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SMALL DATA SCATTERING AND SOLITON STABILITY IN $\dot{H}^{-1/6}$ FOR THE QUARTIC KDV EQUATION

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We prove scattering for perturbations of solitons in the scaling space appropriate for the quartic nonlinearity, namely $\dot{H}^{-1/6}$. The article relies strongly on refined estimates for a KdV equation linearized at the soliton. In contrast to the work of Tao, we are able to work purely in the scaling space without additional regularity assumptions, allowing us to construct wave operators and a weak version of inverse wave operators.

1. Introduction and statement of results

The generalized Korteweg–de Vries (KdV) equation

$$\begin{cases} \partial_t \psi + \partial_x(\partial_x^2 \psi + \psi^p) = 0 & \text{for } t, x \in \mathbb{R}, \\ \psi(0, x) = \psi_0(x) \end{cases} \tag{1-1}$$

has an explicit soliton solution

$$\psi_c(x, t) = Q_{p,c,c^2t+x_0}(x) := c^{2/(p-1)} Q_p(c(x - (x_0 + c^2 t)))$$

with $c > 0$, $x_0 \in \mathbb{R}$ and

$$Q_p = \left(\frac{p+1}{2}\right)^{1/(p-1)} \operatorname{sech}^{2/(p-1)}\left(\frac{p-1}{2} x\right). \tag{1-2}$$

Well-posedness of the generalized KdV equation was established by Kenig, Ponce and Vega [Kenig et al. 1993] in H^s for some s depending on p. The case $p = 4$ (quartic KdV) is particularly interesting as it is the only subcritical power nonlinearity that does not lead to a completely integrable system. The critical space for the quartic KdV equation is $H^{-1/6}$. Grünrock [2005] obtained local wellposedness in H^s for $s > -1/6$ and the endpoint $\dot{H}^{-1/6}$ was reached by Tao [2007]. Though wellposedness is not the main focus of this note, we will return to this question in Section 7 and use spaces of bounded p variation and their predual (see the appendix and [Hadac et al. 2009]) to simplify and strengthen Tao's wellposedness result in the critical space.

The solutions $Q_{c,y}$ are called traveling waves or solitons. These are minimizers of the constrained variational problem

$$\min\{E(w) : w \in H^1, \ \|w\|_{L^2} = \mu > 0\}, \tag{1-3}$$

where

$$E(u) = \int \left(\tfrac{1}{2} u_x^2 - \frac{1}{p+1} u^{p+1} \right) dx.$$

Minimizers also are extremals of the Lagrangian

$$S(u) = E(u) + \frac{\lambda}{2} \int u^2 \, dx, \qquad (1\text{-}4)$$

where λ is a Lagrangian multiplier. Existence of the minimizer has been shown by Berestycki and Lions [1983] using the constrained minimization problem

$$\min\{T(w) : w \in H^1, \ V(w) = \tilde{\mu}\},$$

where

$$T(w) = \int w_x^2 \, dx \quad \text{and} \quad V(w) = \frac{\lambda}{2} \int w^2 \, dx - \frac{1}{p+1} \int w^{p+1} \, dx.$$

The function Q in (1-2) is the unique positive even solution to the Euler–Lagrange equation

$$-Q_{xx} - Q^p + Q = 0 \qquad (1\text{-}5)$$

to (1-4) with $\lambda = 1$. It is a critical point of $S(u)$ again with $\lambda = 1$, a minimizer of E with constraint $\|u\|_{L^2} = \mu$, where

$$\mu^2 = \|Q_p(x)\|_{L^2}^2 = \left(\frac{p+1}{2} \right)^{2/(p-1)} \frac{\Gamma\left(\frac{p+1}{p-1}\right)\sqrt{\pi}}{\Gamma\left(\frac{p+3}{2(p-1)}\right)} \qquad (1\text{-}6)$$

and hence the quadratic form

$$K(\psi) := \int \tfrac{1}{2} w'^2 + \tfrac{1}{2} w^2 - \tfrac{1}{2} p Q^{p-1} w^2 \, dx \geq 0 \quad \text{for } \langle w, Q \rangle = 0 \qquad (1\text{-}7)$$

is nonnegative on the tangent space that is, the functions orthogonal to Q.

The stability of solitons for generic KdV equations has been studied in several seminal works. Orbital stability was first effectively established in the work of Weinstein [1985]. Then asymptotic stability of solitons for KdV was first observed by Pego and Weinstein [1994], who proved that solitons for KdV are stable under perturbations in exponentially weighted spaces. Later, Martel and Merle [2001a; 2005; 2001b] and Martel [2006] refined this result to observe that solitons for generalized KdV equations are indeed stable under perturbations in the energy space, but measured within a moving reference frame. As mentioned above, for the case $p = 4$, building on the multilinear estimates of Grünrock [2005] and the work of Martel and Merle, Tao [2007] assumes smallness in $H^1 \cap \dot{H}^{-1/6}$ and obtains scattering in $\dot{H}^{-1/6}$. We will give a more thorough introduction to previous stability results including rigorous definitions of stability in Section 2.

In the sequel we will focus on the case $p = 4$ and omit p in the notation. It seems that any further progress is tied to an understanding of the linearization, or more precisely of the linear equation

$$u_t + \partial_x \mathscr{L} u = 0 \qquad (1\text{-}8)$$

and its adjoint

$$v_t + \mathcal{L}\partial_x v = 0, \tag{1-9}$$

which have the explicit solutions (with $\tilde{Q} = c\partial_c Q_c|_{c=1}$)

$$u = a(\tilde{Q} + 2t\,Q') + b\,Q' \quad \text{and} \quad v = cQ,$$

where

$$\tilde{Q} := c\frac{d}{dc}c^{2/(p-1)}Q_p(cx)\Big|_{c=1} = \frac{2}{p-1}Q_p + x\,Q'_p, \tag{1-10}$$

usually evaluated at $c = 1$.

Thus both equations (1-8) and (1-9) have linearly growing solutions. It is one of the first contributions of this paper that both equations are uniformly L^2 bounded once we take into account these modes, and, moreover, there are local energy estimates global in time once we remove these modes. In particular the assumption of Pego and Weinstein on the absence of embedded eigenvalues holds.

Our goal is to build on the arguments of Weinstein [1985] and Martel and Merle [2001a; 2005] to establish some type of asymptotic soliton stability for generalized KdV equations by a direct analysis of the equation itself. We apply a variant of Weinstein's and Martel and Merle's arguments to the linear equations (1-8) and (1-9) and their relatives with variable scale and velocity, and control nonlinear terms through estimates for linear equations.

Specifically, we define projection operators related to the spectrum of $\mathcal{L}$:

$$P_{Q'}^{\perp}\psi = \psi - \frac{\langle \psi, Q' \rangle}{\langle Q', Q' \rangle}Q', \quad \tilde{P}\psi = \psi - \frac{\langle \psi, Q \rangle}{\langle Q, \tilde{Q} \rangle}\tilde{Q}. \tag{1-11}$$

We obtain the main linear estimates, which in their simplest form can be written as follows.

Theorem 1. *Let S be the solution operator for* (1-8) *and S^* the solution operator for* (1-9). *Then, we have*

$$\sup_t \|S(t)\tilde{P}^*u_0\|_{L^2} + \|\operatorname{sech}(x)\partial_x P_{Q'}^{\perp}S(t)\tilde{P}^*u_0\|_{L^2(\mathbb{R}^2)} \lesssim \|u_0\|_{L^2}, \tag{1-12}$$

$$\sup_t \|S^*(t)P_{Q'}^{\perp}v(t)\|_{L^2} + \|\operatorname{sech}(x)\partial_x \tilde{P}S^*(t)P^{\perp}\|_{L^2(\mathbb{R}^2)} \lesssim \|v_0\|_{L^2}. \tag{1-13}$$

The linear estimates presented in the sequel may be generalized to any subcritical power $p < 5$. We provide variants of Theorem 1 for linearization at solitons with variable scale and velocity as well as estimates in scales of Banach spaces similar to estimates for the Airy equation.

Even near the trivial solution dominating the nonlinear part globally by the linear parts requires to work in a scale invariant space similar to $\dot{H}^{-1/6}$. On the positive side it will lead to scattering for perturbations of a soliton in $\dot{H}^{-1/6}$, without the smallness condition of Tao in the energy space (2-4). The study of the linear equation will lead to a fairly precise understanding of its properties, which seems to be new — we hope that it will provide a model for many other questions on the stability of solitons.

As is standard in the study of stability, we take

$$\psi(x, t) = Q_{c(t)}(x - y(t)) + w(x, t).$$

Then, we have

$$\partial_t w + \partial_x(\partial_x^2 w + 4Q_c^3 w) = -\dot{c}(\partial_c Q_c)(x-y) + \dot{y}(Q_c')(x-y)$$
$$- \partial_x(\partial_x^2 Q_c - c^2 Q_c + Q_c^4) - c^2(Q_c'(x-y))$$
$$- \partial_x(6Q_c^2(x-y)w^2 + 4Q_c(x-y)w^3 + w^4). \quad (1\text{-}14)$$

The standard choice of $\dot{c}$ and $\dot{y}$ ensures orthogonality conditions for w. Due to low time regularity we are forced to relax the orthogonality conditions to

$$\frac{\dot{c}}{c}\langle Q_c, \widetilde{Q}_c \rangle = \langle w, Q_c \rangle, \qquad (1\text{-}15)$$

$$(\dot{y} - c^2)\langle Q_c', Q_c' \rangle = -\kappa \langle w, Q_c' \rangle, \qquad (1\text{-}16)$$

where $\kappa \gg 1$.

From an implicit function theorem argument similar to that in the proof of [Martel and Merle 2001b, Proposition 1], there exist unique $c(0)$ and $y(0)$ so that $w(\,\cdot\,, 0)$ is orthogonal to $Q_{c(0)}(\,\cdot\, - y(0))$ and $Q_{c(0)}'(\,\cdot\, - y(0))$ provided the distance of ψ to the set of solitons is small in a suitable norm.

We consider the equations above as ordinary differential equations for c and y, coupled with the partial differential equation.

Using the decomposition and linear estimates, in Sections 8.2 and 8.3 we can prove (referring to later sections for the definition of the function spaces, with $\dot{B}_\infty^{-1/6,2}$ slightly larger than $\dot{H}^{-1/6}$) the following global result:

Theorem 2. *There exists $\epsilon > 0$ and $c > 0$ such that given (1-1) with initial data of the form*

$$\min_{c_0, y_0} \|\psi_0 - Q_{c_0}(x - y_0)\|_{\dot{B}_\infty^{-1/6,2}} \le \epsilon,$$

there exist unique functions c and y with

$$\langle w(0), Q_{c(0)} \rangle = \langle w(0), Q_{c(0)}' \rangle = 0, \quad \dot{c} \in L^1 \cap C^0, \quad \dot{y} - c^2 \in L^2 \cap C^0,$$

and a function $w(x, t) \in \dot{X}_\infty^{-1/6}$ such that

$$\psi(x, t) = Q_{c(t), y(t)}(x) + w(x, t)$$

satisfies the quartic KdV equation, and w, c and y satisfy (1-15), (1-16) and (1-14). Moreover,

$$\|\dot{c}\|_{L^1 \cap C^0} + \|\dot{y} - c^2\|_{L^2 \cap C^0} + \|w\|_{\dot{X}_\infty^{-1/6}} \le c\|w_0\|_{\dot{B}_\infty^{-1/6,2}}.$$

In addition, there exists a function $z_0 \in \dot{B}_\infty^{-1/6,2}$ such that

$$\|w(t) - e^{-t\partial_x^3} z_0\|_{\dot{B}_\infty^{-1/6,2}} \to 0$$

and

$$\|w(\,\cdot\,) - e^{-\cdot\,\partial_x^3} z_0\|_{X_\infty^{-1/6}((t,\infty))} \to 0 \quad as \ t \to \infty$$

if $w(0)$ is in the closure of C_0^∞.

In fact, we prove a far stronger result than this, though Theorem 2 captures the main ideas. Finally, in Section 8.4 we show for a function v, there exists a quantity $J(v)$ defined in (8-8) such that we have the following:

Theorem 3. *Let v_0 be in the closure of C_0^∞ in $\dot{B}_\infty^{-1/6,2}$, let $c_\infty > 0$, and let $y_0 \in \mathbb{R}$. Let v be the solution to the linear homogeneous KdV equation. Assume that*

$$J(v) \le \delta \quad \text{for some } \delta = \delta(\|v_0\|_{\dot{B}_\infty^{-1/6,2}}).$$

Then there exists a solution Ψ to the quartic KdV equation, a function $y \in C^1([0, \infty))$, a function $c \in C^1([0, \infty), (0, \infty))$ such that $w = \Psi - Q_{c,y}$, where c and y satisfy equations (1-15), (1-16), (1-14), and

$$\langle w(0), Q_{c(0)}(\cdot - y(0)) \rangle = \langle w(0), Q'_{c(0)}(\cdot - y(0)) \rangle = 0,$$

$$c(t) \to c_\infty, \quad y(0) = y_0, \quad w(t) - v(t) \to 0 \quad \text{in } \dot{B}_\infty^{-1/6,2} \text{ as } t \to \infty.$$

Moreover, if in addition $v_0 \in L^2$, then $\Psi \in C(\mathbb{R}, L^2(\mathbb{R}))$ and

$$\|v_0\|_{L^2}^2 + \|Q_{c_\infty,0}\|_{L^2}^2 = \|\Psi(t)\|_{L^2}.$$

There exists $\varepsilon > 0$ such that the assumptions are satisfied if $\|v_0\|_{\dot{B}_\infty^{-1/6,2}} \le \varepsilon$.

Remark 1.1. The conclusions in Theorems 2 and 3 hold as well in the spaces $\dot{B}_\infty^{-1/6,2} \cap \dot{H}^s \cap H^\sigma$ for any $-1 < s \le 0$ and $\sigma \ge 0$, allowing one to prove uniform bounds in higher Sobolev norms; see Section 7.1. In particular, given initial data in $\dot{B}_\infty^{-1/6,2} \cap \dot{H}^s \cap H^\sigma$, J small will imply stability and scattering in $\dot{B}_\infty^{-1/6,2} \cap \dot{H}^s \cap H^\sigma$. Specifically, we note one can prove boundedness and scattering in the energy space H^1 intersected with $\dot{B}_\infty^{-1/6,2}$.

To motivate the construction of our nonlinear iteration spaces, in Section 3 we first derive some refined estimates for the linear KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u = f, \\ u(0, x) = u_0(x). \end{cases} \tag{1-17}$$

Then, in Section 4 we discuss the spectral and mapping properties of the operator $\mathscr{L}$ and derive linear estimates for the systems (1-8) and (1-9) and their relatives

$$u_t + u_{xxx} + (Q_{c(t)}(x - x(t))u)_x = f.$$

In Section 5, we combine local smoothing estimates as for (1-17), where we treat the Q terms as error terms with the virial identity and energy conservation for (1-8) to prove uniform bounds for a projection of the solution v assuming orthogonality of the initial data to Q'.

With this first result at hand we pursue a standard though nontrivial path and employ pseudodifferential techniques and duality to derive similar estimates in a full scale of function spaces. The Littlewood–Paley decomposition at low frequencies is severely affected by the term containing Q. This is done in Section 6 with main result Proposition 6.7.

Theorems 2 and 3 are proven in the final two sections by combining the wellposedness arguments and the linear estimates.

2. Review of previous soliton stability results

To begin, we consider the linearized operator

$$\mathcal{L}\psi = -\psi'' - pQ^{p-1}\psi + \psi$$

associated to the Euler–Lagrange equation (1-5) of (1-4) with $\lambda = 1$, respectively the constraint variational problem (1-3) with Lagrange multiplier 1. It is one of the remarkable operators for which almost everything is known about the spectrum and scattering; see [Lamb 1980, Section 2.4 and 2.5], and [Titchmarsh 1962, Section 4.19]. The operator

$$\mathcal{L}_M\psi = -\psi_{xx} - M\operatorname{sech}^2(x)\psi$$

has the continuous spectrum $[0, \infty)$ and the ground state $\psi_0(x) = \operatorname{sech}^\alpha(x)$ with eigenvalue α^2 provided $M = \alpha(\alpha + 1)$, with $\alpha > 0$. The other eigenvalues are $(\alpha - j)^2$ for $1 \le j < \alpha$ together with the eigenfunctions can be obtained as follows: Let $\psi_{0,M}$ be the ground state with the constant M. Then,

$$\psi_{j,(\alpha+j)(\alpha+j+1)}(x) = \prod_{l=1}^{j}\left(\frac{d}{dx} - (\alpha + l)\tanh(x)\right)\operatorname{sech}^\alpha(x)$$

is the j eigenfunction to the potential with $M = (\alpha + j)(\alpha + j + 1)$. We consider this information useful, and we will use these results, even if the arguments could easily be adapted to a much larger class of nonlinearities.

Clearly $\mathcal{L}Q' = 0$ and a short calculation or a comparison with the results above shows that $Q^{(p+1)/2}$ is the ground state with eigenvalue $1 - (p+1)^2/4$. There is no other eigenvalue if $p \ge 3$, but there are other eigenvalues in $(0, 1)$ if $p < 3$. As an immediate consequence $K(\psi) \ge \|\psi\|_{L^2}^2$ if $\langle \psi, Q' \rangle = \langle \psi, Q^{(p+1)/2} \rangle = 0$.

We recall that K is positive definite on the orthogonal complement of Q. We follow [Weinstein 1985] and use this bound to establish a lower bound on a different codimension 2 subspace if $p < 5$. There exists $\delta > 0$ such that

$$K(\psi) \ge \delta\|\psi\|_{H^1}^2 \quad \text{for all } \psi \text{ with } \langle \psi, Q^{p-1}Q' \rangle = \langle \psi, Q \rangle = 0. \tag{2-1}$$

It suffices to verify this statement independently for odd and even functions. For odd functions the quadratic form is nonnegative, with a null space spanned by Q'. Positivity follows from $\langle Q', Q^{p-1}Q' \rangle \ne 0$. The argument for even functions is harder, but again the quadratic form is nonnegative since Q is a local minimizer of the constraint variational problem.

Let ψ_j be a minimizing sequence with $\|\psi_j\|_{H^1} = 1$. Suppose that the left hand side of (2-1) converges to 0. The sequence maximizes $\int Q^{p-1}\psi_j^2\,dx$. There exists a weakly converging subsequence which convergences against a nontrivial even limit ψ since $\psi \to \int Q^{p-1}\psi^2\,dx > 0$ is weakly lower semicontinuous. Moreover $\langle \psi, Q \rangle = 0$ and $\|\psi\|_{H^1} \le 1$. Rescaling if necessary we see that $\|\psi\|_{H^1} = 1$.

We want to show that $K(\psi) > 0$ and argue by contradiction. Suppose that $K(\psi) = 0$. Then by (1-7) ψ is a minimizer of K under the sole constraint $\langle Q, \psi \rangle = 0$ and hence it satisfies the Euler–Lagrange equations

$$\mathscr{L}\psi = \lambda Q.$$

But then ψ is a multiple of $\tilde{Q}$ since

$$\mathscr{L}\tilde{Q} = -2Q$$

is the unique symmetric function with this property. However $\langle Q, \tilde{Q} \rangle \neq 0$ if $p \neq 5$, and hence $\psi = 0$, which contradicts our construction and thus implies the existence of $\delta > 0$ with

$$K(\psi) \geq \delta \|\psi\|_{H^1}.$$

Observe that here the subcriticality condition $p < 5$ enters crucially.

Given ψ we define the parameters c_0 and x_0 by the variational problem

$$\|\psi - Q_{c_0, x_0}\|_{H^1}^2 = \inf_{c, x} \|\psi - Q_{c, x}\|_{H^1}^2.$$

Following Weinstein [1985] we claim

$$\|\psi - Q_{c_0, x_0}\|_{H^1}^2 \leq c(E(\psi) - E(Q_c)), \tag{2-2}$$

provided the left hand side is sufficiently small. This is a consequence of the lower bound for the quadratic form (2-1).

Lyapunov stability of solitons has been shown in the seminal work of Weinstein.

Theorem [Weinstein 1985, Theorem 4]. *Let $\varepsilon > 0$. There exists $\delta > 0$ such that*

$$\inf_{x_0} \|\psi(t) - Q_1(x - x_0)\|_{H^1} \leq \varepsilon \quad \text{if } \|\psi_0 - Q_1\|_{H^1} \leq \delta.$$

This is a direct consequence of the conservation of the L^2 norm and the energy, plus (2-2).

The study of asymptotic stability began with Pego and Weinstein [1994] in spaces with growing exponential weights. The effect of the weight is twofold. First, there is not much the soliton could interact with on its path to the right. Secondly, small solitons that are slow and prevent asymptotic stability in L^2 carry a weight that makes them exponentially decreasing in time. A key assumption is the absence of embedded eigenvalues of $\partial_x \mathscr{L}$, other than 0 with eigenfunction Q' and the generalized eigenfunction $\tilde{Q}$. Pego and Weinstein verify this assumption for $p = 2$ and $p = 3$ and show that it fails at at most a finite number of values for p between 2 and 5. It is a consequence of the virial identity below that there are no nonzero purely imaginary eigenvalues of $\partial_x \mathscr{L}$.

The exponential weight pushes the continuous spectrum of $\partial_x \mathscr{L}$ to the left, makes the problem more parabolic, and allows the use of techniques from smooth dynamical systems, in particular of a center manifold reduction that is a restriction of the flow to a two dimensional manifold.

Martel and Merle [2001a; 2005] and Martel [2006] introduced a virial identity or monotonicity formula for the adjoint problem (1-9) as well as for nonlinear problems. Let

$$\eta(x) = -\frac{p+1}{p-1}\frac{Q'}{Q} = \frac{p+1}{2}\tanh\frac{p-1}{2}x$$

and suppose that v satisfies the Equation (1-9). By direct computation we have

$$-\frac{d}{dt}\int \eta v^2\,dx = \langle(3(\mathcal{L} + \tfrac{1}{4}(p+1)^2 - 1)Q^{(p-1)/2}v, Q^{(p-1)/2}v\rangle, \tag{2-3}$$

where the quadratic form is nonnegative and it has by the spectral theory of Schrödinger operators with $\mathrm{sech}^2(x)$ potentials a one-dimensional null space spanned by Q. There are two consequences: the quantity on the left hand side is monotonically decreasing, and the right hand side controls the H^1 norm of $Q^{(p-1)/2}v$ provided v is orthogonal to a vector $\overline{Q}$ with $\langle\overline{Q}, Q\rangle \neq 0$. Hence, if $v(0)$ is orthogonal to Q' and $\tilde{Q}$, which is preserved under the evolution,

$$\|Q^{(p-1)/2}v\|_{H^1} \leq c\sup_t\|v(t)\|_{L^2}.$$

The left hand side is controlled provided we obtain a bound on $\sup_t\|v(t)\|_{L^2}$. Martel and Merle [2001a; 2005] use this and related observations together with the a priori control on the deviation of the solution to the set of solitons in ingenious ways for indirect arguments: The existence of a solution H^1 close to solitons, but not asymptotically converging to the soliton "on the right" leads to the existence of impossible objects.

Later, Côte [2006] constructed solutions with specific asymptotic conditions including many soliton solutions for positive time. This shows that L^2 convergence to a soliton will not be true without restricting the set where convergence is studied.

Already L^2 conservation precludes asymptotic stability of the trivial solution. The relevant notion instead of asymptotic stability is for unitary problems the notion of scattering. Suppose that $\psi(0)$ is close to a soliton. We seek a function w satisfying the Airy equation as well as $c(t)$ and $y(t)$ and a Banach space X such that $\|\psi - Q_{c(t)}(x - y(t)) - w(t)\|_X \to 0$ as $t \to \infty$. Tao [2007] verifies scattering in the following sense: Suppose that

$$\|\psi(0) - Q(0)\|_{H^1} + \|\psi(0) - Q\|_{\dot{H}^{-1/6}} \ll 1. \tag{2-4}$$

Then scattering holds with $X = \dot{H}^{-1/6}$. Tao relies on the work of Martel and Merle, and in particular on Weinstein's a priori estimate of the difference to the soliton.

3. The Airy equation

For purposes of understanding and motivating dispersive estimates for the linearized KdV equation, here we study and collect results for the Airy equation

$$\begin{cases} v_t + v_{xxx} = 0, \\ v(x, 0) = v_0(x). \end{cases} \tag{3-1}$$

The solution operator defines a unitary group $S(t)$ with the kernel

$$K(t, x) = t^{-1/3} Ai(xt^{-1/3}),$$

where as $x \to \infty$ the Airy function is roughly $x^{-1/4}e^{-x^{3/2}}$, and as $x \to -\infty$ the Airy function is roughly $\operatorname{Re}(x^{-1/4}e^{-ix^{3/2}})$. Strichartz estimates for solutions,

$$\|u\|_{L^p L^q} \le c \||D|^{-1/p} u_0\|_{L^2} \tag{3-2}$$

where $L^p L^q$ is the standard space time norm such that the L^p norm in time of the L^q norm in space and

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2},$$

follow as an immediate consequence. Of particular interest for this work are the homogeneous Strichartz pair $(p, q) = (6, 6)$ as well as the endpoint Strichartz pair $(p, q) = (4, \infty)$. For an overview of Airy function asymptotics, see [Fedoryuk 1993].

Local smoothing estimates for (3-1) go back to [Kato 1983]. Here we are interested in a more general version of them. Let $\gamma(t, x) \ge 1$ be a smooth bounded increasing function. We calculate

$$\frac{d}{dt} \int \gamma u^2 \, dx = \int (\gamma_t + \gamma^{(3)}) u^2 - 3\gamma' u_x^2 \, dx \tag{3-3}$$

and search for conditions ensuring that the right hand side is nonpositive. We assume

$$\partial_x^3 \gamma \le -\tfrac{2}{3}\partial_t \gamma \tag{3-4}$$

with the easiest case being $\gamma(t, x) = \gamma_0(x - t)$, for which we assume

$$\gamma_0^{(3)} \le \tfrac{2}{3}\gamma_0'. \tag{3-5}$$

We get

$$\frac{d}{dt} \int \gamma u^2 \, dx + \int \gamma'(u_x^2 + \tfrac{1}{3}u^2) \, dx \le 0. \tag{3-6}$$

Let us fix a particular example,

$$\gamma_0(x) = 1 + \int_{-\infty}^{x} (1 + |y|^2)^{-(1+\varepsilon)/2} dy. \tag{3-7}$$

It satisfies the criteria and, provided ε is sufficiently small, a straightforward calculation gives (3-5). Next, it is instructive to consider a scaling. For $\mu > 0$ and γ_0 as above we define

$$\gamma_\mu(t, x) = \gamma_0(\mu^{-1}(x - \mu^{-2}t)).$$

Then,

$$\frac{d}{dt} \int \gamma_\mu u^2 \, dx + \int \gamma_\mu'(u_x^2 + \frac{1}{3\mu^2}u^2) \, dx \le 0. \tag{3-8}$$

One may easily generalize this inequality by choosing $t \to y(t)$ with $\dot{y} \ge \tfrac{1}{8}\mu^{-2}$, and setting $\gamma(t, x) = \gamma_0(\mu^{-1}(x - y(t)))$. In the sequel we will always restrict ourselves to $\mu = 1$.

The virial identity clearly generalizes to functions spaces with different regularity. To see this, we first define the space H^s_ρ (and similarly L^2_ρ) by the norm

$$\|u\|^2_{H^s_\rho} = \int |\langle D\rangle^s u|^2 \rho^2(x)\, dx < \infty,$$

where $\rho > 0$ with uniformly bounded derivatives of order up to k for some $k \geq |s|$ and $\langle D\rangle^s$ is defined through the Fourier multiplication $(1 + |\xi|^2)^{s/2}$. Similarly we define ρH^s where $u \in \rho H^s$ if and only if

$$u = \rho f \quad \text{for } f \in H^s \quad \text{and} \quad \|u\|_{\rho H^s} = \inf_{u=\rho f} \|f\|_{H^s}.$$

The function ρ will often depend on t. Given a Banach space X, we denote the space of X-valued L^2 functions by $L^2 X$, and give the obvious meaning to $L^2 \rho H^s$ and $L^2 H^s_\rho$. Such spaces will be explored further in Section 4.

Remark 3.1. We note that $\rho H^s = H^s_{\rho^{-1}}$, if ρ is nonnegative, up to equivalent norms. However as we wish to highlight the use of duality throughout the linear analysis and construction of iteration spaces, we adopt the ρH^s convention.

If γ satisfies the assumptions above and

$$\begin{cases} u_t - u_{xxx} = f, & \text{where } f \in L^2\sqrt{\gamma'}H^{-1}, \\ u(0, x) = u_0(x), & \text{where } u_0 \in L^2, \end{cases} \tag{3-9}$$

we obtain by an obvious modification of the argument above

$$\|u\|_{L^\infty L^2} + \|u\|_{L^2 H^1_{\sqrt{\gamma'}}} \leq c \left(\|u(0)\|_{L^2} + \|f\|_{L^2\sqrt{\gamma'}H^{-1}} \right). \tag{3-10}$$

We turn to a useful technical result.

Lemma 3.1. *Let $m \in C^\infty(\mathbb{R})$ satisfy $|m^{(j)}(\xi)| \leq c_j \langle\xi\rangle^{s-j}$ for $j \geq 1$ and let $m(D)$ be the Fourier multiplier defined by m. Suppose that $\gamma \in C^\infty$,*

$$|\gamma^{(j)}(x)| \lesssim \gamma(x) \quad \text{for } j \geq 0, \text{ and}$$
$$|1 - \gamma(x)/\gamma(y)| \lesssim c(|x - y| + |x - y|^N) \quad \text{for some } N.$$

For any $a \in \mathbb{R}$ we have

$$\|\gamma^{-a}[m(D), \gamma^a]\langle D\rangle^{1-s} f\|_{L^2} + \|[m(D), \gamma^a]\gamma^{-a}\langle D\rangle^{1-s} f\|_{L^2} \leq c_{s,a}\|f\|_{L^2}$$

and

$$\|\langle D\rangle^{1-s}[m(D), \gamma^a]\gamma^{-a} f\|_{L^2} + \|\langle D\rangle^{1-s}\gamma^{-a}[m(D), \gamma^a]f\|_{L^2} \leq c_{s,a}\|f\|_{L^2}.$$

The most important example of m is the Fourier multiplier $\langle D\rangle^s$ defined by the function $(1 + |\xi|^2)^{s/2}$.

Proof. We begin with the estimate of the first term in the first inequality, the second term being similar. We decompose $m(D) = m_0(D) + m_1(D)$, where the convolution kernel $m_0(x)$ of $m_0(D)$ is supported in $|x| \leq 2$, and the one for $m_1(D)$ is supported in $|x| \geq 1$. The convolution kernel $m_1(x)$ together with its derivatives decays exponentially.

The integral kernel of $\gamma^{-a}[m_1(D), \gamma^a]$ is

$$K_1(x, y) = m_1(x - y)\left(1 - \left(\frac{\gamma(x)}{\gamma(y)}\right)^a\right).$$

The kernel and its derivatives decay like $\langle x - y \rangle^{-N}$, which implies

$$\|\gamma^{-a}[m_1(D), \gamma^a]f\|_{H^N} \le c_N \|f\|_{H^{-N}}$$

for all $N > 0$ by Schur's lemma. It remains to prove

$$\|\gamma^{-a}[m_0(D), \gamma^a]\langle D \rangle^{1-s} f\|_{L^2} \le c_{s,a}\|f\|_{L^2}.$$

We decompose $\langle D \rangle^s = D_0 + D_1$. The bound for $\gamma^{-a}[m_0(D), \gamma^a]D_0$ follows from standard pseudo-differential calculus. The bound for the term with D_1 follows from

$$\|\gamma^{-a}[m_0(D), \gamma^a]f\|_{L^2} \le c_N \|f\|_{H^N},$$

which again follows easily by standard pseudodifferential calculus. $\qquad\square$

Lemma 3.2. *Suppose that*

$$\begin{cases} u_t + u_{xxx} = f, & \text{where } f \in \sqrt{\gamma'}H^{s-1}, \\ u(0, x) = u_0(x), & \text{where } u_0 \in H^s. \end{cases} \tag{3-11}$$

Then

$$\|u\|_{L^\infty H^s} + \|u\|_{L^2 H^{s+1}_{\sqrt{\gamma'}}} \le c\left(\|u(0)\|_{H^s} + \|f\|_{L^2\sqrt{\gamma'}H^{s-1}}\right). \tag{3-12}$$

Moreover, if

$$\begin{cases} u_t + u_{xxx} = (\text{sech}^2(x - x(t))f)_x + \partial_x g, \\ u(0, x) = u_0(x), \end{cases} \tag{3-13}$$

with $\dot{x} \ge \delta$, then

$$\|u\|_{L^\infty \dot{H}^{-1}} + \|u\|_{L^2 H^0_{\sqrt{\gamma'}}} \lesssim \|u(0)\|_{\dot{H}^{-1}} + \|f\|_{L^2 H^{-1}} + \|g\|_{L^1 L^2}. \tag{3-14}$$

Proof. We set $v = \langle D \rangle^s u$, where u satisfies (3-11); hence

$$v_t + v_{xxx} = \langle D \rangle^s f,$$

and

$$\|u\|_{L^\infty H^s} + \|u\|_{L^2 H^{s+1}_{\sqrt{\gamma'}}} = \|v\|_{L^\infty L^2} + \|v\|_{L^2 H^1_{\sqrt{\gamma'}}} \le c(\|v(0)\|_{L^2} + \|\langle D \rangle^s f\|_{L^2\gamma H^{-1}}),$$

where the first term is equal to $\|u(0)\|_{H^s}$ and

$$\begin{aligned} \|\langle D \rangle^s f\|_{L^2\sqrt{\gamma'}H^{-1}} &= \|\langle D \rangle^{-1}(\gamma')^{-1/2}\langle D \rangle^s f\|_{L^2 L^2} \\ &\le \|f\|_{L^2\sqrt{\gamma'}H^{s-1}} + \|[\langle D \rangle^{-1}, (\gamma')^{-1/2}]\langle D \rangle^s f\|_{L^2 L^2} \\ &\le \|f\|_{L^2\sqrt{\gamma'}H^{s-1}} + \|(\gamma')^{-1/2}\langle D \rangle^{s-2} f\|_{L^2 L^2}. \end{aligned}$$

The last inequality follows from Lemma 3.1 applied with γ' for γ, $a = 1/2$, and -1 for s. This implies the desired estimate (3-12). Now suppose that u satisfies (3-13) and let v be the antiderivative of u with respect to x. It satisfies

$$\begin{cases} v_t + v_{xxx} = \operatorname{sech}^2(x - x(t))f + g, \\ v(0, x) = v_0(x); \end{cases} \tag{3-15}$$

hence

$$\|u\|_{L^\infty \dot{H}^{-1}} + \|u\|_{L^2 H^0_{\sqrt{\gamma'}}} \le c\|g\|_{L^1 L^2} + \|f\|_{L^2 H^{-1}}. \qquad \square$$

4. Properties of the Schrödinger operator

We briefly recall notions from the introduction. Given $p > 1$, solitons of the form $Q_p(x - t)$ satisfy (1-5) and it is not hard to verify that all bounded solutions are translates of $\pm Q_p$ in Equation (1-2). Similarly $Q_{p,c} = c^{2/(p-1)} Q_p(cx)$ satisfies

$$\partial_x^2 (Q_p)_c - c^2 (Q_p)_c + (Q_p)_c^p = 0. \tag{4-1}$$

We will focus on $p = 4$ and omit again p from the notation. Let $'$ denote differentiation with respect to x and $\cdot$ differentiation with respect to time. We recall the definition of $\tilde{Q}$ from (1-10) and $\tilde{Q}_c = c\partial_c Q_c$ respectively $\tilde{\tilde{Q}}_c = c\partial_c c\partial_c Q_c$, the corresponding differentiation at c. There are many explicit calculations, and we collect some of them here. Using the properties of $Q_c(x) = c^{2/3}Q(cx)$, it follows that

$$\|Q_c\|_{L^2} = c^{1/6}\|Q_1\|_{L^2}, \quad \langle \tilde{Q}_c, Q_c \rangle = \tfrac{1}{2}c\partial_c\|Q_c\|_{L^2}^2 = \tfrac{1}{6}\|Q_c\|_{L^2}^2, \tag{4-2}$$

where the L^2 norm is given by (1-6), and

$$\|Q_c'\|_{L^2} = c^{7/6}\|Q_1'\|_{L^2}. \tag{4-3}$$

In addition,

$$\partial_x Q_c = c^{5/3} Q'(cx) \quad \text{and} \quad c\partial_c Q_c = \left(\tfrac{2}{3}Q_c + x Q_c'\right) = \tilde{Q}_c = c^{2/3}\tilde{Q}(cx).$$

The operator $\mathcal{L}_c$ is defined by

$$\mathcal{L}_c u = -u_{xx} + c^2 u - 4Q_c^3 u, \tag{4-4}$$

where we mostly omit y and c if $c = 1$. We recall that virtually everything is known about the spectrum of $\mathcal{L}$; see [Andrews et al. 1999; Lamb 1980; Titchmarsh 1962]. We summarize the findings below. We also refer to [Martel 2006; Weinstein 1985] and the references therein for extensive discussions of these properties for more general operators of type similar to $\mathcal{L}$.

By direct differentiation in x of (1-5), we see $\mathcal{L}Q' = 0$. Hence, the null space of $\mathcal{L}$ consists at least of the space $\alpha Q'$ for all $\alpha \in \mathbb{R}$. Similarly, by differentiation in c of (4-1), we see $\mathcal{L}(\tilde{Q}) = -2Q$, so $\partial_x \mathcal{L}$ has at least a 2-dimensional generalized null space. Also, since $Q' = 0$ only at $x = 0$, we know from the Sturm oscillation theorem that there exists some $\lambda_0 > 0$ and $\mathcal{Q}_0 > 0$ such that $\mathcal{L}\mathcal{Q}_0 = -\lambda_0\mathcal{Q}_0$, the unique negative eigenstate of $\mathcal{L}$. Note, because $\mathcal{L}$ is a sech^2 potential perturbation of the Laplacian, it is

possible to exactly construct $\mathcal{Q}_0 = Q^{5/2}$ and $\lambda_0 = 21/4$ using standard techniques. The analysis above summarizes the entire discrete spectral decomposition for $\mathcal{L}$.

Following the analysis in [Weinstein 1985, Propositions 2.7 and 2.9], if

$$\langle \tilde{u}, Q \rangle = 0 \quad \text{and} \quad \langle \tilde{u}, Q' \rangle = 0,$$

then there exists $k_0 > 0$ such that

$$\langle \tilde{u}, \mathcal{L}\tilde{u} \rangle \geq k_0 \|\tilde{u}\|_{L^2}^2. \tag{4-5}$$

Here k_0 depends only on the power $p = 4$ in (1-1).

We will consider $\rho = e^v$ with $v \in C^{|s|+1}$ with

$$|v^{(j)}(x)| \leq \varepsilon \tag{4-6}$$

for $0 \leq j \leq |s| + 1$ and a small constant ε to be chosen later. Clearly we may regularize v and hence $\rho = e^v$ without changing the spaces. Then

$$u \in H_\rho^s \Longleftrightarrow \rho u \in H^s \Longleftrightarrow u \in \rho^{-1} H^s.$$

It is quite obvious that the dual space of H_ρ^s is ρH^{-s} with isometric norms, and this statement does not depend on the regularity of ρ. We recall the definition of the projectors (1-11).

Lemma 4.1. *For all $s \in \mathbb{R}$, there exists $C > 0$ such that*

$$\|P_Q^\perp u\|_{H^{s+2}} \leq C\|\mathcal{L}u\|_{H^s}, \quad \|P_{Q'}^\perp u\|_{\rho H^{s+2}} \leq C\|\mathcal{L}u\|_{\rho H^s}, \quad \|P_{Q'}^\perp u\|_{H_\rho^{s+2}} \leq C\|\mathcal{L}u\|_{H_\rho^s}.$$

Proof. The first inequality is an immediate consequence of the nature of the spectrum described above along with ellipticity. The second and the third statement are equivalent because $H_\rho^s = \rho^{-1}H^s$, with equivalent norms.

Fix $\mu = 1 - (p+1)^2/4$, where $p = 4$. For $\lambda = \lambda_0 + i\lambda_1$ in the complex half plane left of μ, we obtain the resolvent estimate

$$|\lambda - \mu|\|u\|_{L^2} \leq \|(\mathcal{L} - \lambda)u\|_{L^2}$$

and also for some $1 > \kappa > 0$, we have

$$\begin{aligned}
\text{Re} \int u\overline{(\mathcal{L} - \lambda)u}\, dx &\geq |\lambda - \mu|\|u\|_{L^2}^2 + \langle (\mathcal{L} - \mu)u, u \rangle \\
&\geq \tfrac{1}{2}|\lambda - mu|\|u\|_{L^2}^2 + \kappa\|u_x\|_{L^2}^2 + (1 - \kappa)\langle (\mathcal{L} - \mu)u, u \rangle \\
&\qquad\qquad + \big(\tfrac{1}{2}|\lambda - \mu| + \kappa(1 - \mu) - 4\kappa\|Q\|_{L^\infty}^3\big)\|u\|_{L^2}^2 \\
&\geq \tfrac{1}{2}|\mu - \lambda_0|\|u\|_{L^2}^2 + \min\Big\{\frac{|\mu - \lambda_0|}{8\|Q\|_{L^\infty}^3}, \frac{1}{2}\Big\}\|u_x\|_{L^2}^2
\end{aligned}$$

by the obvious choice of κ.

We obtain the estimate for λ with real part at most μ:

$$|\mu - \lambda|\|u\|_{L^2} + \min\{|\lambda - \mu|, 1\}\|u_x\|_{L^2} \leq C\,\text{Re}\langle (\mathcal{L} - \lambda)u, u \rangle \leq C\|\langle (\mathcal{L} - \lambda)u\|_{H^{-1}}\|u\|_{H^1}.$$

These estimates imply that the resolvent $(\mathscr{L} - \lambda)^{-1}$ defines a continuous uniformly bounded map (for $\operatorname{Re} \lambda \leq \lambda_0 < \mu$) from H^{-1} to H^1. Moreover,

$$\|u\|_{L^2} \leq |\lambda - \mu|^{-1} \|\mathscr{L}u\|_{L^2} \quad \text{and} \quad \|u_x\|_{L^2}^2 \leq |\lambda - \mu|^{-1} \max\left\{\frac{1}{2}, \frac{8\|Q\|_{L^\infty}}{|\lambda - \mu|}\right\} \|(\mathscr{L} - \lambda)u\|_{L^2}^2.$$

We turn to the weighted estimates and calculate formally

$$e^\nu (\mathscr{L} - \lambda) e^{-\nu} = \mathscr{L} - \lambda - |\nu'|^2 + \partial_x \nu' + \nu' \partial_x,$$

and hence, since $\partial_x \nu' + \nu' \partial_x$ is antisymmetric,

$$\operatorname{Re} \int u \overline{e^\nu (\mathscr{L} - \lambda) e^{-\nu} u} \, dx = \operatorname{Re} \int u \overline{(\mathscr{L} - \lambda)u} \, dx - \|\nu' u\|_{L^2}^2 \geq \tfrac{1}{2} |\lambda_0 - \mu| \|u\|_{L^2}^2$$

if $\varepsilon \leq \sqrt{|\lambda_0 - \mu|/2}$, which we assume in the sequel. As above we obtain with an explicit constant C

$$\|u\|_{H^1} \leq C \|e^\nu (\mathscr{L} - \lambda) e^{-\nu} u\|_{H^{-1}}. \tag{4-7}$$

It follows from these estimates that given $\delta > 0$ there is a single resolvent family (for $\operatorname{Re} \lambda < \mu - \delta$) mapping $\rho H^{-1} \to \rho H^1$ and from $H_\rho^{-1} \to H_\rho^1$, provided ε is sufficiently small.

Recall that $\mathscr{L}$ has a zero eigenvalue with eigenfunction Q' and a single negative eigenvalue $-\lambda_0$ with a ground state $\mathfrak{Q}_0$. Let P be the orthogonal projection to the orthogonal complement of these two eigenfunctions. The remaining spectrum is contained in $[\rho, \infty)$, where $\rho > 0$ is either 1 (if $p \geq 3$), or the next positive eigenvalue, which can be easily be calculated. Moreover, $\mathscr{L}$ is selfadjoint. The resolvent $R(\lambda) = (\mathscr{L} - \lambda)^{-1}$ is a holomorphic map in $\mathbb{C} \setminus (1, \infty)$ with simple poles in μ, 0, and possibly some other eigenvalues in $(0, 1)$. In addition, $R_0(\lambda) = R(\lambda)P$ has a continuous and hence holomorphic extension to $\lambda = 0$ and $\lambda = -\lambda_0$, which is uniformly bounded in each half plane strictly left of ρ.

By Equation (4-7) the resolvent is uniformly bounded on the weighted spaces if λ is in the half plane left of $-\mu$. Decreasing ε if necessary (so that the orthogonal projection $P_{Q'}^\perp$ along Q' is bounded in the weighted space), we obtain the same statement for $R_0(\lambda)$. Now complex interpolation implies

$$\|\mathscr{L}^{-1} P f\|_{H_\rho^1} \leq C \|P f\|_{H_\rho^{-1}}.$$

This implies the desired estimates for $s = -1$.

Standard elliptic theory extends this estimate to

$$\|u\|_{H_\rho^{s+2}} \leq C \|(\mathscr{L} - \lambda)u\|_{H_\rho^s}, \tag{4-8}$$

$$\|u\|_{\rho H^{s+2}} \leq C \|(\mathscr{L} - \lambda)u\|_{\rho H^s} \tag{4-9}$$

first to all $s \geq -1$, and then, by duality, to all $s \in \mathbb{R}$. The first estimate is the special situation when ν is constant.

We conclude with the trivial observation that we may replace (4-6) by $\lim_{x \to \in\infty} \nu^j = 0$, which holds for $\rho(x) = (1 + |x|^2)^a$ for all real numbers a, since in that case we may choose an equivalent norm that satisfies (4-6). $\qquad \square$

5. Energy methods for the linearized equation

We turn to a study of what we call the linear u-problem

$$\begin{cases} u_t = \partial_x(\mathscr{L}u), \\ u(0, x) = u_0, \end{cases} \tag{5-1}$$

where

$$\mathscr{L}u = (-\partial_x^2 + 1 - 4Q^3)u = \left(-\partial_x^2 + 1 - 10\operatorname{sech}^2\left(\tfrac{3}{2}x\right)\right)u.$$

We note here that $\mathscr{L}$ is the operator that results from linearization of the KdV equation about Q when we work in a moving reference frame or in other words make the change of variables $x \to x - t$. Indeed, setting $\psi(x, t) = Q(x - t) + u(x - t, t)$ and plugging into (1-1), we get

$$\partial_t u = -\partial_x(\partial_x^2 u - u + (Q + u)^4 - Q^4 + \partial_x^2 Q - Q + Q^4) = \partial_x(\mathscr{L}u) - \partial_x(6Q^2u^2 + 4Qu^3 + u^4).$$

For reasons that will become clear in the sequel, we also consider the linear v-problem

$$\begin{cases} v_t = \mathscr{L}(\partial_x v), \\ v(0, x) = v_0. \end{cases} \tag{5-2}$$

The two equations (5-2) and (5-1) are related in many ways.

(1) They are dual to each other.

(2) If u satisfies the u equation, then $v = \partial_x u$ satisfies the v equation.

(3) If v satisfies the v equation, then $u = \mathscr{L}v$ satisfies the u equation.

We observe that $u = Q'$ is a solution to the u equation, and hence $\langle v, Q' \rangle$ is preserved by the flow for v. In particular orthogonality is preserved by the evolution. Similarly, $v = Q$ is a solution to the v equation and $\langle u, Q \rangle$ is preserved by the u flow. Moreover, $u = aQ' + b(\tilde{Q} + 2tQ')$ satisfies the u equation for all coefficients a and b. As a consequence both equations admit solutions that grow linearly with time. Moreover, if v satisfies the v equation, then

$$\frac{d}{dt}\langle v, \tilde{Q} \rangle + 2t\langle v, Q' \rangle = 0$$

and v is orthogonal to $\tilde{Q}$ and Q' provided it is initially.

Inspired by a set of ideas collected from [Martel and Merle 2008] and the references therein, let us look at a virial identity for (5-2), namely

$$I_\eta(v) = -\int \eta(x)v^2 \, dx,$$

where $\eta(x)$ will be defined in the sequel. We have

$$-\frac{d}{dt}I_\eta(v) = -2\int \eta(x)v(\mathscr{L}v_x)\,dx$$

$$= -2\int \eta(x)v((-\partial_x^2+1-4Q^3)v_x)\,dx$$

$$= 2\int \eta v\partial_x^3 v\,dx - 2\int \eta v\partial_x v\,dx + 8\int \eta Q^3 v\partial_x v\,dx$$

$$= -2\int \eta' v\partial_x^2 v\,dx - 2\int \eta\partial_x v\partial_x^2 v\,dx + \int \eta_x v^2\,dx$$

$$-4\int \eta' Q^3 v^2\,dx - 4\int \eta\partial_x\big(\mathrm{sech}^2\big(\tfrac{3}{2}x\big)\big)v^2\,dx$$

$$= 3\int \eta' v_x^2\,dx + 2\int \eta'' v\partial_x v\,dx + \int \eta' v^2\,dx - 4\int \eta' Q^3 v^2\,dx - 12\int \eta Q^2 Q' v^2\,dx.$$

As in [Martel 2006], we take

$$\eta(x) = -\frac{5}{3}\frac{Q'}{Q} = \tfrac{5}{3}\tanh\big(\tfrac{3}{2}x\big), \tag{5-3}$$

which is similar to x near 0 and bounded at ∞. Note, the sign convention here is chosen to match that of [Martel and Merle 2008]. By direct computation we have

$$\eta'(x) = Q^3(x), \qquad (Q^3\eta)' = -5Q^3 + 3Q^6,$$

$$\frac{\eta'''(x)}{\eta'(x)} = 9\big(1-\tfrac{3}{5}Q^3(x)\big), \qquad \eta^2(x) = \big(\tfrac{5}{3}\big)^2\big(1-\tfrac{2}{5}Q^3(x)\big),$$

$$\bigg(\frac{\eta''(x)}{\eta'(x)}\bigg)^2 = 9\big(1-\tfrac{2}{3}Q^3(x)\big), \qquad |\eta| \le \tfrac{5}{3}.$$

Proposition 5.1. *If v satisfies the v-KdV equation and v is orthogonal to $\tilde{Q}$ and Q', then there exists some $C > 0$ such that given η as in (5-3), we have*

$$\frac{d}{dt}I_\eta(v) + C\|\mathrm{sech}\big(\tfrac{3}{2}x\big)v\|_{H^1}^2 \le 0.$$

Proof of Proposition 5.1. Following the formalism presented above, we see

$$-\frac{d}{dt}I_\eta(v) = -2\int \mathscr{L}(\partial_x v)v\eta\,dx = 3\int (\partial_x v)^2\eta'\,dx + \int v^2(-\eta''' + \eta' - 4(Q^3\eta)')\,dx.$$

Selecting

$$\tilde{w}(t,x) = v(t,x)\sqrt{\eta'(x)}$$

we see

$$-\frac{d}{dt}I_\eta(v) = 3\int (\partial_x\tilde{w})^2\,dx + \int A(x)\tilde{w}^2\,dx,$$

where

$$A(x) = 1 + \frac{1}{2}\frac{\eta'''}{\eta'} - \frac{3}{4}\left(\frac{\eta''}{\eta'}\right)^2 - 4\frac{(Q^3\eta)'}{\eta'} = \frac{75}{4} - 12Q^3.$$

Hence,

$$-\frac{d}{dt}I_\eta(v) = 3\left(\langle \mathscr{L}\tilde{w}, \tilde{w}\rangle + \frac{21}{4}\int \tilde{w}^2\,dx\right).$$

Since $\mathscr{L}\partial_x Q = 0$, we know that given $v = Q$, we have

$$-\frac{d}{dt}I_\eta(v) = 0.$$

However, $v = Q$ corresponds directly to $\tilde{w} = Q^{5/2}$, which is the ground state or $\mathscr{L}$, which has exact eigenvalue $-21/4$. Then, since $\langle Q, \tilde{Q}\rangle \neq 0$, our orthogonality condition $v \perp \tilde{Q}$ is enough to guarantee that there exists $C > 0$ such that

$$\tilde{B}(\tilde{w}, \tilde{w}) \geq C\|\tilde{w}\|_{H^1}^2 = C\|\sqrt{\eta'}v\|_{H^1}^2,$$

which is the desired result. $\qquad\qquad\square$

We note in the case of more general weight functions η, virial identity methods are still applicable even if perhaps analytic proofs of the virial identities are more challenging.

By choosing the multiplier $\gamma(v - v_{xx})$ with $\gamma = \gamma_0(x - t)$ for γ_0 as in (3-7), we see

$$\frac{d}{dt}\int \gamma(v^2 + v_x^2)\,dx = -3\int \gamma'v_x^2\,dx + \int \gamma^{(3)}v^2\,dx - \int \gamma'v^2\,dx$$
$$+ 4\int \gamma'Q^3v^2\,dx + 12\int \gamma Q^2Q'v^2\,dx$$
$$- \int(3\gamma'v_{xx}^2 + \gamma'v_x^2 - \gamma^{(3)}v_x^2)\,dx + \int 4\gamma'Q^3v_x^2\,dx, \quad (5\text{-}4)$$

which consists of a number of negative semidefinite terms. All nonnegative semidefinite terms are easily dominated by a multiple of $\|v\|_{H^1_{\mathrm{sech}(3x/2)}}^2$, the term in Proposition 5.1.

Finally, note that by direct computation

$$\partial_t\langle \mathscr{L}^{-1}v, v\rangle = 0. \quad (5\text{-}5)$$

Now, let us define an energy for the solution v of (5-2) to be

$$E(v) = \int \gamma(x)(v^2 + v_x^2)\,dx + \lambda_E\int \eta(x)v^2\,dx + \Lambda_E\langle \mathscr{L}^{-1}v, v\rangle, \quad (5\text{-}6)$$

where $\eta(x)$ is chosen as in (5-3).

Proposition 5.2. *Let us assume v satisfies the v-KdV equation and v is orthogonal to $\tilde{Q}$ and Q'. There exist $\lambda_E, \Lambda_E, \delta > 0$ such that*

$$E(v) \sim \|v\|_{H^1}^2 \tag{5-7}$$

$$\frac{d}{dt}E(v) + \delta\|v\|_{H^2_{\sqrt{\gamma'}}}^2 \le 0. \tag{5-8}$$

Proof. From (5-4) and the proof of Proposition 5.1, we see easily one may choose a λ_E that depends only on δ and C so that (5-8) holds for all $\Lambda_2 > 0$. We choose Λ_E large to achieve $E(v) \ge C''\|v\|_{H^1}^2$. There exists some constant C' such that $E(v) \le C'\|v\|_{H^1}^2$. Thus the estimate follows given the orthogonality conditions on v. ☐

The assertions of Proposition 5.2 are robust under suitable perturbations. We turn to the analysis of the time dependent problem

$$v_t - (-\partial_x^2 - 4Q_{c(t),y(t)}^3)\partial_x v = \alpha(t)Q_{c(t),y(t)} + \beta(t)Q'_{c(t),y(t)}, \tag{5-9}$$

where

$$\alpha(t) = -\frac{(\dot{c}/c)\langle v, \tilde{\tilde{Q}}_{c(t),y(t)}\rangle + (\dot{y} - c^2)\langle v, \tilde{Q}'_{c(t),y(t)}\rangle}{\langle Q_{c(t),y(t)}, \tilde{Q}_{c(t),y(t)}\rangle}, \tag{5-10}$$

$$\beta(t) = -\frac{(\dot{c}/c)\langle v, \tilde{Q}'_{c(t),y(t)}\rangle + (\dot{y} - c^2)\langle v, \tilde{Q}''_{c(t),y(t)}\rangle}{\langle Q'_{c(t),y(t)}, Q'_{c(t),y(t)}\rangle}. \tag{5-11}$$

Here,

$$\tilde{\tilde{Q}}_{c(t),y(t)} = \tfrac{2}{3}\tilde{Q}_{c(t),y(t)} + x\,\tilde{Q}'_{c(t),y(t)} = c(t)\partial_c\tilde{Q}_{c(t),y(t)}. \tag{5-12}$$

For simplicity of exposition, in the sequel we suppress the t and y dependence and instead write simply $Q_{c(t),y(t)} = Q_c$ unless we want to stress the dependence on $y(t)$ and t. Similarly we recall

$$\mathscr{L}_c v = \mathscr{L}_{c,y} v = -v_{xx} + c^2 v - 4Q_{c,y}^3 v. \tag{5-13}$$

The terms on the right hand side ensure that $\langle v(0), Q'_{c(0),y(0)}\rangle = 0$ implies $\langle v(t), Q'_{c(t),y(t)}\rangle = 0$, and, in addition, $\langle v(0), \tilde{Q}_{c(0),y(0)}\rangle = 0$ implies $\langle v(t), \tilde{Q}_{c(t),y(t)}\rangle = 0$. We choose $\gamma(x,t) = \gamma_0(x - y(t))$ and we prove the following:

Proposition 5.3. *There exists a $\delta, \lambda, \Lambda > 0$ such that the following is true: Suppose that*

$$|c(t) - 1| + |\dot{c}(t)| + |\dot{y}(t) - c^2(t)| < \delta \tag{5-14}$$

for all $t \ge 0$ and define

$$E(v) = \int \gamma(x,t)(v^2 + v_x^2)\,dx + \lambda \int \eta_{1,y(t)}(x)v^2\,dx + \Lambda\langle\mathscr{L}_{1,y(t)}^{-1}v, v\rangle, \tag{5-15}$$

where we suppress the dependence of E and v on t. Then

$$E(v) \sim \|v\|_{H^1}^2, \tag{5-16}$$

for all t > 0 provided

$$\langle v, Q'_c \rangle = \langle v, \tilde{Q}_c \rangle = 0. \tag{5-17}$$

Moreover, if v satisfies the system consisting of (5-9), (5-10) *and* (5-11) *and* $v(\cdot, 0)$ *is orthogonal to* $\tilde{Q}_{c(0),y(0)}$ *and* $Q'_{c(0),y(0)}$ *(which implies the orthogonality for all t) we have*

$$\frac{d}{dt} E(v) + \delta \|v\|^2_{H^2_{\sqrt{\gamma'}}} \leq 0. \tag{5-18}$$

Proof. Since $\langle v(t), Q'_{c(t),y(t)} \rangle = 0$, we have

$$\langle \mathscr{L}^{-1}_{c(t),y(t)} v, v \rangle \geq C \|v\|^2_{H^{-1}}$$

for some $C > 0$, as seen in (4-5). Here and in the remaining part of this section we use the Moore–Penrose inverse, which is by an abuse of notation given by the orthogonal projection to the complement of Q', followed by an inversion of $\mathscr{L}$ on this orthogonal subspace. Let us look at a slightly different quantity (where we replace c by 1) given by $\langle \mathscr{L}^{-1}_{1,y(t)} v, v \rangle$. Then, since $\langle v, Q'_c \rangle = \langle v, \tilde{Q}_c \rangle = 0$, for $|c - 1|$ small enough we have

$$\langle \mathscr{L}^{-1}_{1,y(t)} v, v \rangle \geq 2C \|P^{\perp}_{Q'_{1,y(t)}} v\|^2_{H^{-1}} \geq 2C \|v\|^2_{H^{-1}} - C' |c - 1| \|v\|^2_{H^{-1}} \geq C \|v\|^2_{H^{-1}}$$

for some constants $C, C' > 0$ and $\delta \leq C/C'$. The properties are similar to the previous proposition, but the calculations are more tedious. We consider them to be important for the understanding of the linearization. We recall that we suppress the dependence of Q and $\mathscr{L}$ on y in the notation below. Then we have

$$\frac{d}{dt} \langle \mathscr{L}^{-1}_1 v, v \rangle = 2 \langle v_t, \mathscr{L}^{-1}_1 v \rangle - 12 \dot{y} \langle Q_1^2 Q'_1 \mathscr{L}^{-1}_1 v, \mathscr{L}^{-1}_1 v \rangle + 2 \dot{y} \frac{\langle v, Q'_1 \rangle}{\langle Q'_1, Q'_1 \rangle} \langle Q''_{1,y}, \mathscr{L}^{-1}_1 v \rangle$$

$$= 2I_1 - 12I_2 + 2I_3,$$

where I_2 originates from the differentiation of the inverse and I_3 from the dependence of the implicit projection on time. We have

$$I_1 = \langle \mathscr{L}_c \partial_x v, \mathscr{L}^{-1}_1 v \rangle - c^2 \langle \partial_x v, \mathscr{L}^{-1}_1 v \rangle + \alpha \langle Q'_c, \mathscr{L}^{-1}_1 v \rangle + \beta \langle Q_c, \mathscr{L}^{-1}_1 v \rangle,$$

$$\langle \mathscr{L}_c \partial_x v, \mathscr{L}^{-1}_1 v \rangle = \langle (\mathscr{L}_c - \mathscr{L}_1) \partial_x v, \mathscr{L}^{-1}_1 v \rangle = (c^2 - 1) \langle \partial_x v, \mathscr{L}^{-1}_1 v \rangle + 4 \langle (Q_c^3 - Q_1^3) \partial_x v, \mathscr{L}^{-1} v \rangle,$$

$$[\mathscr{L}^{-1}_1, \partial_x] = -\mathscr{L}^{-1}_1 [\mathscr{L}_1, \partial_x] \mathscr{L}^{-1}_1 = 12 \mathscr{L}^{-1}_1 Q_1^2 Q'_1 \mathscr{L}^{-1}_1 + \mathscr{L}^{-1}_1 [\partial_x, P_{Q'_1}] + [\partial_x, P_{Q'_1}] \mathscr{L}^{-1}_1,$$

$$[\partial_x, P_{Q'_1}] v = -\frac{\langle v, Q'_1 \rangle}{\langle Q'_1, Q'_1 \rangle} Q''_1 + \frac{\langle v, Q''_1 \rangle}{\langle Q'_1, Q'_1 \rangle} Q'_1,$$

$$\langle \partial_x v, \mathscr{L}^{-1}_1 v \rangle = \tfrac{1}{2} \langle v, [\mathscr{L}^{-1}_1, \partial_x] v \rangle = 6 \langle Q_1^2 Q'_1 \mathscr{L}^{-1}_1 v, \mathscr{L}^{-1}_1 v \rangle,$$

$$\langle v, Q'_1 \rangle = \langle v, Q'_1 - Q'_c \rangle$$

by the orthogonality conditions and

$$\langle \mathscr{L}^{-1}_1 v, Q'_c \rangle = \langle \mathscr{L}^{-1}_1 v, Q'_c - Q'_1 \rangle \quad \text{and} \quad \langle \mathscr{L}^{-1}_1 v, Q_c \rangle = \langle \mathscr{L}^{-1}_1 v, (\mathscr{L}^{-1}_1 - \mathscr{L}^{-1}_c) Q_c \rangle$$

because of the orthogonality conditions and since $\mathcal{L}_c^{-1} Q_c = \tilde{Q}_c$.

Altogether, and applying [Lemma 4.1](#), we have

$$\left| \frac{d}{dt} \langle \mathcal{L}_1^{-1} v, v \rangle \right| \le \mathbb{O}(|c^2 - 1| + |\dot{y} - 1| + |\dot{c}|) \|v\|^2_{H^{-1/2}_{\sqrt{\gamma'}}}, \tag{5-19}$$

which we will control by the virial identity below.

We now look at virial weights of the form

$$\eta(x, t) = -\frac{5}{3} \frac{Q_1'(x - y(t))}{Q_1(x - y(t))} = \tfrac{5}{3} \tanh\big(\tfrac{3}{2}(x - y(t))\big),$$

which has properties similar to those of $\eta(x)$ with appropriate changes for the unit scaling.

We have defined v such that

$$v(x, t) \perp \tilde{Q}_{c(t), y(t)} \quad \text{and} \quad v(x, t) \perp Q_{c(t), y(t)}' \quad \text{for all } t \ge 0.$$

Following the formalism presented above and in [Martel 2006], select

$$\tilde{w}(t, x) = v(t, x)\sqrt{\eta'(x)}.$$

Then,

$$-\frac{d}{dt} I_\eta(v) = 3 \int (\partial_x \tilde{w})^2 \, dx + \int A(x)\tilde{w}^2 \, dx - 2 \int \eta_{1,y} \left(\beta(t) Q_{c,y}' + \alpha(t) Q_{c,y} \right) v \, dx$$
$$+ \tfrac{3}{2} c^2 (\dot{y} - c^2) \int \text{sech}^2\big(\tfrac{3}{2}c(x - y(t))\big) v^2 \, dx + \tfrac{3}{2} c^4 \int \text{sech}^2\big(\tfrac{3}{2}c(x - y(t))\big) v^2 \, dx,$$

where

$$A(x, t) = 1 + \frac{1}{2}\frac{\eta'''}{\eta'} - \frac{3}{4}\left(\frac{\eta''}{\eta'}\right)^2 - 4\frac{(Q^3\eta)'}{\eta'} = \frac{75}{4} - 12Q^3.$$

Hence,

$$-\frac{d}{dt} I_\eta(v) > 3\big(\langle \mathcal{L}\tilde{w}, \tilde{w} \rangle + \tfrac{21}{4}c^2 \int \tilde{w}^2 \, dx\big) + \mathbb{O}(|1^2 - c^2| + |\dot{y} - c^2|)\|v\|^2_{L^2_{\sqrt{\gamma'}}}.$$

From above, we know that $3\big(\langle \mathcal{L}\tilde{w}, \tilde{w} \rangle + \tfrac{21}{4} \int \tilde{w}^2 \, dx\big) = 0$ for $v = Q_{1,y(t)}$. This corresponds to $\tilde{w} = Q_{1,y(t)}^{5/2}$, which is the ground state or $\mathcal{L}_{1,y(t)}$. Hence, $v = Q$ is the ground state of the quadratic form

$$3\big(\langle \mathcal{L}\tilde{w}, \tilde{w} \rangle + \tfrac{21}{4} \int w^2 \, dx\big).$$

From [Lemma 4.1](#), our orthogonality condition $v \perp \tilde{Q}_{c(t), y(t)}$ is enough to guarantee there exists $\delta > 0$ such that

$$\frac{d}{dt} I_\eta(v) + \|v\|^2_{H^1_{\sqrt{\gamma'}}} \le 0$$

provided $|c^2 - 1^2| + |\dot{y} - c^2|$ is small for all $t \ge 0$, which follows from our assumptions on the initial perturbation.

The time-dependent version of

$$\frac{d}{dt}\int \gamma(x,t)(v^2 + v_x^2)\,dx$$

is done in full generality in the analysis of (3-13) in Section 3 for the Airy equation. The terms that we have to control are the same as for constant c and $\dot{y}$, plus the terms coming from the right hand side. Those are easy to control. Namely,

$$2\left|\int \left[(\alpha Q + \beta Q')\gamma - \partial_x(\gamma(\alpha Q' + \beta Q''))\right]v\,dx\right| \lesssim (|\dot{c}| + |\dot{y} - c^2|)\|\,|v|^2\|_{L^2_{\sqrt{\gamma'}}}$$

for γ as in Section 3. $\qquad\qquad\square$

Note, above we have always assumed the proper orthogonality conditions, but without them we easily obtain the following estimate for solutions of the v equation:

$$\|v\|_{L^\infty H^1 \cap L^2 H^2_{\sqrt{\gamma'}}} \le C\big(\|v(0)\|_{H^1} + \sup_t |\langle v(\cdot,t), Q'_{1,y(t)}\rangle| + \|\langle v(\cdot,t), \tilde{Q}_{1,y(t)}\rangle\|_{L^2([0,\infty))}\big). \qquad (5\text{-}20)$$

6. Function spaces and projection operators

In this section we construct the function spaces for our nonlinear analysis using properties of the linear evolution we studied in Sections 3–5. Based on the energy functional (5-6) for the v-equation, it seems natural to look at

$$v \in X^1 = L^\infty H^1 \cap L^2 H^2_{\sqrt{\gamma'}},$$

where $\gamma = \gamma_0(x)$ is as in (3-5), and again by convention we set $L^p X$ to be the L^p norm in time of the X norm in space.

Then, as follows naturally from the equation, we define $Y^1 = L^1 H^1 + L^2 \sqrt{\gamma'} L^2$.

Generically, we define

$$X^s = L^\infty H^s \cap L^2 H^{s+1}_{\sqrt{\gamma'}} \quad \text{and} \quad Y^s = L^1 H^s + L^2 \sqrt{\gamma'} H^{s-1},$$

where we note $(Y^s)^* = X^{-s}$.

6.1. *The scale of energy spaces.* Let us study the v-equation

$$\begin{cases} (\partial_t - \mathcal{L}\partial_x)v = f_0 + \sqrt{\gamma'} f_1 = f, \\ v(0, x) = v_0, \end{cases} \qquad (6\text{-}1)$$

where $f_0 \in L^1 H^s$, $f_1 \in L^2 L^{s-1}$ and $v_0 \in H^s$. We assume that the orthogonality conditions

$$v_0 \perp Q', \quad v_0 \perp \tilde{Q} \qquad (6\text{-}2)$$

and

$$(f_0 + \sqrt{\gamma'} f_1) \perp \tilde{Q}, \quad (f_0 + \sqrt{\gamma'} f_1) \perp Q' \quad \text{for all } t \qquad (6\text{-}3)$$

hold.

Proposition 6.1. *There exists a unique solution $v \in X^s$ that satisfies*

$$\|v\|_{X^s} \le c(\|v_0\|_{H^s} + \|f_0 + \sqrt{\gamma'} f_1\|_{Y^s}).$$

Moreover, $v(t)$ is orthogonal to Q' and $\tilde{Q}$.

Note, Theorem 1 is an immediate consequence.

Proof. We begin by considering the case $s = 1$. The previous section implies the estimate

$$\|v\|_{X^s} \le c(\|v_0\|_{H^s} + \|f_0\|_{L^1(H^s)}).$$

if $f_1 = 0$ by a variation of constants argument. We retrace the steps and its modifications needed for f_1. Using the multipliers from the energy inequalities, we need the obvious estimates

$$\left| \int f_0 \gamma (v - v_{xx}) \, dx \, dt \right| + \left| \int f_0 \eta v \, dx \, dt \right| + \left| \int f_0 \mathcal{L}^{-1} v \, dx \, dt \right| \le c \|v\|_{L^\infty H^1} \|f_0\|_{L^1 H^1}$$

and, using Lemma 4.1,

$$\left| \int \sqrt{\gamma'} f_1 (\gamma (v - v_{xx}) \, dx \, dt \right| + \left| \int \sqrt{\gamma'} f_1 \eta v \, dx \, dt \right| + \left| \int \sqrt{\gamma'} f_1 \mathcal{L}^{-1} v \, dx \, dt \right| \le c \|f_1\|_{L^2} \|v\|_{L^2 H^2_{\sqrt{\gamma'}}}.$$

It is not hard to see that $v(t)$ remains orthogonal to Q' and $\tilde{Q}$ so that we can close the argument as in the previous section. We obtain the desired estimate for $s = 1$:

$$\|v\|_{X^1} \le c(\|v_0\|_{H^1} + \|f_0 + \sqrt{\gamma'} f_1\|_{Y^1}).$$

We denote the solution operator for the inhomogeneous v-problem (u-problem) by S_v (S_u) and we write

$$\|S_v f\|_{X^1} \le c \|f\|_{Y^1}. \tag{6-4}$$

The role of the two orthogonality conditions are different: The equation is invariant under the addition of a multiple of Q to v, and orthogonality to Q' is conserved. Orthogonality to $\tilde{Q}$ was needed for the virial identity of Martel and Merle, whereas orthogonality of v and Q' entered the control of the H^{-1} norm by the Moore–Penrose inverse of $\mathcal{L}$. Without orthogonality one still obtains (5-20).

Suppose now that v satisfies

$$\begin{cases} v_t - \mathcal{L} \partial_x v = f, \\ v(x, 0) = v_0. \end{cases} \tag{6-5}$$

Let ε be a small constant. We apply $(1 + \varepsilon^2 D^2)^{(s-1)/2}$ to both sides of the equation and denote $v^s = (1 + \varepsilon^2 D^2)^{(s-1)/2} v$. It satisfies

$$v^s_t - \mathcal{L} \partial_x v^s = (1 + \varepsilon^2 D^2)^{(s-1)/2} f + [(1 + \varepsilon^2 D^2)^{(s-1)/2}, 4Q^3] \partial_x v.$$

Hence, applying (5-20)

$$\|v\|_{X^s} \le c_1 \|v^s\|_{X^1}$$
$$\le c_2 \big(\|(1 + \varepsilon^2 D^2)^{(s-1)/2} f\|_{Y^1} + \|[(1 + \varepsilon^2 D^2)^{(s-1)/2}, 4Q^3] \partial_x v\|_{Y^1} + \sup_t |\langle v^s, Q' \rangle| + \|\langle v^s, \tilde{Q} \rangle\|_{L^2} \big).$$

and we turn to the commutator term.

Lemma 6.2. *Let $\phi \in C^\infty(\mathbb{R})$ satisfy $|\phi| + |\phi'| \le Ce^{-|x|}$. Let $k(x, y)$ be the kernel of the operator*

$$[(1 + \varepsilon^2 D^2)^{s/2}, \phi](1 + \varepsilon^2 D^2)^{-s/2}.$$

Then,

$$|k(x, y)| \le c\varepsilon|s|e^{-(|x|+|y|)/4 - |x-y|/(4\varepsilon)}.$$

We postpone its proof. By Lemma 6.2 (with $\phi = 4Q^3$ and $s - 1$) and Schur's Lemma

$$\begin{aligned}
\|[(1+\varepsilon^2 D^2)^{(s-1)/2}, 4Q^3]\partial_x v\|_{Y^1} &\le \|(\gamma')^{-1/2}[(1+\varepsilon^2 D^2)^{(s-1)/2}, 4Q^3](1+\varepsilon^2 D^2)^{(1-s)/2}\partial_x v_s\|_{L^2} \\
&\le c\varepsilon\|(\gamma')^{1/2}\partial_x v^s\|_{L^2},
\end{aligned} \tag{6-6}$$

and by Lemma 3.1, after rescaling, as for the constant coefficient equation, we have

$$\|(1 + \varepsilon^2 D^2)^{(s-1)/2} f\|_{Y^1} \le c\|f\|_{Y^s}.$$

For all Schwartz functions,

$$\left\|(1 + |x|^2)^N\big((1 + \varepsilon^2 D^2)^{s/2}\phi - \phi\big)\right\|_{L^2} \le C\varepsilon.$$

If $\langle v, \widetilde{Q}\rangle = \langle v, Q'\rangle = 0$, then

$$|\langle v^s, \widetilde{Q}\rangle| = \left|\langle v, \widetilde{Q}\rangle - \langle v^s, \widetilde{Q} - (1 + \varepsilon^2 D^2)^{-\frac{s}{2}}\widetilde{Q}\rangle\right| \le C\varepsilon\|\gamma'^{1/2}v^s\|_{L^2}, \tag{6-7}$$

$$|\langle v^s, Q'\rangle| = \left|\langle v, Q'\rangle - \langle v^s, Q' - (1 + \varepsilon^2 D^2)^{-\frac{s}{2}}Q'\rangle\right| \le C\varepsilon\|v^s\|_{L^2}. \tag{6-8}$$

Suppose that $\langle f, Q'\rangle = \langle f, \widetilde{Q}\rangle = 0$. Then we obtain for all $s \in \mathbb{R}$ from (6-6), (6-7) and (6-8)

$$\|v^s\|_{X^s} \le c(\|f\|_{Y^s} + \varepsilon\|v^s\|_{X^s})$$

and hence

$$\|v\|_{X^s} \lesssim \|f\|_{Y^s}, \tag{6-9}$$

which again implies for solutions v to $v_t - \mathcal{L}\partial_x v = P_{Q'}f$, given by the variation of constants formula, the bound $\|\tilde{P}^* v\|_{X^s} \le C\|f\|_{Y^s}$ or equivalently (recall (1-11))

$$\|\tilde{P}^* S_v P_{Q'}^\perp\|_{Y^s \to X^s} \lesssim 1. \tag{6-10}$$

Using spacetime duality, we consider

$$\begin{cases} (\partial_t - \partial_x \mathcal{L})u = g, \\ u(0, x) = 0. \end{cases}$$

The estimate adjoint to (6-10) is

$$\|P_{Q'}^\perp S_u \tilde{P}\|_{Y^s \to X^s} \lesssim 1, \tag{6-11}$$

which completes the proof. $\qquad\square$

Proof. We turn to the proof of Lemma 6.2.

Let $\hat{\phi}$ be the Fourier transform of ϕ, which, because of the exponential decay extends to a holomorphic function $\hat{\phi}$ in the strip $\{z : |\text{Im } z| < 1\}$. Moreover there exists C such that

$$\int |\hat{\phi}(\xi + i\sigma)| \, d\xi \leq C \quad \text{if } |\sigma| \leq \tfrac{1}{2}.$$

This estimate in turn implies exponential decay. Let $k(x, y)$ be the integral kernel of

$$[(1 + \varepsilon^2 D^2)^{s/2}, \phi](1 + \varepsilon^2 D^2)^{-s/2}.$$

We claim

$$|k(x, y)| \leq c_N \varepsilon |s| e^{-\delta(|x|+|y|)} e^{-\delta|x-y|/\varepsilon}, \tag{6-12}$$

which implies Lemma 6.2.

The symplectic Fourier transform

$$\hat{k}(\xi, \eta) = \frac{1}{2\pi} \int e^{-i\xi x + iy\eta} k(x, y) \, dx \, dy$$

satisfies

$$\hat{k}(\xi, \eta) = \left(\left(\frac{1 + \varepsilon^2 \xi^2}{1 + \varepsilon^2 \eta^2}\right)^{s/2} - 1\right)\hat{\phi}(\xi - \eta).$$

We set $a = \varepsilon(\xi + \eta)/2$ and $b = (\xi - \eta)/2$. Then $\hat{k}(\xi, \eta) = \hat{g}(a, b)$, where

$$\hat{g}(a, b) = \left(\left(\frac{1 + (\varepsilon b + a)^2}{1 + (\varepsilon b - a)^2}\right)^{s/2} - 1\right)\hat{\phi}(2b)$$

and

$$k(x, y) = (2\pi)^{-1} \int e^{i(x\xi - y\eta)} \hat{g}(\varepsilon(\xi + \eta)/2, (\xi - \eta)/2)) \, d\xi \, d\eta$$

$$= 2\varepsilon(2\pi)^{-1} \int e^{i(\frac{x-y}{\varepsilon}a + b(x+y))} \hat{g}(a, b) \, da \, db =: 2\varepsilon g((x - y)/\varepsilon, x + y).$$

The function $\hat{g}$ expands to a holomorphic function in a to the strip $\{z : |\text{Im } z| < 1/2\}$ if $\varepsilon| \text{Im } b| < 1/2$. Clearly,

$$\frac{1 + (a + \varepsilon b)^2}{1 + (a - \varepsilon b)^2} = 1 + \frac{4(\varepsilon b)^2}{1 + (a - \varepsilon b)^2} + 4\varepsilon b \frac{a - \varepsilon b}{1 + (a - \varepsilon b)^2},$$

and hence we define the error term h by the right hand side of

$$\left(\frac{1 + (\varepsilon b + a)^2}{1 + (\varepsilon b - a)^2}\right)^{s/2} - 1 = 2s\varepsilon b \frac{a - \varepsilon b}{1 + (\varepsilon b - a)^2} + h(\varepsilon b, a).$$

It satisfies

$$|h(\varepsilon b, a)| \leq cs^2\varepsilon^2 |b|^2 (1 + |\varepsilon b - a|)^{-2} \quad \text{if } |\varepsilon \text{ Im } b + a| \leq 1/2.$$

Hence,

$$\left|\int e^{i(av + bw)} h(\varepsilon b, a)\hat{\phi}(2b) \, da \, db\right| \leq cs^2 \varepsilon^2 e^{-(|v|+|w|)/4}$$

by the extension of a and b to a suitable complex strip. The leading term contributing to g can be calculated:

$$g_0(v, w) = (2\pi)^{-1} \int e^{i(av+bw)} \frac{a - \varepsilon b}{1 + (a - \varepsilon b)^2} b \hat{\phi}(2b) \, da \, db$$

$$= i \frac{v}{|v|} e^{-|v|} \int e^{i(b(w+\varepsilon v))} b \hat{\phi}(2b) \, db = \sqrt{\pi} 2i \frac{v}{|v|} e^{-|v|} \phi'((w + \varepsilon v)/2)).$$

The leading term for k is

$$k_0(x, y) = \sqrt{\pi} 2i\varepsilon \frac{x - y}{|x - y|} e^{-|x-y|/\varepsilon} \phi'(x). \qquad \square$$

6.2. *U and V space estimates.* In this section, we generalize and improve Theorem 1 using the U^p and V^p spaces as defined in [Hadac et al. 2009] and in the appendix. For notational simplicity, let us define

$$U^p = U^p_{KdV} \quad \text{and} \quad V^p = V^p_{KdV}.$$

We begin with a number of estimates that we will use often in the sequel.

Let $c, y \in C^1$ satisfy (5-14) and let $\gamma(x, t) = \gamma_0(x - y(t))$. Then,

$$\|a Q'_{c(t),y(t)} + b \tilde{Q}_{c(t),y(t)}\|_{Y^0} \lesssim \|a\|_{L^2+L^1} + \|b\|_{L^2+L^1};$$

hence

$$\|P^{\perp}_{Q'} \tilde{P} f\|_{DU^2 + L^2 \sqrt{\gamma'} H^{-1}} \lesssim \|f\|_{DU^2} + \|\langle f, Q \rangle\|_{L^2+L^1} + \|\langle f, Q' \rangle\|_{L^2+L^1}.$$

We consider

$$w_t + w_{xxx} = f, \quad \text{with } w(0) = u_0.$$

Then,

$$\|w\|_{U^2} \lesssim \|u_0\|_{L^2} + \|f\|_{DU^2}$$

and, since $U^2 \subset L^2 H^1_{\sqrt{\gamma'}}$,

$$\|\langle w, Q \rangle\|_{L^2} + \|\langle w, \tilde{Q} \rangle\|_{L^2} \lesssim \|f\|_{DU^2} + \|u_0\|_{L^2}.$$

Hence, with $v = \tilde{P} P^{\perp} w$, we have

$$\|v\|_{L^2 H^1_{\sqrt{\gamma'}}} \lesssim \|f\|_{DU^2} + \|u_0\|_{L^2}.$$

We calculate

$$(\partial_t + c^2 - \partial_x \mathscr{L}) \left(\frac{\langle w, Q \rangle}{\langle Q, \tilde{Q} \rangle} \tilde{Q} + \frac{\langle w, Q' \rangle}{\langle Q', \tilde{Q}' \rangle} Q' \right) = \frac{\dot{c}}{c} \frac{\langle w, Q \rangle}{\langle Q, \tilde{Q} \rangle} \tilde{\tilde{Q}} + (\dot{y} - c^2) \frac{\langle w, Q' \rangle}{\langle Q', \tilde{Q}' \rangle} Q''$$

$$+ \left((\dot{y} - c^2) \frac{\langle w, Q \rangle}{\langle Q, \tilde{Q} \rangle} + \frac{\dot{c}}{c} \frac{\langle w, Q' \rangle}{\langle Q', \tilde{Q}' \rangle} \right) \tilde{Q}' - \tilde{\alpha} \tilde{Q} - \tilde{\beta} Q',$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the time derivatives of the coefficients of $\tilde{Q}$ and Q' and

$$\tilde{\tilde{Q}} = (x - y) \tilde{Q}' + \tfrac{2}{3} \tilde{Q}.$$

Hence, assuming that u_0 satisfies the orthogonality conditions, that w and v are as above, and with g defined through the previous calculations,

$$\partial_t v + c^2 v_x - \partial_x \mathcal{L}_{c,y} v = \tilde{\alpha} \tilde{Q} + \tilde{\beta} Q' + g + f, \quad \text{with } v(0) = u_0, \tag{6-13}$$

where we collect the properties of v and g in the following.

Lemma 6.3. *Assuming* (5-14), *we have* $\langle v(t), Q \rangle = \langle v(t), Q' \rangle = 0$ *and*

$$\|v\|_{V^2 \cap L^2 H^1_{\sqrt{\gamma'}}} + \|g\|_{L^2 \sqrt{\gamma'} H^{-1}} \lesssim \|u_0\|_{L^2} + \|f\|_{DU^2} + \|\langle f, Q \rangle\|_{L^2 + L^1} + \|\langle f, Q' \rangle\|_{L^2 + L^1}. \tag{6-14}$$

Proof. We claim that

$$\|\tilde{\alpha}\|_{L^1 + L^2} + \|\tilde{\beta}\|_{L^1 + L^2} \le c(\|w_0\|_{L^2} + \|f\|_{DU^2}),$$

the proof of which we postpone. Assuming its validity we put the term $4\partial_x Q v$ in (6-13) on the right hand side. We bound $\|v\|_{V^2}$ in terms of $\|w_0\|_{L^2}$ and the right hand side in DV^2. Since $DU^2 \subset DV^2$ and $L^2 \sqrt{\gamma'} H^{-1} \subset DV^2$, we can control all terms on the right hand side.

The only missing piece is the $L^2 + L^1$ bound for α and β. There are two different arguments: Either we can follow the calculation above and calculate $\tilde{\alpha}$ and $\tilde{\beta}$ above, or we can test by Q and Q' and use orthogonality to obtain the standard equations for α and β. We use the first approach and recall the calculations after (1-10). Then

$$\begin{aligned}
\frac{d}{dt}\langle w, Q \rangle &= \langle f, Q \rangle + \dot{y}\langle w, Q' \rangle + \frac{\dot{c}}{c}\langle w, \tilde{Q} \rangle, \\
\frac{d}{dt}\langle w, Q' \rangle &= \langle f, Q' \rangle + \dot{y}\langle w, Q'' \rangle + \frac{\dot{c}}{c}\langle w, \tilde{Q}' \rangle.
\end{aligned} \tag{6-15}$$

There is one more term entering the coefficient of Q' coming from applying the linear operator to $\tilde{Q}$, which gives

$$-2\frac{\langle w, Q \rangle}{\langle Q, \tilde{Q} \rangle} c^2 Q'.$$

All these terms are easily controlled. $\qquad\square$

We return to the analysis of the time dependent v-problem

$$v_t + c^2 c_x - \mathcal{L}\partial_x v = \alpha(t) Q_{c(t),y(t)} + \beta(t) Q'_{c(t),y(t)} + f, \tag{6-16}$$

where

$$\alpha(t) = -\frac{(\dot{c}/c)\langle v, \tilde{Q}_{c(t),y(t)} \rangle + (\dot{y} - c^2)\langle v, \tilde{Q}'_{c(t),y(t)} \rangle}{\langle Q_{c(t),y(t)}, \tilde{Q}_{c(t),y(t)} \rangle} - \frac{\langle \tilde{Q}, f \rangle}{\langle Q, \tilde{Q} \rangle}, \tag{6-17}$$

$$\beta(t) = -\frac{(\dot{c}/c)\langle v, \tilde{Q}'_{c(t),y(t)} \rangle + (\dot{y} - c^2)\langle v, \tilde{Q}''_{c(t),y(t)} \rangle}{\langle Q'_{c(t),y(t)}, Q'_{c(t),y(t)} \rangle} - \frac{\langle Q', f \rangle}{\langle Q', Q' \rangle}, \tag{6-18}$$

with the initial data $v(x, 0) = v_0(x)$ orthogonal to $\tilde{Q}$ and Q'. Then also $v(t)$ satisfies these orthogonality conditions. We combine the arguments of the previous subsection with those of Proposition 5.3:

Lemma 6.4. *Suppose that* (5-14) *holds. There exists a unique solution v to* (6-16) *and* (6-17) *and* (6-18) *that satisfies*

$$\|\langle D\rangle^s v\|_{X^0 \cap V^2} \le c(\|v_0\|_{H^s} + \|\langle D\rangle^s f\|_{Y^0 + DV^2}).$$

Moreover $v(t)$ is orthogonal to Q' and $\tilde{Q}$.

Proof. We begin with $s = 0$. We write $f = f_U + f_Y$ with $f_U \in DU^2$ and $f_Y \in Y^0$. Let $\tilde{v}$ be defined with $f = f_U$ as in Lemma 6.3. It satisfies

$$\|\tilde{v}\|_{V^2 \cap X^0} \le C(\|f_U\|_{DU^2} + \|u_0\|_{L^2}).$$

Let us take $v = \tilde{v} + w$, where w satisfies

$$w_t + w_x - \partial_x \mathcal{L}w = \alpha \tilde{Q} + \beta Q' + f_Y + g, \quad \text{with } w(0) = 0,$$

with g as in Lemma 6.3 and by Lemma 6.4

$$\|w\|_{X^0} \lesssim \|f_Y\|_{Y^0} + \|g\|_{Y^0} \lesssim \|f\|_{DU^2 + Y^0} + \|u_0\|_{L^2}.$$

We put the term $4\partial_x(Q^3 w)$ to the right hand side, which we easily control in Y^0 as well as α and β and we arrive at

$$\|v\|_{V^2 \cap X^0} \le C(\|v_0\|_{L^2} + \|f\|_{Y^0 + DU^2} + \|\langle f, Q\rangle\|_{L^2 + L^1} + \|\langle f, Q'\rangle\|_{L^2 + L^1}). \tag{6-19}$$

The case of general s follows by the same arguments as above. □

Our main interest will be in similar estimates for the u problem below.

We consider the u equations

$$u_t + c^2 u_x - \partial_x(\mathcal{L}_{c,y} u) = \alpha \tilde{Q} + \beta Q' + f, \tag{6-20}$$

with initial data $u(0) = u_0$ that satisfies $\langle u_0, Q\rangle = \langle u_0, Q'\rangle = 0$, together with the modal equations

$$\alpha(t) = -\frac{(\dot{c}/c)\langle u, \tilde{Q}\rangle + \langle f, Q\rangle}{\langle Q, \tilde{Q}\rangle}, \tag{6-21}$$

$$\beta(t) = -\frac{(\dot{y} - c^2)\langle u, Q''\rangle + (\dot{c}/c)\langle u, \tilde{Q}'\rangle + \langle u, \mathcal{L}Q_{xx}\rangle + \langle f, Q'\rangle}{\langle Q', Q'\rangle}, \tag{6-22}$$

which again ensures the orthogonality of $u(t)$ with Q and Q'.

We obtain first the analog of Lemma 6.4.

Lemma 6.5. *Suppose that* (5-14) *holds. There exists a unique solution u to* (6-20), (6-21) *and* (6-22) *that satisfies*

$$\|u\|_{X^0 \cap U^2} \le c(\|u_0\|_{L^2} + \|f\|_{Y^0 + DV^2}).$$

Moreover, $u(t)$ is orthogonal to Q' and Q.

It is not difficult to construct solutions; however we are interested in global estimates. Moreover we may restrict to a finite time interval and assume that all the data as well as u are smooth and decay at infinity.

We set $v = \mathcal{L}u$. It satisfies the orthogonality conditions

$$\langle v, Q' \rangle = 0 = \langle u, Q \rangle = \langle \mathcal{L}^{-1}v, Q \rangle = \langle v, \tilde{Q} \rangle.$$

Moreover, v satisfies

$$v_t + c^2 v_x - \mathcal{L}\partial_x v = -2c^2 \alpha Q + 12Q^2((\dot{c}/c)\tilde{Q} + (\dot{y} - c^2)Q')u + \mathcal{L}f$$

and we may apply Lemma 6.4 with $s = -2$:

$$\|v\|_{X^{-2}} \lesssim \|\mathcal{L}u(0)\|_{H^{-2}} + \|\mathcal{L}f\|_{Y^{-2}} + (|\dot{c}/c| + |\dot{y} - c^2|)\|u\|_{L^2(H^{-3}_{\sqrt{\gamma'}})}.$$

We apply Lemma 4.1 several times to get

$$\|u\|_{X^0} \lesssim \|\mathcal{L}v\|_{X^{-2}} \lesssim \|u_0\|_{L^2} + \|f\|_{Y^0} + \sup_t(|\dot{c}/c| + |\dot{y} - c^2|)\|u\|_{X^{-1}}.$$

To complete the proof we observe that by (5-14) we may subtract the last term on the right hand side from both sides to arrive at the desired estimate. The inclusion of V^2 and DU^2 works now exactly as for the v equation.

We collect the results for the case $s = 0$, which is the only estimate we will need later on.

Proposition 6.6. *Suppose that* (5-14) *holds. There exists a unique solution v to* (6-16), (6-17) *and* (6-18) *that satisfies*

$$\|v\|_{V^2 \cap X^0} \le c(\|v_0\|_{L^2} + \|f\|_{DU^2 + Y^0}).$$

Moreover, $v(t)$ is orthogonal to Q' and $\tilde{Q}$. Similarly there is a unique solution u to (6-20), (6-21) *and* (6-22) *that satisfies*

$$\|u\|_{V^2 \cap X^0} \le c(\|u_0\|_{L^2} + \|f\|_{DU^2 + Y^0}).$$

Moreover, $u(t)$ is orthogonal to Q' and Q.

6.3. Littlewood–Paley decomposition. We consider functions c and y satisfying (5-14) We set $\lambda \in \Lambda_0 = 1.01^{\mathbb{N}}$ and let P_λ be the Littlewood–Paley decomposition with Fourier multipliers supported in the set $\{\xi : 1.01^{-1}\lambda \le |\xi| \le 1.01\lambda\}$ if $\lambda > 1$ and $\{\xi : |\xi| \le 1\}$ if $\lambda = 0$. Then, we denote

$$u_\lambda = P_\lambda u.$$

The Besov spaces are defined as the set of all tempered distributions for which the norm

$$\|v\|_{B_q^{s,p}} = \|\lambda^s\|v_\lambda\|_{L^p}\|_{l^q(\Lambda_0)}$$

is finite. Here $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Similarly we define the homogeneous spaces $\dot{B}_q^{s,p}$ with the summation over $\Lambda = 1.01^{\mathbb{Z}}$, where the frequency $\lambda = 1$ plays no special role. There is an ambiguity

about the meaning of v_0, which differs depending on whether we consider $B_q^{s,p}$ or the homogeneous space $\dot{B}_q^{s,p}$.

We define the spaces X_∞^s and Y_∞^s using the norms

$$\|u\|_{X_\infty^s} = \sup_{\lambda \in \Lambda_0} \lambda^s \|u_\lambda\|_{V^2 \cap X^0} \quad \text{and} \quad \|f\|_{Y_\infty^s} = \sup_{\lambda \in \Lambda_0} \lambda^s \|f_\lambda\|_{DU^2+Y^0}.$$

The homogeneous spaces $\dot{X}_\infty^s$ and $\dot{Y}_\infty^s$ are defined in the same way as the homogeneous Besov space $\dot{B}_q^{s,p}$ with $\Lambda = 1.01^{\mathbb{Z}}$, though with a slight modification for $s < 0$ in the Y spaces due to the ρ multiplier. Namely, we take

$$\|u\|_{\dot{X}_\infty^s} = \sup_{\lambda \in \Lambda} \left(\lambda^s \|u_\lambda\|_{V^2 \cap X^0} \right),$$
$$\|F\|_{\dot{Y}_\infty^s} = \inf_{F=f+g} \left(\sup_{\lambda \in \Lambda} \lambda^s \|f_\lambda\|_{DU^2} + \sup_{\lambda \in \Lambda_0} \lambda^s \|g_\lambda\|_{Y^0} \right),$$

(6-23)

where there is a slight abuse of notation since the operators in f_0 and g_0 are taking on two different meanings, the homogeneous projection for f_0 and the inhomogeneous projection for g_0.

We study

$$\begin{cases} u_t + u_x + \partial_x \mathscr{L} u = \alpha \tilde{Q} + \beta Q_x + f + \partial_x(\rho g), \\ u(x,0) = 0, \end{cases}$$

(6-24)

where α is given by (6-21) and β by (6-22). As a first step we obtain a weighted L^2 bound for u in (6-25) below.

Let $f = f^+ + f^-$ and $g = g^+ + g^-$ be a decomposition into high ($|\xi| > 1$) and low ($|\xi| \leq 1$) frequencies. We define

$$\begin{cases} v_t + c^2 v_x - \mathscr{L} v_x = \alpha_+ Q + \beta_+ Q' + (\partial_x^{-1} f^+ + \rho g^+), \\ v(x,0) = 0, \end{cases}$$

where

$$\langle Q, \tilde{Q} \rangle \alpha_+ = (c^2 - \dot{y})\langle v, \tilde{Q}' \rangle - (\dot{c}/c)\langle v, \tilde{\tilde{Q}} \rangle - \langle \partial_x^{-1} f^+ + \rho g^+, \tilde{Q} \rangle,$$
$$\langle Q', Q' \rangle \beta_+ = (c^2 - \dot{y})\langle v, Q'' \rangle - (\dot{c}/c)\langle v, \tilde{Q}' \rangle + \langle f^+ + \partial_x(\rho g^+), Q \rangle$$

ensure $\tilde{P}^* P_{Q'}^{\perp} v = 0$. Then by Proposition 6.6

$$\|v\|_{X^0} \lesssim \|\partial_x^{-1} f^+ + \rho g^+\|_{DU^2+Y^0} \lesssim \|F^+\|_{\dot{Y}_\infty^s} \lesssim \|F\|_{\dot{Y}_\infty^s},$$

where the second inequality holds for all $s > -1$.

As a simple consequence, we obtain

$$\|P_{Q'} \partial_x v\|_{L^2 L_\rho^2} \lesssim \|F\|_{\dot{Y}^s}$$

and compute similar to arguments above

$$(\partial_t - c^2 \partial_x + \partial_x \mathscr{L})(P_{Q'} \partial_x v)$$
$$= \left(\alpha_+ - \frac{d}{dt} \frac{\langle v, Q'' \rangle}{\langle Q', Q' \rangle} \right) Q' + \left(\beta_+ + \frac{\langle v, Q'' \rangle}{\langle Q', Q' \rangle} (\dot{y} - c^2) \right) Q'' + \frac{\langle v, Q'' \rangle}{\langle Q', Q' \rangle} \frac{\dot{c}}{c} \tilde{Q}' + f_+ + \partial_x(\rho g_+).$$

We make the ansatz $u = P_{Q'} \partial_x v + u_-$ and observe that $\langle \partial_x v, Q \rangle = 0$ by construction. Then,

$$\partial_t u_- + c^2 \partial_x u_- - \partial_x \mathcal{L} u_-$$
$$= \alpha \widetilde{Q} + \beta Q' + f_- + \partial_x (\rho g_-) + \left(\frac{\dot{c}}{c} \frac{\langle v, \widetilde{Q}' \rangle}{\langle Q', Q' \rangle} - \frac{\langle f^+ + \partial_x (\rho g^+), Q \rangle}{\langle Q', Q' \rangle} \right) Q'' - \frac{\langle v, Q'' \rangle}{\langle Q', Q' \rangle} \frac{\dot{c}}{c} \widetilde{Q}',$$

where α and β ensure orthogonality. Later we will need the obvious identity (integrate by parts in the second term)

$$\left\langle f_- + \partial_x (\rho g_-) - \left(\frac{\langle f^+ + \partial_x (\rho g^+), Q \rangle}{\langle Q', Q' \rangle} \right) Q'', Q \right\rangle = \langle F, Q \rangle.$$

Then, $u = \partial_x P_{Q'}^\perp v + u_-$ and hence with $F^+ = f^+ + \partial_x \rho g^+$ we have

$$\|u\|_{L^2 L_\rho^2} \lesssim \|F\|_{Y^s} + \|\langle F, Q \rangle\|_{L^2 + L^1} + \|\langle F^+, Q \rangle\|_{L^2 + L^1}. \tag{6-25}$$

By (6-21) we see

$$\|\alpha\|_{L^1} \lesssim \|\dot{c}\|_{L^2 \cap L^\infty} (\|F\|_{Y^s} + \|\langle F, Q \rangle\|_{L^2 + L^1}) + \|\langle F, Q \rangle\|_{L^1} \tag{6-26}$$

and, using (5-14)

$$\|\alpha\|_{L^2} \lesssim \|F\|_{Y^s} + \|\langle F, Q \rangle\|_{L^2} \tag{6-27}$$

and by (6-22)

$$\|\beta\|_{L^2} \lesssim \|F\|_{Y^s} + \|\langle F, Q' \rangle\|_{L^2}. \tag{6-28}$$

We turn to the frequency localized equation

$$\begin{cases} (u_\lambda)_t + (u_\lambda)_{xxx} = -P_\lambda \partial_x (4Q^3 u) + \alpha P_\lambda \widetilde{Q} + \beta P_\lambda Q_x + P_\lambda f + P_\lambda \partial_x (\rho g), \\ u^\lambda(x, 0) = 0. \end{cases}$$

Observe that by using first the boundedness of Fourier multipliers on U^2, DV^2 and the dual of the embedding $U^2 \subset L^2 H_\rho^1$, we have

$$\|P_\lambda \partial_x (4Q^3 u)\|_{DV^2} \lesssim \lambda \|Q^3 u\|_{DV^2} \lesssim \lambda \|Q^2 u\|_{L^2 L^2} \lesssim \lambda \big(\|F\|_{\dot{Y}_\infty^s} + \|\langle F, Q \rangle\|_{L^2 + L^1} \big).$$

If $\lambda > 1$, then by Lemma 3.1

$$\|[P_\lambda \partial_x, Q^3] u\|_{L^2 L_\rho^2} \lesssim \|u\|_{L^2 L_\rho^2}.$$

Repeating these estimates for the term containing g and using the estimates of the previous section we obtain for $\lambda \leq 1$,

$$\|u_\lambda\|_{V^2 \cap L^2 H_\rho^1} \lesssim \|P_\lambda f\|_{DU^2} + \lambda \big(\|F\|_{Y^s} + \|\langle F, Q \rangle\|_{L^2 + L^1} + \|\langle F^+, Q \rangle\|_{L^2 + L^1} \big) + \lambda^{1/2} \|\alpha\|_{L^1},$$

since

$$\|\alpha \widetilde{Q}\|_{L^1 \dot{B}_\infty^{-1/2,2}} \lesssim \|\alpha\|_{L^1}$$

and, for $\lambda > 1$,

$$\|u_\lambda\|_{V^2 \cap L^2 H_\rho^1} \lesssim \|f_\lambda\|_{DU^2} + \|g_\lambda\|_{L^2} + \|F\|_{Y^s} + \|\langle F, Q \rangle\|_{L^2 + L^1} + \|\langle F^+, Q \rangle\|_{L^2 + L^1} + \|\langle F, Q' \rangle\|_{L^2 + L^1}.$$

As a result, we arrive at the following key fact.

Proposition 6.7. *Suppose* (5-14) *holds for some small δ, that $-1/2 < s < 0$, $F \in Y^s$ and*

$$\begin{cases} u_t + u_{xxx} + 4\partial_x(Q^3 u) = \alpha \widetilde{Q} + \beta Q_x + F, \\ u(x, 0) = 0, \end{cases}$$

where α and β are defined in (6-21) *and* (6-22). *Then,*

$$\|u\|_{\dot{X}_\infty^s} \lesssim \|F\|_{\dot{Y}_\infty^s} + \|\langle F, Q\rangle\|_{L^1} + \|\langle F, Q'\rangle\|_{L^2 + L^1} + \|\langle F, Q^+\rangle\|_{L^2 + L^1}.$$

This result will play a large role in the nonlinear analysis required to prove asymptotic stability.

For future use, we denote by L_I^p, $\dot{X}_{\infty,I}^{-1/6}$, etc. the function spaces on the space time set $I \times \mathbb{R}$, and specifically we set L_T^p, $\dot{X}_{\infty,T}^{-1/6}$ for $I = (0, T)$. All previous constructions carry over to finite time intervals.

7. Local wellposedness for the quartic KdV equation

In this section we study local wellposedness for the quartic generalized KdV equation

$$\begin{cases} \partial_t \psi - \partial_{xxx}\psi - (\psi^4)_x = 0, \\ \psi(0, x) = \psi_0(x). \end{cases} \tag{7-1}$$

Let v be the solution to the Airy equation with the same initial data, that is,

$$\begin{cases} v_t + v_{xxx} = 0, \\ v(0, x) = \psi_0(x). \end{cases} \tag{7-2}$$

The main local wellposedness is the next result.

Theorem 4. *Let $r_0 > 0$. There exist ϵ_0, $\delta_0 > 0$ such that, if $0 < T \le \infty$,*

$$\|\psi_0\|_{\dot{B}_\infty^{-1/6,2}} \le r_0 \tag{7-3}$$

and

$$\sup_\lambda \|v_\lambda\|_{L^6([0,T],\mathbb{R})} \le \delta_0, \tag{7-4}$$

then there is a unique solution $\psi = v + w$ with $\|w\|_{\dot{X}_{\infty,T}^{-1/6}} \le \epsilon_0$. Moreover, the function w (and hence ψ) depends analytically on the initial data.

By the Strichartz estimates for linear KdV (see also (7-5) and (7-6) below), given v as in (7-2) we have

$$\sup_\lambda \|v_\lambda\|_{L^6} \le \kappa_0 \|v\|_{\dot{X}_{\infty,T}^{-1/6}},$$

and by the definition of the spaces

$$\|v\|_{\dot{X}_{\infty,T}^{-1/6}} \le \kappa_1 \big(\|\psi(0)\|_{\dot{B}_\infty^{-1/6,2}} + \|\partial_t v + \partial_{xxx} v\|_{\dot{Y}_{\infty,T}^{-1/6}}\big).$$

Hence, we obtain global existence from Theorem 4 for (7-1) if

$$\|\psi\|_{\dot{B}_\infty^{-1/6,2}} \le \min\Big\{1, \frac{\delta(1)}{(\kappa_0\kappa_1)}\Big\},$$

where $\delta(1)$ is the δ (which depends on r_0) evaluated at $r_0 = 1$.

In any case, if condition (7-4) is satisfied for $T = \infty$, then, since $\psi_\lambda \in V^2$, the function $e^{t\partial_{xxx}}\psi_\lambda$ is of bounded 2-variation with values in L^2 (see the appendix), and hence it has a limit in L^2 as $t \to \infty$. This implies that $\sum_\lambda \lim_{t\to\infty} e^{t\partial_{xxx}}\psi_\lambda =: S(\psi_0)$ exists and is the scattering state. If in addition ψ_0 is in the closure of C_0^∞ in $\dot{B}_\infty^{-1/6,2}$, then we may exchange the summation and limit.

Under the same assumptions we can solve the initial value problem with initial data $\psi_0(T) = e^{-T\partial_{xxx}}\psi_0$, which, by an easy limit as $T \to \infty$, gives the inverse of the map S. We will later see similar constructions for perturbation of the soliton.

It is not hard to see that if ψ_0 is in the closure of C_0^∞ in $\dot{B}_\infty^{-1/6,2}$, then we can achieve condition (7-4) by choosing T small. This implies local existence with smooth dependence on initial data. Moreover, since we obtain smooth dependence on the initial data, if we have any global solution $\psi(t)$ in the closure of C_0^∞ and perturb the initial data by an amount ε, we obtain a solution at least with a life span $T = -c \ln \varepsilon$ by easy perturbation arguments. In particular, if the initial datum lies in an ε neighborhood of a soliton, then the solution exists at least until time $\sim |\ln \varepsilon|$ and remains in a small neighborhood until that time.

Before turning to the proof we remark that in this section we work with the weaker norms

$$\|u\|_{\dot{X}^{-1/6}} = \sup_\lambda \lambda^{-1/6}\|u_\lambda\|_{V^2} \quad \text{and} \quad \|f\|_{\dot{Y}^{-1/6}} = \sup_\lambda \lambda^{-1/6}\|f_\lambda\|_{DU^2}.$$

On the other hand, since the results remain trivially true for the original definition of the spaces we keep the notation.

Proof. First, we recall some estimates for $u \in U_{KdV}^2$. Let $m(\xi, \xi_1) = m(\xi, \xi - \xi_1)$. Then

$$\|u\|_{L_t^6 L_x^6} \lesssim \||D|^{-1/6}u(0)\|_{L^2} \qquad\qquad (L^6 \text{ Strichartz estimate}), \qquad (7\text{-}5)$$

$$\left\| \int_{\mathbb{R}} m(\xi, \xi_1)|\xi_1^2 - (\xi - \xi_1)^2|^{1/2}\hat{u}_1(\xi_1)\hat{u}_2(\xi - \xi_1)\,d\xi_1 \right\|_{L^2(\mathbb{R}^2)}$$

$$\lesssim \sup \frac{|m(\xi, \xi_1)|^2}{|\xi_1^2 - (\xi - \xi_1)^2|^{\frac{1}{2}}} \|u_1(0)\|_{L^2}\|u_2(0)\|_{L^2} \qquad (\text{bilinear estimate}). \qquad (7\text{-}6)$$

The bilinear estimate is a variant of standard estimates as in [Grünrock 2005]. The most important choice is $m = |\xi_1^2 - (\xi - \xi_1)^2|^{1/2}$.

Let $m(\xi, \xi_1)$ be a function that satisfies $m(\xi, \xi - \xi_1) = m(\xi, \xi_1)$. Then,

$$\left\| \int m(\xi, \xi_1)e^{it(\xi_1^3 + (\xi - \xi_1)^3)}\hat{u}_1(0, \xi_1)\hat{u}_2(0, \xi - \xi_1)\,d\xi_1 \right\|_{L^2}^2$$

$$= \int m(\xi, \xi_1)m(\xi, \eta_1)e^{3it(\xi_1^2 - \xi\xi_1 - \eta_1^2 + \xi\eta_1)}\hat{u}_1(\xi_1)\hat{u}_2(\xi - \xi_1)\overline{\hat{u}_2(\xi - \eta_1)\hat{u}_1(\eta_1)}\,dt\,d\xi_1\,d\eta_1\,d\xi$$

$$= \int \frac{|m(\xi, \eta_1)|^2}{|\eta_1^2 - (\xi - \eta_1)^2|}|u_1(\eta_1)|^2|u_2(\xi - \eta_1)|^2\,d\xi\,d\eta_1$$

$$\leq \sup \frac{|m(\xi, \xi_1)|^2}{|\xi_1^2 - (\xi - \xi_1)^2|}\|u_1\|_{L^2}^2\|u_2\|_{L^2}^2$$

since

$$\phi(\xi_1) = \xi\xi_1^2 - \xi^2\xi_1 - \xi\eta_1^2 + \xi^2\eta_1 = \xi(\xi_1 - \eta_1)(\xi_1 + \eta_1 - \xi)$$

vanishes if $\xi_1 = \eta_1$ or $\xi_1 = \xi - \eta_1$ and

$$\phi'(\eta_1) = \xi(2\eta_1 - \xi) = \xi(\eta_1 - (\xi - \eta_1)) \quad \text{and} \quad \phi'(\xi - \eta_1) = \xi((\xi - \eta_1) - \xi_1).$$

These results immediately imply (see the appendix for more information) for $\lambda \geq 1.1\mu$ the estimates

$$\|u_\lambda\|_{L_T^6} \lesssim \lambda^{-1/6}\|u_\lambda\|_{U_T^2}, \tag{7-7}$$

$$\|u_\lambda u_\mu\|_{L_T^2} \lesssim \lambda^{-1}\|u_\lambda\|_{U_T^2}\|u_\mu\|_{U_T^2}. \tag{7-8}$$

By interpolating the bilinear estimate and the Strichartz estimate, if $2 < p \leq 3$,

$$\|u_\lambda u_\mu\|_{L_T^p} \lesssim \lambda^{-1}(\mu^{-1/6}\lambda^{5/6})^{(3p-6)/p}\|u_\lambda\|_{U_T^2}\|u_\mu\|_{U_T^2} \tag{7-9}$$

and, if $\rho \ll \mu \sim \lambda$,

$$\|(u_\lambda u_\mu)_\rho\|_{L_T^2} \lesssim \lambda^{-1/2}\rho^{-1/2}\|u_\lambda\|_{U_T^2}\|u_\mu\|_{U_T^2}. \tag{7-10}$$

Interpolating once again, we have

$$\|(u_\lambda u_\mu)_\rho\|_{L_T^p} \lesssim \lambda^{-1/2}\rho^{-1/2}(\lambda^{1/6}\rho^{1/2})^{(3p-6)/p}\|u_\lambda\|_{U_T^p}\|u_\mu\|_{U_T^p}. \tag{7-11}$$

We proceed with a standard fixed point argument, which requires bounds on the nonlinearity. The solution $\psi = v + w$ is constructed by studying

$$\begin{cases} w_t + w_{xxx} + (v+w)_x^4 = 0, \\ w(0) = 0, \end{cases} \tag{7-12}$$

where again

$$\begin{cases} v_t + v_{xxx} = 0, \\ v(0) = \psi_0. \end{cases}$$

Then, the key estimate is contained in the following.

Lemma 7.1. *There exists $r > 0$ independent of T such that given $v_k \in \dot{X}_{\infty,T}^{-1/6}$ for $k = 1, 2, 3, 4$ we have*

$$\|\partial_x(v_1 v_2 v_3 v_4)\|_{\dot{Y}_{\infty,T}^{-1/6}} \leq r \prod_{k=1}^{4} \|v_k\|_{\dot{X}_{\infty,T}^{-1/6}}, \tag{7-13}$$

and, with v and w defined by (7-2) and (7-12), respectively,

$$\|\partial_x(v^3 w)\|_{\dot{Y}_{\infty,T}^{-1/6}} \leq r \sup_\lambda \|v_\lambda\|_{L^6}\|\psi_0\|_{B_\infty^{-1/6,2}}^2\|w\|_{X_{\infty,T}^{-1/6}}. \tag{7-14}$$

We apply these estimates to $v^4 + 4v^3 w + 6v^2 w^2 + 4vw^3 + w^4$. Either we may choose to estimate one factor v in L^6 or the dependence on w is at least quadratic. Suppose that $\|w\|_{\dot{X}_{\infty,T}^{-1/6}} \leq \mu$. We obtain

$$\|\partial_x(v+w)^4\|_{\dot{Y}_{\infty,T}^{-1/6}} \leq 6r(\kappa_1^3 \delta r_0^3 + \kappa_1^2 \delta\mu r_0^2 + \kappa_1^2 r_0^2 \mu^2 + \kappa_1 r_0 \mu^3 + \mu^4).$$

If $\mu \leq \kappa_1 r_0$, then the right hand side is bounded by

$$20r(\kappa_1^3 \delta r_0^3 + \kappa_1^2 \mu^2 r_0^2).$$

Suppose that

$$\mu \leq \min\left\{\kappa_1 r_0, \frac{1}{40r\kappa_1^2 r_0^2}\right\} \quad \text{and} \quad \delta \leq \frac{\mu}{40r\kappa_1^4 r_0^3}.$$

If w solves

$$\begin{cases} w_t + w_{xxx} + (v+W)_x^4 = 0, \\ w(0) = 0 \end{cases}$$

and $\|W\|_{\dot{X}_{\infty,T}^{-1/6}} \leq \mu$, then w exists and satisfies $\|w\|_{\dot{X}_{\infty,T}^{-1/6}} \leq \mu$.

Standard arguments then allow one to construct a unique solution satisfying the contraction assumption, possibly after decreasing μ by an absolute multiplicative factor. $\qquad\square$

It remains to prove Lemma 7.1. By duality, it suffices to verify that

$$\lambda \left| \int v_1 v_2 v_3 v_4 u_\lambda \, dx \, dt \right| \leq C\lambda^{1/6} \|u_\lambda\|_{V^2} \prod_{k=1}^{4} \|v_k\|_{\dot{X}_{\infty,T}^{-1/6}}$$

and

$$\lambda \left| \int v^3 w u_\lambda \, dx \, dt \right| \leq C\lambda^{1/6} \|u_\lambda\|_{V^2} \sup_\mu \|v_\mu\|_{L^6} \left(\sup_\mu \mu^{-1/6} \|v_\mu\|_{U^2}\right)^2 \|w\|_{\dot{X}_{\infty,T}^{-1/6}},$$

where $u_\lambda \in V^2$ is frequency localized at frequency λ.

By summation, the statement of the lemma holds provided we can prove the following.

Lemma 7.2. *We have for $\lambda_1 \leq \lambda_2 \sim \lambda_3 \sim \lambda_4 \sim \lambda_5$ and $\varepsilon > 0$*

$$\lambda_5 \int v_{1,\lambda_1} v_{2,\lambda_2} v_{3,\lambda_3} v_{4,\lambda_4} v_{5,\lambda_5} \, dx \, dt \lesssim \lambda_{\min}^{-1/3} \lambda_5^{-\frac{1}{6}} \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^\varepsilon \prod_{k=1}^{5} \|v_{k,\lambda_k}\|_{V^2} \tag{7-15}$$

and

$$\lambda_{\max} \int v_{\lambda_1} v_{\lambda_2} v_{\lambda_3} u_{\lambda_4} w_{\lambda_5} \, dx \, dt \lesssim \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{1/6} \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^\varepsilon$$

$$\times \sup_\mu \|v_\mu\|_{L^6} \left(\sup_\mu \mu^{-1/6} \|v_\mu\|_{U^2}\right)^2 \|u_{\lambda_4}\|_{V^2} \|w_{\lambda_5}\|_{V^2}, \tag{7-16}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ respectively are the maximal and minimal λ_j.

Proof. We claim that

$$\left| \int v_{1,\lambda_1} v_{2,\lambda_2} v_{3,\lambda_3} v_{4,\lambda_4} v_{5,\lambda_5} \, dx \, dt \right| \leq C\lambda_{\max}^{-1} \|v_{1,\lambda_1}\|_{U^2} \|v_{2,\lambda_2}\|_{U^2} \|v_{3,\lambda_3}\|_{L^6} \|v_{4,\lambda_4}\|_{L^6} \|v_{5,\lambda_5}\|_{L^6} \tag{7-17}$$

provided

$$|\lambda_1 - \lambda_2| \geq \tfrac{1}{10}\lambda_{\max}. \tag{7-18}$$

This estimate is a consequence of Hölder's inequality and the bilinear estimate (7-6). We recall that

$$\|v_\lambda\|_{L^6} \lesssim \lambda^{-1/6} \|v_\lambda\|_{U^6} \lesssim \lambda^{-1/6} \|v_\lambda\|_{V^2}.$$

To obtain a nontrivial integral there have to be elements in the support of the Fourier transforms that add up to zero. Unless there is at least one pair of (λ_j, λ_k) satisfying (7-18), the integral is zero. Hence, we would obtain (7-15) if we were allowed to replace the V^2 norms there by U^2 norms for the first two factors. Observe that we may reorganize the factors as we wish.

Let us assume $\lambda_1 \lesssim \lambda_2 \lesssim \lambda_3 \lesssim \lambda_4 \lesssim \lambda_5$. We consider first the case when $\lambda_4 \leq 1.05\lambda_1$. Then, if there are elements in the support of the truncations on the Fourier side adding up to zero — otherwise the integral vanishes — either

$$0.8\frac{\lambda_5}{4} \leq \lambda_1 \leq \lambda_4 \leq 1.1\lambda_1 \leq 1.2\frac{\lambda_5}{4} \quad \text{or} \quad 0.6\frac{\lambda_5}{2} \leq \lambda_1 \leq \lambda_4 \leq 1.1\lambda_1 \leq 1.4\frac{\lambda_5}{2}.$$

In this case we can replace the U^2 norms by V^2 norms as follows. We decompose into low and high modulation as

$$v_{j,\lambda_j} = v_{j,\lambda_j}^l + v_{j,\lambda_j}^h,$$

where v_j^l is defined by the Fourier multiplier projecting to $|\tau - \xi^3| \leq \lambda_5^3/1000$. Then, we have

$$\|v_{j,\lambda_j}^l\|_{V^2} + \|v_{j,\lambda_j}^h\|_{V^2} \leq \|v_{j,\lambda_j}\|_{V^2} \quad \text{and} \quad \|v_{j,\lambda_j}^h\|_{L^2} \lesssim \lambda_5^{-\frac{3}{2}} \|v_{j,\lambda_j}\|_{V^2}.$$

We refer to the appendix and [Hadac et al. 2009] for more information.

We expand the product. The integral over the product of the five $v_{j,\lambda}^l$ vanishes because of the support of the Fourier transforms. Hence at least one term has high modulation. We estimate it in L^2, put another term into L^∞ and the others into L^6 using Hölder's inequality. We estimate the L^∞ norm through energy and Bernstein's inequality.

Hence

$$\left| \int v_{1,\lambda_1} v_{2,\lambda_2} v_{3,\lambda_3} v_{4,\lambda_4} v_{5,\lambda_5} \, dx \, dt \right| \lesssim \lambda_5^{-3/2} \prod_{j=1}^{5} \|v_{j,\lambda_j}\|_{V^2}, \tag{7-19}$$

which implies the desired estimate.

It remains to study $\lambda_1 \lesssim \lambda_2 \lesssim \lambda_3 \lesssim \lambda_4 \lesssim \lambda_5$, $\lambda_4 \geq 1.05\lambda_1$. The most difficult case is $\lambda_5 \leq 1.02\lambda_2$ since otherwise we apply two stronger bilinear estimates. For simplicity we consider $\lambda_1 \ll \lambda$ where $\lambda_2 = \lambda_5 = \lambda$. We have to bound

$$\int_{\sum \xi_j = 0} \prod \hat{u}_{j,\lambda_j}(\xi_j) \, d\xi_2 \, d\xi_5 \, dt$$

with $\xi_1 = -\sum_{j=2}^{5} \xi_j$. We may restrict the integration to $\sum_{j=2}^{5} \xi_j \sim \lambda_1$ and $\xi_j \sim \lambda$. By symmetry it suffices to consider

$$\int_{\sum \xi_j = 0} \chi_{||\xi_3| - |\xi_2|| \sim \lambda_1} \prod \hat{v}_{j,\lambda_j}(\xi_j) \, d\xi_2 \, d\xi_5 \, dt.$$

We choose $\varepsilon > 0$ small, p, q so that $1/p = (1-\varepsilon)/2 + \varepsilon/3$, $1/q = \varepsilon/2 + (1-\varepsilon)/3$. By Hölder's inequality

$$\left| \int v_{1,\lambda_1} (v_{2,\lambda} v_{3,\lambda})_{\lambda_1} v_{4,\lambda} v_{5,\lambda} \, dx \, dt \right| \lesssim \lambda^{-1} (\lambda^{5/6} \lambda_1^{-1/6})^{\varepsilon} \lambda^{-1/2} \lambda_1^{-1/2} (\lambda^{1/6} \lambda_1^{1/2})^{1-\varepsilon} \lambda_3^{-1/6} \prod_j \|v_{j,\lambda_j}\|_{U^{12/5}}.$$

$$\lesssim \lambda^{-3/2} (\lambda^{2/3} \lambda_1^{-2/3})^{\varepsilon} \prod_j \|v_{j,\lambda_j}\|_{U^{12/5}}.$$

For the second part we would like to put one v term into L^6, and up to two into U^2. This can be easily be done if there are two frequencies of v that differ by a small constant times $\lambda_{\max}$. If not it is not hard to see that in the argument above we can put one term into L^6. $\qquad\square$

7.1. *Variants and extensions of wellposedness for the quartic KdV equation.* The arguments of the last sections have implications for wellposedness questions in other function spaces. Given $1 \le p \le \infty$, $\omega \in C^1((0, \infty), (0, \infty))$ and $T \in (0, \infty]$, we define the function space $X^{\omega}_{p,T}$ as the set of all distributions for which the norm

$$\|u\|^p_{X^{\omega}_{p,T}} = \sum_{\lambda} (\omega(\lambda) \|u_\lambda\|_{V^2})^p, \tag{7-20}$$

with obvious modifications if $p = \infty$ is finite. We will always assume that

$$\sup |\omega'| / \omega < \infty, \tag{7-21}$$

$$\inf \omega' / \omega > -1. \tag{7-22}$$

This is a Banach space provided for some $C > 0$ we have

$$\liminf_{\lambda \to 0} \omega(\lambda) \lambda^{1/2} > C; \tag{7-23}$$

otherwise we obtain a Banach space of equivalence classes of functions. Similarly, we define the Banach space

$$\|f\|^p_{Y^{\omega}_{p,T}} = \sum_{\lambda} (\omega(\lambda) \|f_\lambda\|_{DU^2})^p. \tag{7-24}$$

The definition of $B^{\omega,p}_q$ follows the same pattern. It is not hard to see that

$$\int uf \, dx \, dt \lesssim \|u\|_{X^{\omega}_{p,T}} \|f\|_{Y^{\omega^{-1}}_{p',T}} \quad \text{and} \quad \|f\|_{Y^{\omega^{-1}}_{p',T}} \lesssim \sup_{\|u\|_{X^{\omega}_{p,T}} \le 1} \int uf \, dx \, dt.$$

Moreover, we may expand the inner product into dyadic pieces and apply uniformly elliptic pseudo-differential operators to the pieces. In particular, we may replace differentiation by multiplication on the dyadic pieces and vice versa.

Proposition 7.3. *The following estimate holds*:

$$\|\partial_x (u^4)\|_{Y^{\omega}_{p,T}} \le C \sup_{\lambda} \|u_\lambda\|_{L^6} \|u\|^2_{\dot{X}^{-1/6}_{\infty,T}} \|u\|_{X^{\omega}_{p,T}}.$$

Proof. Given $v \in X_{p',T}^{\omega^{-1}}$, we expand $\int \partial_x(u^4)v \, dx \, dt$ into dyadic pieces, to which we apply the arguments and estimate (7-16) from the previous section. By symmetry

$$\sum_{\lambda_j} \lambda_5 \left| \int u_{\lambda_1} u_{\lambda_2} u_{\lambda_3} u_{\lambda_4} v_{\lambda_5} \, dx \, dt \right| \lesssim \sum_{\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4, \lambda_5} \lambda_5 \left| \int u_{\lambda_1} u_{\lambda_2} u_{\lambda_3} u_{\lambda_4} v_{\lambda_5} \, dx \, dt \right|.$$

If $\lambda_5 \sim \lambda_4$ we obtain

$$\sum_{\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \sim \lambda_5} \lambda_5 \left| \int u_{\lambda_1} u_{\lambda_2} u_{\lambda_3} u_{\lambda_4} v_{\lambda_5} \, dx \, dt \right|$$

$$\lesssim \sup_{\mu} \|u_\mu\|_{L^6} \left(\sup_{\mu} \mu^{-1/6} \|u_\mu\|_{U^2} \right)^2 \times \sum_{\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \sim \lambda_5} \lambda_1^{1/6} \lambda_5^{-1/6} \left(\frac{\lambda_5}{\lambda_1} \right)^\varepsilon \|u_{\lambda_4}\|_{V^2} \|v_{\lambda_5}\|_{V^2},$$

which is bounded by

$$\sup_{\mu} \|u_\mu\|_{L^6} \left(\sup_{\mu} \mu^{-1/6} \|u_\mu\|_{U^2} \right)^2 \|u\|_{X_{p,T}^\omega} \|v\|_{X_{p',T}^{\omega^{-1}}}.$$

The other extreme is

$$\sum_{\lambda_5 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \sim \lambda_4} \lambda_5 \left| \int u_{\lambda_1} u_{\lambda_2} u_{\lambda_3} u_{\lambda_4} v_{\lambda_5} \, dx \, dt \right|$$

$$\leq \sup_{\mu} \|u_\mu\|_{L^6} \left(\sup_{\mu} \mu^{-1/6} \|u_\mu\|_{U^2} \right)^2 \times \sum_{\lambda_5 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \sim \lambda_4} \lambda_5 \lambda_4^{-1} \left(\frac{\lambda_4}{\lambda_5} \right)^\varepsilon \|u_{\lambda_4}\|_{V^2} \|v_{\lambda_5}\|_{V^2},$$

which satisfies the same estimate provided $\sum_{\lambda \leq \mu} \lambda \omega(\lambda) \lesssim \mu \omega_\mu$. However, this is ensured by (7-22). The remaining cases are similar and the result follows. $\qquad \square$

From Proposition 7.3, we can prove the following corollary to Theorem 4.

Corollary 7.4. *Suppose that ω satisfies (7-21), (7-22) and (7-23). If $\psi_0 \in \dot{B}_\infty^{-1/6,2} \cap B_p^{\omega,2}$ is the initial data for a solution of (7-1) and v satisfies (7-4), then the solution ψ of Theorem 4 is in $X_{p,T}^\omega$ and satisfies*

$$\|\psi\|_{X_{p,T}^\omega} \leq C \|\psi_0\|_{B_p^{\omega,2}}.$$

In addition:

Corollary 7.5. *Suppose that ψ_0 lies in the closure of C_0^∞ in $\dot{B}_{\infty,T}^{-1/6}$. Then, it follows that*

$$(t \to \psi(t)) \in C([0, T], \dot{B}_\infty^{-1/6,2}).$$

If $T = \infty$, then $e^{t \partial_{xxx}^3} \psi$ converges to the scattering data as $t \to \infty$ in $\dot{B}_{\infty,T}^{-1/6}$. If in addition $\psi_0 \in L^2$, then

$$(t \to \psi(t)) \in C([0, T], L^2)$$

and $e^{t \partial_{xxx}^3} \psi$ converges also in L^2.

There exists ω satisfying the assumptions above, with $\omega(\lambda)\lambda^{-1/6} \to \infty$ as $\lambda \to \infty$ and $\lambda \to 0$ and $\|\psi_0\|_{B_\infty^{\omega,2}} < \infty$. By Corollary 7.4 the $X_{\infty,T}^\omega$ is controlled by the initial data. Hence

$$\lambda^{-1/6} \|v_\lambda\|_{DV^2 \cap X^0} \to 0 \text{ as } \lambda \to \infty \text{ or } \lambda \to 0.$$

By the previous argument the deviation of the solution to the linear solution tends to zero as the considered interval shrinks to zero. This implies continuity. Continuity at infinity always holds in V^p.

The second part requires an obvious specialization of Corollary 7.4 to the case $\omega = 1$, plus a repetition of the argument for scattering.

Particular examples for ω are $\langle \lambda \rangle^s$ for $s \geq -1/6$ and $\lambda^s + \lambda^\sigma$ for $-1/2 \leq s \leq -1/6 \leq \sigma$. It is not hard to see that we can replace the homogeneous spaces by inhomogeneous ones if we restrict to finite T and allow the constants to depend on T.

8. Stability and scattering for perturbations of the soliton

8.1. *Setup and main result.* We return now to the full nonlinear problem (7-1). Let us take

$$\psi(x, t) = Q_{c(t)}(x - y(t)) + w(x, t).$$

Then, we have

$$\partial_t w + \partial_x (\partial_x^2 w + 4 Q_c^3 w) = -\dot{c}(\partial_c Q_c)(x - y) + \dot{y}(Q_c')(x - y) - \partial_x(\partial_x^2 Q_c - c^2 Q_c + Q_c^4) - c^2(Q_c'(x - y))$$
$$- \partial_x(6 Q_c^2(x - y)w^2 + 4 Q_c(x - y)w^3 + w^4).$$

Hence,

$$\partial_t w + \partial_x (\partial_{xx}^2 w + 4 Q_c^3 w) = -(\dot{c}/c)\widetilde{Q}_c(x - y) + (\dot{y} - c^2)Q_c'(x - y)$$
$$- \partial_x(6 Q_c^2(x - y)w^2 + 4 Q_c(x - y)w^3 + w^4). \quad (8\text{-}1)$$

To use the dispersive estimates proved in Section 6, we wish to have

$$w \perp Q_c(x - y) \quad \text{and} \quad w \perp Q_c'(x - y). \quad (8\text{-}2)$$

To get more regularity for y and c, we ask for (8-2) only asymptotically and hence take as in (1-15) and (1-16) the modal equations

$$(\dot{c}/c)\langle Q_c, \widetilde{Q}_c \rangle = \langle w, Q_c \rangle, \quad (8\text{-}3)$$
$$(\dot{y} - c^2)\langle Q_c', Q_c' \rangle = -\kappa \langle w, Q_c' \rangle, \quad (8\text{-}4)$$

where $\kappa > 0$ is taken to be large.

We calculate

$$\frac{d}{dt}\langle w, Q \rangle = \langle w_t, Q \rangle + \dot{y}\langle w, Q' \rangle + (\dot{c}/c)\langle w, \widetilde{Q} \rangle$$
$$= \langle w, \mathcal{L}Q' \rangle - (\dot{c}/c)\langle Q, \widetilde{Q} \rangle + \langle 6 Q^2 w^2 + 4 Q w^3 + w^4, Q' \rangle + (\dot{y} - c^2)\langle w, Q' \rangle + (\dot{c}/c)\langle w, \widetilde{Q} \rangle$$

and

$$\frac{d}{dt}\langle w, Q' \rangle = \langle w_t, Q' \rangle + \dot{y}\langle w, Q'' \rangle + (\dot{c}/c)\langle w, \widetilde{Q}' \rangle$$
$$= \langle w, \mathcal{L}Q'' \rangle + (\dot{y} - c^2)\langle Q', Q' \rangle + \langle 6 Q^2 w^2 + 4 Q w^3 + w^4, Q'' \rangle$$
$$+ (\dot{y} - c^2)\langle w, Q'' \rangle + (\dot{c}/c)\langle w, \widetilde{Q}' \rangle.$$

Hence,

$$\frac{d}{dt}\langle w, Q\rangle + \langle w, Q\rangle = -\kappa\frac{\langle w, Q'\rangle^2}{\langle Q', Q'\rangle} - \frac{\langle w, Q\rangle\langle w, \tilde{Q}\rangle}{\langle Q, \tilde{Q}\rangle} + \langle 6Q^2w^2 + 4Qw^3 + w^4, Q'\rangle \tag{8-5}$$

and

$$\frac{d}{dt}\langle w, Q'\rangle + \kappa\langle w, Q'\rangle - \langle w, \mathscr{L}Q''\rangle$$
$$= -\kappa\frac{\langle w, Q'\rangle\langle w, Q''\rangle}{\langle Q', Q'\rangle} + \frac{\langle w, Q\rangle\langle w, \tilde{Q}'\rangle}{\langle Q, \tilde{Q}\rangle} + \langle 6Q^2w^2 + 4Qw^3 + w^4, Q''\rangle. \tag{8-6}$$

The right hand sides are at least quadratic in w, and, as we shall see, small compared to $\|w_0\|$ in a suitable sense. As a consequence the orthogonality conditions are approximately satisfied for large t. In addition, $\dot{c}$ and $\dot{y} - c^2$ are small and continuous.

We study the initial value problem $w(0) = w_0$. Let again v be the solution to the linear problem. We will prove scattering for small perturbations of the soliton in $\dot{B}_\infty^{-1/6,2}$. It will be important for the reverse problem that we will achieve something slightly stronger.

Using the notation

$$\Gamma = \left\{ y \in C([0, \infty)) : y(0) = 0, |\dot{y} - 1| \leq \tfrac{1}{10} \right\}, \tag{8-7}$$

we define for any interval I the quantity

$$J_I(v) = \sup_\lambda\left(\|v_\lambda\|_{L_I^6} + \lambda^{1/4-1/6}\|v_\lambda\|_{L_I^4 L^\infty} + \sup_{y\in\Gamma}\lambda^{-1/6}\int_{\mathbb{R}\times I}\gamma_0'(x - y(t))(v_\lambda^2 + (\partial_x v_\lambda)^2)\,dx\,dt \right). \tag{8-8}$$

Proposition 8.1. *Let v be a solution of* (3-1) *with initial data $v_0 \in \dot{B}_\infty^{-1/6,2}$. Then,*

$$J_{[0,\infty)}(v) \lesssim \|v_0\|_{\dot{B}_\infty^{-1/6,2}}.$$

Moreover, if v_0 is in the closure of C_0^∞ in $\dot{B}_\infty^{-1/6,2}$, then

$$\lim_{t\to\infty} J_{[t,\infty)}(v) = 0.$$

Proof. The first statement is an immediate consequence of the Strichartz estimate and local smoothing. For the second statement we fix $\varepsilon > 0$. There are at most finitely many $v_{0,\lambda}$ of norm larger than ε/c. Hence it suffices to verify the statement for a single λ. Since $v_\lambda \in L^6 L^6$ and $L^4 L^\infty$, we have

$$\lim_{t\to\infty}\|v_\lambda\|_{L^6(\mathbb{R}\times(t,\infty))} = \lim_{t\to\infty}\|v_\lambda\|_{L_{[t,\infty)}^4 L^\infty} = 0.$$

Let I be a bounded interval. Then the map

$$\Gamma \to \lambda^{-1/6}\int_{\mathbb{R}\times I}\gamma_0'(x - y(t))(v_\lambda^2 + (\partial_x v_\lambda)^2)\,dx\,dt$$

is continuous with respect to uniform convergence, hence it assumes its maximum. Given $j \geq 1$, let $y_j : [2^j, 2^{j+1}] \to \mathbb{R}$ be the path for which this quantity is maximal. We choose two paths y_o and y_e with $\gamma(0) = 0$ and the difference between 1 and the derivative at most 0.2, one which coincides with y_j for j

even on the corresponding intervals, and one which does so for j odd. For both paths we have the local smoothing estimate. But this implies the claim. $\qquad\square$

Let $y \in \Gamma$. The function spaces $\dot{X}^s_{\infty;T}$ and $\dot{Y}^s_{\infty;T}$ depend on y but not on c. This dependence is not reflected in the notation. In addition, let $c \in C^1([0, \infty))$. We assume (5-14), $\dot{c} \in L^2$ and $\dot{y} - c^2 \in L^2$ in this section, which we have to verify for the solutions we study, and turn to a study of a priori estimates for solutions to (8-1), (8-3) and (8-4), and we recall (8-5) and (8-6). Because of translation and scaling invariance we may restrict ourselves to a study for $y(0) = 0$ and $c(0) = 1$. Moreover, we may and do assume that the orthogonality conditions hold at time 0, that is,

$$\langle w_0, Q \rangle = \langle w_0, Q' \rangle = 0.$$

The main result is the following sharpened version of Theorem 2.

Proposition 8.2. *Let $C > 0$. There exist $\varepsilon > 0$ and $K > 0$ such that for $\|w_0\|_{\dot{B}^{-1/6,2}_\infty} < C$ and $J_{[0,T)}(v) \leq \varepsilon$ and for v a solution of (3-1) with initial data w_0, the solution w in the system of equations (1-15)–(1-14) satisfies (5-14),*

$$\|w\|_{\dot{X}^{-1/6}_{\infty,T}} \leq K J^{1/2}_{[0,T)}(v),$$

with K depending on C but not on time. Moreover, if $J_{(0,\infty)}(v) \leq \varepsilon$, then there exists a unique $\eta \in \dot{B}^{-1/6,2}_\infty$ such that

$$\lim_{t \to \infty} e^{t\partial^3_{xxx}} w_\lambda(t) = \eta_\lambda,$$

with convergence in L^2. In addition

$$\lim_{t \to \infty} \|w(t)\|_{\dot{B}^{-1/6}_{\infty,T}} = \|\Psi\|_{\dot{B}^{-1/6,2}_\infty}.$$

Remark 8.1. Variants in the spirit of Corollary 7.4 can be easily obtained by including the arguments there, which will establish Theorems 2 and 3 with higher Sobolev regularity as stated in Remark 1.1.

The proof consists of three step,s a preliminary part consisting of an important initialization, multi-linear estimates that are less critical variants of those of the last section, and a priori estimates for the nonlinear equation using the multilinear estimates and the linearized equation.

We recall that v satisfies $v_t + v_{xxx} = 0$ with initial data $v(0) = w_0$. We want to control the difference between v and the solution ν to $\nu_t + c^2 \partial_x \nu - \partial_x \mathscr{L} \nu = \alpha \widetilde{Q} + \beta Q'$ with initial data $\nu(0) = w_0$ with α and β ensuring $\langle \nu, Q \rangle = \langle \nu, Q' \rangle = 0$, which we assume to hold initially. We recall that (5-14) is a standing assumption.

For simplicity, let us define $J = J_{[0,\infty)}(v)$. The following result is the first step of the proof.

Lemma 8.3. *Suppose that $w_0 \in B^{-1/6,2}_\infty$ satisfies the orthogonality conditions. Then*

$$\|\nu\|_{\dot{X}^{-1/6}_\infty} \lesssim \|w_0\|_{\dot{B}^{-1/6,2}_\infty}$$

and

$$\|\nu - \tilde{P} P^\perp_{Q'} v\|_{\dot{X}^{-1/6}_\infty} + \|\alpha\|_{L^1 \cap L^2} + \|\beta\|_{L^2} \lesssim J. \tag{8-9}$$

Proof. The first bound on v is an immediate consequence of Proposition 6.7. The second statement is more delicate. As a first step we consider $u = \tilde{P} P^\perp v$. It satisfies

$$\sup_\lambda \left(\|u_\lambda\|_{L^6} + \lambda^{-1/6+1/4} \|u_\lambda\|_{L^4 L^\infty} + \lambda^{-1/6} \|u_\lambda\|_{L^2 H_\rho^1} \right) \lesssim J, \tag{8-10}$$

since $\|\langle v, Q \rangle\|_{L^2 \cap L^6} + \|\langle v, Q' \rangle\|_{L^2 \cap L^6} \lesssim J$. We calculate

$$\partial_t u + c^2 u_x - \partial_x \mathscr{L}_{c,y} u = G, \quad \text{with } u(0) = w_0, \tag{8-11}$$

where

$$G = -4\partial_x (Q^3 u) - \left(\frac{d}{dt} \frac{\langle v, Q \rangle}{\langle Q, \tilde{Q} \rangle} \right) \tilde{Q} - \left(\frac{d}{dt} \frac{\langle v, Q' \rangle}{\langle Q', Q' \rangle} \right) Q'$$
$$- \frac{\langle v, Q \rangle}{\langle Q, \tilde{Q} \rangle} (\partial_t + c^2 \partial_x - \partial_x \mathscr{L}) \tilde{Q} - \frac{\langle v, Q' \rangle}{\langle Q', Q' \rangle} (\partial_t + c^2 \partial_x - \partial_x \mathscr{L}) \tilde{Q}'.$$

We consider the terms separately. Any derivative falling on $\langle Q, \tilde{Q} \rangle$ or $\langle Q', Q' \rangle$ can be computed using (4-2) and (4-3), yielding a factor $\dot{c}/c$. Next,

$$\frac{d}{dt} \langle v, Q \rangle = \langle \partial_t v + c^2 v', Q \rangle + (c^2 - \dot{y}) \langle v, Q' \rangle + (\dot{c}/c) \langle v, \tilde{Q} \rangle = -\langle Q_x^4, v \rangle + (c^2 - \dot{y}) \langle v, Q' \rangle + (\dot{c}/c) \langle v, \tilde{Q} \rangle$$

and

$$\frac{d}{dt} \langle v, Q' \rangle = \langle \dot{v} + c^2 v', Q' \rangle + (c^2 - \dot{y}) \langle v, Q'' \rangle + (\dot{c}/c) \langle v, \tilde{Q}' \rangle$$
$$= -\langle v, \mathscr{L} Q'' \rangle - 4 \langle Q^3 Q'', v \rangle + (c^2 - \dot{y}) \langle v, Q'' \rangle + (\dot{c}/c) \langle v, \tilde{Q}' \rangle.$$

Moreover,

$$(\partial_t + c^2 \partial_x - \partial_x \mathscr{L}) \tilde{Q} = (\dot{c}/c) \tilde{\tilde{Q}} + (c^2 - \dot{y}) \tilde{Q}' + 2c^2 Q',$$
$$(\partial_t + c^2 \partial_x - \partial_x \mathscr{L}) Q' = (\dot{c}/c) \tilde{Q}' + (c^2 - \dot{y}) Q''.$$

We write $G = \alpha \tilde{Q} + \beta Q' + g$, where, using again (4-2) and (4-3),

$$\langle Q, \tilde{Q} \rangle \alpha = -(\dot{c}/c) \langle v, \tilde{Q} \rangle - (c^2 - \dot{y}) \langle v, Q' \rangle + \tfrac{1}{3} (\dot{c}/c) \langle v, Q \rangle + \langle v, (Q^4)_x \rangle,$$

$$\langle Q', Q' \rangle \beta = -(\dot{c}/c) \langle v, \tilde{Q}' \rangle - (c^2 - \dot{y}) \langle v, Q'' \rangle + \tfrac{10}{3} (\dot{c}/c) \langle v, Q' \rangle + 4 \langle v, Q^3 Q'' \rangle + \langle v, \mathscr{L}_c Q_c'' \rangle - 2c^2 \langle v, Q' \rangle,$$

$$g = -4\partial_x (Q^3 v) - \frac{\langle v, Q \rangle}{\langle Q, \tilde{Q} \rangle} ((\dot{c}/c) \tilde{\tilde{Q}} + (c^2 - \dot{y}) \tilde{Q}') - (\langle v, Q' \rangle / \langle Q', Q' \rangle)(\tfrac{\dot{c}}{c} \tilde{Q}' + (c^2 - \dot{y}) Q'').$$

By Lemma 6.3, we have $\|g\|_{Y^0} \lesssim J$. The difference $w = v - u$ satisfies (abusing the notation slightly by denoting by α and β new quantities)

$$w_t + c^2 w_x - \partial_x \mathscr{L} w = \alpha \tilde{Q} + \beta Q' - g$$

with initial data $w(0) = 0$, and again by Lemma 6.3

$$\|w\|_{X^0 \cap V^2} \lesssim \|g\|_{Y^0} \lesssim J.$$

We rewrite the equation for ν as

$$\nu_t + \nu_{xxx} = -\partial_x(4Q\nu) + \alpha\widetilde{Q} + \beta Q' =: F.$$

Decompose $\nu = u + w$. We recall that

$$\langle Q, \widetilde{Q}\rangle\alpha = -(\dot{c}/c)\langle \nu, \widetilde{Q}\rangle; \qquad (8\text{-}12)$$

hence

$$\|\alpha\|_{L^1} + \|F\|_{\dot{Y}_\infty^{-1/6}} \lesssim (\|\dot{c}/c\|_{L^2} + \|\dot{y} - c^2\|_{L^2})J.$$

The L^2 bound for β is simpler. The estimates for the linear equation imply now (8-9). $\qquad\square$

As it will be used in the sequel, we note the following simple consequence of Lemma 8.3. Namely, we have

$$J_{c,y}(\nu) \lesssim J(\nu), \qquad (8\text{-}13)$$

where we denote by $J_{c,y}$ the quantity analogous to J, but for the given path dictated by the c and y modulation parameters. After this nontrivial preliminary step, we continue with the proof of Proposition 8.2. The strategy is to write the equation in terms of

$$u = \Psi - Q_{c(t),y(t)} - \nu$$

and expand the nonlinearity. In the next step we study multilinear estimates, which in the last step are combined with Proposition 6.7 to obtain the a priori estimates.

8.2. _Multilinear estimates._ We proceed as for the initial value problem and bound multilinear expressions. In this section we collect nonlinear estimates in terms of the V^2 spaces to prove Proposition 8.2.

Lemma 8.4. *Let u be a tempered distribution and u_λ its frequency localization. Let ϕ be a Schwartz function. Then,*

$$\|\phi u_\lambda\|_{L^2} \lesssim \min\{\lambda^{1/2-\varepsilon}, \lambda^{-1}\}\big(\|u_\lambda\|_{L^2(\gamma')} + \|\partial_x u_\lambda\|_{L^2(\gamma')}\big).$$

Here ε is the constant of (3-7).

Proof. We begin with the case $\lambda \geq 1$, in which case we prove the stronger estimate where we replace ϕ by γ' as defined in Section 3. Let $\chi \in C_0^\infty$ be supported in $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Then,

$$\begin{aligned}
\sqrt{\gamma'}u_\lambda &= \sqrt{\gamma'}\partial_x^{-1}\chi\Big(\frac{\partial_x}{\lambda}\Big)\partial_x u_\lambda = \lambda^{-1}\sqrt{\gamma'}\Big(\frac{\partial_x}{\lambda}\Big)^{-1}\chi\Big(\frac{\partial_x}{\lambda}\Big)\partial_x u_\lambda \\
&= \lambda^{-1}\Big(\frac{\partial_x}{\lambda}\Big)^{-1}\chi\Big(\frac{\partial_x}{\lambda}\Big)\sqrt{\gamma'}\partial_x u_\lambda + \lambda^{-1}\Big(\sqrt{\gamma'}, \Big(\frac{\partial_x}{\lambda}\Big)^{-1}\chi\Big(\frac{\partial_x}{\lambda}\Big)\Big)\partial_x u_\lambda,
\end{aligned}$$

where $(\partial_x/\lambda)^{-1}\chi(\partial_x/\lambda)$ is an L^2 bounded Fourier multiplier. As a result,

$$\Big\|\lambda^{-1}\Big(\frac{\partial_x}{\lambda}\Big)^{-1}\chi\Big(\frac{\partial_x}{\lambda}\Big)\sqrt{\gamma'}\partial_x u_\lambda\Big\|_{L^2} \lesssim \lambda^{-1}\|u_\lambda\|_{L^2(\gamma')}.$$

We estimate the second term on the right hand side using the adjoint of Lemma 3.1 with $a = 1$ and $s = 0$:

$$\left\| \lambda^{-1} \left(\sqrt{\gamma'}, \left(\frac{\partial_x}{\lambda} \right)^{-1} \chi \left(\frac{\partial_x}{\lambda} \right) \right) (\gamma')^{-1/2} \sqrt{\gamma'} \partial_x u_\lambda \right\|_{L^2} \lesssim \lambda^{-1} \| u_\lambda \|_{L^2(\gamma')}.$$

We turn to $\lambda < 1$. Clearly,

$$\| \phi u_\lambda \|_{L^2} \leq \| \phi (\gamma')^{-1/2} \|_{L^2} \| \sqrt{\gamma'} u_\lambda \|_{L^\infty}.$$

Let $\tilde{\chi} = \sin(x)/x$, which is the inverse Fourier transform (up to a constant factor) of the characteristic function of the interval $[-1, 1]$. Let $x_0 \in \mathbb{R}$. We define

$$g_\lambda(x) = u_\lambda(x) \tilde{\chi} (\lambda(x - x_0)/100).$$

Then, g_λ satisfies roughly the same frequency localization as v_λ, and it coincides with u_λ at x_0. Thus, by Bernstein's inequalities,

$$|\sqrt{\gamma'(x_0)} u_\lambda(x_0)| \leq c\lambda^{1/2} \sqrt{\gamma'(x_0)} \| g_\lambda \|_{L^2} \leq c\lambda^{1/2} \sup_{x,x_0} \frac{\sqrt{\gamma'(x_0)} \tilde{\chi}(\lambda(x - x_0)/100)}{\sqrt{\gamma'(x)}} \| \sqrt{\gamma'} u_\lambda \|_{L^2}.$$

Now the elementary estimate

$$\sup_{x,x_0} \sqrt{\gamma'(x_0)} \frac{|\tilde{\chi}(\lambda(x - x_0)/100)|}{\sqrt{\gamma'(x)}} \leq c\lambda^{-\varepsilon}$$

completes the proof. $\qquad \square$

We proceed to prove the necessary multilinear estimates.

Lemma 8.5. *Let c, y satisfy (5-14), $u \in \dot{X}_\infty^{-1/6}$ and let v and Q be as in Proposition 8.2. Then, the following estimates hold*:

$$\| \partial_x (u_1 u_2 u_3 Q) \|_{\dot{Y}_\infty^{-1/6}} \lesssim \prod_{j=1}^{3} \| u_j \|_{\dot{X}_\infty^{-1/6}}, \tag{8-14}$$

$$\| \partial_x (u_1 u_2 Q^2) \|_{\dot{Y}_\infty^{-1/6}} \lesssim \prod_{j=1}^{2} \| u_j \|_{\dot{X}_\infty^{-1/6}}, \tag{8-15}$$

$$\| \partial_x (v^2 u Q) \|_{\dot{Y}_\infty^{-1/6}} \lesssim J^{1/2} \| v \|_{\dot{X}_\infty^{-1/6}}^{3/2} \| u \|_{\dot{X}_\infty^{-1/6}} \tag{8-16}$$

$$\| \partial_x (v u Q^2) \|_{\dot{Y}_\infty^{-1/6}} \lesssim J^{1/2} \| v \|_{\dot{X}_\infty^{-1/6}}^{1/2} \| u \|_{\dot{X}_\infty^{-1/6}}. \tag{8-17}$$

Proof. We begin with the dual Strichartz estimate

$$\| f_\lambda \|_{DV^2} \lesssim \lambda^{-1/4} \| f_\lambda \|_{L^{4/3} L^1}.$$

By construction, spatial Fourier multipliers in V^p, U^p, DU^p and DV^p are bounded by the supremum of the multiplier; hence

$$\| P_\lambda \partial_x (Q^2 u_{1,\lambda_1} u_{2,\lambda_2}) \|_{DU^2} \lesssim \lambda^{3/4} \| Q^2 u_{1,\lambda_1} u_{2,\lambda_2} \|_{L^{\frac{4}{3}} L^1}$$

and

$$\|Q^2 u_{1,\lambda_1} u_{2,\lambda_2}\|_{L^{\frac{4}{3}} L^1} \leq \|Qu_{1,\lambda_1}\|_{L^2 L^2} \|Qu_{2,\lambda_2}\|_{L^2 L^2}^1 / 2 \|u_{2,\lambda_2}\|_{L^\infty L^2}^{1/2}$$

$$\lesssim \min\{1, \lambda_1^{-1}\} \min\{1, \lambda_2^{-1/2}\} \lambda_1^{1/6} \lambda_2^{1/6} \|u_1\|_{\dot{X}_\infty^{-1/6}} \|u_2\|_{\dot{X}_\infty^{-1/6}}.$$

This is summable for $\lambda_j \in \Gamma$ and we obtain the desired estimate for $\lambda \leq 1$. Assume now that $\lambda \geq 1$. Then, using Hölder and Bernstein and $|Q'| \lesssim Q$

$$\lambda^{-1/6} \|P_\lambda \partial_x Q^2 u_{1,\lambda_1} u_{2,\lambda_2}\|_{L^2 \gamma H^{-1}} \lesssim \lambda^{-1/6} \|Qu_{1,\lambda_1}\|_{L^\infty}^{1/2} \|Qu_{2,\lambda_2}\|_{L^\infty}^{1/2} \|Qu_{1,\lambda_1}\|_{L^2}^{1/2} \|Qu_{2,\lambda_2}\|_{L^2}^{1/2}$$

$$\lesssim \lambda^{-1/6} \lambda_1^{5/12} \min\{1, \lambda_1^{-1/2}\} \lambda_2^{5/12} \min\{1, \lambda_2^{-1/2}\} \|u_1\|_{\dot{X}_\infty^{-1/6}} \|u_2\|_{\dot{X}_\infty^{-1/6}},$$

which can easily be summed over λ_1 and λ_2 if $\lambda \geq 1$. This implies (8-15) and also (8-17).

We approach estimate (8-14) similarly: We expand u_j and observe that the expressions are symmetric; hence it suffices to sum over $\lambda_1 \leq \lambda_2 \leq \lambda_3$. If $\lambda_1 \lesssim 1$ we argue as above and estimate u_{1,λ_1} in L^∞, followed by Bernstein's inequality. So we restrict to the case $\lambda_1 \gg 1$.

Then, using that Q is integrable,

$$\lambda \|P_\lambda Q u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3}\|_{DU^2} \lesssim \lambda^{3/4} \prod_{j=1}^3 \|u_{j,\lambda_j}\|_{L^4 L^\infty} \lesssim \lambda^{3/4} (\lambda_1 \lambda_2 \lambda_3)^{-1/12} \prod_{j=1}^3 \|u_{j,\lambda_j}\|_{\dot{X}_{\infty,T}^{-1/6}},$$

which is easily summable if $\lambda \lesssim 1 \lesssim \lambda_1, \lambda_2, \lambda_3$. If $\lambda > 1$, we argue differently. To simplify the argument we assume that the Fourier transform of Q is supported in $[-1, 1]$ — handling the tail is straightforward but technical. Instead of bounding $\lambda \|P_\lambda Q u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3}\|_{DU^2}$, we employ duality and study

$$I = \left| \int Q u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} \, dx \, dt \right|$$

assuming that $1 \ll \lambda_1 \leq \lambda_2 \leq \lambda_3$. Then, we have

$$I \leq \|Q u_{3,\lambda_3}\|_{L^2} \|u_{1,\lambda_1}\|_{L^6} \|u_{2,\lambda_2}\|_{L^6} \|u_{4,\lambda_4}\|_{L^6} \lesssim \lambda_3^{-5/6} \lambda_4^{-1/6} \prod_{j=1}^3 \|u_{j,\lambda_j}\|_{\dot{X}_{\infty,T}^{-1/6}} \|u_{4,\lambda_4}\|_{V^2}.$$

The factor $\lambda_3^{-5/6} \lambda_4^{5/6}$ is summable for fixed λ_4 over $1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$, $1 \leq \lambda_4 \lesssim \lambda_3$ — this suffices since $I = 0$ if λ_4 is much larger than λ_3. As a result, we have proven estimate (8-14) and, after checking the proof, (8-16). □

We turn to bounds for inner products occurring as inner products of the right hand side of (8-1) with Q and Q', and at the right hand side of (8-5) and (8-6).

Lemma 8.6. *Let $u \in \dot{X}_\infty^{-1/6}$, and let v and Q be as in Proposition 8.2. In addition, let $\psi_0(t)$ be a one parameter family of Schwartz functions parametrized by t with uniformly bounded seminorms and $\psi(x, t) = \psi_0(t, x - y(t))$. Then for all $1 \leq p < 3/2$*

$$\|\langle \partial_x (u_1 u_2 u_3 u_4), \psi \rangle\|_{L^p} \lesssim \prod_{j=1}^4 \|u_j\|_{\dot{X}_\infty^{-1/6}}, \tag{8-18}$$

where we consider the L^p norm with respect to time and

$$\|\langle \partial_x(v^3 u), \psi\rangle\|_{L^p} \lesssim J\|v\|^2_{\dot{X}_\infty^{-1/6}}\|u\|_{\dot{X}_\infty^{-1/6}}. \tag{8-19}$$

For all $1 \le p < 2$, we have

$$\|\langle \partial_x(u_1 u_2 u_3 Q), \psi\rangle\|_{L^p} \lesssim \prod_{j=1}^{3}\|u_j\|_{\dot{X}_\infty^{-1/6}}, \tag{8-20}$$

$$\|\langle \partial_x(v^2 u Q), \psi\rangle\|_{L^p} \lesssim J\|v\|_{\dot{X}_\infty^{-1/6}}\|u\|_{\dot{X}_\infty^{-1/6}}. \tag{8-21}$$

For all $1 \le p < 3$, we have

$$\|\langle \partial_x(u_1 u_2 Q^2), \psi\rangle\|_{L^p} \lesssim \prod_{j=1}^{2}\|u_j\|_{\dot{X}_\infty^{-1/6}}, \tag{8-22}$$

$$\|\langle \partial_x(v u Q^2), \psi\rangle\|_{L^p} \lesssim J\|u\|_{\dot{X}_\infty^{-1/6}}. \tag{8-23}$$

Proof. We expand the terms in (8-18) and we consider

$$I_p := \|\langle u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4}, \psi\rangle\|_{L^p}.$$

By symmetry it suffices to look at the case $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$. If $p = 1$ we bound the terms using Hölder's and Bernstein's inequalities as above:

$$I_1 \lesssim \|u_{1,\lambda_1}\|_{L^\infty}\|u_{2,\lambda_2}\|_{L^\infty}\||\psi|^{1/2}u_{3,\lambda_3}\|_{L^2}\||\psi|^{1/2}u_{4,\lambda_4}\|_{L^2}$$

$$\lesssim \lambda_1^{2/3}\lambda_2^{2/3}\min\{\lambda_3^{1/6}, \lambda_3^{-5/6}\}\min\{\lambda_4^{1/6}, \lambda_4^{-5/6}\}\prod_{j=1}^{4}\|u_j\|_{\dot{X}_\infty^{-1/6}},$$

which is easily summable. We obtain by Hölder's inequality

$$I_{3/2} \lesssim \prod_{j=1}^{4}\|u_{j,\lambda_j}\|_{L^6} \lesssim \prod_{j=1}^{4}\|u_j\|_{\dot{X}_\infty^{-1/6}},$$

which we use if $1 \le \lambda_1 \le \lambda_4$. If $\lambda_1 \le 1$, we estimate the corresponding term in L^∞, apply Bernstein's inequality, and argue as in the next case. Interpolation with the L^1 estimate yields a summable expression as long as $p < 3/2$.

We turn to estimate (8-20), denote again the p-norms by I_p and expand again

$$I_1 \lesssim \|u_{1,\lambda_1}\|_{L^\infty}\|Qu_{2,\lambda_2}\|_{L^2}\|(\partial_x\psi)u_{3,\lambda_3}\|_{L^2} \lesssim \lambda_1^{2/3}\min\{\lambda_2^{1/6}, \lambda_2^{-5/6}\}\min\{\lambda_3^{1/6}, \lambda_3^{-5/6}\}\prod\|u_j\|_{\dot{X}_\infty^{-1/6}},$$

which again is easily summable over $\lambda_1 \le \lambda_2 \le \lambda_3$. Also

$$I_2 \lesssim \prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{L^6} \lesssim \prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{\dot{X}_\infty^{-1/6}},$$

which is almost summable, and by interpolation we obtain the bounds for any $p < 2$.

The estimate (8-22) with $p = 1$ follows by the same arguments as above. It is even simpler. Again we may restrict ourselves to $\lambda_1 \leq \lambda_2$. For $p = 3$ we put estimate u_{j,λ_j} into L^6 and again the full statement follows by interpolation. A simple check of the proof reveals that the arguments above imply (8-23), (8-21) and (8-19). $\qquad\square$

The right hand sides of (8-5) and (8-6) are functions of t, for which we have bounds in L^p for $1 \leq p < 3/2$ in terms of $\|w\|_{\dot{X}_{\infty,T}^{-1/6}}$. In the second equation, (8-6), the term $\langle w, \mathscr{L}Q_{xx}\rangle$ plays a special role: It is in L^q for $2 \leq q \leq \infty$, but not in L^p for any $p < 2$ in general. In particular we cannot control the deviation of y from the linear movement.

Equation (8-5) and (8-6) can be considered as scalar linear ordinary differential equations for $\langle w, Q\rangle$ and $\langle w, Q'\rangle$. The kernel for the fundamental solution is uniformly bounded in L^p in the first case for all p, and in the second case it is bounded in L^1 by $1/\kappa$, whereas the L^∞ norm is 1.

We collect the consequences as follows.

Lemma 8.7. *Suppose that w solves (8-1) with $\langle w(0), Q\rangle = \langle w(0), Q'\rangle = 0$ and $w = v + u$, where v solves (7-2) with initial data $w(0)$. Then,*

$$\sup_t |\langle w(t), Q\rangle| \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})^2 (1 + \|w\|_{\dot{X}_\infty^{-1/6}})^2, \tag{8-24}$$

$$\sup_t |\langle w(t), Q'\rangle| \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})^2 (1 + \|w\|_{\dot{X}_\infty^{-1/6}})^2 + \kappa^{-1/2} \|w\|_{\dot{X}_{\infty,T}^{-1/6}}. \tag{8-25}$$

Moreover, if $1 \leq p < 3/2$, then

$$\left\| \frac{d}{dt}\langle w(t), Q\rangle \right\|_{L^p(0,\infty)} \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})^2 (1 + \|w\|_{\dot{X}_\infty^{-1/6}})^2.$$

We may write $\frac{d}{dt}\langle w(t), Q'\rangle = \gamma_1 + \gamma_2$ such that

$$\|\gamma_1\|_{L^p(0,\infty)} \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})^2 (1 + \|w\|_{\dot{X}_\infty^{-1/6}})^2 \quad and \quad \|\gamma_2\|_{L^2(0,\infty)} \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}}).$$

Finally, it follows that

$$\sup_t(|c(t) - 1| + |\dot{c}|) + \|\ddot{c}\|_{L^1} \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})^2 (1 + \|w\|_{\dot{X}_\infty^{-1/6}})^2,$$

$$\sup_t |\dot{y} - 1| \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})^2 (1 + \|w\|_{\dot{X}_\infty^{-1/6}})^2 + \kappa^{-1/2}(J + \|u\|_{\dot{X}_\infty^{1/6}}),$$

$$\|\dot{y} - c^2\|_{L^2} \lesssim (J + \|u\|_{\dot{X}_\infty^{-1/6}})(1 + \|w\|_{\dot{X}_\infty^{-1/6}})^3. \tag{8-26}$$

Proof. This is an immediate consequence of Lemma 8.5 and basic properties of the simple ordinary differential equations. $\qquad\square$

The estimates of this subsection remain true if we consider a time integral instead of $(0, \infty)$.

8.3. *Global bounds and scattering near the soliton.* We now complete the proof of Proposition 8.2.

Proof. By the local existence result there exists a local solution in a neighborhood of the soliton.

The decomposition $\psi = Q_{c(t),y(t)} + w$ together with the modal equations (8-3) and (8-4) implies existence of C^1 functions $c(t)$ and $y(t)$ that satisfy (8-1), (8-3) and (8-4) up to fixed time. We recall that after rescaling and shifting, $\langle w_0, Q \rangle = \langle w_0, Q' \rangle = 0$, $c(0) = 1$ and $y(0) = 0$.

As in the first step we denote the solution to the linear equation with initial data $w(0)$ by v. It satisfies the estimates of Lemma 8.3 and (8-13) provided (5-14) is satisfied.

We suppose that (ψ, c, y) is a solution up to time T, such that $u = \psi - Q_{c(t),y(t)} - v$ satisfies for some k_1, k_2 to be chosen later the conditions

$$\|P_{Q'}^{\perp} u\|_{\dot{X}_{\infty,T}^{-1/6}} \leq 2k_1 J^{1/2} \quad \text{and} \quad \|u\|_{\dot{X}_{\infty,T}^{-1/6}} \leq 2k_2 J^{1/2}. \tag{8-27}$$

We shall see that there exist δ, k_1 and k_2 such that if in addition $J \leq \delta$, then

$$\|P_{Q'}^{\perp} u\|_{\dot{X}_{\infty,T}^{-1/6}} \leq k_1 J^{1/2} \quad \text{and} \quad \|u\|_{\dot{X}_{\infty,T}^{-1/6}} \leq k_2 J^{1/2}. \tag{8-28}$$

This implies the estimate conditionally depending on (8-27). Observe that by Lemma 8.7 control of the norms implies validity of (5-14) if δ is sufficiently small. In particular, the estimates on the linear equations hold.

On the other hand, if we fix C and δ we can apply a continuity argument with the initial data τw_0. The estimate clearly holds for small τ and the norms depend (for finite time) continuously on τ. This implies the a priori estimate uniformly for all T. The scattering statement is an immediate consequence: Combine the fact that functions in V^2 are left-continuous at infinity with a frequency envelope argument as above. It remains to derive (8-28) from (8-27) for suitably chosen k_1, k_2 and δ.

We formulate the crucial estimate in the following.

Lemma 8.8. *Let C be given and let v and Q be as in Proposition 8.2. There exist k_1, k_2 and δ such that, if (8-27) holds, and $\|w_0\|_{\dot{B}_{\infty}^{-1/6,2}} \leq C$ and $J_{(0,T)}(v) \leq \delta$ hold, then*

$$\|P_{Q'}^{\perp} u\|_{\dot{X}_{\infty,T}^{-1/6}} \leq c_3 \big(\|u\|_{\dot{X}_{\infty,T}^{-1/6}}^2 + J_{(0,T)}^{1/2}(v) \|w_0\|_{\dot{B}_{\infty}^{-1/6,2}}^{3/2} + J_{(0,T)}(v) \big).$$

We postpone the proof of Lemma 8.8. Clearly $\langle u, Q' \rangle = \langle w, Q' \rangle$ and the same is true for its derivatives. By Lemma 8.7 and simple properties of ODEs, we have with implicit constants depending on the size of the initial data that

$$\left\| \langle u, Q' \rangle \right\|_{L^1 + L^2} + \left\| \frac{d}{dt} \langle u, Q' \rangle \right\|_{L^1 + L^2} \lesssim \|u\|_{\dot{X}_{\infty}^{-1/6}}^2 + J + \|\langle w, Q'' \rangle\|_{L^2}$$

and

$$\|\langle w, Q'' \rangle\|_{L^2} \leq \|\langle v, Q'' \rangle\|_{L^2} + \|\langle P_{Q'}^{\perp} u, Q'' \rangle\|_{L^2}.$$

Hence,

$$\|\langle u, Q' \rangle\|_{L^1 + L^2} + \left\| \frac{d}{dt} \langle u, Q' \rangle \right\|_{L^1 + L^2} \lesssim \|u\|_{\dot{X}_{\infty}^{-1/6}}^2 + J + \|P_{Q'}^{\perp} u\|_{\dot{X}_{\infty}^{-1/6}}.$$

The crucial point is that the right hand side only contains the projection of u, not u itself. We obtain easily

$$\|(\partial_t + \partial_x^3) \gamma(t) Q'\|_{\dot{Y}_{\infty,T}^{-1/6}} \lesssim \|\gamma\|_{L^2} + \|\gamma'\|_{L^1 + L^2}.$$

As a result, using estimates similar to those in Lemma 8.7 we have

$$\left\| \frac{\langle u, Q' \rangle}{\langle Q', Q' \rangle} Q' \right\|_{X_{\infty,T}^{-1/6}} \le k_4 \big(\|u\|_{\dot{X}_{\infty,T}^{-1/6}}^2 + J + \|P^\perp u\|_{\dot{X}_{\infty,T}^{-1/6}} \big). \tag{8-29}$$

By Lemma 8.8 and (8-27),

$$\|P_{Q'}^\perp u\|_{\dot{X}_{\infty,T}^{-1/6}} \le c_3 (\|u\|_{\dot{X}_{\infty,T}^{-1/6}}^2 + J_{(0,T)}^{1/2}(v) + J) \le c_3(4k_2^2 + 1)J + c_3 J^{1/2}$$

using, as we may, $\|u\|_{\dot{X}_{\infty,T}^{-1/6}} \le 1$ and, by the estimate (8-29) and (8-27), we have

$$\|u\|_{\dot{X}_{\infty,T}^{-1/6}} \le k_4 \big(J + c_3(8k_2^2 + 1)J(v) + c_3 J^{1/2} \big).$$

We choose first k_1, then k_2 and finally δ small to complete the proof. $\quad\square$

Proof of Lemma 8.8. We write the equation for $u = w - v$, with $u_t + c^2 u_x - \partial_x \mathscr{L}_{c,y} u =: G$. We have

$$G = (\dot{c}/c + \alpha_1)\widetilde{Q} + (\dot{y} - c^2 + \beta_1)Q' - \partial_x \big(6Q_c^2(u+v)^2 + 4Q_c'(u+v)^3 + (u+v)^4 \big),$$

where α_1 and β_1 ensure the orthogonality conditions for v, that is, (8-12). We recall that they satisfy

$$\|\alpha_1\|_{L^1} \lesssim \|\dot{c}\|_{L^2} J \quad \text{and} \quad \|\alpha_1\|_{L^2} + \|\beta_1\|_{L^2} \lesssim J.$$

To apply Proposition 6.7 we have to project u. This leads to a calculation similar to Lemma 8.3. Let $\mu = P_{Q'}^\perp \tilde{P} u$ and $\mu_t + c^2 \partial_x \mu - \partial_x \mathscr{L} \mu =: H$. Then, using $\langle u, Q \rangle = \langle w, Q \rangle$ and $\langle u, Q' \rangle = \langle w, Q' \rangle$,

$$
\begin{aligned}
H = G &- \left(\frac{d}{dt} \frac{\langle w, Q \rangle}{\langle Q, \widetilde{Q} \rangle} \right) \widetilde{Q} - \left(\frac{d}{dt} \frac{\langle w, Q' \rangle}{\langle Q', Q' \rangle} \right) Q' \\
&- \frac{\langle w, Q \rangle}{\langle Q, \widetilde{Q} \rangle} \left(\frac{\dot{c}}{c} \widetilde{Q} + (c^2 - \dot{y})\widetilde{Q}' + 2Q' \right) - \frac{\langle w, Q' \rangle}{\langle Q', Q' \rangle} \big((\dot{c}/c)\widetilde{Q}' + (c^2 - \dot{y})Q'' \big) \\
= \ &\alpha \widetilde{Q} + \beta Q' + g,
\end{aligned}
$$

where

$$
\begin{aligned}
-g = \partial_x \big(6Q_c^2(u+v)^2 + 4Q_c'(u+v)^3 + (u+v)^4 \big) &+ \frac{\langle w, Q \rangle}{\langle Q, \widetilde{Q} \rangle} \frac{\dot{c}}{c} \widetilde{Q} \\
&+ \left(\frac{\langle w, Q \rangle}{\langle Q, \widetilde{Q} \rangle}(c^2 - \dot{y}) + \frac{\langle w, Q' \rangle}{\langle Q', Q' \rangle} \frac{\dot{c}}{c} \right) \widetilde{Q}' + \frac{\langle w, Q' \rangle}{\langle Q', Q' \rangle}(c^2 - \dot{y})Q''.
\end{aligned}
$$

By construction $u(0) = 0$. We apply Proposition 6.7 to get

$$\|u\|_{\dot{X}_{\infty,T}^s} \lesssim \|g\|_{\dot{Y}_{\infty,T}^{-1/6}} + \|\langle g, Q \rangle\|_{L^1} + \|\langle g^+, Q \rangle\|_{L^2} + \|\langle g, Q' \rangle\|_{L^2}.$$

By Lemma 7.1, Lemma 8.5 and Lemma 8.6, we get

$$\|g\|_{\dot{Y}_\infty^{-1/6}} \lesssim \|u\|_{\dot{X}_\infty^{-1/6}}^2 + J\|w_0\|_{\dot{B}_\infty^{-1/6,2}},$$

and by Lemma 8.7

$$\|\langle g, Q \rangle\|_{L^1} + \|\langle g^+, Q \rangle\|_{L^2} + \|\langle g, Q' \rangle\|_{L^2} \lesssim \|u\|_{\dot{X}_\infty^{-1/6}}^2 + J^{1/2}\|w_0\|_{\dot{B}_\infty^{-1/6,2}}^{3/2}.$$

Together, we have

$$\|\tilde{P}^* P^\perp u\|_{\dot{X}_{\infty,T}^{-1/6}} \lesssim \|u\|_{\dot{X}_{\infty,T}^{-1/6}}^2 + J^{1/2}\|w_0\|_{\dot{B}_\infty^{-1/6,2}}^{3/2}. \qquad \square$$

Proposition 8.2 generalizes straightforwardly to smaller function spaces in the style of Section 7.1.

8.4. *An almost inverse wave operator result.* In this section we will construct solutions with given asymptotic behavior, proving Theorem 3. This is a partial converse statement to Proposition 8.2.

Remark 8.2. Theorem 3 is quite satisfactory in several respects. It shows which asymptotic properties may characterize a solution. The main missing piece is uniqueness of the solution Ψ. It implies existence of a solution for small scattering data, and, for arbitrary scattering states, existence of a solution with given scattering data for large t.

Proof. We turn to the time-reversed equation

$$\partial_t w + \partial_x(\partial_x^2 w + 4Q_c^3 w) = (\dot{y} - c^2)\langle w, Q_{xx}\rangle + (\dot{c}/c)\langle w, \tilde{Q}'\rangle + \langle 6Q^2 w^2 + 4Qw^3 + w^4, Q_{xx}\rangle \quad (8\text{-}30)$$

with

$$(\dot{c}/c)\langle Q_c, \tilde{Q}_c\rangle = -\langle w, Q_c\rangle \quad \text{and} \quad (\dot{y} - c^2)\langle Q_c', Q_c'\rangle = \kappa\langle w, Q_c'\rangle.$$

Let v be the solution to the Airy equation with initial data v_0. We may and do assume that $y_0 = 0$. By Proposition 8.1 we know that $\lim_{t\to\infty} J_{[t,\infty)}(v) = 0$. Given $S > 0$ and y^S satisfying $|y^S(S) - c_\infty^2 S| < \hat{\delta}S$, we solve the backwards initial value problem

$$\Psi(S) = v(S) + Q_{c_\infty, y^S}.$$

We choose $1 \gg \hat{\delta} \gg \delta$ to ensure that $|\dot{y} - c^\infty| \le \hat{\delta}$ for the solutions under consideration. The arguments of the previous section allow one to do that down to a largest time t^{S,y^S} for which

$$|y(t^{S,y^S}) - c_\infty^2 t^{S,y^S}| = \hat{\delta} t^{S,y^S}.$$

We want to show that the infimum of the t^{S,y^S} as a function of y^S is attained for some y^S and it is equal to zero if $\hat{\delta}$ is sufficiently small. Suppose not, and denote the infimum by $\tau > 0$. By continuous dependence on y^S, given $\varepsilon > 0$, there exists an interval $[a, b]$ such that the solution exists down to a time smaller than $(1 + \varepsilon)\tau$, and $y^{S,a}((1 + \varepsilon)\tau) = (c_\infty^2 - \delta)(1 + \varepsilon)\tau$ and $y^{S,b}((1 + \varepsilon)\tau) = (c_\infty^2 + \delta)(1 + \varepsilon)\tau$. Hence, there exists $y^{S,\varepsilon}$ with

$$y^{S,a}((1 + \varepsilon)\tau) = c_\infty^2(1 + \varepsilon)\tau.$$

But then, if $\hat{\delta}$ is sufficiently small, we see that a positive infimum is not possible, and moreover this construction gives a limit that is a solution denoted again by (Ψ^S, y^S) with $y^S(0) = 0$.

We consider the limit $S \to \infty$. Since $\dot{y}^S - c_\infty^2$ and $\dot{c}^S$ are small there exists a converging subsequence $y^{S_j}, c^{S_j}, S_j \to \infty$ that converges to c and y. There are corresponding solutions Ψ_j, u_j and w_j of the corresponding equation. We extend w_j beyond S_j by v. By the stability result, given $\delta > 0$ we find $T > 0$ such that

$$\|w_j - v\|_{\dot{X}_{\infty,[T,\infty)}^{-1/6}} \le \delta.$$

Using a frequency envelope there exists Λ such that

$$\lambda^{-1/6}\|(w_j)_\lambda\|_{V^2(T,\infty)} \lesssim \delta$$

whenever $\lambda > \Lambda$ or $\lambda^{-1} > \Lambda$.

In particular,

$$\|(w_j - w_l)(t)\|_{\dot{B}_\infty^{-1/6,2}} \leq \delta$$

for $t \geq T(\delta)$ and j and l sufficiently big. Again, using J small we are able to deduce that (w_j) is a Cauchy sequence in $\dot{X}_T^{-1/6}$ and the limit is the desired solution. $\qquad\square$

Acknowledgments

Appendix: Setup and properties of the U^p, V^p spaces for the linear KdV equation

To define the function spaces U^2, V^2, we summarize [Hadac et al. 2009, Section 2], where we suggest the reader look for further details. Let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K = \infty$. In the following, we consider functions taking values in $L^2 := L^2(\mathbb{R}^d; \mathbb{C})$, but in the general part of this section L^2 may be replaced by an arbitrary Hilbert space.

Definition A.1. Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1}\|\phi_k\|_{L^2}^p = 1$ we call the function $a : \mathbb{R} \to L^2$ given by

$$a = \sum_{k=1}^K \chi_{[t_{k-1},t_k)}\phi_{k-1}$$

a U^p-atom, where χ_I is the standard cutoff function to interval I. Furthermore, we define the atomic space

$$U^p := \left\{u = \sum_{j=1}^\infty \lambda_j a_j \,\Big|\, a_j \text{ a } U^p\text{-atom,, } \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^\infty |\lambda_j| < \infty\right\}$$

with norm

$$\|u\|_{U^p} := \inf\left\{\sum_{j=1}^\infty |\lambda_j| \,\Big|\, u = \sum_{j=1}^\infty \lambda_j a_j, \ \lambda_j \in \mathbb{C}, \ a_j \text{ a } U^p\text{-atom}\right\}. \tag{A-1}$$

Atoms are bounded in the supremum norm, and hence every convergence here implies uniform convergence.

Proposition A.2. *Let* $1 \leq p < q < \infty$.

(1) *The expression* $\|\cdot\|_{U^p}$ *is a norm. The space* U^p *is complete and hence a Banach space.*

(2) *The embeddings* $U^p \subset U^q$ *have norm* 1.

(3) *For $u \in U^p$, all one sided limits exist, including at $\pm\infty$, u is continuous from the right, and the limit at $-\infty$ is zero.*

(4) *The subspace of continuous functions U_c^p is closed.*

Definition A.3. Let $1 \le p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L^2$ for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathscr{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p} \tag{A-2}$$

is finite. Here we understand $v(\infty)$ as zero. Let V_-^p denote the subspace of all right-continuous functions with limit 0 at $-\infty$.

Taking the partition $\{t, \infty\}$, one sees that the supremum norm is not larger than the V^p norm.

Proposition A.4. Let $1 \le p < q < \infty$.

(1) *The expression $\| \cdot \|_{V^p}$ is a norm and V^p is complete.*

(2) *For $v \in V^p$, all one sided limits including at $\pm\infty$ exist.*

(3) *The subspace V_-^p is closed.*

(4) *The embedding $U^p \subset V_-^p$ is continuous and $\|u\|_{V^p} \le 2^{1/p} \|u\|_{U^p}$.*

(5) *The embeddings $V^p \subset V^q$ are continuous and $\|v\|_{V^q} \le \|v\|_{V^p}$.*

From the proof of [Hadac et al. 2009, Proposition 2.17], we have the following:

Lemma A.5. *Let $f \in V_-^p$, with $q > p$. Then, given $\delta > 0$ and $m > 1$, there exist $f_1 \in U^p$ and $f_2 \in U^q$ such that $f = f_1 + f_2$ and*

$$m^{-1}\|f_1\|_{U^p} + e^{\delta m}\|f_2\|_{U^q} \lesssim \|f\|_{V^p}.$$

The following corollary is obvious.

Corollary A.6. *The space V_-^p is continuously embedded in U^q for $q > p$.*

There is a bilinear map, B, which for $1/p + 1/q = 1$ and $1 < p, q < \infty$ can formally be written as

$$B(f, g) = - \int f_t g \, dt, \quad \text{for } f \in V^p, g \in U^q.$$

It satisfies $|B(f, g)| \le \|f\|_{V^p}\|g\|_{U^q}$, which is natural if we replace g by an atom. The map

$$V^p \ni f \to (g \to B(f, g)) \in (U^q)^*$$

is an isometric bijection. Moreover,

$$\|u\|_{U^p} = \sup\{B(u, v) : v \in C(\mathbb{R}), \|v\|_{V^p} = 1\}.$$

If $v \in V_-^p$, then

$$\|v\|_{V^q} = \sup\{B(u, v) : u \in C(\mathbb{R}), \|u\|_{U^p} = 1\}.$$

If the distributional derivative of u is in L^1 and $v \in V^p$, then

$$B(u, v) = -\int u_t v \, dt.$$

If $f \in L^1$, then $F(t) = \int_{-\infty}^t f \, ds \in V^p$ for all $p \geq 1$, and hence $F \in U^p$. Moreover, $\|f\|_{DU^p} := \|F\|_{U^p} \leq \|f\|_{L^1}$. We denote by DU^p the metric completion of L^1 in the norm given by the duality pairing. Similarly we define DV^q.

There is a close relation to Besov spaces, namely

$$B_1^{1/p,p} \subset U^p \subset V^p \subset B_\infty^{1/p,p} \tag{A-3}$$

with continuous embeddings. These embeddings clarify the relation to $X^{s,b}$ spaces below.

We claim that the convolution with an L^1 function η defines a bounded operator on U^p and V^p with norm $\leq \|\eta\|_{L^1}$. Because of the duality statement it suffices to verify boundedness on U^p. We approximate the characteristic function by a sum of Dirac measures. The convolution with an atom clearly has norm at most 1. Convergence in U^1 to the convolution with the characteristic function is immediate. The full statement is an immediate consequence, as well as the boundedness of the convolution by a Schwarz function on U^p and V^p. In particular smooth projections on high and low frequencies are bounded.

Following Bourgain's strategy for the Fourier restriction spaces, we define the adapted function spaces

$$U_{KdV}^p = S(-t)U^p \quad \text{and} \quad V_{KdV}^p = S(-t)V^p$$

and similarly DU^p and DV^p.

Again, we define a bilinear map B_{KdV} such that for $u \in V_{KdV}^p$ and $v \in U_{KdV}^q$, we have for a function u with $(\partial_t + \partial_x^3)u \in L^1L^2$

$$B_{KdV}(u, v) = -\int \langle (\partial_t + \partial_x^3)u, v \rangle \, dt.$$

Note, this bilinear map is well defined and gives a duality relation. Hence,

$$\|u\|_{DV_{KdV}^p} = \sup_{\|f\|_{U_{KdV}^q} \leq 1} \int uf \, dx \, dt \quad \text{and} \quad \|u\|_{DU_{KdV}^p} = \sup_{\|f\|_{V_{KdV}^q} \leq 1} \int uf \, dx \, dt.$$

Moreover, we may restrict f to suitable subspaces. More details on how the construction of such atomic spaces allows us to put u_t in the dual space are included in [Hadac et al. 2009].

By the construction of our spaces, we obtain for a solution u of the linear KdV equation

$$\begin{cases} u_t + u_{xxx} = f, \\ u(0, x) = u_0(x), \end{cases} \tag{A-4}$$

the estimates

$$\|u\|_{V_{KdV}^2} \lesssim \|u_0\|_{L^2} + \|f\|_{DV_{KdV}^2}, \tag{A-5}$$

$$\|u\|_{U_{KdV}^2} \lesssim \|u_0\|_{L^2} + \|f\|_{DU_{KdV}^2}, \tag{A-6}$$

which follow trivially from the construction of the V_{KdV}^2, DV_{KdV}^2 and U_{KdV}^2, DU_{KdV}^2 spaces.

Spatial Fourier multipliers act on U^p, V^q, DU^p, DV^q in the obvious way and their operator norm is bounded by the supremum of the multiplier.

Let (p, q) be a Strichartz pair. Then,

$$\|u\|_{L^p L^q} \leq c\||D|^{-1/p}u\|_{U^p}$$

and the dual estimate

$$\|f\|_{DV^{p'}} \leq c\||D|^{-1/p}f\|_{L^{p'}L^{q'}}$$

hold. The first estimate is not hard to check on atoms. Since convergence in U^p and in $L^p L^q$ both imply pointwise convergence for subsequences we obtain the full estimate. The second estimate follows by duality.

Similarly the local smoothing estimates carry over to U^p spaces and to DV^q. Let $c(t)$ and $y(t)$ satisfy (5-14). Then

$$\|u\|_{L^2 H^1_{\sqrt{\gamma'}}} \leq c\|u\|_{U^2} \quad \text{and} \quad \|f\|_{DV^2} \leq c\|f\|_{L^2\sqrt{\gamma'}H^{-1}}.$$

In the same fashion the bilinear estimates for solutions to the free equation imply bilinear estimates for functions in U^2.

The smooth decomposition into high and low modulation (that is, the smooth projection of the frequencies to $\tau - \xi^3$ large and respectively small) is bounded in U^2 and V^2, and the L^2 norm of the high modulation part gains the inverse of square root of the truncation as a factor by the embeddings (A-3).

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