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AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH ONE MEASUREMENT AND THE PSEUDORANDOM SOURCE

TAPIO HELIN, MATTI LASSAS AND LAURI OKSANEN

We consider the wave equation $(\partial_t^2 - \Delta_g)u(t,x) = f(t,x)$, in $\mathbb{R}^n$, $u|_{\mathbb{R}_- \times \mathbb{R}^n} = 0$, where the metric $g = (g_{jk}(x))_{j,k=1}^n$ is known outside an open and bounded set $M \subset \mathbb{R}^n$ with smooth boundary ∂M. We define a source as a sum of point sources, $f(t,x) = \sum_{j=1}^\infty a_j \delta_{x_j}(x)\delta(t)$, where the points x_j, $j \in \mathbb{Z}_+$, form a dense set on ∂M. We show that when the weights a_j are chosen appropriately, $u|_{\mathbb{R}\times\partial M}$ determines the scattering relation on ∂M, that is, it determines for all geodesics which pass through M the travel times together with the entering and exit points and directions. The wave $u(t,x)$ contains the singularities produced by all point sources, but when $a_j = \lambda^{-\lambda^j}$ for some $\lambda > 1$, we can trace back the point source that produced a given singularity in the data. This gives us the distance in $(\mathbb{R}^n, g)$ between a source point x_j and an arbitrary point $y \in \partial M$. In particular, if $(\overline{M}, g)$ is a simple Riemannian manifold and g is conformally Euclidian in $\overline{M}$, these distances are known to determine the metric g in M. In the case when $(\overline{M}, g)$ is nonsimple, we present a more detailed analysis of the wave fronts yielding the scattering relation on ∂M.

1. Introduction

In this paper we consider an inverse problem for the wave equation

$$(\partial_t^2 - \Delta_g)u(t,x) = f(t,x) \quad \text{in } (0,\infty)\times\mathbb{R}^n,$$
$$u|_{t=0} = \partial_t u|_{t=0} = 0,$$

where Δ_g is the Laplace–Beltrami operator corresponding to a Riemannian metric $g(x) = [g_{jk}(x)]_{j,k=1}^n$, that is,

$$\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial x^j}\Big(|g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u\Big),$$

where $|g| = \det(g_{jk})$ and $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$ is the inverse matrix of $[g_{jk}(x)]_{j,k=1}^n$. We assume that $g_{jk} \in C^\infty(\mathbb{R}^n)$ and that there are $c_1, c_2 > 0$ such that

$$c_1|\xi|^2 \le \sum_{j.k=1}^n g_{jk}(x)\xi^j\xi^k \le c_2|\xi|^2, \quad x,\xi \in \mathbb{R}^n. \tag{1}$$

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Moreover, we assume that the metric g is known outside an open and bounded set $M \subset \mathbb{R}^n$ having a C^∞-smooth boundary ∂M.

We choose the origin of the time axis so that the source f is active at time $t = 0$. To ensure compatibility with the initial conditions, we let $T_0 < 0 < T$ and define the measurement map $L = L_g$,

$$L : C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n) \to C^\infty((T_0, T) \times \partial M), \quad Lf = u|_{(T_0,T)\times\partial M}, \tag{2}$$

where u is the solution of the wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)u(t, x) &= f(t, x) \quad \text{in } (T_0, T) \times \mathbb{R}^n, \\ u|_{t=T_0} &= \partial_t u|_{t=T_0} = 0. \end{aligned} \tag{3}$$

Above, $C_c^\infty(T_0, T)$ denotes the space of smooth functions having compact support in (T_0, T). Its dual space, the space of generalized functions or distributions, is denoted by $\mathscr{D}'(T_0, T)$. Moreover, for functions $\phi \in C_c^\infty(T_0, T)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$, we denote their pointwise product by $(\phi \otimes \psi)(t, x) = \phi(t)\psi(x)$.

We remark that the assumption (1), together with the finite speed of propagation for the wave equation, implies that the measurement Lf does not depend on $g_{jk}(x)$, for $|x| > R$, when R is sufficiently large. Thus we may assume without loss of generality that all the partial derivatives $\partial_x^\alpha g_{jk}$ are bounded on $\mathbb{R}^n$.

Let $x_j \in \partial M$, $j = 1, 2, \ldots$, be a dense sequence of points in ∂M, and let us consider point sources

$$f_{x_j}(t, x) := \delta(t)\delta_{x_j}(x), \quad j = 1, 2, \ldots.$$

In order to study the measurements Lf_{x_j}, we will use the Sobolev spaces (see [Triebel 1978])

$$\begin{aligned} H_p^s(\mathbb{R}^d) &:= \left\{ f \in \mathscr{S}'(\mathbb{R}^d);\ \|f\|_{H_p^s(\mathbb{R}^d)} := \left\|(1-\Delta)^{s/2} f\right\|_{L^p(\mathbb{R}^d)} < +\infty \right\}, \\ \widetilde{H}_p^s(U) &:= \{ f \in H_p^s(\mathbb{R}^d);\ \operatorname{supp} f \subset \bar{U} \}, \\ H_p^s(U) &:= \{ f \in \mathscr{D}'(U);\ f = h|_U \text{ for some } h \in H_p^s(\mathbb{R}^d) \}, \end{aligned}$$

where $U \subset \mathbb{R}^d$ is open and $s \in \mathbb{R}$. When $p = 2$ we omit the subscript p in our notation, that is, we write $H^s(U) = H_2^s(U)$, etc. Moreover, we use projective topology on the tensor product $X \otimes Y$ of two Banach spaces X and Y, that is, $\|z\|_{X\otimes Y} := \inf \sum_j \|x_j\|_X \|y_j\|_Y$, where the infimum is taken over all representations $z = \sum_j x_j \otimes y_j$. We also use projective topology on tensor products of locally convex spaces; see, e.g., [Trèves 1967, Definition 43.2]. The measurement Lf_{x_j} can be defined in the sense of the following lemma.

Lemma 1.1. *Let $p \in (1, n/(n-1))$ and let $m \in \mathbb{N}$ satisfy $m > (n+1)/4$. Then the measurement operator L defined in (2) has a unique continuous extension*

$$L : \widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to \mathscr{D}'((T_0, T) \times \partial M).$$

We will prove Lemma 1.1 and other results presented in the introduction in Sections 3–6.

In this paper we study a single measurement Lh_0 that simultaneously combines all the measurements Lf_{x_j} by adding them together with appropriate weights. When the measurements Lf_{x_j} are summed together, to the authors' knowledge, there are no algorithms that can filter the value of a particular measurement from the sum. We will ask, however, whether we can find the essential features given by these measurements, like the travel times between points on ∂M, so that the metric could be determined under certain geometric conditions. Our main result is that Lh_0 determines the *scattering relation* $\Sigma_{M,g}$ for the manifold $(\overline{M}, g)$. Here $h_0(t,x)$ is an explicit source that we call pseudorandom; see Definition 1 in Section 2.

The scattering relation has been efficiently used to solve several geometric inverse problems [Dairbekov and Uhlmann 2010; Pestov and Uhlmann 2006; Stefanov and Uhlmann 2008; 2009]. To define the scattering relation, let TM denote the tangent space of M and let $\dot\gamma$ denote the tangent vector of a smooth curve $\gamma : [a,b] \to M$. Let $SM = \{(x,\xi) \in TM;\ \|\xi\|_g = 1\}$ be the unit sphere bundle on M and define

$$\partial_\pm SM = \big\{(x,\xi) \in SM;\ x \in \partial M,\ \mp(\nu,\xi)_g > 0\big\},$$

where ν is the exterior normal vector of ∂M. Moreover, let $\tau_{M,g}(x,\xi)$ be the infimum of the set

$$\big\{t \in (0,\infty];\ \gamma_{x,\xi}(t) \in \partial M\big\},$$

where $\gamma_{x,\xi}$ denotes the geodesic on (M,g) with initial data $(x,\xi) \in TM$. We write $\tau = \tau_{M,g}$ when the manifold (M,g) is clear from the context. We define the infimum of the empty set to be $+\infty$.

The scattering relation is the map $\Sigma = \Sigma_{M,g}$,

$$\Sigma : \mathscr{D}(\Sigma) \to \overline{\partial_+ SM} \times \mathbb{R}, \quad \mathscr{D}(\Sigma) = \big\{(x,\xi) \in \partial_- SM;\ \tau(x,\xi) < \infty\big\},$$

defined by $\Sigma(x,\xi) = \big(\gamma_{x,\xi}(\tau(x,\xi)), \dot\gamma_{x,\xi}(\tau(x,\xi)), \tau(x,\xi)\big)$.

Our main result is the following.

Theorem 1.2. *Let $M \subset \mathbb{R}^n$, $n \ge 2$ be an open and bounded set having a C^∞-smooth boundary. Then there is a generalized function $h_0(t,x)$ supported on $\{0\} \times \partial M$ and having the following properties: Assume that $g_{jk}, g'_{jk} \in C^\infty(\mathbb{R}^n)$ are two Riemannian metric tensors satisfying (1). Moreover, assume that $g_{jk}(x) = g'_{jk}(x)$ for $x \in \mathbb{R}^n \setminus M$. Let*

$$T > \max\Big(\sup_{(x,\xi)\in\partial_- SM} \tau_{M,g}(x,\xi),\ \sup_{(x,\xi)\in\partial_- SM} \tau_{M,g'}(x,\xi)\Big),$$

and assume that

$$L_g h_0 = L_{g'} h_0 \quad \text{on } (T_0,T) \times \partial M.$$

Then the scattering relations $\Sigma_{M,g}$ and $\Sigma_{M,g'}$ of Riemannian manifolds (M,g) and (M,g') are the same. In particular, if $(\overline{M},g)$ and $(\overline{M},g')$ are simple, the restrictions of the distance functions on the boundary satisfy $d_{\overline{M},g}(x,y) = d_{\overline{M},g'}(x,y)$ for $x,y \in \partial M$.

We remark that if $\sup_{\partial_- SM} \tau$ is infinite, then we prove the above result with measurements on an infinite time interval, that is, we prove that the measurement $u|_{(T_0,\infty)\times\partial M}$ determines $\mathscr{D}(\Sigma)$ and Σ.

Recall that a compact Riemannian manifold $(\overline{M},g)$ with boundary is simple if it is simply connected, any geodesic has no conjugate points, and ∂M is strictly convex with respect to the metric g. Any two points of a simple manifold can be joined by a unique geodesic.

The key idea of the proof of Theorem 1.2 is to use source $h_0(t,x)=\sum_{j=1}^{\infty} a_j f_{x_j}$. The point source $a_{j_0} f_{x_{j_0}}$ produces a singularity, which is observed at a point $y \in \mathbb{R}^n \setminus M$ at time $t_0 = d(x_{j_0}, y)$ with a magnitude $a_{j_0}\beta(x_{j_0}, y)$, where β is an unknown nonvanishing smooth function. Appropriate choice of the weights a_j allows us find the index j_0 by looking at nearby singularities. Indeed, when $x_{j_k} \to x_{j_0}$ and $j_k \to \infty$, we see that the asymptotic behavior of the magnitude $a_{j_k}\beta(x_{j_k}, y)$ as $k \to \infty$ will be that of the weights a_{j_k}. Thus it is possible to factor out a_{j_k} in the magnitude and determine a_{j_0}. This argument is presented in Section 7 and gives us the distances $d(x_j, y)$ in $(\mathbb{R}^n, g)$ for arbitrary point $y \in \mathbb{R}^n \setminus M$ and a source point x_j.

Theorem 1.2 and boundary rigidity results for simple manifolds imply the following:

Corollary 1.3. *Let $M \subset \mathbb{R}^n$ and let $g_{jk}, g'_{jk} \in C^\infty(\mathbb{R}^n)$ be two Riemannian metric tensors satisfying the assumptions of Theorem 1.2. Let $(\overline{M}, g)$ and $(\overline{M}, g')$ be simple Riemannian manifolds. Then*:

(i) *If $n = 2$ and*

$$L_g h_0 = L_{g'} h_0 \quad \text{on } (T_0, T) \times \partial M, \tag{4}$$

then there is a diffeomorphism $\Phi : M \to M$ such that $\Phi|_{\partial M} = \mathrm{Id}$ and $g = \Phi_* g'$.

(ii) *For $n \geq 3$, there is $\epsilon = \epsilon_{n,M} > 0$ such that if $\|g_{jk} - \delta_{jk}\|_{C^2(M)} < \epsilon_n$, $\|g'_{jk} - \delta_{jk}\|_{C^2(M)} < \epsilon_n$, and (4) holds, then there is a diffeomorphism $\Phi : M \to M$ such that $\Phi|_{\partial M} = \mathrm{Id}$ and $g = \Phi_* g'$.*

(iii) *If $g_{jk}(x) = a(x)\delta_{jk}$ and $g'_{jk}(x) = a'(x)\delta_{jk}$, that is, the metric tensors are conformally Euclidian, and (4) holds, then $g_{jk}(x) = g'_{jk}(x)$ for $x \in M$.*

Indeed, by Theorem 1.2, case (i) follows from [Pestov and Uhlmann 2005], case (ii) follows from [Burago and Ivanov 2010], and (iii) from [Muhometov 1977; 1981; Muhometov and Romanov 1978].

If Uhlmann's conjecture [2003], that the scattering relation determines the isometry type of nontrapping compact manifolds with nonempty boundary, can be proven, then Corollary 1.3 holds for a more general class of manifolds.

The problem of determining the metric g (possibly up to a diffeomorphism) given the measurement Lh_0 with only one function $h_0(t,x)$ is a formally determined inverse problem. Indeed, the formally computed "dimension of the data," that is, the dimension of $(T_0, T) \times \partial M$, is n and coincides with the dimension of the set M on which the unknown functions $g_{jk}(x)$ are defined.

The formally determined inverse problems have been studied in many cases. For instance, the two-dimensional Calderón inverse problem [Astala and Päivärinta 2006; Astala et al. 2005; Imanuvilov et al. 2010; Nachman 1996; Sylvester 1990] is formally determined. The same is true for the related inverse problem for the Schrödinger equation in two dimensions [Bukhgeim 2008]. The corresponding inverse problems in dimension $n \geq 3$ — see [Calderón 1980; Kenig et al. 2007; Lassas et al. 2003; Nachman 1988; Sylvester and Uhlmann 1987] and references in [Greenleaf et al. 2009a; 2009b], are overdetermined — that is, the dimension of the data is larger than the dimension of the unknown object. Similar classification holds for the elliptic inverse problems on Riemannian manifolds [Guillarmou and Tzou 2010; 2011; Lassas et al. 2003; Lee and Uhlmann 1989; 2001]. Moreover, the boundary rigidity problem [Kurylev et al. 2010; Michel 1981; Muhometov 1977; 1981; Muhometov and Romanov 1978; Romanov 1987; Stefanov and Uhlmann 2005] is formally determined in dimension $n = 2$ and overdetermined for $n \geq 3$.

Inverse problems in the time domain related to the Laplace–Beltrami operator Δ_g, namely the inverse boundary value problem for the wave, heat, and dynamical Schrödinger equations with Dirichlet-to-Neumann as data — see [Anderson et al. 2004; Belishev and Kurylev 1992; Katchalov and Kurylev 1998; Katchalov et al. 2001] — are overdetermined in dimensions $n \geq 2$. However, these problems are equivalent to the inverse boundary spectral problem (see [Katchalov et al. 2004]), and assuming that the eigenvalues are simple, the Dirichlet-to-Neumann map at a generic Dirichlet boundary value determines the boundary spectral data [Lassas 1995; 1998; Ramm 2001]. Thus, under generic conditions on the spectrum and on the boundary value (that is, under the condition that the these data belong in some open and dense set), it is possible to solve a formally determined inverse problem in time domain.

We point out that in this paper we do not impose any generic conditions on the geometry, and we give an explicit construction of the boundary source. The boundary source considered in this paper is based on the idea of imitating a realization of white noise, and due to the many useful properties of the white noise process, we hope that the constructed source may be useful in the study of other inverse problems requiring generic assumptions on the source.

Another formally determined hyperbolic inverse problem, namely measuring Neumann data when the initial data $(u|_{t=0}, \partial_t u|_{t=0})$ is nonzero and satisfies subharmonicity or positivity conditions, has been studied using Carleman estimates [Bellassoued and Yamamoto 2008; Imanuvilov and Yamamoto 2003; Isakov 2006; Klibanov 1992; Stefanov and Uhlmann 2011]. The present paper is closely related to these studies, but we emphasize that we assume that the initial data for u vanishes.

Moreover, there are two approaches to solving the formally determined hyperbolic inverse problem to determine a potential from a single boundary measurement. The first one uses Carleman estimates analogous to the estimates mentioned above and assumes similar conditions on the initial data [Bukhgeim and Klibanov 1981]. The second one relies on an adaptation of the Gelfand–Levitan method to multidimensional problems [Rakesh and Sacks 2011; Rakesh 2003; 2008; Romanov 2002; Sacks and Symes 1985].

2. Pseudorandom source

In this section we define a special source $h_0(t, x)$ which we call *pseudorandom*. The specific assumptions on the amplitudes are explained in Section 7. An important feature of the pseudorandom source is that it is supported only on a single point in time.

Definition 1. Let $x_j \in \partial M$, $j = 1, 2, \dots$, be a dense sequence of distinct points in ∂M, and let $a_j \in \mathbb{R}$, $j = 1, 2, \dots$, with $\sum_{j=1}^{\infty} |a_j| < \infty$, be a sequence of distinct numbers.

We define the pseudorandom source on $(x_j)_{j=1}^{\infty} \subset \partial M$ with coefficients $(a_j)_{j=1}^{\infty} \subset \mathbb{R}$ as the following generalized function on $\mathbb{R} \times \mathbb{R}^n$:

$$h_0(t, x) := \sum_{j=1}^{\infty} a_j \delta(t) \delta_{x_j}(x), \qquad (x, t) \in \mathbb{R}^{n+1},$$

where $\delta(t)$ and $\delta_{x_j}(x)$ are Dirac delta distributions on $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

It is rather straightforward to show that h_0 is well-defined. First, it is well known that $\delta(t) \in H^{-1}(\mathbb{R})$ and $\delta_{x_j}(x) \in C(\mathbb{R}^n)'$. Next, we have $H^1_{p'}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ when $1 > n/p'$, due to [Triebel 1978, Theorem 2.8.1]. According to [ibid., Theorem 2.6.1], the dual space satisfies $(H^1_{p'}(\mathbb{R}^n))' = H^{-1}_p(\mathbb{R}^n)$ with $1/p' = 1 - 1/p$, and hence $C(\mathbb{R}^n)' \subset H^{-1}_p(\mathbb{R}^n)$ for $1 < p < n/(n-1)$. Since $\sum_{j=1}^\infty |a_j| < \infty$, we have

$$\sum_{j=1}^{\infty} a_j \delta_{x_j}(x) \in H^{-1}_p(\mathbb{R}^n).$$

This yields that for any $p \in \left(1, \dfrac{n}{n-1}\right)$ and $\epsilon > 0$, the pseudorandom source h_0 satisfies

$$h_0 \in \widetilde{H}^{-1}(-\epsilon, \epsilon) \otimes \widetilde{H}^{-1}_p(M). \tag{5}$$

The spatial structure of the pseudorandom source can be motivated by the structure of the white noise. In the 1-dimensional radar imaging models, white noise signals are considered to be optimal sources when imaging a stationary scatterer [Toomay and Hannen 2004]. This is due to the fact that different translations of the white noise signal are uncorrelated. In a similar fashion, we have the following property for the pseudorandom source h_0: for each x_{j_0} and each sequence $(x_{j_k})_{k=1}^\infty$ converging to x_{j_0} and satisfying $x_{j_k} \neq x_{j_0}$ for all $k \in \mathbb{Z}_+$, it holds that $a_{j_k} \to 0$. This property will be crucial in what follows.

A natural strategy to choose the points x_j is by random sampling. The term pseudorandom refers to the fact that algorithmic generators of random numbers use, in fact, a deterministic function to produce a sequence of numbers, but the mixingness of the process is such that the user of the algorithm can consider the numbers to be analogous to independent samples of a random variable. In this manner, the pseudorandom source can be seen as an imitation of a realization of a noise process.

Another source of inspiration for us was a rather new measurement paradigm called compressed sensing [Candès et al. 2006; Donoho 2006], where one aims for a sparse reconstruction of a linear problem using a small number of noisy measurements. We point out that by using the pseudorandom source, one can compress the measurements Lf_{x_j} with point sources f_{x_j} into a single measurement Lh_0.

3. Measurement map

In this section we prove that the measurement Lh_0 is well-defined. Let us consider the operator $W : f \mapsto u$ mapping f to the solution of (3). We call such an operator the solution operator for (3). First, we note that by [Hörmander 1985, Theorem 23.2.2], the operator $W : f \mapsto u$ extends in a unique way to a continuous linear operator

$$W : L^1\big((T_0, T); H^s(\mathbb{R}^n)\big) \to C\big([T_0, T]; H^{s+1}(\mathbb{R}^n)\big), \quad s \in \mathbb{R}. \tag{6}$$

Moreover, if $f \in C^\infty([T_0, T] \times \mathbb{R}^n)$ and $\mathrm{supp}(f) \Subset (T_0, T] \times \mathbb{R}^n$, that is, $\mathrm{supp}(f)$ is a compact subset of $(T_0, T] \times \mathbb{R}^n$, then $Wf \in C^\infty([T_0, T] \times \mathbb{R}^n)$.

We will compose the operator W with the one-sided inverse $\mathcal{J}$ of the derivative ∂_t, which is given by

$$\mathcal{J}u(t) := \int_{T_0}^t u(t')dt', \quad u \in C_c^\infty(T_0, T).$$

One sees easily that this operator has a unique continuous linear extension $\mathcal{I} : \widetilde{H}^{-1}(T_0, T) \to L^2(T_0, T)$.

Next we prove Lemma 1.1 formulated in the introduction, that is, we prove that the measurement map L has a unique continuous extension

$$\widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to \mathcal{D}'((T_0, T) \times \partial M). \tag{7}$$

Proof of Lemma 1.1. For sufficiently large $z \in \mathbb{R}_+$, the operator $z - \Delta_g$ is an isomorphism between spaces $H^{s+2}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ as well as between spaces $H_p^{s+2}(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n)$ for all integers s by [Shubin 1992].

By the definition of L, we have that $L = \mathrm{Tr} \circ W$, where Tr is the trace operator

$$\mathrm{Tr}(u) = u|_{(T_0,T)\times\partial M}, \quad u \in C^\infty((T_0, T) \times \mathbb{R}^n).$$

Let $f \in C_c^\infty((T_0, T) \times \mathbb{R}^n)$. Then the solution $u = Wf$ of the wave equation $(\partial_t^2 - \Delta_g)u = f$ can be written in the form

$$Wf = (z - \partial_t^2)^m (z - \Delta_g)^{-m} Wf + \sum_{j=0}^{m-1} (z - \partial_t^2)^j (z - \Delta_g)^{-1-j} f. \tag{8}$$

Now $f = \partial_t \mathcal{I} f$, where $\mathcal{I} f$ is C^∞-smooth and satisfies $\mathrm{supp}(\mathcal{I} f) \Subset (T_0, T] \times \mathbb{R}^n$. By (6), $W\mathcal{I} f$ is C^∞-smooth and $\partial_t W \mathcal{I} f = W \partial_t \mathcal{I} f = Wf$. Hence

$$Lf = \partial_t (z - \partial_t^2)^m \,\mathrm{Tr}(z - \Delta_g)^{-m} W\mathcal{I} f + \sum_{j=0}^{m-1} (z - \partial_t^2)^j \,\mathrm{Tr}(z - \Delta_g)^{-1-j} f. \tag{9}$$

Let us next consider terms appearing in (9). First we consider extension of the operator

$$\begin{aligned} \sum_{k=1}^{N} \phi_k \otimes \psi_k \mapsto & \sum_{k=1}^{N} (z - \partial_t^2)^j \,\mathrm{Tr}(z - \Delta_g)^{-1-j} (\phi_k \otimes \psi_k) \\ &= \sum_{k=1}^{N} \big((z - \partial_t^2)^j \phi_k\big) \otimes \big(\mathrm{Tr}(z - \Delta_g)^{-1-j} \psi_k\big), \quad j = 0, \dots, m-1, \end{aligned} \tag{10}$$

mapping $C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n)$ to $C^\infty((T_0, T) \times \partial M)$. By [Triebel 1978, Theorem 4.7.1], the maps

$$H_p^{-1}(\mathbb{R}^n) \xrightarrow{(z-\Delta_g)^{-1-j}} H_p^1(\mathbb{R}^n) \xrightarrow{\mathrm{Tr}} B_{p,p}^{1-1/p}(\partial M)$$

are continuous, where $B_{p,p}^{1-1/p}(\partial M)$ is the Besov space on ∂M. Thus the operator (10) has a continuous extension in spaces (7).

Next, consider extension of the operator

$$\sum_{k=1}^{N} \phi_k \otimes \psi_k \mapsto \sum_{k=1}^{N} \partial_t (z - \partial_t^2)^m \,\mathrm{Tr}(z - \Delta_g)^{-m} W((\mathcal{I}\phi_k) \otimes \psi_k) \tag{11}$$

mapping $C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n)$ to $C^\infty((T_0, T) \times \partial M)$. As $-1 - n/p > -1 - n$, we have by [Triebel 1978, Theorem 2.8.1] a continuous embedding $H_p^{-1}(\mathbb{R}^n) \hookrightarrow H^{-1-n/2}(\mathbb{R}^n)$. Moreover, the operator

$\mathcal{J} : \widetilde{H}^{-1}(T_0, T) \to L^2(T_0, T)$ and the embedding $L^2(T_0, T) \otimes H^{-1-n/2}(\mathbb{R}^n) \hookrightarrow L^2\big((T_0, T); H^{-1-n/2}(\mathbb{R}^n)\big)$ are continuous. Thus, by (6),

$$W\mathcal{J} : \widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to C\big([T_0, T]; H^{-n/2}(\mathbb{R}^n)\big)$$

is continuous.

As $(1-\Delta_g)^{-m} : C\big([T_0, T]; H^{-n/2}(\mathbb{R}^n)\big) \to C\big([T_0, T]; H^{-n/2+2m}(\mathbb{R}^n)\big)$ is continuous and $-\frac{n}{2}+2m > \frac{1}{2}$, we see that the operator

$$\operatorname{Tr}(1-\Delta_g)^{-m} W\mathcal{J} : \widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to C\big([T_0, T]; L^2(\partial M)\big)$$

is continuous.

Combining the above results, we see that the operator (9) has a continuous extension to the spaces (7). As the spaces $C_c^\infty(T_0, T)$ and $C_c^\infty(\mathbb{R}^n)$ are dense in $\widetilde{H}^{-1}(T_0, T)$ and $H_p^{-1}(\mathbb{R}^n)$, respectively, we see that the continuous extension of L is unique. □

4. Inner product of a solution and a source

Lemma 4.1. *Let $f \in C_c^\infty((T_0, T) \times M)$ and $t_0 \in (T_0, T)$ and let $w \in C^\infty([T_0, t_0] \times \mathbb{R}^n)$ satisfy*

$$(\partial_t^2 - \Delta_g)w = 0, \quad \text{in } (T_0, t_0) \times \mathbb{R}^n.$$

Then

$$\int_{T_0}^{t_0} \int_{\mathbb{R}^n} f(t,x) w(t,x)\, dt\, dV(x) = \int_{\mathbb{R}^n} \big((\partial_t Wf)(t_0,x) w(t_0,x) - (Wf)(t_0,x)(\partial_t w)(t_0,x)\big)\, dV(x),$$

where $dV(x) = |g|^{1/2}\, dx$ is the Riemannian volume measure of $(\mathbb{R}^n, g)$ and $W : f \mapsto u$ is the solution operator of the wave equation (3).

Proof. By finite speed of propagation of waves [Ladyzhenskaya 1985, pp. 150–156], $\operatorname{supp}(Wf(t))$ is compact in $\mathbb{R}^n$. The claim follows by integration by parts:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big((\partial_t u)(t_0,x) w(t_0,x) - u(t_0,x)(\partial_t w)(t_0,x)\big)\, dV(x) \\
&\qquad - \int_{\mathbb{R}^n} \big((\partial_t u)(T_0,x) w(T_0,x) - u(T_0,x)(\partial_t w)(T_0,x)\big)\, dV(x) \\
&\qquad\qquad = \int_{(T_0,t_0)\times\mathbb{R}^n} \big((\partial_t^2 - \Delta_g)u(t,x)\, w(t,x) - u(t,x)\, (\partial_t^2 - \Delta_g)w(t,x)\big)\, dt\, dV(x) \\
&\qquad\qquad = \int_{(T_0,t_0)\times\mathbb{R}^n} f(t,x) w(t,x)\, dt\, dV(x).
\end{aligned}$$ □

Next, we will prove a generalization of the previous lemma for nonsmooth sources f. Denote by $B(0, R) = \{x \in \mathbb{R}^n;\ |x| < R\}$ the Euclidean ball. The finite speed of propagation for the wave equation yields that there is $R > 0$ such that all $f \in C_c^\infty((T_0, T) \times M)$ satisfy $\operatorname{supp}(Wf) \Subset (T_0, T] \times B(0, R)$. We define

$$\Omega := B(0, R) \setminus \overline{M}. \tag{12}$$

Below, we use the fact (see [Evans 1998, Theorems 7.2.3/6 and 5.6.3/6]) that the operator $W_\Omega : h \mapsto v$ mapping h to the solution of the equation

$$\begin{aligned}(\partial_t^2 - \Delta_g)v(t,x) &= 0 \quad \text{in } (T_0,T)\times\Omega,\\ v|_{(T_0,T)\times\partial\Omega} &= h,\\ v|_{t=T_0} = 0, &\quad \partial_t v|_{t=T_0} = 0,\end{aligned} \tag{13}$$

is continuous as a map $W_\Omega : C_c^\infty((T_0,T)\times\partial\Omega) \to C^\infty([T_0,T]\times\bar\Omega)$.

We let $t_0 \in (T_0,T)$ and write

$$\Sigma := \{t_0\}\times\Omega. \tag{14}$$

We denote the trace on Σ by Tr_Σ, that is, we define $(\mathrm{Tr}_\Sigma\, u)(x) := u(t_0,x)$. Let $\nu = \nu(z)$ denote the exterior unit normal vector of ∂M at z.

Moreover, let U be an open subset (or a submanifold) of $\mathbb{R}^n$, and let us denote by dV (or dS) the Riemannian volume measure of (U,g). We embed the test functions into the spaces of distribution by using the inner product of the space $L^2(U; dV)$, that is, we identify $u \in C_0^\infty(U)$ with the distribution

$$\psi \mapsto \int_U u(x)\psi(x)\,dV(x). \tag{15}$$

We will denote the distribution pairing of $u \in \mathscr{D}'(U)$ and $\psi \in C_0^\infty(U)$ by $(u,\psi)_{\mathscr{D}'(U)}$ and use analogous notations for other distribution pairings.

Lemma 4.2. *Let $t_0 \in (T_0,T)$ and define Σ by (14). Then operators $\mathrm{Tr}_\Sigma\, W_\Omega$ and $\mathrm{Tr}_\Sigma\, \partial_t W_\Omega$ have unique continuous extensions $\mathscr{E}'((T_0,t_0)\times\partial\Omega) \to \mathscr{D}'(\Omega)$.*

Proof. Let v satisfy (13). Consider a function $w \in C^\infty([T_0,t_0]\times\bar\Omega)$ such that $(\partial_t - \Delta_g)w = 0$ in $(T_0,t_0)\times\Omega$ and $w|_{(T_0,t_0)\times\partial\Omega} = 0$. Then

$$\begin{aligned}0 &= \int_{\Omega\times(T_0,t_0)} \big((\partial_t - \Delta_g)v\big)w - v\big((\partial_t-\Delta_g)w\big)\,dV(x)\,dt\\ &= \left[\int_\Omega \big((\partial_t v)w - v(\partial_t w)\big)\,dV(x)\right]_{t=T_0}^{t=t_0} + \int_{\partial\Omega\times(T_0,t_0)} \big((\partial_\nu v)w - v(\partial_\nu w)\big)\,dS(x)\,dt\\ &= \int_\Omega \big((\partial_t v)w - v(\partial_t w)\big)\,dV(x)\Big|_{t=t_0} - \int_{\partial\Omega\times(T_0,t_0)} h(\partial_\nu w)\,dS(x)\,dt,\end{aligned}$$

where ∂_ν is the normal derivative on $\partial\Omega$,

Denote by $W_1 : f_1 \mapsto w$ the solution operator of the equation

$$\begin{aligned}(\partial_t - \Delta_g)w(t,x) &= 0 \quad \text{in } (T_0,t_0)\times\Omega,\\ w|_{(T_0,t_0)\times\partial\Omega} &= 0,\\ w|_{t=t_0} = f_1, &\quad \partial_t w|_{t=t_0} = 0.\end{aligned}$$

The operator $W_1 : C_c^\infty(\Omega) \to C^\infty([T_0,t_0]\times\bar\Omega)$ is continuous, as can be seen using Theorems 7.2.3/6 and 5.6.3/6 of [Evans 1998]. Hence, the operator

$$\partial_\nu W_1 : C_c^\infty(\Omega) \to C^\infty([T_0, t_0] \times \partial\Omega), \quad f \mapsto \partial_\nu W_1 f|_{\partial\Omega}$$

is continuous. Moreover,

$$(\mathrm{Tr}_\Sigma\, \partial_t W_\Omega h, f_1)_{L^2(\Omega; dV)} = (h, \partial_\nu W_1 f_1)_{L^2((T_0,t_0)\times\partial\Omega; dt\otimes dS)},$$

where ∂_ν is the normal derivative on $\partial\Omega$. We define the extension of $\mathrm{Tr}_\Sigma\, \partial_t W_\Omega$ by identifying it with the transpose $(\partial_\nu W_1)^t : \mathscr{E}'((T_0, t_0) \times \partial\Omega) \to \mathscr{D}'(\Omega)$ of the operator $\partial_\nu W_1 : C_c^\infty(\Omega) \to C^\infty([T_0, t_0] \times \partial\Omega)$.

Similarly, we define the extension of $\mathrm{Tr}_\Sigma\, W_\Omega$ by the transpose $(\partial_\nu W_2)^t : \mathscr{E}'((T_0, t_0) \times \partial\Omega) \to \mathscr{D}'(\Omega)$ of $\partial_\nu W_2 : C_c^\infty(\Omega) \to C^\infty([T_0, t_0] \times \partial\Omega)$, where $W_2 : f_2 \mapsto w$ is the solution operator of the equation

$$\begin{aligned} (\partial_t - \Delta_g) w(t, x) &= 0 \quad \text{in } (T_0, t_0) \times \Omega, \\ w|_{(T_0,t_0)\times\partial\Omega} &= 0, \\ w|_{t=t_0} &= 0, \quad \partial_t w|_{t=t_0} = -f_2. \end{aligned}$$

□

Denote by $d_\Omega(x, y)$, $x, y \in \overline{\Omega}$, the distance function of Riemannian manifold $(\overline{\Omega}, g|_{\overline{\Omega}})$. Next we generalize the result of Lemma 4.1 for a larger class of functions.

Lemma 4.3. *Let $t_0 \in (0, T)$ and $\epsilon > 0$ satisfy $[-\epsilon, \epsilon] \subset (T_0, t_0)$. Define Σ by (14). Let*

$$f \in \widetilde{H}^{-1}(-\epsilon, \epsilon) \otimes \widetilde{H}_p^{-1}(M) \quad \textit{and} \quad w \in C^\infty([T_0, t_0] \times \mathbb{R}^n)$$

satisfy

$$(\partial_t^2 - \Delta_g) w = 0, \quad \textit{in } (T_0, t_0) \times \mathbb{R}^n.$$

Suppose that $w(t_0), \partial_t w(t_0) \in C_c^\infty(\Omega)$, and let $\chi \in C_c^\infty(T_0, t_0)$ satisfy $\chi = 1$ in a neighborhood of $[-\epsilon, t_0 - r]$, where

$$r := d_\Omega\big(\mathrm{supp}(w(t_0)) \cup \mathrm{supp}(\partial_t w(t_0)), \partial\Omega\big).$$

Then

$$(f, w)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} = (\mathrm{Tr}_\Sigma\, \partial_t W_\Omega \chi L f, w)_{\mathscr{D}'(\Omega)} - (\mathrm{Tr}_\Sigma\, W_\Omega \chi L f, \partial_t w)_{\mathscr{D}'(\Omega)}, \tag{16}$$

where we have defined $Lf = 0$ on $\partial B(0, R)$. Here we regard Ω as a Riemannian manifold $(\Omega, g|_\Omega)$.

Proof. We suppose first that $f \in C_c^\infty((-\epsilon, \epsilon) \times M)$. Recall that W is solution operator of wave equation (3). Then $Wf(\cdot, t) = 0$ if $t < -\epsilon$, and

$$Lf = \mathrm{Tr}_{\partial\Omega}\, Wf = \chi\, \mathrm{Tr}_{\partial\Omega}\, Wf, \quad \text{in } (T_0, t_0 - r) \times \partial\Omega,$$

where $\mathrm{Tr}_{\partial\Omega}$ is the trace on $(T_0, T) \times \partial\Omega$. As $\Omega \cap \overline{M} = \varnothing$, we have that $(\partial_t^2 - \Delta_g) Wf = 0$ in $(T_0, T) \times \Omega$. By uniqueness of the solution of (13),

$$W_\Omega \chi\, \mathrm{Tr}_{\partial\Omega}\, Wf = Wf, \quad \text{in } (T_0, t_0 - r) \times \Omega.$$

By finite speed of propagation,

$$\mathrm{Tr}_\Sigma\, \partial_t^j W_\Omega \chi\, \mathrm{Tr}_{\partial\Omega}\, Wf = \mathrm{Tr}_\Sigma\, \partial_t^j Wf, \quad j = 0, 1,$$

on $\{t_0\} \times \mathrm{supp}(w(t_0)) \cup \mathrm{supp}(\partial_t w(t_0))$. By Lemma 4.1, (16) holds.

Then the claim follows, as the embeddings

$$C_c^\infty(-\epsilon,\epsilon) \hookrightarrow \widetilde{H}^{-1}(-\epsilon,\epsilon), \quad C_c^\infty(M) \hookrightarrow \widetilde{H}_p^{-1}(M)$$

are dense and operators $(\mathrm{Tr}_\Sigma\, \partial_t^j W_\Omega)\chi L : \widetilde{H}^{-1}(-\epsilon,\epsilon) \otimes \widetilde{H}_p^{-1}(M) \to \mathscr{D}'((T_0,t_0)\times\partial\Omega)$, $j=0,1$, are continuous. □

5. Gaussian beams

We consider solutions of the wave equation which are known as Gaussian beams [Babich et al. 1985; Babich and Ulin 1981; Ralston 1982]. These solutions have been constructed to analyze the propagation of singularities for the wave equation in the presence of caustics. Here we use Gaussian beams as an auxiliary technical tool to analyze singularities in the measurements.

Definition 2. Let $\epsilon > 0$, $N \in \mathbb{N}$ and let γ be a unit speed geodesic on $(\mathbb{R}^n, g)$. A formal Gaussian beam of order N propagating along geodesic γ is a function U_ϵ^N of form

$$U_\epsilon^N(t,x) = \epsilon^{-n/4}\exp\bigl\{-(i\epsilon)^{-1}\theta(t,x)\bigr\}\sum_{m=0}^{N} u_m(t,x)(i\epsilon)^m, \quad t\in\mathbb{R},\ x\in\mathbb{R}^n,$$

satisfying the following properties: The phase function θ and the amplitude functions u_m, with $m = 0,1,\dots,N$, are complex-valued smooth functions. The phase function θ satisfies the conditions

$$\theta(t,\gamma(t)) = 0, \quad \mathrm{Im}\,\theta(t,x) \ge C_0(t)d(x,\gamma(t))^2,$$

where $C_0(t)$ is a continuous strictly positive function. The amplitude function u_0 satisfies $u_0(t,\gamma(t)) \ne 0$. Finally, for any compact set $K \Subset \mathbb{R}\times\mathbb{R}^n$, there is a constant $C > 0$ such that the inequality

$$\bigl|(\partial_t^2 - \Delta_g)U_\epsilon^N(t,x)\bigr| \le C\epsilon^{N-n/4}$$

is satisfied uniformly for $(t,x)\in K$.

The construction of a formal Gaussian beam $U_\epsilon^N(t,x)$ is considered in detail in [Katchalov et al. 2001, Section 2.4]. Next, we recall the construction and pay attention to the properties of Gaussian beams which we need later.

Let us write the geodesic γ in the usual coordinates of $\mathbb{R}^n$ as $\gamma(t) = \bigl(\gamma^1(t),\dots,\gamma^n(t)\bigr)$. We construct the phase function $\theta(t,x)$ at each time $t\in\mathbb{R}$ in terms of a finite Taylor expansion in the x variable centered at $\gamma(t)$,

$$\theta(t,x) = \sum_{|\alpha|\le N} \frac{\theta_\alpha(t)}{\alpha!}(x-\gamma(t))^\alpha,$$

where θ_α are smooth functions and $N\in\mathbb{N}$.

Let $e_j = (\delta_{1j},\dots,\delta_{nj})$ be multi-indices with the value 1 at the j-th place. For clarity, we use the notation $p_j(t) = \theta_{e_j}(t)$ for the first-order coefficients and the notation $H_{jk}(t) = \theta_\alpha(t)$, $\alpha = e_j + e_k$, for the second-order coefficients in the expansion of θ.

The construction of a formal Gaussian beam consists of the following steps.

(1) We define $\theta_0(t)=0$ and $p_j(t)=\sum_{k=1}^n g_{jk}(\gamma(t))\dot\gamma^k(t)$, that is, the first-order coefficients $p_j(t)$ are the covariant representation of the velocity vector $\dot\gamma$.

(2) The symmetric matrix $H(t)=[H_{jk}(t)]_{j,k=1}^n$ of the second-order coefficients is obtained by solving a Riccati equation, or an equivalent system of ordinary differential equations. We write $H(t)=Z(t)Y(t)^{-1}$, where the pair of complex $n\times n$ matrices $(Z(t),Y(t))$ is the solution of the system of ordinary differential equations

$$\frac{d}{dt}Y(t)=B(t)^*Y(t)+C(t)Z(t),\qquad Y|_{t=0}=Y^0,$$
$$\frac{d}{dt}Z(t)=-D(t)Y(t)-B(t)Z(t),\quad Z|_{t=0}=Z^0.$$

Here we choose the initial values to be $Z^0=iI$ and $Y^0=I$, where I is the identity matrix and i is the imaginary unit. The matrices $B(t)$, $C(t)$, and $D(t)$ in $\mathbb{R}^{n\times n}$ have components given by the second derivatives of the Hamiltonian $h(x,p)=\left(\sum_{j,k=1}^n g_{jk}(x)p^jp^k\right)^{1/2}$ evaluated in the point $(x,p)=(\gamma(t),p(t))$:

$$B_l^j=\frac{\partial^2 h}{\partial x^l\partial p_j};\quad C^{jl}=\frac{\partial^2 h}{\partial p_j\partial p_l};\quad D_{jl}=\frac{\partial^2 h}{\partial x^j\partial x^l}.$$

The fact that the complex matrix $Y(t)$ is invertible for all $t\in\mathbb{R}$ is crucial for the construction, and is discussed in detail in [Katchalov et al. 2001, Section 2.4].

(3) The coefficients $\theta_\alpha(t)$ of order $|\alpha|=m\geq 3$ are solved inductively, with respect to m. The coefficients $\theta_\alpha(t)$ are constructed using the coefficients $\tilde\theta_\alpha(t)$ defined so that

$$\sum_{|\alpha|=m}\tilde\theta_\alpha(t)\tilde y^\alpha=\sum_{|\alpha|=m}\theta_\alpha(t)(x-\gamma(t))^\alpha,$$

for all $\tilde y=Y^{-1}(t)(x-\gamma(t))$, $y\in\mathbb{C}^n$. We obtain the coefficients $\tilde\theta_\alpha(t)$ by solving successive linear systems of ordinary differential equations

$$\frac{d}{dt}\tilde\theta_\alpha(t)=K_\alpha(t),\quad \tilde\theta_\alpha(0)=0,$$

where $K_\alpha(t)$ depend on $\theta_\beta(t)$ with $|\beta|\leq m-1$, the matrix $H(t)$, the vector $p(t)$, and the metric g_{jk} and its derivatives at $\gamma(t)$.

(4) When the phase function $\theta(t,x)$ is constructed, the amplitude functions $u_n(t,x)$ are solved using the transport equations, or equivalently, the following ordinary differential equations. Let

$$u_m(t,x)=\sum_{|\alpha|\leq N}\tilde u_{m,\alpha}(t)\tilde y^\alpha,\quad \tilde y=Y^{-1}(t)(x-\gamma(t)),$$

where the coefficients $\tilde u_{m,\alpha}(t)$ are obtained by solving the successive equations

$$\frac{d}{dt}\tilde u_{m,\alpha}(t)+r(t)\tilde u_{m,\alpha}(t)=\mathscr{F}_{m,\alpha}(t),\quad \tilde u_{m,\alpha}(0)=\delta_{m,0}\delta_{|\alpha|,0},$$

where $r(t)$ and $\mathscr{F}_{m,\alpha}(t)$ depend on $\tilde u_{m',\beta}$ with $|\beta|\leq|\alpha|+2$ and $m'\leq m-1$, the function $\theta(t,x)$, the metric g_{jk}, and their derivatives at (t,x), $x=\gamma(t)$.

By the above construction, we have the following remark.

Remark 1. The phase function $\theta(t,x)$ and the amplitude functions $u_m(t,x)$ at time $t=0$ have the form

$$\theta(0,x)=\sum_{j,k=1}^{n} g_{jk}(y)\eta^k(x^j-y^j)+i|x-y|^2,$$

where $(y,\eta)=(\gamma(0),\dot\gamma(0))$ is the initial data of the geodesic γ, $u_0(0,x)=1$, and $u_m(0,x)=0$ for $m>0$. Hence, $U_\epsilon^N(0,x)$ is dependent on the metric g_{jk} only via $g_{jk}(y)$. Moreover, $\partial_t U_\epsilon^N(0,x)$, although of more complex form, is dependent on the metric g_{jk} only via $\partial^\alpha g_{jk}(y)$ for a certain finite collection of multi-indices $\alpha\in\mathbb{N}^n$.

If the coefficients of an ordinary differential equation depend smoothly on some parameter, so does the solution [Amann 1990], and thus we see using an induction that the phase function θ and the amplitude functions u_m depend smoothly on the initial data $(y,\eta)=(\gamma(0),\dot\gamma(0))$ of the geodesic γ. In particular, the amplitude function $u_0(t,x;y,\eta)$ satisfies

$$u_0\in C^\infty(\mathbb{R}\times\mathbb{R}^n\times S\mathbb{R}^n). \tag{17}$$

Thus far we have considered a formal Gaussian beam. By using continuous dependency of the solution of the wave equation on the source term, one obtains the following results [Katchalov et al. 2001]:

Let γ be a unit speed geodesic, $N\in\mathbb{N}$, $\epsilon>0$, and let U_ϵ^N be a formal Gaussian beam of order N propagating along geodesic γ. Let $\chi\in C_0^\infty(\mathbb{R}^n)$ be a function which is identically one in a neighborhood of $\gamma(0)$ and let $t_0>0$ and let R be the radius in (12). Then for $j\in\mathbb{N}$ and $\alpha\in\mathbb{N}^n$ satisfying $j+|\alpha|<N-n/4$, there is $C>0$ such that the solution w_ϵ of the wave equation,

$$\begin{aligned}(\partial_t^2-\Delta_g)w_\epsilon(t,x)&=0,\qquad (t,x)\in(T_0,t_0)\times\mathbb{R}^n,\\ w_\epsilon(t_0,x)&=\chi(x)U_\epsilon^N(0,x),\\ \partial_t w_\epsilon(t_0,x)&=-\chi(x)\partial_t U_\epsilon^N(0,x),\end{aligned}\tag{18}$$

satisfies

$$\sup_{x\in B(0,R),t\in(T_0,t_0)}\left|\partial_t^j\partial_x^\alpha\big(w_\epsilon(t_0-t,x)-U_\epsilon^N(t,x)\big)\right|\le C\epsilon^{N-(j+|\alpha|)-n/4}. \tag{19}$$

We call w_ϵ a Gaussian beam of order N propagating along geodesic γ backwards on time interval (T_0,t_0).

6. Determination of the travel times

Lemma 6.1. *Let w_ϵ be a Gaussian beam of order $N\ge 1+n/4$ propagating along geodesic γ backwards on time interval (T_0,t_0), that is, let w_ϵ be the solution of (18). Let h_0 be the pseudorandom source*

$$h_0(t,x)=\sum_{j=1}^{\infty}a_j\delta(t)\delta_{x_j}(x). \tag{20}$$

If $\gamma(t_0)\ne x_j$ for all $j=1,2,\dots$, then

$$\lim_{\epsilon\to 0}\epsilon^{n/4}(h_0,w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))}=0.$$

Moreover, if $\gamma(t_0) = x_j$, *then*

$$\lim_{\epsilon\to 0} \epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} = a_j u_0(t_0, x_j)|g|^{1/2}(x_j),$$

where $u_0(t,x)$ *is the first amplitude function of a formal Gaussian beam propagating along geodesic* γ.

We remind the reader that the test functions are embedded in $\mathscr{E}'(\mathbb{R}^n \times (T_0, T))$ using (15).

Proof. By (19), we have that

$$\begin{aligned}\epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} &= \epsilon^{n/4}\sum_{j=1}^{\infty} a_j U_\epsilon^N(t_0, x_j)|g|^{1/2}(x_j) + O(\epsilon)\\ &= \sum_{j=1}^{\infty} a_j u_0(t_0, x_j)\exp\{-(i\epsilon)^{-1}\theta(t_0, x_j)\}|g|^{1/2}(x_j) + O(\epsilon).\end{aligned}$$

As $\operatorname{Im}\theta(t_0, x_j) \geq C_0(t_0)d(x_j, \gamma(t_0))$, we have that

$$\left|\exp\{-(i\epsilon)^{-1}\theta(t_0, x_j)\}\right| = O(\epsilon), \quad \text{if } \gamma(t_0) \neq x_j.$$

Suppose that $\gamma(t_0) = x_j$. Then $\exp\{-(i\epsilon)^{-1}\theta(t_0, x_j)\} = 1$ and there is a constant $C > 0$ depending on γ and t_0 such that

$$\begin{aligned}&\left|\epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} - a_j u_0(t_0, x_j)|g|^{1/2}(x_j)\right|\\ &\quad\leq C\sum_{k=1}^{j-1}|a_k|\left|\exp\{-(i\epsilon)^{-1}\theta(t_0, x_k)\}\right| + C\sum_{k=j+1}^{l}|a_k|\left|\exp\{-(i\epsilon)^{-1}\theta(t_0, x_k)\}\right| + C\sum_{l+1}^{\infty}|a_l| + O(\epsilon).\end{aligned}$$

We may first choose large $l \in \mathbb{N}$ and then small $\epsilon > 0$ so that the above three sums are arbitrarily small. The case $\gamma(t_0) \neq x_j$ for all $j = 1, 2, \ldots,$ is similar. □

Next we define an auxiliary function $S(y_0, \eta_0, t_0)$ which is nonzero if and only if there is $j \in \mathbb{Z}_+$ such that $\gamma_{y_0,\eta_0}(t_0) = x_j$.

Definition 3. Let $(y_0, \eta_0) \in T\mathbb{R}^n$ be such that $y_0 \in \Omega^{\text{int}}$ and $\|\eta_0\|_g = 1$. We denote by $\gamma(t; y_0, \eta_0) = \gamma_{y_0,\eta_0}(t)$ the geodesic on $(\mathbb{R}^n, g)$ with $\gamma(0) = y_0$, $\dot\gamma(0) = \eta_0$. Moreover, let w_ϵ be a Gaussian beam of order $N \geq 1 + n/4$ propagating along $\gamma(t; y, \eta)$ backwards on time interval (T_0, t_0). We define

$$S(y_0, \eta_0, t_0) := \lim_{\epsilon\to 0} \epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))}.$$

Lemma 6.2. *Let* $(y_0, \eta) \in S\Omega$ *and* $t_0 \in (0, T)$. *Then* Lh_0, *for pseudorandom source* h_0, *and* $(\Omega, g|_\Omega)$, *given as a Riemannian manifold*, *determine* $S(y_0, \eta_0, t_0)$.

Proof. Let w_ϵ be a Gaussian beam of order $N \geq 1 + n/4$ propagating along the geodesic $\gamma(\,\cdot\,; y_0, \eta_0)$ backwards on time interval (T_0, t_0). We may choose the cut-off function χ in (18) so that $w_\epsilon(t_0)$ and $\partial_t w_\epsilon(t_0)$ lie in $C_c^\infty(\Omega)$. As $g|_\Omega$ is known, we have by Remark 1 that the initial data $w_\epsilon(t_0)$, $\partial_t w_\epsilon(t_0)$ are known. Moreover, operators $\operatorname{Tr}_\Sigma \partial_t^j W_\Omega$, $j = 0, 1$, $\Sigma := \{t_0\} \times \Omega$, are known. After choosing a suitable cut-off function χ in Lemma 4.3, we have that the measurement Lh_0 determines the distributional pairing $(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t))}$. Hence $S(y_0, \eta_0, t_0)$ is determined. $\square$

The implicit function theorem yields the following remark. Note that $t_0 \in \mathbb{R}$ in the remark is not necessarily the first intersection time.

Remark 2. Let $(y_0, \eta_0) \in S\mathbb{R}^n$ and $t_0 \in \mathbb{R}$ satisfy

$$\big(\gamma(t_0; y_0, \eta_0), \dot\gamma(t_0; y_0, \eta_0)\big) \in \partial_\pm SM.$$

Then there are neighborhoods $I \subset \mathbb{R}$ and $U \subset S\mathbb{R}^n$ of t_0 and (y_0, η_0) and a smooth map $\ell : U \to I$ such that for $t \in I$ and $(y, \eta) \in U$,

$$\gamma(t; y, \eta) \in \begin{cases} M, & \text{for } \pm t < \pm\ell(y, \eta), \\ \partial M, & \text{for } t = \ell(y, \eta), \\ \Omega, & \text{for } \pm t > \pm\ell(y, \eta). \end{cases}$$

We remind the reader that $\tau(x, \xi)$, $(x, \xi) \in T\mathbb{R}^n$, is defined as the first intersection time with ∂M:

$$\tau(y_0, \eta_0) := \inf\big\{t \in (0, \infty]; \gamma(t; y_0, \eta_0) \in \partial M\big\}.$$

In the following, we use the Sasaki metric on the tangent bundle TM.

Lemma 6.3. *The first intersection times $\tau : S\Omega \to (0, \infty]$ and $\tau : \partial_- SM \to (0, \infty]$ are lower semicontinuous.*

Proof. Let us consider τ on $S\Omega$. Let a sequence $((y_j, \eta_j))_{j=1}^\infty \subset S\Omega$ converge to $(y_0, \eta_0) \in S\Omega$ as $j \to \infty$. We write $\gamma_j(t) := \gamma(t; y_j, \eta_j)$ and $\tau_j := \tau(y_j, \eta_j)$.

We will show next that $\liminf_{j\to\infty} \tau_j \notin (0, \tau_0)$. Let $t \in (0, \tau_0)$. Then $\gamma_0(t) \notin \partial M$ and

$$d(\gamma_0(t), \partial M) > 0.$$

Let $j \in \mathbb{Z}_+$. Suppose for a moment that $\tau_j < \infty$. Noting that γ_j is unit speed and $\gamma_j(\tau_j) \in \partial M$, we have

$$|t - \tau_j| \ge d\big(\gamma_j(t), \gamma_j(\tau_j)\big) \ge d(\gamma_j(t), \partial M).$$

If $\tau_j = \infty$, then $|t - \tau_j| = \infty > d(\gamma_j(t), \partial M)$.

The convergence $\gamma_j(t) \to \gamma_0(t)$, as $j \to \infty$, implies that for large j,

$$|t - \tau_j| \ge \frac{d(\gamma_0(t), \partial M)}{2} > 0.$$

Hence, $\liminf_{j\to\infty} \tau_j \neq t$ for all $t \in (0, \tau_0)$.

Clearly $\liminf_{j\to\infty} \tau_j \ge 0$, and there is $J \in \mathbb{Z}_+$ such that

$$\tau_j \ge d(y_j, \partial M) \ge \frac{d(y_0, \partial M)}{2} > 0, \quad j \ge J.$$

Hence, $\liminf_{j\to\infty} \tau_j \neq 0$ and $\liminf_{j\to\infty} \tau_j \ge \tau_0$.

Let us consider τ on $\partial_- SM$. Let a sequence $((y_j, \eta_j))_{j=1}^\infty \subset \partial_- SM$ converge to $(y_0, \eta_0) \in \partial_- SM$ as $j \to \infty$. We write $\gamma_j(t) := \gamma(t; y_j, \eta_j)$ and $\tau_j := \tau(y_j, \eta_j)$.

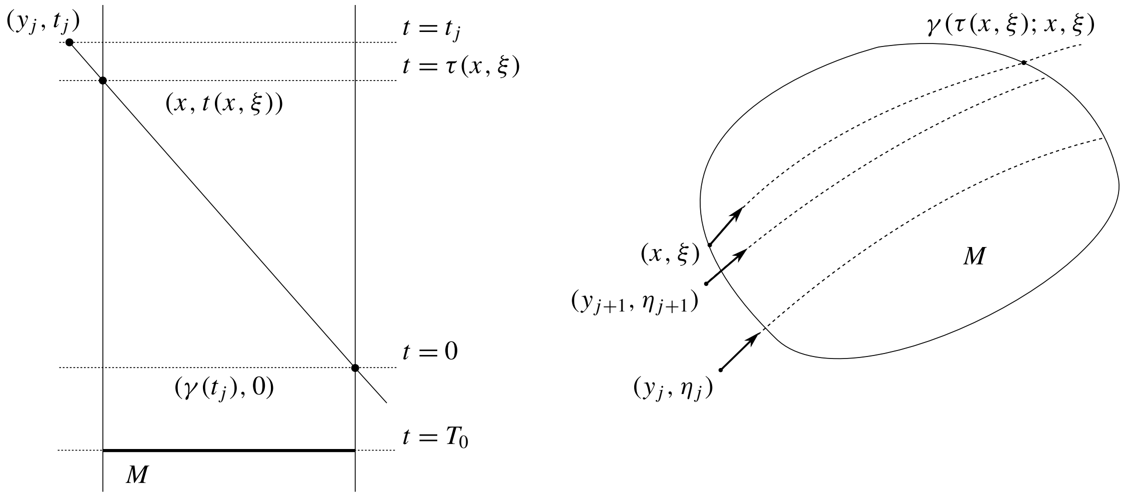

Figure 1. On the left, the trajectory of a Gaussian beam propagating along geodesic $\gamma(t) := \gamma(t; y_j, \eta_j)$ backwards on time interval (T_0, t_j). If $S(y_j, \eta_j, t_j) \neq 0$, then there is a point source at $\gamma(t_j)$. On the right, a sequence $(y_j, \eta_j) \in S\Omega$ converging to $(x, \xi) \in \partial_- SM$ and trajectories of the corresponding geodesics.

Repeating the above argument, we see that $\liminf_{j\to\infty} \tau_j \notin (0, \tau_0)$. Thus it is enough to show that $\liminf_{j\to\infty} \tau_j \neq 0$.

Remark 2 gives neighborhoods $I \subset \mathbb{R}$ and $U \subset S\mathbb{R}^n$ of zero and (y_0, η_0) and a map $\ell : U \to I$ of boundary intersection times. We write $V := U \cap \partial_- SM$. As $\gamma(0; x, \xi) \in \partial M$ for $(x, \xi) \in V$, we have $\ell = 0$ in V. In particular, $r := d\big(\ell(V), \mathbb{R} \setminus I\big) > 0$. For large j, $(\gamma_j(0), \dot\gamma_j(0)) \in V$, and thus

$$\gamma_j(t) \in M, \quad t \in (0, r).$$

Hence, $\tau_j \geq r > 0$ for large j, and $\liminf_{j\to\infty} \tau_j \geq \tau_0$. □

We easily see the following continuity result for τ.

Lemma 6.4. *Let $((y_j, \eta_j))_{j=1}^{\infty} \subset S\Omega$ converge to $(x, \xi) \in \partial_- SM$ in the Sasaki metric. Then*

$$\lim_{j\to\infty} \tau(y_j, \eta_j) = 0.$$

Theorem 6.5. *Let $(x, \xi) \in \partial_- SM$, and denote by $J(x, \xi)$ the set of sequences*

$$((t_j; y_j, \eta_j))_{j=1}^{\infty} \subset (0, \infty) \times S\Omega$$

for which

$$\lim_{j\to\infty} (y_j, \eta_j) = (x, \xi), \quad \lim_{j\to\infty} t_j \in (0, \infty), \quad S(y_j, \eta_j, t_j) \neq 0.$$

The function $S : S\Omega \times (0,\infty) \to \mathbb{C}$ *determines* $\tau : \partial_- SM \to (0,\infty]$ *by the formula*

$$\tau(x,\xi) = \inf\Big\{ \lim_{j\to\infty} t_j;\ ((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi) \textit{ for some } ((y_j,\eta_j))_{j=1}^{\infty} \subset S\Omega \Big\}.$$

Moreover, if $\tau(x,\xi) < \infty$*, then there is a sequence* $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$ *satisfying*

$$\tau(x,\xi) = \lim_{j\to\infty} t_j.$$

Proof. Let $(x,\xi) \in \partial_- SM$ and $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$. Let us show that $\tau(x,\xi) \le \lim_{j\to\infty} t_j$. By Lemma 6.4, $\tau_j := \tau(y_j,\eta_j) \to 0$ as $j \to \infty$. We define

$$\tilde{y}_j := \gamma(\tau_j; y_j, \eta_j), \quad \xi_j := \dot{\gamma}(\tau_j; y_j, \eta_j).$$

As $S(y_j, \eta_j, t_j) \neq 0$, we have

$$\gamma(t_j - \tau_j; \tilde{y}_j, \xi_j) = \gamma(t_j; y_j, \eta_j) \in \partial M.$$

As $\lim_{j\to\infty} t_j > 0$ and $\lim_{j\to\infty} \tau_j = 0$, we have $t_j - \tau_j > 0$ for large j. Thus $\tau(\tilde{y}_j, \xi_j) \le t_j - \tau_j$ for large j. Moreover,

$$\lim_{j\to\infty} (\tilde{y}_j, \xi_j) = \big(\gamma(0; x, \xi), \dot{\gamma}(0; x, \xi)\big) = (x, \xi).$$

In particular, $(\tilde{y}_j, \xi_j) \in \partial_- SM$ for large j. Hence, Lemma 6.3 gives

$$\lim_{j\to\infty} t_j = \lim_{j\to\infty} (t_j - \tau_j) \ge \liminf_{j\to\infty} \tau(\tilde{y}_j, \xi_j) \ge \tau(x,\xi).$$

In particular, we have proved the claim in the case $\tau(x,\xi) = \infty$.

Let us assume that $\tau(x,\xi) < \infty$. It is enough to show that there is a sequence $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$ satisfying $\tau(x,\xi) = \lim_{j\to\infty} t_j$. We write

$$t_0 := \tau(x,\xi), \quad z := \gamma(t_0; x, \xi), \quad \zeta := -\dot{\gamma}(t_0; x, \xi).$$

We have

$$(x,\xi) = \big(\gamma(t_0; z, \zeta), -\dot{\gamma}(t_0; z, \zeta)\big).$$

As $(x,\xi) \in \partial_- SM$, Remark 2 gives neighborhoods I and U of t_0 and (z,ζ) and a map $\ell : U \to I$ of boundary intersection times. After choosing local coordinates around z, we may define

$$(y_j, \eta_j) := \big(\gamma(t_j; x_{k_j}, \zeta), -\dot{\gamma}(t_j; x_{k_j}, \zeta)\big),$$

where $(x_{k_j})_{j=1}^{\infty} \subset U$ is a subsequence of the dense sequence of source points in (20) satisfying $\lim\limits_{j\to\infty} x_{k_j} = z$ and $(t_j)_{j=1}^{\infty} \subset I$ satisfies

$$t_j > \ell(x_{k_j}, \zeta), \quad \lim_{j\to\infty} t_j = \ell(z,\zeta) = t_0.$$

Clearly, $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$ and

$$\lim_{j\to\infty} t_j = t_0 = \tau(x,\xi). \qquad \square$$

7. Determination of the scattering relation

In the next theorem, we consider the pseudorandom source $h_0(t, x)$ with coefficients

$$a_j = \lambda^{-\lambda^j},$$

with some $\lambda > 1$, and make computations "modulo an error in A", where

$$A = \{-\lambda^j : j \in \mathbb{N}\}.$$

To this end, let $m_A(s)$ be the real number r such that $s = r + a$, where $a \in A$ and r has the smallest possible absolute value. In the case when both r and $-r$ satisfy this condition, we choose the positive value.

Lemma 7.1. *Let $(y_0, \eta_0) \in S\Omega$, $t_0 \in (0, T)$, and suppose that $S(y_0, \eta_0, t_0) \neq 0$. Then there is a sequence $((y_j, \eta_j))_{j=1}^\infty \subset S\Omega$ and $(t_j)_{j=1}^\infty \subset (0, T)$ such that*

$$(y_j, \eta_j) \to (y_0, \eta_0), \quad t_j \to t_0, \quad S(y_j, \eta_j, t_j) \to 0, \quad \text{as } j \to \infty, \quad S(y_j, \eta_j, t_j) \neq 0. \tag{21}$$

Suppose, moreover, that the coefficients of the pseudorandom source h_0 are $a_j = \lambda^{-\lambda^j}$. Then for any sequences $((y_j, \eta_j))_{j=1}^\infty \subset T\mathbb{R}^n$ and $(t_j)_{j=1}^\infty \subset (0, T)$ satisfying (21), we have that

$$\lim_{j\to\infty} m_A\big(\log_\lambda |S(y_j, \eta_j, t_j)|\big) = \log_\lambda \big|u_0\big(t_0, \gamma(t_0); y_0, \eta_0\big)|g|^{1/2}(\gamma(t_0))\big|,$$

where $\gamma(t) = \gamma(t; y_0, \eta_0)$ and u_0 is defined as in (17).

Proof. We will use the notation

$$\gamma_j(t) := \gamma(t; y_j, \eta_j), \quad z_j := \gamma_j(t_j), \quad S_j := S(y_j, \eta_j, t_j), \quad \beta_j := \big|u_0(t_j, z_j; y_j, \eta_j)|g|^{1/2}(z_j)\big|.$$

As $S_0 \neq 0$, we have that $z_0 = x_j$ for some $j = 1, 2, \ldots$. By continuity of the geodesic flow and density of $(x_j)_{j=1}^\infty \subset \partial M$, there exist a subsequence $(x_{k_j})_{j=1}^\infty \subset (x_j)_{j=1}^\infty$ and sequences $((y_j, \eta_j))_{j=1}^\infty \subset T\mathbb{R}^n$ and $(t_j)_{j=1}^\infty \subset (0, T)$ such that

$$x_{k_j} \to z_0, \quad (y_j, \eta_j) \to (y_0, \eta_0), \quad t_j \to t_0, \quad \text{as } j \to \infty,$$

and $z_j = x_{k_j} \neq z_0$. Then $|S_j| = |a_{k_j}|\beta_j \neq 0$. As $x_{k_j} \neq z_0$ and $x_{k_j} \to z_0$, we have that $k_j \to \infty$ and thus $a_{k_j} \to 0$. By (17) and continuity of the geodesic flow, it holds that $\beta_j \to \beta_0 > 0$. Hence $S_j \to 0$.

Next we use the assumption that $a_j = \lambda^{-\lambda^j}$. Let $((y_j, \eta_j))_{j=1}^\infty \subset T\mathbb{R}^n$ and $(t_j)_{j=1}^\infty \subset (0, T)$ satisfy (21). As $S_j \neq 0$, we have that $|S_j| = a_{k_j}\beta_j$ for some subsequence $(a_{k_j})_{j=1}^\infty \subset (a_j)_{j=1}^\infty$. As $S_j \to 0$, we have that $a_{k_j} \to 0$. Moreover, sequence $(\log_2 \beta_j)_{j=1}^\infty$ is bounded. This boundedness, together with $\log_\lambda a_{k_j} \in A$ and $\log_\lambda a_{k_j} \to -\infty$, yields

$$m_A(\log_\lambda a_{k_j} + \log_\lambda \beta_j) = \log_\lambda \beta_j$$

for large $j \in \mathbb{N}$. Hence,

$$\lim_{j\to\infty} m_A(\log_\lambda |S_j|) = \lim_{j\to\infty} \log_\lambda \beta_j = \log_\lambda \beta_0. \qquad \square$$

Theorem 7.2. *If the coefficients of the pseudorandom source* h_0 *are* $a_j = \lambda^{-\lambda^j}$, *then the functions* $S : S\Omega \times (0, \infty) \to \mathbb{C}$ *and* $\tau : \partial_- SM \to (0, \infty]$ *determine* $D(\Sigma)$ *and*

$$\gamma\big(\tau(x, \xi); x, \xi\big), \quad (x, \xi) \in D(\Sigma).$$

Proof. Clearly τ on $\partial_- SM$ determines $D(\Sigma)$. Let $(x, \xi) \in D(\Sigma)$. By Theorem 6.5, we may choose $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x, \xi)$ such that $\lim_{j\to\infty} t_j = \tau(x, \xi)$. As $S(y_j, \eta_j, t_j) \neq 0$, we have $\gamma(t_j; y_j, \eta_j) = x_{k_j}$ for some subsequence $(x_{k_j})_{j=1}^{\infty}$ of the sequence of source points. By Lemma 7.1, the function S determines

$$\frac{|S(y_j, \eta_j, t_j)|}{\big|u_0(t_j, x_{k_j}; y_j, \eta_j)|g|^{1/2}(x_{k_j})\big|} = a_{k_j}.$$

As a_j, $j \in \mathbb{Z}_+$, are disjoint, this determines the index k_j and thus also the point x_{k_j}. Moreover,

$$\lim_{j\to\infty} x_{k_j} = \lim_{j\to\infty} \gamma(t_j; y_j, \eta_j) = \gamma\big(\tau(x, \xi); x, \xi\big). \qquad \square$$

The following result follows from Remark 2.

Lemma 7.3. *Let us denote by* X *either* $S\Omega$ *or* $\partial_- SM$. *Let* $(y_0, \eta_0) \in X$ *satisfy*

$$\tau(y_0, \eta_0) < \infty, \quad \dot\gamma\big(\tau(y_0, \eta_0); y_0, \eta_0\big) \notin T_z \partial M,$$

where $z = \gamma\big(\tau(y_0, \eta_0); y_0, \eta_0\big)$. *Then there is a neighborhood* $V \subset X$ *of* (y_0, η_0) *such that* $\tau = \ell$ *in* V, *where* $\ell : U \to I$ *is the map of boundary intersection times defined in Remark 2 for neighborhoods* $U \subset X$ *and* $I \subset \mathbb{R}$ *of* (y_0, η_0) *and* $\tau(y_0, \eta_0)$. *In particular,* τ *is smooth in* V.

Lemma 7.4. *The set of* (x, ξ) *such that* $\gamma(\,\cdot\,; x, \xi)$ *is transverse to* ∂M *is open and dense in*

$$\partial SM := \{(x, \xi) \in SM; x \in \partial M\}.$$

Proof. As $\partial_- SM \cup \partial_+ SM$ is open and dense in ∂SM, it is enough to show that the set of (x, ξ) such that $\gamma(\,\cdot\,; x, \xi)$ is transverse to ∂M is open and dense in $\partial_\pm SM$. By the parametric transversality theorem [Hirsch 1976, Theorem 3.2.7], the claim follows from the fact that the evaluation map

$$F^{\text{ev}} : \partial_\pm SM \times \mathbb{R} \to \mathbb{R}^n,$$
$$F^{\text{ev}} : (x, \xi, t) \mapsto \gamma(t; x, \xi)$$

is transverse to ∂M. $\square$

Lemma 7.5. *Let* $(x_0, \xi_0) \in D(\Sigma)$. *Then there is a sequence* $((x_j, \xi_j))_{j=1}^{\infty} \subset D(\Sigma)$ *such that* $\gamma(\,\cdot\,; x_j, \xi_j)$ *is transverse to* ∂M *and*

$$\lim_{j\to\infty} (x_j, \xi_j) = (x_0, \xi_0), \quad \lim_{j\to\infty} \tau(x_j, \xi_j) = \tau(x_0, \xi_0).$$

Proof. We write $\tau_0 := \tau(x_0, \xi_0)$ and

$$(z_0, \zeta_0) := \big(\gamma(\tau_0; x_0, \xi_0), -\dot\gamma(\tau_0; x_0, \xi_0)\big).$$

Remark 2 gives a map of boundary intersection times $\ell : U \to I$ for neighborhoods $U \subset S\mathbb{R}^n$ and $I \subset \mathbb{R}$ of (z_0, ζ_0) and τ_0. By Lemma 7.4, there is a sequence $((z_j, \zeta_j))_{j=1}^{\infty} \subset SM \cap U$ converging to (z_0, ζ_0) such that $\gamma(\,\cdot\,; z_j, \zeta_j)$ is transverse to ∂M.

We define $t_j := \ell(z_j, \zeta_j)$ and

$$(x_j, \xi_j) := \big(\gamma(t_j; z_j, \zeta_j), -\dot\gamma(t_j; z_j, \zeta_j)\big).$$

Then $(x_j, \xi_j) \to (x_0, \xi_0)$ as $j \to \infty$. In particular, there is $J \ge 1$ such that $(x_j, \xi_j) \in \partial_- SM$ for $j \ge J$. By Lemma 6.3,

$$\begin{aligned}\tau(x_0, \xi_0) &\le \liminf_{j\to\infty} \tau(x_j, \xi_j) \le \limsup_{j\to\infty} \tau(x_j, \xi_j)\\ &\le \lim_{j\to\infty} \ell(z_j, \zeta_j) = \ell(z_0, \zeta_0) = \tau(x_0, \xi_0). \qquad \square\end{aligned}$$

Lemma 7.6. *Let $(x_0, \xi_0) \in D(\Sigma)$ be such that $\gamma(\,\cdot\,; x_0, \xi_0)$ is transverse to ∂M. Then there is $(y_0, \eta_0) \in S\Omega$ lying on the geodesic $\gamma(\,\cdot\,; x_0, \xi_0)$ and a neighborhood $V \subset S_{y_0}\Omega$ of η_0 such that the following conditions hold.*

(C1) *The map $\eta \mapsto \tau(y_0, \eta)$ is smooth $V \to (0, \infty)$.*

(C2) *The map*

$$(x(\eta), \xi(\eta)) := \big(\gamma(\tau(y_0, \eta); y_0, \eta), \dot\gamma(\tau(y_0, \eta); y_0, \eta)\big) \tag{22}$$

is smooth $V \to D(\Sigma)$ and $(x(\eta_0), \xi(\eta_0)) = (x_0, \xi_0)$.

(C3) *The map*

$$\tilde\ell(\eta) := \tau(x(\eta), \xi(\eta)) + \tau(y_0, \eta) \tag{23}$$

is smooth $V \to (0, \infty)$.

(C4) *There is a neighborhood $W \subset \partial M$ of $\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$ such that*

$$\eta \mapsto \gamma\big(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)\big) \tag{24}$$

is a diffeomorphism $V \to W$.

Proof. We write $\gamma(t) := \gamma(t; x_0, \xi_0)$ and $z_0 := \gamma(\tau(x_0, \xi_0))$. By Remark 2, $\gamma(-t) \in \Omega$ for small $t > 0$. Moreover, the points that are conjugate to z_0 along γ are discrete on γ [Jost 2008].

Thus there is $\tau_0 > 0$ such that

$$(y_0, \eta_0) := \big(\gamma(-\tau_0), \dot\gamma(-\tau_0)\big)$$

is in $S\Omega$, y_0 is not conjugate to z_0 along γ, $\tau(y_0, \eta_0) = \tau_0$, and

$$\big(\gamma(\tau_0; y_0, \eta_0), \dot\gamma(\tau_0; y_0, \eta_0)\big) = (x_0, \xi_0).$$

By Lemma 7.3, there is a neighborhood $V_0 \subset S_{y_0}\Omega$ of η_0 such that $\eta \mapsto \tau(y_0, \eta)$ is smooth in V_0. Hence, the function $\eta \mapsto (x(\eta), \xi(\eta))$ maps η_0 to (x_0, ξ_0) and is smooth in V_0. Moreover, this smoothness, transversality of $\gamma(\,\cdot\,, x_0, \xi_0)$, and Lemma 7.3 imply that there is a neighborhood $V_1 \subset V_0$ of η_0 such that

$(x(\eta), \xi(\eta)) \in \partial_- SM$ and $\eta \mapsto \tau(x(\eta), \xi(\eta))$ is smooth $V_1 \to (0, \infty)$. In particular, $(x(\eta), \xi(\eta)) \in D(\Sigma)$ for all $\eta \in V_1$. We have shown that (y_0, η_0) and V_1 satisfy (C1)–(C3).

We have

$$\big(\gamma(s; y_0, \eta), \dot\gamma(s; y_0, \eta)\big)\big|_{s=t+\tau(y_0,\eta)} = \big(\gamma(t; x(\eta), \xi(\eta)), \dot\gamma(t; x(\eta), \xi(\eta))\big). \tag{25}$$

In particular, $\gamma(\tilde\ell(\eta_0); y_0, \eta_0) = z_0$ and

$$\gamma(\tilde\ell(\eta); y_0, \eta) = \gamma\big(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)\big) \in \partial M.$$

Moreover, as y_0 is not conjugate to z_0 along γ, there are neighborhoods $V_2 \subset V_1$, $I_0 \subset (0, \infty)$ and $U_0 \subset \mathbb{R}^n$ of η_0, $\tilde\ell(\eta_0)$ and z_0 such that $(t, \eta) \mapsto \gamma(t; y_0, \eta)$ is a diffeomorphism $V_2 \times I_0 \to U_0$.

There is a neighborhood $V \subset V_2$ of η_0 such that $\tilde\ell(V) \subset I_0$. The graph of $\eta \mapsto \tilde\ell(\eta)$ is an $(n-1)$-dimensional submanifold on $V \times I_0$. Hence, the diffeomorphism $(t, \eta) \mapsto \gamma(t; y_0, \eta)$ maps it onto an $(n-1)$-dimensional submanifold W of U_0. Moreover, $z_0 \in W$ and $W \subset \partial M$. Thus W is a neighborhood of z_0 in ∂M. □

Lemma 7.7. *Let $(x_0, \xi_0) \in D(\Sigma)$ and $(y_0, \eta_0) \in S\Omega$ satisfy conditions (C1)–(C4) of Lemma 7.6 for neighborhoods $V \subset S_{y_0}\Omega$ and $W \subset \partial M$ of η_0 and $z_0 := \gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$. We denote by $F : W \to V$ the inverse map of (24). Then*

$$\operatorname{grad}_{\partial M}(\tilde\ell \circ F)|_{z=z_0} = \dot\gamma_{z_0}^{\top}, \tag{26}$$

where $\tilde\ell : V \to (0, \infty)$ is the function (23) and $\dot\gamma_{z_0}^{\top}$ is the orthogonal projection of $\dot\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$ into $T_{z_0}\partial M$.

Proof. Let $\sigma : (-\epsilon, \epsilon) \to W$ be a smooth curve such that $\sigma(0) = z_0$. We define

$$\Gamma : (-\epsilon, \epsilon) \times \mathbb{R} \to \mathbb{R}^n, \quad \Gamma(s, t) := \gamma\big(t; y_0, F(\sigma(s))\big).$$

We write $\lambda := \tilde\ell \circ F \circ \sigma$ and $\tilde\ell_0 := \tilde\ell(\eta_0)$. By (25),

$$\Gamma(s, \lambda(s)) = \gamma\big(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)\big)\big|_{\eta=F(\sigma(s))} = \sigma(s),$$
$$(\partial_t\Gamma)(0, \tilde\ell_0) = \dot\gamma(\tilde\ell_0; y_0, \eta_0) = \dot\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big).$$

Hence

$$\dot\sigma(0) = \partial_s\Gamma(s, \lambda(s))|_{s=0} = (\partial_s\Gamma)(0, \tilde\ell_0) + (\partial_t\Gamma)(0, \tilde\ell_0)\lambda'(0).$$

The curve $t \mapsto \Gamma(s, t)$ is a unit speed geodesic for all $s \in (-\epsilon, \epsilon)$. Hence

$$\begin{aligned}\big(\dot\sigma(0), \dot\gamma(\tau(x_0, \xi_0); x_0, \xi_0)\big)_g &= \big((\partial_s\Gamma, \partial_t\Gamma)_g + \lambda'(0)(\partial_t\Gamma, \partial_t\Gamma)_g\big)\big|_{s=0, t=\tilde\ell_0} \\ &= (\partial_s\Gamma, \partial_t\Gamma)_g|_{s=0, t=\tilde\ell_0} + \lambda'(0).\end{aligned} \tag{27}$$

We define

$$\mathscr{L}(s, l) := \int_0^l |\partial_t\Gamma(s, t)|_g dt, \quad (s, l) \in (-\epsilon, \epsilon) \times (0, \infty).$$

Then $\mathscr{L}(s,l)$, $s \in (-\epsilon, \epsilon)$ is the length of a unit speed geodesic on the interval $[0,l]$. Thus $\mathscr{L}(s,l) = l$ for all $s \in (-\epsilon, \epsilon)$. We may derive an expression for $\partial_s \mathscr{L}(s,l)|_{s=0}$ as in [Lee 1997, Proposition 6.5]:

$$\partial_s \mathscr{L}(s,l)|_{s=0} = \int_0^l (D_t \partial_s \Gamma, \partial_t \Gamma)_g \, dt|_{s=0}.$$

As $t \mapsto \Gamma(s,t)$ is a geodesic, $D_t \partial_t \Gamma(s,t) = 0$, and thus

$$\partial_t (\partial_s \Gamma, \partial_t \Gamma)_g = (D_t \partial_s \Gamma, \partial_t \Gamma)_g.$$

Moreover, $\Gamma(s,0) = y_0$ for all $s \in (-\epsilon, \epsilon)$, and thus $\partial_s \Gamma(s,0) = 0$. Hence

$$0 = \partial_s \mathscr{L}(s,l)|_{s=0} = \int_0^l \partial_t (\partial_s \Gamma, \partial_t \Gamma)_g \, dt|_{s=0} = (\partial_s \Gamma, \partial_t \Gamma)_g|_{s=0,t=l}, \quad l \in (0,\infty).$$

By (27), we have

$$\begin{aligned}(\dot\sigma(0), \gamma_{z_0}^{\top})_g &= \big(\dot\sigma(0), \dot\gamma(\tau(x_0,\xi_0); x_0, \xi_0)\big)_g \\ &= \lambda'(0) = \big\langle d(\tilde\ell \circ F)|_{z=z_0}, \dot\sigma(0)\big\rangle_{T^*_{z_0}\partial M \times T_{z_0}\partial M} = \big(\dot\sigma(0), \operatorname{grad}_{\partial M}(\tilde\ell \circ F)|_{z=z_0}\big)_g,\end{aligned}$$

for all smooth curves σ in W such that $\sigma(0) = z_0$, which proves the claim. □

Theorem 7.8. *The functions $\tau : \partial_- SM \to (0,\infty]$ and*

$$z : D(\Sigma) \to \partial M, \quad z(x,\xi) := \gamma\big(\tau(x,\xi); x, \xi\big),$$

together with the Riemannian manifold $(\Omega, g|_\Omega)$, determine

$$\dot\gamma\big(\tau(x,\xi); x, \xi\big), \quad (x,\xi) \in D(\Sigma).$$

Proof. The functions τ and z on $D(\Sigma)$ determine the set B of points $(x_0, \xi_0) \in D(\Sigma)$ such that the conditions (C1)–(C4) of Lemma 7.6 hold for some $(y_0, \eta_0) \in S\Omega$.

Let $(x_0,\xi_0) \in B$. We write $\zeta_0 := \dot\gamma\big(\tau(x_0,\xi_0); x_0, \xi_0\big)$. The map

$$\eta \mapsto z(x(\eta), \xi(\eta))$$

determines its local inverse. Hence τ and z determine the function F of Lemma 7.7, and thus they determine $\dot\gamma_{z_0}^{\top}$ by the formula (26). As ζ_0 is a unit vector,

$$\zeta_0 = \dot\gamma_{z_0}^{\top} + (1 - |\dot\gamma_{z_0}^{\top}|^2)^{1/2} \nu_{z_0},$$

where ν_{z_0} is the unit exterior normal vector of ∂M. Hence τ and z determine ζ_0 for all $(x_0,\xi_0) \in B$.

Let $(x_0,\xi_0) \in D(\Sigma)$. By Lemmata 7.5 and 7.6, there is a sequence $((x_j,\xi_j))_{j=1}^\infty \subset B$ such that

$$\lim_{j\to\infty} (x_j,\xi_j) = (x_0,\xi_0), \quad \lim_{j\to\infty} \tau(x_j,\xi_j) = \tau(x_0,\xi_0).$$

Moreover, the functions τ and z determine the set of such sequences, and thus they determine

$$\lim_{j\to\infty} \dot\gamma\big(\tau(x_j,\xi_j); x_j, \xi_j\big) = \dot\gamma\big(\tau(x_0,\xi_0); x_0, \xi_0\big). \qquad \square$$

Theorems 6.5, 7.2 and 7.8 prove Theorem 1.2 formulated in the introduction.

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TWO-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH RANDOM RADIAL DATA

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We study radial solutions of a certain two-dimensional nonlinear Schrödinger (NLS) equation with harmonic potential, which is supercritical with respect to the initial data. By combining the nonlinear smoothing effect of Schrödinger equation with L^p estimates of Laguerre functions, we are able to prove an almost-sure global well-posedness result and the invariance of the Gibbs measure. We also discuss an application to the NLS equation without harmonic potential.

1. Introduction

Burq, Thomann and Tzvetkov [Burq et al. 2010] studied the nonlinear Schrödinger (NLS) equation on $\mathbb{R} \times \mathbb{R}^d$ with harmonic potential

$$i\partial_t u + (\Delta - |x|^2)u = \pm |u|^{p-1}u, \tag{1-1}$$

where the space dimension was one. The purpose of this paper is to extend their results to two space dimensions. We will prove global well-posedness almost surely with respect to a Gaussian measure supported on $\bigcap_{\delta>0} \mathcal{H}^{-\delta}$ (see Section 1.2 for the definition), and we construct the Gibbs measure, absolutely continuous with respect to this Gaussian, which we prove to be invariant.

We also study the NLS equation on $\mathbb{R} \times \mathbb{R}^d$ without harmonic potential, namely

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u. \tag{1-2}$$

In [Burq et al. 2010], it was noticed that using an explicit transform (referred to as the lens transform in [Tao 2009]), we can obtain local and global well-posedness results of (1-2) from the corresponding results of (1-1). This issue is also pursued here.

Like most earlier papers on random data theory of NLS equations in two or more dimensions, ours considers only radial solutions. In the defocusing case in two dimensions, we can prove, when $p \geq 3$ is an odd integer, almost-sure global well-posedness and measure invariance for (1-1) and almost-sure global well-posedness and scattering for (1-2); in the focusing case, we have the same results only for (1-1), when $1 < p < 3$.

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Keywords: nonlinear Schrödinger equation, supercritical NLS, random data, Gibbs measure, global well-posedness.

1.1. ***NLS equation and probabilistic methods.*** The nonlinear Schrödinger equation (1-2) and its periodic variant (which is solved on $\mathbb{R}\times\mathbb{T}^d$) have been extensively studied over the last several decades. Beginning with [Lebowitz et al. 1988; Bourgain 1994; 1996], it has been observed that low regularity local and global solutions to (1-2) on $\mathbb{R}\times\mathbb{T}^d$ can be obtained via randomization of initial data and construction of Gibbs measure. This idea was later developed in a number of papers, for instance [Burq and Tzvetkov 2008a; 2008b; Nahmod et al. 2012; Oh 2009; 2010; Thomann and Tzvetkov 2010; Tzvetkov 2006; 2008]. In [Burq et al. 2010], the method mentioned above was first used to study (1-1).

There are three reasons why (1-1) is worth studying. First, the spectrum of the harmonic oscillator $\boldsymbol{H}=-\Delta+|x|^2$ is discrete, so (1-1) can be approximated by ODEs, and the current techniques of constructing Gibbs measure apply at least formally. Second, (1-1) is solved on $\mathbb{R}\times\mathbb{R}^d$, where the space domain is noncompact, while previous works usually involve a compact manifold. Also (1-1) is related to (1-2) via the lens transform, so results about (1-1) may shed some light on the study of (1-2), where probabilistic methods have not yet entered. Finally, (1-1) also arises naturally from the theory of Bose–Einstein condensates, as noted in [Burq et al. 2010].

The major difficulty in the study of (1-1) is that the support of the Gaussian part of the Gibbs measure contains functions with very low regularity. With radial symmetry the typical element in the support of the Gibbs measure belongs to $\bigcap_{\delta>0}\mathcal{H}^{-\delta}$ but not L^2; without it the typical element does not even belong to $\mathcal{H}^{1-d}$ (the spaces $\mathcal{H}^{\sigma}$, as defined in Section 1.2, are Sobolev spaces associated to $\boldsymbol{H}$; see Section 3 for more details). A consequence of this is that we cannot expect even local well-posedness in the deterministic sense for initial data of such low regularity. In fact in [Thomann 2008] local ill-posedness for $\mathcal{H}^{\sigma}$ initial data was shown[1], provided $\sigma<\sigma_c:=d/2-2/(p-1)$. In particular, we have $\sigma_c\to 1$ as $p\to\infty$ for the two-dimensional defocusing equation; thus deterministic local well-posedness fails completely for regularity below L^2.

In [Burq et al. 2010], the problem was resolved by a probabilistic improvement of (weighted) Strichartz estimate, and it was shown that $\boldsymbol{H}^{\delta/2}e^{-it\boldsymbol{H}}f(\omega)$ almost surely belongs to some weighted Lebesgue space for $\delta<\frac{1}{2}$ (see [Burq et al. 2010, Lemma 6.2] for more details). Since $\sigma_c<\frac{1}{2}$ in one dimension, local well-posedness in this space could be proved. In two dimensions, however, it will be shown in the Appendix that the distribution $\boldsymbol{H}^{\sigma/2}f(\omega)$ is almost surely not a locally integrable function (thus cannot belong to any weighted space) when $\sigma\geq\frac{1}{2}$. Since $\frac{1}{2}$ fails to reach the σ_c threshold when p is large, we have to use different tools to get local well-posedness. Fortunately, the nonlinear smoothing effect of the NLS equation provides such a tool. To fully exploit this effect, we will work in $\mathcal{X}^{\sigma,b}$ spaces (see Section 1.2 for definitions) and use multilinear eigenfunction estimates. This requires p to be an odd integer, but we believe that by more delicate treatment we can remove this restriction and allow for all $1<p<\infty$.

When there is no radial symmetry, the support of the Gaussian will have such low regularity that we cannot even define the Gibbs measure. It would be possible to use alternative Gaussians to get local results, but then we do not have an invariant measure, so global results still seem out of reach. One

[1] The counterexample constructed in [Thomann 2008] was for (1-2), but it could be easily adapted to (1-1) as noted in [Thomann 2009]; also one can check the proof there that the initial data could be made radial.

possible way is to combine the probabilistic local result with the high-low analysis of Bourgain or the I-method of Colliander, Keel, Staffilani, Takaoka and Tao. For progress in this direction, see [Colliander and Oh 2012].

Finally, as we mentioned above, the study of (1-1) is closely related to the study of (1-2). The result we obtain for (1-2) (see Theorem 1.2 below) is an almost-sure global well-posedness and scattering result with supercritical initial data (the critical index of (1-2) is $d/2-2/(p-1)\to 1$ as $p\to\infty$ in two dimensions, while the initial data is below L^2), but due to the use of the lens transform, our result is unsatisfactory in the sense that (i) the space in which uniqueness holds cannot be described in a simple way, and (ii) the Gaussian measure in Theorem 1.2 does not arise naturally from (1-2), and we do not know how to construct the Gibbs measure of (1-2). This may be an interesting problem for further study.

1.2. *Notations and preliminaries.* From now on we assume the spacial dimension $d=2$, and all the functions we consider are radial. Define the Hermite operator $\boldsymbol{H}=-\Delta+|x|^2$. It has a complete series of real L^2_{rad} eigenfunctions

$$e_k(x)=\frac{1}{\sqrt{\pi}}\mathscr{L}_k^0(|x|^2)\quad \text{for } k\geq 0 \tag{1-3}$$

with eigenvalue $4k+2$. Here $\mathscr{L}_k^0$ are Laguerre functions

$$\mathscr{L}_k^0(z)=\frac{e^{z/2}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}(z^ke^{-z}).$$

Concerning these functions we have the basic pointwise estimates

$$\left|\mathscr{L}_k^\alpha(z)\right|\leq\begin{cases} C & \text{if } 0\leq z\leq 1/\nu,\\ C(z\nu)^{-1/4} & \text{if } 1/\nu\leq z\leq \nu/2,\\ C\nu^{-1/4}(\nu^{1/3}+|\nu-z|)^{-1/4} & \text{if } \nu/2\leq z\leq 3\nu/2,\\ Ce^{-cz} & \text{if } z\geq 3\nu/2.\end{cases} \tag{1-4}$$

Here $\nu=4k+2$, C and c (possibly with subscripts) are positive constants varying from line to line, and will be used in this way throughout this paper. For an introduction to Laguerre functions, see [Szegő 1975] or [Thangavelu 1993, Chapter 1]. The proof of (1-4) is also contained in [Erdélyi 1960; Askey and Wainger 1965].

For $\sigma\in\mathbb{R}$ and $1\leq p\leq\infty$, we define the Sobolev spaces associated to $\boldsymbol{H}$:

$$\mathscr{W}^{\sigma,p}_{\mathrm{rad}}=\left\{u\in\mathscr{S}'_{\mathrm{rad}}:\|u\|_{\mathscr{W}^{\sigma,p}}=\left\|\boldsymbol{H}^{\sigma/2}u\right\|_{L^p}<\infty\right\}. \tag{1-5}$$

We also write $\mathscr{W}^{\sigma,2}_{\mathrm{rad}}=\mathscr{H}^{\sigma}_{\mathrm{rad}}$.

We also define a class of spacetime Hilbert spaces associated to $\boldsymbol{H}$, as

$$\mathscr{X}^{\sigma,b}_{\mathrm{rad}}=\left\{u\in\mathscr{S}'_{\mathrm{rad}}(\mathbb{R}\times\mathbb{R}^2):\|u\|_{\mathscr{X}^{\sigma,b}}=\left\|\boldsymbol{H}^{\sigma/2}\langle \mathrm{i}\partial_t-\boldsymbol{H}\rangle^b u\right\|_{L^2_{t,x}}<\infty\right\}, \tag{1-6}$$

or use the radial Hermite expansion and Fourier transform to write

$$\|u\|^2_{\mathscr{X}^{\sigma,b}}=\left(\sum_{k=0}^{\infty}(4k+2)^\sigma\int_{\mathbb{R}}\left(1+(\tau+4k+2)^2\right)^b|\mathscr{F}_t\langle u,e_k\rangle(\tau)|^2\,\mathrm{d}\tau\right)^{1/2},$$

where as usual $\langle t\rangle = (|t|^2+1)^{1/2}$, $\mathscr{F}_t$ denotes the Fourier transform $(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-\mathrm{i}\tau t} f(t)\,\mathrm{d}t$ in t, and $\langle f, g\rangle$ denotes the $L^2(\mathbb{R}^n)$ inner product of f and g. For an interval I we define a localized version of this space by

$$\|u\|_{\mathscr{X}^{\sigma,b,I}} = \inf\bigl\{\|v\|_{\mathscr{X}^{\sigma,b}} : v(t) = u(t),\ t\in I\bigr\}, \tag{1-7}$$

and denote it by $\mathscr{X}^{\sigma,b,I}_{\mathrm{rad}}$. When $I = [-T, T]$, we simply write $\mathscr{X}^{\sigma,b,T}_{\mathrm{rad}}$. Since all the functions will be radial, the "rad" subscript will be dropped from now on. Trivially $\mathscr{X}^{\sigma,b,I}$ is a separable Banach space (simply restrict a countable dense subset of $\mathscr{X}^{\sigma,b}$ to I).

We fix a smooth, nonincreasing function η such that $1 = \eta(1) \geq \eta(x) \geq \eta(2) = 0$ for all x. Using this cutoff, we define Littlewood–Paley projections

$$\Delta_N = \eta\Bigl(\frac{2\boldsymbol{H}}{N^2}\Bigr) - \eta\Bigl(\frac{4\boldsymbol{H}}{N^2}\Bigr) \tag{1-8}$$

for dyadic N. Then $\Delta_N = 0$ for $N \leq 1$, since the first eigenvalue of $\boldsymbol{H}$ is 2. Thus whenever we talk about Δ_N, we always assume $N \geq 2$.

We shall denote by $\#M$ the cardinality of a finite set M and by $|E|$ the Lebesgue measure of a subset set E of Euclidean space. We define $A \lesssim B$ by $A \leq CB$ and define $\gtrsim$ and $\sim$ similarly. The constants C_j and c_j will also be used freely, as indicated above. All these constants will ultimately depend on the only parameter p in (1-1) and (1-2). Finally, we define the finite-dimensional subspace V_k to be the span of $\{e_j\}_{0\leq j\leq k}$. For a function g on $\mathbb{R}^2$ or $I\times\mathbb{R}^2$, where I is an interval, we define g_k° and $g_k^{\perp}$ to be the projection of g on V_k and $V_k^{\perp}$.

1.3. ***Statement of main results and plan for this paper.*** Fix a probability space $(\Omega, \Sigma, \mathbb{P})$ with a sequence of independent normalized complex Gaussians $\{g_k\}$ on Ω (which has density $\pi^{-1}e^{-|z|^2}\mathrm{d}x\mathrm{d}y$, so g_k has mean 0 and variance 1), so that $\omega \mapsto (g_k(\omega))_{k\geq 0}$ is injective, and the series

$$f(\omega) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+2}} g_k(\omega) e_k \tag{1-9}$$

converges[2] in $\mathscr{S}'(\mathbb{R}^2)$ for all $\omega\in\Omega$. Then $f = f(\omega)$ is an $\mathscr{S}'(\mathbb{R}^2)$-valued random variable, and is a bijection between Ω and its range. Our main results can then be stated as follows.

Theorem 1.1. *Consider the Cauchy problem*

$$\begin{cases} \mathrm{i}\partial_t u + (\Delta - |x|^2)u = \pm |u|^{p-1}u, \\ u(0) = f(\omega) \end{cases} \tag{1-10}$$

and distinguish two cases: the sign is $-$ and $1 < p < 3$, or the sign is $+$ and $p \geq 3$ is an odd integer. In the former let $\sigma = 0$, and in the latter let $0 < \sigma < 1$ be sufficiently close to 1, depending on p. In both cases let $1 > b > \frac{1}{2}$ be sufficiently close to $\frac{1}{2}$, depending on σ and p.

[2] For example, we may take the usual product space $\mathbb{C}^{\infty}$ equipped with the product of complex Gaussian measures, and coordinate functions g_j, and choose the (full-measure) subset where $|g_k(\omega)| = O(\langle k\rangle^{10})$ as Ω, this can easily guarantee the convergence of (1-9).

Then almost surely in $\mathbb{P}$, *we have a unique global* (*strong*) *solution* u *in the affine space*

$$\mathcal{Y} = e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \bigcap_{T>0} \mathcal{X}^{\sigma,b,T}, \tag{1-11}$$

and we have continuous embeddings

$$\mathcal{Y} \subset e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{C}(\mathbb{R}, \mathcal{H}^{\sigma}(\mathbb{R}^2)) \subset \mathscr{C}\bigl(\mathbb{R}, \bigcap_{\delta>0} \mathcal{H}^{-\delta}(\mathbb{R}^2)\bigr).$$

We also have a Gibbs measure on $\mathscr{S}'(\mathbb{R}^2)$, *which is absolutely continuous with respect to the push forward of* $\mathbb{P}$ *under* f, *and is invariant under the flow defined by* (1-10).

Theorem 1.2. *Let* σ *and* b *be as in Theorem 1.1. Consider the* (*defocusing*) *Cauchy problem*

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u = |u|^{p-1}u \\ u(0) = f(\omega) \end{cases} \tag{1-12}$$

with $p \geq 3$ *an odd integer. Then almost surely in* $\mathbb{P}$, *we have a global* (*strong*) *solution* u *in the affine space*

$$\mathcal{Z} = e^{\mathrm{i}t\Delta} f(\omega) + \bigcap_{T>0} X^{\sigma,b,T}, \tag{1-13}$$

and we have a continuous embedding

$$\mathcal{Z} \subset e^{\mathrm{i}t\Delta} f(\omega) + \mathscr{C}(\mathbb{R}, H^{\sigma}(\mathbb{R}^2)).$$

Here $X^{\sigma,b,T}$ *is defined in the same way as in* (1-6) *and* (1-7), *but with* $\boldsymbol{H}$ *replaced by* $-\Delta$. *We also have an appropriate affine subspace* $\mathcal{Z}'$ *of* $\mathcal{Z}$ *containing the solution* u, *in which uniqueness holds. Finally we have a scattering result*: *There exist functions* $g_{\pm} \in H^{\sigma}$ *such that*

$$\lim_{t\to\pm\infty} \|u - e^{\mathrm{i}t\Delta}(f(\omega) + g_{\pm})\|_{H^{\sigma}} = 0. \tag{1-14}$$

The rest of this paper is devoted to the proof of Theorems 1.1 and 1.2. In Section 2 we recall the linear Strichartz and L^2-based estimates with respect to the propagator $e^{-\mathrm{i}t\boldsymbol{H}}$. We will rely on the functional calculus of $\boldsymbol{H}$ (thus the results hold for more general Schrödinger operators, though we do not discuss this here). Some results in this section are standard and can be found for example in [Colliander and Oh 2012; Tao 2006]. In Section 3, we prove some large deviation bounds for Gaussian random variables, and use these to construct the Gibbs measure of (1-1). In Section 4, which is the core of this paper, we use a Littlewood–Paley decomposition and hypercontractivity of Gaussians to prove a multilinear estimate in $\mathcal{X}^{\sigma,b}$ spaces, which shows the nonlinear smoothing effect. In Section 5, we put these estimates together to develop a local Cauchy theory. Then in Section 6 we extend this to a global well-posedness result by exploiting the invariance of truncated Gibbs measure under the flow of approximating ODEs. In Section 7 we introduce the lens transform and convert the result on (1-1) to one on (1-2), proving Theorem 1.2. In Section 8, we show the invariance of the Gibbs measure, completing the proof of Theorem 1.1. Finally in the Appendix, we discuss the typical regularity (in terms of $\boldsymbol{H}$) on the support of the Gibbs measure.

2. Functional calculus and Strichartz estimates

We begin with the following kernel estimate about the harmonic oscillator $\boldsymbol{H}$.

Proposition 2.1. *Let ψ be a Schwarz function; then for $t>0$ the operator $\psi(t\boldsymbol{H})$ is an integral operator with kernel $K_t(x,y)$, where*

$$|K_t(x,y)| \lesssim t^{-1}(1+t^{-1/2}|x-y|)^{-N}. \tag{2-1}$$

The implicit constants in $\lesssim$ depend only on N and ψ. In particular, these operators K_t are bounded uniformly in t on $\mathcal{W}^{\sigma,p}$ for all $\sigma\in\mathbb{R}$ and $1\le p\le\infty$.

Proof. It was proved in [Dziubański 1998, Corollary 3.14] that, for any fixed N, the inequality (2-1) holds, provided

$$\psi\in\mathcal{S}_0^m([0,+\infty))=\left\{\psi\in\mathcal{S}([0,+\infty)):\psi^{(k)}(0)=0,\ 0\le k\le m\right\}, \tag{2-2}$$

where m is large enough depending on N (actually the same result was proved for any Schrödinger operator with nonnegative polynomial potential). On the other hand, when $\psi(z)=e^{-\sigma z}$ with $\sigma>0$, we have from Mehler's formula that

$$K_t(x,y)=\frac{e^{-2\sigma t}}{\pi(1-e^{-4\sigma t})}\exp\left(-\frac{1}{2}\frac{1+e^{-4\sigma t}}{1-e^{-4\sigma t}}(|x|^2+|y|^2)+\frac{2e^{-2\sigma t}}{1-e^{-4\sigma t}}x\cdot y\right). \tag{2-3}$$

Writing $2\sigma t=\delta$, we know

$$-\frac{1}{2}\frac{1+e^{-2\delta}}{1-e^{-2\delta}}(|x|^2+|y|^2)+\frac{2e^{-\delta}}{1-e^{-2\delta}}x\cdot y\le-\frac{c}{\delta}|x-y|^2,$$

thus the kernel satisfies

$$0\le K_t(x,y)\le\frac{c_1}{\delta}e^{-(c_2/\delta)|x-y|^2}\lesssim t^{-1}(1+t^{-1/2}|x-y|)^{-N} \tag{2-4}$$

for any N. Now for any fixed m, there exists l such that any function $f\in\mathcal{S}([0,+\infty))$ can be written as

$$f(z)=f_0(z)+\sum_{j=1}^{l}c_je^{-\sigma_jz}, \tag{2-5}$$

where $f_0\in\mathcal{S}_0^m([0,+\infty))$ and $\sigma_j>0$. Combining the two results above, we have proved (2-1). The uniform boundedness now follows from (2-1), Schur's test, and commutativity of $\psi(t\boldsymbol{H})$ and $\boldsymbol{H}^{\sigma/2}$. □

Remark 2.2. The constants in Proposition 2.1 certainly depend on ψ and the Lebesgue or Sobolev exponents, but this dependence can be safely ignored since throughout this paper we only use a finite number of fixed cutoff functions ψ and a finite number of fixed exponents.

Corollary 2.3. *Suppose $1\le p\le\infty$, $\sigma_{1,2}\in\mathbb{R}$, $R>0$ and g is a function.*

(1) *If $\sigma_1\ge\sigma_2$, and $\langle g,e_k\rangle\neq0$ only if $4k+2\gtrsim R^2$ (for example, when $g=\sum_{N>R}\Delta_Nh$ for some h), then $\|g\|_{\mathcal{W}^{\sigma_1,p}}\gtrsim R^{\sigma_1-\sigma_2}\|g\|_{\mathcal{W}^{\sigma_2,p}}$.*

(2) *If* $\sigma_1 \leq \sigma_2$, *and* $\langle g, e_k\rangle \neq 0$ *only if* $4k+2 \lesssim R^2$ (*for example, when* $g = \sum_{N\leq R} \Delta_N h$ *for some* h), *then* $\|g\|_{\mathcal{W}^{\sigma_1,p}} \gtrsim R^{\sigma_1-\sigma_2}\|g\|_{\mathcal{W}^{\sigma_2,p}}$.

(3) *If* $\langle g, e_k\rangle \neq 0$ *only if* $4k+2 \sim R^2$ (*for example, when* $R = N$ *is dyadic and* $g = \Delta_N h$ *for some* h), *then* $\|g\|_{\mathcal{W}^{\sigma_1,p}} \sim R^{\sigma_1-\sigma_2}\|g\|_{\mathcal{W}^{\sigma_2,p}}$.

(4) *All the operators* $\sum_{N>R} \Delta_N$, $\sum_{N\leq R} \Delta_N$ *and* Δ_N *are uniformly bounded from* $\mathcal{W}^{\sigma_1,p}$ *to itself.*

Proof. First (4) is obvious, since $\sum_{N<R} \Delta_N = \eta(t\boldsymbol{H})$ and $\Delta_N = \eta(t'\boldsymbol{H}) - \eta(2t'\boldsymbol{H})$ for some t and t', and $\sum_{N>R} \Delta_N = \mathrm{Id} - \sum_{N\leq R} \Delta_N$. Also it is clear that (1) and (2) implies (3). In proving these we may assume $\min\{\sigma_1, \sigma_2\} = 0$, since $\boldsymbol{H}^{\sigma/2} g$ satisfies the same properties as g.

To prove (1), choose a smooth cutoff ψ_1 that equals 1 for $x \gtrsim 1$ and equals 0 for very small x. Then in (1) we have $g = \psi_1(R^{-2}\boldsymbol{H})g$. Therefore we need to prove that

$$\boldsymbol{H}^{-\sigma/2} R^{\sigma}\psi_1(R^{-2}\boldsymbol{H}) = \sum_{k\geq 0} 2^{-k\sigma/2}\psi_2(2^{-k}R^{-2}\boldsymbol{H}) \tag{2-6}$$

is uniformly bounded on L^p for $\sigma > 0$, where $\psi_2(x) = x^{-\sigma/2}(\psi_1(x) - \psi_1(2^{-1}x))$ is a fixed smooth compactly supported function. Using (2-1), we can estimate the kernel $K(x, y)$ of $\boldsymbol{H}^{-\sigma/2}R^{\sigma}\psi_1(R^{-2}\boldsymbol{H})$ as

$$|K(x, y)| \lesssim \sum_{k\geq 0} 2^{-k\sigma/2} 2^k R^2 \langle 2^{k/2} R|x-y|\rangle^{-N} = R^2\psi_3(R|x-y|), \tag{2-7}$$

where

$$\psi_3(x) = \sum_{k\geq 0} 2^{(1-\sigma/2)k}\langle 2^{k/2}x\rangle^{-N} \lesssim (1+|x|^{\sigma-2})\langle x\rangle^{-N}.$$

The last inequality is easily verified by considering $|x| \geq 1$ and $|x| < 1$ separately. Therefore by Schur's test we have proved the uniform boundedness of the operator, thus proving (1). The proof of (2) is similar and is left as an exercise. □

To get Sobolev and product estimates, we next need a lemma.

Lemma 2.4. *For all* $1 < p < \infty$ *and* $\sigma > 0$, *we have*

$$\|g\|_{\mathcal{W}^{\sigma,p}} \sim \|\langle\nabla\rangle^{\sigma} g\|_{L^p} + \|\langle x\rangle^{\sigma} g\|_{L^p}. \tag{2-8}$$

In particular we have $\|g\|_{\mathcal{W}^{\sigma_1,p}} \lesssim \|g\|_{\mathcal{W}^{\sigma_2,p}}$ *for* $\sigma_1 \leq \sigma_2$.

Proof. See Dziubański and Głowacki [2009], who proved the same result for any Schrödinger operator with nonnegative polynomial potential (the latter inequality also follows from Corollary 2.3). □

Proposition 2.5. *We have the estimate*

$$\|g\|_{\mathcal{W}^{\sigma_1,q}} \lesssim \|g\|_{\mathcal{W}^{\sigma_2,q'}} \tag{2-9}$$

if $1 < q, q' < \infty$ *and* $\sigma_2 - \sigma_1 \geq 2(1/q' - 1/q) \geq 0$, *and the estimate*

$$\Big\|\prod_{j=1}^{k} g_j\Big\|_{\mathcal{W}^{\sigma,p}} \lesssim \sum_{j=1}^{k} \|g_j\|_{\mathcal{W}^{\sigma,q_j}} \prod_{i\neq j} \|g_i\|_{L^{q_i}} \tag{2-10}$$

if $\sigma > 0$ *and* $1 < p, q_j < \infty$ *with* $1 \le j \le k$ *and* $\sum_{j=1}^{k} 1/q_j = 1/p$.

Proof. In considering the first estimate we may assume $\sigma_1 = 0$, and the inequality follows immediately from Lemma 2.4 and the usual Sobolev inequality.

As for the second estimate, if the $\mathscr{W}^{\sigma,p}$ norm is replaced by the usual Sobolev $W^{\sigma,p}$ norm, then (2-10) is a well-known result in Fourier analysis (for $k = 2$, but the general case easily follows from induction). Now using Lemma 2.4, we only need to show

$$\|\langle x\rangle^{\sigma} g_1 \cdots g_k\|_{L^p} \lesssim \|\langle x\rangle^{\sigma} g_1\|_{L^{q_1}} \prod_{j=2}^{k} \|g_j\|_{L^{q_j}},$$

which is simply Hölder's inequality. □

Before proving Strichartz and other estimates, we need a lemma, which gives a representation formula of $\mathscr{X}^{\sigma,b}$ functions.

Lemma 2.6. *Suppose* $\sigma, b \in \mathbb{R}$. *Then for every* u, *if* $\|u\|_{\mathscr{X}^{\sigma,b}} \lesssim 1$, *we have*

$$u(t,x) = \int_{\mathbb{R}} \phi(\lambda) e^{\mathrm{i}\lambda t} \sum_k a_\lambda(k) e^{-\mathrm{i}(4k+2)t} e_k(x)\, \mathrm{d}\lambda, \tag{2-11}$$

where $\sum_k (4k+2)^{\sigma} |a_\lambda(k)|^2 = 1$ *for all* $\lambda \in \mathbb{R}$. *Furthermore, if* $b > \frac{1}{2}$, *then we also have* $\int_{\mathbb{R}} |\phi(\lambda)|\, \mathrm{d}\lambda \lesssim 1$. *If* $b < \frac{1}{2}$ *and* $\mathscr{F}_t\langle u, e_k\rangle(\lambda)$ *is supported in* $\{|\lambda + 4k + 2| \le K\}$ *for each* k, *where* $K \gtrsim 1$, *then we also have* $\int_{\mathbb{R}} |\phi(\lambda)|\, \mathrm{d}\lambda \lesssim K^{1/2-b}$.

Proof. Using radial Hermite expansion and Fourier transform, we can write

$$\begin{aligned} u(t,x) &= (2\pi)^{-1/2} \sum_k \int_{\mathbb{R}} \mathscr{F}_t\langle u, e_k\rangle(\tau) e^{\mathrm{i}t\tau} e_k(x)\, \mathrm{d}\tau \\ &= (2\pi)^{-1/2} \sum_k \int_{\mathbb{R}} \mathscr{F}_t\langle u, e_k\rangle(\lambda - 4k - 2) e^{-\mathrm{i}(4k-2)t} e_k(x) e^{\mathrm{i}t\lambda}\, \mathrm{d}\lambda, \end{aligned}$$

so we may choose

$$a_\lambda(k) = (\mathscr{F}_t\langle u, e_k\rangle)(\lambda - 4k - 2) \cdot \Big(\sum_l (4l+2)^{\sigma} |\mathscr{F}_t\langle u, e_l\rangle(\lambda - 4l - 2)|^2\Big)^{-1/2}, \tag{2-12}$$

and

$$\phi(\lambda) = (2\pi)^{-1/2} \Big(\sum_l (4l+2)^{\sigma} |\mathscr{F}_t\langle u, e_l\rangle(\lambda - 4l - 2)|^2\Big)^{1/2}. \tag{2-13}$$

Then we clearly have $\sum_k (4k+2)^{\sigma} |a_\lambda(k)|^2 = 1$ for each λ, and from the definition of $\mathscr{X}^{\sigma,b}$ norm we see that

$$\int_{\mathbb{R}} \langle\lambda\rangle^{2b} |\phi(\lambda)|^2\, \mathrm{d}\lambda = \frac{1}{2\pi} \|u\|^2_{\mathscr{X}^{\sigma,b}} \lesssim 1. \tag{2-14}$$

If $b > \frac{1}{2}$, then $\langle\lambda\rangle^{-b} \in L^2(\mathbb{R})$, and it follows from Cauchy–Schwartz that $\|\phi\|_{L^1} \le \|\langle\lambda\rangle^b\phi\|_{L^2} \cdot \|\langle\lambda\rangle^{-b}\|_{L^2} \lesssim 1$. If instead $b < \frac{1}{2}$ and u satisfies the support condition, then $\phi(\lambda) = 0$ if $|\lambda| > K$. Again from Cauchy–Schwartz,

$$\|\phi\|_{L^1} \lesssim \left(\int_{|\lambda|\le K} \langle\lambda\rangle^{-2b}\,\mathrm{d}\lambda\right)^{1/2} \sim K^{1/2-b}. \qquad \square$$

Proposition 2.7. *Suppose $b > \frac{1}{2}$, $\sigma_{1,2} \in \mathbb{R}$, and $1 < q_2, r_2 < 2 < q, r, q_1, r_1 < \infty$. We have the following estimates*:

$$\|e^{-\mathrm{i}t\boldsymbol{H}} g\|_{L_t^r L_x^q([-T,T]\times\mathbb{R}^2)} \lesssim \langle T\rangle^{1/r} \|g\|_{L^2} \tag{2-15}$$

if $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and g is defined on $\mathbb{R}^2$;

$$\left\|\int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} u(s)\,\mathrm{d}s\right\|_{L_t^{r_1} L_x^{q_1}([-T,T]\times\mathbb{R}^2)} \lesssim \langle T\rangle^{1+\frac{1}{r_1}-\frac{1}{r_2}} \|u\|_{L_t^{r_2} L_x^{q_2}([-T,T]\times\mathbb{R}^2)} \tag{2-16}$$

if $\frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{2}$, $\frac{1}{q_2} + \frac{1}{r_2} = \frac{3}{2}$, and u is defined on $[-T, T] \times \mathbb{R}^2$;

$$\|u\|_{L_t^r \mathscr{W}_x^{\sigma_1,q}([-T,T]\times\mathbb{R}^2)} \lesssim \langle T\rangle^{1/r} \|u\|_{\mathscr{X}^{\sigma_2,b,T}} \tag{2-17}$$

if $\sigma_2 - \sigma_1 \ge 1 - \frac{2}{q} - \frac{2}{r} \ge 0$, and either u is defined on $[-T, T] \times \mathbb{R}^2$, or u is defined on $\mathbb{R} \times \mathbb{R}^2$ and the right side is replaced by $\|u\|_{\mathscr{X}^{\sigma_2,b}}$;

$$\|u\|_{\mathscr{X}^{\sigma_1,b-1,T}} \lesssim \langle T\rangle^{\frac{1}{q_2}-\frac{1}{2}} \|u\|_{L_t^{q_2} \mathscr{W}_x^{\sigma_1,q_2}([-T,T]\times\mathbb{R}^2)}, \tag{2-18}$$

if $b < 1$, $q_2 > \frac{2}{2-b}$, and either u is defined on $[-T, T] \times \mathbb{R}^2$, or u is defined on $\mathbb{R} \times \mathbb{R}^2$, supported on $[-T, T]$, and the left side is replaced by $\|u\|_{\mathscr{X}^{\sigma_1,b-1}}$; and finally

$$\|u\|_{\mathscr{C}([-T,T],\mathscr{H}^{\sigma_1}(\mathbb{R}^2))} \lesssim \|u\|_{\mathscr{X}^{\sigma_1,b,T}} \tag{2-19}$$

if u is defined on $[-T, T] \times \mathbb{R}^2$. In particular if $T \le 1$, all the implicit constants can be taken 1.

Proof. For (2-15), since $e^{-\mathrm{i}t\boldsymbol{H}}$ is periodic, we may assume $T \lesssim 1$; thus $\langle T\rangle \sim 1$. In addition, by subdividing the interval $[-T, T]$, we may assume T is small enough. Substituting $\sigma = \mathrm{i}$ in Mehler's formula (2-3), we can easily see the integral kernel of $e^{-\mathrm{i}t\boldsymbol{H}}$ is an L^∞ function in the space variables with norm $\lesssim |t|^{-1}$ for $|t| \lesssim T$. Now using the TT^* method we reduce (2-15) to

$$\left\|\int_{-T}^T e^{-\mathrm{i}(t-s)\boldsymbol{H}} u(s)\,\mathrm{d}s\right\|_{L_t^r L_x^q([-T,T]\times\mathbb{R}^2)} \lesssim \|u\|_{L_t^{r'} L_x^{q'}([-T,T]\times\mathbb{R}^2)}. \tag{2-20}$$

Now we interpolate between L^2 conservation and the $L^1 \to L^\infty$ inequality deduced from the L^∞ bound of the integral kernel, to get $\|e^{-\mathrm{i}\delta\boldsymbol{H}} g\|_{L^q} \lesssim |\delta|^{\frac{2}{q}-1} \|u\|_{L^{q'}}$ for $|t| \lesssim T$. Using this and the usual Hardy–Littlewood–Sobolev fractional integral inequality, we immediately get (2-20).

Now from (2-15) and duality we easily get

$$\left\|\int_0^T e^{-\mathrm{i}(t-s)\boldsymbol{H}} u(s)\,\mathrm{d}s\right\|_{L_t^{r_1} L_x^{q_1}([-T,T]\times\mathbb{R}^2)} \lesssim \langle T\rangle^{1+\frac{1}{r_1}-\frac{1}{r_2}} \|u\|_{L_t^{r_2} L_x^{q_2}([-T,T]\times\mathbb{R}^2)}$$

for the exponents q_1, r_1, q_2, r_2; thus from the Christ–Kiselev lemma we get (2-16).

We now prove (2-19) and (2-17), under the assumption $\sigma_2-\sigma_1=1-\frac{2}{q}-\frac{2}{r}=0$. Here we may assume $\sigma_1=0$. By the definition of $\mathscr{X}^{0,b,T}$ we can assume that u is defined for all $t\in\mathbb{R}$, and only need to prove that the left side of each equation is controlled by $\|u\|_{\mathscr{X}^{0,b}}$. We shall use $\|\cdot\|_{\mathfrak{X}}$ to denote either the norm $\langle T\rangle^{-1/r}\|\cdot\|_{L_t^rL_x^q([-T,T]\times\mathbb{R}^2)}$ or $\|\cdot\|_{\mathscr{C}([-T,T],L^2(\mathbb{R}^2))}$, and from what we just proved, we know $\|e^{-\mathrm{i}t\boldsymbol{H}}g\|_{\mathfrak{X}}\lesssim\|g\|_{L^2}$. Assume $\|u\|_{\mathscr{X}^{0,b}}\lesssim 1$; by Lemma 2.6 we may write

$$u(t,x)=\int_{\mathbb{R}}\phi(\lambda)e^{\mathrm{i}\lambda t}\sum_k a_\lambda(k)e^{-\mathrm{i}(4k+2)t}e_k(x)\,\mathrm{d}\lambda \tag{2-21}$$

with $\|\phi\|_{L^1}\lesssim 1$ and $\sum_k|a_\lambda(k)|^2=1$ for each λ. Then we have

$$u=\int_{\mathbb{R}}\phi(\lambda)e^{\mathrm{i}\lambda t}e^{-\mathrm{i}t\boldsymbol{H}}\Big(\sum_k a_\lambda(k)e_k\Big)\,\mathrm{d}\lambda.$$

From Minkowski and Cauchy–Schwartz we see that

$$\begin{aligned}\|u\|_{\mathfrak{X}}&\lesssim\|\phi\|_{L^1}\cdot\sup_\lambda\Big\|e^{\mathrm{i}\lambda t}e^{-\mathrm{i}t\boldsymbol{H}}\Big(\sum_k a_\lambda(k)e_k\Big)\Big\|_{\mathfrak{X}}\\&\lesssim\|\phi\|_{L^1}\cdot\sup_\lambda\Big\|\sum_k a_\lambda(k)e_k\Big\|_{L^2}\lesssim 1,\end{aligned}\tag{2-22}$$

proving (2-19) and this special case of (2-17). To prove (2-17) in general, we use Proposition 2.5 to deduce

$$\|u\|_{L_t^rW_x^{\sigma_1,q}([-T,T]\times\mathbb{R}^2)}\lesssim\|u\|_{L_t^rW_x^{\sigma_2,q'}([-T,T]\times\mathbb{R}^2)}\lesssim\langle T\rangle^{1/r}\|u\|_{\mathscr{X}^{\sigma_2,b,T}},$$

where $\frac{1}{q'}+\frac{1}{r}=\frac{1}{2}$ (so that $2<q,q',r<\infty$ and $\sigma_2-\sigma_1\geq 2(\frac{1}{q'}-\frac{1}{q})\geq 0$), and with obvious modifications when u is globally defined.

Finally we prove (2-18). Again we may assume $\sigma_1=0$. For $v=u$ on $[-T,T]$ and $v=0$ elsewhere, we need to show

$$\|v\|_{\mathscr{X}^{0,b-1}}\lesssim\langle T\rangle^{\frac{1}{q_2}-\frac{1}{2}}\|u\|_{L_{t,x}^{q_2}([-T,T]\times\mathbb{R}^2)}. \tag{2-23}$$

For any w with $\|w\|_{\mathscr{X}^{0,1-b}}\lesssim 1$, we have

$$\left|\int_{\mathbb{R}\times\mathbb{R}^2}v\bar{w}\,\mathrm{d}t\mathrm{d}x\right|=\left|\int_{[-T,T]\times\mathbb{R}^2}u\bar{w}\,\mathrm{d}t\mathrm{d}x\right|\lesssim\|w\|_{L_{t,x}^{q_3}([-T,T]\times\mathbb{R}^2)}\cdot\|u\|_{L_{t,x}^{q_2}([-T,T]\times\mathbb{R}^2)}, \tag{2-24}$$

where $q_3=q_2/(q_2-1)$. Thus by duality, we only need to prove $\|w\|_{L_{t,x}^{q_3}}\lesssim\langle T\rangle^{\frac{1}{2}-\frac{1}{q_3}}\|w\|_{\mathscr{X}^{0,1-b}}$ for all $2<q_3<\frac{2}{b}$. Since the imaginary power $\langle\mathrm{i}\partial_t-\boldsymbol{H}\rangle^{\mathrm{i}\tau}$ is an isometry on $L_{t,x}^2$, we can use Stein's complex interpolation to reduce to the cases $(b,q_3)=(1,2)$ and $(b_1,4)$, where $b_1=(q_3-4+bq_3)/(2q_3-4)<\frac{1}{2}$. The former is trivial by definition, and the latter is a special case of (2-17). □

Lemma 2.8. *Fix $\sigma,b\in\mathbb{R}$, $0<T\leq 1$ and a cutoff function ψ.*

(1) *If* $-\frac{1}{2} < b' \le b < \frac{1}{2}$, *then for* $u \in \mathscr{X}^{\sigma,b}$ *we have*

$$\|\psi(T^{-1}t)u\|_{\mathscr{X}^{\sigma,b'}} \lesssim T^{b-b'}\|u\|_{\mathscr{X}^{\sigma,b}}. \tag{2-25}$$

Also for $u \in \mathscr{X}^{\sigma,b,T}$ *we have*

$$\|u\|_{\mathscr{X}^{\sigma,b',T}} \lesssim T^{b-b'}\|u\|_{\mathscr{X}^{\sigma,b,T}}. \tag{2-26}$$

(2) *If* $\frac{1}{2} < b' = b < 1$, *then for* $u \in \mathscr{X}^{\sigma,b}$ *with* $u(0) = 0$, (2-25) *holds, as well as the limit*

$$\lim_{T\to 0}\|\psi(T^{-1}t)u\|_{\mathscr{X}^{\sigma,b}} = 0. \tag{2-27}$$

Proof. (1) If (2-25) is true, then for any $u \in \mathscr{X}^{\sigma,b,T}$ and any extension $v \in \mathscr{X}^{\sigma,b}$ of u, we have

$$\|u\|_{\mathscr{X}^{\sigma,b',T}} \le \|\psi(T^{-1}t)v\|_{\mathscr{X}^{\sigma,b'}} \lesssim T^{b-b'}\|v\|_{\mathscr{X}^{\sigma,b}},$$

provided $\psi \equiv 1$ on $[-1, 1]$. Taking the infimum over v, we get (2-26). Now we prove (2-25). Define the operator $Mu(t, x) := e^{\mathrm{i}t\boldsymbol{H}}u(t, \cdot)(x)$. We have

$$\mathrm{i}\partial_t(Mu) = e^{\mathrm{i}t\boldsymbol{H}}(\mathrm{i}\partial_t - \boldsymbol{H})u,$$

and therefore we get $\|u\|_{\mathscr{X}^{\sigma,b}} = \|Mu\|_{H_t^b\mathscr{H}_x^\sigma}$. Since M also commutes with multiplication of functions of time, we can reduce to $\|\psi(T^{-1}t)v\|_{H_t^{b'}\mathscr{H}_x^\sigma} \lesssim T^{b-b'}\|v\|_{H_t^b\mathscr{H}_x^\sigma}$. By eigenfunction expansion, we can further reduce to

$$\|\psi(T^{-1}t)g\|_{H^{b'}} \lesssim T^{b-b'}\|g\|_{H^b}. \tag{2-28}$$

By composition we may assume $0 \le b' \le b$ or $b' \le b \le 0$, by duality we may assume $0 \le b' \le b$, and by interpolation we may assume $b' \in \{0, b\}$.

First suppose $b' = b$; we want to prove that multiplication by $\psi(T^{-1}t)$ is bounded, independent of $T > 0$, on H^b. Since it is bounded on L^2, we only need to show that it is also bounded on $\dot{H}^b$. By rescaling we may set $T = 1$. For each $g \in \dot{H}^b$, we split $g = g_1 + g_2$, where $\hat{g}_1$ is supported on $\{|\xi| \le 1\}$ and $\hat{g}_2$ supported on $\{|\xi| \ge 1\}$. Multiplication by ψ is obviously bounded from H^1 to $\dot{H}^1$, and from L^2 to L^2. So it is bounded from H^b to $\dot{H}^b$; thus $\|\psi g_2\|_{\dot{H}^b} \lesssim \|g_2\|_{H^b} \lesssim \|g\|_{\dot{H}^b}$. Since $b < \frac{1}{2}$, we also know

$$\int_{|\tau|\le 1}|\hat{g}_1(\tau)|\,\mathrm{d}\tau \lesssim \big\||\tau|^b\hat{g}_1(\tau)\big\|_{L^2([-1,1])} \cdot \big\||\tau|^{-b}\big\|_{L^2([-1,1])} \lesssim \|g_1\|_{\dot{H}^b} \lesssim \|g\|_{\dot{H}^b}.$$

Thus $(\psi g_1)^\wedge(\tau) = (\hat{\psi} * \hat{g}_1)(\tau)$ is bounded pointwise by $\langle\tau\rangle^{-N}\|g\|_{\dot{H}^b}$, since $\hat{\psi}$ is Schwartz, and the result follows.

Next suppose $b' = 0$, we only need to prove the stronger result

$$\|\psi(T^{-1}t)g\|_{L^2} \lesssim T^b\|g\|_{\dot{H}^b}.$$

By rescaling we can set $T = 1$. Using the same splitting $g = g_1 + g_2$, we have $\|\psi g_2\|_{L^2} \lesssim \|g_2\|_{L^2} \lesssim \|g\|_{\dot{H}^b}$, and $|\psi g_1(\tau)| \lesssim \langle\tau\rangle^{-N}\|g\|_{\dot{H}^b}$. This proves (2-28) and hence (2-25).

(2) We want to prove (2-25), and again we can reduce to (2-28), where we also have $g(0)=0$. Using the same arguments as in (1), we can further reduce to the boundedness on $\dot{H}^b$ and assume $T=1$. Split $g=g_1+g_2$ so that (though we are considering $\dot{H}^b$ norm here, we still assume $g\in H^b$, so $\hat{g}\in L^1$)

$$\hat{g_2}(\tau)=\chi_{|\tau|\geq 1}\cdot\hat{g}(\tau)-\frac{1}{2}\int_{|\lambda|\geq 1}\hat{g}(\lambda)\,\mathrm{d}\lambda\cdot\chi_{1\leq|\tau|\leq 2};$$

then $\hat{g}_1$ is supported in $\{|\tau|\leq 2\}$, $\hat{g}_2$ is supported in $\{|\tau|\geq 1\}$, both the $\hat{g}_i$ have integral zero (since $\hat{g}$ has integral zero), and $\|g_i\|_{\dot{H}^b}\lesssim\|g\|_{\dot{H}^b}$ (since $b>\frac{1}{2}$, we have $\|\hat{g}\|_{L^1(\{|\tau|\geq 1\})}\lesssim\||\tau|^b\hat{g}\|_{L^2}=\|g\|_{\dot{H}^b}$). For g_2 we have $\|\psi g_2\|_{\dot{H}^b}\lesssim\|g_2\|_{H^b}\lesssim\|g\|_{\dot{H}^b}$ as in (1); for g_1 we have

$$(\widehat{\psi g_1})(\tau)=\int_{-2}^{2}(\hat{\psi}(\tau-\xi)-\hat{\psi}(\tau))\hat{g_1}(\xi)\,\mathrm{d}\xi.$$

By Cauchy–Schwartz

$$|(\psi g_1)^\wedge(\tau)|\lesssim\|g_1\|_{\dot{H}^b}\biggl(\int_{-2}^{2}|\xi|^{-2b}|\hat{\psi}(\tau-\xi)-\hat{\psi}(\tau)|^2\,\mathrm{d}\xi\biggr)^{1/2}\lesssim\langle\tau\rangle^{-N}\|g\|_{\dot{H}^b},$$

and (2-28) follows. Finally, to prove (2-27), we first use the operator M and approximation by a finite linear combination of eigenfunctions to reduce to $\|\psi(T^{-1}t)g\|_{H^b}\to 0$ $(T\to 0)$. Since this is easily verified for Schwartz g, we only need to check any $g\in H^b$ with $g(0)=0$ can be approximated by Schwartz h also with $h(0)=0$. But this easily follows since H^b is embedded in L^∞. □

Proposition 2.9. *Suppose $\frac{1}{2}<b<1$. We have*

$$\biggl\|\int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}}u(s)\,\mathrm{d}s\biggr\|_{\mathscr{X}^{\sigma,b,T}}\lesssim\|u\|_{\mathscr{X}^{\sigma,b-1,T}}\tag{2-29}$$

for $T\leq 1$. Also for $u\in\mathscr{X}^{\sigma,b,T}$, the function $\|u\|_{\mathscr{X}^{\sigma,b,\delta}}$ is continuous for $T\geq\delta>0$, and if $u(0)=0$, it tends to 0 as $\delta\to 0$. Moreover, if $p>\frac{1}{2}$ and

$$\|u-e^{-\mathrm{i}(t-k\delta)\boldsymbol{H}}u(k\delta)\|_{\mathscr{X}^{\sigma,b,[(k-1)\delta,(k+1)\delta]}}\leq C\tag{2-30}$$

for $|k|\leq K$, then

$$\|u-e^{-\mathrm{i}t\boldsymbol{H}}u(0)\|_{\mathscr{X}^{\sigma,b,K\delta}}\lesssim c_1K^2\delta^{-b/2}.\tag{2-31}$$

Proof. For the operator M defined in the proof of Lemma 2.8 we have

$$M\biggl(\int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}}u(s)\,\mathrm{d}s\biggr)=\int_0^t Mu(s)\,\mathrm{d}s;\tag{2-32}$$

therefore we can again use an eigenfunction expansion to reduce the problem and see that (2-29) will follow if the operator

$$g(t)\mapsto\mathcal{I}g(t):=\eta(t)\int_0^t g(s)\,\mathrm{d}s\tag{2-33}$$

is bounded from H_t^{b-1} to H_t^b, where η is a fixed smooth function supported on $[-3,3]$ that equals 1 on $[-2,2]$. Choose a smooth compactly supported function ψ that equals 1 on $[-10,10]$, and choose ϕ

supported on $[-5,5]$ that equals 1 on $[-4,4]$. Then we have

$$\mathcal{I}g(t)=\eta(t)\int_{-\infty}^{t}\psi(t-s)\phi(s)g(s)\,\mathrm{d}s-\eta(t)\int_{-5}^{0}\phi(s)g(s)\,\mathrm{d}s. \tag{2-34}$$

We know multiplication by η is bounded on H^b, multiplication by ϕ is bounded on H^{1-b} (to prove these, we first prove them in L^2 and H^1 explicitly, then interpolate), and convolution with $\psi\cdot\chi_{[0,\infty)}$ is bounded from H^{b-1} to H^b, since its Fourier transform is controlled by $\langle\tau\rangle^{-1}$. Thus the first term is bounded. For the second term, we only need to prove $|\langle g,\phi_0\rangle|\lesssim\|g\|_{H^{b-1}}$, where $\phi_0=\phi\cdot\chi_{[0,5]}$ with $|\hat{\phi}_0(\tau)|\lesssim\langle\tau\rangle^{-1}$. But this follows from Plancherel, Cauchy–Schwartz, and the assumption $b>\frac{1}{2}$. This proves (2-29).

Next we consider the function $M(\delta):=\|u\|_{\mathcal{X}^{\sigma,b,\delta}}$, which is clearly nondecreasing. Since we only consider $0<\delta\le T$, we may assume u is defined for $t\in\mathbb{R}$ and belongs to $\mathcal{X}^{\sigma,b}$. For each $\delta>0$, denote by M_0 the left limit of the function M at point δ, and choose a sequence $\delta_n\uparrow\delta$, and (by definition) a sequence of v_n such that $v_n\equiv u$ on $[-\delta_n,\delta_n]$ and $\lim_{n\to\infty}\|v_n\|_{\mathcal{X}^{\sigma,b}}\le M_0$. These v_n have a subsequence converging weakly to some v with $\|v\|_{\mathcal{X}^{\sigma,b}}\le M_0$. Using the embedding $L_t^\infty\mathcal{H}_x^\sigma\supset\mathcal{X}^{\sigma,b}$, we easily see $v\equiv u$ on $[-\delta,\delta]$. This proves left continuity. To prove right continuity at δ, write $M(\delta)=M_1$. For any ϵ, we choose $v\equiv u$ on $[-\delta,\delta]$ and $\|v\|_{\mathcal{X}^{\sigma,b}}<M_1+\epsilon$. Let $u-v=w$ with $w\equiv 0$ on $[-\delta,\delta]$, and define

$$w_\tau=\big(\psi(\tau^{-1}(t-\delta))+\psi(\tau^{-1}(t+\delta))\big)w,$$

for some suitable cutoff that equals 1 on a small neighborhood of 0. From the definition of w_τ, we see that for small τ, we have $v+w_\tau\equiv u$ on a neighborhood of $[-\delta,\delta]$. From Lemma 2.8 we know $\|w_\tau\|_{\mathcal{X}^{\sigma,b}}\to 0$ as $\tau\to 0$, thus $\|v+w_\tau\|_{\mathcal{X}^{\sigma,b}}<M_1+2\epsilon$ if τ is small enough. This proves right continuity. Finally, if $u(0)=0$, then

$$\lim_{\delta\to 0}\|u\|_{\mathcal{X}^{\sigma,b,\delta}}\le\lim_{\tau\to 0}\|\psi(\tau^{-1}t)u\|_{\mathcal{X}^{\sigma,b}}=0$$

for the same cutoff ψ.

Finally we prove (2-31). From (2-30) and the embedding $\|g\|_{L_t^\infty\mathcal{H}_x^\sigma}\lesssim\|g\|_{\mathcal{X}^{\sigma,b,\delta}}$ we see in particular $\|u(k\delta)-e^{-\mathrm{i}k\delta\boldsymbol{H}}u(0)\|_{\mathcal{H}^\sigma}\lesssim K$. Now choose w_k so that $w_k\equiv u-e^{-\mathrm{i}(t-k\delta)\boldsymbol{H}}u(k\delta)$ on $[(k-1)\delta,(k+1)\delta]$ and $\|w_k\|_{\mathcal{X}^{\sigma,b}}\le C$, and choose a partition of unity ψ_k subordinate to the covering $\{((k-1)\delta,(k+1)\delta)\}$ of $[-K\delta,K\delta]$, so that $\psi_k(t)=\tilde{\psi}_k(t/\delta-k)$ and $\tilde{\psi}_k$ have bounded Schwartz norms (this is well known). We then have

$$w=\sum_k\psi_k w_k+\sum_k\psi_k e^{-\mathrm{i}(t-k\delta)\boldsymbol{H}}(u(k\delta)-e^{-\mathrm{i}k\delta\boldsymbol{H}}u(0))\equiv v\ \text{ on }\ [-K\delta,K\delta], \tag{2-35}$$

and $\|w\|_{\sigma,b}\lesssim K^2\delta^{-b/2}$, since it is easy to check (by reducing to estimates of functions of t and interpolating between L^2 and H^1) that multiplication by ψ_k is bounded from $\mathcal{X}^{\sigma,b}$ to itself with norm $\lesssim\delta^{-b/2}$, and that by definition

$$\|\psi_k e^{-\mathrm{i}(t-k\delta)\boldsymbol{H}}(u(k\delta)-e^{-\mathrm{i}k\delta\boldsymbol{H}}u(0))\|_{\mathcal{X}^{\sigma,b}}=\|u(k\delta)-e^{-\mathrm{i}k\delta\boldsymbol{H}}u(0)\|_{\mathcal{H}^\sigma}\|\psi_k\|_{H^b}\lesssim K\delta^{1/2-b}. \qquad\square$$

3. Construction of Gibbs measure

We will construct the Gibbs measure of (1-1) for $1 < p < \infty$ (defocusing case) and $1 < p < 3$ (focusing case). From the definition (1-9) of f, it is obvious that

$$\|f(\omega)\|_{\mathcal{H}^\tau}^2 = \sum_{k=0}^{\infty}(4k+2)^{-1+\tau}|g_k(\omega)|^2. \tag{3-1}$$

This expression is almost surely finite if $\tau < 0$, and is almost surely infinite if $\tau \geq 0$. Thus we have

$$f(\omega) \in \mathcal{H}^{0-} := \bigcap_{\delta>0} \mathcal{H}^{-\delta}, \tag{3-2}$$

almost surely in $\mathbb{P}$. Define $\mu = \mathbb{P} \circ f^{-1}$ to be the push-forward of $\mathbb{P}$ under f; then we see that the typical element in the support of μ belongs to any $\mathcal{H}^{-\delta}$ for all $\delta > 0$, but does not belong to L^2. We also define $\mu_{2^k}^{\circ} = \mathbb{P} \circ (f_{2^k}^{\circ})^{-1}$, and $\mu_{2^k}^{\perp} = \mathbb{P} \circ (f_{2^k}^{\perp})^{-1}$. Now we prove two lemmas concerning linear and multilinear estimates of the eigenfunctions $e_k(x)$ as defined in (1-3).

Lemma 3.1. *For any $2 \leq q \leq \infty$ and $q \neq 4$, write $\nu = 4k+2$ for $k \geq 0$; then we have*

$$\|e_k\|_{L^q(\mathbb{R}^2)} \lesssim \nu^{-\rho(q)}, \tag{3-3}$$

where $\rho(q) = \min\{\frac{1}{2} - \frac{1}{q}, \frac{1}{q}\}$. If $q = 4$ we have

$$\|e_k\|_{L^4(\mathbb{R}^2)} \lesssim \nu^{-\frac{1}{4}} \log^{\frac{1}{4}} \nu. \tag{3-4}$$

Proof. Since $e_k(x) = \pi^{-\frac{1}{2}}\mathcal{L}_k^0(|x|^2)$, we easily see $\|e_k\|_{L^q(\mathbb{R}^2)} \sim \|\mathcal{L}_k^0\|_{L^q(\mathbb{R}^+)}$. Then we can use (1-4) to compute

$$\begin{aligned}\|\mathcal{L}_k^0\|_{L^4(\mathbb{R}^+)}^4 &\lesssim \int_0^{1/\nu} \mathrm{d}z + \int_{1/\nu}^{\nu/2}(z\nu)^{-1}\,\mathrm{d}z + \nu^{-1}\int_{\nu/2}^{3\nu/2}\big(\nu^{1/3}+|\nu-z|\big)^{-1}\,\mathrm{d}z + \int_{3\nu/2}^{\infty} e^{-cz}\,\mathrm{d}z \\ &\lesssim \nu^{-1}\log \nu. \end{aligned}\tag{3-5}$$

This proves (3-4). As for (3-3) we have

$$\begin{aligned}\|\mathcal{L}_k^0\|_{L^q(\mathbb{R}^+)}^q &\lesssim \int_0^{1/\nu} \mathrm{d}z + \int_{1/\nu}^{\nu/2}(z\nu)^{-q/4}\,\mathrm{d}z + \nu^{-q/4}\int_{\nu/2}^{3\nu/2}\big(\nu^{1/3}+|\nu-z|\big)^{-q/4}\,\mathrm{d}z + \int_{3\nu/2}^{\infty} e^{-cz}\,\mathrm{d}z \\ &\lesssim \nu^{-q/4+|1-q/4|} + \nu^{1-q/3} + \nu^{\max(1-q/2,1-q/3)} \\ &\lesssim \nu^{-q\rho(q)}. \end{aligned}$$

□

Lemma 3.2. *Suppose $l \geq 4$ and $n_1, \ldots, n_l \geq 0$. Let $\nu_j = 4n_j + 2$ for $1 \leq j \leq l$, and assume $\nu_1 \gtrsim \cdots \gtrsim \nu_l$. Then we have*

$$\left|\int_{\mathbb{R}^2} e_{n_1}(x)\cdots e_{n_l}(x)\right| \lesssim \nu_1^{-1/2}\nu_3^{-1/4}\log \nu_1. \tag{3-6}$$

Moreover, if $\nu_1 \gtrsim \nu_2^{1+\epsilon}$ *for some* $\epsilon > 0$, *then*

$$\left|\int_{\mathbb{R}^2} e_{n_1}(x)\cdots e_{n_l}(x)\right| \lesssim \nu_1^{-N} \quad \textit{for all } N > 0. \tag{3-7}$$

Proof. Recalling that $\boldsymbol{H} e_{n_j} = \nu_j e_{n_j}$ and $\boldsymbol{H}$ is self-adjoint on $L^2(\mathbb{R}^2)$, we can compute using Proposition 2.5 and Lemma 3.1 that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} e_{n_1}(x)\cdots e_{n_l}(x)\right| &\le \nu_1^{-m}\|\boldsymbol{H}^m(e_{n_2}\cdots e_{n_l})\cdot e_{n_1}\|_{L^1}\\ &\lesssim \nu_1^{-m}\|e_{n_2}\cdots e_{n_l}\|_{\mathscr{H}^{2m}} \lesssim \nu_1^{-m}\sum_{j=2}^{l}\|e_{n_j}\|_{\mathscr{W}^{2m,2(l-1)}}\prod_{2\le i\ne j}\|e_{n_i}\|_{L^{2(l-1)}} \lesssim (\nu_1^{-1}\nu_2)^m.\end{aligned}$$

If $\nu_1 \gtrsim \nu_2^{1+\epsilon}$, we can choose m large enough and prove (3-7). As for (3-6), we choose $m = 1$ and estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^2} e_{n_1}(x)\cdots e_{n_l}(x)\right| &\le \nu_1^{-1}\|\boldsymbol{H}(e_{n_2}\cdots e_{n_l})\cdot e_{n_1}\|_{L^1}\\ &\lesssim \nu_1^{-\frac{5}{4}}\log^{\frac{1}{4}}\nu_1\cdot\|e_{n_2}\cdots e_{n_l}\|_{\mathscr{W}^{2,\frac{4}{3}}}\\ &\lesssim \nu_1^{-\frac{5}{4}}\log^{\frac{1}{4}}\nu_1\cdot\nu_2\|e_{n_2}\|_{L^4}\|e_{n_3}\|_{L^4}\textstyle\prod_{j\ge 4}\|e_{n_i}\|_{L^{4(l-3)}}\\ &\lesssim \nu_1^{-\frac{5}{4}}\nu_2^{\frac{3}{4}}\nu_3^{-\frac{1}{4}}\log^{\frac{3}{4}}\nu_1 \lesssim \nu_1^{-\frac{1}{2}}\nu_3^{-\frac{1}{4}}\log\nu_1.\end{aligned}$$ □

To state and prove the probabilistic L^p estimates for our $\mathscr{S}'$-valued random variable f, we first need a result proved by Fernique.

Lemma 3.3 (Fernique). *There exist absolute constants* c, C *such that for any finite-dimensional normed vector space* $(V, \|\cdot\|)$, *any centered Gaussian random variable* $f(\omega)$ *taking its value in* V, *and any positive constant* A, *we have*

$$\mathbb{E}(e^{cA^{-2}\|f(\omega)\|^2}) \le C \tag{3-8}$$

if $\mathbb{P}(\|f(\omega)\| > A) < \frac{1}{10}$.

Proof. See [Fernique 1975] or [Da Prato and Zabczyk 1992, Theorem 2.6]. □

Proposition 3.4. *Fix* $2 < q < \infty$, $1 < r < \infty$, $0 < \alpha < \min(\frac{2}{q}, 1 - \frac{2}{q})$, *and two positive integers* $M > 10N$. *For any* g, *we define*

$$\Pi g = \sum_{j=N-1}^{M}\langle g, e_j\rangle e_j. \tag{3-9}$$

Then, for the random variable f *as defined in* (1-9), *we have the large deviation estimates*

$$\mathbb{P}\big(\|\Pi f(\omega)\|_{\mathscr{W}^{\alpha,q}} > AN^{-\delta}\big) \le Ce^{-cA^2}, \tag{3-10}$$

$$\mathbb{P}\big(\|e^{-it\boldsymbol{H}}\Pi f(\omega)\|_{L_t^r\mathscr{W}_x^{\alpha,q}([-T,T]\times\mathbb{R}^2)} > AN^{-\delta}T^{1/r}\big) \le Ce^{-cA^2}, \tag{3-11}$$

where $\delta > 0$ *is some small positive exponent.*

Proof. We compute for each $t \in [-\pi, \pi]$

$$\mathbb{E}\big(\|e^{-\mathrm{i}t\boldsymbol{H}}\Pi f(\omega)\|^q_{\mathcal{W}^{\alpha,q}_x}\big) = \int_{\mathbb{R}^2} \mathbb{E}\bigg|\sum_{j=N-1}^{M} (4j+2)^{(\alpha-1)/2} g_j(\omega) e_j(x)\bigg|^q \,\mathrm{d}x. \tag{3-12}$$

Now by Khintchine's inequality (the variant for Gaussians), we have

$$\mathbb{E}\bigg|\sum_{j=N-1}^{M} (4j+2)^{(\alpha-1)/2} g_j(\omega) e_j(x)\bigg|^q \lesssim \bigg(\sum_{j=N-1}^{M} \frac{e_j(x)^2}{(4j+2)^{1-\alpha}}\bigg)^{q/2}. \tag{3-13}$$

Then integrating in x, using Minkowski's inequality (since $q > 2$), we get

$$\mathbb{E}\big(\|e^{-\mathrm{i}t\boldsymbol{H}}\Pi f(\omega)\|^q_{\mathcal{W}^{\alpha,q}_x}\big) \lesssim \bigg(\sum_{j=N-1}^{M} \frac{\|e_j\|^2_{L^q}}{(4j+2)^{1-\alpha}}\bigg)^{q/2} \le CN^{-q\delta}, \tag{3-14}$$

due to Lemma 3.1 and the assumption $\alpha < 2\rho(q)$. Now we can take $t = 0$ in (3-14) and use Markov's inequality and Lemma 3.3, and immediately get (3-10).

As for (3-11), we need a little more work. What we need is

$$\mathbb{P}\big(\|e^{-\mathrm{i}t\boldsymbol{H}}\Pi f(\omega)\|_{L^r_t\mathcal{W}^{\alpha,q}_x} > CN^{-\delta}T^{1/r}\big) < \tfrac{1}{10} \tag{3-15}$$

for large C. If the event in (3-15) happens, then there exists an integer $l \ge 0$ such that

$$\big|\big\{t \in [-T, T] : \|e^{-\mathrm{i}t\boldsymbol{H}}\Pi f(\omega)\|_{\mathcal{W}^{\alpha,q}_x} > 2^l N^{-\delta}\big\}\big| > K2^{-2rl}T. \tag{3-16}$$

For fixed t, due to (3-14) and Lemma 3.3, the probability that $\|e^{-\mathrm{i}t\boldsymbol{H}}\Pi f(\omega)\|_{\mathcal{W}^{\alpha,q}_x} > 2^l N^{-\delta}$ is less than $c_1\exp(-c_2 2^{2l})$. We then use Fubini's theorem to conclude that the probability that (3-16) happens is less than $K^{-1}c_1 2^{2rl}\exp(-c_2 2^{2l})$. Then we sum over $l \ge 0$ and choose K large enough so that this sum is less than $\frac{1}{10}$. □

Corollary 3.5. *For the same parameters q, r, α as in Proposition 3.4, we have*

$$\mathbb{P}\big(\|f(\omega)\|_{\mathcal{W}^{\alpha,q}} > A\big) \le Ce^{-cA^2}, \tag{3-17}$$

$$\mathbb{P}\big(\sup_{k\ge 0}\|f^\circ_{2^k}(\omega)\|_{\mathcal{W}^{\alpha,q}} > A\big) \le Ce^{-cA^2}, \tag{3-18}$$

$$\mathbb{P}\big(\|e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)\|_{L^r_t\mathcal{W}^{\alpha,q}_x([-T,T]\times\mathbb{R}^2)} > AT^{1/r}\big) \le Ce^{-cA^2}, \tag{3-19}$$

$$\mathbb{P}\big(\sup_{k\ge 0}\|e^{-\mathrm{i}t\boldsymbol{H}} f^\circ_{2^k}(\omega)\|_{L^r_t\mathcal{W}^{\alpha,q}_x([-T,T]\times\mathbb{R}^2)} > AT^{1/r}\big) \le Ce^{-cA^2}, \tag{3-20}$$

$$\lim_{k\to\infty}\|f^\circ_{2^k}(\omega) - f(\omega)\|_{\mathcal{W}^{\alpha,q}} + \|e^{-\mathrm{i}t\boldsymbol{H}}(f^\circ_{2^k}(\omega) - f(\omega))\|_{L^r_t\mathcal{W}^{\alpha,q}_x([-T,T]\times\mathbb{R}^2)} = 0 \ \textit{almost surely in } \mathbb{P}. \tag{3-21}$$

Proof. We know $f^\circ_{2^k}(\omega) \to f(\omega)$ and $e^{-\mathrm{i}t\boldsymbol{H}} f^\circ_{2^k}(\omega) \to e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$ in $\mathcal{S}'$. If we can prove (3-18) and (3-20), then almost surely in $\mathbb{P}$, we have

$$\sup_{k\ge 0}\|e^{-\mathrm{i}t\boldsymbol{H}} f^\circ_{2^k}(\omega)\|_{L^r_t\mathcal{W}^{\alpha,q}_x} < \infty, \tag{3-22}$$

and there must be a subsequence of $\{e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\}$ converging weakly in $L^r_t \mathcal{W}^{\alpha,q}_x$. This weak limit must be $e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$, so we know that

$$\|e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)\|_{L^r_t \mathcal{W}^{\alpha,q}_x} \le \sup_{k\ge 0} \|e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\|_{L^r_t \mathcal{W}^{\alpha,q}_x} < \infty, \tag{3-23}$$

almost surely in $\mathbb{P}$. Thus (3-19) also holds true, with the same constants as in (3-20). Clearly (3-17) also follows from (3-18) in the same way.

To prove (3-18) and (3-20), we use (3-10) and (3-11). For any k, the difference $f^{\circ}_{2^k}(\omega) - f^{\circ}_{2^{k-1}}(\omega)$ is of the form $\Pi f(\omega)$ as defined in Proposition 3.4, with the parameter $N \sim 2^k$. We then have, for some $\delta > 0$,

$$\mathbb{P}\big(\|e^{-\mathrm{i}t\boldsymbol{H}}(f^{\circ}_{2^k}(\omega) - f^{\circ}_{2^{k-1}}(\omega))\|_{L^r_t \mathcal{W}^{\alpha,q}_x} > A2^{-k\delta/2}T^{1/r}\big) \le c_1 e^{-c_2 2^{k\delta} A^2}. \tag{3-24}$$

Choose c small enough; then

$$\sup_{k\ge 0} \|e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\|_{L^r_t \mathcal{W}^{\alpha,q}_x} > AT^{1/r} \tag{3-25}$$

implies

$$\|e^{-\mathrm{i}t\boldsymbol{H}}(f^{\circ}_{2^k}(\omega) - f^{\circ}_{2^{k-1}}(\omega))\|_{L^r_t \mathcal{W}^{\alpha,q}_x} > cA2^{-k\delta/2}T^{1/r} \quad \text{for some } k \ge 0. \tag{3-26}$$

Now we can combine this with (3-24) to get

$$\mathbb{P}\big(\sup_{k\ge 0} \|e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\|_{L^r_t \mathcal{W}^{\alpha,q}_x} > AT^{1/r}\big) \le \sum_{k=0}^{\infty} c_3 e^{-c_4 2^{k\delta} A^2} \le c_5 e^{-c_6 A^2}. \tag{3-27}$$

This proves (3-20). Clearly (3-18) also follows from (3-10) in the same way.

Finally we prove (3-21). From the discussion above we see

$$\mathbb{P}\big(\sup_{k\ge 0} 2^{k\delta/2} \|e^{-\mathrm{i}t\boldsymbol{H}}(f^{\circ}_{2^k}(\omega) - f^{\circ}_{2^{k-1}}(\omega))\|_{L^r_t \mathcal{W}^{\alpha,q}_x} < \infty\big) = 1; \tag{3-28}$$

thus with probability 1, the series

$$\sum_{k=0}^{\infty} e^{-\mathrm{i}t\boldsymbol{H}}(f^{\circ}_{2^k}(\omega) - f^{\circ}_{2^{k-1}}(\omega)) \tag{3-29}$$

converges in $L^r_t \mathcal{W}^{\alpha,q}_x$. This can only converge to $e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$, and the same argument works for the space $\mathcal{W}^{\alpha,q}$. □

Equation (1-1) is a hamiltonian PDE with formally conserved mass $\|u\|^2_{L^2}$ and Hamiltonian

$$E(u) = \langle \boldsymbol{H}u, u\rangle \pm \frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}} = \int_{\mathbb{R}^n} \Big(|\nabla u|^2 + |xu|^2 \pm \frac{2}{p+1}|u|^{p+1}\Big)\,\mathrm{d}x. \tag{3-30}$$

Recall that $\mu = \mathbb{P} \circ f^{-1}$ is a probability measure on $\mathcal{S}'(\mathbb{R}^2)$, the push-forward of $\mathbb{P}$ under f. In the defocusing case, for all $1 < p < \infty$, we define the Gibbs measure of (1-1) to be

$$\mathrm{d}\nu = \exp\Big(-\frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}}\Big)\,\mathrm{d}\mu. \tag{3-31}$$

Since the integrand in (3-31) is well defined, bounded and positive, by Corollary 3.5, we know ν is finite and mutually absolutely continuous with μ. We also define the truncated measures

$$d\nu_{2^k} = \exp\Big(-\frac{2}{p+1}\|u_{2^k}^{\circ}\|_{L^{p+1}}^{p+1}\Big)\,d\mu. \tag{3-32}$$

Since $\|u_{2^k}^{\circ}\|_{L^{p+1}} \to \|u\|_{L^{p+1}}$ almost everywhere in μ, thanks to Corollary 3.5, we know $\nu_{2^k} \to \nu$ in the strong sense that the total variation of $\nu_{2^k} - \nu$ tends to 0.

In the focusing case, for $1 < p < 3$, we define the truncated measures $d\nu_{2^k} = \rho_{2^k}\,d\mu$, where

$$\rho_{2^k}(u) = \chi(\|u_{2^k}^{\circ}\|_{L^2}^2 - \alpha_{2^k})\exp\Big(\frac{2}{p+1}\|u_{2^k}^{\circ}\|_{L^{p+1}}^{p+1}\Big). \tag{3-33}$$

Here χ is some compactly supported continuous function on $\mathbb{R}$ that equals 1 on a neighborhood of 0, and

$$\alpha_{2^k} = \mathbb{E}\big(\|f_{2^k}^{\circ}(\omega)\|_{L^2}^2\big) = \sum_{j=0}^{2^k} \frac{1}{4j+2}. \tag{3-34}$$

Clearly $\alpha_{2^k} \lesssim k$ for $k \geq 1$. We define the Gibbs measure ν as the limit of these ν_{2^k}. More precisely:

Proposition 3.6. *The functions ρ_{2^k} converge to a function ρ in $L^r(\mu)$ for all $1 \leq r < \infty$. The measure $d\nu = \rho\,d\mu$ is finite and absolutely continuous with respect to μ. We also know $\nu_{2^k} \to \nu$ in the strong sense that the total variation of $\nu_{2^k} - \nu$ tends to 0. Finally, we can choose a countable number of $\chi_{(m)}$ so that the union of the supports of the corresponding Radon–Nikodym derivatives $\rho_{(m)}$ has full μ measure in $\mathcal{S}'(\mathbb{R}^2)$. If we have fixed χ, we will define ν to be the Gibbs measure of equation (1-1).*

Proof. First we prove that ρ_{2^k} converges almost everywhere in μ, or equivalently, that $\rho_{2^k}(f(\omega))$ converges almost surely in $\mathbb{P}$. Consider

$$\|f_{2^k}^{\circ}(\omega)\|_{L^2}^2 - \alpha_{2^k} = \sum_{j=0}^{k} \frac{|g_j(\omega)|^2 - 1}{4j+2}, \tag{3-35}$$

and see that it is a (partial) independent sum of random variables with zero mean and summable variance (the variance of j-th term is $\sim (j+1)^{-2}$), so it converges almost surely. Thus by the continuity of χ, the first factor $\chi(\|f_{2^k}^{\circ}(\omega)\|_{L^2}^2 - \alpha_{2^k})$ in $\rho_{2^k}(f(\omega))$ converges almost surely. Next, since $f_{2^k}^{\circ}(\omega) \to f(\omega)$ in L^{p+1} for almost surely $\omega \in \Omega$, we know that the second factor also converges almost surely. Therefore, ρ_{2^k} converges almost everywhere in μ, say to some ρ.

To prove $\rho_{2^k}(f) \to \rho(f)$ in $L^r(\mathbb{P})$, we need some uniform integrability conditions. This is provided by the large deviation estimate

$$\mathbb{P}\big(\|f_{2^k}^{\circ}(\omega)\|_{L^2}^2 - \alpha_{2^k} \leq \beta,\ \|f_{2^k}^{\circ}(\omega)\|_{L^{p+1}} > A\big) \leq Ce^{-cA^{\delta}} \tag{3-36}$$

for some $\delta > p+1$ and all large enough A, where β is such that $\chi(z) = 0$ for $|z| \geq \beta$. To prove (3-36) we may assume A is sufficiently large, and set $k_0 \in \mathbb{N}$ so that $2^{k_0} \sim e^{A^{\delta}}$ for some $\delta > 0$ to be determined later.

First we prove (3-36) is true for $k \le k_0 + 1$, with β and A on the left side replaced by 2β and $A/2$. In fact, by Hölder's inequality, if

$$\|f^\circ_{2^k}(\omega)\|^2_{L^2} \le \alpha_{2^k} + 2\beta \lesssim k \lesssim A^\delta \quad \text{and} \quad \|f^\circ_{2^k}(\omega)\|_{L^{p+1}} > A/2, \tag{3-37}$$

then

$$\|f^\circ_{2^k}(\omega)\|_{L^q} \gtrsim A^\sigma \quad \text{and} \quad \sigma = \frac{(q-2)(p+1) - \delta(q-p-1)}{(p-1)q}, \tag{3-38}$$

under the assumption $p + 1 \le q < \infty$. Since $2 < q < \infty$, we know from Corollary 3.5 that

$$\mathbb{P}(\|f^\circ_{2^k}(\omega)\|_{L^q} > A^\sigma) \le Ce^{-cA^{2\sigma}}. \tag{3-39}$$

If $1 < p < 3$, then for q sufficiently large and δ sufficiently small, we have $2\sigma > p + 1$, so (3-36) is true in this case.

Next we assume $k \ge k_0 + 2$. In this case we can prove

$$\mathbb{P}(\|f^\circ_{2^k}(\omega) - f^\circ_{2^{k_0}}(\omega)\|_{L^{p+1}} > A/2) \le c_1 \exp(-c_2 e^{c_3 A^{c_4}}). \tag{3-40}$$

In fact, since $f^\circ_{2^k}(\omega) - f^\circ_{2^{k_0}}(\omega)$ is of the form $\Pi f(\omega)$ as defined in Proposition 3.4, with the parameter $N \sim 2^{k_0}$, by Proposition 3.4 we immediately get (3-40) (notice $N \sim e^{A^\delta}$).

Now if $\|f^\circ_{2^k}(\omega)\|^2_{L^2} \le \alpha_k + \beta$ and $\|f^\circ_{2^k}(\omega)\|_{L^{p+1}} > A$, then we have three possibilities.

(1) If $\|f^\circ_{2^k}(\omega) - f^\circ_{2^{k_0}}(\omega)\|_{L^{p+1}} > A/2$, then we are already done, since this probability is controlled due to (3-40).

(2) If $\|f^\circ_{2^{k_0}}(\omega)\|_{L^{p+1}} > A/2$ and $\|f^\circ_{2^{k_0}}(\omega)\|^2_{L^2} \le \alpha_{k_0} + 2\beta$, then we may set $k = k_0$ in the arguments from (3-37) to (3-39), and again get the desired bound.

(3) If $\|f^\circ_{2^k}(\omega)\|^2_{L^2} \le \alpha_{2^k} + \beta$ as well as $\|f^\circ_{2^{k_0}}(\omega)\|^2_{L^2} > \alpha_{2^{k_0}} + 2\beta$, then

$$\|f^\circ_{2^k}(\omega)\|^2_{L^2} - \|f^\circ_{2^{k_0}}(\omega)\|^2_{L^2} - (\alpha_{2^k} - \alpha_{2^{k_0}}) \le -\beta \tag{3-41}$$

or equivalently

$$Y = \sum_{j=2^{k_0}+1}^{2^k} \frac{1 - |g_j|^2}{4j+2} \ge \beta. \tag{3-42}$$

Noticing that Y is an independent sum with standard deviation

$$\kappa = \left(\sum_{j=2^{k_0}+1}^{2^k} \frac{1}{(4j+2)^2}\right)^{1/2} \lesssim 2^{-k_0/2} \le c_1 e^{-c_2 A^{c_3}}, \tag{3-43}$$

we can compute

$$\mathbb{E}(\exp(Y/2\kappa)) = \prod_{j=2^{k_0}+1}^{2^k} \mathbb{E}\Big(\exp\Big(\frac{\kappa(1-|g_j|^2)}{2(4j+2)}\Big)\Big)$$
$$= \prod_{j=2^{k_0}+1}^{2^k} \Big(e^{\kappa/(2(4j+2))}\Big(1+\frac{\kappa}{2(4j+2)}\Big)^{-1}\Big) \le \prod_{j=2^{k_0}+1}^{2^k} e^{c\theta_j^2/(4(4j+2)^2\kappa^2)} = e^{c/4}. \tag{3-44}$$

Here we have used the fact that $\mathbb{E}(e^{-\lambda|g|^2}) = (1+\lambda)^{-1}$ when $\lambda > -1$, and g is a normalized complex Gaussian; and that $e^x(1+x)^{-1} \le e^{cx^2}$ for large c, and $0 \le x \le \frac{1}{2}$. Therefore we have obtained

$$\mathbb{P}(Y > \beta) \le e^{-c\kappa^{-1}} \le c_1 \exp(-c_2 e^{c_3 A^{c_4}}). \tag{3-45}$$

This completes the proof of (3-36). The other conclusions now follow easily from this large deviation estimate, except the one regarding the support of ρ. We choose a sequence of cutoff functions $\chi_{(m)}$ so that $\chi_{(m)} \equiv 1$ on $[-\gamma_m, \gamma_m]$ with $\gamma_m \uparrow \infty$. By our previous discussions, after discarding null sets, the function $\rho_{(m)}$ will be nonzero wherever

$$\lim_{k\to\infty} \big| \|f_{2^k}^{\circ}(\omega)\|_{L^2}^2 - \alpha_{2^k} \big| \le \gamma_m. \tag{3-46}$$

Since this limit exists almost surely, and $\gamma_m \uparrow \infty$, we know almost surely, (3-46) will hold for at least one m. So the union of support of these ρ_j will have full μ measure. □

Now in both defocusing and focusing case we have defined the Gibbs measure ν and the approximating measure ν_{2^k}. They will be used in Section 6 to obtain global well-posedness, and the invariance of ν will be proved in Section 8.

4. Multilinear analysis in $\mathscr{X}^{\sigma,b}$ spaces

First let us recall the hypercontractivity property of complex Gaussians. To make equations easier to write, we introduce the notation in which u^- represents some element in $\{u, \bar{u}\}$ for any complex number u. This will be used throughout the rest of the paper. The first result about hypercontractivity was proved in [Nelson 1973]. Here we use a formulation of this property taken from [Thomann and Tzvetkov 2010].

Proposition 4.1. *Suppose $l, d \ge 1$, and a random variable S has the form*

$$S = \sum_{0\le n_1,\dots,n_l\le d} c_{n_1,\dots,n_l} \cdot g_{n_1}^-(\omega)\cdots g_{n_l}^-(\omega), \tag{4-1}$$

where $c_{n_1,\dots,n_l} \in \mathbb{C}$, and the $(g_n)_{0\le n\le d}$ are independent normalized complex Gaussians; then we have the estimate

$$(\mathbb{E}|S|^p)^{1/p} \le \sqrt{l+1}(p-1)^{l/2}(\mathbb{E}|S|^2)^{1/2} \quad \textit{for all} \;\; p \ge 2.$$

Proof. This is basically a restatement of [Thomann and Tzvetkov 2010, Proposition 2.4]. There the authors required $n_j \ge 1$ and $n_1 \le \cdots \le n_l$, but an easy modification will immediately settle this. The only difference is that here we may have g_{n_j} or $\bar{g}_{n_j}$, but if we write $g_n = (\gamma_n + \mathrm{i}\tilde{\gamma}_n)/\sqrt{2}$ where γ_n and $\tilde{\gamma}_n$ are mutually independent normalized real Gaussians, then $\bar{g}_n = (\gamma_n - \mathrm{i}\tilde{\gamma}_n)/\sqrt{2}$. So S is again written as a linear combination of products of independent normalized real Gaussians. Then the result follows in the same way as in [Thomann and Tzvetkov 2010]. □

Next we want to adapt the result in Proposition 4.1 to our specific case to yield a large deviation bound on appropriate multilinear expressions of Gaussians.

Proposition 4.2. *Let $N_1 \geq \cdots \geq N_l \geq 2$ be dyadic numbers such that $N_1 \geq 10^3 N_2$. Assume for $n \geq 0$ and $4n+2 \leq 10N_1^2$ that we have independent normalized complex Gaussians $\{w_n\}$. Also let ϱ be any integer, and let $\delta_{n_1,\dots,n_l}$ be arbitrary complex numbers with absolute value ≤ 1. Define*

$$\Xi = \left\{(n_1, \dots, n_l) : n_j \geq 0,\ \frac{1}{10} \leq \frac{4n_j+2}{N_j^2} \leq 10\,(1 \leq j \leq l),\ \sum_{j=1}^{l} \epsilon_j(4n_j+2) = \varrho\right\} \tag{4-2}$$

with $\epsilon_j = \pm 1$; then we have

$$\mathbb{P}\left(\left\{\left|\sum_{(n_1,\dots,n_l)\in\Xi} \delta_{n_1,\dots,n_l} w_{n_1}^{-}(\omega)\cdots w_{n_l}^{-}(\omega)\right| > K\prod_{j=2}^{l} N_j\right\}\right) \leq c_1 \exp(-c_2 K^{c_3}). \tag{4-3}$$

Here all the constants depend only on l.

Proof. We denote the sum on the left side of (4-3) by S. Using Proposition 4.1, we can get

$$(\mathbb{E}|S|^p)^{1/p} \leq \sqrt{l+1}(p-1)^{l/2} A,$$

where we denote $A = (\mathbb{E}|S|^2)^{1/2}$. By Markov's inequality, we in particular have

$$\mathbb{P}(|S| > KA) \leq (KA)^{-p} \cdot \mathbb{E}|S|^p \leq K^{-p}(l+1)^{p/2}(p-1)^{lp/2} \quad \text{for all } p \geq 2.$$

If $K \geq 2\sqrt{l+1}$, we may choose $p = 1 + K^{2/l} 2^{-2/l}(l+1)^{-1/l} \geq 2$ in the inequality above to obtain

$$\mathbb{P}(|S| > KA) \leq 2^{-p} \leq c_1 e^{-c_2 K^{c_3}}.$$

By choosing the constants appropriately, we can guarantee that this also hold for $K < 2\sqrt{l+1}$. Now what remains is to prove that $A \lesssim \prod_{j=2}^{l} N_j$, or equivalently

$$\mathbb{E}|S|^2 \lesssim \prod_{j=2}^{l} N_j^2.$$

Now we expand the square to get

$$\mathbb{E}|S|^2 = \sum \delta_{n_1,\dots,n_l} \bar{\delta}_{m_1,\dots,m_l} \Delta_{n_1,\dots,n_l,m_1,\dots,m_l},$$

where the sum is taken over all $(n_1, \dots, n_l, m_1, \dots, m_l) \in \Xi \times \Xi$, and

$$\Delta_{n_1,\dots,n_l,m_1,\dots,m_l} = \mathbb{E}\left(\prod_{j=1}^{l} w_{n_j}^{-} w_{m_j}^{-}\right).$$

Since each of the δ and Δ has absolute value $\lesssim 1$ (depending on l) in any possible case, we will be done once we establish that

$$\#\big\{(n_1,\ldots,n_l,m_1,\ldots,m_l)\in\Xi\times\Xi:\Delta_{n_1,\ldots,n_l,m_1,\ldots,m_l}\neq 0\big\}\lesssim\prod_{j=2}^{l}N_j^2. \tag{4-4}$$

The crucial observation is that, due to the independence assumption, if the expectation Δ is nonzero, then any integer that appears in $(n_1,\ldots,n_l,m_1,\ldots,m_l)$ must appear at least twice. Next, due to our assumption $N_1\geq 10^3N_2$, we know $n_1=m_1$, and any integer that appears in $(n_2,\ldots,n_l,m_2,\ldots,m_l)$ must appear at least twice. If we permute all the different integers appearing in this $(2l-2)$-tuple as $\sigma_1>\sigma_2>\cdots>\sigma_r$, then with r and all σ_i fixed, we have at most $(2l-2)^{2l-2}$ choices for the $(2l-2)$-tuple; also due to the linear relation enjoyed by both $(n_1,\ldots,n_l)$ and $(m_1,\ldots,m_l)$, the $(2l-2)$-tuple will uniquely determine n_1 and m_1. Thus we only need to show for each possible $1\leq r\leq 2l$, there are $\lesssim\prod_{j=2}^{l}N_j^2$ choices for $(\sigma_1,\ldots,\sigma_r)$. Now for each $1\leq i\leq r$, since each $\sigma_j(1\leq j\leq i)$ appear in the $(2l-2)$-tuple at least twice (and different σ_j cannot appear at the same place), there must exist $1\leq j_1\leq i<i+1\leq j_2$ such that $\sigma_{j_1}\in\{n_{j_2},m_{j_2}\}$. This implies

$$4\sigma_i+2\leq 4\sigma_{j_1}+2\lesssim N_{j_2}^2\lesssim N_{i+1}^2,$$

so for each $1\leq i\leq r$, there are at most N_{i+1}^2 choices for σ_i, and necessarily $1\leq r\leq l-1$. Therefore, for each $r\leq l-1$, we have at most

$$\prod_{i=1}^{r}N_{i+1}^2\lesssim\prod_{j=2}^{l}N_j^2$$

choices for $(\sigma_1,\ldots,\sigma_r)$. $\square$

Proposition 4.3. *Suppose $p\geq 3$ is an odd integer. We choose σ and b so that $0<\sigma<1$ is sufficiently close to 1 depending on p, and $1>b>\frac{1}{2}$ is sufficiently close to $\frac{1}{2}$ depending on σ and p. Let T be small enough depending on b,σ and p. Then we can find a set $\Omega_T\subset\Omega$ and a positive number θ that only depends on σ,b and T, so that $\mathbb{P}(\Omega_T)\leq c_1e^{-c_2T^{-c_3}}$, and that the following holds: For any $t_0\in\mathbb{R}$ and $\omega\in\Omega_T^c$, if for each $1\leq j\leq p$, a function u_j on $[-T,T]\times\mathbb{R}^2$ is given by either*

$$u_j=e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f(\omega), \tag{4-5}$$

or

$$\|u_j\|_{\mathscr{X}^{\sigma,b,T}}\lesssim 1, \tag{4-6}$$

then we have

$$\|u_1^-\cdots u_p^-\|_{\mathscr{X}^{\sigma,b-1,T}}\lesssim T^{\theta}. \tag{4-7}$$

Here all the constants will depend on σ, b and p.

Proof. In what follows, if an estimate holds for ω outside a set with measure ϵ, we simply say it holds "with exceptional probability ϵ". We will use various exponents q_j, and each of them will remain the *same* throughout the proof. First we can use Lemma 2.8 to estimate

$$\|u_1^- \cdots u_p^-\|_{\mathscr{X}^{\sigma,b-1,T}} \lesssim T^{2b-1}\|u_1^- \cdots u_p^-\|_{\mathscr{X}^{\sigma,3b-2,T}},$$

since $-\frac{1}{2} < b-1 < 3b-2 < \frac{1}{2}$. Thus we only need to prove

$$\|u_1^- \cdots u_p^-\|_{\mathscr{X}^{\sigma,3b-2,T}} \lesssim T^{1/2-b}, \tag{4-8}$$

with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3})$. Recalling the Littlewood–Paley projections (1-8), we have

$$u = \sum_{N \geq 2} u_N, \tag{4-9}$$

where for simplicity we write $u_N = \Delta_N u$. Thus we only need to estimate the terms (note $(u_N)^- = (u^-)_N$ since the Littlewood–Paley projectors are real)

$$\prod_{j=1}^{p} (u_j)_{N_j}^-,$$

where we have fixed a choice between u_j and $\bar{u}_j$, and between (4-5) and (4-6), for each u_j. Define

$$A = \{1 \leq j \leq p : u_j \text{ given by (4-5)}\}, \quad \text{and} \quad B = \{1 \leq j \leq p : u_j \text{ given by (4-6)}\}.$$

Let

$$\mathscr{A} = \left\{(N_1, \dots, N_p) : N_j > 10^3 \sum_{i \neq j} N_i \quad \text{for some } j \in B\right\}. \tag{4-10}$$

We first consider the sum of terms with $(N_1, \dots, N_p) \in \mathscr{A}$, and rewrite it as

$$\sum_{j \in B} \sum_{(N_i)_{i \neq j}} \prod_{i \neq j} (u_i)_{N_i}^- \cdot \left(\sum_{N_j > 10^3 \sum_{i \neq j} N_i} (u_j)_{N_j}^- \right). \tag{4-11}$$

To bound this expression we only need to consider a fixed $j_0 \in B$, and without loss of generality, we may assume $j_0 = p$. For each $(N_1, \dots, N_{p-1})$ if we write

$$u_p^{hi} = \sum_{N_p > 10^3 \sum_{i=1}^{p-1} N_i} (u_p)_{N_p}, \tag{4-12}$$

then we only need to prove

$$\mathfrak{S} := \|(u_1)_{N_1}^- \cdots (u_{p-1})_{N_{p-1}}^- (u_p^{hi})^-\|_{\mathscr{X}^{\sigma,3b-2,T}} \lesssim T^{1/2-b} (\max_{j<p} N_j)^{-\theta} \tag{4-13}$$

for some $\theta > 0$, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} (\max_{j<p} N_j)^{c_4})$ (note that when we take the sum over all $(N_1, \dots, N_{p-1})$, we still get an expression $\leq c_1 e^{-c_2 T^{-c_3}}$).

To prove (4-13), we use Propositions 2.5 and 2.7 to estimate (for simplicity, we shall omit the spacetime domain $[-T, T] \times \mathbb{R}^2$ in the following estimates, but one should keep in mind that we are working on a very short time

$$
\begin{aligned}
\mathfrak{S} &\lesssim \|(u_1)^-_{N_1}\cdots(u_{p-1})^-_{N_{p-1}}(u_p^{hi})^-\|_{L_t^{q_1}\mathcal{W}_x^{\sigma,q_1}} && \text{(4-14)}\\
&\lesssim \|(u_p^{hi})^-\|_{L_t^4\mathcal{W}_x^{\sigma,4}}\prod_{j=1}^{p-1}\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}}+\sum_{j=1}^{p-1}\|(u_p^{hi})^-\|_{L_{t,x}^4}\|(u_j)^-_{N_j}\|_{L_t^{q_2}\mathcal{W}_x^{\sigma,q_2}}\prod_{j\neq i<p}\|(u_i)^-_{N_i}\|_{L_{t,x}^{q_2}}\\
&\lesssim \|(u_p^{hi})^-\|_{L_t^4\mathcal{W}_x^{\sigma,4}}\prod_{j=1}^{p-1}\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}}+\sum_{j=1}^{p-1}N_j^{\sigma}\|(u_p^{hi})^-\|_{L_{t,x}^4}\prod_{i=1}^{p-1}\|(u_i)^-_{N_i}\|_{L_{t,x}^{q_2}} && \text{(4-15)}\\
&\lesssim \|u_p^-\|_{L_t^4\mathcal{W}_x^{\sigma,4}}\prod_{j=1}^{p-1}\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}} && \text{(4-16)}\\
&\lesssim \prod_{j=2}^{p}\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}}, && \text{(4-17)}
\end{aligned}
$$

where in (4-15) and (4-16) we have used Corollary 2.3 (recall the definition of u_p^{hi}). In (4-17) we have used Proposition 2.7 and the assumption that $p\in B$. For the parameters, we choose $q_1>\frac{4}{3}$ and sufficiently close to $\frac{4}{3}$ depending on p, and $\frac{p-1}{q_2}=\frac{1}{q_1}-\frac{1}{4}$, and check that (4-14) indeed hold, provided b is sufficiently close to $\frac{1}{2}$, depending on q_1 (see Proposition 2.7, with b there replaced by $3b-1$).

Now we proceed to analyze the expression (4-17). Choose $1\le j\le p-1$ so that $N_j=\max_{i<p}N_i$. If $j\in B$, then from Corollary 2.3 and Proposition 2.7 we have

$$
\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}}\lesssim N_j^{-\epsilon}\|u_j\|_{L_t^{q_2}\mathcal{W}_x^{\epsilon,q_2}}\lesssim N_j^{-\epsilon}\|u_j\|_{\mathcal{X}^{\sigma,b,T}}\lesssim N_j^{-\epsilon}, \tag{4-18}
$$

provided $\sigma-\epsilon>1-\frac{4}{q_2}$ (note $q_2>4$ from our choice of exponents above). This can be achieved if ϵ is small enough depending on q_2, and σ is sufficiently close to 1 depending on q_2 and ϵ. If instead $j\in A$, then from Corollary 2.3 we have

$$
\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}}\lesssim N_j^{-\epsilon}\|u_j\|_{L_t^{q_2}\mathcal{W}_x^{\epsilon,q_2}}=N_j^{-\epsilon}\|e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f(\omega)\|_{L_t^{q_2}\mathcal{W}_x^{\epsilon,q_2}}. \tag{4-19}
$$

The norm in the last expression equals the $L_t^{q_2}\mathcal{W}_x^{\epsilon,q_2}$ norm of $e^{-\mathrm{i}t\boldsymbol{H}}f(\omega)$ on the interval $[t_0-T,t_0+T]$. Since $T<1$, we may expand this interval to an interval with length 2π. Since $e^{-\mathrm{i}t\boldsymbol{H}}f(\omega)$ has period 2π in t, we may replace the enlarged norm by the norm on $[-\pi,\pi]$. Then we could use Corollary 3.5 to bound

$$
N_j^{-\epsilon}\|e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f(\omega)\|_{L_t^{q_2}\mathcal{W}_x^{\epsilon,q_2}}\lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)}N_j^{-\frac{\epsilon}{2}} \tag{4-20}
$$

for all t_0, with exceptional probability $\le c_1\exp(-c_2T^{-c_3}N_j^{c_4})$, provided $0<\epsilon<\frac{2}{q_2}$. Therefore in each case we have

$$
\|(u_j)^-_{N_j}\|_{L_{t,x}^{q_2}}\lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)}N_j^{-\theta}, \tag{4-21}
$$

with exceptional probability $\le c_1\exp(-c_2T^{-c_3}N_j^{c_4})$, for some $\theta>0$.

Then we treat the terms with $i \neq j$. If $i \in B$, we can use Proposition 2.7 to bound $\|(u_i)_{N_i}^{-}\|_{L_{t,x}^{q_2}} \lesssim 1$; if $i \in A$, we can use Corollary 3.5 to bound

$$\|(u_i)_{N_i}^{-}\|_{L_{t,x}^{q_2}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_j^{\frac{\theta}{10p}}$$

for all t_0, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_j^{c_4})$. Putting these together, we have shown

$$\text{(4-17)} \lesssim T^{1/2-b} (\max_{j<p} N_j)^{-\theta} \tag{4-22}$$

for some $\theta > 0$, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} (\max_{j<p} N_j)^{c_4})$. This takes care of the sum of terms with $(N_1, \dots, N_p) \in \mathscr{A}$.

For $(N_1, \dots, N_p) \notin \mathscr{A}$, we are going to prove

$$J = \|v_1^{-} \cdots v_p^{-}\|_{\mathscr{X}^{\sigma,3b-2,T}} \lesssim T^{1/2-b} (\max_{j\geq 1} N_j)^{-\theta}, \tag{4-23}$$

where $v_j = (u_j)_{N_j}$, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} (\max_{j\geq 1} N_j)^{c_4})$. This, together with the analysis above, clearly implies (4-7). Now without loss of generality, assume $N_1 = \max_{j\geq 1} N_j$. If $1 \in B$, then we have $N_1 \sim \max_{j\geq 2} N_j$. By switching the role of 1 and p in the argument above and replacing u_1^{hi} by v_1 (note v_1 also satisfy the estimates about u_1^{hi} that we would use), we can prove (4-22) with the role of 1 and p switched. Since $N_1 \sim \max_{j\geq 2} N_j$, this also proves (4-23).

Now we assume that $N_1 = \max_{j\geq 1} N_j$ and $1 \in A$. If $N_1 \lesssim N_{j_0}^{(1+\sigma)/(3\sigma-1)}$ (note this exponent is > 1) for some $j_0 \geq 2$, then we may assume $j_0 = 2$. Now use the same arguments as in (4-14) (but with different exponents), we have

$$\begin{aligned}
J &\lesssim \|v_1^{-} v_2^{-} \cdots v_p^{-}\|_{L_t^{q_1} \mathscr{W}_x^{\sigma,q_1}} \qquad (4\text{-}24)\\
&\lesssim (\|v_1^{-}\|_{L_t^4 \mathscr{W}_x^{\sigma,4}} \|v_2^{-}\|_{L_{t,x}^4} + \|v_1^{-}\|_{L_{t,x}^2} \|v_2^{-}\|_{L_t^4 \mathscr{W}_x^{\sigma,4}}) \prod_{j=3}^{p} \|v_j^{-}\|_{L_{t,x}^{q_4}} \\
&\qquad + \sum_{j=3}^{p} \|v_1^{-}\|_{L_{t,x}^4} \|v_2^{-}\|_{L_{t,x}^4} \|v_i^{-}\|_{L_t^{q_4} \mathscr{W}_x^{\sigma,q_4}} \prod_{3\leq i\neq j} \|v_j^{-}\|_{L_{t,x}^{q_4}} \\
&\lesssim \left(\sum_{j=1}^{p} N_j^{\sigma}\right) \|v_1^{-}\|_{L_{t,x}^4} \|v_2^{-}\|_{L_{t,x}^4} \prod_{j=3}^{p} \|v_j^{-}\|_{L_{t,x}^{q_4}} \qquad (4\text{-}25)\\
&\lesssim N_1^{\frac{1+\sigma}{4}} N_2^{\frac{1+\sigma}{4}} \|v_1^{-}\|_{L_{t,x}^4} \|v_2^{-}\|_{L_{t,x}^4} \prod_{j=3}^{p} \|v_j^{-}\|_{L_{t,x}^{q_4}} \qquad (4\text{-}26)\\
&\lesssim \|v_1^{-}\|_{L_t^4 W_x^{\frac{1+\sigma}{4},4}} \|v_2^{-}\|_{L_t^4 W_x^{\frac{1+\sigma}{4},4}} \prod_{j=3}^{p} \|v_j^{-}\|_{L_{t,x}^{q_4}}, \qquad (4\text{-}27)
\end{aligned}$$

where $\frac{p-2}{q_4} = \frac{1}{q_1} - \frac{1}{2}$ and $q_4 > 4$. Here in (4-25) and (4-27) we have used Corollary 2.3 and the fact that $v_j = (u_j)_{N_j}$, while in (4-26) we have used $N_j \lesssim N_1 \lesssim N_2^{(1+\sigma)/(3\sigma-1)}$ for all j.

Now we analyze the expression (4-27). If $2 \in B$, then by Corollary 2.3 and Proposition 2.7 we have (note $N_1 \lesssim N_2^2$ when $\sigma > \frac{3}{5}$)

$$\|v_2^-\|_{L_t^4 \mathcal{W}_x^{\frac{1+\sigma}{4},4}} \lesssim N_1^{-\frac{1}{24}} \|v_2^-\|_{L_t^6 \mathcal{W}_x^{\frac{2\sigma}{3},4}} \lesssim N_1^{-\frac{1}{24}} \|u_2\|_{\mathscr{X}^{\sigma,b,T}} \lesssim N_1^{-\frac{1}{24}}, \tag{4-28}$$

provided $\frac{2\sigma}{3} > \frac{1+\sigma}{4} + \frac{1}{12}$ and $\sigma > \frac{2\sigma}{3} + \frac{1}{6}$, which is true for $\sigma > \frac{4}{5}$. If $2 \in A$ (which is the case for 1), we can use the arguments from (4-19) to (4-20) to get

$$\|v_2^-\|_{L_t^4 \mathcal{W}_x^{\frac{1+\sigma}{4},4}} \lesssim N_1^{-\frac{1-\sigma}{16}} \|u_2^-\|_{L_t^4 \mathcal{W}_x^{\frac{3+\sigma}{8},4}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_1^{-\frac{1-\sigma}{32}} \tag{4-29}$$

for all t_0, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$, thanks to Corollary 3.5, and the hypothesis $\sigma < 1$ (hence $\frac{3+\sigma}{8} < \frac{1}{2}$).

Then we treat the terms with $j \geq 3$. If $j \in B$, we can use Proposition 2.7 to bound $\|v_j^-\|_{L_{t,x}^{q_4}} \lesssim 1$; if $j \in A$, we can use Corollary 3.5 to bound

$$\|v_j^-\|_{L_{t,x}^{q_4}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_1^{\frac{1-\sigma}{100p}}$$

for all t_0, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$. Putting these together, we have proved

$$\text{(4-27)} \lesssim T^{1/2-b} N_1^{-\theta}, \tag{4-30}$$

with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$ for some $\theta > 0$. Thus we have proved (4-23) in this case.

In the final case, we assume that $N_1 > (10p)^3 (\max_{j \geq 2} N_j)^{\frac{1+\sigma}{3\sigma-1}}$, which in particular implies $N_1 > 10^3 \sum_{j \geq 1} N_j$, and that $1 \in A$. For each $j \in B$, by definition we can extend u_j to be a function on $\mathbb{R} \times \mathbb{R}^2$ (still denoted by u_j) with $\mathscr{X}^{\sigma,b}$ norm $\lesssim 1$. The relation $v_j = (u_j)_{N_j}$ also extends to $t \in \mathbb{R}$, giving an extension of v_j also. Choose ζ_0 smooth, supported on $[-2, 2]$ and equaling 1 on $[-1, 1]$ and define $\zeta(t) = \zeta_0(T^{-1}t)$. We are to prove

$$\|\zeta \cdot v_1^- \cdots v_p^-\|_{\mathscr{X}^{\sigma,3b-2}} \lesssim T^{1/2-b} N_1^{-\theta} \tag{4-31}$$

for the extended v_j, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$. For a function w on $\mathbb{R} \times \mathbb{R}^2$ radial in x, we split $w = w_{ne} + w_{fa}$, with

$$\mathscr{F}_t \langle w_{ne}, e_k \rangle(\tau) = \chi_{\{|\tau+4k+2| \leq N_1^{\gamma}\}} \cdot \mathscr{F}_t \langle w, e_k \rangle(\tau), \tag{4-32}$$

and w_{fa} by replacing the $\leq$ by $>$. We now split the product in (4-31) into *fa* and *ne* parts and estimate them separately.

We first estimate the *fa* part of product as (due to the presence of ζ, we can work on time interval $[-2T, 2T]$ in the time-Lebesgue norms below, thus gaining powers in T)

$$\|(\zeta \cdot v_1^- \cdots v_p^-)_{fa}\|_{\mathscr{X}^{\sigma,3b-2}} \lesssim N_1^{-\gamma/36} \|\zeta \cdot v_1^- \cdots v_p^-\|_{\mathscr{X}^{\sigma,-4/9}} \tag{4-33}$$

$$\lesssim N_1^{-\gamma/36} \|\zeta \cdot v_1^- \cdots v_p^-\|_{L_t^{3/2}\mathscr{W}_x^{\sigma,3/2}} \tag{4-34}$$

$$\lesssim N_1^{\sigma-\gamma/36} \prod_i \|v_i^-\|_{L_{t,x}^{3p/2}}. \tag{4-35}$$

Here in (4-33) we have used the definition of the *fa*-projection and that b is close to $\frac{1}{2}$ (in particular, $b < \frac{1}{2} + \frac{1}{108}$); in (4-34) we have used Proposition 2.7; in (4-35) we have combined Corollary 2.3 and Proposition 2.5. Now for each i, if $i \in B$ then (provided σ is close to 1 depending on p)

$$\|v_i^-\|_{L_{t,x}^{\frac{3p}{2}}} \lesssim \|v_i\|_{\mathscr{X}^{\sigma,b}} \lesssim 1.$$

If $i \in A$ (such as $i = 1$) we have

$$\|v_i^-\|_{L_{t,x}^{\frac{3p}{2}}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_1^{\frac{1}{p}}$$

for all t_0, with exceptional probability $\le c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$. Therefore, we have (4-35) $\lesssim T^{1/2-b} N_1^{-\theta}$ with exceptional probability $\le c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$, provided $\gamma > 108$.

Now we estimate the *ne* part of the product. Choose v_0 so that $\|v_0\|_{\mathscr{X}^{0,2-3b}} \lesssim 1$. Since we are taking the *ne* part, we may assume $v_0 = v_{0,ne}$. The aim is to estimate $|\mathfrak{J}|$ (recall H is self-adjoint), where

$$\mathfrak{J} = \int_{\mathbb{R}\times\mathbb{R}^2} v_1^- \cdots v_p^- \cdot (\zeta \boldsymbol{H}^{\sigma/2} \bar{v}_0). \tag{4-36}$$

We use Lemma 2.6 to write down

$$v_j(x,t) = \int_{\mathbb{R}} \phi_j(\lambda_j) e^{\mathrm{i}\lambda_j t} \sum_k a_{\lambda_j}^j(k) e^{-\mathrm{i}(4k+2)t} e_k(x)\, \mathrm{d}\lambda_j \tag{4-37}$$

for $j \in B \cup \{0\}$, where the parameters satisfy

$$\sum_k |a_{\lambda_0}^0(k)|^2 \lesssim 1 \tag{4-38}$$

for each λ_0. Since $v_0 = v_{0,ne}$, we also have $\|\phi_0\|_{L^1} \lesssim N_1^{3\gamma(b-\frac{1}{2})}$. For $j \in B$, since $v_j = (u_j)_{N_j}$, we know $a_{\lambda_j}^j(n_j) = 0$ unless $\frac{1}{10} \le (4n_j+2)/N_j^2 \le 10$, and hence

$$\sum_{4n_j+2\sim N_j^2} |a_{\lambda_j}^j(k)|^2 \lesssim N_j^{-2\sigma}. \tag{4-39}$$

Also since $b > \frac{1}{2}$, we have $\|\phi_j\|_{L^1} \lesssim 1$.

For the sake of convenience, in the following proof, we shall use $v^{\sim}(n,\tau)$ to denote $\mathscr{F}_t\langle v, e_n\rangle(\tau)$. Thus from (4-37) we have

$$v_j^{\sim}(n_j,\tau_j)=(2\pi)^{1/2}a^j_{\tau_j+4n_j+2}(n_j)\phi_j(\tau_j+4n_j+2) \tag{4-40}$$

for $j\in B$. If $j\in A$ we have

$$v_j^{\sim}(n_j,\tau_j)=(2\pi)^{1/2}e^{-\mathrm{i}(4n_j+2)t_0}\frac{\theta_j(n_j)g_{n_j}(\omega)}{\sqrt{4n_j+2}}\delta(\tau_j+4n_j+2), \tag{4-41}$$

where

$$\theta_j(n_j)=\eta\Big(\frac{2(4n_j+2)}{N_j^2}\Big)-\eta\Big(\frac{4(4n_j+2)}{N_j^2}\Big).$$

Clearly $|\theta_j|\leq 2$, and $\theta_j\neq 0$ only when $\frac{1}{10}\leq(4n_j+2)/N_j^2\leq 10$ (note we have fixed N_j). Finally, for $j=0$ we have (we may assume ζ is real)

$$(\zeta\boldsymbol{H}^{\sigma/2}v_0)^{\sim}(n_0,\tau_0)=(4n_0+2)^{\sigma/2}\cdot\int_{\mathbb{R}}a^0_{\varrho_0+4n_0+2}(n_0)\phi_0(\varrho_0+4n_0+2)\hat{\zeta}(\tau_0-\varrho_0)\,\mathrm{d}\varrho_0. \tag{4-42}$$

We write $\gamma_j=v_j^{\sim}$ for $j\geq 1$, and $\gamma_0=(\zeta\boldsymbol{H}^{\sigma/2}v_0)^{\sim}$. From the rules of Fourier transform and orthogonality of e_k, we have

$$\mathfrak{J}=(2\pi)^{-(p-2)/2}\sum_{n_1,\dots,n_p,n_0}\kappa^{n_0}_{n_1,\dots,n_p}\int_{\mathbb{D}}\prod_{j=0}^{p}(\gamma_j(n_j,\tau_j))^{-}\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_p, \tag{4-43}$$

where

$$\kappa^{n_0}_{n_1,\dots,n_p}=\int_{\mathbb{R}^2}e_{n_1}(x)\cdots e_{n_p}(x)e_{n_0}(x)\,\mathrm{d}x, \tag{4-44}$$

and

$$\mathbb{D}=\Big\{(\tau_1,\dots,\tau_p,\tau_0):\tau_0=\sum_{j=1}^{p}\epsilon_j\tau_j\Big\}, \tag{4-45}$$

with $\epsilon_j=\pm 1$ depending on the choice of v_j or $\bar{v}_j$. We notice that $\epsilon_j=1$ if and only if the corresponding γ_j^{-} equals γ_j. Now plug in (4-40), (4-41), and (4-42), and use the change of variables $\lambda_j=\tau_j+4n_j+2$ for $j\in B$, $\lambda_0=\varrho_0+4n_0+2$; we get

$$\begin{aligned}\mathfrak{J}=2\pi\sum_{n_1,\dots,n_p,n_0}\kappa^{n_0}_{n_1,\dots,n_p}\int\prod_{j\in B\cup\{0\}}\mathrm{d}\lambda_j\prod_{j\in B}\phi_j(\lambda_j)a^j_{\lambda_j}(n_j)^{-}\prod_{j\in A}\frac{\theta_j(n_j)g^{-}_{n_j}(\omega)}{\sqrt{4n_j+2}}\cdot a^0_{\lambda_0}(n_0)^{-}\phi_0(\lambda_0)\\ \times\hat{\zeta}\Big(\sum_{j\in B}\epsilon_j\lambda_j-\lambda_0-\sum_{j=1}^{p}\epsilon_j(4n_j+2)+(4n_0+2)\Big)^{-}\\ \times(4n_0+2)^{\sigma/2}\exp\Big(-\mathrm{i}t_0\sum_{j\in A}(4n_j+2)\epsilon_j\Big).\end{aligned} \tag{4-46}$$

Here the terms corresponding to $j \in A$ are delta functions and have already been incorporated in the final expression. Letting $\varrho = (4n_0+2) - \sum_{j=1}^{p} \epsilon_j(4n_j+2)$, we can further reduce the expression to

$$\begin{aligned}\mathfrak{J} = (2\pi)^{p+\frac{1}{2}} \sum_{\varrho\in\mathbb{Z}} \int \prod_{j\in B\cup\{0\}} \phi_j(\lambda_j)\,\mathrm{d}\lambda_j \cdot \hat{\zeta}\Big(\sum_{j\in B} \epsilon_j\lambda_j - \lambda_0 + \varrho\Big)^{-} \\ \times \sum_{\mathbb{S}_\varrho} \kappa^{n_0}_{n_1,\dots,n_p} (4n_0+2)^{\sigma/2} \prod_{j\in B\cup\{0\}} a^j_{\lambda_j}(n_j)^{-} \prod_{j\in A} \frac{\theta_j(n_j) g^{-}_{n_j}(\omega)}{\sqrt{4n_j+2}} \\ \times \exp\Big(-\mathrm{i}t_0 \sum_{j\in A}(4n_j+2)\epsilon_j\Big),\end{aligned}$$

$$\mathbb{S}_\varrho = \Big\{(n_0,\dots,n_p) : \frac{1}{10} \le \frac{4n_j+2}{N_j^2} \le 10 \ (j\ge 1),\ (4n_0+2) - \sum_{j=1}^{p} \epsilon_j(4n_j+2) = \varrho\Big\}. \tag{4-47}$$

Noticing that $\hat{\zeta} = T\hat{\zeta}_0(T\,\cdot)$, and that $\hat{\zeta}_0$ is a Schwartz function, we have

$$\sum_{\varrho\in\mathbb{Z}} |\hat{\zeta}(\lambda+\varrho)| \lesssim \sum_{\varrho\in\mathbb{Z}} T\langle T(\lambda+\varrho)\rangle^{-2} \lesssim 1 \tag{4-48}$$

for all $\lambda \in [0, 1]$, and by periodicity, for all $\lambda \in \mathbb{R}$. Therefore

$$\sum_{\varrho\in\mathbb{Z}} \int \prod_{j\in B\cup\{0\}} |\phi_j(\lambda_j)|\,\mathrm{d}\lambda_j \cdot \hat{\zeta}\Big|\Big(\sum_{j\in B} \epsilon_j\lambda_j - \lambda_0 + \varrho\Big)\Big| \lesssim N_1^{3\gamma(b-\frac{1}{2})}. \tag{4-49}$$

Since we choose b close enough to $\frac{1}{2}$ depending on σ and p, and γ does not have any dependence on b whatsoever (we may simply take $\gamma = 200$), (4-31) will follow if

$$\begin{aligned}\Big|\sum_{\mathbb{S}_\varrho} \kappa^{n_0}_{n_1,\dots,n_p} (4n_0+2)^{\sigma/2} \times \prod_{j\in B\cup\{0\}} a^j_{\lambda_j}(n_j)^{-} \prod_{j\in A} \frac{\theta_j(n_j) g^{-}_{n_j}(\omega)}{\sqrt{4n_j+2}} \times \exp\Big(-\mathrm{i}t_0 \sum_{j\in A}(4n_j+2)\epsilon_j\Big)\Big| \\ \lesssim T^{1/2-b} N_1^{-\delta}\end{aligned} \tag{4-50}$$

for *all* possible choices of $t_0 \in \mathbb{R}$, $\varrho \in \mathbb{Z}$, $\lambda_j \in \mathbb{R} (j \in B\cup\{0\})$, $\{a^j_{\lambda_j}(k)\}$ satisfying (4-38) and (4-39), with $\delta > 0$ depending on σ and p, but *not* on b.

Next, by Cauchy–Schwartz in the sum with respect to n_0, we can further estimate the left side of (4-50) by

$$\begin{aligned}\Big(\sum_{n_0} (4n_0+2)^{\sigma} \times \Big|\sum_{\mathbb{S}_{\varrho,n_0}} \kappa^{n_0}_{n_1,\dots,n_p} \prod_{j\in B} b_j(n_j)^{-} \prod_{j\in A} \frac{\theta_j(n_j) g^{-}_{n_j}(\omega)}{\sqrt{4n_j+2}} \\ \times \exp\Big(-\mathrm{i}t_0 \sum_{j\in A}(4n_j+2)\epsilon_j\Big)\Big|^2\Big)^{1/2},\end{aligned} \tag{4-51}$$

where $\mathbb{S}_{\varrho,n_0} = \{(n_1,\dots,n_p) : (n_0,\dots,n_p) \in \mathbb{S}_\varrho\}$, and $b_j(k) = a^j_{\lambda_j}(k)$.

Concerning the inner sum of (4-51), we have (recall that $\frac{1}{10} \le (4n_j+2)/N_j^2 \le 10$ for each $1 \le j \le p$)

$$\left|\sum_{\mathbb{S}_{\varrho,n_0}} \kappa^{n_0}_{n_1,\dots,n_p} \prod_{j\in B} b_j(n_j)^- \prod_{j\in A} \frac{\theta_j(n_j) g^-_{n_j}(\omega)}{\sqrt{4n_j+2}} \cdot e^{-\mathrm{i}t_0 \sum_{j\in A}(4n_j+2)\epsilon_j}\right| \tag{4-52}$$

$$\lesssim \sum_{(n_j)_{j\in B}} \left|\sum_{\Theta} \tau^{n_0}_{n_1,\dots,n_p} \prod_{j\in A} g^-_{n_j}(\omega)\right| \prod_{j\in B} |b_j(n_j)| \tag{4-53}$$

$$\lesssim \sup_{(n_j)_{j\in B}} \left|\sum_{\Theta} \tau^{n_0}_{n_1,\dots,n_p} \prod_{j\in A} g^-_{n_j}(\omega)\right| \prod_{j\in B} \sum_{4n_j+2\sim N_j^2} |b_j(n_j)|$$

$$\lesssim \sup_{(n_j)_{j\in B}} \left|\sum_{\Theta} \tau^{n_0}_{n_1,\dots,n_p} \prod_{j\in A} g^-_{n_j}(\omega)\right| \prod_{j\in B} (N_j^2 N_j^{-2\sigma})^{1/2} \tag{4-54}$$

$$\lesssim \sup_{(n_j)_{j\in B}} \left|\sum_{\Theta} \tau^{n_0}_{n_1,\dots,n_p} \prod_{j\in A} g^-_{n_j}(\omega)\right| \prod_{j\in B} N_j^{1-\sigma},$$

where in (4-53) we write $\Theta = \{(n_j)_{j\in A} : (n_1,\dots,n_p) \in \mathbb{S}_{\varrho,n_0}\}$ for fixed $(n_j)_{j\in B}$, and $\tau^{n_0}_{n_1,\dots,n_p} = \kappa^{n_0}_{n_1,\dots,n_p} \prod_{j\in A} \theta_j(n_j)(4n_j+2)^{-1/2}$. One should notice that for all $(n_j)_{j\in A} \in \Theta$, by definition the expression $\exp(-\mathrm{i}t_0 \sum_{j\in A}(4n_j+2)\epsilon_j)$ is a fixed constant with absolute value 1, which can be extracted. In (4-54) we have used Cauchy–Schwartz and (4-39).

Let us fix ϱ and n_0, and $(n_j)_{j\in B}$. We also assume $|4n_0+2-\varrho| \lesssim N_1^2$ (otherwise $\mathbb{S}_{\varrho,n_0}$ would be empty). Since the set Θ has the form of Ξ in (4-2) and $N_1 > 10^3 \sum_{j\in A-\{1\}} N_j$, we can use Proposition 4.2 to get

$$\left|\sum_{\Theta} \tau^{n_0}_{n_1,\dots,n_p} \prod_{j\in A} g^-_{n_j}(\omega)\right| \le K \prod_{j\in A-\{1\}} N_j \cdot \sup_{\Theta} |\tau^{n_0}_{n_1,\dots,n_p}|, \tag{4-55}$$

with exceptional probability $\le c_1 \exp(-c_2 K^{c_3})$. We choose $K = T^{1/2-b} N_1^{(1-\sigma)/200} (4n_0+2)^{(1-\sigma)/400}$, then the corresponding exceptional probability is $\le c_1 \exp(-c_2 T^{-c_3} N_1^{c_4} (4n_0+2)^{c_5})$. If we add up these probabilities with respect to *all* possible choices of ϱ and $(n_j)_{j\in B\cup\{0\}}$, we still get an expression $\le c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$ (there are $\lesssim N_1^2$ choices for each $n_j (j \in B)$, and for fixed n_0, there are $\lesssim N_1^2$ choices of ϱ). Therefore with exceptional probability $\le c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$, we have

$$\text{(4-51)} \lesssim T^{1/2-b} N_1^{\frac{1-\sigma}{200}} \prod_{j\in B} N_j^{1-\sigma} \prod_{j\in A-\{1\}} N_j \left(\sum_{n_0} (4n_0+2)^{\sigma+\frac{1-\sigma}{200}} \sup_{\mathbb{S}_{\mu,n_0}} |\tau^{n_0}_{n_1,\dots,n_p}|^2\right)^{1/2} \tag{4-56}$$

$$\lesssim T^{1/2-b} N_1^{\frac{1-\sigma}{200}-1} \prod_{j\in B} N_j^{1-\sigma} \left(\sum_{n_0} (4n_0+2)^{\sigma+\frac{1-\sigma}{200}} \sup_{\mathbb{S}_{\mu,n_0}} |\kappa^{n_0}_{n_1,\dots,n_p}|^2\right)^{1/2}. \tag{4-57}$$

To complete the proof of Proposition 4.3, we are going to estimate $\kappa^{n_0}_{n_1,\dots,n_p}$. Let $\nu_{(0)} \ge \cdots \ge \nu_{(p)}$ be the nonincreasing permutation of $\nu_j = 4n_j+2$ (where $0 \le j \le p$). If $\nu_0 \ge N_1^{2(1+(1-\sigma)/200)}$, from Lemma 3.2 we have $|\kappa^{n_0}_{n_1,\dots,n_p}| \lesssim \nu_0^{-100}$. If $\nu_0 < N_1^{2(1+(1-\sigma)/200)}$, since $N_1 \gtrsim (\max_{j\ge2} N_j)^{(\sigma+1)/(3\sigma-1)}$, we see that if $\nu_0 \le \max_{j\ge2} \nu_j$, then $\nu_1 \gtrsim \max_{j\ne1} \nu_j^{(\sigma+1)/(3\sigma-1)}$ and $|\kappa^{n_0}_{n_1,\dots,n_p}| \lesssim N_1^{-100}$; if $\nu_0 > \max_{j\ge2} \nu_j$, then $\nu_{(2)} \geq \max_{j\geq 2} \nu_j$ and from Lemma 3.2 we have

$$|\kappa^{n_0}_{n_1,\dots,n_p}| \lesssim \nu_{(0)}^{-\frac{1}{2}} \nu_{(2)}^{-\frac{1}{4}} \log \nu_{(0)} \lesssim N_1^{-1} (\max_{j\geq 2} N_j)^{-\frac{1}{2}} \log N_1. \tag{4-58}$$

Therefore we have

$$\begin{aligned}
\text{(4-57)} &\lesssim T^{1/2-b} N_1^{\frac{1-\sigma}{200}-1} \prod_{j\in B} N_j^{1-\sigma} \\
&\quad \times \Bigg(\sum_{\nu_0 < N_1^{2(1+\frac{1-\sigma}{200})}} (N_1)^{2(\sigma+\frac{1-\sigma}{200})(1+\frac{1-\sigma}{200})} N_1^{-2} (\max_{j\geq 2} N_j)^{-1} \log^2 N_1 + \sum_{\nu_0 \geq N_1^{2(1+\frac{1-\sigma}{200})}} (4n_0+2)^{-198} \Bigg)^{1/2} \\
&\lesssim T^{1/2-b} N_1^{-\theta_0} \log N_1 \cdot (\max_{j\geq 2} N_j)^{-\frac{1}{2}} \prod_{j\in B} N_j^{1-\sigma} \\
&\lesssim T^{1/2-b} N_1^{-\frac{\theta_0}{2}} (\max_{j\geq 2} N_j)^{-\frac{1}{2}} \prod_{j\in B} N_j^{1-\sigma},
\end{aligned}$$

where

$$\theta_0 = 1 - \frac{1-\sigma}{100} - \sigma - \frac{(1+\sigma)(1-\sigma)}{200} - \frac{(1-\sigma)^2}{40000} > \frac{1-\sigma}{2} > 0. \tag{4-59}$$

Finally, since $1 \in A$, we have

$$(\max_{j\geq 2} N_j)^{-\frac{1}{2}} \prod_{j\in B} N_j^{1-\sigma} \lesssim (\max_{j\geq 2} N_j)^{-\frac{1}{2}+(p-1)(1-\sigma)} \lesssim 1, \tag{4-60}$$

provided $\sigma > 1 - 1/(2(p-1))$.

Having considered all the different cases, we have now finished the proof of Proposition 4.3. □

From now on we will fix σ and b as stated in Proposition 4.3. We have an easy corollary:

Corollary 4.4. *There exist some $\theta > 0$ and $T_0 > 0$, such that the following holds: For all $0 < T < T_0$, there exists a set $\Omega_T \subset \Omega$ such that $\mathbb{P}(\Omega_T) \leq c_1 e^{-c_2 T^{-c_3}}$ and for all $\omega \notin \Omega_T$, the mapping*

$$u \mapsto e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) \mp \mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} (|u(s)|^{p-1} u(s))\, \mathrm{d}s \tag{4-61}$$

is a contraction mapping from the affine ball

$$e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \{v : \|v\|_{\mathscr{X}^{\sigma,b,T}} \leq T^\theta\} \tag{4-62}$$

to itself.

Proof. Suppose $u = e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + v$, where $\|v\|_{\mathscr{X}^{\sigma,b,T}} \leq T^\theta \leq 1$. From Proposition 2.9 we have

$$\begin{aligned}
\mathfrak{M} &:= \Big\| \mp \mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} (|u(s)|^{p-1} u(s))\, \mathrm{d}s \Big\|_{\mathscr{X}^{\sigma,b,T}} \\
&\lesssim \big\| |u|^{p-1} u \big\|_{\mathscr{X}^{\sigma,b-1,T}} \\
&= \big\| (e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + v)^{\frac{p+1}{2}} \cdot (\overline{e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)} + \bar{v})^{\frac{p-1}{2}} \big\|_{\mathscr{X}^{\sigma,b-1,T}}.
\end{aligned}$$

If we expand the product, then each term has the form as in Proposition 4.3 (namely, $u_1^- \cdots u_p^-$ with each u_j either equal to $e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$ or has $\mathscr{X}^{\sigma,b,T}$ norm $\lesssim 1$); thus we have $\mathfrak{M} \lesssim T^{\theta_0}$ for some θ_0 depending only on σ, b and p; thus if we choose $\theta < \theta_0$ and T_0 small enough, then the mapping does map the affine ball to itself.

In addition, if $u_i = e^{-\mathrm{i}t\boldsymbol{H}} f + v_i$ with $\|v_i\|_{\mathscr{X}^{\sigma,b,T}} \le T^\theta$ for $i \in \{1,2\}$, then

$$\begin{aligned}\mathfrak{D} &:= \left\| \mp\mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} (|u_1(s)|^{p-1}u_1(s) - |u_2(x)|^{p-1}u_2(s))\,\mathrm{d}s \right\|_{\mathscr{X}^{\sigma,b,T}} \\ &\lesssim \big\| |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 \big\|_{\mathscr{X}^{\sigma,b-1,T}} \\ &\lesssim \sum_{\mathbb{F}} \|(u_1-u_2)^- \prod_{k=1}^{p-1} u_{j_k}^- \|_{\mathscr{X}^{\sigma,b-1,T}},\end{aligned}$$

where $\mathbb{F}$ is some finite set, and each $j_k \in \{1,2\}$. Since $u_1 - u_2 = v_1 - v_2 \in \mathscr{X}^{\sigma,b,T}$, and each u_j is the sum of two terms, one being $e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$, the other having $\mathscr{X}^{\sigma,b,T}$ norm $\lesssim 1$, we can use Proposition 4.3 to estimate $\mathfrak{D} \lesssim T^{\theta_0} \|v_1 - v_2\|_{\mathscr{X}^{\sigma,b,T}}$ for all $\omega \notin \Omega_T$. Thus the result follows if we choose T small enough. ☐

5. Local well-posedness results

In proving local in time results, we will not care about the $\pm$ sign in (1-10). First we define the truncated Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t u + (\Delta - |x|^2)u = (\pm|u|^{p-1}u)^\circ_{2^k} \\ u(0) = f^\circ_{2^k}(\omega) \end{cases} \tag{5-1}$$

for each $k \ge 1$. When $k = \infty$, we understand that $v^\circ_{2^\infty} = v$, so this is just the original equation (1-10). If $k < \infty$, we solve (5-1) in the finite-dimensional space V_{2^k}. We will consider two cases depending on whether $p \ge 3$ odd or $1 < p < 3$.

5.1. *The algebraic case.* Here we assume $p \ge 3$ is an odd integer, so we can use the estimates is Section 4.

Proposition 5.1. *Suppose $T > 0$ is sufficiently small. There exists a set Ω_T (possibly different from the one in Proposition 4.3), such that $\mathbb{P}(\Omega_T) \le c_1 \exp(-c_2 T^{-c_3})$, and when $\omega \notin \Omega_T$, for each $1 \le k \le \infty$, (5-1) has a unique solution*

$$u \in e^{-\mathrm{i}t\boldsymbol{H}} f^\circ_{2^k}(\omega) + \mathscr{X}^{\sigma,b,T} \tag{5-2}$$

on $[-T,T]$, satisfying

$$\|u - e^{-\mathrm{i}t\boldsymbol{H}} f^\circ_{2^k}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} \le T^\theta. \tag{5-3}$$

Proof. When $k = \infty$, the existence and uniqueness directly follows from 4.4 via Picard iteration. Now we assume $1 \le k < \infty$, then the equation (5-1) is just an ODE, so the solution is unique, and exists until its norm approaches infinity. Thus we only need to obtain the control on each of these solutions, uniformly in k. To this end we need the following modification of Proposition 4.3.

Lemma 5.2. *For each T sufficiently small, we can find a set (still denoted by Ω_T), such that $\mathbb{P}(\Omega_T) \le c_1 \exp(-c_2 T^{-c_3})$, and in Proposition 4.3, if one replaces some u_j by any $(u_j)^{\circ}_{2^{k_j}}$ or $(u_j)^{\perp}_{2^{k_j}}$, the result still holds true. Moreover, if there is at least one $(u_j)^{\perp}_{k_j}$, then the left side of (4-7) tends to zero (uniformly in all choices of u_j) as this $k_j \to \infty$.*

Proof. We use the notations as in Proposition 4.3. Noting that the projections $u^{\circ}_{2^k}$ and $u^{\perp}_{2^k}$ are uniformly bounded on $X^{\sigma,b,T}$, we may assume the modification is only for $j \in A$. Since $f^{\circ}_{2^k}(\omega) = f(\omega) - f^{\perp}_{2^k}(\omega)$ and the result is true when all terms are still u_j, we may assume each term is either u_j or $(u_j)^{\perp}_{2^{k_j}}$, with at least one $(u_j)^{\perp}_{2^{k_j}}$.

For each (k_j), we follow exactly the proof of Proposition 4.3. Suppose $L = \max_j 2^{k_j}$; then in the dyadic decomposition we only need to consider the terms $\max_{j\in A} N_j \gtrsim L$ (for example, if $(N_1, \dots, N_p) \in \mathscr{A}$ with the largest being N_1, then $\max_{j\ge 2} N_j \gtrsim L$; otherwise we have $\max_j N_j \gtrsim L$). On the other hand, all the probabilistic Lebesgue/Sobolev estimates of $f(\omega)$ we used in Proposition 4.3 come from Corollary 3.5; thus they also hold for $f^{\perp}_{2^k}(\omega) = f(\omega) - f^{\circ}_{2^k}(\omega)$ uniformly in k. As for the multilinear estimates of Gaussians (Proposition 4.2), they indeed hold for fixed k_j, because fixing k_j (and replacing $f(\omega)$ by $f(\omega)^{\circ}_{2^{k_j}}$) corresponds to adding constraints $n_j \le 2^{k_j}$ in the set Ξ in (4-2), which does not affect the estimates in (4-4) (which is based on upper bounds of the cardinals of some sets). Therefore for fixed k_j, the estimates about each individual term (including the "grouped" terms in $\mathscr{A}$) in the proof of Proposition 4.3 still hold, with constants *independent* of k_j. Therefore, we have

$$\|\mathrm{Modified}(u_1^- \cdots u_p^-)\|_{X^{\sigma,b,T}} \lesssim \sum_{\max_j N_j \gtrsim L} T^{\theta} (\max_j N_j)^{-\theta} \lesssim T^{\theta} L^{-\theta/2},$$

with exceptional probability not exceeding

$$\sum_{\max_j N_j \gtrsim L} c_1 \exp(-c_2 T^{-c_3} (\max_j N_j)^{c_4}) \le c_5 \exp(-c_6 T^{-c_7} L^{c_8}),$$

which implies

$$\|\mathrm{Modified}(u_1^- \cdots u_p^-)\|_{X^{\sigma,b,T}} \lesssim T^{\theta} (\max_j 2^{k_j})^{-\theta/2},$$

for *all* possible choices of k_j, with exceptional probability not exceeding

$$\sum_{(k_j)} c_5 \exp(-c_6 T^{-c_7} (\max_j 2^{k_j})^{c_8}) \lesssim c_9 \exp(-c_{10} T^{-c_{11}}).$$

If we choose this final exceptional set as our Ω_T, we easily see that all requirements are satisfied. □

Remark 5.3. In Proposition 4.3 and Lemma 5.2, the estimates still hold when the $\mathscr{X}^{\sigma,b,T}$ norm is replaced by $\mathscr{X}^{\sigma,b,I}$ (but with T^{θ} on the right side of (4-7) unchanged), for *any* interval $I \subset [-T, T]$, and ω outside a *single* Ω_T. One can check the proof that all estimates do not become worse with $[-T, T]$ replaced by I. In particular we can get a contraction mapping as in Corollary 4.4 for interval $[-T, 0]$ or $[-T_1, T_1]$ for $T_1 \le T$.

Using Lemma 5.2, we can now proceed with the proof of Proposition 5.1. Suppose for some k that $u = e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + v$ is a maximal solution to (5-1) (strictly speaking the T below should be another T' denoting the lifespan of u, but we will ignore this, in view of Remark 5.3). Then outside the Ω_T constructed in Lemma 5.2 we have

$$\begin{aligned}\|v\|_{\mathscr{X}^{\sigma,b,T}} &= \left\| \mp \mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} \big(|u(s)|^{p-1}u(s)\big)_{2^k}^{\circ}\,\mathrm{d}s \right\|_{\mathscr{X}^{\sigma,b,T}} \\ &\lesssim \big\| |u|^{p-1}u \big\|_{\mathscr{X}^{\sigma,b-1,T}} \\ &= \big\| (e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + v)^{\frac{p+1}{2}} \cdot (\overline{e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega)} + \bar{v})^{\frac{p-1}{2}} \big\|_{\mathscr{X}^{\sigma,b-1,T}}.\end{aligned}$$

Each term in the expansion of the final product has the form as in Lemma 5.2 (namely $\prod_j (u_j^{\pm})_{2^{k_j}}^{\circ}$ with $1 \le k_j \le \infty$, and each u_j either equal to $e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$ or has $\mathscr{X}^{\sigma,b,T}$ norm $\lesssim \|v\|_{\mathscr{X}^{\sigma,b,T}}$). Therefore for some $\theta > 0$ we get

$$\|v\|_{\mathscr{X}^{\sigma,b,T}} \lesssim T^{\theta}(1 + \|v\|_{\mathscr{X}^{\sigma,b,T}})^p;$$

since $v \in \mathscr{X}^{\sigma,b,T}$ and $v(0) = 0$, we know $\|v\|_{\mathscr{X}^{\sigma,b,t}} \to 0$ as $t \to 0$. The local norm is continuous in t; thus we can use a bootstrap argument to get $\|v\|_{\mathscr{X}^{\sigma,b,T}} \le T^{\theta/2}$. Note this also works for the original equation, showing that (5-3) holds for the solution of (1-10) with any k. The uniqueness of (1-10) now follows from Corollary 4.4. □

5.2. *The subcubic case.* Here we assume $1 < p < 3$, and we do not need any multilinear estimate to solve the local problem.

Proposition 5.4. *Suppose $T > 0$ is sufficiently small. There exists a set Ω_T (possibly different from the one in Proposition 4.3), such that $\mathbb{P}(\Omega_T) \le c_1 \exp(-c_2 T^{-c_3})$, and when $\omega \notin \Omega_T$, for each $1 \le k \le \infty$, (5-1) has a unique solution*

$$u \in e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + \mathscr{X}^{0,b,T} \tag{5-4}$$

on $[-T, T]$, satisfying

$$\|u - e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega)\|_{\mathscr{X}^{0,b,T}} \le T^{\theta}. \tag{5-5}$$

Proof. The proof here is almost the same as Proposition 5.1. In fact, once we can obtain

$$\big\| |e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + v|^{p-1}(e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + v) \big\|_{\mathscr{X}^{0,b-1,T}} \lesssim T^{\theta}(1 + \|v\|_{\mathscr{X}^{0,b,T}}^p) \tag{5-6}$$

and

$$\big\| |u|^{p-1}u - |u'|^{p-1}u' \big\|_{\mathscr{X}^{0,b-1,T}} \lesssim T^{\theta} \|v - v'\|_{\mathscr{X}^{0,b,T}} \cdot (1 + \|v\|_{\mathscr{X}^{0,b,T}} + \|v'\|_{\mathscr{X}^{0,b,T}})^{p-1} \tag{5-7}$$

for all $1 \le k \le \infty$ and $\omega \notin \Omega_T$, where $u = e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + v$ and $u' = e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega) + v'$, we can use Proposition 2.9 and argue as in the proof of Corollary 4.4 to show that for $\omega \notin \Omega_T$,

$$u \mapsto e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) \mp \mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} (|u(s)|^{p-1}u(s))\,\mathrm{d}s$$

is a contraction mapping from

$$e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \{v : \|v\|_{\mathscr{X}^{0,b,T}} \le T^{\theta}\}$$

to itself, for some $\theta > 0$. Also we will have the same estimates on solutions to (5-1) as in Proposition 5.1, which is enough for the proof.

To prove (5-7), we simply compute (again we omit the time domain $[-T, T]$ here)

$$\begin{aligned}
&\big\| |u|^{p-1}u - |u'|^{p-1}u' \big\|_{\mathscr{X}^{0,b-1,T}} \\
&\quad \lesssim T^{2b-1} \big\| (u-u')(|u|+|u'|)^{p-1} \big\|_{L^q_{t,x}} && \text{(5-8)}\\
&\quad \lesssim T^{2b-1} \|v-v'\|_{L^{pq}_{t,x}} \cdot (\|u\|_{L^{pq}_{t,x}} + \|u'\|_{L^{pq}_{t,x}})^{p-1} && \text{(5-9)}\\
&\quad \lesssim T^{2b-1} \|v-v'\|_{\mathscr{X}^{0,b,T}} (\|v\|_{\mathscr{X}^{0,b,T}} + \|v'\|_{\mathscr{X}^{0,b,T}} + \|e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\|_{L^{pq}_{t,x}})^{p-1} && \text{(5-10)}\\
&\quad \lesssim T^{b-\frac{1}{2}} \|v-v'\|_{\mathscr{X}^{0,b,T}} \cdot (\|v\|_{\mathscr{X}^{0,b,T}} + \|v'\|_{\mathscr{X}^{0,b,T}} + 1)^{p-1}, && \text{(5-11)}
\end{aligned}$$

outside Ω_T, where $\mathbb{P}(\Omega_T) \le c_1 \exp(-c_2 T^{-c_3})$. In (5-8) we have used Proposition 2.7 and Lemma 2.8, and required $\frac{1}{2} < b < \frac{2}{3}$, $2 > q > \frac{2}{3-3b}$. In (5-9) we have used Hölder and required $1 < pq < \infty$. In (5-10) we have used (Hölder in time and) Proposition 2.7 and required $2 < pq < 4$. In (5-11) we have used Corollary 3.5 to bound

$$\|e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\|_{L^{r_2}_t L^{q_2}_x} \lesssim T^{-\frac{2b-1}{100p}},$$

with exceptional probability $\le c_1 \exp(-c_2 T^{-c_3})$. Therefore, we may choose q so that $\frac{4}{3} < q < 2$ and $2 < pq < 4$ (such q exists because $1 < p < 3$). Then we may choose $\frac{1}{2} < b < 1 - \frac{2}{3q}$, and see that all the requirements indeed hold. This completes the proof of (5-7).

The estimate (5-6) follows from the same choice of exponents and similar arguments. The only difference is that we will have a term $\|e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^k}(\omega)\|_{L^{r_1}_t L^{q_1}_x}$, which is fine as long as $2 < q_1 < \infty$. □

5.3. *Approximating by ODEs.* Here we will prove that almost surely, uniform global bounds on the solutions to the truncated equations (5-1) for infinitely many $k < \infty$ implies the global existence and uniqueness for the original equation (1-10).

Proposition 5.5. *Let $[-T, T]$ be a time interval, where we assume T is large. Suppose for ω belonging to some set E, there exists a subsequence $\{k_j\}_{j\ge 0} \uparrow \infty$ (possibly depending on ω) such that each of the equations (5-1) with $k = k_j$ has a unique solution u_j on $[-T, T]$ and that*

$$\sup_j \|u_j - e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^{k_j}}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} < \infty. \tag{5-12}$$

Then almost surely $\omega \in E$, the equation (1-10) possesses a unique solution u on $[-T, T]$ such that $u \in e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{X}^{\sigma,b,T}$. Moreover for this subsequence we have

$$\lim_{j\to\infty} \|u_j - e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^{k_j}}(\omega) - (u - e^{-\mathrm{i}t\boldsymbol{H}} f(\omega))\|_{\mathscr{X}^{\sigma,b,T}} = 0. \tag{5-13}$$

Proof. For $\omega \in E$, with small exceptional probability (tending to 0 as $A \to \infty$), we may choose a sequence u_j solving (5-1) with $k = k_j \uparrow \infty$, and

$$\|u_j - e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^{k_j}}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} \le A \tag{5-14}$$

for all j. Then we choose an integer M large enough depending on T and A. We are going to prove for each $1 \le m \le M$ that (1-10) has a unique solution $u \in e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{X}^{\sigma,b,mT/M}$ on the interval $[-\frac{mT}{M}, \frac{mT}{M}]$, and
$$\lim_{j\to\infty} \|u_j - e^{-\mathrm{i}t\boldsymbol{H}} f^{\circ}_{2^{k_j}}(\omega) - (u - e^{-\mathrm{i}t\boldsymbol{H}} f(\omega))\|_{\mathscr{X}^{\sigma,b,mT/M}} \to 0, \tag{5-15}$$
for ω outside the fixed set $\Omega_{T/M}$ that is constructed in the proof of Lemma 5.2. Since $\mathbb{P}(\Omega_{T/M}) \to 0$ as $M \to \infty$, this clearly contains the conclusion we need.

Now we proceed by induction on m. First assume $p \ge 3$ is odd. Supposing the conclusion holds for $m-1$ (including $m = 1$), we will prove it for m. Write $\delta = M^{-1}T$ and $t_0 = (m-1)\delta$, we know the solution u exists and is unique on $[-t_0, t_0]$, and we want to extend it to $[-(t_0+\delta), t_0+\delta]$. Without loss of generality we consider the half-line $t > 0$.

From (5-14) and (5-15) we have
$$\lim_{j\to\infty} \|u_j(t_0) - u(t_0) + e^{-\mathrm{i}t_0\boldsymbol{H}} f^{\perp}_{2^{k_j}}(\omega)\|_{\mathscr{H}^\sigma} = 0, \tag{5-16}$$
and $\|u(t_0) - e^{-\mathrm{i}t_0\boldsymbol{H}} f(\omega)\|_{\mathscr{H}^\sigma} \le A$. We would like to solve the equation (1-1) with initial data $u(t_0)$ on $[-\delta, \delta]$, and argue as in Corollary 4.4. Here the linear term is not $e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)$, but
$$e^{-\mathrm{i}t\boldsymbol{H}} u(t_0) = e^{-\mathrm{i}(t+t_0)\boldsymbol{H}} f(\omega) + v,$$
where v is the linear evolution of some function with $\mathscr{H}^\sigma$ norm $\lesssim A$; thus $\|v\|_{\mathscr{X}^{\sigma,b,\delta}} \lesssim A$ (this is easily proved by introducing a cutoff and using $\delta \le 1$). Since $\omega \notin \Omega_{T/M}$, we can use the full strength of Proposition 4.3 and Lemma 5.2. In particular we can proceed as in the proof of Corollary 4.4 and obtain
$$\mathfrak{M} := \left\| \mp\mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} (|w_1(s)|^{p-1} w_1(s))\,\mathrm{d}s \right\|_{\mathscr{X}^{\sigma,b,\delta}} \lesssim \delta^{\theta_0} A^p \le \delta^\theta,$$
and
$$\begin{aligned}\mathfrak{D} &= \left\| \mp\mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}} (|w_1(s)|^{p-1} w_1(s) - |w_2(x)|^{p-1} w_2(s))\,\mathrm{d}s \right\|_{\mathscr{X}^{\sigma,b,\delta}} \\ &\lesssim \delta^{\theta_0} A^{p-1} \|h_1 - h_2\|_{\mathscr{X}^{\sigma,b,\delta}} < \tfrac{1}{2} \|h_1 - h_2\|_{\mathscr{X}^{\sigma,b,\delta}},\end{aligned}$$
for all $w_i = e^{-\mathrm{i}t\boldsymbol{H}} u(t_0) + h_i$ with $\|h_i\|_{\mathscr{X}^{\sigma,b,\delta}} \le \delta^\theta$, provided M is large enough (δ is small enough) depending on T and A. Then we can use Picard iteration and the same bootstrap argument to prove that the original solution u can be uniquely extended to $[t_0, t_0+\delta]$ (and by symmetry, to the other side).

It remains to prove (5-15) for m. First we know
$$\lim_{j\to\infty} \left\| e^{-\mathrm{i}(t-t_0)\boldsymbol{H}} u_j(t_0) - e^{-\mathrm{i}(t-t_0)\boldsymbol{H}} u(t_0) + e^{-\mathrm{i}t\boldsymbol{H}} f^{\perp}_{2^{k_j}}(\omega)) \right\|_{\mathscr{X}^{\sigma,b,[t_0-\delta,t_0+\delta]}} = 0,$$
which is a consequence of (5-16). In view of the induction hypothesis, we only need to prove[3]
$$\lim_{j\to\infty} \left\| u_j - e^{-\mathrm{i}(t-t_0)\boldsymbol{H}} u_j(t_0) - (u - e^{-\mathrm{i}(t-t_0)\boldsymbol{H}} u(t_0)) \right\|_{\mathscr{X}^{\sigma,b,[t_0-\delta,t_0+\delta]}} = 0,$$

[3]Here we have used the following fact: Given two intervals $[x, y]$ and $[z, w]$ with $x < z < y < w$, for some constant C we have $\|u\|_{\mathscr{X}^{\sigma,b,[x,w]}} \le C(\|u\|_{\mathscr{X}^{\sigma,b,[x,y]}} + \|u\|_{\mathscr{X}^{\sigma,b,[z,w]}})$. This is easily proved by using a partition of unity.

which, after a translation of time, is equivalent to

$$\lim_{j\to\infty}\left\|w_j-e^{-\mathrm{i}t\boldsymbol{H}}w_j(0)-(w-e^{-\mathrm{i}t\boldsymbol{H}}w(0))\right\|_{\mathscr{X}^{\sigma,b,\delta}}=0, \tag{5-17}$$

where w_j is a solution of the truncated equation with $k=k_j$, and $w_j(0)=u_j(t_0)$; and w is a solution of the original equation with $w(0)=u(t_0)$. Write $w_j-w=h=h_{li}+h_{no}$, where

$$h_{li}=e^{-\mathrm{i}t\boldsymbol{H}}(u_j(t_0)-u(t_0))=-e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f_{2^{k_j}}^{\perp}(\omega)+e^{-\mathrm{i}t\boldsymbol{H}}\lambda_j, \tag{5-18}$$

with $\|\lambda_j\|_{\mathscr{H}^\sigma}\to 0$, and

$$\begin{aligned}h_{no}&=\mp\mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}}\big((|w_j|^{p-1}w_j-|w|^{p-1}w)_{2^{k_j}}^{\circ}-(|w|^{p-1}w)_{2^{k_j}}^{\perp}(s)\big)\,\mathrm{d}s\\&=\mp\mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}}(|w_j|^{p-1}w_j-|w|^{p-1}w)_{2^{k_j}}^{\circ}(s)\,\mathrm{d}s-(w-e^{-\mathrm{i}t\boldsymbol{H}}w(0))_{2^{k_j}}^{\perp}.\end{aligned} \tag{5-19}$$

Now we need to prove $\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}\to 0$. Since $w-e^{-\mathrm{i}t\boldsymbol{H}}w(0)\in\mathscr{X}^{\sigma,b,\delta}$, the second term in (5-19) tends to zero in $\mathscr{X}^{\sigma,b,\delta}$ as $j\to\infty$. For the first term, we estimate the norm without the final projection. The expression in parentheses can be written as a linear combination of terms like $z_1^-\cdots z_p^-$, where z_1 is either h_{no}, or $e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f_{2^{k_j}}^{\perp}(\omega)$, or $e^{-\mathrm{i}t\boldsymbol{H}}\lambda_j$ which has $\mathscr{X}^{\sigma,b,\delta}$ norm $\to 0$. For $i\geq 2$, each z_i is one of the following:

(1) $e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f_{2^{k_j}}^{\circ}(\omega)$. This is within the applicability of Lemma 5.2 since $\omega\notin\Omega_{T/M}$.

(2) $w_j-e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f_{2^{k_j}}^{\circ}(\omega)$. This has $\mathscr{X}^{\sigma,b,\delta}$ norm $\lesssim A$ since $w_j(t)=u_j(t+t_0)$, due to (5-14).

(3) One of the components of w_j-w. These include h_{no} and $e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f_{2^{k_j}}^{\perp}(\omega)$, as well as another term with $\mathscr{X}^{\sigma,b,\delta}$ norm $\lesssim A$. Since $\omega\notin\Omega_{T/M}$, these terms are controllable using Lemma 5.2.

If $z_1=e^{-\mathrm{i}(t+t_0)\boldsymbol{H}}f_{2^{k_j}}^{\perp}(\omega)$, then from Proposition 2.9 and Lemma 5.2, the corresponding term tends to 0 as $j\to\infty$ (since h_{no} is bounded in $\mathscr{X}^{\sigma,b,\delta}$ independent of j; see below). If z_1 is the term with $\mathscr{X}^{\sigma,b,\delta}$ tending to 0, the same conclusion holds. If $z_1=h_{no}$, then the norm of the corresponding term is bounded by $\delta^\theta\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}(\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}+A)^{p-1}$. Therefore we have

$$\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}\lesssim\delta^\theta\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}(\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}+A)^{p-1}+o(1),$$

as $j\to\infty$. By (5-14) and the Picard argument above, we know $\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}\lesssim A$ independent of j. Therefore, if we choose δ small enough (M large enough), we must have $\|h_{no}\|_{\mathscr{X}^{\sigma,b,\delta}}=o(1)$.

The proof when $1<p<3$ is basically the same, using linear estimates (Corollary 3.5) instead of Proposition 4.3. We will also need a variant of Lemma 5.2, but the proof of this is not hard and is essentially contained in Proposition 3.4 and Corollary 3.5. □

6. Global well-posedness

In what follows, we fix a sufficiently large T and a positive integer M such that $M\gtrsim T^2$.

First let us consider the truncated equation (5-1), which is an ODE on the finite-dimensional space V_{2^k}. If we identify V_{2^k} with $\mathbb{R}^{2^{k+1}+2}$ by the coordinates

$$g = \sum_{j=0}^{2^k} (a_j + \mathrm{i} b_j) e_j, \tag{6-1}$$

then it is easy to check that (5-1) becomes

$$\partial_t a_j = \frac{\partial E_0}{\partial b_j}, \quad \partial_t b_j = -\frac{\partial E_0}{\partial a_j}, \tag{6-2}$$

with Hamiltonian

$$E_0(a_j, b_j) = \sum_{j=0}^{2^k} (2j+1)(a_j^2 + b_j^2) \pm \frac{1}{p+1} \left\| \sum_{j=0}^{2^k} (a_j + \mathrm{i} b_j) e_j \right\|_{L^{p+1}}^{p+1}. \tag{6-3}$$

If we denote the solution flow of this equation by $\Phi_{2^k,t}$, then the following is true by the theory of Hamiltonian ODEs and straightforward computation: The map $(t, x) \mapsto \Phi_{2^k,t}(x)$ is defined on the whole spacetime domain $\mathbb{R} \times V_{2^k}$ (this is a consequence of the conservation of L^2 norm; see (6-4) below). For each $t \in \mathbb{R}$, $\Phi_{2^k,t}$ is a homeomorphism from V_{2^k} to itself. If $p \geq 3$ is odd, it is a diffeomorphism and preserves the quantities

$$\|g\|_{L^2}^2 = \sum_{j=0}^{2^k} (a_j^2 + b_j^2) \quad \text{and} \quad E = 2E_0 \tag{6-4}$$

and the Lebesgue measure. If $1 < p < 3$, it (and its inverse) can be approximated, uniformly on each compact subset of V_{2^k}, by a sequence of pairs of diffeomorphisms that preserve the quantities (6-4) and the Lebesgue measure. Therefore $\Phi_{2^k,t}$ itself also preserves (6-4) and the Lebesgue measure.

From above we know that $\Phi_{2^k,t}$ preserves the measure

$$\nu_{2^k}^{\circ} = \pi^{-1-2^k} \zeta \cdot e^{-E} \prod_{j=0}^{2^k} \mathrm{d}a_j \mathrm{d}b_j \tag{6-5}$$

on V_{2^k}, where $\zeta = 1$ in the defocusing case, and $\zeta = \chi(\|g\|_{L^2}^2 - \alpha_{2^k})$ in the focusing case as in (3-33). By the definition of μ and ν_{2^k} (see Section 3) we have

$$\nu_{2^k} = (\rho_{2^k} \cdot \mu_{2^k}^{\circ}) \otimes \mu_{2^k}^{\perp} = \nu_{2^k}^{\circ} \otimes \mu_{2^k}^{\perp}, \tag{6-6}$$

in both cases, where we understand that $\mu_{2^k}^{\circ}$ and $\mu_{2^k}^{\perp}$ are measures on V_{2^k} and $V_{2^k}^{\perp}$ respectively, and identify V with[4] $V_{2^k} \times V_{2^k}^{\perp}$. From this we immediately see, for each Borel set J of V_{2^k}, that

$$\nu_{2^k}\big(\{g : g_{2^k}^{\circ} \in J\}\big) = \nu_{2^k}\big(\{g : g_{2^k}^{\circ} \in (\Phi_{2^k,t})^{-1}(J)\}\big). \tag{6-7}$$

[4]Here V is some space on which μ is supported. The exact choice of V is unimportant; for example, we may choose $V = \mathcal{S}'(\mathbb{R}^2)$, or $V = \bigcap_{\delta>0} \mathcal{H}^{-\delta}(\mathbb{R}^2)$.

Now we fix the choice

$$J = J_M = \big\{g_{2^k}^{\circ} : g \in f\big(\Omega_{T/M}^c\big)\big\}^c,$$

where $\Omega_{T/M}$ is constructed in the proof of Lemma 5.2. Consider the maximal $m_0 \le M+1$ so that the solution u of equation (5-1) satisfies

$$\big\|u - e^{-\mathrm{i}(t-mT/M)\boldsymbol{H}}u(mT/M)\big\|_{\mathscr{X}^{\sigma,b,[(m-1)T/M,(m+1)T/M]}} \le 1 \tag{6-8}$$

for all $|m| \le m_0 - 1$. If $m_0 = M+1$, from Proposition 2.9 we know that u is defined on $[-T, T]$ and

$$\|u - e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} \lesssim M^3. \tag{6-9}$$

If $m_0 \le M$, then for some choice of $\pm$ sign, we have $\Phi_{2^k,\pm m_0 T/M}(f_{2^k}^{\circ}(\omega)) \in J_M$. In fact, if this fails, then we can use Propositions 5.1 and 5.4 to extend the solution to $[-(m_0+1)T/M, (m_0+1)T/M]$ with (6-8) remaining true, thus contradicting the definition of m_0. Now we use (6-7) and sum over $m_0 \le M$ to get

$$(\nu_{2^k} \circ f)\big(\{\omega : \text{(6-9) fails}\}\big) \lesssim M \cdot \nu_{2^k}\big(\{g : g_{2^k}^{\circ} \in J_M\}\big). \tag{6-10}$$

In the defocusing case we have $\nu_{2^k} \le \mu$. Using Fubini's theorem we get

$$\mu\big(\{g : g_{2^k}^{\circ} \in J_M^c\}\big) \ge \mu(f(\Omega_{T/M}^c)) \ge 1 - c_1 \exp(-c_2 T^{-c_3} M^{c_3}), \tag{6-11}$$

hence

$$(\nu_{2^k} \circ f)\big(\{\omega : \text{(6-9) fails}\}\big) \lesssim c_1 M \exp(-c_2 T^{-c_3} M^{c_3}). \tag{6-12}$$

In the focusing case we have

$$\frac{\mathrm{d}\nu_{2^k}}{\mathrm{d}\mu}(g) = \rho_{2^k}(g) = \chi(\|g_{2^k}^{\circ}\|_{L^2}^2 - \alpha_{2^k}) \exp\Big(\frac{2}{p+1}\|g_{2^k}^{\circ}\|_{L^{p+1}}^{p+1}\Big). \tag{6-13}$$

This function, by Proposition 3.6, has bounded $L^2(\mu)$ norm, so by Cauchy–Schwartz we get

$$\nu_{2^k}\big(\{g : g_{2^k}^{\circ} \in J_M\}\big) \lesssim \big(\mu\big(\{g : g_{2^k}^{\circ} \in J_M\}\big)\big)^{1/2} \le c_1 \exp(-c_2 T^{-c_3} M^{c_3}), \tag{6-14}$$

which again implies (6-12). We summarize our results in the following proposition.

Proposition 6.1. *For fixed T and k, there exists a subset*[5] $\Omega_k \subset \Omega$ *such that* $(\nu_{2^k} \circ f)(\Omega_k^c) = 0$, *and for* $\omega \in \Omega_k$, *equation* (5-1) *has a unique solution* u_k *on* $[-T, T]$, *and that*

$$\sup_k \int_{\Omega_k} \exp\big(\|u_k - e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega)\|_{\mathscr{X}^{\sigma,b,T}}^{\theta}\big)\, \mathrm{d}(\nu_{2^k} \circ f)(\omega) < \infty \tag{6-15}$$

for some $\theta > 0$.

Proof. We choose

$$\Omega_k = \bigcap_{M \gtrsim T^2} Z_M := \bigcap_{M \gtrsim T^2} \big\{\omega : \text{(6-9) fails for } M\big\}.$$

[5] This should not be confused with the Ω_T notation defined above, since our Ω_k is for $k \ge 1$ here!

From the discussion above we easily see $\nu_{2^k}(f(\Omega_k^c)) \leq \lim_{M\to\infty} \nu_{2^k}(f(Z_M)) = 0$. Also for $\omega \notin Z_M$ the solution u_k to (5-1) exists and is unique, and satisfies

$$\|u_k - e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} \lesssim M^3.$$

In other words we have

$$(\nu_{2^k} \circ f)\big(\omega \in \Omega_k : \|u_k - e^{-\mathrm{i}t\boldsymbol{H}} f_{2^k}^{\circ}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} > A\big) \leq \nu_{2^k}(f(Z_M)) \leq c_1 \exp(-c_2 A^{c_3}),$$

for all $A > T^{100}$, where $M \sim A^{1/3}$ is an integer. Since $\nu_{2^k} \circ f$ is uniformly integrable, the part with small A is also under control. The claim then follows. $\square$

With Propositions 5.5 and 6.1, we are ready to prove the global well-posedness part of Theorem 1.1. Denote the integrand in (6-15) by $\eta_k(\omega)$, understanding $\eta_k(\omega) = 0$ when $\omega \notin \Omega_k$. Since $\nu_{2^k} \to \nu$ in the strong sense and $(\nu_{2^k} \circ f)(\Omega_k^c) = 0$, we have $(\nu \circ f)(\Omega_k^c) \to 0$, and we fix a subsequence $\{k_l\}$ such that $\sum_l (\nu \circ f)(\Omega_{k_l}^c) < \infty$ and hence $(\nu \circ f)(\limsup_{l\to\infty} \Omega_{k_l}^c) = 0$. From Proposition 6.1, we get

$$\sup_l \int_\Omega \rho_{2^{k_l}}(f(\omega))\eta_{k_l}(\omega)\, \mathrm{d}\mathbb{P}(\omega) < \infty. \tag{6-16}$$

From the proof of Proposition 3.6, we see $\rho_{2^{k_l}} \circ f \to \rho \circ f$ almost surely, so by Fatou's lemma we get

$$\liminf_{l\to\infty} \eta_{k_l}(\omega) < \infty, \tag{6-17}$$

almost surely in $\mathbb{P}$, on the set where $\rho(f(\omega)) \neq 0$. By the definition of η_k, if (6-17) holds, then either $\omega \in \Omega_{k_l}$ for infinitely many l, or there exists a subsequence $\{k_{l_j}\}_{j\geq 0} \uparrow \infty$, such that (5-1) has a unique solution $u_{k_{l_j}}$ for $k = k_j$ on $[-T, T]$, and

$$\sup_j \|u_{k_{l_j}} - e^{-\mathrm{i}t\boldsymbol{H}} f_{2^{k_{l_j}}}^{\circ}(\omega)\|_{\mathscr{X}^{\sigma,b,T}} < \infty.$$

In the former case we get a null set (actually a set with null $\nu \circ f$ measure, but $\nu \circ f$ is mutually absolutely continuous with $\mathbb{P}$ on the set where $\rho(f(\omega)) \neq 0$), while in the latter case we can use Proposition 5.5 to deduce that, except for another null set, (1-10) also has a unique solution u on $[-T, T]$ such that $u \in e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{X}^{\sigma,b,T}$.

Therefore, for each $T > 0$, except for a null set, the equation (1-10) has a unique solution $u \in e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{X}^{\sigma,b,T}$ for ω in the support of $\rho \circ f$. In the defocusing case, this support itself has full probability in Ω; in the focusing case, it follows from Proposition 3.6 that we can choose a countable number of cutoff χ so that the (countable) union of the support of the corresponding $\rho \circ f$ has full probability. In any case we have found a subset of Ω having full probability, such that when ω does belong to this set, (1-10) has a unique solution $u \in e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{X}^{\sigma,b,T}$. We then take another countable union to get that, almost surely in $\mathbb{P}$, equation (1-10) has a unique solution u on $\mathbb{R} \times \mathbb{R}^2$ such that

$$\begin{aligned} u \in e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{X}^{\sigma,b,T} &\subset e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) + \mathscr{C}([-T, T], \mathscr{H}^{\sigma}(\mathbb{R}^2)) \\ &\subset \mathscr{C}([-T, T], \bigcap_{\delta>0} \mathscr{H}^{-\delta}(\mathbb{R}^2)) \end{aligned}$$

for all $T > 0$. This completes the proof.

Remark 6.2. In fact, from the argument we can extract a polynomial bound on the solution; namely we can prove that for each large A, with exceptional probability $\le c_1 \exp(-c_2 A^{c_3})$ we have

$$\|u - e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)\|_{\mathscr{X}^{\sigma,b,T}} \le A\langle T\rangle^C$$

for all $T > 0$, with some constant C. We omit the details.

7. Transforming into NLS without harmonic potential

As we have mentioned before, the idea of introducing the lens transform and reducing (1-2) to (1-1) is inspired by the arguments in [Burq et al. 2010]. First we define the lens transform [Tao 2009, Section 2; Burq et al. 2010, Section 10]:

$$\mathscr{L}u(t,x) = \frac{1}{\cos(2t)} u\Big(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\Big) e^{-\mathrm{i}|x|^2 \tan(2t)/2}, \tag{7-1}$$

where u is defined on $\mathbb{R} \times \mathbb{R}^2$, and $\mathscr{L}u$ is defined on $(-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^2$. By a simple computation we deduce

$$(\mathrm{i}\partial_t - \boldsymbol{H})(\mathscr{L}u)(t,x) = (\cos(2t))^{-2} \mathscr{L}((\mathrm{i}\partial_t + \Delta)u)(t,x). \tag{7-2}$$

For the inverse transform

$$\mathscr{L}^{-1}u(t,x) = (1+4t^2)^{-\frac{1}{2}} u\Big(\frac{\tan^{-1}(2t)}{2}, (1+4t^2)^{-\frac{1}{2}} x\Big) e^{\mathrm{i}|x|^2 t/(1+4t^2)}, \tag{7-3}$$

we have

$$(\mathrm{i}\partial_t + \Delta)(\mathscr{L}^{-1}u)(t,x) = \frac{1}{1+4t^2} \mathscr{L}^{-1}((\mathrm{i}\partial_t - \boldsymbol{H})u)(t,x). \tag{7-4}$$

Next we prove that the transform $\mathscr{L}^{-1}$ maps the space $\mathscr{X}^{\sigma,b,\delta}$ to $X^{\sigma,b,T}$, where $0 \le \sigma, b \le 1$, $0 < \delta < \frac{\pi}{4}$, and $T = \frac{1}{2}\tan(2\delta)$. First by using a cutoff, we are reduced to proving that $u \mapsto \mathscr{L}^{-1}(\chi \cdot u)$ is bounded from $\mathscr{X}^{\sigma,b}$ to $X^{\sigma,b}$, where $\chi = \chi(t)$ is any smooth function having compact support in $|t| < \frac{\pi}{4}$. First we fix σ. By interpolation, we can assume $b \in \{0, 1\}$. If we can prove the result in the case $b = 0$, then using the identity

$$\|u\|^2_{\mathscr{X}^{\sigma,1}} = \|u\|^2_{\mathscr{X}^{\sigma,0}} + \|(\mathrm{i}\partial_t - \boldsymbol{H})\|^2_{\mathscr{X}^{\sigma,0}}$$

(which remains true with $\mathscr{X}$ replaced by X and $-\boldsymbol{H}$ replaced by Δ) and (7-4), we see

$$\|\mathscr{L}^{-1}(\chi \cdot u)\|_{X^{\sigma,1}} \lesssim \|u\|_{\mathscr{X}^{\sigma,0}} + \|(\mathrm{i}\partial_t - \boldsymbol{H})(\chi u)\|_{\mathscr{X}^{\sigma,0}}, \tag{7-5}$$

because $v = (\mathrm{i}\partial_t - \boldsymbol{H})(\chi u)$ has compact support in $|t| < \frac{\pi}{4}$, and hence equals $\chi_1 v$ for some other χ_1. Since the last term in (7-5) is clearly controlled by $\|u\|_{\mathscr{X}^{\sigma,1}}$, we can conclude the proof for $b = 1$. Therefore we may only consider $b = 0$. Here it is easily seen that we only need to prove that multiplication by $e^{\mathrm{i}\lambda|x|^2}$ is uniformly bounded from $\mathscr{H}^\sigma$ to H^σ for $0 \le \sigma \le 1$ and $|\lambda| \le 1$. By another interpolation we may further reduce to $\sigma \in \{0, 1\}$. The $\sigma = 0$ case is obvious; the $\sigma = 1$ case follows from the observation

$$\nabla(e^{\mathrm{i}\lambda|x|^2} \cdot f) = e^{\mathrm{i}\lambda|x|^2}(\nabla + 2\mathrm{i}\lambda x) \cdot f.$$

Thus we have the desired bound for all $0 \le \sigma, b \le 1$.

Using (7-2) or (7-4) we can compute that u is a solution for the Cauchy problem (1-12) on $\mathbb{R}$, if and only if $v = \mathscr{L}u$ is a solution for the Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t v + (\Delta - |x|^2)v = (\cos(2t))^{p-3} \cdot |v|^{p-1}v, \\ v(0) = f(\omega) \end{cases} \tag{7-6}$$

on $|t| < \frac{\pi}{4}$. Moreover, if

$$v - e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) \in \mathscr{X}^{\sigma,b,\delta} \tag{7-7}$$

with $\delta < \frac{\pi}{4}$, then from the discussion above we see that

$$u - \mathscr{L}^{-1}(e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)) \in X^{\sigma,b,T},$$

with $T = \frac{1}{2}\tan(2\delta) \to \infty$ as $\delta \to \frac{\pi}{4}$. From (7-4) we see that $\mathscr{L}^{-1}(e^{-\mathrm{i}t\boldsymbol{H}} f(\omega))$ has initial value $f(\omega)$ and annihilates $\mathrm{i}\partial_t + \Delta$; thus it must be $e^{\mathrm{i}t\Delta} f(\omega)$. Thus (1-13) will follow[6] if (7-7) holds for all $\delta < \frac{\pi}{4}$. Also from (7-3), the constants in the $\mathscr{H}_x^\sigma \to H_x^\sigma$ boundedness remains under control even near the boundary points $\pm\frac{\pi}{4}$. Thus (1-14) will follow if

$$\lim_{t \to \pm\pi/4} (v(t) - e^{-\mathrm{i}t\boldsymbol{H}} f(\omega)) \quad \text{exists in } \mathscr{H}^\sigma. \tag{7-8}$$

What we will prove is that almost surely in $\mathbb{P}$, (7-6) has a unique (strong) solution v for $|t| \le \frac{\pi}{4}$ such that $v - e^{-\mathrm{i}t\boldsymbol{H}} f(\omega) \in \mathscr{X}^{\sigma,b,\pi/4}$. As is demonstrated above, this implies (7-7) and (7-8), and hence Theorem 1.2.

The proof is basically the same as (1-10). Noticing $m(t) = (\cos(2t))^{p-3}$ has all its derivatives bounded on $\mathbb{R}$, we see that multiplication by $m(t)$ is bounded from any $\mathscr{X}^{\sigma,b}$ (and hence any $\mathscr{X}^{\sigma,b,T}$) to itself. Therefore, the proof from Proposition 4.3 to Lemma 5.2 goes without any difficulty, as if this additional factor were not present. In the proof of Proposition 5.5, when we extend the solution to a larger interval, we must solve another Cauchy problem, which is no longer (7-6), since this equation is not autonomous. This, however, is not a problem; since we just replace $m(t)$ by some $m(t - t_0)$ that obeys the same derivative estimates as $m(t)$, we can use the *same* exceptional set as in Proposition 4.3, Lemma 5.2 and Proposition 5.5, and the other discussions remain unchanged.

The only difficulty we face is the lack of a (formally) invariant measure. This is compensated, however, by a monotonicity property, which was first observed in [Burq et al. 2010].

Lemma 7.1. *Consider the truncated Cauchy problem*

$$\begin{cases} \mathrm{i}\partial_t v + (\Delta - |x|^2)v = (\cos(2t))^{p-3} \cdot (|v|^{p-1}v)^\circ_{2^k}, \\ v(0) = f^\circ_{2^k}(\omega); \end{cases} \tag{7-9}$$

then for its solution v, the quantity

$$\mathscr{E}(t, v(t)) = \langle \boldsymbol{H}v, v\rangle + \frac{2(\cos(2t))^{p-3}}{p+1} \|v\|^{p+1}_{L^{p+1}}$$

[6] The space $\mathscr{X}' \subset \mathscr{X}$ in which we have uniqueness will be the image of the space $\mathscr{Y}$ defined in (1-11) under $\mathscr{L}^{-1}$; as we have said before, we do not have a simple characterization for this.

is monotonically nonincreasing in $|t|$ for $|t| \le \frac{\pi}{4}$.

Proof. We directly compute

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = -\frac{2(p-3)(\cos(2t))^{p-4}\sin(2t)}{p+1}\|v(t)\|_{L^{p+1}}^{p+1},$$

which is nonpositive for $0 \le t \le \frac{\pi}{4}$, and nonnegative for $-\frac{\pi}{4} \le t \le 0$. □

We argue as in Section 6, but we fix $T = \frac{\pi}{4}$ here. If we could prove

$$\mu\big(\{g : g_{2^k}^{\circ} \in J\}\big) \ge \nu_{2^k}\big(\{g : \Phi_{2^k,t}(g_{2^k}^{\circ}) \in J\}\big) \tag{7-10}$$

for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$, where, of course, $\Phi_{2^k,t}$ is now the solution flow of (7-6), then combining this inequality with (6-11) we can get (6-12). Starting from this point, we can follow the argument in Section 6 word by word to get almost surely global well-posedness of (7-6) on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

The proof of (7-10) is also simple. By Lemma 7.1

$$\begin{aligned}\nu_{2^k}(\{g : \Phi_{2^k,t}(g_{2^k}^{\circ}) \in J\}) &= \pi^{-1-2^k}\int_{J_1} e^{-E(g)}\prod_{j=0}^{2^k}\mathrm{d}a_j\mathrm{d}b_j\\ &\le \pi^{-1-2^k}\int_{J_1} e^{-\mathscr{E}(t,g(t))}\prod_{j=0}^{2^k}\mathrm{d}a_j\mathrm{d}b_j \le \int_J \mathrm{d}\mu_{2^k}^{\circ} = \mu(\{g : g_{2^k}^{\circ} \in J\}),\end{aligned}$$

where $J_1 = \{h \in V_{2^k} : \Phi_{2^k,t}(h) \in J\}$. Here we have used the invariance of the Lebesgue measure under $\Phi_{2^k,t}$, which can be directly verified; see [Burq et al. 2010, Lemma 8.3]. Therefore we have completed the proof of Theorem 1.2.

8. Invariance of Gibbs measure

Now we return to the final assertion of Theorem 1.1, and prove the invariance of the Gibbs measure ν under the solution flow of (1-10). More precisely:

Proposition 8.1. *Denote the solution flow of (1-10) by Φ_t. There exists a subset $\Sigma \subset \mathscr{S}'(\mathbb{R}^2)$ such that it has full μ measure, and Φ_t becomes a one-parameter group from Σ to Σ preserving the measure ν (in the focusing case, for each choice of cutoff function χ).*

Proof. We only consider the defocusing case. In the focusing case we need to take another countable intersection corresponding to the cutoff χ chosen, but otherwise the proof is completely analogous. Clearly the set Ω_T in Proposition 4.3 and Lemma 5.2 can be chosen so that $e^{-\mathrm{i}t\boldsymbol{H}}f(\Omega_T^c) = f(\Omega_T^c)$.

We define $\Sigma = \Sigma_1 \cap \Sigma_2$, where Σ_1 is the set of all $g \in \mathscr{S}'(\mathbb{R}^2)$ so that (1-1) (with initial data $u(0) = g$) has a unique solution[7] u on $\mathbb{R}$ that belongs to $e^{-\mathrm{i}t\boldsymbol{H}}g + \mathscr{X}^{\sigma,b,T}$ for all $T > 0$. This has full μ measure due to the global well-posedness part of Theorem 1.1. Also Σ_2 is defined to be $\Sigma_2 = f(\liminf_{i\to\infty}\Omega_{\gamma 2^{-i}}^c) + \mathscr{H}^{\sigma}$, and this also has full μ measure for small enough γ due to our control on $\mathbb{P}(\Omega_T)$. Clearly Σ has full μ measure,

[7] It is a bit vague to say u is a "solution" when g is only a distribution; but since we are considering Σ_2 also, we can assume here $e^{-\mathrm{i}t\boldsymbol{H}}g \in L_{t,x}^q$ on any finite time interval, for appropriate q, and then the definition of Σ_1 becomes rigorous.

and Φ_t is uniquely defined on Σ. If we can prove $\Phi_t(\Sigma) \subset \Sigma$, then they obviously form a (measurable) one-parameter group. Clearly $\Phi_t(\Sigma) \subset \Sigma_2$ since for a solution u we have $u(t) \in e^{-\mathrm{i}t\boldsymbol{H}}u(0) + \mathscr{H}^\sigma$. To prove $\Phi_t(\Sigma) \subset \Sigma_1$, we only need to prove that if u is a solution of (1-10) with $u(0) \in \Sigma_2$, then it is automatically unique. Since all $u(t) \in \Sigma_2$, by bootstrap arguments we only need to prove short time uniqueness. Write $u(0) = f(\omega) + h$ with $\|h\|_{\mathscr{H}^\sigma} = A$ and $\omega \notin \Omega_{c2^{-i}}$ for all large enough i. Repeating the extension argument in Proposition 5.5, we see for i large enough depending on A that $\omega \notin \Omega_{c2^{-i}}$ and the solution is unique for $|t| \le c2^{-i}$. This proves the existence of Σ.

Now we only need to prove that for each measurable set $E \subset \Sigma$ and $t \in \mathbb{R}$, we have

$$\nu(\Phi_t(E)) \ge \nu(E). \tag{8-1}$$

We may assume $|t| \le 1$. Write

$$\Pi_{i_0,A} = \Sigma_1 \cap \Big(\{h : \|h\|_{\mathscr{H}^\sigma} \le A\} + \bigcap_{i \ge i_0} f(\Omega^c_{c2^{-i}}) \Big)$$

and

$$\Pi'_A = \big\{ g \in \Sigma : \big\| u - e^{-\mathrm{i}t\boldsymbol{H}} g \big\|_{\mathscr{X}^{\sigma,b,2}} \le A \big\}.$$

By a limiting argument we can further assume $E \subset \Pi_{i_0,A} \cap \Pi'_A$ for some i_0 and A. Note that this implies $\Phi_t(E) \subset \Pi_{i_0,CA}$ for $|t| \le 2$ and some constant C.

Let T be small enough depending on i_0 and A, we only need to prove (8-1) for $|t| \le T$ and $E \subset \Pi_{i_0,CA}$ (since we can iterate to get the result for $|t| \le 1$). Write $\Pi = \Pi_{i_0,CA}$ and define $\Psi(g) = u - e^{-\mathrm{i}t\boldsymbol{H}} g$, where u is the solution to (1-1) with initial data g, and consider the mapping

$$\Psi_1 : \Pi \to \mathscr{X}^{\sigma,b,T} \times \mathbb{C}^\infty, \quad g \mapsto (\Psi(g), (\langle g, e_k \rangle)_{k \ge 0}),$$

where in $\mathbb{C}^\infty$ we use the standard metric. This mapping is clearly injective (thus it induces a metric on Π) and, as will be explained in Remark 8.2, its image is a Borel set of the product space (denoted by Y). By a theorem in measure theory [Halmos 1950], the finite Borel measure $\nu \circ \Psi_1^{-1}$ on the complete separable metric space Y is regular. For each measurable set $E \subset \Pi$ we can find a compact set $K \subset \Psi_1(E)$ such that $(\nu \circ \Psi_1^{-1})(\Psi_1(E) - K) < \epsilon$; thus $\Psi_1^{-1}(K) \subset E$ is compact in the induced metric and $\nu(E - \Psi_1^{-1}(K)) < \epsilon$. Therefore, we only need to prove (8-1) for compact sets $E \subset \Pi$. Due to Propositions 5.1 and 5.4, when T is small enough depending on i_0 and A, the map $\Phi_{2^k,t}$ will be defined on $E \subset \Pi$ for each k and $|t| \le T$. Thus by the invariance of $\nu^\circ_{2^k}$ under the solution flow $\Phi_{2^k,t}$, we have

$$\nu_{2^k}\big(\big\{ g : g^\circ_{2^k} = \Phi_{2^k,t}(h^\circ_{2^k}),\ h \in E \big\}\big) \ge \nu_{2^k}(E).$$

Let $k \to \infty$, noticing that the total variation of $\nu_{2^k} - \nu$ tends to zero, we only need to prove that

$$\limsup_{k \to \infty} \big\{ g : g^\circ_{2^k} = \Phi_{2^k,t}(h^\circ_{2^k}),\ h \in E \big\} \subset \Phi_t(E).$$

Now suppose that for a subsequence $k_j \uparrow \infty$, we have $g^\circ_{2^{k_j}} = \Phi_{2^{k_j},t}((h_{k_j})^\circ_{2^{k_j}})$, and by compactness, assume $h_{k_j} \to h$ with respect to the induced metric. We are going to prove $g = \Phi_t(h)$.

First of all, we have

$$\lim_{k\to\infty}\left\|\Phi_{2^k,t}(h^{\circ}_{2^k})-\Phi_t(h)+e^{-\mathrm{i}t\boldsymbol{H}}h^{\perp}_{2^k}\right\|_{\mathcal{H}^\sigma}=0,\tag{8-2}$$

uniformly for $|t|\le T$ and $h\in E$. In fact, if T is small enough, we may assume $h=h_1+h_2$, where $h_1\in f(\Omega^c_{T'})$, $2T\le T'\le 4T$, and $\|h_2\|_{\mathcal{H}^\sigma}\le CA$. Since T' is small enough depending on A, we can almost repeat[8] the proof of Proposition 5.5 to get that the $\mathcal{X}^{\sigma,b,T'}$ norm tends to 0. Since the $\mathcal{X}^{\sigma,b,T'}$ norm is not less than the spacial $\mathcal{H}^\sigma$ norm at time t, (8-2) follows.

From (8-2) we get

$$\lim_{j\to\infty}\left\|g^{\circ}_{2^{k_j}}-(\Phi_t(h_{k_j}))^{\circ}_{2^{k_j}}\right\|_{\mathcal{H}^\sigma}=0,$$

and we only need to prove

$$\lim_{k\to\infty}\left\|(\Phi_t(h_{k_j}))^{\circ}_{2^{k_j}}-(\Phi_t(h))^{\circ}_{2^{k_j}}\right\|_{\mathcal{H}^\sigma}=0.\tag{8-3}$$

But since $h_{k_j}\to h$ with respect to the induced metric, we only need to prove that $\|(h_{k_j})^{\circ}_{2^{k_j}}-h^{\circ}_{2^{k_j}}\|_{\mathcal{H}^\sigma}\to 0$.

For $i\ge j$ we have

$$\left(\Phi_{2^{k_i},t}((h_{k_i})^{\circ}_{2^{k_i}})\right)^{\circ}_{2^{k_j}}=g^{\circ}_{2^{k_j}}=\Phi_{2^{k_j},t}((h_{k_j})^{\circ}_{2^{k_j}}),$$

and by using (8-2) once more we see that

$$(\Phi_t(h_{k_i}))^{\circ}_{2^{k_j}}=(\Phi_t(h_{k_j}))^{\circ}_{2^{k_j}}+o(1)$$

as $i\ge j\to\infty$. Again using that $h_{k_j}\to h$, we deduce

$$\lim_{i\ge j\to\infty}\left\|(h_{k_i}-h_{k_j})^{\circ}_{2^{k_j}}\right\|_{\mathcal{H}^\sigma}=0.\tag{8-4}$$

In particular, we see that $\lim_{i\to\infty}(h_{k_i})^{\circ}_{2^{k_j}}$ exists in $\mathcal{H}^\sigma$ for each j. By the definition of the metric, this limit must be $h^{\circ}_{2^{k_j}}$. Therefore we get

$$\lim_{i\to\infty}\left\|(h-h_{k_i})^{\circ}_{2^{k_j}}\right\|_{\mathcal{H}^\sigma}=0.\tag{8-5}$$

Combining (8-4) with (8-5), we finally see that $\lim_{j\to\infty}\|(h_{k_j})^{\circ}_{2^{k_j}}-h^{\circ}_{2^{k_j}}\|_{\mathcal{H}^\sigma}=0$. This completes the proof of Theorem 1.1. □

Remark 8.2. To show that $\Psi_1(\Pi)$ is a Borel set in the product metric space, we only need to show that Ψ is injective, $\Psi(\Pi)$ is a Borel set in $\mathcal{X}^{\sigma,b,T}$, and the map $\Psi^{-1}:\Psi(\Pi)\to\Pi$ is Borelian. To this end we notice

$$\Psi(g)=-\mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)\boldsymbol{H}}(|u(s)|^{p-1}u(s))\,\mathrm{d}s,\tag{8-6}$$

where $u=u(g)$ is the solution map of (1-2), and $g=u(0)$.[9] Then we can decompose Ψ as

$$\Psi: g\mapsto u(g)\mapsto |u(g)|^{p-1}u(g)\mapsto\Psi(g),$$

[8]Actually we do not have the *a priori* bound on the nonlinear part of truncated equations, but since $h_1\in\Omega_{T'}$ with T' small depending on A, it is not hard to get this from scratch.

[9]Here we also require $g\in\mathcal{H}^{-\epsilon}$ for appropriate ϵ, so that $u\in\mathcal{C}(\mathbb{R},\mathcal{H}^{-\epsilon})$ in which $u(0)$ makes sense.

and see that at each step the mapping is injective, and the image of any Borel set is again Borelian (for example, the set $u(\Pi)$ can be characterized as the set of all u such that $u - e^{-\mathrm{i}t\boldsymbol{H}}u(0) \in \mathscr{X}^{\sigma,b}$, that u satisfies equation (1-2), and that $u(0) \in \Pi$, so it is Borelian). Hence the claim.

Appendix. Typical regularity on the support of μ

In this appendix we shall prove that if $\sigma \geq \frac{1}{2}$, then almost surely $\boldsymbol{H}^{\sigma/2} f(\omega)$ is not a (locally integrable) function. More precisely, almost surely in $\mathbb{P}$, we have

$$\psi \cdot \boldsymbol{H}^{\sigma/2} f(\omega) \notin L^1(\mathbb{R}^2) \tag{A-1}$$

for *any* smooth compactly supported ψ that is not identically zero.

To prove this, first notice that we can find a countable number of ψ_j such that each is compactly supported and equals 1 on some annular region $a < |x| < b$, and for any other ψ there exists $\eta \in L^\infty$ and j such that $\psi_j = \psi \cdot \eta$. So we only need to consider a fixed ψ_j (which we write ψ below) and assume it equals 1 for $a < |x| < b$. Here we use an asymptotic formula of $\mathscr{L}_k^0$ proved in [Erdélyi 1960]:

$$\mathscr{L}_k^0(z) = \frac{1}{\sqrt{2\pi}}(\nu z)^{-1/4}\cos\theta + O(\nu^{-3/4}), \tag{A-2}$$

where $a^2 < z < b^2$ and $\nu = 4k+2$ is large, and

$$\theta = \frac{\nu(\phi + \sin\phi) - \pi}{4}, \quad \phi = \cos^{-1}\frac{\nu - 2z}{\nu}.$$

From (A-2) we easily deduce that

$$\mathscr{L}_k^0(z) = 1/\sqrt{2\pi}\,(\nu z)^{-1/4}\cos(\sqrt{\nu z} - \pi/4) + O(\nu^{-3/4}),$$

and hence for each k

$$\|e_k\psi\|_{L^1} \gtrsim \int_{a^2<z<b^2} |\mathscr{L}_k^0(z)|\,\mathrm{d}z \gtrsim \nu^{-1/4}. \tag{A-3}$$

Now we define the Gaussian random variable

$$h_{M,N}(\omega) = \sum_{k=0}^{M}(4k+2)^{(\sigma-1)/2} g_k(\omega)\eta(\boldsymbol{H}/N^2)(e_k\psi),$$

whose range lies in a finite-dimensional space, and use Lemma 3.3 to get the lower bound

$$\mathbb{P}(\|h_{M,N}(\omega)\|_{L^1} \geq cE_{M,N}) \geq \tfrac{1}{10}$$

with some absolute constant c, where

$$\begin{aligned} E_{M,N} = \mathbb{E}(\|h_{M,N}(\omega)\|_{L^1}) &= \int_{\mathbb{R}^2} \mathbb{E}\biggl(\biggl|\sum_{k=0}^{M}(4k+2)^{(\sigma-1)/2} g_k(\omega)e_{k,N}(x)\biggr|\biggr)\,\mathrm{d}x \\ &\sim \int_{\mathbb{R}^2}\biggl(\sum_{k=0}^{M}(4k+2)^{\sigma-1}|e_{k,N}(x)|^2\biggr)^{1/2}\mathrm{d}x \gtrsim \biggl(\sum_{k=0}^{M}(4k+2)^{\sigma-1}\|e_{k,N}\|_{L^1}^2\biggr)^{1/2}, \end{aligned}$$

and $e_{k,N} = \eta(N^{-2}\boldsymbol{H})(e_k\psi)$. Now for fixed N, we let $M \to \infty$ to get

$$h_{M,N} \to \eta(N^{-2}\boldsymbol{H})(\boldsymbol{H}^{\sigma/2} f(\omega)\cdot\psi)$$

in $\mathcal{S}$ almost surely, since for fixed n (say $n \le 3N$), the inner product $\langle e_n, e_k\psi\rangle$ is rapidly decreasing in k (using integration by parts). In particular we have almost surely L^1 convergence and hence (by taking upper limit of a sequence of sets)

$$\mathbb{P}(\|\eta(N^{-2}\boldsymbol{H})(\boldsymbol{H}^{\sigma/2} f(\omega)\cdot\psi)\|_{L^1} \ge cE_N) \ge \tfrac{1}{10},$$

where

$$E_N = \liminf_{M\to\infty} E_{M,N} \gtrsim \left(\sum_{k=0}^{\infty}(4k+2)^{\sigma-1}\|e_{k,N}\|_{L^1}^2\right)^{1/2}.$$

By the uniform boundedness of $\eta(N^{-2}\boldsymbol{H})$, we know $\eta(N^{-2}\boldsymbol{H})g \to g$ in L^1 for any $g \in L^1$, so we have

$$\liminf_{N\to\infty} E_N \gtrsim \left(\sum_{k=0}^{\infty}(4k+2)^{\sigma-1}\|e_k\psi\|_{L^1}^2\right)^{1/2} \gtrsim \left(\sum_{k=0}^{\infty}(4k+2)^{\sigma-3/2}\right)^{1/2} = \infty,$$

due to (A-3). Now we take another upper limit, and see that with probability $\ge \frac{1}{10}$, we have

$$\limsup_{N\to\infty}\|\eta(N^{-2}\boldsymbol{H})(\boldsymbol{H}^{\sigma/2} f(\omega)\cdot\psi)\|_{L^1} = \infty. \tag{A-4}$$

Now (A-4) implies (A-1), again because of the uniform boundedness of $\eta(N^{-2}\boldsymbol{H})$ on L^1. Therefore we have proved that (A-1) holds with positive probability. Since it is clearly a tail event (because $e_k \cdot \psi$ themselves are Schwartz functions), it must hold with probability one.

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SCHRÖDINGER OPERATORS AND THE DISTRIBUTION OF RESONANCES IN SECTORS

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The purpose of this paper is to give some refined results about the distribution of resonances in potential scattering. We use techniques and results from one and several complex variables, including properties of functions of completely regular growth. This enables us to find asymptotics for the distribution of resonances in sectors for certain potentials and for certain families of potentials.

1. Introduction

The purpose of this paper is to prove some results about the distribution of resonances in potential scattering. In particular, we study the distribution of resonances in sectors and give asymptotics of the "expected value" of the number of resonances in certain settings.

More precisely, we consider the operator $-\Delta+V$, where $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ and Δ is the (nonpositive) Laplacian. Then, except for a finite number of values of λ, $R_V(\lambda)=(-\Delta+V-\lambda^2)^{-1}$, $\operatorname{Im}\lambda>0$, is a bounded operator on $L^2(\mathbb{R}^d)$ for λ in the upper half-plane. When d is odd and $\chi \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ satisfies $\chi V=V$, $\chi R_V(\lambda)\chi$ has a meromorphic continuation to the lower half-plane. The poles of $\chi R_V(\lambda)\chi$ are called *resonances*, and are independent of choice of χ satisfying these hypotheses. Resonances are analogous to eigenvalues not only in their appearance as poles of the resolvent, but also because they appear in trace formulas much as eigenvalues do [Bardos et al. 1982; Guillopé and Zworski 1997; Melrose 1982]. Physically, they may be thought of as corresponding to decaying waves.

Let $n_V(r)$ denote the number of resonances of $-\Delta+V$, counted with multiplicity, with norm at most r. When $d=1$, asymptotics of $n_V(r)$ are known:

$$\lim_{r\to\infty}\frac{n_V(r)}{r}=\frac{2}{\pi}\operatorname{diam}(\operatorname{supp}(V))$$

[Zworski 1987]; see also [Froese 1997; Regge 1958; Simon 2000]. Moreover, "most" of the resonances occur in sectors about the real axis, in the sense that for any $\epsilon>0$,

$$\lim_{r\to\infty}\frac{\#\bigl\{\lambda_j \text{ pole of } R_V(\lambda) : |\arg\lambda_j-\pi|<\epsilon \text{ or } |\arg\lambda_j-2\pi|<\epsilon\bigr\}}{r}=\frac{2}{\pi}\operatorname{diam}(\operatorname{supp}(V))$$

[Froese 1997]. These results are valid for complex-valued V. The case $d=1$ is exceptional, though: in higher dimensions much less is known.

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Now we turn to $d \geq 3$ odd, where the question is more subtle. If $V \in L^\infty(\mathbb{R}^d)$ has support in $\overline{B}(0,a) = \{x \in \mathbb{R}^d : |x| \leq a\}$, then

$$d \int_0^r \frac{n_V(t) - n_V(0)}{t}\, dt \leq c_d a^d r^d + o(r^d), \tag{1-1}$$

where c_d is defined in (3-5) and depends only on the dimension. Zworski [1989a] showed that such a bound holds, and Stefanov [2006] identified the optimal constant c_d. There are examples for which equality holds in (1-1) [Zworski 1989b; Stefanov 2006]. Lower bounds have proved more elusive. The current best-known general quantitative lower bound is for nontrivial real-valued $V \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$:

$$\limsup_{r\to\infty} \frac{n_V(r)}{r} > 0 \tag{1-2}$$

[Sá Barreto 2001]. On the other hand, there are nontrivial complex-valued potentials V for which $\chi R_V(\lambda)\chi$ has no poles [Christiansen 2006].

We wish to single out the set for which asymptotics actually hold in (1-1). This is the set defined, for $a > 0$, as

$$\mathfrak{M}_a = \left\{ V \in L^\infty(\mathbb{R}^d) : \operatorname{supp} V \subset \overline{B}(0,a) \text{ and } n_V(r) = c_d a^d r^d + o(r^d) \text{ as } r \to \infty \right\}. \tag{1-3}$$

We remark that it is equivalent to require, as $r \to \infty$, that $n_V(r) = c_d a^d r^d + o(r^d)$ or

$$d \int_0^r t^{-1}\big(n_V(t) - n_V(0)\big)\, dt = c_d a^d r^d + o(r^d).$$

The set $\mathfrak{M}_a$ contains infinitely many radial potentials. By results from [Zworski 1989b; Stefanov 2006], this set contains any potential of the form $V(x) = v(|x|)$, where $v \in C^2([0,a])$ is real-valued, $v(a) \neq 0$, and $V(x) = 0$ for $|x| > a$. Additionally, it contains infinitely many complex-valued potentials which are isoresonant with these real-valued radial potentials [Christiansen 2008].

We now can state some results. For the first, we set, for $\varphi < \theta$, $n_V(r, \varphi, \theta)$ to be the set of poles of $R_V(\lambda)$, counted with multiplicity, with norm at most r and with argument between φ and θ inclusive.

Proposition 1.1. *Let $V \in \mathfrak{M}_a$. Then, if $0 < \varphi < \theta < \pi$,*

$$n_V(r, \pi+\varphi, \pi+\theta) = \frac{1}{2\pi d} \tilde{s}_d(\varphi,\theta) r^d a^d + o(r^d) \quad \text{as } r \to \infty,$$

where

$$\tilde{s}_d(\varphi,\theta) = h_d'(\theta) - h_d'(\varphi) + d^2 \int_\varphi^\theta h_d(s)\, ds,$$

and $h_d(\theta)$ is as defined in (3-4).

If V is real-valued, then λ_0 is a resonance of $-\Delta + V$ if and only if $-\overline{\lambda_0}$ is a resonance. In this case, for $V \in \mathfrak{M}_a$ and $0 < \theta < \pi$,

$$n_V(r, \pi, \pi+\theta) = \frac{1}{2\pi d} \left[h_d'(\theta) + d^2 \int_0^\theta h_d(s)\, ds \right] a^d r^d + o(r^d). \tag{1-4}$$

Here, as elsewhere in this paper, we are concerned with the behavior as $r \to \infty$. Thus, one should understand that statements of the type $f(r) = g(r) + o(r^p)$ are statements which hold for r sufficiently large.

Corollary 1.4 shows that (1-4) holds for any $V \in \mathfrak{M}_a$. These results show that any potential in $\mathfrak{M}_a$ must have resonances distributed regularly in sectors, as well as being distributed regularly in balls centered at the origin. A result like this proposition and Corollary 1.4 is, for the special potentials of the form $V(x) = v(|x|)$ mentioned earlier, implicit in [Zworski 1989b] and [Stefanov 2006]. Here we derive it as a corollary of some complex-analytic results, and note that it holds for *any* potential $V \in \mathfrak{M}_a$. We note that this proposition could, in fact, follow as a corollary to Theorem 1.3. However, we prefer to give a separate proof that uses standard results for functions of completely regular growth.

In the following theorem, we use the notation $N_V(r) = \int_0^r (1/t)\big(n_V(t) - n_V(0)\big)\,dt$ and $N_V(r, \varphi, \theta) = \int_0^r (1/t)\big(n_V(r, \varphi, \theta) - n_V(0, \varphi, \theta)\big)\,dt$. This theorem shows that there are many potentials for which something close to the optimal upper bound on the resonances is achieved.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^p$ be an open connected set. Suppose that $V(z) = V(z, x)$ is holomorphic in $z \in \Omega$, that $V(z, x) \in L^\infty(\mathbb{R}^d)$ for each $z \in \Omega$, and that $V(z, x) = 0$ if $|x| > a$. Suppose in addition that for some $z_0 \in \Omega$, $V(z_0) \in \mathfrak{M}_a$. Then there is a pluripolar set $E \subset \Omega$ so that*

$$\limsup_{r\to\infty} \frac{N_{V(z)}(r)}{r^d} = \frac{c_d a^d}{d} \quad \textit{for all } z \in \Omega \setminus E.$$

Moreover, for any $\theta > 0$, $\theta < \pi$, there is a pluripolar set E_θ so that

$$\limsup_{r\to\infty} \frac{N_{V(z)}(r, \pi, \pi+\theta)}{r^d} \geq \lim_{\epsilon\downarrow 0} \frac{a^d}{2\pi d^2} h'_d(\epsilon)$$

for all $z \in \Omega \setminus E_\theta$.

For example, for a family of potentials satisfying the conditions of the theorem, one may take, for $z \in \mathbb{C}$, $V(z) = zV_1 + (1-z)V_0$, where $V_0 \in \mathfrak{M}_a$ and $V_1 \in L^\infty(\mathbb{R}^d)$ have support in $\overline{B}(0, a)$. Since $h'_d(0+) = \lim_{\epsilon\downarrow 0} h'_d(\epsilon) > 0$ (see Lemma 3.3), the second statement of the theorem is meaningful. This result is of particular interest since resonances near the real axis are considered the more physically relevant ones.

We recall the definition of a pluripolar set in Section 2. Here we mention that a pluripolar set is small. A pluripolar set $E \subset \mathbb{C}^p$ has $\mathbb{R}^{2p}$ Lebesgue measure 0, and if $E \subset \mathbb{C}$ is pluripolar, $E \cap \mathbb{R}$ has one-dimensional Lebesgue measure 0 (see, for example, [Lelong and Gruman 1986; Ransford 1995]). Thus the statements of Theorem 1.2 hold for "most" values of $z \in \Omega$.

If we take a weighted average over a family of potentials, a kind of expected value, we are able to find asymptotics analogous to those which hold for a potential in $\mathfrak{M}_a$. In the statement of the next theorem and later in the paper, we use the notation $d\mathcal{L}(z) = d\,\mathrm{Re}\,z_1 d\,\mathrm{Im}\,z_1 \ldots d\,\mathrm{Re}\,z_p d\,\mathrm{Im}\,z_p$.

Theorem 1.3. *Suppose the hypotheses of Theorem 1.2 are satisfied. Then for any $\psi \in C_c(\Omega)$,*

$$\int_\Omega \psi(z) n_{V(z)}(r)\,d\mathcal{L}(z) = c_d a^d r^d \int_\Omega \psi(z)\,d\mathcal{L}(z) + o(r^d)$$

as $r \to \infty$. Additionally, we have, for $0 < \varphi < \theta < \pi$,

$$\int_\Omega \psi(z) n_{V(z)}(r, \varphi+\pi, \theta+\pi)\,d\mathcal{L}(z) = \frac{1}{2\pi d}\tilde{s}_d(\varphi, \theta) r^d a^d \int_\Omega \psi(z)\,d\mathcal{L}(z) + o(r^d),$$

where $\tilde{s}_d$ *is as defined in Proposition 1.1. Moreover, for* $0 < \theta < \pi$,

$$\int_\Omega \psi(z) n_{V(z)}(r, \pi, \theta+\pi)\, d\mathcal{L}(z) = \frac{1}{2\pi d}\left[h_d'(\theta) + d^2 \int_0^\theta h_d(s)\, ds\right] a^d r^d \int_\Omega \psi(z)\, d\mathcal{L}(z) + o(r^d).$$

Corollary 1.4. *Let* $V \in \mathfrak{M}_a$. *For any* $0 < \theta < \pi$,

$$n_V(r, \pi, \theta+\pi) = \frac{1}{2\pi d}\left[h_d'(\theta) + d^2 \int_0^\theta h_d(s)\, ds\right] a^d r^d + o(r^d) \tag{1-5}$$

and, for any $0 < \varphi < \pi$,

$$n_V(r, \varphi+\pi, 2\pi) = \frac{1}{2\pi d}\left[-h_d'(\varphi) + d^2 \int_\varphi^\pi h_d(s)\, ds\right] a^d r^d + o(r^d) \tag{1-6}$$

as $r \to \infty$.

This corollary follows from Theorem 1.3 by taking $V(z)$ equal to the constant (in z) potential V. We could instead give a more direct proof by, essentially, simplifying the proof of Proposition 5.3 and then applying Lemma 5.4.

It is worth noting that the coefficients of r^d in (1-5) and (1-6) are positive, so that in any sector in the lower half-plane which touches the real axis, the number of resonances grows like r^d.

The proofs of the results here are possible because of the optimal upper bounds on

$$\limsup_{r\to\infty} r^{-d} \ln\left|\det S_V(re^{i\theta})\right|,$$

$0 < \theta < \pi$, proved in [Stefanov 2006] (see Theorem 3.2 here). These, combined with some one-dimensional complex analysis, are used to prove Proposition 1.1, and could be used to give a direct proof of Corollary 1.4. The proofs of the other theorems use, in addition to one-dimensional complex analysis, some facts about plurisubharmonic functions. Many of the complex-analytic results which we shall use are recalled in Section 2.

Again, we emphasize that we are concerned here with large r behavior of resonance counting functions, and consequently of other functions as well. Thus, statements of the type $f(r) = g(r) + o(r^p)$ are to be understood as holding for large values of r.

2. Some complex analysis

In this section we recall some definitions and results from complex analysis in one and several variables. We will mostly follow the notation and conventions of [Levin 1964; Lelong and Gruman 1986]. We also prove a result, Proposition 2.2, for which we are unaware of a proof in the literature.

The *upper relative measure* of a set $E \subset \mathbb{R}_+$ is

$$\limsup_{r\to\infty} \frac{\operatorname{meas}(E \cap (0,r))}{r}.$$

A set $E \subset \mathbb{R}_+$ is said to have *zero relative measure* if it has upper relative measure 0.

If f is a function holomorphic in the sector $\varphi < \arg z < \theta$, we shall say f is of order ρ if

$$\limsup_{r\to\infty} \frac{\ln\ln\left(\max_{\varphi<\phi<\theta} |f(re^{i\phi})|\right)}{\ln r} = \rho.$$

We shall further restrict ourselves to functions of order ρ and *finite type*, so that

$$\limsup_{r\to\infty} \frac{\ln\left(\max_{\varphi<\phi<\theta} |f(re^{i\phi})|\right)}{r^\rho} < \infty.$$

We shall use similar definitions for a function holomorphic in a neighborhood of a closed sector. In this section only, we shall, following [Levin 1964], use the notation h_f for the *indicator function* (or *indicator*) of a function f of order ρ:

$$h_f(\theta) \overset{\text{def}}{=} \limsup_{r\to\infty} \left(r^{-\rho} \ln|f(re^{i\theta})|\right).$$

Suppose f is a function analytic in the angle (θ_1, θ_2) and of order ρ and finite type there. The function f is of *completely regular growth* on some set of rays $R_{\mathfrak{M}}$ ($\mathfrak{M}$ is the set of values of θ) if the function

$$h_{f,r}(\theta) \overset{\text{def}}{=} \frac{\ln|f(re^{i\theta})|}{r^\rho}$$

converges uniformly to $h_f(\theta)$ for $\theta \in \mathfrak{M}$ when r goes to infinity taking on all positive values except possibly for a set $E_{\mathfrak{M}}$ of zero relative measure. The function f is of *completely regular growth in the angle* (θ_1, θ_2) if it is of completely regular growth on every closed interior angle.

Functions of completely regular growth have zeros that are rather regularly distributed. For a function f holomorphic in $\{z : \theta_1 < \arg z < \theta_2\}$ we define $m_f(r, \varphi, \theta)$, for $\theta_1 < \varphi < \theta < \theta_2$, to be the number of zeros of $f(z)$ in the sector $\varphi \le \arg z \le \theta$, $|z| \le r$.[1]

Theorem 2.1 [Levin 1964, Chapter III, Theorem 3]. *If a holomorphic function $f(z)$ of order d and finite type has completely regular growth within an angle (θ_1, θ_2), then for all values of φ and θ with $\theta_1 < \varphi < \theta < \theta_2$, except possibly for a denumerable set, the following limit will exist*:

$$\lim_{r\to\infty} \frac{m_f(r, \varphi, \theta)}{r^d} = \frac{1}{2\pi d}\tilde{s}_f(\varphi, \theta),$$

where

$$\tilde{s}_f(\varphi, \theta) = \left[h'_f(\theta) - h'_f(\varphi) + d^2 \int_\varphi^\theta h_f(s)\, ds\right].$$

The exceptional denumerable set can only consist of points for which $h'_f(\theta + 0) \ne h'_f(\theta - 0)$.

In the following proposition, we use the notation $m_f(r)$ to denote the number of zeros of a function f, counted with multiplicity, with norm at most r. It is likely that some of the hypotheses included here could be relaxed. However, when we apply this proposition, f will be the determinant of the scattering matrix, perhaps multiplied by a rational function, and many of these hypotheses are natural in such applications.

[1] More standard notation would be $n(r, \varphi, \theta)$, but we have already defined $n_V(r, \varphi, \theta)$ to be something else.

Let $f(z)$ be a function meromorphic on $\mathbb{C}$. Then $f(z) = g_1(z)/g_2(z)$, with g_1, g_2 entire. The functions g_1 and g_2 are not uniquely determined. However, the order of f can be defined to be

$$\min\{\max(\text{order of } g_1, \text{order of } g_2) \colon f(z) = g_1(z)/g_2(z) \text{ with } g_1, g_2 \text{ entire}\}.$$

It is possible to define the order of a meromorphic function by using the Nevanlinna characteristic function, yielding the same result.

Proposition 2.2. *Let f be a function meromorphic in the complex plane, with neither zeros nor poles on the real line. Suppose all the zeros of f lie in the open upper half-plane, and all the poles in the open lower half-plane. Furthermore, assume f is of order $d > 1$, h_f is finite for $0 \le \theta \le \pi$, and $h_f(\theta_0) \neq 0$ for some θ_0, $0 < \theta_0 < \pi$. Suppose in addition that*

$$\int_0^r \frac{f'(t)}{f(t)}\, dt = o(r^d) \quad \text{as } r \to \pm\infty, \tag{2-1}$$

and that the number of poles of f with norm at most r is of order at most d. If

$$\liminf_{r\to\infty} \frac{m_f(r)}{r^d} = \frac{d}{2\pi}\int_0^\pi h_f(\theta)\, d\theta,$$

then f is of completely regular growth in the angle $(0, \pi)$.

Before proving the proposition, we note that Govorov [1965; 1967] has studied the issue of completely regular growth of functions holomorphic in an angle. This is discussed in [Levin 1964, Appendix VIII, Section 2]. This is somewhat different than what we consider, since we use the assumption that f is meromorphic and of order d on the plane. Thus Govorov uses different restrictions on the distribution of the zeros of f.

Proof. The proof of this proposition follows in outline the proof of the analogous theorem for entire functions in the plane [Levin 1964, Chapter IV, Theorem 3]. Rather than using Jensen's theorem, though, it uses the equality

$$\int_0^r \frac{m_f(t)}{t}\, dt = \frac{1}{2\pi}\operatorname{Im}\int_0^r \frac{1}{t}\int_{-t}^t \frac{f'(s)}{f(s)}\, ds\, dt + \frac{1}{2\pi}\int_0^\pi \ln|f(re^{i\theta})|\, d\theta \tag{2-2}$$

if $|f(0)| = 1$, which follows using the proof of [Froese 1998, Lemma 6.1].

By [Levin 1964, Property (4), Chapter I, Section 12],

$$\liminf_{r\to\infty} \frac{m_f(r)}{r^d} \le \liminf_{r\to\infty} dr^{-d}\int_0^r \frac{m_f(t)}{t}\, dt. \tag{2-3}$$

We note [ibid., Chapter I, Theorem 28] that for any $\epsilon > 0$, there is an $R > 0$ so that

$$r^{-d}\ln|f(re^{i\theta})| \le h_f(\theta) + \epsilon, \quad \text{for} \quad r > R,\ 0 \le \theta \le \pi. \tag{2-4}$$

Using this, (2-2), and our assumptions on the behavior of f on the real axis, we see that

$$\limsup_{r\to\infty} r^{-d}\int_0^r \frac{m_f(t)}{t}\, dt \le \frac{1}{2\pi}\int_0^\pi h_f(\theta)\, d\theta.$$

Combining this with (2-3) and using our assumptions on $m_f(r)$, we get

$$\lim_{r\to\infty} r^{-d}\int_0^r \frac{m_f(t)}{t}\,dt = \frac{1}{2\pi}\int_0^\pi h_f(\theta)\,d\theta.$$

Thus using (2-2) and (2-1) again, we have

$$\lim_{r\to\infty}\int_0^\pi \big[h_f(\theta) - r^{-d}\ln|f(re^{i\theta})|\big]\,d\theta = 0,$$

and, using (2-4),

$$\lim_{r\to\infty}\int_0^\pi \big|h_f(\theta) - r^{-d}\ln|f(re^{i\theta})|\big|\,d\theta = 0.$$

Since we have assumed f is of order d, we may write f as the quotient of two entire functions, each of order at most d. Then we may apply [Levin 1964, Chapter 2, Theorem 7] to find that for every $\eta > 0$, there is a set E_η of positive numbers of upper relative measure less than η so that if $r \notin E_\eta$, the family of functions of θ,

$$h_{f,r}(\theta) \stackrel{\text{def}}{=} r^{-d}\ln|f(re^{i\theta})|,$$

is equicontinuous in the angle $0 < \epsilon_0 \le \theta \le \pi - \epsilon_0$.

Now let $\theta_2 > \theta_1$, with $[\theta_1, \theta_2] \subset (0, \pi)$. Given $\eta > 0$ and $\epsilon > 0$ we can, by the above result, find a $\delta > 0$ with $[\theta_1 - \delta, \theta_2 + \delta] \subset (0, \pi)$ and a set E_η of upper relative measure at most η so that if $\theta \in [\theta_1, \theta_2]$, $r \notin E_\eta$, and $|\varphi - \theta| < \delta$, then $|h_{f,r}(\theta) - h_{f,r}(\varphi)| < \epsilon/4$ and $|h_f(\theta) - h_f(\varphi)| < \epsilon/4$. Then for $0 < |k| < \delta$, $r \notin E_\eta$, and $\theta \in [\theta_1, \theta_2]$,

$$\begin{aligned} |h_{f,r}(\theta) - h_f(\theta)| &< \frac{\epsilon}{2} + \frac{1}{k}\int_\theta^{\theta+k} |h_{f,r}(\varphi) - h_f(\varphi)|\,d\varphi \\ &\le \frac{\epsilon}{2} + \frac{1}{k}\int_0^{\pi} |h_{f,r}(\varphi) - h_f(\varphi)|\,d\varphi. \end{aligned}$$

Since the integral goes to 0 as $r \to \infty$, we have shown that $|h_{f,r}(\theta) - h_f(\theta)| < \epsilon$ for $r > r_\epsilon$, $r \notin E_\eta$. Since $\eta > 0$ and $\epsilon > 0$ are arbitrary, we have, by [Levin 1964, Chapter III, Lemma 1], that f is of completely regular growth in $[\theta_1, \theta_2]$. Since θ_1, θ_2 were arbitrary except that $[\theta_1, \theta_2] \subset (0, \pi)$, we have proved the proposition. □

We shall also need some basics about plurisubharmonic functions and pluripolar sets. We use notation as in [Lelong and Gruman 1986] and direct the reader to this reference for more details.

Let $\Omega \subset \mathbb{C}^p$ be an open connected set. A function $\Psi : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if $\Psi \not\equiv -\infty$, Ψ is upper semicontinuous, and

$$\Psi(z) \le \frac{1}{2\pi}\int_0^{2\pi} \Psi(z + wre^{i\theta})\,d\theta$$

for all w, r such that $z + uw \in \Omega$ for all $u \in \mathbb{C}$, $|u| \le r$. A classic example of a plurisubharmonic function is $\ln|f(z)|$, where $f(z)$ is holomorphic. A subset $E \subset \Omega \subset \mathbb{C}^p$ is said to be *pluripolar* if there is a function Ψ plurisubharmonic on Ω so that $E \subset \{z : \Psi(z) = -\infty\}$.

For the convenience of the reader, we recall an additional fact from several complex variables that we shall need.

Proposition 2.3 [Lelong and Gruman 1986, Proposition 1.39]. *Let $\{\Psi_q\}$ be a sequence of plurisubharmonic functions uniformly bounded above on compact subsets in an open connected set $\Omega \subset \mathbb{C}^p$, with $\limsup_{q\to\infty} \Psi_q \le 0$, and suppose that there exist $\xi \in \Omega$ such that $\limsup_{q\to\infty} \Psi_q(\xi) = 0$. Then $A = \{z \in \Omega : \limsup_{q\to\infty} \Psi_q(z) < 0\}$ is pluripolar in Ω.*

3. The functions $s_V(\lambda) = \det S_V(\lambda)$ and $h_d(\theta)$

For $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ and $\chi \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ with $\chi V = V$, we have $\chi R_V(\lambda)\chi = \chi R_0(\lambda)\chi(I + V R_0(\lambda)\chi)^{-1}$. Since for any χ with compact support in $\mathbb{R}^d$, $\|\chi R_0(\lambda)\chi\| \le c_\chi/|\lambda|$ when $\operatorname{Im}\lambda \ge 0$, we see that $R_V(\lambda)$ can have only finitely many poles in the closed upper half-plane.

For $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$, let $S_V(\lambda)$ be the associated scattering matrix and $s_V(\lambda) = \det S_V(\lambda)$. With at most finitely many exceptions, the poles of $s_V(\lambda)$ coincide with the poles of $R_V(\lambda)$, and the multiplicities agree. Moreover, $s_V(\lambda)s_V(-\lambda) = 1$.

Lemma 3.1 [Christiansen 2005, Lemma 3.1]. *Let $V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C})$. For $\lambda \in \mathbb{R}$, there is a C_V so that*

$$\left|\frac{d}{d\lambda} \ln s_V(\lambda)\right| \le C_V |\lambda|^{d-2}$$

whenever $|\lambda|$ is sufficiently large.

In fact, if $\operatorname{supp} V \subset \overline{B}(0, a)$, there is a constant $\alpha_d = \alpha_{d,a}$, so that it suffices to take $|\lambda| \ge 2\alpha_d \|V\|_\infty$ for such a bound to hold. We note that for $\lambda \in \mathbb{R}$, $|\lambda| \ge 2\alpha_d\|V\|_\infty$, under these same assumptions on V,

$$\|S_V(\lambda) - I\| \le C|\lambda|^{-1}. \tag{3-1}$$

This is relatively easy to see from an explicit representation of the scattering matrix; see, for example, the proof of the lemma just stated in [Christiansen 2005]. The constants in the statement of that lemma and in (3-1) can be chosen to depend only on the dimension, $\|V\|_\infty$, and the support of V. We note that it follows from Lemma 3.1, (3-1), and (2-2) that as $r \to \infty$,

$$\int_0^r \frac{n_V(t)}{t}\, dt = \int_0^\pi \ln\left|\det S_V(re^{i\theta})\right| d\theta + O(r^{d-1}). \tag{3-2}$$

Let

$$\rho(z) \stackrel{\text{def}}{=} \ln \frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}, \quad 0 < \arg z < \pi. \tag{3-3}$$

This is a function which arises in studying the asymptotics of Bessel functions; see [Olver 1954]. To define the square root which appears here, take the branch cut on the negative real axis and define ρ to be a continuous function in $\{0 < \arg z < \pi\} \cup (0, 1)$ and use the principal branches of the logarithm and the square root when $z \in (0, 1)$.

We use some notation from [Stefanov 2006]. Set, for $0 < \theta < \pi$,

$$h_d(\theta) \stackrel{\text{def}}{=} \frac{4}{(d-2)!} \int_0^\infty \frac{[-\operatorname{Re}\rho]_+(te^{i\theta})}{t^{d+1}}\, dt \tag{3-4}$$

and set $h_d(0)=0,\ h_d(\pi)=0$. Further, define

$$c_d \stackrel{\text{def}}{=} \frac{d}{2\pi}\int_0^\pi h_d(\theta)\,d\theta = \frac{2d}{\pi(d-2)!}\int_{\operatorname{Im} z>0} \frac{[-\operatorname{Re}\rho]_+(z)}{|z|^{d+2}}\,dx\,dy. \tag{3-5}$$

This is the constant c_d that appears in (1-1).

The next result is adapted from [Stefanov 2006, Theorem 5]; the original result covers a much larger class of operators.

Theorem 3.2. *Let $V \in L^\infty(\mathbb{R}^d)$ be supported in $\overline{B}(0,a)$.*

(a) *For any $\theta \in [0,\pi]$,*

$$\ln|s_V(re^{i\theta})| \le h_d(\theta)a^d r^d + o(r^d) \quad \text{as } r\to\infty, \tag{3-6}$$

and the remainder term depends on V and is uniform for $0<\delta\le\theta\le\pi-\delta$ for any $\delta\in(0,\pi)$.

(b) *For any $\delta>0$,*

$$\ln|s_V(re^{i\theta})| \le (h_d(\theta)a^d+\delta)r^d + o(r^d) \quad \text{as } r\to\infty$$

uniformly in $\theta\in[0,\pi]$.

We remark that both of these statements are about "large r" behavior, so that the possibility that s_V has a finite number of poles in the upper half-plane does not affect the validity of the statements.

It is important to note several things about the bounds in this theorem. One is that although Stefanov's theorem is stated only for self-adjoint operators (hence V real), it is equally valid when we allow complex-valued potentials. In fact, the proof of (a) in [Stefanov 2006, Theorem 5] uses self-adjointness only to obtain a bound on the resolvent for λ in the upper half-plane. A similar bound is true for the operator $-\Delta+V$ when V is complex-valued. The proof of (b) uses the fact that $\ln|s_V(\lambda)|=1$ for real V and $\lambda\in\mathbb{R}$. For complex-valued V, the proof in [Stefanov 2006] of (b) can be adapted by using (3-1) and Lemma 3.1 to show that $\bigl|\ln|s_V(\lambda)|\bigr| \le C(1+|\lambda|)^{d-1}$ for $\lambda\in\mathbb{R}$ with $|\lambda|\ge 2\alpha_d\|V\|_\infty$. Here C can be chosen to depend only on d, $\|V\|_\infty$, and the diameter of the support of V.

Likewise, the particulars of the operator enter only through the diameter of the support of the perturbation (for us, the diameter of the support of V, which is $2a$) and the aforementioned bound on the resolvent in the good half-plane $\operatorname{Im}\lambda>0$. Thus, it is easy to see that the estimates of Theorem 3.2 are uniform in V as long as $\operatorname{supp} V\subset \overline{B}(0,a)$, $\|V\|_\infty\le M$, and $r\ge 2\alpha_d M$.

We note that the upper bound (1-1) on the integrated resonance-counting function holds with the constant c_d defined in (3-5) even if V is complex-valued. This follows from the proof in [Stefanov 2006]. In fact, the proof uses the bounds recalled in Theorem 3.2 and the identity (2-2). Together with the bounds in Lemma 3.1 and (3-1), these prove (1-1), even when V is complex-valued.

We shall want to understand the function $h_d(\theta)$ better. Note that for $0<\theta\le\pi/2$,

$$h_d\Bigl(\frac{\pi}{2}+\theta\Bigr) = h_d\Bigl(\frac{\pi}{2}-\theta\Bigr).$$

This can be seen directly using the definition of h_d and ρ.

Lemma 3.3. *The function $h_d(\theta)$, defined in (3-4), is C^1 on $(0,\pi)$. Moreover,*

$$h_d'(0+) \overset{\text{def}}{=} \lim_{\epsilon\downarrow 0} h_d'(\epsilon) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{d-1}{2}\right)}{(d-2)!\,\Gamma\left(1+\frac{d}{2}\right)}.$$

Proof. We note [Olver 1954, Section 4] that $\operatorname{Re}\rho(z) < 0$ if $0 < \arg z < \pi$ and $|z| > |z_0(\arg z)|$, where $z_0(\theta)$ is the unique point in $\mathbb{C}$ with argument θ and which lies on the curve given by

$$\pm(s\coth s - s^2)^{1/2} + i(s^2 - s\tanh s)^{1/2}, \quad 0 \le s \le s_0.$$

Here s_0 is the positive solution of $\coth s = s$. Furthermore, $\operatorname{Re}\rho(z) > 0$ if z is in the upper half-plane but $|z| < |z_0(\arg z)|$. Hence, recalling the definition of h_d, we have

$$h_d(\theta) = \frac{4}{(d-2)!}\int_{|z_0(\theta)|}^{\infty} \frac{[-\operatorname{Re}\rho](te^{i\theta})}{t^{d+1}}\,dt.$$

Using the definition of ρ in (3-3) and the following comments, we see that ρ is in fact a smooth function of z with $0 < \arg z < \pi$, $|z| > 0$. Since $|\rho(z)|/|z| \to 1$ when $|z| \to \infty$ in this region, the integral defining h_d is absolutely convergent. Likewise, since

$$\frac{\partial}{\partial\theta}\rho(te^{i\theta}) = -i\sqrt{1-(te^{i\theta})^2},$$

we have

$$\left|\frac{-\operatorname{Re}\left[\frac{\partial}{\partial\theta}\rho(te^{i\theta})\right]}{t^{d+1}}\right| \le Ct^{-d},$$

and the integral

$$\int_{|z_0(\theta)|}^{\infty} \frac{-\operatorname{Re}\left[\frac{\partial}{\partial\theta}\rho(te^{i\theta})\right]}{t^{d+1}}\,dt$$

converges absolutely. A computation shows that $|z_0|$ is a C^1 function of θ for θ in $(0,\pi)$, and that $\lim_{\epsilon\downarrow 0}(\partial/\partial\theta)|z_0|$ is finite. Thus, using that $\operatorname{Re}\rho(z_0(\theta)) = 0$ and the regularity of the derivative of $|z_0|(\theta)$, we get

$$\frac{d}{d\theta}h_d(\theta) = \frac{4}{(d-2)!}\int_{|z_0(\theta)|}^{\infty} \frac{\operatorname{Re} i\sqrt{1-(te^{i\theta})^2}}{t^{d+1}}\,dt,$$

which is continuous in θ. Thus h_d is C^1 on $(0,\pi)$, we have

$$h_d'(0+) = \frac{4}{(d-2)!}\int_1^{\infty} \frac{\sqrt{t^2-1}}{t^{d+1}}\,dt,$$

and a computation now finishes the proof of the lemma. □

If $d = 3$, we can compute that

$$h_3(\theta) = \tfrac{4}{9}\left(\sin(3\theta) + \operatorname{Re}\frac{(1-z_0^2(\theta))^{3/2}}{|z_0(\theta)|^3}\right),$$

where $z_0(\theta)$ is as in the proof of the lemma. We comment that the $\sin(3\theta)$ term is missing from the first remark following the statement of [Stefanov 2006, Theorem 5].

4. Proof of Proposition 1.1

We can now give the proof of Proposition 1.1, which follows by combining Theorem 2.1, Proposition 2.2, and [Stefanov 2006, Theorem 5].

Recall that $S_V(\lambda)$ is the scattering matrix associated with the operator $-\Delta+V$, and $s_V(\lambda)=\det S_V(\lambda)$. Then s_V has a pole at λ if and only if s_V has a zero at $-\lambda$, and the multiplicities coincide. Moreover, with at most a finite number of exceptions, the poles of $s_V(\lambda)$ coincide, with multiplicity, with the zeros of $R_V(\lambda)$.

If $s_V(\lambda)$ has poles in the closed upper half-plane, it has only finitely many, say $\lambda_1,\dots,\lambda_m$, where the poles are repeated according to multiplicity. Set

$$f(\lambda)=\prod_{j=1}^{m}\frac{(\lambda-\lambda_j)}{\lambda+\lambda_j}s_V(\lambda).$$

We check that f satisfies the hypotheses of Proposition 2.2. Note that f and $s_V(\lambda)$ have the same order and they have the same indicator function for $0\le\theta\le\pi$. We know that s_V has order at most d by [Zworski 1997, Theorem 7]. Moreover, for any M chosen large enough that s_V has no zeros or poles bigger than M on the real line, for $r>M$ we have

$$\int_0^r\frac{f'(t)}{f(t)}\,dt=\int_M^r\frac{s_V'(t)}{s_V(t)}\,dt+O(1).$$

Using (3-1) and Lemma 3.1, we see that

$$\int_M^r\frac{s_V'(t)}{s_V(t)}\,dt=O(r^{d-1})\quad\text{as } r\to\infty,$$

yielding

$$\int_0^r\frac{f'(t)}{f(t)}\,dt=O(r^{d-1})\quad\text{as } r\to\infty. \tag{4-1}$$

A similar argument gives the same bound for $r\to-\infty$. It remains to check the hypotheses on the indicator function; this is done in the next paragraph.

From [Stefanov 2006, Theorem 5], recalled here in Theorem 3.2, for $0\le\theta\le\pi$ and large r,

$$r^{-d}\ln|f(re^{i\theta})|\le a^d h_d(\theta)+o(1),$$

where we have some uniformity in θ. Thus, using (2-2) and (4-1), we get

$$\limsup_{r\to\infty} r^{-d}N_V(r)=\limsup_{r\to\infty} r^{-d}\frac{1}{2\pi}\int_0^\pi \ln|f(re^{i\theta})|d\theta\le\frac{a^d}{2\pi}\int_0^\pi h_d(\theta)d\theta.$$

But since $V\in\mathfrak{M}_a$,

$$\lim_{r\to\infty} r^{-d}N_V(r)=\frac{c_d a^d}{d}=\frac{a^d}{2\pi}\int_0^\pi h_d(\theta)\,d\theta,$$

and we see that we must have

$$\limsup_{r\to\infty} r^{-d}\ln|f(re^{i\theta})|=a^d h_d(\theta),\quad\text{for almost every } \theta\in(0,\pi).$$

The left-hand side of the above equation is the value of the indicator function of f at θ. But the indicator function of f is continuous on $(0,\pi)$ [Levin 1964, Section 16, point (a) on p. 54], and so is $h_d(\theta)$. Thus we must have

$$\limsup_{r\to\infty} r^{-d}\ln|f(re^{i\theta})| = a^d h_d(\theta) \quad \text{for } \theta\in(0,\pi).$$

Applying Proposition 2.2 to $f(\lambda)$, we see that $f(\lambda)$ is a function of completely regular growth in the upper half-plane. Since $h_d(\theta)$ is a C^1 function of θ for $\theta\in(0,\pi)$, we get the proposition from Theorem 2.1.

5. Proof of Theorem 1.3

This section proves Theorem 1.3. We begin by outlining the strategy of the proof.

For $0<\varphi<\theta<2\pi$, recall the notation $n_V(r,\varphi,\theta)$ for the number of poles of $R_V(\lambda)$ in the sector $\{z : |z|\le r,\ \varphi\le \arg z\le\theta\}$. A representative claim of the theorem is that with $V(z)$, Ω as in the statement of the theorem, $0<\theta<\pi$,

$$\int_\Omega \psi(z) n_{V(z)}(r,\pi,\theta+\pi)\, d\mathcal{L}(z) = \frac{1}{2\pi d}\left[h_d'(\theta)+d^2\int_0^\theta h_d(s)\,ds\right] a^d r^d \int_\Omega \psi(z)\, d\mathcal{L}(z) + o(r^d) \quad (5\text{-}1)$$

as $r\to\infty$ for any $\psi\in C_c(\Omega)$. We prove this via the intermediate step of showing that (5-1) holds for ψ which is the characteristic function of any suitable ball in Ω (Proposition 5.7). To get (5-1) for $\psi\in C_c(\Omega)$, we cover the support of ψ with the union of a finite number of small disjoint balls and a set of small volume. On each small ball, we can approximate ψ by its value at the center of the ball and apply Proposition 5.7. This and the necessary estimates are done in the proof of the theorem which ends this section.

The proof of Proposition 5.7 is done in a number of steps. We set

$$N_V(r,\varphi,\theta) = \int_0^r \frac{1}{t}\big(n_V(t,\varphi,\theta)-n_V(0,\varphi,\theta)\big)\,dt.$$

Lemma 5.2 gives $\int_0^\theta N_V(r,\pi,\theta'+\pi)\,d\theta'$ as a sum of two integrals involving $\ln|s_V|$ and an error of order r^{d-1}. This follows from an application of one-dimensional complex analysis, Lemma 3.1, and (3-1). Next we consider the function

$$\Psi(z,r,\rho) \overset{\text{def}}{=} \frac{1}{\operatorname{vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta N_{V(z')}(r,\pi,\theta'+\pi)\,d\theta'\,d\mathcal{L}(z').$$

Here we use $B(z,\rho)$ to be the ball with center z and radius ρ in $\mathbb{C}^p$. Thus the function Ψ is the average over balls of varying center z. Fix ρ small, and consider this as a function of z and r. Lemma 5.2 is used to show that Ψ is the sum of a function Ψ_1 which is plurisubharmonic in z and a function which is $O(r^{d-1})$. The proof of Proposition 5.3 uses a combination of properties of plurisubharmonic functions and the fact that $r^{-d}N_{V(z')}(r,\pi,\theta'+\pi)$ is not negative and can be (locally) uniformly bounded above for large r to prove an "averaged" in θ and r version of (5-1) for ψ the characteristic function of a ball in Ω

satisfying some conditions. Propositions 5.5 and then 5.7 eliminate the need to average in θ and r, using Lemma 5.4.

The proofs of the other claims of Theorem 1.3 are quite similar; the proof of Proposition 5.6 and the final proof of the theorem indicate the differences.

Now we turn to proving the theorem. We shall need an identity related to both (2-2) and to what Levin [1964, Chapter 3, Section 2] calls a generalized formula of Jensen. We define, following [Levin 1964], for a function f meromorphic in a neighborhood of $\arg z = \theta$ and with $|f(0)| = 1$,

$$J_f^r(\theta) \stackrel{\text{def}}{=} \int_0^r \frac{\ln|f(te^{i\theta})|}{t}\,dt. \tag{5-2}$$

This integral is well-defined even if f has a zero or pole with argument θ.

Lemma 5.1. *Let f be holomorphic in $\varphi \le \arg z \le \theta$, let $|f(0)| = 1$, let f have no zeros with argument φ or θ and with norm less than r, and let $m(r,\varphi,\theta)$ be the number of zeros of f in the sector $\varphi < \arg z < \theta$, $|z| \le r$. Then*

$$\int_0^r \frac{m(t,\varphi,\theta)}{t}\,dt$$
$$= \frac{1}{2\pi}\int_0^r \frac{\partial}{\partial\theta} J_f^t(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^r \frac{1}{t}\int_0^t \frac{\partial}{\partial s}\arg f(se^{i\varphi})\,ds\,dt + \frac{1}{2\pi}\int_\varphi^\theta \ln|f(re^{i\omega})|\,d\omega. \tag{5-3}$$

Proof. Using the argument principle and the Cauchy–Riemann equations just as in [Levin 1964, Chapter 3, Section 2], we see that

$$2\pi m(r',\varphi,\theta) = \int_0^{r'} \frac{\partial}{\partial t}\arg f(te^{i\varphi})\,dt + \int_0^{r'} \frac{1}{t}\frac{\partial}{\partial\theta}\ln|f(te^{i\theta})|\,dt + r'\int_\varphi^\theta \frac{\partial}{\partial r'}\ln|f(r'e^{i\omega})|\,d\omega$$

when there are no zeros on the boundary of the sector. As in [Levin 1964], by dividing by $2\pi r'$ and integrating from 0 to r in r', we obtain the lemma. □

We note that $|s_V(0)| = 1$, since $s_V(\lambda)s_V(-\lambda) = 1$.

Lemma 5.2. *Suppose $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$. Then for $0 < \theta < \pi$,*

$$\int_0^\theta N_V(r,\pi,\theta'+\pi)\,d\theta' = \frac{1}{2\pi}\int_0^r J_{s_V}^t(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_V(re^{i\omega})|\,d\omega\,d\theta' + O(r^{d-1})$$

as $r \to \infty$. The error can be bounded by $c\langle r^{d-1}\rangle$, where the constant depends only on $\|V\|_\infty$, the support of V, and d.

Proof. Recall that with at most a finite number of exceptions, λ_0 is a pole of $R_V(\lambda)$ if and only if $-\lambda_0$ is a zero of $s_V(\lambda)$, and the multiplicities coincide. As in the proof of Proposition 1.1, if $s_V(\lambda)$ has poles $\lambda_1,\dots,\lambda_m$ in the closed upper half-plane, we introduce the function

$$f(\lambda) = \frac{(\lambda-\lambda_1)\dots(\lambda-\lambda_m)}{(\lambda+\lambda_1)\dots(\lambda+\lambda_m)}s_V(\lambda),$$

which is holomorphic in the closed upper half-plane. The poles of s_V in the closed upper half-plane correspond to eigenvalues, and the number of such poles can be bounded by a constant depending on d, $\|V\|_\infty$, and the support of V. Note that f has no zeros on the real line and that s_V and f have all but finitely many of the same zeros. Moreover, $\ln|f(re^{i\theta})| = \ln|s_V(re^{i\theta})| + O(1)$ for $r \to \infty$, $0 \le \theta \le \pi$.

Choose $0 < M < \infty$ so that $s_V(\lambda)$ has no zeros in the upper half-plane with norm greater than or equal to M. This constant M can be chosen to depend only on $\|V\|_\infty$, the support of V, and d. Now, by using the relationship between the poles of $R_V(\lambda)$ and the zeros of $s_V = \det S_V$ and the relationships between f and s_V just mentioned, and applying Lemma 5.1 to f, we see that for $r > M$, $0 < \theta' < \pi$,

$$N_V(t,\pi,\theta'+\pi) = \frac{1}{2\pi}\int_M^r \frac{\partial}{\partial\theta'} J^t_{s_V}(\theta')\frac{dt}{t} + \frac{1}{2\pi}\int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg s_V(t')\,dt'dt + \frac{1}{2\pi}\int_0^{\theta'} \ln|s_V(re^{i\omega})|\,d\omega + O((\ln r)^2) \qquad (5\text{-}4)$$

if f has no zeros with argument θ' and norm not exceeding r. Here we are using that

$$\int_0^M \frac{\partial}{\partial\theta'} J^t_f(\theta')\frac{dt}{t} = O(1)$$

and

$$\int_0^r \frac{1}{t}\int_0^t \frac{d}{dt'}\arg f(t')\,dt'dt = \int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg f(t')\,dt'dt + \int_M^r \frac{1}{t}\int_0^M \frac{d}{dt'}\arg f(t')\,dt'dt + \int_0^M \frac{1}{t}\int_0^t \frac{d}{dt'}\arg f(t')\,dt'dt.$$

But

$$\int_M^r \frac{1}{t}\int_0^M \frac{d}{dt'}\arg f(t')\,dt'dt = O(\ln r) \quad \text{and} \quad \int_0^M \frac{1}{t}\int_0^t \frac{d}{dt'}\arg f(t')\,dt'dt = O(1).$$

Additionally, for $t \to \infty$,

$$\frac{d}{dt}\arg f(t) = \frac{d}{dt}\arg s_V(t) + O\Big(\frac{1}{t}\Big).$$

These remainders can be bounded using constants depending only on $\|V\|_\infty$, $\operatorname{supp} V$, and d.

Notice that for fixed value of $r > M$, there are only finitely many values of θ' with s_V having a zero with argument θ' and norm at most r. We integrate (5-4) in θ' from 0 to θ and, as in the proof of Jensen's equality, use the fact that both sides of the equation below are continuous functions of θ, to get

$$\int_0^\theta N_V(r,\pi,\theta'+\pi)\,d\theta' = \frac{1}{2\pi}\int_M^r J^t_{s_V}(\theta)\frac{dt}{t} - \frac{1}{2\pi}\int_M^r J^t_{s_V}(0)\frac{dt}{t} + \frac{\theta}{2\pi}\int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg s_V(t')\,dt'dt + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_V(re^{i\omega})|\,d\omega\,d\theta' + O((\ln r)^2).$$

The bounds of Lemma 3.1 and (3-1) mean that, as $r \to \infty$,

$$\frac{1}{2\pi}\int_M^r J^t_{s_V}(0)\frac{dt}{t} = O(r^{d-1})$$

and
$$\frac{\theta}{2\pi}\int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg s_V(t')\,dt'dt = O(r^{d-1}),$$
where the bounds can be made uniform in V with support contained in a fixed compact set and $\|V\|_\infty$ bounded. Moreover, we note that $\int_0^M J^t_{S_V}(\theta)(dt/t) = O(1)$. □

We shall need some notation for the results which follow. Let $\Omega \subset \mathbb{C}^{d'}$ be an open set containing a point z_0. For $\rho > 0$ small enough that $B(z_0,\rho)\subset\Omega$, we define Ω_ρ to be the connected component of $\{z\in\Omega : \operatorname{dist}(z,\Omega^c) > \rho\}$ which contains z_0.

Proposition 5.3. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, let $\rho > 0$ be small enough that $B(z_0, 2\rho)\subset\Omega$, and let Ω_ρ be as defined above. Then, for $z\in\Omega_{2\rho}$, $0<\theta<\pi$,*
$$\begin{aligned}\Psi(z,r,\rho) &\stackrel{\text{def}}{=} \frac{1}{\operatorname{vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta N_{V(z')}(r,\pi,\theta'+\pi)\,d\theta' d\mathcal{L}(z')\\ &= \frac{1}{2\pi}a^d r^d\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)+o(r^d)\end{aligned}$$
as $r\to\infty$.

Proof. First note that since $0\le dN_{V(z)}(z,\pi,\theta+\pi)\le c_d r^d a^d + o(r^d)$, and the bound is uniform on compact sets of z, we get that holding ρ fixed, $r^{-d}\Psi(\,\cdot\,,r,\rho)$ is a family uniformly continuous in z for z in compact sets of $\overline{\Omega}_{2\rho}$.

We shall use Lemma 5.2. Note that by Stefanov's results recalled in Theorem 3.2, for large r,
$$\frac{1}{2\pi}\int_0^r J^t_{s_{V(z)}}(\theta)\frac{dt}{t}\le\frac{1}{2\pi}\frac{1}{d^2}h_d(\theta)a^d r^d+o(r^d),$$
where the term $o(r^d)$ can be bounded uniformly in z in compact sets of $\overline{\Omega}_\rho$. Recall that this is a statement about large r behavior, and holds even if $s_V(z)$ has poles in the upper half-plane, since it has at most finitely many. By the same argument, for large r,
$$\int_0^\theta\int_0^{\theta'}\ln|s_{V(z)}(re^{i\omega})|\,d\omega\,d\theta'\le\int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta' a^d r^d+o(r^d).$$

Using Lemma 5.2, we find that
$$\begin{aligned}\Psi(z,r,\rho) &= \frac{1}{2\pi\operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^r J^t_{s_{V(z')}}(\theta)\frac{dt}{t}\,d\mathcal{L}(z')\\ &\quad+\frac{1}{2\pi\operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta\int_0^{\theta'}\ln|s_{V(z')}(re^{i\omega})|\,d\omega\,d\theta' d\mathcal{L}(z')+O(r^{d-1}).\end{aligned}$$
Let $M = 2\alpha_d\max_{z\in\overline{\Omega_\rho}}\|V(z)\|_\infty$ and set, for $r>M$,
$$\begin{aligned}\Psi_1(z,r,\rho) &= \frac{1}{2\pi\operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_M^r J^t_{s_{V(z')}}(\theta)\frac{dt}{t}\,d\mathcal{L}(z')\\ &\quad+\frac{1}{2\pi\operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta\int_0^{\theta'}\ln|s_{V(z')}(re^{i\omega})|\,d\omega\,d\theta' d\mathcal{L}(z'),\end{aligned}$$

and note that

$$\Psi(z,r,\rho)=\Psi_1(z,r,\rho)+O(r^{d-1}).$$

By the bounds above,

$$\Psi_1(z,r,\rho)\le\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d r^d+o(r^d). \tag{5-5}$$

Using [Lelong and Gruman 1986, Proposition I.14] and the fact that $\ln|s_{V(z)}(\lambda)|$ is a plurisubharmonic function of $z\in\Omega$ when $|\lambda|>2\alpha_d\|V(z)\|_\infty$ and λ lies in the upper half-plane, we see that $\Psi_1(z,r,\rho)$ is a plurisubharmonic function of $z\in\Omega_{2\rho}$. Since by Proposition 2.2, $s_{V(z_0)}(\lambda)$ is of completely regular growth in $0<\arg\lambda<\pi$, using Lemma 5.2 and [Levin 1964, Chapter III, Section 2, Lemma 2],

$$\lim_{r\to\infty}r^{-d}\int_0^\theta N_{V(z_0)}(r,\pi,\theta'+\pi)\,d\theta'=\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d.$$

By the most basic property of plurisubharmonic functions,

$$\Psi_1(z_0,r,\rho)\ge\frac{1}{2\pi}\int_M^r J^t_{s_{V(z_0)}}(\theta)\frac{dt}{t}+\frac{1}{2\pi}\int_0^\theta\int_0^{\theta'}\ln|s_{V(z_0)}(re^{i\omega})|\,d\omega\,d\theta'.$$

But the right-hand side of this equation is $\int_0^\theta N_{V(z_0)}(r,\pi,\theta'+\pi)\,d\theta'+O(r^{d-1})$, so we see that

$$\liminf_{r\to\infty}r^{-d}\Psi_1(z_0,r,\rho)\ge\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d.$$

Combining this with (5-5), we find

$$\lim_{r\to\infty}r^{-d}\Psi_1(z_0,r,\rho)=\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d. \tag{5-6}$$

Using this and the upper bound (5-5) on Ψ_1, since Ψ_1 is plurisubharmonic in z, it follows from [Lelong and Gruman 1986, Proposition 1.39] (recalled here in Proposition 2.3) that for any sequence $\{r_j\}$, $r_j\to\infty$, there is a pluripolar set $E\subset\Omega_\rho$ (which may depend on the sequence) so that

$$\limsup_{j\to\infty}r_j^{-d}\Psi_1(z,r_j,\rho)=\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d$$

for all $z\in\Omega_\rho\setminus E$. Since $\lim_{r\to\infty}r^{-d}\big(\Psi_1(z,r,\rho)-\Psi(z,r,\rho)\big)=0$, the same conclusion holds for Ψ in place of Ψ_1.

Suppose there is some $z_1\in\Omega_{2\rho}$ and some sequence $r_j\to\infty$ so that

$$\lim_{j\to\infty}r_j^{-d}\Psi(z_1,r_j,\rho)<\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d.$$

Then, using the uniform continuity of $r^{-d}\Psi(z,r,\rho)$ in z, we find there must be an $\epsilon>0$ so that

$$\limsup_{j\to\infty}r_j^{-d}\Psi(z,r_j,\rho)<\frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta)+\int_0^\theta\int_0^{\theta'}h_d(\omega)\,d\omega\,d\theta'\right)a^d$$

for all $z \in B(z_1, \epsilon)$. But since $B(z_1, \epsilon)$ is not contained in a pluripolar set, we have a contradiction. Thus

$$\lim_{r\to\infty} r^{-d}\Psi(z, r, \rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^{\theta}\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d$$

for all $z \in \Omega_{2\rho}$. □

The following lemma will be used to remove the need to average in θ as in Proposition 5.3.

Lemma 5.4. *Let $M(r,\theta)$ be a function so that for any fixed positive $r_0 > C_0$, $M(r_0,\theta)$ is a nondecreasing function of θ, and suppose*

$$\lim_{r\to\infty} r^{-d}\int_0^{\theta} M(r,\theta')\,d\theta' = \alpha(\theta)$$

for $\theta_1 < \theta < \theta_2$. Then if α is differentiable at θ, then

$$\lim_{r\to\infty} r^{-d} M(r,\theta) = \alpha'(\theta).$$

Proof. Let $\epsilon > 0$. Then, since $M(r,\theta)$ is nondecreasing in θ,

$$\int_0^{\theta+\epsilon} M(r,\theta')\,d\theta' - \int_0^{\theta} M(r,\theta')\,d\theta' \geq \epsilon M(r,\theta),$$

which, under rearrangement, yields

$$r^{-d}M(r,\theta) \leq r^{-d}\frac{\int_0^{\theta+\epsilon} M(r,\theta')\,d\theta' - \int_0^{\theta} M(r,\theta')\,d\theta'}{\epsilon}.$$

Thus

$$\limsup_{r\to\infty} r^{-d}M(r,\theta) \leq \frac{\alpha(\theta+\epsilon)-\alpha(\theta)}{\epsilon}.$$

Likewise, we find

$$\liminf_{r\to\infty} r^{-d}M(r,\theta) \geq \frac{\alpha(\theta)-\alpha(\theta-\epsilon)}{\epsilon}.$$

Since both these equalities must hold for all $\epsilon > 0$, the lemma follows from the assumption that α is differentiable at θ. □

The following proposition follows from Proposition 5.3, but is stronger as it does not require averaging in the θ' variables.

Proposition 5.5. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, and let $\rho > 0$ and Ω_ρ be as in Proposition 5.3. Then for $z \in \Omega_{2\rho}$, $0 < \theta < \pi$, as $r \to \infty$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r,\pi,\theta+\pi)\,d\mathcal{L}(z') = \frac{1}{2\pi}a^d r^d\left(\frac{1}{d^2}h_d'(\theta) + \int_0^{\theta} h_d(\omega)\,d\omega\right) + o(r^d).$$

Proof. This follows from applying Lemmas 5.4 and 3.3 to the results of Proposition 5.3. □

Proposition 5.5 does not give results for the counting function for all the resonances (note that we cannot have $\theta = \pi$). The following fills this gap.

Proposition 5.6. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, and let $\rho > 0$ and Ω_ρ be as in Proposition 5.3. Then for $z \in \Omega_{2\rho}$, as $r \to \infty$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r)\, d\mathscr{L}(z') = \frac{1}{2\pi} a^d r^d \int_0^\theta h_d(\omega)\, d\omega + o(r^d).$$

Proof. The proof of this is very similar to that of Proposition 5.3. In fact, the main difference is the use of (2-2), which together with Lemma 3.1 and (3-1) gives us, by handling possible poles in the upper half-plane using a method similar to the proof of Lemma 5.2,

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r)\, d\mathscr{L}(z') = \Psi_1(z,r,\rho) + O(r^{d-1}),$$

where

$$\Psi_1(z,r,\rho) = \frac{1}{\mathrm{Vol}(B(z,\rho))}\frac{1}{2\pi}\int_{z'\in B(z,\rho)}\int_0^\pi \ln|s_{V(z')}(re^{i\theta})|\, d\theta\, d\mathscr{L}(z').$$

Using that Ψ_1 is plurisubharmonic in z, the proof now follows just as in Proposition 5.3. □

The following proposition is much like Propositions 5.5 and 5.6, but eliminates the average in the r variable.

Proposition 5.7. *Let V, Ω, z_0 satisfy the conditions of Theorem 1.2, and let ρ and Ω_ρ be as in Proposition 5.3. Then for $0 < \theta < \pi$, $z \in \Omega_{2\rho}$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} n_{V(z')}(r,\pi,\theta+\pi)\, d\mathscr{L}(z') = \frac{a^d r^d}{2\pi}\left(\frac{1}{d}h_d'(\theta) + d\int_0^\theta h_d(\theta)\, d\theta\right) + o(r^d)$$

and

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} n_{V(z')}(r)\, d\mathscr{L}(z') = \frac{d}{2\pi} a^d r^d \int_0^\pi h_d(\theta)\, d\theta + o(r^d)$$

as $r \to \infty$.

Proof. This proof follows from Propositions 5.5 and 5.6, using, in addition, a result like that of [Stefanov 2006, Lemma 1] or Lemma 5.4. □

Proof of Theorem 1.3. Let $M = \max(1 + |\psi(z)|)$, and for $\rho > 0$ small enough that $B(z_0,\rho) \subset \Omega$, set Ω_ρ to be the connected component of $\{z \in \Omega : \mathrm{dist}(z, \Omega^c) > \rho\}$ which contains z_0. Given $\epsilon > 0$, choose $\rho > 0$ such that $B(z_0, 2\rho) \subset \Omega$ and so that

$$\mathrm{vol}\big(\mathrm{supp}\,\psi \cap (\Omega \setminus \Omega_{2\rho})\big) < \frac{\epsilon}{10Me^d(c_d a^d + 1)}. \tag{5-7}$$

Since ψ is continuous with compact support, we can find a $\delta_1 > 0$, $\delta_1 < \rho$ so that if $|z - z'| < \delta_1$, then

$$|\psi(z) - \psi(z')| < \frac{\epsilon}{10e^d(1 + \mathrm{vol}\,\mathrm{supp}\,\psi)(a^d c_d + 1)}.$$

We may find a finite number J of disjoint balls $B(z_j, \epsilon_j)$ so that $\epsilon_j < \delta_1$, $z_j \subset \Omega_{2\rho}$, and

$$\mathrm{vol}\big(\mathrm{supp}\,\psi \setminus \textstyle\bigcup_1^J B(z_j,\epsilon_j)\big) + \mathrm{vol}\big(\textstyle\bigcup_1^J B(z_j,\epsilon_j) \setminus \mathrm{supp}\,\psi\big) < \dfrac{\epsilon}{4Me^d(a^d c_d + 1)}.$$

Let $\pi \le \varphi' \le \theta' \le 2\pi$. Now

$$\int \psi(z) n_{V(z)}(r,\varphi',\theta')\, d\mathscr{L}(z) = \sum_{j=1}^{J} \int_{B(z_j,\epsilon_j)} \psi(z) n_{V(z)}(r,\varphi',\theta')\, d\mathscr{L}(z) + \int_{\operatorname{supp}\psi\setminus(\cup B(z_j,\epsilon_j))} \psi(z) n_{V(z)}(r,\varphi',\theta')\, d\mathscr{L}(z).$$

We will use that the bound (1-1) implies that $n_V(z) \le e^d c_d a^d r^d + o(r^d)$. By our choice of $B(z_j,\epsilon_j)$,

$$\left| \int_{\operatorname{supp}\psi\setminus(\cup B(z_j,\epsilon_j))} \psi(z) n_{V(z)}(r,\varphi',\theta')\, d\mathscr{L}(z) \right| \le \frac{\epsilon}{4}(r^d + o(r^d)).$$

By our choice of δ_1 and the assumption that $\epsilon_j < \delta_1$, we have

$$\left| \sum_{j=1}^{J} \int_{B(z_j,\epsilon_j)} \psi(z) n_{V(z)}(r,\varphi',\theta')\, d\mathscr{L}(z) - \sum_{j=1}^{J} \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r,\varphi',\theta')\, d\mathscr{L}(z) \right| \le \frac{\epsilon}{5}(r^d + o(r^d)).$$

By Proposition 5.7, if $0 < \theta < \pi$,

$$\sum_{j=1}^{J} \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r,\pi,\pi+\theta) d\mathscr{L}(z) = \left(\sum_{j=1}^{J} \psi(z_j) \operatorname{vol}(B(z_j,\epsilon_j)) \right) \frac{1}{2\pi} a^d r^d \left(\frac{1}{d} h_d'(\theta) + d \int_0^\theta h_d(\omega) d\omega \right) + o(r^d),$$

and

$$\sum_{j=1}^{J} \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r)\, d\mathscr{L}(z) = \left(\sum_{j=1}^{J} \psi(z_j) \operatorname{vol}(B(z_j,\epsilon_j)) \right) \frac{d}{2\pi} a^d r^d \int_0^\pi h_d(\omega)\, d\omega + o(r^d).$$

Again using our choice of δ_1, z_j, and ϵ_j, we have

$$\left| \sum_{j=1}^{J} \psi(z_j) \operatorname{vol}(B(z_j,\epsilon_j)) - \int \psi(z)\, d\mathscr{L}(z) \right| < \frac{2\epsilon}{5(c_d a^d + 1)}.$$

Thus we have shown that given $\epsilon > 0$, if $0 < \theta < \pi$,

$$\left| \int \psi(z) n_{V(z)}(r,\pi,\theta+\pi)\, d\mathscr{L}(z) - \frac{a^d r^d}{2\pi} \int \psi(z) d\mathscr{L}(z) \left(\frac{1}{d} h_d'(\theta) + d \int_0^\theta h_d(\omega)\, d\omega \right) \right| \le \epsilon r^d + o(r^d) \quad (5\text{-}8)$$

and

$$\left| \int \psi(z) n_{V(z)}(r)\, d\mathscr{L}(z) - c_d a^d r^d \int \psi(z)\, d\mathscr{L}(z) \right| \le \epsilon r^d + o(r^d). \quad (5\text{-}9)$$

Thus we have proved the first and third statements of the theorem. The second statement of the theorem follows from the other two. □

6. Proof of Theorem 1.2

This proof uses some ideas similar to those used in the proofs of Propositions 5.3 and 5.6. In fact, because the proofs are so similar, we shall only give an outline.

Note that by (2-2), (3-1), and Lemma 3.1, using an argument similar to the proofs of Lemma 5.2 and Proposition 5.3,

$$N_{V(z)}(r) = \Psi(z,r) + o(r^{d-1}),$$

where

$$\Psi(z,r) = \frac{1}{2\pi}\int_0^{\pi} \ln |s_{V(z)}(re^{i\theta})|\, d\theta$$

is, for fixed (large) r a plurisubharmonic function of $z \in \tilde{\Omega} \Subset \Omega$. Since

$$\limsup_{r\to\infty} r^{-d}\Psi(z,r) \le \frac{a^d}{2\pi}\int_0^{\pi} h_d(\theta)\, d\theta$$

and this maximum is achieved at $z = z_0 \in \Omega$, we get the first part of the Theorem by applying [Lelong and Gruman 1986, Proposition 1.39], recalled in Proposition 2.3.

To obtain the second part, note that as in the proof of Proposition 5.3, for $0 < \theta < \pi$,

$$\int_0^{\theta} N_{V(z)}(r,\pi,\theta'+\pi)\, d\theta' = \Psi_2(z,r,\theta) + o(r^d),$$

where

$$\Psi_2(z,r,\theta) = \frac{1}{2\pi}\int_M^r J^t_{s_{V(z)}}(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^{\theta}\int_0^{\theta'} \ln |s_{V(z)}(re^{i\omega})|\, d\omega\, d\theta'.$$

Since this is a plurisubharmonic function of $z \in \tilde{\Omega}$, $\tilde{\Omega} \Subset \Omega$, if M is chosen so that $M \ge 2\alpha_d \max_{z\in\bar{\tilde{\Omega}}} \|V\|_\infty$, an argument using Proposition 2.3 as in the proof of Proposition 5.3 shows that there exists a pluripolar set $E_\theta \subset \Omega$ so that

$$2\pi \limsup_{r\to\infty} r^{-d}\Psi_2(z,r,\theta) = a^d\left(\frac{1}{d^2}h_d(\theta) + \int_0^{\theta}\int_0^{\theta'} h_d(\omega)\, d\omega\, d\theta'\right)$$

for all $z \in \Omega \setminus E_\theta$. Again, we use that this equality holds when $z = z_0$. Then

$$\limsup_{r\to\infty} r^{-d}\int_0^{\theta} N_{V(z)}(r,\pi,\pi+\theta')\, d\theta' = \frac{a^d}{2\pi}\left(\frac{h_d(\theta)}{d^2} + \int_0^{\theta}\int_0^{\theta'} h_d(\omega)\, d\omega\, d\theta'\right) \quad \text{for } z \in \Omega \setminus E_\theta. \tag{6-1}$$

For $0 < \theta < \pi$,

$$\frac{h_d'(\theta)}{d^2} + \int_0^{\theta} h_d(\omega)\, d\omega$$

is a nondecreasing function of θ. This can be seen by using

$$\lim_{r\to\infty} r^{-d} n_{\tilde{V}}(r,\pi,\pi+\theta) = \frac{1}{2\pi d}\left(h_d'(\theta) + d^2\int_0^{\theta} h_d(\omega)\, d\omega\right)$$

for $\tilde{V} \in \mathfrak{M}_1$, and clearly the left-hand side is a nondecreasing function of θ. This, along with the fact that $\lim_{\theta\downarrow 0} h_d(\theta) = 0$, implies that

$$\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega)\, d\omega\, d\theta' \geq \frac{\theta}{d^2} h_d'(0+)$$

for small $\theta > 0$. Therefore, using (6-1), for $z \in \Omega \setminus E_\theta$,

$$\limsup_{r\to\infty} r^{-d} \int_0^\theta N_{V(z)}(r, \pi, \pi + \theta')\, d\theta' \geq \frac{\theta a^d}{2\pi d^2} h_d'(0+),$$

and so we must have

$$\limsup r^{-d} N_{V(z)}(r, \pi, \pi + \theta) \geq \frac{a^d}{2\pi d^2} h_d'(0+)$$

for the same values of z. □

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WEIGHTED MAXIMAL REGULARITY ESTIMATES AND SOLVABILITY OF NONSMOOTH ELLIPTIC SYSTEMS, II

PASCAL AUSCHER AND ANDREAS ROSÉN

We continue the development, by reduction to a first-order system for the conormal gradient, of L^2 a priori estimates and solvability for boundary value problems of Dirichlet, regularity, Neumann type for divergence-form second-order complex elliptic systems. We work here on the unit ball and more generally its bi-Lipschitz images, assuming a Carleson condition as introduced by Dahlberg which measures the discrepancy of the coefficients to their boundary trace near the boundary. We sharpen our estimates by proving a general result concerning a priori almost everywhere nontangential convergence at the boundary. Also, compactness of the boundary yields more solvability results using Fredholm theory. Comparison between classes of solutions and uniqueness issues are discussed. As a consequence, we are able to solve a long standing regularity problem for real equations, which may not be true on the upper half-space, justifying a posteriori a separate work on bounded domains.

Andreas Rosén was formerly called Andreas Axelsson.

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1. Introduction and main results

This study was initiated in [Auscher and Axelsson 2011] — henceforth referred to as [Part I] — where the reader will find a comprehensive historical account of the theory of boundary value problems for second-order equations of divergence form. Before we come to our work here, let us connect more deeply to even earlier references going back to the seminal work of Stein and Weiss [1960] that paved the way for the development of Hardy spaces H^p on the Euclidean space in several dimensions. Their key discovery was to look at the system of differential equations in the upper half-space satisfied by the gradient $F = (\partial_t u, \nabla_x u)$ of a harmonic function u on the upper half-space, to which they gave the name of conjugate system or M. Riesz system. The system of differential equations is in fact a generalized Cauchy–Riemann system which can be put into a vector-valued ODE form. They did not exploit this ODE structure but used instead subharmonicity properties of $|F|^p$ for $p > \frac{n-1}{n}$ to define the (harmonic) Hardy spaces H^p as the space of those conjugate systems satisfying

$$\sup_{t>0} \int_{\mathbb{R}^n} |F(t,x)|^p \, dx < \infty$$

and to prove that the elements in this space have boundary values

$$F(t,x) \to F(0,x)$$

in the L^p norm and almost everywhere nontangentially. Further, they proved that elements in H^p can be obtained as Poisson integrals of their boundary traces. In other words, there is a one-to-one correspondence between H^p and its trace space $\mathcal{H}^p$. By using Riesz transforms, the trace space $\mathcal{H}^p$ is in one-to-one correspondence with the space defined by taking the first component of trace elements. As they pointed out, it was nothing new for $p > 1$ as we get L^p, but for $p \le 1$ it gave a new space. Over the years, this last space turned out to have many characterizations, including the ones with Littlewood–Paley functionals of [Fefferman and Stein 1972] and the atomic ones of [Coifman 1974] and [Latter 1978], and is now part of a rich and well understood family of spaces.

In our earlier work with McIntosh [Auscher et al. 2010b], and in [Part I], we wrote down the Cauchy–Riemann equations corresponding to the second-order equation and the key point was a further algebraic transformation that transformed this system to a vector-valued ODE. In some sense, we were going back in time since elliptic equations with nonsmooth coefficients have been developed by other methods since then (see [Kenig 1994]). In this respect, it is no surprise in view of the above discussion that we denote our trace spaces by $\mathcal{H}$. They are in a sense generalized Hardy spaces, and this notation was used as well in our earlier work with Hofmann [Auscher et al. 2008]. We shall use again such notation and terminology here. What today allows the methods of Hardy spaces to be applicable in the case of nonsmooth coefficients, are the quadratic estimates related to the solution of the Kato conjecture for square roots. These are a starting point of the analysis. Indeed, the quadratic estimates are equivalent to the fact that two Hardy spaces split the function space topologically, as it is the case for the classical upper and lower Hardy spaces in complex analysis, essentially from the F. and M. Riesz theorem on the boundedness of the Hilbert transform. So in a sense everything looks like the case of harmonic functions (for $p = 2$ at this time). But this is not the case. The difference is in the last step, taking only one component of the trace of a conjugate system. This may or may not be a one-to-one correspondence, which translates to well- or ill-posedness for the boundary value problems of the original second-order equation.

See also [Axelsson et al. 2009] for a different generalization of Stein–Weiss conjugate systems of harmonic functions. There conjugate differential forms on Lipschitz domains were constructed by inverting a generalized double layer potential equation on the boundary.

Let us introduce some notation in order to state our results. Our system of equations is of the form

$$\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u(\boldsymbol{x}) = \left(\sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i (A_{i,j}^{\alpha,\beta} \partial_j u^\beta)(\boldsymbol{x}) \right)_{\alpha=1,\dots,m} = 0, \quad \boldsymbol{x} \in \Omega, \tag{1}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $0 \le i \le n$, and the matrix of coefficients is $A = (A_{i,j}^{\alpha,\beta}(\boldsymbol{x}))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L_\infty(\Omega; \mathscr{L}(\mathbb{C}^{(1+n)m}))$, $n, m \ge 1$. We emphasize that the methods used here work equally well for systems ($m \ge 2$) as for equations ($m = 1$). For the time being, $\Omega = \mathbb{O}^{1+n} := \{\boldsymbol{x} \in \mathbb{R}^{1+n} \ ; \ |\boldsymbol{x}| < 1\}$ for the unit ball in $\mathbb{R}^{1+n}$ (see the end of the introduction for more general Lipschitz domains). The coefficient matrix A is assumed to satisfy the strict accretivity condition

$$\int_{S^n} \operatorname{Re}(A(rx)\nabla_{\boldsymbol{x}} u(rx), \nabla_{\boldsymbol{x}} u(rx))\, dx \ge \kappa \int_{S^n} |\nabla_{\boldsymbol{x}} u(rx)|^2\, dx \tag{2}$$

for some $\kappa > 0$, uniformly for a.e. $r \in (0, 1)$ and $u \in C^1(\mathbb{O}^{1+n}; \mathbb{C}^m)$ where we use polar coordinates $\boldsymbol{x} = rx$, $r > 0$, $x \in S^n$, and dx is the standard (nonnormalized) surface measure on $S^n = \partial\mathbb{O}^{1+n}$. The optimal κ is denoted κ_A. This ellipticity condition is natural when viewing A as a perturbation of its boundary trace. See below.

The boundary value problems we consider are to find $u \in \mathscr{D}'(\mathbb{O}^{1+n}; \mathbb{C}^m)$ solving (1) in distribution sense, with appropriate interior estimates of $\nabla_{\boldsymbol{x}} u$ and Dirichlet data in L_2, or Neumann data in L_2, or regular Dirichlet data with gradient in L_2. Note that since we shall impose distributional $\nabla_{\boldsymbol{x}} u \in L_2^{\text{loc}}$, u can be identified with a function $u \in W_2^{1,\text{loc}}(\mathbb{O}^{1+n}, \mathbb{C}^m)$, i.e., with a weak solution. In order to study these boundary value problems, our task, and this is the first main core of the work, is to obtain L^2 a priori estimates.

As in [Part I], where we worked in the upper half-space $\mathbb{R}^{1+n}_+$, we reduce (1) to a first-order system with the conormal gradient as unknown function, so the strategy and the scale-invariant estimates are similar. See [Part I, Road Map] for an overview. Some changes will arise in the algebraic setup and in the analysis though. Here, the curvature of the boundary (the sphere) will play a role in the algebraic setup, making the unit circle slightly different from the higher dimensional spheres. In addition, owing to the fact that the boundary is compact, we may use Fredholm theory to obtain representations and solvability by only making assumptions on the coefficients near the boundary. We shall focus on this part here and give full details. We also mention that the whole story relies on a quadratic estimate for a first-order bisectorial operator acting on the boundary function space. On the upper half-space, this estimate was already available from [Axelsson et al. 2006b] as a consequence of the strategy to prove the Kato conjecture on $\mathbb{R}^n$. We shall need to prove it on the sphere, essentially by localization and reduction to [Axelsson et al. 2006a], where such estimates were proved for first-order operators with boundary conditions. An implication of independent interest is the solution to the Kato square root on Lipschitz manifolds. This is explained in Section 8.

As is known already for real equations ($m = 1$) from work of Caffarelli, Fabes and Kenig [Caffarelli et al. 1981], solvability requires a Dini square regularity condition on the coefficients in the transverse direction to the boundary. So it is natural to work under a condition of this type. We use the discrepancy function and the Carleson condition introduced in [Dahlberg 1986]. For a measurable function f on $\mathbb{O}^{1+n}$, set

$$f^*(\boldsymbol{x}) := \operatorname*{ess\,sup}_{\boldsymbol{y}\in W^o(\boldsymbol{x})} |f(\boldsymbol{y})|, \tag{3}$$

where $W^o(\boldsymbol{x})$ denotes a Whitney region around $\boldsymbol{x} \in \mathbb{O}^{1+n}$ and

$$\|f\|_C := \sup_{r(Q)<c} \left(\frac{1}{|Q|} \iint_{(e^{-r(Q)},1)Q} f^*(\boldsymbol{x})^2 \frac{d\boldsymbol{x}}{1-|\boldsymbol{x}|} \right)^{1/2} \quad \text{for some fixed } c < 1, \tag{4}$$

where the supremum is over all geodesic balls $Q \subset S^n$ of radius $r(Q) < c$. We make the standing assumption on A throughout that there exists A_1 a measurable coefficient matrix on S^n, identified with radially independent coefficients in $\mathbb{O}^{1+n}$, such that $\mathcal{E}(\boldsymbol{y}) := A(\boldsymbol{y}) - A_1(y)$, $y = \boldsymbol{y}/|\boldsymbol{y}|$, satisfies the large Carleson condition

$$\|\mathcal{E}\|_C < \infty. \tag{5}$$

The choice of c is irrelevant. Note that this means in particular that $\mathcal{E}^*$ vanishes on S^n in a certain sense and so $A_1(y) = A(\boldsymbol{y}/|\boldsymbol{y}|)$. In fact, it can be shown as in [Part I] that if there is one such A_1, it is uniquely defined, $\|A_1\|_\infty \le \|A\|_\infty$ and $\kappa_{A_1} \ge \kappa_A$. So we call A_1 the boundary trace of A. It turns out that this is a very natural assumption with our method, implying a wealth of a priori information about weak solutions as stated in Theorem 1.1. Such a result applies in particular to all systems with radially independent coefficients since $\mathcal{E} = 0$ in that case.

For a function f defined in $\mathbb{O}^{1+n}$, its truncated modified nontangential maximal function is defined as in [Kenig and Pipher 1993] by

$$\widetilde{N}^o_*(f)(x) := \sup_{1-\tau<r<1} \left(|W^o(rx)|^{-1} \int_{W^o(rx)} |f(\boldsymbol{y})|^2 \, d\boldsymbol{y} \right)^{1/2}, \quad x \in S^n, \tag{6}$$

for some fixed $\tau < 1$. Note that changing the value of τ will not affect the results. We shall use the notation $f_r(x) := f(rx)$ for $0 < r < 1$, $x \in S^n$. Our main result is the following.

Theorem 1.1 (*a priori* representations and estimates; existence of a trace; Fatou-type convergence). *Consider coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2) and satisfy (5) with boundary trace A_1. Consider $u \in W^{1,\mathrm{loc}}_2(\mathbb{O}^{1+n}; \mathbb{C}^m)$ which satisfies (1) in $\mathbb{O}^{1+n}$ distributional sense.*

(i) *Suppose $\|\widetilde{N}^o_*(\nabla_{\boldsymbol{x}} u)\|_{L_2(S^n)} < \infty$. Then:*

(a) $\nabla_{\boldsymbol{x}} u$ *has limit*

$$\lim_{r\to 1} \frac{1}{1-r} \int_{r<|\boldsymbol{x}|<(1+r)/2} |\nabla_{\boldsymbol{x}} u(\boldsymbol{x}) - g_1(x)|^2 \, d\boldsymbol{x} = 0 \tag{7}$$

for some $g_1 \in L_2(S^n; \mathbb{C}^{(1+n)m})$ *with* $\|g_1\|_{L_2(S^n;\mathbb{C}^{(1+n)m})} \lesssim \|\widetilde{N}_*^o(\nabla_{\boldsymbol{x}} u)\|_{L_2(S^n)}$.

(b) $r \mapsto u_r$ *belongs to* $C(0, 1; L_2(S^n; \mathbb{C}^m))$ *and has* L_2 *limit* u_1 *at the boundary with*

$$\|u_r - u_1\|_{L_2(S^n;C^m)} \lesssim 1 - r,$$

and $u_1 \in W_2^1(S^n; \mathbb{C}^m)$.

(c) *Fatou-type results*: *For almost every* $x \in S^n$,

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} u(\boldsymbol{y}) \, d\boldsymbol{y} = u_1(x),$$

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} \partial_t u(\boldsymbol{y}) \, d\boldsymbol{y} = (g_1)_\perp(x),$$

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} (A\nabla_{\boldsymbol{x}} u)_\parallel(\boldsymbol{y}) d\boldsymbol{y} = (A_1 g_1)_\parallel(x),$$

and if $m = 1$ (*equations*) *or* $n = 1$ (*unit disk*) *we also have*

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} \nabla_{\boldsymbol{x}} u(\boldsymbol{y}) \, d\boldsymbol{y} = g_1(x),$$

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} (A\nabla_{\boldsymbol{x}} u)(\boldsymbol{y}) \, d\boldsymbol{y} = (A_1 g_1)(x).$$

(ii) *Suppose* $\int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1 - |\boldsymbol{x}|) \, d\boldsymbol{x} < \infty$. *Then*:

(a) $r \mapsto u_r$ *belongs to* $C(0, 1; L_2(S^n; \mathbb{C}^m))$ *and has* L_2 *limit*

$$\lim_{r\to 1} \|u_r - u_1\|_{L_2(S^n;\mathbb{C}^m)} = 0$$

for some $u_1 \in L_2(S^n; \mathbb{C}^m)$.

(b) *We have a priori estimates*

$$\|\widetilde{N}_*^o(u)\|^2_{L_2(S^n)} \lesssim \int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1 - |\boldsymbol{x}|) \, d\boldsymbol{x} + \left| \int_{S^n} u_1(x) \, dx \right|^2, \tag{8}$$

$$\|u_r\|^2_{L_2(S^n;\mathbb{C}^m)} \lesssim r^{-(n-1)} \int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1 - |\boldsymbol{x}|) \, d\boldsymbol{x} + \left| \int_{S^n} u_1(x) \, dx \right|^2, \quad r \in (0, 1). \tag{9}$$

(c) *Fatou-type results*: *For almost every* $x \in S^n$,

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} u(\boldsymbol{y}) \, d\boldsymbol{y} = u_1(x).$$

The definition of the normal component $(\cdot)_\perp$ and tangential part $(\cdot)_\parallel$ of a vector field will be given later. Not stated here are representation formulas giving ansatzes to find solutions as they use a formalism defined later. In particular, we introduce a notion of a pair of conjugate systems associated to a solution. We note that the nontangential maximal estimate (8) was already proved in the $\mathbb{R}^{1+n}_+$ setting of [Part I]. Again, this is an a priori estimate showing that, under the assumption $\|\mathcal{E}\|_C < \infty$, the class of weak solutions with square function estimate $\int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2(1-|\boldsymbol{x}|)\, d\boldsymbol{x} < \infty$ is contained in the class of weak solutions with nontangential maximal estimate $\|\widetilde{N}^o_*(u)\|_2 < \infty$. The almost everywhere convergences of Whitney averages are new. They apply as well to the setup in [Part I].

Theorem 1.1 enables us to make the following rigorous definition of well-posedness of the BVPs.

Definition 1.2. Consider coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2).

- By the Neumann problem with coefficients A being well-posed, we mean that given $\varphi \in L_2(S^n; \mathbb{C}^m)$ with $\int_{S^n} \varphi(x)\, dx = 0$, there is a function $u \in W^{1,\mathrm{loc}}_2(\mathbb{O}^{1+n}; \mathbb{C}^m)$ with estimates $\|\widetilde{N}^o_*(\nabla_{\boldsymbol{x}} u)\|_{L_2(S^n)} < \infty$, unique modulo constants, solving (1) and having trace $g_1 = \lim_{r\to 1}(\nabla_{\boldsymbol{x}} u)_r$ in the sense of (7) such that $(A_1 g_1)_\perp = \varphi$.
- Well-posedness of the regularity problem is defined in the same way, but replacing the boundary condition $(A_1 g_1)_\perp = \varphi$ by $(g_1)_\parallel = \varphi$, for a given $\varphi \in \mathsf{R}(\nabla_S) \subset L_2(S^n; \mathbb{C}^{nm})$.
- By the Dirichlet problem with coefficients A being well-posed, we mean that given $\varphi \in L_2(S^n; \mathbb{C}^m)$, there is a unique function $u \in W^{1,\mathrm{loc}}_2(\mathbb{O}^{1+n}; \mathbb{C}^m)$ with estimates $\int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2(1-|\boldsymbol{x}|)\, d\boldsymbol{x} < \infty$, solving (1) and having trace $\lim_{r\to 1} u_r = \varphi$ in the sense of almost everywhere convergence of Whitney averages.

For the Neumann and regularity problem when $\|\mathcal{E}\|_C < \infty$, for equations ($m = 1$) or in the unit disk ($n = 1$) or any system for which A is strictly accretive in pointwise sense, the trace can also be defined in the sense of almost everywhere convergence of Whitney averages of $\nabla_{\boldsymbol{x}} u$ and the same for the conormal derivative $(A\nabla_{\boldsymbol{x}} u)_\perp$. The operator ∇_S denotes the tangential gradient. See Section 3.

For the Dirichlet problem, the trace is defined for the almost everywhere convergence of Whitney averages. When $\|\mathcal{E}\|_C < \infty$, Theorem 1.1 shows that it is the same as the trace in L_2 sense. We remark that we modified the meaning of the boundary trace in the definition of the Dirichlet problem compared to [Part I]. This modification can be made there as well and the same results hold.

We now come to our general results on these BVPs. A small Carleson condition, but only near the boundary, is further imposed to obtain invertibility of some operators. The second result is on the precise relation between Dirichlet and regularity problems. The first and third are perturbations results for radially dependent and independent perturbations respectively. The last is a well-posedness result for three classes of radially independent coefficients.

Theorem 1.3. *Consider coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2). There exists $\epsilon > 0$ such that, if A satisfies the small Carleson condition*

$$\lim_{\tau\to 1} \|\chi_{\tau<r<1}(A - A_1)\|_C < \epsilon \tag{10}$$

and the Neumann problem with coefficients A_1 is well-posed, then the Neumann problem is well-posed with coefficients A.

The corresponding perturbation result for the regularity and Dirichlet problems also holds. For the Neumann and regularity problems, the solution u for datum φ has estimates

$$\int_{|\boldsymbol{x}|<1/2} |\nabla_{\boldsymbol{x}} u|^2 \, d\boldsymbol{x} \lesssim \|\widetilde{N}_*^o(\nabla_{\boldsymbol{x}} u)\|_2^2 \approx \|\varphi\|_2^2.$$

For the Dirichlet problem, the solution u for datum φ has estimates

$$\|\widetilde{N}_*^o(u)\|_2^2 \approx \sup_{1/2<r<1} \|u_r\|_2^2 \approx \int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1-|\boldsymbol{x}|) \, d\boldsymbol{x} + \left| \int_{S^n} \varphi(x) dx \right|^2 \approx \|\varphi\|_2^2.$$

An ingredient of the proof is the following relation between Dirichlet and regularity problems, in the spirit of [Kenig and Pipher 1993, Theorem 5.4].

Theorem 1.4. *Consider coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2). There exists $\epsilon > 0$ such that, if A satisfies the small Carleson condition (10), then the regularity problem with coefficients A is well-posed if and only if the Dirichlet problem with coefficients A^* is well-posed.*

Theorem 1.5. *Consider radially independent coefficients $A_1 \in L_\infty(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2). If the Neumann problem with coefficients A_1 is well-posed, then there exists $\epsilon > 0$ such that the Neumann problem with coefficients $A_1' \in L_\infty(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ is well-posed whenever $\|A_1 - A_1'\|_\infty < \epsilon$. The corresponding perturbation results for the regularity and Dirichlet problems also hold.*

Theorem 1.6. *Consider radially independent coefficients $A_1 \in L_\infty(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2). The Neumann, regularity and Dirichlet problems with coefficients A_1 are well-posed if*

(1) *either A_1 is Hermitian, i.e., $A_1^* = A_1$,*

(2) *or A_1 has block form, i.e., $(A_1)_{\perp\|} = 0 = (A_1)_{\|\perp}$ in the normal/tangential splitting of $\mathbb{C}^{(1+n)m}$ (see Section 3),*

(3) *or A_1 has Hölder regularity $C^s(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$, $s > \frac{1}{2}$.*

Proof of Theorems 1.1, 1.3, 1.5 and 1.6. For Theorem 1.1, the L_2-limits and L_2-estimates of solutions follow from Theorem 12.4 and Corollary 12.8 respectively. The nontangential maximal estimate (8) is in Theorem 14.1. Almost everywhere convergence of averages follows from Theorems 15.2 and 15.5.

The well-posedness results in Theorem 1.6 are in Propositions 17.16, 17.15 and 17.17. The radially independent perturbation result in Theorem 1.5 is in Theorem 17.13. The well-posedness result for radially dependent coefficients with good boundary trace in Theorem 1.3 is in Theorem 17.14. □

Our next result is the following semigroup representation, analogous to the result in [Auscher 2009] in the upper half-space. It is interesting to note that for harmonic functions u, it gives a direct proof (without passing through nontangential maximal function or sup $-L_2$ estimates) that $\int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1-|\boldsymbol{x}|) \, d\boldsymbol{x} < \infty$

implies a representation by Poisson kernel from its trace (also shown to exist). We have not seen this argument in the literature. Another interesting feature is that it points out the importance of well-posedness of the Dirichlet problem when dealing with more general coefficients.

Theorem 1.7. *Consider radially independent coefficients $A_1 \in L_\infty(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2). Assume that the Dirichlet problem with coefficients A_1 is well-posed. Then the mapping*

$$\mathcal{P}_r : L_2(S^n; \mathbb{C}^m) \to L_2(S^n; \mathbb{C}^m) : u_1 \mapsto u_r,$$

where u is the solution to the Dirichlet problem with datum u_1, defines a bounded operator for each $r \in (0, 1]$. The family $(\mathcal{P}_r)_{r\in(0,1]}$ is a multiplicative C_0-semigroup (i.e., $\mathcal{P}_r\mathcal{P}_{r'} = \mathcal{P}_{rr'}$ and $\mathcal{P}_r \to I$ strongly in L_2 when $r \to 1$) whose infinitesimal generator $\mathcal{A}$ (i.e., $\mathcal{P}_r = e^{(\ln r)\mathcal{A}}$) has domain $D(\mathcal{A})$ contained in $W_2^1(S^n; \mathbb{C}^m)$. Moreover, $D(\mathcal{A}) = W_2^1(S^n; \mathbb{C}^m)$ if and only if the Dirichlet problem with coefficients A_1^* is well-posed.

As mentioned above two classes of weak solutions compare: the one with square function estimates is contained in the one with nontangential maximal control. It is thus interesting to examine this further. Does the opposite containment holds? How do well-posedness in the two classes compare? Clearly uniqueness in the larger class implies uniqueness in the smaller, and conversely for existence. As we shall see, positive answers come a posteriori to solvability.

Definition 1.8. The Dirichlet problem with coefficients A is said to be *well-posed in the sense of Dahlberg* if, given $\varphi \in L_2(S^n; \mathbb{C}^m)$, there is a unique weak solution $u \in W_2^{1,\mathrm{loc}}(\mathbb{O}^{1+n}; \mathbb{C}^m)$ to $\mathrm{div}_{\boldsymbol{x}}\, A\nabla_{\boldsymbol{x}} u = 0$ with estimates $\|\widetilde{N}_*^o(u)\|_2 < \infty$ and convergence of Whitney averages to φ, almost everywhere with respect to surface measure on S^n.

This definition has the merit to be natural not only for equations but for systems as well. For real equations, this is equivalent to the usual one as $\widetilde{N}_*^o$ can be replaced by the usual pointwise nontangential maximal operator by the De Giorgi–Nash–Moser estimates on weak solutions. Even in this case, observe that the control $\|\widetilde{N}_*^o(u)\|_2 < \infty$ does not enforce the almost everywhere convergence property. Thus existence of the limit is part of the hypothesis in Definition 1.8, as compared to Definition 1.2. A first result is the following.

Theorem 1.9. *Consider radially independent coefficients $A_1 \in L_\infty(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2). Assume that the Dirichlet and regularity problems with coefficients A_1 are well-posed in the sense of Definition 1.2. Then, all weak solutions to $\mathrm{div}_{\boldsymbol{x}}\, A_1\nabla_{\boldsymbol{x}} u = 0$ with $\|\widetilde{N}_*^o(u)\|_2 < \infty$ are given by the semigroup of Theorem 1.7. In particular, the Dirichlet problem with coefficients A_1 is well-posed in the sense of Dahlberg.*

Theorem 1.4 implies the same conclusion for the coefficients A_1^*. The next results are only for real equations where the theory based on elliptic measure brings more information. For (complex) equations, the strict accretivity in the sense of (2) is equivalent to the usual pointwise accretivity, which is the same as the strict ellipticity for real coefficients.

Theorem 1.10. *Consider an equation with real coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathscr{L}(\mathbb{R}^{1+n}))$, which are strictly elliptic. Assume further that the small Carleson condition (10) holds. Then the following statements are equivalent.*

(i) *The Dirichlet problems with coefficients A and A^* are well-posed in the sense of Dahlberg.*

(ii) *The Dirichlet problems with coefficients A and A^* are well-posed in the sense of Definition 1.2.*

Moreover, in this case the solutions for coefficients A (resp. A^*) from a same datum are the same.

Note that, by Theorem 1.4, we can replace (ii) by (ii′): the regularity problems with coefficients A and A^* are well-posed. When $A = A^*$, all the problems in (i) and (ii′) are well-posed by [Kenig and Pipher 1993] so there is nothing to prove. For (even nonsymmetric) real coefficients A alone, the direction from (ii′) to (i) was known from [Kenig and Pipher 1993] (without assuming the Carleson condition) and the converse is unknown. It seems that making the statement invariant under taking adjoints solves the issue. We mention the equivalence in [Kilty and Shen 2011] concerning L_p versions of this statement for self-adjoint constant coefficient systems on Lipschitz domains (in this case, the L_2 result is known and used).

Our last result is well-posedness of the regularity problem under a transversal square Dini condition on the coefficients, analogous to the result obtained by Fabes, Jerison and Kenig [Fabes et al. 1984] for the Dirichlet problem with real and symmetric A. This partly answers Problem 3.3.13 in [Kenig 1994].

Theorem 1.11. *Consider an equation with coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathscr{L}(\mathbb{C}^{1+n}))$, which are strictly accretive in the pointwise sense. There exists $\epsilon > 0$ such that, if A satisfies the small Carleson condition (10) and its boundary trace A_1 is real and continuous, then the Dirichlet problem with coefficients A is well-posed in the sense of Definition 1.2 and in the sense of Dahlberg, and the regularity problem with coefficients A is well-posed. In particular, this holds if A is real, continuous in $\overline{\mathbb{O}^{1+n}}$ and the Dini square condition $\int_0 w_A^2(t)\frac{dt}{t} < \infty$ holds, where $w_A(t) = \sup\{|A(rx) - A(x)|\,;\, x \in S^n,\ 1 - r < t\}$. The corresponding results hold in $\mathbb{O}^2$ for the Neumann problem with coefficients A.*

Proofs of Theorems 1.7 and 1.9 are in Sections 18 and proofs of Theorems 1.10 and 1.11 are in Section 19.

We end this introduction with a remark on the Lipschitz invariance of the above results. Let $\Omega \subset \mathbb{R}^{1+n}$ be a domain which is Lipschitz diffeomorphic to $\mathbb{O}^{1+n}$ and let $\rho : \mathbb{O}^{1+n} \to \Omega$ be the Lipschitz diffeomorphism. Denote the boundary by $\Sigma := \partial\Omega$ and the restricted boundary Lipschitz diffeomorphism by $\rho_0 : S^n \to \Sigma$.

Given a function $\tilde{u} : \Omega \to \mathbb{C}^m$, we pull it back to $u := \tilde{u} \circ \rho : \mathbb{O}^{1+n} \to \mathbb{C}^m$. By the chain rule, we have $\nabla_{\boldsymbol{x}} u = \rho^*(\nabla_{\boldsymbol{y}} \tilde{u})$, where the pullback of an m-tuple of vector fields f, is defined as $\rho^*(f)(\boldsymbol{x})^\alpha := \underline{\rho}^t(\boldsymbol{x}) f^\alpha(\rho(\boldsymbol{x}))$, with $\underline{\rho}^t$ denoting the transpose of Jacobian matrix $\underline{\rho}$. If $\tilde{u}$ satisfies $\operatorname{div}_{\boldsymbol{y}} \tilde{A}\nabla_{\boldsymbol{y}}\tilde{u} = 0$ in Ω, with coefficients $\tilde{A} \in L_\infty(\Omega; \mathscr{L}(\mathbb{C}^{(1+n)m}))$, then u will satisfy $\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$, where $A \in L_\infty(\mathbb{O}^{1+n}; \mathscr{L}(\mathbb{C}^{(1+n)m}))$ are the "pulled back" coefficients defined as

$$A(\boldsymbol{x}) := |J(\rho)(\boldsymbol{x})|(\underline{\rho}(\boldsymbol{x}))^{-1}\tilde{A}(\rho(\boldsymbol{x}))(\underline{\rho}^t(\boldsymbol{x}))^{-1}, \quad \boldsymbol{x} \in \mathbb{O}^{1+n}. \tag{11}$$

Here $J(\rho)$ is the Jacobian determinant of ρ.

The Carleson condition, nontangential maximal functions and square functions on Ω correspond to ones on S^n under pullback, so that $1 - |\boldsymbol{x}|$ becomes $\delta(\boldsymbol{y})$ the distance to Σ. In particular, the condition for $\tilde{A}$ amounts to $\|\mathscr{E}\|_C < \infty$ with $\mathscr{E}$ defined from A. We remark that pullbacks allow to replace normal directions by oblique (but transverse) ones to the sphere in the Carleson condition on the coefficients: take $\rho : \mathbb{O}^{1+n} \to \mathbb{O}^{1+n}$ to be a suitable Lipschitz diffeomorphism.

The boundary conditions on $\tilde{u}$ on Σ translate in the following way to boundary conditions on u on S^n.

- The Dirichlet condition $\tilde{u} = \tilde{\varphi}$ on Σ is equivalent to $u = \varphi$ on S^n, where $\varphi := \tilde{\varphi} \circ \rho_0 \in L_2(S^n; \mathbb{C}^m)$.
- The Dirichlet regularity condition $\nabla_\Sigma \tilde{u} = \tilde{\varphi}$ on Σ (∇_Σ denoting the tangential gradient on Σ) is equivalent to $\nabla_S u = \varphi$ on S^n, where $\varphi := \rho_0^*(\tilde{\varphi}) \in \mathsf{R}(\nabla_S) \subset L_2(S^n; \mathbb{C}^{nm})$.
- The Neumann condition $(\nu, \tilde{A}\nabla_y \tilde{u}) = \tilde{\varphi}$ on Σ (ν being the outward unit normal vector field on Σ) with $\int_\Sigma \tilde{\varphi}(y)\, dy = 0$ is equivalent to $(\vec{n}, A\nabla_x u) = \varphi$ on S^n with $\int_{S^n} \varphi(x)\, dx = 0$, where $\varphi := |J(\rho_0)|\tilde{\varphi} \circ \rho_0 \in L_2(S^n; \mathbb{C}^m)$.

In this way the Dirichlet/regularity/Neumann problem with coefficients $\tilde{A}$ in the Lipschitz domain Ω is equivalent to the Dirichlet/regularity/Neumann problem with coefficients A in the unit ball $\mathbb{O}^{1+n}$, and it is straightforward to extend the results on $\mathbb{O}^{1+n}$ above to Lipschitz domains Ω.

The plan of the paper is as follows. In Section 2, we transform the second-order equation (1) into a system of Cauchy–Riemann type equations. In Section 3, the Cauchy–Riemann equations are integrated to a vector-valued ODE for the conormal gradient of u and a second ODE is introduced to construct a vector potential. The infinitesimal generators D_0 and $\widetilde{D}_0$ for these ODE with radially independent coefficients are studied in Sections 4 and 6, and it is shown in Section 7 that D_0 and $\widetilde{D}_0$ have bounded holomorphic functional calculi. Section 5 treats special features of elliptic systems in the unit disk. In Section 9 we define the natural function spaces $\mathscr{X}^o$ and $\mathscr{Y}^o$ for the BVPs and we describe in Section 10 how to construct solutions from the semigroups generated by

$$|D_0| = \sqrt{D_0^2} \quad \text{and} \quad |\widetilde{D}_0| = \sqrt{\widetilde{D}_0^2}.$$

In Section 11, the ODE with radially dependent coefficients for the conormal gradient from Section 4 is reformulated as an integral equation involving an operator S_A, which is shown to be bounded on the natural function spaces for the BVPs. In Section 12, we obtain representation for $\mathscr{X}^o$- and $\mathscr{Y}^o$-solutions. These representations are further developed in Section 13 where we introduce the notion of a pair of conjugate systems for (1), allowing to prove in Sections 14 and 15 nontangential maximal estimates and Fatou-type results. Crucial for the solvability of (1) is the invertibility of $I - S_A$. In Section 16, we apply Fredholm theory to show that $I - S_A$ is invertible on the natural spaces whenever the small Carleson condition (10) holds, which proves that it suffices to assume transversal regularity of A near the boundary only. (For BVPs on the unbounded half-space studied in [Part I], the needed compactness was not available.) We then study well-posedness in Section 17: this is where we prove the perturbation results, the equivalence Dirichlet/regularity up to taking adjoints and obtain classes of radially independent coefficients for which well-posedness holds. The Section 18 deals with uniqueness issues, on comparisons of different classes of solutions upon some well-posedness assumptions. We conclude the article in Section 19 with a discussion in the special case of real equations ($m = 1$) for which we obtain further results.

2. Generalized Cauchy–Riemann system

Following [Auscher et al. 2008; 2010b] and [Part I], the starting point of our analysis is that solving for u the divergence form system (1) amounts to solving for its gradient g a system of Cauchy–Riemann equations.

Proposition 2.1. *Consider coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m}))$. If u is a weak solution to the equation $\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$, then $g := \nabla_{\boldsymbol{x}} u \in L_2^{\text{loc}}(\mathbb{O}^{1+n}; \mathbb{C}^{(1+n)m})$ is a solution of the generalized Cauchy–Riemann system*

$$\begin{cases} \operatorname{div}_{\boldsymbol{x}}(Ag) = 0, \\ \operatorname{curl}_{\boldsymbol{x}} g = 0, \end{cases} \tag{12}$$

in $\mathbb{O}^{1+n} \setminus \{0\}$ distribution sense. Conversely, if $g \in L_2^{\text{loc}}(\mathbb{O}^{1+n}; \mathbb{C}^{(1+n)m})$ is a solution to (12) in $\mathbb{O}^{1+n} \setminus \{0\}$ distribution sense, then there exists a weak solution u to $\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$, such that $g = \nabla_{\boldsymbol{x}} u$ in $\mathbb{O}^{1+n}$ distribution sense.

Proof. If u is given, then $g := \nabla_{\boldsymbol{x}} u$ has the desired properties and the equation is even satisfied in $\mathbb{O}^{1+n}$ distribution sense. Conversely, assume g is given and satisfies (12) in $\mathbb{O}^{1+n} \setminus \{0\}$ distribution sense. Then the next lemma applied to both operators $\operatorname{div}_{\boldsymbol{x}}$ and $\operatorname{curl}_{\boldsymbol{x}}$ implies that 0 is a removable singularity and that (12) holds in $\mathbb{O}^{1+n}$ distribution sense. Thus one can define a distribution u in $\mathbb{O}^{1+n}$ such that $g = \nabla_{\boldsymbol{x}} u$, hence $\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$. That u is a weak solution follows from $g \in L_2^{\text{loc}}(\mathbb{O}^{1+n}; \mathbb{C}^{(1+n)m})$. □

Lemma 2.2. *Let X be a homogeneous first-order partial differential operator on $\mathbb{R}^{1+n}$ mapping $\mathbb{C}^k$-valued distributions to $\mathbb{C}^\ell$-valued distributions, $k, \ell \in \mathbb{Z}_+$. If $h \in L_2^{\text{loc}}(\mathbb{O}^{1+n}; \mathbb{C}^k)$ and $Xh = 0$ in distributional sense on $\mathbb{O}^{1+n} \setminus \{0\}$, then $Xh = 0$ in $\mathbb{O}^{1+n}$-distributional sense.*

Proof. Let $\phi \in C_0^\infty(\mathbb{O}^{1+n}; \mathbb{C}^\ell)$. We need to show that $\int_{\mathbb{O}^{1+n}} (X^*\phi, h)\, d\boldsymbol{x} = 0$. To this end, let η_ϵ be a smooth radial function with $\eta_\epsilon = 0$ on $\{|\boldsymbol{x}| < \epsilon\}$, $\eta_\epsilon = 1$ on $\{2\epsilon < |\boldsymbol{x}| < 1\}$ and $\|\nabla \eta_\epsilon\|_\infty \lesssim \epsilon^{-1}$. Then

$$\int_{\mathbb{O}^{1+n}} \eta_\epsilon (X^*\phi, h)\, d\boldsymbol{x} = \int_{\mathbb{O}^{1+n}} (X^*(\eta_\epsilon \phi), h)\, d\boldsymbol{x} - \int_{\mathbb{O}^{1+n}} ((X^*\eta_\epsilon)\phi, h)\, d\boldsymbol{x}$$
$$= -\int_{\mathbb{O}^{1+n}} ((X^*\eta_\epsilon)\phi, h)\, d\boldsymbol{x}.$$

As $\epsilon \to 0$, the left hand side converges to $\int_{\mathbb{O}^{1+n}} (X^*\phi, h)\, d\boldsymbol{x}$, whereas

$$\left| \int_{\mathbb{O}^{1+n}} ((X^*\eta_\epsilon)\phi, h)\, d\boldsymbol{x} \right| \lesssim \frac{1}{\epsilon} \int_{\epsilon<|\boldsymbol{x}|<2\epsilon} |h| d\boldsymbol{x} \lesssim \epsilon^{(n-1)/2} \left(\int_{\epsilon<|\boldsymbol{x}|<2\epsilon} |h|^2\, d\boldsymbol{x} \right)^{1/2} \to 0.$$

This proves the lemma. □

3. The divergence form equation as an ODE

We introduce a convenient framework to transform the Cauchy–Riemann system into an ODE.

We systematically use boldface letters $\boldsymbol{x}, \boldsymbol{y}, \ldots$ to denote variables in $\mathbb{R}^{1+n}$ and indicate the variable for differential operators in $\mathbb{R}^{1+n}$: for example, $\nabla_{\boldsymbol{x}} \ldots$. We denote points on S^n by $x, y, \ldots$ and the standard (nonnormalized) surface measure on S^n by dx. Polar coordinates are written $\boldsymbol{x} = rx$, with $r > 0$

and $x \in S^n$. For a function f defined in $\mathbb{O}^{1+n}$, we write $f_r(x) := f(rx)$, $x \in S^n$, for the restriction to the sphere with radius $0 < r < 1$, parametrized by S^n.

The radial unit vector field we denote by $\vec{n} = \vec{n}(\boldsymbol{x}) := \boldsymbol{x}/|\boldsymbol{x}|$. Vectors $v \in \mathbb{R}^{1+n}$, we split $v = v_\perp \vec{n} + v_\parallel$, where $v_\perp := (v, \vec{n})$ is the *normal component* and $v_\parallel := v - v_\perp \vec{n}$ is the *angular or tangential part* of v, which is a vector orthogonal to $\vec{n}$. Note that $v_\perp$ is a scalar, but $v_\parallel$ is a vector. In the plane, i.e., when $n = 1$, we denote the counter clockwise angular unit vector field by $\vec{\tau}$, and we have $v = v_\perp \vec{n} + (v, \vec{\tau})\vec{\tau}$. For an m-tuple of vectors $v = (v^\alpha)_{1 \le \alpha \le m}$, we define its normal components and tangential parts componentwise as

$$(v_\perp)^\alpha := (v^\alpha)_\perp, \quad (v_\parallel)^\alpha := (v^\alpha)_\parallel.$$

The tangential gradient, divergence and curl on the unit sphere are denoted by ∇_S, div_S and curl_S respectively. The gradient acts component-wise on tuples of scalar functions, whereas the divergence and curl act vector-wise on tuples of vector fields. In polar coordinates, the $\mathbb{R}^{1+n}$ differential operators are

$$\begin{aligned}
\nabla_{\boldsymbol{x}} u &= (\partial_r u_r)\vec{n} + r^{-1}\nabla_S u_r, \\
\operatorname{div}_{\boldsymbol{x}} f &= r^{-n}\partial_r\big(r^n (f_r)_\perp\big) + r^{-1}\operatorname{div}_S (f_r)_\parallel, \\
\operatorname{curl}_{\boldsymbol{x}} f &= r^{-1}\vec{n} \wedge \big(\partial_r (r (f_r)_\parallel) - \nabla_S (f_r)_\perp\big) + r^{-1}\operatorname{curl}_S (f_r)_\parallel.
\end{aligned}$$

We use the boundary function space $L_2(S^n; \mathcal{V})$, writing the norm $\|\cdot\|_2$, of L_2 sections of the complex vector bundle

$$\mathcal{V} := \begin{bmatrix} \mathbb{C}^m \\ (T_{\mathbb{C}} S^n)^m \end{bmatrix}$$

over S^n, where $\mathbb{C}^m$ is identified with the trivial vector bundle and $T_{\mathbb{C}} S^n$ denotes the complexified tangent bundle of S^n. The elements of this bundle are written in vector form $f = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \end{bmatrix}^t$, and we write $f_\perp := \alpha$, $f_\parallel := \beta$ for the normal component and tangential part. Note that $\mathcal{V}$ is isomorphic to the trivial vector bundle $\mathbb{C}^{(1+n)m}$, when identifying scalar, i.e., $\mathbb{C}^m$-valued, functions and m-tuples of radial vector fields. More precisely, the isomorphism is $\mathcal{V} \ni \begin{bmatrix} \alpha & \beta \end{bmatrix}^t \mapsto \alpha\vec{n} + \beta \in \mathbb{C}^{(1+n)m}$, for $\alpha \in \mathbb{C}^m$ and $\beta \in (T_{\mathbb{C}} S^n)^m$.

The differential operators on S^n can be seen as unbounded operators. We use $\mathsf{D}(A)$, $\mathsf{R}(A)$, $\mathsf{N}(A)$ for the domain, range and null space respectively of unbounded operators. Then

$$\nabla_S : L^2(S^n; \mathbb{C}^m) \to L_2(S^n; (T_{\mathbb{C}} S^n)^m)$$

and its adjoint

$$-\operatorname{div}_S : L_2(S^n; (T_{\mathbb{C}} S^n)^m) \to L^2(S^n; \mathbb{C}^m),$$

with domains $\mathsf{D}(\nabla_S) = W^1_2(S^n; \mathbb{C}^m)$ and $\mathsf{D}(\operatorname{div}_S) = \{g \in L_2(S^n; (T_{\mathbb{C}} S^n)^m)\ ;\ \operatorname{div}_S g \in L^2(S^n; \mathbb{C}^m)\}$, are closed unbounded operators with closed range. The condition $g \in \mathsf{R}(\operatorname{div}_S) = \mathsf{N}(\nabla_S)^\perp$ is that $\int_{S^n} g(x)\, dx = 0$ so $\mathsf{R}(\operatorname{div}_S)$ is of codimension m in $L^2(S^n; \mathbb{C}^m)$. Also when $n \ge 2$, $\mathsf{R}(\nabla_S) = \mathsf{N}(\operatorname{curl}_S)$, and when $n = 1$, $g \in \mathsf{R}(\nabla_S)$ if and only if $\int_{S^1} (g(x), \vec{\tau})\, dx = 0$. Thus $\mathsf{R}(\nabla_S)$ is of codimension m in $L^2(S^1; \mathbb{C}^m)$ when $n = 1$, and infinite codimension when $n \ge 2$.

Definition 3.1. In $L_2(S^n;\mathcal{V})$, we define operators

$$D := \begin{bmatrix} 0 & -\mathrm{div}_S \\ \nabla_S & 0 \end{bmatrix} \quad \text{and} \quad N := \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},$$

where $\mathsf{D}(D) := \begin{bmatrix} \mathsf{D}(\nabla_S) \\ \mathsf{D}(\mathrm{div}_S) \end{bmatrix}$. Write $N^+ f := \frac{1}{2}(I+N)f = \begin{bmatrix} 0 \\ f_\parallel \end{bmatrix}$ and $N^- f := \frac{1}{2}(I-N)f = \begin{bmatrix} f_\perp \\ 0 \end{bmatrix}$.

A basic observation is that the two operators D and N anticommute, i.e.,

$$ND = -DN.$$

Of fundamental importance in this paper are the closed orthogonal subspaces

$$\mathcal{H} := \mathsf{R}(D) = \begin{bmatrix} \mathsf{R}(\mathrm{div}_S) \\ \mathsf{R}(\nabla_S) \end{bmatrix} \quad \text{and} \quad \mathcal{H}^\perp := \mathsf{N}(D) = \begin{bmatrix} \mathsf{N}(\nabla_S) \\ \mathsf{N}(\mathrm{div}_S) \end{bmatrix}.$$

We consistently denote by $P_{\mathcal{H}}$ the orthogonal projection onto $\mathcal{H}$. We remark that

$$N^+\mathcal{H}^\perp = \left\{ \begin{bmatrix} 0 \\ f_\parallel \end{bmatrix} ; \mathrm{div}_S\, f_\parallel = 0 \right\} \quad \text{and} \quad N^-\mathcal{H}^\perp = \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix} ; c \in \mathbb{C}^m \right\},$$

constants being identified to constant functions. It can be checked that (2) is equivalent to A is *strictly accretive* on

$$\mathcal{H}_1 := \{ g \in L_2(S^n; \mathbb{C}^{(1+n)m}) \ ; \ g_\parallel \in \mathsf{R}(\nabla_S)\}, \tag{13}$$

uniformly for a.e. $r \in (0,1)$. More precisely, the accretivity assumption on A rewrites

$$\sum_{i,j=0}^{n} \sum_{\alpha,\beta=1}^{m} \int_{S^n} \mathrm{Re}(A_{i,j}^{\alpha,\beta}(rx) g_j^\beta(x) \overline{g_i^\alpha(x)})\, dx \geq \kappa \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{S^n} |g_i^\alpha(x)|^2\, dx, \tag{14}$$

for all $g \in \mathcal{H}_1$, a.e. $r \in (0,1)$. In fact, as we shall see in Lemma 5.1 this is equivalent to pointwise strict accretivity when $n=1$ (unit disk), but this is in general not the case when $n \geq 2$ except if $m=1$ (equations).

Using the notation above, we can identify $\mathcal{H}_1$ with

$$\begin{bmatrix} L_2(S^n;\mathbb{C}^m) \\ \mathsf{R}(\nabla_S) \end{bmatrix}$$

and see that $\mathcal{H}$ is a subspace of codimension m in $\mathcal{H}_1$.

On identifying $\mathbb{C}^{(1+n)m}$ with $\mathcal{V}$, the space of coefficients $L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ identifies with $L_\infty(\mathbb{O}^{1+n}; \mathcal{L}(\mathcal{V}))$, so that we can split any coefficients A as

$$A(rx) = \begin{bmatrix} A_{\perp\perp}(rx) & A_{\perp\parallel}(rx) \\ A_{\parallel\perp}(rx) & A_{\parallel\parallel}(rx) \end{bmatrix},$$

with $A_{\perp\perp}(rx) \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^m)$, $A_{\perp\parallel}(rx) \in \mathcal{L}((T_x S^n)^m, \mathbb{C}^m)$, $A_{\parallel\perp}(rx) \in \mathcal{L}(\mathbb{C}^m, (T_x S^n)^m)$ and $A_{\parallel\parallel}(rx) \in \mathcal{L}((T_x S^n)^m, (T_x S^n)^m)$. Note also that $A_{\perp\perp}(rx) = (A(rx)\vec{n}, \vec{n})$.

With our accretivity assumption (14), the component $A_{\perp\perp}(r\,\cdot\,)$ seen as a multiplication operator is invertible on $L_2(S^n;\mathbb{C}^m)$, thus as a matrix function it is invertible in $L_\infty(S^n;\mathbb{C}^m)$. This is the reason why strict accretivity on $\mathcal{H}_1$ is needed, and not only on $\mathcal{H}$, so that the transformed coefficient matrix $\hat{A}$ below can be formed in the next result. We make the above identification for coefficients A without mention.

We can now state the two results on which our analysis stands. Proposition 3.3 reformulates this Cauchy–Riemann system (12) further, by solving for the r-derivatives, as the vector-valued ODE (17) for the conormal gradient f defined below. This formulation is well suited for the Neumann and regularity problems. For the Dirichlet problem, we use instead a similar first-order system formulation of the equation; see Proposition 3.5. As explained in [Part I, Section 3], the vector-valued potential v appearing there should be thought of as containing some generalized conjugate functions as tangential part. In the case of the unit disk, we make this rigorous in Section 5 and come back to this in Section 13. The fundamental object is the following.

Definition 3.2. The conormal gradient of a weak solution u to $\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$ is the section $f : \mathbb{R}^+ \times S^n \to \mathcal{V}$ defined by

$$f_t = e^{-(n+1)t/2}\begin{bmatrix}(Ag_r)_\perp \\ (g_r)_\parallel\end{bmatrix}, \tag{15}$$

where $r = e^{-t}$ and $g = \nabla_{\boldsymbol{x}} u$. The map $g_r \mapsto f_t$ is called the gradient-to-conormal gradient map.

Proposition 3.3. *The pointwise transformation*

$$A \mapsto \hat{A} := \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1}A_{\perp\parallel} \\ A_{\parallel\perp}A_{\perp\perp}^{-1} & A_{\parallel\parallel} - A_{\parallel\perp}A_{\perp\perp}^{-1}A_{\perp\parallel}\end{bmatrix}$$

is a self-inverse bijective transformation of the set of bounded matrices which are strictly accretive on $\mathcal{H}_1$.

For a pair of coefficients $A \in L_\infty(\mathbb{O}^{1+n};\mathcal{L}(\mathbb{C}^{(1+n)m}))$ and $B \in L_\infty(\mathbb{R}_+ \times S^n;\mathcal{L}(\mathcal{V}))$ which are strictly accretive on $\mathcal{H}_1$ and such that $B = \hat{A}$, the gradient-to-conormal gradient map gives a one-to-one correspondence, with inverse the conormal gradient-to-gradient map

$$f_t \mapsto g_r = r^{-\frac{n+1}{2}}((Bf_t)_\perp \vec{n} + (f_t)_\parallel), \tag{16}$$

where $t = \ln(1/r)$, between solutions $g \in L_2^{\mathrm{loc}}(\mathbb{O}^{1+n};\mathbb{C}^{(1+n)m})$ to the Cauchy–Riemann system (12) in $\mathbb{O}^{1+n}\setminus\{0\}$ distribution sense, and solutions $f \in L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathcal{H})$, with $\int_1^\infty \|f_t\|_2^2\,dt < \infty$, to the equation

$$\partial_t f + (DB + \tfrac{n-1}{2}N)f = 0, \tag{17}$$

in $\mathbb{R}_+ \times S^n$ distributional sense.

Recall that the Ricci curvature of S^n is $n-1$, so the constant $\frac{n-1}{2}$ is related to curvature. On the other hand, the exponent $\frac{n+1}{2}$ appearing in the correspondence $g_r \leftrightarrow f_t$ is the only exponent for which no powers of r remain in (17). It turns out that this also makes the gradient-to-conormal gradient map an L_2 isomorphism, since

$$\int_{\mathbb{O}^{1+n}} |g|^2\,d\boldsymbol{x} \approx \int_0^1 \|g_r\|_2^2 r^n\,dr \approx \int_0^\infty \|f_t\|_2^2\,dt. \tag{18}$$

Proof. The stated properties of the matrix transformation are straightforward to verify, using the observation that $e^{(n+1)t}\,\mathrm{Re}(B_t f_t, f_t) = \mathrm{Re}(A_r g_r, g_r)$. See [Part I, Proposition 4.1] for details.

(i) Assume first that the equations (12) hold on $\mathbb{O}^{1+n}\setminus\{0\}$. In polar coordinates $\boldsymbol{x} = rx$, the equations $\mathrm{div}_{\boldsymbol{x}}(Ag) = 0$, $\mathrm{curl}_{\boldsymbol{x}}(g) = 0$ give

$$\begin{cases} r^{-n}\partial_r(r^n(Ag)_\perp) + r^{-1}\,\mathrm{div}_S(Ag)_\parallel = 0,\\ \partial_r(rg_\parallel) - \nabla_S g_\perp = 0.\end{cases}$$

Next we pull back the equations to $\mathbb{R}_+\times S^n$. Write $(Ag)_\perp = r^{-(n+1)/2} f_\perp$ and $(Ag)_\parallel = A_{\parallel\perp}g_\perp + A_{\parallel\parallel}g_\parallel$. Then $g_\perp = r^{-(n+1)/2}A_{\perp\perp}^{-1}(f_\perp - A_{\perp\parallel}f_\parallel)$ and $g_\parallel = r^{-(n+1)/2} f_\parallel$, and the equations further become

$$\begin{cases} r^{-n}\partial_r(r^{(n-1)/2} f_\perp) + r^{-(n+3)/2}\,\mathrm{div}_S(B_{\parallel\perp}f_\perp + B_{\parallel\parallel}f_\parallel) = 0,\\ \partial_r(r^{(1-n)/2} f_\parallel) - r^{-(n+1)/2}\nabla_S(B_{\perp\perp}f_\perp + B_{\perp\parallel}f_\parallel) = 0.\end{cases}$$

Using product rule for ∂_r and the chain rule $-r\partial_r = \partial_t$, this yields the equation (17).

It remains to check that $f_t \in \mathcal{H}$ for almost every $t > 0$. This is equivalent to $(A_r g_r)_\perp \in \mathsf{R}(\mathrm{div}_S)$ and $(g_r)_\parallel \in \mathsf{R}(\nabla_S)$ for a.e. $r\in(0,1)$. To see $(A_r g_r)_\perp \in \mathsf{R}(\mathrm{div}_S)$ amounts to seeing that $\int_{S^n}(A_r g_r)_\perp dx = 0$. We apply Gauss's theorem as follows. For any radial function $\phi \in C_0^\infty(\mathbb{O}^{1+n};\mathbb{C}^m)$, the divergence equation gives $\int_{\mathbb{O}^{1+n}}(Ag, \nabla\phi)\,d\boldsymbol{x} = 0$. Taking, for a.e. $r\in(0,1)$, the limit as ϕ approaches the characteristic function for balls $\{|\boldsymbol{x}| < r\}$ shows that $\int_{rS^n}(Ag)_\perp dx = 0$. To check $(g_r)_\parallel \in \mathsf{R}(\nabla_S)$ we distinguish first $n = 1$. In that case, a similar application of Stokes' theorem shows that $\int_{S^1}(\vec{\tau}, g_r)\,dx = 0$ for a.e. $r \in (0,1)$. For $n \geq 2$, that $\mathrm{curl}_S((g_r)_\parallel) = 0$ is a consequence of $\mathrm{curl}_{\boldsymbol{x}}\, g = 0$ and the general fact that pullbacks and the exterior derivative commute. Hence $f_t \in \mathcal{H}$.

(ii) Conversely, assume that (17) holds and $f_t \in \mathcal{H}$ for a.e. $t > 0$. Define the corresponding function $g \in L_2^{\mathrm{loc}}(\mathbb{O}^{1+n};\mathbb{C}^{(1+n)m})$ by the conormal gradient-to-gradient map and note that $\mathrm{curl}_S((g_r)_\parallel) = 0$. Reversing the rewriting of the equations in (i) shows that $\mathrm{div}_{\boldsymbol{x}}(Ag) = 0$, $\mathrm{curl}_{\boldsymbol{x}}(g) = 0$ hold on $\mathbb{O}^{1+n}\setminus\{0\}$. This proves the proposition. ☐

Corollary 3.4. *For any coefficients $A \in L_\infty(\mathbb{O}^{1+n};\mathcal{L}(\mathbb{C}^{(1+n)m}))$ which are strictly accretive in the sense of (2), gradients of weak solutions to (1) in $\mathbb{O}^{1+n}$ are in one-to-one correspondence with $\mathbb{R}_+\times S^n$ distributional solutions to the equation (17), belonging to $L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathcal{H})$ with estimate $\int_1^\infty \|f_t\|_2^2\,dt < \infty$.*

Proof. Combine Proposition 2.1 and Proposition 3.3. ☐

There is a second way of constructing weak solutions, which we now describe.

Proposition 3.5. *Let A and $B = \hat{A}$ be as in Proposition 3.3. Assume that $v \in L_2^{\mathrm{loc}}(\mathbb{R}_+; D(D))$ with $\int_1^\infty \|Dv_t\|_2^2 dt < \infty$ satisfies*

$$\partial_t v + (BD - \tfrac{n-1}{2}N)v = 0 \tag{19}$$

in $\mathbb{R}_+\times S^n$ distributional sense. Then

$$u_r := r^{-\frac{n-1}{2}}(v_t)_\perp, \qquad r = e^{-t} \in (0,1),$$

extends to a weak solution of $\mathrm{div}_{\boldsymbol{x}}\, A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$, and Dv equals the conormal gradient of u.

Proof. By definition of u in the statement, $(Dv)_\| = \nabla_S v_\perp = r^{(n+1)/2}(r^{-1}\nabla_S u)$. On the other hand, taking the normal component of $\partial_t v + (BD - \frac{n-1}{2}N)v = 0$ gives

$$\partial_t v_\perp - A_{\perp\perp}^{-1}(\mathrm{div}_S\, v_\| + A_{\perp\|}\nabla_S v_\perp) + \sigma v_\perp = 0,$$

or equivalently

$$\begin{aligned}(Dv)_\perp = -\mathrm{div}_S\, v_\| &= -A_{\perp\perp}(\partial_t + \sigma)v_\perp + A_{\perp\|}\nabla_S v_\perp\\ &= r^{(n+1)/2}(A_{\perp\perp}\partial_r u + A_{\perp\|} r^{-1}\nabla_S u) = r^{(n+1)/2}(A\nabla_{\boldsymbol{x}} u)_\perp.\end{aligned}$$

These equations hold in $\mathbb{O}^{1+n}\setminus\{0\}$. Next, applying D to (19) yields

$$\big(\partial_t + DB + \tfrac{n-1}{2}N\big)(Dv) = 0.$$

Thus $f := Dv$ satisfies (17) and $f_t \in \mathsf{R}(D) = \mathcal{H}$. By Corollary 3.4, there is a weak solution $\tilde{u}$ in $\mathbb{O}^{1+n}$ of the divergence form equation associated to f. In particular, $f_\| = r^{(n+1)/2}(r^{-1}\nabla_S \tilde{u})$ and $f_\perp = r^{(n+1)/2}(A\nabla_{\boldsymbol{x}}\tilde{u})_\perp$. Applying the conormal gradient-to-gradient map, we deduce $\nabla_{\boldsymbol{x}}\tilde{u} = \nabla_{\boldsymbol{x}} u$ in $\mathbb{O}^{1+n}\setminus\{0\}$ distribution sense. In particular, $u = \tilde{u} + c$ in $\mathbb{O}^{1+n}\setminus\{0\}$ for some constant c. As $\tilde{u}+c$ is also a weak solution in $\mathbb{O}^{1+n}$ to the divergence form equation with coefficients A, this provides us with the desired extension for u. □

For perturbations A of radially independent coefficients, Corollary 12.8(i) proves a converse of this result, i.e., the existence of such a vector-valued potential v containing a given solution u to $\mathrm{div}_{\boldsymbol{x}}\, A\nabla_{\boldsymbol{x}} u = 0$ as normal component. We do not know whether such v can be defined for general coefficients (except in $\mathbb{O}^2$, see Section 5).

Remark 3.6. Assume that the coefficients A are defined in $\mathbb{R}^{1+n}$ and that the accretivity condition (2) or (14) holds for a.e $r \in (0,\infty)$. As in Proposition 3.3, there is also a one-to-one correspondence between solutions $g \in L_2^{\mathrm{loc}}(\mathbb{R}^{1+n}\setminus\overline{\mathbb{O}^{1+n}}; L_2(S^n;\mathcal{V}))$ to $\mathrm{div}_{\boldsymbol{x}}(Ag) = 0$, $\mathrm{curl}_{\boldsymbol{x}}\, g = 0$ in the exterior of the unit ball and solutions $f : \mathbb{R}_- \to \mathcal{H}$ to the equation $\partial_t f + (DB + \frac{n-1}{2}N)f = 0$ for $t < 0$ in $L_2(\mathbb{R}_-;\mathcal{H})$. Also, as in Proposition 3.5, L_2^{loc}-solutions $v : \mathbb{R}_- \to L_2(S^n; \mathsf{D}(D))$ to the equation $\partial_t v + (BD - \frac{n-1}{2}N)v = 0$ for $t < 0$, give weak solutions u to $\mathrm{div}_{\boldsymbol{x}}\, A\nabla_{\boldsymbol{x}} u = 0$ in the exterior of the unit ball.

4. Study of the infinitesimal generator

In this section, we study the *infinitesimal generators* $DB_0 + \frac{n-1}{2}N$ and $B_0 D - \frac{n-1}{2}N$ for the vector-valued ODEs appearing in (17) and (19) for radially independent coefficients

$$B_0 = \widehat{A_1} \in L_\infty(S^n; \mathcal{L}(\mathcal{V})),$$

strictly accretive on $\mathcal{H}$ with constant $\kappa = \kappa_{B_0} > 0$. Note that strict accretivity of A_1 on $\mathcal{H}_1$ is needed for the construction of $B_0 = \widehat{A_1}$ as a multiplication operator. Once we have B_0, only strict accretivity of B_0 on $\mathcal{H}$ is needed in our analysis. This has the following consequences used often in this work. First, $B_0 : \mathcal{H} \to B_0\mathcal{H}$ is an isomorphism. Second, the map $P_{\mathcal{H}} B_0$ is an isomorphism of $\mathcal{H}$.

The first operator will be used to get estimates of $\nabla_{\boldsymbol{x}} u$, needed for the Neumann and regularity problems. The second operator will be used to get estimates of the potential u, needed for the Dirichlet problem.

Definition 4.1. Let $\sigma \in \mathbb{R}$. Define the unbounded linear operators

$$D_0 := DB_0 + \sigma N \quad \text{and} \quad \widetilde{D}_0 := B_0 D - \sigma N$$

in $L_2(S^n; \mathcal{V})$, with domains $\mathsf{D}(D_0) := B_0^{-1}\mathsf{D}(D)$ and $\mathsf{D}(\widetilde{D}_0) := \mathsf{D}(D)$ respectively. Here $B_0^{-1}(X) := \{f \in L_2 \; ; \; B_0 f \in X\}$. When more convenient, we use the notation $D_{A_1} := D_0$ and $\widetilde{D}_{A_1} := \widetilde{D}_0$.

For these two operators, we have the following intertwining and duality relations.

Lemma 4.2. *In the sense of unbounded operators, we have $D_0 D = D\widetilde{D}_0$ and $(\widetilde{D}_{A_1})^* = DB_0^* - \sigma N = -N(D\widehat{A_1^*} + \sigma N)N$.*

Proof. The proof is straightforward, using the identity $B_0^* = N\widehat{A_1^*}N$ for the second statement. □

Proposition 4.3. *In $L_2 = L_2(S^n; \mathcal{V})$, the operator D_0 is a closed unbounded operator with dense domain. There is a topological Hodge splitting*

$$L_2 = \mathcal{H} \oplus B_0^{-1}\mathcal{H}^\perp,$$

i.e the projections $P^1_{B_0}$ and $P^0_{B_0}$ onto $\mathcal{H}$ and $B_0^{-1}\mathcal{H}^\perp$ in this splitting are bounded. The operator D_0 leaves $\mathcal{H}$ invariant, and the restricted operator $D_0 : \mathcal{H} \to \mathcal{H}$, with domain $\mathsf{D}(D_0) \cap \mathcal{H}$, is closed, densely defined, injective, onto, and has a compact inverse.

If $\sigma \neq 0$, then $D_0 : L_2 \to L_2$ is also injective and onto, and $D_0|_{B_0^{-1}\mathcal{H}^\perp} = \sigma N$.

If $\sigma = 0$, then $D_0 = DB_0$, $\mathsf{N}(D_0) = B_0^{-1}\mathcal{H}^\perp$ and $\mathsf{R}(D_0) = \mathcal{H}$ are closed and invariant. In particular, when $n = 1$, $\dim \mathsf{N}(D_0) = 2m = \dim(L_2/\mathsf{R}(D_0))$.

Proof. The splitting is a consequence of the strict accretivity of B_0 on $\mathcal{H}$, and it is clear that $\mathcal{H}$ is invariant under D_0. Note that

$$(iN)(DB_0 + \sigma N) = (iND)B_0 + i\sigma,$$

where iN is unitary on L_2 as well as $\mathcal{H}$, and where $iND = -iDN$ is a self-adjoint operator with range $\mathcal{H}$. This shows that D_0 is closed, densely defined, injective and onto on $\mathcal{H}$, and on L_2 when $\sigma \neq 0$, as a consequence of properties of operators such as $(iND)B_0$ stated in [Auscher et al. 2010b, Proposition 3.3].

Next we show that $D_0 : \mathcal{H} \to \mathcal{H}$ has a compact inverse. Write $D_0 = D(P_{\mathcal{H}} B_0) + \sigma N$. Since $P_{\mathcal{H}} B_0$ is an isomorphism on $\mathcal{H}$, it suffices to prove that the inverse of $D : \mathcal{H} \to \mathcal{H}$ is compact. Note that $\mathsf{D}(\nabla_S) = W^1_2(S^n; \mathbb{C}^m)$ is compactly embedded in $L_2(S^n; \mathbb{C}^m)$ by Rellich's theorem. In particular $\nabla_S : \mathsf{R}(\mathrm{div}_S) \to \mathsf{R}(\nabla_S)$ has compact inverse. Since $\nabla_S^* = -\mathrm{div}_S$, it follows that $\mathrm{div}_S : \mathsf{R}(\nabla_S) \to \mathsf{R}(\mathrm{div}_S)$ has a compact inverse as well. This proves that the inverse of D is compact on $\mathcal{H}$.

The remaining properties when $\sigma \neq 0$ and $\sigma = 0$ are straightforward and are left to the reader. □

Proposition 4.4. *In $L_2 = L_2(S^n; \mathcal{V})$, the operator $\widetilde{D}_0$ is a closed unbounded operator with dense domain. There is a topological Hodge splitting*

$$L_2 = B_0\mathcal{H} \oplus \mathcal{H}^\perp,$$

i.e the projections $\widetilde{P}^1_{B_0}$ and $\widetilde{P}^0_{B_0}$ onto $B_0\mathcal{H}$ and $\mathcal{H}^\perp$ in this splitting are bounded. Here $\mathcal{H}^\perp \subset \mathsf{D}(\widetilde{D}_0)$ and $\widetilde{D}_0$ leaves $\mathcal{H}^\perp$ invariant.

If $\sigma \neq 0$, then $\widetilde{D}_0 : L_2 \to L_2$ is also injective and onto, and $\widetilde{D}_0|_{\mathcal{H}^\perp} = -\sigma N$.

If $\sigma = 0$, then $\widetilde{D}_0 = B_0 D$, $\mathsf{N}(\widetilde{D}_0) = \mathcal{H}^\perp$ and $\mathsf{R}(\widetilde{D}_0) = B_0\mathcal{H}$ is closed. In particular, the subspace $B_0\mathcal{H}$ is invariant under $\widetilde{D}_0$ and when $n = 1$, $\dim \mathsf{N}(\widetilde{D}_0) = 2m = \dim(L_2/\mathsf{R}(\widetilde{D}_0))$.

Proof. These results for $\widetilde{D}_0$ follow from Proposition 4.3 by duality, using Lemma 4.2. □

Remark 4.5. The reader familiar with [Axelsson et al. 2006b] and [Part I] should note carefully the following fundamental difference between the cases $\sigma \neq 0$ and $\sigma = 0$. When $\sigma = 0$, each of the operators D_0 and $\widetilde{D}_0$ is of the type considered in the papers just cited, and each has two complementary invariant subspaces. On the other hand when $\sigma \neq 0$, the operator D_0 has in general only the invariant subspace $\mathcal{H}$, and $\widetilde{D}_0$ only has the invariant subspace $\mathcal{H}^\perp$. One can define an induced operator $\widetilde{D}_0$ on the quotient space $L_2/\mathcal{H}^\perp$, but this cannot be realized as an action in a subspace complementary to $\mathcal{H}^\perp$ in L_2 in general. As σ will be set to $\frac{n-1}{2}$ this means for us a difference in the treatment of $n = 1$ (space dimension 2) and $n \geq 2$ (space dimension 3 and higher).

We prove here a technical lemma for later use.

Lemma 4.6. *There is a unique isomorphism*

$$\mathcal{H} \to L_2/\mathcal{H}^\perp : h \mapsto \tilde{h} \tag{20}$$

such that $D_0 h = D\tilde{h}$ for $h \in \mathcal{H} \cap \mathsf{D}(D_0)$.

Proof. When $\sigma = 0$, we can take $\tilde{h} := B_0 h \in B_0\mathcal{H} \approx L_2/\mathcal{H}^\perp$ as $D_0 h = DB_0 h = D\tilde{h}$.

When $\sigma \neq 0$, we use that $D : L_2 \to \mathcal{H}$ is surjective with null space $\mathcal{H}^\perp$. This defines $\tilde{h}$ for $h \in \mathcal{H} \cap \mathsf{D}(D_0)$. With D^{-1} the compact inverse of $D : \mathcal{H} \to \mathcal{H}$, the equation $D_0 h = D\tilde{h}$ is equivalent to

$$P_{\mathcal{H}} B_0 h + \sigma D^{-1} N h = P_{\mathcal{H}} \tilde{h}. \tag{21}$$

This shows that (20) extends to a bounded map since $\|\tilde{h}\|_{L_2/\mathcal{H}^\perp} \approx \|P_{\mathcal{H}}\tilde{h}\|_2$. Moreover, since $P_{\mathcal{H}} B_0$ is an isomorphism on $\mathcal{H}$, we have also the lower bound $\|h\|_2 \lesssim \|P_{\mathcal{H}} B_0 h\|_2 \lesssim \|\tilde{h}\|_{L_2/\mathcal{H}^\perp} + \|D^{-1}h\|_2$, which shows that (20) is a semi-Fredholm operator. If $\tilde{h} = 0$, then (21) implies $h \in \mathcal{H} \cap \mathsf{D}(D_0)$. Therefore $D_0 h = 0$ and (20) is injective. Since the range contains the dense subspace $\mathsf{D}(D)/\mathcal{H}^\perp$, invertibility follows. □

5. Elliptic systems in the unit disk

In dimension $n = 1$, i.e., for the unit disk $\mathbb{O}^2 \subset \mathbb{R}^2$ with boundary S^1, some special phenomena occurs. In this section we collect these results.

Lemma 5.1. *If $n = 1$ and A is strictly accretive in the sense of (2), then A is pointwise strictly accretive, i.e.,*

$$\mathrm{Re}(A(\boldsymbol{x})v, v) \geq \kappa |v|^2, \quad \textit{for all } v \in \mathbb{C}^{2m}, \textit{ and a.e. } \boldsymbol{x} \in \mathbb{O}^2.$$

Proof. By scaling and continuity, it suffices to consider $v = \big[(z_\alpha) \;\; (w_\alpha)\big]^t \in \mathbb{C}^{2m}$, with $w_\alpha \neq 0$, $\alpha = 1, \dots, m$. In (2), let

$$u^\alpha(re^{i\theta}) := (ik)^{-1} w_\alpha e^{ik\frac{r}{r_0}\frac{z_\alpha}{w_\alpha}} \eta(e^{i\theta}) e^{ik\theta}, \quad \alpha = 1, \dots, m,$$

with a smooth function $\eta : S^1 \to \mathbb{R}$, $k \in \mathbb{Z}_+$ and $r_0 \in (0, 1)$. Using polar coordinates and letting $k \to \infty$ yields

$$\mathrm{Re} \int_{S^1} (A(r_0 x)v, v)\, |\eta(x)|^2 \, dx \geq \kappa |v|^2 \int_{S^1} |\eta(x)|^2 \, dx, \qquad \text{for a.e. } r_0 \in (0, 1).$$

Taking $|\eta|^2$ to be an approximation to the identity at a given point $x \in S^1$ now proves the pointwise strict accretivity in the statement. □

Definition 5.2. Assume that $A \in L_\infty(\mathbb{O}^2; \mathscr{L}(\mathbb{C}^{2m}))$ is pointwise strictly accretive. Given a weak solution $u \in W_2^{1,\mathrm{loc}}(\mathbb{O}^2; \mathbb{C}^m)$ to $\mathrm{div}_{\boldsymbol{x}}\, A\nabla_{\boldsymbol{x}} u = 0$, we say that a solution $\tilde{u} \in W_2^{1,\mathrm{loc}}(\mathbb{O}^2; \mathbb{C}^m)$ to $J\nabla_{\boldsymbol{x}} \tilde{u} = A\nabla_{\boldsymbol{x}} u$ is a *conjugate* of u, where $J := \left[\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right]$.

We note that since $A\nabla_{\boldsymbol{x}} u$ is divergence-free, there always exists a conjugate of u, unique modulo constants in $\mathbb{C}^m$. The notion of conjugate solution for two dimensional divergence form equations, in the scalar case $m = 1$, goes back to Morrey. See [Morrey 1966]. Note that when $A = I$, the system $J\nabla_{\boldsymbol{x}} \tilde{u} = \nabla_{\boldsymbol{x}} u$ is the anti Cauchy–Riemann equations.

Lemma 5.3. *Assume that $A \in L_\infty(\mathbb{O}^2; \mathscr{L}(\mathbb{C}^{2m}))$ is pointwise strictly accretive. Let $u \in W_2^{1,\mathrm{loc}}(\mathbb{O}^2; \mathbb{C}^m)$ be a weak solution to* $\mathrm{div}_{\boldsymbol{x}}\, A\nabla_{\boldsymbol{x}} u = 0$. *Then*

$$A\nabla_{\boldsymbol{x}} u = J\nabla_{\boldsymbol{x}} \tilde{u} \iff \begin{Bmatrix} (A\nabla_{\boldsymbol{x}} u)_\perp = -(\nabla_{\boldsymbol{x}} \tilde{u})_\parallel \\ (\tilde{A}\nabla_{\boldsymbol{x}} \tilde{u})_\perp = (\nabla_{\boldsymbol{x}} u)_\parallel \end{Bmatrix} \iff \tilde{A}\nabla_{\boldsymbol{x}} \tilde{u} = J^t \nabla_{\boldsymbol{x}} u \implies \mathrm{div}_{\boldsymbol{x}}\, \tilde{A}\nabla_{\boldsymbol{x}} \tilde{u} = 0,$$

where $\tilde{A}$ is the conjugate coefficient *defined by*

$$\tilde{A} := J^t A^{-1} J.$$

We have

$$\tilde{A} = \begin{bmatrix} (d - ca^{-1}b)^{-1} & (d - ca^{-1}b)^{-1} ca^{-1} \\ a^{-1}b(d - ca^{-1}b)^{-1} & a^{-1} + a^{-1}b(d - ca^{-1}b)^{-1} ca^{-1} \end{bmatrix} \quad \text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

When $m = 1$, this reduces to $\tilde{A} = (\det A)^{-1} A^t$.

Here, we have identified the tangential part $(\,\cdot\,)_\parallel$ with its component along $\vec{\tau}$. (See below.)

Proof. The equivalences and implication are verified from $\tilde{A} = J^t A^{-1} J$. The explicit formula for $\tilde{A}$ is classical if $m = 1$. If $m \geq 2$, the proposed formula for $\tilde{A}$ can be checked by a straightforward computation. Note that $a, b, c, d \in L_\infty(\mathbb{O}^2; \mathscr{L}(\mathbb{C}^m))$ and all the entries of $\tilde{A}$ as well: the inverses are pointwise multiplications. We omit further details. □

We next show that the vector-valued potential v in Proposition 3.5 contains, along with u as normal component, its conjugate $\tilde{u}$ as tangential component. To do that, it is convenient to identify $\mathscr{V}$ with the trivial bundle $\mathbb{C}^{2m}$ by identifying the tangential component β to the tangential part $\beta\vec{\tau} \in (T_{\mathbb{C}} S^1)^m$.

Given this identification, D becomes

$$D = \begin{bmatrix} 0 & -\partial_{\vec{\tau}} \\ \partial_{\vec{\tau}} & 0 \end{bmatrix},$$

where $\partial_{\vec{\tau}}$ denotes the tangential counter clockwise derivative of m-tuples of scalar functions on S^1. A coefficient $A \in L_\infty(\mathbb{O}^2; \mathcal{L}(\mathbb{C}^{2m}))$ is thus identified with its matrix representation in the moving frame $\{\vec{n}, \vec{\tau}\}$. We remark that this identification commutes with the matrix J.

Proposition 5.4. *Let $A \in L_\infty(\mathbb{O}^2; \mathcal{L}(\mathbb{C}^{2m}))$ be pointwise strictly accretive and let*

$$B := \hat{A} \in L_\infty(\mathbb{O}^2; \mathcal{L}(\mathbb{C}^{2m})).$$

Assume that $v = \begin{bmatrix} u & \tilde{u} \end{bmatrix}^t \in L_2^{\text{loc}}(\mathbb{R}_+; D(D))$ with $\int_1^\infty \|Dv_t\|_2^2\, dt < \infty$ is a $\mathbb{R}_+ \times S^n$ distributional solution to $\partial_t v + BDv = 0$ as in Proposition 3.5, so that $u_r = (v_t)_\perp$, $r = e^{-t}$, is a weak solution to $\operatorname{div}_{\boldsymbol{x}} A \nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^2$. Then $\tilde{u}$ is a conjugate to u.

Conversely, given a weak solution u to $\operatorname{div}_{\boldsymbol{x}} A \nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^2$ and a conjugate $\tilde{u}$, the potential vector $v = \begin{bmatrix} u & \tilde{u} \end{bmatrix}^t$ has the above properties.

Note that the construction of v this way is a feature of two-dimensional systems as compared to higher dimensions.

Proof. Applying J^t to $\partial_t v + BDv = 0$ gives $\partial_t (J^t v) + \widetilde{B} D(J^t v) = 0$ with $\widetilde{B} = J^t BJ$, since $JD = DJ$. A calculation shows that $\widetilde{B} = \widehat{\tilde{A}}$. Applying Proposition 3.5 shows that $\tilde{u}_r = (J^t v_t)_\perp$ is a weak solution to $\operatorname{div}_{\boldsymbol{x}} \tilde{A} \nabla_{\boldsymbol{x}} \tilde{u} = 0$. Also we know that Dv and $D\tilde{v}$ are respectively equal to the conormal gradients of u and $\tilde{u}$, and since $J^t v = \tilde{v}$, this gives the middle term in the equivalence of Lemma 5.3. Thus $\tilde{u}$ is a conjugate of u. The converse is immediate to check and left to the reader. $\square$

We finish this section with the following simple expressions for the projections $P^0_{B_0}$ and $\widetilde{P}^0_{B_0}$ of Propositions 4.3 and 4.4 when $n = 1$. We still make the identification $\mathcal{V} \approx \mathbb{C}^{2m}$.

Lemma 5.5. *Let $A_1 \in L_\infty(\mathbb{O}^2; \mathcal{L}(\mathbb{C}^{2m}))$ be pointwise strictly accretive radially independent coefficients, and let $B_0 := \widehat{A_1} \in L_\infty(\mathbb{O}^2; \mathcal{L}(\mathbb{C}^{2m}))$ the corresponding coefficients. Then*

$$\widetilde{P}^0_{B_0} g = \left(\int_{S^1} B_0^{-1}\, dx \right)^{-1} \int_{S^1} B_0^{-1} g\, dx, \quad g \in L_2(S^1; \mathbb{C}^{2m}),$$

and $P^0_{B_0} = B_0^{-1} \widetilde{P}^0_{B_0} B_0$.

Proof. By accretivity, $(\int_{S^1} B_0^{-1}\, dx)^{-1}$ is a bounded operator (called the harmonic mean of B_0). If $g \in B_0 \mathcal{H}$, then $B_0^{-1} g \in \mathcal{H}$ and $\int_{S^1} B_0^{-1} g\, dx = 0$, hence $\widetilde{P}^0_{B_0} g = 0$, follows. On the other hand, if $g \in \mathcal{H}^\perp$, then g is constant, and therefore the right hand side equals

$$\left(\int_{S^1} B_0^{-1}\, dx \right)^{-1} \left(\int_{S^1} B_0^{-1}\, dx \right) g = g.$$

This proves the expression for $\widetilde{P}^0_{B_0}$. The formula for $P^0_{B_0}$ comes from the similarity relation

$$DB_0 = B_0^{-1} (B_0 D) B_0.$$

$\square$

6. Resolvent estimates

In this section we prove that the spectra of D_0 and $\widetilde{D}_0$ are contained in certain double hyperbolic regions, and we estimate the resolvents. For parameters $0 < \omega < \nu < \pi/2$ and $\sigma \in \mathbb{R}$, define closed and open hyperbolic regions in the complex plane by

$$\begin{aligned} S_{\omega,\sigma} &:= \{x+iy \in \mathbb{C} \ ; \ (\tan^2\omega)x^2 \geq y^2+\sigma^2\}, \\ S^o_{\nu,\sigma} &:= \{x+iy \in \mathbb{C} \ ; \ (\tan^2\nu)x^2 > y^2+\sigma^2\}, \\ S_{\omega,\sigma+} &:= \{x+iy \in \mathbb{C} \ ; \ (\tan\omega)x \geq (y^2+\sigma^2)^{1/2}\}, \\ S^o_{\nu,\sigma+} &:= \{x+iy \in \mathbb{C} \ ; \ (\tan\nu)x > (y^2+\sigma^2)^{1/2}\}. \end{aligned}$$

When $\sigma = 0$, we drop the subscript σ in the notation for the sectorial regions.

Proposition 6.1. *On $L_2 = L_2(S^n;\mathcal{V})$, there is a constant $\omega \in (0,\pi/2)$, depending only on $\|B_0\|_\infty$ and the accretivity constant κ_{B_0}, such that the spectra of the operators D_0 and $\widetilde{D}_0$ are contained in the double hyperbolic region $S_{\omega,\sigma}$. Moreover, there are resolvent bounds*

$$\|(\lambda - D_0)^{-1}\|_{L_2\to L_2}, \|(\lambda-\widetilde{D}_0)^{-1}\|_{L_2\to L_2} \leq \frac{1}{\sqrt{y^2+\sigma^2}/\tan\omega - |x|},$$

for all $\lambda = x+iy \notin S_{\omega,\sigma}$. These same estimates hold for the restriction $D_0 : \mathcal{H} \to \mathcal{H}$.

Proof. (i) To prove the spectral estimates for D_0, assume that

$$(DB_0 + \sigma N - x - iy)u = f.$$

Introduce the auxiliary operator $N_y := i\sigma N - yI$, and note that $\|N_y\| = \|N_y^{-1}\|^{-1} = \sqrt{y^2+\sigma^2}$. Multiply with N_y and rewrite as

$$(N_yD)B_0u + i(y^2+\sigma^2)u = N_yf + xN_yu. \tag{22}$$

Now split the function u as

$$u = u_1 + u_0 \in \mathcal{H} \oplus B_0^{-1}\mathcal{H}^\perp,$$

and note that $\|u\| \approx \|u_1\| + \|u_0\|$. Apply the associated bounded projections $P^i_{B_0}$ to (22) to get

$$\begin{aligned} (N_yD)B_0u_1 + i(y^2+\sigma^2)u_1 &= P^1_{B_0}N_yf + xP^1_{B_0}N_yu, \\ 0 + i(y^2+\sigma^2)u_0 &= P^0_{B_0}N_yf + xP^0_{B_0}N_yu. \end{aligned}$$

Take the imaginary part of the inner product between the first equation and B_0u_1 (using that N_yD is self-adjoint), and the second equation and u_0 to get

$$\begin{aligned} (y^2+\sigma^2)\operatorname{Re}(u_1, B_0u_1) &= \operatorname{Im}(P^1_{B_0}N_yf, B_0u_1) + \operatorname{Im}(xP^1_{B_0}N_yu, B_0u_1), \\ (y^2+\sigma^2)\|u_0\|^2 &= \operatorname{Im}(P^0_{B_0}N_yf, u_0) + \operatorname{Im}(xP^0_{B_0}N_yu, u_0). \end{aligned}$$

Using the strict accretivity of B_0 on $\mathcal{H}$ gives the estimate

$$(y^2+\sigma^2)\|u\|^2 \leq C_1\sqrt{y^2+\sigma^2}(\|f\|\|u\| + |x|\|u\|^2),$$

for some constant $C_1 < \infty$. Thus $\|u\| \le (\sqrt{y^2+\sigma^2}/C_1 - |x|)^{-1}\|f\|$.

(ii) To prove a similar lower bound on $\widetilde{D}_0$, assume that $(B_0 D - \sigma N - x - iy)u = f$, and rewrite as

$$B_0 D N_y^{-1} N_y u + i N_y u = f + xu. \tag{23}$$

Write $N_y u = B_0 u_1 + u_0 \in B_0\mathcal{H} \oplus \mathcal{H}^\perp$. Apply the bounded projections $\widetilde{P}^i_{B_0}$ to (23) to get

$$\begin{aligned} B_0(DN_y^{-1})B_0u_1 + iB_0u_1 &= \widetilde{P}^1_{B_0} f + x\widetilde{P}^1_{B_0} u,\\ 0 + iu_0 &= \widetilde{P}^0_{B_0} f + x\widetilde{P}^0_{B_0} u. \end{aligned}$$

Recall that $B_0 : \mathcal{H} \to B_0\mathcal{H}$ is an isomorphism and apply its inverse $B_0^{-1} : B_0\mathcal{H} \to \mathcal{H}$ to the first equation. Then take the imaginary part of the inner product between the first equation and B_0u_1 (using that DN_y^{-1} is self-adjoint), and the second equation and u_0 to get

$$\begin{aligned} \mathrm{Re}(u_1, B_0u_1) &= \mathrm{Im}\,(B_0^{-1}\widetilde{P}^1_{B_0} f, B_0u_1) + \mathrm{Im}\,(xB_0^{-1}\widetilde{P}^1_{B_0} u, B_0u_1),\\ \|u_0\|^2 &= \mathrm{Im}\,(\widetilde{P}^0_{B_0} f, u_0) + \mathrm{Im}\,(x\widetilde{P}^0_{B_0} u, u_0). \end{aligned}$$

Using the strict accretivity of B_0 on $\mathcal{H}$ gives the estimate

$$(y^2+\sigma^2)\|u\|^2 \le C_2(|x|\|u\| + \|f\|)(y^2+\sigma^2)^{1/2}\|u\|,$$

for some constant $C_2 < \infty$. Thus $\|u\| \le (\sqrt{y^2+\sigma^2}/C_2 - |x|)^{-1}\|f\|$.

(iii) Using that $DB_0 + \sigma N$ and $B_0^* D + \sigma N$ are adjoint operators, combining the results in (i) and (ii) shows that both operators $D_0 - \lambda$ and $\widetilde{D}_0 - \lambda$ are onto, with bounded inverse, when $\lambda \notin S_{\omega,\sigma}$. Here $\omega := \arctan(\max(C_1, C_2))$. The estimates on $\mathcal{H}$ follow. □

We shall also need the following off-diagonal estimates for the resolvents, both in L_2 and in L_p for p near 2.

Lemma 6.2. *(i) There exist $\epsilon, \alpha > 0$ such that for $|\frac{1}{p} - \frac{1}{2}| < \epsilon$, closed sets $E, F \subset S^n$ and $f \in L_p(S^n; \mathcal{V})$ with* supp $f \subset E$ *and $t \in \mathbb{R}$,*

$$\|(I + itD_0)^{-1} f\|_{L_p(F)} \lesssim e^{-\alpha d(E,F)/|t|}\|f\|_{L_p(E)},$$

where $d(E, F)$ is the distance between the sets E and F.

(ii) There exist $q > 2$ with $\frac{1}{2} - \frac{1}{q} < \epsilon$, and $\alpha > 0$ such that for closed sets $E, F \subset S^n$ and $f \in L_2(S^n; \mathcal{V})$ with supp $f \subset E$ *and $f_\parallel = 0$ and $|t| \le 1$,*

$$\|(I + itD_0)^{-1} f\|_{L_q(F)} \lesssim |t[^{-n(\frac{1}{2}-\frac{1}{q})} e^{-\alpha d(E,F)/|t|}\|f\|_{L_2(E)},$$

Proof. We first prove (i). The case $p = 2$ follows the argument in [Auscher et al. 2010a, Proposition 5.1]. It remains to prove L_p boundedness for p near 2 as, the L_q off-diagonal bounds follow by interpolation with the L_2 off-diagonal bounds for q between p and 2.

For $f \in L_p \cap L_2$, we let $h = (I + itD_0)^{-1} f$ and wish to prove $\|h\|_p \lesssim \|f\|_p$ when p is near 2 and uniformly in t. To prove this, we rewrite the equation $(I + itD_0)h = f$ first as $(I + it\sigma N + itDB_0)h = f$

and then in terms of a divergence form equation, with coefficients $A_1 = \widehat{B}_0$. Write $h = \big[(A_1\tilde{h})_\perp \ \ \tilde{h}_\parallel\big]^t$ and $f = \big[(A_1\tilde{f})_\perp \ \ \tilde{f}_\parallel\big]^t$. Then

$$\begin{cases}(1-it\sigma)(A_1\tilde{h})_\perp - it\,\mathrm{div}_S(A_1\tilde{h})_\parallel = (A_1\tilde{f})_\perp,\\ (1+it\sigma)\tilde{h}_\parallel + it\nabla_S\tilde{h}_\perp = f_\parallel.\end{cases}$$

Using the second equation to eliminate $\tilde{h}_\parallel$ in the first equation, and letting $z = (1+it\sigma)^{-1}$, we obtain

$$L\tilde{h}_\perp = \begin{bmatrix}1 & -it\bar{z}\,\mathrm{div}_S\end{bmatrix}\begin{bmatrix}\bar{z}(A_1)_{\perp\perp}\tilde{f}_\perp + (\bar{z}-z)(A_1)_{\perp\parallel}\tilde{f}_\parallel\\ -z(A_1)_{\parallel\parallel}\tilde{f}_\parallel\end{bmatrix}$$

with

$$L := \begin{bmatrix}1 & -it\bar{z}\,\mathrm{div}_S\end{bmatrix} A_1 \begin{bmatrix}1\\ -itz\nabla_S\end{bmatrix} = \begin{bmatrix}1 & -i\tau\,\mathrm{div}_S\end{bmatrix} A_\theta \begin{bmatrix}1\\ -i\tau\nabla_S\end{bmatrix}$$

and $A_\theta = D_{-\theta}A_1D_\theta$ with $tz = e^{i\theta}\tau$, $\tau = |tz|$, and D_θ the diagonal matrix with entries $1, e^{i\theta}$ in the normal/tangential splitting. We note that A_θ is strictly accretive on $\mathscr{H}_1$ with the same constants as A_1, and that $|z| \le 1$ and $|\tau| \le |\sigma|^{-1}$. We claim that L is invertible from the Sobolev space $W^1_p(S^n;\mathbb{C}^m)$ equipped with the scaled norm

$$\|u\|_{W^1_p} := \left(\int_{S^n} (|u(x)|^2 + |\tau\nabla_S u(x)|^2)^{p/2}\,dx\right)^{1/p} \tag{24}$$

to its dual, with bounds independent of τ, θ, for p in a neighborhood of 2.

To prove this, if we rescale from the sphere S^n of radius 1 to the sphere $S^n_{1/\tau}$ of radius $1/\tau$, we obtain the same equation with $A_\theta, A_1, z_\pm$ unchanged, $f(x), h(x)$ replaced by $f(\tau x), h(\tau x)$, and $\tau\,\mathrm{div}_S, \tau\nabla_S$ replaced by $\mathrm{div}_{S^n_{1/\tau}}, \nabla_{S^n_{1/\tau}}$, and we want to show $\|h(\tau\,\cdot)\|_{L_p(S^n_{1/\tau})} \lesssim \|f(\tau\,\cdot)\|_{L_p(S^n_{1/\tau})}$ (with implicit constant uniform in τ, θ). Thus it is enough to set $\tau = 1$ and work on S^n, as long as we only use estimates on S^n which hold (with same constant) on $S^n_{1/\tau}$ as well.

Having set $\tau = 1$, we have, for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, estimates

$$\|Lu\|_{W^{-1}_p} \le \|A_\theta\|_\infty \|u\|_{W^1_p} = \|A_1\|_\infty \|u\|_{W^1_p},$$

where $\|u\|_{W^{-1}_p} := \sup_{\|v\|_{W^1_q}=1} |(u,v)|$ and (u,v) denotes the $L_2(S^n;\mathbb{C}^m)$ pairing extended in the sense of distributions. For $p = q = 2$, the accretivity assumption on A_θ yields $\|Lu\|_{W^{-1}_2} \ge \kappa\|u\|_{W^1_2}$. Applying the extrapolation result of Šneĭberg [1974] to the complex interpolation scale $\{W^1_p\}_{1<p<\infty}$, shows the existence of $\epsilon > 0$ such that

$$\|Lu\|_{W^{-1}_p} \approx \|u\|_{W^1_p},$$

for $\left|\frac{1}{p} - \frac{1}{2}\right| < \epsilon$. (Even for $\tau \ne 1$, one can verify that the Sobolev norms given by (24) on $S^n_{1/\tau}$ (with $\tau\nabla_S$ replaced $\nabla_{S^n_{1/\tau}}$) are equivalent to the ones given by the complex interpolation method, with constant independent of τ. Hence ϵ depends only on the ellipticity constants and dimension, and is thus independent of τ, θ.) Applying this isomorphism, we obtain the resolvent estimate

$$\|h\|_p \approx \|\tilde{h}_\perp\|_p + \|\tilde{h}_\parallel\|_p \lesssim \|\tilde{h}_\perp\|_{W^1_p} + \|\tilde{f}_\parallel\|_p \lesssim \|f\|_p.$$

In the second step we used $\tilde{h}_\parallel = -ie^{i\theta}\nabla_S \tilde{h}_\perp + z\tilde{f}_\parallel$ and $|z| \le 1$ (recall we have rescaled and set $\tau = 1$). In the third step we used the fact that $\begin{bmatrix} 1 & -i\,\mathrm{div}_S \end{bmatrix} : L_p(S^n; \mathbb{C}^{(1+n)m}) \to W_p^{-1}$ is an isometry since $\begin{bmatrix} 1 & -i\nabla_S \end{bmatrix}^t : W_q^1 \to L_q(S^n; \mathbb{C}^{(1+n)m})$ is one. This finishes the proof of (i).

To prove the inequality (ii), the above argument shows that $L\tilde{h}_\perp = z_-(A_1)_{\perp\perp}\tilde{f}_\perp$ and $\tilde{h}_\parallel = -itz_+\nabla_S\tilde{h}_\perp$. Having rescaled in the same way, the Sobolev embedding $L_2 \subset W_q^{-1}$ for some $q > 2$ with $\frac{1}{2} - \frac{1}{q} < \epsilon$, allows us to conclude that $h \in L_q(S^n; \mathcal{V})$ and since we assume $|t| \le 1$, we have $|\tau| \approx |t|$ and obtain $\|h\|_q \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{q})}\|f\|_2$, the power coming from scaling. It suffices to interpolate again with the L_2 off-diagonal decay, and conclude for any exponent between 2 and q. $\square$

We state the following useful corollary. Here and subsequently, $\widetilde{N}_*^p$ is defined as $\widetilde{N}_*$ replacing L_2 averages by L_p averages and M is the Hardy–Littlewood maximal operator.

Corollary 6.3. *For ϵ as above and $\left|\frac{1}{p} - \frac{1}{2}\right| < \epsilon$, we have the pointwise inequalities*

$$\widetilde{N}_*^p((I + itD_0)^{-1}f) \lesssim M(|f|^p)^{1/p},$$
$$\widetilde{N}_*^p((I + it\widetilde{D}_0)^{-1}f) \lesssim M(|f|^p)^{1/p},$$

and, for some $p < 2$ with $\frac{1}{p} - \frac{1}{2} < \epsilon$,

$$\widetilde{N}_*(((I + it\widetilde{D}_0)^{-1}f)_\perp) \lesssim M(|f|^p)^{1/p}.$$

Proof. We fix a Whitney region $W_0 = W(t_0, x_0)$ in $\mathbb{R}_+ \times S^n$. Then

$$|W_0|^{-1}\int_{W_0} \left|(I + itD_0)^{-1}f(x)\right|^p dt\,dx \lesssim M(|f|^p)(x_0)$$

follows directly from the off-diagonal decay of Lemma 6.2 as in [Auscher et al. 2008, Proposition 2.56]. Next, $|W_0|^{-1}\int_{W_0}\left|(I + it\widetilde{D}_0)^{-1}f(x)\right|^p dt\,dx \lesssim M(|f|^p)(x_0)$ follows by testing against $g \in L_q(W_0; \mathcal{V})$, supported in W_0 with $1/p + 1/q = 1$. We have

$$\int_{W_0} \big((I + it\widetilde{D}_0)^{-1}f(x), g(t, x)\big)\,dt\,dx = \int_{t_0/c_0}^{c_0t_0} \big(f, (I - it\widetilde{D}_0^*)^{-1}g_t\big)\,dt$$

so that for each fixed t, using that $\widetilde{D}_0^* = DB_0^* - \sigma N$ has the same form as D_0, we can use the L_q off-diagonal decay for each $t \approx t_0$ and obtain for any $M > 0$,

$$|W_0|^{-1}\int_{W_0} |(I + it\widetilde{D}_0)^{-1}f(x)|^p\,dt\,dx \lesssim \sum_{j\ge 2} 2^{-jM}|B(x_0, 2^jt_0)|^{-1}\int_{B(x_0,2^jt_0)} |f(x)|^p\,dx, \tag{25}$$

using standard computations on annuli around $B(x_0, t_0)$ in S^n. Details are left to the reader.

The last estimate starts in the same way with $g \in L_2(W_0; \mathcal{V})$, but since we want to estimate the normal component of $(I + it\widetilde{D}_0)^{-1}f$ we assume that $(g_t)_\parallel = 0$ for each t. The second estimate in Lemma 6.2, implies that $(I - it\widetilde{D}_0^*)^{-1}g_t =: h_t$ has L_q estimates with decay. Thus using Hölder's inequality on

$\int_{t_0/c_0}^{c_0 t_0}(f, h_t)\,dt$ with exponent q on h_t and dual exponent on f yields

$$\left(|W_0|^{-1}\int_{W_0}\left|((I+it\widetilde{D}_0)^{-1}f)_\perp(x)\right|^2 dt\,dx\right)^{1/2} \lesssim \sum_{j\geq 2} 2^{-jM}\left(|B(x_0,2^j t_0)|^{-1}\int_{B(x_0,2^j t_0)}|f(x)|^p\,dx\right)^{1/p} \tag{26}$$

and the conclusion follows. □

7. Square function estimates and functional calculus

All the remainder of this article rests on the square function estimate below.

Theorem 7.1. *Let $n\geq 1$. The operator $D_0 = DB_0 + \sigma N$, with $\sigma\in\mathbb{R}$ fixed but arbitrary, has square function estimates*

$$\int_0^\infty \|tD_0(1+t^2D_0^2)^{-1}f\|_2^2\,\frac{dt}{t} \approx \|f\|_2^2, \quad \textit{for all } f\in \mathsf{R}(D_0).$$

The estimate $\lesssim$ holds for all $f\in L_2(S^n,\mathbb{C}^m)$. The same estimates hold for $\widetilde{D}_0 = B_0D - \sigma N$.

Proof. Note that equivalence can only hold on $\overline{\mathsf{R}(D_0)} = \mathsf{R}(D_0)$, which equals $L_2(S^n;\mathcal{V})$ if $\sigma\neq 0$ and $\mathcal{H}$ if $\sigma = 0$. By standard duality arguments, the estimates $\gtrsim$ on $\mathsf{R}(D_0)$ follows from the estimates $\lesssim$ for D_0^*. See [Albrecht et al. 1996]. Further D_0^* is of type $\widetilde{D}_0$. Hence it is enough to prove

$$\int_0^\infty \|tD_0(1+t^2D_0^2)^{-1}f\|_2^2\,\frac{dt}{t} \lesssim \|f\|_2^2 \tag{27}$$

for all $f\in L_2(S^n;\mathcal{V})$, and similarly for $\widetilde{D}_0$. Consider first the operator D_0.

(i) We first reduce (27) to

$$\int_0^1 \left\|tDB_0(1+t^2(DB_0)^2)^{-1}f\right\|_2^2\,\frac{dt}{t} \lesssim \|f\|_2^2 \tag{28}$$

for all $f\in L_2(S^n;\mathcal{V})$. First note that

$$\int_1^\infty \left\|tD_0(I+t^2D_0^2)^{-1}f\right\|_2^2\frac{dt}{t} \lesssim \int_1^\infty \left\|t^2D_0^2(I+t^2D_0^2)^{-1}f\right\|_2^2\frac{dt}{t^3} \lesssim \int_1^\infty \|f\|_2^2\,\frac{dt}{t^3} \approx \|f\|_2^2,$$

using that D_0 has bounded inverse by Proposition 4.3. (When $n=1$, write $f = f_1 + f_0\in\mathcal{H}\oplus B_0^{-1}\mathcal{H}^\perp$. The above estimate goes through for f_1, and the contribution from f_0 is zero.) For the integral $\int_0^1$, we may ignore the zero-order term in D_0, using the idea from [Auscher et al. 2010a, Section 9]. Indeed,

$$\left\|(I+itD_0)^{-1}f - (I+itDB_0)^{-1}f\right\|_2 = \left\|(I+itD_0)^{-1}it\sigma N(I+itDB_0)^{-1}f\right\|_2 \lesssim |t|\,\|f\|_2.$$

Since $2itD_0(I+t^2D_0^2)^{-1} = (I-itD_0)^{-1} - (I+itD_0)^{-1}$, and similarly for DB_0, subtraction yields

$$\int_0^1 \left\|tD_0(I+t^2D_0^2)^{-1}f\right\|_2^2\frac{dt}{t} \lesssim \int_0^1 \left\|tDB_0(I+t^2(DB_0)^2)^{-1}f\right\|_2^2\frac{dt}{t} + \int_0^1 t\,dt\,\|f\|_2^2.$$

(ii) Next, using a partition of unity, it suffices to show that

$$\int_0^1 \|\zeta t DB_0(I+t^2(DB_0)^2)^{-1} f\|_2^2 \frac{dt}{t} \lesssim \|f\|_2^2, \tag{29}$$

when ζ is a smooth cutoff that is 1 on a neighborhood of supp f. Indeed, L_2-off diagonal estimates of $tDB_0(1+t^2(DB_0)^2)^{-1}$ from Lemma 6.2 and again

$$2itDB_0(I+t^2(DB_0)^2)^{-1} = (I-itDB_0)^{-1} - (I+itDB_0)^{-1}$$

show in this case that

$$\|(1-\zeta)tDB_0(I+t^2(DB_0)^2)^{-1} f\|_2^2 \lesssim t^2\|f\|_2^2.$$

(iii) To prove (29), we assume that f and ζ are supported inside the lower hemisphere, which we parametrize by $\mathbb{O}^n$ using stereographic coordinates:

$$\rho : \mathbb{R}^n \to S^n : y \mapsto x = \frac{|y|^2-1}{|y|^2+1} e_0 + \frac{2y}{|y|^2+1},$$

where $e_0 \in \mathbb{R}^{1+n}$ is a fixed unit normal vector to $\mathbb{R}^n \subset \mathbb{R}^{1+n}$, which covers all S^n, except the north pole $e_0 \in S^n$. Note that ρ is a conformal map with length dilation d^{-1} and Jacobian determinant $dx/dy = d^{-n}$, where

$$d(y) := (|y|^2+1)/2.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^{1+n} : y \mapsto \partial_y \rho(y)$ be the differential of ρ, and note that $T^t T = d^{-2} I$. Define adjoint rescaled pullbacks and pushforwards

$$\rho^* : L_2(\rho(\mathbb{O}^n); \mathcal{V}) \to L_2(\mathbb{O}^n; \mathbb{C}^{(1+n)m}) : \begin{bmatrix} f_\perp & f_\parallel \end{bmatrix}^t \mapsto \begin{bmatrix} d^{-n}(f_\perp \circ \rho) & T^t(f_\parallel \circ \rho) \end{bmatrix}^t,$$
$$\rho_* : L_2(\mathbb{O}^n; \mathbb{C}^{(1+n)m}) \to L_2(\rho(\mathbb{O}^n); \mathcal{V}) : \begin{bmatrix} g_\perp & g_\parallel \end{bmatrix}^t \mapsto \begin{bmatrix} (g_\perp \circ \rho^{-1}) & (d^n T g_\parallel) \circ \rho^{-1} \end{bmatrix}^t.$$

Note that $(\rho_*)^{-1} = \begin{bmatrix} d^n & 0 \\ 0 & d^{2-n} \end{bmatrix} \rho^*$. We claim that

$$\rho^* D = D_\rho \begin{bmatrix} d^n & 0 \\ 0 & d^{2-n} \end{bmatrix} \rho^*, \quad \text{where } D_\rho := \begin{bmatrix} 0 & -\mathrm{div}_y \\ \nabla_y & 0 \end{bmatrix}.$$

Indeed, the tangential part of the equation is the chain rule, and the normal component is the adjoint statement. We consider D_ρ as a self-adjoint closed unbounded operator in $L_2(\mathbb{O}^n; \mathbb{C}^{(1+n)m})$ with domain

$$\mathsf{D}(D_\rho) := \begin{bmatrix} H_0^1(\mathbb{O}^n; \mathbb{C}^m) \\ \mathsf{D}(\mathrm{div}_y) \end{bmatrix},$$

where H_0^1 denotes the Sobolev W_2^1 functions vanishing at the boundary S^{n-1}.

Next we map coefficients B_0 in $\rho(\mathbb{O}^n)$ to coefficients $B_\rho := (\rho_*)^{-1} B_0 (\rho^*)^{-1}$ in $\mathbb{O}^n$, and claim that B_ρ is strictly accretive on $\mathsf{R}(D_\rho)$. To see this, let $g \in \mathsf{R}(D_\rho)$. Then $\mathrm{curl}_y\, g_\parallel = 0$ and $g_\parallel$ is normal on $\partial\mathbb{O}^n$ (or if $n = 1$ we have $\int_{-1}^1 g_\parallel dy = 0$). Writing $g = \rho^* f$ and extending f by 0 outside $\rho(\mathbb{O}^n)$, it follows that $f \in \mathcal{H}_1$. (To see this, write $g_\parallel = \nabla_y u$ with $u \in H^1_0(\mathbb{O}^n; \mathbb{C}^m)$, and extend u by 0 to an $H^1(\mathbb{R}^n; \mathbb{C}^m)$-function.) The assumed strict accretivity of B_0 on $\mathcal{H}_1$ gives

$$\mathrm{Re}\int_{\mathbb{O}^n} (B_\rho g, g)\, dy = \mathrm{Re}\int_{\mathbb{O}^n} ((\rho_*)^{-1} B_0 f, \rho^* f)\, dy = \mathrm{Re}\int_{S^n} (B_0 f, f)\, dx \geq \kappa \int_{S^n} |f|^2\, dx \approx \int_{\mathbb{O}^n} |g|^2\, dy.$$

Thus we obtain a bisectorial operator $D_\rho B_\rho$ in $L_2(\mathbb{O}^n; \mathbb{C}^{(1+n)m})$, and we observe the intertwining relation

$$\rho^* D B_0 f = D_\rho \begin{bmatrix} d^n & 0 \\ 0 & d^{2-n} \end{bmatrix} \rho^* \rho_* B_\rho \rho^* f = D_\rho B_\rho \rho^* f.$$

for f supported in the lower hemisphere. In $\mathbb{O}^n$, let $K := \{|y| \leq 1/4\}$. By rotational invariance, it is enough to consider those $f = (\rho^*)^{-1} g$ with g supported on K and $\zeta = (\rho^*)^{-1}\eta = \eta \circ \rho^{-1}$ with $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $\eta = 1$ on $\{|y| \leq 1/2\}$ and $\operatorname{supp} \eta \subset \{|y| \leq 3/4\}$. Using $\eta g = g$ and understanding η, ζ as the operators of pointwise multiplication by η, ζ, one can check the identity

$$\begin{aligned}
&\zeta(I + itDB_0)^{-1} f - (\rho^*)^{-1}\eta^2 (I + itD_\rho B_\rho)^{-1} g \\
&\qquad = \zeta(I + itDB_0)^{-1}(\rho^*)^{-1}(\eta g) - \zeta(\rho^*)^{-1}\eta(I + itD_\rho B_\rho)^{-1} g \\
&\qquad = \zeta(I + itDB_0)^{-1}(\rho^*)^{-1}\left(\eta(I + itD_\rho B_\rho) - \rho^*(I + itDB_0)(\rho^*)^{-1}\eta\right)(I + itD_\rho B_\rho)^{-1} g \\
&\qquad = \zeta(I + itDB_0)^{-1}(\rho^*)^{-1} it[\eta, D_\rho] B_\rho (I + itD_\rho B_\rho)^{-1} g.
\end{aligned}$$

As in (i) above, subtracting the corresponding equation with t replaced by $-t$, yields the estimate

$$\|\zeta t D B_0 (I + t^2 (DB_0)^2)^{-1} f - (\rho^*)^{-1}\eta^2 t D_\rho B_\rho (I + t^2 (D_\rho B_\rho)^2)^{-1} g\|_2 \lesssim |t| \|g\|_2,$$

since $[\eta, D_\rho]$ is bounded. As $\|f\|_2 \approx \|g\|_2$ by the support conditions, (29) will follow from

$$\int_0^1 \|t D_\rho B_\rho (I + t^2 (D_\rho B_\rho)^2)^{-1} g\|_2^2 \frac{dt}{t} \lesssim \|g\|_2^2, \quad \text{for all } g \in L_2(\mathbb{O}^n; \mathbb{C}^{(1+n)m}).$$

(iv) The latter square function estimate follows from combining [Axelsson et al. 2006a, Theorem 2] and [Axelsson et al. 2006b, Proposition 3.1(iii)], the latter purely being of functional analytic content. (See [Auscher et al. 2010a, Section 10.1] where this is pointed out.)

(v) Consider now $\widetilde{D}_0$. Similarly one can reduce to prove $\lesssim$ for $B_0 D$. On $\mathsf{N}(B_0 D) = \mathcal{H}^\perp$, this is trivial. On $\mathsf{R}(B_0 D) = B_0 \mathcal{H}$ we use that $B_0 D$ is similar to DB_0 on $\mathsf{R}(DB_0) = \mathcal{H}$ through the isomorphism $B_0 : \mathcal{H} \to B_0 \mathcal{H}$. Thus the square function upper estimate for $B_0 D$ follows by similarity from the one for DB_0. $\square$

The square function estimates from Theorem 7.1 provide bounds on the $S^o_{\nu,\sigma}$-holomorphic functional calculus of the operators D_0 and $\widetilde{D}_0$, adapting the techniques described in [Albrecht et al. 1996]. Write

$$\begin{aligned}
H(S^o_{\nu,\sigma}) &:= \{\text{holomorphic } b\, ;\, S^o_{\nu,\sigma} \to \mathbb{C}\}, \\
H_\infty(S^o_{\nu,\sigma}) &:= \{b \in H(S^o_{\nu,\sigma})\ ;\ \sup\{|b(\lambda)|\ ;\ \lambda \in S^o_{\nu,\sigma}\} < \infty\}, \\
\Psi(S^o_{\nu,\sigma}) &:= \{b \in H(S^o_{\nu,\sigma})\ ;\ |b(\lambda)| \lesssim \min(|\lambda|^a, |\lambda|^{-a}),\ \text{for some } a > 0\}.
\end{aligned}$$

We summarize the result for the $S^o_{\nu,\sigma}$-holomorphic functional calculus in the following corollary. The proof is a straightforward adaption of the results in [Albrecht et al. 1996].

Corollary 7.2. *Assume $\sigma \in \mathbb{R}$ and $D_0 = DB_0 + \sigma N$. Fix $\omega < \nu < \pi/2$. There is a unique continuous Banach algebra homomorphism*

$$H_\infty(S^o_{\nu,\sigma}) \to \mathcal{L}(R(D_0)) : b \mapsto b(D_0),$$

with bounds $\|b(D_0)f\|_2 \le C(\sup_{S^o_{\nu,\sigma}} |b(\lambda)|)\|f\|_2$ for all $f \in R(D_0)$, where C only depends on $\|B_0\|_\infty$, κ_{B_0}, n and σ, and with the following two properties. If $b \in \Psi(S^o_{\nu,\sigma})$ then

$$b(D_0) = \frac{1}{2\pi i}\int_\gamma b(\lambda)(\lambda - D_0)^{-1} d\lambda \in \mathcal{L}(R(D_0)),$$

where $\gamma := \partial S_{\theta,\sigma}$, $\omega < \theta < \nu$, oriented counter clockwise around $S_{\omega,\sigma}$. For any $b \in H_\infty(S^o_{\nu,\sigma})$ we have strong convergence

$$\lim_{k\to\infty} \|b_k(D_0)f - b(D_0)f\|_2 = 0, \quad \textit{for each } f \in R(D_0),$$

whenever $b_k \in \Psi(S^o_{\nu,\sigma})$, $k = 1, 2, \dots$, are uniformly bounded, i.e., $\sup_{k,\lambda} |b_k(\lambda)| < \infty$, and converges pointwise to b.

The corresponding results hold for $\widetilde{D}_0 = B_0 D - \sigma N$ replacing D_0 by throughout.

We remark that the square function estimates in Theorem 7.1 hold when $\psi(z) = z(1+z^2)^{-1}$ is replaced by any $\psi \in \Psi(S^o_\nu)$ which is nonzero on both components of $S^o_{\nu,\sigma}$. We have

$$\int_0^\infty \|\psi(tD_0)f\|_2^2 \frac{dt}{t} \approx \|f\|_2^2, \quad \text{for all } f \in \mathsf{R}(D_0). \tag{30}$$

A similar extension of the square function estimates holds for $\widetilde{D}_0$.

Fundamental operators in this paper are the following.

Definition 7.3. (i) Let $\chi^+(\lambda)$ and $\chi^-(\lambda)$ be the characteristic functions for the right and left half planes. Define spectral projections $E_0^\pm := \chi^\pm(D_0)$ and $\widetilde{E}_0^\pm := \chi^\pm(\widetilde{D}_0)$ on $\mathsf{R}(D_0)$ and $\mathsf{R}(\widetilde{D}_0)$ respectively.

(ii) Define closed and dense defined operators $\Lambda = |D_0| := \operatorname{sgn}(D_0)D_0$ and $\tilde{\Lambda} = |\widetilde{D}_0| := \operatorname{sgn}(\widetilde{D}_0)\widetilde{D}_0$ on $L_2(S^n; \mathcal{V})$. Here $|\lambda| := \lambda \operatorname{sgn}(\lambda)$ and $\operatorname{sgn}(\lambda) := \chi^+(\lambda) - \chi^-(\lambda)$.

Define operators $e^{-t\Lambda}$ and $e^{-t\tilde{\Lambda}}$ on $\mathsf{R}(D_0)$ and $\mathsf{R}(\widetilde{D}_0)$ respectively by applying Corollary 7.2 with $b(\lambda) = e^{-t|\lambda|}$, $t > 0$.

When $\sigma = 0$, $\mathsf{R}(\widetilde{D}_0) = B_0\mathcal{H} = B_0\mathsf{R}(D_0)$ are strict subspaces of L_2 and it is convenient to extend the above operators to all L_2. Using the Hodge splitting $L_2 = B_0\mathcal{H} \oplus \mathcal{H}^\perp$, on $\mathcal{H}^\perp$ the operator $\widetilde{D}_0 = B_0 D$ is already 0 and $\tilde{\Lambda} = |B_0 D|$ is naturally defined by 0. Using the other Hodge splitting $L_2 = \mathcal{H} \oplus B_0^{-1}\mathcal{H}^\perp$, on $B_0^{-1}\mathcal{H}^\perp$ the operator $D_0 = DB_0$ is already 0 and $\Lambda = |DB_0|$ is naturally defined by 0. It follows that $e^{-t\tilde{\Lambda}}$ and $e^{-t\Lambda}$ are naturally extended to L_2, by letting $e^{-t\tilde{\Lambda}}|_{\mathcal{H}^\perp} := I$ and $e^{-t\Lambda}|_{B_0^{-1}\mathcal{H}^\perp} := I$.

However, for the projections the extension is more subtle. Indeed, we see for the functional calculus of $\widetilde{D}_0 = DB_0 - \sigma N$ that

$$b(\widetilde{D}_0) = b(-\sigma N) = \begin{bmatrix} b(\sigma)I & 0 \\ 0 & b(-\sigma)I \end{bmatrix}$$

on $\mathcal{H}^\perp$ when $\sigma \neq 0$ using the definition of N. As we are mainly interested in $\sigma = \frac{n-1}{2}$, it is more natural for consistency of notation towards applications to divergence form equations to define the operators for $\sigma = 0$ by continuity $\sigma \to 0^+$. Thus set

$$b(B_0D) := b(B_0D|_{B_0\mathcal{H}})\widetilde{P}^1_{B_0} + \begin{bmatrix} b(0+)I & 0 \\ 0 & b(0-)I \end{bmatrix}\widetilde{P}^0_{B_0},$$

where $b(0\pm) := \lim_{\pm\lambda\in S_{\omega+},\lambda\to 0} b(\lambda)$, assuming the limits exist, $b(B_0D|_{B_0\mathcal{H}})$ is the operator from Corollary 7.2 and $\widetilde{P}^i_{B_0}$, $i = 0, 1$, denote the projections from Proposition 4.4 onto the subspaces in the Hodge splitting $L_2 = B_0\mathcal{H} \oplus \mathcal{H}^\perp$.

Similarly, for $\sigma \neq 0$, we have $D_0 = DB_0 + \sigma N$ so $D_0 = \sigma N$ on $B_0^{-1}\mathcal{H}^\perp$. For $\sigma = 0$, set

$$b(DB_0) := b(DB_0|_{\mathcal{H}})P^1_{B_0} + P^0_{B_0}\begin{bmatrix} b(0-)I & 0 \\ 0 & b(0+)I \end{bmatrix},$$

where $P^i_{B_0}$, $i = 0, 1$, denote the projections from Proposition 4.3 onto the subspaces in the Hodge splitting $L_2 = \mathcal{H} \oplus B_0^{-1}\mathcal{H}^\perp$. Remark that $P^0_{B_0}$ on the left of the matrix is needed to obtain an element in $B_0^{-1}\mathcal{H}^\perp$. An elementary calculation shows that this extension of the functional calculus coincides with $(\bar{b}(B_0^*D))^*$, where $\bar{b}(\lambda) = \overline{b(\bar{\lambda})}$, and that the extended functional calculi of D_0 and $\widetilde{D}_0$ thus obtained are intertwined by D.

Taking $b(\lambda) = \lambda$ or $\lambda\,\mathrm{sgn}(\lambda)$, this provides us with the zero extension that we already chose so this is consistent. For the projections, this leads to the following definition.

Definition 7.4. When $\sigma = 0$, extend $\widetilde{E}_0^\pm$, $E_0^\pm$ originally defined on $\mathsf{R}(B_0D) = B_0\mathcal{H}$ and $\mathsf{R}(DB_0) = \mathcal{H}$ respectively from Definition 7.3 to operators on all $L_2(S^n; \mathcal{V})$, letting

$$\begin{cases} \widetilde{E}_0^\pm f := N^\mp f & \text{for all } f \in \mathcal{H}^\perp, \\ E_0^\pm f := P^0_{B_0}N^\pm f & \text{for all } f \in B_0^{-1}\mathcal{H}^\perp. \end{cases}$$

Lemma 7.5. *With $L_2 = L_2(S^n; \mathcal{V})$, the spectral projections $E_0^\pm$ and $\widetilde{E}_0^\pm$ are bounded, we have topological spectral splittings*

$$L_2 = E_0^+L_2 \oplus E_0^-L_2,$$

restricting to $\mathcal{H} = E_0^+\mathcal{H} \oplus E_0^-\mathcal{H}$ in the subspace $\mathcal{H}$ invariant under D_0, and

$$L_2 = \widetilde{E}_0^+L_2 \oplus \widetilde{E}_0^-L_2,$$

restricting to $\mathcal{H}^\perp = \widetilde{E}_0^+\mathcal{H}^\perp \oplus \widetilde{E}_0^-\mathcal{H}^\perp$ in the subspace $\mathcal{H}^\perp$ invariant under $\widetilde{D}_0$. We also have the intertwining relation

$$E_0^\pm D = D\widetilde{E}_0^\pm \tag{31}$$

so that $D : \widetilde{E}_0^\pm L_2 \to E_0^\pm\mathcal{H}$ is surjective.

If $\sigma \geq 0$, then in the latter splitting we have $\widetilde{E}_0^{\pm} = N^{\mp}$ in $\mathcal{H}^{\perp}$. Hence $\widetilde{E}_0^{+}\mathcal{H}^{\perp} = N^{-}\mathcal{H}^{\perp}$ and $\widetilde{E}_0^{-}\mathcal{H}^{\perp} = N^{+}\mathcal{H}^{\perp}$. (On the other hand, if $\sigma < 0$, then $\widetilde{E}_0^{\pm} = N^{\pm}$ in $\mathcal{H}^{\perp}$.)

Proof. When $\sigma \neq 0$, $\mathsf{R}(\widetilde{D}_0) = L_2$ and $L_2 = \mathsf{R}(D_0)$ by Proposition 6.1. Boundedness on L_2 follows from Corollary 7.2. The intertwining property is a consequence of Lemma 4.2. The surjectivity of D easily follows from the spectral subspaces and using $D : L_2 \to \mathcal{H}$ surjective and the splittings. That $\widetilde{E}_0^{\pm} = N^{\mp}$ in $\mathcal{H}^{\perp}$ when $\sigma > 0$ comes from $\widetilde{D}_0 = -\sigma N$ in $\mathcal{H}^{\perp}$ and $\chi^{\pm}(-\sigma N) = N^{\mp}$. The case $\sigma = 0$ follows from Definition 7.4. We leave further details to the reader. □

8. A detour to Kato's square root on Lipschitz surfaces

Let Σ be a surface in $\mathbb{R}^{1+n}$, assumed to be Lipschitz diffeomorphic to S^n through a bilipschitz map $\rho_0 : S^n \to \Sigma$. Let $d\sigma$ denote surface measure on Σ. Consider, for $n, m \geq 1$, coefficient matrices $H \in L_\infty(\Sigma; \mathscr{L}((T_{\mathbb{C}}\Sigma)^m))$ (with $T_{\mathbb{C}}\Sigma$ denoting the complexified tangent bundle) and $h \in L_\infty(\Sigma; \mathscr{L}(\mathbb{C}^m))$, assumed to be strictly accretive in the sense that

$$\mathrm{Re} \int_\Sigma (H(x)\nabla_\Sigma u(x), \nabla_\Sigma u(x))\, d\sigma(x) \geq \kappa \int_\Sigma |\nabla_\Sigma u(x)|^2\, d\sigma(x),$$
$$\mathrm{Re}(h(x)z, z) \geq \kappa |z|^2, \quad \text{a.e. } x \in \Sigma,$$

for all $u \in W_2^1(\Sigma; \mathbb{C}^m)$ and $z \in \mathbb{C}^m$, and some $\kappa > 0$. Then $L := -\mathrm{div}_\Sigma H \nabla_\Sigma$, with $\mathrm{div}_\Sigma := -(\nabla_\Sigma)^*$ in $L_2(\Sigma; d\sigma)$, constructed by the method of sesquilinear forms, is a maximal accretive operator and hL is defined on $\mathsf{D}(L)$ and can be shown to be an ω-sectorial operator on $L_2(\Sigma; d\sigma)$ for some $0 < \omega < \pi$. Thus it has a square root and we have

Theorem 8.1. *The square root of the operator $hL = -h\,\mathrm{div}\, H\nabla_\Sigma$ has domain $D(\sqrt{hL}) = W_2^1(\Sigma; \mathbb{C}^m)$, and estimates $\|\sqrt{hL}u\|_2 \approx \|\nabla_\Sigma u\|_2$.*

In particular for $h = 1$, we obtain a version of the Kato square root problem on Lipschitz surfaces Σ. The presence of h makes the theorem invariant under bilipschitz changes of variables as we shall see in the proof.

Our Theorems 7.1 and 8.1 are inspired by [Axelsson et al. 2006b, Theorem 7.1], and a comparison of these two results is in order. The main novelty in Theorems 7.1 and 8.1, is that these do not require the coefficients B_0 or H to be pointwise strictly accretive, which was needed for the localization argument in [Axelsson et al. 2006b, Theorem 7.1]. This theorem considered more general forms on Σ, and more general compact Lipschitz surfaces Σ. It is straightforward to extend our results Theorems 7.1 and 8.1 here to more general compact Lipschitz manifolds. On the other hand, we do not know how to extend our localization argument here to the case of forms, unless pointwise strict accretivity is assumed.

We also mention that A. Morris [2010] proved similar results on embedded (possibly noncompact) Riemannian manifolds with bounds on the second fundamental form and a lower bound on Ricci curvature.

Proof of Theorem 8.1. A calculation shows the pullback formula

$$(h\,\mathrm{div}_\Sigma\, H\nabla_\Sigma u)(\rho_0(x)) = (\tilde{h}\,\mathrm{div}_S\, \widetilde{H}\nabla_S(u \circ \rho_0))(x), \quad x \in S^n,$$

where $\tilde{h}(x) = |J(\rho_0)(x)|^{-1} h(\rho_0(x))$ and $\widetilde{H}(x) := |J(\rho_0)(x)| (\underline{\rho_0}(x))^{-1} H(\rho_0(x)) (\underline{\rho_0}^t(x))^{-1}$. So we assume that $\Sigma = S^n$ from now on. Let D be as in Definition 3.1 and let

$$B_0 := \begin{bmatrix} h & 0 \\ 0 & H \end{bmatrix} \in L_\infty(S^n; \mathcal{L}(\mathcal{V})).$$

Then B_0 is strictly accretive on the space $\mathcal{H}_1$ from (13) and

$$B_0 D = \begin{bmatrix} 0 & -h \operatorname{div}_S \\ H \nabla_S & 0 \end{bmatrix}.$$

Thus by Theorem 7.1, with $\sigma = 0$, we have bounded functional calculus of $B_0 D$ in $B_0 \mathcal{H}$. Following [Auscher et al. 1997b], we have for $u \in \mathsf{D}(\nabla_S)$ that

$$\begin{bmatrix} \sqrt{hL}u \\ 0 \end{bmatrix} = \sqrt{(B_0 D)^2} \begin{bmatrix} u \\ 0 \end{bmatrix} = \operatorname{sgn}(B_0 D) B_0 D \begin{bmatrix} u \\ 0 \end{bmatrix} = \operatorname{sgn}(B_0 D) \begin{bmatrix} 0 \\ H \nabla_S u \end{bmatrix},$$

so that $\|\sqrt{hL}u\|_2 \approx \|H \nabla_S u\|_2 \approx \|\nabla_S u\|_2$, using that $\operatorname{sgn}(B_0 D)$ is bounded and invertible on $B_0 \mathcal{H}$ and that H is bounded above and below on $\mathsf{R}(\nabla_S)$. □

Remark 8.2. It is interesting to note that we apply Theorem 7.1 with $\sigma = 0$ no matter what the dimension is. If $n \geq 2$, Kato's square root problem on S^n is not directly linked to the boundary operator appearing in (17), associated to the equation $\operatorname{div}_{\boldsymbol{x}} A \nabla_{\boldsymbol{x}} u = 0$ on $\mathbb{O}^{1+n}$, with $\hat{A} = \left[\begin{smallmatrix} h & 0 \\ 0 & H \end{smallmatrix}\right]$, i.e., when one can separate in the equation radial derivatives from tangential derivatives. This is different from the case of the half space ($\mathbb{R}^n$ replacing S^n) and emphasizes the role of curvature.

In view of Section 4, the second-order operator on the boundary associated to this $\operatorname{div}_{\boldsymbol{x}} A \nabla_{\boldsymbol{x}}$ on $\mathbb{O}^{1+n}$, comes from

$$(B_0 D - \sigma N)^2 = \begin{bmatrix} -hL + \sigma^2 & 0 \\ 0 & -H \nabla_S h \operatorname{div}_S + \sigma^2 \end{bmatrix},$$

with $\sigma = (n-1)/2$. Thus, the naturally associated Kato square root is $\sqrt{-hL + \sigma^2}$, and one has

$$\left\| \sqrt{-hL + (\tfrac{n-1}{2})^2}\, u \right\|_2 \approx \|\nabla_S u\|_2 + \tfrac{n-1}{2} \|u\|_2.$$

9. Natural function spaces

By Corollary 3.4, our method to study and construct solutions u to the divergence form equation (1) consists in translating this equation to the ODE (17) for the conormal gradient f in $\mathbb{R}_+ \times S^n$. Conormal gradients of variational solutions belong to $L_2(\mathbb{R}_+ \times S^n; \mathcal{V})$ as noted in (18). The appropriate function spaces for f with Dirichlet/Neumann boundary data for u in $L_2(S^n; \mathbb{C}^m)$ are the following.

Definition 9.1. The (truncated) *modified nontangential maximal function* of f defined on $\mathbb{R}_+ \times S^n$, is

$$\widetilde{N}_*(f)(x) := \sup_{0<t<c_0} t^{-(1+n)/2} \|f \chi_{s<1}\|_{L_2(W(t,x))}, \quad x \in S^n,$$

where

$$W(t,x) := \{(s,y) \in \mathbb{R}_+ \times S^n \ ; \ |y-x| < c_1 t,\ c_0^{-1} < s/t < c_0\}$$

for some fixed constants $c_0 > 1$, $c_1 > 0$. We assume that $c_0 \approx 1$ and $c_1 << 1$, so that the *Whitney regions* $W(t,x)$ are nondegenerate for $t < c_0$. For a function f_0 on $\mathbb{O}^{1+n}$, we have $\widetilde{N}_*^o(f_0) = \widetilde{N}_*(f)$ where $f(t,x) := f_0(e^{-t}x)$, which properly defines $\widetilde{N}_*^o$ in the introduction.

The (truncated) *modified Carleson norm* of f in $\mathbb{R}_+ \times S^n$ is

$$\|f\|_C := \left(\sup_{r(Q)<r_0} \frac{1}{|Q|} \iint_{(0,r(Q))\times Q} \operatorname*{ess\,sup}_{W(t,x)} |f|^2 \frac{dt\,dx}{t} \right)^{1/2},$$

and the sup is taken over geodesic balls $Q \subset S^n$ with volume $|Q|$, and with radius $r(Q)$ less than some fixed constant $r_0 << 1$. For a function f_0 on $\mathbb{O}^{1+n}$, we have $\|f_0\|_C = \|f\|_C$ where $f(t,x) := f_0(e^{-t}x)$, which corresponds to $\|f_0\|_C$ as in (4).

Note that changing the parameters c_1, c_1 does not affect the results.

Definition 9.2. (i) For $g : \mathbb{O}^{1+n} \to \mathbb{C}^{(1+n)m}$, define norms

$$\|g\|^2_{\mathcal{Y}^o} := \int_{\mathbb{O}^{1+n}} |g(\boldsymbol{x})|^2 (1-|\boldsymbol{x}|)\, d\boldsymbol{x},$$

$$\|g\|^2_{\mathcal{X}^o} := \|\widetilde{N}_*^o(g)\|_2^2 + \int_{|\boldsymbol{x}|<e^{-1}} |g(\boldsymbol{x})|^2\, d\boldsymbol{x}.$$

Let $\mathcal{Y}^o$ and $\mathcal{X}^o$ be the Hilbert/Banach spaces of functions g for which the respective norm is finite.

(ii) For $f : \mathbb{R}_+ \times S^n \to \mathcal{V}$, define norms

$$\|f\|^2_{\mathcal{Y}} := \int_0^\infty \|f_t\|_2^2 \min(t,1)\, dt,$$

$$\|f\|^2_{\mathcal{X}} := \|\widetilde{N}_*(f)\|_2^2 + \int_1^\infty \|f_t\|_2^2\, dt.$$

Let $\mathcal{Y}$ and $\mathcal{X}$ be the Hilbert/Banach spaces of sections f for which the respective norm is finite.

The gradient-to-conormal gradient map of Proposition 3.3 is an isomorphism $\mathcal{Y}^o \to \mathcal{Y}$ and $\mathcal{X}^o \to \mathcal{X}$.

Lemma 9.3. *There are estimates*

$$\sup_{0<t<1/2} \frac{1}{t} \int_t^{2t} \|f_s\|_2^2\, ds \lesssim \|\widetilde{N}_*(f)\|_2^2 \lesssim \int_0^1 \|f_s\|_2^2 \frac{ds}{s}, \qquad f \in L_2^{\mathrm{loc}}(\mathbb{R}_+ \times S^n; \mathcal{V}).$$

Denoting by $\mathcal{Y}^*$ the dual space of $\mathcal{Y}$ relative to $L_2(\mathbb{R}_+ \times S^n; \mathcal{V})$, i.e., the space of functions f such that $\int_0^\infty \|f_t\|_2^2 \max(t^{-1},1)\, dt < \infty$, we have continuous inclusions of Banach spaces

$$\mathcal{Y}^* \subset \mathcal{X} \subset L_2(\mathbb{R}_+ \times S^n; \mathcal{V}) \subset \mathcal{Y}.$$

Note that Lemma 9.3 shows that another choice of threshold than $t = 1$ in the definition of the norms for $\mathcal{X}$ and $\mathcal{Y}$ would result in equivalent norms.

Proof. The $L_2^{\mathrm{loc}}(L_2)$ estimates of $\|\widetilde{N}_*(f)\|_2$ is an adaption of the corresponding result for $\mathbb{R}^{1+n}_+$, proved in [Part I, Lemma 5.3]. The remaining statements, except possibly that $\mathcal{X} \subset L_2(\mathbb{R}_+ \times S^n; \mathcal{V})$, are straightforward consequences. To verify this embedding of $\mathcal{X}$, we use the lower bound on $\|\widetilde{N}_*(f)\|_2$ to estimate

$$\int_0^\infty \|f_t\|_2^2\,dt = \sum_{k=0}^\infty \int_{2^{-k-1}}^{2^{-k}} \|f_t\|_2^2\,dt + \int_1^\infty \|f_t\|_2^2\,dt \lesssim \sum_{k=0}^\infty 2^{-k-1}\|\widetilde{N}_*(f)\|_2^2 + \int_1^\infty \|f_t\|_2^2\,dt = \|f\|_{\mathscr{X}}^2. \qquad \square$$

The following lemma gives necessary and (different) sufficient conditions for a multiplication operator $\mathscr{E}$ to map $\mathscr{X}$ into $\mathscr{Y}^*$. Write

$$\|\mathscr{E}\|_{C\cap L_\infty} := \|\mathscr{E}\|_C + \|\mathscr{E}\|_{L_\infty(\mathbb{R}_+\times S^n)}.$$

Lemma 9.4. *For functions $\mathscr{E} : \mathbb{R}_+ \times S^n \to \mathbb{C}^{(1+n)m}$, define the multiplicator norm $\|\mathscr{E}\|_* := \|\mathscr{E}\|_{\mathscr{X}\to\mathscr{Y}^*} = \sup_{\|f\|_{\mathscr{X}}=1} \|\mathscr{E}f\|_{\mathscr{Y}^*}$. Then we have estimates*

$$\|\mathscr{E}\|_{L_\infty(\mathbb{R}_+\times S^n)} \lesssim \|\mathscr{E}\|_* \lesssim \|\mathscr{E}\|_{C\cap L_\infty}.$$

Proof. This is an adaption to the unit ball of [Part I, Lemma 5.5]. As in that proof, the estimate $\|\mathscr{E}\|_\infty \lesssim \|\mathscr{E}\|_*$ follows from the L_2^{loc} estimates in Lemma 9.3. For the second estimate we write

$$\|\mathscr{E}f\|_{\mathscr{Y}^*}^2 = \int_0^a \|\mathscr{E}_t f_t\|_2^2 \frac{dt}{t} + \int_a^\infty \|\mathscr{E}_t f_t\|_2^2\,dt.$$

As in [Part I, Lemma 5.5], the first term is estimated with Whitney averaging and Carleson's theorem. The second term is controlled with $\|\mathscr{E}\|_\infty$. In total, this gives the bound $\|\mathscr{E}f\|_{\mathscr{Y}^*} \lesssim \|\mathscr{E}\|_C\|f\|_{\mathscr{X}} + \|\mathscr{E}\|_\infty\|f\|_{\mathscr{X}}$ as desired. $\square$

Remark 9.5. It has been recently proved in [Hytönen and Rosén 2012] that $\|\mathscr{E}\|_* \gtrsim \|\mathscr{E}\|_{C\cap L_\infty}$ so all of our results use in fact the same condition on $\mathscr{E}$.

We end this section by introducing an auxiliary subspace $\mathscr{Y}_\delta$ of $\mathscr{Y}$.

Definition 9.6. For $\delta > 0$, define the norm

$$\|f\|_{\mathscr{Y}_\delta}^2 := \int_0^\infty \|f_t\|_2^2 \min(t, 1) e^{\delta t}\,dt.$$

Let $\mathscr{Y}_\delta$ be the Hilbert spaces of sections $f : \mathbb{R}_+ \times S^n \to \mathscr{V}$ such that $\|f\|_{\mathscr{Y}_\delta}$ is finite.

Clearly $\mathscr{Y}_\delta \subset \mathscr{Y}$. The motivation for introducing $\mathscr{Y}_\delta$ is the following result.

Proposition 9.7. *Given coefficients $A \in L_\infty(\mathbb{O}^{1+n}; \mathscr{L}(\mathbb{C}^{(1+n)m}))$, which are strictly accretive on $\mathscr{H}_1$, there is $\delta > 0$ such that*

$$\int_1^\infty \|f_t\|_2^2 e^{\delta t}\,dt \lesssim \int_{1/2}^\infty \|f_t\|_2^2\,dt,$$

for all $f \in L_2^{\text{loc}}(\mathbb{R}_+; \mathscr{H})$ solving $\partial_t f + (DB + \frac{n-1}{2}N)f = 0$. Hence, if $f \in \mathscr{Y} \cap L_2^{\text{loc}}(\mathbb{R}_+; \mathscr{H})$ and $\partial_t f + (DB + \frac{n-1}{2}N)f = 0$, then $f \in \mathscr{Y}_\delta$ and $\|f\|_{\mathscr{Y}_\delta} \lesssim \|f\|_{\mathscr{Y}}$.

The proof of Proposition 9.7 uses *reverse Hölder inequalities*.

Theorem 9.8. *Fix $c > 1$. There exist $C < \infty$ and $p > 2$ depending only on n, m, the ellipticity constants $\|A\|_\infty$, κ_A of A and c, such that for any ball B with $c\overline{B} \subset \mathbb{O}^{1+n}$ and any weak solution to $\operatorname{div}_{\boldsymbol{x}}(A\nabla_{\boldsymbol{x}} u) = 0$ in $\mathbb{O}^{1+n}$, we have*

$$\left(\fint_B |\nabla_{\boldsymbol{x}} u|^p \, d\boldsymbol{x}\right)^{1/p} \leq C \left(\fint_{cB} |\nabla_{\boldsymbol{x}} u|^2 \, d\boldsymbol{x}\right)^{1/2}.$$

Proof. This result is due to N. Meyers [1963] for equations. Here, we make sure that the result extends to elliptic systems in the sense of Gårding by giving appropriate references. We begin by noting that the usual Caccioppoli inequality for weak solutions

$$\left(\fint_B |\nabla_{\boldsymbol{x}} u|^2 \, d\boldsymbol{x}\right)^{1/2} \leq Cr \left(\fint_{cB} |u|^2 \, d\boldsymbol{x}\right)^{1/2}$$

for any ball B so that $c\overline{B} \subset \mathbb{O}^{1+n}$, with r its radius, holds for any system that is elliptic in the sense of the Gårding inequality (2). Although not stated like this in [Campanato 1980, Theorem 1.5, p. 46], the proof only uses Gårding's inequality. See also [Auscher and Qafsaoui 2000], where the proof is done explicitly for second- and higher-order equations and it is said (p. 315) that this applies in extenso to such systems. The constant C depends only on n, m, κ, $\|A\|_\infty$ and c. Now, this combined with Poincaré inequality yields

$$\left(\fint_B |\nabla_{\boldsymbol{x}} u|^2 \, d\boldsymbol{x}\right)^{1/2} \leq \left(\fint_{cB} |\nabla_{\boldsymbol{x}} u|^q \, d\boldsymbol{x}\right)^{1/q}$$

for $2(n+1)/(n+3) < q < 2$. Finally, Gehring's method for improvement of reverse Hölder inequalities with increase of radii, presented in [Giaquinta 1983, Theorem 6.3], applies. □

Proof of Proposition 9.7. Corollary 3.4 shows that f is the conormal gradient of a weak solution to $\operatorname{div}_{\boldsymbol{x}} A\nabla_{\boldsymbol{x}} u = 0$ in $\mathbb{O}^{1+n}$. By Hölder's inequality and Theorem 9.8, we have for $g = \nabla_{\boldsymbol{x}} u$ the estimate

$$\left(\int_{|\boldsymbol{x}|<e^{-1}} |g(\boldsymbol{x})|^2 |\boldsymbol{x}|^{-\delta} \, d\boldsymbol{x}\right)^{1/2} \lesssim \left(\int_{|\boldsymbol{x}|<e^{-1}} |g(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p} \lesssim \left(\int_{|\boldsymbol{x}|<e^{-1/2}} |g(\boldsymbol{x})|^2 \, d\boldsymbol{x}\right)^{1/2}$$

for $0 < \delta < (n+1)(p-2)/p$. This translates to the stated estimate for f, using the gradient-to-conormal gradient map from Definition 3.2. □

10. Semigroups and radially independent coefficients

In this section and subsequent ones, we set $\boldsymbol{\sigma} = \frac{\boldsymbol{n}-\mathbf{1}}{\mathbf{2}}$.

In this section, fix radially independent coefficients A_1 and $B_0 = \widehat{A_1}$. We show how to obtain weak solutions of $\operatorname{div}_{\boldsymbol{x}} A_1 \nabla_{\boldsymbol{x}} u = 0$ inside and outside $\mathbb{O}^{1+n}$ using the semigroups associated to Λ and $\tilde{\Lambda}$. Later, we show all weak solutions with prescribed growth towards the boundary have a representation in terms of these semigroups.

Theorem 10.1. *Let f_0 belong to the spectral subspace $E_0^+\mathcal{H}$. Then*

$$f_t := e^{-t\Lambda} f_0$$

gives an $\mathcal{H}$-valued solution to $\partial_t f + D_0 f = 0$, in the strong sense $f \in C^1(\mathbb{R}_+; L_2) \cap C^0(\mathbb{R}_+; D(D_0))$ and in $\mathbb{R}^+ \times S^n$ distribution sense. (In particular f is the conormal gradient of a weak solution of $\operatorname{div}_{\boldsymbol{x}} A_1 \nabla_{\boldsymbol{x}} u = 0$ *in $\mathbb{O}^{1+n}$.) The function f has L_2 limit* $\lim_{t\to 0} f_t = f_0$ *and rapid decay $\|\partial_t^j f_t\|_2 \le C_{j,k}/t^k \|f_0\|_2$, for each $k \ge j \ge 0$. Moreover, we have estimates*

$$\|\partial_t f\|_{\mathcal{Y}} \approx \|f_0\|_2 \approx \|f\|_{\mathcal{X}}.$$

If instead f_0 belongs to the spectral subspace $E_0^- \mathcal{H}$, then define $f_t := e^{t\Lambda} f_0$ for $t < 0$. Then $\partial_t f + D_0 f$ vanishes for $t < 0$. (In particular f is the conormal gradient of a weak solution of $\operatorname{div}_{\boldsymbol{x}} A_1 \nabla_{\boldsymbol{x}} u = 0$ *in $\mathbb{R}^n \setminus \overline{\mathbb{O}^{1+n}}$.) Limits and estimates as above hold for f_t, $t < 0$.*

Proof. (i) The rapid decay of f_t follows from the lower bound on $D_0|_{\mathcal{H}}$ from Proposition 4.3, giving

$$\|\partial_t^j f_t\|_2 = \|\Lambda^j e^{-t\Lambda} f_0\|_2 \lesssim \|(D_0)^{k-j} \Lambda^j e^{-t\Lambda} f_0\|_2 \approx t^{-k} \|(t\Lambda)^k e^{-t\Lambda} f_0\|_2 \lesssim t^{-k} \|f_0\|_2.$$

(ii) That f is the conormal gradient of a solution follows from Corollary 3.4 and it is straightforward to show that the ODE $\partial_t f + D_0 f = 0$ is satisfied in the strong and distribution sense.

(iii) Next, $\|\partial_t f\|_{\mathcal{Y}}^2 \le \int_0^\infty \|\partial_t f_t\|_2^2 t\,dt$, and the square function estimate $\int_0^\infty \|\partial_t f_t\|_2^2 t\,dt \approx \|f_0\|_2^2$ follows from (30), since $\partial_t f_t = -\Lambda e^{-t\Lambda} f_0$. This together with the decay from (i) with $j = 1$ shows $\|f_0\|_2 \approx \|\partial_t f\|_{\mathcal{Y}}$.

(iv) It remains to show that $\|f_0\|_2 \approx \|f\|_{\mathcal{X}}$. For this, the decay from (i) with $j = 0$ implies it is enough to prove $\|\widetilde{N}_* f\|_2 \approx \|f_0\|_2$. The proof is an adaptation of the results on $\mathbb{R}^{1+n}_+$ from [Auscher et al. 2008, Proposition 2.56] as follows.

The estimate $\|\widetilde{N}_*(f)\|_2 \gtrsim \|f_0\|_2$ follows from Lemma 9.3. Next consider the estimate $\lesssim$. We follow the argument in [Auscher et al. 2008, Proposition 2.56]. By the reverse Hölder inequalities noted in the proof of Proposition 9.7 applied to a weak solution of the divergence form equation with coefficients A_1 associated with $f = e^{-t|D_0|} f_0$, we can bound L_2 averages by L_p averages for some $p < 2$, i.e., $\widetilde{N}_* f \lesssim \widetilde{N}_*^p f$ in a pointwise sense (up to changing to constants c_0, c_1). Since $\psi(\lambda) = e^{-|\lambda|} - (1 + i\lambda)^{-1} \in \Psi(S^o_{\nu,\sigma})$, it follows from Lemma 9.3 and Theorem 7.1, or more precisely (30), that

$$\|\widetilde{N}_*^p(\psi(tD_0) f_0)\|_2 \lesssim \|\widetilde{N}_*(\psi(tD_0) f_0)\|_2 \lesssim \|f_0\|_2.$$

For $h_t := (I + itD_0)^{-1} f_0$ we have $\|\widetilde{N}_*^p(h)\|_2 \lesssim \|M(|f_0|^p)^{1/p}\|_2 \lesssim \|f_0\|_2$ by Corollary 6.3 and the boundedness of M on $L_{2/p}$. We have proved that $\|\widetilde{N}_* f\|_2 \lesssim \|f_0\|_2$.

(v) The modifications for $f_0 \in E_0^- \mathcal{H}$ are straightforward, and the correspondence with u follows from applying the methods of Proposition 3.3. □

Remark 10.2. The assumption $\sigma = \frac{n-1}{2}$ is used in part (iv) to pass from $\widetilde{N}_*$ to $\widetilde{N}_*^p$ with some $p < 2$. Thus, for any $\sigma \in \mathbb{R}$, $f_0 \in \mathcal{H}$ and $p < 2$, we have $\|\widetilde{N}_*^p(f)\|_2 \lesssim \|f_0\|_2$. The converse, however, is not clear because $p < 2$, and this shows that the value of σ is significant.

Theorem 10.3. *Let $v_0 \in \widetilde{E}_0^+ L_2$. Then*

$$v_t := e^{-t\tilde{\Lambda}} v_0$$

gives a solution to $\partial_t v + \widetilde{D}_0 v = 0$, *in the strong sense* $v \in C^1(\mathbb{R}_+; L_2) \cap C^0(\mathbb{R}_+; D(\widetilde{D}_0))$ *and in* $\mathbb{R}^+ \times S^n$ *distributional sense. (In particular* $r^{-\sigma}(v_t)_\perp$ *extends to a weak solution of* $\operatorname{div}_{\boldsymbol{x}} A_1 \nabla_{\boldsymbol{x}} u = 0$ *in* $\mathbb{O}^{1+n}$ *as in* Proposition 3.5.*) The function* v *has* L_2 *limit* $\lim_{t\to 0} v_t = v_0$ *and rapid decay* $\|\partial_t^j v_t\|_2 \le C_{j,k}/t^k \|v_0\|_2$ *for each* $k \ge j \ge 0$. *(When* $\sigma = 0$, *this estimate for* $j = 0$ *only holds for* $v_0 \in R(\widetilde{D}_0) \cap \widetilde{E}_0^+ L_2$.*) Moreover, for* $p < 2$, *we have estimates*

$$\|\partial_t v\|_{\mathcal{Y}} + \|\widetilde{N}_*^p(v)\|_2 + \|\widetilde{N}_*(v_\perp)\|_2 \lesssim \|v_0\|_2.$$

In dimension $n = 1$, *we have* $\|v\|_{\mathcal{X}} \approx \|v_0\|_2$.

If instead $v_0 \in \widetilde{E}_0^- L_2$, *then define* $v_t := e^{t\tilde{\Lambda}} v_0$ *for* $t < 0$. *Then* $\partial_t v + \widetilde{D}_0 v = 0$ *for* $t < 0$. *(In particular* $r^{-\sigma}(v_t)_\perp$ *satisfies* $\operatorname{div}_{\boldsymbol{x}} A_1 \nabla_{\boldsymbol{x}} u = 0$ *in* $\mathbb{R}^n \setminus \overline{\mathbb{O}^{1+n}}$ *as in* Proposition 3.5.*) Limits and estimates as above hold for* v_t, $t < 0$.

Proof. The proof, except for the nontangential maximal estimates, is identical to that of Theorem 10.1, using Proposition 4.4 and Corollary 6.3. When $n \ge 2$, the estimate of $\|\widetilde{N}_*(v_\perp)\|_2$ follows, using the same ψ as above and reduction to $\|\widetilde{N}_*((I + it\widetilde{D}_0)^{-1} v_0)_\perp)\|_2$, from Corollary 6.3 and the maximal theorem. When $n = 1$, one uses the splitting in Proposition 4.4: we have that $e^{-t\tilde{\Lambda}}$ is the identity on $\mathcal{H}^\perp$ and that $\tilde{\Lambda}$ on $B_0\mathcal{H}$ is similar to Λ on $\mathcal{H}$, so $\|v\|_{\mathcal{X}} \approx \|v_0\|_2$ follows from Theorem 10.1.

The modifications when $v_0 \in \widetilde{E}_0^- L_2$ are straightforward. □

11. The ODE in integral form

Following [Part I], for radially dependent coefficients we solve (17) for f by rewriting it as

$$\partial_t f + (DB_0 + \sigma N) f = D\mathcal{E} f, \quad \text{where } \mathcal{E}_t := B_0 - B_t.$$

Recall that solutions f_t belong to $\mathcal{H}$, where $\mathcal{H}$ splits into $E_0^+\mathcal{H}$ and $E_0^-\mathcal{H}$ by Lemma 7.5, with $E_0^\pm = \chi^\pm(D_0)$ on $\mathcal{H}$. Applying $E_0^\pm$, integrating formally each subequation and subtracting the obtained equations we obtain

$$f_t = e^{-t\Lambda} E_0^+ f_0 + \int_0^t e^{-(t-s)\Lambda} E_0^+ D\mathcal{E}_s f_s \, ds - \int_t^\infty e^{-(s-t)\Lambda} E_0^- D\mathcal{E}_s f_s \, ds, \tag{32}$$

provided $\lim_{t\to 0} f_t = f_0$ and $\lim_{t\to\infty} f_t = 0$ in appropriate sense. We first study proper definition, boundedness of the integral operators in (32) on appropriate spaces and their limits. The justification of (32) is done in Section 12.

Lemma 11.1. *If* $f \in L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$ *satisfies* $\partial_t f + (DB + \sigma N) f = 0$ *in* $\mathbb{R}_+ \times S^n$ *distributional sense, then*

$$-\int_0^t \partial_s \eta_\epsilon^+(t,s) e^{-(t-s)\Lambda} E_0^+ f_s \, ds = \int_0^t \eta_\epsilon^+(t,s) e^{-(t-s)\Lambda} E_0^+ D\mathcal{E}_s f_s \, ds,$$
$$-\int_t^\infty \partial_s \eta_\epsilon^-(t,s) e^{-(s-t)\Lambda} E_0^- f_s \, ds = \int_t^\infty \eta_\epsilon^-(t,s) e^{-(s-t)\Lambda} E_0^- D\mathcal{E}_s f_s \, ds,$$

for all $t > 0$. *The bump functions* $\eta_\epsilon^\pm$ *are constructed as follows. Let* $\eta^0(t)$ *to be the piecewise linear continuous function with support* $[1, \infty)$, *which equals* 1 *on* $(2, \infty)$ *and is linear on* $(1, 2)$. *Then let* $\eta_\epsilon(t) := \eta^0(t/\epsilon)(1 - \eta^0(2\epsilon t))$ *and* $\eta_\epsilon^\pm(t,s) := \eta^0(\pm(t-s)/\epsilon)\eta_\epsilon(t)\eta_\epsilon(s)$.

Proof. Follow [Part I, Proposition 4.4]. □

Define for $f \in L_2^{\mathrm{loc}}(\mathbb{R}^+; L_2(S^n; \mathcal{V}))$,

$$S_A^\epsilon f_t := \int_0^t \eta_\epsilon^+(t,s) e^{-(t-s)\Lambda} E_0^+ D\mathcal{E}_s f_s \, ds - \int_t^\infty \eta_\epsilon^-(t,s) e^{-(s-t)\Lambda} E_0^- D\mathcal{E}_s f_s \, ds.$$

In fact, this formula makes sense by extension thanks to the following algebraic relations.

Lemma 11.2. *We have $S_A^\epsilon f_t = \widehat{S}_A^\epsilon f_t - \sigma \check{S}_A^\epsilon f_t = D\widetilde{S}_A^\epsilon f_t$, where*

$$\widehat{S}_A^\epsilon f_t := \int_0^t \eta_\epsilon^+(t,s) \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s \, ds + \int_t^\infty \eta_\epsilon^-(t,s) \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s \, ds,$$

$$\check{S}_A^\epsilon f_t := \int_0^t \eta_\epsilon^+(t,s) e^{-(t-s)\Lambda} \check{E}_0^+ \mathcal{E}_s f_s \, ds - \int_t^\infty \eta_\epsilon^-(t,s) e^{-(s-t)\Lambda} \check{E}_0^- \mathcal{E}_s f_s ds,$$

$$\widetilde{S}_A^\epsilon f_t := \int_0^t \eta_\epsilon^+(t,s) e^{-(t-s)\tilde{\Lambda}} \widetilde{E}_0^+ \mathcal{E}_s f_s \, ds - \int_t^\infty \eta_\epsilon^-(t,s) e^{-(s-t)\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds.$$

Here $\widehat{E}_0^\pm := E_0^\pm B_0^{-1} \widetilde{P}_{B_0}^1$, $\check{E}_0^\pm := E_0^\pm N B_0^{-1} \widetilde{P}_{B_0}^1$, with $\widetilde{P}_{B_0}^1$ as in Proposition 4.4.

Proof. Here, B_0^{-1} denotes the inverse of the isomorphism $B_0 : \mathcal{H} \to B_0\mathcal{H}$. Since $\mathsf{N}(D) = \mathcal{H}^\perp$, we have

$$E_0^\pm D = E_0^\pm D \widetilde{P}_{B_0}^1 = E_0^\pm\big((DB_0 + \sigma N) - \sigma N\big) B_0^{-1} \widetilde{P}_{B_0}^1 = D_0 \widehat{E}_0^\pm - \sigma \check{E}_0^\pm,$$

Using that $e^{-u\Lambda}$ and $e^{-u\Lambda}\Lambda$ extend to bounded operators on $\mathcal{H}$, this also shows that $e^{-u\Lambda} E_0^+ D$ extend to bounded operators on L_2 for $u > 0$. We now readily obtain $S_A^\epsilon = \widehat{S}_A^\epsilon - \sigma \check{S}_A^\epsilon$. The identity $S_A^\epsilon = D\widetilde{S}_A^\epsilon$ is a consequence of the intertwining relation

$$b(D_0)D = Db(\widetilde{D}_0)$$

between the two functional calculi. □

Theorem 11.3. *Assume $\|\mathcal{E}\|_* < \infty$. We have bounded operators*

$$S_A^\epsilon : \mathcal{X} \to \mathcal{X}, \quad S_A^\epsilon : \mathcal{Y} \to \mathcal{Y},$$

with norms $\lesssim \|\mathcal{E}\|_*$, uniformly for $\epsilon > 0$. In the space $\mathcal{X}$ there is a limit operator $S_A^{\mathcal{X}} \in \mathcal{L}(\mathcal{X}; \mathcal{X})$ such that

$$\lim_{\epsilon \to 0} \|S_A^\epsilon f - S_A^{\mathcal{X}} f\|_{L_2(a,b;L_2)} = 0, \quad \textit{for any } f \in \mathcal{X}, 0 < a < b < \infty.$$

The same bounds and limits hold for $\widehat{S}_A^\epsilon$ and $\check{S}_A^\epsilon$ on $\mathcal{X}$.

In the space $\mathcal{Y}$, there is a limit operator $S_A^{\mathcal{Y}} \in \mathcal{L}(\mathcal{Y}; \mathcal{Y})$ such that

$$\lim_{\epsilon \to 0} \|S_A^\epsilon f - S_A^{\mathcal{Y}} f\|_{\mathcal{Y}} = 0, \quad \textit{for any } f \in \mathcal{Y}.$$

The same bounds and limits hold for $\widehat{S}_A^\epsilon$ and $\check{S}_A^\epsilon$ on $\mathcal{Y}$.

Let $S_A := \lim_{\epsilon\to 0} S_A^\epsilon$, $\widehat{S}_A := \lim_{\epsilon\to 0} \widehat{S}_A^\epsilon$ and $\check{S}_A := \lim_{\epsilon\to 0} \check{S}_A^\epsilon$ denote the limit operators on $\mathcal{Y}$ from Theorem 11.3. Since $\mathcal{X}$ is densely embedded in $\mathcal{Y}$, these limit operators restricts to the corresponding limit operators on $\mathcal{X}$ from Theorem 11.3.

One sees that $S_A = \widehat{S}_A - \sigma \check{S}_A$ holds, and that

$$S_A f_t = \lim_{\epsilon\to 0}\left(\int_\epsilon^{t-\epsilon} e^{-(t-s)\Lambda} E_0^+ D\mathcal{E}_s f_s\, ds - \int_{t+\epsilon}^{\epsilon^{-1}} e^{-(s-t)\Lambda} E_0^- D\mathcal{E}_s f_s ds\right),$$

with convergence in $L_2(a, b; L_2)$ for any $0 < a < b < \infty$, both on $\mathcal{Y}$ and $\mathcal{X}$.

Proof. The proof is essentially an application of [Part I, Section 6], where the results were proved abstractly. Given Theorems 7.1 and 10.1, these results from that paper apply. In particular, this makes use of the holomorphic $S^o_{\omega,\sigma}$ operational calculus of D_0, where more general operator-valued holomorphic functions are applied to D_0. It is straightforward, given Theorem 7.1, to adapt the results in [Part I, Sections 6–7] and construct this $S^o_{\omega,\sigma}$ operational calculus of D_0, and we omit the details.

(i) Consider the operators $\widehat{S}_A^\epsilon : \mathcal{X} \to \mathcal{X}$. Here [Part I, Theorem 6.5] shows that $\widehat{S}_A^\epsilon : L_2(\mathbb{R}_+, dt; L_2) \to L_2(\mathbb{R}_+, dt; L_2)$ are uniformly bounded, with norm $\lesssim \|\mathcal{E}\|_\infty$, and converge strongly in $\mathcal{L}(L_2(\mathbb{R}_+, dt; L_2))$ as $\epsilon \to 0$. Moreover, [Part I, Theorem 6.8] applies and shows that

$$\widehat{S}_A^\epsilon f_t = \widehat{Z}^\epsilon(\mathcal{E} f)_t + \eta_\epsilon(t) e^{-t\Lambda}\int_0^\infty \eta_\epsilon(s)\Lambda e^{-s\Lambda}\widehat{E}_0^+\mathcal{E}_s f_s\, ds$$

where $\widehat{Z}^\epsilon : L_2(\mathbb{R}_+, dt/t; L_2) \to L_2(\mathbb{R}_+, dt/t; L_2)$ are uniformly bounded and converge strongly as $\epsilon \to 0$. These estimates build on the square function estimates and make use of the operational calculus for D_0. On the other hand, using Theorem 10.1 and Theorem 7.1, the last term has estimates

$$\begin{aligned}\left\|\eta_\epsilon(t) e^{-t\Lambda}\int_0^\infty \eta_\epsilon(s)\Lambda e^{-s\Lambda}\widehat{E}_0^+\mathcal{E}_s f_s\, ds\right\|_{\mathcal{X}} &\lesssim \left\|\int_0^\infty \eta_\epsilon(s)\Lambda e^{-s\Lambda}\widehat{E}_0^+\mathcal{E}_s f_s\, ds\right\|_2 \\ &= \sup_{\|h\|_2=1}\left|\int_0^\infty (s\Lambda^* e^{-s\Lambda^*}h, \eta_\epsilon(s)\widehat{E}_0^+\mathcal{E}_s f_s)\frac{ds}{s}\right| \\ &\lesssim \|\eta_\epsilon \mathcal{E} f\|_{\mathcal{Y}^*} \lesssim \|\mathcal{E}\|_* \|f\|_{\mathcal{X}},\end{aligned}$$

and is seen to converge strongly in $\mathcal{L}(\mathcal{X}, L_2(a, b; L_2))$ for any $0 < a < b < \infty$, as in [Part I, Lemma 6.9]. Piecing these estimates together, we obtain

$$\begin{aligned}\|\widehat{S}_A^\epsilon f\|_{\mathcal{X}} &\lesssim \|\widehat{Z}^\epsilon(\mathcal{E} f)\|_{L_2(dt/t;L_2)} + \|\widehat{Z}^\epsilon(\mathcal{E} f)\|_{L_2(dt;L_2)} + \|\widehat{S}_A^\epsilon f - \widehat{Z}^\epsilon(\mathcal{E} f)\|_{\mathcal{X}} \\ &\lesssim \|\mathcal{E}\|_*\|f\|_{\mathcal{X}} + \|\mathcal{E}\|_\infty\|f\|_{L_2(dt;L_2)} + \|\mathcal{E}\|_*\|f\|_{\mathcal{X}},\end{aligned}$$

with strong convergence in $\mathcal{L}(\mathcal{X}, L_2(a, b; L_2))$.

(ii) For the operators $\check{S}_A^\epsilon : \mathcal{X} \to \mathcal{X}$, we note that the estimates for $\widehat{S}_A^\epsilon$ go through when replacing $\widehat{E}_0^\pm$ by $\check{E}_0^\pm$. Since $\check{S}_A^\epsilon = \Lambda^{-1}\widehat{S}_A^\epsilon$ (with $\widehat{E}_0^\pm$ replaced by $\check{E}_0^\pm$) and $\Lambda^{-1} : L_2(dt/t; \mathcal{H}) \to L_2(dt/t; \mathcal{H})$ is bounded, it only remains to estimate the term $\eta_\epsilon(t)e^{-t\Lambda}\int_0^\infty \eta_\epsilon(s)e^{-s\Lambda}\check{E}_0^+\mathcal{E}_s f_s\, ds$. But again using the boundedness of Λ^{-1} gives

$$\left\| \eta_\epsilon(t) e^{-t\Lambda} \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} \check{E}_0^+ \mathcal{E}_s f_s \, ds \right\|_{\mathcal{X}} \lesssim \left\| \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} \check{E}_0^+ \mathcal{E}_s f_s \, ds \right\|_2 \lesssim \left\| \Lambda \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} \check{E}_0^+ \mathcal{E}_s f_s \, ds \right\|_2,$$

and the rest of the estimates go though as for $\widehat{S}_A^\epsilon$. Altogether, this proves the stated bounds and convergence for $S_A^\epsilon : \mathcal{X} \to \mathcal{X}$.

(iii) Next consider the operators $\widehat{S}_A^\epsilon : \mathcal{Y} \to \mathcal{Y}$. We have

$$\begin{aligned} \|\widehat{S}_A^\epsilon f\|_{\mathcal{Y}} \le \|\widehat{S}_A^\epsilon(\chi_{t<1} f)\|_{\mathcal{Y}} + \|\widehat{S}_A^\epsilon(\chi_{t>1} f)\|_{\mathcal{Y}} &\le \|\widehat{S}_A^\epsilon(\chi_{t<1} f)\|_{L_2(tdt;L_2)} + \|\widehat{S}_A^\epsilon(\chi_{t>1} f)\|_{L_2(dt;L_2)} \\ &\lesssim \|\mathcal{E}\|_* \|\chi_{t<1} f\|_{L_2(tdt;L_2)} + \|\mathcal{E}\|_\infty \|\chi_{t>1} f\|_{L_2(dt;L_2)} \\ &\lesssim \|\mathcal{E}\|_* \|f\|_{\mathcal{Y}}, \end{aligned}$$

where the $L_2(tdt; L_2)$ estimate follows from [Part I, Proposition 7.1] and the $L_2(dt; L_2)$ estimate from [Part I, Proposition 6.5], along with convergence. This immediately gives the estimates for $\check{S}_A^\epsilon : \mathcal{Y} \to \mathcal{Y}$ since $\Lambda^{-1} : L_2(tdt; \mathcal{H}) \to L_2(tdt; \mathcal{H})$ and $\Lambda^{-1} : L_2(dt; \mathcal{H}) \to L_2(dt; \mathcal{H})$ are bounded. □

Denote by $C(a, b; L_2)$ the space of continuous functions $(a, b) \ni t \mapsto v_t \in L_2(S^n; \mathcal{V})$.

Theorem 11.4. *Assume $\|\mathcal{E}\|_* < \infty$. If $n \ge 2$, then $\widetilde{S}_A^\epsilon f \in C(0, \infty; L_2)$ for any $f \in \mathcal{Y}$. There are bounds $\|\widetilde{S}_A^\epsilon f_t\|_2 \lesssim \|\mathcal{E}\|_* \|f\|_{\mathcal{Y}}$, uniformly for all $f \in \mathcal{Y}$, $t, \epsilon > 0$, and for each $f \in \mathcal{Y}$ there is a limit function $\widetilde{S}_A f \in C(0, \infty; L_2)$ such that $\lim_{\epsilon \to 0} \|\widetilde{S}_A^\epsilon f_t - \widetilde{S}_A f_t\|_2 = 0$ locally uniformly for $t > 0$. We have the expression*

$$\widetilde{S}_A f_t = \int_0^t e^{-(t-s)\tilde{\Lambda}} \widetilde{E}_0^+ \mathcal{E}_s f_s \, ds - \int_t^\infty e^{-(s-t)\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds, \tag{33}$$

where the integrals are weakly convergent in L_2 for all $f \in \mathcal{Y}$ and $t > 0$. Finally, $S_A f = D\widetilde{S}_A f$ holds in $\mathbb{R}_+ \times S^n$ distributional sense for each $f \in \mathcal{Y}$.

If $n = 1$, then the above results hold if $\mathcal{Y}$ is replaced by $\mathcal{Y}_\delta$, for any fixed $\delta > 0$.

Proof. (i) Consider first the case $n \ge 2$. The proof is a adaption of the proof of [Part I, Proposition 7.2], which we refer to for further details. We split the $(0, t)$-integral

$$\int_0^t \eta_\epsilon^+(t, s) e^{-(t-s)\tilde{\Lambda}} (I - e^{-2s\tilde{\Lambda}}) \widetilde{E}_0^+ \mathcal{E}_s f_s \, ds + e^{-t\tilde{\Lambda}} \int_0^t \eta_\epsilon^+(t, s) e^{-s\tilde{\Lambda}} \widetilde{E}_0^+ \mathcal{E}_s f_s \, ds,$$

The same duality estimate of the second term as in [Part I, Proposition 7.2], given Theorem 10.1 and Lemma 4.2, goes through here. For the first term, we note the estimate

$$\|e^{-(t-s)\tilde{\Lambda}} (I - e^{-2s\tilde{\Lambda}})\| \lesssim \min\Big(\frac{s}{t}, 1, \frac{1}{t-s}\Big).$$

For $t \le 2$, this yields the bound $\|\mathscr{E}\|_\infty \int_0^t (s/t)\|f_s\|_2\, ds \lesssim \|\mathscr{E}\|_\infty \|f\|_{\mathscr{Y}}$. On the other hand, for $t \ge 2$ we have the estimate

$$\|\mathscr{E}\|_\infty \left(\int_0^1 \frac{s}{t}\|f_s\|_2\, ds + \int_1^{t-1} \frac{1}{t-s}\|f_s\|_2\, ds + \int_{t-1}^t \|f_s\|_2\, ds \right) \lesssim \|\mathscr{E}\|_\infty \|f\|_{\mathscr{Y}}.$$

The (t, ∞)-integral is estimated similarly, by splitting it

$$\int_t^\infty \eta_\epsilon^-(t,s) e^{-(s-t)\tilde{\Lambda}}(I - e^{-2t\tilde{\Lambda}})\widetilde{E}_0^- \mathscr{E}_s f_s\, ds + e^{-t\tilde{\Lambda}} \int_t^\infty \eta_\epsilon^-(t,s) e^{-s\tilde{\Lambda}} \widetilde{E}_0^+ \mathscr{E}_s f_s\, ds,$$

The second term is estimated as before, and for the first term we note the estimates $\|e^{-(s-t)\tilde{\Lambda}}(I - e^{-2t\tilde{\Lambda}})\| \lesssim \min(t/s, 1, 1/(s-t))$, which give the bound

$$\|\mathscr{E}\|_\infty \left(\int_t^{t+1} \frac{t}{s}\|f_s\|_2\, ds + \int_{t+1}^\infty \frac{1}{s-t}\|f_s\|_2\, ds \right) \lesssim \|\mathscr{E}\|_\infty \|f\|_{\mathscr{Y}}.$$

(ii) Consider next the case $n = 1$. Since $e^{-t\tilde{\Lambda}} = I$ on $\mathscr{H}^\perp$ and $\widetilde{E}_0^\pm = N^\mp$ on $\mathscr{H}^\perp$, we also need to estimate the L_2-norm of

$$\left(\int_0^t \eta_\epsilon^+(t,s) \widetilde{P}_{B_0}^0 \mathscr{E}_s f_s \right)_\perp - \left(\int_t^\infty \eta_\epsilon^-(t,s) \widetilde{P}_{B_0}^0 \mathscr{E}_s f_s \right)_\parallel,$$

uniformly for $t > 0$, where $\widetilde{P}_{B_0}^0$ is projection onto $\mathscr{H}^\perp$ from Proposition 4.4. So it is enough to obtain the bound

$$\left\| \int_0^\infty |\widetilde{P}_{B_0}^0 \mathscr{E}_s f_s|\, ds \right\|_2 \lesssim \|\mathscr{E}\|_* \|f\|_{\mathscr{Y}_\delta}.$$

On the one hand, we obtain from Proposition 9.7 the estimate

$$\left\| \int_1^\infty |\widetilde{P}_{B_0}^0 \mathscr{E}_s f_s|\, ds \right\|_2 \lesssim \|\mathscr{E}\|_\infty \int_1^\infty \|f_s\|_2\, ds \lesssim \|\mathscr{E}\|_\infty \left(\int_1^\infty \|f_s\|_2^2 e^{\delta s} ds \right)^{1/2} \lesssim \|\mathscr{E}\|_\infty \|f\|_{\mathscr{Y}_\delta}.$$

On the other hand, note that A, hence B_0^{-1}, is pointwise strictly accretive by Lemma 5.1 and by the explicit expression in Lemma 5.5 (expressed in other coordinates), $\widetilde{P}_{B_0}^0$ maps into constant functions and $|\widetilde{P}_{B_0}^0 u| \lesssim \int_{S^1} |u(x)| dx$. Thus

$$\left\| \int_0^1 |\widetilde{P}_{B_0}^0 \mathscr{E}_s f_s|\, ds \right\|_2 \lesssim \int_0^1 \int_{S^1} |\mathscr{E}_s(x)||f_s(x)| dx\, ds.$$

Pick $h : \mathbb{R}_+ \times S^1 \to \mathscr{V}$ such that $|h_s(x)| = 1$ and $|\mathscr{E}_s(x) h_s(x)| = |\mathscr{E}_s(x)|$ when $s < 1$, and $h_s(x) = 0$ when $s > 1$. Cauchy–Schwarz inequality yields

$$\int_0^1 \int_{S^1} |\mathscr{E}_s(x)||f_s(x)|\, ds\, dx \lesssim \|\mathscr{E} h\|_{\mathscr{Y}^*} \|f\|_{\mathscr{Y}} \le \|\mathscr{E}\|_* \|h\|_{\mathscr{X}} \|f\|_{\mathscr{Y}} \lesssim \|\mathscr{E}\|_* \|f\|_{\mathscr{Y}}.$$

This completes the proof of the estimate of $\|\widetilde{S}_A^\epsilon f_t\|_2$.

(iii) As in the proof of [Part I, Proposition 7.2], replacing $\eta_\epsilon^\pm$ by $\eta_\epsilon^\pm - \eta_{\epsilon'}^\pm$ in the estimates shows convergence of $\widetilde{S}_A^\epsilon$ and yield the expression for the limit operator. The relation $S_A = D\widetilde{S}_A$ follows at the limit from the relation in Lemma 11.2. □

We turn to boundary behavior of the integral operators at $t = 0$.

Lemma 11.5. *Assume $\|\mathcal{E}\|_* < \infty$.*

(i) Let $f \in \mathcal{X}$ (or $f \in \mathcal{Y}$) and define $f^0 := S_A f$. Then f^0 and f satisfy

$$(\partial_t + D_0) f^0 = D\mathcal{E} f$$

in $\mathbb{R}_+ \times S^n$ distributional sense. If $f \in \mathcal{X}$, then there are limits

$$\lim_{t\to 0} t^{-1} \int_t^{2t} \|S_A f_s - h^-\|_2^2 \, ds = 0,$$

where $h^- := -\int_0^\infty e^{-s\Lambda} E_0^- D\mathcal{E}_s f_s \, ds \in E_0^- \mathcal{H}$ has bounds $\|h^-\|_2 \lesssim \|f\|_{\mathcal{X}}$.

(ii) Let $n \ge 2$. If $f \in \mathcal{Y}$ and $v := \widetilde{S}_A f$, then

$$(\partial_t + \widetilde{D}_0) v = \mathcal{E} f$$

in $\mathbb{R}_+ \times S^n$ distributional sense, and there are limits

$$\lim_{t\to 0} \|\widetilde{S}_A f_t - \tilde{h}^-\|_2 = 0,$$

where $\tilde{h}^- := -\int_0^\infty e^{-s\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds \in \widetilde{E}_0^- L_2$ has bounds $\|\tilde{h}^-\|_2 \lesssim \|f\|_{\mathcal{Y}}$. If $n = 1$, these results for $\widetilde{S}_A f$ hold when replacing $\mathcal{Y}$ by $\mathcal{Y}_\delta$, for any fixed $\delta > 0$.

Proof. (i) By the convergence properties of S_A^ϵ from Theorem 11.3, it suffices to show that for $\phi \in C_0^\infty(\mathbb{R}_+ \times S^n; \mathbb{C}^{(1+n)m})$ there is convergence

$$\int \Big((-\partial_t \phi_t + B_0^* D + \sigma N)\phi_t, f_t^\epsilon\Big) dt \to \int (D\phi_s, \mathcal{E}_s f_s)\, ds, \quad \epsilon \to 0,$$

where $f_t^\epsilon := S_A^\epsilon f_t$. For the term $(0,t)$-integral, Fubini's theorem and integration by parts gives

$$\begin{aligned}
&\int_0^\infty \int_0^t \eta_\epsilon^+(t,s)((-\partial_t + \Lambda^*)\phi_t, e^{-(t-s)\Lambda} E_0^+ D\mathcal{E}_s f_s)\, ds\, dt \\
&\qquad = -\int_0^\infty \left(\int_s^\infty \eta_\epsilon^+(t,s) D(E_0^+)^* \partial_t (e^{-(t-s)\Lambda^*}\phi_t)\, dt, \mathcal{E}_s f_s\right) ds \\
&\qquad = \int_0^\infty \left(\int_s^\infty (\partial_t \eta_\epsilon^+)(t,s) D(E_0^+)^* e^{-(t-s)\Lambda^*}\phi_t dt, \mathcal{E}_s f_s\right) ds \to \int_0^\infty (D(E_0^+)^*\phi_s, \mathcal{E}_s f_s)\, ds.
\end{aligned}$$

Adding the corresponding limit for the (t,∞)-integral, we obtain in total the limit $\int_0^\infty (D\phi_s, \mathcal{E}_s f_s)\, ds$, since $D((E_0^+)^* + (E_0^-)^*) = ((E_0^+ + E_0^-)D)^* = D^* = D$.

To prove the limit of $S_A f_t$ for $f \in \mathcal{X}$, we note from the proof of Theorem 11.3 that

$$S_A f_t = Z_A f_t + e^{-t\Lambda} \int_0^\infty e^{-s\Lambda} E_0^- D\mathcal{E}_s f_s \, ds,$$

where $Z_A f \in \mathcal{Y}^*$. When taking limits $\epsilon \to 0$, we have used [Part I, Theorem 6.8 and Lemma 6.9]. This proves the stated limit.

(ii) To prove $(\partial_t + \widetilde{D}_0)v = \mathcal{E} f$, we let $t \in (a, b)$ and differentiate $\widetilde{S}_A^\epsilon f$ to get

$$\partial_t \widetilde{S}_A^\epsilon f_t = \frac{1}{\epsilon} \int_\epsilon^{2\epsilon} e^{-s\tilde{\Lambda}} (\widetilde{E}_0^+ \mathcal{E}_{t-s} f_{t-s} + \widetilde{E}_0^- \mathcal{E}_{t+s} f_{t+s}) ds - \widetilde{D}_0(\widetilde{S}_A^\epsilon f_t),$$

for small ϵ. The first term on the right is seen to converge to $\mathcal{E} f$ in $L_2(a, b; L_2)$ as $\epsilon \to 0$, with an argument as in [Part I, Theorem 8.2]. Note that this uses $\widetilde{E}_0^+ + \widetilde{E}_0^- = I$, which holds also when $n = 1$ by Definition 7.4. Letting $\epsilon \to 0$, we obtain $\partial_t v = \mathcal{E} f - \widetilde{D}_0 v$ in distributional sense, since (a, b) was arbitrary.

The limit for $\widetilde{S}_A f_t$ when $f \in \mathcal{Y}$ (or $\mathcal{Y}_\delta$ when $n = 1$) is proved as in [Part I, Proposition 7.2 and Lemma 6.9]. In particular, this uses an identity

$$\widetilde{S}_A f_t = \widetilde{Z}_A f_t + e^{-t\tilde{\Lambda}} \int_0^\infty e^{-s\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds,$$

with $\widetilde{Z}_A f \in C(0, \infty; L_2)$ and $\lim_{t \to 0} \widetilde{Z}_A f_t = 0$ in L_2. □

12. Representation and traces of solutions

We now come to the heart of the matter. The natural classes of solutions for the Dirichlet and Neumann problems, with L_2 boundary data, use the spaces $\mathcal{Y}^o \approx \mathcal{Y}$ and $\mathcal{X}^o \approx \mathcal{X}$ from Definition 9.2.

Definition 12.1. (i) By a $\mathcal{Y}^o$*-solution to the divergence form equation*, with coefficients A, we mean a weak solution u of $\operatorname{div}_{\boldsymbol{x}} A \nabla u = 0$ in $\mathbb{O}^{1+n}$ with $\|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} < \infty$.

(ii) By an $\mathcal{X}^o$*-solution to the divergence form equation*, with coefficients A, we mean the gradient $g := \nabla_{\boldsymbol{x}} u$ of a weak solution u of $\operatorname{div}_{\boldsymbol{x}} A \nabla u = 0$ in $\mathbb{O}^{1+n}$ with $\|g\|_{\mathcal{X}^o} < \infty$.

Note the slight abuse of notation when referring to the gradient $\nabla_{\boldsymbol{x}} u$ rather than u as an $\mathcal{X}^o$-solution. The reason for this convention, here as well as in [Part I], is that the Neumann and regularity problems are BVPs for g (and not for the potential u), and $\mathcal{X}^o$-solutions is the natural class of solutions for these problems. This point of view is the one that lead us to our representations. However, when more convenient we call the potential u itself an $\mathcal{X}^o$-solution.

Remark 12.2. (i) No boundary trace is assumed in our definitions, but will be deduced.

(ii) The seminorm $\|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o}$ on $\mathcal{Y}^o$-solutions is modulo constants, which is unusual for Dirichlet problems. Once we have shown that $\mathcal{Y}^o$-solutions have boundary traces, we will be able to put constants back in the norm in a natural way.

(iii) For any $\mathcal{X}^o$-solution g, the potential u has a boundary trace in appropriate sense (replacing pointwise values by averages) and the trace belongs to $W_2^1(S^n; \mathbb{C}^m)$. This is essentially in [Kenig and Pipher 1993]. We also recover this from our representations. See Section 13.

Here and subsequently, we use the notation $e^{-t\Lambda} g$ to denote the function $(t, x) \mapsto (e^{-t\Lambda} g)(x)$. Similarly for $e^{-t\tilde{\Lambda}} g$.

$\mathcal{X}^o$-solutions. We begin with representation and boundary trace for solutions of the corresponding ODE.

Theorem 12.3. *Assume that $\|\mathcal{E}\|_* < \infty$. Let $f \in \mathcal{X}$. Then $f \in L_2^{\text{loc}}(\mathbb{R}_+; \mathcal{H})$ satisfies*

$$\partial_t f + \big(DB + \tfrac{n-1}{2}N\big) f = 0$$

in $\mathbb{R}_+ \times S^n$ distributional sense if and only if f satisfies the equation

$$f_t = e^{-t\Lambda}h^+ + S_A f_t, \quad \text{for some } h^+ \in E_0^+\mathcal{H}. \tag{34}$$

In this case, f has limit

$$\lim_{t\to 0} t^{-1}\int_t^{2t} \|f_s - f_0\|_2^2\, ds = 0, \tag{35}$$

where $f_0 := h^+ + h^-$ and $h^- := -\int_0^\infty e^{-s\Lambda}E_0^- D\mathcal{E}_s f_s\, ds \in E_0^-\mathcal{H}$, with estimates

$$\max(\|h^+\|_2, \|h^-\|_2) \approx \|f_0\|_2 \lesssim \|f\|_{\mathcal{X}}.$$

If furthermore $I - S_A$ is invertible on $\mathcal{X}$, then

$$f = (I - S_A)^{-1} e^{-t\Lambda}h^+ \tag{36}$$

and $\|f\|_{\mathcal{X}} \lesssim \|h^+\|_2$.

Proof. The proof is an adaption of [Part I, Theorem 8.2], to which we refer for details. Here is a quick summary.

We show that f satisfies (17) if and only if f satisfies (34). Assume (17) and apply Lemma 11.1. Letting $\epsilon \to 0$ and applying Theorem 11.3, we obtain the stated equation for f, with h^+ as a certain weak limit as in part (i) of the proof of [Part I, Theorem 8.2], with $\Lambda = |D_0|$ here.

Conversely, if $f \in \mathcal{X}$ satisfies (34), then we apply Lemma 11.5 with $f^o := f - e^{-t\Lambda}h^+$. Since $(\partial_t + D_0)e^{-t\Lambda}h^+ = 0$ and $e^{-t\Lambda}h^+ \in \mathcal{X}$ by Theorem 10.1, it follows that f satisfies (17).

Lemma 11.5 also shows existence of the limit f_0. The stated estimates follow as in part (iii) of the proof of [Part I, Theorem 8.2].

If $I - S_A$ is invertible, (36) follows immediately from (34), and the estimate $\|f\|_{\mathcal{X}} \lesssim \|h^+\|_2$ follows again from Theorem 10.1. □

Theorem 12.4. *Assume that $\|\mathcal{E}\|_* < \infty$. Then g is an $\mathcal{X}^o$-solution to the divergence form equation with coefficients A if and only if the corresponding conormal gradient $f \in \mathcal{X}$ satisfies the equation*

$$f_t = e^{-t\Lambda}h^+ + S_A f_t, \quad \text{for some } h^+ \in E_0^+\mathcal{H}. \tag{37}$$

In this case, g has limit

$$\lim_{r\to 1}\frac{1}{1-r}\int_{r<|\boldsymbol{x}|<(1+r)/2} |g(\boldsymbol{x}) - g_1(x)|^2\, d\boldsymbol{x} = 0,$$

where $g_1 := (B_0 f_0)_\perp \vec{n} + (f_0)_\parallel$ and $\|g_1\|_2 \lesssim \|g\|_{\mathcal{X}^o}$ holds. If furthermore $I - S_A$ is invertible on $\mathcal{X}$, then $\|h^+\|_2 \approx \|g_1\|_2 \approx \|g\|_{\mathcal{X}^o}$.

Proof. The equivalence follows from Corollary 3.4 and Theorem 12.3. The limit and the estimates follow on applying the conormal gradient-to-gradient map of Proposition 3.3 from the ones satisfied by f. $\square$

It is worth specifying the previous theorem in the case of radially independent coefficients.

Corollary 12.5. *Assume A is radially independent. Then any $\mathcal{X}^o$-solution has corresponding conormal gradient given by $f = e^{-t\Lambda}h^+$ for a unique $h^+ \in E_0^+\mathcal{H}$.*

Remark 12.6. A careful examination of the proof of Theorem 12.3 in the case of radially independent coefficients, shows in fact that for $f \in L_2^{\text{loc}}(\mathbb{R}_+; \mathcal{H})$ the weaker condition $\sup_{0<t<1/2} \frac{1}{t}\int_t^{2t} \|f_s\|_2^2\, ds < \infty$ is sufficient to obtain this corollary, as in this case $S_A = 0$.

$\mathcal{Y}^o$-solutions. We now turn to representations and boundary behavior pertaining to $\mathcal{Y}^o$-solutions.

Theorem 12.7. *Assume that $\|\mathcal{E}\|_* < \infty$ and $f \in \mathcal{Y}$.*

(i) *Then $f \in L_2^{\text{loc}}(\mathbb{R}_+; \mathcal{H})$ satisfies $\partial_t f + (DB + \frac{n-1}{2}N)f = 0$ in $\mathbb{R}_+ \times S^n$ distributional sense if and only if f satisfies the equation*

$$f_t = De^{-t\tilde{\Lambda}}\tilde{h}^+ + S_A f_t, \quad \text{for some } \tilde{h}^+ \in \widetilde{E}_0^+ L_2. \tag{38}$$

Here $\tilde{h}^+$ is unique modulo $\widetilde{E}_0^+\mathcal{H}^\perp$ and $\|\tilde{h}^+\|_{L_2/\mathcal{H}^\perp} \lesssim \|f\|_{\mathcal{Y}}$, and if furthermore $I - S_A$ is invertible on $\mathcal{Y}$ then

$$f = (I - S_A)^{-1} De^{-t\tilde{\Lambda}}\tilde{h}^+ \tag{39}$$

with $\|f\|_{\mathcal{Y}} \lesssim \|\tilde{h}^+\|_{L_2/\mathcal{H}^\perp}$.

(ii) *If (38) holds, let $v_t := e^{-t\tilde{\Lambda}}\tilde{h}^+ + \widetilde{S}_A f_t$. Then $f = Dv$ and $\partial_t v + (BD - \frac{n-1}{2}N)v = 0$, and v_t has L_2 limit*

$$\lim_{t\to 0} \|v_t - v_0\|_2 = 0, \tag{40}$$

where $v_0 := \tilde{h}^+ + \tilde{h}^-$ and $\tilde{h}^- := -\int_0^\infty e^{-s\tilde{\Lambda}}\widetilde{E}_0^-\mathcal{E}_s f_s\, ds \in \widetilde{E}_0^- L_2$, with estimates $\|\tilde{h}^-\|_2 \lesssim \|f\|_{\mathcal{Y}}$ and

$$\|v_t\|_2 \lesssim \|\tilde{h}^+\|_2 + \|f\|_{\mathcal{Y}}, \quad \text{for all } t > 0. \tag{41}$$

Proof. The proof is an adaption, with some modifications, of [Part I, Theorem 9.2], to which we refer for omitted details.

(i) Assume (17). We apply Lemma 11.1 to f. Letting $\epsilon \to 0$ and applying Theorem 11.3, we obtain for f the equation

$$f_t = \tilde{f}_t + S_A f_t,$$

with the limit $\tilde{f}_t := \lim_{\epsilon\to 0} \epsilon^{-1}\int_\epsilon^{2\epsilon} e^{-(t-s)\Lambda} E_0^+ f_s\, ds$. From here, one can proceed as in [Part I, Theorem 9.2] to represent $\tilde{f}_t$ as $D_0 e^{-t\Lambda}h^+$ for some $h^+ \in E_0^+\mathcal{H}$, or use a simpler argument (owing to the boundedness of the boundary here): since $D_0 : E_0^+\mathcal{H} \to E_0^+\mathcal{H}$ is surjective, there exists $h_t \in E_0^+\mathcal{H}$ such that $\tilde{f}_t = D_0 h_t$. From there and $\tilde{f}_{t_0+t} = e^{-t\Lambda}\tilde{f}_{t_0}$, we conclude as in [Part I] that the weak L_2-limit $h^+ := \lim_{t\to 0} h_t$ exists and that $\tilde{f}_t = D_0 e^{-t\Lambda}h^+$.

To write $D_0 e^{-t\Lambda} h^+$ as $De^{-t\tilde{\Lambda}}\tilde{h}^+$ for some $\tilde{h}^+ \in \widetilde{E}_0^+ L_2$, we use Lemma 4.6. Indeed, there is an isomorphism $M : \mathcal{H} \to L_2/\mathcal{H}^\perp$ with $D_0 = D \circ M$ on $\mathsf{D}(D_0)$. It is easy to see that the restriction of M to $E_0^+\mathcal{H}$ maps onto $\widetilde{E}_0^+ L_2/\widetilde{E}_0^+\mathcal{H}^\perp$. Now, on $\mathsf{D}(D_0)$, $D_0 e^{-t\Lambda} = e^{-t\Lambda} D_0 = e^{-t\Lambda} D \circ M = De^{-t\tilde{\Lambda}} \circ M$. By density and boundedness, the left and right terms agree on $\mathcal{H}$. Thus, $\tilde{h}^+ = Mh^+ \in \widetilde{E}_0^+ L_2/\widetilde{E}_0^+\mathcal{H}^\perp$ satisfies $D_0 e^{-t\Lambda} h^+ = De^{-t\tilde{\Lambda}}\tilde{h}^+$.

We conclude that $f_t = De^{-t\tilde{\Lambda}}\tilde{h}^+ + \widetilde{S}_A f_t$, with estimates

$$\|\tilde{h}^+\|_{L_2/\mathcal{H}^\perp} \approx \|h^+\|_2 \approx \|D_0 e^{-t\Lambda} h^+\|_{\mathcal{Y}} = \|f - S_A f\|_{\mathcal{Y}} \lesssim \|f\|_{\mathcal{Y}}. \tag{42}$$

The middle equivalence uses Theorem 10.1.

(i′) Conversely, if $f \in \mathcal{Y}$ satisfies (38) for some $\tilde{h}^+ \in \widetilde{E}_0^+ L_2$, then we apply Lemma 11.5 with

$$f^o = f - De^{-t\tilde{\Lambda}}\tilde{h}^+ = f - D_0 e^{-t\Lambda} h^+,$$

with $h^+ \in E_0^+\mathcal{H}$ given by the isomorphism above. Since $(\partial_t + D_0)D_0 e^{-t\Lambda} h^+ = 0$, it follows that f satisfies (17). For the estimate of $\|f\|_{\mathcal{Y}}$ when $I - S_A$ is invertible on $\mathcal{Y}$, use that the last estimate in (42) in this case is $\approx$.

(ii) Lemma 11.5 and Theorem 11.4 show the ODE satisfied by v, existence of the limit v_0 and the estimates of $\|v_t\|_2$ and $\|\tilde{h}^-\|_2$. This completes the proof. □

Corollary 12.8. *Assume that $\|\mathcal{E}\|_* < \infty$. With the notation from Theorem 12.7, the following holds.*

(i) *Any $\mathcal{Y}^o$-solution u to the divergence form equation has representation $u_r = r^{-\frac{n-1}{2}}(v_t)_\perp$ with $r = e^{-t}$, for some v as in Theorem 12.7, boundary trace in the sense $\lim_{r\to 1}\|u_r - u_1\|_2 = 0$, and there are estimates*

$$\|u_r\|_2 \lesssim r^{-\frac{n-1}{2}}\|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1(x)\,dx\right|, \qquad r \in (0,1).$$

(ii) *The map taking $\mathcal{Y}^o$-solutions u to boundary functions $\tilde{h}^+ = \widetilde{E}_0^+ v_0 \in \widetilde{E}_0^+ L_2$ is well-defined and bounded in the sense that*

$$\|\tilde{h}^+\|_2 \lesssim \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1(x)\,dx\right|.$$

(iii) *If furthermore $I - S_A$ is invertible on $\mathcal{Y}$, then this map is an isomorphism and its inverse $\widetilde{E}_0^+ L_2 \ni \tilde{h}^+ \to u \in \{\mathcal{Y}^o\text{-solutions}\}$ is given by*

$$u_r := r^{-\frac{n-1}{2}}\big((I + \widetilde{S}_A(I - S_A)^{-1}D)e^{-t\tilde{\Lambda}}\tilde{h}^+\big)_\perp, \tag{43}$$

with estimates $\|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + |\int_{S^n} u_1(x)\,dx| \approx \|\tilde{h}^+\|_2$.

Proof. (i) Let f be the conormal gradient of u and define $\tilde{h}^+$ and v applying Theorem 12.7. As in the proof of Proposition 3.5, it follows that

$$u_r = r^{-\sigma}(v_t)_\perp + c$$

for some $c \in \mathbb{C}^m$, where $r = e^{-t} \in (0,1)$ and $\sigma = \frac{n-1}{2}$.

Recall that by (38), $\tilde h^+$ is uniquely defined in $\widetilde{E}_0^+ L_2$ modulo $\widetilde{E}_0^+ \mathcal{H}^\perp$ and we now use this freedom to choose it in $\widetilde{E}_0^+ L_2$ such that $c = 0$. Indeed, by Lemma 7.5, $\widetilde{E}_0^+ \mathcal{H}^\perp = N^- \mathcal{H}^\perp = \{[c\ \ 0]^t; c \in \mathbb{C}^m\}$ and since $\tilde\Lambda = \sigma I$ on $\mathcal{H}^\perp$, we have

$$e^{-t\tilde\Lambda}([c\ \ 0]^t) = e^{-\sigma t}[c\ \ 0]^t, \quad c \in \mathbb{C}^m. \tag{44}$$

(The superscript t of the brackets denotes transpose.) Replacing $\tilde h^+$ by $\tilde h^+ - [c\ \ 0]^t$, then f_t remains unchanged, $e^{-t\tilde\Lambda}\tilde h^+$ is replaced by $e^{-t\tilde\Lambda}\tilde h^+ - e^{-\sigma t}[c\ \ 0]^t$, and $(v_t)_\perp$ by $(v_t)_\perp - e^{-\sigma t}c$. Thus we may assume $c = 0$.

As v_t has an $L_2(S^n; \mathbb{C}^{(1+n)m})$ limit v_0 when $t \to 0$, one can set $u_1 := (v_0)_\perp$ and u_r converges in $L_2(S^n; \mathbb{C}^m)$ to u_1. For the estimate on $\|u_r\|_2$ it suffices to prove

$$\|u_r - m\|_2 \lesssim r^{-\frac{n-1}{2}} \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o}, \quad r \in (0, 1).$$

with m the mean value of u_1 on S^n. We may assume that $m = 0$ as by (44) this amounts to modifying $\tilde h^+$ modulo $N^-\mathcal{H}^\perp$ without changing the conormal gradient f of u. We have

$$\|u_r\|_2 \le r^{-\sigma}\|v_t\|_2 \lesssim r^{-\sigma}(\|\tilde h^+\|_2 + \|f\|_{\mathcal{Y}}).$$

By orthogonal projection onto $N^-\mathcal{H}^\perp$, it follows $\|\tilde h^+\|_2 \approx \|\tilde h^+\|_{L_2/\mathcal{H}^\perp} + |\int_{S^n} (\tilde h^+)_\perp dx|$ since $\tilde h^+ \in \widetilde{E}_0^+ L_2$. We can now conclude since $\|\tilde h^+\|_{L_2/\mathcal{H}^\perp} \lesssim \|f\|_{\mathcal{Y}}$ and, since $m = 0$,

$$\left|\int_{S^n} (\tilde h^+)_\perp(x)\, dx\right| = \left|\int_{S^n} (u_1 - (\tilde h^-)_\perp)(x)\, dx\right| \lesssim \|\tilde h^-\|_2 \lesssim \|f\|_{\mathcal{Y}}.$$

(ii) The argument using (44) shows that given a $\mathcal{Y}^o$-solution u and its conormal gradient f, there exists $\tilde h^+ \in \widetilde{E}_0^+ L_2$ such that $u_r = r^{-\sigma}(e^{-t\tilde\Lambda}\tilde h^+ + \widetilde{S}_A f_t)_\perp$. Moreover, $\tilde h^+ = \widetilde{E}_0^+ v_0$ by construction and the estimate $\|\tilde h^+\|_2 \lesssim \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1(x)\, dx\right|$ follows from the above argument. To define the map and prove its boundedness, it suffices to show uniqueness of such $\tilde h^+ \in \widetilde{E}_0^+ L_2$. So assume $u_r = r^{-\sigma}(e^{-t\tilde\Lambda}\tilde h^+ + \widetilde{S}_A f_t)_\perp = r^{-\sigma}(e^{-t\tilde\Lambda}\tilde h_1^+ + \widetilde{S}_A f_t)_\perp$ with f the conormal gradient of u and $\tilde h^+, \tilde h_1^+ \in \widetilde{E}_0^+ L_2$. This implies that $f_t = De^{-t\tilde\Lambda}\tilde h^+ + S_A f_t = De^{-t\tilde\Lambda}\tilde h_1^+ + S_A f_t$ so we know that $\tilde h^+ - \tilde h_1^+ \in \widetilde{E}_0^+ \mathcal{H}^\perp$ by Theorem 12.7. As $\widetilde{E}_0^+ \mathcal{H}^\perp = N^-\mathcal{H}^\perp$, write $\tilde h^+ - \tilde h_1^+ = [c\ \ 0]^t$, with $c \in \mathbb{C}^m$. We have from (44) that $0 = r^{-\sigma}(e^{-t\tilde\Lambda}(\tilde h^+ - \tilde h_1^+))_\perp = c$.

(iii) Given $\tilde h^+ \in \widetilde{E}_0^+ L_2$, define

$$f_t := (I - S_A)^{-1} De^{-t\tilde\Lambda}\tilde h^+, \quad v_t := e^{-t\tilde\Lambda}\tilde h^+ + \widetilde{S}_A f_t, \quad u_r := r^{-\sigma}(v_t)_\perp.$$

By Theorem 10.3 and Lemma 11.5, v satisfies the equation $\partial_t v + \widetilde{D}_0 v = 0$, and by Proposition 3.5, u extends to a $\mathcal{Y}^o$-solution and f is the conormal gradient of u. For the continuity estimate $\|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1(x)\, dx\right| \lesssim \|\tilde h^+\|_2$, Theorem 12.7 implies $\|f\|_{\mathcal{Y}} \lesssim \|\tilde h^+\|_2$ and $\left|\int_{S^n} u_1(x)\, dx\right| \lesssim \|u_1\|_2 \lesssim \|v_0\|_2 \lesssim \|\tilde h^+\|_2 + \|f\|_{\mathcal{Y}} \lesssim \|\tilde h^+\|_2$. This map clearly inverts the map in (ii). This completes the proof. ☐

It is worth specifying the Corollary 12.8 in the case of radially independent coefficients.

Corollary 12.9. *Assume A is radially independent. Then any $\mathcal{Y}^o$-solution is given by $u = r^{-\frac{n-1}{2}}(e^{-t\tilde\Lambda}\tilde h^+)_\perp$ for a unique $\tilde h^+ \in \widetilde{E}_0^+ L_2$ with $\|\tilde h^+\|_2 \approx \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1\, dx\right|$.*

Conclusion. It is clear from (36) that provided $I - S_A$ is invertible on $\mathcal{X}$, the ansatz

$$E_0^+\mathcal{H} \to \mathcal{X} : h^+ \mapsto f_t = (I - S_A)^{-1} e^{-t\Lambda} h^+$$

maps onto all conormal gradients of $\mathcal{X}^o$-solutions to the divergence form equation with coefficients A.

Similarly, (43) implies that provided $I - S_A$ is invertible on $\mathcal{Y}$, the ansatz

$$\widetilde{E}_0^+\mathcal{H} \to \mathcal{Y}^o : \tilde{h}^+ \mapsto u_r := r^{-\frac{n-1}{2}} \Big((I + \widetilde{S}_A (I - S_A)^{-1} D) e^{-t\tilde{\Lambda}} \tilde{h}^+ \Big)_\perp,$$

maps onto all $\mathcal{Y}^o$-solutions to the divergence form equation with coefficients A.

Thus we have a way of constructing solutions and our two main goals towards well-posedness results are the following.

First understand when invertibility of $I - S_A$ holds. This will be done in Section 16.

Secondly, introduce the boundary maps that connect the traces of solutions to the data for the BVPs and invert them. This is the object of Section 17.

Before we do this, we continue with different a priori representations of solutions in the next section. This will be useful to prove nontangential maximal estimates and obtain convergence of Fatou type at the boundary.

13. Conjugate systems

The results in the preceding section allow to represent $\mathcal{X}^o$-solutions in terms of the conormal gradient f. Actually, if one is interested in u itself, one can try to further describe the corresponding potential vector v. Similarly, representation of $\mathcal{Y}^o$-solutions is embedded into a potential vector v but it could be interesting to describe the properties of the conormal gradient f. Both are related by the rule $Dv = f$. This leads us to the following notion.

Definition 13.1. A *pair of conjugate systems* to the divergence equation with coefficients A is a pair $(v, f) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; L_2(S^n; \mathcal{V})) \times L_2^{\mathrm{loc}}(\mathbb{R}_+; L_2(S^n; \mathcal{V}))$ with

(i) $v_t \in \mathsf{D}(D)$ for almost every t and $\int_1^\infty \|Dv_t\|_2^2 \, dt < \infty$,

(ii) v is an $\mathbb{R}^+ \times S^n$-distributional solution of (19),

(iii) $f_t = Dv_t$ for almost every $t > 0$,

(iv) f is a $\mathcal{H}$-valued $\mathbb{R}^+ \times S^n$-distributional solution of (17).

By Proposition 3.5 and its proof, a pair of conjugate systems is completely determined by v satisfying (i) and (ii). That is, f defined by (iii) automatically satisfies (iv). Moreover, the function

$$u_r := r^{-(n-1)/2} (v_t)_\perp, \quad r = e^{-t} \in (0, 1), \tag{45}$$

extends to a weak solution of $\operatorname{div}_{\mathbf{x}} A \nabla_{\mathbf{x}} u = 0$ in $\mathbb{O}^{1+n}$ and f must be the conormal gradient of u. *We say that a weak solution u and a pair of conjugate systems (v, f) to the divergence form equation for which (45) holds are associated.*

It is our goal to give a description of the pair (not only f or v) in each case. Recall that in integrating $Dv = f$, v_t is only determined by f_t modulo $\mathcal{H}^\perp$ so there is a choice to make.

Theorem 13.2. *Assume $\|\mathcal{E}\|_* < \infty$. Let u be an $\mathcal{X}^o$- or $\mathcal{Y}^o$-solution. Then u has an $L_2(S^n; \mathbb{C}^m)$ trace u_1 at the boundary and there exists an associated pair of conjugate systems given by*

$$\begin{cases} v_t = e^{-t\tilde{\Lambda}} v_0 + \tilde{w}_t, \\ f_t = e^{-t\Lambda} f_0 + w_t, \end{cases} \tag{46}$$

with the following properties.

(i) *If u is an $\mathcal{X}^o$-solution, then $u_1 \in W_2^1(S^n; \mathbb{C}^m)$, $(v_0, f_0) \in D(D) \times \mathcal{H}$ with $Dv_0 = f_0$, $\|\nabla_S u_1\|_2 \lesssim \|f_0\|_2 \lesssim \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{X}^o}$, $\|v_0\|_2 \lesssim \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{X}^o} + \left|\int_{S^n} u_1 \, dx\right|$, $D\tilde{w}_t = w_t \in \mathcal{Y}^*$, $v_t \in C(\mathbb{R}^+; L_2)$ and $\|v_t - v_0\|_2 + \|\tilde{w}_t\|_2 = O(t)$ for $t > 0$.*

(ii) *If u is a $\mathcal{Y}^o$-solution, then $u_1 \in L_2(S^n; \mathbb{C}^m)$, $(v_0, f_0) \in L^2 \times \dot{W}_2^{-1}(S^n; \mathcal{V})$ with $Dv_0 = f_0$, $\|u_1\|_2 \lesssim \|v_0\|_2 + \|f_0\|_{\dot{W}_2^{-1}} \lesssim \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1 dx\right|$, $D\tilde{w}_t = w_t \in \mathcal{Y}$, $v_t \in C(\mathbb{R}^+; L_2)$ and $\|v_t - v_0\|_2 + \|\tilde{w}_t\|_2 = O(1)$ for $t > 0$ and $o(1)$ for $t \to 0$.*

Besides $\mathcal{X}^o$- and $\mathcal{Y}^o$-solutions to the divergence form equation, we shall in the following sections also consider the following classical class of variational solutions.

Definition 13.3. By a *variational solution to the divergence form equation*, with coefficients A, we mean a weak solution of $\operatorname{div}_{\boldsymbol{x}} A\nabla u = 0$ in $\mathbb{O}^{1+n}$ with $\|\nabla_{\boldsymbol{x}} u\|_2 < \infty$.

It is illuminating to see how the representation for variational solutions lies in between the ones for $\mathcal{X}^o$- and $\mathcal{Y}^o$-solutions, independently of solvability issues which are well-known for variational solutions. We state this result without proof as it is not used in this paper. Note that, as compared to Theorem 13.2, the Carleson condition $\|\mathcal{E}\|_* < \infty$ is not needed in the following result.

Proposition 13.4. *Let u be a variational solution to the divergence form equation with coefficients A. Then u has an $L_2(S^n; \mathbb{C}^m)$ trace u_1 at the boundary and there exists an associated pair of conjugate systems given by (46) with the following properties*:

$u_1 \in W_2^{1/2}(S^n; \mathbb{C}^m)$, $(v_0, f_0) \in D(|D|^{1/2}) \times \dot{W}_2^{-1/2}(S^n; \mathcal{V})$ with $Dv_0 = f_0$, $\|v_0\|_2 \lesssim \|\nabla_{\boldsymbol{x}} u\|_2 + \left|\int_{S^n} u_1 \, dx\right|$, $\|u_1\|_{\dot{W}_2^{1/2}} \lesssim \|f_0\|_{\dot{W}_2^{-1/2}} \lesssim \|\nabla_{\boldsymbol{x}} u\|_2$, $D\tilde{w}_t = w_t \in L_2(\mathbb{R}^+; L_2)$, $v_t \in C(\mathbb{R}^+; L_2)$ and $\|v_t - v_0\|_2 + \|\tilde{w}_t\|_2 = O(t^{1/2})$ for $t > 0$.

Here $\dot{W}_2^{1/2}$ is equipped with homogeneous norm and $\dot{W}_2^{-1/2}$ is its dual.

Proof of Theorem 13.2. (i) From Theorem 12.3, we have

$$f_t = e^{-t\Lambda} h^+ + S_A f_t = e^{-t\Lambda} f_0 + w_t, \quad w_t := S_A f_t - e^{-t\Lambda} h^-.$$

with $f_0 = h^+ + h^- \in \mathcal{H}$, $\|f_0\|_2 \lesssim \|\nabla_{\boldsymbol{x}} u\|_{\mathcal{X}^o}$ and $h^- = -\int_0^\infty e^{-s\Lambda} E_0^- D\mathcal{E}_s f_s \, ds$.

We define v_0, $\tilde{h}^+$, $\tilde{h}^-$ and v as follows: $\tilde{h}^+$ is the unique element in $\widetilde{E}_0^+ L_2 / \widetilde{E}_0^+ \mathcal{H}^\perp$ such that $D\tilde{h}^+ = h^+ (= D_0(D_0^{-1} h^+))$, $\tilde{h}^- := -\int_0^\infty e^{-s\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds$, $v_0 = \tilde{h}^+ + \tilde{h}^-$ and

$$v_t := e^{-t\tilde{\Lambda}} \tilde{h}^+ + \widetilde{S}_A f_t = e^{-t\tilde{\Lambda}} v_0 + \tilde{w}_t, \quad \tilde{w}_t := \widetilde{S}_A f_t - e^{-t\tilde{\Lambda}} \tilde{h}^-.$$

Clearly, $\tilde{h}^+ \in \mathsf{D}(D)$. Next, $\tilde{\Lambda}\tilde{h}^- \in L_2$ because $\mathcal{E} f \in \mathcal{Y}^*$, so $\tilde{h}^- \in \mathsf{D}(D) = \mathsf{D}(\tilde{\Lambda})$ and $D\tilde{h}^- = h^-$. So $v_0 \in \mathsf{D}(D)$ and $Dv_0 = f_0$.

The estimate on $\|e^{-t\tilde{\Lambda}} v_0 - v_0\|_2$ follows from $v_0 \in \mathsf{D}(D)$.

Next, $D\tilde{w}_t = w_t$ by construction and $w_t \in \mathcal{Y}^*$ from the proof of Lemma 11.5. (In fact, w_t is nothing but $Z_A f_t$ defined in that proof.)

The estimate on $\|\tilde{w}_t\|_2$ follows from

$$\tilde{w}_t = \int_0^t e^{-(t-s)\tilde{\Lambda}} \widetilde{E}_0^+ \mathcal{E}_s f_s \, ds - \int_t^\infty (e^{-(s-t)\tilde{\Lambda}} - e^{-(t+s)\tilde{\Lambda}}) \widetilde{E}_0^- \mathcal{E}_s f_s \, ds + e^{-t\tilde{\Lambda}} \int_0^t e^{-s\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds,$$

using $\mathcal{E} f \in \mathcal{Y}^*$, the uniform boundedness of the semigroup and its decay at infinity. Details are left to the reader.

Eventually, as in Corollary 12.8, one can adjust $\tilde{h}^+$ by adding an element in $N^- \mathcal{H}^\perp$ such that u and v satisfy (45). In particular, u has an L_2 trace. It also follows that f is the conormal gradient of u with a limit f_0 when $t \to 0$ by (35). So $u_1 \in W_2^1(S^n; \mathbb{C}^m)$ with $\|\nabla_S u_1\|_2 \lesssim \|f_0\|_2$.

(ii) By Corollary 12.8, we have description of

$$v_t = e^{-t\tilde{\Lambda}} \tilde{h}^+ + \widetilde{S}_A f_t = e^{-t\tilde{\Lambda}} v_0 + \tilde{w}_t, \quad \tilde{w}_t = \widetilde{S}_A f_t - e^{-t\tilde{\Lambda}} \tilde{h}^-,$$

with $v_0 = \tilde{h}^+ + \tilde{h}^-$ such that u and v satisfy (45) and of trace and growth estimates for $\|e^{-t\tilde{\Lambda}} v_0 - v_0\|_2 + \|\tilde{w}_t\|_2$. It remains to consider the representation of f. We have by Theorem 12.7,

$$f_t = De^{-t\tilde{\Lambda}} \tilde{h}^+ + S_A f_t = De^{-t\tilde{\Lambda}} v_0 + w_t, \quad w_t = S_A f_t - De^{-t\tilde{\Lambda}} \tilde{h}^- = D\tilde{w}_t.$$

Define $f_0 := Dv_0$ in distribution sense, so that $f_0 \in \dot{W}_2^{-1}(S^n; \mathcal{V})$ and $\|f_0\|_{\dot{W}_2^{-1}} \lesssim \|v_0\|_2$. We obtain

$$f_t = e^{-t\Lambda} f_0 + w_t$$

and here, the action of $e^{-t\Lambda}$ is extended to $\dot{W}_2^{-1}(S^n; \mathcal{V})$ by extending the intertwining formula $De^{-t\Lambda} = e^{-t\tilde{\Lambda}} D$. $\square$

14. Non-tangential maximal estimates

Theorem 14.1. *Assume $\|\mathcal{E}\|_{C \cap L_\infty} < \infty$. Then any $\mathcal{Y}^o$-solution to the divergence form equation with coefficients A satisfies*

$$\|u_1\|_2^2 \lesssim \|\widetilde{N}_*^o(u)\|_2^2 \lesssim \int_{\mathbb{O}^{1+n}} |\nabla_{\mathbf{x}} u|^2 (1 - |\mathbf{x}|) \, d\mathbf{x} + \left| \int_{S^n} u_1(x) \, dx \right|^2.$$

When $n = 1$, the conjugate $\tilde{u}$ of a $\mathcal{Y}^o$-solution u also satisfies the estimates

$$\|\tilde{u}_1\|_2^2 \lesssim \|\widetilde{N}_*^o(\tilde{u})\|_2^2 \lesssim \int_{\mathbb{O}^{1+n}} |\nabla_{\mathbf{x}} u|^2 (1 - |\mathbf{x}|) \, d\mathbf{x} + \left| \int_{S^n} \tilde{u}_1(x) \, dx \right|^2.$$

The proof follows the strategy of [Part I] with a slight modification in view of preparing the proof of almost everywhere nontangential convergence.

Proof. The estimate $\|\widetilde{N}_*^o(u)\|_2 \gtrsim \|u_1\|_2$ follows from Lemma 9.3 and Corollary 12.8(i). For the upper bound, we proceed as follows. From the representation $u_r = r^{-\sigma}(v_t)_\perp$ with $v_t = e^{-t\tilde\Lambda}v_0 + \tilde w_t$ in Theorem 13.2, it is enough to bound $\|\widetilde{N}_*((e^{-t\tilde\Lambda}v_0)_\perp)\|_2$ and $\|\widetilde{N}_*(\tilde w_\perp)\|_2$. Theorem 10.3, and Lemma 14.2 below, show that

$$\|\widetilde{N}_*^o(u)\|_2 \lesssim \|v_0\|_2 + \|f\|_{\mathcal{Y}} \lesssim \|\tilde h^+\|_2 + \|\tilde h^-\|_2 + \|f\|_{\mathcal{Y}} \lesssim \|\tilde h^+\|_{L_2/\mathcal{H}^\perp} + \left|\int_{S^n} \tilde h_\perp^+ dx\right| + \|f\|_{\mathcal{Y}},$$

and $\|\tilde h^+\|_{L_2/\mathcal{H}^\perp} \lesssim \|f\|_{\mathcal{Y}}$, $\left|\int_{S^n} \tilde h_\perp^+ dx\right| = \left|\int_{S^n}(u_1 - \tilde h_\perp^-)\,dx\right| \lesssim \left|\int_{S^n} u_1\,dx\right| + \|f\|_{\mathcal{Y}}$, as in the proof of Corollary 12.8.

When $n = 1$, replacing A by the conjugate coefficients $\tilde A$ defined in Section 5 in the above argument, and using $|\nabla_{\boldsymbol{x}}\tilde u| \approx |\nabla_{\boldsymbol{x}} u|$, proves the estimates for $\|\widetilde{N}_*^o(\tilde u)\|_2$. □

Lemma 14.2. *Assume $\|\mathcal{E}\|_{C\cap L_\infty} < \infty$. Then we have, for each $p < 2$,*

$$\|\widetilde{N}_*^p(\tilde w)\|_2 + \|\widetilde{N}_*(\tilde w_\perp)\|_2 \lesssim \|\mathcal{E}\|_{C\cap L_\infty}\|f\|_{\mathcal{Y}}.$$

Here $\widetilde{N}_*^p$ is defined similarly to $\widetilde{N}_*$, replacing L_2 averages by L_p averages. When $n = 1$, we also have

$$\|\widetilde{N}_*(\tilde w_\parallel)\|_2 \lesssim \|\mathcal{E}\|_{C\cap L_\infty}\|f\|_{\mathcal{Y}}.$$

Furthermore, these estimates hold with $\tilde w$ replaced by the truncation $\chi_{t<\tau}\tilde w$, and $\|f\|_{\mathcal{Y}}^2$ replaced by $\int_0^\infty \|f_t\|_2^2 \min(t,\tau)\,dt$, for any $\tau < 1$.

Proof. The proof will follow closely the strategy of [Part I, Lemma 10.2] on $\mathbb{R}^{1+n}_+$. We remark that $\widetilde{N}_*^p \le \widetilde{N}_*$ pointwise. Thus we will work with $\widetilde{N}_*$, and indicate when we need to consider $\widetilde{N}_*^p$ or the normal component. Recall that $\widetilde{N}_*$ estimates the truncation of the function to $t < 1$.

(i) From $\tilde w_t = \widetilde{S}_A f_t - e^{-t\tilde\Lambda}\tilde h^-$ and the definition of $\tilde h^-$,

$$\begin{aligned}
\tilde w_t &= \int_0^t e^{-(t-s)\tilde\Lambda}\widetilde{E}_0^+\mathcal{E}_s f_s\,ds - \int_t^\infty e^{-(s-t)\tilde\Lambda}\widetilde{E}_0^-\mathcal{E}_s f_s\,ds + e^{-t\tilde\Lambda}\int_0^\infty e^{-s\tilde\Lambda}\widetilde{E}_0^-\mathcal{E}_s f_s\,ds \\
&= \int_0^t e^{-(t-s)\tilde\Lambda}(1-e^{-2s\tilde\Lambda})\widetilde{E}_0^+\mathcal{E}_s f_s ds - \int_t^\infty e^{-(s-t)\tilde\Lambda}(1-e^{-2t\tilde\Lambda})\widetilde{E}_0^-\mathcal{E}_s f_s\,ds + e^{-t\tilde\Lambda}\int_0^t e^{-s\tilde\Lambda}\mathcal{E}_s f_s\,ds \\
&= I_1 - I_2 + I_3.
\end{aligned}$$

Note that $\widetilde{E}_0^+ + \widetilde{E}_0^- = I$ (also in dimension $n = 1$) is used in getting I_3. For the first two terms, we use Schur estimates as follows. Since $\|e^{-(t-s)\tilde\Lambda}(I - e^{-2s\tilde\Lambda})\| \lesssim s/t$, we have, as in [Part I, Lemma 10.2],

$$\|\widetilde{N}_*(I_1)\|_2^2 \lesssim \int_0^1 \left(\int_0^t st^{-1}\|f_s\|_2 ds\right)^2 \frac{dt}{t} \lesssim \|\chi_{t<1} f\|_{\mathcal{Y}}^2.$$

Similarly, since $\|e^{-(s-t)\tilde\Lambda}(I - e^{-2t\tilde\Lambda})\| \lesssim t/s$, we have

$$\begin{aligned}
\|\widetilde{N}_*(I_2)\|_2^2 &\lesssim \int_0^1 \left(\int_t^\infty ts^{-1}\|f_s\|_2\,ds\right)^2 \frac{dt}{t} \lesssim \int_0^1 \left(\int_t^\infty t/s^2\,ds\right)\left(\int_t^\infty t\|f_s\|_2^2\,ds\right)\frac{dt}{t} \\
&\lesssim \int_0^\infty \left(\int_0^{\min(s,1)} t\,\frac{dt}{t}\right)\|f_s\|_2^2\,ds = \|f\|_{\mathcal{Y}}^2.
\end{aligned}$$

Note that the estimates so far hold for all $\tilde w$, not only for its normal component. By inspection, the stated estimates of the truncated maximal function hold for these terms.

(ii) It remains to consider $I_3 = e^{-t\tilde\Lambda}\int_0^t e^{-s\tilde\Lambda}\mathscr{E}_s f_s\,ds$. To make use of off-diagonal estimates in Lemma 6.2, we need to replace $e^{-t\tilde\Lambda}$ by the resolvents $(I+it\widetilde D_0)^{-1}$. To this end, define $\psi_t(z) := e^{-t|z|} - (1+itz)^{-1}$ and split the integral

$$e^{-t\tilde\Lambda}\int_0^t e^{-s\tilde\Lambda}\mathscr{E}_s f_s\,ds = \psi_t(\widetilde D_0)\int_0^\infty e^{-s\tilde\Lambda}\mathscr{E}_s f_s\,ds - \int_t^\infty \psi_t(\widetilde D_0)e^{-s\tilde\Lambda}\mathscr{E}_s f_s\,ds \\ + \int_0^t (I+it\widetilde D_0)^{-1}(e^{-s\tilde\Lambda}-I)\mathscr{E}_s f_s\,ds + (I+it\widetilde D_0)^{-1}\int_0^t \mathscr{E}_s f_s\,ds.$$

For the first term, square function estimates show that $\psi_t(\widetilde D_0) : L_2 \to \mathscr{Y}^* \subset \mathscr{X}$ is continuous, and Theorem 11.4 shows $\|\int_0^\infty e^{-s\tilde\Lambda}\mathscr{E}_s f_s\,ds\|_2 \lesssim \|f\|_{\mathscr{Y}}$ (or $\lesssim \|f\|_{\mathscr{Y}_\delta}$ when $n=1$, but $\|f\|_{\mathscr{Y}_\delta} \lesssim \|f\|_{\mathscr{Y}}$ for conormal gradients of solutions by Proposition 9.7). For the second and third terms, we proceed as above for I_1 and I_2 by Schur estimates using $\|\psi_t(\widetilde D_0)e^{-s\tilde\Lambda}\| \lesssim t/s$, and $\|(I+it\widetilde D_0)^{-1}(e^{-s\tilde\Lambda}-I)\| \lesssim s/t$.

(iii) It remains to estimate $(I+it\widetilde D_0)^{-1}\int_0^t \mathscr{E}_s f_s\,ds$, and this is where we use $\|\mathscr{E}\|_C$. Consider first $\widetilde N_*^p$. Fix a Whitney box $W_0 = W(t_0,x_0)$. We proceed by a duality argument in the spirit of Corollary 6.3, and bound $\|(I+it\widetilde D_0)^{-1}\int_0^t \mathscr{E}_s f_s\,ds\|_{L_p(W_0)}$ by testing against $h \in L_q(W_0;\mathscr{V})$, $1/p+1/q=1$. As in step (iii) of the proof of [Part I, Lemma 10.2], this leads to a pointwise estimate implying

$$\left\|\widetilde N_*^p\left((I+it\widetilde D_0)^{-1}\int_0^t \mathscr{E}_s f_s\,ds\right)\right\|_2 \lesssim \|\mathscr{E}\|_C\|f\|_{\mathscr{Y}}.$$

Since the proof here is essentially the same as there, but replacing $\mathbb{R}^n$ by S^n, using area and maximal functions on S^n instead, we omit the details. The main ingredients are the L_p off-diagonal estimates for $(I+it\widetilde D_0^*)^{-1}$ from Lemma 6.2(i) and the tent space estimate [Coifman et al. 1985, Theorem 1(a)] of Coifman, Meyer and Stein.

To estimate $\widetilde N_*((I+it\widetilde D_0)^{-1}\int_0^t \mathscr{E}_s f_s\,ds)_\perp)$, we proceed by duality as above. We now instead test against $h \in L_2(W_0;\mathscr{V})$ with $h_\parallel = 0$ and use the $L_2 \to L_q$ off-diagonal estimates for $(I+it\widetilde D_0^*)^{-1}$ from Lemma 6.2(ii) to obtain

$$\left\|\widetilde N_*\left(\left((I+it\widetilde D_0)^{-1}\int_0^t \mathscr{E}_s f_s\,ds\right)_\perp\right)\right\|_2 \lesssim \|\mathscr{E}\|_C\|f\|_{\mathscr{Y}}.$$

It remains to see that, when $n=1$, the $\widetilde N_*$ estimate also applies to the tangential part $w_\parallel$. Consider the transformed conjugate coefficients $\widetilde B = \widehat{\widetilde A}$ and $\widetilde B_0 = \widehat{\widetilde A_1}$ from the proof of Proposition 5.4, and let $\widetilde{\mathscr{E}} := \widetilde B_0 - \widetilde B$. Then $\tilde f := J^t f$ solves $(\partial_t + D\widetilde B)\tilde f = 0$, which yields the estimate of $\|\widetilde N_*(w_\parallel)\|_2$ since $(\widetilde S_A f)_\parallel = (J^t\widetilde S_A f)_\perp = (\widetilde S_{\tilde A}\tilde f)_\perp$. This completes the proof. □

Remark 14.3. The proof also shows a priori estimates for the operators $\widetilde S_A$ when f is not supposed to be a conormal gradient of a solution. Assume $\|\mathscr{E}\|_{C\cap L_\infty} < \infty$. If $n \geq 2$, then we have for each $p < 2$,

$$\|\widetilde N_*^p(\widetilde S_A f)\|_2 + \|\widetilde N_*((\widetilde S_A f)_\perp)\|_2 \lesssim \|\mathscr{E}\|_{C\cap L_\infty}\|f\|_{\mathscr{Y}}, \quad f \in \mathscr{Y}.$$

When $n = 1$, we have for each $\delta > 0$,

$$\|\widetilde{N}_*(\widetilde{S}_A f)\|_2 \lesssim \|\mathcal{E}\|_{C\cap L_\infty}\|f\|_{\mathcal{Y}_\delta}, \quad f \in \mathcal{Y}_\delta.$$

15. Almost everywhere nontangential convergence

Since solutions are not defined in a pointwise sense, the classical notion of nontangential convergence at a boundary point x is replaced here by

$$\lim_{r\to 1} |W^o(rx)|^{-1} \int_{W^o(rx)} h(\mathbf{y})\, d\mathbf{y} \;\text{ exists,}$$

which we call *convergence of Whitney averages at* x because the region $W^o(rx)$ is a Whitney ball. Note that since the Whitney balls at x cover a truncated cone with vertex x, it really amounts to a nontangential convergence. Besides, a slight modification of the proofs below yields limits of averages on Whitney regions $W^o(z)$ for z in a fixed cone with vertex at x_0, as $|z| \to 1$. The exact choice of the Whitney balls does not matter.

Definition 15.1. Let h be a function in $\mathbb{O}^{1+n}$ with range in the bundle $\mathcal{V}$ in the sense that $h(rx) \in \mathcal{V}_x$ for all $r > 0$ and $x \in S^n$. Let $x_0 \in S^n$ and $1 \le p < \infty$. We say that the *Whitney averages of* h *converge at* x_0 *in* L_p *sense* to $c \in \mathcal{V}_{x_0}$ if for any/some section $c_{x_0} \in C^\infty(S^n; \mathcal{V})$ with $c_{x_0}(x_0) = c$,

$$\lim_{r\to 1} |W^o(rx_0)|^{-1} \int_{W^o(rx_0)} |h(\mathbf{y}) - c_{x_0}(y)|^p d\mathbf{y} = 0.$$

Here $W^o(\mathbf{x})$ denotes a Whitney ball in $\mathbb{O}^{1+n}$ centered at $\mathbf{x}$. We say that the *Whitney averages of* h *converge in* L_p *sense almost everywhere to* h_0 with respect to surface measure if this happens with $c = h_0(x_0)$ for almost every point $x_0 \in S^n$. For functions with values in a trivial bundle, the sections c_{x_0} are just constant functions.

Note that the limit does not depend on the choice of the section c_{x_0}, so this explains the "any/some" and it suffices to prove the existence of the limit for one chosen section. Clearly this notion entails convergence of Whitney averages.

Theorem 15.2. *Let* A *be coefficients with* $\|\mathcal{E}\|_{C\cap L_\infty} < \infty$. *Let* u *be a* $\mathcal{Y}^o$*-solution to the divergence form equation with coefficients* A *and let* u_1 *be the boundary trace of* u *given by Corollary 12.8. Then Whitney averages of* u *converge in* L_2 *sense almost everywhere to* u_1. *In particular, Whitney averages of* u *converge almost everywhere to* u_1.

The result also holds for the $\mathbb{R}^{1+n}_+$ setup of [Part I], with almost identical proof.

Proof. As in the proof of Theorem 13.2, we can write

$$u(\mathbf{x}) = e^{\sigma t}(e^{-t\tilde{\Lambda}} v_0 + \tilde{w}_t)_\perp(x),$$

where $\mathbf{x} = e^{-t}x$, $\sigma = \frac{n-1}{2}$, $v_0 \in L_2$ with $\|v_0\|_2 \lesssim \|\nabla_{\mathbf{x}} u\|_{\mathcal{Y}^o} + \left|\int_{S^n} u_1 dx\right|$ and $u_1 = (v_0)_\perp$.

Let $p < 2$ as in the third inequality of Corollary 6.3. Let x_0 be a point on S^n, and let $B(x_0, t)$ be the surface ball centered at x_0 with radius t. Adapting the usual Lebesgue point argument for $p = 1$, it is seen that for almost all points x_0

$$\lim_{t\to 0} |B(x_0,t)|^{-1} \int_{B(x_0,t)} |v_0(x) - v_{x_0}(x)|^p dx = 0$$

for any section $v_{x_0} \in C^\infty(S^n; \mathcal{V})$ with $v_{x_0}(x_0) = v_0(x_0)$ and one can further assume $Dv_{x_0} = 0$, which in particular implies that its normal component is the constant scalar function $(v_0(x_0))_\perp = u_1(x_0)$. The key point is the identity

$$u(\boldsymbol{x}) - u_1(x_0) = (e^{\sigma t} e^{-t\tilde{\Lambda}}(v_0 - v_{x_0}))_\perp(x) + e^{\sigma t}(\tilde{w}_t)_\perp(x), \tag{47}$$

which follows since $\widetilde{D}_0 v_{x_0} = -\sigma N v_{x_0}$, and hence $\tilde{\Lambda} v_{x_0} = \sigma v_{x_0}$ and $e^{\sigma t} e^{-t\tilde{\Lambda}} v_{x_0} = v_{x_0}$.

From Theorem 14.1, $\|\widetilde{N}_*(\chi_{t<\tau}\tilde{w}_\perp)\|_2 \to 0$ as $\tau \to 0$. Thus we can assume that the Whitney averages of $\tilde{w}_\perp$ converge to 0 in L_2 sense at x_0. It remains to show, with $h_{x_0} := v_0 - v_{x_0}$,

$$\lim_{t_0\to 0} |W(t_0,x_0)|^{-1} \int_{W(t_0,x_0)} |(e^{\sigma t} e^{-t\tilde{\Lambda}} h_{x_0})_\perp(x)|^2 \, dt \, dx = 0.$$

As in [Stein 1970, Chapter VII, Theorem 4], the rest of the argument consists in using the maximal estimates in Theorem 10.3 with some adaptation. As we do not have pointwise bounds on the operators that substitute the Poisson kernel we also have to handle more technicalities. Let $0 < c_0 t_0 < \tau$ with $t_0, \tau < 1$ to be chosen and $c_0^{-1} t_0 < t < c_0 t_0$. In the L_2 average, write

$$(e^{\sigma t} e^{-t\tilde{\Lambda}} h_{x_0})_\perp = ((1 + it\sigma)(I + it\widetilde{D}_0)^{-1} h_{x_0})_\perp + (e^{\sigma t} e^{-t\tilde{\Lambda}} h_{x_0} - (1 + it\sigma)(I + it\widetilde{D}_0)^{-1} h_{x_0})_\perp.$$

For the first term, we use (26). Fixing t and taking only the L_2 average in x, this gives us a bound

$$\sum_{j\geq 2} 2^{-j} \left(|B(x_0, 2^j t)|^{-1} \int_{B(x_0,2^j t)} |h_{x_0}(x)|^p \, dx \right)^{1/p}.$$

This is controlled by

$$M_\tau^p(h_{x_0})(x_0) + (t_0/\tau) M^p(h_{x_0})(x_0),$$

where M is the Hardy–Littlewood maximal operator over surface balls on S^n, $M^p(h) := M(|h|^p)^{1/p}$, and the subscript τ means that we restrict the maximal operator to balls having radii less than τ. This control is obtained by truncating the sum at $2^j \approx \tau/t$ and using that $t \approx t_0$. The average in t now yields the same bound.

For the second term, we note that $(e^{\sigma t} e^{-t\tilde{\Lambda}} - (1 + it\sigma)(I + it\widetilde{D}_0)^{-1}) v_{x_0} = 0$. Thus we may replace h_{x_0} by v_0 in this term, and write it

$$(e^{\sigma t} \psi(t\widetilde{D}_0) v_0)_\perp + (e^{\sigma t} - (1 + i\sigma t)^{-1})((I + it\widetilde{D}_0)^{-1} v_0)_\perp.$$

with $\psi(\lambda) := e^{-|\lambda|} - (1 + i\lambda)^{-1}$. The first term has estimates

$$\|\widetilde{N}_*(\chi_{t<\tau}\psi(t\widetilde{D}_0)v_0)\|_2^2 \lesssim \int_0^\tau \|\psi(t\widetilde{D}_0)v_0\|_2^2 \frac{dt}{t} \to 0, \quad \tau \to 0,$$

by Lemma 9.3 and square function estimates. Therefore we can assume that Whitney averages of $(e^{\sigma t}\psi(t\widetilde{D}_0)v_0)_\perp$ converge to 0 in L_2 sense at x_0. By Theorem 10.3, the second is controlled by

$$\tau M^p(v_0)(x_0).$$

Thus it remains to show convergence to zero of

$$M^p_\tau(h_{x_0})(x_0)+(t_0/\tau)M^p(h_{x_0})(x_0)+\tau M^p(v_0)(x_0).$$

Since $M^p(v_0)\in L_2(S^n)$ as $p<2$, we can further assume for x_0 that $M^p(v_0)(x_0)<\infty$. For such fixed x_0 it follows that $M^p(h_{x_0})(x_0)\le M^p(v_0)(x_0)+M^p(v_{x_0})(x_0)<\infty$. We now make $M^p_\tau(h_{x_0})(x_0)+\tau M^p(v_0)(x_0)$ small by choosing τ small. Then choose $t_0<\tau$ to make $(t_0/\tau)M^p(h_{x_0})(x_0)$ small. All the constraints on x_0 are met almost everywhere and this completes the proof. □

Remark 15.3. The proof of almost everywhere convergence for averages applies to v (with $\widetilde{N}^p_*$, $p<2$, if $n\ge 2$). The starting point is

$$e^{\sigma t}v_t(x)-v_{x_0}(x)=e^{\sigma t}e^{-t\tilde{\Lambda}}(v_0-v_{x_0})(x)+e^{\sigma t}\tilde{w}_t(x)$$

replacing (47) and the rest of the proof is as above. The only needed modification of the argument is that we now use (25) instead of (26). We obtain almost everywhere convergence of Whitney averages of $e^{\sigma t}v$ in L_p sense to v_0 for $p<2$. Of course, the term $e^{\sigma t}$ can easily be removed in the end. This factor was needed in order to have $e^{\sigma t}e^{-\sigma\tilde{\Lambda}}=I$ on $\mathsf{N}(D)$.

Corollary 15.4. *Assume that A satisfies $\|\mathcal{E}\|_{C\cap L_\infty}<\infty$ and is such that all weak solutions u to the divergence form equation with coefficients A, for some fixed constant $c>1$, satisfy the local boundedness property*

$$\sup_{\boldsymbol{x}\in B}|u(\boldsymbol{x})|\le C\left(|cB|^{-1}\int_{cB}|u(\boldsymbol{y})|^2d\boldsymbol{y}\right)^{1/2},$$

with a constant C independent of u and of closed balls B with $cB\subset\mathbb{O}^{1+n}$. Then any $\mathcal{Y}^o$-solution to the divergence form equation with coefficients A converges nontangentially almost everywhere to its boundary trace.

The local boundedness property is a classical consequence of local Hölder regularity for weak solutions. For real equations ($m=1$), the latter follows from [Moser 1961; De Giorgi 1957]. For small complex L_∞ perturbations of real equations, this is from [Auscher 1996]. For two dimensional systems ($n=1$), local regularity follows immediately from reverse Hölder inequalities described in Theorem 9.8 and Sobolev embeddings. For any dimension and system ($m\ge 1, n\ge 1$), with continuous in $\overline{\mathbb{O}^{1+n}}$ or vmo coefficients, this is explicitly done in [Auscher and Qafsaoui 2000].

Proof. Applying the local boundedness property to $u-u_1(x_0)$ on Whitney balls yields the desired convergence for almost every x_0 from Theorem 15.2. □

We know describe new almost everywhere convergence results for $\mathcal{X}^o$-solutions.

Theorem 15.5. *Let A be coefficients with $\|\mathscr{E}\|_{C\cap L_\infty} < \infty$. Let g be an $\mathscr{X}^o$-solution with potential u to the divergence form equation with coefficients A. Then for any $p < 2$, Whitney averages of $g_\perp = \partial_t u$, and of $(Ag)_\parallel = (A\nabla_{\boldsymbol{x}} u)_\parallel$, converge in L_p sense almost everywhere to $(g_1)_\perp$ and $(A_1 g_1)_\parallel$ respectively, where g_1 is the boundary trace of g given by Theorem 12.4.*

Furthermore, if we have pointwise ellipticity conditions on A, then the Whitney averages of $\nabla_{\boldsymbol{x}} u$ and $\partial_{\nu_A} u$ converge in L_p sense almost everywhere to g_1 and $(A_1 g_1)_\perp$ respectively.

Finally, in all cases, Whitney averages of the potential u converge almost everywhere in L_2 sense to u_1.

Recall that pointwise ellipticity holds when $m = 1$ (equations) or $n = 1$ (two dimensional systems). If A is continuous in $\overline{\mathbb{O}^{1+n}}$, then pointwise accretivity can be deduced from the strict accretivity in the sense of (2), for any m, n. See [Friedman 1976], for example. We do not know if this convergence of $\nabla_{\boldsymbol{x}} u$ and $\partial_{\nu_A} u$ holds when $m \geq 2$ and $n \geq 2$ in general.

Proof. We begin with the convergence for u. It is a straightforward consequence of the growth $\|v_t - v_0\|_2 = O(t)$ for $t > 0$ in Theorem 13.2 and $u(\boldsymbol{x}) - u_1(x) = (e^{-\sigma t} v_t - v_0)_\perp(x)$. Let us turn to the gradient.

By Theorem 13.2 we have $f_t = e^{-t\Lambda} f_0 + w_t$ for some $f_0 \in \mathscr{H}$ and $w \in \mathscr{Y}^*$. From the correspondence between g and f in Proposition 3.3, it follows that, modulo a rescaling, $(g)_\perp \vec{n} + (Ag)_\parallel$ equals Bf. Thus we need to prove convergence of Whitney averages of

$$B_t f_t = e^{-t\tilde{\Lambda}}(B_0 f_0) + (B_0 e^{-t\Lambda} - e^{-t\tilde{\Lambda}} B_0) f_0 - \mathscr{E}_t e^{-t\Lambda} f_0 + B_t w_t.$$

It is clear that any $\mathscr{Y}^*$ element has Whitney averages converging almost everywhere to 0 in L_2 sense. This applies to the last three terms. Indeed, we have $\|w\|_{\mathscr{Y}^*} < \infty$, and hence $\|Bw\|_{\mathscr{Y}^*} < \infty$. Also $\|\mathscr{E}_t e^{-t\Lambda} f_0\|_{\mathscr{Y}^*} \lesssim \|\mathscr{E}\|_* \|e^{-t\Lambda} f_0\|_{\mathscr{X}} < \infty$. Furthermore, using $B_0(I + itDB_0)^{-1} = (I + itB_0 D)^{-1} B_0$, we write

$$\begin{aligned}
&\big(B_0 e^{-t\Lambda} - e^{-t\tilde{\Lambda}} B_0\big) f_0 \\
&\quad = B_0\big(e^{-t|DB_0+\sigma N|} - (I + it(DB_0 + \sigma N))^{-1}\big) f_0 + B_0\big((I + it(DB_0 + \sigma N))^{-1} - (I + itDB_0)^{-1}\big) f_0 \\
&\qquad + \big((I + itB_0 D)^{-1} - (I + it(B_0 D - \sigma N))^{-1}\big) B_0 f_0 + \big((I + it(B_0 D - \sigma N))^{-1} - e^{-t|B_0 D - \sigma N|}\big) B_0 f_0.
\end{aligned}$$

Square-function (that is, $\mathscr{Y}^*$) estimates hold for the first and fourth terms, whereas the second and third terms have L_2 norms bounded by Ct. Hence $\chi_{t<1}(B_0 e^{-t\Lambda} - e^{-t\tilde{\Lambda}} B_0) \in \mathscr{Y}^*$.

For the term $e^{-t\tilde{\Lambda}}(B_0 f_0)$ we proceed as in the proof of Theorem 15.2, modified as in Remark 15.3.

To complete the proof, we now assume that A is pointwise elliptic. Up to rescaling, we have to prove convergence of Whitney averages of the conormal gradient f of u. To see this, write $f = B_0^{-1}(B_0 f)$ using that B_0 is now invertible in $L_\infty(S^n; \mathscr{L}(\mathscr{V}))$, seen as radial coefficients on $\mathbb{O}^{1+n}$. Now the same argument as above replacing B_t by B_0 shows that the Whitney averages of $B_0 f$ converge in L_p sense to $B_0 f_0$ almost everywhere for any $p < 2$. We claim that the notion of convergence in L_p-sense of Whitney averages is stable when $p < 2$ under multiplication by bounded radially independent coefficients. Assume that h has such a convergence property and let $M \in L_\infty(S^n; \mathscr{L}(\mathscr{V}))$. Select smooth sections h_{x_0} and M_{x_0} with $h_{x_0}(x_0) = h(x_0)$ and $M_{x_0}(x_0) = M(x_0)$. Then take the $L_p(W(t_0, x_0)$ average of

$$M(y)h(\boldsymbol{y}) - M_{x_0}(y)h_{x_0}(y) = (M(y) - M_{x_0}(y))h(\boldsymbol{y}) + M_{x_0}(y)(h(\boldsymbol{y}) - h_{x_0}(y))$$

with $\mathbf{y} = e^{-t}y \in W(t_0, x_0)$. For the second term, one uses the assumption on h and that M_{x_0} is bounded. For the first term, use Hölder inequality with exponents $1/p = 1/r + 1/q$ and $p < r < 2$. The exponent q falls on $M(y) - M_{x_0}(y)$ and Lebesgue convergence theorem applies (this is a further almost everywhere constraint on x_0). The exponent r falls on h which has uniform control by assumption. $\square$

16. Fredholm theory for $(I - S_A)^{-1}$

We saw in Section 12 that the invertibility of $I - S_A$ on $\mathcal{X}$ (resp. $\mathcal{Y}$) allows to represent $\mathcal{X}^o$ (resp. $\mathcal{Y}^o$) solutions through Cauchy type extensions

$$f = (I - S_A)^{-1} e^{-t\Lambda} E_0^+ f_0$$

(resp. $f = (I - S_A)^{-1} D e^{-t\tilde{\Lambda}} \widetilde{E}_0^+ v_0)$). Working in the space $\mathcal{X}$ or $\mathcal{Y}$, it is clear from Theorem 11.3 that $I - S_A$ is invertible provided $\|\mathcal{E}\|_*$ is small enough. In this section, we use Fredholm operator theory to relax this condition and show that it suffices to assume this smallness only near the boundary $t = 0$. Our discussion in this section is limited to the specific but relevant case where $\sigma = \frac{n-1}{2}$.

Theorem 16.1. *Assume that $\|\mathcal{E}\|_* < \infty$, so that S_A is bounded on $\mathcal{X}$ and $\mathcal{Y}$. There exists $\epsilon > 0$ such that if $\mathcal{E}$ satisfies the small Carleson condition*

$$\lim_{\tau \to 0} \|\chi_{t<\tau} \mathcal{E}\|_* < \epsilon, \tag{48}$$

then $I - S_A$ is invertible on $\mathcal{X}$ and $\mathcal{Y}$.

We remark that (48) is equivalent to the small Carleson condition (10). The proof of Theorem 16.1 requires the following lemmas.

Lemma 16.2. *Assume $\|\mathcal{E}\|_* < \infty$. Then $I - S_A$ is injective on $\mathcal{X}$.*

Proof. Assume that $f \in \mathcal{X}$ satisfies $f = S_A f$. Lemma 11.5 shows that f has trace $h^- \in E_0^- \mathcal{H}$. As $\mathcal{X} \subset L_2(\mathbb{R}_+; L_2)$ and f is valued in $\mathcal{H}$, we have $f \in L_2(\mathbb{R}_+; \mathcal{H})$. Extend f to $f^1 \in L_2(\mathbb{R}; \mathcal{H})$, letting

$$f_t^1 := \begin{cases} f_t, & t > 0, \\ e^{t\Lambda} h^-, & t \le 0. \end{cases}$$

To verify that f^1 satisfies $\partial_t f^1 + (DB^1 + \sigma N) f^1 = 0$ in $\mathbb{R} \times S^n$ distributional sense, where $B_t^1 := B_t$ for $t > 0$ and $B_t^1 = B_0$ for $t \le 0$, consider a test function $\phi \in C_0^\infty(\mathbb{R} \times S^n; \mathbb{C}^{(1+n)m})$ and let $\xi_\epsilon(t) := 1 - \eta^0(|t|/\epsilon)$, where η^0 is the function from Lemma 11.1. Then

$$\begin{aligned}
&\int_{\mathbb{R}} ((-\partial_t + (B^1)^* D + \sigma N)\phi, f^1)\, dt \\
&= \int_{\mathbb{R}} \Big(((-\partial_t + (B^1)^* D + \sigma N)((1 - \xi_\epsilon)\phi), f^1) + ((-\partial_t + (B^1)^* D + \sigma N)(\xi_\epsilon \phi), f^1) \Big)\, dt \\
&= 0 + \int_{\mathbb{R}} \xi_\epsilon ((-\partial_t + (B^1)^* D + \sigma N)\phi, f^1)\, dt + \epsilon^{-1} \int_\epsilon^{2\epsilon} (\phi_t, f_t^1)\, dt - \epsilon^{-1} \int_{-2\epsilon}^{-\epsilon} (\phi_t, f_t^1)\, dt \\
&\qquad\qquad \to 0 + (\phi_0, h^-) - (\phi_0, h^-) = 0,
\end{aligned}$$

with $\phi_0(x) := \phi(0, x)$, using that the equation holds both in $\mathbb{R}_+$ and $\mathbb{R}_-$. Hence $\partial_t f^1 + (DB^1 + \sigma N) f^1 = 0$ in all $\mathbb{R} \times S^n$. Since $\sigma = \frac{n-1}{2}$, extending Proposition 3.3 from $\mathbb{O}^{1+n}$ to all $\mathbb{R}^{1+n}$ (see Remark 3.6), we see that f^1 corresponds to a function $g^1 \in L_2(\mathbb{R}^{1+n}; \mathbb{C}^{(1+n)m})$ solving $\mathrm{div}_{\boldsymbol{x}}(A^1 g^1) = 0$, $\mathrm{curl}_{\boldsymbol{x}}\, g^1 = 0$ in all $\mathbb{R}^{1+n}$, with A^1 corresponding to B^1. To verify that this forces g^1, and therefore f^1 and f, to vanish, note that for any fixed $R > 0$ we can find u such that $g^1 = \nabla_{\boldsymbol{x}} u$, where $\int_{|\boldsymbol{x}|<2R} |u|^2\, d\boldsymbol{x} \lesssim R^2 \int_{|\boldsymbol{x}|<2R} |g^1|^2\, d\boldsymbol{x}$ by Poincaré's inequality and the implicit constant is independent of R. Take a test function $\eta \in C_0^\infty(|\boldsymbol{x}| < 2R)$ with $\eta = 1$ on $|\boldsymbol{x}| < R$ with $|\nabla_{\boldsymbol{x}} \eta| \lesssim R^{-1}$, and use that $\mathrm{div}_{\boldsymbol{x}}(A^1 g^1) = 0$ in the distributional sense to get

$$\int_{|\boldsymbol{x}|<R} |g^1|^2\, d\boldsymbol{x} \lesssim \mathrm{Re} \int (A^1 g^1, \nabla_{\boldsymbol{x}} u) \eta\, d\boldsymbol{x} = -\,\mathrm{Re} \int (A^1 g^1, \nabla_{\boldsymbol{x}} \eta) u\, d\boldsymbol{x}$$
$$\lesssim \left(\int_{R<|\boldsymbol{x}|<2R} |g^1|^2\, d\boldsymbol{x} \right)^{1/2} \left(\int_{|\boldsymbol{x}|<2R} |g^1|^2\, d\boldsymbol{x} \right)^{1/2} \lesssim \left(\int_{R<|\boldsymbol{x}|<2R} |g^1|^2\, d\boldsymbol{x} \right)^{1/2} \|g^1\|_2.$$

Letting $R \to \infty$ this shows that $g^1 = 0$, which proves the lemma. □

Lemma 16.3. *Assume $\|\mathcal{E}\|_* < \infty$ and fix $\tau > 0$. Then there are lower bounds*

$$\|f\|_{L_2(\tau,\infty;\mathcal{H})} \lesssim \|(I - S_A) f\|_{L_2(\tau/2,\infty;\mathcal{H})},$$

where the implicit constant depends on τ, for all $f \in L_2(\mathbb{R}_+; \mathcal{H})$ such that $f_t = 0$ for $t < \tau$.

Proof. By Lemma 11.5, f and $f^0 := (I - S_A) f$ satisfy $(\partial_t + DB_0 + \sigma N) f^0 = (\partial_t + DB + \sigma N) f$. As in Proposition 3.3 combined with Proposition 2.1, this can be translated to

$$\begin{cases} \mathrm{div}_{\boldsymbol{x}}(A_1 g^0) = \mathrm{div}_{\boldsymbol{x}}(A g), \\ \mathrm{curl}_{\boldsymbol{x}}\, g^0 = \mathrm{curl}_{\boldsymbol{x}}\, g, \end{cases}$$

in $\mathbb{O}^{1+n}$ distributional sense, where $g_r^0 = r^{-(n+1)/2}((B_0 f_t^0)_\perp \vec{n} + (f_t^0)_\parallel)$ and $g_r = r^{-(n+1)/2}((B f_t)_\perp \vec{n} + (f_t)_\parallel)$. Write $\mathbb{O}_\tau^{1+n} := \{|\boldsymbol{x}| < e^{-\tau}\}$, so that $\mathbb{O}_\tau^{1+n} \subset \mathbb{O}_{\tau/2}^{1+n}$. In particular, the last equation implies that there is a potential $u : \mathbb{O}_{\tau/2}^{1+n} \to \mathbb{C}^m$ such that

$$g - g^0 = \nabla_{\boldsymbol{x}} u \quad \text{in } \mathbb{O}_{\tau/2}^{1+n},$$

and we may choose u so that $\|u\|_{L_2(\mathbb{O}_{\tau/2}^{1+n})} \lesssim \|g - g^0\|_{L_2(\mathbb{O}_{\tau/2}^{1+n})}$. Fix $\eta \in C_0^\infty(\mathbb{O}^{1+n})$ such that $\eta|_{\mathbb{O}_\tau^{1+n}} = 1$ and $\mathrm{supp}\, \eta \subset \mathbb{O}_{\tau/2}^{1+n}$. Using the first equation and $\mathrm{supp}\, g \subset \mathbb{O}_\tau^{1+n}$ gives

$$\mathrm{Re} \int (A g, g - g^0)\, d\boldsymbol{x} = \mathrm{Re} \int (A g, \nabla_{\boldsymbol{x}}(\eta u))\, d\boldsymbol{x} = \mathrm{Re} \int (A g^0, \nabla_{\boldsymbol{x}}(\eta u))\, d\boldsymbol{x}$$
$$= \mathrm{Re} \int_{\mathbb{O}_{\tau/2}^{1+n}} \Big(A_1 g^0, \eta(g - g^0) + (\nabla_{\boldsymbol{x}} \eta) u \Big)\, d\boldsymbol{x} \lesssim \|g^0\|_{L_2(\mathbb{O}_{\tau/2}^{1+n})} \|g - g^0\|_{L_2(\mathbb{O}_{\tau/2}^{1+n})}.$$

Note that $(g_r)_\parallel = r^{-(n+1)/2} (f_t)_\parallel \in \mathsf{R}(\nabla_S)$, so that $g_r \in \mathcal{H}_1$. The accretivity (14) of A_r, for each fixed $r \in (0, 1)$, and integration for $0 < r < e^{-\tau}$ imply that

$$\|g\|^2_{L_2(\mathbb{O}^{1+n}_\tau)} \lesssim \mathrm{Re}\int_{\mathbb{O}^{1+n}_\tau}(Ag,g)\,d\boldsymbol{x} \le \mathrm{Re}\int_{\mathbb{O}^{1+n}_\tau}(Ag,g-g^0)\,d\boldsymbol{x} + \|g\|_{L_2(\mathbb{O}^{1+n}_\tau)}\|g^0\|_{L_2(\mathbb{O}^{1+n}_\tau)}$$
$$\lesssim \|g\|_{L_2(\mathbb{O}^{1+n}_\tau)}\|g^0\|_{L_2(\mathbb{O}^{1+n}_{\tau/2})} + \|g^0\|^2_{L_2(\mathbb{O}^{1+n}_{\tau/2})},$$

and hence that $\|g\|_{L_2(\mathbb{O}^{1+n}_\tau)} \lesssim \|g^0\|_{L_2(\mathbb{O}^{1+n}_{\tau/2})}$. By the isomorphism (18), this translates to $\|f\|_{L_2(\tau,\infty;\mathcal{H})} \lesssim \|f^0\|_{L_2(\tau/2,\infty;\mathcal{H})}$ and proves the lemma. □

Lemma 16.4. *Assume $\|\mathcal{E}\|_* < \infty$. Let $\eta : \mathbb{R}_+ \to \mathbb{R}$ be a Lipschitz function, that is $|\eta(t)-\eta(s)| \le C|t-s|$ for all $t, s > 0$. Then the commutator*

$$[\eta, S_A] = \eta S_A - S_A\eta$$

is a compact operator on $L_2(\mathbb{R}_+, dt; L_2)$.

Proof. Write $S_A = \widehat{S}_A - \sigma\check{S}_A$ as in Theorem 11.3. Since $\check{S}_A = \Lambda^{-1}\widehat{S}_A$, except that $\widehat{E}_0^\pm$ are replaced by $\check{E}_0^\pm$, it is enough to show compactness of $[\eta_0, \widehat{S}_A]$. It suffices to verify that

$$F(\Lambda) : f_t \mapsto \int_0^t (\eta(t)-\eta(s))\Lambda e^{-(t-s)\Lambda} f_s\,ds, \tag{49}$$

is a compact operator on $L_2(\mathbb{R}_+, dt; \mathcal{H})$. (The proof below only depends on the fact that Λ has compact resolvents.) Indeed, by duality this implies that also $f_t \mapsto \int_t^\infty (\eta(t)-\eta(s))\Lambda e^{-(s-t)\Lambda} f_s\,ds$ is compact, upon changing Λ to Λ^*. Since $\widehat{E}_0^\pm\mathcal{E}$ are bounded $L_2(\mathbb{R}_+; L_2) \to L_2(\mathbb{R}_+; \mathcal{H})$ and commute with η, we conclude that $[\eta, \widehat{S}_A]$ is compact.

Consider the symbol

$$F(\lambda) : f_t \mapsto \int_0^t (\eta(t)-\eta(s))\lambda e^{-(t-s)\lambda} f_s\,ds.$$

To estimate the norm of this integral operator, acting in $L_2(\mathbb{R}_+; \mathbb{C})$ for fixed $\lambda \in S^o_{\nu,\sigma+}$, we apply Schur estimates as in [Part I, Lemma 6.6]. We need to estimate

$$\sup_{t>0}\int_0^t |(\eta(t)-\eta(s))\lambda e^{-(t-s)\lambda}|\,ds + \sup_{s>0}\int_s^\infty |(\eta(t)-\eta(s))\lambda e^{-(t-s)\lambda}|dt.$$

Using Lipschitz regularity, the first integral has estimate

$$\int_0^t (t-s)\lambda_1 e^{-(t-s)\lambda_1}dt = \lambda_1^{-1}\int_0^{t\lambda_1} xe^{-x}dx \lesssim \lambda^{-1},$$

where $\lambda_1 := \mathrm{Re}\,\lambda \approx |\lambda|$ for $\lambda \in S^o_{\nu,\sigma+}$, and a similar estimate for the second integral gives the bound

$$\|F(\lambda)\|_{L_2(\mathbb{R}_+;\mathbb{C})\to L_2(\mathbb{R}_+;\mathbb{C})} \lesssim \lambda^{-1}.$$

It is also clear that $F(\lambda)$ defines a compact operator on $L_2(\mathbb{R}_+; \mathbb{C})$ (for example truncate the kernel and show from the Schur estimates that $F(\lambda)$ is a uniform limit of Hilbert–Schmidt operators).

Consider now the Dunford integral

$$F(\Lambda) = \frac{1}{2\pi i}\int_{\partial S_{\theta,\sigma+}} F(\lambda)(\lambda-\Lambda)^{-1}d\lambda, \quad \omega < \theta < \nu.$$

From the compactness of $F(\lambda) : L_2(\mathbb{R}_+; \mathbb{C}) \to L_2(\mathbb{R}_+; \mathbb{C})$, and of $(\lambda - \Lambda)^{-1} : \mathcal{H} \to \mathcal{H}$ by Proposition 4.3, we deduce the compactness of $F(\lambda)(\lambda - \Lambda)^{-1} : L_2(\mathbb{R}_+; \mathcal{H}) \to L_2(\mathbb{R}_+; \mathcal{H})$ (for example by approximating $(\lambda - \Lambda)^{-1}$ uniformly by finite rank operators). Since $\|F(\lambda)(\lambda - \Lambda)^{-1}\| \lesssim \lambda^{-2}$, the Dunford integral converges in norm, at least when $\sigma > 0$, and we conclude that $F(\Lambda)$ is a compact operator on $L_2(\mathbb{R}_+; \mathcal{H})$ (for example, approximate with Riemann sums, using norm continuity of $\lambda \mapsto F(\lambda)(\lambda - \Lambda)^{-1}$). In dimension $n = 1$, i.e., $\sigma = 0$, note that $\lambda = 0$ does not belong to the spectrum of D_0 on $\mathcal{H}$. Hence it is not needed to integrate through $\lambda = 0$ in the Dunford integral, in which case the Dunford integral converges in norm also here. This proves the lemma. □

Lemma 16.5. *Assume $\|\mathcal{E}\|_* < \infty$. Let $0 < a < b < \infty$ and write $\chi_0 := \chi_{(0,a)}$ and $\chi_\infty := \chi_{(b,\infty)}$ for the characteristic functions of these intervals. Then*

$$\chi_0 S_A \chi_\infty : \mathcal{X} \to \mathcal{X} \quad \textit{and} \quad \chi_\infty S_A \chi_0 : \mathcal{Y} \to \mathcal{Y}$$

are compact operators.

Proof. As in the proof of Lemma 16.4, we may replace S_A by $\widehat{S}_A$ as straightforward modifications of the proof below give the result for $\check{S}_A$.

(i) We claim that the integral operator

$$F(\lambda) f_t := \int_0^a \lambda e^{-(t-s)\lambda} f_s \, ds$$

is a Hilbert–Schmidt (hence compact) operator $F(\lambda) : L_2(0, a; s ds) \to L_2(b, \infty; dt)$. Indeed, a straightforward calculation shows that

$$\int_b^\infty \int_0^a |\lambda e^{-(t-s)\lambda}|^2 s \, ds \, dt \le \tfrac{a}{4} e^{-2(b-a)\lambda}.$$

As in the proof of Lemma 16.4, it follows by operational calculus that

$$L_2(0, a; s ds; \mathcal{H}) \to L_2(b, \infty; \mathcal{H}) : f_t \mapsto \int_0^a \Lambda e^{-(t-s)\Lambda} f_s \, ds$$

is compact. Since $\widehat{E}_0^- \mathcal{E}$ is bounded on $L_2(0, a; s ds; \mathcal{H})$, this proves that $\chi_\infty \widehat{S}_A \chi_0 : \mathcal{Y} \to \mathcal{Y}$ is compact.

(ii) To prove that $\chi_0 \widehat{S}_A \chi_\infty : \mathcal{X} \to \mathcal{X}$ is compact, it suffices to show that

$$L_2(b, \infty; \mathcal{H}) \to \mathcal{X} : f_t \mapsto \chi_0(t) \int_b^\infty \Lambda e^{-(s-t)\Lambda} f_s \, ds \tag{50}$$

is compact, since $\widehat{E}_0^- \mathcal{E}$ is bounded on $L_2(b, \infty; \mathcal{H})$. To prove this, we write, for $t < a$,

$$\begin{aligned}
&\int_b^\infty \Lambda e^{-(s-t)\Lambda} f_s \, ds \\
&\quad = \int_b^\infty \Lambda e^{-(s+t)\Lambda} f_s \, ds + \int_b^\infty (I - e^{-2t\Lambda}) \Lambda e^{-(s-t)\Lambda} f_s \, ds \\
&\quad = e^{-t\Lambda} e^{-\delta\Lambda} \int_b^\infty \Lambda e^{-(s-\delta)\Lambda} f_s \, ds + \left(\sqrt{t} e^{-(a-t)\Lambda} \frac{I - e^{-2t\Lambda}}{\sqrt{t\Lambda}} \right) e^{-\delta\Lambda} \int_b^\infty \Lambda^{3/2} e^{-(s-a-\delta)\Lambda} f_s \, ds \\
&\quad = I_1 + I_2,
\end{aligned}$$

where $\delta > 0$ is small enough. The Cauchy–Schwarz inequality shows that the integral expressions in both I_1 and I_2 define bounded operators $L_2(b,\infty;\mathcal{H}) \to \mathcal{H}$, whereas $e^{-\delta\Lambda} = D_0^{-1}(D_0 e^{-\delta|D_0|})$ is compact on $\mathcal{H}$ by Proposition 4.3. For I_1, the factor $e^{-t\Lambda} : \mathcal{H} \to \mathcal{X}$ is bounded by Theorem 10.1. Since $\mathcal{Y}^* \subset \mathcal{X}$, boundedness of the first factor in I_2 follows from boundedness of $\sqrt{t}e^{-(a-t)\Lambda}$ for $t \in (0,a)$, and square function estimates for Λ since $\psi(\lambda) = (1-e^{-2\lambda})/\sqrt{\lambda} \in \Psi(S^o_{\nu+})$. This completes the proof. $\square$

Proof of Theorem 16.1. (i) Consider first invertibility in the space $\mathcal{X}$. By Theorem 11.3, we have $\|S_A\|_{\mathcal{X}\to\mathcal{X}} \lesssim \|\mathcal{E}\|_*$, for any perturbation of coefficients $\mathcal{E}$. Thus, for any $\tau > 0$

$$\|S_A f\|_{\mathcal{X}} \le C\|\chi_{t<\tau}\mathcal{E}\|_*\|f\|_{\mathcal{X}}, \quad \text{whenever } f_t = 0 \text{ for } t > \tau,$$

with C independent of τ. This follows upon writing $\mathcal{E}f = (\chi_{t<\tau}\mathcal{E})f$. Under the hypothesis, we can choose $\tau > 0$ such that $C\|\chi_{t<\tau}\mathcal{E}\|_* \le 1/2$. We obtain

$$\|(I-S_A)f\|_{\mathcal{X}} \ge \|f\|_{\mathcal{X}} - \tfrac{1}{2}\|f\|_{\mathcal{X}} = \tfrac{1}{2}\|f\|_{\mathcal{X}}, \quad \text{whenever } f_t = 0 \text{ for } t > \tau.$$

Next consider an arbitrary $f \in \mathcal{X}$. Pick $\eta_0 \in C^\infty(\mathbb{R}_+)$ such supp $\eta_0 \subset [0,\tau]$ and $\eta_0 = 1$ for $t < \tau/2$. Write $\eta_1 := 1-\eta_0$. Then $\|(I-S_A)(\eta_0 f)\|_{\mathcal{X}} \ge \frac{1}{2}\|\eta_0 f\|_{\mathcal{X}}$, and Lemma 16.3 shows that $\|(I-S_A)(\eta_1 f)\|_{\mathcal{X}} \gtrsim \|\eta_1 f\|_{\mathcal{X}}$. This gives

$$\begin{aligned}\|f\|_{\mathcal{X}} &\le \|\eta_0 f\|_{\mathcal{X}} + \|\eta_1 f\|_{\mathcal{X}} \lesssim \|(I-S_A)(\eta_0 f)\|_{\mathcal{X}} + \|(I-S_A)(\eta_1 f)\|_{\mathcal{X}} \\ &\le \|\eta_0(I-S_A)f\|_{\mathcal{X}} + \|[\eta_0, S_A]f\|_{\mathcal{X}} + \|\eta_1(I-S_A)f\|_{\mathcal{X}} + \|[\eta_1, S_A]f\|_{\mathcal{X}} \\ &\lesssim \|(I-S_A)f\|_{\mathcal{X}} + \|[\eta_0, S_A]f\|_{\mathcal{X}}.\end{aligned}$$

To show that $[\eta_0, S_A] : \mathcal{X} \to \mathcal{X}$ is compact, we write

$$[\eta_0, S_A] = \chi_0[\eta_0, S_A] + (1-\chi_0)[\eta_0, S_A] = \chi_0 S_A(1-\eta_0) + (1-\chi_0)[\eta_0, S_A],$$

where $\chi_0 := \chi_{(0,\tau/4)}$. Hence, compactness of the first term is granted from Lemma 16.5. Next, as the $\mathcal{X}$ and L_2 norms are the same away from the boundary, Lemma 16.4 implies that the second term is compact from $\mathcal{X} \to \mathcal{X}$. This shows that $I - S_A : \mathcal{X} \to \mathcal{X}$ is a semi-Fredholm operator.

To see that it is a Fredholm operator with index 0, note that the lower estimate on $I - S_A$ above goes through with $\mathcal{E}$ replaced by $\alpha\mathcal{E}$, $\alpha \in [0,1]$. Apply the method of continuity. Since $I - S_A$ is injective on $\mathcal{X}$ by Lemma 16.2, it follows that it is invertible.

(ii) Consider now invertibility in the space $\mathcal{Y}$. That $I - S_A : \mathcal{Y} \to \mathcal{Y}$ is a Fredholm operator with index 0 follows as in (i), provided we show that $[\eta_0, S_A] : \mathcal{Y} \to \mathcal{Y}$ is compact. Here we write

$$[\eta_0, S_A] = [\eta_0, S_A]\chi_0 + [\eta_0, S_A](1-\chi_0) = (\eta_0 - 1)S_A\chi_0 + [\eta_0, S_A](1-\chi_0),$$

and Lemmas 16.5 and 16.4 are applied in the same way.

To verify bijectivity, we note that $\mathcal{X} \subset \mathcal{Y}$ is a dense continuous inclusion, where $I - S_A : \mathcal{X} \to \mathcal{X}$ is an isomorphism. This implies that $I - S_A : \mathcal{Y} \to \mathcal{Y}$ has dense range, hence is an isomorphism since its index is 0. $\square$

17. Solvability of BVPs

Characterization of well-posedness. For A such that $I - S_A$ is invertible, we introduce boundary maps and characterize well-posedness in terms of their invertibility.

Definition 17.1. For coefficients A such that $\|\mathcal{E}\|_* < \infty$ and $I - S_A : \mathcal{X} \to \mathcal{X}$ is invertible, define the *perturbed Hardy projection*

$$E_A^+ h := E_0^+ h - E_0^- \int_0^\infty e^{-s\Lambda} D\mathcal{E}_s f_s \, ds, \quad h \in L_2(S^n; \mathcal{V}),$$

where $f := (I - S_A)^{-1} e^{-t\Lambda} E_0^+ h$. Write $E_A^- := I - E_A^+$. Here, $E_0^\pm$ denote the Hardy projections associated to the corresponding radially independent coefficients A_1.

Proposition 17.2. *The operators $E_A^\pm : L_2(S^n; \mathcal{V}) \to L_2(S^n; \mathcal{V})$ are bounded projections and the range $E_A^+ \mathcal{H} \subset \mathcal{H}$ consists of all traces f_0 of conormal gradients f of $\mathcal{X}^o$-solutions to the divergence form equation with coefficients A in $\mathbb{O}^{1+n}$.*

Proof. That $E_A^\pm$ are bounded follows from their construction. The projection property $(E_A^\pm)^2 = E_A^\pm$ follows from $E_0^+ E_0^- = 0$. Next, the statement about the range follows from Theorem 12.3. $\square$

Definition 17.3. For coefficients A such that $\|\mathcal{E}\|_* < \infty$ and $I - S_A : \mathcal{Y} \to \mathcal{Y}$ is invertible, define the *perturbed Hardy projection*

$$\widetilde{E}_A^+ \tilde{h} := \widetilde{E}_0^+ \tilde{h} - \widetilde{E}_0^- \int_0^\infty e^{-s\tilde{\Lambda}} \mathcal{E}_s f_s \, ds, \quad \tilde{h} \in L_2(S^n; \mathcal{V}),$$

where $f := (I - S_A)^{-1} D e^{-t\tilde{\Lambda}} \widetilde{E}_0^+ \tilde{h}$. Write $\widetilde{E}_A^- := I - \widetilde{E}_A^+$. Here, $\widetilde{E}_0^\pm$ denote the Hardy projections associated to the corresponding radially independent coefficients A_1.

Proposition 17.4. *The operators $\widetilde{E}_A^\pm : L_2(S^n; \mathcal{V}) \to L_2(S^n; \mathcal{V})$ are bounded projections and $\{(\widetilde{E}_A^+ \tilde{h}^+)_\perp \,;\, \tilde{h}^+ \in \widetilde{E}_0^+ L_2\}$ consists of all traces of $\mathcal{Y}^o$-solutions to the divergence form equation with coefficients A in $\mathbb{O}^{1+n}$.*

Proof. That $\widetilde{E}_A^\pm$ are bounded follows from their construction. The projection property $(\widetilde{E}_A^\pm)^2 = \widetilde{E}_A^\pm$ follows from $\widetilde{E}_0^+ \widetilde{E}_0^- = 0$. Next, the statement about the trace space follows from Corollary 12.8(ii). $\square$

We remark that, unlike the case of r-independent coefficients, the complementary projections E_A^- and $\widetilde{E}_A^-$ are in general not related to solutions of a divergence form equation in the complementary domain $\mathbb{R}^{1+n} \setminus \mathbb{O}^{1+n}$.

Proposition 17.5. *For coefficients A such that $I - S_A$ is invertible on $\mathcal{X}$ for (i) and (ii), or $I - S_A$ is invertible on $\mathcal{Y}$ for (iii), the following hold.*

(i) *The Neumann problem (with coefficients A) is well-posed in the sense of Definition 1.2 if and only if*

$$E_0^+ \mathcal{H} \to \mathcal{H}_\perp : h^+ \mapsto (E_A^+ h^+)_\perp \tag{51}$$

is an isomorphism.

(ii) *The regularity problem (with coefficients A) is well-posed in the sense of Definition 1.2 if and only if*

$$E_0^+\mathcal{H} \to \mathcal{H}_\parallel : h^+ \mapsto (E_A^+ h^+)_\parallel \tag{52}$$

is an isomorphism.

(iii) *The Dirichlet problem (with coefficients A) is well-posed in the sense of Definition 1.2 if and only if*

$$\widetilde{E}_0^+ L_2(S^n;\mathcal{V}) \to L_2(S^n;\mathbb{C}^m) : \tilde{h}^+ \mapsto (\widetilde{E}_A^+ \tilde{h}^+)_\perp \tag{53}$$

is an isomorphism.

Proof. (i) The ansatz (36) in Theorem 12.3 gives is a one-to-one correspondence between $h^+ \in E_0^+\mathcal{H}$ and conormal gradients $f = (I - S_A)^{-1} e^{-t\Lambda} h^+$ of $\mathcal{X}^o$-solutions to the divergence form equation. Moreover, $f_0 = E_A^+ h^+$ by Proposition 17.2. Under this correspondence, invertibility of $h^+ \mapsto (E_A^+ h^+)_\perp$ translates to well-posedness of the Neumann problem. The proof of (ii) is similar.

(iii) The ansatz (43) from Corollary 12.8(iii) gives a one-to-one correspondence between $\tilde{h}^+ \in \widetilde{E}_0^+ L_2$ and $\mathcal{Y}^o$-solutions u to the divergence form equation. Moreover, $(\widetilde{E}_A^+ \tilde{h}^+)_\perp = u_1$ by Proposition 17.4. Under this correspondence, invertibility of $\tilde{h}^+ \mapsto (\widetilde{E}_A^+ \tilde{h}^+)_\perp$ translates to well-posedness of the Dirichlet problem. □

Equivalence between Dirichlet and regularity problems. We show that the Dirichlet and regularity problems are the same up to taking adjoints.

Proposition 17.6. *Assume that A are coefficients such that $I - S_A$ is invertible on $\mathcal{X}$ and $I - S_{A^*}$ is invertible on $\mathcal{Y}$. Then the regularity problem with coefficients A is well-posed if and only if the Dirichlet problem with coefficients A^* is well-posed.*

It is not clear to us whether invertibility of $I - S_A$ on $\mathcal{X}$ implies or is implied by invertibility of $I - S_{A^*}$ on $\mathcal{Y}$. Thus we assume both. We need three lemmas, the first being useful reformulations of invertibility of the Dirichlet boundary map, the second an identity between Hardy projections and the third an abstract principle.

Lemma 17.7. *The maps*

$$\widetilde{E}_0^+ L_2(S^n;\mathcal{V}) \to L_2(S^n;\mathbb{C}^m) : \tilde{h}^+ \mapsto (\widetilde{E}_A^+ \tilde{h}^+)_\perp$$

and

$$\widetilde{E}_0^+ (L_2(S^n;\mathcal{V})/\mathcal{H}^\perp) \to L_2(S^n;\mathbb{C}^m)/\mathbb{C}^m : \tilde{h}^+ \mapsto (\widetilde{E}_A^+ \tilde{h}^+)_\perp$$

are simultaneous isomorphisms.

Proof. This amounts to mod out $\mathcal{H}^\perp$. We recall that $\mathcal{H}^\perp$ is preserved by $\tilde{\Lambda}$ and $\widetilde{E}_0^\pm$, and annihilated by D, so from the definition $\widetilde{E}_A^+ \tilde{h}^+ = \widetilde{E}_0^+ \tilde{h}^+ \in \mathcal{H}^\perp$ for $\tilde{h}^+ \in \mathcal{H}^\perp$. By Lemma 7.5, $(\widetilde{E}_0^+ \tilde{h}^+)_\perp = (\tilde{h}^+)_\perp$ for $\tilde{h}^+ \in \mathcal{H}^\perp$, so $\widetilde{E}_0^+ (L_2(S^n;\mathcal{V})/\mathcal{H}^\perp) \to L_2(S^n;\mathbb{C}^m)/\mathbb{C}^m : \tilde{h}^+ \mapsto (\widetilde{E}_A^+ \tilde{h}^+)_\perp$ is a well defined map. That the two maps simultaneously are isomorphisms can now be verified from $\{(\widetilde{E}_A^+ \tilde{h}^+)_\perp \,;\, \tilde{h}^+ \in \mathcal{H}^\perp\} = \mathbb{C}^m$. □

Lemma 17.8. *On $L_2(S^n; \mathcal{V})$ we have the duality relation*

$$(E_A^-)^* = N \widetilde{E}^+_{A^*} N. \tag{54}$$

Proof. The proof of this duality builds on the formula

$$(D_{A_1})^* = -N \widetilde{D}_{A_1^*} N$$

on $L_2(S^n; \mathcal{V})$ from Lemma 4.2 with A_1 equal to the boundary trace of A and where we used the notation at the end of Definition 4.1. Using this observation and short hand notation $E_0^\pm = E^\pm_{A_1}$, $\Lambda = |D_{A_1}|$, $\widetilde{E}_0^\pm = \widetilde{E}^\pm_{A_1^*}$ and $\tilde{\Lambda} = |\widetilde{D}_{A_1^*}|$, it follows that we have

$$(E_0^\pm)^* = N \widetilde{E}_0^\mp N, \quad \Lambda^* = N \tilde{\Lambda} N.$$

Note that when $n = 1$, these identities can be also checked from the extensions of the projections in Definition 7.4. This implies that

$$\int_0^\infty (N \tilde{f}_t, \mathscr{E}_t (S_A f)_t)\, dt = \int_0^\infty (N (S_{A^*} \tilde{f})_s, \mathscr{E}_s f_s)\, ds, \quad \tilde{f} \in \mathcal{Y},\ f \in \mathcal{X},$$

which follows from Fubini's theorem and the formula defining S_A^ϵ from Lemma 11.2, and then letting $\epsilon \to 0$ using boundedness on $\mathcal{X}$ and $\mathcal{Y}$. Details are left to the reader. Note that S_{A^*} is defined using the coefficients $\widetilde{\mathscr{E}}_t := \widehat{A_1^*} - \widehat{A^*}$, while $\mathscr{E}_t = \widehat{A_1} - \widehat{A}$. This duality relation between S_A and S_{A^*} clearly extends to their resolvents.

For $h, \tilde{h} \in L_2$, using the isomorphism assumption on $I - S_A$ and $I - S_{A^*}$, we let

$$f = (I - S_A)^{-1} e^{-t\Lambda} E_0^+ h \in \mathcal{X} \quad \text{and} \quad \tilde{f} := (I - S_{A^*})^{-1} D e^{-s\tilde{\Lambda}} \widetilde{E}_0^+ \tilde{h} \in \mathcal{Y}$$

and calculate

$$\begin{aligned}
(N\tilde{h}, E_A^+ h) &= (N\tilde{h}, E_0^+ h) - \int_0^\infty (N\tilde{h}, E_0^- e^{-s\Lambda} D \mathscr{E}_s f_s)\, ds \\
&= (N \widetilde{E}_0^- \tilde{h}, h) + \int_0^\infty (N D e^{-s\tilde{\Lambda}} \widetilde{E}_0^+ \tilde{h}, \mathscr{E}_s ((I - S_A)^{-1} e^{-t\Lambda} E_0^+ h)_s)\, ds \\
&= (N \widetilde{E}_0^- \tilde{h}, h) + \int_0^\infty (N ((I - S_{A^*})^{-1} D e^{-s\tilde{\Lambda}} \widetilde{E}_0^+ \tilde{h})_t, \mathscr{E}_t e^{-t\Lambda} E_0^+ h)\, ds \\
&= (N \widetilde{E}_0^- \tilde{h}, h) + \int_0^\infty (N \widetilde{E}_0^- e^{-t\tilde{\Lambda}} \widetilde{\mathscr{E}}_t \tilde{f}_t, h)\, dt = (N \widetilde{E}^-_{A^*} \tilde{h}, h).
\end{aligned}$$

This completes the proof. □

Lemma 17.9. *Assume that $N^\pm$ and $E^\pm$ are two pairs of complementary projections in a Hilbert space $\mathcal{H}$, i.e., $(N^\pm)^2 = N^\pm$ and $N^+ + N^- = I$, and similarly for $E^\pm$. Then the adjoint operators $(N^\pm)^*$ and $(E^\pm)^*$ are also two pair of complementary projections on $\mathcal{H}^*$, and the restricted projection $N^+ : E^+\mathcal{H} \to N^+\mathcal{H}$ is an isomorphism if and only if $(N^-)^* : (E^-)^*\mathcal{H}^* \to (N^-)^*\mathcal{H}^*$ is an isomorphism.*

Proof. This is [Auscher et al. 2008, Proposition 2.52]. □

Proof of Proposition 17.6. We apply the abstract result as follows. Here $\mathcal{H}$ is the Hilbert space $\mathsf{R}(D) \subset L_2(S^n; \mathcal{V}) = L_2$ and we realize its dual $\mathcal{H}^*$ as $L_2/\mathcal{H}^\perp$. The operators $N^\pm$ are those from Definition 3.1:

$$N^+ : f \mapsto \begin{bmatrix} 0 \\ f_\parallel \end{bmatrix} \quad \text{and} \quad N^- : f \mapsto \begin{bmatrix} f_\perp \\ 0 \end{bmatrix}.$$

As both preserve $\mathcal{H}$, their adjoints induce operators on $\mathcal{H}^*$. We choose $E^+ = E_A^+$ and $E^- = E_A^-$. By Proposition 17.5(ii) and reformulating (52) using N^+, well-posedness of the regularity problem for A^* is equivalent to $N^+ : E_{A^*}^+\mathcal{H} \to N^+\mathcal{H}$ being an isomorphism. By Lemma 17.9 this is equivalent to $(N^-)^* : (E_{A^*}^-)^*\mathcal{H}^* \to (N^-)^*\mathcal{H}^*$ being an isomorphism. By (54) with the roles of A and A^* reversed, and written as an identity on $\mathcal{H}^*$ since both terms preserve $\mathcal{H}^\perp$, this translates into $(N^-)^* : \widetilde{E}_A^+\mathcal{H}^* \to (N^-)^*\mathcal{H}^*$ is an isomorphism. Using the definition of $\widetilde{E}_A^+$, $(N^-)^* = N^-$ and $\mathcal{H}^* = L_2/\mathcal{H}^\perp$, this amounts to $\widetilde{E}_0^+(L_2/\mathcal{H}^\perp) \to L_2(S^n; \mathbb{C}^m)/\mathbb{C}^m : \tilde{h}^+ \mapsto (\widetilde{E}_A^+\tilde{h}^+)_\perp$ is an isomorphism. Using Lemma 17.7 and Proposition 17.5(iii), this means that the Dirichlet problem for A is well-posed. □

Perturbation results. Proposition 17.6 shows that it suffices to consider the Neumann and regularity problems and to study invertibility of the maps (51) and (52). Note that for r-independent coefficients $A = A_1$, we have $E_A^+ = E_0^+$ and therefore $(E_A^+h^+)_\perp = h_\perp^+$ and $(E_A^+h^+)_\parallel = h_\parallel^+$.

Lemma 17.10. *Assume that A are coefficients such that $I - S_A$ is invertible on $\mathcal{X}$. Then the maps (51) and (52) are injective.*

Proof. Assume that $h^+ \in E_0^+\mathcal{H}$ is such that $(E_A^+h^+)_\perp = 0$. As in Theorem 12.3, let $f \in \mathcal{X}$ be such that $f_0 = E_A^+h^+$, so that we are assuming $(f_0)_\perp = 0$. For the corresponding $\mathcal{X}^o$-solution $g = \nabla_{\boldsymbol{x}} u$ to $\operatorname{div}_{\boldsymbol{x}} Ag = 0$, Green's formula shows that

$$\int_{\mathbb{O}^{1+n}} (Ag, g)\, d\boldsymbol{x} = \int_{S^n} (A_1 g_1)_\perp u_1 \, dx,$$

where $g \in \mathcal{X}^o \subset L_2(\mathbb{O}^{1+n}; \mathbb{C}^{(1+n)m})$, $(A_1 g_1)_\perp = (f_0)_\perp \in L_2(S^n; \mathbb{C}^m)$ and $u \in H^1(\mathbb{O}^{1+n}; \mathbb{C}^m)$. The accretivity of A then shows that $g = 0$. Hence $f = 0$ and $h^+ = E_0^+ f_0 = 0$.

The proof that the map $h^+ \mapsto (E_A^+h^+)_\parallel$ is injective is similar. In this case, we use that u_1 is constant, and $f_0 \in \mathcal{H}$ so that $\int_{S^n}(f_0)_\perp dx = 0$. □

We can now derive two perturbations results. Our first result is about L_∞ perturbation within the class of radially independent coefficients. We need two preliminary lemmas.

Lemma 17.11. *Let P_t be bounded projections in a Hilbert space $\mathcal{H}$ which depend continuously on a parameter $t \in (-\delta, \delta)$, and let $S : \mathcal{H} \to \mathcal{K}$ be a bounded operator into a Hilbert space $\mathcal{K}$. If $S : P_0\mathcal{H} \to \mathcal{K}$ is an isomorphism, then there exists $0 < \epsilon < \delta$, such that $S : P_t\mathcal{H} \to \mathcal{K}$ is an isomorphism when $|t| < \epsilon$. If each $S : P_t\mathcal{H} \to \mathcal{K}$ is a semi-Fredholm operators with index i_t, then all indices i_t are equal.*

Proof. The first conclusion is in [Axelsson et al. 2006b, Lemma 4.3] and the second one is proved similarly using in addition the continuity method. □

Proposition 17.12. *The operators $\chi^+(DB_0+\sigma N)\in\mathcal{L}(\mathcal{H})$, defined for strictly accretive coefficients $A_1\in L_\infty(S^n;\mathcal{L}(\mathcal{V}))$ and $\sigma\in\mathbb{R}$, depend continuously on A_1 and σ.*

Proof. This is a corollary of Theorem 7.1 and [Auscher et al. 2008, Proposition 2.42]. ☐

Here, note that for fixed σ we called this operator E_0^+. Only its action on $\mathcal{H}$ matters for well-posedness issues. In particular, this does not depend on the extension defined in Definition 7.4 when $\sigma=0$.

Theorem 17.13. *Assume that A_1 are r-independent coefficients for which the Neumann problem is well-posed. Then there exists $\epsilon>0$ such that the Neumann problem is well-posed for any r-independent coefficients A_1' such that $\|A_1-A_1'\|_\infty<\epsilon$. The corresponding results for the regularity and Dirichlet problems hold.*

Proof. Lemma 17.11 and Proposition 17.12 give the result for Regularity and Neumann problems as in [Auscher et al. 2008]. For the Dirichlet problem, apply Proposition 17.6. ☐

The second result is perturbation from radially independent to radially dependent coefficients.

Theorem 17.14. *Assume that A_1 are r-independent coefficients for which the Neumann problem is well-posed. Then there exists $\epsilon>0$ such that the Neumann problem is well-posed for any r-dependent coefficients A such that $\lim_{\tau\to 0}\|\chi_{t<\tau}\mathcal{E}_t\|_*<\epsilon$. The corresponding results for the regularity and Dirichlet problems hold.*

Proof. The condition on the coefficients implies that $I-S_A$ is invertible on $\mathcal{X}$ and $I-S_{A^*}$ invertible on $\mathcal{Y}$ by Theorem 16.1.

We write the map (51) as

$$
\begin{aligned}
(E_A^+h^+)_\perp &= h_\perp^+ + \left(E_0^-\int_0^\tau e^{-s\Lambda}D\mathcal{E}_s f_s\right)_\perp + \left(e^{-(\tau/2)\Lambda}E_0^-\int_\tau^\infty e^{-(s-\tau/2)\Lambda}D\mathcal{E}_s f_s\right)_\perp \\
&=: h_\perp^+ + (h_1)_\perp + (e^{-(\tau/2)\Lambda}h_2)_\perp,
\end{aligned}
$$

for $h^+\in E_0^+\mathcal{H}$, where $\|f\|_{\mathcal{X}}\lesssim\|h^+\|_2$ by Theorem 12.3. By assumption the map $E_0^+\mathcal{H}\to\mathcal{H}_\perp: h^+\mapsto h_\perp^+$ is invertible. By [Part I, Lemma 6.9], the norm of $E_0^+\mathcal{H}\to\mathcal{H}_\perp: h^+\mapsto(h_1)_\perp$ is $\lesssim\|\chi_{t<\tau}\mathcal{E}_t\|_*$. Fix τ small enough so that $E_0^+\mathcal{H}\to\mathcal{H}_\perp: h^+\mapsto(h^++h_1)_\perp$ is invertible. For the last term, we then have estimates

$$
\begin{aligned}
\|h_2\|_2 &\lesssim \int_\tau^\infty \|e^{-(s-\tau/2)\Lambda}D\|_{2\to 2}\|\mathcal{E}\|_\infty\|f_s\|_2\,ds \lesssim \|\mathcal{E}\|_\infty\int_\tau^\infty s^{-1}\|f_s\|_2\,ds \\
&\lesssim \|\mathcal{E}\|_\infty\left(\int_\tau^\infty\|f_s\|_2^2\,ds\right)^{1/2} \lesssim \|\mathcal{E}\|_\infty\|f\|_{\mathcal{X}} \lesssim \|\mathcal{E}\|_\infty\|h^+\|_2.
\end{aligned}
$$

Here we used the estimate

$$
\|e^{-(s-\tau/2)\Lambda}Dg\|_2 \lesssim \|\Lambda e^{-(s-\tau/2)\Lambda}(D_0-\sigma N)B_0^{-1}P_{B_0\mathcal{H}}g\|_2 \lesssim ((s-\tau/2)^{-2}+\sigma(s-\tau/2)^{-1})\|g\|_2.
$$

It follows that $E_0^+\mathcal{H}\to\mathcal{H}_\perp: h^+\mapsto(e^{-(\tau/2)\Lambda}h_2)_\perp$ is a compact operator since $e^{-(\tau/2)\Lambda}$ is compact as a consequence of Proposition 4.3. We conclude that $E_0^+\mathcal{H}\to\mathcal{H}_\perp: h^+\mapsto(E_A^+h^+)_\perp$ is a Fredholm operator with index 0. Lemma 17.10 shows that it is injective, hence an isomorphism.

Replacing normal components $(\cdot)_\perp$ by tangential parts $(\cdot)_\parallel$ in the proof above shows the result for the regularity problem. Proposition 17.6 then gives the result for the Dirichlet problem. $\square$

Positive results. We now give examples of radially dependent coefficients for which one has well-posedness. Given Theorems 17.13 and 17.14, this induces results for perturbed coefficients.

Proposition 17.15. *If A are r-independent coefficients, and if A is a block matrix, i.e., $A_{\perp\parallel} = 0 = A_{\parallel\perp}$, then the Neumann, regularity and Dirichlet problems with coefficients A are well-posed.*

Proof. By Proposition 17.6, it suffices to consider the Neumann and regularity problems. Consider the projections $E_A^\pm = E_0^\pm$. As the maps (51) and (52) act on $E_0^+\mathcal{H} \subset \mathcal{H}$, it suffices to consider their action on $\mathcal{H}$ throughout this proof. In this case, we have $E_0 := \operatorname{sgn}(DB_0 + \sigma N) = E_0^+ - E_0^-$. Consider also the $\mathcal{H}$ preserving projections $N^\pm$ from Definition 3.1. Define the anticommutator

$$C := \tfrac{1}{2}(E_0 N + N E_0).$$

Since B_0 is a block matrix, N commutes with B_0, which shows that $NE_0N = N\operatorname{sgn}(DB_0 + \sigma N)N = \operatorname{sgn}(N(DB_0 + \sigma N)N) = -\operatorname{sgn}(DB_0 - \sigma N)$, using $ND = -DN$. Hence,

$$\begin{aligned} C &= (E_0 + NE_0N)N/2 = (\operatorname{sgn}(DB_0 + \sigma N) - \operatorname{sgn}(DB_0 - \sigma N))N/2 \\ &= ((DB_0)^2 + \sigma^2)^{-1/2}((DB_0 + \sigma N) - (DB_0 - \sigma N))N/2 = \sigma((DB_0)^2 + \sigma^2)^{-1/2}, \end{aligned}$$

and it follows from Proposition 4.3 that C is a compact operator on $\mathcal{H}$.

We claim that

$$\begin{aligned} (2E_0^+)N^+|_{E_0^+\mathcal{H}} &= I + C|_{E_0^+\mathcal{H}}, \quad & N^+(2E_0^+)|_{N^+\mathcal{H}} &= I + C|_{N^+\mathcal{H}}, \\ (2E_0^+)N^-|_{E_0^+\mathcal{H}} &= I - C|_{E_0^+\mathcal{H}}, \quad & N^-(2E_0^+)|_{N^-\mathcal{H}} &= I - C|_{N^-\mathcal{H}}. \end{aligned}$$

The first identity follows from the computation

$$\begin{aligned} (2E_0^+)N^+h^+ &= E_0^+(I+N)h^+ = h^+ + \tfrac{1}{2}(I+E_0)Nh^+ \\ &= h^+ + \tfrac{1}{2}(Nh^+ + 2Ch^+ - NE_0h^+) = h^+ + Ch^+, \quad \text{for all } h^+ \in E_0^+\mathcal{H}, \end{aligned}$$

and the other three identities are proved similarly. This proves that the maps $E_0^+\mathcal{H} \to \mathcal{H}_\perp : h^+ \mapsto h^+_\perp$ and $E_0^+\mathcal{H} \to \mathcal{H}_\parallel : h^+ \mapsto h^+_\parallel$ are Fredholm operators for any $\sigma \in \mathbb{R}$, and for $\sigma = 0$ it follows that they are isomorphisms. By Lemma 17.11, the indices of these operators are zero for any $\sigma \in \mathbb{R}$, and Lemma 17.10 implies that in fact the operators are isomorphisms for $\sigma = (n-1)/2$. $\square$

Proposition 17.16. *If A are r-independent coefficients, and if A is Hermitian, i.e., $A^* = A$, then the Neumann, regularity and Dirichlet problems with coefficients A are well-posed.*

Proof. By Proposition 17.6, it suffices to consider the Neumann and regularity problems. Let $h^+ \in E_0^+\mathcal{H}$ and define $f_t := e^{-t\Lambda}h^+$. By Theorem 10.1, we have $\partial_t f_t + D_0 f_t = 0$, $\lim_{t\to 0} f_t = h^+$ and rapid decay of f_t as $t \to \infty$. We calculate

$$\begin{aligned}(Nh^+, B_0 h^+) &= -\int_0^\infty \partial_t (N f_t, B_0 f_t)\, dt = \int_0^\infty \Big((N D_0 f_t, B_0 f_t) + (N f_t, B_0 D_0 f_t) \Big)\, dt \\ &= \int_0^\infty \Big(((N D B_0 + D B_0^* N) f_t, B_0 f_t) + \sigma (f_t, (B_0 + N B_0 N) f_t) \Big)\, dt \\ &= \sigma \int_0^\infty (f_t, (B_0 + B_0^*) f_t)\, dt.\end{aligned}$$

On the last line, we used that $A^* = A$, or equivalently $B_0^* = N B_0 N$, so that $N D B_0 + D B_0^* N = 0$. This gives the estimate

$$\big| -(h_\perp^+, (B_0 h^+)_\perp) + (h_\parallel^+, (B_0 h^+)_\parallel) \big| \lesssim \sigma \int_0^\infty \|f_t\|_2^2\, dt.$$

From this we deduce the estimate

$$\|h^+\|_2^2 \lesssim \operatorname{Re}(h^+, B_0 h^+) \lesssim |(h_\perp^+, (B_0 h^+)_\perp)| + \|f\|^2_{L_2(\mathbb{R}_+; \mathcal{H})} \lesssim \|h_\perp^+\|_2 \|h^+\|_2 + \|f\|^2_{L_2(\mathbb{R}_+; \mathcal{H})}.$$

This shows that the map (51) is a semi-Fredholm map, if we prove that the map $\mathcal{H} \to L_2(\mathbb{R}_+; \mathcal{H})$ given by $h \mapsto (e^{-t\Lambda} h)_{t>0}$ is compact. To see this, note that square function estimates for D_0 give the estimate

$$\int_0^\infty \|f_t\|_2^2\, dt = \int_0^\infty \|\psi_t(D_0)(\Lambda^{-1/2} f)\|_2^2 \frac{dt}{t} \lesssim \|\Lambda^{-1/2} f\|_2^2,$$

where $\psi_t(z) := \sqrt{t|z|}\, e^{-t|z|}$, and $\Lambda^{-1/2}$ can be seen to be a compact operator on $\mathcal{H}$ by Proposition 4.3. Taking $P_s = \chi^+(D B^s + \sigma N)$ in Lemma 17.11, where B^s, $s \in [0, 1]$, denotes the straight line in $L_\infty(S^n; \mathcal{L}(\mathcal{V}))$ from I to B_0, shows that the index of the map (51) is 0. By Lemma 17.10, this map is in fact an isomorphism.

The proof for the regularity problem is similar, using instead the estimate

$$\|h^+\|_2^2 \lesssim |(h_\parallel^+, (B_0 h^+)_\parallel)| + \|f\|^2_{L_2(\mathbb{R}_+; \mathcal{H})} \lesssim \|h_\parallel^+\|_2 \|h^+\|_2 + \|f\|^2_{L_2(\mathbb{R}_+; \mathcal{H})}. \qquad \square$$

Proposition 17.17. *If A is a Hölder regular $C^{1/2+\varepsilon}(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$, r-independent coefficients, for some $\varepsilon > 0$, then the Neumann, regularity and Dirichlet problems with coefficients A are well-posed.*

For the proof, we need the following lemmas.

Lemma 17.18. *Let $B_0 \in C^{1/2+\varepsilon}(S^n; \mathcal{L}(\mathcal{V}))$ be the matrix associated to A. Then for all $f, g \in \mathcal{H}$, $|([|D|^{1/2}, B_0] f, g)| \lesssim \|f\|_2 \|g\|_2$.*

Lemma 17.19. *Under the same assumptions, $D(|D|^{1/2}) \cap \mathcal{H} = D(|D_0|^{1/2}) \cap \mathcal{H}$ with equivalent graph domain norms.*

Proof of Proposition 17.17. Consider first the Neumann and regularity problems. Let $h^+ \in E_0^+ \mathcal{H}$ and define $f_t := e^{-t\Lambda} h^+$. By Theorem 10.1, we have $\partial_t f_t + D_0 f_t = 0$ and $\lim_{t\to 0} f_t = h^+$ and $\lim_{t\to\infty} f_t = 0$ with rapid decay. We begin with the observation that $(\operatorname{sgn}(D) h^+, h^+) = \operatorname{Re}(\nabla_S(-\operatorname{div}_S \nabla_S)^{-1/2} h_\perp^+, h_\parallel^+)$.

Thus $|(\mathrm{sgn}(D)h^+, h^+)| \le \|h^+_\perp\|_2 \|h^+_\parallel\|_2$. Now, we calculate for fixed $T > 0$

$$
\begin{aligned}
&(\mathrm{sgn}(D)h^+, h^+) - (\mathrm{sgn}(D) f_T, f_T) \\
&\quad = -\int_0^T \partial_t (\mathrm{sgn}(D) f_t, f_t)\, dt = \int_0^T \big((\mathrm{sgn}(D)(DB_0 + \sigma N) f_t, f_t) + (f_t, \mathrm{sgn}(D)(DB_0 + \sigma N) f_t) \big)\, dt \\
&\quad = 2\,\mathrm{Re} \int_0^T (|D| B_0 f_t, f_t)\, dt = 2\,\mathrm{Re} \int_0^T \big((B_0 |D|^{1/2} f_t, |D|^{1/2} f_t)\, dt + ([|D|^{1/2}, B_0] f_t, |D|^{1/2} f_t) \big)\, dt,
\end{aligned}
$$

using that $\mathrm{sgn}(D)D = |D|$ and $\mathrm{sgn}(D)N + N\,\mathrm{sgn}(D) = 0$ in the third equality and Lemma 17.19 in the last since $f_t \in \mathsf{D}(|D_0|^{1/2}) \cap \mathcal{H} \subset \mathsf{D}(|D|^{1/2})$. Accretivity of B_0 and Lemma 17.18 lead to the estimate

$$
\int_0^T \| |D|^{1/2} f_t \|_2^2\, dt \lesssim \|h^+_\perp\|_2 \|h^+_\parallel\|_2 + |(\mathrm{sgn}(D) f_T, f_T)| + \int_0^T \|f_t\|_2 \| |D|^{1/2} f_t \|_2\, dt,
$$

and by absorption, to the same estimate but with last term equal $\int_0^T \|f_t\|_2^2\, dt$. Due the rapid decay of $\|f_t\|_2$ when $t \to \infty$, we conclude that

$$
\int_0^\infty \| |D|^{1/2} f_t \|_2^2\, dt \lesssim \|h^+_\perp\|_2 \|h^+_\parallel\|_2 + \int_0^\infty \|f_t\|_2^2\, dt.
$$

Since $\| |D_0|^{1/2} f_t \|_2 \lesssim \| |D|^{1/2} f_t \|_2 + \|f_t\|_2$ from Lemma 17.19, we may replace D by D_0 in the left hand side. Since square function estimates for D_0 give

$$
\int_0^\infty \| |D_0|^{1/2} f_t \|_2^2\, dt = \int_0^\infty \| (t|D_0|)^{1/2} e^{-t|D_0|} h^+ \|_2^2 \frac{dt}{t} \approx \|h^+\|_2^2,
$$

this implies

$$
\|h^+\|_2^2 \lesssim \|h^+_\perp\|_2 \|h^+_\parallel\|_2 + \int_0^\infty \|f_t\|_2^2\, dt.
$$

Well-posedness of the Neumann and regularity problems now follows as in the proof of Proposition 17.16. Proposition 17.6 then gives the result for the Dirichlet problem. □

Proof of Lemma 17.18. Note that D^2 agrees on $\mathcal{H}$ with the (positive) Hodge–Laplace operator

$$
\Delta_S := -\begin{bmatrix} \mathrm{div}_S \nabla_S & 0 \\ 0 & \nabla_S \mathrm{div}_S - \mathrm{curl}_S^* \mathrm{curl}_S \end{bmatrix},
$$

where $\mathrm{curl}_S : L_2(S^n; (T_{\mathbb{C}} S^n)^m) \to L_2(S^n; \wedge^2 (T_{\mathbb{C}} S^n)^m)$ is the tangential curl/exterior derivative on S^n. Since $f, g \in \mathcal{H}$, we have

$$
([|D|^{1/2}, B_0] f, g) = ([\Delta_S^{1/4}, B_0] f, g)
$$

and it suffices to prove that $[\Delta_S^{1/4}, B_0]$ is bounded on L_2. Since the action of B_0 mixes functions and vector fields, some care has to be taken.

(i) First, by functional calculus we can replace Δ_S by $T_0 = \Delta_S + \lambda$ for any $\lambda \in \mathbb{R}^+$, to be chosen large later, as $\Delta_S^{1/4} - (\Delta_S + \lambda)^{1/4}$ is bounded.

(ii) Next, the commutator estimate is a local problem and by a partition of unity argument and rotational invariance of the assumptions, we can assume that f is supported in the lower hemisphere and it is enough to show that $\|\zeta[T_0^{1/4}, B_0]f\|_2 \lesssim \|f\|_2$ when the smooth scalar function ζ is 1 a neighborhood of the support of f. Indeed $(1-\zeta)[T_0^{1/4}, B_0]f = -[[\zeta, T_0^{1/4}], B_0]f$, where the inner commutator is seen to be bounded on L_2.

(iii) Now using rescaled pullback ρ^* to $\mathbb{R}^n$ from the proof of Theorem 7.1 yields $\rho^*(T_0 f) = T_1(\rho^* f)$ with

$$T_1 := -\begin{bmatrix} \operatorname{div}_{\mathbb{R}^n} d^{2-n} \nabla_{\mathbb{R}^n} d^n & 0 \\ 0 & \nabla_{\mathbb{R}^n} d^n \operatorname{div}_{\mathbb{R}^n} d^{2-n} - d^{n-2} \operatorname{curl}^*_{\mathbb{R}^n} d^{4-n} \operatorname{curl}_{\mathbb{R}^n} \end{bmatrix} + \lambda I$$

in $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$, with $d(y) = (|y|^2+1)/2$ inside $|y|<1$ and extended to a smooth function on $\mathbb{R}^n$, with $d(y) = 2$ for $|y|>2$ and $\frac{1}{2} \le d(y) \le 2$ for all y. Any extension would do since $\rho^* f$ is supported in $|y|<1$. (The proof of this equality builds on the fundamental differential geometric fact that the standard pullback operation intertwines ∇ on S^n and $\mathbb{R}^n$, as well as curl, and the adjoint results for div and curl*. Note that the rescaled pullback ρ^* from Theorem 7.1 equals the standard pullback on vectors, but is d^{-n} times the standard pullback on scalars.) A further calculation shows that $T_1 = -\operatorname{div}_{\mathbb{R}^n} d^2 \nabla_{\mathbb{R}^n} + R + \lambda I$, where R is a first-order differential operator with smooth coefficients and $\operatorname{div}_{\mathbb{R}^n} d^2 \nabla_{\mathbb{R}^n}$ acts componentwise on $\mathbb{C}^{(1+n)m}$-valued functions. Note that the coefficients of R must vanish outside $|y|<2$ by construction. We now choose λ large enough to guarantee the accretivity condition $\operatorname{Re}(T_1 g, g) \ge \delta \|g\|^2_{W^1_2}$ with $\delta > 0$ and all $g \in W^1_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$. Consider K, η and g as in the proof of Theorem 7.1 and $\zeta = (\rho^*)^{-1}\eta$ and $f = (\rho^*)^{-1} g$. We claim that $\|\zeta T_0^{1/4} f - (\rho^*)^{-1}\eta^2 T_1^{1/4} g\|_2 \lesssim \|g\|_2 \approx \|f\|_2$. For both operators T_i, we use the identity

$$T_i^{1/4} = c\int_0^\infty s^{1/2} T_i (I+s^2T_i)^{-1} ds = c\int_0^\infty (I-(I+s^2T_i)^{-1}) \frac{ds}{s^{3/2}}. \tag{55}$$

The part with $s>1$ gives rise to a bounded operator for each T_i. For the integral of the difference over $s<1$, we use the identity obtained as in Theorem 7.1

$$\zeta(I+s^2T_0)^{-1} f - (\rho^*)^{-1}\eta^2 (I+s^2T_1)^{-1} g = \zeta(I+s^2T_0)^{-1}(\rho^*)^{-1} s^2[\eta, T_1](I+s^2T_1)^{-1} g$$

so that

$$\|\zeta(I+s^2T_0)^{-1} f - (\rho^*)^{-1}\eta^2(I+s^2T_1)^{-1} g\|_2 \lesssim s\|g\|_2,$$

using that the commutator $[\eta, T_1]$ is a first-order operator.

(iii) We are reduced to showing that $[T_1^{1/4}, \widetilde{B}_0]$ is bounded on $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ with $\widetilde{B}_0 := \rho^* B_0 (\rho^*)^{-1}$ of B_0 on $|y| \le 1$ extended to a bounded matrix function of class $C^{1/2+\varepsilon}$ on $\mathbb{R}^n$. We now eliminate the R part of T_1. Set $T_2 := -\operatorname{div}_{\mathbb{R}^n} d^2 \nabla_{\mathbb{R}^n} + 1$ acting componentwise in $L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$. The chosen extension of d insures that T_2 is accretive (in fact self-adjoint) as T_1. We claim that $T_1^{1/4} - T_2^{1/4}$ is bounded. We use again (55) for each T_i. The part with $s>1$ gives rise to a bounded operator for each T_i. For the $s<1$ integral of the difference, we use $\|(I+s^2T_1)^{-1} - (I+s^2T_2)^{-1}\| \lesssim s$ by the resolvent formula, because T_i have same second-order term. This proves the claim.

(iv) Hence, it remains to estimate the commutator $C = [T_2^{1/4}, \widetilde{B}_0]$. Since T_2 acts componentwise, so does $T_2^{1/4}$ and the commutator consists of a matrix of commutators with each component of $\widetilde{B}_0$. Thus it suffices to estimate $C = [T_2^{1/4}, b]$ in $L_2(\mathbb{R}^n; \mathbb{C})$, with b scalar-valued. To see this, we use the different representation for $T_2^{1/4}$ to obtain

$$C = c\int_0^\infty [s^2 T_2 e^{-s^2 T_2}, b]\frac{ds}{s^{3/2}}.$$

The $s > 1$ integral is trivially bounded, using boundedness of b and $s^2 T_2 e^{-s^2 T_2}$. For $s < 1$, we have $\|[s^2 T_2 e^{-s^2 T_2}, b]\|_{L_2 \to L_2} \lesssim s^{1/2+\epsilon}$ using pointwise decay and regularity for the kernel of $s^2 T_2 e^{-s^2 T_2}$ and regularity of b. See, for example, [Auscher 1996] where it is proved that under continuity of the coefficients (here d^2), the kernel of the semigroup e^{-sT_2}, $s < 1$, has Gaussian estimates (this is in fact due to Aronson for real measurable coefficients) and Hölder regularity in each variable with any exponent in $(0, 1)$, in particular larger that $\frac{1}{2} + \epsilon$. From here, the same estimates hold for $sT_2 e^{-sT_2} = -s\partial_s e^{-sT_2}$ by analyticity of the semigroup. This takes care of the $s < 1$ integral. Further details are left to the reader. $\square$

Proof of Lemma 17.19. Recall that $D_0 = DB_0 + \sigma N$. As before, by a representation formula it is easy to prove that $|DB_0 + \sigma N|^{1/2} - |DB_0|^{1/2}$ is bounded on L_2. Hence we may replace D_0 by DB_0. We remark that $\mathcal{H}$ is invariant for both D and DB_0.

As $P_{\mathcal{H}} B_0$ is an isomorphism of $\mathcal{H}$, for $f \in \mathcal{H}$, $f \in \mathsf{D}(|DB_0|)$ if and only if $P_{\mathcal{H}} B_0 f \in \mathsf{D}(|D|)$ and in this case

$$\||DB_0| f\|_2 \approx \|DB_0 f\|_2 \approx \|D(P_{\mathcal{H}} B_0 f)\|_2 \approx \||D|(P_{\mathcal{H}} B_0 f)\|_2.$$

Complex interpolation for sectorial operators (see [Auscher et al. 1997a]) shows that for $f \in \mathcal{H}$, $f \in \mathsf{D}(|DB_0|^{1/2})$ if and only if $P_{\mathcal{H}} B_0 f \in \mathsf{D}(|D|^{1/2})$ and

$$\||DB_0|^{1/2} f\|_2 \approx \||D|^{1/2}(P_{\mathcal{H}} B_0 f)\|_2.$$

Next, for $f \in \mathcal{H} \cap \mathsf{D}(|D|^{1/2})$, we have $|D|^{1/2} f \in \mathcal{H}$ so that

$$\||D|^{1/2} f\|_2 \approx \|P_{\mathcal{H}} B_0 |D|^{1/2} f\|_2.$$

Thus it suffices to show that for $f \in \mathcal{H}$, $f \in \mathsf{D}(|D|^{1/2})$ if and only if $P_{\mathcal{H}} B_0 f \in \mathsf{D}(|D|^{1/2})$. This is where we use the regularity of B_0 to yield $\||D|^{1/2}(P_{\mathcal{H}} B_0 f) - P_{\mathcal{H}} B_0 |D|^{1/2} f\|_2 \lesssim \|f\|_2$ when $f \in \mathcal{H}$ as a direct consequence of Lemma 17.18 and the fact that D and $P_{\mathcal{H}}$ commute. $\square$

Remark 17.20. Using the $T1$ theorem, the commutator C of the proof of Lemma 17.18 is bounded on L_2 when $(-\Delta + 1)^{1/4} b \in \mathrm{BMO}$ (and $b \in L_\infty$). The converse is also true. This can be shown to be a regularity condition between $C^{1/2}$ and $C^{1/2+\varepsilon}$. So well-posedness holds under this condition (expressed in local coordinates on the coefficients of B_0). This is probably the best conclusion we can draw from this method. However, we suspect that C^ε should be enough in general.

18. Uniqueness

The following is the class of solutions in Definition 1.8.

Definition 18.1. By a $\mathscr{D}^o$*-solution to the divergence form equation*, with coefficients A, we mean a weak solution of $\operatorname{div}_{\boldsymbol{x}} A\nabla u = 0$ in $\mathbb{O}^{1+n}$ with $\|\widetilde{N}_*^o(u)\|_2 < \infty$.

Note that unlike the previous classes, $\mathscr{D}^o$-solutions are defined through an estimate on u itself, not on the gradient $\nabla_{\boldsymbol{x}} u$.

Under the Carleson control on the discrepancy, we know that $\mathscr{Y}^o$-solutions are $\mathscr{D}^o$-solutions. We would like to know the converse. At this stage we need assumption of well-posedness in the sense of Definition 1.2. It goes via identification with variational solutions for smooth data which will be also useful later.

Lemma 18.2. *Let A be coefficients such that $\|\mathscr{E}\|_* < \infty$ and $I - S_A$ is invertible on $\mathscr{Y}$ and on $\mathscr{X}$, and assume that the regularity problem and the Dirichlet problem in the sense of Definition 1.2 both are well posed. Let $\varphi \in L_2(S^n; \mathbb{C}^m)$ be Dirichlet datum such that $\nabla_S \varphi \in L_2(S^n; (T_{\mathbb{C}} S^n)^m)$. Then the solution u to the Dirichlet problem in the sense of Definition 1.2 coincides with the variational solution with datum φ.*

Proof. By Proposition 17.5, there is a unique $h^+ \in E_0^+ \mathscr{H}$ such that $(E_A^+ h^+)_\parallel = \nabla_S \varphi$, since the regularity problem is well-posed. From Lemma 7.5, we know that $D : \widetilde{E}_0^+ L_2 \to E_0^+ \mathscr{H}$ is surjective. Let $\tilde{h}^+ \in \widetilde{E}_0^+ L_2$ be such that $D\tilde{h}^+ = h^+$. Consider now $\tilde{\varphi} := (\widetilde{E}_A^+ \tilde{h}^+)_\perp$. We claim that $\nabla_S \tilde{\varphi} = \nabla_S \varphi$. Indeed, this follows from taking the tangential part in the intertwining formula

$$D\widetilde{E}_A^+ = E_A^+ D,$$

which is readily verified from Lemma 4.2 and definitions of $\widetilde{E}_A^+$, E_A^+. Thus $\tilde{\varphi} - \varphi$ is constant. As in the proof of Corollary 12.8, by adding a normal constant in $\widetilde{E}_0^+ \mathscr{H}^\perp$ to $\tilde{h}^+$, we may assume that $\tilde{\varphi} = \varphi$.

Given this $\tilde{h}^+$, the solution u to the Dirichlet problem with datum φ is given by the normal component of

$$v := \Big(I + \widetilde{S}_A (I - S_A)^{-1} D\Big) e^{-t\tilde{\Lambda}} \tilde{h}^+$$

as in Corollary 12.8(iii). Next, we have

$$f := Dv = (I - S_A)^{-1} e^{-t\Lambda} h^+$$

and f is the conormal gradient to the solution to the regularity problem with datum $\nabla_S \varphi$. In particular $f \in \mathscr{X} \subset L_2(\mathbb{R}_+ \times S^n; \mathscr{V})$ by Lemma 9.3.

Translated to $\mathbb{O}^{1+n}$, this shows that the solution u to the Dirichlet problem with datum φ has $\nabla_{\boldsymbol{x}} u \in L_2(\mathbb{O}^{1+n}; \mathbb{C}^{(1+n)m})$. This shows that u is a variational solution. Uniqueness of the Dirichlet problem in this class completes the proof. □

Remark 18.3. Note that since $\mathscr{X} \subset L_2(\mathbb{R}_+ \times S^n; \mathscr{V})$, solutions to the regularity and Neumann problem always coincide with the variational solutions, by the uniqueness of such. In the setting of the half-space, as in [Auscher et al. 2010b] and [Part I], it was shown in [Axelsson 2010] that this uniqueness result does not hold. As pointed out in [Auscher et al. 2010b, Remark 5.6], the problem occurs at infinity for the regularity and Neumann problems, which explains why uniqueness holds for the bounded ball. Although the analogue of [Axelsson 2010] for the Dirichlet problem on the ball is not properly understood at the moment, Theorem 19.4 below shows that uniqueness of solutions essentially holds also for the Dirichlet problem on the unit ball.

Proposition 18.4. *Let A be radially independent coefficients and assume that the regularity problem and the Dirichlet problem in the sense of Definition 1.2 are both well-posed. Then all $\mathcal{D}^o$-solutions are given by $u = e^{-\sigma t}(e^{-t\tilde{\Lambda}}\tilde{h}^+)_\perp$ for a unique $\tilde{h}^+ \in \widetilde{E}_0^+ L_2$. In particular, the class of $\mathcal{D}^o$-solutions is the same as the class of $\mathcal{Y}^o$-solutions, and the estimate*

$$\|\widetilde{N}_*^o(u)\|_2^2 \approx \int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1-|\boldsymbol{x}|)\, d\boldsymbol{x} + \left| \int_{S^n} u_1(x)\, dx \right|^2$$

holds for all weak solutions.

Proof. Let u be a $\mathcal{D}^o$-solution. For almost every $\rho \in (0,1)$, $\nabla_S u_\rho \in L_2(S^n; (T_{\mathbb{C}} S^n)^m)$ and $u_\rho \in L_2(S^n; \mathbb{C}^m)$. Fix such ρ. As in the proof of Lemma 18.2, we can find $h_\rho^+ \in E_0^+ \mathcal{H}$, $\tilde{h}_\rho^+ \in \widetilde{E}_0^+ L_2$ with $D\tilde{h}_\rho^+ = h_\rho^+$, $(h_\rho^+)_\parallel = \nabla_S u_\rho$ and $(\tilde{h}_\rho^+)_\perp = u_\rho$ on S^n. Using radial independence, the function $\tilde{u}_\rho(rx) := e^{\sigma t}(e^{-t\tilde{\Lambda}}\tilde{h}_\rho^+)_\perp(x)$ (here, ρ is fixed and $e^{-t} = r \in (0,1)$) thus extends to a solution of the divergence form equation with coefficients A, and it is a variational solution by Lemma 18.2. Since $\boldsymbol{x} \mapsto u(\rho \boldsymbol{x})$ is also a variational solution and agrees with $\tilde{u}_\rho$ on S^n, we conclude by uniqueness that $u(\rho r\,\cdot) = e^{\sigma t}(e^{-t\tilde{\Lambda}}\tilde{h}_\rho^+)_\perp$ as $L^2(S^n; \mathbb{C}^m)$-functions for all $e^{-t} = r \in (0,1]$, and almost every $\rho \in (0,1)$.

From this representation, we see that the right hand side is continuous in t, with range in L_2, so the left hand side is continuous in r. We also have $\|u_{\rho r}\|_2 \lesssim \|\tilde{h}_\rho^+\|_2 \approx \|u_\rho\|_2$ for every $r \in (\frac{1}{2}, 1]$ and almost every $\rho \in (0,1)$. The last equivalence comes from the well-posedness of the Dirichlet problem, and the implicit constants are independent of ρ. Since

$$\sup_{1/2<\rho<1} (1-\rho)^{-1} \int_\rho^{\frac{\rho+1}{2}} \|u_s\|_2 ds \lesssim \|\widetilde{N}_*^o(u)\|_2 < \infty,$$

we conclude that $\|\tilde{h}_\rho^+\|_2$ is bounded for $\frac{1}{2} < \rho < 1$. Consider a weak limit $\tilde{h}^+ \in \widetilde{E}_0^+ L_2$ of a subsequence $\tilde{h}_{\rho_n}^+$ with $\rho_n \to 1$. Reversing the roles of ρ and r, for almost every $r < 1$, $u_{\rho_n r}$ converges in $L_2(S^n; \mathbb{C}^m)$ to u_r, so that $u_r = e^{\sigma t}(e^{-t\tilde{\Lambda}}\tilde{h}^+)_\perp$. Extending to all r, the representation is proved.

In particular, this shows that the classes of $\mathcal{Y}^o$-solutions and of $\mathcal{D}^o$-solutions of $Lu = 0$ coincide under our assumptions. □

Note that the full force of $\|\widetilde{N}_*^o(u)\|_2 < \infty$ is not used and the condition

$$\sup_{1/2<r<1} r^{-1} \int_{1-2r}^{1-r} \|u_\rho\|_2 d\rho < \infty$$

suffices in the proof of Proposition 18.4.

Remark 18.5. If A is not r-independent, we need to know that $A(\rho\,\cdot\,)$ satisfies the large Carleson condition for all $\frac{1}{2} < \rho < 1$ to run the argument. This is not clear if we just assume this for A. However, if we assume that A is continuous on $\overline{\mathbb{O}^{1+n}}$ and satisfies the square Dini condition of Theorem 1.11, then this can be checked.

Proof of Theorem 1.7. We consider $A_1 \in L_\infty(S^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$, radially independent coefficients which are strictly accretive in the sense of (2). Assume that the Dirichlet problem with coefficients A_1 is well-posed. By Corollary 12.9, we have $\mathcal{P}_r u_1 = r^{-\sigma}(e^{-t\tilde{\Lambda}} v_0)_\perp$ with $r = e^{-t}$ and v_0 given by the inverse of the well-posedness map (53) from applied to u_1. The assumed uniqueness of the solution u allows us to prove the product rule of $\mathcal{P}_r$ by considering $\mathcal{P}_r u_1$ as another boundary data. The existence of the generator with domain contained in $W_2^1(S^n; \mathbb{C}^m)$ is as in [Auscher 2009] in the setting of the upper half-space. There, the if direction was deduced using the duality principle between Dirichlet and regularity. An examination of the argument there reveals that the only if direction was implicit. We can repeat the same duality argument using Proposition 17.6. □

Proof of Theorem 1.9. By Proposition 18.4 we know that the two classes of $\mathcal{D}^o$- and $\mathcal{Y}^o$-solutions are the same. Thus the assumed well-posedness for $\mathcal{Y}^o$-solutions carries over to $\mathcal{D}^o$-solutions. This completes the proof. □

19. New well-posedness results for real equations

We now specialize to the case of equations ($m = 1$) with real coefficients, and make this assumption for the coefficients A throughout this section unless mentioned otherwise. For such equations the theory of solvability for the Dirichlet problem using nontangential maximal control is rather complete for real symmetric equations, but not so much for non symmetric equations. In [Kenig et al. 2000], the extensions of the tools for real non symmetric equations are discussed and we refer there for details.

We have developed a strategy using square functions rather than nontangential maximal functions and our goal here is to tie this up. It is convenient to introduce the square function

$$S(u)(x) = \left(\int_{\mathbf{y} \in \Gamma_x} |\nabla u(\mathbf{y})|^2 \frac{d\mathbf{y}}{(1 - |\mathbf{y}|)^{n-1}} \right)^{1/2}, \quad x \in S^n,$$

(Γ_x denoting a truncated cone with vertex x and axis the line $(0, x)$) and the divergence form operator $L := -\operatorname{div}_{\mathbf{x}} A \nabla_{\mathbf{x}}$. We note that a weak solution to $Lu = 0$ is in $\mathcal{Y}^o$ if and only if $S(u) \in L_2(S^n)$, the measure being the surface measure. We have so far studied $\mathcal{Y}^o$-solutions and well-posedness in this class, which is convenient to denote here by *well-posedness in* $\mathcal{Y}^o$. (This was called "in the sense of Definition 1.2" in the introduction.)

Recall that by a $\mathcal{D}^o$*-solution* of $Lu = 0$, we mean a weak solution with $\|\widetilde{N}_*^o(u)\|_2 < \infty$. As said in the introduction, we may replace $\widetilde{N}_*^o(u)$ by the usual pointwise nontangential maximal function. For the Dirichlet problem, we shorten well-posedness in the sense of Dahlberg in Definition 1.8 to *well-posedness in* $\mathcal{D}^o$.

On regular domains such as $\mathbb{O}^{1+n}$, there is always a unique variational solution $u \in W_2^1(\mathbb{O}^{1+n})$, which is in addition continuous in $\overline{\mathbb{O}^{1+n}}$, to the Dirichlet problem with data $\varphi \in C(S^n)$ by results of Littman, Stampacchia and Weinberger [Littman et al. 1963] which extend to real nonsymmetric equations (see [Kenig et al. 2000]). Thus, it is natural to ask whether this solution satisfies $\|\widetilde{N}_*^o(u)\|_2 \le C\|\varphi\|_2$ with C depending on the Lipschitz character of S^n. By a density argument, it suffices to do this for smooth φ, say $\varphi \in C^1(S^n)$. If this is the case, then the Dirichlet problem $(D)_2$ is said to be *solvable*.

From the maximum principle and Harnack's inequalities, one can study the L-elliptic measure ω, say at 0, which is the probability measure $C(S^n) \ni \varphi \mapsto u(0)$ with u the above solution. The question whether ω is absolutely continuous with respect to surface measure is central.

The result, somehow folklore but we have not seen it stated explicitly in the literature, summarizing the state of the art is the following.

Theorem 19.1. *Let $L = -\mathrm{div}_x A\nabla_x$ be a real elliptic operator in $\mathbb{O}^{1+n}$, $n \geq 1$. Then the following statements are equivalent.*

(i) *The Dirichlet problem is well-posed in $\mathcal{D}^o$.*

(ii) *$(D)_2$ is solvable.*

(iii) *The L-elliptic measure w is absolutely continuous with respect to surface measure and its Radon–Nikodym derivative k satisfies the reverse Hölder B_2 condition, i.e., there is a constant $C < \infty$ such that for all surface balls B on S^n,*

$$\left(|B|^{-1}\int_B k^2(x)\,dx\right)^{1/2} \leq C|B|^{-1}\int_B k(x)\,dx.$$

Proof. The proof that (ii) is equivalent to (iii) is stated for real nonsymmetric operators in [Kenig et al. 2000, p. 241]. The proof that (i) implies (ii) is trivial. For $\varphi \in C(S^n)$, the variational solution is bounded, hence satisfies $\|\widetilde{N}^o_*(u)\|_2 < \infty$ since $\mathbb{O}^{1+n}$ is bounded. By uniqueness in (i), it is the unique solution and the continuity estimate that follows from well-posedness shows $\|\widetilde{N}^o_*(u)\|_2 \leq C\|\varphi\|_2$. So (ii) holds. It remains to see (ii) implies (i). Existence and continuity estimate are granted from $(D)_2$. Uniqueness follows the argument in [Fabes et al. 1984, p. 125–126], using the equivalent assumption (iii) instead of (ii). The extension to nonsymmetric real operators is allowed from the details in [Kenig et al. 2000]. □

Theorem 19.2. *Let L be an elliptic operator with real coefficients. Then all weak solutions to $Lu = 0$ satisfy $\|S(u)\|_{L_2(d\mu)} \lesssim \|\widetilde{N}^o_*(u)\|_{L_2(d\mu)}$ for any A_∞ measure μ with respect to L-elliptic measure.*

Proof. This is the result of [Dahlberg et al. 1984] where this is proved when $L = L^*$. Aside from properties of solutions that are valid for all real operators (see [Kenig et al. 2000]), the proof on the use of [Dahlberg et al. 1984, (7), p. 101], which is an integration by parts and is valid as is in the nonsymmetric case. This is why the A_∞ property with respect to L-elliptic measure intervenes in the hypotheses. Further details are in [Dahlberg et al. 1984]. □

Corollary 19.3. *Let L be an elliptic operator with real coefficients. Assume that the Dirichlet problem is well-posed in $\mathcal{D}^o$ for L. Then all weak solutions to $Lu = 0$ satisfy $\|S(u)\|_2 \lesssim \|\widetilde{N}^o_*(u)\|_2$. In particular, $\mathcal{D}^o$-solutions of $Lu = 0$ are $\mathcal{Y}^o$-solutions of $Lu = 0$ under this assumption.*

Proof. By Theorem 19.1, L-elliptic measure is A_∞ with respect to surface measure, and vice-versa by [Coifman and Fefferman 1974]. So $\|S(u)\|_{\mathcal{Y}^o} \lesssim \|\widetilde{N}^o_*(u)\|_2$ follows from Theorem 19.2. □

Note that Corollary 19.3 and Proposition 18.4 are close but incomparable. First, Proposition 18.4 applies to systems of equations, whereas Corollary 19.3 applies to radially dependent coefficient. Secondly, the well-posedness assumptions are different. The next results reconciles the two approaches.

Theorem 19.4. *Let $L = -\operatorname{div}_x A\nabla_x$ be a real elliptic operator in $\mathbb{O}^{1+n}$, $n \geq 1$. Assume further that L has coefficients with $\lim_{\tau\to 0} \|\chi_{t<\tau}\mathcal{E}_t\|_{C\cap L_\infty}$ sufficiently small. The following statements are equivalent.*

(i) *The Dirichlet problem is well-posed in $\mathcal{D}^o$ for L and L^*.*

(ii) *The Dirichlet problem is well-posed in $\mathcal{Y}^o$ for L and L^*.*

Moreover, in this case the solutions for L (resp. L^*) from a same datum are the same.

Proof. It suffices to prove the conclusion for L in each case as the assumptions are invariant under taking adjoints.

Assume (i). Uniqueness in $\mathcal{Y}^0$ is immediate since the class of $\mathcal{D}^o$-solutions a priori contains the class $\mathcal{Y}^o$-solutions when $\|\mathcal{E}\|_{C\cap L_\infty} < \infty$. Next, for the existence, there is by assumption a unique $\mathcal{D}^o$-solution with given boundary datum $\varphi \in L_2(S^n)$. Since the Dirichlet problem is well-posed in $\mathcal{D}^o$ for L, Corollary 19.3 shows that this solution is in fact a $\mathcal{Y}^o$-solution.

Conversely, assume (ii). By Theorem 19.1, it suffices to show that $(D)_2$ is solvable for L. To this end, it suffices to consider $\varphi \in C^1(S^n)$ and the associated variational solution u. By Lemma 18.2, which applies because of Theorem 16.1 ($I - S_A$ is invertible on $\mathcal{X}$ and on $\mathcal{Y}$) and Proposition 17.6, u coincides with the solution in the sense of Definition 1.2, that is, it is a $\mathcal{Y}^o$-solution. Now Theorem 14.1 provides the nontangential maximal estimate that shows that $(D)_2$ is solvable for L. □

Added in proof. In the context of the upper half-space, it was recently shown by Hofmann, Kenig, Mayboroda and Pipher [Hofmann et al. 2012], that the conclusion of Corollary 19.3 is valid a priori for all real operators with transversally independent coefficients, whether or not the Dirichlet problem is well-posed in $\mathcal{D}^o$ for L and without resorting to the A_∞ property of the L-elliptic measure, which they subsequently prove. Hence, for transversally independent coefficients, the two classes of solutions of L are the same, and well-posedness in each class is simultaneous. Presumably, the conclusion should be the same on the ball for operators with radially independent coefficients.

Remark 19.5. In the case of radially independent coefficients (or more generally for continuous, Dini square coefficients) Proposition 18.4 (or the remark that follows it) proves the converse also for systems.

We can generalize results from [Kenig and Pipher 1993] to nonsymmetric perturbations of r-independent real symmetric operators.

Corollary 19.6. *In $\mathbb{O}^{1+n}$, the Dirichlet problem is well-posed in $\mathcal{D}^o$ for all real operators L with coefficients A such that $\lim_{\tau\to 0} \|\chi_{t<\tau}\mathcal{E}_t\|_{C\cap L_\infty}$ is small enough and its boundary trace A_1 real symmetric.*

Proof. Let L_1 be the second-order operator with r-independent coefficients A_1. By Proposition 17.16, we know that the Dirichlet problem for $L_1 = L_1^*$ is well-posed in $\mathcal{Y}^o$. Thus, by Theorem 17.14, it is well-posed in $\mathcal{Y}^o$ for L and L^*. Thus, we conclude with Theorem 19.4. □

We continue with generalizations of results in [Fabes et al. 1984], where well-posedness for Dirichlet was obtained for real symmetric coefficients. Well-posedness for regularity (which we denote here by well-posedness in $\mathcal{X}^o$) is new.

Theorem 19.7. *Assume that A are coefficients with $\lim_{\tau\to 0}\|\chi_{t<\tau}\mathcal{E}_t\|_{C\cap L_\infty}$ small enough and boundary trace A_1 which is real and continuous. Then the Dirichlet problem is well-posed in $\mathcal{D}^o$ and in $\mathcal{Y}^o$, and the regularity problem in $\mathcal{X}^o$ is well-posed. In particular, this holds for real continuous coefficients in $\overline{\mathbb{O}^{1+n}}$ satisfying the Dini square condition*

$$\int_0 w_A^2(t)\frac{dt}{t} < \infty, \quad \text{where } w_A(t) = \sup\{|A(rx) - A(x)|\,;\, x \in S^n,\, 1 - r < t\}.$$

Proof. Let L_1 be the operator with coefficients A_1. Recall that under smallness of $\lim_{\tau\to 0}\|\chi_{t<\tau}\mathcal{E}_t\|_{C\cap L_\infty}$, it suffices to prove the result for L_1 by Theorem 17.14. Next, by Proposition 17.6, the regularity problem (in $\mathcal{X}^o$) for L_1 is well-posed if and only if the Dirichlet problem for L_1^* is well-posed in $\mathcal{Y}^o$. On applying Theorem 19.4, it suffices to prove that the Dirichlet problem with coefficients A_1 is well-posed in $\mathcal{D}^o$, as the same would then hold for A_1^* by symmetry of the assumptions. To do this, we prove that L_1-elliptic measure satisfies property (iii) in Theorem 19.1. The argument is inspired by the one in [Fabes et al. 1984, pp. 139–140].

Assume first we work on some boundary region of $\mathbb{O}^{1+n}$. For r small, set

$$Q_r = \{\rho y \in (0, 1) \times S^n\,;\, 1 - r < \rho < 1,\ y \in B(x_0, r)\},$$

where $B(x_0, r)$ is a surface ball of radius r, with real radially independent coefficients A_1 being the restriction of some matrix defined on $\mathbb{O}^{1+n}$ that we still denote by A_1 and which is close in L_∞ to the constant matrix $A_1(x_0)$. Let g be a C^1 nonnegative function supported on the part of the boundary of $Q_{r/2}$ in S^n. Let v be the variational solution to the Dirichlet problem $L_1 v = 0$ in $Q_{r/2}$ and $v = g$ on the boundary of $Q_{r/2}$ in S^n and $v = 0$ on the part of the boundary that is contained in $\mathbb{O}^{1+n}$. Recall that $v \in W_2^1(Q_{r/2}) \cap C(\overline{Q_{r/2}})$. By Theorem 17.13, because A_1 is L_∞ close to a (constant) matrix for which one knows well-posedness by Proposition 17.17, one can construct the unique solution u in $\mathbb{O}^{1+n}$ to the Dirichlet problem in $\mathcal{Y}^o$ with $u = g$ on S^n, that is $L_1 u = 0$ with $\int_{\mathbb{O}^{1+n}} |\nabla_{\boldsymbol{x}} u|^2 (1 - |\boldsymbol{x}|)\, d\boldsymbol{x} \le C\|g\|_2^2$. As $g \in C^1(S^n)$, we know on applying Lemma 18.2 that the solution u is variational, i.e., $u \in W_2^1(\mathbb{O}^{1+n})$. We can apply Stampacchia's minimum principle to obtain first that $u \ge 0$ in $\mathbb{O}^{1+n}$, and next the maximum principle in $Q_{r/2}$ to conclude that $v \le u$. From there, it remains to repeat the argument in [Fabes et al. 1984], to obtain that $v(\rho y) \le C(1 - \rho)^{-n/2}\|g\|_2$ for all $\rho y \in Q_{r/4}$, which in turn, yields an L_2 estimate on the Radon–Nikodym derivative of the L_1-elliptic measure.

The localization argument as in [Fabes et al. 1984], and using the continuity of A_1 to cover a layer of the boundary with a finite number of such small $Q_{r/2}$, allows us to conclude. □

Corollary 19.8. *With the same assumption as above and $n = 1$, then the Neumann problem with coefficients A is well-posed in $\mathcal{X}^o$.*

Proof. By the results in Section 5, it follows that the Neumann problem for coefficients A is well-posed in $\mathcal{X}^o$ if and only if the regularity problem for conjugate coefficients $\tilde{A}$ is well-posed in $\mathcal{X}^o$. The latter follows from the previous result since $\tilde{A}$ satisfies the same assumption as A. □

Remark 19.9. As in [Fabes et al. 1984], the Dini square condition in the normal direction can be replaced by a Dini square condition in a C^1 transverse direction to the sphere. It suffices to perform locally changes of variables that transform the transverse direction to normal ones.

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THE TWO-PHASE STEFAN PROBLEM: REGULARIZATION NEAR LIPSCHITZ INITIAL DATA BY PHASE DYNAMICS

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In this paper we investigate the regularizing behavior of two-phase Stefan problem near initial Lipschitz data. A description of the regularizing phenomena is given in terms of the corresponding space-time scale.

1. Introduction

Consider $u_0(x) : B_R(0) \to \mathbb{R}$ with $R \gg 1$ and $u_0 \geq -1$, $|\{u_0 = 0\}| = 0$ and $u_0(x) = -1$ on $\partial B_R(0)$ (see Figure 1). The two-phase Stefan problem can be formally written as

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\} \cup \{u < 0\}, \\ u_t/|Du^+| = |Du^+| - |Du^-| & \text{on } \partial\{u > 0\}, \\ u(\cdot, 0) = u_0, & \\ u = -1 & \text{on } \partial B_R(0). \end{cases} \tag{ST2}$$

Here Du denotes the spatial derivative of u, and u^+ and u^- respectively denote the positive and negative parts of u, i.e,

$$u^+ := \max(u, 0) \quad \text{and} \quad u^- := -\min(u, 0).$$

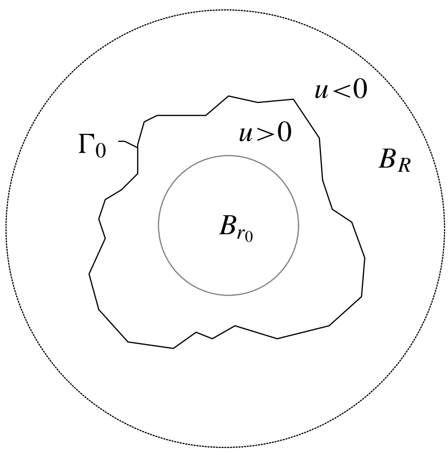

Figure 1. Initial setting of the problem.

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The classical Stefan problem describes the phase transition between solid/liquid or liquid/liquid interface (see [Meirmanov 1992; Oleinik et al. 1993]). In our setting, we consider a bounded domain $\Omega_0 \subset B_R(0)$ and initial data $u_0(x)$ such that

$$\{u_0 > 0\} = \Omega_0 \quad \text{and} \quad \{u_0 < 0\} = B_R(0) - \overline{\Omega}_0.$$

To avoid complications at infinity, we consider the problem in the domain $Q = B_R(0) \times [0, \infty)$. For simplicity we have set $u = -1$ on $\partial B_R(0)$; our analysis presented in this paper applies to (ST2) with the generalized Dirichlet condition

$$u = f(x,t) < 0 \quad \text{on } \partial B_R(0),$$

where $f(x,t)$ is smooth.

Since our initial data will be only locally Hölder continuous, we employ the notion of viscosity solutions to discuss the evolution of the problem. Viscosity solutions for (ST2) were originally introduced by Athanasopoulos et al. [1996] (see also [Caffarelli and Salsa 2005]). As for existence and uniqueness of viscosity solutions for (ST2), we refer to [Kim and Požár 2011].

Note that the second condition of (ST2) states that the normal velocity $V_{x,t}$ at each free boundary point $(x,t) \in \partial\{u > 0\}$ is given by

$$V_{x,t} = (|Du^+| - |Du^-|)(x,t) = (Du^+(x,t) - Du^-(x,t)) \cdot \nu_{x,t},$$

where $\nu_{x,t}$ denotes the spatial unit normal vector of $\partial\{u > 0\}$ at (x,t), pointing inward with respect to the *positive phase* $\{u > 0\}$.

In this paper we investigate the regularizing behavior of the *free boundary* $\partial\{u > 0\}$. Our main result states that when $\Gamma_0 := \partial\{u_0 > 0\}$ is locally a Lipschitz graph with small Lipschitz constant, then the free boundary immediately regularizes and becomes smooth after $t = 0$. Moreover we provide a natural space-time scale for such regularization. More precisely, for $x_0 \in \Gamma_0$, we show that the free boundary regularizes in $B_d(x_0)$ by the time $t(x_0, d)$ given in (1-3) (see Theorem 1.1, and also the heuristic discussion below (1-3)). Corresponding results have been obtained in recent studies on the one-phase free boundary problems [Choi et al. 2007; 2009; Choi and Kim 2010], but the presence of two phases poses new challenges in the analysis. For example there is no generic class of global solutions other than radial solutions where topological changes are ruled out. In the one-phase setting we relied on the fact that solutions with star-shaped initial data stay star-shaped over time: this is no longer true in the two-phase setting (see Remark 3.2). More importantly, the interface motion is no longer monotone and competition between positive and negative fluxes across the free boundary necessitates additional localization procedure (see the remarks below Theorem 1.1).

The celebrated results of [Athanasopoulos et al. 1996; 1998] state that if the solution of (ST2) stays close to a Lipschitz profile in the unit space-time neighborhood $B_1(0) \times [0, 1]$, then the solution is indeed smooth in half of the neighborhood, that is, in $B_{1/2}(0) \times [1/2, 1]$. The main step in our analysis is to prove that the free boundary $\partial\{u > 0\}$ stays close to a locally Lipschitz profile in any given scale. Proving this step corresponds to the derivation of several Harnack-type inequalities for our problem, which are of independent interest.

Before discussing our result in detail, let us introduce precise conditions on the initial data.

(I-a) Ω_0 and u_0 are star-shaped with respect to a ball $B_{r_0}(0) \subset \Omega_0$.

Observe that then the *Lipschitz constant* L of $\partial\Omega_0$ is determined by r_0 and d_0, where

$$d_0 := \sup\{\mathrm{dist}(x, B_{r_0}(0)) : x \in \partial\Omega_0\}.$$

In other words, there exist $h = h(r_0)$ and $L = L(r_0, d_0)$ such that for any $x_0 \in \partial\Omega_0$, after rotation of coordinates, one has the representation

$$B_h(x_0) \cap \Omega_0 = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n \leq f(x)\}, \tag{1-1}$$

where f is a Lipschitz function with $\mathrm{Lip}\, f \leq L$. For simplicity, we set $h = 1$.

For a locally Lipschitz domain such as Ω_0, there exist growth rates $0 < \beta < 1 < \alpha$ such that the following holds: Let H be a positive harmonic function in $\Omega_0 \cap B_2(x)$, $x \in \partial\Omega_0$, with Dirichlet condition on $\partial\Omega_0 \cap B_2(x)$, and with value 1 at $x - e_n$. (Here e_n is the direction of the axis for the Lipschitz graph near x.) Then for $x - se_n \in \Omega_0 \cap B_1(x)$,

$$s^\alpha \leq H(x - se_n) \leq s^\beta. \tag{1-2}$$

Below we list conditions on the range of the Lipschitz constant L of the initial positive phase Ω_0.

(I-b) $L < L_n$ for a sufficiently small dimensional constant L_n so that

$$5/6 \leq \beta < \alpha \leq 7/6.$$

The remaining conditions are on the regularity of u_0.

(I-c) $-N_0 \leq \Delta u_0 \leq N_0$ in $\Omega_0 \cup (B_R(0) - \overline{\Omega}_0)$, where N_0 is some constant.

(I-d) For $x \in \partial\Omega_0$, we may let $e_n = x/|x|$ after a rotation. Then for small $s > 0$ (for $0 < s < 1/10$),

$$|Du_0(x \pm se_n)| \geq Cs^{\alpha-1}.$$

Note that (I-c) and (I-d) hold for u_0 which is smooth in its positive and negative phases and is harmonic near the initial free boundary, that is, $-\Delta u_0 = 0$ in the set $(\{u_0 > 0\} \cup \{u_0 < 0\}) \cap \{x : \mathrm{dist}(x, \partial\Omega_0) \leq 1\}$.

We mention that, roughly speaking, the series of the hypothesis (Ia)–(Id) suggests that we have in mind an initial positive phase Ω_0 whose boundary is "almost" C^1 (that is, a small perturbation of a C^1 boundary in its Lipschitz norm), and initial data u_0 whose rescaled profile is "almost" harmonic near $\partial\Omega_0$. The smallness assumption on L given in (I-b) is to avoid waiting time phenomena (see [Athanasopoulos et al. 1996; Choi and Kim 2006]), and is most natural in the spirit of previous results [Athanasopoulos et al. 1996; 1998]. The assumption on u_0 is introduced to ensure that the initial data does not perturb the initial geometry of Ω_0 too much (see the discussion in [Choi and Kim 2010]). We expect that regularization of the interface over time should hold for general continuous initial data u_0.

For a function $u(x, t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, let us write

$$\Omega(u) := \{u > 0\}, \quad \Omega_t(u) := \{u(\,\cdot\,, t) > 0\},$$

and

$$\Gamma(u) := \partial\{u > 0\}, \quad \Gamma_t(u) := \partial\{u(\cdot, t) > 0\}.$$

Since $\Gamma_0 = \partial\{u(\cdot, 0) > 0\} = \partial\{u(\cdot, 0) < 0\}$ in our setting, this property is preserved for later times, i.e.,

$$\Gamma_t(u) = \partial\{u(\cdot, t) > 0\} = \partial\{u(\cdot, t) < 0\} \quad \text{for all } t > 0;$$

see [Rogers and Berger 1984; Götz and Zaltzman 1991; Kim and Požár 2011].

For $x_0 \in \Gamma_0 = \Gamma_0(u)$, we may let $e_n = x_0/|x_0|$ after a rotation. Then we define

$$t(x_0, r) := \min\left\{\frac{r^2}{u^+(x_0 - re_n, 0)}, \frac{r^2}{u^-(x_0 + re_n, 0)}\right\}. \tag{1-3}$$

Some remarks concerning $t(x_0, r)$ are in order. In one-phase case (where $u^- \equiv 0$), it was shown in [Choi et al. 2007] that

$$t(x_0, r) \sim \sup\{t > 0 : u(x_0 + re_n, t) = 0\},$$

i.e., $t(x_0, r)$ is the time it takes for the free boundary to reach $x_0 + re_n$. In our (two-phase) case $t(x_0, r)$ is the time it takes for the free boundary to reach $x_0 + re_n$ if we evolve the free boundary only according to the dominant phase with bigger size of u. In particular $\Gamma(u)$ moves at most by distance r by the time $t(x_0, r)$. It turns out that $t(x_0, r)$ is the correct time scale for the solutions in r-neighborhood of x_0 to "mix" and regularize the interface (Theorem 1.1(3)). See the paragraph below Theorem 1.1 for further heuristics based on scaling properties of our problem.

Suppose u is a solution of (ST2) with initial data u_0 satisfying (Ia)–(Id) with $\Omega_0(u) \subset B_R(0)$. Due to (Ia)–(Ib), for sufficiently small r and given $x_0 \in \Gamma_0$ the initial free boundary Γ_0 is given by the graph of a Lipschitz function in $B_r(x_0)$. After a rotation if necessary, we may assume that

$$\Omega_0 \cap B_r(x_0) = \{x + x_0 : x = (x', x_n),\ x_n \le f(x')\},$$

where f is a Lipschitz function with Lipschitz constant $L < L_n$. We summarize our main results:

Theorem 1.1 (cf. Theorem 5.6, Theorem 5.7 and Corollary 5.8). *Let u, Ω_0, r and f be as above. There exists $d_0 > 0$ depending only on n and N_0 such that the following holds for $r \le d_0$:*

(1) *In $\Sigma_r := B_{2r}(x_0) \times [t(x_0, r)/2, t(x_0, r)]$, we have*

$$\Gamma(u) = \{(x + x_0, t) : x = (x', x_n),\ x_n \le f(x', t)\},$$

where $f(x', t)$ is a C^1 function of space and time. Moreover, there exists a positive dimensional constant c_0 and $1 < m < 2$ such that

$$|D_{x'} f(x', t) - D_{x'} f(y', t)| \le c_0\left(-\log\left|\frac{x'}{r} - \frac{y'}{r}\right|\right)^{-m},$$
$$|\partial_t f(x', t) - \partial_t f(x', s)| \le c_0\left(-\log\left|\frac{t}{t(x_0, r)} - \frac{s}{t(x_0, r)}\right|\right)^{-1/3}.$$

(2) *u is a classical solution of (ST2) in Σ_r in the sense that*

(i) *Du^+ exists in $\Omega(u)$ and is continuous up to $\overline{\Omega}(u)$;*

(ii) *Du^- exists in $\Omega(u)$ and is continuous up to $\Sigma_r \cap (\mathbb{R}^n - \Omega(u))$;*

(iii) *the free boundary condition is satisfied in the classical sense, i.e.,*

$$V_{x,t} = (|Du^+| - |Du^-|)(x,t) \quad \text{on } \Gamma(u) \cap \Sigma_r.$$

(3) *There exists a positive dimensional constant M such that*

$$M^{-1}\frac{u^+(x_0 - re_n, 0)}{r} \le |Du^+|(x,t) \le M\frac{u^+(x_0 - re_n, 0)}{r}$$

and

$$M^{-1}\frac{u^-(x_0 + re_n, 0)}{r} \le |Du^-|(x,t) \le M\frac{u^-(x_0 + re_n, 0)}{r}$$

in Σ_r.

Remark 1.2. Our result extends to the case where the star-shaped condition (I-a)–(I-b) is replaced by:

(I-ab) Ω_0 is locally Lipschitz with a sufficiently small Lipschitz constant.

See the discussion in Section 6.

The one phase version of the above result was proved in [Choi and Kim 2010] (see Theorem 2.16 in Section 2). Let us briefly motivate our result below in the context of the existing literature.

For a given reference point $(x_0, t_0) \in \mathbb{R}^n \times [0, \infty)$ and positive constants r and c, one can rescale the solution u of (ST2) as

$$\tilde{u} := \frac{1}{c}u\Big(x_0 + rx,\, t_0 + \frac{r^2}{c}t\Big). \tag{1-4}$$

Then $\tilde{u}$ satisfies the free boundary problem

$$\begin{cases} r\tilde{u}_t - \Delta\tilde{u} = 0 & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} < 0\}, \\ V = |D\tilde{u}^+| - |D\tilde{u}^-| & \text{on } \partial\{\tilde{u} > 0\}, \end{cases} \tag{$\tilde{P}$}$$

in a corresponding neighborhood of the origin. Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, so that $x \in \mathbb{R}^n$ can be written as $x = (x', x_n)$, with $x_n = x \cdot e_n$. Choose $(x_0, t_0) = (x_0, 0)$ with $x_0 \in \Gamma_0(u)$. By our hypothesis, after a change of coordinates if necessary, there exists a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with a small Lipschitz constant such that

$$\Omega_0(u) \cap B_{2r}(x_0) = \{x : x_n \le f(x')\}.$$

Let us choose

$$c = \max\{u^+(x_0 - re_n, 0), u^-(x_0 + re_n, 0)\} \tag{1-5}$$

so that one of $\tilde{u}^+(-e_n, 0)$ and $\tilde{u}^-(+e_n, 0)$ equals 1, and the other is less than 1.

Now suppose that we can show the following two conditions:

(A) $|\tilde{u}|(x,t) \le C$ in $B_1(0) \times [0, 1]$ with a constant $C > 0$ independent of x_0 and r.

(B) The level sets of $\tilde{u}$ are Lipschitz graphs in space and time with small Lipschitz constant in $B_1(0) \times [0, 1]$.

Then Theorem 1.1 follows from the results of [Athanasopoulos et al. 1996] applied to $\tilde{u}$. Indeed, (B) can be replaced by a relaxed version (B′) as stated below, which is sufficient to derive Theorem 1.1 due to the results of [Athanasopoulos et al. 1998].

(B′) The level sets of $\tilde{u}$ are ϵ-monotone with respect to cones of directions $W_x(\theta^x, e)$ and $W_t(\theta^t, \nu)$ with $\nu \in \operatorname{span}(e_n, e_t)$, and $\pi/2 - \theta^x$ and ϵ sufficiently small.

(For the meaning of ϵ-monotonicity and the space and time cones W_x and W_t, see Definition 2.1.)

In our case (A) can be verified using previously known results in the one-phase Stefan problem (Lemma 3.3 and Lemma 3.4). Unfortunately, as shown in [Choi and Kim 2010], verifying (B′) for all scales r turns out to be as difficult as showing (B) or the full regularity of u. Since $\tilde{u}$ no longer satisfies the heat equation, one loses control of the change of u over time. For this reason it is necessary to show (B′) for all level sets of $\tilde{u}$, not just for the free boundary $\Gamma(\tilde{u})$. Indeed, in this article we will first show that $\tilde{u}$ (scaled correspondingly for the two-phase) is ϵ-monotone in the space variable (Lemma 3.1), and then we show that $\Gamma(\tilde{u})$ is ϵ-monotone in the space-time variables (Corollary 4.4 and Lemma 4.7). Then in Section 5 we use the ϵ-monotonicity obtained from previous sections, the almost-harmonicity of $\tilde{u}$ (Lemma 3.6), as well as the iteration methods originating from [Athanasopoulos et al. 1996; 1998] to show directly that $\tilde{u}$ is a classical solution and u satisfies (B) and (B′) (Section 5). The arguments in Section 5 are mostly drawn from [Athanasopoulos et al. 1996; 1998] as well as [Choi et al. 2007; 2009].

Let us now illustrate the underlying ideas in the analysis in Section 4, where we show the ϵ-monotonicity of the solution over time. In terms of the original solution u, verifying (A) and (B′) corresponds to analyzing u over the time interval $[0, t(x_0, r)]$, where $t(x_0, r)$ is given by

$$t(x_0, r) := r^2/c,$$

and c is as given in (1-5). Note that $t(x_0, r)$ coincides with the one given by (1-3).

Heuristically speaking, there are two possible scenarios for interface regularization, depending on its initial configuration in the local neighborhood:

(1) One of the phases has much bigger flux than the other, i.e.,

$$u^+(x_0 - se_n, 0) \gg u^-(x_0 + se_n, 0) \quad \text{or} \quad u^-(x_0 - se_n, 0) \ll u^-(x_0 + se_n, 0)$$

for s comparable to r.

In this case one-phase-like phenomena (regularization by the dominant phase as obtained in Theorem 2.16) are expected. As mentioned above, in this case the time interval for regularization of the free boundary in r-neighborhood is proportional to the distance it has traveled.

(2) Both phases are in balance, i.e.,

$$u^+(x_0 - se_n, 0) \sim u^-(x_0 + se_n, 0) \tag{1-6}$$

for s comparable to r.

In this case one expects regularization due to competition between two phases, resulting in Lipschitz-like behavior over time. Again the corresponding time interval for regularization amounts to $t(x_0, r)$ given in (1-3).

To make the above heuristics rigorous, in Section 4 we will introduce a decomposition procedure based on Harnack-type inequalities, which illustrates local dynamics near the free boundary: roughly speaking, for a given $r > 0$ we divide $B_r(x_0) \times \{t = 0\}$ into regions where (1-6) holds for $0 < s \ll r$ (*balanced* region) and the rest of domain (*unbalanced* region). (See detailed definitions of these regions in Section 4.) Of course the main issue is whether the dynamics of one region affect the other, in particular whether the one-phase-type dynamics of the unbalanced region breaks the property (1-6) in the balanced region for future times. We will show that this does not happen (Proposition 4.3), due to a fast regularization property in the unbalanced region (Proposition 3.7 and Lemma 4.7) as well as Harnack-type inequalities (Lemmas 4.5 and 4.6) in the balanced region.

Let us finish this section with an outline of the paper. In Section 2 we introduce preliminary results and notation, including the regularity results in the one-phase Stefan problem (Theorem 2.16). Sections 3 to 5 consist of the proof of Theorem 1.1; in Section 3 we prove some properties on the evolution of solutions of (ST2) with star-shaped data. In addition to Harnack inequalities, we show that the solution stays near the star-shaped profile for a unit time (Lemma 3.1), which in turn yields that the solution stays very close to harmonic functions (Lemma 3.6). This establishes that (B′) holds in the space variable. Making use of the results in Section 3, we perform a decomposition procedure in Section 4, to show that (A) holds for $\tilde{u}$ (Proposition 4.3) and that (B′) holds for $\Gamma(\tilde{u})$ (Corollary 4.4). This completes the main step in our analysis. In Section 5 we describe the rather technical iteration procedure leading to further regularization, and we complete the proof of Theorem 1.1 by combining arguments from [Athanasopoulos et al. 1996; 1998; Choi et al. 2007; 2009] (Theorem 5.7 and Corollary 5.8). In Section 6 we discuss a generalized proof of the corresponding regularization result (Theorem 6.1) when the star-shapedness of the initial data (I-a) and (I-b) are replaced by the local version (I-ab).

2. Preliminary lemmas and notation

We introduce some notation.

- For $x \in \mathbb{R}^n$, denote by $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $x_n = x \cdot e_n$.
- Let $B_r(x)$ be the space ball of radius r, centered at x.
- Let $Q_r := B_r(0) \times [-r^2, r^2]$ be the parabolic cube and let $K_r := B_r(0) \times [-r, r]$ be the hyperbolic cube.
- A caloric function in $\Omega \cap Q_r$ will denote a nonnegative solution of the heat equation, vanishing along the lateral boundary of Ω.
- For $x_0 \in \Gamma_0$ and $e_n = x_0/|x_0|$, define

$$t(x_0, d) := \min\left\{\frac{d^2}{u^+(x_0 - de_n, 0)}, \frac{d^2}{u^-(x_0 + de_n, 0)}\right\}.$$

• C is called a universal constant if it depends only on the dimension n and the regularity constant N_0 of u_0.

• We say $a \sim b$ if there exists a dimensional constant $C > 0$ such that $C^{-1}b \leq a \leq Cb$.

• Lastly let us recall the definition of ϵ-monotonicity. Let $W_x(\theta^x, e)$ and $W_t(\theta^t, \nu)$ with $e \in \mathbb{R}^n$ and $\nu \in \operatorname{span}(e_n, e_t)$, respectively, denote a spatial circular cone of aperture $2\theta^x$ and axis in the direction of e, and a two-dimensional space-time cone in (e_n, e_t) plane of aperture $2\theta^t$ and axis in the direction of ν.

Definition 2.1. (a) Given $\epsilon > 0$, a function w is called *ϵ-monotone* in the direction τ if

$$u(p + \lambda\tau) \geq u(p) \quad \text{for any } \lambda \geq \epsilon.$$

(b) w is *ϵ-monotone* in a cone of directions $W_x(\theta^x, e)$ or $W_t(\theta^t, \nu)$ if w is ϵ-monotone in every direction in the cone.

Next we state preliminary results that are important in our analysis. The first lemma is a direct consequence of the interior Harnack inequalities proved in [Caffarelli and Cabré 1995].

Lemma 2.2. *Suppose $w(x) : \mathbb{R}^n \to \mathbb{R}$ has bounded Laplacian. Then w is Hölder continuous with its constant depending on the Laplacian bound.*

Lemma 2.3 [Fabes et al. 1984, Theorem 3]. *Let Ω be a domain in $\mathbb{R}^n \times \mathbb{R}$ such that $(0, 0)$ is on its lateral boundary. Suppose Ω is a* $\mathrm{Lip}^{1,1/2}$ *domain, i.e.,*

$$\Omega = \{(x', x_n, t) : |x'| < 1, |x_n| < 2L, |t| < 1, x_n \leq f(x', t)\},$$

where f satisfies $|f(x', t) - f(y', s)| \leq L(|x' - y'| + |t - s|^{1/2}$.) If u is a caloric function in Ω, then there exists $C = C(n, L)$, where L is the Lipschitz constant for Ω, such that

$$\frac{u(x, t)}{v(x, t)} \leq C \frac{u(-Le_n, \frac{1}{2})}{v(-Le_n, -\frac{1}{2})}$$

for $(x, t) \in Q_{1/2}$.

Lemma 2.4 [Athanasopoulos et al. 1996, Theorem 1]. *Let Ω be a Lipschitz domain in $\mathbb{R}^n \times \mathbb{R}$, i.e.,*

$$Q_1 \cap \Omega = Q_1 \cap \{(x, t) : x_n \leq f(x', t)\},$$

where f satisfies $|f(x, t) - f(y, s)| \leq L(|x - y| + |t - s|)$. Let u be a caloric function in $Q_1 \cap \Omega$ with $(0, 0) \in \partial\Omega$ and $u(-e_n, 0) = m > 0$ and $\sup_{Q_1} u = M$. Then there exists a constant C, depending only on n, L, m/M such that

$$u(x, t + \rho^2) \leq Cu(x, t - \rho^2)$$

for all $(x, t) \in Q_{1/2} \cap \Omega$ and for $0 \leq \rho \leq d_{x,t}$.

Lemma 2.5 [Athanasopoulos et al. 1996, Lemma 5]. *Let u and Ω be as in Lemma 2.4. Then there exist $a, \delta > 0$ depending only on n, L, m/M such that*

$$w_+ := u + u^{1+a} \quad \textit{and} \quad w_- := u - u^{1+a}$$

are subharmonic and superharmonic, respectively, in $Q_\delta \cap \Omega \cap \{t = 0\}$.

Next we state several properties of harmonic functions:

Lemma 2.6 [Dahlberg 1979]. *Let* u_1, u_2 *be two nonnegative harmonic functions in a domain* D *of* $\mathbb{R}^n$ *of the form*

$$D = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 2, |x_n| < 2L, x_n > f(x')\},$$

with f *a Lipschitz function with constant less than* L *and* $f(0) = 0$. *Assume further that* $u_1 = u_2 = 0$ *along the graph of* f. *Then in*

$$D_{1/2} = \{|x'| < 1, |x_n| < L, x_n > f(x')\}$$

we have

$$0 < C_1 \leq \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, L)}{u_1(0, L)} \leq C_2,$$

with C_1, C_2 *depending only on* L.

Lemma 2.7 [Jerison and Kenig 1982]. *Let* D, u_1 *and* u_2 *be as in* Lemma 2.6. *Assume further that*

$$\frac{u_1(0, L/2)}{u_2(0, L/2)} = 1.$$

Then, $u_1(x', x_n)/u_2(x', x_n)$ *is Hölder continuous in* $\overline{D}_{1/2}$ *for some coefficient* α, *both* α *and the* C^α *norm of* u_1/u_2 *depending only on* L.

Lemma 2.8 [Caffarelli 1987]. *Let* u *be as in* Lemma 2.6. *Then there exists* $c > 0$ *depending only on* L *such that for* $0 < d < c$, $\frac{\partial}{\partial x_n} u(0, d) \geq 0$ *and*

$$C_1 \frac{u(0, d)}{d} \leq \frac{\partial u}{\partial x_n}(0, d) \leq C_2 \frac{u(0, d)}{d},$$

where $C_i = C_i(M)$.

Lemma 2.9 [Jerison and Kenig 1982, Lemma 4.1]. *Let* Ω *be Lipschitz domain contained in* $B_{10}(0)$. *There exists a dimensional constant* $\beta_n > 0$ *such that for any* $\zeta \in \partial\Omega$, $0 < 2r < 1$ *and positive harmonic function* u *in* $\Omega \cap B_{2r}(\zeta)$, *if* u *vanishes continuously on* $B_{2r}(\zeta) \cap \partial\Omega$, *then for* $x \in \Omega \cap B_r(\zeta)$,

$$u(x) \leq C\left(\frac{|x - \zeta|}{r}\right)^{\beta_n} \sup\{u(y) : y \in \partial B_{2r}(\zeta) \cap \Omega\},$$

where C *depends only on the Lipschitz constants of* Ω.

Next, we point out that we use the notion of viscosity solutions for our investigation. When $\{u_0 = 0\}$ is of zero Lebesgue measure, it was proved in [Kim and Požár 2011] that the viscosity solution of (ST2) is unique and coincides with the usual weak solutions. (See [Kim and Požár 2011] for the definition as well as other properties of viscosity solutions.) Below we state important properties of viscosity solutions for (ST2) that relate our solutions to the one-phase version of our problem,

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ u_t/|Du| = |Du| & \text{on } \partial\{u > 0\}, \\ u(\cdot\,, 0) = u_0 \geq 0. \end{cases} \tag{ST1}$$

Lemma 2.10. *Suppose u is a viscosity solution of (ST2). Then*:

(a) *u is caloric in its positive and negative phases.*

(b) *$-u$ is also a viscosity solution of (ST2) with boundary data $-g$.*

(c) *$u^+ = \max(u, 0)$ (or $u^- = -\min(u, 0)$) is a viscosity subsolution of (ST1) with initial data u_0^+ (or u_0^-).*

We say that a pair of functions $u_0, v_0 : \overline{D} \to [0, \infty)$ are (strictly) separated (denoted by $u_0 \prec v_0$) in $D \subset \mathbb{R}^n$ if:

(i) The support of u_0, $\operatorname{supp}(u_0) = \overline{\{u_0 > 0\}}$, restricted to $\overline{D}$ is compact.

(ii) $u_0(x) < v_0(x)$ in $\operatorname{supp}(u_0) \cap \overline{D}$.

Lemma 2.11 [Kim and Požár 2011, Comparison principle]. *Let u, v be, respectively, viscosity sub- and supersolutions of (ST2) in $D \times (0, T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \overline{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot\,, t) \prec v(\cdot\,, t)$ in D for $t \in [0, T)$.*

Below we state a distance estimate for the free boundary and Harnack inequality for the one-phase solution u of (ST1).

Lemma 2.12 [Choi and Kim 2010, Lemma 2.2]. *Let u be given as in Theorem 2.16. There exists $t_0 = t_0(N_0, M_0, n) > 0$ such that if $x_0 \in \Gamma_0$ and $t \leq t_0$, then*

$$\frac{1}{C} t^{1/(2-\alpha)} \leq d(x_0, t) \leq C t^{1/(2-\beta)}, \tag{2-1}$$

where α and β are given in (1-2), C depends on N_0, M_0 and n, and $d(x_0, t)$ denotes the distance that Γ moved from the point x_0 during the time t, i.e.,

$$d(x_0, t) := \sup\{d : u(x_0 + de_n, t) > 0\}.$$

Lemma 2.13 [Choi and Kim 2010, Lemma 2.3]. *Let u be given as in Theorem 2.16. There exists d_0 depending on N_0, M_0 and n such that if $x_0 \in \Gamma_0$ and $d \leq d_0$, then*

$$u(x_0 - de_n, t) \leq C u(x_0 - de_n, 0) \quad \text{for } 0 \leq t \leq t(x_0, d),$$

where C depends on N_0, M_0 and n.

The following monotonicity formula by Alt–Caffarelli–Friedman prevents the scenario that both phases compete with large pressure in our problem.

Lemma 2.14 [Alt et al. 1984]. *Let h_+ and h_- be nonnegative continuous functions in $B_1(0)$ such that $\Delta h_\pm \geq 0$ and $h_+ \cdot h_- = 0$ in $B_1(0)$. Then the functional*

$$\phi(r) = \frac{1}{r^4}\int_{B_r(0)} \frac{|Dh_+|^2}{|x|^{n-2}}\,dx \int_{B_r(0)} \frac{|Dh_-|^2}{|x|^{n-2}}\,dx$$

is monotone increasing in r, $0 < r < 1$.

Corollary 2.15. *Let $\partial\Omega_0 \subset \mathbb{R}^n$ be star-shaped with respect to the ball $B_1(0) \subset \Omega_0$ and suppose that $B_{4/3}(0) \subset \Omega_0 \subset B_{5/3}(0)$. Let h_+ be the harmonic function in $\Omega_0 - B_1(0)$ with boundary values $h_+ = 0$ on $\partial\Omega_0$, and $h_+ = 1$ on $\partial B_1(0)$. Let h_- be the harmonic function in $B_2(0) - \Omega_0$ with boundary values $h_- = 0$ on $\partial\Omega_0$, and $h_- = 1$ on $\partial B_2(0)$. Then there exists a sufficiently large dimensional constant $M > 0$ such that*

$$\frac{h_+(x_0 - re_n)}{r} \geq M \ \textit{implies}\ \frac{h_-(x_0 + re_n)}{r} \leq 1$$

for $x_0 \in \partial\Omega_0$, $e_n = x/|x|$ and $0 \leq r \leq 1/6$.

Proof. This follows from Lemma 2.14, since

$$\begin{aligned}\Bigl(\frac{h_+(x_0 - re_n)}{r}\cdot\frac{h_-(x_0 + re_n)}{r}\Bigr)^2 &\sim \frac{1}{(2r)^4}\int_{B_{r/2}(x_0 - re_n)} \frac{|Dh_+|^2}{|x - x_0|^{n-2}}\,dx \cdot \int_{B_{r/2}(x_0 + re_n)} \frac{|Dh_-|^2}{|x - x_0|^{n-2}}\,dx \\ &\leq \frac{1}{(2r)^4}\int_{B_{2r}(x_0)} \frac{|Dh_+|^2}{|x - x_0|^{n-2}}\,dx \cdot \int_{B_{2r}(x_0)} \frac{|Dh_-|^2}{|x - x_0|^{n-2}}\,dx \\ &= \phi(2r) \leq \phi(1/3) \leq C_n. \end{aligned}$$

□

Lastly, let us finish this section by stating the results obtained in [Choi and Kim 2010] for the one-phase version of our problem in the local setting:

Theorem 2.16 [Choi and Kim 2010, Theorem 0.1]. *Suppose a nonnegative function $u(x, t)$ is a solution of (ST1) in $B_2(0) \times [0, 1]$, $0 \in \Gamma_0(u)$, with the initial data $u_0 \geq 0$ satisfying (I-b), (I-c) and (I-d) in $B_2(0)$. Suppose the initial data satisfies*

$$\{u(x, 0) \geq 0\} = \{x + x_0 : x_n \leq f(x')\}$$

in $B_1(0)$, where f is a Lipschitz function with Lipschitz constant $L < L_n$. Further, suppose $u_0(-e_n) = 1$ and $\sup_{B_2(0)\times[0,1]} u \leq M_0$.

For given $r > 0$, let us define

$$t(x_0, r) := \frac{r^2}{u(x_0 + re_n, 0)}.$$

Then there exists a small $c_0 > 0$ depending on M_0 and n such that in $\Sigma_r = B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$ for $r \leq c_0$, we have:

(1) *Theorem 1.1(1) holds for u.*

(2) *u is a classical solution of (ST1) in Σ_r in the sense that the spatial derivative Du exists in $\Omega(u)$ and is continuous up to $\overline{\Omega}(u)$, and the free boundary condition is satisfied in the classical sense, i.e.,*

$$V_{x,t} = |Du|(x,t) \quad \text{on } \Gamma(u) \cap \Sigma_r.$$

(3) *There exists a positive constant M depending on M_0 and n such that*

$$M^{-1}\frac{u(x_0 - re_n, 0)}{r} \leq |Du|(x,t) \leq M\frac{u(x_0 - re_n, 0)}{r}.$$

(4) *If $x \in \Gamma_0(u) \cap B_{c_0}(0)$ and $x + re_n \in \Gamma_t(u) \cap B_{c_0}(0)$, then*

$$M^{-1}\frac{u(x - re_n, 0)}{r} \leq |Du(x + re_n, t)| = V_{x+re_n,t} \leq M\frac{u(x - re_n, 0)}{r},$$

where M depends on n and M_0. In particular,

$$\frac{r}{t} \sim |Du(x + re_n, t)| \sim \frac{u(x - re_n, 0)}{r}.$$

Theorem 2.16 states that the free boundary regularizes in a scale proportional to the distance it has traveled. Note that the regularity results hold up to the initial time and all the regularity assumptions are imposed only on the initial data.

3. Properties of solutions with star-shaped initial data

Lemma 3.1. *If Ω_0 and u_0 are star-shaped with respect to the ball $B_{r_0}(0) \subset \Omega_0$, then $\Omega_t(u)$ and $u(\cdot, t)$ stays σ-close to star-shaped for all $0 \leq t \leq \frac{1}{3}\sigma^{1/5}$ (see Figure 2).*

Proof. Step 1. For any $a > 0$, the parabolic scaling $(x,t) \to (ax, a^2 t)$ preserves both the heat operator

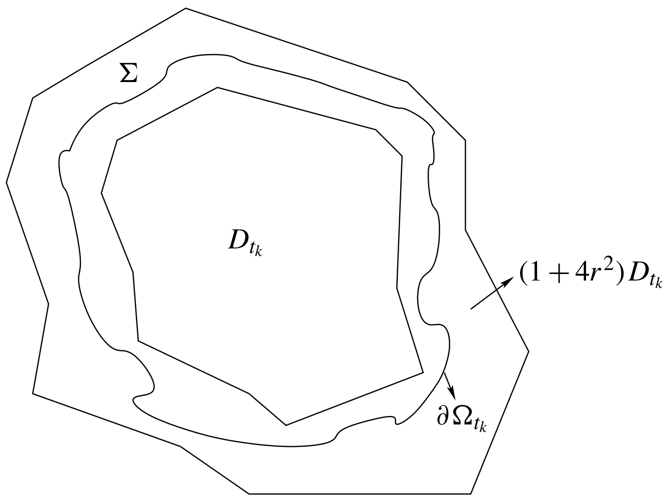

Figure 2. Approximation of the positive phase by a star-shaped domain.

and the boundary motion law in (ST2). Therefore, for any $\sigma > 0$ the function

$$u_1(x,t) := u\big((1+\sigma)(x-x_0)+x_0,\,(1+\sigma)^2 t\big)$$

is also a viscosity solution of (ST2) with corresponding initial data.

Step 2. Choose $x_0 \in B_{r_0}(0)$. Take a small $c_0 > 0$ such that $B_{r_0+c_0}(0) \subset \Omega_0$. We claim that for $0 \le \delta \le \sigma^{6/5}$,

$$u_1(x,0) \le u(x,\delta) \quad \text{in } B_R(0) - B_{r_0+c_0}(0) \tag{3-1}$$

if σ is small enough. To show (3-1), let us introduce another function

$$\tilde{u}(x,0) := u\big((1+\tfrac{1}{2}\sigma)(x-x_0)+x_0,0\big).$$

Also let v^* be the solution of the one phase problem (ST1) with initial data u_0^-, and with $v^* = 1$ on $\partial B_R(0)$. Note also that, due to Lemma 2.10, u^- is a subsolution of (ST1) with initial data $v^*(x,0) = u^-(x,0)$.

Thus by Lemma 2.11, $v^* \le u^-$. It follows that $\Omega_t(v^*) \subset \Omega_t(u) \subset \Omega_t(u)$. Hence by Lemma 2.12 applied to $-v^*$,

$$\Omega_0(\tilde{u}) \subset \Omega_t(u) \quad \text{for } 0 \le t \le \sigma^{7/6}.$$

Moreover, due to our assumption,

$$\tilde{u}(x,0) \le u_0(x).$$

Therefore, the maximum principle for caloric functions implies

$$w(x,t) \le u(x,t),$$

where w solves the heat equation in the cylindrical domain $D = \Omega_0(\tilde{u}) \times [0,\sigma^{7/6}]$ with initial data $\tilde{u}(x,0)$ and zero boundary data on $\partial\Omega_0(\tilde{u}) \times [0,\sigma^{7/6}]$.

Now w_t solves the heat equation in D,

$$w_t = \Delta w \ge -C \text{ at } t = 0 \quad \text{and} \quad w_t = 0 \text{ on } \partial\Omega_0(\tilde{u}).$$

Therefore we conclude that $w_t \ge -C$ in D. In particular

$$w(x,\delta) \ge \tilde{u}(x,0) - C\delta. \tag{3-2}$$

Next we compare $u_1(x,0)$ with $w(x,\delta)$. Observe that for $x \in B_R(0) - B_{r_0+c_0}(0)$,

$$\begin{aligned} u_1(x,0) = \tilde{u}(x,0) + \int_{\sigma/2}^{\sigma} \big((x-x_0)\cdot Du((1+s)(x-x_0)+x_0,0)\big)\,ds &\le \tilde{u}(x,0) - c_0\sigma^{7/6} \\ &\le \tilde{u}(x,0) - C\sigma^{6/5} \\ &\le w(x,\delta) \le u(x,\delta) \end{aligned}$$

for $0 \le \delta \le \sigma^{6/5}$, where the first inequality follows from our assumption (I-d) on u_0, the second inequality follows if σ is sufficiently small, and the third inequality follows from (3-2). Hence we conclude (3-1).

Step 3. Our goal is to prove that for $0 \le \delta \le \sigma^{6/5}$,

$$u_1(x,t) \le u_2(x,t) := u(x,t+\delta) \tag{3-3}$$

in $(B_R(0) - B_{r_0+c_0}(0)) \times [0, \sigma^{1/5}]$. Note that the inequality holds at $t = 0$ by Step 2. However, we need a few more arguments since we do not know yet if the lateral boundary data on $\partial B_{r_0+c_0}(0)$ is properly ordered.

Suppose

$$\Omega(u_1) \subset \Omega(u) \quad \text{for } 0 \le t \le t_0$$

and $\Omega(u_1)$ contacts $\partial\Omega(u)$ for the first time at $t = t_0$. Observe then that

$$f(x, t) := u(x, t + \delta) - u_1(x, t)$$

solves the heat equation in $\Omega(u_1)$ with nonnegative boundary data for $0 \le t \le t_0$, with

$$f(x, 0) \ge 0 \quad \text{in } B_R(0) - B_{r_0+c_0}(0).$$

Indeed, following the computation given above, it follows that

$$f(x, 0) \ge c_0\sigma \quad \text{in } B_{r_0+c_0}(0) - B_{r_0+c_0/2}(0).$$

On the other hand, due to the fact that $w_t \ge -C$ and $\delta \le \sigma^{6/5}$, we have

$$f(x, 0) \ge (w(x, \delta) - w(x, 0)) + (w(x, 0) - u_1(x, 0)) \ge -C\sigma^{6/5} \quad \text{in } B_{r_0+c_0/2}(0).$$

Therefore we have

$$f(x, t) > 0 \quad \text{on } \partial B_{r_0+c_0}(0) \times [0, t_0]$$

if $t_0 \ll 1$. But then this contradicts Lemma 2.11 applied to the region $(B_R(0) - B_{r_0+c_0}(0)) \times [0, t_0]$.

Step 4. From (3-3) of Step 3, we obtain

$$u\big((1+\sigma)(x - x_0) + x_0, (1+\sigma)^2 t\big) \le u(x, t + \delta) \tag{3-4}$$

in $(B_R(0) - B_{r_0+c_0}(0)) \times [0, \sigma^{1/5}]$ for any $x_0 \in B_{r_0}(0)$, as long as σ and δ are sufficiently small and satisfy $0 \le \delta \le \sigma^{6/5}$. As a result, for $0 \le t \le \frac{1}{3}\sigma^{1/5}$, we can choose $\delta = \sigma(2+\sigma)t \le \sigma^{6/5}$ such that

$$(1+\sigma)^2 t = t + \delta.$$

It follows then from (3-4) that the function $u(\cdot, t)$ is σ-monotone with respect to the cone of directions W_x in $(B_R(0) - B_{r_0+c_0}(0))$ for $t \in [0, \frac{1}{3}\sigma^{1/5}]$. (Here $W_x = \{\nu \in S^n : \nu = (x - x_0)/|x - x_0| \text{ for some } x_0 \in B_{r_0}(0)\}$.) □

Remark 3.2. For $x \in \Gamma_0$, we may let $e_n = x/|x|$ after a rotation. Then, due to (I-b),

$$t(x, d) := \min\left\{\frac{r^2}{u^+(x - re_n, 0)}, \frac{r^2}{u^-(x + re_n, 0)}\right\} \in [r^{7/6}, r^{5/6}] \ll r^{4/5}, \tag{3-5}$$

where $t(x, r)$ is the time it takes for the free boundary to regularize in $B_r(0)$. Therefore, we have, for $0 \le t \le t(x_0, r)$,

$$u(\cdot, t) \text{ is } r^4\text{-monotone with respect to } W_x \text{ in } (B_R(0) - B_{r_0+c_0}(0)).$$

This property will ensure that our solution u has its level sets close to Lipschitz graphs in the space variable in an appropriate scale, which serves as the first step towards the regularization argument; see Lemma 3.6.

Lemma 3.3 (Harnack at $t=0$). *Let u be as in Theorem 1.1. For $x \in \Gamma_0$, we may set $e_n = x/|x|$ after a rotation. Then for all $s > 0$ and for $0 \le t \le t(x,s)$ we have*

$$u^+(x - se_n, t) \le C_1 u^+(x - se_n, 0) \quad \text{and} \quad u^-(x + se_n, t) \le C_1 u^-(x + se_n, 0),$$

where $e_n = x/|x|$.

Proof. Let v^{**} solve the one-phase Stefan problem (ST1) with initial data $v_0^{**}(x) = u_0^+(x)$. Then v^{**} is also a solution of (ST2) with $u_0(x) \le v_0^{**}(x)$, and thus by Lemma 2.11 we have

$$u(x,t) \le v^{**}(x,t).$$

Therefore it follows from one-phase Harnack inequality applied for $v^{**}(x,t)$ that

$$u^+(x - se_n, t) \le v^{**}(x - se_n, t) \le C_1 v^{**}(x - se_n, 0) = C_1 u(x - se_n, 0)$$

for $0 \le t \le t_0$, where $t_0 = s^2/u(x - se_n, 0) \ge t(x,s)$.

As for $u^-(x,t)$, we compare u^- with the solution v^* of (ST1) with initial data $v_0^*(x) = u_0^-(x)$ and with boundary data $v^* = 1$ on $\partial B_R(0)$. The rest of the argument is parallel to the above one. □

Lemma 3.4 (backward Harnack at $t=0$). *Let u be as in Theorem 1.1. Let $x \in \Gamma_0$ and let $e_n = x/|x|$ after a rotation. Then for $s > 0$ and for $0 \le t \le t(x,s)$,*

$$u^+(x - se_n, 0) \le C_1 u^+(x - se_n, t) \quad \text{and} \quad u^-(x + se_n, 0) \le C_1 u^-(x + se_n, t).$$

Proof. We will only show the lemma for u^+. The other part follows by a parallel argument. Let v^* solve the one phase problem (ST1) with initial data u_0^- and with boundary data 1 on $\partial B_R(0)$. Then $-v^*$ is also a solution of (ST2) with $-v_0^* \le u_0$, and thus by Lemma 2.11, $-v^* \le u$. This inequality implies that

$$\{v^* = 0\} \subset \{u \ge 0\}.$$

Note that $\Omega(v^*)$ moves according to the one-phase dynamics, which have been studied in detail in [Choi and Kim 2006]. In particular we know that $\Omega(v^*)$ will be Lipschitz at each time. Moreover, for a boundary point $(x,t) \in \Gamma(v^*)$ and $d := \text{dist}(x, \Gamma_0(v^*))$, the normal velocity $V_{x,t}$ satisfies

$$V_{x,t} = |Dv^*(x,t)| \sim \frac{v^*(x + 2de_n, 0)}{2d} \le d^{\beta-1} \le t^{(\beta-1)/(2-\alpha)}, \tag{3-6}$$

where the last inequality follows from Lemma 2.12. Let $v_*(x,t)$ solve the heat equation in $\{v^* = 0\}$ with initial data $u_0(x)$ and boundary data 0 on the lateral boundary of $\partial\{v^* = 0\}$, i.e., v_* solves

$$\begin{cases} \partial_t v_* - \Delta v_* = 0 & \text{in } \{v^* = 0\} = B_R(0) \times [0,1] - \Omega(v^*), \\ v_*(x,0) = u_0(x) & \text{on } \{v^* = 0\} \cap \{t = 0\}, \\ v_* = 0 & \text{on } \partial\{v^* = 0\} \cap \{t > 0\}. \end{cases}$$

Since

$$\Omega(v_*) = \{v^* = 0\} \subset \{u \ge 0\},$$

we have $v_*(x,t) \le u(x,t)$ in $\{v^* = 0\}$. Moreover, for any given $t > 0$, $\tilde{v}^-(x,s) := v^*(\sqrt{t}x, ts)$ satisfies the assumptions of Lemma 2.5. Thus it follows that $v^*(\cdot, t)$ is t^a-close to a harmonic function in $B_{\sqrt{t}}(x)$ for some $a > 0$, where $x \in \Gamma_0$. Moreover, due to the assumption on the initial data, $(v_*)_t = \Delta v_* \ge -C$ at $t = 0$. Also on $\Gamma(v_*)$,

$$(v_*)_t/|Dv_*| = -(v^*)_t/|Dv^*| = -|Dv^*| \ge -t^{(\beta-1)/(2-\alpha)}.$$

Here the first equality follows since $(v_*)_t/|Dv_*|$ and $-(v^*)_t/|Dv^*|$ are the normal velocities of their respective level sets $\Gamma(v_*)$ and $\Gamma(v^*)$, but $\Gamma(v^*) = \Gamma(v_*)$ by definition. The second equality follows since v^* solves the one phase problem (ST1), and the last inequality follows from (3-6).

Since $\Omega(v_*)$ is Lipschitz and $\Gamma_t(v_*) = \Gamma_t(v^*)$ is regularized in space over time (see Theorem 2.16), (3-6) also holds for $|Dv_*|$.

Hence on $\Gamma(v_*)$,

$$(v_*)_t = -|Dv^*||Dv_*| \ge -t^{2(\beta-1)/(2-\alpha)} > -t^{-2/5},$$

where α and β are the growth rates defined in (1-2), and the last inequality follows from the assumption (I-b). Since $(v_*)_t$ solves a heat equation in $\Omega(v_*)$, it follows that for $x \in \Gamma_0$,

$$(v_*)_t \ge -t^{-2/5} \quad \text{in } B_{\sqrt{t}/2}(x - \sqrt{t}e_n) \times [0, t]. \tag{3-7}$$

Then since $v_*(x - \sqrt{t}e_n, 0) \ge (\sqrt{t})^\alpha \ge (\sqrt{t})^{7/6} = t^{7/12}$, for $x \in \Gamma_0$ we have

$$\begin{aligned} v_*(x - \sqrt{t}e_n, t) = v_*(x - \sqrt{t}e_n, 0) + \int_0^t (v_*)_t(x - \sqrt{s}e_n, s)\, ds &\ge v_*(x - \sqrt{t}e_n, 0) - \tfrac{5}{3}t^{3/5} \\ &\ge \tfrac{1}{2}v_*(x - \sqrt{t}e_n, 0) + \tfrac{1}{2}t^{7/12} - \tfrac{5}{3}t^{3/5} \\ &\ge \tfrac{1}{2}v_*(x - \sqrt{t}e_n, 0) \end{aligned}$$

if t is sufficiently small. It follows that

$$u^+(x - \sqrt{t}e_n, 0) = v_*(x - \sqrt{t}e_n, 0) \le 2v_*(x - \sqrt{t}e_n, t) \le 2u^+(x - \sqrt{t}e_n, t),$$

where the first inequality follows from (3-7).

Since $\Gamma(v_*) = \Gamma(v^*)$ is Lipschitz in a parabolic scaling, v_* is almost harmonic. Hence $v_*(\cdot, t)$ is bigger than the harmonic function $\omega^t(x)$ in $\Omega_t(v_*) \cap B_{\sqrt{t}}(x)$ with its value

$$\omega^t(x - \sqrt{t}e_n) = (C_1)^{-1}u^+(x - \sqrt{t}e_n, 0).$$

Note that if $0 \le t \le t(x,s)$, then $s < \sqrt{t}$. Hence for $0 \le t \le t(x,s)$,

$$C_1u^+(x - se_n, t) \ge C_1v_*(x - se_n, t) \ge C_1\omega^t(x - se_n) \ge Cu^+(x - se_n, 0),$$

where the last inequality follows since the one-phase result implies a power law on the movement of $\Gamma(v^*) = \Gamma(v_*)$ (see Lemma 2.5 of [Choi et al. 2007]), and this yields a bound on $u^+(x - se_n, 0)/\omega^t(x - se_n)$.

Similar arguments apply to u^-, if we consider the function v^{**} solving (ST1) with initial data u_0^+, and the function $v^\star$ solving the heat equation in $\{v^{**}=0\}$ with initial data u_0 and with boundary data 0 on $\Gamma(v^{**})$ and -1 on $\partial B_R(0)$. $\square$

Lemma 3.5 (distance estimate at $t=0$). *Let u be as in Theorem 1.1. Let $x \in \Gamma_0$ and let $e_n = x/|x|$ after a rotation. Let s be a sufficiently small positive constant. If*

$$\frac{|u^+(x-se_n,0)|}{s} \le m \quad \textit{and} \quad \frac{|u^-(x+se_n,0)|}{s} \le m,$$

then for $t \in [0, s/m]$,

$$d(x,t) = \sup\{r : x+re_n \text{ or } x-re_n \in \Gamma_t(u)\} \le s.$$

Proof. Let v^{**} solve (ST1) with initial data u_0^+, and let v^* solve (ST1) with initial data u_0^- and with $v^*=1$ on $\partial B_R(0)$. Then by comparison, $-v^* \le u \le v^{**}$ and the lemma follows from the one-phase result of Theorem 2.16. $\square$

In the next lemma, we approximate our solution by harmonic functions.

Note that, due to Lemma 3.1, We know that the rescaled function $\tilde{u}(x,t)$ as given in (1-4) satisfies the condition (B$'$) in the space variable. On the other hand, it is not clear if the level sets of u are close to Lipschitz graphs in the time variable. The approximation by harmonic functions given by Lemma 3.6, as well as Harnack-type inequalities obtained at $t=0$ and at future times, will ensure us that $\Gamma(u)$ is almost Lipschitz in the time variable as well (Corollary 4.4). This fact will serve as the first step towards the regularization procedure in Section 5.

Lemma 3.6 (spatial regularity in the whole domain). *Let u be as in Theorem 1.1. Then there exists a positive constant r_0 depending only on n such that for $x_0 \in \Gamma_0$ and $0 < r < r_0$, there exists a function $\omega(x,t) := \omega^+(x,t) - \omega^-(x,t)$ that satisfies:*

(a) *$\omega(\cdot\,,t)$ is harmonic in its positive and negative phase in $(1+r)\Omega_t(u) - (1-r)\Omega_t(u)$, and $\Omega(\omega^+)$, $\Omega(\omega^-)$ are star-shaped with respect to $B_{r_0}(0)$ given in* (I-a).

(b) *For a dimensional constant $C > 0$, we have*

$$\omega^+(x,t) \le u^+(x,t) \le C\omega^+((1-r^{5/4})x,t) \quad \textit{and} \quad \omega^-(x,t) \le u^-(x,t) \le C\omega^-((1+r^{5/4})x,t)$$

in $B_r(x_0) \times [r^2, t(x_0,r)]$.

Note that $t(x_0,r) \ge r^{7/6} \ge r^2$, and $\partial\{\omega^+ > 0\}$ need not be $\partial\{\omega^- > 0\}$.

Proof. Step 1. We will only show the lemma for u^+. For a given $x_0 \in \Gamma_0$, we may assume that $e_n = x_0/|x_0|$ after a rotation.

First we will construct a barrier function v_1 which will serve as a supersolution of (ST2). For this, let us first consider the viscosity solution $u^\star$ of (ST1) with the initial data u_0^+ for $0 \le t \le t_0$. We may assume that for t_0 small compared to R the support of $u^\star$ stays inside $B_R(0)$. Let us define

$$\Omega_+^\star := \{u^\star > 0\}, \quad \Gamma^\star := \partial\{u^\star = 0\}, \quad \Omega_-^\star := B_R(0) \times [0,t_0] - \overline{\Omega}_+^\star.$$

Now let v_1 solve the heat equation in $\Omega^\star_+$ and in $\Omega^\star_-$, with initial data u_0 and with $v_1 = -1$ on $\partial B_R(0)$. In other words, $v_1 = v_1^+ - v_1^-$, where

$$\begin{cases} \partial_t v_1^+ - \Delta v_1^+ = 0 & \text{in } \Omega^\star_+, \\ v_1^+(x,0) = u_0^+(x) & \text{on } \{t=0\}, \\ v_1^+ = 0 & \text{on } \Gamma^\star, \end{cases}$$

and

$$\begin{cases} \partial_t v_1^- - \Delta v_1^- = 0 & \text{in } \Omega^\star_-, \\ v_1^-(x,0) = u_0^-(x) & \text{on } \{t=0\}, \\ v_1^- = 0 & \text{on } \Gamma^\star, \\ v_1^- = 1 & \text{on } \partial B_R(0) \times [0,1]. \end{cases}$$

Note that v_1 solves the heat equation in two regions $\Omega^\star_+$ and $\Omega^\star_-$, with free boundary $\Gamma^\star$. Also note that $v_1^+ = u^\star$ and $\partial_t v_1^+ = |Dv_1^+|^2$ on $\Gamma^\star$ since the boundary $\Gamma^\star$ is obtained from the one phase problem with initial data u_0^+. Hence we can observe that v_1 is a supersolution of the two-phase problem (ST2).

Similarly one can construct a subsolution of (ST2): let us consider $\tilde{u}^\star$: the viscosity solution of (ST1) in $B_R(0) \times [0, t_0]$ with the initial data u_0^- and fixed boundary data 1 on $\partial B_R(0) \times [0, t_0]$. Let us define

$$\widetilde{\Omega}^\star_- := \{\tilde{u}^\star > 0\}, \quad \widetilde{\Gamma}^\star := \partial\{\tilde{u}^\star = 0\}, \quad \widetilde{\Omega}^\star_+ := B_R(0) \times [0, t_0] - \overline{\widetilde{\Omega}^\star_-}.$$

Now let v_2 solve the heat equation in two regions $\widetilde{\Omega}^\star_-$ and $\widetilde{\Omega}^\star_+$, with boundary data 0 on $\widetilde{\Gamma}^\star$ and -1 on $\partial B_R(0)$, and with initial data u_0. Note that $v_2^- = \tilde{u}^\star$. Then v_2 is a subsolution of (ST2), and by comparison,

$$v_2 \le u \le v_1. \tag{3-8}$$

Hence the free boundary of u is trapped between the free boundaries of v_1 and v_2. Note that the free boundaries $\Gamma^\star$ and $\widetilde{\Gamma}^\star$ of v_1 and v_2 are obtained from the one-phase problem (ST1). Hence by Theorem 2.16(a), $\Gamma^\star$ and $\widetilde{\Gamma}^\star$ are Lipschitz in space in $B_d(x_0)$ for a small constant $d > 0$. Also, Theorem 2.16(c) implies that for $\delta \in [d/2, d]$ and $x_0 + \delta e_n \in \Gamma^\star_t$, the normal velocity $V_{x_0+\delta e_n, t}$ of $\Gamma^\star$ at $(x_0 + \delta e_n, t)$ satisfies

$$V_{x_0+\delta e_n,t} = |Dv_1^+(x_0 + \delta e_n, t)| \sim \frac{d}{t} \sim \frac{u_0^+(x_0 - de_n)}{d} \le d^{\beta-1}.$$

Since $d/t \le d^{\beta-1}$, we obtain

$$t \ge d^{2-\beta} > d^2.$$

Hence the above speed bound of $\Gamma^\star$ implies that $\Omega^\star_+$ and $\Omega^\star_-$ are Lipschitz in space and time, in parabolic scaling. Then by Lemma 2.5, v_1^+ and v_1^- are almost harmonic up to a d-neighborhood of their free boundaries for $t \ge d^2$. Similarly, we obtain that v_2^+ and v_2^- are almost harmonic up to a d-neighborhood of their free boundaries for $t \ge d^2$.

Next we fix $r \le d$. Note that if $t \le t(x_0, r)$, then by Theorem 2.16(c), both of the sets $\Gamma_t(v_1)$ and $\Gamma_t(v_2)$ are within distance r of $\Gamma_0(u)$ in $B_r(x_0)$ during this time. In particular, arguments parallel to the ones in the proofs of Lemmas 2.1 and 2.3 in [Choi and Kim 2010] yield that

$$\sup\{u(y,s) : (y,s) \in B_d(x_0) \times [0, d^2]\} \sim u(x - de_n, 0).$$

Now using the almost harmonicity of v_1^+ and v_2^+, we conclude that for $0 \le t \le t(x_0, r)$,

$$v_2(x_0 - 2re_n, t) \sim u_0(x_0 - 2re_n, 0) \sim v_1(x_0 - 2re_n, t). \tag{3-9}$$

Step 2. Observe that by the definition of $t(x_0, r)$ and the assumption on the growth rates of u_0,

$$r^{2-\beta} \le t(x_0, r) \le r^{2-\alpha} \le r^{5/6} := \tau. \tag{3-10}$$

Due to Lemma 3.1, we know that at each time, $\Omega_t(u)$ is τ^5-close to a star-shaped domain D_t up to the time $t = \tau$, i.e.,

$$D_t \subset \Omega_t(u) \subset (1+\tau^5)D_t \subset (1+r^4)D_t \tag{3-11}$$

for $0 \le t \le \tau$.

Also note that by the first inequality of (3-10) with $\beta \ge 5/6$,

$$t(z, r^{13/20}) \ge r^{13(2-\beta)/20} > \tau \quad \text{for any } z \in \Gamma_0.$$

Hence we can apply Lemma 3.3 for $s = r^{13/20}$ up to the time τ. Then by Lemma 3.3 and (3-11) with $\beta \ge 5/6$,

$$u(x,t) \le r^{(13/20)(5/6)} = r^{13/24}$$

for $x \in \partial(1 - r^{13/20})D_0$ and for $0 \le t \le \tau$. Then by the τ^5-monotonicity of u,

$$u(x,t) \le r^{13/24} \quad \text{on } B_R(0) - (1 - r^{13/20} + r^4)D_0 \tag{3-12}$$

for $0 \le t \le \tau$. Since $\Gamma_t(u)$ is located between the free boundaries $\Gamma^\star$ and $\widetilde{\Gamma}^\star$ of one-phase problem, Lemma 2.12 with $\beta \ge 5/6$ implies that $\Gamma(u)$ stays in the $\tau^{6/7}$-neighborhood of $\Gamma_0(u)$ up to τ. Also (3-11) implies that ∂D_t stays in the τ^5-neighborhood of $\Gamma_t(u)$ up to τ. Hence we obtain that ∂D_t stays in the $\tau^{5/6}$-neighborhood of ∂D_0 up to the time τ. Since $\tau^{5/6} = r^{25/36} < r^{13/20}$, (3-12) implies

$$u(x,t) \le r^{13/24} \quad \text{on } B_R(0) - D_s \tag{3-13}$$

for any $0 \le s, t \le \tau$.

Step 3. Let

$$t_0 = 0 \le t_1 = r^2 \le t_2 = 2r^2 \le \cdots \le t_{k_0} = k_0 r^2 \le \tau$$

and fix a number b such that

$$5/4 \le b < 61/48.$$

We will construct a supersolution of (ST2) in

$$(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}].$$

Let $w^k(x)$ be the harmonic function in

$$\Sigma := (1+4r^b)D_{t_k} - D_{t_k},$$

with boundary data zero on $\partial(1+4r^b)D_{t_k}$ and $C_n r^{13/24}$ on ∂D_{t_k}, where C_n is a sufficiently large dimensional constant. Extend $w^k(x)$ by 0 to $\mathbb{R}^n - \Sigma$. Next define

$$\Phi^k(x,t) := \inf\{w^k(y) : |x-y| \le r^b - (t-t_k)\tfrac{1}{2}r^{b-2}\}$$

in $(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}]$. We claim that the function Φ^k is a supersolution of (ST2) in $(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}]$, since our constant b satisfies

$$r^{b-2} > r^{13/24-b}. \tag{3-14}$$

For simplicity, write $\Phi = \Phi^k$. To check that Φ is a supersolution, first note that $\Phi(\,\cdot\,,t)$ is superharmonic in its positive set and $\Phi_t \ge 0$. Hence we only need to show that

$$\frac{\Phi_t}{|D\Phi|} \ge |D\Phi| \quad \text{on } \Gamma(\Phi). \tag{3-15}$$

Due to the definition of Φ, $\Gamma_t(\Phi)$ has an interior ball of radius at least $r^b/2$ for $t_k \le t \le t_{k+1}$. This and the superharmonicity of Φ in the positive set yield that

$$|D\Phi| \le \frac{Cr^{13/24}}{r^b} \quad \text{on } \Gamma(\Phi)$$

for a dimensional constant $C > 0$. Moreover $\Gamma(\Phi)$ evolves with normal velocity $\frac{1}{2}r^{b-2}$. Since (3-14) holds for our choice of b (i.e., for $5/4 \le b < 61/48$), we conclude that (3-15) holds for r smaller than a dimensional constant $r(n)$. Now we compare u with Φ on

$$(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}].$$

Note that by (3-13),

$$u^+ \le \Phi \quad \text{on } \partial(1+r^b)D_{t_k}$$

if C_n is chosen sufficiently large. Also at $t = t_k$, (3-11) implies

$$u(\,\cdot\,,t_k) \le 0 \le \Phi(\,\cdot\,,t_k) \quad \text{on } B_R(0) - (1+r^b)D_{t_k}.$$

Hence we get $u \le \Phi$ in $(\mathbb{R}^n - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}]$. This implies

$$\Omega(u) \subset \Omega(\Phi) \cup ((1+r^b)D_{t_k} \times [t_k, t_{k+1}]) := \widetilde{\Omega}(\Phi) \tag{3-16}$$

for $t_k \le t \le t_{k+1}$.

Step 4. Next we let $v(x,t)$ solve the heat equation in

$$\widetilde{\Omega}(\Phi) - ((1-3r)\Omega_0(u) \times [t_k, t_{k+1}])$$

with initial data $v(\cdot, t_k) = u(\cdot, t_k)$ and boundary data zero on $\Gamma(\Phi)$ and $v = u$ on $(1-3r)\Gamma_0(u)$. Observe that, due to (3-16), we have

$$u^+ \le v \quad \text{for } t_k \le t \le t_{k+1}. \tag{3-17}$$

Since $\widetilde{\Omega}(\Phi)$ is star-shaped and expands with its normal velocity $< r^{b-2}$, which is less than r^{-1}, Lemma 2.5 applies to $\tilde{v}(x,t) := v(rx, r^2 t)$. In particular there exists a constant $C > 0$ such that

$$(1/C)v(x,t) \le h_1(x,t) \le Cv(x,t)$$

for $(t_k + t_{k+1})/2 \le t \le t_{k+1}$, where $h_1(\cdot, t)$ is the harmonic function in $\Omega_t(v) - (1-2r)\Omega_0(u)$ with boundary data zero on $\Gamma_t(v)$ and v on $(1-2r)\Gamma_0(u)$.

Hence we conclude that

$$u^+ \le v \le Ch_1$$

in $(B_R(0) - (1-2r)\Omega_0(u)) \times [(t_k + t_{k+1})/2, t_{k+1}]$.

Step 5. Similar arguments, now pushing the boundary purely by the minus phase given by the harmonic function, yield that

$$\Pi_t := \{x \in D_{t_k} : \operatorname{dist}(x, \partial D_{t_k}) \ge 3r^b + \tfrac{1}{2} r^{b-2}(t - t_k)\} \subset \Omega_t(u)$$

for $t_k \le t \le t_{k+1}$. Let $w(x,t)$ solve the heat equation in

$$\Pi - ((1-3r)\Omega_0(u) \times [t_k, t_{k+1}])$$

with initial data $u(\cdot, t_k)$ and boundary data zero on $\partial\Pi$, and u on $(1-3r)\Gamma_0(u)$. Then $u \ge w(x,t)$. Since Π is star-shaped and it shrinks with its normal velocity $< r^{b-2}$, which is less than r^{-1}, Lemma 2.5 applies to $\tilde{w}(x,t) := w(rx, r^2 t)$. In particular there exists $C > 0$ such that

$$u^+ \ge w \ge (1/C)h_2$$

for $(t_k + t_{k+1})/2 \le t \le t_{k+1}$, where $h_2(\cdot, t)$ is the harmonic function in $\Pi_t - (1-2r)\Omega_0(u)$ with boundary data coinciding with that of w.

Step 6. Lastly we will show that h_1 and h_2 are not too far apart, i.e.,

$$h_1(x,t) \le Ch_2(x - 8r^b e_n, t), \tag{3-18}$$

with a dimensional constant $C > 0$. Since u is between $(1/C)h_2$ and Ch_1, this will conclude our proof for $(t_k + t_{k+1})/2 \le t \le t_{k+1}$. Then by changing the time intervals $[t_k, t_{k+1}]$ to $[t_k + r^2/2, t_{k+1} + r^2/2]$, we obtain the lemma for any $t \in [r^2, t(x_0, r)]$.

To prove (3-18), observe that by the construction of v and w,

$$\Omega_t(w) \subset \Omega_t(v) \subset (1 + 8r^b)\Omega_t(w).$$

Since $t_{k+1} - t_k = r^2$, Lemma 2.12 implies

$$\sup\{d(x, \Gamma_t(u)) : x \in \Gamma_{t_k}(u)\} \le r^{12/7}$$

for $t \in [t_k, t_{k+1}]$. Then by (3-11),

$$\sup\{d(x, \Gamma_t(u)) : x \in \partial D_{t_k}\} \le r^{12/7} + r^4 \ll r^b \tag{3-19}$$

for $t \in [t_k, t_{k+1}]$. Then we obtain

$$v_2(x,t) \le v(x,t) \le v_1\big((1-4r^b)x, (1-4r^b)^2(t-t_k)+t_k\big) \tag{3-20}$$

for $t_k \le t \le t_{k+1}$, where the first inequality follows from (3-8) and (3-17), and the second inequality follows from the comparison principle along with (3-8), $v(\cdot, t_k) = u(\cdot, t_k)$ and (3-19). Similarly,

$$v_2\big((1+4r^b)x, (1+4r^b)^2(t-t_k)+t_k\big) \le w(x,t) \le v_1(x,t). \tag{3-21}$$

Combing (3-20) and (3-21), we get

$$\begin{aligned} v_2\big((1+4r^b)x, (1+4r^b)^2(t-t_k)+t_k\big) &\le w(x,t), v(x,t) \\ &\le v_1\big((1-4r^b)x, (1-4r^b)^2(t-t_k)+t_k\big). \end{aligned}$$

This and (3-9) yield

$$v(x_0 - 2re_n, t) \sim w(x_0 - 2re_n, t) \sim u(x_0 - 2re_n, 0).$$

It follows that

$$w(x,t) \le v(x,t) \le Cw(x - 8r^b e_n, t) \quad \text{on } (1-2r)\Gamma_0 \times [t_k, t_{k+1}].$$

Hence due to Dahlberg's lemma, we conclude that

$$h_1(x,t) \le C_1 v(x,t) \le C_2 w(x - 8r^b e_n, t) \le C_3 h_2(x - 8r^b e_n, t)$$

in $B_r(x_0) \times [(t_k + t_{k+1})/2, t_{k+1}]$. Since the inequality holds for any $5/4 \le b < 61/48$, we obtain the lemma. □

Next we show that in the "unbalanced" region, where one phase has much larger flux than the other, the regularization process occurs similarly to the one in the one-phase problem. This observation will be useful for the analysis in Section 4.

Proposition 3.7 (regularization in unbalanced region I). *Let u be as given in Theorem 1.1. For a fixed $x_0 \in \Gamma_0(u)$, we may let $e_n = x_0/|x_0|$ after a rotation. Suppose that either*

$$u^+(x_0 - re_n, 0) \ge Mu^-(x_0 + re_n, 0) \quad \textit{or} \quad u^-(x_0 + re_n, 0) \ge Mu^+(x_0 - re_n, 0)$$

for $M > M_n$, where M_n is a sufficiently large dimensional constant. Then, for $r \le 1/M_n$, there exists a dimensional constant $C > 0$ such that

$$|Du^+(x,t)| \le C\frac{u^+(x_0 - re_n, 0)}{r} \quad \textit{and} \quad |Du^-(x,t)| \le C\frac{u^-(x_0 + re_n, 0)}{r}$$

in $B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$.

Remark 3.8. 1. In the next section, we will extend Proposition 3.7 to later times, i.e., to $x_0 \in \Gamma_{t_0}$ (see Lemma 4.7).

2. The situation given in Proposition 3.7 is essentially a perturbation of the one-phase case in [Choi and Kim 2010]. The main step in the proof is the verification of this observation; by barrier arguments we will show that our solution is very close to a rescaled version of the one-phase solution, for which the regularity of solutions is well-understood (see Theorem 2.16).

Proof of Proposition 3.7. Without loss of generality, we may assume that

$$u^+(x_0 - re_n, 0) \geq Mu^-(x_0 + re_n, 0).$$

Step 1. First we will show that after a small amount of time u becomes almost harmonic near the free boundary. Lemmas 3.3 and 3.4 imply that for $0 \leq t \leq t(x_0, r)$,

$$u^+(x_0 - re_n, t) \sim u^+(x_0 - re_n, 0), \quad u^-(x_0 + re_n, t) \sim u^-(x_0 + re_n, 0). \tag{3-22}$$

Also note that, by the assumption on the initial data u_0, Lemma 3.6 holds at $t = 0$. In other words, there exists a function $\omega(x, 0) = \omega_0(x)$ such that:

(a) ω_0 is harmonic in its positive and negative phases in $(1+r)\Omega_0(u) - (1-r)\Omega_0(u)$.

(b) $\Omega(\omega_0^+)$ and $\Omega(\omega_0^-)$ are star-shaped.

(c) In $B_r(x_0)$, we have

$$\omega_0^+(x) \leq u_0^+(x) \leq C\omega_0^+((1 - r^{5/4})x), \tag{3-23}$$

$$\omega_0^-(x) \leq u_0^-(x) \leq C\omega_0^-((1 + r^{5/4})x). \tag{3-24}$$

Next we will improve (3-23) and (3-24) for later times to obtain the inequalities with $C = (1 + r^a)$ for $t \geq r^{3/2}$. By the distance estimate in Lemma 2.12, the free boundary of u moves less that $r^{9/7} < r^{5/4}$ during the time $t = r^{3/2}$. Then we let v_1 solve

$$\begin{cases} \partial_t v_1 = \Delta v_1 & \text{in } (1 + 2r^{5/4})\Omega_0(\omega^+) \times [0, r^{3/2}], \\ \partial_t v_1 = \Delta v_1 & \text{in } (B_R(0) - (1 + 2r^{5/4})\overline{\Omega}_0(\omega^+)) \times [0, r^{3/2}], \\ v_1(\,\cdot\,, 0) = u_0^+ & \text{on } (1 + 2r^{5/4})\Omega_0(\omega^+), \\ v_1(\,\cdot\,, 0) = -u_0^- & \text{on } B_R(0) - (1 + 2r^{5/4})\overline{\Omega}_0(\omega^+), \\ v_1 = 0 & \text{on } (1 + 2r^{5/4})\Gamma_0(\omega^+) \times [0, r^{3/2}], \\ v_1 = -1 & \text{on } \partial B_R(0) \times [0, r^{3/2}]. \end{cases}$$

Similarly, we let v_2 solve the heat equation in two cylindrical regions,

$$(1 - 2r^{5/4})\Omega_0(\omega^+) \times [0, r^{3/2}], \quad \text{and} \quad (B_R(0) - (1 - 2r^{5/4})\overline{\Omega}_0(\omega^+)) \times [0, r^{3/2}]$$

with initial data u_0^+ and $-u_0^-$, and with lateral boundary data zero on $(1 - 2r^{5/4})\Gamma_0(\omega^+) \times [0, r^{3/2}]$ and -1 on $\partial B_R(0) \times [0, r^{3/2}]$. Then, by comparison,

$$v_2 < u < v_1. \tag{3-25}$$

Also by Lemma 2.5 with $\beta \geq 5/6$,

$$|v_1 - v_2| \leq r^{5/4\times 5/6} = r^{25/24}.$$

Note that on $(1-r^{6/7})\Gamma_0(\omega^+)$,

$$|v_1| \geq r^{(6/7)\alpha} \geq r^{6/7\times 7/6} = r$$

and thus for $a_1 = 1/24$,

$$|v_1 - v_2| \leq r^{a_1}|v_1| \quad \text{on } (1-r^{6/7})\Gamma_0(\omega^+). \tag{3-26}$$

Similarly,

$$|v_1 - v_2| \leq r^{a_1}|v_2| \quad \text{on } (1+r^{6/7})\Gamma_0(\omega^+). \tag{3-27}$$

Now, v_1 and v_2 are almost harmonic in the $r^{3/4}$-neighborhood of their boundaries for $\frac{1}{2}r^{3/2} \leq t \leq r^{3/2}$ by Lemma 2.5. Then the almost harmonicity of v_1 and v_2 with (3-25)–(3-27) imply the following: For $\frac{1}{2}r^{3/2} \leq t \leq r^{3/2}$, there exist positive harmonic functions $\tilde{\omega}^+(\cdot\,,t)$ and $\tilde{\omega}^-(\cdot\,,t)$ defined respectively in

$$\Omega_t(v_2^+) \cap (B_R(0) - (1-r^{1-b})\Omega_0(\omega^+)) \quad \text{and} \quad \Omega_t(v_1^-) \cap (1+r^{1-b})\Omega_0(\omega^+)),$$

where $b = 1/7$, such that for some $a > 0$,

$$\tilde{\omega}^+(x,t) \leq u^+(x,t) \leq (1+r^a)\tilde{\omega}^+((1-4r^{5/4})x,t) \tag{3-28}$$

and

$$\tilde{\omega}^-(x,t) \leq u^-(x,t) \leq (1+r^a)\tilde{\omega}^-((1+4r^{5/4})x,t). \tag{3-29}$$

Now on the time interval $[0, r^{3/2}] + \frac{1}{2}kr^{3/2}$, $1 \leq k \leq m$, we construct v_1 and v_2 so that they solve the heat equation in the cylindrical domains with

$$\Gamma(v_1) = (1+2r^{5/4})\Gamma_{\frac{1}{2}kr^{3/2}}(\omega^+) \times \left[\tfrac{1}{2}kr^{3/2}, (1+\tfrac{1}{2}k)r^{3/2}\right],$$
$$\Gamma(v_2) = (1-2r^{5/4})\Gamma_{\frac{1}{2}kr^{3/2}}(\omega^+) \times \left[\tfrac{1}{2}kr^{3/2}, (1+\tfrac{1}{2}k)r^{3/2}\right].$$

By a similar argument to the one above, we then obtain harmonic functions $\tilde{\omega}^{\pm}(\cdot\,,t)$ satisfying (3-28) and (3-29) for

$$\tfrac{1}{2}(1+k)r^{3/2} \leq t \leq (1+\tfrac{1}{2}k)r^{3/2}.$$

Hence we conclude (3-28) and (3-29) for $r^{3/2} \leq t \leq t(x_0, r)$.

Step 2. Next we rescale $u(x,t)$ as

$$\tilde{u}(x,t) := \alpha^{-1}u(rx + x_0, r^2\alpha^{-1}t) \quad \text{in } 2Q_{x_0},$$

where $\alpha := u^+(x_0 - re_n, 0) \ll r^{1/2}$. Then $\tilde{u}(x,t)$ solves

$$\begin{cases} (\alpha\partial_t - \Delta)\tilde{u} = 0 & \text{in } \Omega(\tilde{u}), \\ V = |D\tilde{u}^+| - |D\tilde{u}^-| & \text{on } \Gamma(\tilde{u}), \\ \tilde{u}(-e_n, 0) = 1, & \\ \tilde{u}(e_n, 0) = -1/N, & \text{where } N \geq M. \end{cases}$$

Furthermore, (3-22) implies that for $0 \leq t \leq 1$,

$$\tilde{u}^+(-e_n, t) \sim 1, \quad \tilde{u}^-(e_n, t) \sim 1/N.$$

Let $\tilde{w}$ be the corresponding rescaled version of $\widetilde{\omega}$ given in (3-28) and (3-29), then in $B_{r^{-b}}(0) \cap \Omega_0(\tilde{u})$ we have

$$(1 - r^a)\tilde{w}^+((1 + 4r^{5/4})x, \alpha r^{-1/2}) \leq \tilde{u}^+(x, \alpha r^{-1/2}) \leq \tilde{w}^+(x, \alpha r^{-1/2}) \tag{3-30}$$

and

$$(1 - r^a)\tilde{w}^-(x, \alpha r^{-1/2}) \leq \tilde{u}^-(x, \alpha r^{-1/2}) \leq \tilde{w}^-((1 + 4r^{5/4})x, \alpha r^{-1/2}). \tag{3-31}$$

Here note that

$$\alpha r^{-1/2} = \sqrt{r} \cdot \frac{u^+(x_0 - re_n, t_0)}{r} \leq r^{1/3}.$$

Lastly, for given $x_0 \in \Gamma(\tilde{u}) \cap B_1(0)$, an argument similar to the one in (3-7) implies that

$$\tilde{u}(x,t) \leq (1 + r^b)\tilde{u}(x, 0) \quad \text{in } \partial B_{(1/2)r^{-b}}(r^{-b}e_n) \times [0, 1]. \tag{3-32}$$

Step 3. We claim that we can construct a supersolution U_1 and a subsolution U_2 of (ST2) such that

$$U_2(x,t) \leq \tilde{u}(x,t) \leq U_1(x,t) \leq U_2(x - \sqrt{\epsilon}e_n, t) \quad \text{in } B_1(0) \times [\alpha r^{-1/2}, 1],$$

and so that U_2 is a smooth solution with uniformly Lipschitz boundary in space and time. Then for sufficiently small $r > 0$ the lemma will follow from analysis parallel to that of [Athanasopoulos et al. 1998].

To illustrate the main ideas, let us first assume that:

(a) (3-30) and (3-31) hold in the entire ring domain $\mathcal{R} \times [0, 1]$, where

$$\mathcal{R} := \{x : d(x, \Gamma_0(\tilde{u})) \leq r^{-b}\}.$$

(b) $\tilde{u}(x,t) \leq (1 + r^b)\tilde{u}(x, 0)$ on $\partial\mathcal{R} \times [0, 1]$.

Let

$$\Sigma := \{x : d(x, \mathbb{R}^n - \Omega_0) \leq r^{-b}\} \times [\alpha r^{-1/2}, 1],$$

and let U_1^+ be the solution of the one-phase Hele-Shaw problem in Σ,

$$\begin{cases} \Delta U_1^+ = 0 & \text{in } \{U_1^+ > 0\} \cap \Sigma, \\ \partial_t U_1^+ = |DU_1^+|^2 & \text{on } \partial\{U_1^+ > 0\} \cap \Sigma, \\ U_1^+(x, \alpha r^{-1/2}) = \tilde{w}^+(x, \alpha r^{-1/2}), & \\ U_1^+(x,t) = (1+r^b)\tilde{u}(x,0) & \text{for } x \in \partial\Sigma. \end{cases} \tag{HS}$$

Let

$$U_1 = U_1^+ - U_1^- \quad \text{in } \mathcal{R} \times [\alpha r^{-1/2}, 1],$$

where $U_1^-(\,\cdot\,, t)$ is the harmonic function in $\mathcal{R} - \Omega(U_1^+)$ with boundary data

$$U_1^- = 0 \quad \text{on } \Gamma(U_1^+), \qquad U_1^- = C/N \quad \text{on } \partial\mathcal{R} - \Omega(U_1^+).$$

Then U_1 is a supersolution of (ST2) in Σ, and thus by Lemma 2.11 and the assumptions (a)–(b) we have $\tilde{u} \le U_1$ in Σ.

Step 4. The construction of the subsolution U_2 is a bit less straightforward. We use

$$U_2^+(x,t) := (1-\epsilon) \sup_{|y-x| \le \sqrt{\epsilon}(1-c(t))} U_1^+((1+\sqrt{\epsilon})y, t),$$

where $\epsilon = 1/N$ and $c(t) := t^{4/5}$. Then we define

$$U_2 = U_2^+ - U_2^- \quad \text{in } R \times [\alpha r^{-1/2}, 1],$$

where $\mathcal{R}$ is the ring domain as given above and $U_2^-(\,\cdot\,, t)$ is the harmonic function in $R - \Omega(U_2^+)$ with fixed boundary data zero on $\Gamma(U_2^+)$ and C/N on $\partial\mathcal{R} - \Omega(U_2^+)$. Then U_2 satisfies the free boundary condition

$$V_{U_2} \le (1+\epsilon)|DU_2^+| - \sqrt{\epsilon}c'(t).$$

Therefore, U_2 is a subsolution of (ST2) if we can show that

$$\sqrt{\epsilon}c'(t) \ge \epsilon|DU_2^+| + |DU_2^-| \quad \text{on } \Gamma(U_2) \tag{3-33}$$

and $\int_0^1 c'(s)\,ds \le 1$.

The analysis performed in [Choi and Kim 2010], as in the proof of (c) of Theorem 2.16, yields that at a fixed time t, $\Gamma(U_1)$ regularizes in the scale of $d := d(t)$ that solves

$$t = \frac{d^2}{U_1(-de_n, 0)}.$$

Therefore,

$$|DU_2^+| \sim \frac{U_2^+(-de_n, 0)}{d} \quad \text{and} \quad |DU_2^-| \sim \frac{U_2^-(de_n, 0)}{d}$$

on

$$\Gamma(U_2) \times [t/2, t].$$

Observe that since $\beta \geq 5/6$,

$$U_2^+(-de_n, 0) \leq d^{5/6} \quad \text{and} \quad U_2^-(de_n, 0) \leq \epsilon d^{5/6},$$

then we have

$$\epsilon \frac{U_2^+(-de_n, 0)}{d} + \frac{U_2^-(de_n, 0)}{d} \leq \epsilon d^{-1/6} \leq \sqrt{\epsilon} t^{-1/5}.$$

where the last inequality follows from

$$t = d^2 / U_1(-de_n, 0) \leq d^2/d^\alpha \leq d^{5/6}.$$

Hence $c(t) = t^{4/5}$ satisfies (3-33), and we conclude that U_2 is a subsolution of (ST2) in Σ.

Now we can use the fact that

$$U_2 \leq \tilde{u} \leq U_1 \quad \text{in } B_c(0) \times [\alpha r^{-1/2}, c]$$

to conclude that $\tilde{u}$ is $\sqrt{\epsilon}$- close to a Lipschitz (and smooth) solution U_1 in $B_1(0) \times [1/2, 1]$, confirming (B′). Moreover (A) holds due to Lemma 3.3 and Lemma 3.4. Once we can confirm this, we can conclude our proof by using the results of [Athanasopoulos et al. 1998] with the choice of a sufficiently small ϵ.

Step 5. Now we proceed to the general proof without the simplified assumptions (a) and (b) in Step 3, which are replaced by the local inequalities (3-30)–(3-32). For this we need to perturb the initial data outside of $B_1(0)$ (see Section 4, pages 2781–2783 of [Choi et al. 2009]), to obtain functions $W_1(x)$ and $W_2(x)$ that satisfy:

(a) $\{W_k > 0\}$ with $k = 1, 2$ is star-shaped and coincides with $\Omega_{\alpha r^{-1/2}}(\tilde{w})$ in $B_{r^{-b}}(0)$.

(b) $\{W_2 > 0\} \subset \Omega_{\alpha r^{-1/2}}(\tilde{w}) \subset \{W_1 > 0\}$.

(c) $d(x, \{W_k > 0\}) \geq r^{-b}$ with $k = 1, 2$ for $x \in \Gamma_{\alpha r^{-1/2}}(\tilde{w}) \cap (\mathbb{R}^n - B_{2r^{-b}}(0))$.

(d) W_k is harmonic in $\{W_k > 0\} - K$ with boundary data zero on $\Gamma(W_k)$ and $(1 + r^b)\tilde{w}(x, \alpha r^{-1/2})$ on ∂K, where

$$K = \{x : d(x, \Gamma(W_k)) \geq r^{-b}\}.$$

Let U_k be the solution of Hele-Shaw problem in

$$\mathbb{R}^n - \tfrac{1}{2}\{W_k > 0\} \times [\alpha r^{-1/2}, 1]$$

with initial data W_1 and with lateral boundary data $(1 + r^b)\tilde{w}(x, \alpha r^{-1/2})$. Due to Proposition 4.1 of [Choi et al. 2009], for sufficiently small $r > 0$, the level sets of U_1 are then ϵc-close to those of U_2 in $B_1(0) \times [0, 1]$. Hence we can use U_2 instead of U_1 in Step 4 and proceed as in Step 4 to conclude. □

4. Decomposition based on local phase dynamics

Throughout the rest of the paper, let u be as in Theorem 1.1, and fix $x_0 \in \Gamma_0$ and a sufficiently small constant $r > 0$. We will prove the regularization of the solution u in $B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$. After a rotation if necessary, we may assume that $x_0/|x_0| = e_n$.

Let us fix a constant $M \geq M_n$, where M_n is a sufficiently large dimensional constant. If the ratio between $u^+(x_0 - re_n, 0)$ and $u^-(x_0 + re_n, 0)$ is bigger than M, then we can directly apply Proposition 3.7 to prove the main theorem. Therefore we assume that

$$M^{-1}u^-(x_0 + re_n, 0) \leq u^+(x_0 - re_n, 0) \leq Mu^-(x_0 + re_n, 0). \tag{4-1}$$

Let

$$C_0 := \max\left\{\frac{u^+(x_0 - re_n, 0)}{r}, \frac{u^-(x_0 + re_n, 0)}{r}\right\}. \tag{4-2}$$

Then since u_0^+ and u_0^- are comparable with harmonic functions, C_0 is less than a constant depending on n and M (See Corollary 2.15). Also note that

$$C_0 \geq r^{\alpha-1} \geq r^{1/6}.$$

Let us now sort out the initial free boundary points where the flux from one phase dominates the flux from the other phase. Let us define

$$A^+ = \left\{x \in \Gamma_0 \cap B_{2r}(x_0) : \frac{u^+(x - se_n, 0)}{s} \geq MC_0 \text{ for some } s \text{ with } r^{5/4} \leq s \leq r\right\},$$
$$A^- = \left\{x \in \Gamma_0 \cap B_{2r}(x_0) : \frac{u^-(x + se_n, 0)}{s} \geq MC_0 \text{ for some } s \text{ with } r^{5/4} \leq s \leq r\right\}.$$

We then write

$$A = A^+ \cup A^-.$$

Throughout the paper we will let $e_n = x/|x|$ for any boundary point x, after a necessary rotation.

Lemma 4.1. *Let u be as given in Theorem 1.1, and let M and C as given above.*

(a) *If* $\dfrac{u^+(x - se_n, 0)}{s} \geq MC_0$ *for some $s \leq r$, then* $\dfrac{u^+(x - se_n, 0)}{s} \leq C_0$.

(b) *If* $\dfrac{u^-(x + se_n, 0)}{s} \geq MC_0$ *for some $s \leq r$, then* $\dfrac{u^-(x + se_n, 0)}{s} \leq C_0$.

Proof. Since $u_0^\pm$ are comparable with harmonic functions $h^\pm$, we can argue similarly as in Corollary 2.15. Observe that

$$\frac{u_0^+(x - se_n)}{s} \cdot \frac{u_0^-(x + se_n)}{s} \sim \frac{h^+(x - se_n)}{s} \cdot \frac{h^-(x + se_n)}{s} \lesssim \sqrt{\phi(r)} \lesssim C_0^2. \qquad \square$$

Now for $x \in A^+$, there exists a largest constant $r_x < r$ such that

$$\frac{u^+(x - r_x e_n, 0)}{r_x} = MC_0.$$

We then define

$$Q_x = B_{r_x}(x) \times \left[0, \frac{r_x}{MC_0}\right].$$

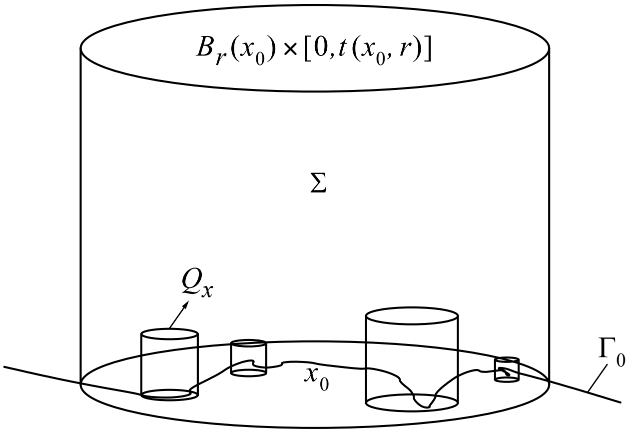

Figure 3. Decomposition of the domain.

Also for $x \in A^-$, we can similarly define r_x and Q_x. Now we define

$$\Sigma := B_r(x_0) \times [0, t(x_0, r)] - \bigcup_{x \in A} Q_x;$$

see Figure 3. Σ is then the region where the fluxes from both sides are initially balanced. Our aim in this section is to prove that the balance is kept over time, so that the interface remains close to a Lipschitz graph over time.

The following statement is a direct consequence of the definition of Σ.

Lemma 4.2. *If $x \in \Gamma_0 \cap \Sigma_0$, then for all $r^{5/4} \le s \le r$,*

$$\frac{u^+(x - se_n, 0)}{s}, \frac{u^-(x + se_n, 0)}{s} \le MC_0.$$

The next proposition, the main result in this section, states that the solution is "well-behaved" in Σ.

Proposition 4.3. *There exists a dimensional constant $K > 0$ such that for all $(x, t) \in \Gamma \cap \Sigma$,*

$$\frac{u^+(x - se_n, t)}{s}, \frac{u^-(x + se_n, t)}{s} < KMC_0 \quad \textit{for } r^{5/4} \le s \le r. \tag{4-3}$$

Before proving Proposition 4.3, we show an immediate consequence of it; we are ready to show that $\Gamma(u)$ is close to a Lipschitz graph in time as well as in space.

Corollary 4.4. *For $(x, t) \in \Gamma \cap \Sigma$, suppose $(x + ke_n, t + \tau) \in \Gamma$. Then there exists a dimensional constant $K_1 > 0$ such that*

$$|k| \le r^{5/4} \quad \textit{if } \tau \in \left[0, \frac{r^{5/4}}{K_1 MC_0}\right].$$

Proof. Due to Lemma 3.6, at any time $0 \le t \le t(x_0, r)$, we have

$$h^+(x, t) \le u^+(x, t) \le C_1 h^+(x - r^{5/4} e_n, t) \tag{4-4}$$

and

$$h^-(x,t) \le u^-(x,t) \le C_1 h^-(x + r^{5/4} e_n, t), \tag{4-5}$$

in $B_r(x_0)$, where the function $h := h^+(\cdot\,,t) - h^-(\cdot\,,t)$ is harmonic in its positive and negative phases in $(1+r)\Omega_t(u) - (1-r)\Omega_t(u)$, and the domains $\Omega(h^+)$ and $\Omega(h^-)$ are both star-shaped with respect to $B_{r_0}(0)$.

Let us pick $(y_0, t_0) \in \Gamma \cap \Sigma$. Due to Proposition 4.3, (4-4) and the Harnack inequality for harmonic functions, we have

$$\sup_{y \in B_{10 r^{5/4}}(y_0)} u(y, t_0) \le C C_1 K M C_0 r^{5/4}, \tag{4-6}$$

where C is a dimensional constant. On the other hand, due to Lemma 3.1 and $t_0^5 \le r^{25/6}$, we have

$$u(\cdot\,, t_0) \le 0 \quad \text{in } B_{(1/2) r^{5/4}}(y_0 + r^{5/4} e_n). \tag{4-7}$$

Let

$$y_1 := y_0 + r^{5/4} e_n, \quad C_2 := C C_1 K M C_0, \quad r(t) := \tfrac{1}{2} r^{5/4} - C_3 (t - t_0),$$

where $C_3 = C C_2$. Next we define $\phi(x,t)$ in the domain

$$\Pi := B_{2r^{5/4}}(y_1) \times \left[t_0, t_0 + \frac{r^{5/4}}{C_3}\right]$$

such that

$$\begin{cases} -\Delta \phi(\cdot\,, t) = 0 & \text{in } B_{2r^{5/4}}(y_1) - B_{r(t)}(y_1), \\ \phi = 2 C_2 r^{5/4} & \text{on } \partial B_{2r^{5/4}}(y_1), \\ \phi = 0 & \text{in } B_{r(t)}(y_1). \end{cases}$$

Then by (4-4)–(4-7), $u \prec \phi$ at $t = t_0$ in Π. Let T_0 be the first time when u hits ϕ from below in Π. Since (4-6) also holds for any $(x,t) \in \Gamma \cap \Sigma$ in place of (y_0, t_0), we have $u < \phi$ on the parabolic boundary of $\Pi \cap \{t_0 \le t \le T_0\}$. On the other hand, if C is chosen sufficiently large, then

$$\frac{\phi_t}{|D\phi|} = C_3 \ge |D\phi| \quad \text{on } \partial B_{r(t)}(y_1) \times \left[t_0, t_1 := t_0 + \frac{r^{5/4}}{4C_3}\right],$$

and thus ϕ is a supersolution of (ST1). This and Lemma 2.11 applied to u and ϕ in Π yields a contradiction, and we conclude that $\Gamma(u)$ lies outside of $B_{\frac{1}{4} r^{5/4}}(y_0 + r^{5/4} e_n)$ for $t_0 \le t \le t_1$.

Similarly, by constructing a negative radial barrier and comparing it with u, one can show that $\Gamma(u)$ lies outside of $B_{\frac{1}{4} r^{5/4}}(y_0 - r^{5/4} e_n)$ for $t_0 \le t \le t_1$. This concludes the proof. □

For $x_0 \in \Gamma_{t_0}$, define

$$t(x_0, r) := \min\left\{ \frac{r^2}{u^+(x_0 - r e_n, t_0)}, \frac{r^2}{u^-(x_0 + r e_n, t_0)} \right\}.$$

We now proceed to show our main result, Proposition 4.3. First we show Harnack-type inequalities for positive times.

Lemma 4.5 (Harnack at later times). *Fix $s \in [r^{5/4}, r]$. If $(y_0, t_0) \in \Gamma \cap \Sigma$, then*

$$u^+(y_0 - se_n, t_0) \geq c_1 u^+(y_0 - se_n, t_0 + \tau) \quad \textit{and} \quad u^-(y_0 + se_n, t_0) \geq c_1 u^-(y_0 + se_n, t_0 + \tau)$$

for $0 \leq \tau \leq t(y_0, s)/2$ and $c_1 > 0$.

Proof. We will show the lemma for u^+, the statement for u^- follows via parallel arguments.

Step 1. Let $(y_0, t_0) \in \Gamma \cap \Sigma$ and let $s \in [r^{5/4}, r]$. Let h^+ be given as in (4-4). Due to Lemma 3.3 and Lemma 3.4, we have

$$h^+(y_0 - 2re_n, t_1) \leq u^+(y_0 - 2re_n, t_1) \leq Cu^+(y_0 - 2re_n, t_2) \leq Ch^+(y_0 - (2r + r^{5/4})e_n, t_2)$$

for $0 \leq t_1, t_2 \leq t_0 + t(y_0, r)/2$. (Here note that $y_0 \in B_r(x_0)$.) In particular

$$u^+(y_0 - 2re_n, t) \leq Ch^+(y_0 - (2r + r^{5/4})e_n, t_0) \leq C_1 h^+(y_0 - 2re_n, t_0) \tag{4-8}$$

for $t \leq t_0 + t(y_0, s)/2$.

Step 2. Now let v^{**} solve (ST1) in $(\mathbb{R}^n - (1 - 2r)D_{t_0}) \times [t_0, t_0 + t(y_0, s)/2]$ with initial and boundary data $C_2 h^+(x - 2se_n, t)$. Since $s \geq r^{5/4}$, (4-4) implies

$$\Omega_t(u) \subset \Omega_{t_0}(v^{**}) \subset \Omega_t(v^{**}) \quad \text{in } B_{2s}(y_0) \times [t_0, t_0 + t(y_0, s)/2]. \tag{4-9}$$

Then by (4-9), (4-8) and (4-4),

$$u^+ \leq v^{**} \quad \text{in } B_s(y_0) \times [t_0, t_0 + t(y_0, s)/2]$$

if we choose C_2 as a multiple of C_1 by a dimensional constant. Moreover, due to the Harnack inequality for one-phase (ST1), one can conclude that

$$\begin{aligned} u^+(y_0 - se_n, t_0 + \tau) &\leq v^{**}(y_0 - se_n, t_0 + \tau) \\ &\leq Cv^{**}(y_0 - se_n, t_0) \\ &= CC_2 h^+(y_0 - 3se_n, t_0) \\ &\leq C_3 h^+(y_0 - se_n, t_0) \leq C_3 u^+(y_0 - se_n, t_0) \end{aligned}$$

for

$$0 \leq \tau \leq \frac{s^2}{v^{**}(y_0 - se_n, t_0)} \sim t(y_0, s)/2.$$

Here the first inequality uses the fact $u^+ \leq v^{**}$, the second uses the Harnack inequality for v^{**}, the third one uses the Harnack inequality for harmonic functions and the last one uses (4-4). □

Lemma 4.6 (backward Harnack). *Suppose that (4-3) holds up to time $t = T_0 \leq t(x_0, r)$. If $(y_0, t_0) \in \Gamma$ and $t_0 \leq T_0$, then for $0 \leq \tau \leq t(y_0, s)/2$,*

$$u^+(y_0 - se_n, t_0) \leq Cu^+(y_0 - se_n, t_0 + \tau) \quad \textit{and} \quad u^-(y_0 + se_n, t_0) \leq Cu^-(y_0 + se_n, t_0 + \tau),$$

where $0 \leq s \leq r$ and C is a universal constant.

Proof. We will show the argument for u^+, due to the symmetric nature of the claim. The argument here will be similar to that of Lemma 3.4, replacing the initial data u_0^+ and u_0^- (used in the construction of barriers) by $h^+(x,t_0)$ and $h^-(x,t_0)$ given in (4-4)–(4-5).

We consider a solution v_1 of (ST1) in

$$\Pi := (1+r)\Omega_{t_0} \times [t_0, t_0 + t(y_0, s)/2]$$

with initial and lateral boundary data $C_1 h^-$. Then $v_1 \le u$ in Π. Now let v_2 solve the heat equation in $\{v_1 = 0\} \times [t_0, t_0 + t(y_0, s)/2]$ with initial data

$$v_2(\cdot\,, t_0) = \begin{cases} h^+(\cdot\,, t_0) & \text{in } \{v_1(\cdot\,, t_0) = 0\} - (1-r)\{h^+(\cdot\,, t_0) > 0\}, \\ \tilde{h}(\cdot) & \text{in } (1-r)\{h^+(\cdot\,, t_0) > 0\}, \end{cases}$$

where $\tilde{h}(\cdot)$ is a C^2 extension function of $h^+(\cdot\,, t_0)$ chosen so that $\tilde{h}(\cdot) \le u^+(\cdot\,, t_0)$. The rest of the proof is the same as that of Lemma 3.4. □

Next we show that in the unbalanced region, possibly forming at positive times, the fast regularization phenomena still holds. This lemma will be used in the proof of Proposition 4.3 to show that there cannot be a severe unbalance of flux in the initially balanced region Σ.

Lemma 4.7 (regularization in unbalanced region II). *For a fixed $(x_0, t_0) \in \Gamma(u)$, suppose that*

$$u^+(x_0 - re_n, t_0) \ge M u^-(x_0 + re_n, t_0) \quad \textit{or} \quad u^-(x_0 + re_n, t_0) \ge M u^+(x_0 - re_n, t_0)$$

for $M > M_n$, where M_n is a dimensional constant. Then for $r \le 1/M_n$, there exists a dimensional constant $C > 0$ such that

$$|Du^+| \le C \frac{u^+(x_0 - re_n, t_0)}{r} \quad \textit{and} \quad |Du^-| \le C \frac{u^-(x_0 + re_n, t_0)}{r}$$

in $B_r(x_0) \times [t_0 + t(x_0, r)/2, t_0 + t(x_0, r)]$.

Proof. The proof of this lemma is parallel to that of Proposition 3.7. We use the Harnack and backward Harnack inequalities (Lemmas 4.5 and 4.6) instead of Lemmas 3.3 and 3.4. □

We are now ready to prove our main result, Proposition 4.3. Observe that (4-3) holds up to some $T_0 > 0$ by Lemma 4.2 and Lemma 3.3.

Proof of Proposition 4.3. Let K be a sufficiently large dimensional constant such that $K \gg M$. Let us assume that (4-3) breaks down for u^+ for the first time at $t = T_0$. Then

$$\frac{u^+(z_0 - se_n, T_0)}{s} = KMC_0 \tag{4-10}$$

for some $(z_0, T_0) \in \Gamma \cap \Sigma$ and $r^{5/4} \le s \le r$. Let

$$h = \sup\Big\{h : \frac{u^+(z_0 - ke_n, T_0)}{k} \ge M^2 C_0 \quad \text{for } s \le k \le h\Big\}. \tag{4-11}$$

Note that $h < r/2$ due to Lemma 3.3 and the definition of C_0, and $h > 2s$ due to Lemma 3.6. By the definition of h we have

$$\frac{u^+(z_0 - he_n, T_0)}{h} = M^2 C_0. \tag{4-12}$$

Let us find the largest time t_0 before T_0 such that for some $(y_0, t_0) \in \Gamma$

$$T_0 - t_0 = \frac{t(y_0, h)}{2} \quad \text{and} \quad \frac{y_0}{|y_0|} = \frac{z_0}{|z_0|}.$$

Then Lemma 4.5 implies

$$\frac{u^+(y_0 - he_n, t_0)}{h} \sim \frac{u^+(y_0 - he_n, T_0)}{h} \sim \frac{u^+(z_0 - he_n, T_0)}{h} = M^2 C_0.$$

Since $u^+(\cdot\,, t_0)$ and $u^-(\cdot\,, t_0)$ are comparable to harmonic functions (Lemma 3.6), a similar argument as in Lemma 4.1 implies that

$$\frac{u^-(y_0 + he_n, t_0)}{h} \lesssim C_0 \lesssim \frac{1}{M^2}\frac{u^+(y_0 - he_n, t_0)}{h}.$$

Hence by Lemma 4.7, we have

$$|Du^+(\cdot\,, T_0)| \sim M^2 C_0 \quad \text{in } B_h(y_0).$$

Since $B_s(z_0) \subset B_h(y_0)$, this would contradict (4-10) as $K \gg M$. □

Due to Lemma 3.6, Proposition 4.3 and Corollary 4.4, we have shown that condition (A) holds and that the level sets of u are close to a Lipschitz graph, and $\Gamma(u)$ is close to a Lipschitz graph in space and time (see the detailed description of this fact in the next section). However, we do not yet have sufficient control of the change of u over time to verify the condition (B′). We will therefore prove Theorem 1.1 by carrying out a modified argument, combining arguments from [Athanasopoulos et al. 1996; 1998] and [Choi et al. 2007; 2009].

5. Further regularization based on flatness

Let u, Γ_0 be as given in Theorem 1.1. Recall that $x_0 \in \Gamma_0$ and $r > 0$ are fixed, and they satisfy (4-1). Let C_0 be as given in (4-2) and $t(x_0, r)$ as given in (1-3).

Our goal is to prove the regularization of the free boundary after the time $t(x_0, r)/2$ in $B_r(x_0)$. Define

$$\Sigma_r(x_0) := B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)] \subset \Sigma.$$

Let us briefly review the information we have on u so far. As a result of Proposition 4.3, condition (A) holds up to

$$t = t(x_0, r) \le Cr^{2-\alpha} < r^{3/4}.$$

Also due to Lemma 3.6, our solution u is ϵ-monotone in $Q_r(x_0)$ with respect to a space cone $W_x(e_n, \theta_0)$ satisfying

$$|\theta_0 - \pi| = O(L),$$

where L is the Lipschitz constant of the initial domain Ω_0 given by (1-1).

Moreover $Q_r(x_0) \subset \Sigma$, and thus Corollary 4.4 and Lemma 3.1 yield that the free boundary $\Gamma(u)$ is $r^{4/3}$-monotone in $Q_r(x_0)$ with respect to the time cone $W_t(e_n, \tan^{-1}(1/K_1 M C_0))$ and the space cone $W_x(e_n, \theta_0)$. Here θ_0 is the angle corresponding to the Lipschitz constant of Γ_0, and $t(x_0, r) = r/C_0$.

On the other hand, by Lemma 3.3 and the definition of C_0,

$$\frac{u(x_0 - re_n, \frac{1}{2}t(x_0, r))}{C_0 r} \sim 1.$$

Since $Q_r(x_0) \subset \Sigma$, Proposition 4.3 implies

$$\frac{u(x,t)}{C_0 r} \lesssim KM \quad \text{in } B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)].$$

The main difficulty in applying the method of [Athanasopoulos et al. 1996; 1998] lies in the fact that we cannot guarantee the ϵ-monotonicity of the solution u in the time variable (although we can obtain, as above, the $r^{4/3}$-monotonicity of the free boundary $\Gamma(u)$). To go around this difficulty, we will first use the parabolic scale to improve the regularity of the solution in space. Consider the function

$$\bar{u}(x,t) := \frac{1}{C_0 r} u\bigl(rx + x_0, r^2 t + \tfrac{1}{2}t(x_0, r)\bigr). \tag{5-1}$$

In [Athanasopoulos et al. 1996; 1998], it was important that initially the time derivative of the solution was assumed to be controlled by the spatial derivative, i.e.,

$$|u_t| \le C(|Du^+| + |Du^-|). \tag{5-2}$$

Using (5-2) one can prove that the *direction vectors*

$$\frac{Du^+}{|Du^+|}(-le_n, t) \quad \text{and} \quad \frac{Du^-}{|Du^-|}(le_n, t)$$

do not change much for $0 \le t \le l$. This is pivotal in the regularization procedure since then $\Gamma(u)$ regularizes along the direction of the "common gain" obtained by those two direction vectors, the regularity of $\Gamma(u)$ then makes the above two vectors line up better in a smaller scale, which contributes to further regularization of $\Gamma(u)$ in a finer scale. In our case we do not know *a priori* that $\Gamma(u)$ is Lipschitz in either space or in time; in fact the Lipschitz continuity of $\Gamma(u)$ in time will be proved in the very last stage of Section 5 (see Theorem 5.7). Therefore, we do not have (5-2), and thus extra care is required to show that the spatial gradients $Du^{\pm}$ do not change their directions too rapidly.

In the following series of results, we will assume that $\bar{u}$ is given by (5-1). The lemmas and theorems will be proved to in the order they are stated, to improve the regularity of $\bar{u}$ in multiple steps.

Lipschitz continuity in space. First we prove that the ϵ-monotonicity of $\Gamma(\bar{u})$ improves to Lipschitz continuity. Let $a = C_0 r$. Then, in the domain $B_1(0) \times [-1/a, 1/a]$, $\bar{u}(x,t)$ solves

$$\begin{cases} \bar{u}_t - \Delta \bar{u} = 0 & \text{in } \{\bar{u} > 0\}, \\ V = a(|D\bar{u}^+| - |D\bar{u}^-|) & \text{on } \partial\{\bar{u} > 0\}. \end{cases}$$

Here note that

$$r^{7/6} \le r^{\alpha} \le a \le r^{\beta} \le r^{5/6}.$$

In this scale, since $\bar{u}$ is caloric and $\Gamma(\bar{u})$ is $r^{1/3}$-close to a Lipschitz graph in space and time, it follows that so is $\bar{u}$ in $B_{1/2}(0) \times [-1/a+1, 1/a]$.

Note that in above step we are losing a lot of information over time; $\Gamma(\bar{u})$ is in fact $r^{1/3}$-close to a Lipschitz graph moving very slowly in time, but this does not guarantee that $\bar{u}$ also changes slowly in time.

We then follow the iteration process in Lemma 7.2 of [Athanasopoulos et al. 1996] to show this:

Lemma 5.1. *If r is sufficiently small, then there exist $0 < c, d < 1/2$ such that $\bar{u}$ is $\lambda r^{1/3}$-monotone in the cone of directions $W_x(\theta_x - r^d, e_n)$ and $W_t(\theta_t - r^d, \nu)$ in the domain $B_{1-r^c}(0) \times [(-1+r^c)/a, 1/a]$.*

One can then iterate above lemma to improve the ϵ-monotonicity to full monotonicity, and state the result in terms of $\bar{u}$:

Lemma 5.2. *$\bar{u}$ is fully monotone in $B_{1/2}(0) \times [0, 1/a]$ for the cone*

$$\mathscr{C}_1 := W_x(\theta_x - r^d, e_n) \cup W_t(\theta_t - r^d, \nu)$$

for some constant $0 < d < 1/2$.

Regularity in time away from the free boundary. Now we suppose that $\bar{u}$ is Lipschitz in space and time. Then in particular, we have the Lipschitz regularity of u in space (and very weak Lipschitz regularity of u in time). We are interested in proving the following type of statement:

Lemma 5.3 (enlargement for the cone of monotonicity). *There exists $\lambda > 0$ such that if $\bar{u}$ is Lipschitz with respect to the cone of monotonicity $\Lambda_x(e_n, \theta_0)$ in $B_1(0) \times [-1/a, 1/a]$, then in the half domain $B_{1/2}(0) \times [-1/(2a), 1/(2a)]$, $\bar{u}$ is Lipschitz with respect to the cone of monotonicity $\Lambda_x(\nu, (1+\lambda)\theta_0)$ with some unit vector ν.*

To prove the enlargement of the cone, we take a closer look at the change of $\bar{u}$ over time, in the interior region. More precisely, we need the following lemma, which follows the approach taken in [Choi et al. 2007; 2009].

Lemma 5.4. *We have*

$$|\bar{u}_t| \le a|D\bar{u}|^2 \le Ca \quad \text{in } [B_{1/2}(e_n) \cup B_{1/2}(-e_n)] \times [-1/(2a), 1/(2a)],$$

where C is a dimensional constant.

Proof. Step 1. The proof is similar to that of Lemma 8.3 of [Choi et al. 2009]. Note that $\bar{u}_t$ is a caloric function in $\Omega^+(\bar{u})$ and $\Omega^-(\bar{u})$. Let us prove the lemma for $\bar{u}^+$, since parallel arguments apply to $\bar{u}^-$.

Step 2. We divide $\bar{u}_t$ into two parts. More precisely, let

$$\bar{u}_t = v_1 + v_2,$$

where both functions v_1 and v_2 are caloric in $\Omega^+(\bar{u})$, v_1 has the initial data zero and the boundary data $a|D\bar{u}^+|(|D\bar{u}^+|-|D\bar{u}^-|)$ on $\Gamma(\bar{u})$, and v_2 has the initial data $\bar{u}_t(\cdot,-1/a)$ and the boundary data zero on $\Gamma(\bar{u})$.

Step 3. For v_1, we need to use the absolute continuity of the caloric measure with respect to the harmonic measure, as well as the Lipschitz continuity of the free boundary. We proceed as in Lemma 8.3 of [Choi et al. 2007]. Note that we have

$$|D\bar{u}^+|\sim|D\bar{u}^-|\sim 1$$

in $[B_{1/2}(e_n)\cup B_{1/2}(-e_n)]\times[-1/a,1/a]$: this follows from the assumption in (4-1), and Lemmas 3.3 and 3.4. Therefore we can proceed as in Lemma 8.3 of [Choi et al. 2007] to obtain

$$v_1(x,t)\le a\int_{\Gamma(\bar{u})\cap\{-1/a\le s\le t\}}|D\bar{u}^+|^2d\omega^{(x,t)}\le a|D\bar{u}|^2(x,t),$$

where $\omega^{(x,t)}$ is the caloric measure for $\Omega(\bar{u})$, and

$$v_1(x,t)\ge a\int_{\Gamma(\bar{u})\cap\{-1/a\le s\le t\}}-|D\bar{u}^-|^2d\omega^{(x,t)}\ge -a|D\bar{u}|^2(x,t).$$

Step 4. As for v_2, we conclude that it must be smaller than a caloric function solved in the whole domain with the absolute value of its initial data. The advantage is that then we can use the heat kernel. Note that the initial data is given at $t=-1/a$ and has compact support. The initial data is given by $v_t\le(C/a)v_{e_n}$, where $v_{e_n}(x,t)$ is comparable to the derivative of a harmonic function in a Lipschitz domain.

Therefore the heat kernel representation is given as

$$\frac{1}{(t+1/a)^{n/2+1}}\int|x_n-y_n|\exp^{-|x-y|^2/(t+1/a)}v(y,-1/a)\,dy.$$

Since $t\in[0,1/a]$ and $k\exp^{-ak^2}\le C\exp^{-(a/2)k^2}$, we get the effect of $O(a)$. □

Further regularity in space. Now that we have sufficient information on the change of u over time, we change the scale following the one introduced in (1-4), and consider the function

$$v(x,t):=\frac{1}{C_0r}u\Big(rx+x_0,\ \frac{r}{C_0}t+1\Big). \tag{5-3}$$

Note that $C_0=r^{-1}c(x_0,r)$, and thus v coincides with $\tilde{u}$ defined in (1-4) with the choice of $c=rC_0$.

Due to the previous results, this function is Lipschitz continuous, in space and time, *away* from the free boundary. The following lemma suggests that the cone of monotonicity improves away from the free boundary, as we look at smaller scales. The proof is parallel to that of Lemma 8.4 in [Athanasopoulos et al. 1998].

Lemma 5.5. *Let v given by (5-3). Suppose that there exist constants $\delta>0$ and $0\le A\le B$, $\mu:=B-A$, such that*

$$\alpha(Dv,-e_n)\le\delta\quad\textit{and}\quad A\le\frac{v_t}{-e_n\cdot Dv}\le B$$

in $B_{1/6}(-\frac{3}{4}e_n) \times (-\delta/\mu, \delta/\mu)$ *with* $\delta/\mu < r$. *Then there exist a unit vector* $\nu \in \mathbb{R}^n$ *and positive constants* $r_0, b_0 < 1$ *depending only on* A, B *and* n *such that*

$$\alpha(Dv(x,t), \nu) \le b_0\delta \quad \text{in } B_{1/8}(-\tfrac{3}{4}e_n) \times \Bigl(-r_0\frac{\delta}{\mu}, r_0\frac{\delta}{\mu}\Bigr).$$

Now we can proceed as in Section 6 of [Choi et al. 2009] to obtain further regularity, using Lemma 5.4 instead of the uniform upper bound on $|Du|$ up to the free boundary.

Theorem 5.6. $\Gamma(v)$ *is* C^1 *in space in* $Q_{1/2}$. *In particular, there exist dimensional constants* $l_0, C_0 > 0$ *such that for a free boundary point* $(x_0, t_0) \in \Gamma(v)$, $\Gamma(v) \cap (B_{2^{-l}}(x_0) \times [t_0 - 2^{-l}, t_0 + 2^{-l}]$ *is a Lipschitz graph in space with Lipschitz constant less than* C_0/l *if* $l \ge l_0$.

Regularity in time up to the free boundary. Lastly, proceeding as in Sections 7–8 of [Choi et al. 2009] yields the differentiability of $\Gamma(v)$ in time. The main step in the argument is the following proposition: the statement and its proof is parallel to those of Theorem 7.2 in [Choi et al. 2009] and the blow-up argument as in Section 8 of [Choi et al. 2009]:

Theorem 5.7. $\Gamma(v)$ *is differentiable in space and time. More precisely there exist dimensional constants* $l_0 > 0$ *and* $1 < \gamma < 2$ *such that for* $(x_0, t_0) \in \Gamma(v) \cap Q_1$, *if* $l > l_0$ *then* $\Gamma(v) \cap (B_{2^{-l}}(x_0) \times [t_0 - 2^{-l}, t_0 + 2^{-l}]$ *is a Lipschitz graph in space with Lipschitz constant less than* $l^{-\gamma}$, *and Lipschitz graph in time with Lipschitz constant less than* $l^{-1/3}$.

Corollary 5.8.

$$C^{-1} \le |Dv^+|(x,t) \le C, \qquad C^{-1} \le \frac{|Dv^-|(x,t)}{v(-e_n, t)} \le C$$

in $Q_{1/2}$, *where* $C = C(n)$.

6. General case: solutions with locally Lipschitz initial data

In this section, we present how to extend the result of the main theorem to solutions with locally Lipschitz initial data. Our setting is as follows. Suppose Ω_0 is a bounded region in $B_R(0)$. Suppose u is a solution of (ST2) with $u_0 \ge -1$, $u_0 = -1$ in $B_R(0)$ and $u_0 \le M_0$. Further suppose that Ω_0 is locally Lipschitz, that is, for any $x_0 \in \Gamma_0$, $\Gamma_0 \cap B_1(x_0)$ is Lipschitz with a Lipschitz constant $L \le L_n$.

Let the initial data u_0 solve $\Delta u_0 = 0$ in $B_1(x_0)$. Then we claim that the parallel statements as in Theorem 1.1 hold in $B_{2d_0}(x_0) \times [t(x_0, d_0)/2, t(x_0, d_0)]$, where d_0 is a constant depending on n and M_0. More precisely:

Theorem 6.1. *Suppose* u *is a solution of* (ST2) *with initial data* u_0 *such that* $-1 \le u_0 \le M_0$. *Further suppose that for* $x_0 \in \Gamma_0$, $\Gamma_0 \cap B_1(x_0)$ *is Lipschitz with a Lipschitz constant* $L \le L_n$ *and* $\Delta u_0 = 0$ *in the positive and negative phases of* u_0 *in* $B_1(x_0)$. *Then there exists a constant* $d_0 > 0$ *depending on* n *and* M_0 *such that* (a) *and* (b) *of Theorem 1.1 hold for* u *and* $d \le d_0$.

The proof of the above theorem is parallel to that of Theorem 1.1 in Section 5, after proving the following lemma.

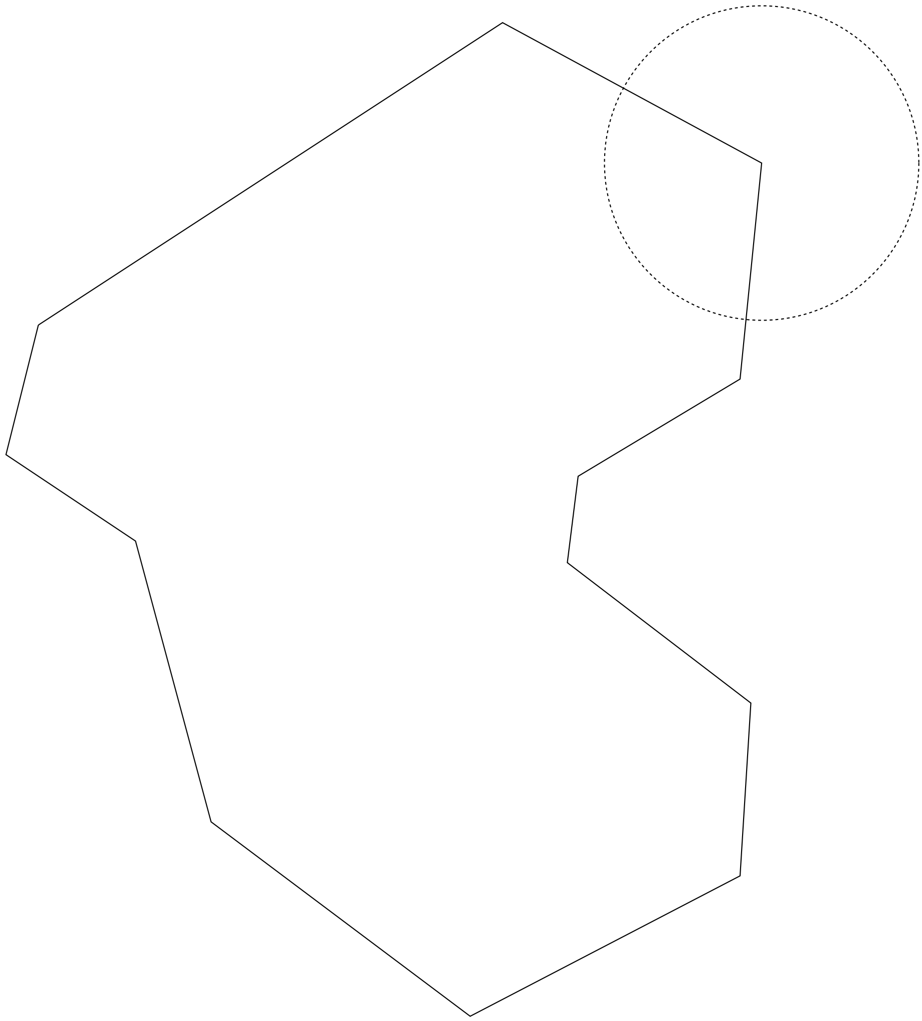

Figure 4. Locally Lipschitz initial domain.

Lemma 6.2. *There exists a solution v of (ST2) with star-shaped initial data such that the level sets of u and v are ϵd_0-close to each other in $B_{2d_0}(x_0)$ up to the time $t(x_0, d_0; u)$, where $d_0 > 0$ is sufficiently small. In particular, u and $\Gamma(u)$ are ϵ-monotone in a cone of W_x and W_t in $B_{2d_0}(x_0) \times [t(x_0, d_0)/2, t(x_0, d_0)]$.*

Even though our equation is nonlocal, the behavior in a far-away region would not affect much the behavior of the solution in the unit ball, if the solution behaves "reasonably" outside the unit ball. For example, in the star-shaped case, we know at least that the free boundary is almost locally Lipschitz at each time. In the locally Lipschitz case, we control the solution by putting an upper bound M_0 on the initial data u_0. We will argue that in a sufficiently small subregion of $B_1(x_0) \times [0, 1]$, the solution is mostly determined by the local initial data in $B_1(x_0)$. The perturbation method in the proof of Lemma 2.4 in [Choi et al. 2007] will be adopted here. Write $B_1(x_0) = B_1$.

Step 1. Construct a star-shaped region $\Omega' \subset B_R(0)$ such that:

(a) $\Omega' \cap B_1 = \Omega_0 \cap B_1$.

(b) Ω' is star-shaped with respect to every $x \in K \subset \Omega'$ for a sufficiently large ball K.

Let v_0^+ be the harmonic function in $\Omega' - K$ with boundary data 1 on ∂K, and 0 on $\partial\Omega'$. Next, let v_0^- be the harmonic function in $B_R(0) - \Omega'$ with boundary data 1 on $\partial B_R(0)$, and 0 on $\partial\Omega'$. Let B_2 be a concentric ball in B_1 with the radius of ϵ^{k_0}, i.e.,

$$B_2 = B_{\epsilon^{k_0}}(x_0) \subset B_1(x_0) = B_1.$$

Let k_0 be sufficiently large. Then by Lemma 2.7, a normalization of $v_0^{\pm}$ by a suitable constant multiple yields that for any $x \in B_2$,

$$1-\epsilon \le \frac{u_0(x)}{v_0(x)} \le 1+\epsilon. \tag{6-1}$$

Let v solve (ST2) with initial data $v_0 = v_0^+ - v_0^-$. Then Theorem 1.1 applies for v since v_0 is star-shaped with respect to K.

For the proof of the claim, we will find a sufficiently small d_0 such that v is ϵd_0-close to u in $B_{2d_0}(x_0)$ up to the time $t(x_0, d_0)$. More precisely, we will construct a supersolution w_1 and a subsolution w_2 of (ST2) such that in some small ball $B_h(x_0)$, we have

$$w_2 \le u \le w_1$$

and the level sets of w_1 and w_2 are $h\epsilon$ close to the level sets of v.

Step 2. Let k_1 and k_2 be large constants which will be determined later. Define

$$H^{\pm} := (\Gamma_0(v) \pm \epsilon^{k_0+k_1} e_n) \cap B_2.$$

Let

$$d_0 := \epsilon^{k_0+k_1+k_2}$$

and let $t(d_0) := t(x_0, d_0; v) = t(x_0, d_0; u)$. First note that

$$t(d_0) \ge d_0^{2-\beta} \ge \epsilon^{7(k_0+k_1+k_2)/6}.$$

Hence for v to be almost harmonic in a scale much larger than $\epsilon^{k_0+k_1}$, we need $\sqrt{t(d_0)} > \epsilon^{k_0}$, i.e.,

$$7(k_0+k_1+k_2)/12 < k_0.$$

Observe that by the construction of $H^{\pm}$ and d_0,

$$\sqrt{t(d_0)} \gg \text{radius}(B_2) \gg \text{dist}(H^{\pm}, \Gamma_0) \gg \max_{x \in \Gamma_t \cap B_2, 0 \le t \le t(d_0)} \text{dist}(x, \Gamma_0), \tag{6-2}$$

where the last inequality follows from Lemma 2.12 if we choose $k_2 \ge 2k_1$. If k_2 is sufficiently large, then one can prove from the last inequality of (6-2) and the bound on v_t that

$$1-\epsilon \le \frac{|v(x,t)|}{|v_0(x)|} = \frac{|v(x,t)|}{|u_0(x)|} \le 1+\epsilon \quad \text{on } H^{\pm} \times [0, t(d_0)]. \tag{6-3}$$

Step 3. We do have an estimate, Lemma 2.12, on how far the boundaries move away for the local one-phase case. If we take the one-phase versions with initial data u_0^+ and u_0^-, and compare with u, then we obtain that $\Gamma(u) \cap B_2$ stays in the $d_0^{(2-\alpha)/(2-\beta)}$-neighborhood of $\Gamma_0(u) \cap B_2$ up to the time $t(d_0) = t(x_0, d_0)$. In other words, the free boundary of u moves less than $d_0^{5/7}$ in B_2 up to the time $t(d_0)$.

Now we let S be the region between H^+ and H^-. To construct a subsolution (or supersolution) in S, we take the fixed boundary data $(1-\epsilon)v_0(x)$ on H^- (or H^+), and $(1+\epsilon)v_0(x)$ on H^+ (or H^-). To control the effect from the side $\partial B_2 \cap S$, we bend the free boundary $\Gamma_t(v)$ by $d_0^{5/7}$ on each side of $\partial B_2 \cap S$, using the conformal mapping $\hat{\Phi}$ (or $\check{\Phi}$). (See Section 4 of for the definitions of $\hat{\Phi}$ and $\check{\Phi}$.) More precisely, we bend the free boundary of v downward (or upward) using the conformal map $\hat{\Phi}$ (or $\check{\Phi}$), and solve the heat equation in there. Then similar arguments as in Lemmas 4.1 and 4.3 of [Choi and Kim 2010] yield that the solution is still (almost) a supersolution, and it stays close to the original solution.

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C^∞ SPECTRAL RIGIDITY OF THE ELLIPSE

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We prove that ellipses are infinitesimally spectrally rigid among C^∞ domains with the symmetries of the ellipse.

An isospectral deformation of a plane domain Ω_0 is a one-parameter family Ω_ϵ of plane domains for which the spectrum of the Euclidean Dirichlet (or Neumann) Laplacian Δ_ϵ is constant (including multiplicities). We say that Ω_ϵ is a C^1 curve of C^∞ plane domains if there exists a C^1 curve of diffeomorphisms φ_ϵ of a neighborhood of $\Omega_0 \subset \mathbb{R}^2$ with $\varphi_0 = \text{id}$ and with $\Omega_\epsilon = \varphi_\epsilon(\Omega_0)$. The infinitesimal generator $X = d\varphi_\epsilon/d\epsilon$ is a vector field in a neighborhood of Ω_0 which restricts to a vector field along $\partial\Omega_0$; we denote by $X_\nu = \dot\rho\nu$ its outer normal component. With no essential loss of generality we may assume that $\varphi_\epsilon|_{\partial\Omega_0}$ is a map of the form

$$x \in \partial\Omega_0 \to x + \rho_\epsilon(x)\nu_x, \tag{1}$$

where $\rho_\epsilon \in C^1([0, \epsilon_0], C^\infty(\partial\Omega_0))$, $\epsilon_0 > 0$ and $\rho_0 = 0$. We put

$$\dot\rho(x) = \delta\rho\,(x) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \rho_\epsilon(x).$$

An isospectral deformation is said to be trivial if $\Omega_\epsilon \simeq \Omega_0$ (up to isometry) for sufficiently small ϵ. A domain Ω_0 is said to be spectrally rigid if all isospectral deformations Ω_ϵ are trivial. The domain Ω_0 is called infinitesimally spectrally rigid if $\dot\rho = 0$ (up to rigid motions) for all isospectral deformations.

In this article, we use the Hadamard variational formula of the wave trace (apparently for the first time) to study spectral rigidity problems (Theorem 2). Our main application is the infinitesimal spectral rigidity of ellipses among C^1 curves of C^∞ plane domains with the symmetries of an ellipse. We orient the domains so that the symmetry axes are the x-y axes. The symmetry assumption is then that each φ_ϵ is invariant under $(x, y) \to (\pm x, \pm y)$.

Theorem 1. *Suppose that Ω_0 is an ellipse and that Ω_ϵ is a C^1 Dirichlet (or Neumann) isospectral deformation of Ω_0 through C^∞ domains with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Then $X_\nu = 0$ or equivalently $\dot\rho = 0$.*

As discussed in Sections 0.2 and 3.2, Theorem 1 implies that ellipses admit no isospectral deformations for which the Taylor expansion of ρ_ϵ at $\epsilon = 0$ is nontrivial. A function such as e^{-1/ϵ^2} for which the Taylor series at $\epsilon = 0$ vanishes is called *flat* at $\epsilon = 0$.

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Corollary 1. *Suppose that* Ω_0 *is an ellipse and that* $\epsilon \to \Omega_\epsilon$ *is a* C^∞ *Dirichlet* (*or Neumann*) *isospectral deformation through* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *symmetric* C^∞ *domains. Then* ρ_ϵ *must be flat at* $\epsilon = 0$. *In particular*, *there exist no nontrivial real analytic curves* $\epsilon \to \Omega_\epsilon$ *of* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *symmetric* C^∞ *domains with the spectrum of an ellipse.*

Spectral rigidity of the ellipse has been expected for a long time and is a kind of model problem in inverse spectral theory. Ellipses are special since their billiard flows and maps are completely integrable. It was conjectured by G. D. Birkhoff that the ellipse is the only convex smooth plane domain with a completely integrable billiard. We cannot assume that the deformed domains Ω_ϵ have this property, although the results of [Siburg 2000; Zelditch 1998] come close to showing that they do. The results are somewhat analogous to the spectral rigidity of flat tori or the sphere in the Riemannian setting.

The main novel step in the proof is the Hadamard variational formula for the wave trace (Theorem 2), which holds for all smooth Euclidean domains $\Omega \subset \mathbb{R}^n$ satisfying standard "cleanliness" assumptions. It is of independent interest and may have applications to spectral rigidity beyond the setting of ellipses. We therefore present the proof in detail. (See also [Golse and Lochak 2003], where a variational formula for the Selberg's trace formula on compact Riemann surfaces is derived.)

The main advance over prior results is that the domains Ω_ϵ are allowed to be C^∞ rather than real analytic. Much less than C^∞ could be assumed for the domains Ω_ϵ, but we do not belabor the point. For real analytic domains a length spectral rigidity result for analytic domains with the symmetries of the ellipse was proved in [Colin de Verdière 1984]. The method does not apply directly to Δ-isospectral deformations of ellipses since the length spectrum of the ellipse may have multiplicities and the full length spectrum might not be a Δ-isospectral invariant. If it were, then Siburg's results would imply that the marked length spectrum is preserved [Siburg 1999; 2000; 2004]. In [Zelditch 2009; 2000] it is shown that analytic domains with one symmetry are spectrally determined if the length of the minimal bouncing ball orbit and one iterate is a Δ-isospectral invariant. The prior results on Δ-isospectral deformations that we are aware of are contained in the articles [Guillemin and Melrose 1979a; Popov and Topalov 2003; 2012] and concern deformations of boundary conditions. To our knowledge, the only prior results on Δ-isospectral deformations of the domain are contained in [Marvizi and Melrose 1982]. Marvizi and Melrose [1982] introduce new spectral invariants and prove certain rigidity results, but they do not apparently settle the case of the ellipse (see also [Amiran 1993; 1996] for further attempts to apply them to the ellipse). It would be desirable to remove the symmetry assumption (to the extent possible), but symmetry seems quite necessary for our argument. Further discussion of prior results can be found in the earlier arXiv posting of this article [Hezari and Zelditch 2010].

0.1. ***Theorem on variation of the wave trace.*** We now state a general result on the variation of the wave trace on a domain with boundary under variations of the boundary.

To state the result, we need some notation. We denote by

$$E_B(t) = \cos\left(t\sqrt{-\Delta_B}\,\right) \quad \text{and} \quad S_B(t) = \frac{\sin\left(t\sqrt{-\Delta_B}\,\right)}{\sqrt{-\Delta_B}} \tag{2}$$

the even and odd wave operators of a domain Ω with boundary conditions B. We recall that $E_B(t)$ has a distribution trace as a tempered distribution on $\mathbb{R}$. That is, $E_B(\varphi) = \int_{\mathbb{R}} \varphi(t) E_B(t)\, dt$ is of trace class for any $\varphi \in C_0^\infty(\mathbb{R})$; we refer to [Guillemin and Melrose 1979b; Petkov and Stoyanov 1992] for background.

The Poisson relation of a manifold with boundary gives a precise description of the singularities of this distribution trace in terms of periodic transversal reflecting rays of the billiard flow, or equivalently periodic points of the billiard map. For the definitions of "billiard map", "clean", "transversal reflecting rays", etc., we refer to [Guillemin and Melrose 1979a; 1979b; Petkov and Stoyanov 1992]. A periodic point of the billiard map $\beta : B^*\partial\Omega \to B^*\partial\Omega$ on the unit ball bundle $B^*\partial\Omega = \{(q,\zeta) \in T^*\partial\Omega;\ |\zeta| < 1\}$ of the boundary corresponds to a billiard trajectory, i.e an orbit of the billiard flow Φ^t on $S^*\Omega$. We define the *length* of the periodic orbit of β to be the length of the corresponding billiard trajectory in $S^*\Omega$. Note that the *period* of a periodic point of β is ambiguous since it could refer to this length or to the power of β. We also denote by $\operatorname{Lsp}(\Omega)$ the length spectrum of Ω, that is, the set of lengths of closed billiard trajectories. The perimeter of Ω is denoted by $|\partial\Omega|$.

In the following deformation theorem, the boundary conditions are fixed during the deformation and we therefore do not include them in the notation. We also do not include ϵ in our notation for Δ even though all Laplacians below are associated with Ω_ϵ and hence dependent on ϵ.

Theorem 2. *Let $\Omega_0 \subset \mathbb{R}^n$ be a C^∞ convex Euclidean domain with the property that the fixed point sets of the billiard map are clean. Then, for any C^1 variation of Ω_0 through C^∞ domains Ω_ϵ, the variation of the wave traces $\delta \operatorname{Tr} \cos(t\sqrt{-\Delta}\,)$, with Dirichlet (or Neumann) boundary conditions is a classical conormal distribution for $t \neq m|\partial\Omega_0|$ $(m \in \mathbb{Z})$ with singularities contained in $\operatorname{Lsp}(\Omega_0)$. For each $T \in \operatorname{Lsp}(\Omega_0)$ for which the set F_T of periodic points of the billiard map β of length T is a d-dimensional clean fixed point set consisting of transverse reflecting rays, there exist nonzero constants C_Γ independent of $\dot\rho$ such that, near T, the leading order singularity is*

$$\delta \operatorname{Tr}\cos(t\sqrt{-\Delta}\,) \sim \frac{t}{2}\,\Re\left\{\left(\sum_{\Gamma\subset F_T} C_\Gamma \int_\Gamma \dot\rho\,\gamma_1\, d\mu_\Gamma\right)(t - T + i0^+)^{-2-(d/2)}\right\},$$

modulo lower order singularities. The sum is over the connected components Γ of F_T. Here $\delta = \left.\dfrac{d}{d\epsilon}\right|_{\epsilon=0}$ and $\gamma_1(q,\zeta) = \sqrt{1-|\zeta|^2}$.

The function γ_1 on $B^*\partial\Omega$ is defined in (27) and appeared earlier in [Hassell and Zelditch 2004]. The densities $d\mu_\Gamma$ on the fixed point sets of β and its powers are very similar to the canonical densities defined in Lemma 4.2 of [Duistermaat and Guillemin 1975], and further discussed in [Guillemin and Melrose 1979a; Popov and Topalov 2003; 2012]. The constants C_Γ are explicit and depend on the boundary conditions. We suppress the exact formulae since we do not need them, but their definition is reviewed in the course of the proof.

To clarify the dimensional issues, we note that there are four closely related definitions of the set of closed billiard trajectories (or closed orbits of the billiard map). The first is the fixed point set of the billiard flow Φ^T at time T in $T^*\Omega$. The second is the set of unit vectors in the fixed point set. The third is the fixed point set of the billiard flow restricted to $T^*_{\partial\Omega}\Omega$, the set of covectors with foot points at the boundary. The fourth is the set of periodic points of the billiard map β on $B^*\partial\Omega$ of length T, where as above the length is defined by the length of the corresponding billiard trajectory. The dimension d refers to the dimension of the latter which we show by F_T. In the case of the ellipse, for instance, $d = 1$; the periodic points of a given length form invariant curves for β.

To prove Theorem 2, we use the Hadamard variational formula for the Green's kernel to give an exact formula for the wave trace variation (Lemma 1). We then prove that it is a classical conormal distribution and calculate its principal symbol.

It is verified in [Guillemin and Melrose 1979a] that the ellipse satisfies the cleanliness assumptions.

Corollary 2. *For any C^1 variation of an ellipse through C^∞ domains Ω_ϵ, the leading order singularity of the wave trace variation is*

$$\delta \operatorname{Tr} \cos(t\sqrt{-\Delta}) \sim \frac{t}{2} \Re\left\{\left(\sum_{\Gamma \subset F_T} C_\Gamma \int_\Gamma \dot{\rho}\, \gamma_1 \, d\mu_\Gamma\right) (t - T + i0^+)^{-5/2}\right\},$$

modulo lower-order singularities, where the sum is over the connected components Γ of the set F_T of periodic points of β (and its powers) of length T.

0.2. ***Flatness issues.*** We now discuss an apparently new flatness issue in isospectral deformations. The rather technical assumption that Ω_ϵ is a C^1 family of C^∞ domains rather than a C^∞ family in the ϵ variable is made to deal with a somewhat neglected and obscure point about isospectral deformations. Isospectral deformations are curves in the "manifold" of domains. The curve might be a nontrivial C^∞ family in ϵ but the first derivative $\dot{\rho}$ might vanish at $\epsilon = 0$. Thus, infinitesimal spectral rigidity is at least apparently weaker than spectral rigidity. We impose the C^1 regularity to allow us to reparametrize the family and show that the first derivative of any C^1 reparametrization must be zero. This is not the primary focus of Theorem 1, but with no additional effort the proof extends to the C^1 case.

This flatness issue does not seem to have arisen before in inverse spectral theory, even when the main conclusions are derived from infinitesimal rigidity. The main reason is that first-order perturbation theory very often requires analytic perturbations (i.e., analyticity in the deformation parameter ϵ), and so most (if not all) prior results on isospectral deformations assume that the deformation is real analytic. But our proof is based on Hadamard's variational formula, which is valid for C^1 perturbations of domains and so we can study this more general situation. Further, the prior spectral rigidity results [Guillemin and Kazhdan 1980] are proved for an open set of domains and metrics and therefore flatness at all points implies triviality of the deformations. We are only deforming the one-parameter family of ellipses and therefore cannot eliminate flat isospectral deformations by that kind of argument. We also note that there could exist continuous but nondifferentiable isospectral deformations.

0.3. ***Pitfalls and complications.*** The route taken in the proof of Theorem 1, and the flatness issues just discussed, reflect certain technical issues that arise in the inverse problem.

First is the issue of multiplicities in the eigenvalue spectrum or in the length spectrum. The multiplicities of the Δ-eigenvalues of the ellipse (for either Dirichlet or Neumann boundary conditions) appear to be almost completely unknown. If a sufficiently large portion of the eigenvalue spectrum were simple (i.e., of multiplicity one), one could simplify the proof of Theorem 1 by working directly with the eigenfunctions and their semiclassical limits (as in the first arXiv posting of this article, [Hezari and Zelditch 2010]).

The dual multiplicity of the length spectrum is also largely unknown for the ellipse. Without length spectral simplicity one cannot work with the wave trace invariants. Our proof relies on the observation in [Guillemin and Melrose 1979a] that the multiplicities have to be one (modulo the symmetry) for periodic orbits that creep close enough to the boundary.

Second is the issue of cleanliness. Theorem 2 and Corollary 2 would apply to any of the deformed domains Ω_ϵ if the fixed points sets were known to be clean. One could then use the conclusion of Corollary 2 to rule out flat isospectral deformations. However, we do not know that the fixed point sets are clean for the deformed domains even though we do know that they have the same wave trace singularities as the ellipse. Equality of the wave traces for isospectral deformations of ellipses shows that the periodic points of β of Ω_ϵ can never be nondegenerate. Hence the deformations are very nongeneric. It is plausible that equality of wave traces forces the sets of periodic points to be clean invariant curves of dimension one. But we do not know how to prove this kind of inverse result at this time.

1. Hadamard variational formula for wave traces

In this section we consider the Dirichlet and Neumann eigenvalue problems for a C^1 one-parameter family of smooth Euclidean domains $\Omega_\epsilon \subset \mathbb{R}^n$,

$$\begin{cases} -\Delta_{B_\epsilon}\Psi_j(\epsilon) = \lambda_j^2(\epsilon)\Psi_j(\epsilon) \ \text{ in } \ \Omega_\epsilon, \\ B_\epsilon \Psi_j(\epsilon) = 0, \end{cases} \tag{3}$$

where the boundary condition B_ϵ could be $B_\epsilon\Psi_j(\epsilon) = \Psi_j(\epsilon)|_{\partial\Omega_\epsilon}$ (Dirichlet) or $\partial_{\nu_\epsilon}\Psi_j(\epsilon)|_{\partial\Omega_\epsilon}$ (Neumann). Here, $\lambda_j^2(\epsilon)$ are the eigenvalues of $-\Delta_{B_\epsilon}$, enumerated in order and with multiplicity, and ∂_{ν_ϵ} is the interior unit normal to Ω_ϵ. We do not assume that $\lambda_j^2(\epsilon)$ and $\Psi_j(\epsilon)$ are C^1 in ϵ.

We will use Hadamard's variational formula for the variation of Green's kernels, and adapt the formula to give the variation of the (regularized) trace of the wave kernel. Our references are [Garabedian 1964; Peetre 1980; Fujiwara et al. 1978; Ozawa 1982; Fujiwara and Ozawa 1978].

To state our main variational Lemma 1 we introduce some notation. We denote by dq the surface measure on the boundary $\partial\Omega$ of a domain Ω, and by $ru = u|_{\partial\Omega}$ the trace operator. We use $S_B^b(t, q', q) \in \mathcal{D}'(\mathbb{R}\times\partial\Omega\times\partial\Omega)$ for the following boundary traces of the Schwartz kernel $S_B(t, x, y) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$ of $S_B(t)$ defined in (2):

$$S_B^b(t, q', q) = \begin{cases} r_{q'} r_q \partial_{\nu_{q'}}\partial_{\nu_q} S_D(t, q', q) & \text{(Dirichlet)}, \\ \nabla_{q'}^T\nabla_q^T r_{q'} r_q S_N(t, q', q) + r_{q'} r_q \ \Delta_{q'} \ S_N(t, q', q) & \text{(Neumann)}. \end{cases} \tag{4}$$

Here, the subscripts q', q refer to the variable involved in the differentiating or restricting. According to convenience, we may also indicate this by subscripting with indices $1, 2$, referring to the first and second variables in the kernel. For instance,

$$\frac{\partial}{\partial\nu_{q'}} K(q', q) = \frac{\partial}{\partial\nu_1} K(q', q).$$

We may also use the notations ∂_ν and $\partial/\partial\nu$ interchangeably to refer to the inward normal derivative. Here, ∇^T corresponds to tangential differentiation which is the gradient associated to the hypersurface $\partial\Omega$.

Lemma 1. *The variation of the wave trace with boundary conditions B is given by*

$$\delta \operatorname{Tr} E_B(t) = \frac{t}{2} \int_{\partial\Omega_0} S^b_{B_0}(t, q, q)\dot{\rho}(q)\, dq.$$

We summarize by writing

$$\delta \operatorname{Tr} E_B(t) = \frac{t}{2} \operatorname{Tr}_{\partial\Omega_0} \dot{\rho}\, S^b_B.$$

Here, $\delta = \frac{d}{d\epsilon}|_{\epsilon=0}$ and the equality is understood in the sense of distributions; meaning if $\varphi \in C_0^\infty(\mathbb{R})$ then

$$\delta \operatorname{Tr}\Big(\int \varphi(t) E_B(t)\, dt\Big) = \int_{\partial\Omega_0} \Big(\int \frac{t}{2}\varphi(t) S^b_{B_0}(t, q, q)\, dt\Big)\dot{\rho}(q)\, dq.$$

We note that the right hand side is well defined because the kernel of the operator $\int \varphi(t) S_{B_0}(t)\, dt$ is smooth up to the boundary.

We prove the lemma by relating the variation of the wave trace to the known variational formula for the Green's function (resolvent kernel). We now review the latter.

1.1. ***Hadamard variational formula for Green's function.*** Here by the Green's function $G_{B_\epsilon}(\lambda, x, y)$ of Ω_ϵ, with the boundary condition B_ϵ, we mean the integral kernel of the resolvent $R_B(\lambda) = (-\Delta_{B_\epsilon} - \lambda^2)^{-1}$ where $\Im\lambda > 0$. We also define $R_B(\lambda)$ for $\lambda \in \mathbb{R}$ by $R_B(\lambda + i0^+)$ (that the limit exists follows, for example, from Theorem 3.1.11 of [Hörmander 1983]). The variational formula below is valid for both of these resolvents (also for $\Im\lambda < 0$). Since the domains of $G_{B_\epsilon}(\lambda, x, y)$ depend on ϵ we first have to make our definition of δ precise.

Definition. Let $u_\epsilon \in C^1([0, \epsilon_0], \mathscr{D}'(\Omega_\epsilon))$ with $\epsilon_0 > 0$, be a C^1 family of distributions in Ω_ϵ. We use δu_ϵ or $\dot{u}$ to represent the first variation of u_ϵ at $\epsilon = 0$ as a distribution in Ω_0:

$$\delta u_\epsilon = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_\epsilon.$$

We note that if $\alpha \in C_0^\infty(\Omega_0)$ then for ϵ small $\operatorname{supp}(\alpha) \subset \Omega_\epsilon$, and therefore we can define δu_ϵ by

$$(\delta u_\epsilon)(\alpha) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (u_\epsilon(\alpha)).$$

However, the problem with this definition is that it defines $\dot{u}$ only in the interior of Ω_0 and not at the boundary even if u_ϵ is defined there. Below we will see another definition of $\dot{u}$, using diffeomorphisms, which resolves this issue.

In the statement of the formulas we will not include ϵ in our notation. In the Dirichlet case, the classical Hadamard variational formula states that, under a C^1 deformation Ω_ϵ,

$$\delta G_D(\lambda, x, y) = \int_{\partial\Omega_0} \frac{\partial}{\partial\nu_2} G_D(\lambda, x, q) \frac{\partial}{\partial\nu_1} G_D(\lambda, q, y)\dot{\rho}(q)\, dq. \tag{5}$$

In the Neumann case,

$$\delta G_N(\lambda, x, y) = \int_{\partial\Omega_0} \nabla_2^T G_N(\lambda, x, q) \cdot \nabla_1^T G_N(\lambda, q, y)\dot{\rho}(q)\, dq - \lambda^2 \int_{\partial\Omega_0} G_N(\lambda, x, q) G_N(\lambda, q, y)\dot{\rho}(q)\, dq. \tag{6}$$

We briefly review the proof of the Hadamard variational formula to clarify the definition of $\delta G_B(\lambda, x, y)$ and of the other kernels. We give the proof for the variation of the resolvent $R_B(\lambda)$ with $\Im\lambda > 0$. From this we can obtain the analogous formula for $\delta R_B(\lambda + i0^+)$ by taking $\Im\lambda \to 0^+$. Following [Peetre 1980], we write the inhomogeneous problem

$$\begin{cases} (-\Delta - \lambda^2)u = f & \text{in } \Omega\,(\lambda \in \mathbb{C},\ \Im\lambda > 0), \\ u = 0 \ \ (\text{resp. } \partial_\nu u = 0) & \text{on } \partial\Omega, \end{cases}$$

in terms of the energy integral

$$E(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx - \lambda^2 \int_\Omega uv \, dx = \int_\Omega v(-\Delta - \lambda^2)u \, dx - \int_{\partial\Omega} v \partial_\nu u \, dq,$$

where ∂_ν is the inward unit normal. The inhomogeneous problem is to solve

$$E(u, v) = \int_\Omega f v \, dx,$$

where v is a smooth test function which vanishes to order 1 (resp. 0) on $\partial\Omega$ for the Dirichlet (resp. Neumann) problem. We denote the energy density by $e(u, v) = \nabla u \cdot \nabla v - \lambda^2 uv$.

We now vary the problems over a one-parameter family of domains. We use one-parameter families of smooth diffeomorphisms φ_ϵ of a neighborhood of $\Omega_0 \subset \mathbb{R}^n$ to define the one-parameter families $\Omega_\epsilon = \varphi_\epsilon(\Omega_0)$ of domains. We assume φ_ϵ to be a C^1 curve of diffeomorphisms with $\varphi_0 = \mathrm{id}$.

The variational derivative of the solution is defined as follows: Let u_ϵ be a C^1 curve of functions in $H^s(\Omega_\epsilon)$. Then $\varphi_\epsilon^* u_\epsilon \in H^s(\Omega_0)$ and $d(\varphi_\epsilon^* u_\epsilon)/d\epsilon$ is a continuous curve in $H^s(\Omega_0)$. Put

$$X = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon \quad \text{and} \quad \theta_X u = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon^* u_\epsilon.$$

Assume that $u_0 \in H^{s+1}(\Omega_0)$. Then $\dot{u}$, defined by

$$\dot{u} = \theta_X u - X u_0,$$

exists in $H^s(\Omega_0)$. This gives a new definition of $\dot{u}$ which has a well-defined restriction to $\partial\Omega_0$ (for $s \geq 1$), and it agrees with $\dot{u}$ defined above in the interior of Ω_0. Further, let v be a test function on Ω_0 and use $\varphi_\epsilon^{-1*} v$ as a test function on Ω_ϵ. Now rewrite the boundary problems as

$$\int_{\Omega_\epsilon} e(u_\epsilon, (\varphi_\epsilon^{-1})^* v) \, dx = \int_{\Omega_\epsilon} f_\epsilon((\varphi_\epsilon^{-1})^* v) \, dx.$$

Changing variables, one pulls back the equation to Ω_0 as

$$\int_{\Omega_0} e_\epsilon(\varphi_\epsilon^* u_\epsilon, v) \varphi_\epsilon^* dx = \int_{\Omega_0} (\varphi_\epsilon^* f_\epsilon) v \varphi_\epsilon^* dx,$$

where

$$e_\epsilon(w, v) := \varphi_\epsilon^*(e(\varphi_\epsilon^{-1*} w, \varphi_\epsilon^{-1*} v)).$$

Then, by the computations of [Peetre 1980, (8) and (10)] we have

$$\begin{aligned}\int_{\Omega_0} \dot{u}(-\Delta-\lambda^2)v\,dx \\ =\int_{\Omega_0} \dot{f}v\,dx+\int_{\partial\Omega_0} fv\dot{\rho}\,dq+\int_{\partial\Omega_0}(\nabla u_0\cdot\nabla v-\lambda^2 u_0 v)\dot{\rho}\,dq+\begin{cases}\lambda^2\int_{\partial\Omega_0} u_0 v\dot{\rho}dq & \text{(Dirichlet)},\\ 0 & \text{(Neumann)}.\end{cases}\end{aligned}\tag{7}$$

To obtain (5)–(6), at least formally, one puts

$$u_\epsilon(x)=G_{B_\epsilon}(\lambda,z,x),\qquad v(x)=G_{B_0}(\lambda,y,x),\qquad f_\epsilon(x)=\delta_z(x)$$

where $z\in\mathring{\Omega}$. Thus $\dot{u}(x)=\delta G_B(\lambda,z,x)$ and $\dot{f}=0$. Since $z\in\mathring{\Omega}$ we have $z\in\Omega_\epsilon$ for sufficiently small ϵ and one easily verifies that (7) implies (5)–(6). The Green's kernel depends on ϵ as smoothly as the coefficients of operator $\tilde{\Delta}_\epsilon$ on Ω_0 defined by the pulled back energy form.

1.2. *Proof of Lemma 1.* Rather than the Green's function, we are interested in the Hadamard variational formula for the wave kernels $E_B(t)$, $S_B(t)$ in (2), or more precisely, for their distribution traces. We will give two proofs for the lemma.

First proof. By the definition of the distribution trace, we only need the variational formula for traces of variations $\delta\int_{\mathbb{R}} e^{i\lambda t}\hat{\psi}(t)E_B(t)\,dt$ of integrals of these kernels against test functions $\hat{\psi}(t)e^{i\lambda t}\in C_0^\infty(\mathbb{R})$, which are simpler because the Schwartz kernels are smooth.

We derive the Hadamard variational formula for wave traces from that of the Green's function by using the identities

$$-i\lambda R_B(\lambda)=\int_0^\infty e^{i\lambda t}E_B(t)\,dt,\qquad \frac{d}{dt}S_B(t)=E_B(t).\tag{8}$$

Using integration by parts (and $\Im\lambda>0$), we get

$$R_B(\lambda)=\int_0^\infty e^{i\lambda t}\;S_B(t)\,dt.\tag{9}$$

We will assume that $\hat{\psi}$ is supported in $\mathbb{R}_+$ since in the wave trace we localize its support to the length of a closed geodesic. Hence by (8),

$$\begin{aligned}\int_{\mathbb{R}}\hat{\psi}(t)e^{i\lambda t}\;E_B(t)\,dt&=\int_{\mathbb{R}}\psi(\mu)\int_0^\infty e^{i(\lambda-\mu)t}E_B(t)\,dt\,d\mu\\ &=-i\int_{\mathbb{R}}\psi(\mu)(\lambda-\mu)R_B(\lambda-\mu)\,d\mu.\end{aligned}\tag{10}$$

This implies that

$$\delta\int_{\mathbb{R}}\hat{\psi}(t)e^{i\lambda t}\;E_B(t)\,dt=-i\int_{\mathbb{R}}\psi(\mu)(\lambda-\mu)\delta R_B(\lambda-\mu)\,d\mu.$$

That we can pass δ under the integral sign can be justified using the dominated convergence theorem and we leave the proof to the reader. In the Dirichlet case, it follows from (10), (5), (8) and (9) that

$$
\begin{aligned}
&\delta \int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t}\, E_D(t,x,y)\,dt \\
&\qquad = -i \int_{\mathbb{R}} (\lambda-\mu)\psi(\mu) \int_{\partial\Omega_0} \partial_{\nu_2} G_D(\lambda-\mu,x,q)\partial_{\nu_1} G_D(\lambda-\mu,q,y)\dot{\rho}(q)\,dq\,d\mu \\
&\qquad = \int_{\mathbb{R}} \int_0^\infty e^{i(\lambda-\mu)t}\psi(\mu) \int_{\partial\Omega_0} \partial_{\nu_2} E_D(t,x,q)\partial_{\nu_1} G_D(\lambda-\mu,q,y)\dot{\rho}(q)\,dq\,d\mu\,dt \\
&\qquad = \int_{\mathbb{R}} \int_0^\infty \int_0^\infty e^{i(\lambda-\mu)(t+t')}\psi(\mu) \int_{\partial\Omega_0} \partial_{\nu_2} E_D(t,x,q)\partial_{\nu_1} S_D(t',q,y)\dot{\rho}(q)\,dq\,d\mu\,dt\,dt' \\
&\qquad = \int_0^\infty \int_0^\infty e^{i\lambda(t+t')}\hat{\psi}(t+t') \int_{\partial\Omega_0} \partial_{\nu_2} E_D(t,x,q)\partial_{\nu_1} S_D(t',q,y)\dot{\rho}(q)\,dq\,dt\,dt' \\
&\qquad = \int_0^\infty \int_{\partial\Omega_0} e^{i\lambda\tau}\hat{\psi}(\tau)\Big(\int_0^\tau \partial_{\nu_2} E_D(\tau-t',x,q)\partial_{\nu_1} S_D(t',q,y)dt'\Big)\dot{\rho}(q)\,dq\,d\tau.
\end{aligned}
$$

The inner integral is the same if we change the argument of E_D to t' and that of S_D to $\tau - t'$. We then average the two, set $x = y$, integrate over Ω_0 and use the angle addition formula for sin to obtain

$$
\delta\,\mathrm{Tr} \int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t} E_D(t)\,dt = \frac{1}{2} \int_{\partial\Omega_0} \int_{\mathbb{R}} t\hat{\psi}(t)e^{i\lambda t}\partial_{\nu_1}\partial_{\nu_2} S_D(t,q,q)\dot{\rho}(q)\,dt\,dq. \tag{11}
$$

The proof in the Neumann case is similar and left to the reader. We notice that in the above argument we have commuted the operations δ and Tr:

$$
\delta\,\mathrm{Tr} \int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t} E_D(t)\,dt = \mathrm{Tr}\,\delta \int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t} E_D(t)\,dt. \tag{12}
$$

To show this we first put $K_\epsilon(x,y) = \int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t} E_D(t,x,y)\,dt$. We then note that $K_\epsilon(x,y)$ is a C^1 curve in $C^\infty(\overline{\Omega}_\epsilon \times \overline{\Omega}_\epsilon)$, in the sense that $(d/d\epsilon)\varphi_\epsilon^* K_\epsilon(x,y)$ is a continuous curve in $C^\infty(\overline{\Omega}_0 \times \overline{\Omega}_0)$. Therefore both traces in (12) are the integrals of their corresponding kernels on the diagonal and hence (12) is equivalent to

$$
\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{\Omega_\epsilon} K_\epsilon(x,x)\,dx = \int_{\Omega_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} K_\epsilon(x,x)\,dx.
$$

However we have to be careful since the domain of integration on the left hand side depends on ϵ and under the variation it contributes an integral along the boundary. More precisely, since $(d/d\epsilon)\varphi_\epsilon^*(K_\epsilon(x,x))$ is a continuous curve in $C^\infty(\overline{\Omega}_0)$ and hence uniformly bounded, by the dominated convergence theorem

$$
\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{\Omega_\epsilon} K_\epsilon(x,x)\,dx &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{\Omega_0} \varphi_\epsilon^*(K_\epsilon(x,x))\varphi_\epsilon^*(dx) \\
&= \int_{\Omega_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} K_\epsilon(x,x)dx + \int_{\partial\Omega_0} K_0(q,q)\dot{\rho}(q)\,dq.
\end{aligned}
$$

But the second integral is zero in the Dirichlet case because $K_0(q,q) = 0$ for all $q \in \partial\Omega_0$. (Note: this term does not vanish in the Neumann case but it cancels out with a term which appears in the analogous computations). This concludes the first proof of Lemma 1. □

Second proof. This derivation is based on the Hadamard variational formulas for eigenvalues. When $\lambda_j^2(0)$ is a simple eigenvalue (i.e., of multiplicity one), Hadamard's variational formula for Dirichlet eigenvalues of Euclidean domains states that if $\epsilon \to \Omega_\epsilon$ is C^1 then

$$\delta(\lambda_j^2(\epsilon)) = -\int_{\partial\Omega_0} (\partial_\nu \Psi_j(q))^2 \dot{\rho}(q)\, dq,$$

where Ψ_j is an L^2 normalized eigenfunction for the eigenvalue $\lambda_j^2(0)$. See [Garabedian 1964]. However if the eigenvalue $\lambda_j^2(0)$ is multiple with multiplicity $m(\lambda_j(0))$ and if $\{\lambda_{j,k}^2(\epsilon)\}_{k=1}^{m(\lambda_j(0))}$ is the perturbed set of eigenvalues, then we cannot assume that $\lambda_{j,k}^2(\epsilon)$ are C^1 in ϵ (although this is known to be true for symmetric operators on finite-dimensional spaces. See, for example, Theorem II.6.8 of [Kato 1980]). But as we shall see, the sum $\sum_{k=1}^{m(\lambda_j(0))} \lambda_{j,k}^2(\epsilon)$ is C^1 in ϵ and there exists a Hadamard's variational formula for it which can be derived from the one for Green's function. In fact we prove a slightly more general statement. For the sake of convenience we let $\widetilde{R}_{B_\epsilon}(z) = (-\Delta_{B_\epsilon} - z)^{-1}$ where $z \notin \mathrm{Spec}(-\Delta_{B_\epsilon})$ and we use $\widetilde{G}_{B_\epsilon}(z, x, y)$ for its integral kernel. Now let $g(z)$ be a holomorphic function on the right half-plane $\Re(z) > 0$. We will show that

$$\delta \sum_{k=1}^{m(\lambda_j(0))} g(\lambda_{j,k}^2(\epsilon)) = -g'(\lambda_j^2(0)) \sum_{k=1}^{m(\lambda_j(0))} \int_{\partial\Omega_0} (\partial_\nu \Psi_{j,k}(q))^2 \dot{\rho}(q)\, dq, \tag{13}$$

where $\{\Psi_{j,k}\}_{k=1}^{m(\lambda_j(0))}$ is an orthonormal basis for the eigenspace of the multiple eigenvalue $\lambda_j^2(0)$. Lemma 1 follows easily from (13) by putting $g(z) = \cos(t\sqrt{z})$:

$$\begin{aligned}\delta \,\mathrm{Tr}\, E_B(t) = \delta \sum \cos(t\lambda_{j,k}) &= -t \sum_j \frac{\sin(t\lambda_j(0))}{2\lambda_j(0)} \left(\sum_{k=1}^{m(\lambda_j(0))} \int_{\partial\Omega_0} (\partial_\nu \Psi_{j,k})^2 \dot{\rho}(q)\, dq \right) \\ &= \frac{t}{2} \int_{\partial\Omega_0} \partial_{\nu_1}\partial_{\nu_2} S_B(t, q, q) \dot{\rho}(q)\, dq.\end{aligned}$$

We have pushed the operation δ under the sum. This can be done because for a test function $\varphi(t)$ the sums

$$\sum \int \cos(t\lambda_{j,k}(\epsilon))\varphi(t)\, dt \quad \text{and} \quad \sum \int \frac{d}{d\epsilon} \cos(t\lambda_{j,k}(\epsilon))\varphi(t) dt$$

are (by Weyl's law) uniformly convergent in ϵ.

It remains to prove (13). Let γ be a circle in $\mathbb{C}$ centered at $\lambda_j^2(0)$ such that no other eigenvalues of $-\Delta_{B_0}$ are in the interior of γ or on γ. We define

$$T_{g,\epsilon} = -\frac{1}{2\pi i} \int_\gamma g(z) \widetilde{R}_{B_\epsilon}(z)\, dz.$$

By the Cauchy integral formula, it is clear that at $\epsilon = 0$ we have $T_{g,0} = g(P_{\lambda_j^2(0)})$ where $P_{\lambda_j^2(0)}$ is the orthogonal projector on the eigenspace of $\lambda_j^2(0)$. Since the eigenvalues $\lambda_{j,k}^2(\epsilon)$ vary continuously in ϵ, for ϵ small these are the only eigenvalues of $-\Delta_{B_\epsilon}$ in γ. Therefore $T_{g,\epsilon}$ is the total projector (the direct sum of projectors) associated with $\{\lambda_{j,k}^2(\epsilon)\}_{k=1}^m$. The operator $T_{g,\epsilon}$ is C^1 in ϵ. See, for example, Theorem II.5.4

of [Kato 1980]. Although this theorem is stated for operators on finite dimensional spaces but the same proof works for our case. It is basically because the resolvent (and so the Green's function) is C^1 in ϵ. We now write

$$\begin{aligned}\delta \sum_{k=1}^{m(\lambda_j(0))} g(\lambda_{j,k}^2(\epsilon)) &= \delta\,\mathrm{Tr}(T_{g,\epsilon}) = -\,\mathrm{Tr}\,\frac{1}{2\pi i}\int_\gamma g(z)\delta\widetilde{R}_{D_\epsilon}(z)\,dz \\ &= -\int_{\Omega_0}\int_{\partial\Omega_0}\int_\gamma \frac{1}{2\pi i} g(z)\frac{\partial}{\partial\nu_2}G_D(z,x,q)\frac{\partial}{\partial\nu_1}G_D(z,q,x)\dot\rho(q)\,dz\,dq\,dx \\ &= -\int_{\Omega_0}\int_{\partial\Omega_0}\int_\gamma \frac{1}{2\pi i}\frac{g(z)}{(\lambda_j^2(0)-z)^2}\sum_{k=1}^{m}(\partial_\nu\Psi_{j,k}(q))^2(\Psi_{j,k}(x))^2\dot\rho(q)\,dz\,dq\,dx \\ &= -g'(\lambda_j^2(0))\sum_{k=1}^{m(\lambda_j(0))}\int_{\partial\Omega_0}(\partial_\nu\Psi_{j,k}|_{\partial\Omega_0})^2\dot\rho(q)\,dq.\end{aligned}$$

We leave it to the reader to show that, on the first line one can commute δ with Tr by means of the dominated convergence theorem.

There exist similar Hadamard variational formulas in the Neumann case. When the eigenvalue is simple, we have

$$\delta(\lambda_j^2) = \int_{\partial\Omega_0}\big(|\nabla_q^T(\Psi_j(q))|^2 - \lambda_j^2(0)(\Psi_j(q))^2\big)\dot\rho(q)\,dq, \tag{14}$$

For a multiple eigenvalue we sum over the expressions over an orthonormal basis of the eigenspace. The result does not depend on a choice of orthonormal basis. Similar computation using (14) follows to show Lemma 1 for the Neumann case. □

2. Proof of Theorem 2

We now study the singularity expansion of $\delta\,\mathrm{Tr}\cos(t\sqrt{-\Delta_B})$ and prove Theorem 2. At first sight, one could do this in two ways: by taking the variation of the spectral side of the formula, or by taking the variation of the singularity expansion. It seems simpler and clearer to do the former since we do not know how the invariant tori of the integrable elliptical billiard deform under an isospectral deformation. For example, one difficulty in taking the variation of the singularity expansion is that we do not know whether the fixed point set of an isospectral deformation Ω_ϵ of domain Ω_0 (satisfying the conditions of Theorem 2) is necessarily clean. Hence, even though we know that the wave trace of Ω_ϵ has the same type of singularity as the one for Ω_0, but we cannot apply the method of stationary phase and compute the principal term in the singularity expansion of the wave trace of Ω_ϵ.

In this section we will drop the subscript 0 in Ω_0 and we assume Ω is a smooth convex domain.

The variational formula for $\delta\,\mathrm{Tr}\cos(t\sqrt{-\Delta_B})$ is given in Lemma 1. In the Dirichlet case, by (4),

$$\mathrm{Tr}_{\partial\Omega}\ \dot\rho\ S_D^b = \pi_*\,\Delta^*\dot\rho\ (r_1 r_2 N_{\nu_1} N_{\nu_2} S_D(t,x,y)), \tag{15}$$

where N_ν is any smooth vector field in Ω extending ν, and where the subscripts indicate the variables on which the operator acts. In the Neumann case by (4),

$$\mathrm{Tr}_{\partial\Omega}\ \dot\rho\ S_N^b = \pi_*\ \Delta^*\dot\rho\ \big((\nabla_1^T\nabla_2^T r_1 r_2 + r_1 r_2\ \Delta_x)S_N(t,x,y)\big). \tag{16}$$

Here, $\Delta : \partial\Omega \to \partial\Omega\times\partial\Omega$ is the diagonal embedding $q \to (q,q)$ and π_* (the pushforward of the natural projection $\pi : \partial\Omega\times\mathbb{R}\to\mathbb{R}$) is the integration over the fibers with respect to the surface measure dq. The duplication in notation between the Laplacian and the diagonal is regrettable, but both are standard and should not cause confusion. Since $S_B(t,x,y)$ is microlocally a Fourier integral operator near the transversal periodic reflecting rays of F_T, it will follow from (15) that the trace is locally a Fourier integral distribution near $t = T$.

We are assuming that the set of periodic points of the billiard map corresponding to space-time billiard trajectories of length $T \in \mathrm{Lsp}(\Omega)$ is a submanifold F_T of $B^*\partial\Omega$. We thus fix $T \in \mathrm{Lsp}(\Omega)$ consisting only of periodic reflecting rays, that is, we assume $T \neq m|\partial\Omega|$ ($|\partial\Omega|$ being the perimeter) for $m \in \mathbb{Z}$. In order to study the singularity of the boundary trace near a component F_T of the fixed point set, we construct a pseudo-differential cutoff $\chi_T = \chi_T(t, D_t, q, D_q) \in \Psi^0(\mathbb{R}\times\partial\Omega)$ whose complete symbol $\chi_T(t,\tau,q,\zeta)$ has the form $\chi_T(q,\zeta/\tau)$ with $\chi_T(y,\zeta)$ supported in a small neighborhood of the fixed point set $F_T \subset B^*\partial\Omega$, equals one in a smaller neighborhood, and in particular vanishes in a neighborhood of the glancing directions in $S^*\partial\Omega = \partial(B^*\partial\Omega)$. Since the symbol of χ_T is independent of t we will instead use $\chi_T(D_t, q, D_q)$. We may assume that the support of the cutoff is invariant under the billiard map β. Therefore we need to study the operator

$$\pi_*\Delta^*\ \dot\rho\ \chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b, \tag{17}$$

and compute its symbol. To do this we first study the operators r and $S_B(t)$ and review their basic properties. Next we study the composition

$$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b,$$

and compute its symbol. Finally in Lemma 7 we take composition with $\pi_*\Delta^*\ \dot\rho$ and calculate the symbol of (17).

2.1. ***FIOs and their symbol.*** We recall that the principal symbol σ_I of a Fourier integral distribution

$$I = \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)}a(x,\theta)\,d\theta, \qquad I \in I^m(M,\Lambda_\varphi),$$

of order m is defined in terms of the parametrization

$$\iota_\varphi : C_\varphi = \{(x,\theta) : d_\theta\varphi = 0\} \to (x, d_x\varphi) \in \Lambda_\varphi \subset T^*M$$

of the associated Lagrangian Λ_φ. It is a half-density on Λ_φ given by $\sigma_I = (\iota_\varphi)_*(a_0|d_{C_\varphi}|^{1/2})$, where a_0 is the leading term of the classical symbol $a \in S^{m+\frac{n}{4}-\frac{N}{2}}(M\times\mathbb{R}^N)$, $n = \dim M$ and

$$d_{C_\varphi} := \frac{dc}{|D(c,\varphi'_\theta)/D(x,\theta)|}$$

is the Gelfand–Leray form on C_φ, where c is a system of coordinates on C_φ. For notation and background we refer to [Hörmander 1985b, Chapter XXV]. When $I(x, y) \in I^m(X \times Y, \Lambda)$ is the kernel of an FIO it is very standard to use the symplectic form $\omega_X - \omega_Y$ on $X \times Y$ and define

$$\iota_\varphi : C_\varphi = \{(x, y, \theta) : d_\theta \varphi = 0\} \to (x, d_x\varphi, y, -d_y\varphi) \in \Lambda_\varphi \subset T^*X \times T^*Y.$$

We will call Λ_φ the *canonical relation* of $I(x, y)$.

2.2. ***The restriction operator r as an FIO.*** The restriction r to the boundary lies in $I^{1/4}(\partial\Omega \times \mathbb{R}^n, \Gamma_{\partial\Omega})$, with the canonical relation

$$\Gamma_{\partial\Omega} = \big\{(q, \zeta, q, \xi) \in T^*\partial\Omega \times T^*_{\partial\Omega}\mathbb{R}^n;\ \xi|_{T_q\partial\Omega} = \zeta\big\}. \tag{18}$$

The adjoint then satisfies $r^* \in I^{1/4}(\mathbb{R}^n \times \partial\Omega, \Gamma^*_{\partial\Omega})$, where

$$\Gamma^*_{\partial\Omega} = \big\{(q, \xi, q, \zeta) \in T^*_{\partial\Omega}\mathbb{R}^n \times T^*\partial\Omega;\ \xi|_{T_q\partial\Omega} = \zeta\big\}.$$

Here, $T^*_{\partial\Omega}\mathbb{R}^n$ is the set of covectors to $\mathbb{R}^n$ with footpoint on $\partial\Omega$. We parametrize $\Gamma_{\partial\Omega}$ (18) by $T^{*+}_{\partial\Omega}(\Omega)$, the inward pointing covectors, using the Lagrange immersion

$$\iota_{\Gamma_{\partial\Omega}}(q, \xi) = (q, \xi|_{T_q(\partial\Omega)}, q, \xi). \tag{19}$$

To prove these statements, we introduce Fermi normal coordinates (q, x_n) along $\partial\Omega$, that is, $x = \exp_q(x_n\nu_q)$ where ν_q is the interior unit normal at q. Let $\xi = (\zeta, \xi_n) \in T^*_{(q,x_n)}\mathbb{R}^n$ denote the corresponding symplectically dual fiber coordinates. In these coordinates, the kernel of r is given by

$$r(q, (q', x_n')) = C_n \int_{\mathbb{R}^n} e^{i\langle q - q', \zeta\rangle - i x_n'\xi_n}\, d\xi_n\, d\zeta. \tag{20}$$

The phase $\varphi(q, (q', x_n'), (\zeta, \xi_n)) = \langle q - q', \zeta\rangle - x_n'\xi_n$ is nondegenerate and its critical set is $C_\varphi = \{(q, q', x_n', \xi_n, \zeta);\ q' = q, x_n' = 0\}$. The Lagrange map $\iota_\varphi : (q, q, 0, \xi_n, \zeta) \to (q, \zeta, q, \zeta, \xi_n)$ embeds $C_\varphi \to T^*\partial\Omega \times T^*\mathbb{R}^n$ and maps onto $\Gamma_{\partial\Omega}$. The adjoint kernel has the form $K^*(x, q) = \overline{K}(q, x)$ and therefore has a similar oscillatory integral representation. It is clear from ((20)) that the order of r as an FIO is $\frac{1}{4}$. Also, in the parametrization (19), the principal symbol of r is $\sigma_r = |dq \wedge d\zeta \wedge d\xi_n|^{1/2}$.

2.3. ***Background on parametrices for $S_B(t)$.*** We first review the Fourier integral description of $E_B(t)$, $S_B(t)$ microlocally near transversal reflecting rays. This is partly for the sake of completeness, but mainly because we need to compute their principal symbols (and related ones) along the boundary. Although the principal symbols are calculated in the interior in [Guillemin and Melrose 1979b, Proposition 5.1; Marvizi and Melrose 1982, Section 6; Petkov and Stoyanov 1992, Section 6], the results do not seem to be stated along the boundary (i.e., the symbols are not calculated at the boundary). The statements we need are contained in Theorem 3.1 of [Chazarain 1976] (and its proof), and we largely follow its presentation.

We need to calculate the canonical relation and principal symbol of the wave group, its derivatives and their restrictions to the boundary. We begin by recalling that the propagation of singularities theorem for the mixed Cauchy–Dirichlet (or Neumann) problem for the wave equation states that the wave front set of the wave kernel satisfies

$$WF(S_B(t,x,y)) \subset \bigcup_{\pm} \Lambda_{\pm},$$

where $\Lambda_{\pm} = \{(t,\tau,x,\xi,y,\eta) : (x,\xi) = \Phi^t(y,\eta),\ \tau = \pm|\eta|_y\} \subset T^*(\mathbb{R}\times\Omega\times\Omega)$ is the graph of the generalized (broken) geodesic flow, that is, the billiard flow Φ^t. For background we refer to [Guillemin and Melrose 1979b; Petkov and Stoyanov 1992; Chazarain 1973; 1976; Hörmander 1985a, Theorem 23.1.4; 1985b, Proposition 29.3.2]. For the application to spectral rigidity, we only need a microlocal description of wave kernels away from the glancing set, that is, in the hyperbolic set microlocally near periodic transversal reflecting rays. In these regions, there exists a microlocal parametrix due to Chazarain [1976], which is more fully analyzed in [Guillemin and Melrose 1979b; Petkov and Stoyanov 1992] and applied to the ellipse in [Guillemin and Melrose 1979a].

The microlocal parametrices for E_B and S_B are constructed in the ambient space $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. Since $E_B = dS_B/dt$ it suffices to consider the latter. Then there exists a Fourier integral (Lagrangian) distribution,

$$\tilde{S}_B(t,x,y) = \sum_{j=-\infty}^{\infty} S_j(t,x,y), \quad \text{with } S_j \in I^{-\frac{1}{4}-1}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n, \Gamma_{\pm}^j),$$

which microlocally approximates $S_B(t,x,y)$ modulo a smooth kernel near a transversal reflecting ray. The sum is locally finite hence well-defined. The canonical relation of $\tilde{S}_B$ is contained in a union

$$\Gamma = \bigcup_{\pm, j\in\mathbb{Z}} \Gamma_{\pm}^j \subset T^*(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$$

of canonical relations $\Gamma_{\pm}^j$ corresponding to the graph of the broken geodesic flow with j reflections. Notice we let $j\in\mathbb{Z}$ which is different from [Chazarain 1976] where j goes from 0 to ∞ and where the two graphs $\Gamma_{\pm}^j$ and $\Gamma_{\pm}^{-j}$ are combined.

We know discuss these graphs more precisely. We first recall some useful notation from [Chazarain 1976] with a slight adjustment. We have two Hamiltonian flows $g^{\pm t}$ corresponding to the Hamiltonians $\pm|\eta|$. For (y,η) in $T^*\Omega$ or (y,η) in $T^*_{\partial\Omega}\mathbb{R}^n$ where η is transversal to $\partial\Omega$ and is pointing inward, we define

$$t_{\pm}^1(y,\eta) = \inf\{t>0 : \pi g^{\pm t}(y,\eta) \in \partial\Omega\},$$
$$t_{\pm}^{-1}(y,\eta) = \sup\{t<0 : \pi g^{\pm t}(y,\eta) \in \partial\Omega\}.$$

In this notation we have $t_{\pm}^{-1} = -t_{\mp}^1$. We define $t_{\pm}^j$ inductively for $j>0$ (resp. $j<0$) to be the time of j-th reflection for the flow $g^{\pm t}$ as t increases (resp. decreases). Then we put

$$\lambda_{\pm}^1(y,\eta) = g^{\pm t_{\pm}^1(y,\eta)}(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^n,$$
$$\lambda_{\pm}^{-1}(y,\eta) = g^{\pm t_{\pm}^{-1}(y,\eta)}(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^n.$$

Next we define $\widehat{\lambda_{\pm}^1(y,\eta)}$ to be the reflection of $\lambda_{\pm}^1(y,\eta)$ at the boundary. That is, it has the same foot point y and the same tangential projection as $\lambda_{\pm}^1(y,\eta)$ but opposite normal component. Similarly we define $\widehat{\lambda_\pm^{-1}(y,\eta)}$. Flowing $\widehat{\lambda_\pm^{1}(y,\eta)}$ (resp. $\widehat{\lambda_\pm^{-1}(y,\eta)}$) by $g^{\pm t}$ as t increases (resp. decreases) and continuing the same procedure we get $t_\pm^j(y,\eta)$ and $\lambda_\pm^j(y,\eta)$ for all $j \in \mathbb{Z}$. We also set $T_\pm^j = \sum_{k=1}^j t_\pm^k$ for $j > 0$ and $T_\pm^j = \sum_{k=-1}^j t_\pm^k$ for $j < 0$.

The canonical graph $\Gamma_\pm^j$ can now be written as

$$\Gamma_\pm^j = \begin{cases} \{(t,\tau, g^{\pm t}(y,\eta), y,\eta) : \ \tau = \pm|\eta|_y\} & j = 0, \\ \{(t,\tau, g^{\pm(t - T_\pm^j(y,\eta))}\widehat{\lambda_\pm^j(y,\eta)}, y,\eta) : \tau = \pm|\eta|_y\} & j \in \mathbb{Z},\ j \neq 0. \end{cases} \tag{21}$$

For each $j \in \mathbb{Z}$, $\bigcup_\pm \Gamma_\pm^j$ is the union of two canonical graphs, which we refer to as its *branches* or *components* (see Figure 3.2 of [Guillemin and Melrose 1979b] for an illustration). These two branches arise because

$$S_B(t) = \frac{1}{2i\sqrt{-\Delta_B}}\left(e^{it\sqrt{-\Delta_B}} - e^{-it\sqrt{-\Delta_B}}\right)$$

is the sum of two terms whose canonical relations are respectively the graphs of the forward/backward broken geodesic flow and which correspond to the two halves $\tau > 0$, $\tau < 0$ of the characteristic variety $\tau^2 - |\eta|^2 = 0$ of the wave operator.

2.3.1. *Symbol of $S_B(t,x,y)$ in the interior.* In the boundaryless case of [Duistermaat and Guillemin 1975], the half-density symbol of $e^{it\sqrt{-\Delta_g}}$ is a constant multiple (Maslov factor) of the canonical graph volume half-density $\sigma_{\text{can}} = |dt \wedge dy \wedge d\eta|^{1/2}$ on Γ_+ in the graph parametrization $(t,y,\eta) \to \Gamma_+ = (t, |\eta|_g, g^t(y,\eta), y, \eta)$. In the boundary case for $E_B(t)$ the symbol in the interior is computed in Corollary 4.3 of [Guillemin and Melrose 1979b] as a scalar multiple of the graph half-density. It is a constant multiple of the graph half-density

$$\sigma_{\text{can},\pm} = |dt \wedge dy \wedge d\eta|^{1/2} \tag{22}$$

in the obvious graph parametrization of $\Gamma_\pm^j$ in (21); the constant equals $\frac{1}{2}$ in the Neumann case and $\frac{1}{2}(-1)^j$ in the Dirichlet case. However in [Guillemin and Melrose 1979b] the symbols are not calculated at the boundary.

Remark. We will have four modes of propagation at the boundary: in addition to the two $\pm$ branches corresponding to $\tau > 0$ and $\tau < 0$, at the boundary, the boundary condition requires two modes of propagation corresponding to the two "sides" of $\partial\Omega$. To illustrate this we first discuss a simple model of the upper half space.

2.3.2. *Upper half space; a local model for one reflection.* Let $\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \geq 0\}$ be the upper half space. Denote by $S_0(t,x,y)$ the kernel of $\sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$ of Euclidean $\mathbb{R}^n$. Then

$$\begin{cases} S_D(t,x,y) = S_0(t,x,y) - S_0(t,x,y^*), \\ S_N(t,x,y) = S_0(t,x,y) + S_0(t,x,y^*), \end{cases}$$

where $y^* \in \mathbb{R}_-^n$ is the reflection of y through the boundary $\mathbb{R}^{n-1} \times \{0\}$. Indeed, $y \to y^*$ is an isometry, so both kernels satisfy $\Box E_B = 0$ (in either the x or y variable) and have the correct initial conditions since $y^* \notin \mathbb{R}_+^n$. Further they satisfy the correct boundary conditions: it is clear that $S_D(t,x,y) = 0$ if

$y \in \mathbb{R}_+^{n-1} \times \{0\}$ since $y^* = y$ for such points. Also, if $x_n = 0$ then $S_D(t, x, y) = 0$ since $S_D(t, x, y)$ is a function of the distance $|x - y|$ and $|x - y| = |x - y^*|$ if $x_n = 0$. Similarly, the normal derivative is $\partial/\partial y_n$, so the normal derivatives cancel for $S_N(t, x, y)$ when $y_n = 0$. Also, $S_0(t, x, y^*) = S_0(t, x^*, y)$ and $S_0(t, x, y) = S_0(t, y, x)$, so the same calculation applies in the x variable. The canonical relation associated to S_N and S_D is the union of the canonical relations of S_0 and of $S_0^* = S_0(t, x, y^*)$. More precisely, by our notation in (21),

$$WF(S_B(t, x, y)) \subset \Gamma_\pm^0 \cup \Gamma_\pm^1 \cup \Gamma_\pm^{-1}.$$

Note that this example is asymmetric in past and future: the forward trajectory may intersect boundary, but then backward one does not. Also, in this example for $j > 1$ and $j < -1$ the graphs $\Gamma_\pm^j$ are empty.

2.3.3. *Symbol of $S_B(t, x, y)$ at the boundary.* Since we want to restrict kernels and symbols to the boundary, we introduce further notation for the subset of the canonical relations lying over boundary points. Following [Chazarain 1976], we denote by

$$A_\pm^0 = \{(0, \tau, y, \eta, y, \eta) : \tau = \pm|\eta|_y\}$$

the subset of $\Gamma_\pm^0$ with $t = 0$. Under the flow $\psi_\pm^t$ of the Hamiltonian $\tau \pm |\xi|_x$ on $\mathbb{R} \times \mathbb{R}^n$, it flows out to the graph $\Gamma_\pm^0$ (denoted by $C_\pm^0$ in [loc. cit., (2.11)]). One then defines $A_\pm^1 \subset \Gamma_\pm^0$ (resp. $A_\pm^{-1} \subset \Gamma_\pm^0$) as the subset lying over $\mathbb{R}_+ \times \partial\Omega \times \Omega$ (resp. $\mathbb{R}_- \times \partial\Omega \times \Omega$). Still following Chazarain, we denote by $\xi \to \widehat{\xi}$ the reflection map for $(q, \xi) \in T_q^*\mathbb{R}^n$, $q \in \partial\Omega$. That is, $\widehat{\xi}$ has the same tangential projection as ξ but opposite normal component. We then have

$$\Gamma_\pm^1 = \bigcup_{t \in \mathbb{R}} \psi_\pm^t \widehat{A}_\pm^1 \quad \text{and} \quad \Gamma_\pm^{-1} = \bigcup_{t \in \mathbb{R}} \psi_\pm^t \widehat{A}_\pm^{-1},$$

as the flow out under the Euclidean space-time geodesic flow of $\widehat{A}_\pm^1$ and $\widehat{A}_\pm^{-1}$. Thus, along the boundary, for $t > 0$ (resp. $t < 0$) $A_\pm^1$ and $\widehat{A}_\pm^1$ (resp. $A_\pm^{-1}$ and $\widehat{A}_\pm^{-1}$) both lie in the canonical relation of $E_B(t)$, $S_B(t)$. In a similar way one defines $A_\pm^2$ to be the subset of $\Gamma_\pm^1$ lying over $\mathbb{R}_+ \times \partial\Omega \times \Omega$ and $\widehat{A}_\pm^2$ to be its reflection. Then also $A_\pm^2 \cup \widehat{A}_\pm^2$ lies in the canonical relation. Similarly one defines $A_\pm^j$ and $\widehat{A}_\pm^j$ for all $j \in \mathbb{Z}$.

Remark. Since we are interested in the singularity of the trace at $t = T > 0$ we will only consider the graphs $\Gamma_\pm^j$ for $j \geq 0$. Regardless of this, because $\delta \operatorname{Tr} E_B(t)$ is even in t it has the same singularity at $t = T$ and $t = -T$.

The symbols of $E_B(t)$ and $S_B(t)$ are half-densities on the associated canonical relations, and therefore are sums of four terms at boundary points, that is, there is a contribution from each of $A_\pm^j$ and $\widehat{A}_\pm^j$. In the interior, there is only a contribution from the $\pm$ components.

The following lemma gives formulas for the principal symbol of S_B (and therefore E_B) on $\Gamma_\pm^j$ and its restriction to $\Gamma_{\partial\Omega} \circ (A_\pm^j \cup \widehat{A}_\pm^j)$.

Lemma 2. *Let $e_\pm$ be the principal symbol of $\tilde{S}_B$ when restricted to $\Gamma_\pm = \bigcup_j \Gamma_\pm^j$. Let σ_r be the principal symbol of the boundary restriction operator r.*

1. *In the interior, on* $\Gamma^j_\pm$, *up to Maslov factors we have*:

Dirichlet case:
$$e_\pm = \frac{(-1)^j}{2\tau}\sigma_{\mathrm{can},\pm} = \pm\frac{(-1)^j}{2|\eta|}\sigma_{\mathrm{can},\pm}.$$

Neumann case:
$$e_\pm = \frac{1}{2\tau}\sigma_{\mathrm{can},\pm} = \pm\frac{1}{2|\eta|}\sigma_{\mathrm{can},\pm}.$$

2. *At the boundary, on* $\Gamma_{\partial\Omega}\circ A^j_\pm = \Gamma_{\partial\Omega}\circ \widehat{A}^j_\pm$ *we have*:

Dirichlet case:
$$\sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \widehat{\lambda^j_\pm(y,\eta)}, y, \eta) = -\sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \lambda^j_\pm(y,\eta), y, \eta).$$

Neumann case:
$$\sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \widehat{\lambda^j_\pm(y,\eta)}, y, \eta) = \sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \lambda^j_\pm(y,\eta), y, \eta).$$

Proof. These formulas are obtained from the transport equations in [Chazarain 1976, (b'_0)–(e'_0), p. 175]. We now sketch the proof.

The transport equations for the symbols of E_B, S_B determine how they propagate along broken geodesics. As in the boundaryless case, the principal symbol has a zero Lie derivative, $\mathscr{L}_{H_{\tau+|\xi|}}\sigma_E = 0$, in the interior along geodesics. The important point for us is the rule by which they are reflected at the boundary. Let σ_B be the principal symbol of the boundary restriction operator B defined in (3) ($B = r$ under Dirichlet and $B = rN$ under Neumann boundary conditions) and let σ_0 be the principal symbol of the restriction operator to $t = 0$. Then:

$$\begin{aligned}
&(b_0) : (d^2/dt^2 - \Delta_B)\tilde{S}_B \sim 0 \Longrightarrow (b'_0) : \mathscr{L}_{\psi^t_\pm} e_\pm = 0;\\
&(c_0) : \tilde{S}_B|_{t=0} \sim 0 \qquad\qquad \Longrightarrow (c'_0) : \sigma_0\circ e_+(0,\tau,y,\eta,y,\eta) + \sigma_0\circ e_-(0,-\tau,y,\eta,y,\eta) = 0;\\
&(d_0) : \frac{d}{dt}\Big|_{t=0}\tilde{S}_B \sim \delta(x-y) \Longrightarrow (d'_0) : \tau\big(\sigma_0\circ e_+(0,\tau,y,\eta,y,\eta) - \sigma_0\circ e_-(0,-\tau,y,\eta,y,\eta)\big) = \sigma_I;\\
&(e_0) : B\tilde{S}_B \sim 0 \qquad\qquad \Longrightarrow (e'_0) : \sigma_B\circ e_\pm = \sigma_B\circ\big(e_\pm|_{A^j_\pm}\big) + \sigma_B\circ\big(e_\pm|_{\widehat{A}^j_\pm}\big) = 0.
\end{aligned} \tag{23}$$

Here σ_I is the principal symbol of the identity operator. The implication $(b_0) \Longrightarrow (b'_0)$ follows, for example, from Theorem 5.3.1 of [Duistermaat and Hörmander 1972]. The other implications are obvious. From (c'_0) and (d'_0) we get

$$(\sigma_0\circ e_\pm)(y,\eta,y,\eta) = \frac{(-1)^j}{2\tau}\sigma_I \quad \text{on} \quad T^*\Omega.$$

But by (b'_0), the symbol $e_\pm$ is invariant under the flow $\psi^t_\pm$ and therefore the first part of the lemma follows but only on $\Gamma^0_\pm$. The second part of the lemma follows from (e'_0). The first term of (e'_0) is known from the previous transport equations. Hence (e'_0) determines the "reflected symbol" at the j-th impact time and impact point. In the Dirichlet case, B is just r the restriction to the boundary and so the reflected principal symbol is simply the opposite of the direct principal symbol. In the Neumann case, B is the product of the symbol $\langle\lambda^1_\pm(y,\eta), \nu_y\rangle$ of the inward normal derivative times restriction r. The reflected symbol thus equals the direct symbol since the sign is canceled by the sign of the $\widehat{\langle \lambda^1_\pm(y,\eta), \nu_y\rangle} = -\langle \lambda^1_\pm(y,\eta), \nu_y\rangle$ factor. Thus, the volume half-density is propagated unchanged in the Neumann case and has a sign change at each impact point in the Dirichlet case. Thus on $\Gamma^j_\pm$ and after j reflections, the Dirichlet wave group symbol is $(-1)^j$ times $1/2\tau$ times the graph half-density and the Neumann symbol is $1/2\tau$ times the graph half-density. □

2.4. *$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_B(t, q', q)$ is a Fourier integral operator.*

Lemma 3. *We have*

$$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_B(t, q', q) \in I^{(1/2)+1-(1/4)}(\mathbb{R}\times\partial\Omega\times\partial\Omega, \Gamma_{\partial,\pm}).$$

Here, $\Gamma_{\partial,\pm} = \bigcup_{j\in\mathbb{Z}} \Gamma^j_{\partial,\pm}$, *with*

$$\Gamma^j_{\partial,\pm} := \big\{(t,\tau,q',\zeta',q,\zeta) \in T^*(\mathbb{R}\times\partial\Omega\times\partial\Omega) : \exists\, \xi' \in T^*_{q'}\mathbb{R}^n, \xi \in T^*_q\mathbb{R}^n : \\ (t,\tau,q',\xi',q,\xi) \in \Gamma^j_\pm,\ \ \xi'|_{T_{q'}\partial\Omega} = \zeta',\ \xi|_{T_q\partial\Omega} = \zeta\big\}.$$

Proof. We only show the proof in the Dirichlet case. The Neumann case is very similar. The kernel $\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_D(t, q', q)$ for fixed t is the Schwartz kernel of the composition

$$\chi_T \circ (r\ N) \circ S_D(t) \circ (N^*\, r^*) \circ \chi_T^* : L^2(\partial\Omega) \to L^2(\partial\Omega), \tag{24}$$

where r^* is the adjoint of $r : H^{1/2}(\overline{\Omega}) \to L^2(\partial\Omega)$.

To prove the lemma, we use that r is a Fourier integral operator with a folding canonical relation, and that the composition (24) is transversal away from the tangential directions to $\partial\Omega$, where $S_B(t)$ fails to be a Fourier integral operator. The cutoff χ_T removes the part of the canonical relation near the fold locus and near the normal directions $N^*\partial\Omega$ (where the composition $(r\ N)\circ S_D(t)\circ(N^*\, r^*)$ fails to be well-behaved as an FIO), hence the composition is a standard Fourier integral operator.

By the results cited above in [Chazarain 1976; Guillemin and Melrose 1979b; Petkov and Stoyanov 1992; Marvizi and Melrose 1982], microlocally away from the gliding directions, the wave operator $S_B(t)$ is a Fourier integral operator associated to the canonical relations $\Gamma^j_\pm$. Since $\Gamma^j_\pm$ is a union of graphs of canonical transformations, its composition (away from the normal bundle $N^*\partial\Omega$), with the canonical relation of $r^D := rN$ is automatically transversal. The further composition with the canonical relation of r^{D*} is also transversal. Hence, the composition is a Fourier integral operator with the composed wave front relation and the orders add. Taking into account that we have two boundary derivatives, we need to add $\frac{1}{2}$ to the order.

To determine the composite relation, we note that

$$\begin{aligned} &\Phi_\pm : \mathbb{R}\times T^*_{\partial\Omega}\mathbb{R}^n \to T^*\mathbb{R}\times T^*\Omega\times T^*_{\partial\Omega}\mathbb{R}^n, \\ &\Phi_\pm(t,q,\zeta,\xi_n) := (t, \pm|\zeta+\xi_n|, \Phi^t(q,\zeta,\xi_n), q, \zeta, \xi_n) \end{aligned} \tag{25}$$

parametrizes the graph of the (space-time) billiard flow with initial condition on $T^*_{\partial\Omega}\mathbb{R}^n$. Here, $\zeta \in T^*\partial\Omega$ and $\xi_n \in N^*_+\partial\Omega$, the inward pointing (co)normal bundle. $\Phi_\pm$ is a homogeneous folding map with folds along $\mathbb{R} \times T^*\partial\Omega$ (see, e.g., [Hörmander 1985a] for background). It follows that $S_D(t) \circ (N^* r^*)\chi_T^*$ is a Fourier integral operator of order one associated to the canonical relation

$$\left\{(t, \pm|\xi|, \Phi^t(q,\xi), q, \xi|_{T^*\partial\Omega}\right\} \subset T^*(\mathbb{R} \times \Omega \times \partial\Omega),$$

and is a local canonical graph away from the fold singularity along $T^*\partial\Omega$. Composing on the left by the restriction relation produces a Fourier integral operator with the stated canonical relation. The two normal derivatives N of course do not change the relation. □

2.5. *Symbol of $\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b(t, q', q)$.* The next step is to compute the principal symbols of the operators in Lemma 3.

To state the result, we need some further notation. We denote points of $T^*_{\partial\Omega}\mathbb{R}^n$ by $(q, 0, \zeta, \xi_n)$ as above, and put $\tau = \sqrt{|\zeta|^2 + \xi_n^2}$. We note that ξ_n is determined by (q, ζ, τ) by $\xi_n = \sqrt{\tau^2 - |\zeta|^2}$, since it is inward pointing. The coordinates q, ζ are symplectic, so the symplectic form on $T^*\partial\Omega$ is $d\sigma = dq \wedge d\zeta$. Also, below when we write $|\beta^j(q, \zeta/\tau)|$ we mean the norm of the fiber component of $\beta^j(q, \zeta/\tau)$ or when we write $\tau\beta^j(q, \zeta/\tau)$ we mean that τ is multiplied in the fiber component only. We now relate the graph of the billiard flow (25) with initial and terminal point on the boundary to the billiard map (after j reflections) by the formula

$$\Phi^{T_j}(q, 0, \zeta, \xi_n) = \left(\tau\beta^j\left(q, \frac{\zeta}{\tau}\right), \xi_n'(q, \zeta, \xi_n)\right), \tag{26}$$

where $\xi_n' = \tau\sqrt{1 - \left|\beta^j(q, \zeta/\tau)\right|^2}$. We also put

$$\gamma(q, \zeta, \tau) = \sqrt{1 - \frac{|\zeta|^2}{\tau^2}} \quad \text{and} \quad \gamma_1(q, \zeta) = \sqrt{1 - |\zeta|^2}. \tag{27}$$

It is the homogeneous (of degree zero) analogue of the function denoted by γ in [Hassell and Zelditch 2004].

Further, we parametrize the canonical relation $\Gamma^j_{\partial,+}$ of Lemma 3 using the billiard map β and its powers. We define the j-th return time $T^j(q, \xi)$ of the billiard trajectory in a codirection $(q, \xi) \in T^*_q\Omega$ to be the length the j-link billiard trajectory starting at (q, ξ) and ending at a point $\Phi^{T^j(q,\xi)}(q, \xi) \in T^*_{\partial\Omega}\Omega$. It is the same as $T^j_+(q, \xi)$. Then we define

$$\begin{aligned} &\iota_{\partial,j,+} : \mathbb{R}_+ \times T^*\partial\Omega \to T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega), \\ &\iota_{\partial,j,+}(\tau, q, \zeta) = \left(T^j(q, \xi(q, \zeta, \tau)), \tau, \left(\tau\beta^j\left(q, \frac{\zeta}{\tau}\right)\right), q, \zeta\right), \end{aligned} \tag{28}$$

where

$$\xi(q, \zeta, \tau) = \zeta + \xi_n \nu_q, \quad |\zeta|^2 + |\xi_n|^2 = \tau^2.$$

The map (28) parametrizes $\Gamma^j_{\partial,+}$ of Lemma 3.

Proposition 4. *In the coordinates $(\tau, q, \zeta) \in \mathbb{R}_+ \times T^*\partial\Omega$ of (28), the principal symbol of*

$$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b(t, q', q)$$

on $\Gamma^j_{\partial,+}$ *is as follows*:

- *in the Dirichlet case*:

$$\sigma_{j,+}(q,\zeta,\tau)=C^D_{j,+}\chi_T\Big(q,\frac{\zeta}{\tau}\Big)\chi_T\Big(\beta^j\Big(q,\frac{\zeta}{\tau}\Big)\Big)\gamma^{1/2}(q,\zeta,\tau)\gamma^{1/2}\Big(\tau\beta^j\Big(q,\frac{\zeta}{\tau}\Big),\tau\Big)\tau\,|dq\wedge d\zeta\wedge d\tau|^{1/2};$$

- *in the Neumann case*:

$$\sigma_{j,+}(q,\zeta,\tau)=C^N_{j,+}\chi_T\Big(q,\frac{\zeta}{\tau}\Big)\chi_T\Big(\beta^j\Big(q,\frac{\zeta}{\tau}\Big)\Big)\gamma^{-1/2}(q,\zeta,\tau)\gamma^{-1/2}\Big(\tau\beta^j\Big(q,\frac{\zeta}{\tau}\Big),\tau\Big) \times\Big(\Big\langle\zeta,\beta^j\Big(q,\frac{\zeta}{\tau}\Big)\Big\rangle-\tau\Big)\,|dq\wedge d\zeta\wedge d\tau|^{1/2}, \tag{29}$$

where the $C^B_{j,+}$ *are certain constants* (*Maslov factors*).

Proof. We only show the computations in the Dirichlet case. The Neumann case is very similar and uses (4) which will produce an additional factor of $\tau\langle\zeta,\beta^j(q,\zeta/\tau)\rangle-\tau^2$.

By Lemma 2, the principal symbol of $S_B(t)$ consists of four pieces at the boundary, one for each mode $A^j_\pm$, $\widehat{A}^j_\pm$. The symbol for the $-$ mode of propagation is equal to that for the $+$ mode of propagation under the time reversal map $\xi\to-\xi$. Further by part 2 of Lemma 2, the symbol at the boundary (adjusted by taking normal derivatives in the Dirichlet case) is invariant under the reflection map $\xi\to\hat{\xi}$ at the boundary due to the boundary conditions. Hence we only calculate the A^j_+ component and use the invariance properties to calculate the symbol on the other components.

We therefore assume that the symbol of S_B is $1/2\tau$ times the graph half-density $|dt\wedge dx\wedge d\xi|^{1/2}$ on Γ^j_+. We need to compose this graph half-density on the left by the symbol $\xi_n\,|dq\wedge d\zeta\wedge d\xi_n|^{1/2}$ of $r^D=r\,N$, and on the right by the symbol $\xi'_n\,|dq'\wedge d\zeta'\wedge d\xi'_n|^{1/2}$ of the adjoint $r^{D*}=N^*r^*$. Therefore we compute the restriction of the Γ^j_+ component onto $\Gamma^j_{\partial,+}$ and we remember to multiply the symbol by $\xi_n\xi'_n=\tau^2\gamma(q,\zeta,\tau)\gamma(\tau\beta^j(q,\frac{\zeta}{\tau}),\tau))$ and also by $1/2\tau$ at the end.

It is simplest to use symbol algebra and pullback formulae to calculate it [Duistermaat and Guillemin 1975]. One can also try to compute the symbol of this composition directly by using the oscillatory integral representations of these operators but that computation is more complicated. The composition is equivalent to the pullback of the symbol under the pullback

$$\Gamma^j_\partial=(i_{\partial\Omega}\times i_{\partial\Omega})^*\Gamma^j, \tag{30}$$

of the canonical relation of the S_B by the canonical inclusion map

$$i_{\partial\Omega}\times i_{\partial\Omega}:\mathbb{R}\times\partial\Omega\times\partial\Omega\to\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n.$$

We recall that a map $f:X\to Y$ is transversal to $W\subset T^*Y$ if $df^*\eta\neq 0$ for any $\eta\in W$. If $f:X\to Y$ is smooth and $\Gamma\subset T^*Y$ is Lagrangian, and if f and $\pi:T^*Y\to Y$ are transverse then $f^*\Gamma$ is Lagrangian. Since

$$(i_{\partial\Omega}\times i_{\partial\Omega})^*(t,\tau,\Phi^t(q,\xi),q,\xi)=(t,\tau,\Phi^t(q,\xi)|_{T\partial\Omega},q,\xi|_{T\partial\Omega})$$

at a point over $(i_{\partial\Omega}\times i_{\partial\Omega})(t,q',q)$, and since $\tau=|\xi|\neq 0$, it is clear that $i_{\partial\Omega}\times i_{\partial\Omega}$ is transversal to π.

We now claim that on the pullback of Γ^j, using the parametrization (28),

$$(i_{\partial\Omega} \times i_{\partial\Omega})^* |dt \wedge dx \wedge d\xi|^{1/2} = \gamma^{-1/2}(q, \zeta, \tau)\gamma^{-1/2}\Big(\tau\beta^j\Big(q, \frac{\zeta}{\tau}\Big), \tau\Big)\, |dq \wedge d\zeta \wedge d\tau|^{1/2}, \tag{31}$$

where γ is defined in (27). To see this, we use the pullback diagram

$$\begin{tikzcd}
\Gamma^j \arrow[d, "i"'] & F \arrow[l, "\pi"'] \arrow[r, "\alpha"] \arrow[d, "\pi"] & (i_{\partial\Omega} \times i_{\partial\Omega})^*\Gamma^j \subset T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega) \\
T^*(\mathbb{R} \times \mathbb{R}^n \times \Omega) & \mathcal{N}^*(\operatorname{graph}(i_{\partial\Omega} \times i_{\partial\Omega})) \arrow[l, "\pi"'] &
\end{tikzcd}$$

Here, F is the fiber product, $\mathcal{N}^* \operatorname{graph}(i_{\partial\Omega} \times i_{\partial\Omega})$ is the conormal bundle to the graph, and the map $\alpha : F \to (i_{\partial\Omega} \times i_{\partial\Omega})^*\Gamma^j$ is the natural projection to the composition [Duistermaat and Guillemin 1975]. Since the composition is transversal, $D\alpha$ is an isomorphism [loc. cit.]. The graph of $i_{\partial\Omega} \times i_{\partial\Omega}$ is the set $\{(t, q, q', t, q, q') : (t, q, q') \in \mathbb{R} \times \partial\Omega \times \partial\Omega\}$ and its conormal bundle is (in the Fermi normal coordinates),

$$\begin{aligned}
\mathcal{N}^*(\operatorname{graph}(i_{\partial\Omega} \times i_{\partial\Omega})) = \Big\{(t, \tau, q, \zeta, q', \zeta', t, -\tau, q, -\zeta + \xi_n, q', -\zeta' + \xi_n'),\ (q, \zeta, \xi_n),\ (q', \zeta', \xi_n') \in T^*_{\partial\Omega}\mathbb{R}^n\Big\} \\
\subset T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n).
\end{aligned}$$

The half-density produced by the pullback diagram takes the exterior tensor product of the canonical half-density

$$\big|dt \wedge d\tau \wedge dq \wedge d\zeta \wedge d\xi_n \wedge d\xi_n' \wedge dq' \wedge d\zeta'\big|^{1/2}$$

on $\mathcal{N}^*(\operatorname{graph}(i_{\partial\Omega} \times i_{\partial\Omega}))$ and

$$|dt' \wedge dx' \wedge d\xi'|^{1/2} \qquad \text{on } \Gamma^j \subset T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$$

at a point of the fiber product (where the $T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ components are equal) and divides by the canonical half-density

$$\big|dt' \wedge d\tau' \wedge dq' \wedge d\zeta' \wedge dx_n' \wedge d\xi_n' \wedge dx' \wedge d\xi'\big|^{1/2}$$

on the common $T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ component.

Since $\tau' = \tau$, the factors of $|dt' \wedge d\tau' \wedge dq' \wedge d\zeta' \wedge d\xi_n' \wedge dx' \wedge d\xi'|^{1/2}$ cancel in the quotient half-density, leaving the half-density

$$\frac{|dt \wedge dq \wedge d\zeta \wedge d\xi_n|^{1/2}}{|dx_n'|^{1/2}}$$

on the composite. The numerator is a half-density on $\mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n$. We write it more intrinsically in the following lemma. Note that it explains the first of our two γ factors.

Lemma 5. *Let $\Phi = \Phi_+$ be the parametrization (25). Then*

$$|dt \wedge dq \wedge d\zeta \wedge d\xi_n|^{1/2} = \left|\frac{\xi_n}{\sqrt{|\zeta|^2 + \xi_n^2}}\right|^{-1/2} \left|\Phi^*\Omega_{T^*\mathbb{R}^n}\right|^{1/2}$$

as half-densities on $\mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n$.

Proof. We have

$$\begin{aligned}\frac{\Phi^*\Omega_{T^*\mathbb{R}^n}}{dt\wedge dq\wedge d\zeta\wedge d\xi_n} &= \Omega_{T^*\mathbb{R}^n}\Big(\frac{d}{dt}\Phi^t(q,\zeta,\xi_n), d\Phi^t\frac{\partial}{\partial q_j}, d\Phi^t\frac{\partial}{\partial \zeta_j}, d\Phi^t\frac{\partial}{\partial \xi_n}\Big)\\ &= \Omega_{T^*\mathbb{R}^n}\Big(H_g, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial \zeta_j}, \frac{\partial}{\partial \xi_n}\Big)\\ &= \frac{\xi_n}{\sqrt{|\zeta|^2+\xi_n^2}}\Omega_{T^*\mathbb{R}^n}\Big(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial \zeta_j}, \frac{\partial}{\partial \xi_n}\Big) = \frac{\xi_n}{\sqrt{|\zeta|^2+\xi_n^2}},\end{aligned}$$

since

$$\frac{d}{dt}\Phi^t(q,\eta,\xi_n) = H_g = \frac{\xi_n}{\sqrt{|\zeta|^2+\xi_n^2}}\frac{\partial}{\partial x_n}+\cdots$$

is the Hamilton vector field of $g=\sqrt{g^2}$, $g^2=\xi_n^2+(g')^2$ where $\cdots$ represent vector fields in the span of $\partial/\partial q_j$, $\partial/\partial\zeta_j$, $\partial/\partial\xi_n$. Finally, we use that $d\Phi^t$ is a symplectic linear map and that q, x_n, ζ, ξ_n are symplectic coordinates. Note that we have evaluated the symplectic volume form at the domain point, not the image point. □

Next we consider the points in the image of Φ on $\mathbb{R}\times T^*_{\partial\Omega}\mathbb{R}^n$ where $x_n'=0$ and take the quotient by $|dx_n'|^{1/2}$, resulting in a half-density on Γ_∂^j. The next result explains the origin of the second γ factor.

Lemma 6. *In the subset $\Gamma_\partial^j\subset\Phi(\mathbb{R}\times T^*_{\partial\Omega}\mathbb{R}^n)$ where $x_n'=0$ and where $t=T^j$, we have (in the parametrizing coordinates (28)),*

$$\frac{|dt\wedge dq\wedge d\zeta\wedge d\xi_n|^{1/2}}{|dx_n'|^{1/2}} = \big|((\beta^j)^*\gamma^{-1})\,dq\wedge d\eta\wedge d\tau\big|^{1/2}.$$

Proof. By Lemma 5, it suffices to rewrite

$$|dx_n'|^{-1/2}\,\big|\Phi^*\Omega_{T^*\mathbb{R}^n}\big|^{1/2}$$

in the coordinates (τ,q,η) of $\iota_{\partial,j,+}$ in (28). We observe that $x_n'=\Phi^*x_n$. Hence

$$|dx_n'|^{-1/2}\big|\Phi^*\Omega_{T^*\mathbb{R}^n}\big|^{1/2} = \left|\Phi^*\frac{\Omega_{T^*\mathbb{R}^n}}{|dx_n|}\right|^{1/2} = \big|((\beta^j)^*\gamma^{-1})\,dq\wedge d\zeta\wedge d\tau\big|^{1/2}.$$

In the last equality, we have used (26), the equality $\dfrac{\Omega_{T^*\mathbb{R}^n}}{|dx_n|}=|dq\wedge d\zeta\wedge d\xi_n|$, and the fact that β is symplectic. Indeed, by (26),

$$\begin{aligned}\Phi^*(dq\wedge d\zeta\wedge d\xi_n) &= \Big(\tau(\beta^j)^*\Big(dq\wedge d\frac{\zeta}{\tau}\Big)\wedge\Phi^*d\xi_n\Big)\\ &= \Big(\tau(\beta^j)^*\Big(dq\wedge d\frac{\zeta}{\tau}\Big)\wedge\Phi^*d\sqrt{\tau^2-|\zeta|^2}\Big)\\ &= dq\wedge d\zeta\wedge\Phi^*\frac{\tau\,d\tau}{\sqrt{\tau^2-|\zeta|^2}} = \big((\beta^j)^*\gamma^{-1}\big)\,dq\wedge d\zeta\wedge d\tau.\end{aligned}$$

Note that $\tau(\beta^j)^*(dq\wedge d\frac{\zeta}{\tau}) = dq\wedge d\zeta|_{\beta^j(q,\zeta)}$. □

Combining Lemma 6 with Lemma 5 completes the proof of (31) and Proposition 4. □

2.6. *Trace along the boundary: composition with $\pi_*\Delta^*$.* We now take the trace along the boundary of this operator. Analogously to [Duistermaat and Guillemin 1975; Guillemin and Melrose 1979a; Marvizi and Melrose 1982], we define $\Delta : \mathbb{R} \times \partial\Omega \to \mathbb{R} \times \partial\Omega \times \partial\Omega$ to be the diagonal embedding and π_* to be integration over $\partial\Omega$.

Lemma 7. *If the fixed point sets of period T of β^k are clean for all k and form a submanifold F_T of $B^*\partial\Omega$ of dimension d (with connected components Γ), then*

$$\pi_*\Delta^*\dot{\rho}\,\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b(t, q', q) \in I^{(d/2)+(1/2)+1-(1/4)}(\mathbb{R},\ T_T^*\mathbb{R}),$$

where

$$T_T^*\mathbb{R} = \bigcup\nolimits_\pm \Lambda_{T,\pm} = \bigcup\nolimits_\pm \{(T, \pm\tau) : \tau \in \mathbb{R}_+\},$$

and its principal symbol on $\Lambda_{T,\pm}$ is given by

$$c^\pm \tau^{(d+2)/2}\sqrt{d\tau},$$

where

$$c^\pm = \sum_{\Gamma\subset F_T} C_\Gamma^\pm \int_\Gamma \dot{\rho}\,\gamma_1\,d\mu_\Gamma$$

and $c^- = \overline{c^+}$ the complex conjugate of c^+.

Proof. The calculation of the principal symbol of the trace of a Fourier integral operator in [Duistermaat and Guillemin 1975] is valid for the boundary restriction of the wave kernel, since it only uses that it is $\pi_*\Delta^*$ composed with a Fourier integral kernel with a known symbol and canonical relation. Hence we follow the proof closely and refer there for further details.

As in [Guillemin and Melrose 1979a], the composition of $\pi_*\Delta^*$ with

$$\dot{\rho}\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b(t, q, q') \tag{32}$$

is clean if and only if the fixed point set of β^k corresponding to periodic orbits of period T is clean. When the fixed point set has dimension d in the ball bundle $B^*\partial\Omega$, composition with $\pi_*\Delta^*$ adds $d/2$ to the order [Duistermaat and Guillemin 1975, (6.6)]. Combining with Lemma 3, we obtain the order

$$\frac{d}{2} + \frac{1}{2} + 1 - \frac{1}{4}.$$

Hence under the cleanliness assumption, it follows that $\delta\,\mathrm{Tr}\cos t\sqrt{-\Delta_B}$ is a Lagrangian distribution on $\mathbb{R}$ with singularities at $t \in \mathrm{Lsp}(\Omega)$. As discussed in [loc. cit.] for the upper/lower half lines $\Lambda_{T,\pm}$ in $T_T^*\mathbb{R}$, $I^{\frac{d}{2}+\frac{5}{4}}(\mathbb{R}, \Lambda_{T,\pm})$ consists of multiples of the distribution

$$\int_0^\infty \tau^{(d+2)/2} e^{\pm i\tau(t-T)}\,d\tau = (t - T \pm i0)^{-(d+4)/2}.$$

The principal symbol of this Fourier integral distribution is $\tau^{(d+2)/2}\sqrt{d\tau}$. Therefore to conclude the Lemma we only need to compute the coefficients of this symbol in the trace.

This coefficient is computed in a universal way from the principal symbol of (32) computed from Proposition 4. Following the proof in [loc. cit.], the coefficient of $\tau^{(d+2)/2}\sqrt{d\tau}$ is

$$c^{\pm} = \sum_{\Gamma \subset F_T} C_{\Gamma}^{\pm} \int_{\Gamma} \dot{\rho}\, \gamma_1 \, d\mu_{\Gamma},$$

where F_T is the fixed point set of β (and its powers) in $B^*\partial\Omega$. The sum is over the connected components Γ of F_T. Here, $d\mu_\Gamma$ is the restriction to Γ of a density $d\mu$ on F_T which is the pushforward (under the natural projection map) of the canonical density defined on the fixed point set of Φ^T on $S^*_{\partial\Omega}\Omega$. This canonical density is defined in Lemma 4.2 of [Duistermaat and Guillemin 1975]. We note that the distribution $c^+(t-T+i0)^{-(d+4)/2} + c^-(t-T-i0)^{-(d+4)/2}$ is real only if $c^- = \bar{c^+}$. This completes the proof of the lemma. □

The lemma also completes the proof of the Theorem 2.

Remark. As a check on the order, we note that for the wave trace in the interior and for nondegenerate closed trajectories, the singularities are of order $(t-T+i0)^{-1}$. When the periodic orbits are degenerate and the unit vectors in the fixed point sets have dimension d, the singularity increases to order

$$(t-T+i0)^{-1-\frac{d}{2}}.$$

If we formally take the variation of the wave trace, the singularity should increase to order

$$(t-T+i0)^{-1-\frac{d}{2}-1}.$$

In comparison, the boundary trace in the Dirichlet case involves two extra derivatives of the wave kernel and composition with $(-\Delta)^{-1/2}$. Compared to the interior trace, this adds one net derivative and order to the trace singularity. We claim that the restriction to the boundary does not further change the order compared to the interior trace. This can be seen by considering the method of stationary phase for oscillatory integrals with Bott–Morse phase functions, whose nondegenerate critical manifolds are transverse to the boundary. If we restrict the integral to the boundary, we do not change the number of phase variables in the integral, but we simultaneously decrease the number of variables by one and the dimension of the fixed point set by one. The number of nondegenerate directions stays the same. It follows that the singularity order of the variational trace goes up by one overall unit compared to the interior trace, consistently with the formal variational calculation.

3. Case of the ellipse and the proof of Theorem 1

In this section we let Ω_0 be an ellipse. In this case, the fixed point sets are clean fixed point sets for Φ^t in $T^*\Omega_0$ and for β in $B^*\partial\Omega_0$ [Guillemin and Melrose 1979a, Proposition 4.3]. In fact the fixed point sets F_T of β in $B^*\partial\Omega_0$ form a one dimensional manifold. Thus $d=1$ and Corollary 2 follows.

As is well-known, both the billiard flow and billiard map of the ellipse are completely integrable. In particular, except for certain exceptional trajectories, the periodic points of period T form a Lagrangian tori in $S^*\Omega_0$, and the homogeneous extensions of the Lagrangian tori are cones in $T^*\Omega_0$. The exceptions are the two bouncing ball orbits through the major/minor axes and the trajectories which intersect the foci or glide along the boundary. The fixed point sets of Φ^T intersect the coball bundle $B^*\partial\Omega_0$ of the boundary in the fixed point sets of the billiard map $\beta : B^*\partial\Omega_0 \to B^*\partial\Omega_0$ (for background we refer to [Petkov and Stoyanov 1992; Guillemin and Melrose 1979a; 1979b; Hassell and Zelditch 2004; Toth and Zelditch 2012] for instance). Except for the exceptional orbits, the fixed point sets are real analytic curves. For the bouncing ball rays, the associated fixed point sets are nondegenerate fixed points of β.

Since the final step of the proof uses results of [Guillemin and Melrose 1979a], we briefly review the description of the billiard map of the ellipse $\Omega_0 := x^2/a + y^2/b = 1$ (with $a > b > 0$) in that article. In the interior, there exist for each $0 < Z \le b$ a caustic set given by a confocal ellipse

$$\frac{x^2}{E+Z} + \frac{y^2}{Z} = 1,$$

where $E = a - b$, or for $-E < Z < 0$ by a confocal hyperbola. Let (q, ζ) be in $B^*\partial\Omega_0$ and let (q, ξ) in $S^*\Omega_0$ be the unique inward unit normal to boundary that projects to (q, ζ). The line segment $(q, r\xi)$ will be tangent to a unique confocal ellipse or hyperbola (unless it intersects the foci). We then define the function $Z(q, \zeta)$ on $B^*\partial\Omega_0$ to be the corresponding Z. Then Z is a β-invariant function and its level sets $\{Z = c\}$ are the invariant curves of β. The invariant Leray form on the level set is denoted by du_Z [loc. cit., (2.17)]; thus the symplectic form of $B^*\partial\Omega_0$ is $dq \wedge d\zeta = dZ \wedge du_Z$. A level set has a rotation number and the periodic points live in the level sets with rational rotation number. As it is explained in [loc. cit., p. 143] the Leray form du_Z restricted to a connected component Γ of F_T is a constant multiple of the canonical density $d\mu_\Gamma$.

As mentioned in the introduction, the well-known obstruction to using trace formula calculations such as in Theorem 2 is multiplicity in the length spectrum, that is, existence of several connected components of F_T. A higher dimensional component is not itself a problem, but there could exist cancellations among terms coming from components with different Morse indices, since the coefficients C_Γ are complex. This problem arose earlier in the spectral theory of the ellipse in [loc. cit.]. The key Proposition 4.3 there shows that there is a sufficiently large set of lengths T for which F_T has one component up to $(q, \zeta) \to (q, -\zeta)$ symmetry. Since it is crucial here as well, we state the relevant part:

Proposition 8 [Guillemin and Melrose 1979a, Proposition 4.3]. *Let $T_0 = |\partial\Omega_0|$. Then for every interval $(mT_0 - \epsilon,\ mT_0)$, for $m = 1, 2, 3, \ldots,$ there exist infinitely many periods $T \in \mathrm{Lsp}(\Omega_0)$ for which F_T is the union of two invariant curves which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$.*

Since for an isospectral deformation $\delta\,\mathrm{Tr}\cos(t\sqrt{-\Delta}) = 0$, we obtain from Theorem 2:

Corollary 9. *Suppose we have an isospectral deformation of an ellipse Ω_0 with velocity $\dot\rho$. Then for each T in Proposition 8 for which F_T is the union of two invariant curves Γ_1 and Γ_2 which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$ we have*

$$\int_{\Gamma_j} \dot\rho\, \gamma_1\, du_Z = 0, \qquad j = 1, 2.$$

Proof. From Theorem 2 we get

$$\Re\left\{\left(\sum_{j=1}^{2} C_{\Gamma_j}\int_{\Gamma_j}\dot\rho\,\gamma_1\,d\mu_{\Gamma_j}\right)(t-T+i0)^{-2-(d/2)}\right\}=0.$$

Since $\dot\rho$ and γ_1 are invariant under the time reversal map $(q,\zeta)\to(q,-\zeta)$, the two integrals are identical. Also by directly looking at the stationary phase calculations it can be shown that the Maslov coefficients C_{Γ_1} and C_{Γ_2} are also the same. Thus the corollary follows. □

3.1. ***Abel transform.*** The remainder of the proof of Theorem 1 is identical to that of Theorem 4.5 of [Guillemin and Melrose 1979a] (see also [Popov and Topalov 2003]). For the sake of completeness, we sketch the proof.

Proposition 10. *The only $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant function $\dot\rho$ satisfying the equations of Corollary 9 is $\dot\rho=0$.*

Proof. First, we may assume $\dot\rho=0$ at the endpoints of the major/minor axes, since the deformation preserves the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry and we may assume that the deformed bouncing ball orbits will not move and are aligned with the original ones. Thus $\dot\rho(\pm\sqrt{a})=\dot\rho(\pm\sqrt{b})=0$.

The Leray measure may be explicitly evaluated [Guillemin and Melrose 1979a, eq. 2.18]. By a change of variables with Jacobian J, and using the symmetric properties of $\dot\rho$, the integrals become

$$A(Z)=\int_b^a\frac{\dot\rho(t)\,\gamma_1\,J(t)\,dt}{\sqrt{t-(b-Z)}},\tag{33}$$

for an infinite sequence of Z accumulating at b. The function $A(Z)$ is smooth in Z for Z near b. It vanishes infinitely often in each interval $(b-\epsilon,b)$, hence is flat at b. The k-th Taylor coefficient at b is

$$A^{(k)}(b)=\int_b^a\dot\rho(t)\,\gamma_1\,J(t)t^{-k-(1/2)}dt=0.\tag{34}$$

Since the functions t^{-k} span a dense subset of $C[b,a]$, it follows that $\dot\rho\equiv0$. □

3.2. ***Infinitesimal rigidity and flatness.*** We now show that infinitesimal rigidity implies flatness and prove Corollary 1. As mentioned, the Hadamard variational formula is valid for any C^1 parametrization $\Omega_{\alpha(\epsilon)}$ of the domains Ω_ϵ. For each one we have $\delta\rho_{\alpha(\epsilon)}(x)\equiv0$.

Assume $\rho_\epsilon(x)$ is not flat at $\epsilon=0$ and let ϵ^k be the first nonvanishing term in the Taylor expansion of $\rho_\epsilon(x)$ at $\epsilon=0$. Then

$$\rho_\epsilon(x)=\epsilon^k\frac{\rho^{(k)}(x)}{k!}+\epsilon^{k+1}\frac{\rho^{(k+1)}(x)}{(k+1)!}+\cdots.\tag{35}$$

We then reparametrize the family by $\epsilon\to\alpha(\epsilon):=\epsilon^{1/k}$ so that

$$\rho_{\alpha(\epsilon)}(x)=\frac{\rho^{(k)}(x)}{k!}\epsilon+O(\epsilon^{1+1/k}).$$

By Hadamard's variational formulae we get $\delta\rho_{\alpha(\epsilon)}(x)=\rho^{(k)}(x)\equiv0$, a contradiction.

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A NATURAL LOWER BOUND FOR THE SIZE OF NODAL SETS

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We prove that, for an n-dimensional compact Riemannian manifold (M, g), the $(n-1)$-dimensional Hausdorff measure $|Z_\lambda|$ of the zero-set Z_λ of an eigenfunction e_λ of the Laplacian having eigenvalue $-\lambda$, where $\lambda \geq 1$, and normalized by $\int_M |e_\lambda|^2 dV_g = 1$ satisfies

$$C|Z_\lambda| \geq \lambda^{\frac{1}{2}} \left(\int_M |e_\lambda| \, dV_g \right)^2$$

for some uniform constant C. As a consequence, we recover the lower bound $|Z_\lambda| \gtrsim \lambda^{(3-n)/4}$.

The purpose of this brief note is to prove a natural lower bound for the $(n-1)$-dimensional Hausdorff measure of nodal sets of eigenfunctions. To wit:

Theorem 1. *Let (M, g) be a compact manifold of dimension n and e_λ an eigenfunction satisfying*

$$-\Delta_g e_\lambda = \lambda e_\lambda, \text{ and } \int_M |e_\lambda|^2 \, dV_g = 1.$$

Then if $Z_\lambda = \{x \in M : e_\lambda(x) = 0\}$ is the nodal set and $|Z_\lambda|$ its $(n-1)$-dimensional Hausdorff measure, we have

$$\lambda^{\frac{1}{2}} \left(\int_M |e_\lambda| \, dV_g \right)^2 \leq C|Z_\lambda|, \quad \lambda \geq 1, \tag{1}$$

for some uniform constant C. Consequently,

$$\lambda^{\frac{3-n}{4}} \lesssim |Z_\lambda|, \quad \lambda \geq 1. \tag{2}$$

Inequality (2) follows from (1) and the lower bounds in [Sogge and Zelditch 2011a]

$$\lambda^{\frac{1-n}{8}} \lesssim \int_M |e_\lambda| \, dV_g. \tag{3}$$

The lower bound (2) is due to Colding and Minicozzi [2011]. Yau [1982] conjectured that $\lambda^{\frac{1}{2}} \approx |Z_\lambda|$. This lower bound $\lambda^{\frac{1}{2}} \lesssim |Z_\lambda|$ was verified in the two-dimensional case by Brüning [1978] and independently by Yau (unpublished). The bounds in (2) seem to be the best known ones for higher dimensions, although Donnelly and Fefferman [1988; 1990] showed that, as conjectured, $|Z_\lambda| \approx \lambda^{\frac{1}{2}}$, if (M, g) is assumed to be real analytic.

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The first "polynomial type" lower bounds appear to be those given in [Colding and Minicozzi 2011] and [Sogge and Zelditch 2011a] (see also [Mangoubi 2011]). As we shall point out, inequality (1) cannot be improved and it to some extent unifies the approaches in [Colding and Minicozzi 2011] and [Sogge and Zelditch 2011a]. As was shown in the latter paper, the L^1-lower bounds in (3) follow from Hölder's inequality and the L^p eigenfunction estimates of [Sogge 1988] for the range where $2 < p \leq 2(n+1)/(n-1)$. These too cannot be improved, but it is thought better L^p-bounds hold for a typical eigenfunction or if one makes geometric assumptions such as negative curvature (cf. [Sogge and Zelditch 2010; 2011b]). Thus, it is natural to expect to be able to improve (3) and hence the lower bounds (2) for all eigenfunctions on manifolds with negative curvature, or for "typical" eigenfunctions on any manifold. Of course, Yau's conjecture that $|Z_\lambda| \approx \lambda^{\frac{1}{2}}$ would be the ultimate goal, but understanding when (3) can be improved is a related problem of independent interest.

Let us now turn to the proof of Theorem 1. We shall use an identity from [Sogge and Zelditch 2011a]:

$$\int_M |e_\lambda|\,(\Delta_g+\lambda)f\,dV_g = 2\int_{Z_\lambda} |\nabla_g e_\lambda|\,f\,dS_g, \tag{4}$$

Here dS_g is the Riemannian surface measure on Z_λ, and ∇_g is the gradient coming from the metric and $|\nabla_g u|$ is the norm coming from the metric, meaning that in local coordinates

$$|\nabla_g u|_g^2 = \sum_{jk=1}^{n} g_{jk}(x)\partial_j u \partial_k u. \tag{5}$$

Identity (4) follows from the Gauss–Green formula and a related earlier identity was proved by Dong [1992].

As in [Hezari and Wang 2011], if we take $f \equiv 1$ and apply Schwarz's inequality we get

$$\lambda \int_M |e_\lambda|\,dV_g \leq 2|Z_\lambda|^{1/2}\left(\int_{Z_\lambda} |\nabla_g e_\lambda|^2\,dS_g\right)^{1/2}. \tag{6}$$

Thus we would have (1) if we could prove that the energy of e_λ on its nodal set satisfies the natural bounds

$$\int_{Z_\lambda} |\nabla_g e_\lambda|^2\,dS_g \lesssim \lambda^{\frac{3}{2}}. \tag{7}$$

We shall do this by choosing a different auxiliary function f. This time we want to use

$$f = \big(1+\lambda e_\lambda^2 + |\nabla_g e_\lambda|_g^2\big)^{\frac{1}{2}}. \tag{8}$$

If we plug this into (4) we get that

$$2\int_{Z_\lambda} |\nabla_g e_\lambda|_g^2 dS_g \leq \int_M |e_\lambda|\,(\Delta_g+\lambda)\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\big)^{\frac{1}{2}}\,dV_g.$$

Since we have the L^2-Sobolev bounds

$$\|e_\lambda\|_{H^s(M)} = O(\lambda^{\frac{s}{2}}), \tag{9}$$

it is clear that

$$\lambda \int_M |e_\lambda| \left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\right)^{\frac{1}{2}} dV_g = O(\lambda^{\frac{3}{2}}),$$

and thus to prove (7), it suffices to show that

$$\int_M |e_\lambda|\, \Delta_g\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\right)^{\frac{1}{2}} dV_g = O(\lambda^{\frac{3}{2}}). \tag{10}$$

To prove this we first note that

$$\partial_k\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\right)^{\frac{1}{2}} = \frac{\lambda e_\lambda \partial_k e_\lambda + \frac{1}{2}\partial_k |\nabla_g e_\lambda|_g^2}{\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}}};$$

from this and (9) we deduce that

$$\int_M |e_\lambda| \left|\nabla_g\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}}\right| dV_g = O(\lambda).$$

This means that the contribution of the first order terms of the Laplace–Beltrami operator (written in local coordinates) to (10) are better than required, and so it suffices to show that in a compact subset K of a local coordinate patch we have

$$\int_K |e_\lambda| \left|\partial_j\partial_k\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}}\right| dV_g = O(\lambda^{\frac{3}{2}}). \tag{11}$$

A calculation shows that $\partial_j\partial_k\left(\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}}$ equals

$$-\frac{\left(\lambda e_\lambda \partial_j e_\lambda + \frac{1}{2}\partial_j|\nabla_g e_\lambda|_g^2\right)\left(\lambda e_\lambda \partial_k e_\lambda + \frac{1}{2}\partial_k|\nabla_g e_\lambda|_g^2\right)}{\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{3}{2}}} + \frac{\lambda \partial_j e_\lambda \partial_k e_\lambda + \lambda e_\lambda \partial_j\partial_k e_\lambda + \frac{1}{2}\partial_j\partial_k|\nabla_g e_\lambda|_g^2}{\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}}}.$$

If $|D^m f| = \sum_{|\alpha|=m} |\partial^\alpha f|$, then by (5)

$$\partial_k|\nabla_g e_\lambda|^2 = O(|D^2 e_\lambda|\,|De_\lambda| + |De_\lambda|^2),$$

and

$$\partial_j\partial_k|\nabla_g e_\lambda|_g^2 = O(|D^3 e_\lambda|\,|De_\lambda| + |D^2 e_\lambda|^2 + |D^2 e_\lambda|\,|De_\lambda| + |De_\lambda|^2).$$

Therefore,

$$\begin{aligned}
&\partial_j\partial_k\left(\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}} \\
&\quad = O\left(\frac{\lambda^2|e_\lambda|^2|De_\lambda|^2 + |D^2 e_\lambda|^2|De_\lambda|^2 + |De_\lambda|^4}{\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{3}{2}}}\right) \\
&\qquad + O\left(\frac{\lambda|De_\lambda|^2 + \lambda|e_\lambda|\,|D^2 e_\lambda| + |D^3 e_\lambda|\,|De_\lambda| + |D^2 e_\lambda|^2 + |D^2 e_\lambda|\,|De_\lambda| + |De_\lambda|^2}{\left(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\right)^{\frac{1}{2}}}\right).
\end{aligned}$$

This implies that the integrand in the left side of (11) is dominated by

$$\big(\lambda^{\frac{1}{2}}|De_\lambda|^2+\lambda^{-\frac{1}{2}}|D^2e_\lambda|^2+|De_\lambda|^2\big) \\ +\big(\lambda^{\frac{1}{2}}|De_\lambda|^2+\lambda^{\frac{1}{2}}|e_\lambda|\,|D^2e_\lambda|+|e_\lambda|\,|D^3e_\lambda|+\lambda^{-\frac{1}{2}}|D^2e_\lambda|^2+|D^2e_\lambda|\,|e_\lambda|+|De_\lambda|\,|e_\lambda|\big),$$

leading to (11) after applying (9). ☐

Remarks.

- We could also have taken f to be $(\lambda+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2)^{\frac{1}{2}}$ and obtained the same upper bounds, but there does not seem to be any advantage to doing this.
- Inequality (1) cannot be improved. There are many cases when the L^1 and L^2-norms of eigenfunctions are comparable. For instance, for the sphere the zonal functions have this property and it is easy to check that their nodal sets satisfy $|Z_\lambda|\approx\lambda^{\frac{1}{2}}$, which means that for zonal functions (1) cannot be improved.
- There are many cases where inequality (1) can be improved. For instance, the L^2-normalized highest weight spherical harmonics Q_k have eigenvalues $\lambda=\lambda_k\approx k^2$, and L^1-norms $\approx k^{-\frac{n-1}{4}}$ (see e.g., [Sogge 1986]). This means that for the highest weight spherical harmonics the left side is proportional to $\lambda^{\frac{3-n}{4}}$ even though here too $|Z_\lambda|\approx\lambda^{\frac{1}{2}}$. Similarly, the highest weight spherical harmonics saturate (7). It is because of functions like the highest weight spherical harmonics that the current techniques only seem to yield (2). Note that inequality (2) gives the correct lower bound in the trivial case where the dimension n is one. As the dimension increases, the bound gets worse and worse due to the fact that (3) is saturated by functions like the highest weight spherical harmonics ("Gaussian beams") whose mass is supported on a $\lambda^{-\frac{1}{4}}$ neighborhood of a geodesic and the volume of such a tube decreases geometrically as n increases. (See [Bourgain 2009; Sogge 2011] for related work on this phenomena.)
- W. Minicozzi pointed out to us that (7) also follows from the identity

$$2\int_{Z_\lambda}|\nabla_g e_\lambda|^2\,dS_g=-\int_M \operatorname{sgn}(e_\lambda)\operatorname{div}_g\big(|\nabla_g e_\lambda|\,\nabla_g e_\lambda\big)\,dV_g. \tag{12}$$

  and (9). Like the proof of (4) in [Sogge and Zelditch 2011a], the identity (12) follows from an application of the divergence theorem applied to each of the nodal domains of e_λ.

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EFFECTIVE INTEGRABLE DYNAMICS FOR A CERTAIN NONLINEAR WAVE EQUATION

PATRICK GÉRARD AND SANDRINE GRELLIER

We consider the following degenerate half-wave equation on the one-dimensional torus:

$$i\partial_t u - |D|u = |u|^2 u, \quad u(0, \cdot) = u_0.$$

We show that, on a large time interval, the solution may be approximated by the solution of a completely integrable system — the cubic Szegő equation. As a consequence, we prove an instability result for large H^s norms of solutions of this wave equation.

1. Introduction

Let us consider, on the one-dimensional torus $\mathbb{T}$, the "half-wave" equation

$$i\partial_t u - |D|u = |u|^2 u, \quad u(0, \cdot) = u_0. \tag{1}$$

Here $|D|$ denotes the pseudodifferential operator defined by

$$|D|u = \sum |k| u_k e^{ikx}, \quad u = \sum_k u_k e^{ikx}.$$

This equation can be seen as a toy model for nonlinear Schrödinger equations on degenerate geometries leading to lack of dispersion. For instance, it has the same structure as the cubic nonlinear Schrödinger equation on the Heisenberg group, or associated with the Grušin operator. We refer to [Gérard and Grellier 2010a; 2010b] for more detail.

We endow $L^2(\mathbb{T})$ with the symplectic form

$$\omega(u, v) = \operatorname{Im}(u, v),$$

where (u, v) denotes the inner product on $L^2(\mathbb{T})$. Equation (1) may be seen as the Hamiltonian system related to the energy function $H(u) := \frac{1}{2}(|D|u, u) + \frac{1}{4}\|u\|_{L^4}^4$. In particular, H is invariant by the flow, which also admits the conservation laws

$$Q(u) := \|u\|_{L^2}^2, \quad M(u) := (Du, u).$$

However, Equation (1) is a nondispersive equation. Indeed, it is equivalent to the system

$$i(\partial_t \pm \partial_x)u_\pm = \Pi_\pm(|u|^2 u), \quad u_\pm(0, \cdot) = \Pi_\pm(u_0), \tag{2}$$

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where $u_\pm = \Pi_\pm(u)$. Here, Π_+ denotes the orthogonal projector from $L^2(\mathbb{T})$ onto

$$L^2_+(\mathbb{T}) := \left\{ u = \sum_{k\geq 0} u_k e^{ikx},\ (u_k)_{k\geq 0} \in \ell^2 \right\}$$

and $\Pi_- := I - \Pi_+$.

Though the scaling is L^2-critical, the first iteration map of the Duhamel formula

$$u(t) = e^{-it|D|}u_0 - i\int_0^t e^{-i(t-\tau)|D|}(|u(\tau)|^2 u(\tau))d\tau$$

is not bounded on H^s for $s < \frac{1}{2}$. Indeed, such boundedness would require the inequality

$$\int_0^1 \|\mathrm{e}^{-it|D|} f\|^4_{L^4(\mathbb{T})}\, dt \lesssim \|f\|^4_{H^{s/2}}.$$

However, testing this inequality on functions localized on positive modes, for instance, shows that this fails if $s < \frac{1}{2}$ (see the Appendix for more detail).

Proceeding as in the case of the cubic Szegő equation [Gérard and Grellier 2010a, Theorem 2.1],

$$i\partial_t w = \Pi_+(|w|^2 w), \tag{3}$$

one can prove the global existence and uniqueness of solutions of (1) in H^s for any $s \geq \frac{1}{2}$. The proof uses in particular the a priori bound of the $H^{1/2}$-norm provided by the energy conservation law.

Proposition 1. *Given $u_0 \in H^{\frac{1}{2}}(\mathbb{T})$, there exists $u \in C(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{T}))$ unique such that*

$$i\partial_t u - |D|u = |u|^2 u, \quad u(0,x) = u_0(x).$$

Moreover if $u_0 \in H^s(\mathbb{T})$ for some $s > \frac{1}{2}$, then $u \in C(\mathbb{R}, H^s(\mathbb{T}))$.

Similarly to the cubic Szegő equation, the proof of Proposition 1 provides only bad large time estimates:

$$\|u(t)\|_{H^s} \lesssim \mathrm{e}^{\mathrm{e}^{C_s t}}.$$

This naturally leads to the question of the large time behavior of solutions of (1). In order to answer this question, a fundamental issue is the decoupling of nonnegative and negative modes in system (2). Assuming that initial data are small and spectrally localized on nonnegative modes, a first step in that direction is given by the next simple proposition, which shows that $u_-(t)$ remains smaller in $H^{1/2}$ uniformly in time.

Proposition 2. *Assume*

$$\Pi_+ u_0 = u_0 = \mathcal{O}(\varepsilon) \quad \text{in } H^{\frac{1}{2}}(\mathbb{T}).$$

Then, the solution u of (1) satisfies

$$\sup_{t\in\mathbb{R}} \|\Pi_- u(t)\|_{H^{\frac{1}{2}}} = \mathcal{O}(\varepsilon^2).$$

Proof. By the energy and momentum conservation laws, we have

$$(|D|u,u)+\tfrac{1}{2}\|u\|_{L^4}^4=(|D|u_0,u_0)+\tfrac{1}{2}\|u_0\|_{L^4}^4,\ (Du,u)=(Du_0,u_0).$$

Subtracting these equalities, we get

$$2(|D|u_-,u_-)+\tfrac{1}{2}\|u\|_{L^4}^4=\tfrac{1}{2}\|u_0\|_{L^4}^4=\mathbb{O}(\varepsilon^4);$$

hence

$$\|u_-\|_{H^{\frac{1}{2}}}^2=\mathbb{O}(\varepsilon^4). \qquad \square$$

This decoupling result suggests neglecting u_- in the system (2) and hence comparing the solutions of (1) to those of

$$i\partial_t v-Dv=\Pi_+(|v|^2v),$$

which can be reduced to (3) by the transformation $v(t,x)=w(t,x-t)$.

Our main result is the following.

Theorem 1.1. *Let $s>1$ and $u_0=\Pi_+(u_0)\in L^2_+(\mathbb{T})\cap H^s(\mathbb{T})$ with $\|u_0\|_{H^s}=\varepsilon$, for $\varepsilon>0$ small enough. Denote by v the solution of the cubic Szegő equation*

$$i\partial_t v-Dv=\Pi_+(|v|^2v),\quad v(0,\cdot)=u_0. \tag{4}$$

Then, for any $\alpha>0$, there exists a constant $c=c_\alpha<1$ such that

$$\|u(t)-v(t)\|_{H^s}=\mathbb{O}(\varepsilon^{3-\alpha})\quad \textit{for } t\le \frac{c_\alpha}{\varepsilon^2}\log\frac{1}{\varepsilon}. \tag{5}$$

Furthermore, there exists $c>0$ such that

$$\|u(t)\|_{L^\infty}=\mathbb{O}(\varepsilon)\quad \textit{for all } t\le \frac{c}{\varepsilon^3}. \tag{6}$$

Remarks. 1. If we rescale u as εu, Equation (1) becomes

$$i\partial_t u-|D|u=\varepsilon^2|u|^2u,\quad u(0,\cdot)=u_0,$$

with $\|u_0\|_{H^s}=1$. On the latter equation, it is easy to prove that $u(t)=e^{-it|D|}u_0+o(1)$ for $t\ll 1/\varepsilon^2$, so that nonlinear effects only start for $1/\varepsilon^2\lesssim t$. Rescaling v as εv in (4), Theorem 1.1 states that the cubic Szegő dynamics appear as the effective dynamics of (1) on a time interval where nonlinear effects are taken into account.

2. As pointed out before, (4) reduces to (3) by a simple Galilean transformation. Equation (3) has been studied in [Gérard and Grellier 2010a; 2010b; 2012], where its complete integrability is established together with an explicit formula for its generic solutions. Consequently, the first part of Theorem 1.1 provides an accurate description of solutions of (1) for a reasonably large time. Moreover, the second part of Theorem 1.1 claims an L^∞ bound for the solution of (1) on an even larger time. This latter bound is closely related to a special conservation law of (3), namely, some Besov norm of v — see Section 2 below.

3. In the case of small Cauchy data localized on nonnegatives modes, system (2) can be reformulated as a — singular — perturbation of the cubic Szegő equation (3). Indeed, write $u_0 = \varepsilon w_0$ and $u(t,x) = \varepsilon w(\varepsilon^2 t, x - t)$; then $w = w_+ + w_-$ solves the system

$$\begin{aligned} i\partial_t w_+ &= \Pi_+(|w|^2 w), \\ i(\varepsilon^2\partial_t - 2\partial_x)w_- &= \varepsilon^2 \Pi_-(|w|^2 w). \end{aligned} \tag{7}$$

Notice that, for $\varepsilon = 0$ and $\Pi_+ w_0 = w_0$, the solution of this system is exactly the solution of (3). It is therefore natural to ask how much, for $\varepsilon > 0$ small, the solution of system (7) stays close to the solution of Equation (3). Since Equation (3) turns out to be completely integrable, this problem appears as a perturbation of a completely integrable infinite-dimensional system. There is a lot of literature on this subject (see the books [Kuksin 1993; Craig 2000; Kappeler and Pöschel 2003] for KAM theory). In the case of the 1D cubic NLS equation and the modified KdV equation, with special initial data such as solitons or 2-solitons, we refer to [Holmer and Zworski 2007; 2008; Holmer et al. 2007; 2011] and references therein. Here we emphasize that our perturbation is more singular and that we deal with general Cauchy data.

4. The proof of Theorem 1.1 is based on a Poincaré–Birkhoff normal form approach, similarly to [Bambusi 2003; Grébert 2007] for instance. More specifically, we prove that (4) turns out to be a Poincaré–Birkhoff normal form of (1), for small initial data with only nonnegative modes.

As a corollary of Theorem 1.1, we get the following instability result.

Corollary 1. *Let $s > 1$. There exist a sequence of data u_0^n and a sequence of times $t^{(n)}$ such that, for any r,*

$$\|u_0^n\|_{H^r} \to 0,$$

while the corresponding solution of (1) satisfies

$$\|u^n(t^{(n)})\|_{H^s} \simeq \|u_0^n\|_{H^s} \left(\log \frac{1}{\|u_0^n\|_{H^s}}\right)^{2s-1}.$$

It is interesting to compare this result to what is known about the cubic NLS. In the one-dimensional case, the cubic NLS is integrable [Zakharov and Shabat 1972] and admits an infinite number of conservation laws which control the regularity of the solution in Sobolev spaces. As a consequence, no such norm inflation occurs. This is in contrast with the 2D cubic NLS case for which Colliander, Keel, Staffilani, Takaoka, and Tao [2010] exhibited small initial data in H^s which give rise to large H^s solutions after a large time.

In our case, the situation is different. Although the cubic Szegő equation is completely integrable, its conservation laws do not control the regularity of the solutions, which allows a large time behavior similar to the one proved in [Colliander et al. 2010] for 2D cubic NLS [Gérard and Grellier 2010a, Section 6, Corollary 5]. Unfortunately, the time interval on which the approximation (5) holds does not allow to infer large solutions for (1), but only solutions with large relative size with respect to their Cauchy data —

see Section 3 below. A time interval of the form $[0, 1/\varepsilon^{2+\beta}]$ for some $\beta > 0$ would be enough to construct large solutions for (1) for some H^s-norms.

We close this introduction by mentioning that O. Pocovnicu solved a similar problem for Equation (1) on the line by using the renormalization group method instead of the Poincaré–Birkhoff normal form method. Moreover, she improved the approximation in Theorem 1.1 by introducing a quintic correction to the Szegő cubic equation [Pocovnicu 2011].

The paper is organized as follows. In Section 2 we recall some basic facts about the Lax pair structure for the cubic Szegő equation (3). In Section 3, we deduce Corollary 1 from Theorem 1.1. Finally, the proof of Theorem 1.1 is given in Section 4.

2. The Lax pair for the cubic Szegő equation and some of its consequences

In this section, we recall some basic facts about Equation (3) (see [Gérard and Grellier 2010a] for more detail). Given $w \in H^{1/2}(\mathbb{T})$, we define (see, e.g., [Peller 1982; Nikolski 2002]) the Hankel operator of symbol w by

$$H_w(h) = \Pi_+(w\bar{h}), \quad h \in L^2_+.$$

It is easy to check that H_w is a $\mathbb{C}$-antilinear Hilbert–Schmidt operator. In [Gérard and Grellier 2010a], we proved that the cubic Szegő flow admits a Lax pair in the following sense. For simplicity let us restrict ourselves to the case of H^s solutions of (3) for $s > \frac{1}{2}$. By [ibid., Theorem 3.1], there exists a mapping $w \in H^s \mapsto B_w$, valued into $\mathbb{C}$-linear bounded skew-symmetric operators on L^2_+, such that

$$H_{-i\Pi_+(|w|^2w)} = [B_w, H_w]. \tag{8}$$

Moreover,

$$B_w = \frac{i}{2}H_w^2 - iT_{|w|^2},$$

where T_b denotes the Toeplitz operator of symbol b, given by $T_b(h) = \Pi_+(bh)$. Consequently, w is a solution of (3) if and only if

$$\frac{d}{dt}H_w = [B_w, H_w]. \tag{9}$$

An important consequence of this structure is that the cubic Szegő equation admits an infinite number of conservation laws. Indeed, denoting by $W(t)$ the solution of the operator equation

$$\frac{d}{dt}W = B_w W, \quad W(0) = I,$$

the operator $W(t)$ is unitary for every t, and

$$W(t)^* H_{w(t)} W(t) = H_{w(0)}.$$

Hence, if w is a solution of (3), then $H_{w(t)}$ is unitarily equivalent to $H_{w(0)}$. Consequently, the spectrum of the $\mathbb{C}$-linear positive self-adjoint trace class operator H_w^2 is conserved by the evolution. In particular, the trace norm of H_w is conserved by the flow. A theorem by Peller [1982, Theorem 2, p. 454] states that the trace norm of a Hankel operator H_w is equivalent to the norm of w in the Besov space $B^1_{1,1}(\mathbb{T})$. Recall that the Besov space $B^1 = B^1_{1,1}(\mathbb{T})$ is defined as the set of functions w such that $\|w\|_{B^1_{1,1}}$ is finite, where

$$\|w\|_{B^1_{1,1}} = \|S_0(w)\|_{L^1} + \sum_{j=0}^{\infty} 2^j \|\Delta_j w\|_{L^1};$$

here $w = S_0(w) + \sum_{j=0}^{\infty} \Delta_j w$ stands for the Littlewood–Paley decomposition of w. It is standard that B^1 is an algebra included into L^∞ (in fact into the Wiener algebra). The conservation of the trace norm of H_w therefore provides an L^∞ estimate for solutions of (3) with initial data in B^1.

The space B^1 and formula (8) will play an important role in the proof of Theorem 1.1. In particular, the last part will follow from the fact that $\|u(t)\|_{B^1}$ remains bounded by ε for $t \ll 1/\varepsilon^3$. The fact that $H^s(\mathbb{T}) \subset B^1$ for $s > 1$, explains why we assume $s > 1$ in the statement.

3. Proof of Corollary 1

As observed in [Gérard and Grellier 2010a, Section 6.1, Proposition 7, and Section 6.2, Corollary 5], the equation

$$i\partial_t w = \Pi_+(|w|^2 w)\ , \qquad w(0,x) = \frac{a_0\,\mathrm{e}^{ix} + b_0}{1 - p_0 \mathrm{e}^{ix}}$$

with $a_0, b_0, p_0 \in \mathbb{C}$, $|p_0| < 1$ can be solved as

$$w(t,x) = \frac{a(t)\,\mathrm{e}^{ix} + b(t)}{1 - p(t)\mathrm{e}^{ix}}$$

where a, b, p satisfy an explicitly solvable ODE system.

In the particular case when

$$a_0 = \varepsilon, \quad b_0 = \varepsilon\delta, \quad p_0 = 0, \quad w_\varepsilon(0,x) = \varepsilon(\mathrm{e}^{ix} + \delta)\ ,$$

one finds

$$1 - \left| p\left(\frac{\pi}{2\varepsilon^2\delta}\right)\right|^2 \simeq \delta^2,$$

so that, for $s > \frac{1}{2}$,

$$\left\| w_\varepsilon\left(\frac{\pi}{2\varepsilon^2\delta}\right)\right\|_{H^s} \simeq \frac{\varepsilon}{\delta^{2s-1}}.$$

Let v_ε be the solution of

$$i(\partial_t + \partial_x) v_\varepsilon = \Pi_+(|v_\varepsilon|^2 v_\varepsilon)\ , \qquad v_\varepsilon(0,x) = \varepsilon(\mathrm{e}^{ix} + \delta).$$

Then $v_\varepsilon(t,x) = w_\varepsilon(t, x-t)$, so that

$$\left\| v_\varepsilon\left(\frac{\pi}{2\varepsilon^2\delta}\right)\right\|_{H^s} \simeq \frac{\varepsilon}{\delta^{2s-1}}.$$

Choose

$$\varepsilon = \frac{1}{n}, \qquad \delta = \frac{C}{\log n},$$

with C large enough that if $t^{(n)} := \pi/(2\varepsilon^2\delta)$ then $t^{(n)} < c\log(1/\varepsilon)/\varepsilon^2$, where $c = c_\alpha$ in Theorem 1.1 for $\alpha = 1$, say. Set $u_0^n := v_\varepsilon(0,\cdot)$. As $\|u_0^n\|_{H^s} \simeq \varepsilon$, the previous estimate reads

$$\left\| v_\varepsilon\left(\frac{\pi}{2\varepsilon^2\delta}\right)\right\|_{H^s} \simeq \|u_0^n\|_{H^s}\left(\log\frac{1}{\|u_0^n\|_{H^s}}\right)^{2s-1}.$$

Applying Theorem 1.1, we get the same information about $\|u_n(t^{(n)})\|_{H^s}$.

4. Proof of Theorem 1.1

First of all, we rescale u as εu so that Equation (1) becomes

$$i\partial_t u - |D|u = \varepsilon^2|u|^2u, \quad u(0,\cdot) = u_0 \tag{10}$$

with $\|u_0\|_{H^s} = 1$.

4.1. ***Study of the resonances.*** We write the Duhamel formula as

$$u(t) = \mathrm{e}^{-it|D|}\underline{u}(t),$$

with

$$\underline{\hat{u}}(t,k) = \hat{u}_0(k) - i\varepsilon^2 \sum_{k_1-k_2+k_3-k=0} I(k_1,k_2,k_3,k),$$

where

$$I(k_1,k_2,k_3,k) = \int_0^t \mathrm{e}^{-i\tau\Phi(k_1,k_2,k_3,k)}\underline{\hat{u}}(\tau,k_1)\overline{\underline{\hat{u}}(\tau,k_2)}\underline{\hat{u}}(\tau,k_3)\,d\tau,$$

and

$$\Phi(k_1,k_2,k_3,k_4) := |k_1| - |k_2| + |k_3| - |k_4|.$$

If $\Phi(k_1,k_2,k_3,k_4) \neq 0$, an integration by parts in $I(k_1,k_2,k_3,k_4)$ provides an extra factor ε^2; hence the set of (k_1,k_2,k_3,k_4) such that $\Phi(k_1,k_2,k_3,k_4) = 0$ is expected to play a crucial role in the analysis. This set is described in the following lemma.

Lemma 1. *Given* $(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4$,

$$k_1 - k_2 + k_3 - k_4 = 0 \quad \text{and} \quad |k_1| - |k_2| + |k_3| - |k_4| = 0$$

if and only if at least one of the following properties holds:

(a) $k_j \geq 0$ *for all* j.

(b) $k_j \leq 0$ *for all* j.

(c) $k_1 = k_2$ *and* $k_3 = k_4$.

(d) $k_1 = k_4$ *and* $k_3 = k_2$.

Proof. Consider $(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$ such that $k_1 - k_2 + k_3 - k_4 = 0$, $|k_1| - |k_2| + |k_3| - |k_4| = 0$, and the k_j are not all nonnegative or all nonpositive. Let us prove in that case that either $k_1 = k_2$ and $k_3 = k_4$, or $k_1 = k_4$ and $k_3 = k_2$. Without loss of generality, we can assume that at least one of the k_j is positive, for instance k_1. Then, subtracting both equations, we get that $|k_3| - k_3 = |k_2| - k_2 + |k_4| - k_4$. If k_3 is nonnegative, both k_2 and k_4 must be nonnegative; hence all the k_j are nonnegative. Assume now that k_3 is negative. At least one among k_2, k_4 is negative. If both of them are negative, then $k_3 = k_2 + k_4$ but this would imply $k_1 = 0$ which is impossible by assumption. So we get either that $k_3 = k_2$ (and so $k_1 = k_4$) or $k_3 = k_4$ (and so $k_1 = k_2$). This completes the proof of the lemma. □

4.2. *First reduction.* We get rid of the resonances corresponding to cases (c) and (d) by applying the transformation

$$u(t) \mapsto \mathrm{e}^{2it\varepsilon^2\|u_0\|^2_{L^2}} u(t) \tag{11}$$

which, since the L^2 norm of u is conserved, leads to the equation

$$i\partial_t u - |D|u = \varepsilon^2(|u|^2 - 2\|u\|^2_{L^2})u, \quad u(0,\cdot) = u_0. \tag{12}$$

Notice that this transformation does not change the H^s norm. The Hamiltonian function associated to (12) is given by

$$H(u) = \tfrac{1}{2}(|D|u, u) + \tfrac{1}{4}\varepsilon^2(\|u\|^4_{L^4} - 2\|u\|^4_{L^2}) = H_0(u) + \varepsilon^2 R(u),$$

where

$$\begin{aligned} H_0(u) &:= \tfrac{1}{2}(|D|u, u), \\ R(u) &:= \tfrac{1}{4}(\|u\|^4_{L^4} - 2\|u\|^4_{L^2}) = \frac{1}{4}\Bigg(\sum_{\substack{k_1-k_2+k_3-k_4=0\\ k_1\neq k_2,k_4}} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}} - \sum_{k\in\mathbb{Z}} |u_k|^4\Bigg). \end{aligned}$$

4.3. *The Poincaré–Birkhoff normal form.* We claim that under a suitable canonical transformation on u, H can be reduced to the Hamiltonian

$$\tilde{H}(u) = H_0(u) + \varepsilon^2\tilde{R}(u) + O(\varepsilon^4),$$

where

$$\tilde{R}(u) = \frac{1}{4}\sum_{(k_1,k_2,k_3,k_4)\in\mathcal{R}} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}},$$

with

$$\mathcal{R} = \big\{(k_1, k_2, k_3, k_4) : k_1 - k_2 + k_3 - k_4 = 0;\ k_1 \neq k_2;\ k_1 \neq k_4;\ k_j \geq 0 \text{ for all } j \text{ or } k_j \leq 0 \text{ for all } j\big\}.$$

We look for a canonical transformation as the value at time 1 of some Hamiltonian flow. In other words, we look for a function F such that its Hamiltonian vector field is smooth on H^s and on B^1, so that our canonical transformation is φ_1, where φ_σ is the solution of

$$\frac{d}{d\sigma}\varphi_\sigma(u) = \varepsilon^2 X_F(\varphi_\sigma(u)), \quad \varphi_0(u) = u. \tag{13}$$

Recall that, given a smooth real valued function F, its Hamiltonian vector field X_F is defined by

$$dF(u).h =: \omega(h, X_F(u)),$$

and, given functions F, G admitting Hamiltonian vector fields, their Poisson bracket $\{F, G\}$ is defined by

$$\{F, G\}(u) = \omega(X_F(u), X_G(u)).$$

Let us make some preliminary remarks about the Poisson brackets.

In view of the expression of ω, we have

$$\{F, G\} := dG.X_F = \frac{2}{i}\sum_k (\partial_{\bar{k}} F \partial_k G - \partial_{\bar{k}} G \partial_k F)$$

where $\partial_k F$ stands for $\partial F/\partial u_k$ and $\partial_{\bar{k}} F$ for $\partial F/\partial \overline{u_k}$. In particular, if F and G are respectively homogeneous of order p and q, then their Poisson bracket is homogeneous of order $p+q-2$.

Lemma 2. *Set*

$$F(u) := \sum_{k_1-k_2+k_3-k_4=0} f_{k,k_2,k_3,k_4} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}},$$

where

$$f_{k_1,k_2,k_3,k_4} = \begin{cases} \dfrac{i}{4(|k_1|-|k_2|+|k_3|-|k_4|)} & \textit{if } |k_1|-|k_2|+|k_3|-|k_4| \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then X_F is smooth on H^s, $s > \frac{1}{2}$, as well as on B^1, and

$$\{F, H_0\} + R = \tilde{R},$$

$$\|DX_F(u)h\| \lesssim \|u\|^2\|h\|,$$

where the norm is taken either in H^s, $s > \frac{1}{2}$, or in B^1.

Proof. First we make a formal calculation with F given by

$$F(u) := \sum_{k_1-k_2+k_3-k_4=0} f_{k_1,k_2,k_3,k_4} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}}$$

for some coefficients f_{k_1,k_2,k_3,k_4} to be determined later. We compute

$$\{F, H_0\} = \frac{1}{i}\sum_{k_1-k_2+k_3-k_4=0} (-|k_1|+|k_2|-|k_3|+|k_4|) f_{k_1,k_2,k_3,k_4} u_{k_1}\overline{u_{k_2}}u_{k_3}\overline{u_{k_4}},$$

so that equality $\{F, H_0\} + R = \tilde{R}$ requires

$$f_{k_1,k_2,k_3,k_4} = \begin{cases} \dfrac{i}{4(|k_1|-|k_2|+|k_3|-|k_4|)} & \text{if } |k_1|-|k_2|+|k_3|-|k_4| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that the function F is explicitly given by

$$F(u)=\frac{1}{2}\mathrm{Im}\big((D_0^{-1}u_-,|u_+|^2u_+)-(D_0^{-1}u_+,|u_-|^2u_-)-(D_0^{-1}|u_+|^2,|u_-|^2)\big),$$

where D_0^{-1} is the operator defined by

$$D_0^{-1}u(x)=\sum_{k\neq 0}\frac{u_k}{k}\,\mathrm{e}^{ikx}.$$

Notice that, since functions $|u_+|^2$ and $|u_-|^2$ are real valued, the quantity $(D_0^{-1}|u_+|^2,|u_-|^2)$ is purely imaginary, and therefore is equal to i times its imaginary part.

In view of the formula above, the Hamiltonian vector field $X_F(u)$ is a sum of products of terms involving the maps $f\mapsto\bar f$, $f\mapsto D_0^{-1}f$, $f\mapsto\Pi_\pm f$, $(f,g)\mapsto fg$. These maps are continuous on H^s and on B^1. Hence, X_F is smooth and its differential satisfies the claimed estimate on H^s, $s>\frac{1}{2}$, and B^1. $\square$

The proof of the following technical lemma is based on straightforward calculations.

Lemma 3. *The function $\tilde R$ and its Hamiltonian vector field are given by*

$$\tilde R(u)=\frac{1}{4}(\|u_+\|_{L^4}^4+\|u_-\|_{L^4}^4)+\mathrm{Re}((u,1)\,(u_-,u_-^2))-\frac{1}{2}(\|u_+\|_{L^2}^4+\|u_-\|_{L^2}^4),$$

$$\begin{aligned}iX_{\tilde R}(u)=\Pi_+(|u_+|^2u_+)+\Pi_-(|u_-|^2u_-)-2\|u_+\|_{L^2}^2u_+-2\|u_-\|_{L^2}^2u_-+(u_-^2,u_-)\\+2(1,u)\Pi_-(|u_-|^2)+(1,u)u_-^2,\end{aligned}$$

where we have set $u_\pm:=\Pi_\pm(u)$.

The maps $X_{\{F,R\}}$ and $X_{\{F,\tilde R\}}$ are smooth homogeneous polynomials of degree five on B^1 and on H^s for every $s>\frac{1}{2}$.

We now perform the canonical transformation

$$\chi_\varepsilon:=\exp(\varepsilon^2X_F).$$

Lemma 4. *Set $\varphi_\sigma:=\exp(\varepsilon^2\sigma X_F)$ for $-1\le\sigma\le 1$. There exist $m_0>0$ and $C_0>0$ so that, for any $u\in B^1$ so that $\varepsilon\|u\|_{B^1}\le m_0$, $\varphi_\sigma(u)$ is well defined for $\sigma\in[-1,1]$ and*

$$\|\varphi_\sigma(u)\|_{B^1}\le\frac{3}{2}\|u\|_{B^1},$$

$$\|\varphi_\sigma(u)-u\|_{B^1}\le C_0\varepsilon^2\|u\|_{B^1}^3,$$

$$\|D\varphi_\sigma(u)\|_{B^1\to B^1}\le e^{C_0\varepsilon^2\|u\|_{B^1}^2}.$$

Moreover, the same estimates hold in H^s, $s>\frac{1}{2}$, with some constants $m(s)$ and $C(s)$.

Proof. Write φ_σ as the integral of its derivative and use Lemma 2 to get

$$\sup_{|\sigma|\le\tau}\|\varphi_\sigma(u)\|_{B^1}\le\|u\|_{B^1}+C\varepsilon^2\sup_{|\sigma|\le\tau}\|\varphi_\sigma(u)\|_{B^1}^3,\quad 0\le\tau\le 1. \tag{14}$$

We now use the following standard bootstrap lemma.

Lemma 5. *Let $a, b, T > 0$ and $\tau \in [0,T] \mapsto M(\tau) \in \mathbb{R}_+$ be a continuous function satisfying*

$$M(\tau) \le a + bM(\tau)^3 \quad \textit{for all } \tau \in [0,T].$$

Assume that $\sqrt{3b}\,M(0) < 1$ and $\sqrt{3b}\,a < \frac{2}{3}$. Then $M(\tau) \le \frac{3}{2}a$ for all $\tau \in [0,T]$.

Proof. For the convenience of the reader, we give the proof of Lemma 5. The function $f : z \ge 0 \mapsto z - bz^3$ attains its maximum at $z_c = 1/\sqrt{3b}$, equal to $f_m = 2/(3\sqrt{3b})$. Consequently, since a is smaller than f_m by the second inequality,

$$\{z \ge 0 : f(z) \le a\} = [0, z_-] \cup [z_+, +\infty)$$

with $z_- < z_c < z_+$ and $f(z_-) = a$. Since $M(\tau)$ belongs to this set for every τ and since $M(0)$ belongs to the first interval by the first inequality, we conclude by continuity that $M(\tau) \le z_-$ for every τ. By the concavity of f, $f(z) \ge \frac{2}{3}z$ for $z \in [0, z_c]$, hence $z_- \le \frac{3}{2}a$. □

Let us come back to the proof of Lemma 4. If $\varepsilon\|u\|_{B^1} < \dfrac{2}{3\sqrt{3C}}$, Equation (14) and Lemma 5 imply that

$$\sup_{|\sigma|\le 1} \|\varphi_\sigma(u)\|_{B^1} \le \tfrac{3}{2}\|u\|_{B^1}, \tag{15}$$

which is the first estimate. For the second one, we write for $|\sigma| \le 1$,

$$\|\varphi_\sigma(u) - u\|_{B^1} = \|\varphi_\sigma(u) - \varphi_0(u)\|_{B^1} \le |\sigma| \sup_{|s|\le|\sigma|} \left\| \frac{d}{ds}\varphi_s(u) \right\|_{B^1} \le C_0 \varepsilon^2 \|u\|_{B^1}^3,$$

where the last inequality comes from Lemma 2 and estimate (15).

It remains to prove the last estimate. We differentiate the equation satisfied by φ_σ and use again Lemma 2 to obtain

$$\begin{aligned}\|D\varphi_\sigma(u)\|_{B^1\to B^1} &\le 1 + \varepsilon^2 \left| \int_0^\sigma \|DX_F(\varphi_\tau(u))\|_{B^1\to B^1} \|D\varphi_\tau(u)\|_{B^1\to B^1}\, d\tau \right| \\ &\le 1 + C_0\varepsilon^2\|u\|_{B^1}^2 \left| \int_0^\sigma \|D\varphi_\tau(u)\|_{B^1\to B^1} d\tau \right|,\end{aligned}$$

and Gronwall's lemma yields the result. Analogous proofs give the estimates in H^s. □

Let u satisfy the assumption of Lemma 4 in B^1 or in H^s for some $s > \frac{1}{2}$.

Let us compute $H \circ \chi_\varepsilon = H \circ \varphi_1$ as the Taylor expansion of $H \circ \varphi_\sigma$ at time 1 around 0. One gets

$$\begin{aligned} H \circ \chi_\varepsilon &= H \circ \varphi_1 = H_0 \circ \varphi_1 + \varepsilon^2 R \circ \varphi_1 \\ &= H_0 + \frac{d}{d\sigma}[H_0 \circ \varphi_\sigma]_{\sigma=0} + \varepsilon^2 R + \int_0^1 \left((1-\sigma)\frac{d^2}{d\sigma^2}[H_0 \circ \varphi_\sigma] + \varepsilon^2 \frac{d}{d\sigma}[R \circ \varphi_\sigma] \right) d\sigma \\ &= H_0 + \varepsilon^2(\{F, H_0\} + R) + \varepsilon^4 \int_0^1 ((1-\sigma)\{F,\{F,H_0\}\} + \{F,R\}) \circ \varphi_\sigma\, d\sigma \\ &= H_0 + \varepsilon^2 \tilde{R} + \varepsilon^4 \int_0^1 \Big((1-\sigma)\{F,\tilde{R}\} + \sigma\{F,R\} \Big) \circ \varphi_\sigma\, d\sigma \\ &=: H_0 + \varepsilon^2 \tilde{R} + \varepsilon^4 \int_0^1 G(\sigma) \circ \varphi_\sigma\, d\sigma. \end{aligned}$$

By Lemma 3, one gets

$$\sup_{0\le\sigma\le 1} \|X_{G(\sigma)}(w)\| \le C\|w\|^5$$

where the norm stands for the B^1 norm or the H^s norm. Since

$$X_{G(\sigma)\circ\varphi_\sigma}(u) = D\varphi_{-\sigma}(\varphi_\sigma(u)).X_{G(\sigma)}(\varphi_\sigma(u)),$$

we conclude from Lemma 4 that, if $\varepsilon\|u\|_{B^1} \le m_0$,

$$\|X_{G(\sigma)\circ\varphi_\sigma}(u)\|_{B^1} \le C\|u\|_{B^1}^5.$$

As a consequence, one can write

$$X_{H\circ\chi_\varepsilon} = X_{H_0} + \varepsilon^2 X_{\tilde{R}} + \varepsilon^4 Y,$$

where, if $\varepsilon\|u\|_{B^1} \le m_0$, then

$$\|Y(u)\|_{B^1} \lesssim \|u\|_{B^1}^5.$$

An analogous estimate holds in H^s, $s > \frac{1}{2}$.

4.4. *End of the proof.* We first deal with the B^1-norm of a solution u of (12). We are going to prove that $\|u(t)\|_{B^1} = \mathcal{O}(1)$ for $t \ll 1/\varepsilon^3$ by the following bootstrap argument. We assume that for some K large enough with respect to $\|u_0\|_{B^1}$, for some $T > 0$, for all $t \in [0, T]$, we have $\|u(t)\|_{B^1} \le 10K$, and we prove that if $T \ll 1/\varepsilon^3$, $\|u(t)\|_{B^1} \le K$ for $t \in [0, T]$. This will prove the result by continuity.

Set, for $t \in [0, T]$,

$$\tilde{u}(t) := \chi_\varepsilon^{-1}(u(t)),$$

so that $\tilde{u}$ is a solution of

$$i\partial_t\tilde{u} - |D|\tilde{u} = \varepsilon^2 i X_{\tilde{R}}(\tilde{u}) + \varepsilon^4 i Y(\tilde{u}).$$

Moreover, by Lemma 4,

$$\|\tilde{u}(t) - u(t)\|_{B^1} \lesssim \varepsilon^2\|u\|_{B^1}^3$$

and so by the hypothesis, $\|\tilde{u}(t)\|_{B^1} \le 11K$ if ε is small enough. In view of the expression of the Hamiltonian vector field of $\tilde{R}$ in Lemma 3, the equation for $\tilde{u}$ reads

$$\begin{cases} i\partial_t\tilde{u}_+ - D\tilde{u}_+ = \varepsilon^2\left(\Pi_+(|\tilde{u}_+|^2\tilde{u}_+) - 2\|\tilde{u}_+\|_{L^2}^2\tilde{u}_+ + \textstyle\fint_{\mathbb{T}} |\tilde{u}_-|^2\tilde{u}_-\right) + \varepsilon^4 i Y_+(\tilde{u}),\\ i\partial_t\tilde{u}_- + D\tilde{u}_- = \varepsilon^2\left(\Pi_-(|\tilde{u}_-|^2\tilde{u}_-) - 2\|\tilde{u}_-\|_{L^2}^2\tilde{u}_- + 2(1,\tilde{u})\Pi_-(|\tilde{u}_-|^2) + (1,\tilde{u})\tilde{u}_-^2\right) + \varepsilon^4 i Y_-(\tilde{u}).\end{cases}$$

Notice that all the Hamiltonian functions we have dealt with so far are invariant by multiplication by complex numbers of modulus 1, hence their Hamiltonian vector fields satisfy

$$X(\mathrm{e}^{i\theta}z) = \mathrm{e}^{i\theta}z,$$

so that the corresponding Hamiltonian flows conserve the L^2 norm. Hence $\tilde{u}$ has the same L^2 norm as u, which is the L^2 norm of u_0. In particular, $|(1,\tilde{u})| \le \|u_0\|_{L^2}$.

Moreover, as $\|u_0\|_{B^1} \lesssim \|u_0\|_{H^s} = \mathbb{O}(1)$ since $s > 1$, $\tilde{u}_0$ satisfies

$$\|\tilde{u}_0 - u_0\|_{B^1} \lesssim \varepsilon^2$$

by Lemma 4, so that, as $u_{0-} = 0$, we get $\|\tilde{u_0}_-\|_{B^1} = \mathbb{O}(\varepsilon^2)$. Then we obtain from the second equation

$$\sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1} \lesssim \varepsilon^2 + \varepsilon^2 t\big(\sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1}^3 + \sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1}^2\big) + \varepsilon^4 t K^5.$$

Let $M(t) = \dfrac{1}{\varepsilon}\sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{B^1}$, so that, if $t \le T$,

$$M(t) \lesssim \varepsilon + \varepsilon^3 T M(t)^2(1+\varepsilon M(t)) + \varepsilon^3 T.$$

As $3m^2 \le 1 + 2m^3$ for any $m \ge 0$, we get

$$M(t) \lesssim \varepsilon + \varepsilon^3 T M(t)^3 + \varepsilon^3 T.$$

Using Lemma 5, we conclude that, if $T \ll 1/\varepsilon^3$,

$$\sup_{0\le\tau\le T} \|\tilde{u}_-(\tau)\|_{B^1} \ll \varepsilon.$$

For further reference, notice that, if $T \lesssim \dfrac{1}{\varepsilon^2}\log\dfrac{1}{\varepsilon}$, this estimate can be improved to

$$\sup_{0\le\tau\le T} \|\tilde{u}_-(\tau)\|_{B^1} \lesssim \varepsilon^{2-\alpha} \quad \text{for all } \alpha > 0.$$

We come back to the case $T \ll 1/\varepsilon^3$. From the estimate on $\tilde{u}_-$, we infer

$$\|\tilde{u}_+\|_{L^2}^2 = \|\tilde{u}\|_{L^2}^2 + \mathbb{O}(\varepsilon^2) = \|u_0\|_{L^2}^2 + \mathbb{O}(\varepsilon^2),$$

and the equation for $\tilde{u}_+$ reads

$$i\partial_t\tilde{u}_+ - D\tilde{u}_+ = \varepsilon^2\left(\Pi_+(|\tilde{u}_+|^2\tilde{u}_+) - 2\|u_0\|_{L^2}^2\tilde{u}_+\right) + \varepsilon^4 i Y_+(\tilde{u}) + \mathbb{O}(\varepsilon^5) + \mathbb{O}(\varepsilon^4)\tilde{u}_+.$$

Since $\tilde{u_0}_+$ is not small in B^1, we have to use a different strategy to estimate $\tilde{u}_+$. We use the complete integrability of the cubic Szegő equation, especially its Lax pair and the conservation of the B^1-norm.

At this stage it is of course convenient to cancel the linear term $\|u_0\|_{L^2}^2\tilde{u}_+$ by multiplying $\tilde{u}_+(t)$ by $\mathrm{e}^{2i\varepsilon^2 t\|u_0\|_{L^2}^2}$. As pointed out before, this change of unknown is completely transparent to the above system. This leads to

$$i\partial_t\tilde{u}_+ - D\tilde{u}_+ = \varepsilon^2\Pi_+(|\tilde{u}_+|^2\tilde{u}_+) + \varepsilon^4 Y_+(\tilde{u}) + \mathbb{O}(\varepsilon^5) + \mathbb{O}(\varepsilon^4)\tilde{u}_+.$$

Notice that all the $\mathbb{O}$ terms above are measured in B^1 norm. We now appeal to the results recalled in Section 2. We introduce the unitary family $U(t)$ defined by

$$i\partial_t U - DU = \varepsilon^2(T_{|\tilde{u}_+|^2} - \tfrac{1}{2}H_{\tilde{u}_+}^2)U, \quad U(0) = I,$$

so that, using formula (8),

$$i\partial_t(U(t)^* H_{\tilde{u}_+(t)} U(t)) = \varepsilon^4 U(t)^* H_{Y_+(\tilde{u}) + \mathbb{O}(\varepsilon) + \mathbb{O}(1)\tilde{u}_+} U(t).$$

Then, we use the theorem from [Peller 1982] that states, as recalled in Section 2, that the trace norm of a Hankel operator of symbol b is equivalent to the B^1-norm of b to obtain

$$\begin{aligned}\|\tilde{u}_+(t)\|_{B^1} &\simeq \mathrm{Tr}|H_{\tilde{u}_+(t)}| \\ &\lesssim \mathrm{Tr}|H_{\tilde{u}_{0+}}| + \varepsilon^4 \int_0^t (\mathrm{Tr}|H_{Y_+(\tilde{u})}(\tau)| + \mathrm{Tr}|H_{\tilde{u}_+}(\tau)| + \varepsilon)\, d\tau \\ &\lesssim \|\tilde{u}_{0+}\|_{B^1} + \varepsilon^4 \int_0^t (\|\tilde{u}(\tau)\|_{B^1}^5 + \|\tilde{u}_+(\tau)\|_{B^1} + \varepsilon)\, d\tau\end{aligned}$$

so that as $\|\tilde{u}(t)\|_{B^1} \le 11K$,

$$\|\tilde{u}_+(t)\|_{B^1} \lesssim \|\tilde{u_0}_+\|_{B^1} + \varepsilon^4 t(11K)^5,$$

and, if $t \ll 1/\varepsilon^3$ and ε is small enough,

$$\|\tilde{u}(t)\|_{B^1} \le \frac{K}{10}.$$

Using again the second estimate in Lemma 4, we infer

$$\|u(t)\|_{B^1} \le K.$$

Finally, using the inverse of transformation (11) and multiplying u by ε, we obtain estimate (6) of Theorem 1.1.

We now estimate the difference between the solution of the wave equation and the solution of the cubic Szegő equation. Since we have applied transformation (11), we have to compare in B^1 the solution u of (12) to the solution v of equation

$$i\partial_t v - Dv = \varepsilon^2(\Pi_+(|v|^2 v) - 2\|u_0\|_{L^2}^2 v)\ , \quad v(0) = u_0.$$

Notice that, as u_0 is bounded in H^s, $s > 1$, and as the B^1 norm is conserved by the cubic Szegő flow,

$$\|v(t)\|_{B^1} \simeq \|u_0\|_{B^1} \lesssim \|u_0\|_{H^s} = \mathbb{O}(1).$$

We shall prove that, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that,

$$\|u(t) - v(t)\|_{B^1} \le \varepsilon^{2-\alpha} \quad \text{for all } t \le \frac{c_\alpha}{\varepsilon^2} \log \frac{1}{\varepsilon}.$$

In view of the previous estimates, it is enough to prove that, on the same time interval,

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \le \varepsilon^{2-\alpha},$$

where $\tilde{u}_+$ satisfies

$$\begin{cases} i\partial_t \tilde{u}_+ - D\tilde{u}_+ = \varepsilon^2 \left(\Pi_+(|\tilde{u}_+|^2 \tilde{u}_+) - 2\|u_0\|_{L^2}^2 \tilde{u}_+\right) + \mathbb{O}(\varepsilon^4), \\ \qquad \tilde{u}_+(0) = \tilde{u}_{0,+}. \end{cases} \tag{16}$$

As $\|\tilde{u}(t)\|_{B^1} \lesssim 1$, $\|v(t)\|_{B^1} \lesssim 1$, $\|\tilde{u}_{0,+} - u_0\|_{B^1} \le \varepsilon^2 \|u_0\|_{B^1} \lesssim \varepsilon^2$ and

$$(i\partial_t - D)(\tilde{u}_+ - v) = \varepsilon^2 \Pi_+(|\tilde{u}_+|^2 \tilde{u}_+ - |v|^2 v - 2\|u_0\|_{L^2}^2(\tilde{u}_+ - v)) + \mathbb{O}(\varepsilon^4),$$

we get, using that B^1 is an algebra on which Π_+ acts,

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \lesssim \varepsilon^2 + \varepsilon^4 t + \varepsilon^2 \int_0^t \|\tilde{u}_+(\tau) - v(\tau)\|_{B^1}\, d\tau.$$

This yields

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \lesssim (\varepsilon^2 + \varepsilon^4 t)\mathrm{e}^{\varepsilon^2 t};$$

hence, for $t \le \dfrac{c_\alpha}{\varepsilon^2} \log \dfrac{1}{\varepsilon}$,

$$\|\tilde{u}_+(t) - v(t)\|_{B^1} \le \varepsilon^{2-\alpha}.$$

We now turn to the estimates in H^s for $s > 1$.

From the equation on v and the a priori estimate in B^1, it follows that $\|v(t)\|_{H^s} \le Ae^{A\varepsilon^2 t}$, $t > 0$, so that $\|v(t)\|_{H^s} \le N(\varepsilon)$ for $t \le (c/\varepsilon^2)\log(1/\varepsilon)$, $0 < c \ll 1$, where $N(\varepsilon) := A\varepsilon^{-cA}$.

Let us assume that for some $T > 0$,

$$\|u(t)\|_{H^s} \le 10N(\varepsilon) \quad \text{for all } t \in [0, T].$$

We are going to prove that, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that, if

$$T \le \frac{c_\alpha}{\varepsilon^2} \log \frac{1}{\varepsilon},$$

then

$$\|u(t) - v(t)\|_{H^s} \le \varepsilon^{2-\alpha} \quad \text{for all } t \in [0, T]$$

Since $\|v(t)\|_{H^s} \le N(\varepsilon)$ for $t \le (c/\varepsilon^2)\log(1/\varepsilon)$, this will prove the result by a bootstrap argument.

As before, we perform the same canonical transformation

$$\tilde{u}(t) := \chi_\varepsilon^{-1}(u(t)),$$

to get the solution of

$$i\partial_t \tilde{u} - |D|\,\tilde{u} = \varepsilon^2 i X_{\tilde{R}}(\tilde{u}) + \varepsilon^4 i Y(\tilde{u}).$$

By Lemma 4,

$$\|\tilde{u}(t) - u(t)\|_{H^s} \lesssim \varepsilon^2 N(\varepsilon)^3$$

and so $\|\tilde{u}(t)\|_{H^s} \lesssim N(\varepsilon)$. Therefore it suffices to prove that

$$\|\tilde{u}(t) - v(t)\|_{H^s} \le \varepsilon^{2-\alpha} \quad \text{for all } t \in [0, T].$$

We first deal with $\tilde{u}_-$. A similar argument to the one developed in B^1 gives that for, for $0 \le t \lesssim \dfrac{1}{\varepsilon^2} \log \dfrac{1}{\varepsilon}$,

$$\sup_{0\le\tau\le t} \|\tilde{u}_-(\tau)\|_{H^s} \le C_\alpha \varepsilon^{2-\alpha}$$

for every $\alpha > 0$.

It remains to estimate the H^s norm of $\tilde{u}_+ - v$. Notice that

$$\|\tilde{u}_{0,+} - u_0\|_{H^s} \le \varepsilon^2$$

by Lemma 4. We use the following inequality — recall that $B^1 \subset L^\infty$:

$$\left\| \Pi_+(|u|^2u-|v|^2v) \right\|_{H^s} \lesssim \left(\|u\|_{B^1}^2 + \|v\|_{B^1}^2 \right) \|u-v\|_{H^s} + \left(\|v\|_{H^s} + \|u-v\|_{H^s} \right) \left(\|u\|_{B^1} + \|v\|_{B^1} \right) \|u-v\|_{B^1}.$$

Plugging this into a Gronwall inequality, in view of the previous estimates, we finally get

$$\|\tilde{u}_+(t) - v(t)\|_{H^s} \le \varepsilon^{2-\alpha}$$

for $t \le \frac{c_\alpha}{\varepsilon^2} \log \frac{1}{\varepsilon}$. This completes the proof.

Appendix: A necessary condition for wellposedness

In this section, we justify that the boundedness in H^s of the first iteration map of the Duhamel formula

$$F(t) = e^{-it|D|} f - i \int_0^t e^{-i(t-\tau)|D|} (|F(\tau)|^2 F(\tau))\, d\tau$$

implies

$$\int_0^1 \|e^{-it|D|} f\|_{L^4(\mathbb{T})}^4 dt \lesssim \|f\|_{H^{s/2}}^4.$$

Indeed, assume the inequality

$$\left\| \int_0^1 e^{-i(1-\tau)|D|} (|e^{-i\tau|D|} f|^2 e^{-i\tau|D|} f) d\tau \right\|_{H^s} \lesssim \|f\|_{H^s}^3.$$

We compute the scalar product of the expression in the left hand side with $e^{-i|D|} f$ and we get

$$\int_0^1 \|e^{-i\tau|D|} f\|_{L^4}^4 d\tau \lesssim \|f\|_{H^s}^3 \|f\|_{H^{-s}}.$$

If we assume first that f is spectrally supported, that is if $f = \Delta_N f$ for some N, then $\|f\|_{H^{\pm s}} \simeq N^{\pm s} \|f\|_{L^2}$ and the preceding inequality becomes

$$\int_0^1 \|e^{-i\tau|D|} f\|_{L^4}^4 d\tau \lesssim N^{2s} \|f\|_{L^2}^4.$$

Finally, for general $f = \sum_N \Delta_N(f)$, we used the Littlewood–Paley estimate

$$\|g\|_{L^4}^4 \lesssim \sum_N \|\Delta_N g\|_{L^4}^4$$

to get

$$\int_0^1 \|e^{-i\tau|D|} f\|_{L^4}^4 d\tau \lesssim \|f\|_{H^{s/2}}^4.$$

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NONLINEAR SCHRÖDINGER EQUATION AND FREQUENCY SATURATION

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We propose an approach that permits to avoid instability phenomena for the nonlinear Schrödinger equations. We show that by approximating the solution in a suitable way, relying on a frequency cut-off, global well-posedness is obtained in any Sobolev space with nonnegative regularity. The error between the exact solution and its approximation can be measured according to the regularity of the exact solution, with different accuracy according to the cases considered.

1. Introduction

We consider the nonlinear Schrödinger equation

$$i\partial_t u+\Delta u=\epsilon|u|^{2\sigma}u,\quad (t,x)\in I\times\mathbb{R}^d,\quad u_{|t=0}=u_0, \tag{1-1}$$

for some time interval $I\ni 0$, with $\epsilon=1$ (defocusing case) or $\epsilon=-1$ (focusing case). The aim of this paper is to propose an approach to overcome the lack of local well-posedness in Sobolev spaces with nonnegative regularity.

Recall two important invariances associated to (1-1):

- Scaling: if u solves (1-1), then for $\lambda>0$, so does $u_\lambda(t,x):=\lambda^{1/\sigma}u(\lambda^2t,\lambda x)$. This scaling leaves the $\dot H^{s_c}_x$-norm invariant, with $s_c=d/2-1/\sigma$.
- Galilean: if u solves (1-1), then for $v\in\mathbb{R}^d$, so does $e^{iv\cdot x-i|v|^2t/2}u(t,x-vt)$. This transform leaves the L^2_x-norm invariant.

These two arguments suggest that the critical Sobolev regularity to solve (1-1) is $\max(s_c,0)$. Indeed, if $s_c\geqslant 0$, local well-posedness from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for $s\geqslant s_c$ has been established in [Cazenave and Weissler 1990], and if $s_c<0$, local well-posedness from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for $s\geqslant 0$ has been established in [Tsutsumi 1987].

If $s_c>0$, pathological phenomena have been exhibited for initial data in $H^s(\mathbb{R}^d)$ with $0<s<s_c$; Gilles Lebeau has proved a "norm inflation" phenomenon for the wave equation $\partial_t^2u-\Delta u+u^p=0$, $x\in\mathbb{R}^3$, $p\in 2\mathbb{N}+1$, $p\geqslant 7$ [Lebeau 2001; Métivier 2004]. The analogous result for (1-1) is this:

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Theorem 1.1 [Christ et al. 2003; Burq et al. 2005]. *Let $\sigma \geqslant 1$. Assume that $s_c = d/2 - 1/\sigma > 0$, and let $0 < s < s_c$. There exists a family $(u_0^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^d)$ with*

$$\|u_0^h\|_{H^s(\mathbb{R}^d)} \to 0 \quad \textit{as } h \to 0,$$

a solution u^h to (1-1) and $0 < t^h \to 0$, such that

$$\|u^h(t^h)\|_{H^s(\mathbb{R}^d)} \to +\infty \quad \textit{as } h \to 0.$$

The argument of the proof consists in considering concentrated initial data

$$u_0(x) = h^{s-d/2}(\log 1/h)^{-\alpha} a_0\Big(\frac{x}{h}\Big), \quad \text{with } h \to 0,$$

and showing that for very short time, the Laplacian can be neglected in (1-1). The above result then stems from its (easy) counterpart in the ODE case, by choosing a suitable $\alpha > 0$. In the spirit of [Lebeau 2005], the above result has been strengthened to a "loss of regularity" in [Alazard and Carles 2009; Carles 2007; Thomann 2008]. The assumptions and conclusion are similar to that in Theorem 1.1; the only difference is that $u^h(t^h, \cdot)$ is measured in $H^k(\mathbb{R}^d)$ for any $k > s/(1+\sigma(s_c - s))$, so k is allowed to be smaller than s. In all the cases mentioned here, the lack of uniform continuity of the nonlinear flow map near the origin is due to the appearance of higher and higher frequencies on a very short time scale. If $s_c < 0$, similar pathological phenomena have been established in $H^s(\mathbb{R}^d)$ with $s < 0$, where on the contrary, low frequencies are ignited; see [Bejenaru and Tao 2006; Carles et al. 2012; Christ et al. 2003; Kenig et al. 2001]. In the rest of this paper, we focus on nonnegative regularity, $s \geqslant 0$.

The goal of this paper is twofold. First, we want to investigate a method to remove the pathology mentioned above, causing a lack of well-posedness for (1-1), in a deterministic way, as opposed to the probabilistic approach initiated in [Burq and Tzvetkov 2008a; 2008b] for the wave equation. The other motivation is related to numerical simulations for (1-1), where high frequencies may be a source of important errors; see for instance [Ignat and Zuazua 2012], a reference which will be discussed in further detail in Sections 3 and 4.

We show that with a suitable cut-off on the high frequencies of the nonlinearity, the obstructions to local well-posedness vanish, and the problem becomes globally well-posed: the nonlinear evolution of any initial datum in $L^2(\mathbb{R}^d)$ can be controlled *a priori*, an information which may be useful for numerics, since we do not have to decide if the initial datum belongs to a full measure set or not. This strategy is validated inasmuch as this procedure yields a good approximation of the solution to (1-1) as the cut-off tends to the identity. Note that this approach can be viewed as a deterministic counterpart of the one presented in [Burq et al. 2012] (see also [Burq 2011]). There, for the one-dimensional L^2-supercritical defocusing nonlinear Schrödinger equation, the authors construct a Gibbs measure such that, among other features, the pathological phenomenon described in Theorem 1.1 occurs for a set of initial data whose measure is zero: on the support of the Gibbs measure, the Cauchy problem is globally well-posed, and a scattering theory is available. Both points of view aim at showing that norm inflation in the sense of Theorem 1.1 is a rare phenomenon: in [Burq et al. 2012], the authors give a rigorous meaning to this statement in an abstract way, while we are rather interested in a recipe to avoid instabilities for sure, by a suitable approximation of the equation, which can be used typically for numerical simulations.

Our choice of cutting off the high frequencies instead of, for instance, the values of the function itself is indeed motivated by numerics, where it is standard to filter out high frequencies (sometimes without even saying so). In an appendix, we discuss another approach, consisting in saturating the values of the nonlinearity. One could of course combine both approaches, frequency and physical saturations.

Notation. We define the Fourier transform by the formula

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

We write $a \lesssim b$ if there exists C such that $a \leqslant Cb$. In the presence of a small parameter h, the notation indicates that C is independent of $h \in (0, 1]$.

2. From instability to global well-posedness

Let $\chi : \mathbb{R}^d \to [0, 1]$ be a smooth function, equal to one on the unit ball, and even: $\chi(-x) = \chi(x)$ for all $x \in \mathbb{R}^d$. It may be compactly supported, in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, or with a slower decay at infinity. For simplicity, we will not discuss sharp assumptions on χ. We define the frequency "cut-off" Π as the Fourier multiplier

$$\widehat{\Pi(f)}(\xi) = \chi(\xi)\widehat{f}(\xi).$$

As pointed out in the introduction, in the examples constructed to prove the lack of local well-posedness, the mechanism of high frequencies amplification occurs at the level of the ordinary differential equation. We discuss some strategies to saturate high frequencies at the ODE level first, with $\epsilon = 1$ for simplicity.

2A. ***Candidates at the ODE level.*** The first possibility to prevent the appearance of high frequencies by nonlinear self-interaction consists in saturating the whole nonlinearity:

$$i\partial_t v = \Pi(|v|^{2\sigma} v). \tag{2-1}$$

This can be viewed as an extremely simplified version of the I-method (see [Colliander et al. 2002]). Another choice consists in saturating the high frequencies in the "nonlinear multiplicative potential" only, that is $|v|^{2\sigma}$. For $\sigma \in \mathbb{N}$, we propose two possibilities,

$$i\partial_t v = \Pi(|v|^{2\sigma})v, \tag{2-2}$$

$$i\partial_t v = \big(\Pi(|v|^2)\big)^{\sigma} v. \tag{2-3}$$

In the cubic case $\sigma = 1$, the last two approaches obviously coincide. These approaches have two advantages over (2-1):

- They preserve gauge invariance. If v solves the equation, then so does $ve^{i\theta}$ for any constant $\theta \in \mathbb{R}$.
- They preserve conservation of mass.

To see the second point, rewrite $\Pi(f)$ as $K * f$, with $K(x)=(2\pi)^{-d/2}\widehat{\chi}(-x)$. Since χ is even and real-valued, so is K, and therefore $\partial_t|v|^2=0$ in (2-2) and (2-3). This identity leads to the conservation of the L^2-norm at the PDE level.

Before passing to the PDE case, we conclude this section by showing that even at the ODE level, cutting off high frequencies in the initial data does not suffice to prevent the appearance of higher frequencies in the solution for positive time. For $a \in \mathcal{S}(\mathbb{R}^d)$ and $s>0$, consider the solution v^h to

$$i\partial_t v^h = |v^h|^{2\sigma} v^h, \quad v^h(0,x)=h^{s-d/2}a\left(\frac{x}{h}\right).$$

Then $v^h{}_{|t=0}$ is bounded in $H^s(\mathbb{R}^d)$, uniformly in $h\in(0,1]$, and if $\widehat{a}$ is compactly supported (in $B(0,R)$), then $\widehat{v^h}_{|t=0}$ is compactly supported (in $B(0,R/h)$). Since $\partial_t|v^h|^2=0$, we have the explicit formula

$$v^h(t,x)=h^{s-d/2}a\left(\frac{x}{h}\right)\exp\left(-ith^{2\sigma(s-d/2)}\left|a\left(\frac{x}{h}\right)\right|^{2\sigma}\right).$$

We check that for $t>0$, as $h\to 0$, the homogeneous Sobolev norms behave like

$$\|v^h(t)\|_{\dot{H}^k} \approx h^{s-2k\sigma(s-d/2)-k}t^k,$$

at least for $k\in\mathbb{N}$. The above quantity is unbounded as $h\to 0$ if

$$k>\frac{s}{1+2\sigma(s-d/2)}.$$

Therefore, if $s<d/2$, $v^h(t,\cdot)$ is unbounded in $H^s(\mathbb{R}^d)$ for $t>0$, as $h\to 0$: cutting off the high frequencies in the initial data does not suffice to control the frequency support of the solution. On the other hand, the models (2-2) and (2-3) prevent the appearance of high frequencies by nonlinear self-interaction. The above mechanism is essentially the one that leads to the norm inflation phenomenon in [Burq et al. 2005; Christ et al. 2003; Lebeau 2001], except that in those papers, the approximation by an ODE is used only on a time interval where the H^s-norm becomes unbounded, but not the H^k-norm for any $k<s$. The above mechanism at the PDE level leads to the loss of regularity [Alazard and Carles 2009; Carles 2007; Lebeau 2005; Thomann 2008], where indeed k is allowed to be smaller than s, as recalled in the introduction. Roughly speaking, the appearance of oscillations is quite similar to the above ODE example; in the PDE case, the numerology is different, and the proof is more intricate.

2B. ***Choice at the PDE level.*** We consider now the equations

$$i\partial_t u + P(D)u = \epsilon\Pi(|u|^{2\sigma})u, \tag{2-4}$$

$$i\partial_t u + P(D)u = \epsilon\left(\Pi(|u|^2)\right)^{\sigma}u, \tag{2-5}$$

where $P(D)$ is a Fourier multiplier with a real-valued symbol $P:\mathbb{R}^d\to\mathbb{R}$,

$$\widehat{P(D)f}=P(\xi)\widehat{f}(\xi).$$

The L^2-norm of u is formally independent of time:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx=0. \tag{2-6}$$

In view of this conservation and of the Young inequality

$$\|\Pi(f)\|_{L^\infty} \leqslant \|K\|_{L^\infty}\|f\|_{L^1}, \tag{2-7}$$

the option (2-5) seems more interesting than (2-4), and we have the following result.

Theorem 2.1. *Let $\sigma \in \mathbb{N}$, $\epsilon \in \{\pm 1\}$, $P : \mathbb{R}^d \to \mathbb{R}$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ even and real-valued.*

- *For any $u_0 \in L^2(\mathbb{R}^d)$, (2-5) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^d))$ such that $u_{|t=0} = u_0$. Its L^2-norm is independent of time; (2-6) holds.*
- *If in addition $u_0 \in H^s(\mathbb{R}^d)$, $s \in \mathbb{N}$, then $u \in C(\mathbb{R}; H^s(\mathbb{R}^d))$.*
- *The flow map $u_0 \mapsto u$ is uniformly continuous from the balls in $L^2(\mathbb{R}^d)$ to $C(\mathbb{R}; L^2(\mathbb{R}^d))$. More precisely, for all $u_0, v_0 \in L^2(\mathbb{R}^d)$, there exists C depending on σ, $\|K\|_{L^\infty}$, $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$ such that, for all $T > 0$,*
$$\|u - v\|_{L^\infty([-T,T];L^2(\mathbb{R}^d))} \leqslant \|u_0 - v_0\|_{L^2(\mathbb{R}^d)} e^{CT}, \tag{2-8}$$
where u and v denote the solutions to (2-5) with initial data u_0 and v_0, respectively.
- *More generally, let $s \in \mathbb{N}$. For all $u_0, v_0 \in H^s(\mathbb{R}^d)$, there exists C depending on σ, $\|K\|_{W^{s,\infty}}$, $\|u_0\|_{H^s}$ and $\|v_0\|_{H^s}$ such that for all $T > 0$,*
$$\|u - v\|_{L^\infty([-T,T];H^s(\mathbb{R}^d))} \leqslant \|u_0 - v_0\|_{H^s(\mathbb{R}^d)} e^{CT}. \tag{2-9}$$

Remark 2.2. As pointed out in [Cazenave et al. 2011], even if the solution is constructed by a fixed point argument, the continuity of the flow map is not trivial in general. In the case of Schrödinger equations (1-1), continuity of the flow map in $H^s(\mathbb{R}^d)$ is known only in a limited number of cases: see [Tsutsumi 1987] for $s = 0$, [Kato 1987] for $s = 1$ and $s = 2$, and [Cazenave et al. 2011] for $0 < s < 1$.

Proof. First, recall that $S(t) = e^{-itP(D)}$ is a unitary group on $\dot{H}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$. Duhamel's formula associated to (2-5) reads

$$u(t) = S(t)u_0 - i\epsilon \int_0^t S(t-\tau)\big((K * |u|^2)^\sigma u\big)(\tau)\, d\tau. \tag{2-10}$$

The local existence in L^2 stems from a standard fixed point argument in

$$X(T) = \big\{u \in C([-T,T]; L^2(\mathbb{R}^d)) \;\big|\; \|u\|_{L^\infty([-T,T];L^2)} \leqslant 2\|u_0\|_{L^2}\big\}.$$

Denote by $\Phi(u)(t)$ the right hand side of (2-10). In view of (2-7), for $t \in [-T, T]$,

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^2} &\leqslant \|u_0\|_{L^2} + \int_{-T} \big\|\big((K * |u|^2)^\sigma u\big)(\tau)\big\|_{L^2}\, d\tau \\
&\leqslant \|u_0\|_{L^2} + \int_{-T}^T \big\|K * |u(\tau)|^2\big\|_{L^\infty}^\sigma \|u(\tau)\|_{L^2}\, d\tau \\
&\leqslant \|u_0\|_{L^2} + \|K\|_{L^\infty}^\sigma \int_{-T}^T \|u(\tau)\|_{L^2}^{2\sigma+1}\, d\tau.
\end{aligned}$$

By choosing $T > 0$ sufficiently small, we see that $X(T)$ is stable under the action of Φ. Note that in the case of the model (2-4), the above estimate would have to be adapted, forcing us to work in a space smaller than $X(T)$ (L^2 regularity in space would no longer be sufficient in general). Contraction is established in the same way:

$$\|\Phi(u)(t)-\Phi(v)(t)\|_{L^2} \leqslant \int_{-T}^{T} \left\| \left((K*|u|^2)^\sigma u\right)(\tau) - \left((K*|v|^2)^\sigma v\right)(\tau) \right\|_{L^2} d\tau$$
$$\leqslant \int_{-T}^{T} \left\| \left((K*|u|^2)^\sigma - (K*|v|^2)^\sigma\right) u \right\|_{L^2} d\tau + \int_{-T}^{T} \left\| \left((K*|v|^2)^\sigma\right)(u-v) \right\|_{L^2} d\tau.$$

Using the estimate $|a^\sigma - b^\sigma| \lesssim (|a|^{\sigma-1}+|b|^{\sigma-1})|a-b|$, and (2-7) again, we infer

$$\|\Phi(u)(t)-\Phi(v)(t)\|_{L^2}$$
$$\lesssim \|K\|_{L^\infty}^\sigma \int_{-T}^{T} \left(\|u\|_{L^2}^{2\sigma-1} + \|v\|_{L^2}^{2\sigma-1}\right) \|u-v\|_{L^2} \|u\|_{L^2}\, d\tau + \|K\|_{L^\infty}^\sigma \int_{-T}^{T} \|v\|_{L^2}^{2\sigma} \|u-v\|_{L^2}\, d\tau,$$

where all the functions inside the integrals are implicitly evaluated at time τ. Choosing $T>0$ possibly smaller, Φ is a contraction on $X(T)$. Note that this small time T depends only on σ, $\|K\|_{L^\infty}$ and $\|u_0\|_{L^2}$. Since the L^2-norm of u is preserved (see [Cazenave 2003] for a rigorous justification), the construction of a local solution can be repeated indefinitely, hence global existence and uniqueness at the L^2 level.

Global existence in $H^s(\mathbb{R}^d)$ for $s\in\mathbb{N}$ then follows easily, thanks to the estimate

$$\left\|(K*|u|^2)^\sigma u\right\|_{H^s} \lesssim \sum_{|\alpha|+|\beta|=s} \left\|\partial^\alpha (K*|u|^2)^\sigma \partial^\beta u\right\|_{L^2} \lesssim \|K\|_{W^{s,\infty}}^\sigma \|u\|_{L^2}^\sigma \|u\|_{H^s}.$$

The continuity of the flow map in L^2 is obtained by resuming the estimate written to establish the contraction of Φ: For $t>0$,

$$\|u(t)-v(t)\|_{L^2} \leqslant \|u_0-v_0\|_{L^2} + \|K\|_{L^\infty}^\sigma \int_0^t \left(\|u\|_{L^2}^{2\sigma} + \|v\|_{L^2}^{2\sigma}\right)\|u-v\|_{L^2}\, d\tau$$
$$\leqslant \|u_0-v_0\|_{L^2} + \|K\|_{L^\infty}^\sigma \left(\|u_0\|_{L^2}^{2\sigma} + \|v_0\|_{L^2}^{2\sigma}\right) \int_0^t \|u-v\|_{L^2}\, d\tau,$$

where we have used the conservation of the L^2-norm. Proceeding similarly for $t<0$, Gronwall's lemma then yields (2-8) for C depending only of σ, $\|K\|_{L^\infty}$, $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$. Finally, (2-9) is obtained in a similar fashion. □

Remark 2.3. The proof of continuity of the flow map is easy. This is in sharp contrast with the case of the equation without frequency cut-off. In the case of Schrödinger equations ($P(\xi)=-|\xi|^2$), continuity is more intricate to establish (see [Tsutsumi 1987]), and is true only for L^2-subcritical nonlinearities, $\sigma\leqslant 2/d$, from [Christ et al. 2003].

We note that even for large σ, global well-posedness in L^2 is available, in sharp contrast with the nonlinear Schrödinger equation (1-1). Even in the focusing case $\epsilon=-1$, the high frequency cut-off prevents finite time blow-up. In (2-9), consider $v_0=v=0$ and $s=1$ for instance: by comparison with the case of (1-1), we see that the constant C necessarily depends on K (or equivalently on χ), and is unbounded as χ converges to the Dirac mass. The frequency cut-off Π removes the instabilities, and prevents finite time blow-up.

Remark 2.4 (Hamiltonian structure in the cubic case). If $\sigma = 1$, (2-4) and (2-5) coincide. We have the equivalence

$$\chi \text{ even and real-valued } \iff K \text{ even and real-valued.}$$

This implies that under the assumption of Theorem 2.1, (2-5) has an Hamiltonian structure, and the conserved energy is

$$H(u) = \int_{\mathbb{R}^d} \overline{u}(x) P(D) u(x)\, dx + \frac{\epsilon}{2} \iint K(x-y)|u(y)|^2 |u(x)|^2 \, dx dy.$$

3. Convergence in the smooth case

Suppose that $P(D)$ converges to Δ and that Π converges to Id, does the solution to (2-5) then converge to the solution of NLS? We show that this is the case under suitable assumptions on these convergences, at least in the case where the solution to the limiting Equation (1-1) is very smooth. In the sequel, the convergence is indexed by $h \in (0, 1]$.

Proposition 3.1. *Let $\sigma \in \mathbb{N}$. We assume that P and Π verify the following properties*:

- *There exist $\alpha, \beta \geqslant 0$ such that $P_h(\xi) = -|\xi|^2 + \mathcal{O}(h^\alpha \langle \xi \rangle^\beta)$.*
- *$\chi_h(\xi) = \chi(h\xi)$, with $\chi \in \mathcal{S}(\mathbb{R}^d; [0, 1])$ even, real-valued, $\chi = 1$ on the unit ball.*

Denote by u^h the solution to (2-5) with P_h and χ_h, such that $u^h_{|t=0} = u_{|t=0}$. Suppose that the solution to (1-1) satisfies $u \in L^\infty([0, T]; H^{s+\beta})$, for some $s > d/2$. Then

$$\|u - u^h\|_{L^\infty([0,T];H^s)} \lesssim h^{\min(\alpha,\beta)}.$$

Example 3.2. The above assumption on P_h is satisfied with $\alpha = 1$ and $\beta = 2$ in the following cases:

- $P_h(\xi) = \dfrac{-|\xi|^2}{1+h|\xi|^2}$.
- $P_h(\xi) = -\dfrac{1}{h} \arctan(h|\xi|^2)$.

The second example is borrowed from [Debussche and Faou 2009], where this truncated operator appears naturally when discretizing the Laplacian for numerical schemes.

Remark 3.3. In this result, no assumption is needed on the possible decay of χ at infinity.

Proof. Let $w^h = u - u^h$. It satisfies $w^h_{|t=0} = 0$ and

$$\begin{aligned} i\partial_t w^h + P_h(D) w^h = {} & \epsilon (\Pi_h(|u|^2))^\sigma u - \epsilon (\Pi_h(|u^h|^2))^\sigma u^h \\ & + (P_h(D) - \Delta)\, u + \epsilon \big(|u|^{2\sigma} - (\Pi_h(|u|^2))^\sigma\big) u, \end{aligned}$$

where we have denoted by Π_h the Fourier multiplier of symbol χ_h. Denote by $R^h(u)$ the second line, which corresponds to a source term. In view of the assumption on P_h, there exists C independent of $h \in (0, 1]$ such that

$$\|P_h(D) f - \Delta f\|_{H^s} \leqslant C h^\alpha \|f\|_{H^{s+\beta}} \quad \text{for all } f \in H^{s+\beta}(\mathbb{R}^d).$$

We also have, by the Plancherel formula,

$$\|(1-\Pi_h)f\|_{H^s}^2 = \int_{\mathbb{R}^d} (1-\chi(h\xi))^2 \langle\xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi$$
$$\leqslant \int_{|\xi|>1/h} \langle\xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \leqslant h^{2\beta} \int_{|\xi|>1/h} \langle\xi\rangle^{2s+2\beta} |\widehat{f}(\xi)|^2 d\xi \leqslant h^{2\beta} \|f\|_{H^{s+\beta}}^2.$$

Therefore,

$$\|R^h(u)\|_{L^\infty([0,T];H^s)} \lesssim h^{\min(\alpha,\beta)} \|u\|_{L^\infty([0,T];H^{s+\beta})}.$$

Now since $s > d/2$, $H^s(\mathbb{R}^d)$ is an algebra, and there exists C independent of h such that

$$\left\|(\Pi_h(|u|^2))^\sigma u - (\Pi_h(|u^h|^2))^\sigma u^h\right\|_{H^s} \leqslant C\|\widehat{\chi}\|_{L^1}^\sigma \left(\|u\|_{H^s}^{2\sigma} + \|u^h\|_{H^s}^{2\sigma}\right)\|u-u^h\|_{H^s},$$

where the Young inequality that we have used is not the same as in Section 2:

$$\|K * f\|_{L^2} \leqslant \|K\|_{L^1}\|f\|_{L^2}.$$

This is essentially the only way to obtain an estimate independent of $h \in (0, 1]$. Indeed, $\Pi_h(f) = K_h * f$, with

$$K_h(x) = \frac{1}{(2\pi)^{d/2}h^d}\widehat{\chi}\left(\frac{-x}{h}\right).$$

The result then stems from a bootstrap argument: so long as

$$\|u^h\|_{L^\infty([0,t];H^s)} \leqslant 1 + \|u\|_{L^\infty([0,T];H^s)},$$

Gronwall's lemma yields

$$\|u-u^h\|_{L^\infty([0,t];H^s)} \lesssim h^{\min(\alpha,\beta)} \|u\|_{L^\infty([0,T];H^{s+\beta})}.$$

Therefore, up to choosing h sufficiently small, this estimate is valid up to $t = T$. □

Such a convergence result can be compared to the one proved in [Ignat and Zuazua 2012] to prove the convergence of numerical approximations. The approach there is a bit different though, inasmuch as the frequency cut-off does not affect the nonlinearity (as in (2-5)), but the initial data: consider a solution v^h to

$$i\partial_t v^h + P_h(D)v^h = \epsilon|v^h|^{2\sigma}v^h, \quad v^h_{|t=0} = \Pi_h u_0.$$

Ignat and Zuazua proved that the discrete analogue of $\Pi_h u - v_h$ is small. Proposition 3.1 differs from their results in several aspects:

- The context in [Ignat and Zuazua 2012] is discrete.
- Only the low frequency part of u, $\Pi_h u$, is shown to be well approximated.
- The regularity assumption on u may be much weaker.

As mentioned above, the second point is due to the choice of the model. However, as discussed in Section 2A, controlling the high frequencies of the initial data must not be expected to ensure a control of high frequencies of the solution v^h for positive time.

The third point is due to the use of Strichartz estimates in [Ignat and Zuazua 2012]. In the next section, we show that in the presence of dispersion (with $P_h(\xi) = -|\xi|^2$), Proposition 3.1 can be adapted to rougher data.

4. Convergence using dispersive estimates

We first recall a standard definition.

Definition 4.1. A pair $(p, q) \neq (2, \infty)$ is admissible if $p \geqslant 2$, $q \geqslant 2$, and

$$\frac{2}{p} = d\Big(\frac{1}{2} - \frac{1}{q}\Big).$$

We shall consider (2-5) when $P(D)$ is exactly the Laplacian, and not an approximation as in Proposition 3.1. The reason is that when P is bounded, then no Strichartz estimate is available, as we now recall. Let $S(\cdot)$ be bounded on H^s for all $s \geqslant 0$. By the Sobolev embedding, for all (p, q) (not necessarily admissible) with $2 \leqslant q < \infty$, there exists $C > 0$ such that for all $u_0 \in H^{d/2-d/q}(\mathbb{R}^d)$, and all finite time interval I,

$$\|S(\cdot)u_0\|_{L^p(I;L^q(\mathbb{R}^d))} \leqslant C\|S(\cdot)u_0\|_{L^p(I;H^{d/2-d/q}(\mathbb{R}^d))} \leqslant C\|u_0\|_{L^p(I;H^{d/2-d/q}(\mathbb{R}^d))} = C|I|^{1/p}\|u_0\|_{H^{d/2-d/q}(\mathbb{R}^d)}.$$

If the Fourier multiplier P is bounded, the above estimate cannot be improved, in sharp contrast with the result provided by the Strichartz estimates.

Proposition 4.2 [Carles 2011]. *Let $d \geqslant 1$, and $P \in L^\infty(\mathbb{R}^d; \mathbb{R})$. Write $S(t) = e^{-itP(D)}$. Suppose that there exist an admissible pair (p, q), an index $k \in \mathbb{R}$, a time interval $I \ni 0$, $|I| > 0$, and a constant $C > 0$ such that*

$$\|S(\cdot)u_0\|_{L^p(I;L^q(\mathbb{R}^d))} \leqslant C\|u_0\|_{H^k(\mathbb{R}^d)} \quad \textit{for all } u_0 \in H^k(\mathbb{R}^d).$$

Then necessarily $k \geqslant 2/p = d/2 - d/q$.

We now state the main result of this section.

Theorem 4.3. *Let $\sigma \in \mathbb{N}$ and $T > 0$. We assume that $\chi_h(\xi) = \chi(h\xi)$, with $\chi \in \mathcal{S}(\mathbb{R}^d)$ even, real-valued, $\chi = 1$ on $B(0, 1)$. Let u solve (1-1), and consider the solution u^h to*

$$i\partial_t u^h + \Delta u^h = \epsilon\big(\Pi_h\big(|u^h|^2\big)\big)^\sigma u^h, \quad u^h_{|t=0} = u_0.$$

1. *Suppose that $\sigma = 1$ and $d \leqslant 2$. If $u \in L^\infty([0, T]; L^2) \cap L^{8/d}([0, T]; L^4)$, then*

$$\|u - u^h\|_{L^\infty([0,T];L^2)} \underset{h\to 0}{\longrightarrow} 0.$$

2. *Suppose that $\sigma = 1$ and $d = 3$.*

- *If* $u, \nabla u \in L^\infty([0,T];L^2)\cap L^{8/d}([0,T];L^4)$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

- *If* $u\in L^\infty([0,T];H^s)$, *with* $s>3/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h^{s-1}.$$

3. *Suppose that* $\sigma\geqslant 1$ *and* $d\leqslant 2$. *If* $u\in L^\infty([0,T];H^s)$, *with* $s\geqslant 1$ *and* $s>d/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|u-u^h\|_{L^\infty([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

If in addition $s>1$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h^{s-1}.$$

Remark 4.4. Suppose u_0 sufficiently smooth. If $\epsilon=+1$ (defocusing case), the bounds for u are known in several cases, with $T>0$ arbitrarily large. On the contrary, if $\epsilon=-1$ (focusing case), T may have to be finite, bounded by a blow-up time. See [Cazenave 2003; Ginibre and Velo 1984]. Typically, if $\sigma=d=1$, then the assumption of the first point is fulfilled for all $T>0$ as soon as $u_0\in L^2(\mathbb{R})$, for $\epsilon\in\{\pm 1\}$, from [Tsutsumi 1987], and if $\sigma\geqslant 1$, $d\leqslant 2$, the assumption of the third point is fulfilled for all $T>0$ as soon as $u_0\in H^s(\mathbb{R}^d)$, for $\epsilon=+1$, from [Ginibre and Velo 1984].

Proof. For fixed $h>0$, Theorem 2.1 shows that $u^h\in C(\mathbb{R};H^k)$, with $k=0$, 1 or s according to the cases considered in the assumptions of the theorem. Of course, the bounds provided by Theorem 2.1 blow up as $h\to 0$ if $k>0$.

As in the proof of Proposition 3.1, let $w^h=u-u^h$. The equation satisfied by w^h is simpler than in the proof of Proposition 3.1, since $P_h(D)=\Delta$:

$$i\partial_t w^h+\Delta w^h=\epsilon\left(\Pi_h(|u|^2)\right)^\sigma u-\epsilon\left(\Pi_h(|u^h|^2)\right)^\sigma u^h+\epsilon\big(|u|^{2\sigma}-\left(\Pi_h(|u|^2)\right)^\sigma\big)u.$$

We resume the notation

$$R^h(u)=\epsilon\big(|u|^{2\sigma}-\left(\Pi_h(|u|^2)\right)^\sigma\big)u \quad \text{and} \quad \Pi_h(f)=K_h * f,$$

with $K_h(x)=(2\pi)^{-d/2}h^{-d}\widehat{\chi}(-x/h)$. From the Young inequality, we have, for all $q\in[1,\infty]$,

$$\|\Pi_h(f)\|_{L^q}\leqslant \|K_h\|_{L^1}\|f\|_{L^q}\leqslant \|\widehat{\chi}\|_{L^1}\|f\|_{L^q}, \tag{4-1}$$

an estimate which is uniform in $h>0$. Introduce the Lebesgue exponents

$$q=2\sigma+2,\qquad p=\frac{4\sigma+4}{d\sigma},\qquad \theta=\frac{2\sigma(2\sigma+2)}{2-(d-2)\sigma}.$$

The pair (p,q) is admissible, and

$$\frac{1}{q'}=\frac{2\sigma}{q}+\frac{1}{q},\qquad \frac{1}{p'}=\frac{2\sigma}{\theta}+\frac{1}{p}. \tag{4-2}$$

For $t>0$, write $L_t^jL^k=L^j([0,t];L^k(\mathbb{R}^d))$. From the Strichartz estimates (see [Cazenave 2003]),

$$\begin{aligned}\|w^h\|_{L_t^pL^q\cap L_t^\infty L^2} &\lesssim \left\|\left(\Pi_h(|u|^2)\right)^\sigma u-\left(\Pi_h(|u^h|^2)\right)^\sigma u^h\right\|_{L_t^{p'}L^{q'}}+\|R^h(u)\|_{L_t^{p_1'}L^{q_1'}}\\ &\lesssim \left(\|u\|^{2\sigma}_{L_t^\theta L^q}+\|u^h\|^{2\sigma}_{L_t^\theta L^q}\right)\|w^h\|_{L_t^pL^q}+\|R^h(u)\|_{L_t^{p_1'}L^{q_1'}},\end{aligned}$$

where we have used the Hölder inequality and (4-1), and where (p_1,q_1) is an admissible pair whose value will be given later.

If $\sigma=1$ and $d\leqslant 2$, then $\theta\leqslant p$, and we infer

$$\|w^h\|_{L_t^pL^q\cap L_t^\infty L^2}\lesssim t^{1/\theta-1/p}\left(\|u\|^{2\sigma}_{L_t^pL^q}+\|u^h\|^{2\sigma}_{L_t^pL^q}\right)\|w^h\|_{L_t^pL^q}+\|R^h(u)\|_{L_t^{p_1'}L^{q_1'}}.$$

In the first case of the theorem, we assume $u\in L^p([0,T];L^q)$, since $p=8/d$ and $q=4$ for $\sigma=1$. We use again a bootstrap argument: so long as $\|u^h\|_{L_t^pL^q}\leqslant 2\|u\|_{L_t^pL^q}$, we divide the interval $[0,T]$ into finitely many small intervals so the first term of the right hand side is absorbed by the left hand side (recall that p is finite), and we have

$$\|w^h\|_{L_t^pL^q\cap L_t^\infty L^2}\lesssim \|R^h(u)\|_{L_t^{p_1'}L^{q_1'}}.$$

The bootstrap argument is validated provided that $\|R^h(u)\|_{L_T^{p_1'}L^{q_1'}}\to 0$ as $h\to 0$.

If we have only $\sigma<2/(d-2)$, then by the Sobolev embedding,

$$\|u\|_{L_t^\theta L^q}\leqslant t^{1/\theta}\|u\|_{L_t^\infty H^1}.$$

In the same way as above,

$$\|\nabla w^h\|_{L_t^pL^q\cap L_t^\infty L^2}\lesssim \left\|\nabla\left(\left(\Pi_h(|u|^2)\right)^\sigma u-\left(\Pi_h(|u^h|^2)\right)^\sigma u^h\right)\right\|_{L_t^{p'}L^{q'}}+\|\nabla R^h(u)\|_{L_t^{p_1'}L^{q_1'}}.$$

The first term of the right hand side is controlled by

$$\left\|\left(\Pi_h(|u|^2)\right)^\sigma \nabla u-\left(\Pi_h(|u^h|^2)\right)^\sigma \nabla u^h\right\|_{L_t^{p'}L^{q'}}+\left\|u\nabla\left(\Pi_h(|u|^2)\right)^\sigma-u^h\nabla\left(\Pi_h(|u^h|^2)\right)^\sigma\right\|_{L_t^{p'}L^{q'}}. \tag{4-3}$$

Introducing the factor $(\Pi_h(|u|^2))^\sigma\nabla u^h$, the first term is estimated by

$$\begin{aligned}&\left\|\left(\Pi_h(|u|^2)\right)^\sigma \nabla w^h\right\|_{L_t^{p'}L^{q'}}+\left\|\left(\left(\Pi_h(|u|^2)\right)^\sigma-\left(\Pi_h(|u^h|^2)\right)^\sigma\right)\nabla u^h\right\|_{L_t^{p'}L^{q'}}\\ &\qquad\lesssim \left\|\Pi_h(|u|^2)\right\|^\sigma_{L_t^{\theta/2}L^{q/2}}\|\nabla w^h\|_{L_t^pL^q}+\left(\|u\|^{2\sigma-2}_{L_t^\theta L^q}+\|u^h\|^{2\sigma-2}_{L_t^\theta L^q}\right)\left\||u|^2-|u^h|^2\right\|_{L_t^{\theta/2}L^{q/2}}\|\nabla u^h\|_{L_t^pL^q}\\ &\qquad\lesssim \|u\|^{2\sigma}_{L_t^\theta L^q}\|\nabla w^h\|_{L_t^pL^q}+\left(\|u\|^{2\sigma-1}_{L_t^\theta L^q}+\|u^h\|^{2\sigma-1}_{L_t^\theta L^q}\right)\left\|w^h\right\|_{L_t^\theta L^q}\|\nabla u^h\|_{L_t^pL^q}\\ &\qquad\lesssim t^{2\sigma/\theta}\|u\|^{2\sigma}_{L_t^\infty H^1}\|\nabla w^h\|_{L_t^pL^q}+t^{2\sigma/\theta}\left(\|u\|^{2\sigma-1}_{L_t^\infty H^1}+\|u^h\|^{2\sigma-1}_{L_t^\infty H^1}\right)\left\|w^h\right\|_{L_t^\infty H^1}\|\nabla u^h\|_{L_t^pL^q}.\end{aligned}$$

Proceeding similarly for the other term in (4-3), splitting $[0,T]$ into finitely many time intervals where the terms containing w^h on the right hand side can be absorbed by the left hand side, and using a bootstrap argument, we end up with

$$\|w^h\|_{L_t^pW^{1,q}\cap L_t^\infty H^1}\lesssim \|R^h(u)\|_{L_t^{p_1'}W^{1,q_1'}}.$$

Therefore, it suffices to show that for some admissible pair (p_1, q_1), the source term converges to 0 in $L^{p_1'}([0,T]; L^{q_1'})$ (if $\sigma = 1$ and $d \leqslant 2$) or in $L^{p_1'}([0,T]; W^{1,q_1'})$ (in the other cases), so the bootstrap argument is completed. In addition, the rate of converge of the source term, if any, yields a rate of convergence for w^h. The theorem then stems from the following lemma, in which (p,q) is given by (4-2).

Lemma 4.5. *Let $T > 0$. The source term $R^h(u)$ can be controlled as follows.*

1. *Suppose that $\sigma = 1$ and $d \leqslant 2$. If $u \in L^\infty([0,T]; L^2) \cap L^{8/d}([0,T]; L^4)$, then*

$$\|R^h(u)\|_{L^{p'}([0,T]; L^{q'})} \underset{h\to 0}{\longrightarrow} 0.$$

2. *Suppose that $\sigma = 1$ and $d = 3$.*

- *If $u, \nabla u \in L^\infty([0,T]; L^2) \cap L^{8/d}([0,T]; L^4)$, then*

$$\|R^h(u)\|_{L^{p'}([0,T]; W^{1,q'})} \underset{h\to 0}{\longrightarrow} 0.$$

- *If $u \in L^\infty([0,T]; H^s)$, with $s > 3/2$, then*

$$\|R^h(u)\|_{L^1([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|R^h(u)\|_{L^1([0,T];H^1)} \lesssim h^{s-1}.$$

3. *Suppose that $\sigma \geqslant 1$ and $d \leqslant 2$. If $u \in L^\infty([0,T]; H^s)$, with $s \geqslant 1$ and $s > d/2$, then*

$$\|R^h(u)\|_{L^1([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|R^h(u)\|_{L^1([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

If in addition $s > 1$, then

$$\|R^h(u)\|_{L^1([0,T];H^1)} \lesssim h^{s-1}.$$

Proof of Lemma 4.5. For the first case, we use the Hölder inequality, in view of (4-2):

$$\|R^h(u)\|_{L_T^{p'}L^{q'}} = \|(1-\Pi_h)(|u|^2)u\|_{L_T^{p'}L^{q'}} \leqslant \|(1-\Pi_h)(|u|^2)\|_{L_T^{\theta/2}L^{q/2}} \|u\|_{L_T^p L^q}.$$

We note that for $\sigma = 1$, $q = 4$, so by the Plancherel theorem,

$$\|(1-\Pi_h)(|u|^2)\|_{L^2}^2 = \int_{\mathbb{R}^d} (1-\chi(h\xi))^2 |\mathcal{F}(|u|^2)(\xi)|^2 d\xi \leqslant \int_{|\xi|>1/h} |\mathcal{F}(|u|^2)(\xi)|^2 d\xi.$$

By assumption, $u \in L^p([0,T]; L^4) \subset L^\theta([0,T]; L^4)$, thus $|u|^2 \in L^{\theta/2}([0,T]; L^2)$, and by the Plancherel theorem, $\mathcal{F}(|u|^2) \in L^{\theta/2}([0,T]; L^2)$. The first point of the lemma then stems from the dominated convergence theorem.

For the first case of the second point, we note that now $\theta > p$, so the above argument must be adapted, and we have to estimate the gradient of $R^h(u)$ in the same space as above. Since we have $L^\infty([0,T]; H^1(\mathbb{R}^3)) \subset L^\theta([0,T]; L^4(\mathbb{R}^3))$, the dominated convergence theorem yields

$$\|R^h(u)\|_{L_T^{p'}L^{q'}} \underset{h\to 0}{\longrightarrow} 0.$$

We now estimate $\nabla R^h(u)$. Write

$$\begin{aligned}\|\nabla R^h(u)\|_{L^{p'}_T L^{q'}} &\leqslant \|(1-\Pi_h)(|u|^2)\|_{L^{\theta/2}_T L^2}\|\nabla u\|_{L^p_T L^2} + \|(1-\Pi_h)\nabla\left(|u|^2\right)\|_{L_T^{(1/\theta+1/p)^{-1}}L^2}\|u\|_{L^\theta_T L^2}\\ &\lesssim \|(1-\Pi_h)(|u|^2)\|_{L^\infty_T L^2}\|\nabla u\|_{L^p_T L^2} + \|(1-\Pi_h)\nabla(|u|^2)\|_{L_T^{(1/\theta+1/p)^{-1}}L^2}\|u\|_{L^\infty_T L^2}.\end{aligned}$$

By the same argument as above,

$$\|(1-\Pi_h)(|u|^2)\|_{L^\infty_T L^2}\|\nabla u\|_{L^p_T L^2} \underset{h\to 0}{\longrightarrow} 0.$$

We note that u bounded in $L^\infty([0,T];H^1(\mathbb{R}^3)) \subset L^\theta([0,T];L^4(\mathbb{R}^3))$, and ∇u bounded in $L^p_T L^4$, so $\nabla|u|^2$ is bounded in $L_T^{(1/\theta+1/p)^{-1}}L^2$. Invoking Plancherel theorem and the dominated convergence theorem like above, we infer that

$$\|(1-\Pi_h)\nabla(|u|^2)\|_{L_T^{(1/\theta+1/p)^{-1}}L^2}\|u\|_{L^\infty_T L^2} \underset{h\to 0}{\longrightarrow} 0.$$

This completes the proof for the first case of the second point.

For the remaining cases, we use that $H^s(\mathbb{R}^d)$ is embedded into $L^\infty(\mathbb{R}^d)$: for fixed t,

$$\begin{aligned}\|R^h(u)(t)\|_{L^2} &\lesssim \left(\|u(t)\|_{L^\infty}^{2\sigma-2} + \|\Pi_h(|u(t)|^2)\|_{L^\infty}^{\sigma-1}\right)\|(1-\Pi_h)(|u(t)|^2)\|_{L^2}\|u(t)\|_{L^\infty}\\ &\lesssim \|u(t)\|_{L^\infty}^{2\sigma-1}\|(1-\Pi_h)(|u(t)|^2)\|_{L^2} \lesssim \|u(t)\|_{H^s}^{2\sigma-1}\|(1-\Pi_h)(|u(t)|^2)\|_{L^2}.\end{aligned}$$

Like in the proof of Proposition 3.1, we use the estimate

$$\|(1-\Pi_h)f\|_{L^2} \leqslant h^s\|f\|_{H^s}, \tag{4-4}$$

and since $H^s(\mathbb{R}^d)$ is an algebra,

$$\|R^h(u)\|_{L^\infty([0,T];L^2)} \lesssim h^s\|u\|_{L^\infty([0,T];H^s)}^{2\sigma+1}.$$

To conclude the proof, we estimate $\nabla R^h(u)$ in $L^2(\mathbb{R}^d)$. We compute

$$\begin{aligned}\nabla R^h(u) = \sigma|u|^{2\sigma-2}\big((1-\Pi_h)(\nabla(|u|^2))\big)u + \left(|u|^{2\sigma} - (\Pi_h(|u|^2))^\sigma\right)\nabla u,&\\ +\sigma\left(|u|^{2\sigma-2} - (\Pi_h(|u|^2))^{\sigma-1}\right)\Pi_h(\nabla(|u|^2))u,&\end{aligned}$$

where the second line is zero if $\sigma = 1$. We estimate successively, thanks to (4-1),

$$\begin{aligned}\left\||u|^{2\sigma-2}\left((1-\Pi_h)\left(\nabla(|u|^2)\right)\right)u\right\|_{L^2} &\leqslant \|u\|_{L^\infty}^{2\sigma-1}\left\|(1-\Pi_h)(|u|^2)\right\|_{H^1},\\ \left\|\left(|u|^{2\sigma} - (\Pi_h(|u|^2))^\sigma\right)\nabla u\right\|_{L^2} &\leqslant \|u\|_{L^\infty}^{2\sigma-2}\left\|(1-\Pi_h)\left(|u|^2\right)\right\|_{L^\infty}\|\nabla u\|_{L^2},\end{aligned}$$

and, if $\sigma \geqslant 2$,

$$\begin{aligned}\left\|\left(|u|^{2\sigma-2} - (\Pi_h(|u|^2))^{\sigma-1}\right)\Pi_h(\nabla(|u|^2))u\right\|_{L^2} &\lesssim \|u\|_{L^\infty}^{2\sigma-4}\|(1-\Pi_h)(|u|^2)\|_{L^2}\|\nabla(|u|^2)\|_{L^2}\|u\|_{L^\infty}\\ &\lesssim \|u\|_{L^\infty}^{2\sigma-2}\|(1-\Pi_h)(|u|^2)\|_{L^2}\|\nabla u\|_{L^2}.\end{aligned}$$

Since we have $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, we end up with

$$\|\nabla R^h(u)\|_{L^2} \lesssim \|u\|_{H^s}^{2\sigma-2}\|(1-\Pi_h)(|u|^2)\|_{H^1}.$$

If $s > 1$, (4-4) yields, since in addition $s > d/2$,

$$\|(1-\Pi_h)(|u|^2)\|_{H^1} \lesssim h^{s-1}\||u|^2\|_{H^s} \lesssim h^{s-1}\|u\|_{H^s}^2.$$

If $s = 1$ (a case which may occur only if $d = 1$, since $s > d/2$), we write

$$\|\nabla(1-\Pi_h)(|u|^2)\|_{L^2}^2 \leqslant \int_{|\xi|>1/h} |\mathcal{F}(\nabla(|u|^2))(\xi)|^2 d\xi.$$

Now since $\nabla(|u|^2) = 2\operatorname{Re}\bar{u}\nabla u$ and $u \in H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, $\nabla u \in L^2(\mathbb{R})$, we conclude thanks to the dominated convergence theorem. □

This completes the proof of Theorem 4.3, by choosing $(p_1, q_1) = (p, q)$ or $(\infty, 2)$. □

Appendix: Physical saturation of the nonlinearity

Instead of cutting off the high frequencies, one may be tempted to saturate the nonlinear potential, by replacing $|u|^2$ not by $\Pi(|u|^2)$ but by $f(|u|^2)$ where f is smooth, equal to the identity near the origin, and constant at infinity. Note also that a saturated nonlinearity may be in better agreement with physical models (recall however that (1-1) appears in rather different physical contexts, such as quantum mechanics, optics, and even fluid mechanics), since typically the power-like nonlinearity in (1-1) may stem from a Taylor expansion; see [Lannes 2011; Sulem and Sulem 1999]. More precisely, let $f \in C^\infty(\mathbb{R};\mathbb{R})$ such that

$$f(s) = \begin{cases} s & \text{if } 0 \leqslant s \leqslant 1, \\ 2 & \text{if } s \geqslant 2. \end{cases} \tag{A-1}$$

The analogue of the Fourier multiplier Π_h is defined as

$$f_h(|u|^2) = \frac{1}{h} f(h|u|^2),$$

and we replace (2-5) with

$$i\partial_t u^h + P_h(D)u^h = \epsilon \left(f_h(|u^h|^2)\right)^\sigma u^h, \tag{A-2}$$

so the formal conservation of the L^2-norm still holds. We could also consider

$$f_h(|u|^2) = \frac{|u|^2}{1+h|u|^2}. \tag{A-3}$$

In both cases, the main aspect to notice is that f_h is bounded and $z \mapsto f_h(|z|^2)^\sigma z$ is globally Lipschitzian. We infer the analogue of Theorem 2.1, at least in the L^2 case.

Proposition A.1. *Let $\sigma \in \mathbb{N}$, $\epsilon \in \{\pm 1\}$, $P : \mathbb{R}^d \to \mathbb{R}$ and f given either by (A-1) or by (A-3).*

- *For any $u_0 \in L^2(\mathbb{R}^d)$, (A-2) has a unique solution $u^h \in C(\mathbb{R}; L^2(\mathbb{R}^d))$ such that $u^h_{|t=0} = u_0$. Its L^2-norm is independent of time.*

- *The flow map $u_0 \mapsto u^h$ is uniformly continuous from the balls in $L^2(\mathbb{R}^d)$ to $C(\mathbb{R}; L^2(\mathbb{R}^d))$. More precisely, for all $u_0, v_0 \in L^2(\mathbb{R}^d)$, there exists C depending on σ, h, $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$ such that for all $T > 0$,*

$$\|u^h - v^h\|_{L^\infty([-T,T];L^2(\mathbb{R}^d))} \leqslant \|u_0 - v_0\|_{L^2(\mathbb{R}^d)} e^{CT},$$

where u^h and v^h denote the solutions to (A-2) with data u_0 and v_0, respectively.

Introduce

$$F_h(s) = \int_0^s f_h(y)^\sigma \, dy.$$

We check that the following conservation of energy holds:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \bar{u}^h(t,x) P_h(D) u^h(t,x)\, dx + \epsilon \int_{\mathbb{R}^d} F_h(|u(t,x)|^2)\, dx\right) = 0.$$

Proving the analogue of Proposition 3.1 is easy in the case (A-1), since the last source term for the error w^h is now

$$R^h(u) = \big(|u|^{2\sigma} - f_h(|u|^2)^\sigma\big)u,$$

and under the assumptions of Proposition 3.1, $u \in L^\infty([0,T] \times \mathbb{R}^d)$, so there exists $h_0 > 0$ such that for $0 < h \leqslant h_0$,

$$|u(t,x)|^{2\sigma} = f_h(|u(t,x)|^2)^\sigma \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^d.$$

Therefore, this source term simply vanishes for h sufficiently small. In the case (A-3), we can use the relation

$$|R_h(u)| = \left|\big(|u|^{2\sigma} - f_h(|u|^2)^\sigma\big)u\right| \lesssim \frac{h|u|^2}{1+h|u|^2}|u|^{2\sigma+1}, \tag{A-4}$$

and the Schauder lemma to get a source term which is $\mathbb{O}(h)$ in $H^s(\mathbb{R}^d)$, for $s > d/2$.

Proposition A.2. *Let $\sigma \in \mathbb{N}$. We assume that P is such that $P_h(\xi) = -|\xi|^2 + \mathbb{O}(h^\alpha \langle \xi \rangle^\beta)$ for some $\alpha, \beta \geqslant 0$. Denote by u^h the solution to (A-2) with P_h and f_h, such that $u^h_{|t=0} = u_{|t=0}$. Suppose that the solution to (1-1) satisfies $u \in L^\infty([0,T]; H^{s+\beta})$, for some $s > d/2$.*

- *In the case (A-1), $\|u - u^h\|_{L^\infty([0,T];H^s)} \lesssim h^\alpha$.*
- *In the case (A-3), $\|u - u^h\|_{L^\infty([0,T];H^s)} \lesssim h^{\min(\alpha,1)}$.*

In the case (A-1), proving an analogue to Theorem 4.3 seems to be more delicate though, and we choose not to investigate this aspect here. On the other hand, in the case (A-3), using the estimate (A-4), Strichartz estimates and Hölder inequalities with the “standard” Lebesgue exponents (in the same fashion as in the proof of Theorem 4.3, see [Cazenave 2003]), we have, with steps similar to those presented in the proof of Theorem 4.3:

Theorem A.3. *Let $\sigma \in \mathbb{N}$ and $T > 0$. Let u solve (1-1), and consider a solution u^h to*

$$i\partial_t u^h + \Delta u^h = \epsilon \left(\frac{|u^h|^2}{1+h|u^h|^2}\right)^\sigma u^h, \quad u^h_{|t=0} = u_0.$$

1. *If* $\sigma \leqslant 2/d$, *and* $u \in L^\infty([0,T];L^2)\cap L^{(4\sigma+4)/d\sigma}([0,T];L^{2\sigma+2})$, *then*

$$\|u-u^h\|_{L^\infty([0,T];L^2)} \underset{h\to 0}{\longrightarrow} 0.$$

2. *Suppose that* $\sigma = 1$ *and* $d = 3$.
   - *If* $u, \nabla u \in L^\infty([0,T];L^2)\cap L^{8/d}([0,T];L^4)$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

   - *If* $u \in L^\infty([0,T];H^s)$, *with* $s > 3/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h.$$

3. *Suppose that* $\sigma \geqslant 1$ *and* $d \leqslant 2$. *If* $u \in L^\infty([0,T];H^s)$, *with* $s \geqslant 1$ *and* $s > d/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h.$$

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