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PERIODICITY OF THE SPECTRUM IN DIMENSION ONE

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A bounded measurable set Ω, of Lebesgue measure 1, in the real line is called *spectral* if there is a set Λ of real numbers ("frequencies") such that the exponential functions $e_\lambda(x) = \exp(2\pi i \lambda x)$, $\lambda \in \Lambda$, form a complete orthonormal system of $L^2(\Omega)$. Such a set Λ is called a *spectrum* of Ω. In this note we prove that any spectrum Λ of a bounded measurable set $\Omega \subseteq \mathbb{R}$ must be periodic.

1. Tilings, spectral sets and periodicity

Spectra of domains in Euclidean space and the Fuglede conjecture. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set and let us assume for simplicity that Ω has Lebesgue measure 1. The concept of a spectrum of Ω that we deal with in this paper may be interpreted as a way of using Fourier series for functions defined on Ω with nonstandard frequencies. It was introduced by Fuglede [1974] who was studying a problem of Segal on the extendability of the partial differential operators

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d}$$

on $C_c(\Omega)$ to commuting operators on all of $L^2(\Omega)$.

Definition 1.1. A set $\Lambda \subseteq \mathbb{R}^d$ is called a *spectrum* of Ω (and Ω is said to be a *spectral set*) if the set of exponentials

$$E(\Lambda) = \{e_\lambda(x) = e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$$

is a complete orthonormal set in $L^2(\Omega)$.

(The inner product in $L^2(\Omega)$ is $\langle f, g\rangle = \int_\Omega f\overline{g}$.)

It is an easy result (see [Kolountzakis 2004], for instance) that the orthogonality of $E(\Lambda)$ is equivalent to the *packing condition*

$$\sum_{\lambda \in \Lambda} \left|\widehat{\chi_\Omega}\right|^2 (x - \lambda) \le 1, \quad \text{a.e. } (x), \tag{1}$$

as well as to the condition

$$\Lambda - \Lambda \subseteq \{0\} \cup \left\{\widehat{\chi_\Omega} = 0\right\}. \tag{2}$$

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The orthogonality and completeness of $E(\Lambda)$ is in turn equivalent to the *tiling condition*

$$\sum_{\lambda\in\Lambda} \left|\widehat{\chi_\Omega}\right|^2(x-\lambda) = 1, \quad \text{a.e. } (x). \tag{3}$$

These equivalent conditions follow from the identity

$$\langle e_\lambda, e_\mu\rangle = \int_\Omega e_\lambda \overline{e_\mu} = \widehat{\chi_\Omega}(\mu-\lambda) \tag{4}$$

and from the completeness of all the exponentials in $L^2(\Omega)$. Condition (1) roughly expresses the validity of Bessel's inequality for the system of exponentials $E(\Lambda)$, while condition (3) says that Bessel's inequality holds as an equality.

If Λ is a spectrum of Ω then so is any translate of Λ but there may be other spectra as well.

Example. If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $\mathbb{Z}^d$ is a spectrum of Q_d. Let us remark here that there are spectra of Q_d that are very different from translates of the lattice $\mathbb{Z}^d$ [Iosevich and Pedersen 1998; Lagarias et al. 2000; Kolountzakis 2000].

In the one-dimensional case, which will concern us in this paper, condition (2) implies that the set Λ has gaps bounded below by a positive number: the smallest positive zero of $\widehat{\chi_\Omega}$. (Note that since Ω is a bounded set, the function $\widehat{\chi_\Omega}(\xi)$ can be defined for all complex ξ and is an entire function. This guarantees that its zeros are a discrete set.)

The Fuglede or spectral set conjecture. Research on spectral sets has been driven for many years by a conjecture of Fuglede [1974] which stated that a set Ω is spectral if and only if it is a translational tile. A set Ω is a translational tile if we can translate copies of Ω around and fill space without overlaps. More precisely, there exists a set $S \subseteq \mathbb{R}^d$ such that

$$\sum_{s\in S} \chi_\Omega(x-s) = 1, \quad \text{a.e. } (x). \tag{5}$$

One can extend the definition of translational tiling to functions from sets.

Definition 1.2. We say that a nonnegative function $f : \mathbb{R}^d \to \mathbb{R}$ *tiles by translation with the set* $S \subseteq \mathbb{R}^d$ if

$$\sum_{s\in S} f(x-s) = \ell \quad \text{for almost every } x \in \mathbb{R}^d,$$

where ℓ is a constant (the *level* of the tiling).

Thus the question of spectrality for a set Ω is essentially a tiling question for the function $\left|\widehat{\chi_\Omega}\right|^2$ (the *power spectrum*). Taking into account the equivalent condition (3) one can now, more elegantly, restate the Fuglede conjecture as the equivalence

$$\chi_\Omega \text{ tiles } \mathbb{R}^d \text{ by translation at level } 1 \iff \left|\widehat{\chi_\Omega}\right|^2 \text{ tiles } \mathbb{R}^d \text{ by translation at level } 1. \tag{6}$$

In this form the conjectured equivalence is perhaps more justified. However this conjecture is now known to be false in both directions if $d \geq 3$ [Tao 2004; Matolcsi 2005; Kolountzakis and Matolcsi 2006a; 2006b;

Farkas et al. 2006; Farkas and Révész 2006], but remains open in dimensions 1 and 2 and it is not out of the question that the conjecture is true if one restricts the domain Ω to being convex. (It is known that the direction "tiling $\Rightarrow$ spectrality" is true in the case of convex domains; see [Kolountzakis 2004].) The equivalence (6) is also known, from the time of [Fuglede 1974], to be true if one adds the word *lattice* to both sides (that is, lattice tiles are the same as sets with a lattice spectrum).

Periodicity of spectra and tilings. The property of periodicity is very important for a tiling.

Definition 1.3. A set $S \subseteq \mathbb{R}^d$ is called (*fully*) *periodic* if there exists a lattice $L \subseteq \mathbb{R}^d$ (a discrete subgroup of $\mathbb{R}^d$ with d linearly independent generators: the period lattice) such that $S + t = S$ for all $t \in L$. We call a translation tiling *periodic* if the set of translations is periodic.

The so-called *periodic tiling conjecture* [Grünbaum and Shephard 1989; Lagarias and Wang 1997] should be mentioned at this point: if a set Ω tiles $\mathbb{R}^d$ by translations (at level 1) then it can also tile $\mathbb{R}^d$ by a periodic set of translations.

As an example of the importance of periodicity for a tiling we mention its connection to decidability [Robinson 1971], a question to which the study of tilings has provided several examples and problems. Although the general problem of tiling (not restricting the motions to be translations or allowing more than one tile) is undecidable, it is not hard to see that when the assumption of periodicity is added, the problem becomes decidable. Let us make this connection more clear by stating it in the discrete case:

> Assume that the *periodic tiling conjecture* is true. Then one can algorithmically decide if a given *finite* $\Omega \subseteq \mathbb{Z}^d$ admits tilings by translation or not.

Roughly, if one knows a priori that a set Ω admits periodic tilings, if it admits any, then the question "Does Ω admit a tiling?" can be answered algorithmically by simultaneously enumerating all possible counterexamples to tiling (if a tiling does not exist then the obstacle will show up at some finite stage) as well as all possible tilings of finite regions. If a tiling does not exist then the first enumeration will produce a counterexample. Otherwise, if a tiling exists then, by the periodic tiling conjecture, a periodic tiling exists and one of the finite regions that can be tiled with Ω will show this periodicity and can therefore be extended to all space. More details of this argument can be found in [Robinson 1971]. Both the periodic tiling conjecture and the question of decidability of tilings by translation are open for $d \geq 2$ (but see [Szegedy 1998; Wijshoff and van Leeuwen 1984] for some special cases). For $d = 1$ all translational tilings by finite subsets of $\mathbb{Z}$ are necessarily periodic [Newman 1977] and the problem is decidable. Another class of tilings where the periodic tiling conjecture holds is the case when Ω is assumed to be a convex polytope in $\mathbb{R}^d$, for any d [Venkov 1954; McMullen 1980].

In dimension $d = 1$ it is known [Leptin and Müller 1991; Lagarias and Wang 1996; Kolountzakis and Lagarias 1996] that all translational tilings by a bounded measurable set are necessarily periodic. More generally it is known that whenever $f \geq 0$ is an integrable function on the real line that tiles the real line by translation with a set of translates S, then S is of the form

$$S = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j), \tag{7}$$

where the real numbers α_j are necessarily commensurable (and S is in that case periodic) if the tiling is indecomposable (cannot be made up by superimposing other tilings). But this result is not applicable to the periodicity of spectra, as the power-spectrum $\left|\widehat{\chi_\Omega}\right|^2$ is never of compact support when Ω is bounded (a qualitative expression of the *uncertainty principle*).

The question of periodicity of one-dimensional spectra was explicitly raised in [Łaba 2002]. It was recently proved (first in [Bose and Madan 2011] and then a simplified proof was given in [Kolountzakis 2012]) that if Ω is a finite union of intervals in the real line then any spectrum of Ω is periodic. See also [Lagarias and Wang 1997], where periodicity of spectra and of tilings plays an important role.

Theorem 1.4 [Bose and Madan 2011; Kolountzakis 2012]. *If $\Omega = \bigcup_{j=1}^n (a_j, b_j) \subseteq \mathbb{R}$ is a finite union of intervals of total length 1 and $\Lambda \subseteq \mathbb{R}$ is a spectrum of Ω, then there exists a positive integer T such that $\Lambda + T = \Lambda$.*

Our purpose in this note is to improve this result by removing the assumption that Ω is a finite union of intervals.

Theorem 1.5. *Suppose that Λ is a spectrum of $\Omega \subseteq \mathbb{R}$, where Ω is a bounded measurable set of measure 1. Then Λ is periodic and any period is a positive integer.*

The proof of Theorem 1.5 is given in Section 2.

Corollary 1.6. *If Ω, a bounded measurable set of measure 1, is spectral then Ω tiles the real line at some integer level T when translated at the locations $T^{-1}\mathbb{Z}$.*

Proof. Let Λ is a spectrum of Ω. By Theorem 1.5 we know that Λ is a periodic set and let T be one of its periods: $\Lambda + T = \Lambda$. Then we have $\Lambda = T\mathbb{Z} + \{\ell_1, \dots, \ell_T\}$ (the number of elements in each period must be T in order for Λ to have density 1, hence T is an integer), and, by (2), this implies that $\widehat{\chi_\Omega}(nT) = 0$ for all nonzero $n \in \mathbb{Z}$. Hence Ω tiles $\mathbb{R}$ when translated at $T^{-1}\mathbb{Z}$ (see [Kolountzakis 2004]) at level T. □

Theorem 1.5 is not true in dimensions higher than 1. For instance, even when Ω is as simple as a cube, it may have spectra that are not periodic [Lagarias et al. 2000; Iosevich and Pedersen 1998; Kolountzakis 2000].

2. Proof of periodicity for spectra in dimension 1

The spectrum as a double sequence of symbols. Because of (2) we have that the gap between any two elements of Λ is bounded below by $\delta > 0$: the smallest positive zero of $\widehat{\chi_\Omega}$. Let us now observe that the gap between successive elements of Λ is also bounded above by a constant that depends only on Ω.

Lemma 2.1. *If $\Omega \subseteq \mathbb{R}$ is a bounded measurable set of measure 1 then there is a finite number $\Delta > 0$ such that if Λ is any spectrum of Ω then the gap between any two successive elements of Λ is at most Δ.*

Proof. Lemma 2.1 is essentially a special case of Lemma 2.3 of [Kolountzakis and Lagarias 1996]. In that lemma it is proved that if $0 \le f \in L^1(\mathbb{R})$ tiles the line with a set A,

$$\sum_{a \in A} f(x - a) = w \quad \text{for almost all } x \in \mathbb{R}, \text{ with } w > 0 \text{ a constant,}$$

then the set A has asymptotic density equal to $\rho = w / \int f$. This means that the ratio

$$|A \cap I| / |I|$$

tends to ρ as the length of the interval I tends to infinity. The convergence is *uniform* over the choice of the set A and the location of the interval I.[1] This uniformity of course implies that the maximum gap of A is bounded by a quantity that depends on f only.

Since $\sum_{\lambda \in \Lambda} |\chi_\Omega|^2 (x - \lambda) = 1$ is a tiling and $0 \le |\chi_\Omega|^2 \in L^1(\mathbb{R})$ we deduce that Λ has gaps bounded above by a function of Ω alone. □

Let now

$$Z = \left\{ \xi \in \mathbb{R} : \widehat{\chi_\Omega}(\xi) = 0 \right\}$$

and define the finite set (as Z is discrete)

$$\Sigma = Z \cap (0, \Delta] = \{s_1, s_2, \ldots, s_k\}, \tag{8}$$

where Δ is the quantity given by Lemma 2.1.

We now view the set Σ as a finite set of symbols (alphabet) and consider the set $\Sigma^{\mathbb{Z}}$ of all bidirectional sequences of elements of Σ equipped with the product topology. A sequence x^n of elements of $\Sigma^{\mathbb{Z}}$ converges to $x \in \Sigma^{\mathbb{Z}}$ if for all $k = 1, 2, \ldots$ the double sequences x^n and x agree in the window $[-k, k]$ for large enough n. More precisely, for all $k = 1, 2, \ldots$ there is n_0 such that for $n \ge n_0$ we have

$$x_j^n = x_j \quad \text{for } -k \le j \le k.$$

$\Sigma^{\mathbb{Z}}$ is a metrizable compact space so that each sequence $x^n \in \Sigma^{\mathbb{Z}}$ has a convergent subsequence. This is just another way of phrasing a diagonal argument that is somewhat more convenient to use. The proof below may of course be phrased avoiding topological notions altogether and replacing the convergence of each subsequence with a diagonal argument.

The space $\Sigma^{\mathbb{Z}}$ is the natural space in which to view a spectrum Λ of Ω, as the set Λ is locally of *finite complexity*: because of (2) the difference of any two successive elements of Λ can be only be an element of Σ. By demanding, as we may, that 0 is always in Λ we can therefore represent any set Λ with the sequence of its successive differences. More precisely, we map any set $\Lambda \subseteq \mathbb{R}$ whose successive differences are in Σ and which contains 0,

$$\Lambda = \{\cdots < -\lambda_2 < -\lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots\},$$

to the element $(\Lambda_n) \in \Sigma^{\mathbb{Z}}$ given by

$$\Lambda_n = \lambda_{n+1} - \lambda_n \quad (n \in \mathbb{Z}).$$

This correspondence is a bijection and we will use one or the other form of the set Λ as it suits us.

[1] Inequality (2.4) in [Kolountzakis and Lagarias 1996] speaks of $N_A(T) = |A \cap [-T, T]|$, but none of the other quantities that appear in it depend on A. This means that (2.4) holds even if we take $N_A(T)$ to be the number of elements of A in *any* interval of length $2T$. In fact, one can prove that $N_A(T)$ cannot be 0 if T is sufficiently large, depending on Ω, without taking the limit in (2.4) and without talking about asymptotic density.

Symbolic sequences determined by their values in a half-line. Suppose $X \subseteq \Sigma^{\mathbb{Z}}$. We say that X is determined by left half-lines if knowing an element of X to the left of any index n suffices to determine the element in the remaining positions to the right of n, i.e., if for any $x, y \in X$ and $n \in \mathbb{Z}$ we have

$$(x_i = y_i \text{ for } i \le n) \Longrightarrow (x_i = y_i \text{ for all } i \in \mathbb{Z}).$$

Determination of X by right half-lines is defined analogously.

We similarly say that X is determined by any window of size w (a positive integer) if for any $x \in X$ and any $n \in \mathbb{Z}$ knowing x_i for $i = n, n+1, \ldots, n+w-1$ completely determines x.

Theorem 2.2. *Suppose $X \subseteq \Sigma^{\mathbb{Z}}$ is a closed, shift-invariant set that is determined by left half-lines and by right half-lines. Then there is a finite number w such that X is determined by windows of size w.*

Proof. It is enough to show that there is a finite window size w such that whenever two elements of X agree on a window of size w, then they necessarily agree at the first index to the right of that window. For in that case they necessarily agree at the entire right half-line to the right of the window and are by assumption equal elements of X.

Assume this is not true. Then there are elements x^n, y^n of X, $n = 1, 2, \ldots$, which agree at some window of width n but disagree at the first location to the right of that window. Using the shift-invariance of X we may assume that

$$x^n_{-n} = y^n_{-n},\ x^n_{-n+1} = y^n_{-n+1},\ \ldots,\ x^n_{-1} = y^n_{-1} \quad \text{and} \quad x^n_0 \neq y^n_0.$$

By the compactness of the space there are $x, y \in X$ and a subsequence (n_k) such that $x^{n_k} \to x$ and $y^{n_k} \to y$. By the meaning of convergence in the space $\Sigma^{\mathbb{Z}}$ we have that the sequences x and y agree for all negative indices and disagree at 0. This contradicts the assumption that X is determined by left half-lines. □

Theorem 2.3. *If $X \subseteq \Sigma^{\mathbb{Z}}$ is shift-invariant and is determined by windows of size w then all elements of X are periodic, and the period can be chosen to be at most $|\Sigma|^w$.*

Proof. Fix $x \in X$. Since there are at most $|\Sigma|^w$ different window-contents of length w, it follows that there are two indices $i, j \in \{0, 1, \ldots, |\Sigma|^w\}$, $i < j$, such that

$$x_i = x_j,\ x_{i+1} = x_{j+1},\ \ldots,\ x_{i+w-1} = x_{j+w-1}.$$

Writing Tx for the left shift of $x \in X$ (i.e., $(Tx)_n = x_{n+1}$) we have that x and $T^{j-i}x$ agree at the window $i, i+1, \ldots, i+w-1$. By assumption then $x = T^{j-i}x$, which is another way of saying that the sequence x has period $j - i \le |\Sigma|^w$. □

Symbolic sequences with spectral gaps. Suppose $\Lambda \subseteq \mathbb{R}$ is a spectrum of the bounded set $\Omega \subseteq \mathbb{R}$ of measure 1. Write $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, where δ_λ is a unit point mass at point λ. It is well known (see [Kolountzakis 2004]) that the Fourier transform of the tempered distribution δ_Λ is supported by 0 plus the zeros of the function

$$\left(|\widehat{\chi_\Omega}|^2\right)^\wedge = \chi_\Omega * \chi_{-\Omega},$$

which is a continuous function with value 1 at the origin. Therefore there is an interval $(0, a)$, with $a = a(\Omega) > 0$, such that δ_Λ has a spectral gap:

$$\operatorname{supp} \widehat{\delta_\Lambda} \cap (0, a) = \varnothing. \tag{9}$$

With $\Sigma = \Sigma(\Omega)$ defined by (8), let $X \subseteq \Sigma^{\mathbb{Z}}$ consist of all sequences which correspond to sets Λ with gaps from Σ such that (9) holds. The set X is obviously shift-invariant, as shifting a sequence in X corresponds to translation of the set Λ and translation will not affect the support of $\widehat{\delta_\Lambda}$.

Lemma 2.4. *The set X is closed in $\Sigma^{\mathbb{Z}}$.*

Proof. Suppose $\Lambda^n \in X$ and $\Lambda^n \to \Lambda \in \Sigma^{\mathbb{Z}}$ and that $\phi \in C^\infty(0, a)$. It is enough to show that $\widehat{\delta_\Lambda}(\phi) = 0$, as this is what it means for $\widehat{\delta_\Lambda}$ to have no support in $(0, a)$ and therefore $\Lambda \in X$. By the definition of the Fourier transform,

$$\widehat{\delta_\Lambda}(\phi) = \delta_\Lambda(\widehat{\phi}) = \sum_{\lambda \in \Lambda} \widehat{\phi}(\lambda) \stackrel{*}{=} \lim_{n\to\infty} \sum_{\lambda \in \Lambda^n} \widehat{\phi}(\lambda) = \lim_{n\to\infty} \delta_{\Lambda^n}(\widehat{\phi}) = \lim_{n\to\infty} \widehat{\delta_{\Lambda^n}}(\phi) = 0.$$

The justification for the starred equality above is very easy given the rapid decay of $\widehat{\phi}$, and the fact that all Λ^n have the same positive minimum gap. Indeed, these properties imply that for any $\epsilon > 0$ we can find an $R > 0$ such that

$$\left| \sum_{\substack{\lambda \in \Lambda \\ |\lambda| > R}} \widehat{\phi}(\lambda) \right| < \epsilon \quad \text{for } L = \Lambda \text{ or } L = \Lambda^n,$$

and also an n_0 such that $\Lambda^n \cap [-R, R] = \Lambda \cap [-R, R]$ for $n \geq n_0$. It follows that for $n \geq n_0$ we have

$$\left|\widehat{\delta_\Lambda}(\phi)\right| = \left|\widehat{\delta_\Lambda}(\phi) - \widehat{\delta}_{\Lambda^n}(\phi)\right| = \left| \sum_{\substack{\lambda \in \Lambda \\ |\lambda| > R}} \widehat{\phi}(\lambda) - \sum_{\substack{\lambda \in \Lambda \\ |\lambda| > R}} \widehat{\phi}(\lambda) \right| \leq 2\epsilon.$$

This implies that $\widehat{\delta_\Lambda}(\phi) = 0$, as we had to show. ☐

Theorem 2.5. *The sequences in X are determined by both left half-lines and right half-lines.*

Proof. Suppose that X is not determined by left half-lines (the argument is similar for right half-lines). Then there are distinct $\Lambda^1, \Lambda^2 \in X$ such that $\Lambda^1_i = \Lambda^2_i$ for all negative integers i. Both δ_{Λ^1} and δ_{Λ^2} have a spectral gap at $(0, a)$ and therefore so does their difference

$$\mu = \delta_{\Lambda^1} - \delta_{\Lambda^2}.$$

Notice that μ is supported in the half-line $[0, +\infty)$. Suppose $\psi \in C^\infty(-a/10, a/10)$. It follows from the rapid decay of $\widehat{\psi}$ that the measure

$$\nu = \widehat{\psi} \cdot \mu$$

is totally bounded and still has a spectral gap at the interval $(a/10, 9a/10)$. But the measure ν is also supported in the half-line $[0, +\infty)$ and by the F. and M. Riesz theorem [Havin and Jöricke 1994] its

Fourier transform is mutually absolutely continuous with respect to the Lebesgue measure on the line.[2] But this is incompatible with the vanishing of $\widehat{\nu}$ in some interval. Therefore ν must be identically 0 and, since $\psi \in C^\infty(-a/10, a/10)$ is otherwise arbitrary, it follows that $\mu \equiv 0$, or $\Lambda^1 = \Lambda^2$, a contradiction. It follows that X is indeed determined by left half-lines. □

Conclusion of the argument. By Lemma 2.4 and Theorem 2.5 the set X defined above, right after (9), given Ω is a closed shift-invariant subset of $\Sigma^{\mathbb{Z}}$ and its elements are determined by half-lines. By Theorem 2.2 there exists a finite number w such that the elements of X are determined by their values at any window of width w. By Theorem 2.3 all elements of X are therefore periodic sequences. Since all spectra of Ω can also be viewed as elements of X, the periodicity of any spectrum of Ω follows from the periodicity of the sequence of its successive differences.

The fact that any period of Λ is a positive integer is a consequence of the fact that Λ has density 1: if T is a period of Λ this implies that there are exactly T elements of Λ in each interval $[x, x+T)$ hence T is an integer.

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