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ON SOME MICROLOCAL PROPERTIES OF THE RANGE OF A PSEUDODIFFERENTIAL OPERATOR OF PRINCIPAL TYPE

JENS WITTSTEN

We obtain microlocal analogues of results by L. Hörmander about inclusion relations between the ranges of first order differential operators with coefficients in C^∞ that fail to be locally solvable. Using similar techniques, we study the properties of the range of classical pseudodifferential operators of principal type that fail to satisfy condition (Ψ).

1. Introduction

We shall study the properties of the range of a classical pseudodifferential operator $P \in \Psi^m_{\mathrm{cl}}(X)$ that is not locally solvable, where X is a C^∞ manifold of dimension n. Here, classical means that the total symbol of P is an asymptotic sum of homogeneous terms,

$$\sigma_P(x, \xi) = p_m(x, \xi) + p_{m-1}(x, \xi) + \cdots,$$

where p_k is homogeneous of degree k in ξ and p_m denotes the principal symbol of P. When no confusion can occur we will simply refer to σ_P as the symbol of P. We shall restrict our study to operators of principal type, which means that the Hamilton vector field H_{p_m} and the radial vector field are linearly independent when $p_m = 0$. We shall also assume that all operators are properly supported, that is, both projections from the support of the kernel in $X \times X$ to X are proper maps. For such operators, local solvability at a compact set $M \subset X$ means that for every f in a subspace of $C^\infty(X)$ of finite codimension there is a distribution u in X such that

$$Pu = f \tag{1-1}$$

in a neighborhood of M. We can also define microlocal solvability at a set in the cosphere bundle, or equivalently, at a conic set in $T^*(X) \smallsetminus 0$, the cotangent bundle of X with the zero section removed. By a conic set $K \subset T^*(X) \smallsetminus 0$ we mean a set that is conic in the fiber, that is,

$$(x, \xi) \in K \quad \text{implies} \quad (x, \lambda\xi) \in K \quad \text{for all } \lambda > 0.$$

If, in addition, $\pi_x(K)$ is compact in X, where $\pi_x : T^*(X) \to X$ is the projection, then K is said to be compactly based. Thus, we say that P is solvable at the compactly based cone $K \subset T^*(X) \smallsetminus 0$ if there

is an integer N such that for every $f \in H_{(N)}^{\mathrm{loc}}(X)$ there exists a $u \in \mathscr{D}'(X)$ with $K \cap WF(Pu - f) = \varnothing$ (see Definition 2.1).

The famous example due to Hans Lewy [1957] of the existence of functions $f \in C^\infty(\mathbb{R}^3)$ such that the equation

$$\partial_{x_1} u + i \partial_{x_2} u - 2i(x_1 + i x_2)\partial_{x_3} u = f$$

does not have any solution $u \in \mathscr{D}'(\Omega)$ in any open nonvoid subset $\Omega \subset \mathbb{R}^3$ contradicted the assumption that partial differential equations with smooth coefficients behave as analytic partial differential equations, for which existence of analytic solutions is guaranteed by the Cauchy–Kovalevsky theorem. This example led to an extension due to Hörmander [1960b; 1960a] in the sense of a necessary condition for a differential equation $P(x, D)u = f$ to have a solution locally for every $f \in C^\infty$. In fact (see [Hörmander 1963, Theorem 6.1.1]), if Ω is an open set in $\mathbb{R}^n$, and P is a differential operator of order m with coefficients in $C^\infty(\Omega)$ such that the differential equation $P(x, D)u = f$ has a solution $u \in \mathscr{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$, then $\{p_m, \bar{p}_m\}$ must vanish at every point $(x, \xi) \in \Omega \times \mathbb{R}^n$ for which $p_m(x, \xi) = 0$, where

$$\{a, b\} = \sum_{j=1}^n \partial_{\xi_j} a \, \partial_{x_j} b - \partial_{x_j} a \, \partial_{\xi_j} b$$

denotes the Poisson bracket.

In addition to his example, Lewy conjectured that differential operators that fail to have local solutions are essentially uniquely determined by the range. Later Hörmander [1963, Chapter 6.2] proved that if P and Q are two first order differential operators with coefficients in $C^\infty(\Omega)$ and in $C^1(\Omega)$, respectively, such that the equation $P(x, D)u = Q(x, D)f$ has a solution $u \in \mathscr{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$, and x is a point in Ω such that

$$p_1(x, \xi) = 0 \quad \text{and} \quad \{p_1, \bar{p}_1\}(x, \xi) \neq 0 \tag{1-2}$$

for some $\xi \in \mathbb{R}^n$, then there is a constant μ such that (at the fixed point x)

$${}^t Q(x, D) = \mu \, {}^t P(x, D),$$

where ${}^t Q$ and ${}^t P$ are the formal adjoints of Q and P. If (1-2) holds for a dense set of points x in Ω and if the coefficients of $p_1(x, D)$ do not vanish simultaneously in Ω, then there is a function $\mu \in C^1(\Omega)$ such that

$$Q(x, D)u = P(x, D)(\mu u). \tag{1-3}$$

Furthermore, for such an operator P and function μ, the equation $P(x, D)u = \mu P(x, D)f$ has a solution $u \in \mathscr{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$ if and only if $p_1(x, D)\mu = 0$.

Hörmander also showed that this result extends to operators of higher order in the following way (see [1963, Theorem 6.2.4]). If P is a differential operator of order m with coefficients in $C^\infty(\Omega)$ and μ is a function in $C^m(\Omega)$ such that the equation

$$P(x, D)u = \mu P(x, D)f$$

has a solution $u \in \mathcal{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$, then it follows that

$$\sum_{j=1}^{n} \partial_{\xi_j} p_m(x, \xi) \partial_{x_j} \mu(x) = 0$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ such that

$$\{p_m, \bar{p}_m\}(x, \xi) \neq 0 \quad \text{and} \quad p_m(x, \xi) = 0. \tag{1-4}$$

This means that the derivative of μ must vanish along every bicharacteristic element with initial data (x, ξ) satisfying (1-4), that is, giving rise to nonexistence of solutions.

If P is a pseudodifferential operator such that P is microlocally elliptic near (x_0, ξ_0), then there exists a microlocal inverse, called a parametrix P^{-1} of P, such that in a conic neighborhood of (x_0, ξ_0) we have $PP^{-1} = P^{-1}P = \text{Identity}$ modulo smoothing operators. P is then trivially seen to be microlocally solvable near (x_0, ξ_0), and for any pseudodifferential operator Q we can write $Q = PP^{-1}Q + R = PE + R$, where R is a smoothing operator. When the range of Q is microlocally contained in the range of P, we will show the existence of this type of representation for Q in the case when P is a nonsolvable pseudodifferential operator of principal type, although we will have to content ourselves with a weaker statement concerning the Taylor coefficients of the symbol of the operator R (see Theorem 2.19 for the precise formulation of the result). Note that when P is solvable but nonelliptic we cannot hope to obtain such a representation in general; see the remark on page 440.

For pseudodifferential operators of principal type, Hörmander [1985b] proved that local solvability in the sense of (1-1) implies that M has an open neighborhood Y in X where p_m satisfies condition (Ψ), which means that

Im ap_m does not change sign from $-$ to $+$ along the oriented bicharacteristics of Re ap_m (1-5)

over Y for any $0 \neq a \in C^\infty(T^*(Y) \setminus 0)$. The oriented bicharacteristics are the positive flow-outs of the Hamilton vector field $H_{\text{Re } ap_m}$ on Re $ap_m = 0$. The proof relies on an idea due to Moyer [1978], and uses the fact that condition (1-5) is invariant under multiplication of p_m with nonvanishing factors, and conjugation of P with elliptic Fourier integral operators.

Rather recently Dencker [2006] proved that condition (Ψ) is also sufficient for local and microlocal solvability for operators of principal type. To get local solvability at a point x_0, Dencker assumed the strong form of the nontrapping condition at x_0,

$$p_m = 0 \quad \text{implies} \quad \partial_\xi p_m \neq 0. \tag{1-6}$$

This was the original condition for principal type of Nirenberg and Treves [1970a; 1970b; 1971], which is always obtainable microlocally after a canonical transformation. Thus, we shall study pseudodifferential operators that fail to satisfy condition (Ψ) in place of the condition given by (1-4), and show that such operators are, in analogue with the inclusion relations between the ranges of differential operators that fail to be locally solvable, essentially uniquely determined by the range. However, even though (1-4) is a microlocal condition, we get the mentioned local results for differential operators because of analyticity

in ξ of the corresponding symbol. Since this is not generally true for pseudodifferential operators, our results will be inherently microlocal. We will combine the techniques used in [Hörmander 1963] to prove the inclusion relations for differential operators with the approach used in [Hörmander 1985b] to prove the necessity of condition (Ψ) for local solvability of pseudodifferential operators of principal type.

It is possible to extend these results to certain systems of pseudodifferential operators, which will be addressed in a forthcoming joint paper with Nils Dencker.

2. Nonsolvable operators of principal type

Let X be a C^∞ manifold of dimension n. In what follows, C will be taken to be a new constant every time unless stated otherwise. We let $\mathbb{N} = \{0, 1, 2, \dots\}$, and if $\alpha \in \mathbb{N}^n$ is a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, we let

$$D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n},$$

where $D_{x_j} = -i \partial_{x_j}$. We shall also employ the standard notation $f_{(\alpha)}^{(\beta)}(x, \xi) = \partial_x^\alpha \partial_\xi^\beta f(x, \xi)$ for multi-indices α, β.

In this section we will follow the outline of [Hörmander 1985b, Chapter 26, Section 4]. Recall that the Sobolev space $H_{(s)}(X)$ for $s \in \mathbb{R}$ is a local space, that is, if $\varphi \in C_0^\infty(X)$ and $u \in H_{(s)}(X)$, then $\varphi u \in H_{(s)}(X)$, and the corresponding operator of multiplication is continuous. Thus we can define

$$H_{(s)}^{\mathrm{loc}}(X) = \{u \in \mathscr{D}'(X) : \varphi u \in H_{(s)}(X) \text{ for all } \varphi \in C_0^\infty(X)\}.$$

This is a Fréchet space, and its dual with respect to the inner product on L^2 is

$$H_{(-s)}^{\mathrm{comp}}(X) = H_{(-s)}^{\mathrm{loc}}(X) \cap \mathscr{E}'(X).$$

Definition 2.1. If $K \subset T^*(X) \smallsetminus 0$ is a compactly based cone, we shall say that the range of $Q \in \Psi_{\mathrm{cl}}^m(X)$ is microlocally contained in the range of $P \in \Psi_{\mathrm{cl}}^k(X)$ at K if there exists an integer N such that for every $f \in H_{(N)}^{\mathrm{loc}}(X)$, there exists a $u \in \mathscr{D}'(X)$ with $WF(Pu - Qf) \cap K = \varnothing$.

If $I \in \Psi_{\mathrm{cl}}^0(X)$ is the identity on X, we obtain from Definition 2.1 the definition of microlocal solvability for a pseudodifferential operator (see [Hörmander 1985b, Definition 26.4.3]) by setting $Q = I$. Thus, the range of the identity is microlocally contained in the range of P at K if and only if P is microlocally solvable at K. Note also that if P and Q satisfy Definition 2.1 for some integer N, then due to the inclusion

$$H_{(t)}^{\mathrm{loc}}(X) \subset H_{(s)}^{\mathrm{loc}}(X) \quad \text{if } s < t,$$

the statement also holds for any integer $N' \geq N$. Hence N can always be assumed to be positive. Furthermore, the property is preserved if Q is composed with a properly supported pseudodifferential operator $Q_1 \in \Psi_{\mathrm{cl}}^{m'}(X)$ from the right. Indeed, let g be an arbitrary function in $H_{(N+m')}^{\mathrm{loc}}(X)$. Then $f = Q_1 g \in H_{(N)}^{\mathrm{loc}}(X)$ since the map

$$Q_1 : H_{(s)}^{\mathrm{loc}}(X) \to H_{(s-m')}^{\mathrm{loc}}(X)$$

is continuous for every $s \in \mathbb{R}$, so by Definition 2.1 there exists a $u \in \mathcal{D}'(X)$ with $WF(Pu - Qf) \cap K = \varnothing$. Hence the range of QQ_1 is microlocally contained in the range of P at K with the integer N replaced by $N + m'$.

The property given by Definition 2.1 is also preserved under composition of both P and Q with a properly supported pseudodifferential operator from the left. This follows immediately from the fact that properly supported pseudodifferential operators are microlocal, that is,

$$WF(Au) \subset WF(u) \cap WF(A) \quad \text{for } u \in \mathcal{D}'(X).$$

Remark. In Definition 2.1 we may always assume that $f \in H_{(N)}^{\mathrm{comp}}(X)$ and $u \in \mathcal{E}'(X)$ when considering a fixed cone K. In fact, assume

$$Qf = Pu + g,$$

where $f \in H_{(N)}^{\mathrm{loc}}(X)$ and $u, g \in \mathcal{D}'(X)$ with $WF(g) \cap K = \varnothing$, and let $Y \Subset X$ satisfy $K \subset T^*(Y) \smallsetminus 0$. (We write $Y \Subset X$ when $\overline{Y}$ is compact and contained in X.) Since P and Q are properly supported we can find $Z_1, Z_2 \subset X$ such that $Pv = 0$ in Y if $v = 0$ in Z_1, and $Qv = 0$ in Y if $v = 0$ in Z_2. We may of course assume that $Y \Subset Z_j$ for $j = 1, 2$. Fix $\phi_j \in C_0^\infty(X)$ with $\phi_j = 1$ on Z_j. Then we have $Pu = P(\phi_1 u)$ and $Qf = Q(\phi_2 f)$ in Y, so

$$\varnothing = WF(Qf - Pu) \cap K = WF(Q(\phi_2 f) - P(\phi_1 u)) \cap K$$

where $\phi_1 u$ and $\phi_2 f$ have compact support. Hence we may assume that $u \in \mathcal{E}'(X)$ and $f \in H_{(N)}^{\mathrm{comp}}(X) = H_{(N)}^{\mathrm{loc}}(X) \cap \mathcal{E}'(X)$ to begin with. Note that this also implies $g = Qf - Pu \in \mathcal{E}'(X)$ since P and Q are properly supported.

The following easy example will prove useful when discussing inclusion relations between the ranges of solvable but nonelliptic operators.

Example 2.2. If $X \subset \mathbb{R}^n$ is open, and $K \subset T^*(X) \smallsetminus 0$ is a compactly based cone, then the range of $D_1 = -i\partial/\partial x_1$ is microlocally contained in the range of D_2 at K. In fact, this is trivially true since both operators are surjective $\mathcal{D}'(X) \to \mathcal{D}'(X)/C^\infty(X)$. To see that for example D_1 is surjective, we note that by the remark on page 427 it suffices to show that there exists a number $N \in \mathbb{Z}$ such that the equation $D_1 u = f$ has a solution $u \in \mathcal{D}'(X)$ for every $f \in H_{(N)}^{\mathrm{comp}}(X) = H_{(N)}^{\mathrm{loc}}(X) \cap \mathcal{E}'(X)$. By [Hörmander 1983b, Theorem 10.3.1] this is satisfied for every $N \in \mathbb{Z}$ if $u \in H_{(N+1)}^{\mathrm{loc}}(X)$ is given by $E * f$, where E is the regular fundamental solution of D_1.

Just as the microlocal solvability of a pseudodifferential operator P gives an a priori estimate for the adjoint P^*, we have the following result for operators satisfying Definition 2.1.

Lemma 2.3. *Let $K \subset T^*(X) \smallsetminus 0$ be a compactly based cone. Let $Q \in \Psi_{\mathrm{cl}}^m(X)$ and $P \in \Psi_{\mathrm{cl}}^k(X)$ be properly supported pseudodifferential operators such that the range of Q is microlocally contained in the range of P at K. If $Y \Subset X$ satisfies $K \subset T^*(Y)$ and if N is the integer in Definition 2.1, then for every positive integer κ we can find a constant C, a positive integer ν and a properly supported pseudodifferential*

operator A with $WF(A) \cap K = \varnothing$ such that

$$\|Q^*v\|_{(-N)} \leq C(\|P^*v\|_{(v)} + \|v\|_{(-N-\kappa-n)} + \|Av\|_{(0)}) \tag{2-1}$$

for all $v \in C_0^\infty(Y)$.

Since (2-1) holds for any κ, it is actually superfluous to include the dimension n in the norm $\|v\|_{(-N-\kappa-n)}$. However, for our purposes, it turns out that this is the most convenient formulation.

Proof. We shall essentially adapt the proof of [Hörmander 1985b, Lemma 26.4.5]. Let $\| \cdot \|_{(s)}$ denote a norm in $H_{(s)}^{\text{comp}}(X)$ that defines the topology in $H_{(s)}^c(M) = H_{(s)}^{\text{loc}}(X) \cap \mathscr{E}'(M)$ for every compact set $M \subset X$. (The reason we change notation from $H_{(s)}^{\text{comp}}(M)$ to $H_{(s)}^c(M)$ when M is compact is to signify that $H_{(s)}^c(M)$ is a Hilbert space for each fixed compact set M.) Let $Y \Subset Z \Subset X$, and take $\chi \in C_0^\infty(X)$ with supp $\chi = \overline{Z}$ to be a real-valued cutoff function identically equal to 1 in a neighborhood of Y. Then $\chi Qf \in H_{(N-m)}^c(\overline{Z})$ for all $f \in H_{(N)}^{\text{comp}}(X)$ since Q is properly supported, and we claim that for fixed $f \in H_{(N)}^{\text{comp}}(X)$ we have for some C, v and A as in the statement of the lemma

$$|(\chi Qf, v)| \leq C(\|P^*v\|_{(v)} + \|v\|_{(-N-\kappa-n)} + \|Av\|_{(0)}) \tag{2-2}$$

for all $v \in C_0^\infty(Y)$. Indeed, by hypothesis and the remark on page 427 we can find u and $\tilde{g}$ in $\mathscr{E}'(X)$ with $WF(\tilde{g}) \cap K = \varnothing$ such that

$$\chi Qf = Qf - (1-\chi)Qf = Pu + \tilde{g} - (1-\chi)Qf.$$

Since $K \subset T^*(Y)$ and $\chi \equiv 1$ near Y we get $WF((1-\chi)Qf) \cap K = \varnothing$, so $\chi Qf = Pu + g$ for some $g \in \mathscr{E}'(X)$ with $WF(g) \cap K = \varnothing$. Thus

$$(\chi Qf, v) = (u, P^*v) + (g, v) \quad \text{for } v \in C_0^\infty(Y).$$

Now choose properly supported pseudodifferential operators B_1 and B_2 of order 0 with $I = B_1 + B_2$ and $WF(B_1) \cap WF(g) = \varnothing$ and $WF(B_2) \cap K = \varnothing$, which is possible since $WF(g) \cap K = \varnothing$. Since $g \in \mathscr{E}'(X)$ and $B_1 : \mathscr{E}'(X) \to \mathscr{E}'(X)$ is continuous and microlocal we get $B_1 g \in C_0^\infty(X)$, so $(B_1 g, v)$ can be estimated by $C\|v\|_{(-N-\kappa-n)}$. Also, $g \in H_{(-\mu)}^{\text{loc}}(X)$ for some $\mu > 0$ so if B is properly supported and elliptic of order μ, and $B' \in \Psi_{\text{cl}}^{-\mu}(X)$ is a properly supported parametrix of B, then

$$B_2^*v = B'BB_2^*v + LB_2^*v, \tag{2-3}$$

where $L \in \Psi^{-\infty}(X)$ and both B' and L are continuous $H_{(s)}^{\text{comp}}(X) \to H_{(s+\mu)}^{\text{comp}}(X)$. Hence

$$|(B_2 g, v)| \leq C\|B_2^*v\|_{(\mu)} \leq C(\|BB_2^*v\|_{(0)} + \|B_2^*v\|_{(0)}),$$

and if we apply the identity (2-3) to $\|B_2^*v\|_{(0)}$, $\|B_2^*v\|_{(-\mu)}$, ... sufficiently many times, and then recall that B_2^* is properly supported and of order 0, we obtain

$$|(B_2 g, v)| \leq C(\|BB_2^*v\|_{(0)} + \|v\|_{(-N-\kappa-n)}).$$

Since we chose B to be properly supported this gives (2-2) with $A = BB_2^*$.

For fixed κ, suppose V is the space $C_0^\infty(Y)$ equipped with the topology defined by the seminorms $\|v\|_{(-N-\kappa-n)}$, $\|P^*v\|_{(\nu)}$ for $\nu = 1, 2, \ldots$, and $\|Av\|_{(0)}$, where A is a properly supported pseudodifferential operator with $K \cap WF(A) = \varnothing$. It suffices to use a countable sequence $A_1, A_2, \ldots$, where A_ν is noncharacteristic of order ν in a set that increases to $(T^*(X) \smallsetminus 0) \smallsetminus K$ as $\nu \to \infty$. Thus V is a metrizable space. The sesquilinear form $(\chi Qf, v)$ in the product of the Hilbert space $H_{(N-m)}^c(\overline{Z})$ and the metrizable space V is obviously continuous in χQf for fixed v, and by (2-2) it is also continuous in v for fixed f. Hence it is continuous, which means that for some ν and C

$$|(\chi Qf, v)| \le C\|Qf\|_{(N-m)}(\|P^*v\|_{(\nu)} + \|v\|_{(-N-\kappa-n)} + \|Av\|_{(0)})$$

for all $f \in H_{(N)}^{\text{comp}}(X)$ and $v \in C_0^\infty(Y)$. Now Q is continuous from $H_{(N)}^{\text{comp}}(X)$ to $H_{(N-m)}^{\text{comp}}(X)$ so we have $\|Qf\|_{(N-m)} \le C\|f\|_{(N)}$. Since $\chi \equiv 1$ near Y and $(\chi Q)^* = Q^*\chi$, this yields the estimate

$$|(f, Q^*v)| \le C\|f\|_{(N)}(\|P^*v\|_{(\nu)} + \|v\|_{(-N-\kappa-n)} + \|Av\|_{(0)}). \tag{2-4}$$

For $v \in C_0^\infty(Y)$ and Q^* properly supported we have $Q^*v \in C_0^\infty(X)$, and therefore also $Q^*v \in H_{(-N)}^{\text{loc}}(X)$. Viewing Q^*v as a functional on $H_{(N)}^{\text{comp}}(X)$, the dual of $H_{(-N)}^{\text{loc}}(X)$ with respect to the standard inner product on L^2, we obtain (2-1) after taking the supremum over all $f \in H_{(N)}^{\text{comp}}(X)$ with $\|f\|_{(N)} = 1$. □

We will need the following analogue of [Hörmander 1985b, Proposition 26.4.4]. Recall that $\mathcal{H}: T^*(Y) \smallsetminus 0 \to T^*(X) \smallsetminus 0$ is a canonical transformation if and only if its graph $C_\mathcal{H}$ in the product $(T^*(X) \smallsetminus 0) \times (T^*(Y) \smallsetminus 0)$ is Lagrangian with respect to the difference $\sigma_X - \sigma_Y$ of the symplectic forms of $T^*(X)$ and $T^*(Y)$ lifted to $T^*(X) \times T^*(Y) = T^*(X \times Y)$. This differs in sign from the symplectic form $\sigma_X + \sigma_Y$ of $T^*(X \times Y)$ so it is the *twisted graph*

$$C'_\mathcal{H} = \{(x, \xi, y, -\eta) : (x, \xi, y, \eta) \in C_\mathcal{H}\},$$

which is Lagrangian with respect to the standard symplectic structure in $T^*(X \times Y)$.

Proposition 2.4. *Let $K \subset T^*(X) \smallsetminus 0$ and $K' \subset T^*(Y) \smallsetminus 0$ be compactly based cones and let χ be a homogeneous symplectomorphism from a conic neighborhood of K' to one of K such that $\chi(K') = K$. Let $A \in I^{m'}(X \times Y, \Gamma')$ and $B \in I^{m''}(Y \times X, (\Gamma^{-1})')$, where Γ is the graph of χ, and assume that A and B are properly supported and noncharacteristic at the restriction of the graphs of χ and χ^{-1} to K' and to K respectively, while $WF'(A)$ and $WF'(B)$ are contained in small conic neighborhoods. Then the range of the pseudodifferential operator Q in X is microlocally contained in the range of the pseudodifferential operator P in X at K if and only if the range of the pseudodifferential operator BQA in Y is microlocally contained in the range of the pseudodifferential operator BPA in Y at K'.*

Proof. Choose $A_1 \in I^{-m''}(X \times Y, \Gamma')$ and $B_1 \in I^{-m'}(Y \times X, (\Gamma^{-1})')$ properly supported such that

$$K' \cap WF(BA_1 - I) = \varnothing, \qquad K \cap WF(A_1B - I) = \varnothing,$$
$$K' \cap WF(B_1A - I) = \varnothing, \qquad K \cap WF(AB_1 - I) = \varnothing.$$

Assume that the range of Q is microlocally contained in the range of P at K and choose N as in Definition 2.1. Let $g \in H_{(N+m')}^{\text{loc}}(Y)$ and set $f = Ag \in H_{(N)}^{\text{loc}}(X)$. Then we can find $u \in \mathcal{D}'(X)$ such that

$K \cap WF(Pu - Qf) = \varnothing$. Let $v = B_1 u \in \mathscr{D}'(Y)$. Then

$$WF(Av - u) = WF((AB_1 - I)u)$$

does not meet K, so $K \cap WF(PAv - Qf) = \varnothing$. Recalling that $f = Ag$ this implies

$$K' \cap WF(BPAv - BQAg) = \varnothing,$$

so the range of BQA is microlocally contained in the range of BPA at K'. Conversely, if the range of BQA is microlocally contained in the range of BPA at K', it follows that the range of $A_1 BQAB_1$ is microlocally contained in the range of $A_1 BPAB_1$ at K. Since

$$K \cap WF(A_1 BPAB_1 u - A_1 BQAB_1 f) = K \cap WF(Pu - Qf),$$

this means that the range of Q is microlocally contained in the range of P at K, which proves the proposition. $\square$

Before we can state our main theorem, we need to study the geometric situation that occurs when p fails to satisfy condition (Ψ). Recall that by [Hörmander 1985b, Theorem 26.4.12] we may always assume that the nonvanishing factor in condition (1-5) is a homogeneous function. We begin with a lemma concerning a reduction of the general case.

Lemma 2.5. *Let p and q be homogeneous smooth functions on $T^*(X) \setminus 0$, and let $t \mapsto \gamma(t)$, for $a \le t \le b$, be a bicharacteristic interval of $\operatorname{Re} qp$ such that $q(\gamma(t)) \ne 0$ for $a \le t \le b$. If*

$$\operatorname{Im} qp(\gamma(a)) < 0 < \operatorname{Im} qp(\gamma(b)), \tag{2-5}$$

then there exists a proper subinterval $[a', b'] \subset [a, b]$, possibly reduced to a point, such that

 (i) *$\operatorname{Im} qp(\gamma(t)) = 0$ for $a' \le t \le b'$,*

 (ii) *for every $\varepsilon > 0$ there exist $a' - \varepsilon < s_- < a'$ and $b' < s_+ < b' + \varepsilon$ such that $\operatorname{Im} qp(\gamma(s_-)) < 0 < \operatorname{Im} qp(\gamma(s_+))$.*

If $\gamma(t)$ is defined for $a \le t \le b$ we shall in the sequel say that $\operatorname{Im} qp$ changes sign from $-$ to $+$ on γ if (2-5) holds. If $\gamma|_{[a',b']}$ is the restriction of γ to $[a', b']$ and (i) and (ii) hold we shall say that $\operatorname{Im} qp$ *strongly* changes sign from $-$ to $+$ on $\gamma|_{[a',b']}$.

Proof. It suffices to regard the case that $q = 1$, $X = \mathbb{R}^n$, p is homogeneous of degree 1 with $\operatorname{Re} p = \xi_1$, and the bicharacteristic of $\operatorname{Re} p$ is given by

$$a \le x_1 \le b, \quad x' = (x_2, \ldots, x_n) = 0, \quad \xi = \varepsilon_n. \tag{2-6}$$

Here $\varepsilon_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$, and we shall in what follows write ξ^0 in place of ε'_n. The proof of this fact is taken from [Hörmander 1985b, page 97] and is given here for the purpose of reference later, in particular in connection with Definition 2.11 below.

Choose a pseudodifferential operator Q with principal symbol q. If we let $P_1 = QP$, then the principal symbol of P_1 is $p_1 = qp$ so $\operatorname{Im} p_1$ changes sign from $-$ to $+$ on the bicharacteristic γ of $\operatorname{Re} p_1$. Now choose Q_1 to be of order $1 - \operatorname{degree} P_1$ with positive, homogeneous principal symbol. If p_2 is the

principal symbol of $P_2 = Q_1 P_1$, it follows that Re p_1 and Re p_2 have the same bicharacteristics, including orientation, and since p_2 is homogeneous of degree 1 these can be considered to be curves on the cosphere bundle $S^*(X)$. Moreover, Im p_1 and Im p_2 have the same sign, so Im p_2 changes sign from $-$ to $+$ along $\gamma \subset S^*(X)$. If γ is a closed curve on $S^*(X)$ we can pick an arc that is not closed where the sign change still occurs. If we assume this to be done, then [Hörmander 1985b, Proposition 26.1.6] states that there exists a C^∞ homogeneous canonical transformation χ from an open conic neighborhood of (2-6) to one of γ such that $\chi(x_1, 0, \varepsilon_n) = \gamma(x_1)$ and $\chi^*(\text{Re } p_2) = \xi_1$. Since the Hamilton field is symplectically invariant it follows that the equations of a bicharacteristic are invariant under the action of canonical transformations, that is, $\tilde{\gamma}$ is a bicharacteristic of $\chi^*(\text{Re } p_2)$ if and only if $\chi(\tilde{\gamma})$ is a bicharacteristic of Re p_2. This proves the claim.

In accordance with the notation in [Hörmander 1985b, page 97], let $(x', \xi') = (0, \xi^0)$ and consider

$$L(0, \xi^0) = \inf\{t - s : a < s < t < b, \text{ Im } p(s, 0, \varepsilon_n) < 0 < \text{Im } p(t, 0, \varepsilon_n)\}.$$

For every small $\delta > 0$ there exist s_δ and t_δ such that $a < s_\delta < t_\delta < b$, Im $p(s_\delta, 0, \varepsilon_n) < 0 < \text{Im } p(t_\delta, 0, \varepsilon_n)$ and $t_\delta - s_\delta < L(0, \xi^0) + \delta$. Choose a sequence $\delta_j \to 0$ such that the limits $a' = \lim s_{\delta_j}$ and $b' = \lim t_{\delta_j}$ exist. Then $b' - a' = L(0, \xi^0)$ and in view of (2-5) we have $a < a' \le b' < b$ by continuity. Moreover, Im $p(t, 0, \varepsilon_n) = 0$ for $a' \le t \le b'$. This is clear if $a' = b'$. If on the other hand Im $p(t, 0, \varepsilon_n)$ is, say, strictly positive for some $a' < t < b'$, then $L(0, \xi^0) \le t - s_{\delta_j} \to t - a' < b' - a'$, a contradiction. Thus (i) holds.

To prove (ii), let $\varepsilon > 0$. After possibly reducing to a subsequence we may assume that the sequences $\{s_{\delta_j}\}$ and $\{t_{\delta_j}\}$ given above are monotone increasing and decreasing, respectively. It then follows by (i) that $s_{\delta_j} < a' \le b' < t_{\delta_j}$ for all j. Since $s_{\delta_j} \to a'$ and $t_{\delta_j} \to b'$ we can choose j so that $a' - \varepsilon < s_{\delta_j} < a'$ and $b' < t_{\delta_j} < b' + \varepsilon$. By construction we have Im $p(s_{\delta_j}, 0, \varepsilon_n) < 0 < \text{Im } p(t_{\delta_j}, 0, \varepsilon_n)$. This completes the proof. $\square$

Although it will not be needed here, we note that if $[a', b']$ is the interval given by Lemma 2.5 and $a' < b'$, then in addition to (i) and (ii) we also have

(iii) there exists a $\delta > 0$ such that Im $qp(\gamma(s)) \le 0 \le \text{Im } qp(\gamma(t))$ for all $a' - \delta < s < a'$ and $b' < t < b' + \delta$.

Indeed, the infimum $L(0, \xi^0) = b' - a'$ would otherwise satisfy $L(0, \xi^0) < \delta$ for every δ in view of (ii), which is a contradiction when $a' < b'$.

We next recall the definition of a one-dimensional bicharacteristic.

Definition 2.6. A one-dimensional bicharacteristic of the pseudodifferential operator with homogeneous principal symbol p is a C^1 map $\gamma : I \to T^*(X) \setminus 0$, where I is an interval on $\mathbb{R}$, such that

(i) $p(\gamma(t)) = 0$ for $t \in I$,

(ii) $0 \ne \gamma'(t) = c(t)H_p(\gamma(t))$ if $t \in I$

for some continuous function $c : I \to \mathbb{C}$.

Let P be an operator of principal type on a C^∞ manifold X with principal symbol p, and suppose p fails to satisfy condition (Ψ) in X. By (1-5) there is a function q in $C^\infty(T^*(X) \setminus 0)$ such that Im qp

changes sign from $-$ to $+$ on a bicharacteristic γ of $\operatorname{Re} qp$, where $q \neq 0$. As can be seen in [Hörmander 1985b, pages 96–97], we can then find a compact one-dimensional bicharacteristic interval $\Gamma \subset \gamma$ or a characteristic point $\Gamma \in \gamma$ such that the sign change occurs on bicharacteristics of $\operatorname{Re} qp$ arbitrarily close to Γ. What we mean by this will be clear from the following discussion, although we will not use this terminology in the sequel. By the proof of Lemma 2.5 it suffices to regard the case that $q = 1$, $X = \mathbb{R}^n$, p is homogeneous of degree 1 with $\operatorname{Re} p = \xi_1$, and the bicharacteristic of $\operatorname{Re} p$ is given by (2-6).

We shall now study a slightly more general situation is some detail. If $\gamma = I \times \{w_0\}$, where $I = [a, b]$, we shall by $|\gamma|$ denote the usual arc length in $\mathbb{R}^{2n}$, so that $|\gamma| = b - a$. Furthermore, we will assume that all curves are bicharacteristics of $\operatorname{Re} p = \xi_1$, that is, $w_0 = (x', 0, \xi') \in \mathbb{R}^{2n-1}$. We owe parts of this exposition to Nils Dencker.

Lemma 2.7. *Assume that* $\operatorname{Im} p$ *strongly changes sign from* $-$ *to* $+$ *on* $\gamma = [a, b] \times \{w_0\}$. *Then for any* $\delta > 0$ *there exist* $\varepsilon > 0$, $a - \delta < s_- < a$ *and* $b < s_+ < b + \delta$ *such that* $\pm \operatorname{Im} p(s_\pm, w) > 0$ *for any* $|w - w_0| < \varepsilon$.

Proof. Since $t \mapsto \operatorname{Im} p(t, w_0)$ strongly changes sign on $[a, b]$ we can find $s_\pm$ satisfying the conditions so that $\pm \operatorname{Im} p(s_\pm, w_0) > 0$. By continuity we can find $\varepsilon_\pm > 0$ so that $\pm \operatorname{Im} p(s_\pm, w) > 0$ for any $|w - w_0| < \varepsilon_\pm$. The lemma now follows if we take $\varepsilon = \min(\varepsilon_-, \varepsilon_+)$. $\qquad\square$

Definition 2.8. Let $\gamma = [a, b] \times \{w_0\}$ and $\gamma_j = [a_j, b_j] \times \{w_j\}$. If $\liminf_{j \to \infty} a_j \geq a$, $\limsup_{j \to \infty} b_j \leq b$ and $\lim_{j \to \infty} w_j = w_0$, then we shall write $\gamma_j \dashrightarrow \gamma$ as $j \to \infty$. If in addition $\lim_{j \to \infty} a_j = a$ and $\lim_{j \to \infty} b_j = b$ then we shall write $\gamma_j \to \gamma$ as $j \to \infty$.

Definition 2.9. If γ is a bicharacteristic of $\operatorname{Re} p = \xi_1$ and there exists a sequence $\{\gamma_j\}$ of bicharacteristics of $\operatorname{Re} p$ such that $\operatorname{Im} p$ strongly changes sign from $-$ to $+$ on γ_j for all j and $\gamma_j \dashrightarrow \gamma$ as $j \to \infty$, we set

$$L_p(\gamma) = \inf_{\{\gamma_j\}} \{\liminf_{j \to \infty} |\gamma_j| : \gamma_j \dashrightarrow \gamma \text{ as } j \to \infty\}, \tag{2-7}$$

where the infimum is taken over all such sequences. We shall write $L_p(\gamma) \geq 0$ to signify the existence of such a sequence $\{\gamma_j\}$.

Remark. The definition of $L_p(\gamma)$ corresponds to what is denoted by L_0 in [Hörmander 1985b, page 97], when $\gamma = [a, b] \times \{w_0\}$ is given by (2-6) and

$$\operatorname{Im} p(a, w_0) < 0 < \operatorname{Im} p(b, w_0). \tag{2-8}$$

To prove this claim, we begin by showing that $L_p(\gamma) \leq L_0$, after having properly defined L_0. To this end, let $\tilde{\gamma} = [\tilde{a}, \tilde{b}] \times \{\tilde{w}\}$ be a bicharacteristic of $\operatorname{Re} p$ such that $\operatorname{Im} p$ changes sign on $\tilde{\gamma}$. For w close to w_0 we set

$$\mathscr{L}_p(\tilde{\gamma}, w) = \inf\{t - s : \tilde{a} < s < t < \tilde{b}, \ \operatorname{Im} p(\tilde{a}, w) < 0 < \operatorname{Im} p(\tilde{b}, w)\}.$$

(Using the notation in [Hörmander 1985b, page 97] we would have $\mathscr{L}_p(\gamma, w) = L(x', \xi')$ if $w = (x', 0, \xi')$.) Then

$$L_0 = \liminf_{w \to w_0} \mathscr{L}_p(\gamma, w).$$

By an adaptation of the arguments in [Hörmander 1985b, page 97] it follows from the definition of L_0 that we can find a sequence $\{\gamma_j\}$ of bicharacteristics of Re p with $\gamma_j = [a_j, b_j] \times \{w_j\}$ such that

$$\operatorname{Im} p(a_j, w_j) < 0 < \operatorname{Im} p(b_j, w_j) \quad \text{for all } j,$$

where $\lim w_j = w_0$ and the limits $a_0 = \lim a_j$ and $b_0 = \lim b_j$ exist, belong to the interval (a, b) and satisfy $b_0 - a_0 = L_0$. If we for each j apply Lemma 2.5 to γ_j we obtain a sequence of bicharacteristics $\Gamma_j \subset \gamma_j$ of Re p such that Im p strongly changes sign from $-$ to $+$ on Γ_j, where $|\Gamma_j| = \mathscr{L}_p(\gamma_j, w_j) < |\gamma_j|$. Clearly $\Gamma_j \dashrightarrow \gamma$ as $j \to \infty$. Since $a < a_j \leq b_j < b$ if j is sufficiently large it follows that for such j we have $\mathscr{L}_p(\gamma, w_j) \leq \mathscr{L}_p(\gamma_j, w_j)$ by definition. This implies

$$
\begin{aligned}
L_0 = \liminf_{w \to w_0} \mathscr{L}_p(\gamma, w) &\leq \liminf_{j \to \infty} \mathscr{L}_p(\gamma, w_j) \\
&\leq \liminf_{j \to \infty} |\Gamma_j| \leq \limsup_{j \to \infty} |\Gamma_j| \leq \lim_{j \to \infty} |\gamma_j| = L_0,
\end{aligned}
\tag{2-9}
$$

so $|\Gamma_j| \to L_0$ as $j \to \infty$. Thus $L_p(\gamma) \leq L_0$.

For the reversed inequality, suppose $\{\tilde{\gamma}_j\}$ is any sequence satisfying the properties of Definition 2.9, with $\tilde{\gamma}_j = [\tilde{a}_j, \tilde{b}_j] \times \{\tilde{w}_j\}$. By assumption we have Im $p(\tilde{a}_j, \tilde{w}_j) = \operatorname{Im} p(\tilde{b}_j, \tilde{w}_j) = 0$ for all j, which together with (2-8) and a continuity argument implies the existence of a positive integer j_0 such that

$$a < \tilde{a}_j \leq \tilde{b}_j < b \quad \text{for all } j \geq j_0.$$

If $\tilde{\gamma}_{j,\delta} = [\tilde{a}_j - \delta, \tilde{b}_j + \delta] \times \{\tilde{w}_j\}$, this means that for small $\delta > 0$ and sufficiently large j we have

$$\mathscr{L}_p(\gamma, \tilde{w}_j) \leq \mathscr{L}_p(\tilde{\gamma}_{j,\delta}, \tilde{w}_j).$$

Since Im p strongly changes sign from $-$ to $+$ on $\tilde{\gamma}_j$, the infimum in the right side exists for every $\delta > 0$ and is bounded from above by $\tilde{b}_j - \tilde{a}_j + 2\delta$. Taking the limit as $\delta \to 0$ yields $\mathscr{L}_p(\gamma, \tilde{w}_j) \leq |\tilde{\gamma}_j|$. Since $\tilde{w}_j \to w_0$ as $j \to \infty$ the definition of L_0 now gives

$$L_0 \leq \liminf_{j \to \infty} \mathscr{L}_p(\gamma, \tilde{w}_j) \leq \liminf_{j \to \infty} |\tilde{\gamma}_j|, \tag{2-10}$$

and since the sequence $\{\tilde{\gamma}_j\}$ was arbitrary, we obtain $L_0 \leq L_p(\gamma)$ by Definition 2.9. This proves the claim.

When no confusion can occur we will omit the dependence on p in Definition 2.9. We note that if $L_p(\gamma)$ exists, then $L_p(\gamma) \leq |\gamma|$ by definition. Also, if Im p strongly changes sign from $-$ to $+$ on γ then Lemma 2.7 implies that the conditions of Definition 2.9 are satisfied. This proves the first part of the following result.

Corollary 2.10. *Let $\gamma = [a, b] \times \{w_0\}$ be a bicharacteristic of Re $p = \xi_1$. If Im p strongly changes sign from $-$ to $+$ on γ, then $0 \leq L_p(\gamma) \leq |\gamma|$. Moreover, for every $\delta, \varepsilon > 0$ there exists a bicharacteristic $\tilde{\gamma} = \tilde{\gamma}_{\delta,\varepsilon}$ of Re p with*

$$\tilde{\gamma} = [\tilde{a}, \tilde{b}] \times \{\tilde{w}\}, \quad a - \varepsilon < \tilde{a} \leq \tilde{b} < b + \varepsilon, \quad |\tilde{w} - w_0| < \varepsilon,$$

such that Im p strongly changes sign from $-$ to $+$ on $\tilde{\gamma}$ and $|\tilde{\gamma}| < L_p(\gamma) + \delta$.

Proof. The existence of the sequence $\{\Gamma_j\}$ in the preceding remark can after some adjustments be used to prove the second part of Corollary 2.10, but we prefer the following direct proof.

Given $\delta > 0$ we can by Definition 2.9 find a sequence $\gamma_j = [a_j, b_j] \times \{w_j\}$ of bicharacteristics of Re p such that $\gamma_j \dashrightarrow \gamma$ as $j \to \infty$, Im p strongly changes sign from $-$ to $+$ on γ_j and $\liminf_{j\to\infty} |\gamma_j| < L(\gamma) + \delta$. After reducing to a subsequence we may assume $|\gamma_j| < L(\gamma) + \delta$ for all j. We have $\liminf_{j\to\infty} a_j \geq a$, so for every ε there exists a $j_1(\varepsilon)$ such that $a_j > a - \varepsilon$ for all $j \geq j_1$. Similarly there exists a $j_2(\varepsilon)$ such that $b_j < b + \varepsilon$ for all $j \geq j_2$. Also, $w_j \to w_0$ as $j \to \infty$, so there exists a $j_3(\varepsilon)$ such that $|w_j - w_0| < \varepsilon$ for all $j \geq j_3$. Hence we can take $\tilde{\gamma} = \gamma_{j_0}$, where $j_0 = \max(j_1, j_2, j_3)$. $\qquad\square$

Consider now the general case when Im qp changes sign from $-$ to $+$ on a bicharacteristic $\gamma \subset T^*(X) \smallsetminus 0$ of Re qp, where $q \neq 0$, that is, (2-5) holds. In view of the proof of Lemma 2.5 we can by means of (2-7) define a minimality property of a subset of the curve γ in the following sense.

Definition 2.11. Let $I \subset \mathbb{R}$ be a compact interval possibly reduced to a point and let $\tilde{\gamma} : I \to T^*(X) \smallsetminus 0$ be a characteristic point or a compact one-dimensional bicharacteristic interval of the homogeneous function $p \in C^\infty(T^*(X) \smallsetminus 0)$. Suppose that there exists a function $q \in C^\infty(T^*(X) \smallsetminus 0)$ and a C^∞ homogeneous canonical transformation χ from an open conic neighborhood V of

$$\Gamma = \{(x_1, 0, \varepsilon_n) : x_1 \in I\} \subset T^*(\mathbb{R}^n)$$

to an open conic neighborhood $\chi(V) \subset T^*(X) \smallsetminus 0$ of $\tilde{\gamma}(I)$ such that

(i) $\chi(x_1, 0, \varepsilon_n) = \tilde{\gamma}(x_1)$ and Re $\chi^*(qp) = \xi_1$ in V,

(ii) $L_{\chi^*(qp)}(\Gamma) = |\Gamma|$.

Then we say that $\tilde{\gamma}(I)$ is a minimal characteristic point or a minimal bicharacteristic interval if $|I| = 0$ or $|I| > 0$, respectively.

The definition of the arclength is of course dependent of the choice of Riemannian metric on $T^*(\mathbb{R}^n)$. However, since we are only using the arclength to compare curves where one is contained within the other and both are parametrizable through condition (i), the results here and Definition 2.11 in particular are independent of the chosen metric. By choosing a Riemannian metric on $T^*(X)$, one could therefore define the minimality property given by Definition 2.11 through the corresponding arclength in $T^*(X)$ directly, although there, the notion of convergence of curves is somewhat trickier. We shall not pursue this any further.

Note that condition (i) implies that $q \neq 0$ and Re $H_{qp} \neq 0$ on $\tilde{\gamma}$, and that by definition, a minimal bicharacteristic interval is a compact one-dimensional bicharacteristic interval. Moreover, if Im qp changes sign from $-$ to $+$ on a bicharacteristic $\gamma \subset T^*(X) \smallsetminus 0$ of Re qp, where $q \neq 0$, then we can always find a minimal characteristic point $\tilde{\gamma} \in \gamma$ or a minimal bicharacteristic interval $\tilde{\gamma} \subset \gamma$. In view of the proof of Lemma 2.5, this follows from the conclusion of the extensive remark beginning on page 432 together with (2-9). The following proposition shows that this continues to hold even when the assumption (2-5) is relaxed in the sense of Definition 2.9. We will state this result only in the (very weak) generality needed here.

Proposition 2.12. *Let $\gamma = [a, b] \times \{w_0\}$ be a bicharacteristic of* Re $p = \xi_1$, *and assume that $L(\gamma) \geq 0$. Then there exists a minimal characteristic point $\Gamma \in \gamma$ of p or a minimal bicharacteristic interval $\Gamma \subset \gamma$ of p of length $L(\gamma)$ if $L(\gamma) = 0$ or $L(\gamma) > 0$, respectively. If $\Gamma = [a_0, b_0] \times \{w_0\}$ and $a_0 < b_0$, that is, $L(\gamma) > 0$, then*

$$\operatorname{Im} p_{(\alpha)}^{(\beta)}(t, w_0) = 0 \tag{2-11}$$

for all α, β with $\beta_1 = 0$ if $a_0 \leq t \leq b_0$. Conversely, if γ is a minimal characteristic point or a minimal bicharacteristic interval then $L(\gamma) = |\gamma|$.

Lemma 2.13. *Let γ and γ_j for $j \geq 1$ be bicharacteristics of* Re $p = \xi_1$, *and assume that* Im p *strongly changes sign from $-$ to $+$ on γ_j for each j. If $\gamma_j \dashrightarrow \gamma$ as $j \to \infty$ then $L(\gamma) \leq \liminf_{j \to \infty} L(\gamma_j)$.*

Proof. Let $\gamma_j = [a_j, b_j] \times \{w_j\}$ and $\gamma = [a, b] \times \{w_0\}$. Since Im p strongly changes sign from $-$ to $+$ on γ_j we can by Corollary 2.10 for each j find a bicharacteristic $\tilde{\gamma}_j = [\tilde{a}_j, \tilde{b}_j] \times \{\tilde{w}_j\}$ of Re p with

$$a_j - 1/j < \tilde{a}_j \leq \tilde{b}_j < b_j + 1/j \quad \text{and} \quad |\tilde{w}_j - w_j| < 1/j,$$

such that Im p strongly changes sign from $-$ to $+$ on $\tilde{\gamma}_j$ and $|\tilde{\gamma}_j| < L(\gamma_j) + 1/j$. Now $|\tilde{w}_j - w_0| \leq |\tilde{w}_j - w_j| + |w_j - w_0|$, and since $\liminf_{j \to \infty} \tilde{a}_j \geq \liminf_{j \to \infty}(a_j - 1/j) \geq a$ and correspondingly for $\tilde{b}_j$, we find that $\tilde{\gamma}_j \dashrightarrow \gamma$ as $j \to \infty$. Thus

$$L(\gamma) \leq \liminf_{j \to \infty} |\tilde{\gamma}_j| \leq \liminf_{j \to \infty}(L(\gamma_j) + 1/j). \qquad \square$$

Proof of Proposition 2.12. We may without loss of generality assume that $w_0 = (0, \varepsilon_n) \in \mathbb{R}^{2n-1}$. The last statement is then an immediate consequence of Definition 2.11. To prove the theorem it then also suffices to show that we can find a characteristic point $\Gamma \in \gamma$ of p, or a compact one-dimensional bicharacteristic interval $\Gamma \subset \gamma$ of p of length $L(\gamma)$, with the property that in any neighborhood of Γ there is a bicharacteristic of Re p where Im p strongly changes sign from $-$ to $+$. This is done by adapting the arguments in [Hörmander 1985b, page 97], which also yields (2-11).

For small $\delta > 0$ we can find $\varepsilon(\delta)$ with $0 < \varepsilon < \delta$ such that $L(\tilde{\gamma}) > L(\gamma) - \delta/2$ for any bicharacteristic $\tilde{\gamma} = [\tilde{a}, \tilde{b}] \times \{\tilde{w}\}$ with $a - \varepsilon < \tilde{a} \leq \tilde{b} < b + \varepsilon$ and $|\tilde{w} - w_0| < \varepsilon$ such that Im p strongly changes sign from $-$ to $+$ on $\tilde{\gamma}$. Indeed, otherwise there would exist a $\delta > 0$ such that for each (sufficiently large) k we can find a bicharacteristic $\gamma_k = [a_k, b_k] \times \{w_k\}$ with $a - 1/k < a_k \leq b_k < b + 1/k$ and $|w_k - w_0| < 1/k$ such that Im p strongly changes sign from $-$ to $+$ on γ_k and $L(\gamma_k) \leq L(\gamma) - \delta/2$. This implies that $\gamma_k \dashrightarrow \gamma$ as $k \to \infty$, so by Lemma 2.13 we obtain

$$L(\gamma) \leq \liminf_{k \to \infty} L(\gamma_k) \leq L(\gamma) - \delta/2,$$

a contradiction. Since $L(\gamma) \geq 0$ we have by Corollary 2.10 for some $|w_\delta - w_0| < \varepsilon$ and $a - \varepsilon < a_\delta \leq b_\delta < b + \varepsilon$ with $w_\delta = (x_\delta', 0, \xi_\delta')$ that Im p strongly changes sign from $-$ to $+$ on the bicharacteristic $\gamma_\delta = [a_\delta, b_\delta] \times \{w_\delta\}$, and $|\gamma_\delta| < L(\gamma) + \delta/4$. Thus,

$$L(\gamma) - \delta/2 < |\gamma_\delta| < L(\gamma) + \delta/4. \tag{2-12}$$

We claim that Im p and all derivatives with respect to x' and ξ' must vanish at (t, w_δ) if $a_\delta + \delta < t < b_\delta - \delta$. Indeed, by Lemma 2.7 we can find a $\rho > 0$, $a_\delta - \delta/4 < s_- < a_\delta$ and $b_\delta < s_+ < b_\delta + \delta/4$ such that

$$\operatorname{Im} p(s_-, w) < 0 < \operatorname{Im} p(s_+, w) \quad \text{for all } |w - w_\delta| < \rho.$$

If Im p and all derivatives with respect to x' and ξ' do not vanish at (t, w_δ) if $a_\delta + \delta < t < b_\delta - \delta$, then we can choose $w = (x', 0, \xi')$ so that $|w - w_\delta| < \rho$, $|w - w_0| < \varepsilon$ and Im $p(t, w) \neq 0$ for some $a_\delta + \delta < t < b_\delta - \delta$. It follows that the required sign change of Im $p(x_1, w)$ must occur on one of the intervals (s_-, t) and (t, s_+), which are shorter than $L(\gamma) - \delta/2$. This contradiction proves the claim.

Now choose a sequence $\delta_j \to 0$ as $j \to \infty$ such that $\lim a_{\delta_j}$ and $\lim b_{\delta_j}$ exist. If we denote these limits by a_0 and b_0, respectively, then $L(\gamma) = b_0 - a_0$ by (2-12), and (2-11) holds if $a_0 < b_0$. In particular, if $a_0 < b_0$ then

$$H_p(\gamma(t)) = (1 + i\partial \operatorname{Im} p(\gamma(t))/\partial\xi_1)\gamma'(t) \quad \text{for } a_0 \leq t \leq b_0,$$

so if $\Gamma = \{(t, w_0) : t \in [a_0, b_0]\}$ then Γ is a compact one-dimensional bicharacteristic interval of p with the function c in Definition 2.6 given by

$$c(t) = (1 + i\partial \operatorname{Im} p(\Gamma(t))/\partial\xi_1)^{-1}. \qquad \square$$

Proposition 2.12 allows us to make some additional comments on the implications of Definition 2.11. With the notation in the definition, we note that condition (ii) implies that there exists a sequence $\{\Gamma_j\}$ of bicharacteristics of Re $\chi^*(qp)$ on which Im $\chi^*(qp)$ strongly changes sign from $-$ to $+$, such that $\Gamma_j \to \Gamma$ as $j \to \infty$. By our choice of terminology, the sequence $\{\Gamma_j\}$ may simply be a sequence of points when $L(\Gamma) = 0$. Conversely, if $\{\Gamma_j\}$ is a point sequence then $L(\Gamma) = 0$. Also note that if $\tilde{\gamma}(I)$ is minimal, and condition (i) in Definition 2.11 is satisfied for some other choice of maps q' and χ', then condition (ii) also holds for q' and χ'; in other words,

$$L_{\chi^*(qp)}(\Gamma) = |\Gamma| = L_{(\chi')^*(q'p)}(\Gamma).$$

This follows by an application of Proposition 2.12 together with [Hörmander 1985b, Lemma 26.4.10]. It is then also clear that $\tilde{\gamma}(I)$ is a minimal characteristic point or a minimal bicharacteristic interval of the homogeneous function $p \in C^\infty(T^*(X) \smallsetminus 0)$ if and only if $\Gamma(I)$ is a minimal characteristic point or a minimal bicharacteristic interval of $\chi^*(qp) \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$ for any maps q and χ satisfying condition (i) in Definition 2.11.

The proof of [Hörmander 1985b, Theorem 26.4.7] stating that condition (Ψ) is necessary for local solvability relies on the imaginary part of the principal symbol satisfying (2-11). By Proposition 2.12, it is clear that (2-11) holds on a minimal bicharacteristic interval Γ in the case $q = 1$ and Re $p = \xi_1$. However, we shall require that we can find bicharacteristics arbitrarily close to Γ for which the following stronger result is applicable, at least if Im p does not depend on ξ_1 as is the case for the standard normal form. This will be made precise below.

Proposition 2.14. *Let $p = \xi_1 + i \operatorname{Im} p$. Assume that* $\operatorname{Im} p$ *strongly changes sign from $-$ to $+$ on $\gamma = [a, b] \times \{w\}$ and that $L(\gamma) \geq |\gamma| - \varrho$ for some $0 < \varrho < |\gamma|/2$. If* $\operatorname{Im} p$ *does not depend on ξ_1 then for any $\kappa > \varrho$ we find that* $\operatorname{Im} p$ *vanishes identically in a neighborhood of $I_\kappa \times \{w\}$, where $I_\kappa = [a + \kappa, b - \kappa]$.*

The statement would of course be void if the hypotheses hold only for $\varrho \geq |\gamma|/2$, for then $I_\kappa = \varnothing$.

Proof. If the statement is false, there exists a $\kappa > 0$ such that $\operatorname{Im} p \not\equiv 0$ near $I_\kappa \times \{w\}$. Thus there exists a sequence $(s_j, w_j) \dashrightarrow I_\kappa \times \{w\}$ such that $\operatorname{Im} p(s_j, w_j) \neq 0$ for all j. Since $\operatorname{Im} p$ does not depend on ξ_1 we can choose w_j to have ξ_1 coordinate equal to zero for all j, so that (s_j, w_j) is contained in a bicharacteristic of $\operatorname{Re} p$. We may choose a subsequence so that for some $s \in I_\kappa$ we have $|s_j - s| \to 0$ and $|w_j - w| \to 0$ monotonically, and either $\operatorname{Im} p(s_j, w_j) > 0$ or $-\operatorname{Im} p(s_j, w_j) > 0$ for all j. We shall consider the case with positive sign, the negative case works similarly.

Choose $\delta < (\kappa - \varrho)/3$ and use Lemma 2.7. We find that there exists $a - \delta < s_- < a$ and $\varepsilon > 0$ such that $\operatorname{Im} p(s_-, v) < 0$ for any $|v - w| < \varepsilon$. Choose $k > 0$ so that $|s_j - s| < \delta$ and $|w_j - w| < \varepsilon$ when $j > k$. Then $t \mapsto \operatorname{Im} p(t, w_j)$ changes sign from $-$ to $+$ on $I_j = [s_-, s_j]$, which has length

$$|I_j| = s_j - s_- \leq |s_j - s| + s - a + a - s_- < |\gamma| - \kappa + 2\delta < |\gamma| - \varrho - \delta.$$

If we for each j apply Lemma 2.5 to $I_j \times \{w_j\}$ and let $j \to \infty$ we obtain a contradiction to the hypothesis $L(\gamma) \geq |\gamma| - \varrho$. $\qquad\square$

One could state Proposition 2.14 without the condition that the imaginary part is independent of ξ_1. The invariant statement would then be that the restriction of the imaginary part to the characteristic set of the real part vanishes in a neighborhood of γ.

The fact that Proposition 2.14 assumes that $\operatorname{Im} p$ strongly changes sign from $-$ to $+$ on γ means that the conditions are not in general satisfied when γ is a minimal bicharacteristic interval. As mentioned above, we will instead show that arbitrarily close to a minimal bicharacteristic interval one can always find bicharacteristics for which Proposition 2.14 is applicable. Before we state the results we introduce a helpful definition together with some (perhaps contrived but illustrative) examples.

Definition 2.15. A minimal bicharacteristic interval $\Gamma = [a_0, b_0] \times \{w_0\} \subset T^*(\mathbb{R}^n) \setminus 0$ of the homogeneous function $p = \xi_1 + i \operatorname{Im} p$ of degree 1 is said to be ϱ-minimal if there exists a $\varrho \geq 0$ such that $\operatorname{Im} p$ vanishes in a neighborhood of $[a_0 + \kappa, b_0 - \kappa] \times \{w_0\}$ for any $\kappa > \varrho$.

By a 0-minimal bicharacteristic interval Γ we thus mean a minimal bicharacteristic interval such that the imaginary part vanishes in a neighborhood of any proper closed subset of Γ. Note that this does not hold for minimal bicharacteristic intervals in general.

Example 2.16. Let $f \in C^\infty(\mathbb{R})$ be given by

$$f(t) = \begin{cases} -e^{-1/t^2} & \text{if } t < 0, \\ 0 & \text{if } 0 \leq t \leq 2, \\ e^{-1/(t-2)^2} & \text{if } t > 2 \end{cases} \tag{2-13}$$

and let $\phi \in C^\infty(\mathbb{R})$ be a smooth cutoff function with $\operatorname{supp} \phi = [0, 2]$ such that $\phi > 0$ on $(0, 2)$. If $\xi = (\xi_1, \xi')$ then

$$p_1(x, \xi) = \xi_1 + i|\xi'|(f(x_1) + x_2\phi(x_1))$$

is homogeneous of degree 1. If we write $x = (x_1, x_2, x'')$ then for any fixed $(x'', \xi') \in \mathbb{R}^{n-2} \times \mathbb{R}^{n-1}$ with $\xi' \neq 0$ we find that $\{(x_1, x_2, x'', 0, \xi') : x_1 = a, x_2 = c\}$ is a minimal characteristic point of p_1 if $c \geq 0$ and $a = 0$ or if $c \leq 0$ and $a = 2$. Note that if $\xi' \neq 0$ then $\operatorname{Im} p_1$ changes sign from $-$ to $+$ on the bicharacteristic $\gamma(x_1) = \{(x_1, 0, x'', 0, \xi')\}$ of $\operatorname{Re} p_1$, but that none of the points $\{\gamma(x_1) : 0 < x_1 < 2\}$ are minimal characteristic points.[1] On the other hand, if f is given by (2-13) let

$$h(x, \xi') = \begin{cases} |\xi'|f(x_1 - 1)e^{1/x_2} & \text{if } x_2 < 0, \\ 0 & \text{if } x_2 = 0, \\ |\xi'|f(x_1)e^{-1/x_2} & \text{if } x_2 > 0 \end{cases}$$

be the imaginary part of $p_2(x, \xi)$. If $\operatorname{Re} p_2 = \xi_1$ then p_2 is homogeneous of degree 1 and

$$\Gamma_c = \{(x_1, x_2, x'', 0, \xi') : x_2 = c, x_1 \in I_c\}$$

is a minimal bicharacteristic interval of p_2 for any $(x'', \xi') \in \mathbb{R}^{n-2} \times \mathbb{R}^{n-1}$ with $\xi' \neq 0$ if $c \geq 0$ and $I_c = [0, 2]$ or if $c \leq 0$ and $I_c = [1, 3]$. Moreover, if $c \lessgtr 0$ then Γ_c is a 0-minimal bicharacteristic interval. However, there is no $\varrho > 0$ such that the minimal bicharacteristic interval $\Gamma = \{(x_1, 0, x'', 0, \xi') : 0 \leq x_1 \leq 2\}$ is ϱ-minimal. The same holds for the minimal bicharacteristic interval $\tilde{\Gamma} = \{(x_1, 0, x'', 0, \xi') : 1 \leq x_1 \leq 3\}$. Figure 1 shows a cross-section of the characteristic sets of $\operatorname{Im} p_1$ and $\operatorname{Im} p_2$.

Lemma 2.17. *Let $p = \xi_1 + i \operatorname{Im} p$, and assume that $L(\gamma) > 0$ and that $\operatorname{Im} p$ does not depend on ξ_1. Then one can find $\tilde{\gamma}_j \subset \gamma_j \dashrightarrow \gamma$ such that $|\tilde{\gamma}_j| \to L(\gamma)$, $\operatorname{Im} p$ strongly changes sign from $-$ to $+$ on γ_j and $\operatorname{Im} p$ vanishes in a neighborhood of $\tilde{\gamma}_j$.*

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[1]If the factor x_2 in $\operatorname{Im} p_1$ is raised to the power 3 for example, then it turns out that $\{\gamma(x_1) : 0 < x_1 < 2\}$ is a one-dimensional bicharacteristic interval of p_1, and not only a bicharacteristic of the real part. It is obviously not minimal though, nor does it contain any minimal characteristic points.
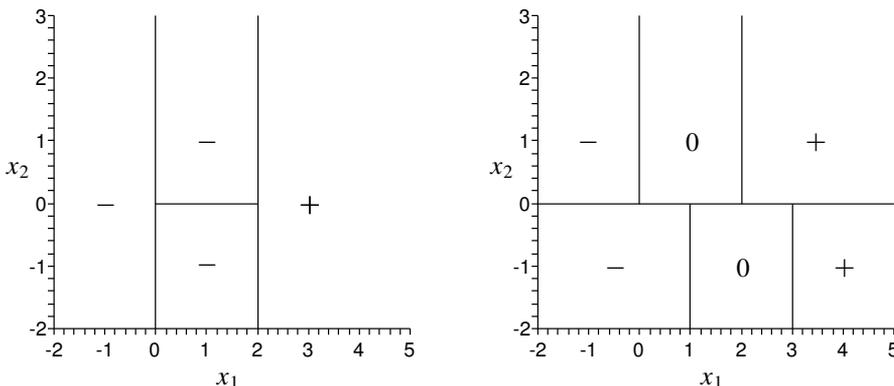

Figure 1. Cross-sections of the characteristic sets of $\operatorname{Im} p_1$ and $\operatorname{Im} p_2$, respectively.

Note that the conditions imply that $\tilde{\gamma}_j \dashrightarrow \gamma$ as $j \to \infty$.

Proof. Choose $\gamma_j \dashrightarrow \gamma$ when $j \to \infty$ as in the proof of Proposition 2.12, so that Im p strongly changes sign from $-$ to $+$ on γ_j and $L(\gamma) = \lim_{j \to \infty} |\gamma_j|$. By Lemma 2.13 and Corollary 2.10 we have

$$L(\gamma) \leq \liminf_{j \to \infty} L(\gamma_j) \leq \liminf_{j \to \infty} |\gamma_j| = L(\gamma).$$

Thus we can for every $\varepsilon > 0$ choose j so that $|L(\gamma) - |\gamma_j|| < \varepsilon$ and $|L(\gamma_j) - |\gamma_j|| < \varepsilon$. If we choose $\varepsilon < L(\gamma)/5$ then

$$2\varepsilon < (L(\gamma) - \varepsilon)/2 < |\gamma_j|/2.$$

Hence, if $\gamma_j = [a_j, b_j] \times w_j$ then by using Proposition 2.14 on γ_j we find that Im p vanishes identically in a neighborhood of $\tilde{\gamma}_j = [a_j + 2\varepsilon, b_j - 2\varepsilon] \times \{w_j\}$. Now choose a sequence $\varepsilon_k \to 0$ as $k \to \infty$. Then $\tilde{\gamma}_{j(k)} \subset \gamma_{j(k)}$ and assuming as we may that $j(k) > j(k')$ if $k > k'$ we obtain $|\tilde{\gamma}_{j(k)}| \to L(\gamma)$ as $k \to \infty$, which completes the proof. $\square$

If $\Gamma \subset \gamma$ is a minimal bicharacteristic interval in $T^*(\mathbb{R}^n) \smallsetminus 0$ of the homogeneous function $p = \xi_1 + i$ Im p of degree 1, where the imaginary part is independent of ξ_1, then by Definition 2.11 and Proposition 2.12 we have $0 < |\Gamma| = L(\Gamma)$. By the proof of Lemma 2.17 there exists a sequence $\gamma_j \to \Gamma$ of bicharacteristics of Re p such that Im p strongly changes sign from $-$ to $+$ on γ_j and vanishes identically in a neighborhood of a subinterval $\tilde{\gamma}_j \subset \gamma_j$. Moreover, $\tilde{\gamma}_j \to \Gamma$ as $j \to \infty$. By Lemma 2.13 we have $L(\gamma_j) > 0$ for sufficiently large j, so according to Proposition 2.12 we can for each such j find a minimal bicharacteristic interval $\Gamma_j \subset \gamma_j$. We have $\gamma_j \to \Gamma$ as $j \to \infty$ and since

$$|\Gamma| = L(\gamma) \leq \liminf_{j \to \infty} L(\gamma_j) = \liminf_{j \to \infty} |\Gamma_j| \leq \limsup_{j \to \infty} |\Gamma_j| \leq \lim_{j \to \infty} |\gamma_j| = |\Gamma|,$$

it follows that $\Gamma_j \to \Gamma$ as $j \to \infty$. Since also $\tilde{\gamma}_j \subset \gamma_j$ and $\tilde{\gamma}_j \to \Gamma$ as $j \to \infty$, the intersection $\tilde{\gamma}_j \cap \Gamma_j$ must be nonempty for large j. For such j it follows that $\tilde{\gamma}_j$ must be a proper subinterval of Γ_j, for if not, this would contradict the fact that Γ_j is a minimal bicharacteristic interval. Hence we can find a sequence $\{\varrho_j\}$ of positive numbers with $\varrho_j \to 0$ as $j \to \infty$, such that Γ_j is a ϱ_j-minimal bicharacteristic interval. We have thus proved the following theorem, which concludes our study of the bicharacteristics.

Theorem 2.18. *If Γ is a minimal bicharacteristic interval in $T^*(\mathbb{R}^n) \smallsetminus 0$ of the homogeneous function $p = \xi_1 + i$ Im p of degree 1, where the imaginary part is independent of ξ_1, then there exists a sequence $\{\Gamma_j\}$ of ϱ_j-minimal bicharacteristic intervals of p such that $\Gamma_j \to \Gamma$ and $\varrho_j \to 0$ as $j \to \infty$.*

We can now state our main theorem, which yields necessary conditions for inclusion relations between the ranges of operators that fail to be microlocally solvable.

Theorem 2.19. *Let $K \subset T^*(X) \smallsetminus 0$ be a compactly based cone. Let $P \in \Psi_{\mathrm{cl}}^k(X)$ and $Q \in \Psi_{\mathrm{cl}}^{k'}(X)$ be properly supported pseudodifferential operators such that the range of Q is microlocally contained in the range of P at K, where P is an operator of principal type in a conic neighborhood of K. Let p_k be the homogeneous principal symbol of P, and let $I = [a_0, b_0] \subset \mathbb{R}$ be a compact interval possibly reduced to a point. Suppose that K contains a conic neighborhood of $\gamma(I)$, where $\gamma : I \to T^*(X) \smallsetminus 0$ is either*

(a) *a minimal characteristic point of p_k, or*

(b) *a minimal bicharacteristic interval of p_k with injective regular projection in $S^*(X)$.*

Then there exists a pseudodifferential operator $E \in \Psi_{\mathrm{cl}}^{k'-k}(X)$ such that the terms in the asymptotic sum of the symbol of $Q - PE$ have vanishing Taylor coefficients at $\gamma(I)$.

The hypotheses of Theorem 2.19 imply that P is not solvable at the cone K. Indeed, solvability at $K \subset T^*(X) \smallsetminus 0$ implies solvability at any smaller closed cone, and in view of Definition 2.11 it follows by [Hörmander 1985b, Theorem 26.4.7′] together with [Hörmander 1985b, Proposition 26.4.4] that P is not solvable at the cone generated by $\gamma(I)$. Conversely, suppose that P is an operator of principal type that is not microlocally solvable in any neighborhood of a point $(x_0, \xi_0) \in T^*(X) \smallsetminus 0$. Then the principal symbol p_k fails to satisfy condition (1-5) in every neighborhood of (x_0, ξ_0) by [Dencker 2006, Theorem 1.1]. In view of the alternative version of condition (1-5) given by [Hörmander 1985b, Theorem 26.4.12], it is then easy to see using [Hörmander 1985a, Theorem 21.3.6] and [Hörmander 1985b, Lemma 26.4.10] that (x_0, ξ_0) is a minimal characteristic point of p_k, so Theorem 2.19 applies there.

We also mention that if P is of principal type and γ is a minimal bicharacteristic interval of the principal symbol p_k contained in a curve along which p_k fails to satisfy condition (1-5), then γ has injective regular projection in $S^*(X)$ by the proof of [Hörmander 1985b, Theorem 26.4.12].

Remark. As pointed out in the introduction, we cannot hope to obtain a result such as Theorem 2.19 for solvable nonelliptic operators in general. Indeed, Example 2.2 shows that if $X \subset \mathbb{R}^n$ is open, and $K \subset T^*(X) \smallsetminus 0$ is a compactly based cone, then the range of D_2 is microlocally contained in the range of D_1 at K. If there were to exist a pseudodifferential operator $e(x, D) \in \Psi_{\mathrm{cl}}^0(X)$ such that all the terms in the symbol of $R(x, D) = D_2 - D_1 \circ e(x, D)$ have vanishing Taylor coefficients at a point $(x_0, \xi_0) \in K$ contained in a bicharacteristic of the principal symbol $\sigma(D_1) = \xi_1$ of D_1, then in particular this would hold for the principal symbol

$$\sigma(R)(x, \xi) = \xi_2 - \xi_1 e_0(x, \xi),$$

if e_0 denotes the principal symbol of $e(x, D)$. However, taking the ξ_2 derivative of the equation above and evaluating at (x_0, ξ_0) then immediately yields the contradiction $0 = 1$ since (x_0, ξ_0) belongs to the hypersurface $\xi_1 = 0$.

In the proof of the theorem we may assume that P and Q are operators of order 1. In fact, the discussion following Definition 2.1 shows that if the conditions of Theorem 2.19 hold and $Q_1 \in \Psi_{\mathrm{cl}}^{k-k'}(X)$ and $Q_2 \in \Psi_{\mathrm{cl}}^{1-k}(X)$ are properly supported, then the range of $Q_2 Q Q_1 \in \Psi_{\mathrm{cl}}^1(X)$ is microlocally contained in the range of $Q_2 P \in \Psi_{\mathrm{cl}}^1(X)$ at K. If the theorem holds for operators of the same order k then there exists an operator $E \in \Psi_{\mathrm{cl}}^0(X)$ such that all the terms in the asymptotic expansion of the symbol of $QQ_1 - PE$ have vanishing Taylor coefficients at $\gamma(I)$. If we choose Q_1 to be elliptic, then we can find a parametrix Q_1^{-1} of Q_1 such that

$$Q - PEQ_1^{-1} \equiv (QQ_1 - PE) \circ Q_1^{-1} \mod \Psi^{-\infty}(X)$$

has symbol

$$\sigma_{A \circ Q_1^{-1}}(x, \xi) \sim \sum \partial_\xi^\alpha \sigma_A(x, \xi) D_x^\alpha \sigma_{Q_1^{-1}}(x, \xi)/\alpha! \tag{2-14}$$

with $A = QQ_1 - PE$. Clearly, all the terms in the asymptotic expansion of the symbol of $Q - PEQ_1^{-1}$ then have vanishing Taylor coefficients at $\gamma(I)$, and $E_1 = EQ_1^{-1} \in \Psi_{cl}^{k'-k}(X)$, so the theorem holds with E replaced by E_1. If the theorem holds for operators of order 1 we can choose Q_2 elliptic and use the same argument to show that if all the terms in the asymptotic expansion of the symbol of $Q_2QQ_1 - Q_2PE$ have vanishing Taylor coefficients at $\gamma(I)$, then the same holds for

$$Q - PEQ_1^{-1} \equiv Q_2^{-1} \circ (Q_2QQ_1 - Q_2PE) \circ Q_1^{-1} \quad \mod \Psi^{-\infty}(X),$$

where Q_2^{-1} is a parametrix of Q_2. Here we use the fact that if $\gamma(I)$ is a minimal characteristic point or a minimal bicharacteristic interval of the principal symbol of P, then this also holds for the principal symbol of Q_2P by Definition 2.11.

For pseudodifferential operators, the property that all terms in the asymptotic expansion of the total symbol have vanishing Taylor coefficients is preserved under conjugation with Fourier integral operators associated with a canonical transformation (see Lemma A.1). Thus we will be able to prove Theorem 2.19 by local arguments and an application of Proposition 2.4.

Let $\gamma: I \to T^*(X) \setminus 0$, with $I = [a_0, b_0] \subset \mathbb{R}$, be the map given by Theorem 2.19. By using [Hörmander 1985a, Theorem 21.3.6] or [Hörmander 1985b, Theorem 26.4.13], when γ is a characteristic point or a one-dimensional bicharacteristic, respectively, we can find a C^∞ canonical transformation χ from a conic neighborhood of $\Gamma = \{(x, \varepsilon_n) : x_1 \in I, \ x' = 0\}$ in $T^*(\mathbb{R}^n) \setminus 0$ to a conic neighborhood of $\gamma(I)$ in $T^*(X) \setminus 0$ and a C^∞ homogeneous function b of degree 0 with no zero on $\gamma(I)$ such that $\chi(x_1, 0, \varepsilon_n) = \gamma(x_1)$ for $x_1 \in I$ and

$$\chi^*(bp_1) = \xi_1 + if(x, \xi'), \tag{2-15}$$

where f is real-valued, homogeneous of degree 1 and independent of ξ_1. Thus, by the hypotheses of Theorem 2.19 one can in any neighborhood of Γ find an interval in the x_1 direction where f changes sign from $-$ to $+$ for increasing x_1. Also, if I is an interval then f vanishes of infinite order on Γ by (2-11), and by Theorem 2.18 there exists a sequence $\{\Gamma_j\}$ of ϱ_j-minimal bicharacteristics of $\chi^*(bp_1)$ such that $\varrho_j \to 0$ and $\Gamma_j \to \Gamma$ as $j \to \infty$.

The existence of the canonical transformation χ together with Proposition 2.4 implies that we can find Fourier integral operators A and B such that the range of BQA is microlocally contained in the range of BPA at a cone K' containing Γ, where the principal symbol of BPA is given by (2-15). In view of Lemma A.1 we may therefore reduce the proof to the case that $P, Q \in \Psi_{cl}^1(\mathbb{R}^n)$ and the principal symbol p of P is given by (2-15). In accordance with the notation in Proposition 2.4 we will assume that the range of Q is microlocally contained in the range of P at a cone K containing Γ, thus renaming K' to K. If

$$\sigma_Q = q_1 + q_0 + \cdots$$

is the asymptotic sum of homogeneous terms of the symbol of Q, we can then use the Malgrange preparation theorem (see [Hörmander 1983a, Theorem 7.5.6]) to find $e_0, r_1 \in C^\infty$ near Γ such that

$$q_1(x, \xi) = (\xi_1 + i f(x, \xi'))e_0(x, \xi) + r_1(x, \xi'),$$

where r_1 is independent of ξ_1. Restricting to $|\xi| = 1$ and extending by homogeneity we can make e_0 and r_1 homogeneous of degree 0 and 1, respectively. The term of degree 1 in the symbol of $Q - P \circ e_0(x, D)$ is $r_1(x, \xi')$. Again, by Malgrange's preparation theorem we can find $e_{-1}, r_0 \in C^\infty$ near Γ such that

$$q_0(x, \xi) - \sigma_0(P \circ e_0(x, D))(x, \xi) = (\xi_1 + i f(x, \xi'))e_{-1}(x, \xi) + r_0(x, \xi'),$$

where e_{-1} and r_0 are homogeneous of degree -1 and 0, respectively, and r_0 is independent of ξ_1. The term of degree 0 in the symbol of

$$Q - P \circ e_0(x, D) - P \circ e_{-1}(x, D)$$

is $r_0(x, \xi')$. Repetition of the argument allows us to write

$$Q = P \circ E + R(x, D_{x'}) \tag{2-16}$$

where $\sigma_R(x, \xi') = r_1(x, \xi') + r_0(x, \xi') + \cdots$ is an asymptotic sum of homogeneous terms, all independent of ξ_1. Thus $R(x, D_{x'})$ is a pseudodifferential operator in the $n - 1$ variables x' depending on x_1 as a parameter. Furthermore, the range of $R(x, D_{x'})$ is microlocally contained in the range of P at K. Indeed, suppose N is the integer given by Definition 2.1. If $g \in H_{(N)}^{\mathrm{loc}}(\mathbb{R}^n)$, then $Rg = PEg - Qg = Pv - Qg$ for some $v \in \mathcal{D}'(\mathbb{R}^n)$, and there exists a $u \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$K \cap WF(Qg - Pu) = \varnothing.$$

Hence, $WF(P(v - u) - Rg)$ does not meet K, so the range of R is microlocally contained in the range of P at K. We claim that under the assumptions of Theorem 2.19, this implies that all terms in the asymptotic sum of the symbol of the operator $R(x, D_{x'})$ in (2-16) have vanishing Taylor coefficients at Γ, thus proving Theorem 2.19. The proof of this claim will be based on the two theorems stated below. As we have seen, the principal symbol p of P may be assumed to have the normal form given by (2-15). By means of Theorem 2.20 below, we shall also use the fact that an even simpler normal form exists near a point where $p = 0$ and $\{\mathrm{Re}\, p, \mathrm{Im}\, p\} \neq 0$. To prove these two theorems, we will use techniques that actually require the lower order terms of P to be independent of ξ_1 near Γ. However, we claim that this may always be assumed. In fact, Malgrange's preparation theorem implies that

$$p_0(x, \xi) = a(x, \xi)(\xi_1 + i f(x, \xi')) + b(x, \xi')$$

where a is homogeneous of degree -1 and b homogeneous of degree 0, as demonstrated in the construction of the operators E and R above. The term of degree 0 in the symbol of $(I - a(x, D))P$ is equal to $b(x, \xi')$. Repetition of the argument implies that there exists a classical operator $\tilde{a}(x, D)$ of order -1 such that $(I - \tilde{a}(x, D))P$ has principal symbol $\xi_1 + i f(x, \xi')$ and all lower order terms are independent of ξ_1. The microlocal property of pseudodifferential operators immediately implies that the

range of $(I - \tilde{a}(x, D))Q$ is microlocally contained in the range of $(I - \tilde{a}(x, D))P$ at K. Hence, if there are operators E and R with

$$R = (I - \tilde{a}(x, D))Q - (I - \tilde{a}(x, D))PE$$

such that all terms in the asymptotic expansion of the symbol of R have vanishing Taylor coefficients at Γ, then this also holds for the symbol of $Q - PE \equiv (I - \tilde{a}(x, D))^{-1}R \mod \Psi^{-\infty}$, since this property is preserved under composition with elliptic pseudodifferential operators by (2-14).

Theorem 2.20. *Suppose that in a conic neighborhood Ω of*

$$\Gamma' = \{(0, \varepsilon_n)\} \subset T^*(\mathbb{R}^n) \smallsetminus 0$$

P has the form $P = D_1 + ix_1 D_n$ and the symbol of $R(x, D_{x'})$ is given by the asymptotic sum

$$\sigma_R = \sum_{j=0}^{\infty} r_{1-j}(x, \xi'),$$

with r_{1-j} homogeneous of degree $1 - j$ and independent of ξ_1. If there exists a compactly based cone $K \subset T^*(\mathbb{R}^n) \smallsetminus 0$ containing Ω such that the range of R is microlocally contained in the range of P at K, then all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients at Γ'.

Theorem 2.21. *Suppose that in a conic neighborhood Ω of*

$$\Gamma' = \{(x_1, x', 0, \xi') : a \le x_1 \le b\} \subset T^*(\mathbb{R}^n) \smallsetminus 0$$

the principal symbol of P has the form

$$p(x, \xi) = \xi_1 + if(x, \xi'),$$

where f is real-valued and homogeneous of degree 1, and suppose that if $b > a$ then f vanishes of infinite order on Γ' and there exists a $\varrho \ge 0$ such that for any $\varepsilon > \varrho$ one can find a neighborhood of

$$\Gamma'_\varepsilon = \{(x_1, x', 0, \xi') : a + \varepsilon \le x_1 \le b - \varepsilon\}, \tag{2-17}$$

where f vanishes identically. Suppose also that

$$f(x, \xi') = 0 \quad \text{implies} \quad \partial f(x, \xi')/\partial x_1 \le 0 \tag{2-18}$$

in Ω and that in any neighborhood of Γ' one can find an interval in the x_1 direction where f changes sign from $-$ to $+$ for increasing x_1. Furthermore, suppose that in Ω the symbol of $R(x, D_{x'})$ is given by the asymptotic sum

$$\sigma_R = \sum_{j=0}^{\infty} r_{1-j}(x, \xi')$$

with r_{1-j} homogeneous of degree $1 - j$ and independent of ξ_1. If the lower order terms $p_0, p_{-1}, \ldots$ in the symbol of P are independent of ξ_1 near Γ', and there exists a compactly based cone $K \subset T^*(\mathbb{R}^n) \smallsetminus 0$ containing Ω such that the range of R is microlocally contained in the range of P at K, then all the

terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients on Γ'_{ϱ} if $a < b$, and at Γ' if $a = b$.

Assuming these results for the moment, we can now show how Theorem 2.19 follows.

End of proof of Theorem 2.19. Recall that

$$\Gamma = \{(x_1, 0, \varepsilon_n) : a_0 \le x_1 \le b_0\} \subset T^*(\mathbb{R}^n) \setminus 0.$$

By what we have shown, it suffices to regard the case $Q = PE + R$, where we may assume that the conditions of Theorem 2.21 are all satisfied in a conic neighborhood Ω of Γ, with the exception of (2-18) and the condition concerning the existence of a neighborhood of (2-17) in which f vanishes identically when $a_0 < b_0$. We consider three cases.

(i) Γ is an interval. We then claim that condition (2-18) imposes no restriction. Indeed, if there is no neighborhood of Γ in which (2-18) holds, then there exists a sequence $\{\gamma_j\} = \{(t_j, x'_j, 0, \xi'_j)\}$ such that $a_0 \le \liminf t_j \le \limsup t_j \le b_0$, $(x'_j, \xi'_j) \to (0, \xi^0) \in \mathbb{R}^{2n-2}$,

$$f(t_j, x'_j, \xi'_j) = 0 \quad \text{and} \quad \partial f(t_j, x'_j, \xi'_j)/\partial x_1 > 0 \tag{2-19}$$

for each j. By (2-19) we can choose a sequence $0 < \delta_j \to 0$ such that

$$f(t_j - \delta_j, x'_j, \xi'_j) < 0 < f(t_j + \delta_j, x'_j, \xi'_j).$$

In view of Definition 2.9 we must therefore have $L(\Gamma) = 0$. Since Γ is minimal, this implies that $|\Gamma| = 0$, so $\gamma_j \to \Gamma$. Thus, if there is no neighborhood of Γ in which (2-18) holds, then Γ is a point, and we will in this case use the existence of the sequence $\{\gamma_j\}$ satisfying (2-19) to reduce the proof of Theorem 2.19 to Theorem 2.20, as demonstrated in case (iii) below. In the present case however, Γ is assumed to be an interval, so there exists a neighborhood $\mathcal{U}$ of Γ in which (2-18) holds. We may assume that $\mathcal{U} \subset \Omega$ and since f is homogeneous of degree 1 we may also assume that $\mathcal{U}$ is conic.

By Theorem 2.18, there exists a sequence $\{\Gamma_j\}$ of ϱ_j-minimal bicharacteristic intervals such that $\varrho_j \to 0$ and $\Gamma_j \to \Gamma$ as $j \to \infty$. For sufficiently large j we have $\Gamma_j \subset \mathcal{U}$. Hence, if

$$\Gamma_j = \{(x_1, x'_j, 0, \xi'_j) : a_j \le x_1 \le b_j\}$$

then all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients on

$$\Gamma_{\varrho_j} = \{(x_1, x'_j, 0, \xi'_j) : a_j + \varrho_j \le x_1 \le b_j - \varrho_j\}$$

by Theorem 2.21. Since $\Gamma_{\varrho_j} \to \Gamma$ as $j \to \infty$, and all the terms in the asymptotic sum of the symbol of R are smooth functions, it follows that all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients on Γ. This proves Theorem 2.19 in this case.

(ii) Γ is a point and condition (2-18) holds. Then all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients on Γ by Theorem 2.21, so Theorem 2.19 follows.

(iii) Γ is a point and (2-18) is false. Let $\{\gamma_j\}$ be the sequence satisfying (2-19). Then $\{\text{Re } p, \text{Im } p\}(\gamma_j) > 0$ and $p(\gamma_j) = 0$ for each j since $\gamma_j = (t_j, x'_j, 0, \xi'_j)$. For fixed j we may assume that $\gamma_j = (0, \eta)$ and use

[Hörmander 1985a, Theorem 21.3.3] to find a canonical transformation χ together with Fourier integral operators A, B, A_1 and B_1 as in Proposition 2.4 such that $\chi(0, \varepsilon_n) = \gamma_j$, and $BPA = D_1 + i x_1 D_n$ in a conic neighborhood Ω of $\{(0, \varepsilon_n)\}$. Repetition of the arguments above allows us to write

$$BQA = BPAE + R(x, D_{x'}),\tag{2-20}$$

where the range of R is microlocally contained in the range of BPA at some compactly based cone K' containing Ω with $\chi(K') = K$. As before, E and R have classical symbols. Then all the terms in the asymptotic expansion of the symbol of R have vanishing Taylor coefficients at $\{(0, \varepsilon_n)\}$ by Theorem 2.20, and therefore all the terms in the asymptotic expansion of the symbol of $A_1 R B_1$ have vanishing Taylor coefficients at γ_j by Lemma A.1 in the appendix. Since the Fourier integral operators are chosen so that

$$K \cap WF(A_1 B - I) = \varnothing \quad \text{and} \quad K \cap WF(A B_1 - I) = \varnothing,$$

we have

$$\varnothing = K \cap WF(A_1 B Q A B_1 - A_1 B P A E B_1 - A_1 R B_1) = K \cap WF(Q - PAEB_1 - A_1 R B_1)$$

in view of (2-20). Hence, all the terms in the asymptotic expansion of the symbol of

$$Q - PE_1 = A_1 R B_1 + S, \quad \text{where} \quad WF(S) \cap K = \varnothing,\tag{2-21}$$

have vanishing Taylor coefficients at γ_j if $E_1 = AEB_1$. (Strictly speaking, the change of base variables $\gamma_j \mapsto (0, \eta)$ should be represented in (2-21) by conjugation of a linear transformation $\kappa : \mathbb{R}^n \to \mathbb{R}^n$, but this could be integrated in the Fourier integral operators A_1 and B_1 so it has been left out since it will not affect the arguments below.) It is clear that $E_1 \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n)$.

We have now shown that for each j there exists an operator $E_j \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n)$ such that all the terms in the asymptotic expansion of the symbol of $Q - PE_j$ have vanishing Taylor coefficients at γ_j. To construct the operator E in Theorem 2.19, we do the following. For each j, denote the symbol of E_j by

$$e^j(x, \xi) \sim \sum_{l=0}^{\infty} e^j_{-l}(x, \xi)$$

where $e^j_0(x, \xi)$ is the principal part, and $e^j_{-l}(x, \xi)$ is homogeneous of degree $-l$. If q is the principal symbol of Q, then by Proposition A.3 there exists a function $e_0 \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$, homogeneous of degree 0, such that $q - pe_0$ has vanishing Taylor coefficients at Γ.

This argument can be repeated for lower order terms. Indeed, if $\sigma_Q = q + q_0 + \cdots$, then the term of degree 0 in σ_{Q-PE_j} is

$$\sigma_0(Q - PE_j) = \tilde{q}_j - pe^j_{-1},$$

where (see (2-25) below)

$$\tilde{q}_j(x, \xi) = q_0(x, \xi) - p_0(x, \xi)e^j_0(x, \xi) - \sum_k \partial_{\xi_k} p(x, \xi) D_{x_k} e^j_0(x, \xi).$$

We can write

$$p(x, \xi)e^j_{-1}(x, \xi) = p(x, \xi/|\xi|)e^j_{-1}(x, \xi/|\xi|),$$

so that $\tilde{q}_j(x, \xi)$, $p(x, \xi/|\xi|)$ and $e^j_{-1}(x, \xi/|\xi|)$ are all homogeneous of degree 0. Since

$$\partial_x^\alpha \partial_\xi^\beta e_0(\Gamma) = \lim_{j \to \infty} \partial_x^\alpha \partial_\xi^\beta e_0^j(\gamma_j)$$

it follows by Proposition A.3 that there is a function $g \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$, homogeneous of degree 0, such that

$$q_0(x, \xi) - p_0(x, \xi)e_0(x, \xi) - \sum_k \partial_{\xi_k} p(x, \xi)D_{x_k}e_0(x, \xi) - p(x, \xi/|\xi|)g(x, \xi)$$

has vanishing Taylor coefficients at Γ. Putting $e_{-1}(x, \xi) = |\xi|^{-1}g(x, \xi)$ we find that

$$\partial_x^\alpha \partial_\xi^\beta e_{-1}(\Gamma) = \lim_{j \to \infty} \partial_x^\alpha \partial_\xi^\beta e^j_{-1}(\gamma_j),$$

and that

$$\sigma_0(Q - P \circ e_0(x, D) - P \circ e_{-1}(x, D))$$

has vanishing Taylor coefficients at Γ. Continuing this way we successively obtain functions $e_m(x, \xi) \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$, homogeneous of degree m for $m \leq 0$, such that

$$\sigma_Q - \Big(\sum_{m=0}^M e_{-m}\Big)\sigma_P \quad \mathrm{mod} \ S^{-M}_{\mathrm{cl}}$$

has vanishing Taylor coefficients at Γ. If we let E have symbol

$$\sigma_E(x, \xi) \sim \sum_{m=0}^\infty (1 - \phi(\xi))e_{-m}(x, \xi)$$

with $\phi \in C_0^\infty$ equal to 1 for ξ close to 0, then $E \in \Psi_{\mathrm{cl}}^0(\mathbb{R}^n)$ and all terms in the asymptotic expansion of the symbol of $Q - PE$ have vanishing Taylor coefficients at Γ. We have proved Theorem 2.19. $\square$

Remark. Instead of reducing to the study of the normal form $P = D_{x_1} + ix_1 D_{x_n}$ when condition (2-18) does not hold, as in case (iii) above, one could show that the terms in the asymptotic expansion of the operator R given by (2-16) has vanishing Taylor coefficients at every point in the sequence $\{\gamma_j\}$ satisfying (2-19) using techniques very similar to those used to prove Theorem 2.21. Theorem 2.19 would then follow by continuity, but the proof of the analogue of Theorem 2.20 would be more involved. In particular, we would have to construct a phase function w solving the eikonal equation

$$\partial w/\partial x_1 - if(x, \partial w/\partial x') = 0$$

approximately instead of explicitly (confer the proofs of Theorems 2.21 and 2.20, respectively). For fixed j this could be accomplished by adapting the approach in [Hörmander 1963; Hörmander 1966] (for a brief discussion, see [Hörmander 1981, p. 83]), where one has $f = 0$ and $\partial f/\partial x_1 > 0$ at $(0, \xi^0)$ instead of at γ_j.

We shall now show how our results relate to the ones referred to in the introduction, beginning with (1-3). There, it sufficed to have the coefficients of P and Q in C^∞ and C^1, respectively. However, in order for Theorem 2.19 to qualify, we must require both P and Q to have smooth coefficients. On the other hand, we shall only require the equation $Pu = Qf$ to be microlocally solvable (at an appropriate cone K) as given by Definition 2.1. Note that if P is a first order differential operator on an open set $\Omega \subset \mathbb{R}^n$, such that the principal symbol p of P satisfies condition (1-4) at a point $(x, \xi) \in T^*(\Omega) \setminus 0$, then either $\{\operatorname{Re} p, \operatorname{Im} p\} > 0$ at (x, ξ), or $\{\operatorname{Re} p, \operatorname{Im} p\} > 0$ at $(x, -\xi)$. (The order of the operator is not important; the statement is still true for a differential operator of order m, since the Poisson bracket is then homogeneous of order $2m - 1$.) Assuming the former, this implies that (x, ξ) satisfies condition (a) in Theorem 2.19 by an application of [Hörmander 1985a, Theorem 21.3.3] and Lemma 2.7. In order to keep the formulation of the following result as simple as possible, we will assume that there exists a compactly based cone $K \subset T^*(\Omega) \setminus 0$ with nonempty interior such that K contains the appropriate point $(x, \pm\xi)$, and such that the equation $Pu = Qf$ is microlocally solvable at K. This is clearly the case if the equation $Pu = Qf$ is locally solvable in Ω in the weak sense suggested by (1-1).

Corollary 2.22. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $P(x, D)$ and $Q(x, D)$ be two first order differential operators with coefficients in $C^\infty(\Omega)$. Let p be the principal symbol of P, and let x_0 be a point in Ω such that*

$$p(x_0, \xi_0) = 0 \quad and \quad \{\operatorname{Re} p, \operatorname{Im} p\}(x_0, \xi_0) > 0 \tag{2-22}$$

for some $\xi_0 \in \mathbb{R}^n$. If $K \subset T^*(\Omega) \setminus 0$ is a compactly based cone containing (x_0, ξ_0) such that the range of Q is microlocally contained in the range of P at K, then there exists a constant μ such that (at the fixed point x_0)

$$Q^*(x_0, D) = \mu P^*(x_0, D), \tag{2-23}$$

where Q^* and P^* are the adjoints of Q and P.

Proof. By (2-22), $P \in \Psi^1_{\text{cl}}(\Omega)$ is an operator of principal type microlocally near (x_0, ξ_0). P and Q therefore satisfy the hypotheses of Theorem 2.19, and in view of the discussion above regarding the point (x, ξ) we find that there exists an operator $E \in \Psi^0_{\text{cl}}(\Omega)$ such that all the terms in the asymptotic expansion of the symbol of $Q - PE$ have vanishing Taylor coefficients at (x_0, ξ_0). By the discussion following (3-7) on page 452 below, it follows that the same must hold for the adjoint $Q^* - E^*P^*$. If we let Q^* and P^* have symbols $\sigma_{Q^*}(x, \xi) = q_1(x, \xi) + q_0(x)$ and $\sigma_{P^*}(x, \xi) = p_1(x, \xi) + p_0(x)$, then E^*P^* has principal symbol $e_0 p_1$ if $\sigma_{E^*} = e_0 + e_{-1} + \dots$ denotes the symbol of E^*. Hence

$$\partial q_1(x_0, \xi_0)/\partial \xi_k = e_0(x_0, \xi_0)\partial p_1(x_0, \xi_0)/\partial \xi_k \quad \text{for } 1 \le k \le n$$

and $p_1(x_0, \xi_0) = \overline{p(x_0, \xi_0)} = 0$. Since q_1 and p_1 are polynomials in ξ of degree 1, this means that at the fixed point x_0 we have $q_1(x_0, \xi) = \mu p_1(x_0, \xi)$ for $\xi \in \mathbb{R}^n$, where the constant μ is given by the value of e_0 at (x_0, ξ_0). Moreover,

$$0 = \partial_{\xi_j} \partial_{\xi_k} q_1(x_0, \xi_0) = \partial_{\xi_j} e_0(x_0, \xi_0)\partial_{\xi_k} p_1(x_0, \xi_0) + \partial_{\xi_k} e_0(x_0, \xi_0)\partial_{\xi_j} p_1(x_0, \xi_0). \tag{2-24}$$

By assumption, the coefficients of $p(x, D)$ do not vanish simultaneously, so the same is true for $p_1(x, D)$. Hence $\partial_{\xi_j} p_1(x_0, \xi_0) \neq 0$ for some j. Assuming this holds for $j = 1$, we find by choosing $j = k = 1$ in (2-24) that $\partial_{\xi_1} e_0(x_0, \xi_0) = 0$. But this immediately yields

$$\partial_{\xi_k} e_0(x_0, \xi_0) = -\partial_{\xi_1} e_0(x_0, \xi_0) \partial_{\xi_k} p_1(x_0, \xi_0) / \partial_{\xi_1} p_1(x_0, \xi_0) = 0$$

for $2 \leq k \leq n$. Now

$$\sigma_{E^* P^*}(x, \xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{E^*} D_x^\alpha (p_1(x, \xi) + p_0(x)),$$

and since we have a bilinear map

$$S_{cl}^{m'} / S^{-\infty} \times S_{cl}^{m''} / S^{-\infty} \ni (a, b) \mapsto a \# b \in S_{cl}^{m'+m''} / S^{-\infty}$$

with

$$(a \# b)(x, \xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) \, D_x^\alpha b(x, \xi),$$

we find that the term of order 0 in the symbol of $E^* P^*$ is

$$\sigma_0(E^* P^*)(x, \xi) = e_{-1}(x, \xi) p_1(x, \xi) + e_0(x, \xi) p_0(x) + \sum_{k=1}^n \partial_{\xi_k} e_0(x, \xi) \, D_k p_1(x, \xi). \tag{2-25}$$

Since $\partial_{\xi_k} e_0$ and p_1 vanish at (x_0, ξ_0) we find that $q_0(x_0) = \mu p_0(x_0)$ at the fixed point x_0, which completes the proof. $\qquad \square$

Having proved this result, we immediately obtain the following after making the obvious adjustments to [Hörmander 1963, Theorem 6.2.2]. The fact that we require higher regularity on the coefficients of Q then yields higher regularity on the propertionality factor. Since the proof remains the same, it is omitted.

Corollary 2.23. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $P(x, D)$ and $Q(x, D)$ be two first order differential operators with coefficients in $C^\infty(\Omega)$. Let p be the principal symbol of P, and assume that the coefficients of $p(x, D)$ do not vanish simultaneously in Ω. If for a dense set of points x in Ω one can find $\xi \in \mathbb{R}^n$ such that (2-22) is fulfilled, and if for each (x, ξ) there is a compactly based cone $K \subset T^*(\Omega) \setminus 0$ containing (x, ξ) such that the range of Q is microlocally contained in the range of P at K, then there exists a function $e \in C^\infty(\Omega)$ such that*

$$Q(x, D)u \equiv P(x, D)(eu). \tag{2-26}$$

In stating Corollary 2.23 we could replace the assumption that the coefficients of $p(x, D)$ do not vanish simultaneously in Ω with the condition that P is of principal type. Indeed, if $dp \neq 0$ then by a canonical transformation we find that condition (1-6) holds. Since $p \neq 0$ implies $\partial_\xi p \neq 0$ by the Euler homogeneity equation we then have $\partial_\xi p \neq 0$ everywhere, that is, the coefficients of $p(x, D)$ do not vanish simultaneously in Ω. The converse is obvious.

As shown in Example 2.25 below, we also recover the result for higher order differential operators mentioned in the introduction as a special case of the following corollary to Theorem 2.19, although we again need to assume higher regularity in order to apply our results.

Proposition 2.24. *Let X be a smooth manifold, and let $P \in \Psi_{\mathrm{cl}}^k(X)$ and $Q \in \Psi_{\mathrm{cl}}^{k'}(X)$ be properly supported such that the range of $Q \circ P$ is microlocally contained in the range of P at a compactly based cone $K \subset T^*(X) \setminus 0$. Let p and q be the principal symbols of P and Q, respectively, and assume that P is of principal type microlocally near K. If $\gamma : I \to T^*(X) \setminus 0$ is a minimal characteristic point or a minimal bicharacteristic interval of p contained in K then it follows that*

$$H_p^m(q) = 0 \quad \text{for all } (x, \xi) \in \gamma(I) \text{ and } m \geq 1.$$

Here $H_p^m(q)$ is defined recursively by $H_p(q) = \{p, q\}$ and $H_p^m(q) = \{p, H_p^{m-1}(q)\}$ for $m \geq 2$.

Proof. First note that if the range of $Q \in \Psi_{\mathrm{cl}}^{k'}(X)$ is microlocally contained in the range of $P \in \Psi_{\mathrm{cl}}^k(X)$ at K and both operators are properly supported, then it follows that the range of $Q \circ P$ is microlocally contained in the range of P at K. (The converse is not true in general.) Indeed, let N be the integer given by Definition 2.1, and let $f \in H_{(N+k)}^{\mathrm{loc}}(X)$. Since $P : H_{(N+k)}^{\mathrm{loc}}(X) \to H_{(N)}^{\mathrm{loc}}(X)$ is continuous, we have $g = Pf \in H_{(N)}^{\mathrm{loc}}(X)$. Thus, there exists a $u \in \mathcal{D}'(X)$ such that

$$\varnothing = K \cap WF(Qg - Pu) = K \cap WF(QPf - Pu),$$

so the conditions of Definition 2.1 are satisfied with N replaced with $N + k$.

Let $(x, \xi) \in \gamma(I)$. The range of PQ is easily seen to be microlocally contained in the range of P for any properly supported pseudodifferential operator Q. The assumptions of the proposition therefore imply that the range of the commutator

$$R_1 = P \circ Q - Q \circ P \in \Psi_{\mathrm{cl}}^{k+k'-1}(X) \tag{2-27}$$

is microlocally contained in the range of P at K. Hence, by Theorem 2.19 there exists an operator $E \in \Psi_{\mathrm{cl}}^{k'-1}(X)$ such that, in particular, the principal symbol of $R_1 - PE$ vanishes at (x, ξ). If e is the principal symbol of E, homogeneous of degree $k' - 1$, then the principal symbol of PE satisfies $p(x, \xi)e(x, \xi) = 0$ since $p \circ \gamma = 0$. Since the principal symbol of R_1 is

$$\sigma_{k+k'-1}(R_1) = \frac{1}{i}\{p, q\},$$

the result follows for $m = 1$.

Let R_m be defined recursively by $R_m = [P, R_{m-1}]$ for $m \geq 2$ with R_1 given by (2-27). Arguing by induction, we conclude in view of the first paragraph of the proof that the range of R_m is microlocally contained in the range of P at K for $m = 1, 2 \ldots$ since this holds for R_1. Assuming the proposition holds for some $m \geq 1$, we can repeat the arguments above to show that the principal symbol of R_{m+1} must vanish at (x, ξ). Since the principal symbol of R_{m+1} equals $-i\{p, H_p^m(q)\}$, this completes the proof. $\square$

Example 2.25. Let $\Omega \subset \mathbb{R}^n$ be open, $P(x, D)$ be a differential operator of order m with coefficients in $C^\infty(\Omega)$, and let μ be a function in $C^\infty(\Omega)$ such that the equation

$$P(x, D)u = \mu P(x, D)f$$

has a solution $u \in \mathscr{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$. If p is the principal symbol of P then it follows that

$$\sum_{j=1}^{n} \partial_{\xi_j} p(x, \xi) D_{x_j} \mu(x) = 0 \qquad (2\text{-}28)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ such that

$$\{p, \bar{p}\}(x, \xi) \neq 0 \quad \text{and} \quad p(x, \xi) = 0. \qquad (2\text{-}29)$$

Indeed, if (x, ξ) satisfies (2-29) then we may assume that

$$\{\operatorname{Re} p, \operatorname{Im} p\}(x, \xi) = -\frac{1}{2i}\{p, \bar{p}\}(x, \xi) > 0$$

since otherwise we just regard $(x, -\xi)$ instead, as per the remarks preceding Corollary 2.22. By the same discussion it is also clear that (x, ξ) is a minimal characteristic point of p. Now the conditions above imply that there exists a compactly based cone $K \subset T^*(\Omega) \setminus 0$ containing (x, ξ) such that the range of μP is microlocally contained in the range of P at K. By condition (2-29), P is of principal type near (x, ξ), so Proposition 2.24 implies that $\{p, \mu\} = 0$ at (x, ξ), that is,

$$\sum_{j=1}^{n} \partial_{\xi_j} p(x, \xi) \partial_{x_j} \mu(x) - \partial_{x_j} p(x, \xi) \partial_{\xi_j} \mu(x) = 0.$$

Since μ is independent of ξ we find that (2-28) holds at (x, ξ). By homogeneity it then also holds at $(x, -\xi)$.

3. Proof of Theorem 2.20

Throughout this section we assume that the hypotheses of Theorem 2.20 hold. We shall prove the theorem by using Lemma 2.3 on approximate solutions of the equation $P^*v = 0$ concentrated near $\Gamma' = \{(0, \varepsilon_n)\}$. We take as starting point the construction on [Hörmander 1985b, page 103], but some modifications need to be made in particular to the amplitude function ϕ, so the results there concerning the estimates for the right side of (2-1) cannot be used immediately. To obtain the desired estimates we will instead have to use [Hörmander 1985b, Lemma 26.4.15]. Set

$$v_\tau(x) = \phi(x) e^{i\tau w(x)}, \qquad (3\text{-}1)$$

where

$$w(x) = x_n + i(x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + (x_n + ix_1^2/2)^2)/2$$

satisfies $P^*w = 0$ and $\phi \in C_0^\infty(\mathbb{R}^n)$. By the Cauchy–Kovalevsky theorem we can solve $D_1\phi - ix_1 D_n\phi = 0$ in a neighborhood of 0 for any analytic initial data $\phi(0, x') = f(x') \in C^\omega(\mathbb{R}^{n-1})$; in particular we are free to specify the Taylor coefficients of $f(x')$ at $x' = 0$. We take ϕ to be such a solution. If need be we can reduce the support of ϕ by multiplying by a smooth cutoff function χ, where χ is equal to 1 in

some smaller neighborhood of 0 so that $\chi\phi$ solves the equation there. We assume this to be done and note that if $\operatorname{supp}\phi$ is small enough then

$$\operatorname{Im} w(x) \geq |x|^2/4 \quad \text{for } x \in \operatorname{supp}\phi. \tag{3-2}$$

Since

$$d \operatorname{Re} w(x) = -x_1 x_n dx_1 + (1 - x_1^2/2)dx_n,$$

we may similarly assume that $d \operatorname{Re} w(x) \neq 0$ in the support of ϕ. We then have the following result.

Lemma 3.1. *Suppose $P = D_1 + i x_1 D_n$ and let v_τ be defined by* (3-1). *Then ϕ and w can be chosen so that for any $f \in C^\omega(\mathbb{R}^{n-1})$ and any positive integers k and m we have $\phi(0, x') = f(x')$ in a neighborhood of $(0, 0)$, $\tau^k \|P^* v_\tau\|_{(m)} \to 0$ as $\tau \to \infty$, and*

$$\|v_\tau\|_{(-m)} \leq C_m \tau^{-m}. \tag{3-3}$$

If $\tilde{\Gamma}$ is the cone generated by

$$\{(x, w'(x)) : x \in \operatorname{supp}\phi, \ \operatorname{Im} w(x) = 0\},$$

then $\tau^k v_\tau \to 0$ in $\mathcal{D}'_{\tilde{\Gamma}}$ as $\tau \to \infty$; hence $\tau^k A v_\tau \to 0$ in $C^\infty(\mathbb{R}^n)$ if A is a pseudodifferential operator with $WF(A) \cap \tilde{\Gamma} = \varnothing$.

Here $\mathcal{D}'_{\tilde{\Gamma}}(X) = \{u \in \mathcal{D}'(X) : WF(u) \subset \tilde{\Gamma}\}$, equipped with the topology given by all the seminorms on $\mathcal{D}'(X)$ for the weak topology, together with all seminorms of the form

$$P_{\phi, V, N}(u) = \sup_{\xi \in V} |\widehat{\phi u}(\xi)|(1 + |\xi|)^N$$

where $N \geq 0$, $\phi \in C_0^\infty(X)$, and $V \subset \mathbb{R}^n$ is a closed cone with $(\operatorname{supp}\phi \times V) \cap \tilde{\Gamma} = \varnothing$. Note that $u_j \to u$ in $\mathcal{D}'_{\tilde{\Gamma}}(X)$ is equivalent to $u_j \to u$ in $\mathcal{D}'(X)$ and $A u_j \to A u$ in C^∞ for every properly supported pseudodifferential operator A with $\tilde{\Gamma} \cap WF(A) = \varnothing$; see the remark following [Hörmander 1985a, Theorem 18.1.28].

Proof. We observe that $\tau^k P^* v_\tau = \tau^k (P^*\phi)e^{i\tau w} \to 0$ in $C_0^\infty(\mathbb{R}^n)$ for any k as $\tau \to \infty$, if w and ϕ are chosen in the way given above. Hence $\tau^k \|P^* v_\tau\|_{(m)} \to 0$ for any positive integers k and m. In view of (3-2) and the fact that $d \operatorname{Re} w \neq 0$ in the support of ϕ we can apply [Hörmander 1985b, Lemma 26.4.15] to v_τ. This immediately yields (3-3) and also that $\tau^k v_\tau \to 0$ in $\mathcal{D}'_{\tilde{\Gamma}}$ as $\tau \to \infty$, proving the lemma. $\square$

We are now ready to proceed with a tool that will be instrumental in proving Theorem 2.21. The idea is based on techniques found in [Hörmander 1963].

Let R be the operator given by Theorem 2.20. By assumption there exists a compactly based cone $K \subset T^*(\mathbb{R}^n) \setminus 0$ such that the range of R is microlocally contained in the range of P at K. If N is the integer given by Definition 2.1, let $H(x) \in C_0^\infty(\mathbb{R}^n)$ and set

$$h_\tau(x) = \tau^{-N} H(\tau x). \tag{3-4}$$

Since $\hat{h}_\tau(\xi) = \tau^{-N-n}\hat{H}(\xi/\tau)$ it is clear that for $\tau \geq 1$ we have $h_\tau \in H_{(N)}(\mathbb{R}^n)$ and $\|h_\tau\|_{(N)} \leq C\tau^{-n/2}$. In particular, $\|h_\tau\|_{(N)} \leq C$ for $\tau \geq 1$, where the constant depends on H but not on τ. Now denote by I_τ the integral

$$I_\tau = \tau^n \int H(\tau x) R^* v_\tau(x)\, dx = \tau^{N+n}(R^* v_\tau, \overline{h_\tau}), \tag{3-5}$$

where R^* is the adjoint of R. For any κ we then have by the second equality and Lemma 2.3 that

$$|I_\tau| \leq \tau^{N+n}\|h_\tau\|_{(N)}\|R^* v_\tau\|_{(-N)} \leq C_\kappa \tau^{N+n}(\|P^* v_\tau\|_{(\nu)} + \|v_\tau\|_{(-N-\kappa-n)} + \|Av_\tau\|_{(0)})$$

for some positive integer ν and properly supported pseudodifferential operator A with $WF(A) \cap K = \varnothing$. By Lemma 3.1 this implies

$$|I_\tau| \leq C_\kappa \tau^{-\kappa} \tag{3-6}$$

for any positive integer κ if τ is sufficiently large.

Recall that $R(x, D_{x'})$ is a pseudodifferential operator in x' depending on x_1 as a parameter. Its symbol is given by the asymptotic sum

$$\sigma_R(x, \xi') = r_1(x, \xi') + r_0(x, \xi') + \cdots,$$

where $r_{-j}(x, \xi')$ is homogeneous of degree $-j$ in ξ'. The symbol of R^* has the asymptotic expansion

$$\sigma_{R^*} = \sum \partial_\xi^\alpha D_x^\alpha \overline{\sigma_R(x, \xi')}/\alpha!,$$

which shows that R^* is also a pseudodifferential operator in x' depending on x_1 as a parameter. If we sort the terms above with respect to homogeneity we can write

$$\sigma_{R^*} = q_1(x, \xi') + q_0(x, \xi') + \cdots, \tag{3-7}$$

where q_{-j} is homogeneous of order $-j$, $q_1(x, \xi') = \overline{r_1(x, \xi')}$ and

$$q_0(x, \xi') = \overline{r_0(x, \xi')} + \sum_{k=2}^n \partial_{\xi_k} D_{x_k} \overline{r_1(x, \xi')}.$$

A moment's reflection shows that if all the terms in (3-7) have vanishing Taylor coefficients at some point (x, ξ'), then the same must hold for σ_R.

Our goal is to show that if $q_{-j\,(\alpha)}^{(\beta)}(0, \xi^0)$ does not vanish for all $j \geq -1$ and all $\alpha, \beta \in \mathbb{N}^n$, then (3-6) cannot hold. For this purpose, we introduce a total well-ordering $>_t$ on the Taylor coefficients by means of an ordering of the indices (j, α, β) as follows.

Definition 3.2. Let $\alpha_i, \beta_i \in \mathbb{N}^n$ and $j_i \geq -1$ for $i = 1, 2$. We say that

$$q_{-j_1\,(\alpha_1)}^{(\beta_1)}(0, \xi^0) >_t q_{-j_2\,(\alpha_2)}^{(\beta_2)}(0, \xi^0) \quad \text{if } j_1 + |\alpha_1| + |\beta_1| > j_2 + |\alpha_2| + |\beta_2|.$$

To "break ties", we say that if $j_1 + |\alpha_1| + |\beta_1| = j_2 + |\alpha_2| + |\beta_2|$, then

$$q_{-j_1\,(\alpha_1)}^{(\beta_1)}(0, \xi^0) >_t q_{-j_2\,(\alpha_2)}^{(\beta_2)}(0, \xi^0) \quad \text{if } |\beta_2| > |\beta_1|.$$

Note the reversed order. If also $|\beta_1| = |\beta_2|$, then we use a monomial ordering on the β index to break ties. Recall that this is any relation $>$ on $\mathbb{N}^n$ such that $>$ is a total well-ordering on $\mathbb{N}^n$ and $\beta_1 > \beta_2$ and $\gamma \in \mathbb{N}^n$ implies $\beta_1 + \gamma > \beta_2 + \gamma$. Having come this far, the actual order turns out not to matter for the proof of Theorem 2.20, but it will have bearing on the proof of Theorem 2.21. Which monomial ordering we use on the β index will not be important, but for completeness let us choose lexicographic order since this will be used at a later stage in the definition. Here we by lexicographic order refer to the usual one, corresponding to the variables being ordered $x_1 > \cdots > x_n$. That is to say, if $\alpha_i \in \mathbb{N}^n$ for $i = 1, 2$, then $\alpha_1 >_{\text{lex}} \alpha_2$ if, in the vector difference $\alpha_1 - \alpha_2 \in \mathbb{Z}^n$, the leftmost nonzero entry is positive. Thus, if $j_1 + |\alpha_1| + |\beta_1| = j_2 + |\alpha_2| + |\beta_2|$ and $\beta_1 = \beta_2$, then we first say that

$$q_{-j_1\,(\alpha_1)}^{(\beta_1)}(0, \xi^0) >_t q_{-j_2\,(\alpha_2)}^{(\beta_2)}(0, \xi^0) \quad \text{if } |\alpha_2| > |\alpha_1| \tag{3-8}$$

and then use lexicographic order on the n-tuples α to break ties at this stage. Using the lexicographic order on both multiindices (separately) we get

$$q_1 <_t q_1^{(\varepsilon_n)} <_t \cdots <_t q_1^{(\varepsilon_1)} <_t q_{1(\varepsilon_n)} <_t \cdots <_t q_{1(\varepsilon_1)} <_t q_0 <_t \cdots.$$

As indicated above we will prove Theorem 2.20 by a contradiction argument, so in the sequel we let κ denote an integer such that

$$j + |\alpha| + |\beta| < \kappa \tag{3-9}$$

if $q_{-j\,(\alpha)}^{(\beta)}(0, \xi^0)$ is the first nonvanishing Taylor coefficient with respect to the ordering $>_t$. Since $j \geq -1$ we will thus have $\kappa \geq 0$.

To simplify notation, we shall in what follows write t instead of x_1 and x instead of x'. Then v_τ takes the form

$$v_\tau(t, x) = \phi(t, x)e^{i\tau w(t,x)},$$

where

$$w(t, x) = x_{n-1} + i(t^2 + x_1^2 + \cdots + x_{n-2}^2 + (x_{n-1} + it^2/2)^2)/2. \tag{3-10}$$

We shall as before use the notation $\xi^0 = (0, \ldots, 0, 1) \in \mathbb{R}^{n-1}$ when in this context. To interpret the integral I_τ we will need a formula for how $R^*(t, x, D)$ acts on the functions v_τ. This is given by the following lemma, where the parameter t has been suppressed to simplify notation.

Lemma 3.3 [Hörmander 1985b, Lemma 26.4.16]. *Let* $q(x, \xi) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, *let* $\phi \in C_0^\infty(\mathbb{R}^{n-1})$ *and* $w \in C^\infty(\mathbb{R}^{n-1})$, *and assume that* $\text{Im}\, w > 0$ *except at a point* y *where* $w'(y) = \eta \in \mathbb{R}^{n-1} \setminus 0$ *and* $\text{Im}\, w''$ *is positive definite. Then*

$$\left| q(x, D)(\phi e^{i\tau w}) - \sum_{|\alpha|<k} q^{(\alpha)}(x, \tau\eta)(D - \tau\eta)^\alpha(\phi e^{i\tau w})/\alpha! \right| \leq C_k \tau^{\mu-k/2} \tag{3-11}$$

for $\tau > 1$ *and* $k = 1, 2, \ldots$.

An inspection of the proof of [Hörmander 1985b, Lemma 26.4.16] shows that the result is still applicable if $\text{Im}\, w > 0$ everwhere. This is also used without mention in [Hörmander 1985b] when proving the necessity of condition (Ψ). Thus the statement holds if $\text{Im}\, w > 0$ except *possibly* at a point y where

$w'(y) = \eta \in \mathbb{R}^{n-1} \setminus 0$ and $\operatorname{Im} w''$ is positive definite. We will also use this fact, but we have refrained from altering the statement of the lemma.

If q is homogeneous of degree μ, then the sum in (3-11) consists (apart from the factor $e^{i\tau w}$) of terms that are homogeneous in τ of degree $\mu, \mu - 1, \ldots$. The terms of degree μ are those in

$$\phi \sum q^{(\alpha)}(x, \tau\eta)(\tau w'(x) - \tau\eta)^{\alpha}/\alpha!, \tag{3-12}$$

which is the Taylor expansion at $\tau\eta$ of $q(x, \tau w')$. In this way one can give meaning to the expression $q(x, \tau w')$ even though $q(x, \xi)$ may not be defined for complex ξ. The terms of degree $\mu - 1$ where ϕ is differentiated are similarly

$$\sum_{k=1}^{n-1} q^{(k)}(x, \tau w'(x)) D_k \phi,$$

where $q^{(k)}$ should be replaced by the Taylor expansion at $\tau\eta$ representing the value at $\tau w'(x)$, as in (3-12). In the present case we have

$$w'_x(t, x) - \xi^0 = ix - (t^2/2)\xi^0,$$

so the expression $q_{-j}(t, x, w'_x(t, x))$ is given meaning if it is replaced by a finite Taylor expansion

$$\sum_{\beta} q_{-j}^{(\beta)}(t, x, \xi^0)(w'_x(t, x) - \xi^0)^{\beta}/|\beta|!$$

of sufficiently high order.

Using the classicality of R^* we have

$$\sigma_{R^*}(t, x, \xi) - \sum_{j=-1}^{M} q_{-j}(t, x, \xi) \in \Psi_{\mathrm{cl}}^{-M-1}(\mathbb{R}^n),$$

so there is a symbol $a \in S_{\mathrm{cl}}^{-M-1}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ such that

$$a(t, x, D) = R^*(t, x, D) - \sum_{j=-1}^{M} q_{-j}(t, x, D) \mod \Psi^{-\infty}(\mathbb{R}^n).$$

By (3-2) and (3-10) it is clear that w satisfies the conditions of Lemma 3.3, so

$$a(t, x, D)v_{\tau} = a(t, x, \tau\xi^0)v_{\tau} + \mathbb{O}(\tau^{-M-3/2}) = \tau^{-M-1}a(t, x, \xi^0)v_{\tau} + \mathbb{O}(\tau^{-M-3/2}),$$

which implies that $|a(t, x, D)v_{\tau}| \le C\tau^{-M-1}$. If we for each $-1 \le j \le M$ write

$$\left| q_{-j}(t, x, D)v_{\tau} - \sum_{|\alpha| < k_j} q_{-j}^{(\alpha)}(t, x, \tau\xi^0)(D_x - \tau\xi^0)^{\alpha}v_{\tau}/\alpha! \right| \le C_{k_j}\tau^{-j-k_j/2}$$

with $k_j = 2M - 2j + 1$, then

$$R^*(t, x, D)v_{\tau} = \sum_{j=-1}^{M} \sum_{|\alpha| < k_j} q_{-j}^{(\alpha)}(t, x, \tau\xi^0)(D_x - \tau\xi^0)^{\alpha}v_{\tau}/\alpha! + \mathbb{O}(\tau^{-M-1/2}).$$

Now recall the discussion above regarding the homogeneity of the terms in (3-11), and choose $M \geq \kappa$, where κ is an integer satisfying (3-9). Then

$$R^*(t, x, D)v_\tau = e^{i\tau w} \sum_{j=-1}^{M} \sum_{|\alpha| \leq 2M - 2j} q_{-j}^{(\alpha)}(t, x, \tau w_x'(t, x)) D^\alpha \phi$$

$$= e^{i\tau w} \sum_{j=-1}^{M} \sum_{|\alpha| \leq 2M - 2j} \tau^{-j-|\alpha|} q_{-j}^{(\alpha)}(t, x, w_x'(t, x)) D^\alpha \phi$$

$$= e^{i\tau w} \sum_{J=-1}^{M} \tau^{-J} \lambda_J(t, x)$$

with an error of order $\mathbb{O}(\tau^{-\kappa - 1/2})$, where

$$\lambda_J(t, x) = \sum_{j + |\alpha| = J} q_{-j}^{(\alpha)}(t, x, w_x'(t, x)) D^\alpha \phi \quad \text{for } j \geq -1. \tag{3-13}$$

As before, $q_{-j}^{(\alpha)}(t, x, w_x'(t, x))$ should be replaced by a finite Taylor expansion at ξ^0 of sufficiently high order representing the value at $w_x'(t, x)$. In view of (3-5), this yields

$$I_\tau = \tau^n \int H(\tau t, \tau x) e^{i\tau w(t, x)} \left(\sum_{J=-1}^{\kappa} \tau^{-J} \lambda_J(t, x) + \mathbb{O}(\tau^{-\kappa - 1/2}) \right) dt\, dx.$$

After the change of variables $(\tau t, \tau x) \mapsto (t, x)$ we find that

$$I_\tau = \int H(t, x) e^{i\tau w(t/\tau, x/\tau)} \left(\sum_{J=-1}^{\kappa} \tau^{-J} \lambda_J(t/\tau, x/\tau) + \mathbb{O}(\tau^{-\kappa - 1/2}) \right) dt\, dx. \tag{3-14}$$

To illustrate how we will proceed to prove Theorem 2.20 by contradiction, let us for the moment assume that $q_1(0, 0, \xi^0) \neq 0$, where $\xi^0 = (0, \ldots, 0, 1) \in \mathbb{R}^{n-1}$. Since

$$\lambda_{-1}(t/\tau, x/\tau) = \phi(t/\tau, x/\tau) \sum_\beta q_1^{(\beta)}(t/\tau, x/\tau, \xi^0)(w_x'(t/\tau, x/\tau) - \xi^0)^\beta / |\beta|! \tag{3-15}$$

where

$$w_x'(t/\tau, x/\tau) - \xi^0 = ix/\tau - (t^2/(2\tau^2))\xi^0 = \mathbb{O}(\tau^{-1}), \tag{3-16}$$

and (3-10) implies that $\tau w(t/\tau, x/\tau) \to x_{n-1}$ as $\tau \to \infty$, we obtain

$$\lim_{\tau \to \infty} I_\tau / \tau = \int H(t, x) e^{ix_{n-1}} \phi(0, 0) q_1(0, 0, \xi^0) dt\, dx.$$

Since we may choose $\phi \neq 0$ at the origin, the limit above will then not be equal to 0 for a suitable choice of H. However, this contradicts (3-6).

Now assume that $\partial_t^{k_0} q_{-j_0(\alpha_0)}^{(\beta_0)}(0, 0, \xi^0)$ is the first nonvanishing Taylor coefficient with respect to the ordering $>_t$, and let

$$m = j_0 + k_0 + |\alpha_0| + |\beta_0| \tag{3-17}$$

so that $m < \kappa$ by (3-9). Note that $\alpha_0, \beta_0 \in \mathbb{N}^{n-1}$ and that the integer k_0 accounts for derivatives in t while there is no corresponding term for derivatives in the Fourier transform of t since the q_{-j} are independent of this variable. Note also that since j_0 is permitted to be -1, we have $0 \le k_0, |\alpha_0|, |\beta_0| \le m + 1$.

To use our assumption we will need for each term $q_{-j}^{(\beta+\gamma)}(t/\tau, x/\tau, \xi^0)$ in the Taylor expansion of $q_{-j}^{(\gamma)}(t/\tau, x/\tau, w_x'(t/\tau, x/\tau))$ (as it appears in (3-13)) at ξ^0 to consider Taylor expansions in t and x at the origin. Note that for given j and γ, it suffices to consider finite Taylor expansions of $q_{-j}^{(\gamma)}$ of order $\kappa - j - |\gamma|$ by (3-14) and (3-16). For each j and γ we thus write

$$q_{-j}^{(\gamma)}(t/\tau, x/\tau, w_x'(t/\tau, x/\tau))$$
$$= \sum_{k+|\alpha|+|\beta|\le \kappa-j-|\gamma|} (\partial_t^k q_{-j\,(\alpha)}^{(\beta+\gamma)})(0,0,\xi^0)\tau^{-k-|\alpha|}t^k x^\alpha (w_x'(t/\tau, x/\tau) - \xi^0)^\beta/(k!|\alpha|!|\beta|!) + \mathbb{O}(\tau^{-\kappa-1+j+|\gamma|}),$$

where $(w_x'(t/\tau, x/\tau) - \xi^0)^\beta$ should be interpreted by means of (3-16). As we shall see, the term $(t^2/(2\tau^2))\xi^0$ will not pose any problem, since it is $\mathbb{O}(\tau^{-2})$. We have

$$\lambda_J(t/\tau, x/\tau) = \sum_{j+|\gamma|=J}\ \sum_{k+|\alpha|+|\beta|\le\kappa-J} (\partial_t^k q_{-j\,(\alpha)}^{(\beta+\gamma)})(0,0,\xi^0)D^\gamma\phi(t/\tau, x/\tau)$$
$$\times \tau^{-k-|\alpha|}t^k x^\alpha (w_x'(t/\tau, x/\tau) - \xi^0)^\beta/(k!|\alpha|!|\beta|!) + \mathbb{O}(\tau^{-\kappa-1+J}),$$

where $-1 \le j \le J$. If we are only interested in terms of order τ^{-m} in (3-14), we can use the assumption that $\partial_t^k q_{-j\,(\alpha)}^{(\beta+\gamma)}(0,0,\xi^0) = 0$ for all $-1 \le j + k + |\alpha| + |\beta| + |\gamma| < m$ to let the term $(t^2/(2\tau^2))\xi^0$ from (3-16) be absorbed by the error term in the expression above. This yields

$$\sum_{J=-1}^m \tau^{-J}\lambda_J(t/\tau, x/\tau) = \sum_{j+k+|\alpha|+|\beta|+|\gamma|=m} (\partial_t^k q_{-j\,(\alpha)}^{(\beta+\gamma)})(0,0,\xi^0)$$
$$\times D^\gamma\phi(t/\tau, x/\tau)\tau^{-m}t^k x^\alpha (ix)^\beta/(k!|\alpha|!|\beta|!) + \mathbb{O}(\tau^{-m-1}),$$

where we use $J = j + |\gamma|$ together with the fact that we get a factor $\tau^{-|\beta|}$ from $(w_x'(t/\tau, x/\tau) - \xi^0)^\beta$ by (3-16). Thus,

$$\lim_{\tau\to\infty}\tau^m I_\tau = \int H(t,x)e^{ix_{n-1}}$$
$$\times\left(\sum_{j+k+|\alpha|+|\beta|+|\gamma|=m} t^k x^\alpha (ix)^\beta(\partial_t^k q_{-j\,(\alpha)}^{(\beta+\gamma)})(0,0,\xi^0)D^\gamma\phi(0,0)/(k!|\alpha|!|\beta|!)\right)dt\,dx.$$

Now choose ϕ such that $D^{\beta_0}\phi(0,0) = 1$, but $D^\gamma\phi(0,0) = 0$ for all other γ such that $|\gamma| \le |\beta_0|$. This is possible by the discussion following (3-1). By (3-17) and our choice of the ordering $>_t$, we have $\partial_t^k q_{-j\,(\alpha)}^{(\beta+\beta_0)}(0,0,\xi^0) = 0$ for all β such that $|\beta| > 0$ as long as $j + k + |\alpha| + |\beta| + |\beta_0| = m$. Hence, with this choice of ϕ, the last expression takes the form

$$\lim_{\tau\to\infty}\tau^m I_\tau = \int H(t,x)e^{ix_{n-1}}\left(\sum_{j+k+|\alpha|+|\beta_0|=m} t^k x^\alpha(\partial_t^k q_{-j\,(\alpha)}^{(\beta_0)})(0,0,\xi^0)/(k!|\alpha|!)\right)dt\,dx, \tag{3-18}$$

where as usual j is allowed to be -1 so that $j \in [-1, m - |\beta_0|]$ in (3-18). Now some of the Taylor coefficients in (3-18) may be zero, in particular, the expression may well contain Taylor coefficients that preceed $\partial_t^{k_0} q_{-j_0\,(\alpha_0)}^{(\beta_0)}(0,0,\xi^0)$, and those are by assumption zero. However, we claim that if at least one of

the Taylor coefficients above are nonzero, then we may choose H so that the limit is nonzero. Indeed, if that were not the case then the expression within brackets in (3-18) would be a polynomial with infinitely many zeros, and thus it would have to have vanishing coefficients. Since this violates our assumption, we conclude that the limit is nonzero. However, this contradicts (3-6), which proves Theorem 2.20.

4. Proof of Theorem 2.21

In this section we shall give the proof of Theorem 2.21, using ideas taken from [Hörmander 1963] together with the approach used to prove [Hörmander 1985b, Theorem 26.4.7′]. As in the previous section, we aim to use Lemma 2.3 to estimate the operator $R(x, D_{x'})$ on approximate solutions of the equation $P^*v = 0$, concentrated near

$$\Gamma' = \{(x_1, x', 0, \xi') : x_1 \in I'\} \subset T^*(\mathbb{R}^n) \smallsetminus 0. \tag{4-1}$$

The proofs will be similar, but the situation is more complicated now, which will affect the construction of the approximate solutions. We will also have to make some adjustments to the proof of [Hörmander 1985b, Theorem 26.4.7′] to make it work, so a lot of the details will have to be revisited. Our approximate solutions will also differ slightly from the ones used to prove [Hörmander 1985b, Theorem 26.4.7′], so although we will refer directly to results in [ibid.] whenever possible, the formulation of some of these results will be affected. For a more complete description of the approximate solutions, we refer the reader to [Hörmander 1981] or [Hörmander 1985b], where their construction is carried out in greater detail. When proving Theorem 2.21 we may without loss of generality assume that $x' = 0$ and $\xi' = \xi^0$ in (4-1). In accordance with the notation in the proof of Theorem 2.19, we shall therefore throughout this section refer to Γ' simply by Γ, and we will let $I' = [a_0, b_0]$.

To simplify notation we shall in what follows write t instead of x_1 and x instead of x'. If N is the integer given by Definition 2.1, and n is the dimension, the approximate solutions v_τ will be taken of the form

$$v_\tau(t, x) = \tau^{N+n} e^{i\tau w(t,x)} \sum_0^M \phi_j(t, x) \tau^{-j}. \tag{4-2}$$

Here $\phi_0, \phi_1, \ldots$ are amplitude functions, and w is a phase function that should satisfy the eikonal equation

$$\partial w/\partial t - if(t, x, \partial w/\partial x) = 0 \tag{4-3}$$

approximately, where f is the imaginary part of the principal symbol of P. We take w of the form

$$w(t, x) = w_0(t) + \langle x - y(t), \eta(t) \rangle + \sum_{2 \leq |\alpha| \leq M} w_\alpha(t)(x - y(t))^\alpha / |\alpha|!, \tag{4-4}$$

where M is a large integer to be determined later and $x = y(t)$ is a smooth real curve. When discussing the functions w_α we shall permit ourselves to use the notation $\alpha = (\alpha_1, \ldots, \alpha_s)$ for a sequence of $s = |\alpha|$ indices between 1 and the dimension $n - 1$ of the x variable, and w_α will be symmetric in these indices. If we take $\eta(t)$ to be real-valued and make sure the matrix $(\mathrm{Im}\, w_{jk})$ is positive definite, then $\mathrm{Im}\, w$ will have a strict minimum when $x = y(t)$ as a function of the x variables.

On the curve $x = y(t)$ the eikonal equation (4-3) is reduced to

$$w'_0(t) = \langle y'(t), \eta(t) \rangle + i f(t, y(t), \eta(t)), \tag{4-5}$$

which is the only equation where w_0 occurs. Hence it can be used to determine w_0 after y and η have been chosen. In particular

$$d \operatorname{Im} w_0(t)/dt = f(t, y(t), \eta(t)). \tag{4-6}$$

In the proof of Theorem 2.20 we could solve the corresponding eikonal equation explicitly. Here this is not possible, so our goal will instead be to make (4-3) valid apart from an error of order $M + 1$ in $x - y(t)$. Note that $f(t, x, \xi)$ is not defined for complex ξ, but since

$$\partial w(t, x)/\partial x_j - \eta_j(t) = \sum w_{\alpha, j}(t)(x - y(t))^\alpha/|\alpha|!,$$

(4-3) is given meaning if $f(t, x, \partial w/\partial x)$ is replaced by the finite Taylor expansion

$$\sum_{|\beta| \leq M} f^{(\beta)}(t, x, \eta(t))(\partial w(t, x)/\partial x - \eta(t))^\beta/|\beta|!. \tag{4-7}$$

To compute the coefficient of $(x - y(t))^\alpha$ in (4-7) we just have to consider the terms with $|\beta| \leq |\alpha|$. Since

$$\partial w/\partial t = w'_0 - \langle y', \eta \rangle + \langle x - y, \eta' \rangle$$
$$+ \sum_{2 \leq |\alpha| \leq M} w'_\alpha(t)(x - y)^\alpha/|\alpha|! - \sum_k \sum_{1 \leq |\alpha| \leq M-1} w_{\alpha, k}(t)(x - y)^\alpha dy_k/dt/|\alpha|!,$$

the first order terms in the equation (4-3) give

$$d\eta_j/dt - \sum_k w_{jk}(t)dy_k/dt = i \left(f_{(j)}(t, y, \eta) + \sum_k f^{(k)}(t, y, \eta) w_{jk}(t) \right). \tag{4-8}$$

Note that this is a system of $2n$ equations

$$d\eta_j/dt - \sum_k \operatorname{Re} w_{jk}(t)dy_k/dt = - \sum_k \operatorname{Im} w_{jk}(t) f^{(k)}(t, y, \eta), \tag{4-8$'$}$$

$$\sum_k \operatorname{Im} w_{jk}(t)dy_k/dt = -f_{(j)}(t, y, \eta) - \sum_k \operatorname{Re} w_{jk}(t) f^{(k)}(t, y, \eta), \tag{4-8$''$}$$

since y and η are real, and under the assumption that $\operatorname{Im} w_{jk}$ is positive definite these equations can be solved for dy/dt and $d\eta/dt$. We observe that at a point where $f = df = 0$ they just mean that $dy/dt = d\eta/dt = 0$.

When $2 \leq |\alpha| \leq M$ we obtain a differential equation

$$dw_\alpha/dt - \sum_k w_{\alpha, k} dy_k/dt = F_\alpha(t, y, \eta, \{w_\beta\}) \tag{4-9}$$

from (4-3). Here F_α is a linear combination of the derivatives of f of order $|\alpha|$ or less, multiplied with polynomials in w_β with $2 \leq |\beta| \leq |\alpha| + 1$. Of course, when $|\alpha| = M$ the sum on the left side of (4-9) should be dropped, and β should satisfy $|\beta| \leq |\alpha|$ instead. Altogether (4-8)$'$, (4-8)$''$ and (4-9) form a quasilinear system of differential equations with as many equations as unknowns. Hence we have local

solutions with prescribed initial data. According to [Hörmander 1985b, pages 105–106] we can find a $c > 0$ such that the equations (4-8) and (4-9) with initial data

$$w_{jk} = i\delta_{jk}, \qquad w_\alpha = 0, \qquad \text{when } 2 < |\alpha| \leq M \text{ and } t = (a_0 + b_0)/2, \tag{4-10}$$

$$y = x, \qquad \eta = \xi, \qquad \text{when } t = (a_0 + b_0)/2 \tag{4-11}$$

have a unique solution in $(a_0 - c, b_0 + c)$ for all x and ξ with $|x| + |\xi - \xi^0| < c$. (Here δ_{jk} is the Kronecker delta.) Moreover,

(i) $(\text{Im } w_{jk} - \delta_{jk}/2)$ is positive definite,

(ii) the map

$$(x, \xi, t) \mapsto (y, \eta, t), \quad \text{where } |x| + |\xi - \xi^0| < c, \ a_0 - c < t < b_0 + c,$$

is a diffeomorphism.

   In the range X_c of the map (ii) we let v denote the image of the vector field $\partial/\partial t$ under the map. Thus v is the tangent vector field of the integral curves, and when $f = df = 0$ we have $v = \partial/\partial t$. By assumption $f = 0$ implies $\partial f/\partial t \leq 0$ in a neighborhood of Γ (see (2-18)), so if c is small enough this also holds in X_c. An application of [Hörmander 1985b, Lemma 26.4.11] now yields that f must have a change of sign from $-$ to $+$ along an integral curve of v in X_c, for otherwise there would be no such sign change for increasing t and fixed (x, ξ), and that contradicts the hypothesis in Theorem 2.21. By (4-6) this means that $\text{Im } w_0(t)$ will start decreasing and end increasing, so the minimum is attained at an interior point. We can normalize the minimum value to zero and have then for a suitable interval of t that $\text{Im } w_0 > 0$ at the end points and $\text{Im } w_0 = 0$ at some interior point. Since $\text{Re } w_0$ is given by (4-5) we can at this interior point also normalize the value of $\text{Re } w_0$ to zero. This completes the proof of [Hörmander 1985b, Lemma 26.4.14]. However, in order to prove Theorem 2.21 when $a_0 < b_0$ we shall need the following stronger result.

Lemma 4.1. *Assume that the hypotheses of Theorem 2.21 are fulfilled, the variables being denoted (t, x) now. Then given $M \in \mathbb{N}$ we can find*

(i) *a curve $t \mapsto (t, y(t), 0, \eta(t)) \in \mathbb{R}^{2n}$, with $a' \leq t \leq b'$ as close to Γ as desired,*

(ii) *C^∞ functions $w_\alpha(t)$ for $2 \leq |\alpha| \leq M$, with $(\text{Im } w_{jk} - \delta_{jk}/2)$ positive definite when $a' \leq t \leq b'$,*

(iii) *a function $w_0(t)$ with $\text{Im } w_0(t) \geq 0$ for $a' \leq t \leq b'$, $\text{Im } w_0(a') > 0$, $\text{Im } w_0(b') > 0$ and $\text{Re } w_0(c') = \text{Im } w_0(c') = 0$ for some $c' \in (a', b')$*

such that (4-4) is a formal solution to (4-3) with an error of order $\mathbb{O}(|x - y(t)|^{M+1})$. If $a_0 < b_0$ then (iii) can be improved in the sense that if $\varrho \geq 0$ is the number given by Theorem 2.21, then we can for any $\varepsilon > \varrho$ find

(iii)' *a function $w_0(t)$ with $\text{Im } w_0(t) \geq 0$, $a' \leq t \leq b'$, $\text{Im } w_0(a') > 0$, $\text{Im } w_0(b') > 0$ and $\text{Re } w_0(t) = \text{Im } w_0(t) = 0$ for all $t \in [a_0 + \varepsilon, b_0 - \varepsilon]$.*

Proof. In view of [Hörmander 1985b, Lemma 26.4.14] we only need to prove (iii)′.

Let $\varepsilon > \varrho$, and let $I_\varepsilon = [a_0 + \varepsilon, b_0 - \varepsilon]$. By the hypotheses of Theorem 2.21, there is a neighborhood $\mathcal{U}$ of

$$\Gamma_\varepsilon = \{(t, 0, 0, \xi^0) : t \in I_\varepsilon\}$$

where f vanishes identically. Take $\delta > 0$ sufficiently small so that

$$t \in I_\varepsilon, \quad |x| + |\xi - \xi^0| < \delta \quad \text{implies} \quad (t, x, 0, \xi) \in \mathcal{U}.$$

As above we can find $c > 0$ such that the equations (4-8) and (4-9) with initial data (4-10) and (4-11) have a unique solution in $(a_0 - c, b_0 + c)$ for all x and ξ with $|x| + |\xi - \xi^0| < c$. Since the map

$$(x, \xi, t) \mapsto (y, \eta, t), \quad \text{where } |x| + |\xi - \xi^0| < c, \ a_0 - c < t < b_0 + c,$$

is a diffeomorphism, we can choose c small enough so that if (y, η, t) is in the range X_c of this map, then $|y| + |\eta - \xi^0| < \delta$. As we have seen, f must change sign from $-$ to $+$ along an integral curve of v in X_c if c is small enough, where in X_c we denote by v the image of the vector field $\partial/\partial t$ under the map. Let this integral curve be given by

$$\gamma(t) = (t, y(t), 0, \eta(t)) \in \mathbb{R}^{2n} \quad \text{for } a' \le t \le b',$$

for some choice of a' and b' such that $a_0 - c < a'$, $b' < b_0 + c$ and

$$f(a', y(a'), \eta(a')) < 0 < f(b', y(b'), \eta(b')).$$

Recall that at a point where $f = df = 0$ the equations (4-8)′ and (4-8)″ imply that $dy/dt = d\eta/dt = 0$. Since f vanishes identically on γ for $t \in I_\varepsilon$ and the function w_0 is determined by (4-5), this proves the lemma after a suitable normalization. $\qquad\square$

Note that if Γ is a point then by Lemma 4.1 we can obtain a sequence $\{\gamma_j\}$ of curves

$$\gamma_j(t) = (t, y_j(t), 0, \eta_j(t)) \quad \text{for } a'_j \le t \le b'_j$$

approaching Γ, which implies that at $t = c'_j$ we have

$$(c'_j, y_j(c'_j), 0, \eta_j(c'_j)) \to \Gamma \quad \text{as } j \to \infty$$

in $T^*(\mathbb{R}^n) \setminus 0$, where c'_j is the point where $\operatorname{Re} w_{0j} = \operatorname{Im} w_{0j} = 0$. Similarly, if Γ is an interval and $\varrho \ge 0$ is the number given by Theorem 2.21, then for any point ω in the interior of Γ_ϱ we can use Lemma 4.1 to obtain a sequence $\{\gamma_j\}$ of curves approaching Γ and a sequence $\{w_{0j}\}$ of functions such that for each j there exists a point $\omega_j \in \gamma_j$ with $\omega_j = \gamma_j(t_j)$ that can be chosen so that $\operatorname{Re} w_{0j}(t_j) = \operatorname{Im} w_{0j}(t_j) = 0$ and $\omega_j \to \omega$ as $j \to \infty$. This will be crucial in proving Theorem 2.21. Our strategy is to show that all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients at ω_j, or at $(c'_j, y_j(c'_j), 0, \eta_j(c'_j))$ when Γ is a point. Theorem 2.21 will then follow by continuity. In what follows we will suppress the index j to simplify notation.

Let K and Ω be the cones given by Theorem 2.21, and suppose that the function w given by (4-4) is a formal solution to (4-3) with an error of order $\mathcal{O}(|x - y(t)|^{M+1})$ in a neighborhood Y of

$$\{(t, 0) : a_0 \leq t \leq b_0\} \subset \mathbb{R}^n$$

with $K \subset T^*(Y)$, such that Im $w > 0$ in Y except on a compact nonempty subset T of the curve $x = y(t)$, with $(t_0, y(t_0)) \in T$ and $w = 0$ on T. We want to show that all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients at $(t_0, y(t_0), 0, \eta(t_0))$. By part (i) of Lemma 4.1 we can choose w so that

$$\Gamma_0 = \{(t, x, \partial w(t, x)/\partial t, \partial w(t, x)/\partial x) : (t, x) \in T\} \tag{4-12}$$

is contained in Ω. This is done to ensure that if A is a given pseudodifferential operator with wavefront set contained in the complement of K, then $WF(A)$ does not meet the cone generated by Γ_0.

We now turn our attention to the amplitude functions ϕ_j. With the exception of ϕ_0, which will be of great interest to us, we will not be very thorough in describing them. Suffice it to say that these functions can be chosen so that if P^* is the adjoint of P then

$$\|P^* v_\tau\|_{(\nu)} \leq C\tau^{N+n+\nu+(1-M)/2}, \tag{4-13}$$

where M is the number given by (4-2). The procedure begins by setting

$$\phi_0(t, x) = \sum_{|\alpha| < M} \phi_{0\alpha}(t)(x - y(t))^\alpha$$

with $y(t)$ as above, and having $\phi_{0\alpha}$ satisfy the linear system of ordinary differential equations

$$D_t \phi_{0\alpha} + \sum_{|\beta| < M} a_{\alpha\beta} \phi_{0\beta} = 0. \tag{4-14}$$

In the same way we then successively choose ϕ_j and obtain (4-13). The precise details can be found in [Hörmander 1981, pages 87–89], or in [Hörmander 1985b, pages 107–110]. Note that we for any positive integer $J < M$ can solve the equations that determine ϕ_0 so that at the point $(t_0, y(t_0)) \in T$ we have $D_x^\alpha \phi_0(t_0, y(t_0)) = 0$ for all $|\alpha| \leq J$ except for one index α, $|\alpha| = J$. This will be important later on. Note also that the estimate (4-13) is not affected if the functions ϕ_j are multiplied by a cutoff function in $C_0^\infty(Y)$ that is 1 in a neighborhood of T. Since the ϕ_j will be irrelevant outside of Y for large τ by construction, we can in this way choose them to be supported in Y so that $v_\tau \in C_0^\infty(Y)$.

Having completed the construction of the approximate solutions, we are now ready to start to follow the proof of Theorem 2.20. To get the estimates for the right side of (2-1) when v is an approximate solution, we shall need the following two results. The first, corresponding to Lemma 3.1, is taken from [Hörmander 1985b]. Observe that here it is stated for our approximate solutions which differ from those in [ibid.] by a factor of τ^{N+n}, which explains the difference in appearance. Note also that although we will not use the lower bound for the approximate solutions, that estimate is included so as not to alter the statement.

Lemma 4.2 [Hörmander 1985b, Lemma 26.4.15]. *Let $X \subset \mathbb{R}^n$ be open, and let v_τ be defined by (4-2), where $w \in C^\infty(X)$, $\phi_j \in C_0^\infty(X)$, $\operatorname{Im} w \geq 0$ in X and $d \operatorname{Re} w \neq 0$. For any positive integer m we then have*

$$\|v_\tau\|_{(-m)} \leq C\tau^{N+n-m} \quad for \ \tau > 1. \tag{4-15}$$

If $\operatorname{Im} w(t_0, x_0) = 0$ and $\phi_0(t_0, x_0) \neq 0$ for some $(t_0, x_0) \in X$ then

$$\|v_\tau\|_{(-m)} \geq c\tau^{N+n/2-m} \quad for \ \tau > 1$$

and for some $c > 0$. If $\tilde{\Gamma}$ is the cone generated by

$$\left\{ (t, x, \partial_t w(t, x), \partial_x w(t, x)) : (t, x) \in \bigcup_j \operatorname{supp} \phi_j, \ \operatorname{Im} w(t, x) = 0 \right\},$$

then $\tau^k v_\tau \to 0$ in $\mathscr{D}'_{\tilde{\Gamma}}$ as $\tau \to \infty$; hence $\tau^k A v_\tau \to 0$ in $C^\infty(\mathbb{R}^n)$ if A is a pseudodifferential operator with $WF(A) \cap \tilde{\Gamma} = \varnothing$ and k is any real number.

Proposition 4.3. *Assume that the hypotheses of Theorem 2.21 are fulfilled, the variables being denoted (t, x) now, and let v_τ be given by (4-2), where $w \in C^\infty(Y)$, $\phi_j \in C_0^\infty(Y)$, $\operatorname{Im} w \geq 0$ in Y and $d \operatorname{Re} w \neq 0$. Here Y is a neighborhood of $\{(t, 0) : a_0 \leq t \leq b_0\}$ such that $K \subset T^*(Y)$. Let $H(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{n-1})$ and set*

$$h_\tau(t, x) = \tau^{-N} H(\tau(t - t_0), \tau(x - y(t))), \tag{4-16}$$

where N is the positive integer given by Definition 2.1 for the operators R and P in Theorem 2.21. Then $h_\tau \in H_{(N)}(\mathbb{R}^n)$ for all $\tau \geq 1$ and $\|h_\tau\|_{(N)} \leq C$, where the constant depends on H but not on τ. Furthermore, if M is the integer given by the definition of v_τ in (4-2) so that (4-13) holds, and I_τ is the integral

$$I_\tau = (R^* v_\tau, \overline{h_\tau}), \tag{4-17}$$

where R^* is the adjoint of $R(t, x, D)$, then for any positive integer κ there exists a constant C such that $|I_\tau| \leq C\tau^{-\kappa}$ if $M = M(\kappa)$ is sufficiently large.

Proof. In Section 3, one easily obtains a formula for the Fourier transform of the corresponding function h_τ (see (3-4)), which yields the estimates needed to show that $h_\tau \in H_{(N)}$. Here we shall instead use the equality

$$\iint |h_\tau(t, x)|^2 dt \, dx = \tau^{-2N} \iint |H(\tau(t - t_0), \tau(x - y(t)))|^2 dt \, dx$$

which shows that if $\tau \geq 1$ then $D_t^j D_x^\alpha h_\tau \in L^2(\mathbb{R}^n)$ for all $(j, \alpha) \in \mathbb{N} \times \mathbb{N}^{n-1}$ such that $j + |\alpha| \leq N + [n/2]$. Hence, by using the equivalent norm on $H_{(N)}(\mathbb{R}^n)$ given by

$$\|h_\tau\|_{(N)} = \sum_{j+|\alpha| \leq N} \|D_t^j D_x^\alpha h_\tau\|_{(0)},$$

we find that $\{h_\tau\}_{\tau \geq 1}$ is a bounded one parameter family in $H_{(N)}(\mathbb{R}^n)$, which proves the first assertion of the proposition.

To prove the second part, let κ be an arbitrary positive integer, and let ν be the positive integer given by Lemma 2.3 (applied to the operator R instead of Q) so that (2-1) holds for the choice of seminorm $\|P^*v\|_{(\nu)}$ in the right side. If we choose

$$(1 - M)/2 \leq -N - n - \nu - \kappa, \tag{4-18}$$

and recall (4-13), then

$$\|P^*v_\tau\|_{(\nu)} \leq C\tau^{-\kappa}. \tag{4-19}$$

Since supp H is compact, we can find a bounded open ball containing supp h_τ for all $\tau \geq 1$. Hence $h_\tau \in H_{(N)}(\mathbb{R}^n)$ has compact support and $v_\tau \in C_0^\infty(Y)$, so the result now follows by the estimate (2-4) together with Lemma 4.2. $\qquad\square$

To shorten the notation we will from now on assume that $t_0 = 0$, so that $w(0, y(0)) = 0$. As in the proof of Theorem 2.20 it suffices to show that all terms in the asymptotic expansion of the symbol of R^*, given by

$$\sigma_{R^*} = q_1(t, x, \xi) + q_0(t, x, \xi) + \cdots,$$

with q_j homogeneous of degree j in ξ, have vanishing Taylor coefficients at $(0, y(0), \eta(0))$. The method will be to argue by contradiction that if not, then Proposition 4.3 does not hold. Therefore, let us assume that $\partial_t^{k_0} q_{-j_0(\alpha_0)}^{(\beta_0)}(0, y(0), \eta(0))$ is the first nonvanishing Taylor coefficient with respect to the ordering $>_t$ given by Definition 3.2, and let

$$m = j_0 + k_0 + |\alpha_0| + |\beta_0|. \tag{4-20}$$

Now let κ be a positive integer such that $m < \kappa$, and sort the terms in I_τ, given by (4-17), with respect to homogeneity degree in τ. We can use Lemma 3.3 and the classicality of the symbol σ_{R^*} to write

$$R^*(t, x, D)v_\tau = \sum_{j=-1}^{M'} q_{-j}(t, x, D)v_\tau + \mathbb{O}(\tau^{N+n-M'-1})$$

$$= \sum_{j=-1}^{M'} \sum_{l=0}^{M} \tau^{N+n-l} q_{-j}(t, x, D)(e^{i\tau w}\phi_l) + \mathbb{O}(\tau^{N+n-M'-1})$$

for some large number M'. Note that (4-18) implies a lower bound on M, but as we shall see below, we must also make sure to pick $M > 2M' + 1$. For each j we then estimate $q_{-j}(t, x, D)(e^{i\tau w}\phi_l)$ using (3-11) with $k = M - 1 - 2j$, so that

$$q_{-j}(t, x, D)(e^{i\tau w}\phi_l) = \sum_{|\alpha| < M-1-2j} q_{-j}^{(\alpha)}(t, x, \tau\eta)(D - \tau\eta)^\alpha(\phi_l e^{i\tau w})/\alpha!$$

with an error of order $\mathbb{O}(\tau^{(1-M)/2})$. Recalling (4-18) and the discussion following Lemma 3.3 regarding the homogeneity of the terms in (3-11), this yields, for sufficiently large M',

$$R^*(t, x, D)v_\tau = \sum_{j=-1}^{M'} \sum_{l=0}^{M} \tau^{N+n-l} e^{i\tau w} \sum_{|\alpha|<M-1-2j} q_{-j}^{(\alpha)}(t, x, \tau w_x') D^\alpha \phi_l + \mathbb{O}(\tau^{-\kappa-1})$$

$$= \tau^{N+n} e^{i\tau w} \sum_{j=-1}^{M'} \sum_{l=0}^{M} \sum_{|\alpha|<M-1-2j} \tau^{-j-|\alpha|-l} q_{-j}^{(\alpha)}(t, x, w_x') D^\alpha \phi_l + \mathbb{O}(\tau^{-\kappa-1}). \quad (4\text{-}21)$$

Note that $\tau^{-j-|\alpha|-l} q_{-j}^{(\alpha)}(t, x, w_x') D^\alpha \phi_l$ is now homogeneous of order $-j-|\alpha|-l$ in τ, and that as before, $q_{-j}^{(\alpha)}(t, x, w_x')$ should be replaced by a finite Taylor expansion at η of sufficiently high order. For each $-1 \le J \le \kappa$, collect all terms of the form $\tau^{-j-|\alpha|-l} q_{-j}^{(\alpha)}(t, x, w_x') D^\alpha \phi_l$ in (4-21) that are homogeneous of order $-J$ in τ, that is, all terms that satisfy $j+|\alpha|+l = J$ for $j \ge -1$, and $|\alpha|, l \ge 0$. If

$$\lambda_J(t, x) = \sum_{j+|\alpha|+l=J} q_{-j}^{(\alpha)}(t, x, w_x'(t, x)) D^\alpha \phi_l(t, x)$$

for the permitted values of j and l, then

$$I_\tau = \tau^n \iint H(\tau t, \tau(x - y(t))) \left(e^{i\tau w(t,x)} \sum_{J=-1}^{\kappa} \tau^{-J} \lambda_J(t, x) + \mathbb{O}(\tau^{-\kappa-1}) \right) dt \, dx.$$

After the change of variables $(\tau t, \tau(x - y(t))) \mapsto (t, x)$ we obtain

$$I_\tau = \iint H(t, x) \left(e^{i\tau w(t/\tau, x/\tau + y(t/\tau))} \sum_{J=-1}^{\kappa} \tau^{-J} \lambda_J(t/\tau, x/\tau + y(t/\tau)) + \mathbb{O}(\tau^{-\kappa-1}) \right) dt \, dx, \quad (4\text{-}22)$$

where

$$\lambda_J(t/\tau, x/\tau + y(t/\tau))$$
$$= \sum_{j+|\alpha|+l=J} D^\alpha \phi_l(t/\tau, x/\tau + y(t/\tau)) q_{-j}^{(\alpha)}(t/\tau, x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau))). \quad (4\text{-}23)$$

Recall that $w_0(0) = 0$, which together with (4-4) implies

$$i\tau w(t/\tau, x/\tau + y(t/\tau)) = it w_0'(0) + i \langle x, \eta(t/\tau) \rangle + \mathbb{O}(\tau^{-1}).$$

Hence

$$\lim_{\tau \to \infty} e^{i\tau w(t/\tau, x/\tau + y(t/\tau))} = e^{it w_0'(0) + i \langle x, \eta(0) \rangle}. \quad (4\text{-}24)$$

In the sequel we shall also need

$$\partial w / \partial x_j(t/\tau, x/\tau + y(t/\tau)) - \eta_j(t/\tau) = \sum_{k=1}^{n-1} w_{j,k}(t/\tau)(x_k/\tau) + \mathbb{O}(\tau^{-2}), \quad (4\text{-}25)$$

which follows from the definition of w and the fact that w_α is symmetric in these special indices α. In particular, $w_{j,k}(t) = w_{k,j}(t)$ for all $j, k \in [1, n-1]$.

Recall that we chose the integer κ such that $m < \kappa$. By Proposition 4.3 there is a constant C such that

$$|I_\tau| \leq C\tau^{-\kappa}, \tag{4-26}$$

and we shall now show that if $\partial_t^{k_0} q_{-j_0(\alpha_0)}^{(\beta_0)}(0, y(0), \eta(0))$ is the first nonvanishing Taylor coefficient with respect to the ordering $>_t$, where $m = j_0 + k_0 + |\alpha_0| + |\beta_0|$, then (4-26) cannot hold. (Since we are denoting the variables by (t, x) now, the index α in Definition 3.2 will be replaced by the pair $(k, \alpha) \in \mathbb{N} \times \mathbb{N}^{n-1}$.) We will do this by determining the limit of $\tau^m I_\tau$ as $\tau \to \infty$. To see what is needed, consider $\lambda_{-1}(t/\tau, x/\tau + y(t/\tau))$ and recall that this is

$$q_1(t/\tau, x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau)))\phi_0(t/\tau, x/\tau + y(t/\tau)),$$

which should be regarded as a Taylor expansion in ξ of q_1 at $\eta(t/\tau)$ of finite order. The same applies to all the other terms of the form $q_{-j}^{(\alpha)}$. For given j and α, we only ever need to consider Taylor expansions of $q_{-j}^{(\alpha)}$ of order $\kappa - j - |\alpha|$ in view of (4-22) and (4-25). To keep things simple, we shall first only consider q_1; it will be clear by symmetry what the corresponding expressions for the other terms should be. Thus,

$$q_1(t/\tau, x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau)))$$
$$= \sum_{|\beta| \leq \kappa+1} q_1^{(\beta)}(t/\tau, x/\tau + y(t/\tau), \eta(t/\tau))(w_x'(t/\tau, x/\tau + y(t/\tau)) - \eta(t/\tau))^\beta / |\beta|! + \mathcal{O}(\tau^{-\kappa-2}), \tag{4-27}$$

which shows that to use our assumption regarding the Taylor coefficient $\partial_t^{k_0} q_{-j_0(\alpha_0)}^{(\beta_0)}(0, y(0), \eta(0))$, we have to for each β write $q_1^{(\beta)}(t/\tau, x/\tau + y(t/\tau), \eta(t/\tau))$ as a Taylor series at $\eta(0)$, in addition to having to expand each term as a Taylor series in t and x. However, it is immediate from (4-25) that if β is an $(n-1)$-tuple corresponding to a given differential operator D_ξ^β, then there is a sequence $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_s)$ of $s = |\beta|$ indices between 1 and the dimension $n-1$ of the x variable, such that

$$g_\tau^\beta(t, x) = (w_x'(t/\tau, x/\tau + y(t/\tau)) - \eta(t/\tau))^\beta, \tag{4-28}$$

as it appears in (4-27), satisfies

$$g_\tau^\beta(t, x) = c_\beta(t/\tau, x/\tau) + \mathcal{O}(\tau^{-|\beta|-1}),$$

where

$$c_\beta(t/\tau, x/\tau) = \prod_{j=1}^s \left(\sum_{k=1}^{n-1} w_{k, \tilde{\beta}_j}(t/\tau) x_k / \tau \right) \quad \text{and} \quad c_\beta(0, x/\tau) = \tau^{-|\beta|} c_\beta(0, x).$$

These expressions make sense if we choose the sequence $\tilde{\beta}$ to be increasing, for then it is uniquely determined by β. If for instance $D_\xi^\beta = -\partial^2 / \partial \xi_i \partial \xi_j$, then $\tilde{\beta} = (i, j)$ if $i \leq j$ (see the indices α used in connection with w_α in (4-4)). Thus (4-27) takes the form

$$q_1(t/\tau, x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau)))$$
$$= \sum_{|\beta| \leq \kappa+1} q_1^{(\beta)}(t/\tau, x/\tau + y(t/\tau), \eta(t/\tau)) g_\tau^\beta(t, x) / |\beta|! + \mathcal{O}(\tau^{-\kappa-2}),$$

and if we expand each term in this expression as a Taylor series at $\eta(0)$ we obtain

$$
\begin{aligned}
q_1(t/\tau, & x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau))) \\
&= \sum_{|\beta| \le \kappa+1} \sum_{|\gamma| \le \kappa+1-|\beta|} q_1^{(\beta+\gamma)}(t/\tau, x/\tau + y(t/\tau), \eta(0)) g_\tau^\beta(t, x)(\eta(t/\tau) - \eta(0))^\gamma/(|\beta|!|\gamma|!) \\
&\hspace{10cm} + \mathbb{O}(\tau^{-\kappa-2}), \quad (4\text{-}29)
\end{aligned}
$$

where we regard $\eta(t/\tau) - \eta(0)$ as a finite Taylor series $\eta'(0)t/\tau + \eta''(0)t^2/(2\tau^2) + \cdots$ of sufficiently high order to maintain control of the error term in (4-29). If we for each multiindex β let $G_\tau^\beta(t, x)$ be given by

$$
G_\tau^\beta(t, x) = \sum_{\gamma_1+\gamma_2=\beta} (\eta(t/\tau) - \eta(0))^{\gamma_1} g_\tau^{\gamma_2}(t, x)/(|\gamma_1|!|\gamma_2|!) \quad \text{for } \gamma_j \in \mathbb{N}^{n-1},
$$

then the required order of the Taylor expansion $\eta(t/\tau) - \eta(0)$ will ultimately depend on β, so we can write

$$
\begin{aligned}
q_1(t/\tau, & x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau))) \\
&= \sum_{|\beta| \le \kappa+1} q_1^{(\beta)}(t/\tau, x/\tau + y(t/\tau), \eta(0)) G_\tau^\beta(t, x) + \mathbb{O}(\tau^{-\kappa-2}) \quad (4\text{-}30)
\end{aligned}
$$

and we can always bound $G_\tau^\beta(t, x)$ by a constant times $\tau^{-|\beta|}$. As will be evident in a moment, the value of $G_\tau^\beta(t, x)$ for $|\beta| > 0$ is not important. For notational purposes, denote by $G_0^\beta(t, x)$ the limit of $\tau^{|\beta|} G_\tau^\beta(t, x)$ as $\tau \to \infty$. Since $G_\tau^\beta(t, x) = 1$ when $\beta = 0$ it is clear that $G_0^0(t, x) = 1$.

For each β we must now write $q_1^{(\beta)}(t/\tau, x/\tau + y(t/\tau), \eta(0))$ as a Taylor expansion in t and x at 0 and $y(0)$, respectively. As before, for given j and α, we will only have to consider Taylor expansions of $q_{-j}^{(\alpha)}$ of order $\kappa - j - |\alpha|$. By (4-23) and (4-30) we have

$$
\begin{aligned}
\lambda_{-1}(t/\tau, & x/\tau + y(t/\tau)) \\
&= \sum_{k+|\alpha|+|\beta| \le \kappa+1} \phi_0(t/\tau, x/\tau + y(t/\tau)) \big((t/\tau)^k (x/\tau + y(t/\tau) - y(0))^\alpha G_\tau^\beta(t, x) \\
&\hspace{4cm} \times \partial_t^k q_{1(\alpha)}^{(\beta)}(0, y(0), \eta(0))/(k!|\alpha|!) + \mathbb{O}(\tau^{-\kappa-2})\big), \quad (4\text{-}31)
\end{aligned}
$$

where we in $(x/\tau + y(t/\tau) - y(0))^\alpha$ regard $y(t/\tau) - y(0)$ as a finite Taylor series of sufficiently high order to maintain control of the error terms.

In the way that we expressed the term $q_1(t/\tau, x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau)))$ by (4-31), we can get similar expressions of appropriate order for the terms $q_{-j}^{(\gamma)}(t/\tau, x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau)))$ that appear in (4-23). For each j and γ we have

$$
\begin{aligned}
q_{-j}^{(\gamma)}(t/\tau, & x/\tau + y(t/\tau), w_x'(t/\tau, x/\tau + y(t/\tau))) \\
&= \sum_{k+|\alpha|+|\beta| \le \kappa-j-|\gamma|} (t/\tau)^k (x/\tau + y(t/\tau) - y(0))^\alpha G_\tau^\beta(t, x) \partial_t^k q_{-j(\alpha)}^{(\beta+\gamma)}(0, y(0), \eta(0))/(k!|\alpha|!) \\
&\hspace{9cm} + \mathbb{O}(\tau^{-\kappa-1+j+|\gamma|}). \quad (4\text{-}32)
\end{aligned}
$$

This together with (4-23) gives

$$
\lambda_J(t/\tau, x/\tau + y(t/\tau)) = \sum_{j+l+|\gamma|=J} \sum_{k+|\alpha|+|\beta|\le\kappa-j-|\gamma|} (t/\tau)^k (x/\tau + y(t/\tau) - y(0))^\alpha G_\tau^\beta(t, x)
$$
$$
\times D_x^\gamma \phi_l(t/\tau, x/\tau + y(t/\tau)) \frac{\partial_t^k q_{-j(\alpha)}^{(\beta+\gamma)}(0, y(0), \eta(0))}{k!\,|\alpha|!}
$$
$$
+ \mathbb{O}(\tau^{-\kappa-1+j+|\gamma|}),
$$

where $-1 \le j \le J$ and $l \ge 0$. Using that by assumption the Taylor coefficients $\partial_t^k q_{-j(\alpha)}^{(\beta+\gamma)}(0, y(0), \eta(0))$ vanish for all $-1 \le j+k+|\alpha|+|\beta|+|\gamma| < m$, and $\tau^{-J-k-|\alpha|} = \tau^{|\beta|}\tau^{-j-k-|\alpha|-|\beta|-|\gamma|-l}$ when $J = j+l+|\gamma|$, the equation above yields

$$
\sum_{J=-1}^m \tau^{-J}\lambda_J(t/\tau, x/\tau + y(t/\tau)) = \sum_{j+l+|\gamma|=-1}^m \sum_{j+k+|\alpha|+|\beta|+|\gamma|=m} \tau^{-m-l} t^k (x + y'(0)t)^\alpha \tau^{|\beta|} G_\tau^\beta(t, x)
$$
$$
\times D_x^\gamma \phi_l(t/\tau, x/\tau + y(t/\tau)) \frac{\partial_t^k q_{-j(\alpha)}^{(\beta+\gamma)}(0, y(0), \eta(0))}{k!\,|\alpha|!}
$$
$$
+ \mathbb{O}(\tau^{-m-1-l}),
$$

where $\tau^{|\beta|}G_\tau^\beta(t, x) \to G_0^\beta(t, x)$ as $\tau \to \infty$. As we can see, the expression above is $\mathbb{O}(\tau^{-m-1})$ as soon as $l > 0$, so in view of (4-22) and (4-24) we obtain

$$
\lim_{\tau\to\infty} \tau^m I_\tau = \iint H(t, x)e^{itw_0'(0)+i\langle x, \eta(0)\rangle}
$$
$$
\times \left(\sum_{j+k+|\alpha|+|\beta|+|\gamma|=m} t^k (x + y'(0)t)^\alpha G_0^\beta(t, x) D_x^\gamma \phi_0(0, y(0)) \right.
$$
$$
\left. \times \partial_t^k q_{-j(\alpha)}^{(\beta+\gamma)}(0, y(0), \eta(0))/(k!\,|\alpha|!) \right) dt\, dx. \quad (4\text{-}33)
$$

Recall (4-20) and choose ϕ_0 so that $D_x^{\beta_0}\phi_0(0, y(0)) = 1$, but so that $D_x^\gamma \phi_0(0, y(0)) = 0$ for all other γ such that $|\gamma| \le |\beta_0|$ (see (4-14)). By the choice of our ordering $>_t$ we have $\partial_t^k q_{-j(\alpha)}^{(\beta+\beta_0)}(0, y(0), \eta(0)) = 0$ for all β such that $|\beta| > 0$ as long as $j+k+|\alpha|+|\beta|+|\beta_0| = m$. Hence, with this choice of ϕ_0, equation (4-33) takes the form

$$
\lim_{\tau\to\infty} \tau^m I_\tau = \iint H(t, x)e^{itw_0'(0)+i\langle x, \eta(0)\rangle}
$$
$$
\times \left(\sum_{j+k+|\alpha|+|\beta_0|=m} t^k (x + y'(0)t)^\alpha \partial_t^k q_{-j(\alpha)}^{(\beta_0)}(0, y(0), \eta(0))/(k!\,|\alpha|!) \right) dt\, dx, \quad (4\text{-}34)
$$

so as promised, the value of $G_0^\beta(t, x)$ for $|\beta| > 0$ does not matter. (Note that $G_0^0(t, x)$ is present in (4-34) as the constant factor 1.) As in the proof of Theorem 2.20, some of the Taylor coefficients in (4-34) may be zero. In particular, the expression may well contain Taylor coefficients that preceed $\partial_t^{k_0} q_{-j_0(\alpha_0)}^{(\beta_0)}(0, y(0), \eta(0))$ in the ordering, and those are by assumption zero. In contrast to the proof of Theorem 2.20 we shall have to exploit this fact, since the coefficients of most of the monomials in (4-34) will be linear combinations of the Taylor coefficients due to the factor $(x + y'(0)t)^\alpha$. However, the ordering $>_t$ was chosen so that there can be no nonzero Taylor coefficient $\partial_t^k q_{-j(\alpha)}^{(\beta_0)}(0, y(0), \eta(0))$ such that $k + |\alpha| > k_0 + |\alpha_0|$, or $k + |\alpha| = k_0 + |\alpha_0|$ and $k < k_0$. This follows immediately from the

choice of lexicographic order on the n-tuple $(k, \alpha) \in \mathbb{N}^n$. (Recall that in the definition of the ordering $>_t$, x denoted all the variables in $\mathbb{R}^n$, while here we denote those variables by (t, x).) Hence, the only coefficient of the monomial $t^{k_0} x^{\alpha_0}$ in (4-34) is $\partial_t^{k_0} q_{-j_0 (\alpha_0)}^{(\beta_0)}(0, y(0), \eta(0))$. We may therefore, as in the proof of Theorem 2.20, choose H so that the limit in (4-34) is nonzero. Since this contradicts (4-26), Theorem 2.21 follows in view of the discussion following Lemma 4.1.

Appendix A.

Here we prove a few results used in the main text, related to how the property that all terms in the asymptotic expansion of the total symbol have vanishing Taylor coefficients is affected by various operations.

Lemma A.1. *Suppose X and Y are two C^∞ manifolds of the same dimension n. Let $K \subset T^*(X) \smallsetminus 0$ and $K' \subset T^*(Y) \smallsetminus 0$ be compactly based cones and let χ be a homogeneous symplectomorphism from a conic neighborhood of K' to one of K such that $\chi(K') = K$. Let $A \in I^{m'}(X \times Y, \Gamma')$ and $B \in I^{m''}(Y \times X, (\Gamma^{-1})')$, where Γ is the graph of χ, and assume that A and B are properly supported and noncharacteristic at the restriction of the graphs of χ and χ^{-1} to K' and to K respectively, while $WF'(A)$ and $WF'(B)$ are contained in small conic neighborhoods. If R is a properly supported classical pseudodifferential operator in Y, then each term in the asymptotic expansion of the total (left) symbol of R has vanishing Taylor coefficients at a point $(y, \eta) \in K'$ if and only if each term in the asymptotic expansion of the total (left) symbol of the pseudodifferential operator ARB in X has vanishing Taylor coefficients at $\chi(y, \eta) \in K$.*

Proof. We may assume that we have a homogeneous generating function $\varphi \in C^\infty$ for the symplectomorphism χ; see [Grigis and Sjöstrand 1994, pages 101–103]. Then χ is locally of the form

$$(\partial\varphi(x, \eta)/\partial\eta, \eta) \mapsto (x, \partial\varphi(x, \eta)/\partial x),$$

and A and B are given by

$$Au(x) = \frac{1}{(2\pi)^n} \iint e^{i(\varphi(x,\zeta)-z\cdot\zeta)} a(x, z, \zeta) u(z) \, dz \, d\zeta,$$

$$Bv(y) = \frac{1}{(2\pi)^n} \iint e^{i(y\cdot\theta-\varphi(s,\theta))} b(y, s, \theta) v(s) \, ds \, d\theta.$$

Since R is properly supported we may assume that

$$Ru(z) = \frac{1}{(2\pi)^n} \int e^{iz\cdot\eta} r(z, \eta) \hat{u}(\eta) \, d\eta \quad \text{for } u \in C_0^\infty(Y), \tag{A-1}$$

where $r(z, \eta) = \sigma_R$ is the total symbol of R. Hence

$$ARBu(x) = \frac{1}{(2\pi)^{3n}} \int e^{i(\varphi(x,\zeta)-z\cdot\zeta+(z-y)\cdot\sigma+y\cdot\theta-\varphi(s,\theta))} a(x, z, \zeta) r(z, \sigma) b(y, s, \theta) u(s) \, ds \, d\theta \, dy \, d\sigma \, dz \, d\zeta, \tag{A-2}$$

since B being properly supported implies that $Bu \in C_0^\infty(Y)$ when $u \in C_0^\infty(Y)$. Using integration by parts in z, we see that we can insert a cutoff $\phi((\zeta - \sigma)/|\sigma|)$ in the last integral without changing the operator

ARB mod $\Psi^{-\infty}$. If we make the change of variables $\tau = \zeta - \sigma$, then (A-2) takes the form

$$ARBu(x) = \frac{1}{(2\pi)^{3n}} \int \phi(\tau/|\sigma|) e^{i(\varphi(x,\tau+\sigma)-z\cdot(\tau+\sigma)+(z-y)\cdot\sigma+y\cdot\theta-\varphi(s,\theta))}$$
$$\times a(x, z, \tau+\sigma) r(z, \sigma) b(y, s, \theta) u(s) \, ds \, d\theta \, dy \, d\sigma \, dz \, d\tau + Lu,$$

with $L \in \Psi^{-\infty}$. If $\Omega \subset \mathbb{R}^{2n}$ is open and $\tilde{\varphi} \in C^{\infty}(\Omega, \mathbb{R})$ is a phase function with a nondegenerate critical point $x_0 \in \Omega$ such that $d\tilde{\varphi} \neq 0$ everywhere else, then [Grigis and Sjöstrand 1994, Proposition 2.3] states, in particular, that for every compact $M \subset \Omega$ and every $u \in C^{\infty}(\Omega) \cap \mathscr{E}'(M)$ we have

$$\left| \int e^{i\lambda\tilde{\varphi}(x)} u(x) dx - e^{i\lambda\tilde{\varphi}(x_0)} A_0 u(x_0) \lambda^{-n} \right| \leq C_M \lambda^{-n-1} \sum_{|\alpha| \leq 2n+3} \sup |\partial^{\alpha} u(x)| \quad \text{for } \lambda \geq 1, \qquad \text{(A-3)}$$

where

$$A_0 = \frac{(2\pi)^n \cdot e^{i\pi \, \text{sgn} \, \tilde{\varphi}''(x_0)/4}}{|\det \tilde{\varphi}''(x_0)|^{1/2}}. \qquad \text{(A-4)}$$

It is clear that the result extends to the setting $\Omega = T^*(\mathcal{N}) \setminus 0$, where $\mathcal{N}$ is a C^{∞} manifold of dimension n. In order to apply the result, we put $\sigma = \lambda\omega$, and make the change of variables $\tau = \lambda\tilde{\tau}$. After dropping the tilde we obtain

$$ARBu(x) = \frac{\lambda^{2n}}{(2\pi)^{3n}} \int \phi(\tau/|\omega|) e^{i\lambda(\varphi(x,\tau+\omega)-z\cdot(\tau+\omega)+y\cdot\theta/\lambda+(z-y)\cdot\omega-\varphi(s,\theta)/\lambda)}$$
$$\times a(x, z, \lambda(\tau+\omega)) r(z, \lambda\omega) b(y, s, \theta) u(s) \, ds \, d\theta \, dy \, d\omega \, dz \, d\tau + Lu,$$

where we have used the fact that φ is homogeneous of degree 1 in the fiber. For the z, τ-integration we have the nondegenerate critical point given by $\tau = 0$, $z = \varphi'_{\zeta}(x, \tau+\omega)$. Note that since φ'_{ζ} is homogeneous of degree 0 in the fiber we have $\varphi'_{\zeta}(x, \sigma/\lambda) = \varphi'_{\zeta}(x, \sigma)$, so this critical point corresponds to the critical point for the z, ζ-integration given by $\zeta = \sigma$, $z = \varphi'_{\zeta}(x, \sigma)$. Hence the expression above together with (A-3) imply that

$$ARBu(x) = C\lambda^{2n} \int e^{i(\varphi(x,\lambda\omega)+y\cdot\theta-y\cdot\lambda\omega-\varphi(s,\theta))} w(x, y, s, \omega, \theta) u(s) \, ds \, d\theta \, dy \, d\omega + Lu,$$

where

$$w(x, y, s, \omega, \theta) = \frac{A_0}{\lambda^n} a(x, z, \lambda(\tau+\omega)) r(z, \lambda\omega) b(y, s, \theta) \phi(\tau/|\omega|) \Big|_{\substack{\tau=0, \\ z=\varphi'_{\zeta}(x,\omega)}}$$

$$= \frac{A_0}{\lambda^n} a(x, \varphi'_{\zeta}(x, \omega), \lambda\omega) r(\varphi'_{\zeta}(x, \omega), \lambda\omega) b(y, s, \theta)$$

with an error of order $\mathbb{O}(\lambda^{-n-1})$. Note that A_0 is now a function of x and ω, since the matrix corresponding to $\tilde{\varphi}''(x_0)$ in (A-4) is given by the block matrix

$$F = \begin{pmatrix} 0 & -\text{Id}_n \\ -\text{Id}_n & \varphi''_{\zeta\zeta}(x, \omega) \end{pmatrix}, \qquad \text{(A-5)}$$

where Id_n is the identity matrix on $\mathbb{R}^n$. Clearly the determinant of F is either 1 or -1, so F is nonsingular. Furthermore, F depends smoothly on the parameters x and ω since $\varphi \in C^{\infty}$, so the eigenvalues of F

are continuous in x and ω. Hence it follows that the signature of F is constant, for if not there has to exist an eigenvalue vanishing at some point (x, ω), contradicting the nonsingularity of F. Reverting to the variable $\sigma = \lambda\omega$ we thus obtain

$$ARBu(x) = C \int e^{i(\varphi(x,\sigma)+y\cdot(\theta-\sigma)-\varphi(s,\theta))} \tilde{w}(x, y, s, \sigma, \theta)u(s)\, ds\, d\theta\, dy\, d\sigma + Lu,$$

where

$$\tilde{w}(x, y, s, \sigma, \theta) = a(x, \varphi'_\zeta(x, \sigma), \sigma)r(\varphi'_\zeta(x, \sigma), \sigma)b(y, s, \theta)$$

with an error of order $\mathbb{O}(\lambda^{-1})$. Taking the limit as $\lambda \to \infty$ yields

$$ARBu(x) = C \int e^{i(\varphi(x,\sigma)+y\cdot(\theta-\sigma)-\varphi(s,\theta))} a(x, \varphi'_\zeta(x, \sigma), \sigma)r(\varphi'_\zeta(x, \sigma), \sigma)b(y, s, \theta)u(s)\, ds\, d\theta\, dy\, d\sigma + Lu.$$

We can now repeat the procedure. Indeed, we can insert a cutoff $\phi((\sigma - \theta)/|\theta|)$ without changing the operator mod $\Psi^{-\infty}$, and after making the corresponding changes of variables in order to apply [Grigis and Sjöstrand 1994, Proposition 2.3] we find that for the y, σ-integration we have the nondegenerate critical point given in the original variables by $\sigma = \theta$, $y = \varphi'_\sigma(x, \sigma)$. After taking the limit as $\lambda \to \infty$ we obtain

$$ARBu(x) = C \int e^{i(\varphi(x,\theta)-\varphi(s,\theta))} w_1(x, s, \theta)u(s)\, ds\, d\theta + L_1 u,$$

where $L_1 \in \Psi^{-\infty}$ and

$$w_1(x, s, \theta) = a(x, \varphi'_\theta(x, \theta), \theta)r(\varphi'_\theta(x, \theta), \theta)b(\varphi'_\theta(x, \theta), s, \theta). \tag{A-6}$$

As before we let the factor A_0 from (A-4) be included in the constant C. In a conic neighborhood of $\operatorname{supp} w_1$ we can write

$$\varphi(x, \theta) - \varphi(s, \theta) = (x - s)\Xi(x, s, \theta).$$

Then $\Xi(x, x, \theta) = \varphi'_x(x, \theta)$ so $\partial\Xi(x, x, \theta)/\partial\theta = \varphi''_{x\theta}(x, \theta)$ is invertible, since $\varphi''_{x\theta}(x, \theta) \neq 0$ is equivalent to the fact that the graph of χ is (locally) the graph of a smooth map. Hence $\theta \mapsto \Xi(x, s, \theta)$ is C^∞, homogeneous of degree 1 and with an inverse having the same properties. For s close to x, the equation $\Xi(x, s, \theta) = \xi$ then defines $\theta = \Theta(x, s, \xi)$. After a change of variables, the last integral therefore takes the form

$$ARBu(x) = C \int e^{i(x-s)\cdot\xi} \tilde{w}_1(x, s, \xi)u(s)\, ds\, d\xi + L_1 u, \tag{A-7}$$

where $\tilde{w}_1(x, s, \xi)$ is just $w_1(x, s, \Theta(x, s, \xi))$ multiplied by a Jacobian. We note in passing that evaluating $\tilde{w}_1$ at a point (x, x, ξ) where ξ is of the form $\xi = \varphi'_x(x, \eta)$ therefore involves evaluating w_1 at the point (x, x, η). The integral (A-7) defines a pseudodifferential operator with total symbol $\rho(x, \xi)$ satisfying

$$\rho(x, \xi) \sim \sum \frac{i^{-|\alpha|}}{\alpha!}(\partial^\alpha_\xi \partial^\alpha_y \tilde{w}_1(x, y, \xi))|_{y=x}. \tag{A-8}$$

If the total symbol $r = \sigma_R$ of R has vanishing Taylor coefficients at a point $(y, \eta) = (\varphi'_\eta(x, \eta), \eta)$, then by examining (A-8) in decreasing order of homogeneity we find that each term of ρ must have vanishing

Taylor coefficients at $(x, \xi) = (x, \varphi'_x(x, \eta))$, since by what we have shown this would involve evaluating $r(z, \sigma)$ and its derivatives at $(\varphi'_\eta(x, \eta), \eta)$.

To prove the converse, choose $A_1 \in I^{-m''}(X \times Y, \Gamma')$ and $B_1 \in I^{-m'}(Y \times X, (\Gamma^{-1})')$ properly supported such that

$$K' \cap WF(BA_1 - I) = \varnothing, \qquad K \cap WF(A_1B - I) = \varnothing,$$
$$K' \cap WF(B_1A - I) = \varnothing, \qquad K \cap WF(AB_1 - I) = \varnothing.$$

Then a repetition of the arguments above shows that all the terms in the asymptotic expansion of the total symbol of B_1ARBA_1 has vanishing Taylor coefficients at a point $(y, \eta) = (\varphi'_\eta(x, \eta), \eta)$ if all the terms in the asymptotic expansion of the total symbol of ARB has vanishing Taylor coefficients at $(x, \xi) = (x, \varphi'_x(x, \eta))$. Since R and B_1ARBA_1 have the same total symbol in K' mod $\Psi^{-\infty}$, the same must hold for the total symbol of R. $\qquad\square$

Let $\{e_k : k = 1, \ldots, n\}$ be a basis for $\mathbb{R}^n$, let (U, x) be local coordinates on a smooth manifold M of dimension n, and let

$$\left\{\frac{\partial}{\partial x_k} : k = 1, \ldots, n\right\}$$

be the induced local frame for the tangent bundle TM. Since the local frame fields commute, we can use standard multiindex notation to express the partial derivatives $\partial_x^\alpha f$ of $f \in C^\infty(U)$.

Lemma A.2. *Let M be a smooth manifold of dimension n, and for $j \geq 1$ let $p, q_j, g_j \in C^\infty(M)$. Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence in M such that $\gamma_j \to \gamma$ as $j \to \infty$, and assume that $p(\gamma) = p(\gamma_j) = 0$ for all j, and that $dp(\gamma) \neq 0$. Let (U, x) be local coordinates on M near γ, and suppose that there exists a smooth function $q \in C^\infty(M)$ such that*

$$\partial_x^\alpha q(\gamma) = \lim_{j \to \infty} \partial_x^\alpha q_j(\gamma_j) \quad \text{for all } \alpha \in \mathbb{N}^n.$$

If $q_j - pg_j$ vanishes of infinite order at γ_j for all j, then there exists a smooth function $g \in C^\infty(M)$ such that $q - pg$ vanishes of infinite order at γ. Furthermore,

$$\partial_x^\alpha g(\gamma) = \lim_{j \to \infty} \partial_x^\alpha g_j(\gamma_j) \quad \text{for all } \alpha \in \mathbb{N}^n. \tag{A-9}$$

Proof. We have stated the result for a manifold, but since the result is purely local we may assume that $M \subset \mathbb{R}^n$ in the proof. It is also clear that we may assume that there exists an open neighborhood $\mathcal{U}$ of γ such that $\gamma_j \in \mathcal{U}$ for $j \geq 1$, and that $dp \neq 0$ in $\mathcal{U}$. By shrinking $\mathcal{U}$ if necessary, we can then find a unit vector $v \in \mathbb{R}^n$ such that $\partial_v p(w) = \langle v, dp(w) \rangle \neq 0$ for $w \in \mathcal{U}$. (We will identify a tangent vector $v \in \mathbb{R}^n$ at γ with $\partial_v \in T_\gamma \mathbb{R}^n$ through the usual vector space isomorphism.) Hence $\partial_v p(w)$ is invertible in $\mathcal{U}$, and we let $(\partial_v p(w))^{-1} \in C^\infty(\mathcal{U})$ denote its inverse. By an orthonormal change of coordinates we may even assume that $\partial_v p(w) = \partial_{e_1} p(w)$. In accordance with the notation used in the statement of the lemma, we shall write $\partial_{x_k} p(w)$ for the partial derivatives $\partial_{e_k} p(w)$ and denote by $(\partial_{x_1} p(w))^{-1}$ the inverse of $\partial_v p(w) = \partial_{x_1} p(w)$ in $\mathcal{U}$.

Now

$$0 = \partial_{x_1}(q_j - pg_j)(\gamma_j) = \partial_{x_1} q_j(\gamma_j) - \partial_{x_1} p(\gamma_j)g_j(\gamma_j) \tag{A-10}$$

for all j since $p(\gamma_j) = 0$. Since $\lim_j \partial_{x_1} q_j(\gamma_j) = \partial_{x_1} q(\gamma)$ by assumption, equation (A-10) yields

$$\lim_{j \to \infty} g_j(\gamma_j) = (\partial_{x_1} p(\gamma))^{-1} \partial_{x_1} q(\gamma) = a \in \mathbb{C}. \tag{A-11}$$

We claim that we can in the same way determine

$$\lim_{j \to \infty} (\partial_x^\alpha g_j)(\gamma_j) = a_{(\alpha)} \in \mathbb{C} \quad \text{for any } \alpha \in \mathbb{N}^n.$$

We start by determining $\lim_{j \to \infty} \partial g_j(\gamma_j)/\partial x_k = a_{(k)}$ for $1 \le k \le n$. By the hypotheses of the lemma we have

$$\begin{aligned}
0 &= \partial_{x_k} \partial_{x_l}(q_j - pg_j)(\gamma_j) \\
&= \partial_{x_k} \partial_{x_l} q_j(\gamma_j) - \partial_{x_k} \partial_{x_l} p(\gamma_j) g_j(\gamma_j) - \partial_{x_k} p(\gamma_j) \partial_{x_l} g_j(\gamma_j) - \partial_{x_l} p(\gamma_j) \partial_{x_k} g_j(\gamma_j)
\end{aligned} \tag{A-12}$$

since $p(\gamma_j) = 0$. For $k = l = 1$ we obtain from (A-11) and (A-12)

$$\lim_{j \to \infty} \partial_{x_1} g_j(\gamma_j) = (\partial_{x_1} p(\gamma))^{-1} \big(\partial_{x_1}^2 q(\gamma) - \partial_{x_1}^2 p(\gamma) a\big)/2. \tag{A-13}$$

This allows us to solve for $\partial_{x_k} g_j(\gamma_j)$ in (A-12) by choosing $l = 1$. If $b \in \mathbb{C}$ denotes the limit in (A-13) and $a \in \mathbb{C}$ is given by (A-11) we thus obtain

$$\lim_{j \to \infty} \partial_{x_k} g_j(\gamma_j) = (\partial_{x_1} p(\gamma))^{-1} \big(\partial_{x_1} \partial_{x_k} q(\gamma) - \partial_{x_1} \partial_{x_k} p(\gamma) a - \partial_{x_k} p(\gamma) b\big) \quad \text{for } 2 \le k \le n.$$

Now assume that for some $m \ge 3$ we have in this way determined

$$\lim_{j \to \infty} \partial_{x_{k_1}} \ldots \partial_{x_{k_{m-2}}} g_j(\gamma_j), \quad \text{for } k_i \in [1, n], \text{ with } i \in [1, m-2].$$

To shorten notation, we will use the (nonstandard) multiindex notation introduced on page 465; to every $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$ corresponds precisely one m-tuple $\beta = (k_1, \ldots, k_m)$ of nondecreasing numbers $1 \le k_1 \le \cdots \le k_m \le n$ such that ∂_x^β equals ∂_x^α. Throughout the rest of this proof we shall let β represent such an m-tuple, and we let

$$\hat{\beta}_i = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_m).$$

As before we have

$$0 = \partial_x^\beta(q_j - pg_j)(\gamma_j) = \partial_x^\beta q_j(\gamma_j) - \partial_x^\beta p(\gamma_j) g_j(\gamma_j) - \cdots - \sum_{i=1}^m \partial_{x_{k_i}} p(\gamma_j) \partial_x^{\hat{\beta}_i} g_j(\gamma_j) \tag{A-14}$$

by assumption. If we choose $k_i = 1$ for all $1 \le i \le m$, the last sum is just $m \partial_{x_1} p(\gamma_j) \partial_{x_1}^{m-1} g_j(\gamma_j)$, and since the limit of all other terms on the right side are known by the induction hypothesis, we thus obtain the value of the limit of $\partial_{x_1}^{m-1} g_j(\gamma_j)$ from (A-14) by first multiplying by $m^{-1}(\partial_{x_1} p(\gamma_j))^{-1}$ and then letting $j \to \infty$. Denote this limit by $c \in \mathbb{C}$. If we choose $k_i \ne 1$ for precisely one $i \in [1, m]$, say $k_m = k$, then the last sum in (A-14) satisfies

$$\sum_{i=1}^m \partial_{x_{k_i}} p(\gamma_j) \partial_x^{\hat{\beta}_i} g_j(\gamma_j) = \partial_{x_k} p(\gamma_j) \partial_{x_1}^{m-1} g_j(\gamma_j) + (m-1) \partial_{x_1} p(\gamma_j) \partial_{x_1}^{m-2} \partial_{x_k} g_j(\gamma_j),$$

so by the same argument as before we can obtain the value of $\lim_{j\to\infty}\partial_{x_1}^{m-2}\partial_{x_k}g_j(\gamma_j)$ for $2 \leq k \leq n$ by multiplying by $(m-1)^{-1}(\partial_{x_1}p(\gamma_j))^{-1}$ and using $\partial_{x_1}^{m-1}g_j(\gamma_j) \to c$ when taking the limit as $j \to \infty$ in (A-14). Continuing this way it is clear that we can successively determine

$$\lim_{j\to\infty} \partial_{x_{k_1}}\dots\partial_{x_{k_{m-1}}}g_j(\gamma_j) \quad \text{for any } 1 \leq k_1 \leq \dots \leq k_{m-1} \leq n,$$

which completely determines $\lim_{j\to\infty}\partial_x^\alpha g_j(\gamma_j) = a_{(\alpha)}$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| = m-1$, proving the claim.

By Borel's theorem there exists a smooth function $g \in C^\infty(M)$ such that

$$\partial_x^\alpha g(\gamma) = a_{(\alpha)} = \lim_{j\to\infty} \partial_x^\alpha g_j(\gamma_j) \quad \text{for all } \alpha \in \mathbb{N}^n.$$

Since $q - pg$ vanishes of infinite order at γ by construction, this completes the proof. $\qquad\square$

The lemma will be used to prove the following result for homogeneous smooth functions on the cotangent bundle.

Proposition A.3. *For $j \geq 1$ let $p, q_j, g_j \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$, where p and q_j are homogeneous of degree m and the g_j are homogeneous of degree 0. Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence in $T^*(\mathbb{R}^n) \smallsetminus 0$ such that $\gamma_j \to \gamma$ as $j \to \infty$, and assume that $p(\gamma) = p(\gamma_j) = 0$ for all j, and that $dp(\gamma) \neq 0$. If there exists a smooth function $q \in C^\infty(T^*(\mathbb{R})^n \smallsetminus 0)$, homogeneous of degree m, such that*

$$\partial_x^\alpha \partial_\xi^\beta q(\gamma) = \lim_{j\to\infty} \partial_x^\alpha \partial_\xi^\beta q_j(\gamma_j) \quad \text{for all } (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n,$$

and if $q_j - pg_j$ vanishes of infinite order at γ_j for all j, then there exists a $g \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$, homogeneous of degree 0, such that $q - pg$ vanishes of infinite order at γ. Furthermore,

$$\partial_x^\alpha \partial_\xi^\beta g(\gamma) = \lim_{j\to\infty} \partial_x^\alpha \partial_\xi^\beta g_j(\gamma_j) \quad \text{for all } (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n. \tag{A-15}$$

Proof. Let $\pi : T^*(\mathbb{R}^n) \smallsetminus 0 \to S^*(\mathbb{R}^n)$ be the projection. Since $dp(\gamma) \neq 0$ it follows from homogeneity that $dp(\pi(\gamma)) \neq 0$. By using the homogeneity of q, q_j and g_j we may even assume that γ and γ_j belong to $S^*(\mathbb{R}^n)$ for $j \geq 1$ to begin with.

Now, the radial vector field $\xi \partial_\xi$ applied k times to $a \in C^\infty(T^*(\mathbb{R}^n) \smallsetminus 0)$ equals $l^k a$ if a is homogeneous of degree l. For any point $w \in S^*(\mathbb{R}^n)$ with $w = (w_x, w_\xi)$ in local coordinates on $T^*(\mathbb{R}^n)$ it is easy to see that

$$T_w S^*(\mathbb{R}^n) = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : \langle w_\xi, v \rangle = 0\}.$$

Therefore a basis for $T_w S^*(\mathbb{R}^n)$ together with the radial vector field $(\xi \partial_\xi)_w$ at w constitutes a basis for $T_w T^*(\mathbb{R}^n)$. This implies that if we can find a homogeneous function g such that $q - pg$ vanishes of infinite order in the directions $T_\gamma S^*(\mathbb{R}^n)$, then $q - pg$ vanishes of infinite order at γ, for the derivatives involving the radial direction are determined by lower order derivatives in the directions $T_\gamma S^*(\mathbb{R}^n)$.

By the hypotheses of the proposition together with an application of Lemma A.2, we find that there exists a function $\tilde{g} \in C^\infty(T^*(\mathbb{R}^n))$, not necessarily homogeneous, such that $q - p\tilde{g}$ vanishes of infinite order at γ and (A-15) holds for $\tilde{g}$. The function $g(x, \xi) = \tilde{g}(x, \xi/|\xi|)$ coincides with $\tilde{g}$ on $S^*(\mathbb{R}^n)$. In particular, all derivatives of g and $\tilde{g}$ in the directions $T_\gamma S^*(\mathbb{R}^n)$ are equal at γ. Thus, by the arguments

above we conclude that $q - pg$ vanishes of infinite order at γ. Since g and g_j are homogeneous of degree 0, the same arguments also imply that (A-15) holds for g, which completes the proof. □

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References

[Dencker 2006] N. Dencker, "The resolution of the Nirenberg–Treves conjecture", *Ann. of Math.* (2) **163**:2 (2006), 405–444. MR 2006i:35386 Zbl 1104.35080

[Grigis and Sjöstrand 1994] A. Grigis and J. Sjöstrand, *Microlocal analysis for differential operators: An introduction*, London Mathematical Society Lecture Note Series **196**, Cambridge University Press, 1994. MR 95d:35009 Zbl 0804.35001

[Hörmander 1960a] L. Hörmander, "Differential equations without solutions", *Math. Ann.* **140** (1960), 169–173. MR 26 #5279 Zbl 0093.28903

[Hörmander 1960b] L. Hörmander, "Differential operators of principal type", *Math. Ann.* **140** (1960), 124–146. MR 24 #A434 Zbl 0090.08101

[Hörmander 1963] L. Hörmander, *Linear partial differential operators*, Grundlehren der mathematischen Wissenschaften **116**, Springer, Berlin, 1963. Zbl 0108.09301

[Hörmander 1966] L. Hörmander, "Pseudo-differential operators and non-elliptic boundary problems", *Ann. of Math.* (2) **83** (1966), 129–209. MR 38 #1387 Zbl 0132.07402

[Hörmander 1981] L. Hörmander, "Pseudo-differential operators of principal type", pp. 69–96 in *Singularities in boundary value problems: Proc. NATO Adv. Study Inst., 1980* (Maratea, Italy: September 22–October 3, 1980), NATO Adv. Study Inst., Ser. C: Math. Phys. Sci. **65**, Reidel, Dordrecht-Boston, MA, 1981. Zbl 0459.35096

[Hörmander 1983a] L. Hörmander, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, Grundlehren der Mathematischen Wissenschaften **256**, Springer, Berlin, 1983. MR 85g:35002a

[Hörmander 1983b] L. Hörmander, *The analysis of linear partial differential operators, II: Differential operators with constant coefficients*, Grundlehren der Mathematischen Wissenschaften **257**, Springer, Berlin, 1983. MR 85g:35002b

[Hörmander 1985a] L. Hörmander, *The analysis of linear partial differential operators, III: Pseudodifferential operators*, Grundlehren der Mathematischen Wissenschaften **274**, Springer, Berlin, 1985. MR 87d:35002a Zbl 0601.35001

[Hörmander 1985b] L. Hörmander, *The analysis of linear partial differential operators, IV: Fourier integral operators*, Grundlehren der Mathematischen Wissenschaften **275**, Springer, Berlin, 1985. MR 87d:35002b

[Lewy 1957] H. Lewy, "An example of a smooth linear partial differential equation without solution", *Ann. of Math.* (2) **66** (1957), 155–158. MR 19,551d Zbl 0078.08104

[Moyer 1978] R. D. Moyer, "Local solvability in two dimensions: Necessary conditions for the principal-type case", mimeographed manuscript, University of Kansas, 1978.

[Nirenberg and Treves 1970a] L. Nirenberg and F. Treves, "On local solvability of linear partial differential equations, I: Necessary conditions", *Communications on Pure and Applied Mathematics* **23**:1 (1970), 1–38. Zbl 0191.39103

[Nirenberg and Treves 1970b] L. Nirenberg and F. Treves, "On local solvability of linear partial differential equations, II: Sufficient conditions", *Communications on Pure and Applied Mathematics* **23**:3 (1970), 459–509.

[Nirenberg and Treves 1971] L. Nirenberg and F. Treves, "Corrections to On local solvability of linear partial differential equations", *Communications on Pure and Applied Mathematics* **24**:2 (1971), 279–288.

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