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We establish a sharp geometric constant for the upper bound on the resonance counting function for surfaces with hyperbolic ends. An arbitrary metric is allowed within some compact core, and the ends may be of hyperbolic planar, funnel, or cusp type. The constant in the upper bound depends only on the volume of the core and the length parameters associated to the funnel or hyperbolic planar ends. Our estimate is sharp in that it reproduces the exact asymptotic constant in the case of finite-area surfaces with hyperbolic cusp ends, and also in the case of funnel ends with Dirichlet boundary conditions.

1. Introduction

For a compact Riemannian surface, the Weyl law shows that the asymptotic distribution of eigenvalues is determined by global geometric quantities. In the compact hyperbolic case, Weyl asymptotics follow easily from the Selberg trace formula; see, e.g, [McKean 1972], This approach extends also to noncompact hyperbolic surfaces of finite area [Venkov 1990]. Some reinterpretation of the spectral counting is needed for the noncompact case, however. One can either supplement the counting function for the discrete spectrum by a term related to the scattering phase, or else use the counting function for resonances instead of eigenvalues. Weyl asymptotics, in this extended sense, were established for general finite-area surfaces with hyperbolic cusp ends by Müller [1992] and Parnovski [1995].

For infinite-area surfaces with hyperbolic ends, the discrete spectrum is finite and possibly empty, and therefore plays no role in the spectral asymptotics. One could look for analogies to the finite-area results in the asymptotics of either the scattering phase or the resonance counting function. For the scattering phase of a surface with hyperbolic ends, Weyl asymptotics were proven by Guillopé and Zworski [1997]. One does not necessarily expect a corresponding result to hold for the resonance counting function — see e.g., [Guillopé and Zworski 1997, Remark 1.6] — but neither can we rule out the possibility at this point. Understanding the role that global geometric properties play in the distribution of resonances remains a compelling problem.

In the context of infinite-area hyperbolic surfaces, only the order of growth of the resonance counting function is currently well understood. Guillopé and Zworski [1995; 1997] showed the resonance counting function for infinite-area surfaces with hyperbolic ends satisfies $N_g(t) \asymp t^2$ (with the caveat that the lower bound is proportional to the 0-volume, which might be zero in exceptional cases). These results have been

extended to higher-dimensional manifolds with hyperbolic ends in [Borthwick 2008]. Unfortunately, the methods used in these proofs yield only an ineffective constant for the upper bound, with no clear geometric content. Moreover, the derivation of the lower bound depends explicitly on the upper bound, so the geometric dependence of the lower bound is likewise undetermined.

In this paper we present a geometric constant for the upper bound on the resonance counting function for infinite-area surfaces with hyperbolic ends. This constant is sharp in the sense that it agrees with the exact asymptotics in the cases of finite area surfaces or truncated funnels. Our approach is inspired by Stefanov's recent paper [2006] on compactly supported perturbations of the Laplacian on $\mathbb{R}^n$ for n odd, and similar techniques were applied to compactly supported perturbations of $\mathbb{H}^{n+1}$ in [Borthwick 2010].

We can state the cleanest result for a hyperbolic surface $(X, g) \cong \mathbb{H}^2 / \Gamma$. Let $\mathcal{R}_g$ denote the associated resonance set (poles of the meromorphic continuation of $(\Delta_g - s(1-s))^{-1}$), with counting function

$$N_g(t) := \#\big\{\zeta \in \mathcal{R}_g : |\zeta - \tfrac{1}{2}| \leq t\big\}.$$

The sharp version of our bound involves a regularization of the counting function,

$$\widetilde{N}_g(a) := \int_0^a \frac{2(N_g(t) - N_g(0))}{t^2}\, dt. \tag{1-1}$$

This type of regularization is standard in the theory of zeros of entire functions, and there is a natural connection to the asymptotics of $N_g(t)$,

$$\widetilde{N}_g(a) \sim Ba^2 \quad \Longleftrightarrow \quad N_g(t) \sim Bt^2;$$

see [Stefanov 2006, Lemma 1]. If we have only the upper bound on $\widetilde{N}_g$, then we lose some sharpness in the estimate of N_g:

$$\widetilde{N}_g(a) \leq Ba^2 \quad \Longrightarrow \quad N_g(t) \leq eBt^2.$$

Theorem 1.1. *Suppose (X, g) is a smooth geometrically finite hyperbolic surface with $\chi(X) < 0$. Let $\ell_1, \ldots, \ell_{n_{\mathrm{f}}}$ denote the diameters of the geodesic boundaries of the funnels of X. The regularized counting function for the resonances of Δ_g satisfies*

$$\frac{\widetilde{N}_g(a)}{a^2} \leq |\chi(X)| + \sum_{j=1}^{n_{\mathrm{f}}} \frac{\ell_j}{4} + o(1). \tag{1-2}$$

We can see that this result is sharp in two extreme cases. For a finite-area hyperbolic surface (that is, with $n_{\mathrm{f}} = 0$), our upper bound agrees with the known asymptotic $N_g(t)/t^2 \sim |\chi(X)|$. On the other hand, for an isolated hyperbolic funnel F_ℓ of boundary length ℓ, under Dirichlet boundary conditions, the resonances form a half lattice. It is easy to see that $N_{F_\ell}(t)/t^2 \sim \ell/4$, so the funnel portion of (1-2) is also sharp.

The restriction to $\chi(X) < 0$ in Theorem 1.1 leaves out just a few cases. The complete (smooth) hyperbolic surfaces for which $\chi(X) \geq 0$ are the hyperbolic plane $\mathbb{H}^2$, the hyperbolic cylinder $C_\ell := \mathbb{H}^2 / \langle z \mapsto e^\ell z \rangle$, and the parabolic cylinder $C_\infty := \mathbb{H}^2 / \langle z \mapsto z + 1 \rangle$. Resonance sets can be computed explicitly in these cases (see [Borthwick 2007, Sections 4–5]), and exact asymptotics for the counting

function are easily obtained:

$$N_{\mathbb{H}^2}(t) \sim t^2, \quad N_{C_\ell}(t) \sim \tfrac{1}{2}\ell t^2, \quad N_{C_\infty}(t) = 1.$$

If we interpret C_ℓ as the union of 2 funnel ends, then (1-2) would also give a sharp estimate for this case.

Using Theorem 1.1 in conjunction with the argument of Guillopé and Zworski [1997] for the lower bound, we can deduce the following:

Corollary 1.2. *For $k \in \mathbb{N}$ there exists a constant c_k such that for any geometrically finite hyperbolic surface (X, g) with $\chi(X) < 0$,*

$$\frac{N_g(t)}{t^2} \geq c_k |\chi(X)| \left(1 + \frac{1}{|\chi(X)|} \sum_{j=1}^{n_{\mathrm{f}}} \frac{\ell_j}{4}\right)^{-2/k} \quad \textit{for } t \geq 1.$$

The constant c_k obtained in this way (see Section 4 for the derivation) is rather ineffective; the point here is just that there exists a lower bound that depends only on $\chi(X)$ and $\{\ell_j\}$.

We will obtain Theorem 1.1 as a consequence of a somewhat more general estimate. Consider a smooth Riemannian surface (X, g), possibly with boundary, that has finitely many ends that are assumed to be of hyperbolic planar, funnel, or cusp type. That is, X admits the decomposition

$$X = K \sqcup Y_1 \sqcup \cdots \sqcup Y_{n_{\mathrm{f}}} \sqcup C_{n_{\mathrm{f}}+1} \sqcup \cdots \sqcup C_{n_{\mathrm{f}}+n_{\mathrm{c}}} \tag{1-3}$$

illustrated in Figure 1, where the core K is a smooth compact manifold with boundary. The metric in K is arbitrary. The Y_j are infinite-area ends: either hyperbolic planar,

$$Y_j \cong [b_j, \infty) \times S^1, \quad g|_{Y_j} = dr^2 + \sinh^2 r \, d\theta^2, \quad \text{where } b_j > 0, \tag{1-4}$$

or hyperbolic funnels,

$$Y_j \cong [b_j, \infty) \times S^1, \quad g|_{Y_j} = dr^2 + \ell_j^2 \cosh^2 r \frac{d\theta^2}{(2\pi)^2}, \quad \text{where } b_j \geq 0 \text{ and } \ell_j > 0. \tag{1-5}$$

The C_j are hyperbolic cusps,

$$C_j \cong [b_j, \infty) \times S^1, \quad g|_{C_j} = dr^2 + e^{-2r} \frac{d\theta^2}{(2\pi)^2}, \quad \text{where } b_j \geq 0. \tag{1-6}$$

The finite-area portion of X consisting of the core plus the cusps is denoted by

$$X_{\mathrm{c}} := K \sqcup C_{n_{\mathrm{f}}+1} \sqcup \cdots \sqcup C_{n_{\mathrm{f}}+n_{\mathrm{c}}}. \tag{1-7}$$

Any geometrically finite hyperbolic surface, with the exception of the parabolic cylinder C_∞, admits a decomposition of the form (1-3). In such surfaces, aside from $\mathbb{H}^2$ itself, only funnel or cusp ends can occur.

We let Δ_g denote the positive Laplacian on (X, g). In general we may consider the operator

$$P := \Delta_g + V,$$
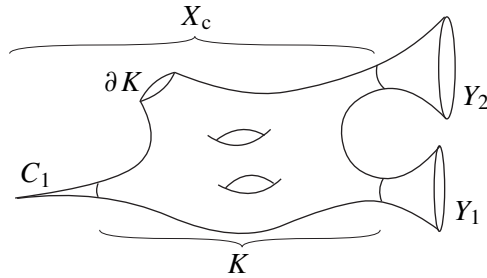
Figure 1. Surface X with boundary and hyperbolic ends.

where $V \in C_0^\infty(X)$ with $\mathrm{supp}(V) \subset K$. We denote by $\mathscr{R}_P$ the resonance set associated to P. These resonances are the poles of the analytically continued resolvent

$$R_P(s) := (P - s(1-s))^{-1},$$

counted according to multiplicity. The associated resonance counting function is

$$N_P(t) := \#\big\{\zeta \in \mathscr{R}_P : |\zeta - \tfrac{1}{2}| \le t\big\}.$$

Our context is essentially that of Guillopé and Zworski [1995; 1997], and so we already know that $N_P(t) \asymp t^2$ (see Section 2 for details). It is thus natural to define the regularized counting function $\widetilde{N}_P(a)$ just as in (1-1).

Before stating the upper bound, we introduce the asymptotic constants associated to the resonance count for isolated hyperbolic planar or funnel ends.

Theorem 1.3. *For a hyperbolic planar or funnel end $Y \cong [b, \infty) \times S^1$, with metric as in (1-4) or (1-5), the resonance counting function for the Laplacian with Dirichlet boundary conditions at $r = b$ satisfies an asymptotic as $t \to \infty$,*

$$N_Y(t) \sim A(Y)t^2.$$

We will write these constants $A(Y)$ explicitly in a moment. First let us state our main result.

Theorem 1.4. *For (X, g) a surface with hyperbolic ends as in (1-3) and $V \in C_0^\infty(X)$, the regularized counting function for $P = \Delta_g + V$ satisfies*

$$\frac{\widetilde{N}_P(a)}{a^2} \le \frac{1}{2\pi} \, \mathrm{vol}\big(X_c, g\big) + \sum_{j=1}^{n_f} A(Y_j) + o(1), \tag{1-8}$$

where X_c is the subset (1-7).

If (X, g) is a finite-area surface with hyperbolic cusp ends (and arbitrary metric in the interior), Parnovski [1995] proved that

$$N_g(t) \sim \frac{1}{2\pi} \, \mathrm{vol}(X, g)t^2.$$

This shows that Theorem 1.4 is sharp in the case $n_f = 0$. It also suggests an intriguing interpretation of the constants appearing in (1-8). Suppose we split X into a disjoint union $X_c \cup Y_1 \cup \cdots \cup Y_{n_f}$ at the boundary of X_c and impose Dirichlet boundary conditions at the newly created boundaries. The constant on the right side of (1-8) is the sum of the asymptotic constants for the resonance counting function of the resulting components.

To obtain Theorem 1.1 from Theorem 1.4, we take the Y_j to be standard funnels with boundaries at $b_j = 0$, in which case $A(Y_j) = \ell_j/4$. And under the assumptions that X_c has geodesic boundary and hyperbolic interior, the Gauss–Bonnet theorem gives $\mathrm{vol}(X_c, g) = -2\pi\chi(X)$.

As in Corollary 1.2, combining Theorem 1.4 with the Guillopé–Zworski argument gives a lower bound on $N_P(t)$ with a constant that depends only on 0-vol(X, g) and the end parameters ℓ_j and b_j for $j = 1, \ldots, n_f$, assuming that 0-vol$(X, g) \neq 0$.

The asymptotic constants $A(Y)$ appearing in Theorem 1.3 have a somewhat complicated form. Consider first a model funnel end F_{ℓ, r_0} defined by

$$F_{\ell, r_0} \cong [r_0, \infty) \times S^1 \quad \text{and} \quad ds^2 = dr^2 + \ell^2 \cosh^2 r \frac{d\theta^2}{(2\pi)^2}. \tag{1-9}$$

The case $r_0 = 0$, a standard funnel with geodesic boundary, is simply denoted by F_ℓ. The resonance set for the Laplacian on F_{ℓ, r_0} with Dirichlet boundary conditions at $r = r_0$ is denoted $\mathcal{R}_{F_{\ell, r_0}}$.

In Section 7 we will show that for $r_0 \geq 0$,

$$A(F_{\ell, r_0}) = -\frac{\ell}{2\pi}\sinh r_0 + \frac{4}{\pi}\int_0^{\pi/2}\int_0^\infty \frac{[I(xe^{i\theta}, \ell, r_0)]_+}{x^3}\, dx\, d\theta, \tag{1-10}$$

where $[\cdot]_+$ denotes the positive part and, with $\omega := 2\pi/\ell$,

$$I(\alpha, \ell, r) := \mathrm{Re}\left(2\alpha\log\left(\frac{\alpha\sinh r + \sqrt{\omega^2 + \alpha^2\cosh^2 r}}{\sqrt{\omega^2 + \alpha^2}}\right)\right)$$
$$+ \omega\arg\left(\frac{\sqrt{\omega^2 + \alpha^2\cosh^2 r} - i\omega\sinh r}{\sqrt{\omega^2 + \alpha^2\cosh^2 r} + i\omega\sinh r}\right) + \pi(\mathrm{Im}\,\alpha - \omega). \tag{1-11}$$

(We will use the principal branch of log in all such formulas.) The integral in (1-10) is explicitly computable in the case $r_0 = 0$, since $I(xe^{i\theta}, \ell, 0) = \pi(x\sin\theta - \omega)$. In this case we recover the asymptotic constant for the standard funnel, $A(F_\ell) = \ell/4$.

It is interesting to compare the resonance sets of truncated funnels F_{ℓ, r_0} with $r_0 > 0$ to extended funnels with $r_0 < 0$. The two cases are quite different in terms of the classical dynamics; an extended funnel contains a trapped geodesic, while truncated funnels are nontrapping. Because of this change in dynamics, we expect the distribution of resonances near the critical line to change dramatically as r_0 switches from positive to negative. Figure 2 illustrates these differences. In the nontrapping case at left, the distance from the resonances to the critical line increases logarithmically as $\mathrm{Im}\,s \to \infty$. For the trapping case at right, the distance decreases exponentially. These behaviors are consistent with results on resonance-free regions for asymptotically hyperbolic manifolds by Guillarmou [2005].
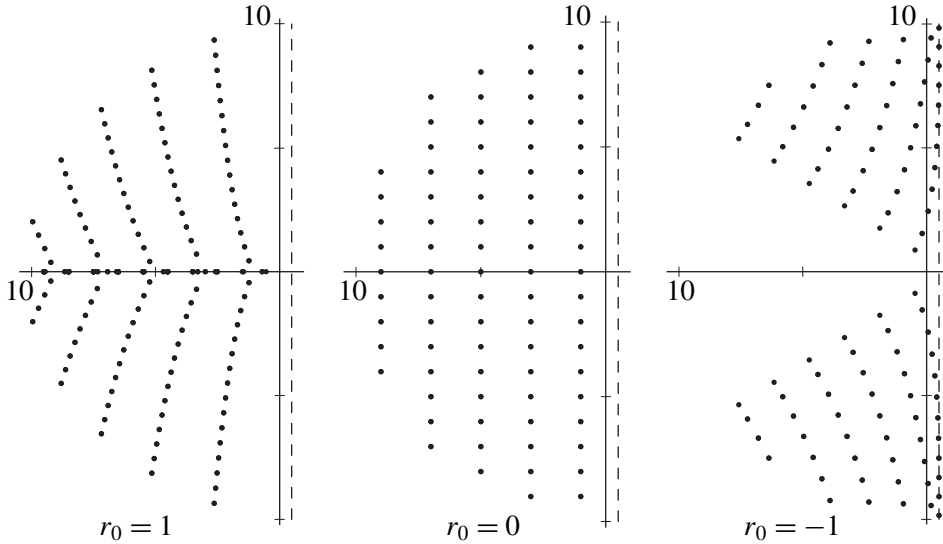
Figure 2. Resonance sets of the funnel F_{ℓ,r_0} with different boundary locations r_0, shown for $\ell = 2\pi$.

Of course, the asymptotics of the global counting function $N_P(t)$ are not expected to be sensitive to the dynamics. Indeed, we will show in Section 8 that the formula (1-10) for the asymptotic constant of $N_{F_{\ell,r_0}}(t)$ remains valid for $r_0 < 0$. This exact asymptotic can be compared to the upper bound obtained for the extended funnel from Theorem 1.4, which is

$$\frac{\widetilde{N}_{F_{\ell,r_0}}(a)}{a^2} \leq -\frac{\ell}{2\pi}\sinh r_0 + \frac{\ell}{4} \quad \text{for } r_0 \leq 0. \tag{1-12}$$
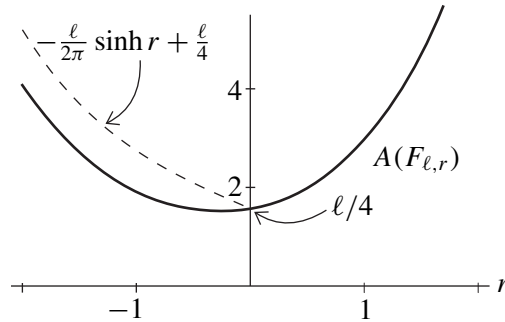
Figure 3 illustrates the difference between the upper bound (1-12) and the sharp asymptotic in this situation. Given this discrepancy, one might think that the bound in Theorem 1.4 could be improved by moving the boundary of K further into the interior of the surface (that is, by allowing $b_j < 0$ in the definition (1-5)). Unfortunately, for reasons that we will explain in Section 4, it does not seem possible to obtain any improvement this way.

In the hyperbolic planar case, the model problem for Y_j is scattering by a spherical obstacle in $\mathbb{H}^2$, that is, on the exterior Dirichlet domain $\Omega_{r_0} := \{r \geq r_0\} \subset \mathbb{H}^2$. The resonance asymptotics for this spherical obstacles in $\mathbb{H}^{n+1}$ were worked out in Borthwick [2010, Theorem 1.2]. In two dimensions the result is

$$A(\Omega_{r_0}) = 2 - \cosh r_0 + \frac{4}{\pi}\int_0^{\pi/2}\int_0^{\infty}\frac{[H(xe^{i\theta},r_0)]_+}{x^3}\,dx\,d\theta, \tag{1-13}$$

where

$$H(\alpha,r) := \operatorname{Re}\left(2\alpha\log\left(\frac{\alpha\cosh r + \sqrt{1+\alpha^2\sinh^2 r}}{\sqrt{\alpha^2-1}}\right)\right) + \log\left|\frac{\cosh r - \sqrt{1+\alpha^2\sinh^2 r}}{\cosh r + \sqrt{1+\alpha^2\sinh^2 r}}\right|. \tag{1-14}$$

Figure 3. The exact asymptotic constant for $F_{\ell,r}$ as a function of boundary location r, shown for $\ell = 2\pi$. The dotted line shows the bound from Theorem 1.4.

The paper is organized as follows. The basic material on the resolvent and resonances of the operator P is reviewed in Section 2. In Section 3 we present the factorization formula for the relative scattering determinant and show that this leads to Weyl asymptotics for the scattering phase and a counting formula for resonances based on contour integration. The growth estimates on the scattering determinant and the resulting proof of Theorem 1.4 are given in Section 4, assuming certain estimates to be developed in later sections. The derivation of Corollary 1.2 is also given in Section 4. In Section 5, we develop the asymptotic analysis of Dirichlet eigenmodes on hyperbolic funnels. These asymptotics are applied in Section 6 to prove the Poisson operator estimates needed for Section 4. Finally, in Section 7 and Section 8 we establish the exact asymptotic constant (1-10) for the truncated and extended funnel cases, respectively, and prove the funnel part of Theorem 1.3 in particular.

2. Resonances

The context introduced in Section 1 differs from that of Guillopé and Zworski [1995; 1997] in two relatively minor ways: Hyperbolic planar ends are allowed in addition to funnels, and a compactly supported potential V is possibly added to Δ_g. The latter addition really is trivial, but the inclusion of hyperbolic planar ends requires a few extra estimates on model terms. In this section we will briefly review the theory [Guillopé and Zworski 1995; 1997], in order to explain those additional estimates.

To define resonances we need analytic continuation of the resolvent, $R_P(s) := (P - s(1-s))^{-1}$, from its original domain $\operatorname{Re} s > \frac{1}{2}$. Each end Y_j is isometric to a portion of either $\mathbb{H}$ or the model funnel F_{ℓ_j}, and we can use this identification to pullback model resolvents $R^0_{Y_j}(s)$. After appropriate cutoffs are applied, we can treat these model terms as operators on X, whose kernels have support only in the corresponding ends Y_j. Similarly, we define $R^0_{C_j}(s)$ by pullback from the model cusp. Suppose that $\chi^j_k \in C^\infty(X)$ are cutoff functions for $j = 1, \ldots, n_{\mathrm{f}} + n_{\mathrm{c}}$ and $k = 0, 1, 2$, such that

$$\chi^j_k = \begin{cases} 0 & \text{for } r \geq k+1 \text{ in end } j, \\ 1 & \text{for } r \leq k \text{ in end } j, \\ 1 & \text{outside of end } j. \end{cases}$$

We also set $\chi_k := \prod_j \chi^j_k$.

For some s_0 with Re s_0 sufficiently large, so that $R_P(s_0)$ is defined, we set

$$M(s) := \chi_2 R_P(s_0)\chi_1 + \sum_{j=1}^{n_{\mathrm{f}}}(1 - \chi_0^j)R_{Y_j}^0(s)(1 - \chi_1^j) + \sum_{j=n_{\mathrm{f}}+1}^{n_{\mathrm{f}}+n_{\mathrm{c}}}(1 - \chi_0^j)R_{C_j}^0(s)(1 - \chi_1^j).$$

This parametrix satisfies

$$(P - s(1 - s))M(s) = I - L(s),$$

where

$$L(s) := -[\Delta_g, \chi_2]R_P(s_0)\chi_1 + (s(1 - s) - s_0(1 - s_0))\chi_2 R_P(s_0)\chi_1$$

$$+ \sum_{j=1}^{n_{\mathrm{f}}}[\Delta_g, \chi_0^j]R_{Y_j}^0(s)(1 - \chi_1^j) + \sum_{j=n_{\mathrm{f}}+1}^{n_{\mathrm{f}}+n_{\mathrm{c}}}[\Delta_g, \chi_0^j]R_{C_j}^0(s)(1 - \chi_1^j).$$

There are two differences here from the construction of [Guillopé and Zworski 1995]. First of all, some of our model terms $R_{Y_j}^0(s)$ will be copies of $R_{\mathbb{H}}(s)$ instead of the funnel resolvent. Second, we follow the treatment in [Borthwick 2007] in using the model resolvent for a full cusp, rather than modifying the original Hilbert space.

Let $\rho \in C^\infty(X)$ be proportional to e^{-r} in the ends Y_j and C_j, with respect to the coordinate systems given in (1-4)–(1-6). The operator $L(s)$ is compact on $\rho^N L^2(X, dg)$ for Re $s > \frac{1}{2} - N$ and defines a meromorphic family with poles of finite rank. (The structure of the kernel of $R_{Y_j}^0(s)$ at infinity is the same whether Y_j is a funnel or hyperbolic planar, so this part of the argument is unaffected by the addition of hyperbolic planar ends.)

By choosing s and s_0 appropriately we can insure that $I - L(s)$ is invertible at some s, and then the analytic Fredholm yields

$$R_P(s) = M(s)(I - L(s))^{-1}. \tag{2-1}$$

This proves the following result, a slight generalization of [Guillopé and Zworski 1995, Theorem 1]:

Theorem 2.1 (Guillopé and Zworski). *The formula* (2-1) *defines a meromorphic extension of $R_P(s)$ to a bounded operator on $\rho^N L^2(X, dg)$ for* Re $s > \frac{1}{2} - N$, *with poles of finite rank.*

Meromorphic continuation allows us to define $\mathcal{R}_P$ as the set of poles of $R_P(s)$, listed according to multiplicities given by

$$m_P(\zeta) := \operatorname{rank} \operatorname{Res}_\zeta R_P(s).$$

The same parametrix construction also leads to an estimate of the order of growth of the resonance counting function. The following is a slight generalization of [Guillopé and Zworski 1995, Theorem 2]:

Theorem 2.2 (Guillopé and Zworski). *The resonance counting function satisfies a bound*

$$N_P(t) = O(t^2).$$

Our version requires just a few additional estimates. To obtain this bound on the counting function, Guillopé and Zworski [1995] introduced a Fredholm determinant

$$D(s) := \det(I - L_3(s)^3), \quad \text{where } L_3(s) := L(s)\chi_3.$$

Using the relation

$$R_P(s)\chi_3 = M(s)\chi_3(I + L_3(s) + L_3(s)^2)(I - L_3(s)^3)^{-1},$$

and a result of Vodev [1994, Appendix], they showed that $\mathfrak{R}_P$ is included in the union of the set of poles of $D(s)$ with 3 copies of the union of the sets of poles of $M(s)$ and $L_3(s)$.

The only change that the inclusion of hyperbolic planar ends requires in this argument is that for each hyperbolic planar end we include a copy of $\mathfrak{R}_{\mathbb{H}}$ among the possible poles of $M(s)$ and $L_3(s)$. Since $N_{\mathbb{H}}(t) = O(t^2)$, just as for funnels, the problem reduces as in [Guillopé and Zworski 1995] to an estimate of the growth of $D(s)$. Through Weyl's inequality, the estimate of $D(s)$ is broken up into estimates on the singular values of various model terms. We must check that the relevant estimates are satisfied by the hyperbolic planar model terms.

There are three estimates to consider. The first concerns the resolvent $R_{\mathbb{H}}(s)$. If Q_1 and Q_2 are compactly supported differential operators of orders q_1 and q_2, with disjoint supports, then for $\varepsilon > 0$,

$$\|Q_1 R_{\mathbb{H}}(s) Q_2\| \leq C(q_j, \varepsilon)\langle s \rangle^{q_1+q_2} \quad \text{for } \operatorname{Re} s > \varepsilon, \tag{2-2}$$

and

$$\|Q_1 R_{\mathbb{H}}(s) Q_2\| \leq C(q_j, \varepsilon)\langle s \rangle^{q_1+q_2-1} \quad \text{for } \operatorname{Re} s > \tfrac{1}{2} + \varepsilon. \tag{2-3}$$

To prove either of these, one can simply use the explicit formula

$$R_{\mathbb{H}}(s; z, z') = \frac{1}{4\pi} \int_0^1 \frac{(t(1-t))^{s-1}}{(t + \sinh^2 d(z, z'))^s} \, dt,$$

and repeat the argument from [Guillopé and Zworski 1995, Lemma 3.2].

The next estimate is for the Poisson kernel $E_{\mathbb{H}}(s)$. In the Poincaré ball model $\mathbb{B}$, this kernel is given by

$$E_{\mathbb{B}}(s; z, \theta) = \frac{1}{4\pi} \frac{\Gamma(s)^2}{\Gamma(2s)} \frac{(1 - |z|^2)^s}{|e^{i\theta} - z|^{2s}} \quad \text{for } z \in \mathbb{B}, \ \theta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Given a compact set $K \subset \mathbb{B}$ and $\varepsilon > 0$, we have

$$\left| \partial_\theta^k E_{\mathbb{B}}(s; z, \theta) \right| \leq C(K, \varepsilon)^k k! \, e^{c\langle s \rangle} \quad \text{for } z \in K, \ k \in \mathbb{N}. \tag{2-4}$$

This is not difficult to prove directly by induction, or one can use an analyticity argument as in [Guillopé and Zworski 1995, Lemma 3.1].

Finally, we must estimate the scattering matrix $S_{\mathbb{H}}(s)$. We can write this explicitly in terms of Fourier modes,

$$S_{\mathbb{H}}(s) = \sum_{k \in \mathbb{Z}} [S_{\mathbb{H}}(s)]_k e^{ik(\theta - \theta')}, \quad \text{where } [S_{\mathbb{H}}(s)]_k = 2^{1-2s} \frac{\Gamma(\tfrac{1}{2} - s)}{\Gamma(s - \tfrac{1}{2})} \frac{\Gamma(s + |k|)}{\Gamma(1 - s + |k|)}$$

Using Stirling's formula, it is easy to use this expression for the eigenvalues to estimate the singular values of $S_{\mathbb{H}}(s)$. Assuming that $\operatorname{Re} s < \frac{1}{2} - \varepsilon$ and $\operatorname{dist}(s, -\mathbb{N}_0) > \eta$, we have

$$\mu_j(S_{\mathbb{H}}(s)) \le \exp(C(\eta)\langle s\rangle + \operatorname{Re}(1-2s)\log(\langle s\rangle/j)). \tag{2-5}$$

This is the analog of [Guillopé and Zworski 1997, Lemma 4.2].

With these model estimates in place, one can simply apply Guillopé and Zworski's original argument (treating the cusp contributions as in [Borthwick 2007, Section 9.4]) to prove that

$$|g(s)D(s)| \le e^{C\langle s\rangle^2},$$

where $g(s)$ is a entire function of order 2 and finite type, with zeros derived from $\mathcal{R}_{\mathbb{H}}$ and the model resolvent sets for the funnels and cusps. This yields the proof of Theorem 2.2.

3. Relative scattering determinant

To define scattering matrices, we will fix a function $\rho \in C^\infty(X)$ that serves as a boundary defining function for a suitable compactification of X. We start with smooth positive functions ρ_{f}, ρ_{c} satisfying

$$\rho_{\mathrm{f}} = \begin{cases} 2e^{-r} & \text{in each } Y_j, \\ 1 & \text{in each } C_j \end{cases} \quad \text{and} \quad \rho_{\mathrm{c}} = \begin{cases} 1 & \text{in each } Y_j, \\ e^{-r} & \text{in each } C_j. \end{cases}$$

Then we set $\rho = \rho_{\mathrm{f}}\rho_{\mathrm{c}}$ for the global boundary defining function.

The ends Y_j are conformally compact, and we distinguish between the internal boundary ∂Y_j, and the boundary at infinity $\partial_\infty Y_j$ induced by the conformal compactification. The funnel ends Y_j come equipped with a length parameter ℓ_j, the length of the closed geodesic bounding the finite end. If we assign length $\ell_j = 2\pi$ to a hyperbolic planar end, for consistency, then the metric induced by $\rho^2 g$ on the boundary of Y_j at infinity gives an isometry

$$\partial_\infty Y_j \cong \mathbb{R}/\ell_j\mathbb{Z}.$$

The cusp ends can be compactified naturally by lifting to $\mathbb{H}$ and invoking the Riemann-sphere topology, as described in [Borthwick 2007, Section 6.1]. The resulting boundary $\partial_\infty C_j$ consists of a single point.

Despite the discrepancy in dimensions, it will be convenient to group all of the infinite boundaries together as

$$\partial_\infty X := \partial_\infty Y_1 \cup \cdots \cup \partial_\infty Y_{n_{\mathrm{f}}} \cup \partial_\infty C_{n_{\mathrm{f}}+1} \cup \cdots \cup \partial_\infty C_{n_{\mathrm{f}}+n_{\mathrm{c}}}.$$

Then we have

$$C^\infty(\partial_\infty X) := C^\infty(\mathbb{R}/\ell_1\mathbb{Z}) \oplus \cdots \oplus C^\infty(\mathbb{R}/\ell_{n_{\mathrm{f}}}\mathbb{Z}) \oplus \mathbb{C}^{n_{\mathrm{c}}},$$

and similarly for $L^2(\partial_\infty X)$.

In Section 2, $R^0_{Y_j}(s)$ denoted the pullback of the model resolvent in the parametrix construction. Carrying on with this notation, we also define the model Poisson operators

$$E^0_{Y_j}(s) : C^\infty(\partial_\infty Y_j) \to L^2(Y_j),$$

and scattering matrices

$$S^0_{Y_j}(s) : C^\infty(\partial_\infty Y_j) \to C^\infty(\partial_\infty Y_j).$$

Similarly, for the cusp ends we have the Poisson kernels

$$E^0_{C_j}(s) : \mathbb{C} \to L^2(C_j).$$

There is no analog of the model scattering matrix for a cusp; see [Borthwick 2007, Section 7.5] for an explanation of this.

The scattering matrix $S_P(s)$ is defined as a map on $C^\infty(\partial_\infty X)$, which we can write as

$$S_P(s) = \begin{pmatrix} S^{\mathrm{ff}}(s) & S^{\mathrm{fc}}(s) \\ S^{\mathrm{cf}}(s) & S^{\mathrm{cc}}(s) \end{pmatrix}, \tag{3-1}$$

where the blocks are split between the "funnel-type" ends Y_j and the cusps C_j. The block $S^{\mathrm{ff}}(s)$ is a matrix of pseudodifferential operators; all other blocks have finite rank. To define a scattering determinant, we normalize using the background operator

$$S_0(s) = \begin{pmatrix} S^0_Y(s) & 0 \\ 0 & I \end{pmatrix}, \quad \text{where } S^0_Y(s) = S^0_{Y_1}(s) \oplus \cdots \oplus S^0_{Y_{n_{\mathrm{f}}}}(s).$$

The relative scattering determinant is then defined by

$$\tau(s) = \det S_P(s) S_0(s)^{-1}. \tag{3-2}$$

The poles of the background scattering matrix $S_0(s)$ define a background resonance set

$$\mathcal{R}_0 = \bigcup_{j=1}^{n_{\mathrm{f}}} \begin{cases} \mathcal{R}_{F_{\ell_j}} & \text{for a funnel end,} \\ \mathcal{R}_{\mathbb{H}} & \text{for a hyperbolic planar end.} \end{cases} \tag{3-3}$$

For $* = 0$ or P let $H_*(s)$ denote the Hadamard product over $\mathcal{R}_*$,

$$H_*(s) := \prod_{\zeta \in \mathcal{R}_*} (1 - s/\zeta) e^{s/\zeta + s^2/(2\zeta^2)}.$$

Theorem 2.2 implies that the product for $H_P(s)$ converges, and for $H_0(s)$ this is clear from the definition of $\mathcal{R}_0$.

Proposition 3.1. *For $P = \Delta_g + V$, the relative scattering determinant admits a factorization*

$$\tau(s) = e^{q(s)} \frac{H_P(1-s)}{H_P(s)} \frac{H_0(s)}{H_0(1-s)},$$

where $q(s)$ is a polynomial of degree at most 2.

Proof. If the ends Y_j are all hyperbolic funnels, then Guillopé and Zworski [1997, Proposition 3.7] proved the factorization formula of with $q(s)$ a polynomial of degree at most 4. The first part of the proof, the characterization of the divisor of $\tau(s)$ obtained in [Guillopé and Zworski 1997, Proposition 2.14], remains valid if hyperbolic planar ends are included.

To extend the more difficult part of the argument, which is the estimate that shows $q(s)$ is polynomial, we require only the extra estimates on model terms given in (2-2), (2-3), (2-4), and (2-5). With these estimates one can easily extend the proof of [Guillopé and Zworski 1997, Proposition 3.7]. We refer the reader also to [Borthwick 2007, Section 10.5], for an expository treatment of these details.

To see that the maximal order of $q(s)$ is 2, we could prove an estimate analogous to [Borthwick 2008, Lemma 5.2]. However, we will be proving a sharper version of this estimate later in this paper. From (4-12) in the proof of Theorem 4.1, it will follow that for some sequence $a_i \to \infty$,

$$\log|\tau(s)| \le O(a_i^2) \quad \text{for } |s - \tfrac{1}{2}| = a_i, \; |\arg(s - \tfrac{1}{2})| \le \tfrac{1}{2}\pi - \delta.$$

Because the Hadamard products $H_*(s)$ have order 2, this implies a bound $|q(s)| = O(|s|^{2+\varepsilon})$ in the sector $|\arg(s - \tfrac{1}{2})| \le \tfrac{1}{2}\pi - \delta$. Hence $q(s)$ has degree at most 2, since it is already known to be polynomial. (The derivations leading to (4-12) require only that $q(s)$ is polynomial, so this argument is not circular.) $\qquad\square$

To apply the factorization of $\tau(s)$ to resonance counting we introduce the relative scattering phase of P, defined as

$$\sigma(\xi) := \frac{i}{2\pi} \log \tau(\tfrac{1}{2} + i\xi), \tag{3-4}$$

with branches of the log chosen so that $\sigma(\xi)$ is continuous and $\sigma(0) = 0$. By the properties of the relative scattering matrix, $\sigma(\xi)$ is real and $\sigma(-\xi) = -\sigma(\xi)$.

To state the relative counting formula, we let N_0 denote the counting function associated to $\mathscr{R}_0$,

$$N_0(t) := \#\big\{\zeta \in \mathscr{R}_0 : |\zeta - \tfrac{1}{2}| \le t\big\},$$

and $\widetilde{N}_0(a)$ the corresponding regularized counting function.

Corollary 3.2. *As $a \to \infty$,*

$$\widetilde{N}_P(a) - \widetilde{N}_0(a) = 4 \int_0^a \frac{\sigma(t)}{t}\, dt + \frac{2}{\pi} \int_0^{\pi/2} \log|\tau(\tfrac{1}{2} + ae^{i\theta})|\, d\theta + O(\log a). \tag{3-5}$$

The proof is by contour integration of $\tau'/\tau(s)$ around a half-circle centered at $s = \tfrac{1}{2}$. See [Borthwick 2010, Proposition 3.2] for the details of the derivation of (3-5) from Proposition 3.1. This is the analog of a formula developed by Froese [1998] for Schrödinger operators in the Euclidean setting.

The other consequence we need from Proposition 3.1 is essentially also already proven. To analyze the first term on the right side of (3-5), we will invoke the Weyl-type asymptotics satisfied by the scattering phase:

Theorem 3.3 (Guillopé and Zworski). *As $\xi \to +\infty$,*

$$\sigma(\xi) = \Big(\frac{1}{4\pi} \, 0\text{-vol}(X, g) - \frac{n_{\mathrm{hp}}}{2}\Big) \xi^2 - \frac{n_{\mathrm{c}}}{\pi} \xi \log \xi + O(\xi),$$

where n_{hp} denotes the number of the Y_j that are hyperbolic planar.

For surfaces with hyperbolic funnel or cusp ends, this result was established by Guillopé and Zworski [1997, Theorem 1.5]. As in the other cases discussed above, the modifications needed to adapt the proof to our slightly more general setting are fairly simple. The first point is that the addition of a compactly

supported potential V does not change the argument at all, since it does not affect the leading term in the wave trace asymptotics as derived in [Guillopé and Zworski 1997, Lemma 6.2]. The second issue is that we allow hyperbolic planar ends in addition to funnels. However, for $|t| < \ell$ the restriction to the diagonal of the wave kernel on a model funnel F_ℓ is identical to that of $\mathbb{H}^2$. This is the content of [Guillopé and Zworski 1997, Equation (6.1)]. So hyperbolic planar ends may also be included without modifying the argument. Such ends do affect the final calculation, however, because $0\text{-vol}(\mathbb{H}^2) = -2\pi$ whereas the model funnels had $0\text{-vol}(F_\ell) = 0$. This difference accounts for the n_{hp} term.

4. Scattering determinant asymptotics

To state the asymptotic estimate for the scattering determinant contribution to the resonance counting formula (3-5), we introduce the following constants. If Y_j is a funnel with parameters ℓ_j, b_j, then we set

$$B(Y_j) := \frac{4}{\pi} \int_0^{\pi/2} \int_0^\infty \frac{[I(xe^{i\theta}, \ell_j, b_j)]_+}{x^3} \, dx \, d\theta - \frac{\ell_j}{4},$$

where $I(\alpha, \ell, r)$ was defined in (1-11). If Y_j is a hyperbolic planar end with parameter b_j, then

$$B(Y_j) := \frac{4}{\pi} \int_0^{\pi/2} \int_0^\infty \frac{[H(xe^{i\theta}, b_j)]_+}{x^3} \, dx \, d\theta,$$

where $H(\alpha, \ell, r)$ was defined in (1-14). The cusps do not contribute to the asymptotics of $\tau(s)$ to leading order, so we make no analogous definition for C_j.

Theorem 4.1. *For (X, g) a surface with hyperbolic ends as in* (1-3), *there exists an unbounded set $\Lambda \subset [1, \infty)$ such that*

$$\frac{2}{\pi} \int_0^{\pi/2} \log|\tau(\tfrac{1}{2} + ae^{i\theta})| \, d\theta \le \sum_{j=1}^{n_{\mathrm{f}}} B(Y_j)a^2 + o(a^2) \quad \text{for all } a \in \Lambda.$$

Before undertaking the proof of Theorem 4.1, we will show how this theorem leads to the proof of the main result stated in Section 1:

Proof of Theorem 1.4. Starting from the counting formula from Corollary 3.2, we apply Theorem 3.3 to the scattering phase term and Theorem 4.1 to the scattering determinant contribution. This yields

$$\widetilde{N}_P(a) \le \widetilde{N}_0(a) + \frac{1}{2\pi} 0\text{-vol}(X, g)a^2 + \sum_{j=1}^{n_{\mathrm{f}}} B(Y_j)a^2 + o(a^2), \tag{4-1}$$

as $a \to \infty$. From the explicit definition (3-3) of $\mathcal{R}_0$, we see that

$$\frac{N_0(t)}{t^2} \sim \sum_{j=1}^{n_{\mathrm{f}}} \begin{cases} 1 & \text{for a hyperbolic planar end,} \\ \ell_j/4 & \text{for a funnel end,} \end{cases}$$

and so $\widetilde{N}_0(a)$ satisfies the same asymptotic. Also, we have

$$0\text{-vol}(X, g) = \text{vol}(X_c, g) + \sum_{j=1}^{n_f} 0\text{-vol}(Y_j, g).$$

The 0-volumes of the Y_j are easily computed. For a hyperbolic planar end,

$$0\text{-vol}(Y_j, g) = 2\pi \operatorname*{FP}_{\varepsilon \to 0} \int_{b_j}^{\log(2/\varepsilon)} \sinh r \, dr = -2\pi \cosh b_j,$$

and for a funnel end,

$$0\text{-vol}(Y_j, g) = \ell_j \operatorname*{FP}_{\varepsilon \to 0} \int_{b_j}^{\log(2/\varepsilon)} \cosh r \, dr = -\ell_j \sinh b_j.$$

By the formulas (1-13) and (1-10) for $A(Y_j)$, we see that (4-1) is equivalent to the claimed estimate. $\square$

The derivation of Theorem 1.1 from Theorem 1.4 was already explained in Section 1. To prove Corollary 1.2 we simply recall a few details of the proof of the lower bound in Guillopé and Zworski [1997, Theorem 1.3]. For a test function $\phi \in C_0^\infty(\mathbb{R}_+)$ with $\phi \geq 0$ and $\phi(1) > 0$, we have estimates

$$|\hat{\phi}(\xi)| \leq C_k(1 + |\xi|)^{-k-2} \quad \text{for } k \in \mathbb{N} \text{ and } \operatorname{Im} \xi \leq 0.$$

Pairing the distributional Poisson formula [Guillopé and Zworski 1997, Theorem 5.7] with $\lambda\phi(\lambda \cdot)$ yields

$$|0\text{-vol}(X, g)| \lambda^2 \leq C_k \int_0^\infty (1+r)^{-k-3} N_P(\lambda r) \, dr.$$

If we have $N_P(t) \leq At^2$ for $t \geq 1$, then splitting the integral at a gives

$$|0\text{-vol}(X, g)| \lambda^2 \leq C_k \big(N(\lambda a) + A\lambda^2 a^{-k}\big).$$

Setting $t = \lambda a$, we have

$$N(t) \geq \big(c_k |0\text{-vol}(X, g)| a^{-2} - Aa^{-2-k}\big) t^2,$$

and optimizing with respect to a then yields

$$N(t) \geq c_k |0\text{-vol}(X, g)|^{1+k/2} A^{-k/2}.$$

Corollary 1.2 is then proven by substituting the constant obtained in Theorem 1.1 for A.

The rest of this section is devoted to the proof of Theorem 4.1. To produce a formula convenient for estimation, we introduce cutoff functions as follows. Fix some $\eta \in (0, 1)$. For $j = 1, \ldots, n_f + n_c$ and $k = 1, 2$, we define $\chi_k^j \in C^\infty(X)$ so that $\chi_k^j = 1$ outside the j-th end (Y_j or C_j), and inside the j-th end we have

$$\chi_k^j = \begin{cases} 0 & \text{for } r \geq b_j + (k+1)\eta, \\ 1 & \text{for } r \leq b_j + k\eta. \end{cases} \tag{4-2}$$

Note that $\chi_2^j = 1$ on the support of χ_1^j, as illustrated in Figure 4.
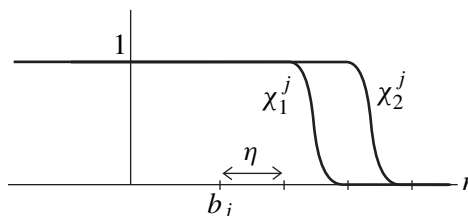
Figure 4. The cutoff functions χ_k^j in the j-th end.

Proposition 4.2. *With cutoffs defined as in* (4-2), *we have*

$$S_X(s)S_0(s)^{-1} = I + Q(s),$$

where the components of $Q(s)$, in terms of the block decomposition introduced in (3-1), *are*

$$Q_{ij}^{\mathrm{ff}}(s) = (2s-1)E_{Y_i}^0(s)^t[\Delta_{Y_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(1-s),$$

$$Q_{ij}^{\mathrm{cf}}(s) = (2s-1)E_{C_i}^0(s)^t[\Delta_{C_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(1-s),$$

$$Q_{ij}^{\mathrm{fc}}(s) = -(2s-1)E_{Y_i}^0(s)^t[\Delta_{Y_i}, \chi_2^i]R_P(s)[\Delta_{C_j}, \chi_1^j]E_{C_j}^0(s),$$

$$Q_{ij}^{\mathrm{cf}}(s) = -(2s-1)E_{C_i}^0(s)^t[\Delta_{C_i}, \chi_2^i]R_P(s)[\Delta_{C_j}, \chi_1^j]E_{C_j}^0(s),$$

Proof. One can characterize the scattering matrix $S_X(s)$ through the boundary behavior of solutions of $(\Delta_g - s(1-s))u = 0$. For $\psi \in C^\infty(\partial_\infty X)$ and $\mathrm{Re}\, s \geq \frac{1}{2}$, with $s \neq \mathbb{N}/2$, there is a unique generalized eigenfunction $u \in C^\infty(X)$ with the asymptotic behavior

$$u \sim \rho_{\mathrm{f}}^{1-s}\rho_{\mathrm{c}}^{-s}\psi + \rho_{\mathrm{f}}^s\rho_{\mathrm{c}}^{s-1}S_X(s)\psi. \tag{4-3}$$

For hyperbolic surfaces with cusps, a proof is given in Borthwick [2007, Proposition 7.13]. The essential analysis takes place in the ends, so including smooth metric or potential perturbations within K requires only trivial modifications to the proof. Likewise, hyperbolic planar ends may be included without much change to the argument.

Suppose $f_j \in C^\infty(\partial_\infty Y_j)$. Then we can use the model Poisson kernel $E_{Y_j}^0(s)$ to create a partial solution $(1 - \chi_1^j)E_{Y_j}^0(s)f_j$ supported in Y_j. As $\rho \to 0$ in Y_j, this function has the asymptotic behavior

$$(1 - \chi_1^j)E_{Y_j}^0(s)f_j \sim \frac{1}{2s-1}(\rho_{\mathrm{f}}^{1-s}f_j + \rho_{\mathrm{f}}^s S_{Y_j}^0(s)f_j). \tag{4-4}$$

To create a full solution, we will take the ansatz

$$u = (1 - \chi_1^j)E_{Y_j}^0(s)f_j + u'$$

and then solve $(\Delta_g - s(1-s))u = 0$ for u' by applying the resolvent. The result is

$$u' = R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s)f_j.$$

In the end Y_i, we can use the fact that $(1 - \chi_2^i)[\Delta_{Y_j}, \chi_1^j] = 0$ to deduce

$$(\Delta_{Y_i} - s(1-s))(1 - \chi_2^i)u' = -[\Delta_{Y_i}, \chi_2^i]u',$$

and hence that

$$(1 - \chi_2^i)u' = -R_{Y_i}^0(s)[\Delta_{Y_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s)f_j.$$

This gives the asymptotic behavior in Y_i:

$$u' \sim -\rho_{\rm f}^s E_{Y_i}^0(s)^t[\Delta_{Y_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s)f_j. \qquad (4\text{-}5)$$

By comparing the asymptotics (4-4) and (4-5) to the general form (4-3), we see that

$$S_{ij}^{\rm ff}(s) = \delta_{ij}S_{Y_j}^0(s) - (2s-1)E_{Y_i}^0(s)^t[\Delta_{Y_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s)$$

We then obtain $Q_{ij}^{\rm ff}(s)$ by noting that

$$E_{Y_j}^0(s)S_{Y_j}^0(s)^{-1} = -E_{Y_j}^0(1-s).$$

To find $Q_{ij}^{\rm cf}(s)$ we use the same setup starting from $f_j \in C^\infty(\partial_\infty Y_j)$, but then analyze u' by restricting to the cusp end C_i. This yields

$$(1 - \chi_2^i)u' = -R_{C_i}^0(s)[\Delta_{C_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s)f_j.$$

The asymptotic behavior in C_i is given by

$$(1 - \chi_2^i)u' \sim -\rho^{s-1}E_{C_i}^0(s)^t[\Delta_{C_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s)f_j,$$

so that

$$S_{ij}^{\rm cf}(s) = -(2s-1)E_{C_i}^0(s)^t[\Delta_{C_i}, \chi_2^i]R_P(s)[\Delta_{Y_j}, \chi_1^j]E_{Y_j}^0(s).$$

Next take $a_j \in C^\infty(\partial_\infty C_j) = \mathbb{C}$. Since $E_{C_j}^0(s; r) = \rho_{\rm c}^{-s}/(2s-1)$, our ansatz for a generalized eigenfunction satisfying (4-3) starts from

$$(1 - \chi_1^j)E_{C_j}^0(s)a_j \sim \frac{1}{2s-1}\rho_{\rm c}^{-s}a_j.$$

The corresponding generalized eigenfunction is

$$u = (1 - \chi_1^j)E_{C_j}^0(s)a_j + u', \quad \text{where } u' = R_P(s)[\Delta_{C_\infty}, \chi_1^j]E_{C_j}^0(s)a_j.$$

arguing as above, we find that

$$u' \sim -\rho_{\rm f}^s E_{Y_i}^0(s)^t[\Delta_{Y_i}, \chi_2^i]R_P(s)[\Delta_{C_\infty}, \chi_1^j]E_{C_j}^0(s)a_j$$

in the funnel Y_i, and

$$u' \sim -\rho_{\rm c}^{1-s}E_{C_i}^0(s)^t[\Delta_{C_i}, \chi_2^i]R_P(s)[\Delta_{C_j}, \chi_1^j]E_{C_j}^0(s)a_j$$

in the cusp C_i. We can then read off the matrix elements, $S_{ij}^{\rm fc}(s)$ and $S_{ij}^{\rm cc}(s)$, as above. $\square$

In conjunction with the cutoffs defined in (4-2), we introduce projections $\mathbb{1}_k^j$ on $L^2(X, dg)$, where

$$\mathbb{1}_k^j f = \begin{cases} f & \text{for } r \in [b_j + k\eta, b_j + (k+1)\eta] \text{ in end } j, \\ 0 & \text{otherwise.} \end{cases} \tag{4-6}$$

As with the cutoffs, these projections depend on b_j and also on the choice of $\eta > 0$. We then introduce operators on $L^2(X, dg)$ given by

$$G_j(s) := (2s - 1)\mathbb{1}_1^j E_{Y_j}^0(1 - s)E_{Y_j}^0(s)^t \mathbb{1}_2^j \quad \text{for } j = 1, \ldots, n_{\mathrm{f}}, \tag{4-7}$$

$$G_j(s) := -(2s - 1)\mathbb{1}_1^j E_{C_j}^0(s)E_{C_j}^0(s)^t \mathbb{1}_2^j \quad \text{for } j = n_{\mathrm{f}} + 1, \ldots, n_{\mathrm{f}} + n_{\mathrm{c}}. \tag{4-8}$$

Proposition 4.3. *The relative scattering phase is bounded by*

$$\log|\tau(s)| \le \sum_{j=1}^{n_{\mathrm{f}}+n_{\mathrm{c}}} \log \det(I + C(\eta, \varepsilon)|G_j(s)|)$$

for $\operatorname{Re} s \ge \frac{1}{2}$ *with* $\operatorname{dist}(s(1 - s), \sigma(P)) \ge \varepsilon$.

Proof. In the formula for the relative scattering matrix given in Proposition 4.2, we can write $Q(s)$ as the composition of three operators,

$$Q(s) : L^2(\partial_\infty X) \xrightarrow{Q_3} L^2(X, dg) \xrightarrow{Q_2} L^2(X, dg) \xrightarrow{Q_1} L^2(\partial_\infty X),$$

where

$$Q_1 := \sum_{j=1}^{n_{\mathrm{f}}} E_{Y_j}^0(s)^t \mathbb{1}_2^j + \sum_{j=n_{\mathrm{f}}+1}^{n_{\mathrm{f}}+n_{\mathrm{c}}} E_{C_j}^0(s)^t \mathbb{1}_2^j, \quad Q_2 := \sum_{i,j=1}^{n_{\mathrm{f}}+n_{\mathrm{c}}} [\Delta_g, \chi_2^i] R_P(s)[\Delta_g, \chi_1^j],$$

$$Q_3\big|_{L^2(\partial_\infty Y_j)} := \mathbb{1}_1^j E_{Y_j}^0(1 - s), \quad Q_3\big|_{L^2(\partial_\infty C_j)} := \mathbb{1}_1^j E_{C_j}^0(s).$$

By the cyclicity of the trace,

$$\tau(s) = \det(I + Q(s)) = \det(I + Q_2 \circ Q_3 \circ Q_1).$$

Under the assumptions $\operatorname{Re} s \ge \frac{1}{2}$ with $\operatorname{dist}(s(1 - s), \sigma(P)) \ge \varepsilon$, we can apply the spectral theorem and standard elliptic estimates to prove that $\|Q_2\| \le C(\eta, \varepsilon)$. By the Weyl estimate this then gives

$$|\tau(s)| \le \prod_{j=1}^{\infty}(1 + C(\eta, \varepsilon)\mu_j(Q_3 \circ Q_1)) = \det(1 + C(\eta, \varepsilon)|Q_3 \circ Q_1|).$$

The result follows immediately from

$$Q_3 \circ Q_1 = G_1 \oplus \cdots \oplus G_{n_{\mathrm{f}}+n_{\mathrm{c}}},$$

where the $G_j(s)$ are given by (4-7) and (4-8). $\square$

The right side of the estimate from Proposition 4.3 is always positive. It is therefore impossible to obtain a sharp estimate by this approach in cases where the leading asymptotic behavior of $\log|\tau(s)|$ is

negative. The extended funnel, whose resonance asymptotics are studied in Section 8, gives an example of this situation.

Proof of Theorem 4.1. Let $\mathfrak{R}_0$ be the background resonance set as defined in (3-3). To avoid poles, we will restrict our attention to radii in the set

$$\Lambda := \left\{ a \geq 1 : \mathrm{dist}\big(\{|s - \tfrac{1}{2}| = a\}, \mathfrak{R}_0 \cup \mathfrak{R}_P\big) \geq a^{-3} \right\}.$$

Since $N_0(t)$ and $N_P(t)$ are $O(t^2)$, the density of Λ in $[1, r)$ approaches 1 as $r \to \infty$.

If we assume that $0 \leq \theta \leq \pi/2 - \varepsilon a^{-2}$, then $s = \frac{1}{2} + ae^{i\theta}$ will satisfy the hypothesis that

$$\mathrm{dist}(s(1-s), \sigma(P)) \geq \varepsilon$$

for Proposition 4.3. We also assume $a \in \Lambda$ throughout this argument. If Y_j is a funnel end, then Proposition 6.3 gives

$$\log \det(I + C(\eta, \varepsilon)|G_j(\tfrac{1}{2} + ae^{i\theta})|) \leq \kappa_j(\theta, b_j + 4\eta)a^2 + C(\eta, \varepsilon, b_j)a \log a, \qquad (4\text{-}9)$$

where

$$\kappa_j(\theta, r) := 2 \int_0^\infty \frac{[I(xe^{i\theta}, \ell_j, r)]_+}{x^3} \, dx - \tfrac{1}{2}\ell_j \sin^2 \theta,$$

If Y_j is hyperbolic planar, the corresponding estimate follows from [Borthwick 2010, Proposition 5.4], with

$$\kappa_j(\theta, r) := 2 \int_0^\infty \frac{[H(xe^{i\theta}, r)]_+}{x^3} \, dx,$$

(A slight modification of the original proof is required, replacing the assumption $a \in \mathbb{N}$ with an estimate based on $\mathrm{dist}(\frac{1}{2} - ae^{i\theta}, -\mathbb{N})$.)

For a cusp end C_j, it is easy to estimate directly since

$$E_{C_j}^0(s) = \frac{e^{sr}}{2s - 1},$$

which gives

$$G_j(s; r, \theta, r', \theta') = -\frac{1}{2s - 1}\mathbb{1}_{j,1}(r)e^{s(r+r')}\mathbb{1}_{j,2}(r').$$

This operator has rank one, so that

$$\det(I + c|G_j(s)|) = 1 + c\mu_1(G_j(s)),$$

where the sole singular value is given by

$$\mu_1(G_j(s)) = \frac{1}{|2s - 1|}\Big(\int_{b_j+\eta}^{b_j+2\eta} e^{2r \,\mathrm{Re}\, s}e^{-r} \, dr\Big)^{1/2}\Big(\int_{b_j+2\eta}^{b_j+3\eta} e^{2r \,\mathrm{Re}\, s}e^{-r} \, dr\Big)^{1/2}.$$

Hence we have

$$\det(I + c|G_j(\tfrac{1}{2} + ae^{i\theta})|) \leq 1 + \frac{c}{2a}e^{2a(b_j+3\eta)}.$$

For a sufficiently large,

$$\log \det(I + C(\eta, \varepsilon)|G_j(\tfrac{1}{2} + ae^{i\theta})|) \le C(\eta, \varepsilon, b_j)a \quad \text{for all } |\theta| \le \pi/2. \tag{4-10}$$

From (4-9) and (4-10) we conclude that

$$\frac{\log|\tau(\tfrac{1}{2} + ae^{i\theta})|}{a^2} \le \sum_{j=1}^{n_f} \kappa_j(\theta, b_j + 4\eta) + C(\eta, \varepsilon, b_j)a^{-1}\log a \tag{4-11}$$

for $a \in \Lambda$ and $0 \le \theta \le \pi/2 - \varepsilon a^{-2}$. Since the $\kappa_j(\theta, r)$ are uniformly continuous on $[0, \pi/2] \times [b_j, b_j + 1]$, we can take $\eta \to 0$ in (4-11), to obtain

$$\frac{\log|\tau(\tfrac{1}{2} + ae^{i\theta})|}{a^2} \le \sum_{j=1}^{n_f} \kappa_j(\theta, b_j) + o(a^2), \tag{4-12}$$

uniformly for $0 \le \theta \le \pi/2 - \varepsilon a^{-2}$.

By integrating the estimate (4-12) over θ, we obtain

$$\frac{2}{\pi} \int_0^{\pi/2 - \varepsilon a^{-2}} \log|\tau(\tfrac{1}{2} + ae^{i\theta})| \, d\theta \le \sum_{j=1}^{n_f} B(Y_j)a^2 + o(a^2).$$

It remains to fill in the small gap where $|\theta|$ is close to $\pi/2$. The factorization given by Proposition 3.1, together with the minimum modulus theorem [Boas 1954, Theorem 3.7.4], implies that for any $\eta > 0$,

$$|\tau(\tfrac{1}{2} + ae^{i\theta})| \le C_\eta \exp(a^{2+\eta}), \tag{4-13}$$

provided $a \in \Lambda$. (This was the reason that $\mathcal{R}_P$ was included in the definition of Λ.) Thus,

$$\frac{2}{\pi} \int_{\pi/2 - \varepsilon a^{-2}}^{\pi/2} \log|\tau(\tfrac{1}{2} + ae^{i\theta})| \, d\theta = O(a^\eta \varepsilon),$$

and so this term can be absorbed into the $o(a^2)$ error. $\qquad \square$

To conclude this section, we'll derive some uniform upper and lower bounds on the growth of $\tau(s)$ for $s \in \mathbb{C}$, refining the estimates that one could obtain directly from Proposition 3.1. These will prove useful in Section 7 and Section 8, in particular.

Lemma 4.4. *Let $\mathcal{Q}$ denote the joint set of zeros and poles of $\tau(\tfrac{1}{2} + z)$ and $\tau(\tfrac{1}{2} - iz)$. Assuming $|z| \ge 1$ and $\mathrm{dist}(z, \mathcal{Q}) > |z|^{-\beta}$ with $\beta > 2$, we have*

$$-c(\beta)|z|^2 \le \log|\tau(\tfrac{1}{2} + z)| \le C(\beta)|z|^2.$$

Proof. Since $\tau(\tfrac{1}{2} - z) = 1/\tau(\tfrac{1}{2} + z)$ and $\tau(\tfrac{1}{2} + \bar{z}) = \overline{\tau(\tfrac{1}{2} + z)}$, it suffices to prove the bounds for z in the first quadrant.

For $\mathrm{Re}\, z \ge \delta$ with $\delta > 0$, the upper bound is given in (4-11). As long as $\delta < 1$, the function $\tau(s)$ is analytic in the strip $\mathrm{Re}\, z \in [0, \delta]$. And since $\log|\tau(\tfrac{1}{2} + z)| = 1$ for $\mathrm{Re}\, z = 0$, the bound $\log|\tau(\tfrac{1}{2} + z)| = O(|z|^2)$ extends to the strip $\mathrm{Re}\, z \in [0, \delta]$ by (4-13) and the Phragmén–Lindelöf theorem.

To prove the lower bound, consider the Hadamard products appearing in the factorization of $\tau(s)$ given in Proposition 3.1. These products are of order 2 but not finite type, so applying the minimum modulus theorem directly would give $-\log|\tau(\frac{1}{2}+z)| = O(|z|^{2+\eta})$, away from the zeros. However, Lindelöf's theorem (see e.g., [Boas 1954, Theorem 2.10.1]) shows that products of the form $H_*(\frac{1}{2}+z)H_*(\frac{1}{2}\pm iz)$ are of finite type. In other words,

$$\log|H_*(\tfrac{1}{2}+z)H_*(\tfrac{1}{2}\pm iz)| \leq C|z|^2,$$

as $|z| \to \infty$. Using these estimates, and their implications via the minimum modulus theorem, we can prove a lower bound

$$\log|\tau(\tfrac{1}{2}+z)| \geq -c(\beta)|z|^2 - \log|\tau(\tfrac{1}{2}\pm iz)|, \tag{4-14}$$

provided $\frac{1}{2}+z$ and $\frac{1}{2}\pm iz$ stay at least a distance $|z|^{-\beta}$ away from the sets $1-\mathcal{R}_{F_\ell,r_0}$ and $\mathcal{R}_{F_\ell}$, with $\beta > 2$.

Assuming $\arg z \in [0, \pi/2]$, we already know $\log|\tau(\frac{1}{2}-iz)| \leq C(\beta)|z|^2$ from above, provided $\frac{1}{2}-iz$ stays at least a distance $|z|^{-\beta}$ away from the sets $\mathcal{R}_{F_\ell,r_0}$ and $1-\mathcal{R}_{F_\ell}$. The lower bound in the first quadrant then follows from (4-14). $\qquad\square$

5. Funnel eigenmodes

Let F_ℓ be a hyperbolic funnel of diameter ℓ. In geodesic coordinates $(r, \theta) \in \mathbb{R}_+ \times S^1$, defined with respect to the closed geodesic neck, the metric is

$$g_0 = dr^2 + \cosh^2 r\,\frac{d\theta^2}{\omega^2}, \quad \text{where } \omega := \frac{2\pi}{\ell}. \tag{5-1}$$

The Laplacian is given by

$$\Delta_{F_\ell} = -\partial_r^2 - \tanh r\,\partial_r - \frac{\omega^2}{\cosh^2 r}\partial_\theta^2. \tag{5-2}$$

In this section we will consider asymptotic properties of the Fourier modes of generalized eigenfunctions of Δ_{F_ℓ}.

The restriction of eigenvalue equation $(\Delta_{F_\ell} - s(1-s))u = 0$ to the k-th Fourier mode, $u = w(r)e^{ik\theta}$, yields the equation

$$-\partial_r^2 w - \tanh r\,\partial_r w + \left(\frac{k^2\omega^2}{\cosh^2 r} - s(1-s)\right)w = 0. \tag{5-3}$$

This is essentially a hypergeometric equation. With respect to the symmetry $r \mapsto -r$, we have an even solution,

$$w_k^+(s; r) := (\cosh r)^{i\omega k}\mathbf{F}(\tfrac{1}{2}(s+i\omega k), \tfrac{1}{2}(1-s+i\omega k); \tfrac{1}{2}; -\sinh^2 r), \tag{5-4}$$

and an odd solution,

$$w_k^-(s; r) := \sinh r(\cosh r)^{i\omega k}\mathbf{F}(\tfrac{1}{2}(1+s+i\omega k), \tfrac{1}{2}(2-s+i\omega k); \tfrac{3}{2}; -\sinh^2 r). \tag{5-5}$$

(We follow Olver's convention in using $\mathbf{F}(a, b; c; z) := F(a, b; c; z)/\Gamma(c)$, where $F(a, b; c; z)$ is the standard Gauss hypergeometric function.)

By symmetry, we can and will assume that $k \geq 0$. If we substitute $w = (\cosh r)^{-1/2} U$ and introduce the parameter α defined by $s = \frac{1}{2} + k\alpha$, the coefficient equation (5-3) becomes

$$\partial_r^2 U = (k^2 f + g) U, \tag{5-6}$$

where

$$f := \frac{\omega^2 + \alpha^2 \cosh^2 r}{\cosh^2 r} \quad \text{and} \quad g := \frac{1}{4 \cosh^2 r}.$$

This equation has turning points when $\alpha = \pm i\omega / \cosh r$. We will restrict our attention to $\arg \alpha \in [0, \frac{1}{2}\pi]$, so that we only consider the upper turning point. The Liouville transformation involves a new variable ζ defined by integrating

$$\sqrt{\zeta} \, d\zeta := \sqrt{f} \, dr, \tag{5-7}$$

on a contour that starts from the upper turning point. Integrating (5-7) yields

$$(2/3)\zeta^{3/2} = \phi, \tag{5-8}$$

where $\phi(\alpha, r)$, the integral of $\sqrt{f} \, dr$ from the turning point, is given explicitly by

$$\phi(\alpha, r) := \alpha \log \left(\frac{\alpha \sinh r + \sqrt{\omega^2 + \alpha^2 \cosh^2 r}}{\sqrt{\omega^2 + \alpha^2}} \right)$$
$$+ \frac{i\omega}{2} \log \left(\frac{\sqrt{\omega^2 + \alpha^2 \cosh^2 r} - i\omega \sinh r}{\sqrt{\omega^2 + \alpha^2 \cosh^2 r} + i\omega \sinh r} \right) + \phi_0(\alpha) \tag{5-9}$$

for $\alpha \neq i\omega$, where

$$\phi_0(\alpha) = \phi(\alpha; 0) = -\frac{1}{2}\pi(i\alpha + \omega). \tag{5-10}$$

By continuity, the definition of ϕ extends to $\alpha = i\omega$, with

$$\phi(i\omega, r) = i\omega \log \cosh r.$$

To complete the Liouville transformation, we set $W = (f/\zeta)^{1/4} U$, so that (5-6) becomes an approximate Airy equation,

$$\partial_\zeta^2 W = (k^2 \zeta + \psi) W, \tag{5-11}$$

with the extra term given by

$$\psi = \frac{\zeta}{4f^2} \partial_r^2 f - \frac{5\zeta}{16f^3} (\partial_r f)^2 + \frac{\zeta g}{f} + \frac{5}{16\zeta^2}. \tag{5-12}$$

The solutions of (5-11) are of the form

$$W_\sigma := \text{Ai}(k^{2/3} e^{2\pi i\sigma/3} \zeta) + h_\sigma(k, \alpha, r), \tag{5-13}$$

where $\sigma = 0$ or ± 1, and the error term satisfies the differential equation

$$\partial_\zeta^2 h_\sigma - k^2 \zeta h_\sigma = (h_\sigma + \text{Ai}(k^{2/3} e^{2\pi i\sigma/3} \zeta)) \psi. \tag{5-14}$$

Using methods from Olver [1974] we can control this error term.

Lemma 5.1. *The error equation* (5-14) *admits solutions that satisfy* $\lim_{r \to \infty} h_\sigma(r) = 0$ *and*

$$|h_\sigma| \leq Ck^{-1}|\alpha|^{-2/3}(1 + |k\phi|^{1/6})^{-1}e^{(-1)^{\sigma+1}k\,\text{Re}\,\phi},$$

with C independent of r, k and α.

We will defer the rather technical proof of Lemma 5.1 to the end of this section, in order to concentrate on the implications of (5-13). The asymptotics of the Airy function are well known; see for example [Olver 1974, Section 11.8]. Uniformly for $|\arg z| < \pi - \varepsilon$, we have

$$\text{Ai}(z) = \frac{1}{2\pi^{1/2}}z^{-1/4}\exp(-\tfrac{2}{3}z^{3/2})(1 + O(|z|^{-3/2})). \tag{5-15}$$

And uniformly for $|\arg z| \geq \tfrac{1}{3}\pi + \varepsilon$,

$$\text{Ai}(z) = \frac{1}{\pi^{1/2}}(-z)^{-1/4}\cos(\tfrac{2}{3}(-z)^{3/2} - \tfrac{1}{4}\pi)(1 + O(|z|^{-3/2})). \tag{5-16}$$

These asymptotics make it convenient to introduce a pair of solutions of the eigenvalue equation (5-3) defined by

$$w_\sigma = 2\pi^{1/2}e^{i\pi\sigma/6}k^{1/6}\zeta^{1/4}(\omega^2 + \alpha^2\cosh^2 r)^{-1/4}W_\sigma, \tag{5-17}$$

where W_σ is the ansatz (5-13) for $\sigma = 0$ or 1.

Proposition 5.2. *Consider the solutions of the equation*

$$(\Delta_{F_\ell} - \tfrac{1}{4} - k^2\alpha^2)e^{ik\theta}w_\sigma(r) = 0$$

given by (5-17) *with* $\sigma = 0$ *or* 1. *Assuming* $k \geq 1$ *and* $\arg\alpha \in [0, \tfrac{1}{2}\pi - \varepsilon]$, *we have asymptotics*

$$w_\sigma = (\omega^2 + \alpha^2\cosh^2 r)^{-1/4}\exp((-1)^{\sigma+1}k\phi)(1 + O(|k\alpha|^{-1})), \tag{5-18}$$

with constants that depend only on ε. *In addition, for* $\arg\alpha \in [0, \pi/2]$ *and* $|k\alpha|$ *sufficiently large, we have the upper bounds*

$$|w_\sigma| \leq Ck^{1/6}\exp((-1)^{\sigma+1}k\,\text{Re}\,\phi), \tag{5-19}$$

and the lower bound

$$|w_0| \geq ce^{-k\,\text{Re}\,\phi}. \tag{5-20}$$

Proof. The assumption that $\arg\alpha \in [0, \pi/2 - \varepsilon]$ implies that $\arg\zeta \in [-2\pi/3, \pi/3 - \varepsilon]$, so that (5-15) applies to both w_0 and w_1 in this case. It also implies that $|\phi| \geq c(\varepsilon)(|\alpha| + 1)$, so that the error term $O(|w|^{-3/2})$ from (5-15) becomes $O(|k\alpha|^{-1})$ when applied to $|w| = k^{2/3}|\zeta|$. In combination with Lemma 5.1, this proves (5-18), and also (5-19) and (5-20) in the case where $\arg\alpha$ is bounded away from $\pi/2$.

If $\arg\alpha \in [\pi/2 - \varepsilon, \pi/2]$, then (5-15) and (5-16), together with Lemma 5.1, give the estimates

$$|k^{1/6}\zeta^{1/4}W_\sigma| \leq C\exp((-1)^{\sigma+1}k\,\text{Re}\,\phi), \tag{5-21}$$

and

$$|k^{1/6}\zeta^{1/4}W_0| \geq ce^{-k\operatorname{Re}\phi}, \tag{5-22}$$

If $|\omega^2 + \alpha^2 \cosh^2 r| \geq 1$, which bounds ϕ away from 0, then this gives (5-19) immediately. This leaves the case $|\omega^2 + \alpha^2 \cosh^2 r| \leq 1$, which puts ϕ close to zero. In this case, $\zeta \asymp (\omega^2 + \alpha^2 \cosh^2 r)$, so that $w_\sigma \asymp k^{1/6}W_\sigma$. Then if $|k\phi| \geq 1$ we can derive the estimates from (5-21) and (5-22), while for $|k\phi| \leq 1$ we simply note that W_σ is bounded and nonzero near the origin. $\qquad\square$

Another detail we will need later is the asymptotic behavior of w_σ as $r \to \infty$.

Lemma 5.3. *For* $\operatorname{Re}\alpha \geq 0$, *as* $r \to \infty$,

$$w_0 \sim \alpha^{-1/2}e^{-k(\phi_0(\alpha)+\gamma(\alpha))}\rho^{1/2+k\alpha},$$

and

$$w_1 \sim \alpha^{-1/2}e^{k(\phi_0(\alpha)+\gamma(\alpha))}(\rho^{1/2-k\alpha} + i\rho^{1/2+k\alpha})$$

where $\rho := 2e^{-r}$, *and*

$$\gamma(\alpha) := \alpha \log \frac{2\alpha}{\sqrt{\omega^2+\alpha^2}} + \frac{i\omega}{2}\log \frac{\alpha-i\omega}{\alpha+i\omega}. \tag{5-23}$$

Proof. The results follow immediately from (5-15) and (5-16), in combination with the asymptotic

$$\phi(\alpha; r) = \alpha r + \phi_0(\alpha) + \alpha \log \frac{\alpha}{\sqrt{\omega^2+\alpha^2}} + \frac{i\omega}{2}\log \frac{\alpha-i\omega}{\alpha+i\omega} + O(r^{-1}), \tag{5-24}$$

as $r \to \infty$. $\qquad\square$

We conclude the section with the proof of the error estimate that is the basis of Proposition 5.2 and Lemma 5.3.

Proof of Lemma 5.1. The cases of different σ are all very similar, so we consider only $\sigma = 0$. In this case combining the boundary condition with variation of parameters allows us to transform (5-14) into an integral equation:

$$h_0(k,\alpha,r) = \frac{2\pi e^{-i\pi/6}}{k^{2/3}}\int_r^\infty K_0(r,r')\psi(r')\big(h_0(k,\alpha,r') + \operatorname{Ai}(k^{2/3}\zeta(r'))\big)\frac{f(r')^{1/2}}{\zeta(r')^{1/2}}\,dr',$$

where

$$K_0(r,r') := \operatorname{Ai}(k^{2/3}\zeta(r'))\operatorname{Ai}(k^{2/3}e^{-2\pi i/3}\zeta(r)) - \operatorname{Ai}(k^{2/3}e^{-2\pi i/3}\zeta(r'))\operatorname{Ai}(k^{2/3}\zeta(r)).$$

Then, using the method of successive approximations as in [Olver 1974, Theorem 6.10.2], together with the bounds on the Airy function and its derivatives developed in [Olver 1974, Section 11.8], we obtain the bound

$$|h_0| \leq Ce^{-k\operatorname{Re}\phi}(1 + k^{1/6}|\zeta|^{1/4})^{-1}(e^{ck^{-1}\Psi(r)} - 1), \tag{5-25}$$

where

$$\Psi(r) := \int_r^\infty |\psi f^{1/2}\zeta^{-1/2}|dr'. \tag{5-26}$$

From (5-12), we compute

$$\psi f^{1/2}\zeta^{-1/2} = \left(\frac{\alpha^4\cosh^2 r + 4\alpha^2\omega^2\sinh^2 r - \omega^4}{4(\omega^2 + \alpha^2\cosh^2 r)^{5/2}}\right)\zeta^{1/2}\cosh r + \frac{5}{16}\frac{(\omega^2 + \alpha^2\cosh^2 r)^{1/2}}{\zeta^{5/2}\cosh r}. \qquad (5\text{-}27)$$

The estimate must be broken into various regions. Fix some $c > 0$.

Case 1. Assume $|\alpha| \geq 1$ and $|\omega^2 + \alpha^2\cosh^2(r)| \geq c$. Under these conditions, we can estimate

$$|\phi| \asymp |\alpha|(r+1).$$

Then from (5-27), we find

$$|\psi f^{1/2}\zeta^{-1/2}| \leq C_1|\alpha|^{-2/3}e^{-2r}(r+1)^{1/3} + C_2|\alpha|^{-2/3}(r+1)^{-5/3}.$$

We easily conclude that for $|\alpha| \geq 1$,

$$\int_{|\omega^2 + \alpha^2\cosh^2(r)| \geq c} |\psi f^{1/2}\zeta^{-1/2}|\,dr = O(|\alpha|^{-2/3}). \qquad (5\text{-}28)$$

Case 2. Assume $|\alpha| \leq 1$ and $|\omega^2 + \alpha^2\cosh^2(r)| \geq c$. The behavior of ϕ is now slightly more complicated, depending on the size of r relative to $|\alpha|$,

$$|\phi| \asymp \begin{cases} |\alpha| + e^{-r} & \text{for } |\alpha|\sinh r \leq 1, \\ |\alpha|(r + \log|\alpha|) & \text{for } |\alpha|\sinh r \geq 1. \end{cases}$$

In this case, we estimate (5-27) by

$$|\psi f^{1/2}\zeta^{-1/2}| \leq \begin{cases} C_1(|\alpha| + e^{-r})^{1/3}e^r + C_2 e^{-r}(|\alpha| + e^{-r})^{-5/3} & \text{for } |\alpha|\sinh r \leq 1, \\ C_1(1 + |\alpha|e^r)^{-3}|\alpha|^{1/3}(r + \log|\alpha|)^{1/3}e^r + C_2|\alpha|^{-5/2}r^{-5/3}e^{-r} & \text{for } |\alpha|\sinh r \geq 1. \end{cases}$$

It is then straightforward to bound, for $|\alpha| \leq 1$,

$$\int_{|\omega^2 + \alpha^2\cosh^2(r)| \geq c} |\psi f^{1/2}\zeta^{-1/2}|\,dr = O(|\alpha|^{-2/3}). \qquad (5\text{-}29)$$

Case 3. Assume $|\omega^2 + \alpha^2\cosh^2(r)| \leq c$. In this case we are near the turning point, where ϕ and ζ are small. Since $|\omega^2 + \alpha^2\cosh^2(r)| \leq c$ implies $|\alpha|^2 \leq \omega^2 + c$, we are only concerned with small $|\alpha|$ here. We proceed as in [Borthwick 2010, Appendix]. In the coordinate $z = \sinh r$, the turning point occurs at

$$z_0 = \sqrt{-1 - \frac{\omega^2}{\alpha^2}}.$$

Set

$$p(z) := \left(\frac{f}{z - z_0}\right)^{1/2} = \frac{\alpha\sqrt{z + z_0}}{\sqrt{z^2 + 1}}. \qquad (5\text{-}30)$$

Because $|\omega^2 + \alpha^2\cosh^2(r)| = |\alpha^2(z^2 - z_0^2)|$, the assumption $|\omega^2 + \alpha^2\cosh^2(r)| \leq c$ implies

$$z \asymp z_0 \asymp |\alpha|^{-1}, \qquad (5\text{-}31)$$

with constants that depend only on c. This makes it easy to estimate

$$|\partial_z^k p(z)| \asymp |\alpha|^{3/2+k}, \tag{5-32}$$

with constants that depend only on c and k. If we define

$$q(z) := \frac{\phi}{(z-z_0)^{3/2}},$$

then by writing

$$q(z) = \int_0^1 t^{1/2} \frac{p(z_0+t(z-z_0))}{\sqrt{((1-t)z_0+tz)^2+1}} \, dt,$$

we can deduce from (5-32) that

$$|\partial_z^k q(z)| \asymp |\alpha|^{5/2+k}. \tag{5-33}$$

To apply these estimates, we note that $f/\zeta = p^2(\frac{3}{2}q)^{-2/3}$. We can use this identification to apply the bounds (5-32) and (5-33) to the formula (5-31) for ψ, obtaining

$$|\psi f^{1/2}\zeta^{-1/2}| \asymp |\alpha|^{-2/3} \quad \text{for } |\omega^2+\alpha^2\cosh^2(r)| \leq c.$$

The bound

$$\int_{|\omega^2+\alpha^2\cosh^2(r)|\leq c} |\psi f^{1/2}\zeta^{-1/2}| \, dr = O(|\alpha|^{-2/3}), \tag{5-34}$$

follows immediately, since the range of integration for r is $O(1)$.

Combining the bounds (5-28), (5-29), and (5-34) gives

$$\Phi(0) = O(|\alpha|^{-2/3}),$$

and the claimed estimate follows from (5-25). $\square$

6. Funnel determinant estimates

For the model funnel F_ℓ, fix $r_0 \geq 0$ and for some $\eta > 0$ set

$$r_k = r_0 + k\eta.$$

Let $\mathbb{1}_k$ denote the multiplication operator for the characteristic function of the interval $r \in [r_k, r_{k+1}]$ in $L^2(F_\ell)$. The operator $G_j(s)$ defined in (4-7) can be represented in the model funnel case by

$$G(s) := (2s-1)\mathbb{1}_1 E_{F_\ell}(1-s)E_{F_\ell}(s)^t \mathbb{1}_2 \tag{6-1}$$

Our goal in this section is to prove the sharp bound on $\log\det(1+c|G(s)|)$ used in the proof of Theorem 4.1.

To proceed we must analyze the Fourier decomposition of $E_{F_\ell}(s)$. Because of the circular symmetry, the Poisson kernel on F_ℓ admits a diagonal expansion into Fourier modes:

$$E_{F_\ell}(s; r, \theta, \theta') = \frac{1}{\ell} \sum_{k\in\mathbb{Z}} a_k(s; r) e^{ik(\theta-\theta')} \tag{6-2}$$

The coefficients $a_k(s; r)$ satisfy (5-3) with the boundary condition $a_k(s; 0) = 0$, so we must have

$$a_k(s; r) = c_k(s) w_k^-(s; r), \tag{6-3}$$

where w_k^- is the odd solution (5-5). To compute the normalization constant $c_k(s)$, we use the fact that

$$(2s - 1) a_k(s; r) \sim \rho^{1-s} + [S_{F_\ell}(s)]_k \rho^s \quad \text{as } \rho \to 0, \tag{6-4}$$

where $[S_{F_\ell}(s)]_k$ is the k-th matrix element of the scattering matrix $S_\ell(s)$. Applying the appropriate Kummer identity [Olver 1974, Equation (5.10.16)] to the hypergeometric function in (5-5) gives

$$a_k(s; r) \sim c_k(s)\left(\Gamma(\tfrac{1}{2} - s)\beta_k(2 - s)\rho^s + \Gamma(s - \tfrac{1}{2})\beta_k(1 + s)\rho^{1-s}\right),$$

where

$$\beta_k(s) := \frac{1}{\Gamma(\tfrac{1}{2}(s + ik\omega)))\Gamma(\tfrac{1}{2}(s - ik\omega)))}. \tag{6-5}$$

By comparing this asymptotic to (6-4), we can read off the coefficient

$$c_k(s) = \frac{2s - 1}{\Gamma(s - \tfrac{1}{2})\beta_k(1 + s)},$$

as well as the scattering matrix element

$$[S_{F_\ell}(s)]_k = \frac{\Gamma(\tfrac{1}{2} - s)\beta_k(2 - s)}{\Gamma(s - \tfrac{1}{2})\beta_k(1 + s)}. \tag{6-6}$$

For future reference we note also that

$$a_k(1 - s; r) = -\frac{a_k(s; r)}{[S_{F_\ell}(s)]_k}. \tag{6-7}$$

and

$$a_k(s; r) = a_{-k}(s; r) \tag{6-8}$$

We can express the singular values of $G(s)$ in terms of the coefficients $a_k(s; r)$. Up to reordering, these singular values are given by

$$\lambda_k(s) := |2s - 1|\left(\int_{r_1}^{r_2} |a_k(1 - s; r)|^2 \cosh r \, dr\right)^{1/2}\left(\int_{r_2}^{r_3} |a_k(s; r)|^2 \cosh r \, dr\right)^{1/2} \quad \text{for } k \in \mathbb{Z}. \tag{6-9}$$

To prove this, we note that $\lambda_k(s)^2$ is the eigenvalue of $G^*G(s)$ corresponding to the eigenfunction $\chi_{[r_2, r_3]}(r)\overline{a_k(s; r)}e^{-ik\theta}$. Also, it is easy to see from (6-1) and (6-2) that these are the only nonzero eigenvalues.

Using (6-7) to replace $a_k(1 - s)$ by $a_k(s)$, and assuming $\eta \leq 1$, we can estimate

$$\lambda_k(\tfrac{1}{2} + k\alpha) \leq \left|2k\alpha a_k(\tfrac{1}{2} + k\alpha; r_3)^2[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k \cosh r_3\right|. \tag{6-10}$$

We will first estimate the various components. Recall that the matrix elements of $S_{F_\ell}(s)$ were expressed in terms of the function β_k defined in (6-5).

Lemma 6.1. *For $k > 0$ and $\arg \alpha \in [0, \frac{1}{2}\pi]$, if we assume $\mathrm{dist}(k\alpha, \mathbb{N}_0) \geq \delta$ then we have*

$$\log\big|[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k\big| \geq 2k \operatorname{Re} \gamma + 2k[\operatorname{Re} \phi_0]_- - C(\delta),$$

where $\gamma(\alpha)$ was defined in (5-23). If instead we assume that $\mathrm{dist}(\frac{1}{2} - k\alpha, \mathfrak{R}_{F_\ell}) \leq |k\alpha|^{-\beta}$, then

$$\log\big|[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k\big| \leq 2k \operatorname{Re} \gamma + 2k[\operatorname{Re} \phi_0]_- + C(\beta) \log|k\alpha|.$$

Proof. Consider the matrix element (6-6). For $\operatorname{Re}\alpha \geq 0$, we can apply Stirling's formula directly to obtain

$$\log \Gamma(k\alpha)\beta_k(\tfrac{3}{2} + k\alpha) = k\gamma(\alpha) - \tfrac{1}{2}\log \pi k^2 \alpha \sqrt{\omega^2 + \alpha^2} + O(|k\alpha|^{-1}),$$

To estimate the other term, we must avoid zeros and poles. For $\operatorname{Re} z \leq 0$, applying Stirling via the reflection formula gives

$$\log|\Gamma(z)| \leq \operatorname{Re}\big((z - \tfrac{1}{2}) \log(-z) - z\big) - \pi|\operatorname{Im} z| + \log(1 + \mathrm{dist}(z, -\mathbb{N}_0)^{-1}) + O(1),$$

and

$$\log|\Gamma(z)| \geq \operatorname{Re}\big((z - \tfrac{1}{2}) \log(-z) - z\big) - \pi|\operatorname{Im} z| + O(1).$$

If we assume that $\mathrm{dist}(k\alpha, \mathbb{N}_0) \geq \delta$, then we obtain the upper bound

$$\log\big|\Gamma(-k\alpha)\beta_k(\tfrac{3}{2} - k\alpha)\big| \leq -k \operatorname{Re} \gamma(\alpha) - 2k[\operatorname{Re} \phi_0]_- - \tfrac{1}{2}\log k^2 \alpha \sqrt{\omega^2 + \alpha^2} + C(\delta).$$

For a lower bound, we need to assume that $\mathrm{dist}(k\alpha, \mathfrak{R}_{F_\ell}) \geq |k\alpha|^{-\beta}$, and then we find that

$$\log\big|\Gamma(-k\alpha)\beta_k(\tfrac{3}{2} - k\alpha)\big| \geq -k \operatorname{Re} \gamma(\alpha) - 2k[\operatorname{Re} \phi_0]_- - \tfrac{1}{2}\log k^2 \alpha \sqrt{\omega^2 + \alpha^2} - C(\beta) \log|k\alpha|. \qquad \square$$

Lemma 6.2. *Assuming that $\operatorname{Re}\alpha \geq 0$, $k > 0$, and $\mathrm{dist}(\frac{1}{2} - k\alpha, \mathfrak{R}_{F_\ell}) \leq |k\alpha|^{-\beta}$, we have*

$$\log \lambda_k(\tfrac{1}{2} + k\alpha) \leq 2k \operatorname{Re} \phi(\alpha; r_3) - 2k[\operatorname{Re} \phi_0(\alpha)]_+ + O(\log |k\alpha|).$$

Proof. By conjugation we can assume $\arg\alpha \in [0, \frac{1}{2}\pi]$. Then $a_k(\frac{1}{2} + k\alpha; r)$ can be expressed in terms of the solutions w_σ from Proposition 5.2. To satisfy the Dirichlet boundary condition, it must be a constant multiple of $w_0(0)w_1(r) - w_1(0)w_0(r)$. Lemma 5.3 gives the asymptotic behavior of this expression as $r \to \infty$, allowing us to deduce the constant. After comparing to (6-4), we find that

$$a_k(\tfrac{1}{2} + k\alpha; r) = \frac{1}{2kw_0(0)}\alpha^{-1/2}e^{-k(\phi_0(\alpha)+\gamma(\alpha))}\big(w_0(0)w_1(r) - w_1(0)w_0(r)\big) \qquad (6\text{-}11)$$

The estimate

$$\big|a_k(\tfrac{1}{2} + k\alpha; r)\big| \leq Ck^{1/6}e^{k \operatorname{Re}(\phi(\alpha,r)-\phi_0(\alpha)-\gamma(\alpha))}, \qquad (6\text{-}12)$$

for $|k\alpha|$ sufficiently large, then follows immediately from (5-19) and (5-20). The result now follows from applying Lemma 6.1 and (6-12) in (6-10). $\qquad \square$

Proposition 6.3. *Assuming that $\eta \leq 1$, $0 \leq \theta \leq \pi/2$, and $\mathrm{dist}(\frac{1}{2} - ae^{i\theta}, \mathfrak{R}_{F_\ell}) \geq a^{-\beta}$ for some fixed $\beta > 1$, we have*

$$\log \det\big(I + c|G(\tfrac{1}{2} + ae^{i\theta})|\big) \leq \kappa(\theta, r_4)a^2 + C(c, r_0, \beta)a \log a,$$
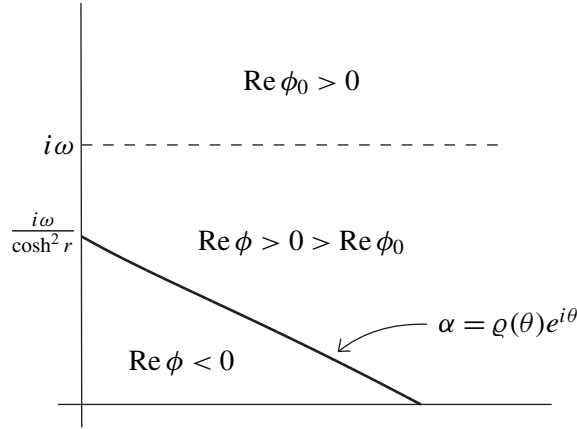
Figure 5. Positive and negative regions for $\operatorname{Re}\phi(\alpha; r)$ and $\operatorname{Re}\phi_0(\alpha)$, shown for $r = 1$.

where

$$\kappa(\theta, r) = 2\int_0^\infty \frac{[I(xe^{i\theta}, \ell, r)]_+}{x^3}\, dx - \tfrac{1}{2}\ell\sin^2\theta, \tag{6-13}$$

with $I(xe^{i\theta}, \ell, r) := 2\operatorname{Re}\phi(xe^{i\theta}; r)$, *which agrees with the definition* (1-11).

Proof. We start from the expression for the determinant in terms of the singular values,

$$\det(I + c|G(\tfrac{1}{2} + ae^{i\theta})|) = \prod_{k\in\mathbb{Z}}(1 + c\lambda_k(\tfrac{1}{2} + ae^{i\theta})).$$

By the conjugation symmetry, we can assume $\theta \in [0, \tfrac{1}{2}\pi]$. Let $\varrho(\theta)$ be the implicit solution of the equation $\operatorname{Re}\phi(\varrho(\theta)e^{i\theta}, r_3) = 0$, as illustrated in Figure 5.

Note that $\operatorname{Re}\phi_0(xe^{i\theta}) = 0$ in a neighborhood of $x = \varrho(\theta)$. For some $\delta > 0$, we subdivide the sum in

$$\log\det\big(I + c|G(\tfrac{1}{2} + ae^{i\theta})|\big) = 2\sum_{k=1}^\infty \log(1 + c\lambda_k(\tfrac{1}{2} + a_i e^{i\theta})) + O(a\log a)$$

at values where $a_i/k = \varrho(\theta)$ and $(1 - \delta)\varrho(\theta)$. The dominant part of the sum is

$$\Sigma_+ := \sum_{1\le k\le a/\varrho(\theta)} \log(1 + c\lambda_k(\tfrac{1}{2} + ae^{i\theta})).$$

Assuming that $a \in \{a_i\}$, Lemma 6.2 gives the bound

$$\Sigma_+ \le \sum_{1\le k\le a/\varrho(\theta)} 2k\big(\operatorname{Re}\phi(ae^{i\theta}/k; r_3) - [\operatorname{Re}\phi_0(ae^{i\theta}/k)]_+\big) + C(c, r_0, \beta)a\log a.$$

Because the summand is a decreasing function of k, we may estimate the sum by the integral

$$\Sigma_+ \le \int_0^{a/\varrho(\theta)} 2k\big(\operatorname{Re}\phi(ae^{i\theta}/k; r_3) - [\operatorname{Re}\phi_0(ae^{i\theta}/k)]_+\big) + C(c, r_0, \beta)a\log a$$

Substituting $x = a/k$ gives

$$\int_0^{a/\varrho(\theta)} 2k \operatorname{Re} \phi(ae^{i\theta}/k; r_3)\, dk = 2a^2 \int_{\varrho(\theta)}^\infty \frac{\operatorname{Re} \phi(xe^{i\theta}; r_3)}{x^3}\, dx.$$

We can also compute that

$$\int_0^{a/\varrho(\theta)} 2k[\operatorname{Re} \phi_0(ae^{i\theta}/k)]_+\, dk = \pi a^2 \int_{\omega/\sin(\theta)}^\infty \frac{x \sin\theta - \omega}{x^3}\, dx = \frac{\pi a^2}{2\omega} \sin\theta.$$

Comparing to (6-13), we conclude that

$$\Sigma_+ \le \kappa(\theta, r_3)a^2 + C(c, r_0, \beta)a \log a.$$

The middle term is given by

$$\Sigma_0 := \sum_{a/\varrho(\theta) \le k \le a/(1-\delta)\varrho(\theta)} \log(1 + c\lambda_k(\tfrac{1}{2} + ae^{i\theta})),$$

Since $I(\alpha, \ell, r_3) = O(\delta)$ for k in this range, the same integral estimate used for Σ_+ gives

$$|\Sigma_0| \le C(c, r_0, \beta)\delta a^2 + C(c, r_0, \beta)a \log a.$$

Finally, we set

$$\Sigma_- := \sum_{k \ge a/(1-\delta)\varrho(\theta)} \log(1 + c\lambda_k(\tfrac{1}{2} + ae^{i\theta})).$$

For k in this range, $I(\alpha, \ell, r_3) \le -C\delta$ and we can estimate

$$|\Sigma_-| \le C(c, r_0, \beta, \delta)e^{-ca} \quad \text{for some } c > 0.$$

Adding together the estimates for Σ_+, Σ_0, and Σ_- gives

$$\log \det\bigl(I + C|G(\tfrac{1}{2} + ae^{i\theta})|\bigr) \le \kappa(\theta, r_3)a^2 + C(c, r_0, \beta)(\delta a^2 + a \log a) + C(c, r_0, \beta, \delta)e^{-ca}$$

We can absorb the δa^2 term into the first term by replacing r_3 by r_4, assuming that $\eta = O(\delta)$, since $\kappa(\theta, \cdot)$ is strictly increasing. This yields the claimed estimate. $\qquad\square$

7. Resonance asymptotics for truncated funnels

Inside the model funnel F_ℓ, with metric given by (5-1), we let F_{ℓ, r_0} denote the truncated region $\{r \ge r_0\}$, with the Laplacian defined by imposing Dirichlet boundary conditions at $r = r_0$. To compute the associated scattering matrix elements exactly, we consider the solutions of the Fourier mode equation (5-3) given by (5-4) and (5-5). To impose the boundary condition at $r = r_0$, we set

$$u_k(s; r) := w_k^+(s; r_0)w_k^-(s; r) - w_k^-(s; r_0)w_k^+(s; r). \tag{7-1}$$

The scattering matrix element may be obtained from the asymptotics of $u_k(s; r)$ as $r \to \infty$ be noting that for any generalized eigenmode we have

$$u_k(s; r) \sim c_{k,s}(\rho^{1-s} + [S_{F_{\ell,r_0}}(s)]_k \rho^s) \tag{7-2}$$

as $r \to \infty$, where $\rho := 2e^{-r}$ as before. The solutions $w_k^{\pm}$ have leading asymptotics,

$$
\begin{aligned}
w_k^+(s; r) &\sim \Gamma(s - \tfrac{1}{2})\beta_k(s)\rho^{1-s} + \Gamma(\tfrac{1}{2} - s)\beta_k(1 - s)\rho^s, \\
w_k^-(s; r) &\sim \Gamma(s - \tfrac{1}{2})\beta_k(1 + s)\rho^{1-s} + \Gamma(\tfrac{1}{2} - s)\beta_k(2 - s)\rho^s
\end{aligned}
\tag{7-3}
$$

as $r \to \infty$, where $\beta_k(s)$ was defined in (6-5).

If we set

$$f_k(s; r) := \Gamma(s - \tfrac{1}{2})\big(\beta_k(1 + s)w_k^+(s; r) - \beta_k(s)w_k^-(s; r)\big), \tag{7-4}$$

Then from (7-2) we can read off that

$$[S_{F_{\ell,r_0}}(s)]_k = \frac{f_k(1 - s; r_0)}{f_k(s; r_0)}. \tag{7-5}$$

The k-th Fourier mode thus contributes scattering poles at the values of s for which

$$\beta_k(1 + s)w_k^+(s; r_0) - \beta_k(s)w_k^-(s; r_0) = 0.$$

This function can be written in terms of a single normalized hypergeometric function, via the standard identities, yielding

$$\mathscr{R}_{F_{\ell,r_0}} = \bigcup_{k \in \mathbb{Z}} \big\{ s : \mathbf{F}(\tfrac{1}{2}(1 + s + i\omega k), \tfrac{1}{2}(s + i\omega k); \tfrac{1}{2} + s; -\sinh^{-2} r_0) = 0 \big\}.$$

A sample resonance counting function is shown in Figure 6.

Theorem 7.1. *For the truncated funnel with Dirichlet boundary conditions,*

$$N_{F_{\ell,r_0}}(t) \sim A(F_{\ell,r_0})t^2,$$

where $A(F_{\ell,r_0})$ is given by (1-10).

In conjunction with [Borthwick 2010, Theorem 1.2] for the hyperbolic planar case, this will complete the proof of Theorem 1.3. Before giving the proof, we need some estimates of scattering matrix elements.

Lemma 7.2. *Assuming that $\arg \alpha \in [0, \pi/2 - \varepsilon]$ with $\mathrm{dist}(k\alpha, \mathbb{N}_0) \geq \eta$, we can have*

$$\log \left| \frac{[S_{F_{\ell,r_0}}(\tfrac{1}{2} + k\alpha)]_k}{[S_{F_\ell}(\tfrac{1}{2} + k\alpha)]_k} - 1 \right| \geq 2k(\mathrm{Re}\, \phi(\alpha; r_0) - [\mathrm{Re}\, \phi_0(\alpha)]_+) - C(\eta)$$

for $|k\alpha|$ sufficiently large.
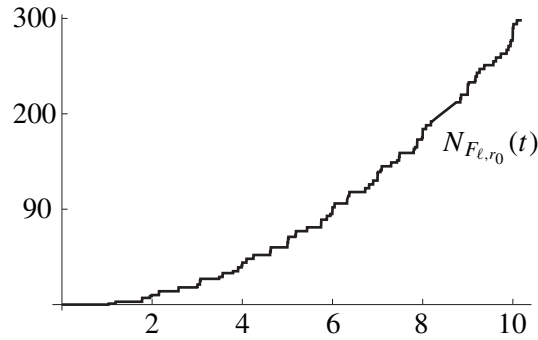
Figure 6. The resonance counting function for F_{ℓ, r_0}, shown for $\ell = 2\pi$ and $r_0 = 1$.

Proof. To estimate $[S_{F_{\ell, r_0}}(s)]_k$, as given in (7-5), we must connect f_k to the solutions w_σ introduced in (5-17). Since $f_k(\frac{1}{2} + k\alpha; r)$ is recessive as $r \to \infty$, this solution must be proportional to w_0. From (7-3), we can use the reflection formula for the gamma function to see that

$$f_k(\tfrac{1}{2} + k\alpha; r) \sim \frac{\rho^s}{\pi k \alpha} \quad \text{as } r \to \infty.$$

By comparing this to the asymptotic from Lemma 5.3, we find that

$$f_k(\tfrac{1}{2} + k\alpha; r) = A_0^+ w_0(r). \tag{7-6}$$

where

$$A_0^+ := \frac{1}{\pi k \sqrt{\alpha}} e^{k(\phi_0 + \gamma)}.$$

We may also express $f_k(\frac{1}{2} - k\alpha; r)$ in terms of the w_σ,

$$f_k(\tfrac{1}{2} - k\alpha; r) = A_0^- w_0(r) + A_1^- w_1(r), \tag{7-7}$$

for some coefficients A_0^- and A_1^- that are independent of r but do depend on k and α. By (7-3),

$$f_k(\tfrac{1}{2} - k\alpha; r) \sim -\frac{\rho^{1-s}}{\pi k \alpha},$$

and so by Lemma 5.3 we have

$$A_1^- = -\pi^{-1} k^{-1} \alpha^{-1/2} e^{-k(\phi_0 + \gamma)}. \tag{7-8}$$

The other coefficient can then be computed by comparing values at $r = 0$,

$$A_0^- = \frac{1}{w_0(0)} \left(f_k(\tfrac{1}{2} - k\alpha; 0) - A_1^- w_1(0) \right). \tag{7-9}$$

Using (7-6) to relate $w_0(0)$ to $f_k(\frac{1}{2} + k\alpha; 0)$, we can then deduce that

$$[S_{F_{\ell, r}}(\tfrac{1}{2} + k\alpha)]_k = [S_{F_\ell}(\tfrac{1}{2} + k\alpha)]_k - e^{-2k(\phi_0 + \gamma)} \left(\frac{w_1(r)}{w_0(r)} - \frac{w_1(0)}{w_0(0)} \right). \tag{7-10}$$
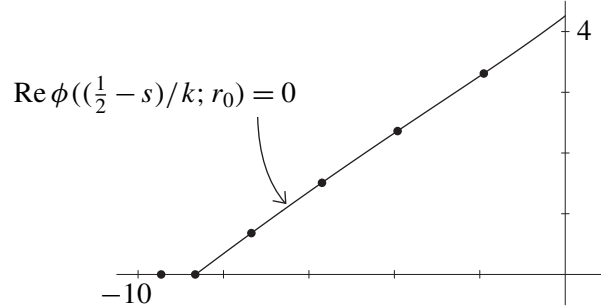
Figure 7. Using the equation $\mathrm{Re}\,\phi = 0$ to locate the resonances of F_{ℓ,r_0} occurring in the $k = 7$ Fourier mode, shown for $\ell = 2\pi$ and $r_0 = 1$.

Hence

$$\frac{[S_{F_{\ell,r}}(\tfrac{1}{2} + k\alpha)]_k}{[S_{F_\ell}(\tfrac{1}{2} + k\alpha)]_k} - 1 = -e^{-2k(\phi_0+\gamma)}\left(\frac{w_1(r)}{w_0(r)} - \frac{w_1(0)}{w_0(0)}\right)[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k \tag{7-11}$$

For $\arg\alpha \in [0, \pi/2 - \varepsilon]$, we deduce from (5-18) (using also the fact that $\mathrm{Re}(\phi - \phi_0) > c(\varepsilon, r)$) that

$$\left(\frac{w_1(r)}{w_0(r)} - \frac{w_1(0)}{w_0(0)}\right) = e^{2k\phi}(1 + O(|k\alpha|^{-1})). \tag{7-12}$$

The result then follows from (7-11) and the lower bound on $[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k$ provided by Lemma 6.1. $\square$

The estimates in Lemma 7.2 give approximate locations for the resonances in $\mathcal{R}_{F_{\ell,r_0}}$ arising from the k-th Fourier mode. The zeros of (7-10) correspond to resonances at $s = \tfrac{1}{2} - k\alpha$. This requires a cancellation between the two terms on the right side of (7-10). If $\mathrm{Re}\,\phi > 0$, then the second term is larger by approximately $e^{2k\phi}$ and cancellation only occurs near the poles of $[S_{F_\ell}(s)]_k$; this explains the poles of $[S_{F_{\ell,r_0}}(s)]_k$ on the negative real axis. For $\mathrm{Re}\,\phi = 0$, the two terms in (7-10) have the same magnitude; the resonances off the real axis in $\mathcal{R}_{F_{\ell,r_0}}$ thus occur near the line $\mathrm{Re}\,\phi((\tfrac{1}{2} - s)/k; r_0) = 0$ (and its conjugate). Figure 7 illustrates this phenomenon. For $\mathrm{Re}\,\phi < 0$, the first term in (7-10) is always larger than the second and no zeros occur.

Since $[S_{F_{\ell,r}}(\tfrac{1}{2} + k\alpha)]_k$ may indeed have zeros near the line $\mathrm{Re}\,\phi = 0$, proving a lower bound is more delicate in this region. By focusing on a relatively narrow strip, we can settle for a cruder estimate on the matrix elements in the vicinity of the zeros.

Lemma 7.3. *For $k \geq 0$ and $\mathrm{Re}\,s \geq \tfrac{1}{2}$ and assuming* $\mathrm{dist}(1 - s, \mathcal{R}_{F_\ell}) \geq |s|^{-\beta}$ *with $\beta > 2$,*

$$\log\left|\frac{[S_{F_{\ell,r_0}}(s)]_k}{[S_{F_\ell}(s)]_k}\right| \leq C(r_0, \beta)(k + |s|)\log|s|.$$

If $\mathrm{dist}(1 - s, \mathcal{R}_{F_{\ell,r_0}}) \geq |s|^{-\beta}$ *with $\beta > 2$, then we have*

$$\log\left|\frac{[S_{F_{\ell,r_0}}(s)]_k}{[S_{F_\ell}(s)]_k}\right| \geq -c(r_0, \beta)(k + |s|)\log|s|.$$

Proof. From (7-4), we note that $f_k(s; r_0)/\Gamma(s - \frac{1}{2})$ is an entire function of s. By Stirling's formula and the estimate (5-19), we can estimate its growth for large $|s|$ and $k \neq 0$ by

$$\log\left|\frac{f_k(s; r_0)}{\Gamma(s - \frac{1}{2})}\right| \leq C(r_0)(k + |s|) \log|s|),$$

where C is independent of k. The same estimate holds for $k = 0$, by the classical asymptotics of the hypergeometric function due to Watson [Erdélyi et al. 1953, Section 2.3.2]. Assuming that $\mathrm{dist}(s, \mathfrak{R}_{F_{\ell, r_0}}) \geq |s|^{-\beta}$, where $\beta > 2$, the minimum modulus theorem gives

$$\log\left|\frac{f_k(s; r_0)}{\Gamma(s - \frac{1}{2})}\right| \geq -c(r_0, \beta)(k + |s|) \log|s| \quad \text{for large } |s|.$$

The results follow from applying these estimates to

$$\frac{[S_{F_{\ell, r_0}}(s)]_k}{[S_{F_\ell}(s)]_k} = \frac{f_k(1 - s; r_0)}{f_k(s; r_0)} \frac{f_k(s; 0)}{f_k(1 - s; 0)}. \qquad \square$$

Proof of Theorem 7.1. We note that

$$N_{F_\ell}(t) \sim \tfrac{1}{4}\ell t^2 \quad \text{and} \quad 0\text{-vol}(F_{\ell, r_0}) = -\ell \sinh r_0.$$

By Corollary 3.2 and Theorem 4.1, the claimed asymptotic will be proved if we can show that there exists an unbounded set $\Lambda \subset [1, \infty)$ such that

$$\frac{2}{\pi} \int_0^{\pi/2} \log|\tau(\tfrac{1}{2} + ae^{i\theta})| \, d\theta \geq \frac{4a^2}{\pi} \int_0^{\pi/2} \int_0^\infty \frac{[I(xe^{i\theta}, \ell, r_0)]_+}{x^3} \, dx - \tfrac{1}{4}\ell a^2 - o(a^2) \qquad (7\text{-}13)$$

for all $a \in \Lambda$. We take

$$\Lambda := \left\{a \geq 1 : \mathrm{dist}\big(\{|s - \tfrac{1}{2}| = a\}, \mathfrak{R}_{F_\ell} \cup \mathfrak{R}_{F_{\ell, r_0}} \cup \mathbb{N}_0\big) \geq a^{-3}\right\}. \qquad (7\text{-}14)$$

Using the symmetry of coefficients under $k \to -k$, and estimating the $k = 0$ term by Lemma 7.3, we have

$$\log|\tau(\tfrac{1}{2} + ae^{i\theta})| = 2 \sum_{k=1}^\infty \log\left|\frac{[S_{F_{\ell, r_0}}(\tfrac{1}{2} + ae^{i\theta})]_k}{[S_{F_\ell}(\tfrac{1}{2} + ae^{i\theta})]_k}\right| + O(a \log a). \qquad (7\text{-}15)$$

Define $\varrho(\theta)$ by $\mathrm{Re}\,\phi(\varrho(\theta)e^{i\theta}, r_0) = 0$, as in the proof of Proposition 6.3, and assume for now that $\theta \leq \frac{1}{2}\pi - \varepsilon$. For $\delta > 0$, we will split the sum (7-15) at $a/k = \varrho(\theta)(1 \pm a^{-1/2})$. Let Σ_+ denote the portion of the sum with $a/k \geq \varrho(\theta)(1 + a^{-1/2})$. Under this condition, we want to derive a lower bound from Lemma 7.2 using the inequality

$$\log|1 + \lambda| \geq \log|\lambda| - \log 2 \quad \text{for } |\lambda| \geq 2.$$

For a sufficiently large, we will have $\mathrm{Re}\,\phi(xe^{i\theta}, r_0) \geq ca^{-1/2}$ for $x \geq \varrho(\theta)(1 + a^{-1/2})$. Thus, for $k \geq c\sqrt{a}$ we can deduce from Lemma 7.2 that

$$\log\left|\frac{[S_{F_{\ell, r_0}}(\tfrac{1}{2} + ae^{i\theta})]_k}{[S_{F_\ell}(\tfrac{1}{2} + ae^{i\theta})]_k}\right| \geq 2k\big(\mathrm{Re}\,\phi(ae^{i\theta}/k; r_3) - [\mathrm{Re}\,\phi_0(ae^{i\theta}/k)]_+\big) + O(1).$$

Arguing as in the proof of Proposition 6.3, we can then obtain

$$\sum_{c\sqrt{a}\leq k\leq a/(\varrho(\theta)(1+a^{-1/2}))} \log\left|\frac{[S_{F_{\ell,r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right|$$
$$\geq 2a^2\int_{\varrho(\theta)(1+a^{-1/2})}^{C\sqrt{a}} \frac{\mathrm{Re}\,\phi(xe^{i\theta},r_0)-[\mathrm{Re}\,\phi_0(xe^{i\theta})]_+}{x^3}\,dx - O(a\log a).$$

For $k\leq c\sqrt{a}$, Lemma 7.3 gives the estimate

$$\sum_{1\leq k\leq c\sqrt{a}} \log\left|\frac{[S_{F_{\ell,r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| \geq -O(a^{3/2}\log a).$$

On the other hand, since $|\mathrm{Re}\,\phi(\alpha,r)|=O(|\alpha|)$ for large $|\alpha|$, we also have

$$2a^2\int_{C\sqrt{a}}^{\infty} \frac{\mathrm{Re}\,\phi(xe^{i\theta},r_0)-[\mathrm{Re}\,\phi_0(xe^{i\theta})]_+}{x^3}\,dx = O(a^{3/2}).$$

We can also estimate

$$2a^2\int_{\varrho(\theta)}^{\varrho(\theta)(1+a^{-1/2})} \frac{\mathrm{Re}\,\phi(xe^{i\theta},r_0)-[\mathrm{Re}\,\phi_0(xe^{i\theta})]_+}{x^3}\,dx = O(a^{3/2})$$

since $\mathrm{Re}\,\phi(\alpha,\ell,r_0)$ is $O(\delta)$ in the range of integration. In combination, these estimates give

$$\Sigma_+ \geq 2a^2\int_{\varrho(\theta)}^{\infty} \frac{\mathrm{Re}\,\phi(xe^{i\theta},r_0)}{x^3}\,dx - \frac{\pi a^2}{2\omega}\sin^2\theta - O(a^{3/2}\log a) \quad \text{for } a\in\Lambda. \tag{7-16}$$

Let Σ_0 denote the portion of the sum in (7-15) for which $\varrho(\theta)(1-a^{-1/2}) < a/k < \varrho(\theta)(1+a^{-1/2})$. Since there are $O(a^{1/2})$ values of k in this range, Lemma 7.3 gives the estimate

$$\Sigma_0 \geq -O(a^{3/2}\log a). \tag{7-17}$$

Finally, we have Σ_-, defined as the portion of (7-15) with $a/k\leq\varrho(\theta)(1-a^{-1/2})$. Now we wish to apply Lemma 7.2 using

$$\log|1+\lambda| \geq -|\lambda|\log 4 \quad \text{for } |\lambda|\leq\tfrac{1}{2}.$$

Note that $I(xe^{i\theta},\ell,r_0)\leq -ca^{-1/2}$ for $x\leq\varrho(\theta)(1-a^{-1/2})$ and a sufficiently large, and that $k\geq ca$ in the range of Σ_-. Thus for large a Lemma 7.2 yields

$$\log\left|\frac{[S_{F_{\ell,r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| \geq -O(e^{-cka^{-1/2}}),$$

within the scope of Σ_-. We conclude that

$$\Sigma_- \geq -O(e^{-ca^{1/2}}). \tag{7-18}$$

Applying the estimates (7-16), (7-17), and (7-18) to the sum (7-15) now proves the lower bound

$$\frac{2}{\pi} \int_0^{\pi/2-\varepsilon} \log|\tau(\tfrac{1}{2}+ae^{i\theta})|\, d\theta \geq \frac{4a^2}{\pi} \int_0^{\pi/2-\varepsilon} \int_0^{\infty} \frac{[2\operatorname{Re}\phi(xe^{i\theta},r_0)]_+}{x^3}\, dx - \frac{2a^2}{\omega} \int_0^{\pi/2-\varepsilon} \sin^2\theta\, d\theta - o(a^2),$$

For the missing sectors, we appeal to Lemma 4.4 to see that

$$\frac{2}{\pi} \int_{\pi/2-\varepsilon}^{\pi/2} \log|\tau(\tfrac{1}{2}+ae^{i\theta})|\, d\theta \geq -c\varepsilon a^2.$$

We can thus take $\varepsilon \to 0$ to complete the proof of (7-13). □

Remark. In the proof of (1-13) given in [Borthwick 2010, Theorem 1.2], the Σ_- term was estimated incorrectly. This term is not necessarily positive, so the upper bound $O(e^{-ca})$ does not imply a corresponding lower bound. Instead, one needs to argue as in the derivation of (7-18) above. The estimates needed for the correct argument were given in [Borthwick 2010, Equations (6.8)–(6.10)].

8. Resonance asymptotics for extended funnels

Using the same notation as in Section 7, we now consider $F_{\ell,-r_0}$, defined as the subset $r \geq -r_0$ in a hyperbolic cylinder of diameter ℓ, where $r_0 \geq 0$. The metric and Laplacian are still given by (5-1) and (5-2), so that the scattering matrix elements are easily computed in terms of hypergeometric functions as before.

With reference to the even/odd solutions $w_k^{\pm}$ defined in (5-4) and (5-5), a solution $u_k(s;r)$ to the k-th eigenmode equation (5-3) satisfying $u_k(s;-r_0)=0$ can be written

$$u_k(s;r) = w_k^+(s;r_0)w_k^-(s;r) + w_k^-(s;r_0)w^+(s;r),$$

where $w_k^{\pm}(s;r)$ are the even/odd hypergeometric solutions defined in (5-4) and (5-5). Using the asymptotic expansions (7-3) as $r \to \infty$, we can read off the scattering matrix elements

$$[S_{F_{\ell,-r_0}}(s)]_k = \frac{\Gamma(\tfrac{1}{2}-s)}{\Gamma(s-\tfrac{1}{2})} \frac{\beta_k(2-s)w_k^+(s;r_0) + \beta_k(1-s)w_k^-(s;r_0)}{\beta_k(1+s)w_k^+(s;r_0) + \beta_k(s)w_k^-(s;r_0)}, \tag{8-1}$$

where $\beta_k(s)$ was defined in (6-5).

This shows in particular that

$$\mathscr{R}_{F_{\ell,-r_0}} = \bigcup_{k\in\mathbb{Z}}\{s : \beta_k(1+s)w_k^+(s;r_0) + \beta_k(s)w_k^-(s;r_0) = 0\}.$$

Theorem 8.1. *For the extended funnel with Dirichlet boundary conditions imposed at $r=-r_0$, for $r_0 \geq 0$, we have*

$$N_{F_{\ell,-r_0}}(t) \sim A(F_{\ell,-r_0})t^2,$$

where

$$A(F_{\ell,-r_0}) = \frac{\ell}{2\pi} \sinh r_0 + \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\infty} \frac{[I(xe^{i\theta},\ell,-r_0)]_+}{x^3}\, dx\, d\theta, \tag{8-2}$$

and $I(\alpha,\ell,r)$ was defined in (1-11).

Proof. Since $N_{F_\ell}(t) \sim \frac{1}{4}\ell t^2$ and $0\text{-vol}(F_{\ell,-r_0}) = \ell \sinh r_0$, Theorem 8.1 will follow from Corollary 3.2 and Theorem 3.3, once we establish

$$\frac{2}{\pi} \int_0^{\pi/2} \log|\tau(\tfrac{1}{2} + ae^{i\theta})| \, d\theta = \frac{4a^2}{\pi} \int_0^{\pi/2} \int_0^\infty \frac{[I(xe^{i\theta}, \ell, -r_0)]_+}{x^3} \, dx \, d\theta - \tfrac{1}{4}\ell a^2 - o(a^2), \qquad (8\text{-}3)$$

where Λ is defined again by (7-14).

As in the proof of Theorem 7.1, we start with the Fourier decomposition of the scattering matrices and use Lemma 7.3 to estimate the $k = 0$ term, leaving

$$\log|\tau(\tfrac{1}{2} + ae^{i\theta})| = 2 \sum_{k=1}^\infty \log \left| \frac{[S_{F_{\ell,-r_0}}(\tfrac{1}{2} + ae^{i\theta})]_k}{[S_{F_\ell}(\tfrac{1}{2} + ae^{i\theta})]_k} \right| + O(a \log a). \qquad (8\text{-}4)$$

If we define

$$g_k(s; r) := \Gamma(s - \tfrac{1}{2})\big(\beta_k(1 + s)w_k^+(s; r) + \beta_k(s)w_k^-(s; r)\big),$$

then by (8-1),

$$[S_{F_{\ell,-r_0}}(\tfrac{1}{2} + ae^{i\theta})]_k = g_k(\tfrac{1}{2} - ae^{i\theta})/g_k(\tfrac{1}{2} + ae^{i\theta}).$$

Assuming $k > 0$, we set $k\alpha = ae^{i\theta}$. Since $g_k(s; \cdot)$ solves (5-3), for $\operatorname{Re} \alpha \geq 0$, we can write

$$g_k(\tfrac{1}{2} \pm k\alpha; r) = B_0^\pm w_0(r) + B_1^\pm w_1(r),$$

where w_σ are the solutions given in (5-17).

As $r \to \infty$, the coefficient of ρ^{1-s} in the expansion of $g_k(\tfrac{1}{2} + k\alpha; r)$ is

$$2\Gamma(k\alpha)^2 \beta_k(\tfrac{1}{2} + k\alpha)\beta_k(\tfrac{3}{2} + k\alpha) = \frac{1}{\pi k\alpha}\left(1 - \frac{\cosh \pi k\omega}{\sin \pi k\alpha}\right)[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k. \qquad (8\text{-}5)$$

The coefficient of ρ^{1-s} in $g_k(\tfrac{1}{2} - k\alpha; r)$ is

$$\Gamma(k\alpha)\Gamma(-k\alpha)\big(\beta_k(\tfrac{1}{2} + k\alpha)\beta_k(\tfrac{3}{2} - k\alpha) + \beta_k(\tfrac{1}{2} - k\alpha)\beta_k(\tfrac{3}{2} + k\alpha)\big) = -\frac{1}{\pi k\alpha}\frac{\cosh \pi k\omega}{\sin \pi k\alpha}. \qquad (8\text{-}6)$$

Comparing these to the asymptotics for w_σ, as given in Lemma 5.3, we see that

$$B_1^+ = \frac{e^{-k(\phi_0+\gamma)}}{\pi k\sqrt{\alpha}}\left(1 - \frac{\cosh \pi k\omega}{\sin \pi k\alpha}\right)[S_{F_\ell}(\tfrac{1}{2} - k\alpha)]_k, \qquad (8\text{-}7)$$

and

$$B_1^- = -\frac{e^{-k(\phi_0+\gamma)}}{\pi k\sqrt{\alpha}}\frac{\cosh \pi k\omega}{\sin \pi k\alpha} \qquad (8\text{-}8)$$

We then find the B_0 coefficients by evaluating at $r = 0$,

$$B_0^\pm = \frac{1}{w_0(0)}\big(g_k(\tfrac{1}{2} \pm k\alpha; 0) - B_1^\pm w_1(0)\big). \qquad (8\text{-}9)$$

Since f_k and g_k agree at $r = 0$, (7-6) shows that

$$w_0(0) = A_0^+ g_k(\tfrac{1}{2} + k\alpha; 0), \quad \text{where } A_0^+ := \frac{1}{\pi k\sqrt{\alpha}}e^{k(\phi_0+\gamma)}.$$
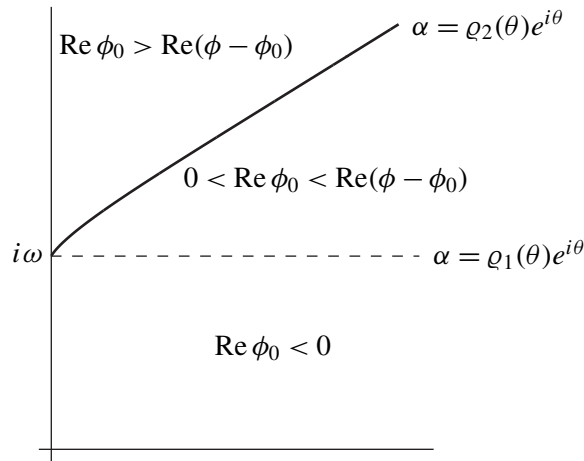
Figure 8. Positive and negative regions for $\mathrm{Re}(\phi(\alpha; r) - \phi_0(\alpha))$, shown for $r = 1$.

Combining these formulas gives

$$g_k(\tfrac{1}{2} + k\alpha; r) = A_0^+ w_0(r) + B_1^+\left(w_1(r) - \frac{w_1(0)}{w_0(0)} w_0(r)\right), \tag{8-10}$$

and

$$g_k(\tfrac{1}{2} - k\alpha; r) = [S_{F_\ell}(\tfrac{1}{2} + k\alpha)]_k A_0^+ w_0(r) + B_1^-\left(w_1(r) - \frac{w_1(0)}{w_0(0)} w_0(r)\right). \tag{8-11}$$

The asymptotic analysis of (8-10) is straightforward. The $B_1^+ w_1(r)$ term always dominates for $|k\alpha|$ large and $\arg\alpha \in [0, \pi/2 - \varepsilon]$, by Proposition 5.2. By applying Stirling's formula to (8-5) we find that

$$g_k(\tfrac{1}{2} + k\alpha; r) = \frac{1}{\pi k\sqrt{\alpha}}(\omega^2 + \alpha^2 \cosh^2 r)^{-1/4} e^{k(\phi - \phi_0 + \gamma)}(1 + O(|k\alpha|^{-1}). \tag{8-12}$$

The analysis of (8-11) more complicated. This term has both zeros and poles, and different terms can dominate for α in different regions. For $\alpha = xe^{i\theta}$, the borders between these regions will be denoted $x = \varrho_j(\theta)$ for $j = 1, 2$, where

$$\mathrm{Re}\,\phi_0(\varrho_1(\theta)e^{i\theta}) = 0 \quad \text{and} \quad \mathrm{Re}\big(\phi(\varrho_2(\theta)e^{i\theta}; r) - 2\phi_0(\varrho_2(\theta)e^{i\theta}; r)\big) = 0.$$

For the first curve we can be explicit, with $\varrho_1(\theta) = \omega\csc\theta$.

Consider first the portion of the sum (8-4) with $a/k \geq \varrho_2(\theta)$. In this region, $\mathrm{Re}\,\phi_0 > \mathrm{Re}(\phi - \phi_0)$ and the first term in (8-11) dominates the asymptotics. In this case, provided $|k\alpha| \in \Lambda$,

$$\log|g_k(\tfrac{1}{2} - k\alpha; r)| = k\,\mathrm{Re}(-\phi + \phi_0 - \gamma) + O(\log|k\alpha|).$$

For $k \leq a/\varrho_2(\theta)$, we thus have

$$\log\left|\frac{[S_{F_\ell, -r_0}(\tfrac{1}{2} + ae^{i\theta})]_k}{[S_{F_\ell}(\tfrac{1}{2} + ae^{i\theta})]_k}\right| = -2k\,\mathrm{Re}\left(\phi\left(\frac{ae^{i\theta}}{k}; r_0\right) - \phi_0\left(\frac{ae^{i\theta}}{k}\right)\right) + O(\log a).$$

This gives the estimate

$$\sum_{1 \le k \le a/\varrho_2(\theta)} \log\left|\frac{[S_{F_{\ell,-r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| = a^2 \int_{\varrho_2(\theta)}^{\infty} \frac{2\operatorname{Re}[\phi_0(xe^{i\theta})-\phi(xe^{i\theta};r_0)]}{x^3}\,dx + O(a\log a). \quad (8\text{-}13)$$

The region $\varrho_1(\theta) < a/k < \varrho_2(\theta)$ corresponds to $0 < \operatorname{Re}\phi_0 < \operatorname{Re}(\phi - \phi_0)$. In this case, the $B_1^- w_1(r)$ term dominates the asymptotics of (8-11), and we have

$$\log|g_k(\tfrac{1}{2}-k\alpha;r)| = k\operatorname{Re}(\phi - 3\phi_0 - \gamma) + O(\log|k\alpha|).$$

Using this along with (8-12) gives

$$\log\left|\frac{[S_{F_{\ell,-r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| = -2k\operatorname{Re}\phi_0(ae^{i\theta}/k) + O(\log a) \quad \text{for } k \le a/\varrho_2(\theta).$$

We conclude that

$$\sum_{a/\varrho_2(\theta)\le k\le a/\varrho_1(\theta)} \log\left|\frac{[S_{F_{\ell,-r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| = -a^2 \int_{\varrho_2(\theta)}^{\infty} \frac{2\operatorname{Re}\phi_0(xe^{i\theta})}{x^3}\,dx + O(a\log a). \quad (8\text{-}14)$$

The terms with $\operatorname{Re}\phi_0 \le 0$ make only lower order contributions. First of all, we can prove a general estimate,

$$\log\left|\frac{[S_{F_{\ell,-r_0}}(s)]_k}{[S_{F_\ell}(s)]_k}\right| = O((k+|s|)\log|s|),$$

just as in Lemma 7.3, to show that

$$\sum_{\varrho_1(\theta)(1-a^{-1/2})\le a/k\le\varrho_1(\theta)} \log\left|\frac{[S_{F_{\ell,-r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| = O(a^{3/2}\log a). \quad (8\text{-}15)$$

For the remaining terms, we use (8-10) and (8-11) to write

$$\frac{[S_{F_{\ell,-r}}(\frac{1}{2}+k\alpha)]_k}{[S_{F_\ell}(\frac{1}{2}+k\alpha)]_k} = 1 + \frac{e^{-k(\phi_0+\gamma)}}{\pi k\sqrt{\alpha}} \frac{[S_{F_\ell}(\frac{1}{2}-k\alpha)]_k}{g_k(\frac{1}{2}+k\alpha;r)}\left(w_1(r) - \frac{w_1(0)}{w_0(0)}w_0(r)\right).$$

This gives the estimate

$$\log\left|\frac{[S_{F_{\ell,-r}}(\frac{1}{2}+k\alpha)]_k}{[S_{F_\ell}(\frac{1}{2}+k\alpha)]_k} - 1\right| \le 2k\operatorname{Re}\phi_0(\alpha) + O(\log|k\alpha|).$$

For a sufficiently large, this gives

$$\sum_{a/k\le\varrho_1(\theta)(1-a^{-1/2})} \log\left|\frac{[S_{F_{\ell,-r_0}}(\frac{1}{2}+ae^{i\theta})]_k}{[S_{F_\ell}(\frac{1}{2}+ae^{i\theta})]_k}\right| = O(e^{-c\sqrt{a}}). \quad (8\text{-}16)$$

The estimates (8-14)–(8-16) cover all terms in the sum (8-4), and together yield

$$\log|\tau(\tfrac{1}{2}+ae^{i\theta})| = 2a^2 \int_{\varrho_2(\theta)}^{\infty} \frac{2\operatorname{Re}(2\phi_0(xe^{i\theta})-\phi(xe^{i\theta};r_0))}{x^3}\,dx - \frac{\pi a^2}{\omega}\sin^2\theta + O(a\log a)$$

for $a \in \Lambda$ and $0 \le \theta \le \pi/2 - \varepsilon$.

We now integrate over $\theta \in [0, \frac{1}{2}\pi - \varepsilon]$ and use Lemma 4.4 to control the limit $\varepsilon \to 0$, as in the proof of Theorem 7.1. This yields

$$\frac{2}{\pi} \int_0^{\pi/2} \log|\tau(\tfrac{1}{2} + ae^{i\theta})| \, d\theta = \frac{4a^2}{\pi} \int_0^{\pi/2} \int_{\varrho_2(\theta)}^{\infty} \frac{2 \operatorname{Re}(2\phi_0(xe^{i\theta}) - \phi(xe^{i\theta}; r_0))}{x^3} \, dx \, d\theta - \tfrac{1}{4}\ell a^2 - o(a^2).$$

To complete the proof of (8-3), recall the definition of $\phi(\alpha; r)$ as the integral of $\sqrt{f}\, dr$ in (5-7). Since the function f occurring there is an even function of r, the function $\phi - \phi_0$ will be odd in r. (This is not readily apparent from the definition (5-9).) This parity implies that

$$I(\alpha, \ell, -r_0) = 2 \operatorname{Re}(2\phi_0(\alpha) - \phi(\alpha; r_0)). \qquad \square$$

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