Porous Media: The Muskat Problem in Three Dimensions

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The Muskat problem involves filtration of two incompressible fluids through a porous medium. We consider the problem in three dimensions, discussing the relevance of the Rayleigh–Taylor condition and the topology of the initial interface, in order to prove the local existence of solutions in Sobolev spaces.

1. Introduction

The Muskat problem [Muskat and Wickoff 1937; Bear 1972] involves filtration of two incompressible fluids through a porous medium, characterized by a positive constant $\kappa$ quantifying its porosity and permeability. The two fluids, having velocity fields $v^1$ and $v^2$, occupy disjoint regions $D^1$ and $D^2 = \mathbb{R}^3 - D^1$, with a common boundary (interface) given by the surface $S = \partial D^1 = \partial D^2$. Naturally, those domains change with time, as does the interface. We denote by $p^j$ ($j = 1, 2$) the corresponding pressures, and we will also assume that the dynamical viscosities $\mu^j$ and the densities $\rho^j$ are constants with $\mu^1 \neq \mu^2$, $\rho^1 \neq \rho^2$.

Conservation of mass in this setting is given by the equation $\nabla \cdot v = 0$ (in the distribution sense), where $v = v^1 \chi_{D^1} + v^2 \chi_{D^2}$.

The momentum equation, obtained experimentally by Darcy [1856] (see also [Bear 1972]), is

$$\frac{\mu^j}{\kappa} v^j = -\nabla p^j - (0, 0, \rho^j g), \quad j = 1, 2,$$

where $g$ is the acceleration due to gravity.

One can find in the literature several attempts to derive Darcy’s law from Navier–Stokes [Tartar 1980; Sánchez-Palencia and Zaoui 1987] through the process of homogenization under the hypothesis of a periodic, or almost periodic, porosity. In any case, the presence of the porous medium justifies the elimination of the inertial terms in the motion, leaving friction (viscosity) and gravity as the only relevant forces, to which one has to add pressure as it appears in the formulation of Darcy’s law. There are three scales involved in the analysis: the macroscopic or bulk mass, the microscopic size of the fluid particle, and the mesoscopic scale corresponding to the pores. In the references above, one finds descriptions of the velocity $v$ as an average over the mesoscopic cells of the fluid particle velocities. Taking into account that each cell contains a solid part where the particle velocity vanishes, it is then natural to get the viscous

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forces associated to that average velocity, which is a scaled approximation of the laplacian term appearing in the Navier–Stokes equation.

In this paper, we shall consider the case of a homogeneous and isotropic porous material. Porosity is the fraction of the volume occupied by pores or void space. But it is important to distinguish between two kinds of pores — the kind that forms a continuous interconnected phase within the medium, and the kind that is isolated — because non-interconnected pores cannot contribute to fluid transport. Permeability is the term used to describe the conductivity of a porous medium with respect to a newtonian fluid, and it depends upon the properties of the medium and the fluid. Darcy’s law indicates this dependence, allowing us to define the notion of specific permeability $\kappa$ and its units. In the case of an anisotropic material, $\kappa$ will be a symmetric and positive definite tensor, and the methods of our proof can be modified to get local existence; but for a nonhomogeneous medium, the properties of the tensor $\kappa(x)$ will have to be specified in a very precise manner in order to allow an interesting theory.

The Muskat problem and related problems [Saffman and Taylor 1958] have been studied recently [Constantin and Pugh 1993; Siegel et al. 2004; Córdoba and Gancedo 2007; 2009; Córdoba et al. 2011]. The first natural question is about the evolution of the system (existence of solutions), at least for a short time $t > 0$, and the persistence of smoothness of the interface $S(t)$ if we begin with a smooth enough surface at time $t = 0$. One can easily deduce from this formulation that in the event of smooth evolution, both pressures can be taken to be equal at the interface:

$$p^1|_{S(t)} = p^2|_{S(t)}.$$

Therefore, we look at the case without surface tension (see [Escher and Simonett 1997], where the regularizing effect of surface tension is considered). The normal component of the velocity fields must also agree at the free boundary; that is, if $v^j$ is the unit normal to $S$ pointing into $D^j$, we have

$$(v^1 - v^2) \cdot v^j = 0 \quad \text{at} \quad S(t), \quad j = 1, 2$$

(note that $v^2 = -v^1$). Furthermore, the vorticity will be concentrated at the interface, having the form

$$\text{curl}(v) = \omega(z) \, dS(z),$$

where $\omega$ is tangent to $S$ at the point $z$ and $dS(z)$ is surface measure.

This paper extends to the three-dimensional case the results obtained in [Córdoba et al. 2011] for the case of two dimensions, by proving local existence in the scale of Sobolev spaces of the initial value problem if the Rayleigh–Taylor (R-T) condition is initially satisfied (see [Saffman and Taylor 1958], where this issue is studied from a physical point of view). In our case, that condition amounts to the positivity of the function

$$\sigma = (\nabla p^2 - \nabla p^1) \cdot (v^2 - v^1)$$

at the interface $S$. The R-T property also appears in other fluid interface problems, such as water waves [Cordoba et al. 2009].

Together with that hypothesis, one also assumes that the initial surface $S$ is connected and simply connected. In the presence of a global parametrization $X : \mathbb{R}^2 \to S$, the preservation of that character will
be controlled by the gauge
\[
F(X)(\alpha, \beta) = \frac{|\alpha - \beta|}{|X(\alpha) - X(\beta)|}, \quad \|F(X)\|_{L^\infty} = \sup_{\alpha \neq \beta} \frac{|\alpha - \beta|}{|X(\alpha) - X(\beta)|} < \infty.
\]

Section 2 of this paper contains the derivation of the evolution equations for the interface \( S \). In Section 3, we prove the existence of global isothermal parametrization as a consequence of the Koebe–Poincaré uniformization theorem of Riemann surfaces in the geometric scenarios considered in our work, namely, double periodicity in the horizontal variables and asymptotic flatness. Let us add that given the nonlocal character of the operator involved, to obtain a global isothermal parametrization is an important step in the proof, whose main components are sketched in Section 4.

In closing our system (Section 2), we need to control the norm of the inverse operator \((I + \lambda \mathcal{D})^{-1}\), where \( \mathcal{D} \) is the double-layer potential and \( |\lambda| \leq 1 \). It is well-known from Fredholm’s theory that those operators are bounded on \( L^2(S) \). However, since the surface \( S = S(t) \) is moving, a precise control of its norm is needed in order to proceed with our proof. That is the purpose of Section 5, where the estimates for the double-layer potential are revisited.

In Sections 6 and 7, we develop the energy estimates needed to conclude local existence. Let us mention that at a crucial point (more precisely, just at that step where the positivity of \( \sigma(\alpha, t) \) (R-T) plays its role), we use the pointwise estimate \( \theta(x) \Lambda \theta(x) \geq \frac{1}{2} \Lambda \theta^2(x) \) of [Córdoba and Córdoba 2003], with \( \Lambda = \sqrt{-\Delta} \).

In the strategy of our proof, it is crucial to analyze the evolution of both quantities \( \sigma \) and \( F \) (Section 8) at the same time as the interface \( X \) and vorticity \( \omega \). There are several publications (see, for example, [Ambrose 2007]) where different authors have treated these problems assuming that the Rayleigh–Taylor condition is preserved during the evolution. Under such a hypothesis the proof can be considerably simplified, especially if one also assumes the appropriate bounds for the resolvent of the double-layer potential with respect to a moving domain, or the existence of global isothermal coordinates, etc. It is our purpose to carefully go over such items, which are responsible for the more delicate and intricate parts of this paper.

### 2. The contour equation

We consider the following evolution problem for the active scalars \( \rho = \rho(x, t) \) and \( \mu = \mu(x, t) \), with \( x \in \mathbb{R}^3 \) and \( t \geq 0 \):

\[
\begin{align*}
\rho_t + v \cdot \nabla \rho &= 0, \\
\mu_t + v \cdot \nabla \mu &= 0,
\end{align*}
\]

with a velocity \( v = (v_1, v_2, v_3) \) satisfying the momentum equation

\[
\mu v = -\nabla p - (0, 0, \rho)
\]

and the incompressibility condition \( \nabla \cdot v = 0 \), where, without loss of generality, we have prescribed the values \( \kappa = g = 1 \).
The vector \((\mu, \rho)\) is defined by
\[
(\mu, \rho)(x_1, x_2, x_3, t) = \begin{cases}
(\mu^1, \rho^1) & x \in D^1(t), \\
(\mu^2, \rho^2) & x \in D^2(t) = \mathbb{R}^3 \setminus D^1(t),
\end{cases}
\]
where \(\mu^1 \neq \mu^2\) and \(\rho^1 \neq \rho^2\). Darcy’s law (2.1) implies that the fluid is irrotational in the interior of each domain \(D^j\), and because of the jump of densities and viscosities on the free boundary, we may assume a velocity field such that
\[
\text{curl } v = \omega(\alpha, t) \delta(x - X(\alpha, t)),
\]
where \(\delta D^j(t) = \{X(\alpha, t) \in \mathbb{R}^3 : \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2\}\); that is,
\[
\langle \text{curl } v, \varphi \rangle = \int_{\mathbb{R}^2} \omega(\alpha, t) \cdot \varphi(X(\alpha, t)) \, d\alpha,
\]
for any \(\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) vector field in \(C_c^\infty(\mathbb{R}^3)\).

The incompressibility hypothesis (\(\langle \nabla \cdot v, \varphi \rangle \equiv -\langle v, \nabla \varphi \rangle = 0\), for any \(\varphi \in C_c^\infty(\mathbb{R}^3)\)), yields
\[
v^1(X(\alpha, t), t) \cdot N(\alpha, t) = v^2(X(\alpha, t), t) \cdot N(\alpha, t),
\]
with \(N(\alpha, t) = \partial_{\alpha_1} X(\alpha, t) \wedge \partial_{\alpha_2} X(\alpha, t)\), and Equation (2.2) gives us the identity
\[
\omega(\alpha, t) = (v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)) \wedge N(\alpha, t).
\]
Defining the potential \(\phi\) by \(v(x, t) = \nabla \phi(x, t)\) for \(x \in \mathbb{R}^2 \setminus \partial D^j(t)\), we get
\[
\Omega(\alpha, t) = \phi^2(X(\alpha, t), t) - \phi^1(X(\alpha, t), t),
\]
\[
\partial_{\alpha_1} \Omega(\alpha, t) = (v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)) \cdot \partial_{\alpha_1} X,
\]
\[
\partial_{\alpha_2} \Omega(\alpha, t) = (v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)) \cdot \partial_{\alpha_2} X.
\]
Then one has the equality
\[
\omega(\alpha, t) = (v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)) \wedge (\partial_{\alpha_1} X(\alpha, t) \wedge \partial_{\alpha_2} X(\alpha, t)),
\]
and therefore
\[
\omega(\alpha, t) = \partial_{\alpha_2} \Omega(\alpha, t) \partial_{\alpha_1} X(\alpha, t) - \partial_{\alpha_1} \Omega(\alpha, t) \partial_{\alpha_2} X(\alpha, t),
\]
(2.3)
implying that \(\nabla \cdot \text{curl } v = 0\) in a weak sense.

Using the law of Biot–Savart, we have for \(x\) not lying in the free surface \((x \neq X(\alpha, t))\) the following expression for the velocity:
\[
v(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x - X(\beta, t)}{|x - X(\beta, t)|^3} \wedge \omega(\beta) \, d\beta.
\]
It follows that
\[
X_t(\alpha) = \text{BR}(X, \omega)(\alpha, t) + C_1(\alpha) \partial_{\alpha_1} X(\alpha) + C_2(\alpha) \partial_{\alpha_2} X(\alpha),
\]
(2.4)
where BR is the well-known Birkhoff–Rott integral:
\[
\text{BR}(X, \omega)(\alpha, t) = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge \omega(\beta) \, d\beta.
\]
(2.5)
Next we will close the system using Darcy’s law. Since
\[
\nabla \phi = v(x, t) - \Omega(\alpha, t) N(\alpha, t) \delta(x - X(\alpha, t)),
\]
we have
\[
\langle \Delta \phi, \varphi \rangle = -\langle \nabla \phi, \nabla \varphi \rangle = \int_{\mathbb{R}^2} \Omega(\alpha, t) N(\alpha, t) \cdot \nabla \varphi(X(\alpha, t)) \, d\alpha,
\]
and taking \(\varphi(y) = -1/(4\pi|x - y|)\), one obtains \(\phi\) in terms of the double layer potential:
\[
\phi(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x - X(\alpha)}{|x - X(\alpha)|^3} \cdot N(\alpha) \Omega(\alpha) \, d\alpha.
\]
Darcy’s law yields
\[
\Delta p(x, t) = -\text{div}(\mu(x, t)v(x, t)) - \partial_{x_3} \rho(x, t),
\]
that is,
\[
\Delta p(x, t) = P(\alpha, t) \delta(x - X(\alpha, t)),
\]
where \(P(\alpha, t)\) is given by
\[
P(\alpha, t) = (\mu^2 - \mu^1)v(X(\alpha, t), t) \cdot N(\alpha, t) + (\rho^2 - \rho^1)N_3(\alpha, t),
\]
implying the continuity of the pressure at the free boundary.

Next, if \(x \neq X(\alpha, t)\), i.e., \(x\) is not placed at the interface, we can write Darcy’s law in the form
\[
\mu \phi(x, t) = -p(x, t) - \rho x_3,
\]
and taking limits in both domains \(D^j\), we get at \(S\) the equality
\[
(\mu^2 \phi^2(X(\alpha, t), t) - \mu^1 \phi^1(X(\alpha, t), t)) = -(\rho^2 - \rho^1)X_3(\alpha, t).
\]
Then the formula for the double-layer potential gives
\[
\frac{\mu^2 + \mu^1}{2} \Omega(\alpha, t) - (\mu^2 - \mu^1) \frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \cdot N(\beta) \Omega(\beta) \, d\beta = -(\rho^2 - \rho^1)X_3(\alpha, t),
\]
that is,
\[
\Omega(\alpha, t) - A_\mu \mathcal{D}(\Omega)(\alpha, t) = -2A_\rho X_3(\alpha, t), \tag{2-6}
\]
where
\[
\mathcal{D}(\Omega)(\alpha) = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \cdot N(\beta) \Omega(\beta) \, d\beta, \quad A_\mu = \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1}, \quad A_\rho = \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1}. \tag{2-7}
\]
The evolution equations are then given by (2-3)–(2-7), where the functions \(C_1\) and \(C_2\) will be chosen in the next section.

Furthermore, taking limits, we get from Darcy’s law the following two formulas:
\[
\partial_{\alpha_1} \Omega(\alpha, t) + 2A_\mu \text{BR}(X, \omega)(\alpha, t) \cdot \partial_{\alpha_1} X(\alpha, t) = -2A_\rho \partial_{\alpha_1} X_3(\alpha, t), \tag{2-8}
\]
\[
\partial_{\alpha_2} \Omega(\alpha, t) + 2A_\mu \text{BR}(X, \omega)(\alpha, t) \cdot \partial_{\alpha_2} X(\alpha, t) = -2A_\rho \partial_{\alpha_2} X_3(\alpha, t). \tag{2-9}
\]
3. Isothermal parametrization: choosing the tangential terms

Although the normal component of the velocity vector field is the relevant one in the evolution of the interface, it is however very important to choose an adequate parametrization in order to uncover and handle properly the cancellations contained in the equations of motion. Fortunately for our task, we can rely upon the ideas of H. Lewy [1951], and many other authors, who discovered the convenience of using isothermal coordinates in different PDEs for understanding how a minimal surface leaves an obstacle and also in several fluid mechanical problems.

Let us recall that an isothermal parametrization must satisfy

\[ |X_{\alpha_1}(\alpha, t)|^2 = |X_{\alpha_2}(\alpha, t)|^2, \quad X_{\alpha_1}(\alpha, t) \cdot X_{\alpha_2}(\alpha, t) = 0, \]

for \( t \geq 0 \).

Next we define

\[
C_1(\alpha) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1 \cdot \text{BR}_{\beta_2} \cdot \text{BR}_{\beta_1} \cdot |X_{\beta_1}|^2}{|\alpha - \beta|^2} \, d\beta - \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_2 - \beta_2 \cdot \text{BR}_{\beta_1} \cdot |X_{\beta_2}|^2}{|\alpha - \beta|^2} \, d\beta \quad (3-1)
\]

and

\[
C_2(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_2 - \beta_2 \cdot \text{BR}_{\beta_2} \cdot |X_{\beta_1}|^2}{|\alpha - \beta|^2} \, d\beta - \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1 \cdot \text{BR}_{\beta_2} \cdot |X_{\beta_1}|^2}{|\alpha - \beta|^2} \, d\beta. \quad (3-2)
\]

That is, \( X_t = \text{BR} + C_1 X_{\alpha_1} + C_2 X_{\alpha_2} \) and

\[
X_{\alpha_1 t} = \text{BR}_{\alpha_1} + C_1 X_{\alpha_1 \alpha_1} + C_2 X_{\alpha_1 \alpha_2} + C_{1\alpha_1} X_{\alpha_1} + C_{2\alpha_1} X_{\alpha_2},
\]

\[
X_{\alpha_2 t} = \text{BR}_{\alpha_2} + C_1 X_{\alpha_2 \alpha_1} + C_2 X_{\alpha_2 \alpha_2} + C_{1\alpha_2} X_{\alpha_1} + C_{2\alpha_2} X_{\alpha_2}.
\]

Writing \( f = (|X_{\alpha_1}|^2 - |X_{\alpha_2}|^2)/2 \) and \( g = X_{\alpha_1} \cdot X_{\alpha_2} \), we have

\[
f_t = \left( \text{BR}_{\alpha_1} \cdot X_{\alpha_1} - \text{BR}_{\alpha_2} \cdot X_{\alpha_2} \right) + C_1 f_{\alpha_1} + C_2 f_{\alpha_2} + (C_{2\alpha_1} - C_{1\alpha_2}) g + 2C_{1\alpha_1} f + (C_{1\alpha_1} - C_{2\alpha_2}) |X_{\alpha_2}|^2.
\]

The expressions for \( C_1 \) and \( C_2 \) yield the vanishing of the sum of the first and the last terms in the identity above. Therefore, we get

\[
f_t = C_1 f_{\alpha_1} + C_2 f_{\alpha_2} + (C_{2\alpha_1} - C_{1\alpha_2}) g + 2C_{1\alpha_1} f. \quad (3-3)
\]

Similarly, we have

\[
g_t = \left( \text{BR}_{\alpha_2} \cdot X_{\alpha_1} + \text{BR}_{\alpha_1} \cdot X_{\alpha_2} \right) + C_1 g_{\alpha_1} + C_2 g_{\alpha_2} + (C_{1\alpha_1} + C_{2\alpha_2}) g - 2C_{2\alpha_1} f + (C_{1\alpha_2} + C_{2\alpha_1}) |X_{\alpha_1}|^2
\]

and

\[
g_t = C_1 g_{\alpha_1} + C_2 g_{\alpha_2} + (C_{1\alpha_1} + C_{2\alpha_2}) g - 2C_{2\alpha_1} f. \quad (3-4)
\]

The linear character of equations (3-3) and (3-4) allows us to conclude that if there is a solution of the system \( X_t = \text{BR} + C_1 X_{\alpha_1} + C_2 X_{\alpha_2} \) and we start with isothermal coordinates at time \( t = 0 \), then they will continue to be isothermal so long as the evolution equations provide us with a smooth enough interface.
The fact that one can always prescribe such coordinates at time \( t = 0 \) follows from the following argument: in the double periodic setting we have a \( C^2 \) simply connected surface, homeomorphic to the euclidean plane \( \mathbb{R}^2 \), which, by the Riemann–Koebe–Poincaré uniformization theorem, is conformally equivalent to either the Riemann sphere, the plane, or the unit disc. The sphere is easily eliminated by compactness, but we can also rule out the unit disc because the assumption of double periodicity in the horizontal variables implies the existence of a discrete abelian subgroup of rank two in the group of conformal transformations, and that cannot happen in the case of the unit disc.

Therefore, we have an orientation-preserving conformal (isothermal) equivalence

\[
\phi : \mathbb{R}^2 \longrightarrow S.
\]

Since \( S \) is invariant under translations \( \tau_\nu(x) = x + 2\pi \nu \), where \( \nu \in \mathbb{Z}^2 \times \{0\} \), it follows that \( f_\nu(z) = \phi^{-1} \circ \tau_\nu \circ \phi(z) \) must be a diffeoholomorphism of \( \mathbb{C} = \mathbb{R}^2 \), and therefore it has to be of the form

\[
f_\nu(z) = a_\nu z + b_\nu,
\]

for certain \( a_\nu, b_\nu \in \mathbb{C} \). Clearly, the family \( f_\nu \) is generated by \( f_1 = f(1,0,0), f_2 = f(0,1,0) \). Let

\[
f_1(z) = a_1z + b_1, \quad f_2(z) = a_2z + b_2.
\]

We claim that \( a_1 = a_2 = 1 \). Suppose that \( |a_1| < 1 \); then we get \( f_1^n(z) = a_1^n z + b_1 (1 + a_1 + \cdots + a_1^{n-1}) \), a sequence converging to \( b_1/(1 - a_1) \), contradicting the discrete character of the group action. On the other hand, if \( |a_1| > 1 \), then since

\[
f_1^{-1}(z) = f_{(-1,0,0)}(z) = \frac{z}{a_1} - \frac{b_1}{a_1},
\]

we get a contradiction with the sequence \( f_1^{-n}(z) \). Therefore, we must have \( a_1 = e^{2\pi i \theta} \) for some \( 0 \leq \theta < 1 \).

Assume that \( 0 < \theta < 1 \); then

\[
f_1^{(n)}(z) = e^{2\pi i n \theta} z + b_1 \left( 1 + e^{2\pi i \theta} + \cdots + e^{2\pi i (n-1) \theta} \right) = e^{2\pi i \theta} z + b_1 \frac{1 - e^{2\pi i \theta}}{1 - e^{2\pi i \theta}},
\]

so the sequence \( f^n(z) \) is bounded and satisfies \( |f^n(z)| \leq |z| + |b_1|/\sin \pi \theta \). Therefore it contains a converging subsequence, again contradicting discreteness. It follows that \( f_1(z) = z + b_1 \) and, similarly, \( f_2(z) = z + b_2 \), which leads easily to the double periodicity of the isothermal parametrization \( \phi \).

In the asymptotically flat case, we start with an orientable simply connected surface \( S \) that, outside a ball \( B \) in \( \mathbb{R}^3 \), is the graph of a \( C^2 \)-function \( x_3 = \varphi(x_1, x_2) \) such that \( |D^2 \varphi(x)| = o(|x|^{-N}) \) for every \( N \) and \( |\alpha| \leq 2 \). In particular, the normal vector \( \nu(x) = (-\nabla \varphi, 1)/\sqrt{1 + |\nabla \varphi|^2} \) is roughly vertical and \( 1/\sqrt{1 + |\nabla \varphi|^2} \) is close to 1 for \( |x| \) big enough.

Then one can find isothermal coordinates whose first fundamental form \( \lambda(\alpha, \beta)(d\alpha^2 + d\beta^2) \) converges asymptotically to the identity.

Again by the uniformization theorem, \( S \) must be conformally equivalent to either \( \mathbb{C} \) or the unit disc. But since outside \( B \), the surface \( S \) is conformally equivalent to \( \mathbb{C} - B \cap \{x_3 = 0\} \), it cannot be also conformally equivalent to \( D - K \), for any regular compact set \( K \) contained in the unit disc \( D \), because the harmonic measure of the ideal boundary is 1 in the case of \( D \) and 0 for \( \mathbb{R}^2 \).
4. Main theorem and outline of the proof

The proof of local existence requires the following:

(1) A connected and simply connected surface \( S = S(t) \) parametrized by isothermal coordinates
\[
X : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad X = X(\alpha, t),
\]
with normal vector \( N(\alpha, t) = X_{\alpha_1} \wedge X_{\alpha_2} \) and gauge
\[
F(X)(\alpha, \beta) = \frac{|\beta|}{|X(\alpha) - X(\alpha - \beta)|},
\]
such that \( \| F(X) \|_{L^\infty} < \infty \) and \( \| |N|^{-1} \|_{L^\infty} < \infty \).

(2) The positivity of
\[
\sigma(\alpha, t) = -\left( \nabla p^2(X(\alpha, t), t) - \nabla p^1(X(\alpha, t), t) \right) \cdot N(\alpha, t)
\]
\[
= (\mu^2 - \mu^1) \text{BR}(X, \omega)(\alpha, t) \cdot N(\alpha, t) + (\rho^2 - \rho^1)N_3(\alpha, t),
\]
where the last equality is a consequence of Darcy’s law after taking limits in both domains \( D^j \). This is the Rayleigh–Taylor condition to be imposed at time \( t = 0 \), it being a part of the problem to prove that it remains true as time passes.

(3) The estimates on the norm of \((I - \lambda \mathcal{D})^{-1}, |\lambda| < 1, \mathcal{D} = \) double-layer potential (see Section 5), allowing us to obtain the inequalities
\[
\| \Omega \|_{H^{k+1}} \leq P \left( \| X \|_{k+1}^2 + \| F(X) \|_{L^\infty}^2 + \| |N|^{-1} \|_{L^\infty} \right),
\]
\[
\| \omega \|_{H^k} \leq P \left( \| X \|_{k+1}^2 + \| F(X) \|_{L^\infty}^2 + \| |N|^{-1} \|_{L^\infty} \right),
\]
for \( k \geq 3 \), where \( P \) is a polynomial function and the norm \( \| \cdot \|_k \) is given by
\[
\| X \|_k = \| X_1 - \alpha_1 \|_{L^3} + \| X_2 - \alpha_2 \|_{L^3} + \| X_3 \|_{L^2} + \| \nabla(X - (\alpha, 0)) \|_{H^{k-1}}^2,
\]
as in (7-1) below, and \( \| \cdot \|_{H^j} \) denotes the norm in the Sobolev space \( H^j \).

(4) A control of the Birkhoff–Rott integral \( \text{BR}(X, \omega) \):
\[
\| \text{BR}(X, \omega) \|_{H^k} \leq P \left( \| X \|_{k+1}^2 + \| F(X) \|_{L^\infty}^2 + \| |N|^{-1} \|_{L^\infty} \right),
\]
for \( k \geq 3 \).

(5) Energy estimates: the properties of isothermal parametrizations help us to reorganize the terms in such a way that
\[
\frac{d}{dt} \| X \|_k^2(t) \leq P \left( \| X \|_k^2(t) + \| F(X) \|_{L^\infty}(t) + \| |N|^{-1} \|_{L^\infty}(t) \right) \]
\[ - \sum_{i=1,2} \frac{2^{3/2}}{\mu_1 + \mu_2} \int_{\mathbb{R}^2} \frac{\sigma(\alpha, t)}{|\nabla X(\alpha, t)|^3} \partial_{a_i} X(\alpha, t) \cdot \Lambda(\partial_{a_i} X)(\alpha, t) \, d\alpha, \]
where \( k \geq 4 \), \(|\nabla X(\alpha)|^3 = (|\partial_{\alpha_1} X(\alpha)|^2 + |\partial_{\alpha_2} X(\alpha)|^2)^{3/2} \), and \( \Lambda = (-\Delta)^{1/2} = R_1(\partial_{\alpha_1}) + R_2(\partial_{\alpha_2}) \). Then the pointwise inequality

\[
\theta \Lambda(\theta) - \frac{1}{2} \Lambda(\theta^2) \geq 0,
\]

together with the condition \( \sigma > 0 \), allows us to get rid of the dangerous terms in the inequality above (those involving \((k + 1)\)-derivatives of \( X \)) to obtain the estimate

\[
\frac{d}{dt} \|X\|_k^2(t) \leq P\left(\|X\|_k^2(t) + \|F(X)\|_{L^\infty}(t) + \|N\|_{L^\infty}^{-1}(t)\right).
\]

(6) Finally, we need to control the evolution of \( \|F(X)\|_{L^\infty}(t) \) and \( \inf(t) = \inf_{\alpha \in \mathbb{R}^2} \sigma(\alpha, t) \), which is obtained via the estimates

\[
\frac{d}{dt} \|F(X)\|_{L^\infty}(t) \leq P\left(\|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \|N\|_{L^\infty}^{-1}(t)\right),
\]

\[
\frac{d}{dt} \inf(t) \leq \frac{1}{\inf(t)^2} P\left(\|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \|N\|_{L^\infty}^{-1}(t)\right).
\]

(7) All those facts together yield the inequality

\[
\frac{d}{dt} E(t) \leq CP(E(t))
\]

for the energy

\[
E(t) = \|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \|N\|_{L^\infty}^{-1}(t) + \inf(t)^{-1},
\]

where \( k \geq 4 \), \( C \) is a universal constant, and \( P \) has polynomial growth (depending upon \( k \)).

At this point it is not difficult to prove the existence of a solution, locally in time, so long as the initial data \( X(0) \) is in the appropriate Sobolev space of order \( k \geq 4 \), and the Rayleigh–Taylor and no-self-intersection conditions \((\sigma_0 > c > 0, \|F(X(0))\|_{L^\infty} < \infty) \) are satisfied.

The main theorem presented in this paper is the following:

**Theorem 4.1.** Let \( X(0) \) with \( \|X(0)\|_k < \infty \) for \( k \geq 4 \), \( \|F(X(0))\|_{L^\infty} < \infty \), \( \|N(\alpha, 0)\|_{L^\infty}^{-1} < \infty \), and

\[
\sigma(\alpha, 0) = -\left(\nabla p^2(X(0), 0) - \nabla p^1(X(0), 0)\right) \cdot N(\alpha, 0) \geq 0.
\]

Then there exists a time \( \tau > 0 \) such that there is a solution to (2-3), (2-4), (2-6) in \( C([0, \tau]; H^k) \) with \( X(\alpha, 0) = X(0) \).

Finally, let us point out that since our existence proof is based upon energy inequalities, an extra argument is needed to prove uniqueness. Nevertheless, that task is much easier than proving existence. (The interested reader may consult [Córdoba et al. \( \geq 2013 \), where the details of the proof have been written out for some important cases, such as Muskat and SQG patches.)

Let us remark that, at the end, we have to work with a coupled system involving the evolution of the surface \( X \), the “vorticity density” \( \omega \), the Rayleigh–Taylor condition \( \sigma \), the non-self-intersecting character of \( S \) quantified by the gauge \( F(X) \), and the tangential parts \( C_1 X_{\alpha_1} + C_2 X_{\alpha_2} \) of the velocity field.
Remark. This paper is a continuation of [Córdoba et al. 2011], where the two-dimensional case was considered. Many of the needed estimates can be obtained following exactly the same methods that were used in [Córdoba et al. 2011] for the lower-dimensional case. Therefore, in order to simplify our presentation, we shall avoid here many details which were carefully proven there. This is especially the case in Section 6 (control of the Birkhoff–Rott integral) and Section 8 (energy estimates), and also for the approximation schemes which are identical to those developed in [Córdoba et al. 2011]. Therefore, in the following, we shall focus our attention on the more innovative parts of the proof, namely the evolution of the Rayleigh–Taylor condition, the non-self-intersecting property of the free boundary, and the needed estimates for double-layer potentials.

5. Inverting the operator: the single- and double-layer potentials revisited

In this proof, we need to consider the properties of single- and double-layer potentials, which are well-known characters in finding solutions to the Dirichlet and Neumann problems in domains $D$ of $\mathbb{R}^n$.

For our purposes, these domains will be of three different types, namely: bounded, periodic in the “horizontal” variables, and asymptotically flat. We shall also assume that their boundaries are smooth enough (say $C^2$) and do not present self-intersections. Therefore, one has tangent balls at every point of the boundary, one completely contained in $D$ and the other in $D^c$. We shall denote by $\nu(x)$ the unit inner normal at the point $x \in \partial D$; then under our hypothesis we have that, for $r > 0$ small enough, the parallel surfaces $\partial D_r = \{ x + r \nu(x) \mid x \in \partial D \}$ are also $C^2$ surfaces with curvatures controlled by those of $\partial D$. Furthermore, the vector field $\nu$ can be extended smoothly up to a collar neighborhood of $\partial D$, allowing us to write the formula

$$\Delta u(x) = \frac{\partial^2 u}{\partial \nu^2}(x) - h(x) \frac{\partial u}{\partial \nu}(x) + \Delta_x u(x),$$

where $\Delta$ denotes the ordinary laplacian in $\mathbb{R}^n$, $\Delta_x$ is the Laplace–Beltrami operator in $\partial D$, $h(x)$ is the mean curvature of $\partial D$ at the point $x$, and $u$ is any $C^2$-function defined in a neighborhood of $\partial D$.

For convenience, we will use the notation $D_1 = D$, $D_2 = D^c$, $S = \partial D_j$, and $\nu_j(x)$ (for $j = 1, 2$) the inner normal at $x \in S$ pointing inside $D_j$. Let $dS$ be the surface measure in $S$ induced by Lebesgue measure in ambient space. Given integrable functions $\varphi, \psi$ on $S$, we call

$$V(x) = c_n \int_S \psi(y) \frac{1}{\|x - y\|^{n-2}} dS(y)$$

the single-layer potential of $\psi$, and we call

$$W(x) = c_n \int_S \varphi(y) \frac{\partial}{\partial \nu_y} \left( \frac{1}{\|x - y\|^{n-2}} \right) dS(y)$$

the double-layer potential of $\varphi$. In both cases, $c_n$ is a normalizing constant chosen so that $\frac{c_n}{\|x\|^{n-2}}$ is a fundamental solution of $\Delta$ in $\mathbb{R}^n$, $n \geq 3$. 

For $x \in S$ and $j = 1, 2$, denote by $W_j(x)$ and $V_j(x)$ the corresponding limits of the potentials in $D_j$. We have

\[
W_1(x) = \frac{1}{2} \left( \varphi(x) - \int_S \varphi(y) K(x, y) \, d\sigma(y) \right) = \frac{1}{2} (\varphi(x) - D \varphi(x)),
\]

\[
W_2(x) = \frac{1}{2} \left( \varphi(x) + \int_S \varphi(y) K(x, y) \, d\sigma(y) \right) = \frac{1}{2} (\varphi(x) + D^* \varphi(x)),
\]

\[
\frac{\partial V}{\partial v_1}(x) = -\frac{1}{2} \left( \psi(x) + \int_S \psi(y) K(y, x) \, d\sigma(y) \right) = -\frac{1}{2} (\psi(x) + D^* \psi(x)),
\]

\[
\frac{\partial V}{\partial v_2}(x) = -\frac{1}{2} \left( \psi(x) - \int_S \psi(y) K(y, x) \, d\sigma(y) \right) = -\frac{1}{2} (\psi(x) - D^* \psi(x)),
\]

where

\[
K(x, y) = 2c_n \frac{\partial}{\partial y} \left( \frac{1}{\|x - y\|^{n-2}} \right) = \tilde{c}_n \frac{\langle x - y, \nu(y) \rangle}{|x - y|^n}.
\]

It is well-known that in the scenarios considered above, the boundary operators $D$ (and $D^*$) are smoothing of order $-1$, and therefore compact. Furthermore, all their eigenvalues are real numbers having absolute value strictly less than 1. Therefore, by the standard Fredholm theory, the operators $I - \lambda D$, $I - \lambda D^*$ are invertible when $|\lambda| \leq 1$. However, in our case, the domains are moving, and the evolution of their common boundary $S$ involves the inverse operators, making it necessary to estimate their norms in terms of the geometry and smoothness of $S$.

Although there is a vast literature about single- and double-layer potentials, we have not been able to point out a precise statement giving the information needed for our results. Therefore, in this section, we provide arguments to prove that the norms of such inverse operators grow at most polynomially: $P(\|S\|)$, where $\|S\|$ is just $\|S\|_{C^2}$ plus a term of chord-arc type controlling the non-self-intersecting character of the boundary. The term has the form $r(S)^{-1}$, where $r(S)$ is the sup over all the positive $r$ such that $S$ admits tangent balls of radius $r$ in both domains $D_j$:

\[
\|S\| = \|S\|_{C^2} + (r(S))^{-1}.
\]

We shall write $P(\|S\|)$ to denote $\leq C(\|S\|^p)$ for certain positive constants $C$, $p$ which are independent of the characters whose evolution is being controlled, but the size of both constants may change during the proof and we shall make no effort to obtain their best values.

We will consider the case of bounded domains in $\mathbb{R}^n$, $n \geq 3$, because the needed modifications when $n = 2$, namely taking $\log |x|$ as fundamental solution for the laplacian, as well as the changes for the periodic or asymptotically flat domains, are left to the reader.

Let $D$ and $D^*$ be the potential defined above, with kernel

\[
K(x, y) = c_n \frac{\partial}{\partial y} \left( \frac{1}{\|x - y\|^{n-2}} \right) = c_n \frac{\langle x - y, \nu(y) \rangle}{|x - y|^n}
\]

and $K(y, x)$ respectively. In the study of the inverse operators $(I - \lambda D)^{-1}$, $|\lambda| \leq 1$, it is convenient to consider first the particular values $\lambda = \pm 1$. 
Proposition 5.1. The following estimate holds, where $P$ is a polynomial function:

$$\|(I \pm \partial D)^{-1}\|_{L^2(S)} = P(\|S\|).$$

Since the boundedness of $(I \pm \partial D)^{-1}$ in $L^2(S)$ is well-known from the general theory, we can simplify the proof, considering only functions $f \in L^2(S)$ whose support lies inside a region of $S$ where the normal $v(x)$ is close enough to a fixed direction. Then for a general $f$, an appropriate partition of unity would allow us to add the local estimates, so long as the number of pieces is controlled by $\|S\|$. We shall use the following observation, whose proof is immediate.

Lemma 5.2 (Rellich). Let $u$ be a harmonic function and $h$ a smooth vector field in the domain $D$; then we have

(i) $\text{div}(|\nabla u|^2 h) = 2 \text{div}((\nabla u \cdot h) \nabla u) + O(|\nabla u|^2 |\nabla h|),$

(ii) $\int_{\partial D}(\nu, h)|\nabla u|^2 d\sigma = 2 \int_{\partial D}(\partial u / \partial \nu)(\nabla u \cdot h)d\sigma + O(\int_{D} |\nabla u|^2 |\nabla h|).$

Given a function $f \in C^1(S)$, we may define $\nabla_{\tau} f$, choosing at each point $x \in S$ an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of the tangent space $T_x(S)$ (we can consider also $\nabla_{\tau} f$ to be the gradient naturally associated to the induced Riemannian metric by the ambient space). In both ways, although different, we have that $|\nabla_{\tau} f| \equiv \Lambda_{\tau} f$ is an elliptic pseudodifferential operator of order 1 in $S$. Solving the Dirichlet problem $\Delta u = 0$ in $D$, $u|_S = f$, we obtain the operator $D_{\nu} \equiv (\partial u / \partial \nu)|_S$, which is also a pseudodifferential operator of order 1 in $S$.

Lemma 5.3. Let $f \in L^2(S)$ having support on the region $\frac{1}{2} \leq \langle v(x), \eta \rangle \leq 1$ (for a fixed unit vector $\eta$); then we have

$$\int_S |D_{\nu} f|^2 d\sigma \simeq \int_S |\nabla_{\tau} f|^2 d\sigma,$$

where the constants involved in the stated equivalence $\simeq$ are $P(\|S\|)$.

Proof. Let $u$ be harmonic in $D$ so that $u|_S = f$. Under our hypothesis about $f$, and since $|\nabla u|^2 = |D_{\nu} u|^2 + |\nabla_{\tau} u|^2$ and $\nabla_{\tau} u$ is a local operator (supp$_S(\nabla_{\tau} f) \subset$ supp$(f)$), Lemma 5.2 yields:

$$\frac{1}{2} \int_S |\nabla_{\tau} f|^2 d\sigma \leq \int_S \langle v(x), \eta \rangle |\nabla_{\tau} u|^2 d\sigma \leq 3 \int_S |D_{\nu} u|^2 d\sigma + 2 \int_S |\nabla_{\tau} u||D_{\nu} u|d\sigma,$$

from which we easily obtain

$$\int_S |\nabla_{\tau} f|^2 d\sigma \leq P(\|S\|) \int_S |D_{\nu} f|^2 d\sigma.$$
In order to obtain the estimate

\[ \int_S |D_v f|^2 d\sigma \leq P(||S||) \int_S |\nabla_x f|^2 d\sigma, \]

we may assume, without loss of generality, that \( \text{supp}(f) \subset B_R(x) \), for some \( x \in S \), and then prove that

\[ \int_{B_R(y_0)} |D_v f|^2 d\sigma \leq P(||S||) \int_S |\nabla_x f|^2 d\sigma \]

uniformly on \( y_0 \in S \).

With the vector field \( h \) defined above in \( \Delta_{2R}(y) \), let us apply Rellich’s estimate to get

\[ \int_S |D_v f|^2 \langle h, \nu(x) \rangle d\sigma(x) = \int_S \langle v, h \rangle |\nabla_x f|^2 d\sigma - 2 \int_S D_v f \nabla_x f \cdot h d\sigma + O\left( \int_D |\nabla u|^2 |\nabla h| \right), \]

where \( u \) satisfies \( \Delta u = 0 \) in \( D, u|_S = f \). We get easily

\[ \int_{B_R(y_0)} |D_v f|^2 \langle h, \nu(x) \rangle d\sigma(x) = O\left( \int_S |\nabla_x f|^2 d\sigma + \int_D |\nabla u|^2 |\nabla h| dx \right). \]

Then the proof will be finished if we can show that

\[ \int_D |\nabla u|^2 |\nabla h| dx \leq P(||S||) \int_S |\nabla_x f|^2 d\sigma. \]

To see this, let us consider the parallel surfaces \( S_r = \{ x + r \nu(x) \mid x \in S \} \) \((0 \leq r \leq ||S||)\) and observe that

\[ \int_{S_r} u^2 d\sigma_r \simeq \int_S u^2(x + r \nu(x)) d\sigma \]

and

\[ \int_S \left[ u^2(x + r \nu(x)) - u^2(x) \right] d\sigma(x) = \int_S \int_0^r \nabla u^2(x + t \nu(x)) \cdot \nu(x) dt d\sigma \\
= 2 \int_{L_r} u(y) \nabla u(y) \cdot \nu(y) \leq 2 \left( \int_{L_r} u^2(y) \right)^{1/2} \left( \int_{L_r} |\nabla u|^2(y) \right)^{1/2}, \]

where \( L_r = \{ x + \rho \nu(x) \mid x \in S, 0 \leq \rho \leq r \} \).

Let \( \mathcal{X} \) be a smooth cut-off function. Taking

\[ F(x + r \nu(x)) = f(x)\mathcal{X}(x), \]

as a comparison function, Dirichlet’s principle and Poincaré’s inequality give us the estimate

\[ \int_D |\nabla u|^2 \leq \int_D |\nabla F|^2 \leq C \left( \int_S |\nabla_x f|^2 + \int_S |f|^2 \right) = O\left( \int_S |\nabla_x f|^2 d\sigma \right). \]

Therefore

\[ \int_{S_r} u^2 d\sigma_r \simeq \int_S u^2(x + r \nu(x)) d\sigma \leq \int_S f^2(x) d\sigma + \left( \int_{L_r} u^2(y) \right)^{1/2} \left( \int_S |\nabla_x f|^2 \right)^{1/2}. \]
Integration in $r$ in the range $0 \leq r \leq R = \|S\|^{-1}$ yields
\[
\int_{L_r} u^2 dx \leq R \left( \int_S f^2(x) \, d\sigma + \left( \int_{L_r} u^2(y) \right)^{1/2} \left( \int_S |\nabla \tau f|^2 \right)^{1/2} \right).
\]
That is,
\[
\int_{L_r} u^2 dx \leq CR \int_S |\nabla \tau f|^2 \, d\sigma.
\]
To conclude, let us observe that
\[
\int_D |\nabla u|^2 \, |\nabla h| = \frac{1}{2} \int_D \Delta u |\nabla h| = \frac{1}{2} \int_D (\Delta u |\nabla h| - u^2 \Delta (|\nabla h|)) + \frac{1}{2} \int_D u^2 (|\nabla h|)
\]
\[
= \frac{1}{2} \int_S u \frac{\partial u}{\partial v} \cdot |\nabla h| \, d\sigma - \frac{1}{2} \int_S f^2 (|\nabla h|) \, d\sigma + \frac{1}{2} \int_D u^2 \nabla |\nabla h|
\]
\[
\leq \left( \int_S f^2 d\sigma \right)^{1/2} \left( \int |\frac{\partial u}{\partial v}|^2 |\nabla h|^2 \, d\sigma \right)^{1/2} + C \int_S f^2 d\sigma + C \int_{L_r} u^2.
\]

**Proof of Proposition 5.1.** As before, let $f \in C^1(S)$, supp$(f) \subset U_0$, and let $u$ be its single-layer potential:
\[
u(x) = c_n \int_S \frac{f(y)}{\|x - y\|^{n-2}} \, dS(y).
\]
Taking derivatives on each domain $D_j$ with respect to the normal direction and evaluating at $S$, we get
\[
\frac{\partial u}{\partial v_1} = -\frac{1}{2} (f(x) + \mathcal{D}^* f(x)), \quad \frac{\partial u}{\partial v_2} = -\frac{1}{2} (f(x) - \mathcal{D}^* f(x)).
\]
By Lemma 5.3, we know that
\[
\int_S \left| \frac{\partial v}{\partial v_1} \right|^2 \, d\sigma \asymp \int_S |\nabla \tau v|^2 \, d\sigma \asymp \int_S \left| \frac{\partial v}{\partial v_2} \right|^2 \, d\sigma,
\]
where the constants involved in the equivalences are all controlled by above by $P(\|S\|)$ and below by $1/P(\|S\|)$.

Since $\partial v/\partial v_1 + \partial v/\partial v_2 = -f$, these estimates imply that
\[
\min\left( \|f - \mathcal{D}^* f\|_2, \|f + \mathcal{D}^* f\|_2 \right) \geq \frac{1}{P(\|S\|)},
\]
that is, $\|(I \pm \mathcal{D})^{-1}\| = P(\|S\|)$. Then using an appropriate partition of unity, that estimate extends to a general $f \in L^2(S)$. \hfill \Box

Next we shall consider Sobolev spaces $H^s(S)$, $0 \leq s \leq 1$, defined in the usual manner throughout local coordinate charts. We have also the elliptic pseudodifferential operator $\Lambda^s = (-\Delta)^{s/2}$ in such a way that
\[
\|f\|_{H^s(S)} \asymp \|f\|_{L^2} + \|\Lambda^s f\|_{L^2}.
\]
Then $H^{-s}(S) \equiv (H^s(S))^*$ allows us to consider the negative case by duality, under the pairing
\[
\int_S \phi \psi \, d\sigma, \quad \phi \in H^{-s}, \psi \in H^s,
\]
and we have

$$\|\phi\|_{H^{-1}} = \sup_{\|\psi\|_{H^1} = 1} \int_S \phi \psi \, d\sigma.$$  

Since both $\mathcal{D}$ and $\mathcal{D}^*$ are compact and smoothing operators of degree $-1$, the commutators $[\Lambda^*, \mathcal{D}], [\Lambda, \mathcal{D}^*]$ are then bounded in $L^2(S)$ $(0 \leq s \leq 1)$ with norms controlled by $\|\|S\||$, allowing us to extend Proposition 5.1 to the chain of Sobolev spaces:

**Corollary 5.4.** The norm of the operators $(I \pm \mathcal{D})^{-1}, (I \pm \mathcal{D}^*)^{-1}$ in the space $H^s(S), -1 \leq s \leq 1$, is bounded by $P(\|\|S\||)$.

**Estimates for $(I + \lambda \mathcal{D})^{-1}, |\lambda| \leq 1$.** With the same notation used before, we have

$$\frac{1 - \lambda}{2} \frac{\partial V}{\partial v_1} + \frac{1 + \lambda}{2} \frac{\partial V}{\partial v_2} = -\frac{1}{2} (\phi(x) - \lambda \mathcal{D}^* \phi(x)) \quad \text{and} \quad \frac{1 + \lambda}{2} \frac{\partial V}{\partial v_1} + \frac{1 - \lambda}{2} \frac{\partial V}{\partial v_2} = -\frac{1}{2} (\phi(x) + \lambda \mathcal{D}^* \phi(x)),$$

where

$$V(x) = c_n \int_S \frac{\phi(y)}{\|x - y\|^{n-2}} \, dS(y).$$

Then the identity $\phi - \lambda \mathcal{D}^* \phi = 0$ yields

$$0 = (1 - \lambda) \int_{\partial D_1} V \frac{\partial V}{\partial v_1} \, dS + (1 + \lambda) \int_{\partial D_2} V \frac{\partial V}{\partial v_2} \, dS = (1 - \lambda) \int_{D_1} |\nabla V|^2 + (1 + \lambda) \int_{D_2} |\nabla V|^2,$$

which implies $\phi \equiv 0$. Similarly for $\phi + \lambda \mathcal{D}^* \phi = 0$, $-1 \leq \lambda \leq 1$.

**Remark.** This observation can be improved applying the following fact (whose proof we skip because it will not be used in our theorem):

$$\int_{D_1} |\nabla u|^2 \sim \int_{D_2} |\nabla u|^2,$$

where, again, the $\sim$ is controlled by $P(\|\|S\||)$. In particular, it implies that the spectral radius of the operators $\mathcal{D}, \mathcal{D}^*$ is less than $1 - (P(\|\|S\||))^{-1}$.

**Theorem 5.5.** The operator norms $\| (I + \lambda \mathcal{D})^{-1} \|_{H^s(S)}, \| (I + \lambda \mathcal{D}^*)^{-1} \|_{H^s(S)}, |s| \leq 1, |\lambda| \leq 1$, are $P(\|\|S\||)$ (growth at most polynomially with $\|\|S\||$).

**Proof.** The identity $(I - \mathcal{D})^{-1} (I - \lambda \mathcal{D}) = I + (1 - \lambda)(I - \mathcal{D})^{-1} \mathcal{D}$ shows that the conclusion of the theorem follows easily when $|1 - \lambda| \leq 1/P(\|\|S\||)$, and similarly when $|1 + \lambda| \leq 1/P(\|\|S\||)$.

Therefore, without loss of generality, we may assume that

$$1 - |\lambda| \geq \frac{1}{P(\|\|S\||)}.$$

Assume now that $\phi \in H^{-1/2}(S)$ satisfies $\|\phi\|_{H^{-1/2}} = 1$ and

$$\|\phi - \lambda \mathcal{D}^* \phi\|_{H^{-1/2}} \leq \frac{1}{P(\|\|S\||)}.$$

Then the single-layer potential

$$V(x) = c_n \int_S \frac{\phi(y)}{\|x - y\|^{n-2}} \, dS(y).$$
satisfies the inequality
\[ \left| \int_S V(\phi - \lambda \mathcal{D}^* \phi) \, dS \right| \leq \frac{1}{P(||S||)}. \]

On the other hand, one has
\[ \int_S V(\phi - \lambda \mathcal{D}^* \phi) \, dS = (1 - \lambda) \int_{D_1} |\nabla V|^2 + (1 + \lambda) \int_{D_2} |\nabla V|^2, \]
implying the estimate
\[ \int_S V(\phi + \lambda \mathcal{D}^* \phi) \, dS = (1 + \lambda) \int_{D_1} |\nabla V|^2 + (1 - \lambda) \int_{D_2} |\nabla V|^2 \leq \frac{1}{P(||S||)}. \]

Adding both inequalities together, we would obtain
\[ \int_S V \phi \, d\sigma \leq \frac{1}{P(||S||)}, \]
which is impossible because of the following:

**Lemma 5.6.** If \( V \) is the single-layer potential of \( \phi \), then
\[ \int_S V(x) \phi(x) \, dS(x) = \int_S \int_S \phi(x) \phi(y) \, dS(x) \, dS(y) \geq \frac{1}{P(||S||)} \|\phi\|_{H^{-1/2}(S)}^2, \]

Let us first observe that
\[ \int_S \int_S \frac{\phi(x) \phi(y)}{\|x - y\|^{n-2}} \, d\sigma(x) \, d\sigma(y) = \int_{\mathbb{R}^n} \frac{1}{|\xi|^2} |\hat{\phi}(\xi)|^2 \, d\xi \geq 0, \]
where \( \hat{\phi} \, dS \) denotes the Fourier transform of the measure \( \phi \, dS \) supported on \( S \). This implies that
\[ \langle \phi, \psi \rangle = \int_S \int_S \frac{\phi(x) \phi(y)}{\|x - y\|^{n-2}} \, dS(x) \, dS(y) \]
is an inner product satisfying
\[ |\langle \phi, \psi \rangle| \leq \langle \phi, \phi \rangle^{1/2} \langle \psi, \psi \rangle^{1/2}, \]
and we wish to show that
\[ \langle \phi, \phi \rangle \asymp \|\phi\|_{H^{-1/2}(S)}^2, \]
where \( \asymp \) denotes equivalence modulo a factor \( P(||S||) \). To see this, observe first that given \( \phi \in H^{-1/2}(S) \), its single-layer potential \( u|S \) belongs to the space \( H^{1/2}(S) \), satisfying
\[ \|u\|_{H^{1/2}(S)} \leq P(||S||) \|\phi\|_{H^{-1/2}(S)}, \]
which can be proved easily using local coordinates. As a consequence, we have
\[ \int_S \int_S \frac{\phi(x) \phi(y)}{\|x - y\|^{n-2}} \, dS(x) \, dS(y) \leq P(||S||) \|\phi\|_{H^{-1/2}(S)}^2. \]
In the opposite direction, since $H^{-s} = (H^s)^*$, we have

$$\|\phi\|_{H^{-s}} = \sup_{f \in H^s} \int_S \phi(x) f(x) d\sigma(x).$$

Let us assume, for the moment, that given $f \in H^s$, there exists $g \in H^{s-1}$ such that

$$f(x) = c_n \int_S \frac{g(y)}{\|x - y\|^{n-2}} dS(y) \quad \text{and} \quad \|f\|_{H^s} \leq \|g\|_{H^{s-1}}.$$

Then

$$\|\phi\|_{H^{-s}} \leq \sup_{\|g\|_{H^{s-1}} = 1} \langle \phi, g \rangle,$$

and taking $s = \frac{1}{2}$, $s - 1 = -\frac{1}{2}$, we get

$$\|\phi\|_{H^{-1/2}} \leq P(|||S|||) \langle \phi, \phi \rangle^{1/2} \langle g, g \rangle^{1/2} \leq P(|||S|||) \langle \phi, \phi \rangle^{1/2} \|g\|_{H^{-1/2}} \leq P(|||S|||) \langle \phi, \phi \rangle^{1/2}.$$

To close our argument, it remains to solve the equation

$$f(x) = c_n \int_S \frac{g(y)}{\|x - y\|^{n-2}} dS(y),$$

that is, to prove that given $f \in H^s$, there exists $g \in H^{s-1}$ satisfying the this equation.

To see that, let us consider the solution of the Dirichlet problem

$$\begin{cases}
\Delta u = 0 & \text{in } D_1, \\
|S| = f
\end{cases}$$

and the equation

$$-2 \frac{\partial u}{\partial v_1} = g - \mathcal{D}^* g,$$

that is, $g = (I - \mathcal{D}^*)^{-1}(-2\partial u/\partial v_1)$. Then we claim that such $g$ verifies the identity

$$f(x) = c_n \int_S \frac{g(y)}{\|x - y\|^{n-2}} dS(y).$$

This is because the function

$$V(x) = c_n \int_S \frac{g(y)}{\|x - y\|^{n-2}} dS(y)$$

is harmonic in $D_1$ and satisfies

$$-2 \frac{\partial V}{\partial v_1} = g - \mathcal{D}^* g = -2 \frac{\partial u}{\partial v_1},$$

which implies that $V = u$ in $D_1$, and therefore, taking limits up to the boundary, we obtain

$$f(x) = c_n \int_S \frac{g(y)}{\|x - y\|^{n-2}} dS(y).$$

To finish the proof of Theorem 5.5, let us consider, for every $0 \leq \tau \leq 1$, the identity

$$(I - \lambda \mathcal{D})^{-1} A^\tau = A^\tau (I - \lambda \mathcal{D})^{-1} + (I - \lambda \mathcal{D})^{-1} C_\tau (I - \lambda \mathcal{D})^{-1},$$
where the commutator $C_\tau = [\partial^\tau \Lambda^\tau - \Lambda^\tau \partial^\tau]$ is a pseudodifferential operator of order $\tau - 2$ whose bounds are controlled by $\|\|S\||$. Then
\[
\|(I - \lambda \partial)\)^{-1} f \|_{H^s} \leq \|(I - \lambda \partial)\)^{-1} f \|_{H^{-1/2}} + \|\Lambda^{s+1/2}(I - \lambda \partial)\)^{-1} f \|_{H^{-1/2}} \\
\lesssim \|f\|_{H^{-1/2}} + \|\Lambda^{s+1/2}f\|_{H^{-1/2}} \\
\lesssim \|f\|_{L^2} + \|\Lambda^{s+1/2}f\|_{H^{-1/2}} \leq P(\|\|S\||\|f\|_{H^s}). \quad \Box
\]

**Remark 5.7.** In the particular case of the sphere $S = S^{n-1}$ ($n \geq 2$), the estimate of Lemma 5.6 becomes an identity:
\[
\int_{S^{n-1}} \int_{S^{n-1}} \frac{\phi(x)\phi(y)}{\|x - y\|^{n-2}} dS(x) dS(y) = c_n \|\phi\|_{H^{-1/2}(S^{n-1})}^2
\]
for $n \geq 3$, and
\[
- \int_{S^{n-1}} \int_{S^{n-1}} \log \|x - y\| \phi(x)\phi(y) dS(x) dS(y) = c_2 \|\phi\|_{H^{-1/2}(S^{1})}^2
\]
for $n = 2$.

**Proof.** We present the details when $n \geq 3$. The case $n = 2$ follows similarly. Let $\phi(x) = \sum a_k Y_k(x)$, where $Y_k$ is a spherical harmonic of degree $k$, normalized so that $\|Y_k\|_{L^2(S^{n-1})} = 1$; then we have
\[
|a_0|^2 + \sum_{k \geq 1} \frac{|a_k|^2}{2k + n - 2} = \|\phi\|_{H^{-1/2}(S^{n-1})}^2 < \infty.
\]

Claim: if $k \neq j$, then
\[
\int_{S^{n-1}} \int_{S^{n-1}} \frac{Y_k(x)Y_j(y)}{\|x - y\|^{n-2}} dS(x) dS(y) = 0.
\]

Taking the Fourier transform and using Plancherel, we get
\[
\int_{S^{n-1}} \int_{S^{n-1}} \frac{Y_k(x)Y_j(y)}{\|x - y\|^{n-2}} dS(x) dS(y) = \int_{\mathbb{R}^n} \frac{1}{|\xi|^2} Y_k(\overline{dS(\xi)}) Y_j(\overline{dS(\xi)}) d\xi.
\]
But it turns out that
\[
Y_k(\overline{dS(\xi)}) = 2\pi i^{-k} |\xi|^{(n-2)/2} J_{(n+2k-2)/2}(|\xi|) Y_k\left(\frac{\xi}{|\xi|}\right),
\]
where $J_\nu$ designates Bessel’s function of order $\nu$, implying the claim.

Therefore our estimate diagonalizes:
\[
\int_{\mathbb{R}^n} \frac{1}{|\xi|^2} \left|\widehat{Y_k}(\overline{dS(\xi)})\right|^2 d\xi = c \int_0^\infty \frac{1}{r} \left|J_{k+(n-2)/2}(r)\right|^2 dr,
\]
and the well-known identity for Bessel’s functions
\[
\int_0^\infty \frac{J_\mu^2(r)}{r} dr = \frac{1}{2\mu}
\]
allows us to finish the proof. \box
Estimates for $\Omega$ and $\omega$. In the following, we shall consider asymptotically flat domains, leaving to the reader the details of the periodic case. Since we have controlled the norms of the operator relating $\Omega$ and $X$, we are in a position to obtain the inequality
\[
\|\Omega\|_{H^1} \leq P\left(\|X\|_k^2 + \|F(X)\|_{L^\infty}^2 + \|N\|_{L^\infty}^{-1}\right),
\] (5-1)
for $k \geq 4$, with $P$ a polynomial function. Then Sobolev’s embedding implies
\[
\|\omega\|_{H^k} \leq P\left(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \|N\|_{L^\infty}^{-1}\right),
\] (5-2)
for $k \geq 3$. We will present the proof of (5-1) when $k = 4$, because the case $k > 4$ can be obtained with the same method.

Theorem 5.5 applied to (2-6) yields
\[
\|\Omega\|_{H^1} = \|(I - A_\mu\overline{\Omega})^{-1}(-2A_\rho X_3)\|_{H^1} \leq C\|(I - A_\mu\overline{\Omega})^{-1}\|_{H^1} \|X_3\|_{H^1} \leq P(\|S\|)\|X_3\|_{H^1},
\]
implicating that
\[
\|\Omega\|_{H^1} \leq P\left(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \|N\|_{L^\infty}^{-1}\right).
\]
Next we will show that
\[
\|\partial_{\alpha_1}^2 \Omega\|_{L^2} \leq P\left(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \|N\|_{L^\infty}^{-1}\right)\|\Omega\|_{H^1},
\] (5-3)
which together with the estimate for $\|\Omega\|_{H^1}$ above, will allow us to control $\partial_{\alpha_1}^2 \Omega$ in terms of the free boundary.

In order to do that, we start with formula (2-8) to get $\partial_{\alpha_1}^2 \Omega = I_1 + I_2 + I_3 + I_4 - 2A_\rho \partial_{\alpha_1}^2 X_3$, where
\[
I_1 = \frac{A_\mu}{2\pi} \operatorname{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) d\beta \cdot \partial_{\alpha_1}^2 X(\alpha),
\]
\[
I_2 = \frac{A_\mu}{2\pi} \operatorname{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]
\[
I_3 = -\frac{3A_\mu}{4\pi} \operatorname{PV} \int_{\mathbb{R}^2} A(\alpha, \beta) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^5} \wedge \omega(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]
with $A(\alpha, \beta) = (X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha - \beta))$, and
\[
I_4 = \frac{A_\mu}{2\pi} \operatorname{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \omega(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha).
\]

Our next objective is to introduce the operators $\mathcal{F}_k$ (A-5) defined in the Appendix in the analysis of the integrals $I_j$. Formula (2-3) gives us $\omega = \partial_{\alpha_2} (\Omega \partial_{\alpha_1} X) - \partial_{\alpha_1} (\Omega \partial_{\alpha_2} X)$, and from standard Sobolev’s estimates we get
\[
\|I_j\|_{L^2} \leq P\left(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \|N\|_{L^\infty}^{-1}\right)\|\Omega\|_{H^1}, \quad j = 1, 2,
\]
and similarly with $I_3$. 

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Regarding
\[ I_4 = \int_{|\beta|>1} d\beta + \int_{|\beta|<1} d\beta = J_1 + J_2, \]
we integrate by parts in \( J \) and the fact that the kernel in the integral \( K \) to show that it can be estimated via an integration by parts in the variable \( \beta \). We rewrite
\[ J_1 = \frac{A_\mu}{2\pi} \int_{|\beta|>1} \partial_{\beta_1} \left( \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \right) \wedge \omega(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha) \]
\[ \quad - \frac{A_\mu}{2\pi} \int_{|\beta|=1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha). \]
From this last expression, it is easy to deduce the inequality
\[ J_1 \leq C \| F(X) \|^3_{L^\infty} \| X - (\alpha, 0) \|^2_{C^1} \left( \int_{|\beta|>1} \frac{|\omega(\alpha - \beta)|}{|\beta|^3} d\beta + \int_{|\beta|=1} |\omega(\alpha - \beta)| d\beta, \right) \]
providing us with an appropriate control (see the Appendix for more details).

Next let us consider \( J_2 = K_1 + K_2 + K_3 + K_4 \), where
\[ K_1 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_2} \Omega(\alpha - \beta) \partial_{\alpha_1}^2 X(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha), \]
\[ K_2 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \partial_{\alpha_2} \Omega(\alpha - \beta) \partial_{\alpha_1} X(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha), \]
\[ K_3 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \Omega(\alpha - \beta) \partial_{\alpha_1} \partial_{\alpha_2} X(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha), \]
\[ K_4 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1}^2 \Omega(\alpha - \beta) \partial_{\alpha_2} X(\alpha - \beta) d\beta \cdot \partial_{\alpha_1} X(\alpha). \]

Then the terms \( K_1 \) and \( K_3 \) are handled with the same approach used for \( I_2 \) — see (A-13) in the Appendix — and we rewrite \( K_2 \) in the form
\[ K_2 = \frac{A_\mu}{2\pi} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \partial_{\alpha_2} \Omega(\alpha - \beta) (\partial_{\alpha_1} X(\alpha - \beta) - \partial_{\alpha_1} X(\alpha)) d\beta \cdot \partial_{\alpha_1} X(\alpha), \]
to show that it can be estimated via an integration by parts in the variable \( \beta_1 \), using the identity
\[ \partial_{\alpha_1} \partial_{\alpha_2} \Omega(\alpha - \beta) = -\partial_{\beta_1} (\partial_{\alpha_2} \Omega(\alpha - \beta)) \]
and the fact that the kernel in the integral \( K_2 \) has degree \(-1\).

It remains to deal with \( K_4 \): to do that, let us consider \( K_4 = L_1 + L_2 \), where
\[ L_1 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1}^2 \Omega(\alpha - \beta) (\partial_{\alpha_2} X(\alpha - \beta) - \partial_{\alpha_2} X(\alpha - \beta)) d\beta \cdot \partial_{\alpha_1} X(\alpha) \]
and
\[ L_2 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \partial_{\alpha_1}^2 \Omega(\alpha - \beta) d\beta \cdot N(\alpha). \]
The term $L_2$ can be controlled like $K_2$, and $L_2$ can be rewritten in the form

$$L_2 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \left( \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} - \frac{\nabla X(\alpha) \cdot \beta}{|\nabla X(\alpha) \cdot \beta|^3} \right) \partial_{\alpha_1}^2 \Omega(\alpha - \beta) \, d\beta \cdot N(\alpha),$$

showing that it can be estimated as we did with $\mathcal{F}_4$ (A-8), that is, we obtain (5-3). Similarly, Equation (2-9) yields

$$\|\partial^2_{\alpha_2} \Omega\|_{L^2} \leq P \left( \|X\|_4^2 + \|F(X)\|_{L^4}^2 + \|N\|^{-1}_{L^\infty} \right) \|\Omega\|_{H^1},$$

and then the inequality $2\|\partial_{\alpha_1} \partial_{\alpha_2} \Omega\|_{L^2} \leq \|\partial^2_{\alpha_1} \Omega\|_{L^2} + \|\partial^2_{\alpha_2} \Omega\|_{L^2}$ gives us the desired control upon $\|\Omega\|_{H^2}$.

Next we will show that

$$\|\partial^3_{\alpha_1} \Omega\|_{L^2} \leq P \left( \|X\|_4^2 + \|F(X)\|_{L^4}^2 + \|N\|^{-1}_{L^\infty} \right) \|\Omega\|_{H^2},$$

allowing us to use the estimates for $\|\Omega\|_{H^2}$ above. In order to do that, we start with formula (2-8), to get

$$J_3 = \frac{A_\mu}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial^3_{\alpha_1} X(\alpha),$$

$$J_4 = \frac{A_\mu}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial^2_{\alpha_1} X(\alpha) - \partial^2_{\alpha_1} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$J_5 = -\frac{3A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} B(\alpha, \beta) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

with $B(\alpha, \beta) = (X(\alpha) - X(\alpha - \beta)) \cdot (\partial^2_{\alpha_1} X(\alpha) - \partial^2_{\alpha_1} X(\alpha - \beta))$, and

$$J_6 = \frac{A_\mu}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial^2_{\alpha_1} \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

and where the remainder terms can be estimated with the same method used before.

Now we write

$$J_3 = \frac{A_\mu}{2\pi} \mathcal{F}_1 \left( \partial_{\alpha_2} (\Omega \partial_{\alpha_1} X) - \partial_{\alpha_1} (\Omega \partial_{\alpha_2} X) \right) \cdot \partial^3_{\alpha_1} X$$

to obtain

$$\|J_3\|_{L^2} \leq C \|\mathcal{F}_1 \left( \partial_{\alpha_2} (\Omega \partial_{\alpha_1} X) - \partial_{\alpha_1} (\Omega \partial_{\alpha_2} X) \right)\|_{L^4} \|\partial^3_{\alpha_1} X\|_{L^4}.$$

Next observe that in the proof of estimate (A-9), one can replace $L^2$ by $L^p$ for $1 < p < \infty$ [Stein 1993]. In particular, we have

$$\|J_3\|_{L^2} \leq P \left( \|X - (\alpha, 0)\|_{C^{1,1}} + \|F(X)\|_{L^\infty} + \|N\|^{-1}_{L^\infty} \right) \left( \|\Omega \partial_{\alpha_1} X\|_{L^4} + \|\Omega \partial_{\alpha_2} X\|_{L^4} + \|\omega\|_{L^4} \right) \|\partial^3_{\alpha_1} X\|_{L^4},$$

and then Sobolev’s embedding in dimension two, $\|g\|_{L^4} \leq C \|g\|_{H^1}$, yields the desired control. Regarding $J_4$, we follow the approach taken before for $\mathcal{F}_3$, but now using the $L^4$ norm. That is, we split

$$J_4 = \int_{|\beta| > 1} d\beta + \int_{|\beta| < 1} d\beta = K_5 + K_6,$$
and since
\[ K_5 \leq \|X - (\alpha, 0)\|_{C^2}^2 \|F(X)\|_L^3 \int_{|\beta| > 1} \frac{|\omega(\alpha - \beta)|}{|\beta|^3} \, d\beta, \]
that term can be estimated as above.

Next we introduce the splitting \( K_6 = L_3 + L_4 \), where
\[
L_3 = \frac{A_\mu}{2\pi} \int_{|\beta| < 1} \left( \frac{\partial_\alpha^2 X(\alpha) - \partial_\alpha^2 X(\alpha - \beta)}{|\nabla X(\alpha - \beta)|^3} - \frac{1}{|X(\alpha) - X(\alpha - \beta)|^3} \right) \cdot \omega(\alpha - \beta) \cdot \partial_\alpha X(\alpha),
\]
\[
L_4 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{\partial_\alpha^2 X(\alpha) - \partial_\alpha^2 X(\alpha - \beta)}{|\nabla X(\alpha - \beta)|^3} \cdot \omega(\alpha - \beta) \cdot \partial_\alpha X(\alpha).
\]
We have
\[
L_3 \leq C \|X - (\alpha, 0)\|_{C^2}^2 (\|F(X)\|_L^4 + \|X - (\alpha, 0)\|_{C^1}^4 \|N\|^{-1} \|L\|_L^4) \int_{|\beta| < 1} \frac{|\omega(\alpha - \beta)|}{|\beta|^{2-\delta}} \, d\beta
\]
(see the Appendix for more details), giving us the appropriate estimate. Regarding \( L_4 \), we use identity (A-16), which, after a careful integration by parts, yields
\[
L_4 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{\beta \cdot \nabla \left( (\partial_\alpha^2 X(\alpha) - \partial_\alpha^2 X(\alpha - \beta)) \cdot \omega(\alpha - \beta) \cdot \partial_\alpha X(\alpha) \right)}{|\nabla X(\alpha - \beta)|^3} \, d\beta
\]
\[
- \frac{A_\mu}{2\pi} \int_{|\beta| = 1} \frac{|\beta| (\partial_\alpha^2 X(\alpha) - \partial_\alpha^2 X(\alpha - \beta)) \cdot \omega(\alpha - \beta) \cdot \partial_\alpha X(\alpha)}{|\nabla X(\alpha - \beta)|^3} \, d\beta,
\]
helping us to prove the inequality
\[
\|L_4\|_{L^2} \leq P \left( \|X - (\alpha, 0)\|_{C^2} + \|F(X)\|_{L^\infty} + \|N\|^{-1} \|L\|_L^4 \right) (\|\partial_\alpha^3 X\|_{L^4} \|\omega\|_{L^4} + \|\omega\|_{L^2}).
\]
Clearly, \( J_5 \) can be approached with the same method used for \( J_4 \). Regarding the term \( J_6 \), we have to decompose further: first, its most singular terms, which are given by
\[
L_5 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot \partial_\alpha \Omega(\alpha - \beta) \partial_\alpha^3 X(\alpha - \beta) \, d\beta \cdot \partial_\alpha X(\alpha),
\]
\[
L_6 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot \partial_\alpha^2 \partial_\alpha \Omega(\alpha - \beta) \partial_\alpha X(\alpha - \beta) \, d\beta \cdot \partial_\alpha X(\alpha),
\]
\[
L_7 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot \partial_\alpha \Omega(\alpha - \beta) \partial_\alpha^2 \partial_\alpha X(\alpha - \beta) \, d\beta \cdot \partial_\alpha X(\alpha),
\]
\[
L_8 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot \partial_\alpha^3 \Omega(\alpha - \beta) \partial_\alpha^2 X(\alpha - \beta) \, d\beta \cdot \partial_\alpha X(\alpha).
\]
Second, let us observe that the remainder is easy to handle: the terms \( L_5 \) and \( L_7 \) can be estimated as we did with \( K_1 \) and \( K_3 \), using the \( L^4 \) norm and, finally, \( L_6 \) and \( L_8 \) are like \( K_2 \) and \( K_4 \), respectively. Putting all these facts together, we obtain (5-4).
Similarly to the case of lower derivatives, Equation (2-9) yields
\[
\|\Omega\|_{H^3} \leq P(\|X\|_4^2 + \|F(X)\|_L^{\infty} + \|N^{-1}\|_{L^{\infty}})\|\Omega\|_{H^3}.
\]
To finish, it remains to show the corresponding inequality for derivatives of fourth order:
\[
\|\Omega\|_{H^4} \leq P(\|X\|_4^2 + \|F(X)\|_L^{\infty} + \|N^{-1}\|_{L^{\infty}})\|\Omega\|_{H^3}.
\]
(5-5)

Identity (2-8) allows us to point out the most singular terms in \(\partial_{\alpha_1}^4 \Omega\):

\[
M_1 = \frac{A_{\mu}}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1}^4 X(\alpha),
\]

\[
M_2 = \frac{A_{\mu}}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

\[
M_3 = -\frac{3A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} C(\alpha, \beta) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

with \(C(\alpha, \beta) = (X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta))\), and

\[
M_4 = \frac{A_{\mu}}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1}^3 \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha).
\]

Then, in order to estimate \(M_1\), we start with \(\|M_1\|_{L^2} \leq CK \|\partial_{\alpha_1}^4 X\|_{L^2}\), where

\[
K = \sup_{\alpha} \left| \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \right|.
\]

Following [Córdoba and Gancedo 2007], we have

\[
K \leq O_1 + O_2 + O_3 + O_4 + O_5,
\]
where

\[
O_1 = \sup_{\alpha} \left| \text{PV} \int_{|\beta|>1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \right|,
\]

\[
O_2 = \sup_{\alpha} \left| \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta) - \nabla X(\alpha) \cdot \beta}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \right|,
\]

\[
O_3 = \sup_{\alpha} \left| \int_{|\beta|<1} \nabla X(\alpha) \cdot \beta \left[ \frac{1}{|X(\alpha) - X(\alpha - \beta)|^3} - \frac{1}{|\nabla X(\alpha) \cdot \beta|^3} \right] \wedge \omega(\alpha - \beta) \, d\beta \right|,
\]

\[
O_4 = \sup_{\alpha} \left| \int_{|\beta|<1} \frac{\nabla X(\alpha) \cdot \beta}{|\nabla X(\alpha) \cdot \beta|^3} \wedge (\omega(\alpha - \beta) - \omega(\alpha)) \, d\beta \right|,
\]

\[
O_5 = \sup_{\alpha} \left| \text{PV} \int_{|\beta|<1} \frac{\nabla X(\alpha) \cdot \beta}{|\nabla X(\alpha) \cdot \beta|^3} \wedge \omega(\alpha) \, d\beta \right|.
\]
An integration by parts in $O_1$ yields
\[
O_1 \leq C \| \nabla X \|_{L^\infty}^2 \| F(X) \|_{L^\infty}^3 \sup_{\alpha} \left( \int_{|\beta| > 1} \frac{|\Omega(\alpha - \beta)|}{|\beta|^3} \, d\beta + \int_{|\beta| = 1} |\Omega(\alpha - \beta)| \, dl(\beta) \right)
\leq C \| \nabla X \|_{L^\infty}^2 \| F(X) \|_{L^\infty}^3 \| \Omega \|_{L^\infty},
\]
and Sobolev's embedding allows us to conclude.

Regarding $O_2$, we have
\[
O_2 \leq \| X - (\alpha, 0) \|_{C^{2,\delta}} \| F(X) \|_{L^\infty}^3 \| \omega \|_{L^\infty} \int_{|\beta| < 1} \| \beta \|^{2-\delta} \, d\beta,
\]
and the estimate $\| \omega \|_{C^i} \leq C \| \omega \|_{H^2}$, for $0 < \delta < 1$, gives the desired control. Using (A-15) and some straightforward algebraic manipulations, we get a similar inequality for $O_3$. Next, we have
\[
O_4 \leq C \| X - (\alpha, 0) \|_{C^i} \| |N|^{-1} \|_{L^\infty}^3 \| \omega \|_{C^i} \int_{|\beta| < 1} \| \beta \|^{2-\delta} \, d\beta,
\]
giving us also the same estimate. Furthermore, it is easy to prove that $O_5 = 0$.

Next we consider the term $M_2$ with the splitting $M_2 = Q_1 + Q_2 + Q_3$, where
\[
Q_1 = \frac{A_\mu}{2\pi} \int_{|\beta| > 1} \frac{\partial^{3}_{\alpha_1} X(\alpha) - \partial^{3}_{\alpha_1} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
Q_2 = \frac{A_\mu}{2\pi} \int_{|\beta| < 1} \frac{\partial^{3}_{\alpha_1} X(\alpha) - \partial^{3}_{\alpha_1} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge (\omega(\alpha - \beta) - \omega(\alpha)) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
Q_3 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{\partial^{3}_{\alpha_1} X(\alpha) - \partial^{3}_{\alpha_1} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta \wedge \omega(\alpha) \cdot \partial_{\alpha_1} X(\alpha).
\]
The term $Q_1$ can be estimated as before; regarding $Q_2$, we can use the identity
\[
\partial^{3}_{\alpha_1} X(\alpha) - \partial^{3}_{\alpha_1} X(\alpha - \beta) = \int_0^1 \nabla \partial^{3}_{\alpha_1} X(\alpha + (s - 1)\beta) \, ds \cdot \beta,
\]
and the control of $Q_3$ can be approached as we did with the operator in (A-7). Similarly with $M_3$, while $M_4$ is analogous to $J_6$, and all these observations together allow us to obtain (5-5).

### 6. Controlling the Birkhoff–Rott integral

Here we consider estimates for the Birkhoff–Rott integral along a non-self-intersecting surface. Let us assume that $\nabla (X(\alpha) - (\alpha, 0)) \in H^k(\mathbb{R}^2)$ for $k \geq 3$, and that both $F(X)$ and $|N|^{-1}$ are in $L^\infty$, where
\[
F(X)(\alpha, \beta) = |\beta|/|X(\alpha) - X(\alpha - \beta)| \quad \text{and} \quad N(\alpha) = \partial_{\alpha_1} X(\alpha) \wedge \partial_{\alpha_2} X(\alpha).
\]
The main purpose of this section is to prove the estimate
\[
\| \text{BR}(X, \omega) \|_{H^{k-1}} \leq P \left( \| X \|_{L^k}^2 + \| F(X) \|_{L^\infty}^2 + \| |N|^{-1} \|_{L^\infty} \right),
\]
(6-1)
for \( k \geq 4 \). Here we shall show it when \( k = 4 \), because the other cases, \( k > 4 \), follow by similar arguments. We rewrite \( BR \) in the following manner:

\[
BR(X, \omega)(\alpha, t) = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge (\partial_{\beta_2}(\Omega \partial_{\beta_1} X) - \partial_{\beta_1}(\Omega \partial_{\beta_2} X))(\beta) \, d\beta,
\]

which, together with the estimates about \( \Omega \) in Section 5 and also about the operator \( T_1 \) in the Appendix, yields

\[
\| BR(X, \omega) \|_{L^2} \leq P \left( \| X \|_{4}^2 + \| F(X) \|_{L^\infty}^2 + \| N \|_{L^\infty}^{-1} \right).
\]

To estimate derivatives of order 3, we consider \( \partial_{\alpha_1}^3 (BR(X, \omega)) \), and observe that the most dangerous terms are given by

\[
I_1 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta)) \wedge \omega(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta,
\]

\[
I_2 = \frac{3}{4\pi} \text{PV} \int_{\mathbb{R}^2} (X(\alpha) - X(\alpha - \beta)) \wedge \omega(\alpha - \beta) \frac{(X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^5} \, d\beta,
\]

\[
I_3 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\alpha - \beta)) \wedge (\partial_{\alpha_1}^3 \omega)(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta.
\]

In the Appendix, we find all the ingredients needed to estimate these terms \( I_j \), while the remainder in \( \partial_{\alpha_1}^3 (BR(X, \omega)) \) is easily bounded: in \( I_3 \) we can recognize an operator with the form of \( T_1 \) in (A-5), so the estimate for \( \omega \) in Section 5 gives the desired control for \( I_3 \). Regarding \( I_1 \), we may use the splitting \( I_1 = J_1 + J_2 \), where

\[
J_1 = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta)) \wedge (\omega(\alpha) - \omega(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta,
\]

\[
J_2 = \frac{\omega(\alpha)}{4\pi} \wedge \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta.
\]

Then the identity \( \partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha - \beta) = \beta \cdot \int_0^1 \nabla \partial_{\alpha_1}^3 X(\alpha + (s - 1)\beta) \, ds \) allows us to find in \( J_1 \) a kernel of degree \(-1\) which we know how to handle (see the Appendix). One uses the estimate for \( T_3 \) (A-7) to deal with \( J_2 \), and we proceed similarly to control \( I_2 \).

### 7. In search of the Rayleigh–Taylor condition

As was pointed out in Section 4 (outline of the proof), our approach is based on energy estimates, and a crucial step is to characterize those terms involving higher derivatives which are controlled because they have the appropriate sign. In our terminology, they constitute the Rayleigh–Taylor condition, which is supposed to hold at time \( T = 0 \), it being an important part of the proof to show that it prevails under the evolution.

Let us introduce the notation

\[
\| X \|_k^2 = \| X \|_k^2 + \| F(X) \|_{L^\infty}^2 + \| N \|_{L^\infty}^{-1},
\]
where
\[ \|X\|_k = \|X_1 - \alpha_1\|_{L^3} + \|X_2 - \alpha_2\|_{L^3} + \|X_3\|_{L^2} + \|\nabla(X - (\alpha, 0))\|_{H^{k-1}}^2 \tag{7-1} \]
and
\[ \|\nabla(X - (\alpha, 0))\|_{H^{k-1}}^2 = \|\nabla(X - (\alpha, 0))\|_{L^2}^2 + \|\partial^k_{\alpha_1}(X - (\alpha, 0))\|_{L^2}^2 + \|\partial^k_{\alpha_2}(X - (\alpha, 0))\|_{L^2}^2. \]

In order to justify the formula
\[ \frac{d}{dt}\|X\|_k^2(t) \leq -\sum_{i=1,2} \frac{2^{3/2}}{(\mu_1 + \mu_2)} \int_{\mathbb{R}^2} \frac{\sigma(\alpha, t)}{\|\nabla X(\alpha, t)\|^3} \partial^k_{\alpha_i} X(\alpha, t) \cdot \Delta(\partial^k_{\alpha_i} X)(\alpha, t) \, d\alpha + P(\|X\|_k(t)) \]
(here \(k \geq 4\), although for the sake of simplicity we will present the explicit computations when \(k = 4\), leaving the other cases as an exercise for the interested reader), it will be convenient to make use of the following tools, which give us different kinds of cancellations, and which constitute our particular bestiary of formulas for this paper.

From the definition of the isothermal parametrization, we have the identities
\[ |\partial_{\alpha_1} X|^2 = |\partial_{\alpha_2} X|^2, \tag{7-2} \]
\[ \partial_{\alpha_1} X \cdot \partial_{\alpha_2} X = 0, \tag{7-3} \]
which yield
\[ \frac{1}{2} \Delta(|\partial_{\alpha_1} X|^2) = |\partial_{\alpha_1} \partial_{\alpha_2} X|^2 - \partial^2_{\alpha_1} X \cdot \partial^2_{\alpha_2} X, \tag{7-4} \]
\[ \partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X = -3\partial^3_{\alpha_1} X \cdot \partial^2_{\alpha_1} X + (\partial^2_{\alpha_1} \Delta^{-1} \partial_{\alpha_1})(|\partial_{\alpha_1} \partial_{\alpha_2} X|^2 - \partial^2_{\alpha_1} X \cdot \partial^2_{\alpha_2} X), \tag{7-5} \]
\[ \partial^4_{\alpha_2} X \cdot \partial_{\alpha_2} X = -3\partial^3_{\alpha_2} X \cdot \partial^2_{\alpha_2} X + (\partial^2_{\alpha_2} \Delta^{-1} \partial_{\alpha_2})(|\partial_{\alpha_1} \partial_{\alpha_2} X|^2 - \partial^2_{\alpha_1} X \cdot \partial^2_{\alpha_2} X). \tag{7-6} \]

Using (7-3) and (7-4), we obtain
\[ \partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X = -2\partial^3_{\alpha_1} X \cdot \partial_{\alpha_1} \partial_{\alpha_2} X - \partial^2_{\alpha_1} \partial_{\alpha_2} X \cdot \partial^2_{\alpha_1} X - (\partial_{\alpha_1} \partial_{\alpha_2} \Delta^{-1} \partial_{\alpha_1})(|\partial_{\alpha_1} \partial_{\alpha_2} X|^2 - \partial^2_{\alpha_1} X \cdot \partial^2_{\alpha_2} X), \tag{7-7} \]
\[ \partial^4_{\alpha_2} X \cdot \partial_{\alpha_2} X = -2\partial^3_{\alpha_2} X \cdot \partial_{\alpha_1} \partial_{\alpha_2} X - \partial^2_{\alpha_2} \partial_{\alpha_1} X \cdot \partial^2_{\alpha_2} X - (\partial_{\alpha_1} \partial_{\alpha_2} \Delta^{-1} \partial_{\alpha_2})(|\partial_{\alpha_1} \partial_{\alpha_2} X|^2 - \partial^2_{\alpha_1} X \cdot \partial^2_{\alpha_2} X). \tag{7-8} \]

And Sobolev inequalities imply that if \(\nabla(X - (\alpha, 0)) \in H^3\), then \(\partial^4_{\alpha_i} X \cdot \partial_{\alpha_j} X \in H^3\) for \(i, j = 1, 2\).

With the help of the estimates above, we may now determine \(\sigma\). There is a part that may be considered a mere “algebraic” manipulation to detect the relevant characters and, in so doing, we disregard many terms because they are of lower order in the sense of Sobolev spaces. At the end, we shall present how to deal with those lower-order terms—if not for the whole collection of them, at least for the ones that we may consider to be the most “dangerous” characters. Here it is convenient to recommend to the reader our previous works [Córdoba and Gancedo 2007; Córdoba et al. 2011], where similar estimates were carried out.

**Low-order norms.** Since \(X_i(\alpha) \to \alpha_i\) for \(i = 1, 2\) at infinity, let us consider the evolution of the \(L^3\) norm. That is,
\[ \frac{1}{3} \frac{d}{dt} \|X_1 - \alpha_1\|_{L^3}^3(t) = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)X_1, d\alpha = I_1 + I_2 + I_3, \]
where
\[
I_1 = \int_{\mathbb{R}^2} |X_1 - \alpha_1| |X_1 - \alpha_1| BR_1 d\alpha,
\]
\[
I_2 = \int_{\mathbb{R}^2} |X_1 - \alpha_1| |X_1 - \alpha_1| C_1 \partial\alpha_1 X_1 d\alpha,
\]
\[
I_3 = \int_{\mathbb{R}^2} |X_1 - \alpha_1| |X_1 - \alpha_1| C_2 \partial\alpha_2 X_1 d\alpha.
\]

Then we have
\[
I_1 \leq \|X_1 - \alpha_1\|^2_{L^3} \|BR\|_{L^3} \leq C(\|X_1 - \alpha_1\|^3_{L^3} + \|BR\|_{L^\infty} \|BR\|^2_{L^2}),
\]
and Sobolev estimates, together with (6.1), yield the appropriate control in terms of $P(\|X\|_k)$.

Next, since $\partial\alpha_1 X_1 \to 1$ as $\alpha \to \infty$, we have
\[
I_2 \leq \|\partial\alpha_1 X_1\|_{L^\infty} \|X_1 - \alpha_1\|^2_{L^3} \|C_1\|_{L^3},
\]
and it remains to get control of $C_1$. Using (3.1), we introduce the splitting $C_1 = \sum_{j=1}^4 C_1^j$, where
\[
C_1^1(\alpha) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha - \beta|^2} BR_{\beta_1} \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|} d\beta,
\]
\[
C_1^2(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha - \beta|^2} BR_{\beta_1} \cdot \frac{X_{\beta_1}}{|X_{\beta_2}|} d\beta,
\]
\[
C_1^3(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_2 - \beta_2}{|\alpha - \beta|^2} BR_{\beta_1} \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|} d\beta,
\]
\[
C_1^4(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha - \beta|^2} BR_{\beta_2} \cdot \frac{X_{\beta_1}}{|X_{\beta_2}|} d\beta.
\]

We shall show how to control $C_1^1$, because the estimates for the other terms follow by similar arguments.

Integrating by parts, one obtains $C_1^1 = D_1 + D_2$, where
\[
D_1 = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha - \beta|^2} BR \cdot \partial\beta_1 \left( \frac{X_{\beta_2}}{|X_{\beta_2}|} \right) d\beta,
\]
\[
D_2 = -\frac{1}{\pi} \text{PV} \int_{\mathbb{R}^2} \frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)}{|\alpha - \beta|^4} BR \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|} d\beta.
\]

Regarding $D_1$, we write $D_1 = E_1 + E_2$, where
\[
E_1 = -\frac{1}{2\pi} \int_{|\beta|<1} \frac{\beta_1}{|\beta|^2} BR(\alpha - \beta) \cdot \partial\beta_1 \left( \frac{X_{\beta_2}}{|X_{\beta_2}|} \right) (\alpha - \beta) d\beta,
\]
\[
E_2 = -\frac{1}{2\pi} \int_{|\beta|>1} \frac{\beta_1}{|\beta|^2} BR(\alpha - \beta) \cdot \partial\beta_1 \left( \frac{X_{\beta_2}}{|X_{\beta_2}|} \right) (\alpha - \beta) d\beta.
\]

The Minkowski and Young inequalities yield, respectively,
\[
\|E_1\|_{L^3} \leq C \left\| BR \cdot \partial\beta_2 \left( \frac{X_{\beta_2}}{|X_{\beta_2}|} \right) \right\|_{L^3} \leq P(\|X\|_4),
\]
\[
\|E_2\|_{L^3} \leq C \left\| BR \cdot \partial\beta_2 \left( \frac{X_{\beta_2}}{|X_{\beta_2}|} \right) \right\|_{L^1} \leq C \|BR\|_{L^2} \left\| \partial\beta_2 \left( \frac{X_{\beta_2}}{|X_{\beta_2}|} \right) \right\|_{L^2} \leq P(\|X\|_4),
\]
and the desired control is achieved. In the term $D_2$, we have a double Riesz transform, and the standard
Calderón–Zygmund theory yields
\[ \| D_2 \|_{L^3} \leq C \left\| \text{BR} \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|^2} \right\|_{L^3} \leq C \| X_{\beta_2} \|^{-1}_{L^\infty} \| \text{BR} \|_{L^3} \leq P(\|X\|_4). \]

The estimate for \( I_3 \) follows on a similar path, and the case of the second coordinate is also identical:
\[ \frac{1}{3} \frac{d}{dt} \| X_2 - \alpha_2 \|^3_{L^3}(t) \leq P(\|X\|_4). \]

Regarding the third coordinate, we have stronger decay because of the asymptotic flatness hypothesis:
\[ \frac{1}{2} \frac{d}{dt} \| X_3 \|^2_{L^2}(t) = \int_{\mathbb{R}^2} X_3 \text{BR}_3 \, d\alpha + \int_{\mathbb{R}^2} X_3 C_1 \partial_{\alpha_1} X_3 \, d\alpha + \int_{\mathbb{R}^2} X_3 C_2 \partial_{\alpha_2} X_3 \, d\alpha \\
= \int_{\mathbb{R}^2} X_3 \text{BR}_3 \, d\alpha - \frac{1}{2} \int_{\mathbb{R}^2} (\partial_{\alpha_1} C_1 + \partial_{\alpha_2} C_2) |X_3|^2 \, d\alpha, \]
and therefore the use of Sobolev’s embedding in the formulas for \( C_1 \) (3-1) and \( C_2 \) (3-2), together with the estimates for \( \text{BR} \) (6-1), allows us to obtain:
\[ \frac{1}{2} \frac{d}{dt} \| X_3 \|^2_{L^2}(t) \leq P(\|X\|_4). \]

Once we have control of higher-order derivatives, we can use the estimates of the Appendix to get
\[ \frac{1}{2} \frac{d}{dt} \| \nabla (X - (\alpha, 0)) \|^2_{L^2}(t) \leq P(\|X\|_4). \]

**Higher-order norms.** Let us now consider
\[ \frac{1}{2} \frac{d}{dt} \| \partial^4_{\alpha_1} X \|^2_{L^2}(t) \]
\[ = \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1} \text{BR}(X, \omega) \, d\alpha + \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1} (C_1 \partial_{\alpha_1} X) \, d\alpha + \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1} (C_2 \partial_{\alpha_2} X) \, d\alpha \\
= I_1 + I_2 + I_3. \quad (7-9) \]

The higher-order terms in \( I_2 \) and \( I_3 \) are given by
\[ J_1 = \int_{\mathbb{R}^2} C_1 \partial^4_{\alpha_1} X \cdot \partial^5_{\alpha_1} X \, d\alpha, \quad J_2 = \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X \partial^4_{\alpha_1} C_1 \, d\alpha, \]
\[ J_3 = \int_{\mathbb{R}^2} C_2 \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1} \partial_{\alpha_2} X \, d\alpha, \quad J_4 = \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial_{\alpha_2} X \partial^4_{\alpha_1} C_2 \, d\alpha. \]

Integration by parts yields
\[ J_1 + J_3 = -\frac{1}{2} \int_{\mathbb{R}^2} (\partial_{\alpha_1} C_1 + \partial_{\alpha_2} C_2) |\partial^4_{\alpha_1} X|^2 \, d\alpha, \]
and therefore
\[ J_1 + J_3 \leq \frac{1}{2} \left( \| \partial_{\alpha_1} C_1 \|_{L^\infty} + \| \partial_{\alpha_2} C_2 \|_{L^\infty} \right) \| \partial^4_{\alpha_1} X \|^2_{L^2} \leq P(\|X\|_4). \]

Then in \( I_2 \) we use (7-5) to get
\[ J_2 = -\int_{\mathbb{R}^2} \partial_{\alpha_1} (\partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X) \partial^3_{\alpha_1} C_1 \, d\alpha \leq \| \partial_{\alpha_1} (\partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X) \|_{L^2} \| \partial^3_{\alpha_1} C_1 \|_{L^2}. \]
In $J_4$, we use (7-7) to obtain

$$J_4 = -\int_{\mathbb{R}^2} \partial_{\alpha_1} (\partial_{\alpha_1}^4 X \cdot \partial_{\alpha_2} X) \partial_{\alpha_1}^3 C_2 \, d\alpha \leq \|\partial_{\alpha_1} (\partial_{\alpha_1}^4 X \cdot \partial_{\alpha_2} X)\|_{L^2} \|\partial_{\alpha_1}^3 C_2\|_{L^2}.$$ 

From formulas (3-1), (3-2), one realizes that $C_1$ and $C_2$ are at the same level as Birkhoff–Rott (2-5), and therefore, we can use the estimates for BR (6-1) to control $\|\partial_{\alpha_1}^3 C_i\|_{L^2}$, $i = 1, 2$. Then formulas (7-5) and (7-7) indicate how to estimate $\|\partial_{\alpha_1} (\partial_{\alpha_1}^4 X \cdot \partial_{\alpha_2} X)\|_{L^2}$, $i = 1, 2$. That is, we have

$$J_2 + J_4 \leq P(\|X\|_4).$$

In $I_1$, the most singular terms are given by

$$J_5 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1} X(\alpha) \cdot \frac{(\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta)) \wedge \omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,$$

$$J_6 = \frac{3}{4\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge \omega(\beta) \frac{(X(\alpha) - X(\beta)) \cdot (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta))}{|X(\alpha) - X(\beta)|^5} \, d\alpha \, d\beta,$$

$$J_7 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge (\partial_{\alpha_1}^4 \omega(\beta)) \frac{(X(\alpha) - X(\beta)) \cdot (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta))}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta.$$ 

Let us consider now the splitting $J_5 = K_1 + K_2$:

$$K_1 = -\frac{1}{8\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \wedge (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta)) \cdot \frac{\omega(\beta) + \omega(\alpha)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,$$

$$K_2 = \frac{1}{8\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \wedge (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta)) \cdot \frac{\omega(\alpha) - \omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,$$

Next we exchange $\alpha$ and $\beta$ in $K_1$ to get

$$K_1 = \frac{1}{8\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\beta) \wedge (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta)) \cdot \frac{\omega(\beta) + \omega(\alpha)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,$$

$$= -\frac{1}{16\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta)) \wedge (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta)) \cdot \frac{\omega(\beta) + \omega(\alpha)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,$$

and therefore we can conclude that $K_1 = 0$. In $K_2$ we find a singular integral with a kernel of degree $-2$:

$$K_2 = -\frac{1}{8\pi} \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\beta) \wedge \frac{\omega(\alpha) - \omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\beta \, d\alpha,$$

and as is proved in the Appendix, we have

$$K_2 \leq P(\|X\|_4).$$

Let us now decompose $J_6 = K_3 + K_4^1 + K_4^2 + K_5^1 + K_5^2$, where

$$K_3 = \frac{3}{4\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge \omega(\beta) \frac{A(\alpha, \beta) \cdot (\partial_{\alpha_1}^4 X(\alpha) - \partial_{\alpha_1}^4 X(\beta))}{|X(\alpha) - X(\beta)|^5} \, d\alpha \, d\beta,$$
with $A(\alpha, \beta) = X(\alpha) - X(\beta) - \nabla X(\alpha)(\alpha - \beta)$,

$$K_4 = -\frac{3}{4\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge \omega(\beta) \frac{(\alpha_i - \beta_i)(\partial_{a_i} X(\alpha) - \partial_{a_i} X(\beta)) \cdot \partial_{a_1}^4 X(\beta)}{|X(\alpha) - X(\beta)|^5} \, d\alpha \, d\beta$$

$$K_5^i = \frac{3}{4\pi} \text{PV} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge \omega(\beta)$$

$$\times \frac{(\alpha_i - \beta_i)(\partial_{a_i} X(\alpha) \cdot \partial_{a_1}^4 X(\alpha) - \partial_{a_i} X(\beta) \cdot \partial_{a_1}^4 X(\beta))}{|X(\alpha) - X(\beta)|^5} \, d\alpha \, d\beta.$$ 

In $K_3$ and $K_4^i$ we find kernels of degree $-2$, and as shown in the Appendix, they behave as a Riesz transform acting on $\partial_{a_1}^4 X$. In $K_5^i$ the kernels have degree $-3$ and act as a $\Lambda$ operator on $\partial_{a_1} X \cdot \partial_{a_1}^4 X$. Then using formulas (7-5) and (7-7), we get finally the desired estimate.

We will find the R-T condition in $J_7$. Let us take $J_7 = K_6 + K_7$, where

$$K_6 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta))}{|X(\alpha) - X(\beta)|^3} - \frac{\nabla X(\alpha)(\alpha - \beta)}{|\nabla X(\alpha)(\alpha - \beta)|^3} \wedge (\partial_{a_1}^4 \omega)(\alpha) \, d\beta \, d\alpha,$$

$$K_7 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \int_{\mathbb{R}^2} \frac{\nabla X(\alpha)(\alpha - \beta)}{|\nabla X(\alpha)(\alpha - \beta)|^3} \wedge (\partial_{a_1}^4 \omega)(\alpha) \, d\beta \, d\alpha.$$ 

The term $K_6$ is controlled by (A-8) in the Appendix. Using (7-2) and (7-3), we get

$$K_7 = -\frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{a_1}^4 X(\alpha)}{|\partial_{a_1} X(\alpha)|^3} \cdot (\partial_{a_1} X(\alpha) \wedge R_1(\partial_{a_1}^4 \omega)(\alpha) + \partial_{a_2} X(\alpha) \wedge R_2(\partial_{a_1}^4 \omega)(\alpha)) \, d\alpha.$$ 

Formula (2-3) helps us to detect the most singular terms inside $K_7$, which will be denoted by $L_i$, $i = 1, \ldots, 8$, and are the following:

$$L_1 = -\frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_1} X(\alpha)}{|\partial_{a_1} X(\alpha)|^3} \wedge R_1(\partial_{a_1}^4 \partial_{a_2} \Omega \partial_{a_1} X)(\alpha) \, d\alpha,$$

$$L_2 = -\frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_1} X(\alpha)}{|\partial_{a_1} X(\alpha)|^3} \wedge R_1(\partial_{a_2} \Omega \partial_{a_1}^5 X)(\alpha) \, d\alpha,$$

$$L_3 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_1} X(\alpha)}{|\partial_{a_1} X(\alpha)|^3} \wedge R_1(\partial_{a_1}^5 \Omega \partial_{a_2} X)(\alpha) \, d\alpha,$$

$$L_4 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_1} X(\alpha)}{|\partial_{a_1} X(\alpha)|^3} \wedge R_1(\partial_{a_1} \Omega \partial_{a_1}^4 \partial_{a_2} X)(\alpha) \, d\alpha,$$

$$L_5 = -\frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_2} X(\alpha)}{|\partial_{a_2} X(\alpha)|^3} \wedge R_2(\partial_{a_1}^4 \partial_{a_2} \Omega \partial_{a_1} X)(\alpha) \, d\alpha,$$

$$L_6 = -\frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_2} X(\alpha)}{|\partial_{a_2} X(\alpha)|^3} \wedge R_2(\partial_{a_2} \Omega \partial_{a_1}^5 X)(\alpha) \, d\alpha,$$

$$L_7 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_2} X(\alpha)}{|\partial_{a_2} X(\alpha)|^3} \wedge R_2(\partial_{a_1}^5 \Omega \partial_{a_2} X)(\alpha) \, d\alpha,$$

$$L_8 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{a_1}^4 X(\alpha) \cdot \frac{\partial_{a_2} X(\alpha)}{|\partial_{a_2} X(\alpha)|^3} \wedge R_2(\partial_{a_1} \Omega \partial_{a_1}^4 \partial_{a_2} X)(\alpha) \, d\alpha.$$
In $L_1$ we get a kernel of degree $-1$ of the form

$$L_1 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot \int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha - \beta|^3} \partial_{\alpha_1} X(\alpha) \wedge (\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\beta)) \partial_{\alpha_2} \Omega(\beta) \, d\beta \, d\alpha,$$

which can be estimated integrating by parts throughout $\partial_{\alpha_1}^4 \partial_{\alpha_2} \Omega$; the term $L_7$ also follows in a similar manner. In order to estimate $L_2$, $L_4$, $L_6$ and $L_8$, we realize that they can be written like (A-3) in the Appendix plus commutators of the form (A-1). Next we have to deal with $L_3$ and $L_5$: with $L_3$, we proceed as follows:

$$L_3 \leq \tilde{L}_3 + \|\partial_{\alpha_1} X\|^{-2}_{L^\infty} \|\partial_{\alpha_1}^2 X\|_{L^2} \|R_1(\partial_{\alpha_1}^5 \Omega \partial_{\alpha_2} X) - R_1(\partial_{\alpha_1}^5 \Omega \partial_{\alpha_2} X)\|_{L^2},$$

where $\tilde{L}_3$ is given by

$$\tilde{L}_3 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot \frac{N(\alpha)}{|\partial_{\alpha_1} X(\alpha)|^3}(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 \Omega(\alpha)) \, d\alpha,$$

and the commutator estimates (A-1) show that it only remains to control $\tilde{L}_3$. We now use formula (2-8) to get $\tilde{L}_3 = M_1 + M_2$, where

$$M_1 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot \frac{N(\alpha)}{|\partial_{\alpha_1} X(\alpha)|^3}(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3(\alpha)) \, d\alpha$$

and

$$M_2 = -A_\mu \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X(\alpha) \cdot \frac{N(\alpha)}{|\partial_{\alpha_1} X(\alpha)|^3}(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3(\alpha)) \, d\alpha.$$

Then we write $M_1 = O_1 + O_2 + O_3$, where

$$O_1 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^4 X_1}{|\partial_{\alpha_1} X|^3} \left(\partial_{\alpha_1} X_2 \partial_{\alpha_2} X_3 - \partial_{\alpha_1} X_3 \partial_{\alpha_2} X_2\right)(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3) \, d\alpha,$$

$$O_2 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^4 X_2}{|\partial_{\alpha_1} X|^3} \left(\partial_{\alpha_1} X_3 \partial_{\alpha_2} X_2 - \partial_{\alpha_1} X_2 \partial_{\alpha_2} X_3\right)(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3) \, d\alpha,$$

$$O_3 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X_3(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3) \, d\alpha.$$

Next we consider $O_1 = P_1 + P_2 + P_3$, with

$$P_1 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^4 X_1}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1} X_2(R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial_{\alpha_1}^4 X_3) \, d\alpha,$$

$$P_2 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^4 X_1}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_2} X_2(R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial_{\alpha_1}^4 X_3) \, d\alpha,$$

$$P_3 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^4 X_1}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1} X_2 \left[(R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial_{\alpha_1}^4 X_3) - \partial_{\alpha_2} X_3(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3)\right] \, d\alpha$$

$$+ A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^4 X_1}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_2} X_2 \left[\partial_{\alpha_1} X_3(R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3) - (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_3 \partial_{\alpha_1}^4 X_3)\right] \, d\alpha,$$

and the commutator estimate allows us to control the term $P_3$. 


Now we use (7-7) to write $P_1 = Q_1 + Q_2 + Q_3$:

\[
Q_1 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 (R_1 \partial_\alpha)(\partial_\alpha X_1 (\partial^4_\alpha X_1)) \, d\alpha,
\]

\[
Q_2 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 (R_1 \partial_\alpha)(\partial_\alpha X_2 (\partial^4_\alpha X_2)) \, d\alpha,
\]

\[
Q_3 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 (R_1 \partial_\alpha)(\text{lower-order terms}) \, d\alpha.
\]

The term $Q_3$ is easily estimated. Regarding $P_2$, equality (7-5) allows us to write $P_2 = Q_4 + Q_5 + Q_6$, where

\[
Q_4 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 (R_1 \partial_\alpha)(\partial_\alpha X_1 (\partial^4_\alpha X_1)) \, d\alpha,
\]

\[
Q_5 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 (R_1 \partial_\alpha)(\partial_\alpha X_2 (\partial^4_\alpha X_2)) \, d\alpha,
\]

\[
Q_6 = -A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 (R_1 \partial_\alpha)(\text{lower-order terms}) \, d\alpha.
\]

Let us recall the identity $P_1 + P_2 = (Q_4 + Q_1) + (Q_2 + Q_5) + (Q_3 + Q_6)$, where $Q_3$ and $Q_6$ are easily estimated. With respect to $Q_2 + Q_5$, we have

\[
Q_2 + Q_5 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 \left[ (R_1 \partial_\alpha)(\partial_\alpha X_2 (\partial^4_\alpha X_2)) - \partial_\alpha X_2 (R_1 \partial_\alpha)(\partial^4_\alpha X_2) \right] \, d\alpha
\]

\[
+ A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 \left[ (R_1 \partial_\alpha)(\partial_\alpha X_2 (\partial^4_\alpha X_2)) - (R_1 \partial_\alpha)(\partial_\alpha X_2 (\partial^4_\alpha X_2)) \right] \, d\alpha,
\]

and again the commutator estimates yield the desired control.

Next we have

\[
Q_4 + Q_1 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 \left[ \partial_\alpha X_1 (R_1 \partial_\alpha)(\partial^4_\alpha X_1) - (R_1 \partial_\alpha)(\partial_\alpha X_1 (\partial^4_\alpha X_1)) \right] \, d\alpha
\]

\[
+ A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_\alpha X_1}{|\partial_\alpha X|^3} \partial_\alpha X_2 \left[ (R_1 \partial_\alpha)(\partial_\alpha X_1 (\partial^4_\alpha X_1)) - \partial_\alpha X_1 (R_1 \partial_\alpha)(\partial^4_\alpha X_1) \right] \, d\alpha
\]

\[
- A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{N_3}{|\partial_\alpha X|^3} \partial^4_\alpha X_1 (R_1 \partial_\alpha)(\partial^4_\alpha X_1) \, d\alpha.
\]

The first two integrals above are easily handled, allowing us to get

\[
O_1 = P_1 + P_2 + P_3 \leq P(\|X\|_4) - A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{N_3}{|\partial_\alpha X|^3} \partial^4_\alpha X_1 (R_1 \partial_\alpha)(\partial^4_\alpha X_1) \, d\alpha. \tag{7-13}
\]

For the term $O_2$, we proceed in a similar manner, first checking that $O_2 = P_4 + P_5 + P_6$:
We control $P_{6}$ as before. Regarding $P_{4}$, we use (7-7) to write it in the form $P_{4} = S_{1} + S_{2} + S_{3}$, where

$$S_{1} = -A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\partial X_{1}) \alpha_{1} X_{1} \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} d\alpha,$$

$$S_{2} = -A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\partial X_{1}) \alpha_{1} X_{1} \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} d\alpha,$$

$$S_{3} = -A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\text{lower-order terms}) \kappa_{1} \lambda_{1} d\alpha.$$

The identity (7-5) allows us to write $P_{5} = S_{4} + S_{5} + S_{6}$, where

$$S_{4} = A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\partial X_{1}) \alpha_{1} X_{1} \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} d\alpha,$$

$$S_{5} = A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\partial X_{1}) \alpha_{1} X_{1} \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} d\alpha,$$

$$S_{6} = A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\text{lower-order terms}) \kappa_{1} \lambda_{1} d\alpha.$$

Next, we reorganize the sum in the form

$$P_{4} + P_{6} = (S_{1} + S_{4}) + (S_{2} + S_{5}) + (S_{3} + S_{6}),$$

where the term $S_{3} + S_{6}$ can be easily estimated. Regarding $S_{1} + S_{4}$, we have

$$S_{1} + S_{4} = A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\partial X_{1}) \alpha_{1} X_{1} \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} d\alpha$$

$$+ A_{\rho} \text{PV} \int_{\mathbb{R}^2} \frac{\partial X_{2}}{\partial \alpha} \alpha \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} X_{1} \alpha_{1} X_{1} (R_{1} \partial X_{1}) (\partial X_{1}) \alpha_{1} X_{1} \beta_{1} \gamma_{1} \delta_{1} \epsilon_{1} \kappa_{1} \lambda_{1} d\alpha,$$

and the commutator estimates give us precise control.
Let us consider now

\[ S_2 + S_3 = A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1} X_1 \left[ \partial_{\alpha_2} X_2(R_1 \partial_{\alpha_1}) (\partial^4_{\alpha_1} X_2) - (R_1 \partial_{\alpha_1}) (\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2) \right] d\alpha \]

\[ + A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_2} X_1 \left[ (R_1 \partial_{\alpha_1}) (\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2) - \partial_{\alpha_1} X_2(R_1 \partial_{\alpha_1}) (\partial^4_{\alpha_1} X_2) \right] d\alpha \]

\[ - A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X_2(R_1 \partial_{\alpha_1}) (\partial^4_{\alpha_1} X_2) d\alpha. \]

Here again the commutator estimates control the first two integrals above, allowing us to conclude that

\[ O_2 = P_4 + P_5 + P_6 \leq P(||X||_4) - A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X_2(R_1 \partial_{\alpha_1}) (\partial^4_{\alpha_1} X_2) d\alpha. \] (7-14)

Furthermore, inequalities (7-13), (7-14) and (7-12) yield

\[ M_1 = O_1 + O_2 + O_3 \leq P(||X||_4) - A_\rho \text{PV} \int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X \cdot (R_1 \partial_{\alpha_1}) (\partial^4_{\alpha_1} X) d\alpha, \] (7-15)

and at this point we begin to recognize the Rayleigh–Taylor condition in the nonintegrable terms. Let us return now to the term \( M_2 \), which can be written in the form

\[ M_2 = A_\mu \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \partial^4_{\alpha_1} (BR(X, \omega) \cdot \partial_{\alpha_1} X) d\alpha, \] (7-16)

and whose most dangerous components are given by

\[ O_4 = -\frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) (\alpha) \int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge \omega(\beta) \cdot \partial_{\alpha_1} X(\alpha) d\alpha, \]

\[ O_5 = \frac{3A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) (\alpha) \int_{\mathbb{R}^2} B(\alpha, \beta)(X(\alpha) - X(\beta)) \wedge \omega(\beta) \cdot \partial_{\alpha_1} X(\alpha) d\alpha, \]

with

\[ B(\alpha, \beta) = \frac{(X(\alpha) - X(\beta)) \cdot (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_2} X(\beta))}{|X(\alpha) - X(\beta)|^5}. \]

\[ O_6 = -\frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) (\alpha) \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge \partial^4_{\alpha_1} \omega(\beta) \cdot \partial_{\alpha_1} X(\alpha) d\alpha, \]

and

\[ O_7 = A_\mu \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) (\alpha) \partial_{\alpha_1} (BR(X, \omega) \cdot \partial^4_{\alpha_1} X)(\alpha) d\alpha. \]

The remainder terms are less singular and can be estimated with the same methods used before.
To deal with $O_4$, we decompose it further as $O_4 = P_7 + P_8$:

$$P_7 = \frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right)(\alpha) \int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)}{|X(\alpha) - X(\beta)|^3} \cdot \omega(\beta) \wedge (\partial_{\alpha_1} X(\beta) - \partial_{\alpha_1} X(\alpha)) \, d\beta \, d\alpha,$$

$$P_8 = \frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right)(\alpha) \int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)}{|X(\alpha) - X(\beta)|^3} \cdot N(\beta)\partial_{\alpha_1} \Omega(\beta) \, d\beta \, d\alpha,$$

where in $P_8$, we have used formula (2-3) to get

$$\omega \wedge \partial_{\alpha_1} X = N\partial_{\alpha_1} \Omega.$$

In the integral (with respect to $\beta$) of $P_7$, we have a kernel of degree $-2$ applied to 4 derivatives, which can be estimated easily. Next let us consider $P_8 = Q_7 + Q_8 + Q_9$, where

$$Q_7 = -\frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right)(\alpha) \partial^4_{\alpha_1} X(\alpha) \cdot \int_{\mathbb{R}^2} \frac{N(\alpha)\partial_{\alpha_1} \Omega(\alpha) - N(\beta)\partial_{\alpha_1} \Omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\beta \, d\alpha,$$

$$Q_8 = \frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right)(\alpha) \int_{\mathbb{R}^2} (\partial_{\alpha_1} \Omega N \cdot \partial^4_{\alpha_1} X(\alpha) - (\partial_{\alpha_1} \Omega N \cdot \partial^4_{\alpha_1} X(\beta))) C(\alpha, \beta) \, d\beta \, d\alpha,$$

and

$$C(\alpha, \beta) = \frac{1}{|X(\alpha) - X(\beta)|^3} - \frac{1}{|
abla X(\alpha) - \nabla X(\beta)|^3},$$

$$Q_9 = \frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right)(\alpha) \frac{1}{|\partial_{\alpha_1} X(\alpha)|^3} \Lambda(\partial_{\alpha_1} \Omega N \cdot \partial^4_{\alpha_1} X(\alpha)) \, d\alpha.$$

In $Q_7$, we have

$$Q_7 \leq \left\| R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \right\|_{L^2} \|\partial^4_{\alpha_1} X\|_{L^2} \sup_{\alpha} \left| \int_{\mathbb{R}^2} \frac{N(\alpha)\partial_{\alpha_1} \Omega(\alpha) - N(\beta)\partial_{\alpha_1} \Omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\beta \right|,$$

giving us the appropriate control, which can be also obtained in $Q_8$ because the corresponding kernel has degree $-2$. Regarding $Q_9$, we have the expression

$$Q_9 = \frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \left[ \frac{1}{|\partial_{\alpha_1} X|^3} \Lambda(\partial_{\alpha_1} \Omega N \cdot \partial^4_{\alpha_1} X) - \Lambda \left( \frac{\partial_{\alpha_1} \Omega N \cdot \partial^4_{\alpha_1} X}{|\partial_{\alpha_1} X|^3} \right) \right] \, d\alpha$$

$$+ \frac{A_\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \Lambda \left( \frac{\partial_{\alpha_1} \Omega N}{|\partial_{\alpha_1} X|^3} \right) \, d\alpha.$$

Then we use (A-2) to control the first integral above, and since $\Lambda = R_1 \partial_{\alpha_1} + R_2 \partial_{\alpha_2}$ by (A-4), we can also take care of the second term.

With $O_5$, one proceeds as we did with $J_6$ (7-10) to get the desired estimate.
Next, we use (2.3) to catch the most singular terms in $O_6$, which are given by

$$S_7 = -\frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta)) \cdot \partial_{\alpha_1} X(\alpha)}{|X(\alpha) - X(\beta)|^3} \partial_{\alpha_1}^4 \partial_{\alpha_2} \Omega(\beta) \, d\alpha,$$

$$S_8 = -\frac{A_{\mu}}{8\pi^2} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta)) \cdot \partial_{\alpha_1} X(\alpha)}{|X(\alpha) - X(\beta)|^3} \cdot \partial_{\alpha_2} \Omega(\beta) \partial_{\alpha_1}^5 X(\beta) \, d\alpha,$$

$$S_9 = \frac{A_{\mu}}{8\pi^2} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta)) \cdot \partial_{\alpha_2} X(\beta) \cdot \partial_{\alpha_1} X(\alpha)}{|X(\alpha) - X(\beta)|^3} \partial_{\alpha_1}^5 \Omega(\beta) \, d\alpha,$$

$$S_{10} = \frac{A_{\mu}}{8\pi^2} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta)) \cdot \partial_{\alpha_2} X(\beta) \cdot \partial_{\alpha_1} X(\alpha)}{|X(\alpha) - X(\beta)|^3} \cdot \partial_{\alpha_1} \Omega(\beta) \partial_{\alpha_1}^4 \partial_{\alpha_2} X(\beta) \, d\alpha.$$

One may write

$$S_7 = \frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta)) \cdot \partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\beta)}{|X(\alpha) - X(\beta)|^3} \partial_{\alpha_1}^4 \partial_{\alpha_2} \Omega(\beta) \, d\alpha,$$

expressing the fact that we have a kernel of degree $-1$ applied to $\partial_{\alpha_1}^4 \partial_{\alpha_2} \Omega$, and therefore an integration by parts gives us the desired control, as before. To treat $S_8$, we further decompose $S_8 = T_1 + T_2$:

$$T_1 = -\frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \int_{\mathbb{R}^2} D(\alpha, \beta) \cdot \partial_{\alpha_2} \Omega(\beta) \partial_{\alpha_1}^5 X(\beta) \, d\alpha,$$

where

$$D(\alpha, \beta) = \frac{(X(\alpha) - X(\beta))}{|X(\alpha) - X(\beta)|^3} - \frac{\nabla X(\alpha) - \nabla X(\beta)}{|\nabla X(\alpha) - \nabla X(\beta)|^3} \cdot \partial_{\alpha_1} X(\alpha)$$

and

$$T_2 = \frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \alpha \frac{N(\alpha)}{|\partial_{\alpha_1} X(\alpha)|^3} \cdot R_2(\partial_{\alpha_2} \Omega \partial_{\alpha_1}^5 X(\alpha)) \, d\alpha.$$

In $T_1$, we use the estimate for the operator (A-8). The term $T_2$ reads as follows:

$$T_2 = -\frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \frac{N}{|\partial_{\alpha_1} X|^3} \cdot R_2(\partial_{\alpha_2} \partial_{\alpha_1} \Omega \partial_{\alpha_1}^4 X) \, d\alpha
+ \frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \frac{N}{|\partial_{\alpha_1} X|^3} \cdot (R_2 \partial_{\alpha_1})(\partial_{\alpha_2} \Omega \partial_{\alpha_1}^4 X) - (R_2 \partial_{\alpha_1})(\partial_{\alpha_2} \Omega \frac{N \cdot \partial_{\alpha_1}^4 X}{|\partial_{\alpha_1} X|^3}) \, d\alpha
- \frac{A_{\mu}}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial_{\alpha_1}^4 X \cdot N}{|\partial_{\alpha_1} X|^3} \right) (R_2 \partial_{\alpha_1})(\partial_{\alpha_2} \Omega \frac{N \cdot \partial_{\alpha_1}^4 X}{|\partial_{\alpha_1} X|^3}) \, d\alpha.$$
For the next term, one has $S_9 = T_3 + T_4$, where
\[ T_3 = \frac{A\mu}{4\pi} \text{PV} \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) (\alpha) \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\beta)) \cdot \partial_{\alpha_2} X(\beta) \wedge (\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\beta))}{|X(\alpha) - X(\beta)|^3} \partial^5_{\alpha_1} \Omega(\beta) \, d\alpha, \]
\[ T_4 = -A\mu \int_{\mathbb{R}^2} R_1 \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \mathcal{B}(\partial^5_{\alpha_1} \Omega) \, d\alpha, \]
Proceeding as before, we get bounds for $T_3$, and the double-layer potential estimates help us to control $T_4$.

For $S_{10}$, one can adapt exactly the same approach used for $S_8$. Finally, we have to deal with $O_7$, which is given by
\[ O_7 = -A\mu \text{PV} \int_{\mathbb{R}^2} \mathcal{B}(X, \omega) \cdot \partial^4_{\alpha_1} X(R_1 \partial_{\alpha_1}) \left( \frac{\partial^4_{\alpha_1} X \cdot N}{|\partial_{\alpha_1} X|^3} \right) \, d\alpha, \]
after an integration by parts. Let us introduce the splitting $O_7 = \sum_{j,k=1}^{3} U^j_k$, where
\[ U^j_k = -A\mu \text{PV} \int_{\mathbb{R}^2} \mathcal{B}(X, \omega) \partial^4_{\alpha_1} X_j(R_1 \partial_{\alpha_1}) \left( \frac{\partial^4_{\alpha_1} X_k N_k}{|\partial_{\alpha_1} X|^3} \right) \, d\alpha. \]
Then the commutator estimates allow us to write $U^j_k = V^j_k + \text{lower order terms}$, where
\[ V^j_k = -A\mu \text{PV} \int_{\mathbb{R}^2} \mathcal{B}(X, \omega) \partial^4_{\alpha_1} X_j N_k \left( R_1 \partial_{\alpha_1} \right) \left( \partial^4_{\alpha_1} X_k \right) \, d\alpha. \]
Using (7-5) and (7-7), one has
\[ N_1 \partial^4_{\alpha_1} X_2 = N_2 \partial^4_{\alpha_1} X_1 + \text{lower-order terms}, \]
so that $V^2_1$ becomes
\[ V^2_1 = -A\mu \text{PV} \int_{\mathbb{R}^2} \frac{\mathcal{B}_2(X, \omega) N_2}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X_1(R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) \, d\alpha - A\mu \text{PV} \int_{\mathbb{R}^2} f(R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) \, d\alpha, \]
where $f$ is at the level of $\partial^3_{\alpha_1} X$. Integration by parts in the last integral allows us to conclude that
\[ V^2_1 \leq -A\mu \text{PV} \int_{\mathbb{R}^2} \frac{\mathcal{B}_2(X, \omega) N_2}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X_1(R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) \, d\alpha + P(||X||_4). \]
With the help of (7-5) and (7-7), we also get
\[ N_1 \partial^4_{\alpha_1} X_3 = N_3 \partial^4_{\alpha_1} X_1 + \text{lower-order terms}, \]
and therefore
\[ V^3_1 \leq -A\mu \text{PV} \int_{\mathbb{R}^2} \frac{\mathcal{B}_3(X, \omega) N_3}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X_1(R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) \, d\alpha + P(||X||_4). \]
Using the two inequalities above, we obtain
\[ V^1_1 + V^2_1 + V^3_1 \leq -A\mu \text{PV} \int_{\mathbb{R}^2} \frac{\mathcal{B}(X, \omega) \cdot N}{|\partial_{\alpha_1} X|^3} \partial^4_{\alpha_1} X_1(R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) \, d\alpha + P(||X||_4). \]
Next, let us observe that
\[ N_2 \partial_{\alpha_1}^4 X_1 = N_1 \partial_{\alpha_1}^4 X_2 + \text{lower-order terms}, \quad N_2 \partial_{\alpha_1}^4 X_3 = N_3 \partial_{\alpha_1}^4 X_2 + \text{lower-order terms}, \]
which implies the estimate
\[
V_1^2 + V_2^2 + V_3^2 \leq -A_\mu \text{PV} \int_{\mathbb{R}^2} \frac{\text{BR}(X, \omega) \cdot N}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X_2 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_2) \, d\alpha + P(||X||_4). \tag{7-18}
\]

Regarding \( V_1^3 \) and \( V_2^3 \), the identities
\[ N_3 \partial_{\alpha_1}^4 X_1 = N_1 \partial_{\alpha_1}^4 X_3 + \text{lower-order terms}, \quad N_3 \partial_{\alpha_1}^4 X_3 = N_2 \partial_{\alpha_1}^4 X_3 + \text{lower-order terms} \]
yield
\[
V_1^3 + V_2^3 + V_3^3 \leq -A_\mu \text{PV} \int_{\mathbb{R}^2} \frac{\text{BR}(X, \omega) \cdot N}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X_3 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3) \, d\alpha + P(||X||_4). \tag{7-19}
\]

Finally (7-17), (7-18) and (7-19) imply
\[
\sum_{j,k=1}^{3} V_j^k \leq -A_\mu \text{PV} \int_{\mathbb{R}^2} \frac{\text{BR}(X, \omega) \cdot N}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X \cdot (R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X) \, d\alpha + P(||X||_4).
\]

Now we put together the estimates (7-16)–(7-19) to conclude that
\[
M_2 \leq -A_\mu \text{PV} \int_{\mathbb{R}^2} \frac{\text{BR}(X, \omega) \cdot N}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X \cdot (R_1 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X) \, d\alpha + P(||X||_4),
\]
and taking into account (7-15), we obtain
\[
\tilde{L}_3 = M_1 + M_2 \leq -\frac{1}{\mu_2 + \mu_1} \text{PV} \int_{\mathbb{R}^2} \frac{\sigma}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X \cdot (R_2 \partial_{\alpha_1})(\partial_{\alpha_1}^4 X) \, d\alpha + P(||X||_4). \tag{7-20}
\]

Finally, we have to work with \( L_5 \), which can be written in the following manner:
\[
L_5 = \tilde{L}_5 - \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X \cdot \frac{\partial_{\alpha_2} X}{|\partial_{\alpha_2} X|^3} \wedge [R_2(\partial_{\alpha_1}^4 \partial_{\alpha_2} \Omega \partial_{\alpha_1} X) - R_2(\partial_{\alpha_1}^4 \partial_{\alpha_2} \Omega) \partial_{\alpha_1} X] \, d\alpha,
\]
where
\[
\tilde{L}_5 = \frac{1}{2} \text{PV} \int_{\mathbb{R}^2} \partial_{\alpha_1}^4 X \cdot N |\partial_{\alpha_2} X|^3 (R_2 \partial_{\alpha_2})(\partial_{\alpha_1}^4 \Omega) \, d\alpha.
\]

Using the commutator estimate, once more, it remains only to consider \( \tilde{L}_5 \), but let us point out that replacing the operator \( R_1 \partial_{\alpha_1} \) by \( R_2 \partial_{\alpha_2} \), the term \( \tilde{L}_3 \) (7-11) becomes \( \tilde{L}_5 \). Therefore, proceeding exactly as we did before, one obtains the inequality
\[
\tilde{L}_5 \leq -\frac{1}{\mu_2 + \mu_1} \text{PV} \int_{\mathbb{R}^2} \frac{\sigma}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X \cdot (R_2 \partial_{\alpha_2})(\partial_{\alpha_1}^4 X) \, d\alpha + P(||X||_4). \tag{7-21}
\]

Introducing now the identity \( \Lambda = (R_1 \partial_{\alpha_1}) + (R_2 \partial_{\alpha_2}) \) in (7-20) and (7-21), we get
\[
\tilde{L}_3 + \tilde{L}_5 \leq -\frac{1}{\mu_2 + \mu_1} \text{PV} \int_{\mathbb{R}^2} \frac{\sigma}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X \cdot \Lambda (\partial_{\alpha_1}^4 X) \, d\alpha + P(||X||_4).
\]
Finally, all the estimates so far obtained, beginning with (7-9), allow us to write

\[
\frac{1}{2} \frac{d}{dt} \| \partial_{\alpha_1}^4 X \|_{L^2}^2(t) \leq - \frac{1}{\mu_2 + \mu_1} \text{PV} \int_{\mathbb{R}^2} \frac{\sigma}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_1}^4 X \cdot \Lambda(\partial_{\alpha_1}^4 X) \, d\alpha + P(\|X\|_4). \tag{7-22}
\]

In a similar manner, now using equations (2-9), (7-6) and (7-8) instead of (2-8), (7-5) and (7-7) respectively, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \partial_{\alpha_2}^4 X \|_{L^2}^2(t) \leq - \frac{1}{\mu_2 + \mu_1} \text{PV} \int_{\mathbb{R}^2} \frac{\sigma}{|\partial_{\alpha_1} X|^3} \partial_{\alpha_2}^4 X \cdot \Lambda(\partial_{\alpha_2}^4 X) \, d\alpha + P(\|X\|_4). \tag{7-23}
\]

The two inequalities (7-22) and (7-23) are the main purpose of this section.

8. Estimates for the evolution of \( \|F(X)\|_{L^\infty} \) and R-T

In this section we analyze the evolution of the no-self-intersection condition of the free surface as well as the Rayleigh–Taylor property, but in order to do that, we shall need precise bounds for both \( \nabla X_t \) and \( \Omega_t \).

We shall estimate \( \|\nabla X_t\|_{H^k} \) by means of equality (2-4) to get

\[
\|\nabla X_t\|_{H^k} \leq P\left(\|X\|_{k+2}^2 + \|F(X)\|_{L^\infty}^2 + \|N\|^{-1}_{L^\infty}\right). \tag{8-1}
\]

for \( k \geq 2 \). In fact

\[
\|\nabla X_t\|_{H^k} \leq \|\nabla BR(X, \omega)\|_{H^k} + \|\nabla (C_1 \partial_{\alpha_1} X + C_2 \partial_{\alpha_2} X)\|_{H^k},
\]

and with the help of (6-1), we can handle both terms on the right.

Next we shall consider the norms \( \|\Omega_t\|_{H^k} \) to obtain the inequality

\[
\|\Omega_t\|_{H^k} \leq P\left(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \|N\|^{-1}_{L^\infty}\right), \tag{8-2}
\]

for \( k \geq 3 \). To do that, let us take a time derivative in the identity (2-6) to get

\[
\Omega_t(\alpha, t) - A_\mu \mathcal{D}(\Omega_t)(\alpha, t) = A_\mu I_1(\alpha, t) - 2A_\rho \partial_t X_3(\alpha, t),
\]

which yields

\[
\|\partial_t\|_{H^1} \leq C\|I - A_\mu \mathcal{D}\|^{-1}_{H^1}\left(\|I_1\|_{H^1} + \|\partial_t X_3\|_{H^1}\right),
\]

and since we have control of \( \|I - A_\mu \mathcal{D}\|^{-1}_{H^1} \) and \( \|\partial_t X_3\|_{H^1} \), it only remains to estimate \( \|I_1\|_{H^1} \). For that purpose, let us consider the splitting \( I_1 = J_1 + J_2 + J_3 \), where

\[
J_1 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X_t(\alpha) - X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot N(\alpha - \beta) \Omega(\alpha - \beta) \, d\beta,
\]

\[
J_2 = -\frac{3}{4\pi} \int_{\mathbb{R}^2} \left(\frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3}\right) \cdot \frac{X_t(\alpha) - X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^5} \cdot N(\alpha - \beta) \Omega(\alpha - \beta) \, d\beta,
\]

\[
J_3 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot N_t(\alpha - \beta) \Omega(\alpha - \beta) \, d\beta.
\]

Proceeding as we did with the operator \( \mathcal{T}_2 \) (A-6) (with \( X_t \) instead of \( \partial_{\alpha_j} X_k \)), one gets

\[
\|J_1\|_{L^2} + \|J_2\|_{L^2} \leq P\left(\|X\|_4 + \|F(X)\|_{L^\infty} + \|N\|^{-1}_{L^\infty}\right).
\]
Regarding $J_3$, we split further:

$$J_3 = \frac{1}{2\pi} \int_{|\beta|>1} d\beta + \frac{1}{2\pi} \int_{|\beta|<1} d\beta = K_1 + K_2.$$ 

Since

$$|K_1(\alpha)| \leq \|F(X)\|_{L^\infty}^2 \int_{|\beta|>1} \frac{|N_\mu(\alpha - \beta)| |\Omega(\alpha - \beta)|}{2\pi |\beta|^2} d\beta,$$

Young’s inequality yields

$$\|K_1\|_{L^2} \leq \|F(X)\|_{L^\infty}^2 \|N_\mu \Omega\|_{L^1} \leq C\|F(X)\|_{L^\infty}^2 \|N_\mu\|_{L^2} \|\Omega\|_{L^2},$$

and since we know that $\|N_\mu\|_{L^2} \leq \|\nabla X\|_{L^\infty} \|\nabla X_t\|_{L^2}$, estimate (8-1) allows us to handle the terms $K_1$. The estimate for $K_2$ is similar to the one obtained for $I_2$ (A-13) in the Appendix.

Next we consider the most singular terms in $\partial_{\alpha_t} I_1$, which are given by

$$J_4 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_t} X_t(\alpha) - \partial_{\alpha_t} X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot N(\alpha - \beta) \Omega(\alpha - \beta) d\beta,$$

$$J_5 = -\frac{3}{4\pi} \int_{\mathbb{R}^2} (X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_t} X_t(\alpha) - \partial_{\alpha_t} X_t(\alpha - \beta)) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^5} \cdot N(\alpha - \beta) \Omega(\alpha - \beta) d\beta,$$

$$J_6 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot N(\partial_{\alpha_t} X_t(\alpha - \beta) \Omega(\alpha - \beta) d\beta,$$

because the remainder terms are easier to handle. Let us write $J_4 = K_3 + K_4$, where

$$K_3 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_t} X_t(\alpha) - \partial_{\alpha_t} X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot (N(\alpha - \beta) \Omega(\alpha - \beta) - N(\alpha) \Omega(\alpha)) d\beta,$$

$$K_4 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_t} X_t(\alpha) - \partial_{\alpha_t} X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot N(\alpha) \Omega(\alpha) d\beta.$$

In $K_3$, the identity $\partial_{\alpha_t} X_t(\alpha) - \partial_{\alpha_t} X_t(\alpha - \beta) = \int_0^1 \nabla \partial_{\alpha_t} X_t(\alpha + (s - 1)\beta) \cdot ds \cdot \beta$ together with (8-1) gives us the desired control. Regarding $K_4$, we may observe its similarity with $\mathcal{T}_3$ (A-7), so that an application to (8-1) yields the appropriate bound; $J_5$ can be treated in a similar manner, and $J_6$ is analogous to $J_3$. By symmetry, one could get the same estimate for $\partial_{\alpha_v} I_1$, so that finally

$$\|\Omega_t\|_{H^1} \leq P(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \|N\|^{-1}_{L^\infty}).$$

(8-3)

Next, we will show how to deal with $\|\Omega_t\|_{H^2}$. Using Equation (2-8), one gets

$$\partial_{\alpha_t}^2 \Omega_t = -2A_\mu \partial_{\alpha_t} \partial_t (B \mathbf{R}(X, \omega) \cdot \partial_{\alpha_t} X) - 2A_\mu \partial_{\alpha_t}^2 \partial_t X,$$
and with the help of (8-1), the last term above is properly controlled. To continue, we shall consider the most singular remainder terms. Namely, in \(-\partial_{\alpha_1} \partial_t (BR(X, \omega) \cdot \partial_{\alpha_1} X)\), we have

\[
L_1 = -BR(X, \omega) \cdot \partial_{\alpha_1}^2 X_t,
\]

\[
L_2 = \frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1} X_t(\alpha) - \partial_{\alpha_1} X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

\[
L_3 = -\frac{3}{8\pi} \text{PV} \int_{\mathbb{R}^2} A(\alpha, \beta) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

where \(A(\alpha, \beta) = (X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_1} X_t(\alpha) - \partial_{\alpha_1} X_t(\alpha - \beta))\).

Let us observe that \(\|L_1\|_{L^2} \leq \|BR(X, \omega)\|_{L^\infty} \|\partial_{\alpha_1}^2 X_t\|_{L^2}\), where both quantities have been appropriately controlled before. In \(L_2\) and \(L_3\), we have kernels of degree \(-2\), and therefore operators analogous to \(T_3\) (A-7) acting on \(\partial_{\alpha_1} X_t\). Therefore, using (8-1), its control follows easily. In \(L_4\), we use the decomposition

\[
L_4 = \frac{1}{2\pi} \text{PV} \int_{|\beta|<1} d\beta + \frac{1}{2\pi} \text{PV} \int_{|\beta|>1} d\beta = M_1 + M_2.
\]

Thus, an integration by parts yields

\[
\|M_1\|_{L^2} \leq C \|F(X)\|_{L^\infty} \|\nabla X\|^2_{L^\infty} \|w_t\|_{L^2}.
\]

Formula (2-3), together with estimates (8-1) and (8-3), provides the appropriate bound.

Next, let us expand (2-3) to obtain the most singular terms in \(M_2\), which are given by the integrals

\[
O_1 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_2} \Omega(\alpha - \beta) \partial_{\alpha_1}^2 X_t(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

\[
O_2 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \partial_{\alpha_2} \Omega_t(\alpha - \beta) \partial_{\alpha_1} X(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

\[
O_3 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \Omega(\alpha - \beta) \partial_{\alpha_1} \partial_{\alpha_2} X_t(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

\[
O_4 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1}^2 \Omega_t(\alpha - \beta) \partial_{\alpha_1} X(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha).
\]

Estimate (8-1) help us with the terms \(O_1\) and \(O_3\), which can be treated with the same approach used for \(I_2\) (A-13) in the Appendix. Let us write \(O_2\) as

\[
O_2 = \frac{A_\mu}{2\pi} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1} \partial_{\alpha_2} \Omega_t(\alpha - \beta) \left(\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha - \beta)\right) \, d\beta \cdot \partial_{\alpha_1} X(\alpha),
\]

which can be estimated integrating by parts in the variable \(\beta_1\) using the identity

\[
\partial_{\alpha_1} \partial_{\alpha_2} \Omega_t(\alpha - \beta) = -\partial_{\beta_1} (\partial_{\alpha_2} \Omega_t(\alpha - \beta)).
\]
Let us point out that the kernel in the integral $O_2$ has degree $-1$, and therefore one can use (8-3) to control it. It remains to deal with $O_4$, which is decomposed in the form $O_4 = P_1 + P_2$, where

\[ P_1 = \frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1}^2 \Omega_l(\alpha - \beta)(\partial_{\alpha_2} X(\alpha - \beta) - \partial_{\alpha_2} X(\alpha)) \, d\beta \cdot \partial_{\alpha_1} X(\alpha), \]

\[ P_2 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \partial_{\alpha_1}^2 \Omega_l(\alpha - \beta) \, d\beta \cdot N(\alpha). \]

$P_1$ is estimated like $O_2$. We rewrite $P_2$ as follows:

\[ P_2 = -\frac{A_\mu}{2\pi} \text{PV} \int_{|\beta| < 1} \left( \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} - \frac{\nabla X(\alpha) \cdot \beta}{|\nabla X(\alpha) \cdot \beta|^3} \right) \partial_{\alpha_1}^2 \Omega_l(\alpha - \beta) \, d\beta \cdot N(\alpha), \]

and this expression shows that the above integral can be estimated like $T_4$ (A-8).

Using (8-3), we obtain

\[ \| \partial_{\alpha_1}^2 \Omega_l \|_{L^2} \leq P \left( \| X \|_4^2 + \| F(X) \|_{L^\infty}^2 + \| N \|_{L^\infty}^{-1} \right), \]

and the identity

\[ \partial_{\alpha_2}^2 \Omega_l = -2A_\mu \partial_{\alpha_2} \partial_t (BR(X, \omega) \cdot \partial_{\alpha_2} X) - 2A_\rho \partial_{\alpha_2} \partial_t X \]

yields

\[ \| \partial_{\alpha_2}^2 \Omega_l \|_{L^2} \leq P \left( \| X \|_4^2 + \| F(X) \|_{L^\infty}^2 + \| N \|_{L^\infty}^{-1} \right), \]

that is,

\[ \| \Omega_l \|_{H^2} \leq P \left( \| X \|_4^2 + \| F(X) \|_{L^\infty}^2 + \| N \|_{L^\infty}^{-1} \right). \]

Next we consider third-order derivatives:

\[ \partial_{\alpha_1}^3 \Omega_l = -2A_\mu \partial_{\alpha_1}^2 \partial_t (BR(X, \omega) \cdot \partial_{\alpha_1} X) - 2A_\rho \partial_{\alpha_1}^3 \partial_t X. \]

Since (8-1) gives us control of the last term, we will concentrate on the other one, which is of a much more difficult character. In particular, for $-\partial_{\alpha_1}^2 \partial_t (BR(X, \omega) \cdot \partial_{\alpha_1} X)$, the most singular components are given by

\[ L_5 = -BR(X, \omega) \cdot \partial_{\alpha_1}^3 X_t, \]

\[ L_6 = \frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^2 X_t(\alpha) - \partial_{\alpha_1}^2 X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha), \]

\[ L_7 = -\frac{3}{8\pi} \text{PV} \int_{\mathbb{R}^2} B(\alpha, \beta) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^5} \wedge \omega(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha), \]

where $B(\alpha, \beta) = (X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_1}^2 X_t(\alpha) - \partial_{\alpha_1}^2 X_t(\alpha - \beta))$.

\[ L_8 = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \partial_{\alpha_1}^2 \omega_l(\alpha - \beta) \, d\beta \cdot \partial_{\alpha_1} X(\alpha). \]

Inequalities (8-1) and (8-4) show how to handle $L_i$, $i = 5, \ldots, 8$ as $L_j$, $j = 1, \ldots, 4$ respectively, and then a similar approach for $\partial_{\alpha_2}^3 \Omega_l$ allows us to get finally (8-2) for $k = 3$. The cases $k > 3$ are similar to deal with.
Our next goal is to obtain estimates for the evolution of \( \| F(X) \|_{L^\infty} \) and R-T. Regarding the quantity \( F(X) \), we have

\[
\frac{d}{dt} F(X)(\alpha, \beta, t) = -\frac{\beta}{\|X(\alpha, t) - X(\alpha - \beta, t)\|^3} \cdot (X_t(\alpha, t) - X_t(\alpha - \beta, t))
\leq (F(X)(\alpha, \beta, t))^2 \| \nabla X_t \|_{L^\infty}(t).
\]

Then Sobolev inequalities in \( \| \nabla X_t \|_{L^\infty}(t) \), together with (8-1), yield

\[
\frac{d}{dt} F(X)(\alpha, \beta, t) \leq F(X)(\alpha, \beta, t) P\left( \|X\|^2_4(t) + \|F(X)\|^2_{L^\infty}(t) + \|N\|^{-1}_{L^\infty}(t) \right),
\]

and an integration in time gives us

\[
F(X)(\alpha, \beta, t + h) \leq F(X)(\alpha, \beta, t) \exp\left( \int_t^{t+h} P(s) \, ds \right),
\]

for \( h > 0 \), where

\[
P(s) = P\left( \|X\|^2_4(s) + \|F(X)\|^2_{L^\infty}(s) + \|N\|^{-1}_{L^\infty}(s) \right).
\]

Hence

\[
\| F(X) \|_{L^\infty}(t + h) \leq \| F(X) \|_{L^\infty}(t) \exp\left( \int_t^{t+h} P(s) \, ds \right).
\]

This inequality, applied to the limit

\[
\frac{d}{dt} \| F(X) \|_{L^\infty}(t) = \lim_{h \to 0^+} \frac{\| F(X) \|_{L^\infty}(t + h) - \| F(X) \|_{L^\infty}(t)}{h},
\]

allows us to get

\[
\frac{d}{dt} \| F(X) \|_{L^\infty}(t) \leq \| F(X) \|_{L^\infty}(t) P\left( \|X\|^2_4 + \|F(X)\|^2_{L^\infty} + \|N\|^{-1}_{L^\infty} \right).
\]

Next we search for an a priori estimate for the evolution of the infimum of the difference of the gradients of the pressure in the normal direction to the interface. Let us recall the formula

\[
\sigma(\alpha, t) = (\mu^2 - \mu^1) \text{BR}(X, \omega)(\alpha, t) \cdot N(\alpha, t) + (\rho^2 - \rho^1) N_3(\alpha, t)
\]

to obtain

\[
\frac{d}{dt} \left( \frac{1}{\sigma(\alpha, t)} \right) = -\frac{\sigma_t(\alpha, t)}{\sigma^2(\alpha, t)},
\]

with \( \sigma_t(\alpha, t) = I_1 + I_2 \), where

\[
I_1 = ((\mu^2 - \mu^1) \text{BR}(X, \omega)(\alpha, t) + (\rho^2 - \rho^1)(0, 0, 1)) \cdot N_t(\alpha, t),
\]
\[
I_2 = (\mu^2 - \mu^1) \text{BR}_t(\omega)(\alpha, t) \cdot N(\alpha, t).
\]
First we deal with $\|I_1\|_{L^\infty}$ using the estimates (8-1) for $\nabla X_t$, and then we focus our attention on $I_2$ using the splitting $I_2 = J_1 + J_2 + J_3$, where

$$J_1 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X_t(\alpha) - X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta,$$

$$J_2 = \frac{3}{4\pi} \text{PV} \int_{\mathbb{R}^2} (X(\alpha) - X(\alpha - \beta)) \wedge \omega(\alpha - \beta) \frac{(X(\alpha) - X(\alpha - \beta)) \cdot (X_t(\alpha) - X_t(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^5} \, d\beta,$$

$$J_3 = -\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega_t(\alpha - \beta) \, d\beta.$$

The terms $J_1$ and $J_2$ are similar and can be treated with the same method. Let us consider $J_1 = K_1 + K_2 + K_3 + K_4$, where

$$K_1 = -\frac{1}{4\pi} \int_{|\beta| > 1} \frac{X_t(\alpha) - X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega(\alpha - \beta) \, d\beta,$$

$$K_2 = \frac{1}{4\pi} \int_{|\beta| < 1} \frac{X_t(\alpha) - X_t(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge (\omega(\alpha) - \omega(\alpha - \beta)) \, d\beta,$$

$$K_3 = -\frac{1}{4\pi} \int_{|\beta| < 1} \left[ \frac{1}{|X(\alpha) - X(\alpha - \beta)|^3} - \frac{1}{|\nabla X(\alpha) \cdot \beta|^3} \right] (X_t(\alpha) - X_t(\alpha - \beta)) \wedge \omega(\alpha) \, d\beta,$$

$$K_4 = -\frac{1}{4\pi} \text{PV} \int_{|\beta| < 1} \frac{X_t(\alpha) - X_t(\alpha - \beta)}{|\nabla X(\alpha) \cdot \beta|^3} \wedge \omega(\alpha) \, d\beta.$$

First we have

$$\|K_1\|_{L^\infty} \leq C \|F(X)\|_{L^3}^3 \|\nabla X_t\|_{L^\infty} \|\omega\|_{L^2} \left( \int_{|\beta| > 1} |\beta|^{-4} \, d\beta \right)^{1/2},$$

giving us an appropriate control. Next, we get

$$\|K_2\|_{L^\infty} \leq C \|F(X)\|_{L^\infty}^3 \|\nabla X_t\|_{L^\infty} \|\omega\|_{L^\infty} \int_{|\beta| < 1} |\beta|^{-1} \, d\beta,$$

and an analogous estimate for $K_3$. Therefore, Sobolev’s embedding helps us to obtain the desired control. Regarding $K_4$, we have

$$K_4 = -\frac{1}{4\pi} \int_{|\beta| < 1} \frac{X_t(\alpha) - X_t(\alpha - \beta) - \nabla X_t(\alpha) \cdot \beta}{|\nabla X(\alpha) \cdot \beta|^3} \wedge \omega(\alpha) \, d\beta.$$

Inequality (A-15) yields

$$\|K_4\|_{L^\infty} \leq C \|\nabla X\|_{L^\infty}^3 \|N\|_{L^\infty}^{-1} \|\nabla X_t\|_{L^\infty} \|\omega\|_{L^\infty} \|\nabla X_t\|_{C^3} \int_{|\beta| < 1} |\beta|^{-2+\delta} \, d\beta,$$
and the control $\|\nabla X_t\|_{C^\delta}$ follows again by (8-1) and Sobolev’s embedding. Next let us continue with $J_3 = K_5 + K_6$, where

$$K_5 = - \frac{1}{4\pi} \text{PV} \int_{|\beta|>1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge (\partial_{\beta_1}((\Omega \partial_{\alpha_2} X)_t(\alpha - \beta)) - \partial_{\beta_2}((\Omega \partial_{\alpha_1} X)_t(\alpha - \beta))) \, d\beta,$$

$$K_6 = - \frac{1}{4\pi} \text{PV} \int_{|\beta|<1} \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \wedge \omega_t(\alpha - \beta) \, d\beta.$$ 

Integration by parts yields

$$\|K_5\|_{L^\infty} \leq C \|F(X)\|_{L^3}^3 \|\nabla X\|_{L^\infty} (\|\Omega\|_{L^\infty} \|\nabla X_t\|_{L^\infty} + \|\Omega_t\|_{L^\infty} \|\nabla X\|_{L^\infty}),$$

where $4\pi C = \int_{|\beta|>1} |\beta|^{-3} \, d\beta + \int_{|\beta|=1} dl(\beta)$, and we may use (8-2) to estimate $\|\Omega_t\|_{L^\infty}$. With $K_6$, we introduce a similar splitting to obtain

$$\|K_6\|_{L^\infty} \leq P(\|X - (\alpha, 0)\|_{C^2} + \|F(X)\|_{L^\infty} + \|N\|^{-1}_{L^\infty}) \|\omega_t\|_{C^\delta}.$$ 

Then it remains to estimate $\|\omega_t\|_{C^\delta}$, for which purpose we use formula (2-3) and inequalities (8-1), (8-2).

Therefore, we have the estimate

$$\frac{d}{dt} \left( \frac{1}{\sigma(\alpha, t)} \right) \leq \frac{1}{\sigma^2(\alpha, t)} \frac{d}{dt} \left[ \|X\|_{4(t)} + \|F(X)\|_{L^\infty(t)} + \|N\|^{-1}_{L^\infty(t)} \right],$$

and proceeding similarly as we did for $F(X)$, we finally get

$$\frac{d}{dt} \|\sigma^{-1}\|_{L^\infty(t)} \leq \|\sigma^{-1}\|_{L^\infty(t)}^2 \frac{d}{dt} \left[ \|X\|_{4(t)} + \|F(X)\|_{L^\infty(t)} + \|N\|^{-1}_{L^\infty(t)} \right].$$

**Remark 8.1.** Having obtained the a priori bounds of the preceding sections, we are in position to successfully implement the same approximation scheme developed in [Córdoba et al. 2011] to conclude local existence.

**Appendix**

Here we prove first some helpful inequalities regarding commutators of the Riesz transform ($R_j$, $j = 1, 2$) with several differential operators. Next we analyze the singular integral operators associated to the non-self-intersecting surface which appears throughout the paper. But the main goal of this section is to simplify the presentation of the main result.

**Lemma A.1.** Consider $f \in L^2(\mathbb{R}^2)$ and $g \in C^{1,\delta}(\mathbb{R}^2)$, with $0 < \delta < 1$. Then for any $k, l = 1, 2$, we have the estimate

$$\|(R_k \partial_{\alpha_l})(gf) - g(R_k \partial_{\alpha_l})(f)\|_{L^2} \leq C \|g\|_{C^{1,\delta}} \|f\|_{L^2}. \quad \text{(A-1)}$$

An application of these inequalities to the operator $\Lambda = (R_1 \partial_{\alpha_1}) + (R_2 \partial_{\alpha_2})$ yields

$$\|\Lambda(gf) - g\Lambda(f)\|_{L^2} \leq C \|g\|_{C^{1,\delta}} \|f\|_{L^2}. \quad \text{(A-2)}$$

For vector fields, we have:
Lemma A.2. Consider \( f, g : \mathbb{R}^2 \to \mathbb{R}^3 \) vector fields, where \( f \in L^2(\mathbb{R}^2) \) and \( g \in C^{1,\delta}(\mathbb{R}^2) \), with \( 0 < \delta < 1 \). Then for any \( k, l = 1, 2 \), the following inequality holds:

\[
\left| \int_{\mathbb{R}^2} (g \wedge f) \cdot (R_k \partial_{\alpha l})(f) \, d\alpha \right| \leq C \| g \|_{C^{1,\delta}} \| f \|_{L^2}^2. \tag{A-3}
\]

Proof. Denoting by \( I \) the integral above, and since the operator \( R_k \partial_{\alpha l} \) is self-adjoint, we may write

\[
I = \int_{\mathbb{R}^2} f_1 \left[ (R_k \partial_{\alpha l})(g_2 f_3) - g_2 (R_k \partial_{\alpha l})(f_3) \right] \, d\alpha
+ \int_{\mathbb{R}^2} f_2 \left[ (R_k \partial_{\alpha l})(g_3 f_1) - g_3 (R_k \partial_{\alpha l})(f_1) \right] \, d\alpha
+ \int_{\mathbb{R}^2} f_3 \left[ (R_k \partial_{\alpha l})(g_1 f_2) - g_1 (R_k \partial_{\alpha l})(f_2) \right] \, d\alpha.
\]

Then estimate (A-1) yields (A-3). \( \square \)

Lemma A.3. Consider \( f \in L^2(\mathbb{R}^2) \) and \( g \in C^{1,\delta}(\mathbb{R}^2) \), with \( 0 < \delta < 1 \). Then for any \( j, k, l = 1, 2 \), the following inequality holds:

\[
\left| \int_{\mathbb{R}^2} R_j(f)(R_k \partial_{\alpha l})(g f) \, d\alpha \right| \leq C \| g \|_{C^{1,\delta}} \| f \|_{L^2}^2. \tag{A-4}
\]

Proof. Let \( J \) be the integral to be bounded; then we have

\[
J = \int_{\mathbb{R}^2} R_j(f) \left[ (R_k \partial_{\alpha l})(g f) - g (R_k \partial_{\alpha l})(f) \right] \, d\alpha
- \int_{\mathbb{R}^2} \left[ R_j(f g) - g R_j(f) \right] (R_k \partial_{\alpha l})(f) \, d\alpha + \int_{\mathbb{R}^2} R_j(f g)(R_k \partial_{\alpha l})(f) \, d\alpha.
\]

Since \( R_j^* = -R_j \) and \( R_k \partial_{\alpha l} \) is self-adjoint, we get

\[
J = \frac{1}{2} \int_{\mathbb{R}^2} R_j(f) \left[ (R_k \partial_{\alpha l})(g f) - g (R_k \partial_{\alpha l})(f) \right] \, d\alpha - \frac{1}{2} \int_{\mathbb{R}^2} \left[ R_j(f g) - g R_j(f) \right] (R_k \partial_{\alpha l})(f) \, d\alpha.
\]

An integration by parts in the second integral above yields

\[
J = \frac{1}{2} \int_{\mathbb{R}^2} R_j(f) \left[ (R_k \partial_{\alpha l})(g f) - g (R_k \partial_{\alpha l})(f) \right] \, d\alpha
+ \frac{1}{2} \int_{\mathbb{R}^2} \left[ (R_j \partial_{\alpha l})(f g) - g (R_j \partial_{\alpha l})(f) \right] (R_k)(f) \, d\alpha - \frac{1}{2} \int_{\mathbb{R}^2} (\partial_{\alpha l} g) R_j(f) R_k(f) \, d\alpha,
\]

allowing us to conclude the proof. \( \square \)
Lemma A.4. Let us define, for any \( j = 1, 2 \) and \( k = 1, 2, 3 \), the following operators:

\[
\mathcal{T}_1(\partial_{\alpha_j} f)(\alpha) = \text{PV} \int_{\mathbb{R}^2} \frac{X_k(\alpha) - X_k(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \partial_{\alpha_j} f(\alpha - \beta) \, d\beta, \quad (A-5)
\]

\[
\mathcal{T}_2(f)(\alpha) = \text{PV} \int_{\mathbb{R}^2} \frac{\partial_{\alpha_j} X_k(\alpha) - \partial_{\alpha_j} X_k(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} f(\alpha - \beta) \, d\beta, \quad (A-6)
\]

\[
\mathcal{T}_3(f)(\alpha) = \text{PV} \int_{\mathbb{R}^2} \frac{f(\alpha) - f(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta, \quad (A-7)
\]

\[
\mathcal{T}_4(\partial_{\alpha_j} f)(\alpha) = \text{PV} \int_{\mathbb{R}^2} \left( \frac{(X(\alpha) - X(\beta))}{|X(\alpha) - X(\beta)|^3} \cdot \frac{\nabla X(\alpha) \cdot (\alpha - \beta)}{|\nabla X(\alpha) \cdot (\alpha - \beta)|^3} \right) \partial_{\alpha_j} f(\beta) \, d\beta \, d\alpha, \quad (A-8)
\]

where \( \nabla X(\alpha) \cdot \beta = \partial_{\alpha_1} X(\alpha) \beta_1 + \partial_{\alpha_2} X(\alpha) \beta_2 \). Assume that \( X(\alpha) \in C^{2,\delta}(\mathbb{R}^2) \), and that both \( F(X) \) and \( |N|^{-1} \) are in \( L^\infty \), where

\[
F(X)(\alpha, \beta) = |\beta|/|X(\alpha) - X(\alpha - \beta)| \quad \text{and} \quad N(\alpha) = \partial_{\alpha_1} X(\alpha) \wedge \partial_{\alpha_2} X(\alpha).
\]

Then the following estimates hold:

\[
\|\mathcal{T}_1(\partial_{\alpha_j} f)\|_{L^2} \leq P \left( \| X - (\alpha, 0) \|_{C^{1,\delta}} + \| F(X) \|_{L^\infty} + \| |N|^{-1} \|_{L^\infty} \right) (\| f \|_{L^2} + \| \partial_{\alpha_j} f \|_{L^2}), \quad (A-9)
\]

\[
\|\mathcal{T}_2(f)\|_{L^2} \leq P \left( \| X - (\alpha, 0) \|_{C^{2,\delta}} + \| F(X) \|_{L^\infty} + \| |N|^{-1} \|_{L^\infty} \right) \| f \|_{L^2}, \quad (A-10)
\]

\[
\|\mathcal{T}_3(f)\|_{L^2} \leq P \left( \| X - (\alpha, 0) \|_{C^{2,\delta}} + \| F(X) \|_{L^\infty} + \| |N|^{-1} \|_{L^\infty} \right) \| f \|_{H^1}, \quad (A-11)
\]

\[
\|\mathcal{T}_4(f)\|_{L^2} \leq P \left( \| X - (\alpha, 0) \|_{C^{2,\delta}} + \| F(X) \|_{L^\infty} + \| |N|^{-1} \|_{L^\infty} \right) \| f \|_{L^2}, \quad (A-12)
\]

with \( P \) a polynomial function.

Proof. To estimate the first set of operators, we first consider the splitting

\[
\mathcal{T}_1(\partial_{\alpha_j} f) = \text{PV} \int_{|\beta| > 1} d\beta + \text{PV} \int_{|\beta| < 1} d\beta = I_1 + I_2, \quad (A-13)
\]

and an integration by parts allows us to write \( I_1 = J_1 + J_2 + J_3 \), where

\[
J_1 = \int_{|\beta| > 1} - \frac{\partial_{\alpha_j} X_k(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} f(\alpha - \beta) \, d\beta,
\]

\[
J_2 = 3 \int_{|\beta| > 1} \frac{(X_k(\alpha) - X_k(\alpha - \beta))(X(\alpha) - X(\alpha - \beta)) \cdot \partial_{\alpha_j} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^5} \, d\beta,
\]

\[
J_3 = \int_{|\beta| = 1} \frac{X_k(\alpha) - X_k(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} f(\alpha - \beta) \, dl(\beta).
\]

The above decomposition shows that

\[
|I_1| \leq C \| X - (\alpha, 0) \|_{C^1} \| F(X) \|_{L^\infty}^3 \left( \int_{|\beta| > 1} \frac{|f(\alpha - \beta)|}{|\beta|^3} \, d\beta + \int_{|\beta| = 1} |f(\alpha - \beta)| \, dl(\beta), \right)
\]

and then Minkowski’s inequality gives the desired control.
Regarding $I_2$, we write $I_2 = J_4 + J_5 + J_6$, with

$$J_4 = \int_{|\beta|<1} \frac{X_k(\alpha) - X_k(\alpha - \beta) - \nabla X_k(\alpha) \cdot \beta}{|X(\alpha) - X(\alpha - \beta)|^3} \partial_{\alpha_j} f(\alpha - \beta) \, d\beta,$$

$$J_5 = \nabla X_k(\alpha) \cdot \int_{|\beta|<1} \beta \left[ \frac{1}{|X(\alpha) - X(\alpha - \beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot \beta|^3} \right] \partial_{\alpha_j} f(\alpha - \beta) \, d\beta,$$

$$J_6 = \nabla X_k(\alpha) \cdot \text{PV} \int_{|\beta|<1} \frac{\beta}{|\nabla X(\alpha)\cdot \beta|^3} \partial_{\alpha_j} f(\alpha - \beta) \, d\beta.$$

It is easy to see that

$$J_4 \leq \|X - (\alpha, 0)\|_{C^{1,\delta}} \|F(X)\|_{L^\infty} \int_{|\beta|<1} \frac{|\partial_{\alpha_j} f(\alpha - \beta)|}{|\beta|^{2-\delta}} \, d\beta,$$

(A-14)

and therefore that term can also be estimated with the use of Minkowski’s inequality.

Some elementary algebraic manipulations allow us to get

$$J_5 \leq C \|X - (\alpha, 0)\|_{C^{1,\delta}} \|F(X)\|_{L^\infty} \int_{|\beta|<1} \left( \frac{\|F(X)(\alpha, \beta)\|}{|\nabla X(\alpha)\cdot \beta|^4} \right)^{4+\frac{|\beta|}{|\nabla X(\alpha)\cdot \beta|^4}} |\partial_{\alpha_j} f(\alpha - \beta)| \, d\beta,$$

and then the inequality

$$\frac{|\beta|}{|\nabla X(\alpha)\cdot \beta|} \leq 2 \|\nabla X\|_{L^\infty} \||N|^{-1}\|_{L^\infty}$$

(A-15)

yields for $J_5$ the same estimate (A-14).

The term $J_6$ can be written as

$$J_6 = \nabla X_k(\alpha) \cdot \text{PV} \int_{|\beta|<1} \frac{\Sigma(\alpha, \beta)}{|\beta|^2} \partial_{\alpha_j} f(\alpha - \beta) \, d\beta,$$

where

(i) $\Sigma(\alpha, \lambda \beta) = \Sigma(\alpha, \beta)$ for all $\lambda > 0$,

(ii) $\Sigma(\alpha, -\beta) = -\Sigma(\alpha, \beta)$,

(iii) $\sup_\alpha |\Sigma(\alpha, \beta)| \leq 8 \|\nabla X\|_{L^\infty}^3 \||N|^{-1}\|_{L^\infty}^3$,

as a consequence of (A-15).

Here we have a singular integral operator with odd kernel [Córdoba and Gancedo 2007; Stein 1993], and therefore a bounded linear map on $L^2(\mathbb{R}^2)$, giving us

$$\|J_6\|_{L^2} \leq C \|\nabla X\|_{L^\infty}^4 \||N|^{-1}\|_{L^\infty}^3 \|\partial_{\alpha_j} f\|_{L^2}.$$

For the family of operators $\mathcal{F}_2(f)(\alpha)$, we use the splitting $\mathcal{F}_2(f) = I_3 + I_4$, where

$$I_3 = \int_{|\beta|>1} \frac{\partial_{\alpha_j} X_k(\alpha) - \partial_{\alpha_j} X_k(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} f(\alpha - \beta) \, d\beta.$$

We easily get

$$I_3 \leq 2 \|X - (\alpha, 0)\|_{C^{1,\delta}} \|F(X)\|_{L^\infty} \int_{|\beta|>1} \frac{|f(\alpha - \beta)|}{|\beta|^3} \, d\beta,$$
while for \( I_4 \), we proceed with the same method used with \( I_2 \), now replacing \( X_k(\alpha) \) by \( \partial_{\alpha_j} X_k(\alpha) \) and \( \partial_{\alpha_j} f(\alpha - \beta) \) by \( f(\alpha - \beta) \).

Next we shall show that the operator \( \mathcal{T}_3 \) behaves like \( \Lambda = (-\Delta)^{1/2} \). To do that, we split it as \( I_5 + I_6 \), where

\[
I_5 = \int_{|\beta| > 1} \frac{f(\alpha) - f(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \, d\beta
\]
can be easily estimated by

\[
I_5 \leq \|F(X)\|_L^3 \left( 2\pi |f(\alpha)| + \int_{|\beta| > 1} \frac{|f(\alpha - \beta)|}{|\beta|^3} \, d\beta \right).
\]

The other term is written in the form \( I_6 = J_7 + J_8 \), where

\[
J_7 = \int_{|\beta| < 1} \left( \frac{1}{|X(\alpha) - X(\alpha - \beta)|^3} - \frac{1}{|\nabla X(\alpha) \cdot \beta|^3} \right) (f(\alpha) - f(\alpha - \beta)) \, d\beta.
\]

The identity

\[
f(\alpha) - f(\alpha - \beta) = \beta \cdot \int_0^1 \nabla f(\alpha + (s - 1)\beta) \, ds
\]

allows us to treat \( J_7 \) as we did with \( J_5 \). To estimate \( J_8 \), the equality

\[
\frac{1}{|\nabla X(\alpha) \cdot \beta|^3} = -\partial_{\beta_1} \left( \frac{\beta_1}{|\nabla X(\alpha) \cdot \beta|^3} \right) - \partial_{\beta_2} \left( \frac{\beta_2}{|\nabla X(\alpha) \cdot \beta|^3} \right)
\]

will be very useful. After a careful integration by parts, it yields

\[
J_8 = \text{PV} \int_{|\beta| < 1} \frac{\nabla f(\alpha - \beta) \cdot \beta}{|\nabla X(\alpha) \cdot \beta|^3} \, d\beta - \int_{|\beta| = 1} \frac{(f(\alpha) - f(\alpha - \beta))|\beta|}{|\nabla X(\alpha) \cdot \beta|^3} \, dl(\beta).
\]

The principal value in \( J_8 \) is treated with the same method used for \( J_6 \), and since the integral on the circle is inoffensive, so long as \(|N|^{-1}\) is in \( L^\infty \), the estimate for \( \mathcal{T}_3 \) follows.

For the remaining operator, one integrates by parts to get \( \mathcal{T}_4 = I_7 + I_8 \), where

\[
I_7 = \text{PV} \int_{\mathbb{R}^2} P_1(\alpha, \beta) f(\alpha - \beta) \, d\beta, \quad I_8 = \text{PV} \int_{\mathbb{R}^2} P_2(\alpha, \beta) f(\alpha - \beta) \, d\beta,
\]

with

\[
P_1(\alpha, \beta) = \frac{\partial_{\alpha_j} X(\alpha)}{|\nabla X(\alpha) \cdot \beta|^3} - \frac{\partial_{\alpha_j} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3}
\]

and

\[
P_2(\alpha, \beta) = 3 \left( \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^5} \right) \frac{X(\alpha) - X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3} \cdot \partial_{\alpha_j} X(\alpha - \beta) - 3 \frac{\nabla X(\alpha) \cdot \beta (\nabla X(\alpha) \cdot \beta) \cdot \partial_{\alpha_j} X(\alpha)}{|\nabla X(\alpha) \cdot \beta|^5}.
\]

Next we will show how to treat \( I_7 \), because the estimate for \( I_8 \) follows similarly. For \( P_1 \) we introduce the decomposition \( P_1 = Q_1 + Q_2 \), where

\[
Q_1 = \partial_{\alpha_j} X(\alpha) \left( \frac{1}{|\nabla X(\alpha) \cdot \beta|^3} - \frac{1}{|X(\alpha) - X(\alpha - \beta)|^3} \right), \quad Q_2 = \frac{\partial_{\alpha_j} X(\alpha) - \partial_{\alpha_j} X(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3}.
\]
Since the kernel $Q_2$ has already appeared in the operator $\mathcal{T}_1$, it only remains to control $J_9$, which is given by

$$J_9 = \partial_{\alpha_j} X(\alpha) \text{PV} \int_{\mathbb{R}^2} Q_1(\alpha, \beta) f(\alpha - \beta) \, d\beta.$$  

The decomposition

$$J_9 = \partial_{\alpha_j} X(\alpha) \int_{|\beta| > 1} d\beta + \partial_{\alpha_j} X(\alpha) \text{PV} \int_{|\beta| < 1} d\beta = K_1 + K_2$$

shows that the term $K_1$ trivializes. Regarding $K_2$, let us write

$$Q_1 = \frac{(|A|^4 + |B|^2 |A|^2 + |B|^4)(A + B) \cdot (A - B)}{|A|^3 |B|^3 (|A|^3 + |B|^3)},$$

where

$$A(\alpha, \beta) = X(\alpha) - X(\alpha - \beta), \quad B(\alpha, \beta) = \nabla X(\alpha) \cdot \beta.$$  

This formula shows that inside $Q_1$ lies a kernel of degree $-2$. Then let us take $Q_1 = S_1 + S_2$, where

$$S_2 = \frac{3 |B|^4 B \cdot (A - B)}{|B|^6} = \frac{3 B \cdot (A - B)}{|B|^5}.$$  

Next we check that the kernel $S_1$ has degree $-1$, and is therefore easy to handle. Finally, we have to consider the kernel $S_2$ appearing in the integral

$$L = 3 \partial_{\alpha_j} X(\alpha) \text{PV} \int_{|\beta| < 1} \frac{(\nabla X(\alpha) \cdot \beta) \cdot (X(\alpha) - X(\alpha - \beta) - \nabla X(\alpha) \cdot \beta)}{|\nabla X(\alpha) \cdot \beta|^5} f(\alpha - \beta) \, d\beta.$$  

To do that, we introduce a further decomposition $L = M_1 + M_2$, with

$$M_1 = 3 \partial_{\alpha_j} X(\alpha) \int_{|\beta| < 1} \frac{(\nabla X(\alpha) \cdot \beta) \cdot (X(\alpha) - X(\alpha - \beta) - \nabla X(\alpha) \cdot \beta - \frac{1}{2} \beta \cdot \nabla^2 X(\alpha) \cdot \beta)}{|\nabla X(\alpha) \cdot \beta|^5} f(\alpha - \beta) \, d\beta$$

and

$$M_2 = \frac{3}{2} \partial_{\alpha_j} X(\alpha) \text{PV} \int_{|\beta| < 1} \frac{(\nabla X(\alpha) \cdot \beta) \cdot (\beta \cdot \nabla^2 X(\alpha) \cdot \beta)}{|\nabla X(\alpha) \cdot \beta|^5} f(\alpha - \beta) \, d\beta,$$

where $\frac{1}{2} \beta \cdot \nabla^2 X(\alpha) \cdot \beta$ is the second-order term in the Taylor expansion of $X$. It is now easy to check that

$$M_1 \leq C \|\nabla X\|_{L^\infty}^5 \|X - (\alpha, 0)\|_{C^{2, \delta}} \|N\|_{L^\infty}^{-1} \|N\|_{L^\infty}^{-1} \|\nabla X\|_{L^\infty}^{4} \|f\|_{L^2} \int_{|\beta| < 1} \frac{|f(\alpha - \beta)|}{|\beta|^{2 - \delta}} \, d\beta.$$  

Then we also check that $M_2$ is controlled like $J_6$ through the estimate

$$\|M_2\|_{L^2} \leq C \|\nabla X\|_{L^\infty}^5 \|\nabla^2 X\|_{L^\infty} \|N\|_{L^\infty}^{-1} \|\nabla X\|_{L^\infty}^{4} \|f\|_{L^2},$$

which allows us to finish the proof.

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References


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