \to \infty$ we conclude

$$\theta_i \ge \mathscr{E}_i(v;\, \mathbb{R}^4) + \sum_{x\in\Sigma^{(1)}} \theta_i^{(1)}(x) \quad \text{for both } i = 1, 2. \tag{28}$$

Hence

$$\theta \geq \mathscr{E}(v;\mathbb{R}^4) + \sum_{x\in\Sigma^{(1)}} \theta^{(1)}(x). \tag{29}$$

Since $\mathscr{E}(v;\mathbb{R}^4) \geq \alpha$ this implies that $\theta^{(1)}(x) \leq \theta - \alpha$ for all $x \in \Sigma^{(1)}$. If $m \geq 2$ we can thus apply the induction hypothesis to conclude that

$$\theta_i^{(1)}(x) = \sum_{j=1}^{M_x} \mathscr{E}_i(v_x^j;\mathbb{R}^4) \quad \text{for both } i = 1, 2. \tag{30}$$

Here $v_x^1, \dots, v_x^{M_x} \in C^\infty(\mathbb{R}^4, N)$ are biharmonic and $M_x \in (0, m-1]$ is a natural number. (If $m = 1$, then (29) implies that $\Sigma^{(1)} = \varnothing$ and that $\theta = \alpha = \mathscr{E}(v;\mathbb{R}^4)$. This concludes the proof of the case $m = 1$.)

On the other hand, for all $\rho \in (0, 1)$ and all $R > 1$,

$$\begin{aligned}\theta &\leq \lim_{k\to\infty}\big(\mathscr{E}(u_k; B_\rho \setminus B_{RR_k}) + \mathscr{E}(u_k; B_{RR_k})\big)\\ &\leq \liminf_{k\to\infty} \mathscr{E}(u_k; B_\rho \setminus B_{RR_k}) + \lim_{k\to\infty} \mathscr{E}(v_k; B_R)\\ &= \liminf_{k\to\infty} \mathscr{E}(u_k; B_\rho \setminus B_{RR_k}) + \mathscr{E}(v; B_R) + \sum_{x\in\Sigma^{(1)}} \theta^{(1)}(x)\delta_{\{x\}}.\end{aligned} \tag{31}$$

We used that $\Sigma^{(1)} \subset \overline{B}_1$, so $\lim_{k\to\infty} \mathscr{E}(v_k; B_R) = \mathscr{E}(v; B_R) + \sum_{x\in\Sigma^{(1)}} \theta^{(1)}(x)\delta_{\{x\}}$. Now let $\rho \to 0$ and $R \to \infty$ in (31) using Claim #1. We conclude that $\theta \leq \mathscr{E}(v;\mathbb{R}^4) + \sum_{x\in\Sigma^{(1)}} \theta^{(1)}(x)$. Thus by (29) and (30),

$$\theta = \mathscr{E}(v;\mathbb{R}^4) + \sum_{x\in\Sigma^{(1)}} \sum_{j=1}^{M_x} \mathscr{E}(v_x^j;\mathbb{R}^4).$$

Combining this with the inequalities (28) immediately implies that

$$\theta_i = \mathscr{E}(v;\mathbb{R}^4) + \sum_{x\in\Sigma^{(1)}} \sum_{j=1}^{M_x} \mathscr{E}_i(v_x^j;\mathbb{R}^4) \quad \text{for both } i = 1, 2. \qquad \square$$

3. Energy estimates on the "neck" region

The purpose of this section is to prove the following proposition.

Proposition 3.1. *There exists a constant C_1 such that the following holds: For all $R \in (0, 1/2)$ and for all biharmonic $u \in C^\infty(B_1, N)$ satisfying*

$$\varepsilon := \sup_{x\in B_1\setminus\overline{B}_R} |x|[u]_{C^3}(x) < 1, \tag{32}$$

we have

$$\int_{B_1\setminus B_R} |\nabla^u Du|^2 \leq C_1(\varepsilon + \mathscr{E}(u; B_1 \setminus B_R))\varepsilon. \tag{33}$$

Corollary 3.1. *There exists a constant C_2 such that the following holds: For all $R \in (0, 1/2)$ and for all biharmonic $u \in C^\infty(B_1, N)$ satisfying* (32), *we have*

$$\int_{B_1 \setminus B_R} \frac{|Du|^2}{|x|^2} \le C_2(\varepsilon + \mathscr{E}(u; B_1 \setminus B_R))\varepsilon. \tag{34}$$

If, in addition, $\varepsilon \le 1/(2(C_1 + C_2))$, then

$$\mathscr{E}(u; B_1 \setminus B_R) \le 2(C_1 + C_2)\varepsilon^2. \tag{35}$$

Proof. Set $\varepsilon = \sup_{x \in B_1 \setminus \overline{B}_R} |x|[u]_{C^3}(x)$. By (33) and by (63) from Lemma 5.2, we have

$$\int_{B_1 \setminus \overline{B}_R} \frac{|Du|^2}{|x|^2} \le C_1(\varepsilon + \mathscr{E}(u; B_1 \setminus \overline{B}_R))\varepsilon + 2\mathscr{H}^3(\partial B_1)\varepsilon^2.$$

This implies (34) because $\varepsilon < 1$. We clearly have

$$\int_{B_1 \setminus \overline{B}_R} |Du|^4 \le \varepsilon^2 \int_{B_1 \setminus \overline{B}_R} \frac{|Du|^2}{|x|^2}.$$

Thus (34) implies that

$$\int_{B_1 \setminus \overline{B}_R} |Du|^4 \le C_2(\varepsilon + \mathscr{E}(u; B_1 \setminus \overline{B}_R))\varepsilon^3.$$

Adding this to (33) yields

$$\mathscr{E}(u; B_1 \setminus \overline{B}_R) \le (C_1 + C_2)\varepsilon^2 + (C_1 + C_2)\mathscr{E}(u; B_1 \setminus \overline{B}_R)\varepsilon,$$

because $\varepsilon < 1$. Since $\varepsilon \le 1/(2(C_1 + C_2))$, we can absorb the second term into the left hand side. This yields (35). □

As a consequence of Corollary 3.1 we obtain Lemma 2.5:

Proof of Lemma 2.5. Set $\varepsilon = \sup_{\rho \in (R, 1/2)} \mathscr{E}(u; B_{2\rho} \setminus B_\rho)$. We claim that

$$|x|[u]_{C^3}(x) \le 4\omega(\varepsilon) \quad \text{for all } x \in B_{1/2} \setminus \overline{B}_{4R/3}. \tag{36}$$

In fact, let $x \in B_{1/2} \setminus \overline{B}_{4R/3}$ and apply Lemma 2.4 to the ball $B_{|x|/4}(x)$. This yields

$$\operatorname{dist}_{\partial B_{|x|/4}(x)}(x)[u]_{C^3}(x) \le \omega\Big(\int_{B_{|x|/4}(x)} |Du|^4\Big).$$

Since $B_{|x|/4}(x) \subset B_{3|x|/2} \setminus \overline{B}_{3|x|/4}$, this implies (36).

Applying (35) (with $B_{1/2}$ instead of B_1 and $B_{4R/3}$ instead of B_R) to (36) implies

$$\mathscr{E}(u; B_{1/2} \setminus B_{4R/3}) \le C\omega^2(\varepsilon) \tag{37}$$

for some constant C, provided that ε is small enough (since then $\omega(\varepsilon)$ is small, and so $|x|[u]_{C^3}(x)$ is small by (36)). Finally, note that by definition of ε we have $\mathscr{E}(u; B_1 \setminus B_{1/2}) + \mathscr{E}(u; B_{2R} \setminus B_R) \le 2\varepsilon$. Together with (37) and smallness of $\omega(\varepsilon)$ this implies (14). □

The rest of this section will be devoted to the proof of Proposition 3.1. We will use the notation

$$\partial_r u = e_r^\alpha \partial_\alpha u, \quad D_r u = \partial_r u \otimes e_r, \quad D_{S^3} u = Du - D_r u, \quad D^2 u = (\partial_\alpha \partial_\beta u) \otimes e_\alpha \otimes e_\beta.$$

Above and in what follows we tacitly sum over repeated indices. A short calculation shows that

$$D_{S^3} u = \big(|x| \partial_{\partial_\alpha e_r} u\big) \otimes e_\alpha. \tag{38}$$

Proof of Proposition 3.1. Since $u \in C^\infty(B_1, N)$, [Scheven 2009, Lemma 4.2] implies that (1) is equivalent to

$$\Delta^2 u = -\partial_\alpha E_\alpha[u] + G[u], \tag{39}$$

where $E_\alpha[u] = -\partial_\beta(A(u)(\partial_\alpha u, \partial_\beta u)) + F_\alpha[u]$, and $F_\alpha[u] : S \to (\mathbb{R}^4)^* \otimes \mathbb{R}^n$ and $G[u] : S \to \mathbb{R}^n$ are as in [Scheven 2009, Lemma 4.2], that is, $F_\alpha[u] = f_\alpha(u, \nabla Du \otimes Du)$ for functions f_α that are smooth in the first and linear in the second argument, and $G[u] = g_1(u, \nabla Du \otimes \nabla Du) + g_2(u, \nabla Du \otimes Du \otimes Du)$ for functions g_1 and g_2 that again are smooth in the first and linear in the second argument. Therefore,

$$|G[u]| \le C(|D^2 u|^2 + |Du|^4), \tag{40}$$

$$|E_\alpha[u]| \le C(|D^2 u||Du| + |Du|^3). \tag{41}$$

For $r_1 < r_2$ define the open annulus $A(r_1, r_2) = B_{r_2} \setminus \overline{B}_{r_1}$ and set $A = A(R, 1)$. (This should not be confused with the second fundamental form of N.) As we will show at the end of this proof, we may assume without loss of generality that $R = 2^{-L}$ for some integer $L > 1$.

Define $R_k = 2^k R$ and set $A_k = A(R_k, R_{k+1})$. Set

$$\varepsilon = \sup_{x \in B_1 \setminus \overline{B}_R} |x| [u]_{C^3}(x). \tag{42}$$

Following an idea used in [Sacks and Uhlenbeck 1981] and [Ding and Tian 1995] in the context of harmonic mappings, we introduce the unique radial mapping $q : A \to \mathbb{R}^n$ solving the following boundary value problem for all $k = 0, \dots, L$:

$$\Delta^2 q = 0 \quad \text{on } A_k, \tag{43}$$

$$q(R_k) = \frac{1}{\mathcal{H}^3(\partial B_{R_k})} \int_{\partial B_{R_k}} u \quad \text{and} \quad q'(R_k) = \frac{1}{\mathcal{H}^3(\partial B_{R_k})} \int_{\partial B_{R_k}} \partial_r u. \tag{44}$$

(For a radial function of the form $q(x) = \tilde{q}(|x|)$, we often write q instead of $\tilde{q}$.) Notice that q is indeed well and uniquely defined on each A_k by (43) and (44) because (43) is simply a fourth order ordinary differential equation on (R_k, R_{k+1}), since q is radial. (See Lemma 5.1 below for details.) The rest of this proof is divided into Lemma 3.1 and Lemma 3.2 below. Combining their conclusions one obtains that of Proposition 3.1.

Let us finally check that the case of arbitrary $R \in (0, 1)$ follows from the case when $R = 2^{-L}$. In fact, for general R let L be such that $2^L R \in [\frac{1}{2}, 1)$. The definition of ε implies that

$$\int_{A(2^L R, 1)} |\nabla Du|^2 \le \varepsilon^2 \int_{A(2^L R, 1)} |x|^{-4} \le \varepsilon^2 \mathcal{H}^3(\partial B_1) \log 2.$$

Applying Proposition 3.1 with $B_{2^L R}$ instead of B_1, the estimate (33) follows. □

Lemma 3.1. *For u, q and R as in the proof of Proposition 3.1 we have*

$$\int_A |D^2(u-q)|^2 \le C\Big(\varepsilon + \int_A |\nabla^u Du|^2 + |Du|^4\Big)\varepsilon, \tag{45}$$

$$\int_A \frac{|D(u-q)|^2}{|x|^2} \le C\Big(\varepsilon + \int_A |\nabla^u Du|^2 + |Du|^4\Big)\varepsilon. \tag{46}$$

Proof. Since $q|_{A_k}$ is a solution of a linear ordinary differential equation with smooth coefficients, it is C^∞ up to the boundary of A_k. Moreover, for $r \in (R_k, R_{k+1})$, by Lemma 5.1 there exists a universal constant C such that

$$|q'(r)| \le C(|q'(R_k)| + |q'(R_{k+1})| + R_k^{-1}|q(R_{k+1}) - q(R_k)|). \tag{47}$$

By (44) and by (42) this implies that $|u(x) - q(R_k)| \le \|Du\|_{L^\infty(\partial B_{R_k})} \cdot \operatorname{diam}(\partial B_{R_k})$ for all $x \in \partial B_{R_k}$ and all k. Therefore,

$$|q(R_{k+1}) - q(R_k)| \le \|Du\|_{L^\infty(A_k)} \operatorname{diam} A_k \le C\varepsilon \tag{48}$$

by (42) and because $\operatorname{diam} A_k \le C R_k$. Since $|x|$ is comparable to R_k on A_k and since k was arbitrary, we conclude from (47) and (48) and from (44) and (42) that $|x||Dq(x)| \le C\varepsilon$ for all $x \in A$. By (44) and by (42) this implies that $|u-q| \le C\varepsilon$. Summarizing, we have shown that

$$|(u-q)(x)| + |x||D(u-q)(x)| \le C\varepsilon \quad \text{for all } x \in A. \tag{49}$$

Notice that while (44) implies that $q \in C^1(A, \mathbb{R}^n)$ and that $q|_{A_k} \in C^\infty(\bar{A}_k, \mathbb{R}^n)$ for all k, in general $q \notin C^2(A; \mathbb{R}^n)$.

By partial integration one obtains, for arbitrary $v \in C^2(\bar{A}_k, \mathbb{R}^n)$,

$$\int_{A_k} |D^2 v|^2 = \int_{A_k} (\partial_\alpha \partial_\beta v) \cdot (\partial_\alpha \partial_\beta v) = \int_{A_k} (\Delta^2 v) \cdot v + \Big[\int_{\partial A_k} (\partial_r \partial_\beta v) \cdot \partial_\beta v - (\partial_r \Delta v) \cdot v\Big]_{r=R_k}^{R_{k+1}}.$$

Here and below we use the notation

$$\big[f(r)\big]_{r=t_1}^{t_2} := f(t_2) - f(t_1)$$

for functions $f \in C^0([t_1, t_2])$. Inserting $v = u - q$ and summing over $k = 0, \dots, L$ yields

$$\begin{aligned}
\int_A |D^2(u-q)|^2 &= \int_A (\Delta^2 u) \cdot (u-q) \\
&\quad + \sum_{k=0}^{L} \Big[\int_{\partial B_\rho} (\partial_r \partial_\beta (u-q)) \cdot \partial_\beta (u-q) - (\partial_r \Delta (u-q)) \cdot (u-q)\Big]_{\rho=R_k}^{R_{k+1}} \\
&= \int_A (\Delta^2 u) \cdot (u-q) + \Big[\int_{\partial B_\rho} \partial_r \partial_\beta u \cdot \partial_\beta (u-q) - \partial_r \Delta u \cdot (u-q)\Big]_{\rho=R}^{1} \\
&\quad - \sum_{k=0}^{L} \Big[\int_{\partial B_\rho} (\partial_r \partial_r q)(\rho) \cdot \partial_r (u-q)(x) - (\partial_r \Delta q)(\rho) \cdot (u-q)(x)\, d\mathcal{H}^3(x)\Big]_{\rho=R_k}^{R_{k+1}}.
\end{aligned} \tag{50}$$

In the first step we used that $\Delta^2 q = 0$ on A_k. In the last step we used that the boundary integrals with continuous integrands cancel successively, and we used that q is radial. Since q is radial, the same is true for $\partial_r\partial_r q$ and $\partial_r\Delta q$; see (60). The choice of boundary conditions (44) implies that

$$(\partial_r\partial_r q)(\rho)\cdot\int_{\partial B_\rho}\partial_r(u-q)(x)\,d\mathcal{H}^3(x)=0 \quad\text{and}\quad (\partial_r\Delta q)(\rho)\cdot\int_{\partial B_\rho}(u-q)(x)\,d\mathcal{H}^3(x)=0$$

for all $\rho\in\{R_0, R_1, \dots, R_L\}$. So the sum in the last term in (50) is zero. (The discontinuous expressions $q''=\partial_r\partial_r q$ and q''' occurring in $\partial_r\Delta q$ must be understood in the trace sense: If ∂B_{R_k} belongs to ∂A_k then $q''(R_k)=\lim_{r\uparrow R_k} q''(r)$ and if ∂B_{R_k} belongs to ∂A_{k+1} then $q''(R_k)=\lim_{r\downarrow R_k} q''(r)$. These limits exists because, as noted above, $q|_{A_k}$ is smooth up to the boundary of A_k.)

To estimate the second term in (50) we use (49) and (42). This gives

$$\int_{\partial B_r}|\partial_r\partial_\beta u||\partial_\beta(u-q)|\le C\mathcal{H}^3(\partial B_r)\frac{\varepsilon}{r^2}\frac{\varepsilon}{r}\le C\varepsilon^2.$$

Similarly, $\int_{\partial B_r}|\partial_r\Delta u||u-q|\le C\varepsilon^2$. Thus (50) implies

$$\int_A|D^2(u-q)|^2\le\left|\int_A(\Delta^2 u)\cdot(u-q)\right|+C\varepsilon^2. \tag{51}$$

To estimate the term $\left|\int_A(\Delta^2 u)\cdot(u-q)\right|$ in (51), we use (39) to replace $\Delta^2 u$. We obtain

$$\begin{aligned}\int_A(\Delta^2 u)\cdot(u-q)&=\int_A(-\partial_\alpha E_\alpha[u])\cdot(u-q)+G[u]\cdot(u-q)\\&=\int_A E_\alpha[u]\cdot\partial_\alpha(u-q)+\int_A G[u]\cdot(u-q)-\left[\int_{\partial B_r}\frac{x_\alpha}{|x|}E_\alpha[u]\cdot(u-q)\right]_{r=R}^1.\end{aligned} \tag{52}$$

To estimate the last term in (52) we simply use that $|E_\alpha[u]|\le|D^2u||Du|+|Du|^3\le C\varepsilon^2/|x|^3$ pointwise by (41). Thus

$$\int_{\partial B_r}|E_\alpha[u]||u-q|\le C\varepsilon^3\mathcal{H}^3(\partial B_r)r^{-3}\le C\varepsilon^3$$

for both $r=1$ and $r=R$.

To estimate the second term in (52), we use (40) and (49) to find

$$\int_A|G[u]||u-q|\le C\varepsilon\int_A(|D^2u|^2+|Du|^4).$$

To estimate the first term in (52) notice that by (41) and by (49) we have

$$\int_A|E_\alpha[u]||D(u-q)|\le C\varepsilon\int_A|D^2u|\frac{|Du|}{|x|}+\frac{|Du|^3}{|x|}\le C\varepsilon\int_A\Big(|D^2u|^2+|Du|^4+\frac{|Du|^2}{|x|^2}\Big). \tag{53}$$

Applying Lemma 5.2 to $v=u$ with $r_1=R$ and $r_2=1$, we have

$$\int_A\frac{|Du|^2}{|x|^2}\le\int_A|D^2u|^2+\left[\frac{1}{r}\int_{\partial B_r}|Du|^2\right]_{r=R}^1.$$

The boundary terms can be estimated as above using the definition of ε, Thus

$$\int_A \frac{|Du|^2}{|x|^2} \le \int_A |D^2u|^2 + C\varepsilon^2.$$

So (53) implies

$$\int_A |E_\alpha[u]||D(u-q)| \le C\varepsilon\Big(\varepsilon^2 + \int_A |D^2u|^2 + |Du|^4\Big).$$

Since $|D^2u|^2 \le C(N)(|\nabla Du|^2 + |Du|^4)$ for some constant $C(N)$ depending only on the immersion $N \hookrightarrow \mathbb{R}^n$, this concludes the proof of (45).

To prove (46) we apply Lemma 5.2 to each restriction $(u-q)|_{A_k}$. This yields

$$\int_{A_k} \frac{|D(u-q)|^2}{|x|^2} \le \int_{A_k} |D^2(u-q)|^2 + \left[\frac{1}{r}\int_{\partial B_r} |D(u-q)|^2\right]_{r=R_k}^{R_{k+1}}.$$

When we sum over $k = 0, \dots, L$, the terms in square brackets cancel successively because $D(u-q)$ is continuous. After estimating the boundary terms on ∂B_1 and on ∂B_R using (42), this yields (46). □

Lemma 3.2. *For u, q and R as in the proof of Proposition 3.1 we have*

$$\int_{A(R,1)} |D^2(u-q)|^2 \ge \Big(\frac{1}{2} - \frac{\sqrt{2}}{3}\Big)\int_{A(R,1)} |\nabla^u Du|^2 - C\Big(\varepsilon + \int_A |\nabla^u Du|^2 + |Du|^4\Big)\varepsilon.$$

Proof. For $v \in C^\infty(S, \mathbb{R}^n)$ we have

$$D^2 v = DD_{S^3}v + DD_r v,$$

where $D_{S^3}v = Dv - D_r v$. Thus

$$|D^2v|^2 \ge |DD_{S^3}v|^2 + 2D(Dv - D_r v)\cdot DD_r v. \tag{54}$$

Now $D(Dv - D_r v)\cdot DD_r v$ equals

$$\begin{aligned}
&\partial_\alpha((\partial_\beta v)\otimes(e_\beta - e_r^\beta e_r))\cdot\partial_\alpha(\partial_\gamma v\otimes e_r^\gamma e_r)\\
&\quad= \big((\partial_\alpha\partial_\beta v)\otimes(e_\beta - e_r^\beta e_r) - (\partial_\beta v)\otimes\partial_\alpha(e_r^\beta e_r)\big)\cdot\big((\partial_\alpha\partial_\gamma v)\otimes e_r^\gamma e_r + (\partial_\gamma v)\otimes\partial_\alpha(e_r^\gamma e_r)\big)\\
&\quad= \big((\partial_\alpha\partial_\beta v)\otimes(e_\beta - e_r^\beta e_r)\big)\cdot\big((\partial_\gamma v)\otimes\partial_\alpha(e_r^\gamma e_r)\big)\\
&\qquad\qquad - |(\partial_\beta v)\otimes\partial_\alpha(e_r^\beta e_r)|^2 - (\partial_\beta v)\otimes\partial_\alpha(e_r^\beta e_r)\cdot(\partial_\alpha\partial_\gamma v)\otimes e_r^\gamma e_r\\
&\quad= \big((\partial_\alpha\partial_\beta v)\otimes(e_\beta - e_r^\beta e_r)\big)\cdot\big((\partial_r v)\otimes(\partial_\alpha e_r)\big)\\
&\qquad\qquad - |(\partial_\beta v)\otimes\partial_\alpha(e_r^\beta e_r)|^2 - (\partial_\beta v)\otimes(\partial_\alpha e_r^\beta)e_r\cdot(\partial_\alpha\partial_\gamma v)\otimes e_r^\gamma e_r\\
&\quad= (\partial_{\partial_\alpha e_r}\partial_\alpha v)\cdot(\partial_r v) - |\partial_\beta v|^2\,|\partial_\alpha(e_r^\beta e_r)|^2 - \partial_{\partial_\alpha e_r}v\cdot(\partial_r\partial_\alpha v).
\end{aligned}$$

This shows that

$$\begin{aligned}
D(Dv - D_r v)\cdot DD_r v &\ge -2|De_r||D^2v||Dv| - 2|Dv|^2|De_r|^2\\
&\ge -(|D^2v|^2 + C|De_r|^2|Dv|^2)
\end{aligned} \tag{55}$$

for some universal constant $C > 0$. Since $|De_r(x)|^2 = 3/|x|^2$, inserting (55) into the estimate (54) yields

$$3|D^2v|^2 \geq |DD_{S^3}v|^2 - C|Dv|^2/|x|^2.$$

Inserting $v = u - q$, integrating and using that $D_{S^3}q = 0$ gives

$$\begin{aligned}3\int|D^2(u-q)|^2 &\geq \int|DD_{S^3}u|^2 - C\int\frac{|D(u-q)|^2}{|x|^2}\\ &\geq \int|\nabla Du - \nabla D_r u|^2 - C\int\frac{|D(u-q)|^2}{|x|^2}\\ &\geq \Big(1-\frac{1}{\sqrt{2}}\Big)\int|\nabla Du|^2 + (1-\sqrt{2})\int|\nabla D_r u|^2 - C\int\frac{|D(u-q)|^2}{|x|^2}.\end{aligned}$$

In the second step we used that

$$Du = D_{S^3}u + D_r u$$

and the trivial estimate $|Df| \geq |\nabla^u f|$. By (58) the last line equals

$$\begin{aligned}&\big(\tfrac{3}{2}-\sqrt{2}\big)\int|\nabla Du|^2 + (\sqrt{2}-1)\int\frac{|\nabla^u(|x|\partial_r u)|^2}{|x|^2} - C\int\frac{|D(u-q)|^2}{|x|^2}\\ &\qquad + \frac{1-\sqrt{2}}{2}\bigg[\int_{\partial B_r}\Big(\frac{3}{r}|Du|^2 + 2(\nabla^u_r\partial_r u)\cdot\partial_r u - \frac{2}{r}|\partial_r u|^2\Big)d\mathscr{H}^3\bigg]_{r=R}^1.\end{aligned}$$

The claim follows by dropping the second term, which is nonnegative, and noticing that the fourth term is dominated by ε^2 by (42) while, by (46), the third term is dominated by

$$\varepsilon\Big(\varepsilon + \int_A|\nabla Du|^2 + |Du|^4\Big). \qquad \square$$

4. An equality for stationary biharmonic mappings

The following lemma is true for mappings that are stationary with respect to the energy E_2 in the sense of [Moser 2008]. We do not need the precise definition here. We only remark that every smooth biharmonic mapping is also stationary. Therefore by Remark (ii) to Theorem 1.1, every $u \in W^{2,2}(S, N)$ that is biharmonic is also stationary. To recall the monotonicity formula from [Moser 2008], for $u \in W^{2,2}(B_1, N)$ we define

$$\mathscr{F}(r) = \frac{1}{4}\int_{B_r}|\nabla Du|^2 + \frac{1}{4}\int_{\partial B_r}\Big(\frac{3}{r}|Du|^2 + 2(D_r\partial_r u\cdot\partial_r u)\Big)d\mathscr{H}^3.$$

Theorem 3.1 in [Moser 2008] (see also [Hornung and Moser 2012]) then states that, if $u \in W^{2,2}(S, N)$ is stationary, then

$$\mathscr{F}(r_2) - \mathscr{F}(r_1) = \int_{B_{r_2}\setminus B_{r_1}}\Big(\frac{|\nabla^u|x|\partial_r u(x)|^2}{|x|^2} + 2\frac{|\partial_r u(x)|^2}{|x|^2}\,dx\Big) \tag{56}$$

for almost all r_1, r_2 with $0 < r_1 \leq r_2 \leq 1$. As a corollary to this fact we obtain the following lemma:

Lemma 4.1. *Let $u \in W^{2,2}(B_1, N)$ be stationary and let $R \in (0, 1)$. Then*

$$\begin{aligned}\int_{B_1 \setminus B_R} |\nabla^u D_r u|^2 &= \int_{B_1 \setminus B_R} \Big(\frac{1}{4}|\nabla^u Du|^2 + 2\frac{|\partial_r u|^2}{|x|^2}\Big) \\ &\quad + \frac{1}{4}\Big[\int_{\partial B_r} \Big(\frac{3}{r}|Du|^2 - \frac{4}{r}|\partial_r u|^2 + 2(\nabla_r^u \partial_r u)\cdot \partial_r u\Big) d\mathcal{H}^3\Big]_{r=R}^1 \qquad (57)\\ &= \int_{B_1 \setminus B_R} \Big(\frac{1}{2}|\nabla^u Du|^2 - \frac{|\nabla^u(|x|\partial_r u)|^2}{|x|^2}\Big) \\ &\quad + \frac{1}{2}\Big[\int_{\partial B_r} \Big(\frac{3}{r}|Du|^2 + 2(\nabla_r^u \partial_r u)\cdot \partial_r u - \frac{2}{r}|\partial_r u|^2\Big) d\mathcal{H}^3\Big]_{r=R}^1. \qquad (58)\end{aligned}$$

We remark that Lemma 4.1 can be regarded as a biharmonic counterpart of [Sacks and Uhlenbeck 1981, Lemma 3.5].

Proof. First notice that $|\nabla D_r u|^2 = |\nabla \partial_r u|^2 + |De_r|^2 |\partial_r u|^2$ and that $|De_r|^2 = 3/|x|^2$. Moreover, a short calculation using (38) shows that $|x|\nabla \partial_r u = \nabla(|x|\partial_r u) - D_r u$. Using these facts we calculate

$$\begin{aligned}|\nabla D_r u|^2 &= \Big|\frac{\nabla(|x|\partial_r u)}{|x|} - \frac{D_r u}{|x|}\Big|^2 + |De_r|^2 |\partial_r u|^2 \\ &= \frac{|\nabla(|x|\partial_r u)|^2}{|x|^2} + 4\frac{|\partial_r u|^2}{|x|^2} - \frac{2}{|x|^2} D(|x|\partial_r u)\cdot D_r u \\ &= \frac{|\nabla(|x|\partial_r u)|^2}{|x|^2} + 4\frac{|\partial_r u|^2}{|x|^2} - \operatorname{div}\Big(\frac{|\partial_r u|^2}{|x|^2} x\Big). \qquad (59)\end{aligned}$$

Integrating over $B_1 \setminus B_R$ and using (56) we obtain (57). On the other hand, (59) clearly equals

$$2\Big(\frac{|\nabla(|x|\partial_r u)|^2}{|x|^2} + 2\frac{|\partial_r u|^2}{|x|^2}\Big) - \frac{|\nabla(|x|\partial_r u)|^2}{|x|^2} - \operatorname{div}\Big(\frac{|\partial_r u|^2}{|x|^2} x\Big).$$

Integrating this over $B_1 \setminus B_R$ and using (56) we obtain (58). □

5. Appendix

Lemma 5.1. *There exists a universal constant C_4 such that for all $R > 0$ and for all radial solutions $q \in C^\infty(B_{2R} \setminus \overline{B}_R, \mathbb{R}^n)$ of the equation $\Delta^2 q = 0$ on $B_{2R} \setminus \overline{B}_R$, the following estimate holds*:

$$\|q'\|_{C^0(B_{2R}\setminus \overline{B}_R, \mathbb{R}^n)} \le C_4\big(|q'(R)| + |q'(2R)| + R^{-1}|q(2R) - q(R)|\big).$$

Proof. After rescaling we may assume without loss of generality that $R = 1$. Since

$$\Delta q(x) = 3\frac{q'(|x|)}{|x|} + q''(|x|), \qquad (60)$$

we see that $\Delta^2 q = 0$ is equivalent to q' being a solution of the third order system

$$\frac{3}{t}\Big(\frac{3f(t)}{t} + f'(t)\Big)' + \Big(\frac{3f(t)}{t} + f'(t)\Big)'' = 0. \qquad (61)$$

Denote by $X \subset C^\infty(B_2 \setminus B_1, \mathbb{R}^n)$ the (at most three dimensional) subspace of solutions to (61). Denote by $L : X \to \mathbb{R}^3$ the functional given by $Lf = (f(1), f(2), \int_1^2 f)$. We claim that L is surjective.

In fact, let $a \in \mathbb{R}^3$. By the direct method it is easy to see that the functional $v \mapsto \int_{B_2 \setminus B_1} |\nabla^2 v|^2$ has a minimizer in the class of all radial $v \in W^{2,2}$ satisfying $v'(1) = a_1$ and $v'(2) = a_2$ and $v(2) - v(1) = a_3$. This minimizer q satisfies the Euler–Lagrange equation $\Delta^2 q = 0$, so its radial derivative q' solves the ODE (61). Thus $q' \in X$ and $Lq' = a$. This proves surjectivity of L.

Hence X is three dimensional and L is in fact bijective. Since all norms on X are equivalent and since the inverse of L is of course bounded, we conclude that $\|f\|_{C^0((1,2),\mathbb{R}^n)} \le C|Lf|$ for all $f \in X$. This implies the claim. □

Lemma 5.2. *Let* $0 < r_1 < r_2 \le 1$ *and assume that* $v \in W^{2,2}(B_{r_2} \setminus \overline{B}_{r_1}, \mathbb{R}^n)$. *Then*

$$\int_{B_{r_2} \setminus B_{r_1}} \frac{|Dv|^2}{|x|^2} \le \int_{B_{r_2} \setminus B_{r_1}} |D^2 v|^2 + \left[\frac{1}{r} \int_{\partial B_r} |Dv|^2\right]_{r=r_1}^{r_2}. \tag{62}$$

If $v \in W^{2,2}(B_{r_2} \setminus \overline{B}_{r_1}, N)$ *then*

$$\int_{B_{r_2} \setminus B_{r_1}} \frac{|Dv|^2}{|x|^2} \le \int_{B_{r_2} \setminus B_{r_1}} |\nabla^v Dv|^2 + \left[\frac{1}{r} \int_{\partial B_r} |Dv|^2\right]_{r=r_1}^{r_2}. \tag{63}$$

Proof. For $v \in C^2(A(r_1, r_2), \mathbb{R}^n)$ we have

$$2\frac{|Dv|^2}{|x|^2} = \operatorname{div}\Big(\frac{|Dv|^2}{|x|^2}\, x\Big) - \frac{\partial_r |Dv|^2}{|x|}.$$

Hence if Dv is continuous up to the boundary of $A(r_1, r_2)$ then

$$\begin{aligned} 2\int_{A(r_1,r_2)} \frac{|Dv|^2}{|x|^2} &= -\int_{A(r_1,r_2)} \frac{\partial_r |Dv|^2}{|x|} + \left[\int_{\partial B_r} \frac{|Dv|^2}{|x|^2} x \cdot \frac{x}{|x|}\right]_{r=r_1}^{r_2} \\ &= -2\int_{A(r_1,r_2)} (\partial_r \partial_\alpha v) \cdot \frac{\partial_\alpha v}{|x|} + \left[\frac{1}{r} \int_{\partial B_r} |Dv|^2\right]_{r=r_1}^{r_2}. \end{aligned} \tag{64}$$

By density and by continuity of the trace operator, this equality remains true for $v \in W^{2,2}(A(r_1, r_2), \mathbb{R}^n)$. We conclude that

$$2\int_{A(r_1,r_2)} \frac{|Dv|^2}{|x|^2} \le \int_{A(r_1,r_2)} |D^2 v|^2 + \int_{A(r_1,r_2)} \frac{|Dv|^2}{|x|^2} + \left[\frac{1}{r} \int_{\partial B_r} |Dv|^2\right]_{r=r_1}^{r_2}.$$

Absorbing the second term on the right into the left hand side yields (62).

If v takes values in N then the first term on the right hand side of (64) equals

$$-2\int_{A(r_1,r_2)} (\nabla_r^v \partial_\alpha v) \cdot \frac{\partial_\alpha v}{|x|}$$

because $\partial_\alpha v(x) \in T_{v(x)} N$ for all x. Estimating as above yields (63). □

References

[Chang et al. 1999] S.-Y. A. Chang, L. Wang, and P. C. Yang, "A regularity theory of biharmonic maps", *Comm. Pure Appl. Math.* **52**:9 (1999), 1113–1137. MR 2000j:58025 Zbl 0953.58013

[Ding and Tian 1995] W. Ding and G. Tian, "Energy identity for a class of approximate harmonic maps from surfaces", *Comm. Anal. Geom.* **3**:3-4 (1995), 543–554. MR 97e:58055 Zbl 0855.58016

[Hélein 1990] F. Hélein, "Régularité des applications faiblement harmoniques entre une surface et une sphère", *C. R. Acad. Sci. Paris Sér. I Math.* **311**:9 (1990), 519–524. MR 92a:58034 Zbl 0728.35014

[Hélein 1991] F. Hélein, "Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne", *C. R. Acad. Sci. Paris Sér. I Math.* **312**:8 (1991), 591–596. MR 92e:58055 Zbl 0728.35015

[Hornung and Moser 2012] P. Hornung and R. Moser, "Intrinsically p-biharmonic maps", in preparation, 2012.

[Jost 1991] J. Jost, *Two-dimensional geometric variational problems*, Wiley, Chichester, 1991. MR 92h:58045 Zbl 0729.49001

[Jost 2005] J. Jost, *Riemannian geometry and geometric analysis*, 4th ed., Springer, Berlin, 2005. MR 2006c:53002 Zbl 1083.53001

[Lamm and Rivière 2008] T. Lamm and T. Rivière, "Conservation laws for fourth order systems in four dimensions", *Comm. Partial Differential Equations* **33**:1-3 (2008), 245–262. MR 2009h:35095 Zbl 1139.35328

[Lin and Rivière 2002] F.-H. Lin and T. Rivière, "Energy quantization for harmonic maps", *Duke Math. J.* **111**:1 (2002), 177–193. MR 2002k:58036 Zbl 1014.58008

[Montaldo and Oniciuc 2006] S. Montaldo and C. Oniciuc, "A short survey on biharmonic maps between Riemannian manifolds", *Rev. Un. Mat. Argentina* **47**:2 (2006), 1–22. MR 2008a:53063 Zbl 1140.58004

[Moser 2008] R. Moser, "A variational problem pertaining to biharmonic maps", *Comm. Partial Differential Equations* **33**:7-9 (2008), 1654–1689. MR 2009h:58034 Zbl 1154.58007

[Parker 1996] T. H. Parker, "Bubble tree convergence for harmonic maps", *J. Differential Geom.* **44**:3 (1996), 595–633. MR 98k:58069 Zbl 0874.58012

[Sacks and Uhlenbeck 1981] J. Sacks and K. Uhlenbeck, "The existence of minimal immersions of 2-spheres", *Ann. of Math.* (2) **113**:1 (1981), 1–24. MR 82f:58035 Zbl 0462.58014

[Scheven 2009] C. Scheven, "An optimal partial regularity result for minimizers of an intrinsically defined second-order functional", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:5 (2009), 1585–1605. MR 2010i:49061 Zbl 05612918

[Struwe 2008] M. Struwe, "Partial regularity for biharmonic maps, revisited", *Calc. Var. Partial Differential Equations* **33**:2 (2008), 249–262. MR 2009b:35068 Zbl 1151.58011

[Wang 2004a] C. Wang, "Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold", *Math. Z.* **247**:1 (2004), 65–87. MR 2005c:58030 Zbl 1064.58016

[Wang 2004b] C. Wang, "Remarks on biharmonic maps into spheres", *Calc. Var. Partial Differential Equations* **21**:3 (2004), 221–242. MR 2005e:58026 Zbl 1060.58011

[Wang 2004c] C. Wang, "Stationary biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold", *Comm. Pure Appl. Math.* **57**:4 (2004), 419–444. MR 2005e:58027 Zbl 1055.58008

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