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DISCRETE FOURIER RESTRICTION ASSOCIATED WITH KDV EQUATIONS
In this paper, we consider a discrete restriction associated with KdV equations. Some new Strichartz estimates are obtained. We also establish the local well-posedness for the periodic generalized Korteweg–de Vries equation with nonlinear term $F(u)\partial_x u$ provided $F \in C^5$ and the initial data $\phi \in H^s$ with $s > 1/2$.

1. Introduction

The discrete restriction problem associated with KdV equations is a problem asking the best constant $A_{p,N}$ satisfying

$$
\sum_{n=-N}^{N} |\hat{f}(n,n^3)|^2 \leq A_{p,N} \|f\|_{p'}^2,
$$

(1-1)

where $f$ is a periodic function on $\mathbb{T}^2$, $\hat{f}$ is the Fourier transform of $f$ on $\mathbb{T}^2$, $p \geq 2$, and $p' = p/(p-1)$. It is natural to pose a conjecture asserting that for any $\varepsilon > 0$, $A_{p,N}$ satisfies

$$
A_{p,N} \leq \begin{cases} 
C_p N^{1-8/p+\varepsilon} & \text{for } p \geq 8, \\
C_p & \text{for } 2 \leq p < 8.
\end{cases}
$$

(1-2)

It was proved by Bourgain that $A_{6,N} \leq N^{\varepsilon}$. The desired upper bound for $A_{8,N}$ is not yet obtained; however, we are able to establish an affirmative answer for large $p$.

**Theorem 1.1.** Let $A_{p,N}$ be defined as in (1-1). If $p \geq 14$, for any $\varepsilon > 0$, there exists a constant $C_p$ independent of $N$ such that

$$
A_{p,N} \leq C_p N^{1-8/p+\varepsilon}.
$$

(1-3)

The periodic Strichartz inequality associated to KdV equations is the inequality seeking the best constant $K_{p,N}$ satisfying

$$
\left\| \sum_{n=-N}^{N} a_n e^{2\pi i n t^3 + 2\pi i x n} \right\|_{L^p_{t,x}(\mathbb{T} \times \mathbb{T})} \leq K_{p,N} \left( \sum_{n=-N}^{N} |a_n|^2 \right)^{1/2}.
$$

(1-4)

By duality, we immediately see that

$$
K_{p,N} \sim \sqrt{A_{p,N}}.
$$

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Hence, Theorem 1.1 is equivalent to Strichartz estimates,

$$K_{p,N} \leq C N^{1/2-4/p+\varepsilon}, \quad \text{for } p \geq 14.$$  \hspace{1cm} (1-5)

It was observed by Bourgain that the periodic Strichartz inequalities (1-4) for $p = 4, 6$ are crucial for obtaining the local well-posedness of periodic KdV (mKdV or gKdV). The local (global) well-posedness of periodic KdV for $s \geq 0$ was first studied by Bourgain [1993b]. Via a bilinear estimate approach, Kenig, Ponce, and Vega [Kenig et al. 1996] established the local well-posedness of periodic KdV for $s > -1/2$. The sharp global well-posedness of the periodic KdV was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2003], by utilizing the I-method.

Inspired by Bourgain’s work, we can obtain the following theorem on gKdV. Here the gKdV is the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0, \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, \ t \in \mathbb{R}, \end{cases}$$  \hspace{1cm} (1-6)

where $k \in \mathbb{N}$ and $k \geq 3$.

**Theorem 1.2.** The Cauchy problem (1-6) is locally well-posed if the initial data $\phi \in H^s$ for $s > 1/2$.

Theorem 1.2 is not new. It was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2004], but our method is different. The method used by those authors is based on a rescaling argument and the bilinear estimates proved by Kenig, Ponce and Vega [Kenig et al. 1996]. Our method is more straightforward and does not need the rescaling argument, the bilinear estimates, or the multilinear estimates in the earlier papers. This allows us to extend Theorem 1.2 to a very general setting. More precisely, consider the Cauchy problem for periodic generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + F(u) u_x = 0, \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, \ t \in \mathbb{R}. \end{cases}$$  \hspace{1cm} (1-7)

Here $F$ is a suitable function. Then the following theorem can be established.

**Theorem 1.3.** The Cauchy problem (1-7) is locally well-posed provided $F$ is a $C^5$ function and the initial data $\phi \in H^s$ for $s > 1/2$.

For sufficiently smooth $F$, say $F \in C^{15}$, the existence of a local solution of (1-7) for $s \geq 1$ and the global well-posedness of (1-7) for small data $\phi \in H^s$ with $s > 3/2$ were proved by Bourgain [1995]. The index $1/2$ is sharp because the ill-posedness of (1-6) for $s < 1/2$ is known; see [Colliander et al. 2004]. In order to make Theorem 1.3 well-posed for the initial data $\phi \in H^s$ with $s > 1/2$, the sharp regularity condition for $F$ is perhaps $C^4$. But the method utilized in this paper, with a small modification, seems only to be able to reach an affirmative result for $F \in C^{(9/2)+}$ and $s > 1/2$. Moreover, the endpoint $s = 1/2$ case could possibly be done by combining the ideas from [Colliander et al. 2004] and this paper. We do not pursue this here.
2. Proof of Theorem 1.1

Proof. To prove Theorem 1.1, we need to introduce a level set. Since \( \sqrt{A_{p,N}} \sim K_{p,N} \), it suffices to prove the Strichartz estimates (1-4). Let \( F_N \) be a periodic function on \( \mathbb{T}^2 \) given by

\[
F_N(x, t) = \sum_{n=-N}^{N} a_n e^{2\pi i n x} e^{2\pi i n^3 t}, \quad (2-1)
\]

where \( \{a_n\} \) is a sequence with \( \sum_n |a_n|^2 = 1 \) and \( (x, t) \in \mathbb{T}^2 \). For any \( \lambda > 0 \), set a level set \( E_\lambda \) to be

\[
E_\lambda = \{(x, t) \in \mathbb{T}^2 : |F_N(x, t)| > \lambda \}. \quad (2-2)
\]

To obtain the desired estimate for the level set, let us first state a lemma on Weyl’s sums.

**Lemma 2.1.** Suppose that \( t \in \mathbb{T} \) satisfies \( |t - a/q| \leq 1/q^2 \), where \( a \) and \( q \) are relatively prime. Then if \( q \geq N^2 \),

\[
\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + bn^2 + cn)} \right| \leq C N^{1/4+\epsilon} q^{1/4}. \quad (2-3)
\]

Here \( b \) and \( c \) are real numbers, and the constant \( C \) is independent of \( b, c, t, a, q, \) and \( N \).

The proof of Lemma 2.1 relies on Weyl’s squaring method. See [Hua 1965] or [Montgomery 1994] for details. We also need the following lemma.

**Lemma 2.2 [Bourgain 1993a].** For any integer \( Q \geq 1 \) and any integer \( n \neq 0 \), and any \( \epsilon > 0 \),

\[
\sum_{Q < q < 2Q} \left| \sum_{a \in \mathcal{P}_q} e^{2\pi i (a/q)n} \right| \leq C_\epsilon d(n, Q) Q^{1+\epsilon}. \quad (2-4)
\]

Here \( \mathcal{P}_q \) is given by

\[
\mathcal{P}_q = \{ a \in \mathbb{N} : 1 \leq a \leq q \text{ and } (a, q) = 1 \}, \quad (2-4)
\]

and \( d(n, Q) \) denotes the number of divisors of \( n \) less than \( Q \) and \( C_\epsilon \) is a constant independent of \( Q, n \).

Lemma 2.2 can be proved by observing that the arithmetic function defined by \( f(q) = \sum_{a \in \mathcal{P}_q} e^{2\pi i (a/q)n} \) is multiplicative, and then utilizing the prime factorization for \( q \) to conclude the lemma.

**Proposition 2.3.** Let \( K_N \) be a kernel defined by

\[
K_N(x, t) = \sum_{n=-N}^{N} e^{2\pi i mn^3 + 2\pi i xn}. \quad (2-5)
\]

For any given positive number \( Q \) with \( N^2 \leq Q \leq N^3 \), the kernel \( K_N \) can be decomposed into \( K_{1,Q} + K_{2,Q} \) such that

\[
\| K_{1,Q} \|_{\infty} \leq C_1 \sqrt[4]{Q^{1/4}} Q^{1/4}. \quad (2-6)
\]

and

\[
\| \hat{K}_{2,Q} \|_{\infty} \leq \frac{C_2 N^{\epsilon}}{Q}. \quad (2-7)
\]
Here the constants \( C_1, C_2 \) are independent of \( Q \) and \( N \).

**Proof.** We can assume that \( Q \) is an integer, since otherwise we can take the integer part of \( Q \). For a standard bump function \( \varphi \) supported on \([1/200, 1/100] \), we set

\[
\Phi(t) = \sum_{Q \leq q \leq 5Q} \sum_{a \in \mathcal{P}_q} \varphi\left(\frac{t - a/q}{1/q^2}\right).
\]

(2-8)

Clearly \( \Phi \) is supported on \([0, 1]\). We can extend \( \Phi \) to other intervals periodically to obtain a periodic function on \( \mathbb{T} \). This periodic function, generated by \( \Phi \), will also be denoted by \( \Phi \). It is easy to see that

\[
\hat{\Phi}(0) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{\mathcal{F}_R \varphi(0)}{q^2} = \sum_{q \sim Q} \frac{\phi(q)}{q^2} \mathcal{F}_R \varphi(0)
\]

(2-9)

is a constant independent of \( Q \). Here \( \phi \) is Euler’s phi function, and \( \mathcal{F}_R \) denotes the Fourier transform of a function on \( \mathbb{R} \). Also we have

\[
\hat{\Phi}(k) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{1}{q^2} e^{-2\pi i (a/q)k} \mathcal{F}_R \varphi(k/q^2).
\]

(2-10)

Applying Lemma 2.2 and the fact that \( Q \leq N^3 \), we obtain

\[
|\hat{\Phi}(k)| \leq \frac{N^\varepsilon}{Q},
\]

(2-11)

if \( k \neq 0 \).

We now define

\[
K_{1,Q}(x, t) = \frac{1}{\Phi(0)} K_N(x, t) \Phi(t) \quad \text{and} \quad K_{2,Q} = K_N - K_{1,Q}.
\]

Equation (2-6) follows from Lemma 2.1 since the intervals \( J_{a/q} = \left[ \frac{a}{q} + \frac{1}{100q^2}, \frac{a}{q} + \frac{1}{50q^2} \right] \) are pairwise disjoint for all \( Q \leq q \leq 5Q \) and \( a \in \mathcal{P}_q \).

We now prove (2-7). In fact, represent \( \Phi \) as its Fourier series to get

\[
K_{2,Q}(x, t) = -\frac{1}{\Phi(0)} \sum_{k \neq 0} \hat{\Phi}(k) e^{2\pi i kt} K_N(x, t).
\]

Thus its Fourier coefficient is

\[
\hat{K}_{2,Q}(n_1, n_2) = -\frac{1}{\Phi(0)} \sum_{k \neq 0} \hat{\Phi}(k) \mathbf{1}_{\{n_2 = n_1^3 + k\}}(k).
\]

Here \( (n_1, n_2) \in \mathbb{Z}^2 \) and \( \mathbf{1}_A \) is the indicator function of a set \( A \). This implies that \( \hat{K}_{2,Q}(n_1, n_2) = 0 \) if \( n_2 = n_1^3 \), and if \( n_2 \neq n_1^3 \),

\[
\hat{K}_{2,Q}(n_1, n_2) = -\frac{1}{\Phi(0)} \hat{\Phi}(n_2 - n_1^3).
\]
Applying (2-11), we estimate $\hat{K}_{2,Q}(n_1, n_2)$ by
\[ |\hat{K}_{2,Q}(n_1, n_2)| \leq \frac{C N^\varepsilon}{Q}, \]
since $N \leq Q \leq N^2$. Hence we obtain (2-7), completing the proof. \qed

Now we can state our theorem on the level set estimates.

**Theorem 2.4.** For any positive numbers $\varepsilon$ and $Q \geq N^2$, the level set defined as in (2-2) satisfies
\[ \lambda^2 |E_\lambda|^2 \leq C_1 N^{1/4+\varepsilon} Q^{1/4} |E_\lambda|^2 + \frac{C_2 N^\varepsilon}{Q} |E_\lambda| \] (2-12)
for all $\lambda > 0$. Here $C_1$ and $C_2$ are constants independent of $N$ and $Q$.

**Proof.** Notice that if $Q \geq N^3$, (2-12) becomes trivial, since $E_\lambda = \varnothing$ if $\lambda \geq CN^{1/2}$. So we can assume that $N^2 \leq Q \leq N^3$. For the function $F_N$ and the level set $E_\lambda$ given in (2-1) and (2-2), respectively, we define $f$ to be
\[ f(x, t) = \frac{F_N(x, t)}{|F_N(x, t)|} 1_{E_\lambda}(x, t). \]
Clearly
\[ \lambda |E_\lambda| \leq \int_{T^2} F_N(x, t) f(x, t) dx dt. \]
By the definition of $F_N$, we get
\[ \lambda |E_\lambda| \leq \sum_{n=-N}^{N} \hat{a}_n \hat{f}(n, n^3). \]
Utilizing the Cauchy–Schwarz inequality, we have
\[ \lambda^2 |E_\lambda|^2 \leq \sum_{n=-N}^{N} |\hat{f}(n, n^3)|^2. \]
The right hand side can be written as
\[ (K_N * f, f). \] (2-13)
For any $Q$ with $N^2 \leq Q \leq N^3$, we employ Proposition 2.3 to decompose the kernel $K_N$. We then have
\[ \lambda^2 |E_\lambda|^2 \leq |(K_{1,Q} * f, f)| + |(K_{2,Q} * f, f)|. \] (2-14)
From (2-6) and (2-7), we then obtain
\[ \lambda^2 |E_\lambda|^2 \leq C_1 N^{1/4+\varepsilon} Q^{1/4} \| f \|^2_1 + \frac{C_2 N^\varepsilon}{Q} \| f \|^2_2 \leq C_1 N^{1/4+\varepsilon} Q^{1/4} |E_\lambda|^2 + \frac{C_2 N^\varepsilon}{Q} |E_\lambda|. \] \qed

**Corollary 2.5.** If $\lambda \geq 2C_1 N^{3/8+\varepsilon}$,
\[ |E_\lambda| \leq \frac{C N^{1+\varepsilon}}{\lambda^{10}}. \] (2-15)
Here $C_1$ is the constant $C_1$ in Theorem 2.4 and $C$ is a constant independent of $N$ and $\lambda$. 
Proof. Since \( \lambda \geq 2C_1N^{3/8+\varepsilon} \), we simply take \( Q \) satisfying \( 2C_1N^{1/4+\varepsilon}Q^{1/4} = \lambda^2 \). Then Corollary 2.5 follows from Theorem 2.4. \( \square \)

We are now ready to finish the proof of Theorem 1.1. In fact, let \( p \geq 14 \) and write
\[
p \int_0^{2C_1N^{3/8+\varepsilon}} \lambda^{p-1}|E_\lambda| d\lambda + p \int_{2C_1N^{3/8+\varepsilon}}^{2N^{1/2}} \lambda^{p-1}|E_\lambda| d\lambda. \tag{2-16}
\]
Observe that \( A_{6,N} \leq N^\varepsilon \) implies
\[
|E_\lambda| \leq \frac{N^\varepsilon}{\lambda^6}. \tag{2-17}
\]
Thus the first term in (2-16) is bounded by
\[
CN^{3(p-6)/8+\varepsilon} \leq CN^{p/2-4+\varepsilon}, \tag{2-18}
\]
since \( p \geq 14 \). From (2-15), the second term is majorized by
\[
CN^{p/2-4+\varepsilon}. \tag{2-19}
\]
Putting both estimates together, we complete the proof of Theorem 1.1. \( \square \)

3. A Lower bound of \( A_{p,N} \)

In this section we show that \( N^{1-8/p} \) is the best upper bound of \( A_{p,N} \) if \( p \geq 8 \). Hence (1-3) can not be improved substantially, and it is sharp up to a factor of \( N^\varepsilon \).

For \( b \in \mathbb{N} \), let \( J(N; b) \) be defined by
\[
S(N; b) = \int_{T^2} \left| \sum_{n \in \mathbb{Z}} e^{2\pi im_1+2\pi i n} \right|^{2b} dx dt. \tag{3-1}
\]

Proposition 3.1. Let \( S(N; b) \) be defined as in (3-1). Then
\[
S(N; b) \geq C(N^b + N^{2b-4}). \tag{3-2}
\]
Here \( C \) is a constant independent of \( N \).

Proof. Clearly \( S(N; b) \) is equal to the number of solutions of
\[
\begin{align*}
    &n_1 + \cdots + n_b = m_1 + \cdots + m_b, \\
    &n_1^3 + \cdots + n_b^3 = m_1^3 + \cdots + m_b^3
\end{align*} \tag{3-3}
\]
with \( n_j, m_j \in \{-N, \ldots, N\} \) for all \( j \in \{1, \ldots, b\} \). For each \( (m_1, \ldots, m_b) \), we may obtain a solution of (3-3) by taking \( (n_1, \ldots, n_b) = (m_1, \ldots, m_b) \). Thus
\[
S(N; b) \geq N^b. \tag{3-4}
\]
To derive a further lower bound for \( S(N; b) \), we set \( \Omega \) to be
\[
\Omega = \left\{ (x, t) : |x| \leq \frac{1}{60N}, |t| \leq \frac{1}{60N^3} \right\}. \tag{3-5}
\]
If \((x, t) \in \Omega\) and \(|n| \leq N\),
\[
|tn^3 + xn| \leq \frac{1}{30}.
\] (3-6)

Hence, if \((x, t) \in \Omega\),
\[
\left| \sum_{n=-N}^{N} e^{2\pi itn^3 + 2\pi i xn} \right| \geq \left| \text{Re} \sum_{n=-N}^{N} e^{2\pi itn^3 + 2\pi i xn} \right| \geq \sum_{n=-N}^{N} \cos(2\pi (tn^3 + xn)) \geq CN. \] (3-7)

Consequently, we have
\[
S(N; b) \geq \int_{\Omega} \left| \sum_{n=-N}^{N} e^{2\pi itn^3 + 2\pi i xn} \right|^{2b} dx dt \geq CN^{2b}|\Omega| \geq CN^{2b-4}. \] \(\square\)

**Proposition 3.2.** Let \(p \geq 2\) be even. Then \(A_{p,N}\) satisfies
\[
A_{p,N} \geq C(1 + N^{1-8/p}). \] (3-8)

Here \(C\) is a constant independent of \(N\).

**Proof.** Let \(p = 2b\) since \(p\) is even. Setting \(a_n = 1\) for all \(n\) in the definition of \(K_{p,N}\), we get
\[
S(N; b) \leq K_{p,N}^P (2N)^b. \] (3-9)

By Proposition 3.1, we have
\[
K_{p,N} \geq C(1 + N^{1/2-4/p}). \] (3-10)

Consequently, we conclude (3-8) since \(A_{p,N} \sim K_{p,N}^2\). \(\square\)

4. An estimate of Hua

The following theorem was proved by Hua [1965] by an arithmetic argument. We provide a different proof.

**Theorem 4.1.** Let \(S(N; b)\) be defined as in (3-1). Then
\[
S(N; 5) \leq CN^{6+\varepsilon}. \] (4-1)

By Proposition 3.1, we see that the estimate (4-1) is (almost) sharp. \(S(N; 4) \leq N^{4+\varepsilon}\) is still open.

**Proof of Theorem 4.1.** Let \(G_\lambda\) be the level set given by
\[
G_\lambda = \{(x, t) \in \mathbb{T}^2 : |K_N(x, t)| \geq \lambda\}. \] (4-2)

Here \(K_N\) is the function defined as in (2-5).

Letting \(f = 1_{G_\lambda} K_N / |K_N|\), we have
\[
\lambda |G_\lambda| \leq \sum_{n=-N}^{N} \hat{f}(n, n^3) = \langle f_N, K_N \rangle, \] (4-3)
where $f_N$ is a rectangular Fourier partial sum defined by
\[
f_N(x, t) = \sum_{|n_1| \leq N} \sum_{|n_2| \leq N^3} \hat{f}(n_1, n_2) e^{2\pi n_1 x} e^{2\pi i n_2 t}.
\] (4-4)

Employing Proposition 2.3 for $K_N$, we estimate the level set $G_\lambda$ by
\[
\lambda |G_\lambda| \leq |\langle f_N, K_1, Q \rangle| + |\langle f_N, K_2, Q \rangle|.
\] (4-5)

for any $Q \geq N^2$. From (2-6) and (2-7), $\lambda |G_\lambda|$ can be bounded further by
\[
C \left( N^{1/4+\varepsilon} Q^{1/4} \|f_N\|_1 + \sum_{|n_1| \leq N} \sum_{|n_2| \leq N^3} |\hat{K}_2, Q(n_1, n_2) \hat{f}(n_1, n_2)| \right).
\] (4-6)

Thus, from the fact that the $L^1$ norm of Dirichlet kernel $D_N$ is comparable to $\log N$, (2-7), and the Cauchy–Schwarz inequality, we have
\[
\lambda |G_\lambda| \leq CN^{1/4+\varepsilon} Q^{1/4} |G_\lambda| + \frac{CN^{2+\varepsilon}}{Q} |G_\lambda|^{1/2},
\] (4-7)

for all $Q \geq N^2$. For $\lambda \geq 2CN^{3/4+\varepsilon}$, take $Q$ to be a number satisfying
\[
2CN^{1/4+\varepsilon} Q^{1/4} = \lambda,
\]
and obtain
\[
|G_\lambda| \leq \frac{CN^{6+\varepsilon}}{\lambda^{10}}.
\] (4-8)

Notice that
\[
\|K_N\|_6 \leq N^{1/2} K_{6, p} \leq N^{1/2+\varepsilon}.
\] (4-9)

Hence, by (4-3), we majorize $|G_\lambda|$ by
\[
|G_\lambda| \leq \frac{CN^{3+\varepsilon}}{\lambda^6}.
\] (4-10)

We now estimate $S(N; 5)$ by
\[
S(N; 5) \leq C \int_{2CN^{3/4+\varepsilon}}^{2N} \lambda^{10-1} |G_\lambda| d\lambda + C \int_0^{2CN^{3/4+\varepsilon}} \lambda^{10-1} |G_\lambda| d\lambda.
\] (4-11)

From (4-8), the first term in the right hand side of (4-11) can be bounded by $CN^{6+\varepsilon}$. From (4-10), the second term is clearly bounded by $N^{6+\varepsilon}$. Putting both estimates together,
\[
S(N; 5) \leq CN^{6+\varepsilon},
\] (4-12)
as desired.
5. Estimates for the nonlinear term and Local well-posedness of (1-6)

For any integrable function $u$ on $\mathbb{T} \times \mathbb{R}$, we define the space-time Fourier transform by

$$\hat{u}(n, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{T}} u(x, t) e^{-inx} e^{-i\lambda t} \, dx \, dt \quad (5-1)$$

and set

$$\langle x \rangle := 1 + |x|.$$ 

We now introduce the $X_{s,b}$ space, initially used by Bourgain.

**Definition 5.1.** Let $I$ be a time interval in $\mathbb{R}$ and $s, b \in \mathbb{R}$. Let $X_{s,b}(I)$ be the space of functions $u$ on $\mathbb{T} \times I$ that may be represented as

$$u(x, t) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \hat{u}(n, \lambda) e^{inx} e^{i\lambda t} \, d\lambda \quad \text{for } (x, t) \in \mathbb{T} \times I \quad (5-2)$$

with the space-time Fourier transform $\hat{u}$ satisfying

$$\|u\|_{X_{s,b}(I)} = \left( \sum_n \int \langle n \rangle^{2s} \langle \lambda - n^3 \rangle^{2b} |\hat{u}(n, \lambda)|^2 \, d\lambda \right)^{1/2} < \infty. \quad (5-3)$$

Here the norm should be understood as a restriction norm.

We take the time interval to be $[0, \delta]$ for a small positive number $\delta$ and abbreviate $\|u\|_{X_{s,b}(I)}$ as $\|u\|_{s,b}$ for any function $u$ restricted to $\mathbb{T} \times [0, \delta]$. In this section, we always restrict the function $u$ to $\mathbb{T} \times [0, \delta]$.

Let $w$ be the nonlinear function defined by

$$w = \left( u^k - \int u^k \, dx \right) u_x. \quad (5-4)$$

We also define

$$\|u\|_{Y_s} := \|u\|_{s,1/2} + \left( \sum_n \langle n \rangle^{2s} \left( \int |\hat{u}(n, \lambda)| \, d\lambda \right)^2 \right)^{1/2}. \quad (5-5)$$

We need the following estimate on the nonlinear function $w$, in order to establish a contraction on the space $\{u : \|u\|_{Y_s} \leq M\}$ for some $M > 0$.

**Proposition 5.2.** For $s > 1/2$, there exists $\theta > 0$ such that, for the nonlinear function $w$ given by (5-4),

$$\|w\|_{s,-1/2} + \left( \sum_n \langle n \rangle^{2s} \left( \int \frac{|\hat{w}(n, \lambda)|}{\langle \lambda - n^3 \rangle} \, d\lambda \right)^2 \right)^{1/2} \leq C \delta^\theta \|u\|_{Y_s}^{k+1}. \quad (5-6)$$

Here $C$ is a constant independent of $\delta$ and $u$.

The proof of Proposition 5.2 will appear in Section 6, and is based on the idea applied by Bourgain [1993b] while proving the special case $k = 2$. In the proof, we write out the detailed treatment to some subcases, and omit the similar treatment of other subcases (but it is very easy to figure out). The main
reason we include the proof of Proposition 5.2 in Section 6 is to provide the preparation so the readers can follow the (more technical) proof of the general case \( F \in C^5 \) more easily.

We now start to derive the local well-posedness of (1-6). For this purpose, we only need to consider the well-posedness of the Cauchy problem

\[
\begin{cases}
    u_t + u_{xxx} + (u^k - \int_T u^k \, dx) u_x = 0, \\
    u(x, 0) = \phi(x), \quad x \in \mathbb{T}, \; t \in \mathbb{R}.
\end{cases}
\]  

(5-7)

This is because if \( v \) is a solution of (5-7), the gauge transform

\[
u(x, t) := v\left(x - \int_0^t \int_{\mathbb{T}} u^k(y, \tau) \, dy \, d\tau, \; t\right)
\]

is a solution of (1-6) with the same initial value \( \phi \). Notice that this transform is invertible and preserves the initial data \( \phi \). The inverse transform is

\[
v(x, t) := u\left(x + \int_0^t \int_{\mathbb{T}} u^k(y, \tau) \, dy \, d\tau, \; t\right).
\]

(5-9)

It is easy to see that for any solution \( u \) of (1-6), this inverse transform of \( u \) defines a solution of (5-7).

By Duhamel’s principle, the corresponding integral equation associated to (5-7) is

\[
u(x, t) = e^{-t \partial_x^3} \phi(x) - \int_0^t e^{-(t-\tau) \partial_x^3} w(x, \tau) \, d\tau,
\]

(5-10)

where \( w \) is defined as in (5-4).

Since we are only seeking the local well-posedness, we may use a bump function to truncate the time variable. Let \( \psi \) be a bump function supported in \([-2, 2]\) with \( \psi(t) = 1, |t| \leq 1 \), and let \( \psi_\delta \) be

\[
\psi_\delta(t) = \psi(t/\delta).
\]

Then it suffices to find a local solution of

\[
u(x, t) = \psi_\delta(t) e^{-t \partial_x^3} \phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau) \partial_x^3} w(x, \tau) \, d\tau.
\]

Let \( T \) be an operator given by

\[
Tu(x, t) := \psi_\delta(t) e^{-t \partial_x^3} \phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau) \partial_x^3} w(x, \tau) \, d\tau.
\]

(5-11)

We denote the first term (the linear term) in (5-11) by \( Lu \) and the second term (the nonlinear term) by \( Nu \). Henceforth we represent \( Tu \) as \( Lu + Nu \). The following two lemmas deal with \( Lu \) and \( Nu \) separately.

**Lemma 5.3.** The linear term \( L \) satisfies

\[
\|Lu\|_{Y_\delta} \leq C \|\phi\|_{H^1}.
\]

Here \( C \) is a constant independent of \( \delta \).
Lemma 5.4. The nonlinear term $\mathcal{N}$ satisfies
\[ \|\mathcal{N} u\|_{Y_s} \leq C\left(\|w\|_{X, -1/2} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\hat{w}(n, \lambda)|}{|\lambda - n|^3} \, d\lambda \right)^2\right)^{1/2}\right), \] (5-13)
where $C$ is a constant independent of $\delta$.

Lemmas 5.3 and 5.4 are considered classical and their proofs can be found in many references, such as [Colliander et al. 2004].

Proposition 5.5. Let $s > 1/2$ and $T$ be the operator defined as in (5-11). Then there exists a positive number $\theta$ such that
\[ \|T u\|_{Y_s} \leq C(\|\phi\|_{H^s} + \|u\|_{Y_s}^{k+1}). \] (5-14)
Here $C$ is a constant independent of $\delta$.

Proof. Since $T u = Lu + Nu$, Proposition 5.5 follows from Lemmas 5.3, 5.4, and Proposition 5.2. \qed

Proposition 5.5 yields that for $\delta$ sufficiently small, $T$ maps a ball in $Y_s$ into itself. Moreover, we write
\[ \left( u^k - \int u^k \, dx \right) u_x - \left( v^k - \int v^k \, dx \right) v_x = \left( u^k - \int u^k \, dx \right) (u - v)_x + \left( (u^k - v^k) - \int (u^k - v^k) \, dx \right) v_x \]
which equals
\[ \left( u^k - \int u^k \, dx \right) (u - v)_x + \sum_{j=0}^{k-1} \left( (u - v) u^{k-1-j} v^j - \int (u - v) u^{k-1-j} v^j \, dx \right) v_x. \] (5-15)

For $k + 1$ terms in (5-15), repeating similar argument as in the proof of Proposition 5.2, one obtains, for $s > 1/2$,
\[ \|T u - T v\|_{Y_s} \leq C \delta^\theta \left( \|u\|_{Y_s}^k + \sum_{j=1}^{k-1} \|u\|_{Y_s}^{k-1-j} \|v\|_{Y_s}^{j+1} \right) \|u - v\|_{Y_s}. \] (5-16)
Hence, for $\delta > 0$ small enough, $T$ is a contraction and the local well-posedness follows from Picard’s fixed-point theorem.

6. Proof of Proposition 5.2

Proof. From the definition of $w$ in (5-4), we may write $\hat{w}(n, \lambda)$ as
\[ \sum_{m+n_1+\cdots+n_k=n \atop n_1+\cdots+n_k \neq 0} m \int \hat{w}(m, \lambda - \lambda_1 - \cdots - \lambda_k) \hat{w}(n_1, \lambda_1) \cdots \hat{w}(n_k, \lambda_k) \, d\lambda_1 \cdots d\lambda_k. \] (6-1)

By duality, there exists a sequence $\{A_{n, \lambda}\}$ satisfying
\[ \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |A_{n, \lambda}|^2 \, d\lambda \leq 1, \] (6-2)
and \( \|w\|_{s,-1/2} \) is bounded by
\[
\sum_{m+n_1+\cdots+n_k=n \atop n_1+\cdots+n_k \neq 0} \int \frac{\langle n \rangle^s |m|}{\langle \lambda-n^3 \rangle^{1/2}} |\hat{u}(m, \lambda-\lambda_1-\cdots-\lambda_k)| |\hat{u}(n_1, \lambda_1)| \cdots |\hat{u}(n_k, \lambda_k)| |A_{n,\lambda}| \, d\lambda_1 \cdots d\lambda_k \, d\lambda. \tag{6-3}
\]

Since the \( X_{s,b} \) is a restriction norm, we may assume that \( u \) is supported in \( \mathbb{T} \times [0, \delta] \). However, the inverse space-time Fourier transform \( |\hat{u}|^\vee \) in general may not be a function with compact support. The following standard trick allows us to assume \( |\hat{u}|^\vee \) has a compact support too. In fact, let \( \eta \) be a bump function supported on \([−2\delta, 2\delta]\) and with \( \eta(t) = 1 \) in \( |t| \leq \delta \). Also \( \hat{\eta} \) is positive. Then \( u = u\eta \) and \( \hat{u} = \hat{u} \ast \hat{\eta} \). Thus \( |\hat{u}| \leq |\hat{u}| \ast |\hat{\eta}| = (|\hat{u}|^\vee \eta)^\wedge \). Whenever we need to make \( |\hat{u}|^\vee \) supported in a small time interval, we replace \( |\hat{u}| \) by \((|\hat{u}|^\vee \eta)^\wedge \) since \( |\hat{u}|^\vee \eta \) clearly is supported on \( \mathbb{T} \times [−2\delta, 2\delta] \). This will help us gain a positive power of \( \delta \) in our estimates. Moreover, without loss of generality we can assume \( |n_1| \geq |n_2| \geq \cdots \geq |n_k| \).

The trouble occurs mainly because of the factor \( |m| \) resulting from \( \partial_t u \). The idea (inspired by Bourgain [1993b]) is that either the factor \( \langle \lambda - n^3 \rangle^{-1/2} \) can be used to cancel \( |m| \), or \( |m| \) can be distributed to some of the \( \hat{u} \). More precisely, we consider three cases:

\[
|m| < 1000k^2 |n_2|, \tag{6-4}
\]
\[
100k^2 |n_2| \leq |m| \leq 100k |n_1|, \tag{6-5}
\]
\[
|m| > 100k |n_1|. \tag{6-6}
\]

**Case 1:** \( |m| < 1000k^2 |n_2| \). This is the simplest case. In fact, in this case, it is easy to see that
\[
\langle n \rangle^s |m| \leq C \langle n_1 \rangle^s \langle n_2 \rangle^{1/2} / m^{1/2}. \tag{6-7}
\]

Let
\[
F_1(x, t) = \sum_n \int |A_{n,\lambda}| \langle \lambda-n^3 \rangle^{1/2} e^{i\lambda t} e^{inx} \, d\lambda; \tag{6-8}
\]
\[
G(x, t) = \sum_n \int \langle n \rangle^{1/2} |\hat{u}(n, \lambda)| e^{i\lambda t} e^{inx} \, d\lambda; \tag{6-9}
\]
\[
H(x, t) = \sum_n \int \langle n \rangle^s |\hat{u}(n, \lambda)| e^{i\lambda t} e^{inx} \, d\lambda; \tag{6-10}
\]
\[
U(x, t) = \sum_n \int |\hat{u}(n, \lambda)| e^{i\lambda t} e^{inx} \, d\lambda. \tag{6-11}
\]

Using (6-7), we can estimate (6-3) by
\[
C \sum_{m+n_1+\cdots+n_k=n} \int \hat{F}(n, \lambda) \hat{G}(m, \lambda-\lambda_1-\cdots-\lambda_k) \hat{H}(n_1, \lambda_1) \hat{G}(n_2, \lambda_2) \prod_{j=3}^k \hat{U}(n_j, \lambda_j) \, d\lambda_1 \cdots d\lambda_k \, d\lambda,
\]
which clearly equals
\[
C \int_{\mathbb{T} \times \mathbb{R}} F_1(x, t) G(x, t)^2 H(x, t) U(x, t)^{k-2} \, dx \, dt. \tag{6-12}
\]

Apply Hölder’s inequality to majorize it by
\[ C \| F_1 \|_4 \| G \|_{6+}^2 \| H \|_4 \| U \|_{6(k-2)-}^{k-2}. \]
Since \( U \) is supported on \( \mathbb{T} \times [-2\delta, 2\delta] \), one more use of Hölder inequality yields
\[ (6-3) \leq C \delta^\theta \| F_1 \|_4 \| G \|_{6+}^2 \| H \|_4 \| U \|_{6(k-2)-}^{k-2}. \quad (6-13) \]

Let us recall some useful local embedding facts on \( X_{s,b} \).
\[ X_{0,1/3} \subseteq L^4_{x,t}, \quad X_{0+.1/2+} \subseteq L^6_{x,t} \quad (t \text { local}), \quad (6-14) \]
\[ X_{\alpha,1/2} \subseteq L^q_{x,t}, \quad 0 < \alpha < \frac{1}{2}, \quad 2 \leq q < \frac{6}{1-2\alpha} \quad (t \text { local}), \quad (6-15) \]
\[ X_{1/2-\alpha,1/2-} \subseteq L^q_{t} L^r_{x}, \quad 0 < \alpha < \frac{1}{2}, \quad 2 \leq q, r < 1/\alpha. \quad (6-16) \]
The two embedding results in (6-14) are consequences of the discrete restriction estimates on \( L^4 \) and \( L^6 \), respectively (see \[ Bourgain 1993b \] for details). (6-15) and (6-16) follow by interpolation (see \[ Colliander et al. 2004 \] for details). (6-14) yields
\[ \| F_1 \|_4 \leq C \| F_1 \|_{0+.1/3} \leq C \left( \sum_n \int |A_{n,\lambda}|^2 d\lambda \right)^{1/2} \leq C, \]
and
\[ \| H \|_4 \leq C \| H \|_{0.1/3} \leq C \| u \|_{s,1/2} \leq C \| u \|_{Y_s}. \]
From (6-15) we have
\[ \| G \|_{6+} \leq C \| G \|_{0+.1/2} \leq C \| u \|_{s,1/2} \leq C \| u \|_{Y_s}. \]
Using (6-16), we get
\[ \| U \|_{6(k-2)-} \leq C \| U \|_{1/2-.1/2-} \leq C \| u \|_{s,1/2} \leq C \| u \|_{Y_s}. \]
Hence, for Case 1, we have
\[ (6-3) \leq C \delta^\theta \| u \|_{Y_s}^{k+1}. \quad (6-17) \]

Case 2: \( 100k^2|n_2| \leq |m| \leq 100k|n_1| \). In this case, we further consider two subcases:
\[ |m+n_1| \leq 100k^2|n_2|, \quad (6-18) \]
\[ |m+n_1| > 100k^2|n_2|. \quad (6-19) \]
If \( |m+n_1| \leq 100k^2|n_2| \), we use the triangle inequality to get
\[ |n| = |m+n_1+n_2+\cdots+n_k| \leq C |n_2|. \quad (6-20) \]
Hence we have
\[ \langle n \rangle^s |m| \leq C \langle n_2 \rangle^s \langle m \rangle^{1/2} \langle n_1 \rangle^{1/2}. \quad (6-21) \]
Thus this subcase can be treated exactly the same as Case 1. We omit the details.
In the second subcase, $|m + n_1| > 1000k^2|n_2|$, the crucial arithmetic observation is
\[ n^3 - (m^3 + n_1^3 + \cdots + n_k^3) = 3(m + n_1)(m + a)(n_1 + a) + a^3 - (n_2^3 + \cdots + n_k^3), \] (6-22)
where $a = n_2 + \cdots + n_k$. This observation can be easily verified since $n = m + n_1 + \cdots + n_k$. From (6-5) and (6-19), we get
\[ |n^3 - (m^3 + n_1^3 + \cdots + n_k^3)| \geq Ck^2|n_2||m||n_1| \geq Ck|m|^2. \] (6-23)
This implies that at least one of following statements holds:
\[ |\lambda - n^3| \geq C|m|^2, \] (6-24)
\[ |\langle \lambda - \lambda_1 - \cdots - \lambda_k \rangle - m^3| \geq C|m|^2, \] (6-25)
there exists an $i \in \{1, \ldots, k\}$ such that $|\lambda_i - n_i^3| \geq C|m|^2$. (6-26)

For (6-24), (6-3) can be bounded by
\[ \sum_{m+n_1+\cdots+n_k=n} \int \langle n_1 \rangle^s |\hat{u}(m, \lambda - \lambda_1 - \cdots - \lambda_k)| |\hat{u}(n_1, \lambda_1)| \cdots |\hat{u}(n_k, \lambda_k)| |A_{n,\lambda}| d\lambda_1 \cdots d\lambda_k d\lambda. \] (6-27)
Let $F_2$ be defined by
\[ F_2(x, t) = \sum_n \int |A_{n,\lambda}| e^{i\lambda t} e^{inx} d\lambda. \] (6-28)
Then we represent (6-27) as
\[ \sum_{m+n_1+\cdots+n_k=n} \int \hat{F}_2(n, \lambda) \hat{U}(m, \lambda - \lambda_1 - \cdots - \lambda_k) \hat{H}(n_1, \lambda_1) \prod_{j=2}^k \hat{U}(n_j, \lambda_j) d\lambda_1 \cdots d\lambda_k d\lambda. \] (6-29)
Here $H$ and $U$ are the functions defined in (6-10) and (6-11). Clearly (6-29) equals
\[ \int_{\mathbb{T} \times \mathbb{R}} F_2(x, t) H(x, t) U(x, t)^k \ dx \ dt. \] (6-30)
Utilizing Hölder’s inequality, we estimate it further by
\[ \|F_2\|_2 \|H\|_4 \|U\|_{4k}^k \leq C \delta^\theta \|u\|_{Y_{4k}}^{k+1}. \] (6-31)
This yields the desired estimate for subcase (6-24).

One can similarly complete the proofs of subcases (6-25) and (6-26), and hence the proof of Case 2.

Case 3: $|m| > 100k|n_1|$. The arithmetic observation (6-22) again plays an important role. In this case, let us further consider two subcases:
\[ |m|^2 \leq 1000k^2|n_2|^2|n_3|, \] (6-32)
\[ |m|^2 > 1000k^2|n_2|^2|n_3|. \] (6-33)

For the first subcase, we observe that, from (6-32),
\[ |m|^2 \leq C|n_1||n_2||n_3|, \]

since \(|n_2| \leq |n_1|\). Hence we have

\[ |m| = |m|^{1/2}|m|^{2/3} \leq C|m|^{1/3}|n_1|^{1/3}|n_2|^{1/3}|n_3|^{1/3}. \]  

(6-34)

This immediately implies

\[ \langle n \rangle^s|m| \leq C|m|^{s+1} \leq \langle m \rangle^{(s+1)/3} \langle n_1 \rangle^{(s+1)/3} \langle n_2 \rangle^{(s+1)/3} \langle n_3 \rangle^{(s+1)/3}. \]  

(6-35)

Note that \((s+1)/3 < s\) for \(s > 1/2\). By distributing the four factors to the corresponding functions, one can mimic the proof of Case 1 to finish subcase (6-32).

We now turn to the contribution of (6-33). Clearly we have

\[ |(n_2 + \cdots + n_k)^3 - (n_2^3 + \cdots + n_k^3)| \leq 10k|n_2|^2|n_3|, \]  

(6-36)

since \(|n_2| \geq |n_3| \geq \cdots \geq |n_k|\). From the crucial arithmetic observation (6-22), (6-36), and (6-33), we have

\[ |n^3 - (m^3 + n_1^3 + \cdots + n_k^3)| \geq Ck|m|^2. \]  

(6-37)

This is the same as (6-23). Hence we again reduce the problems to (6-24), (6-25), and (6-26), which were all done in Case 2. Therefore Case 3 is finished.

Putting all the cases together, we obtain

\[ \|w\|_{s,-1/2} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}. \]  

(6-38)

Finally we need to estimate

\[ \left( \sum_n \langle n \rangle^{2s} \left( \int \frac{|\hat{w}(n, \lambda)|}{\langle \lambda - n^3 \rangle} d\lambda \right)^2 \right)^{1/2}. \]  

(6-39)

Let \(\{A_n\}\) be a sequence with

\[ \left( \sum_n |A_n|^2 \right)^{1/2} \leq 1. \]

By duality, it suffices to estimate

\[ \sum_{m+n_1+\cdots+n_k=n \atop n_1+\cdots+n_k \neq 0} \int \frac{\langle n \rangle^s|m|}{\langle \lambda - n^3 \rangle} |\hat{u}(m, \lambda - \lambda_1 - \cdots - \lambda_k)| |\hat{u}(n_1, \lambda_1)| \cdots |\hat{u}(n_k, \lambda_k)| |A_n| d\lambda_1 \cdots d\lambda_k d\lambda. \]  

(6-40)

By the same idea and similar techniques, one can bound (6-40) by mimicking the treatment of (6-3) and get

\[ \left( \sum_n \langle n \rangle^{2s} \left( \int \frac{|\hat{w}(n, \lambda)|}{\langle \lambda - n^3 \rangle} d\lambda \right)^2 \right)^{1/2} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}. \]  

(6-41)

We complete the proof of Proposition 5.2 by combining (6-38) and (6-41). \qed
7. Proof of Theorem 1.3

The argument is similar to that in Section 5. By using a gauge transform as in (5-8) with $u^k$ replaced by $F(v)$, the well-posedness of (1-7) is equivalent to the well-posedness of the following equation:

\[
\begin{cases}
  u_t + u_{xxx} + (F(u) - \int_T F(u) \, dx)u_x = 0, \\
  u(x, 0) = \phi(x), \quad x \in \mathbb{T}, \; t \in \mathbb{R}.
\end{cases}
\]  

(7-1)

Now the nonlinear function $w$ is defined by

\[ w = \partial_x u \left( F(u) - \int_T F(u) \, dx \right). \]  

(7-2)

Let $T_F$ be an operator given by

\[ T_F u(x, t) := \psi_{\delta}(t) e^{-t \partial_x^3} \phi(x) - \psi_{\delta}(t) \int_0^t e^{-(t-\tau)\partial_x^3} w(x, \tau) \, d\tau. \]  

(7-3)

As in Section 5, the local well-posedness is a consequence of the following proposition.

**Proposition 7.1.** Let $s > 1/2$. There exists $\theta > 0$ such that, for the nonlinear function $w$ given by (7-2) and any $u$ satisfying $\|u\|_{Y_s} \leq C_0 \|\phi\|_{H^s}$,

\[ \|w\|_{s-1/2} + \left( \sum_n \langle n \rangle^{2s} \left( \int \frac{|\hat{w}(n, \lambda)|}{ \langle \lambda - n^3 \rangle} \, d\lambda \right)^2 \right)^{1/2} \leq C(\|\phi\|_{H^s}, F) \delta^\theta \|u\|_{Y_s}^4, \]  

(7-4)

provided $F \in C^5$. Here $C_0$ is a suitably large constant, and $C(\|\phi\|_{H^s}, F)$ is a constant independent of $\delta$ and $u$, but which may depend on $\|\phi\|_{H^s}$ and $F$.

The constant $C(\|\phi\|_{H^s}, F)$ will be specified in the proof of Proposition 7.1, which we postpone to Section 8. We now return to the proof of Theorem 1.3. Proposition 7.1 implies that for $\delta$ sufficiently small, $T$ maps a ball

\[ \{u \in Y_s : \|u\|_{Y_s} \leq C_0 \|\phi\|_{H^s}\} \]

into itself. Moreover, using Lemma 5.4 and repeating similar argument as in the proof of Proposition 7.1, one obtains, for $s > 1/2$ and $F \in C^5$,

\[ \|T_F u - T_F v\|_{Y_s} \leq \delta^\theta C(\|\phi\|_{H^s}, F) \|u - v\|_{Y_s} \]  

(7-5)

for all $u, v$ in the ball $\{u \in Y_s : \|u\|_{Y_s} \leq C_0 \|\phi\|_{H^s}\}$. Therefore, for $\delta > 0$ small enough, $T_F$ is a contraction on the ball and the local well-posedness again follows from Picard’s fixed-point theorem. This completes the proof of Theorem 1.3.

8. Proof of Proposition 7.1

First we introduce a decomposition of $F(u)$ which was used by Bourgain. Let $K$ be a dyadic number, and define a Fourier multiplier operator $P_K$ by setting

\[ P_K u(x, t) = \int \psi_K(y)u(x - y, t) \, dy. \]  

(8-1)
Here the Fourier transform of $\psi_K$ is a standard bump function supported on \([-2K, 2K]\) and $\hat{\psi}_K(x) = 1$ for $x \in [-K, K]$. Let $u_K$ denote the Littlewood–Paley Fourier multiplier, that is,

$$u_K = P_K u - P_{K/2} u.$$  \hfill (8-2)

Then we may decompose $F(u)$ by

$$F(u) = \sum_K (F(P_K u) - F(P_{K/2} u)) = \sum_K F_1(P_K u, P_{K/2} u) u_K + R_1,$$

where $R_1$ is a function independent of the space variable $x$. Repeating this procedure for $F_1$, we obtain

$$F(u) = \sum_{K_1 \geq K_2} F_2(P_{2K_2} u, \ldots, P_{K_2/4} u) u_{K_1} u_{K_2} + \sum_{K_1} R_{2u} u_{K_1} + R_1$$

$$= \sum_{K_1 \geq K_2 \geq K_3} F_3(P_{4K_3} u, \ldots, P_{K_3/8} u) u_{K_1} u_{K_2} u_{K_3} + \sum_{K_1 \geq K_2} R_3 u_{K_1} u_{K_2} + \sum_{K_1} R_{2u} u_{K_1} + R_1$$

where $R_1, R_2, R_3$ are functions independent of the space variable. Set

$$G_{K_3}(x, t) = F_3(P_{4K_3} u, \ldots, P_{K_3/8} u).$$  \hfill (8-3)

Hence we represent $w$, defined in (7-2), as

$$w = \sum_{K_0, K_1 \geq K_2 \geq K_3} \partial_x u_{K_0} \left( u_{K_1} u_{K_2} u_{K_3} G_{K_3} - \int_{\mathbb{T}} u_{K_1} u_{K_2} u_{K_3} G_{K_3} \, dx \right)$$

$$+ \sum_{K_0, K_1 \geq K_2} \partial_x u_{K_0} \left( u_{K_1} u_{K_2} - \int_{\mathbb{T}} u_{K_1} u_{K_2} \, dx \right) R_3 + \sum_{K_0, K_1} \partial_x u_{K_0} \left( u_{K_1} - \int_{\mathbb{T}} u_{K_1} \, dx \right) R_2.$$  \hfill (8-4)

The main contribution of $w$ is from the first term. The remaining terms can be handled by the method presented in Section 6, because $R_2, R_3$ are functions independent of the space variable $x$ (actually they only depend on the conserved quantity $\int_{\mathbb{T}} u \, dx$). Hence in what follows we only focus on estimating the first term — the most difficult one. Denote the first term by $w_1$:

$$w_1 = \sum_{K_0, K_1 \geq K_2 \geq K_3} \partial_x u_{K_0} \left( u_{K_1} u_{K_2} u_{K_3} G_{K_3} - \int_{\mathbb{T}} u_{K_1} u_{K_2} u_{K_3} G_{K_3} \, dx \right).$$  \hfill (8-5)

We should prove

$$\|w_1\|_{s, -1/2} + \left( \sum_n \langle n \rangle^{2s} \left( \int_{\mathbb{T}} |\hat{w}_1(n, \lambda)|^4 \, d\lambda \right)^2 \right)^{1/4} \leq C \delta^s C(\|\phi\|_{H^s}, F) \|u\|_{Y_s}^4.$$  \hfill (8-6)

In order to specify the constant $C(\|\phi\|_{H^s}, F)$, we define $\mathcal{M}$ by setting

$$\mathcal{M} = \sup \{ |D^\alpha F_3(u_1, \ldots, u_6)| : u_j \text{ satisfies } \|u_j\|_{Y_s} \leq C_0 \|\phi\|_{H^s} \text{ for all } j = 1, \ldots, 6; \alpha \}.$$  \hfill (8-6)

Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_6}^{\alpha_6}$ and $\alpha$ is taken over all tuples $(\alpha_1, \ldots, \alpha_6) \in (\mathbb{N} \cup \{0\})^6$ with $\sum |\alpha_j| \leq 2$. $\mathcal{M}$ is a real number. This is because, for $s > 1/2$, $\|u\|_{Y_s} \leq C_0 \|\phi\|_{H^s}$ yields that $u$ is bounded by $C \|\phi\|_{H^s}$, and the previous claim follows from $F_3 \in C^2$. 


In order to bound \( \|w_1\|_{s,-1/2} \), by duality, it suffices to bound
\[
\sum_{K_0, K_1 \geq K_3} \int \frac{A_{n, \lambda} (n)^s n_0}{(\lambda - n^3)^{1/2}} \hat{u}_{K_0} (n_0, \lambda - \lambda_1 - \lambda_2 - \lambda_3 - \mu) \prod_{j=1}^{3} \hat{u}_{K_j} (n_j, \lambda_j) \hat{G}_{K_3} (m, \mu) d\lambda_1 \cdots d\lambda_4 d\lambda d\mu, \tag{8-7}
\]
where \( A_{n, \lambda} \) satisfies
\[
\sum_n \int |A_{n, \lambda}|^2 d\lambda = 1.
\]

The trouble maker is \( G_{K_3} \) since there is no way to find a suitable upper bound for its \( X_{s,b} \) norm. Because of this, the method in Section 6 is no longer valid, and we have to treat \( m \) and \( \mu \) differently from \( n \) and \( \lambda \), respectively. A delicate analysis must be done to overcome the difficulty caused by \( G_{K_3} \).

For simplicity, we assume that \( \delta = 1 \). One can modify the argument to gain a decay of \( \delta^\theta \) by using the technical treatment from Section 6.

For a dyadic number \( M \), define the Littlewood–Paley Fourier multiplier by
\[
g_{K_3, M} = P_M G_{K_3} - P_{M/2} G_{K_3} = (G_{K_3})_M. \tag{8-8}
\]

Let \( v \) be defined by
\[
v(x, t) = \sum_n \int \frac{A_{n, \lambda}}{(\lambda - n^3)^{1/2}} e^{i\lambda t} e^{ins} d\lambda. \tag{8-9}
\]

To estimate (8-7), it suffices to estimate
\[
\sum_{K, K_0, K_1 \geq K_2 \geq K_3, M} \int (\partial_x)^s v_K (n, \lambda) \partial_x \hat{u}_{K_0} (n_0, \lambda - \lambda_1 - \lambda_2 - \lambda_3 - \mu) \prod_{j=1}^{3} \hat{u}_{K_j} (n_j, \lambda_j) g_{K_3, M} (m, \mu) d\lambda_1 \cdots d\lambda_4 d\lambda d\mu. \tag{8-10}
\]

Here \( K \) is a dyadic number.

As we did in Section 6, we consider three cases:
\[
K_0 < 2^{100} K_2; \tag{8-11}
\]
\[
2^{100} K_2 \leq K_0 \leq 2^{10} K_1; \tag{8-12}
\]
\[
K_0 > 2^{10} K_1. \tag{8-13}
\]

The rest of the paper is devoted to a proof of these three cases. In what follows, we will only provide the details for the estimates of \( \|w_1\|_{s,-1/2} \) with \( 1/2 < s < 1 \) (the case \( s \geq 1 \) is easier). For the desired estimate of
\[
\left( \sum_n (n)^{2s} \left( \int \frac{|\hat{w}_1(n, \lambda)|^2}{(\lambda - n^3)^s} d\lambda \right)^2 \right)^{1/2},
\]
simply replace \( v \) by
\[
v_1(x, t) = \sum_n \int \frac{C_n \lambda}{(\lambda - n^2)} e^{i \lambda t} e^{inx} d\lambda,
\]
(8-14)
and then the desired estimate follows similarly. Here \( C_n, \lambda \in \mathbb{C} \) satisfies \( \sup_{\lambda} |C_n, \lambda| \leq 1 \) and \( \{A_n\} \) satisfies \( \sum_n |A_n|^2 \leq 1 \).

9. Proof of case (8-11)

In this case, we should consider further two subcases:

\[
M \leq 2^{10} K_1, \quad (9-1)
\]
\[
M > 2^{10} K_1. \quad (9-2)
\]

For the contribution of \( (9-1) \), noticing that \( K \leq CK_1 \) in this subcase, we estimate (8-10) by

\[
\sum_{K_1 \geq K_2 \geq K_3} \int_{T \times \mathbb{R}} \left| \left( \sum_{K \leq CK_1} \partial_x v_K \right) \left( \sum_{K_0 \leq CK_2} \partial_x u_{K_0} \right) u_{K_1} u_{K_2} u_{K_3} (P_{2^{10} K_1} G_{K_3}) \right| \, dx \, dt,
\]
(9-3)
which is bounded by

\[
\sum_{K_3} \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \int_{T \times \mathbb{R}} \sum_{K \leq CK_1} \sum_{K_0 \leq CK_2} K^s v^*_K |u_{K_1}| \sum_{K_2} K^s u^*_K |u_{K_2}| \, dx \, dt,
\]
(9-4)
where \( f^* \) stands for the Hardy–Littlewood maximal function of \( f \). By the Schur’s test, (9-4) can be estimated by

\[
\sum_{K_3} K_3^{-(2s-1)/2} \|u\|_{Y_4} \mathcal{O} \int \left( \sum_{K} |v^*_K|^2 \right)^{1/2} \left( \sum_{K_1} K^{2s} |u_{K_1}|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2s} |u_{K_0}|^2 \right)^{1/2} \left( \sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} \, dx \, dt.
\]
(9-5)
Since \( s > 1/2 \), we obtain, by a use of Hölder’s inequality, that (9-4) is majorized by

\[
C \mathcal{O} \|u\|_{Y_4} \left( \sum_{K} |v^*_K|^2 \right)^{1/4} \left( \sum_{K_1} K^{2s} |u_{K_1}|^2 \right)^{1/4} \left( \sum_{K_0} K_0^{2s} |u_{K_0}|^2 \right)^{1/4} \left( \sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/4}.
\]
(9-6)
Observe that

\[
\left( \sum_{K} |v^*_K|^2 \right)^{1/4} \leq \left( \sum_{K} |v_K|^2 \right)^{1/4} \leq C \|v\|_4 \leq C \|v\|_{0, 1/3} \leq C.
\]
(9-7)
Here the first inequality is obtained by using Fefferman and Stein’s vector-valued inequality on the maximal function, and the second is a consequence of the classical Littlewood–Paley theorem. Similarly,

\[
\left( \sum_{K_0} K_0 |u_{K_0}|^2 \right)^{1/4} \leq \left( \sum_{K_0} K_0 |u_{K_0}|^2 \right)^{1/4} \leq C \|\partial_x^{1/2} u\|_4 \leq C \|u\|_{1/2, 1/3} \leq C \|u\|_{Y_4}.
\]
(9-8)
and
\[
\left\| \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \right\|_4 \leq C \| \partial_x^s u \|_4 \leq C \| u \|_{s, 1/3} \leq C \| u \|_{Y_s}. \tag{9-9}
\]

Hence, from (9-7), (9-8) and (9-9), we have
\[
\frac{(8-10)}{\| u \|_{Y_s}^4} \leq C M \| u \|_{Y_s}^4. \tag{9-10}
\]

For the contribution of (9-2), since in this subcase \( K \leq C M \), we estimate (8-10) by
\[
\sum_{K_1} \| u_{K_1} \|_\infty \int_{T \times R} \sum_{K_3 \leq K_1} \sum_{M} \sum_{K \leq C M} K^2 v^*_K |g_{K_3, M}| \sum_{K_2} \sum_{K_0 \leq C K_2} K_0 u^*_K |u_{K_2}| \, dx \, dt, \tag{9-11}
\]
which is bounded by
\[
\sum_{K_1} K_1^{-(2s-1)/2} \| u \|_{Y_s} \int_{T \times R} \sum_{K_3 \leq K_1} \sum_{K} \left( \sum_{K_0} |u^*_K|^2 \right)^{1/2} \left( \sum_{M} M^{2s} |g_{K_3, M}|^2 \right)^{1/2}
\cdot \left( \sum_{K_0} K_0 |u^*_K|^2 \right)^{1/2} \left( \sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} \, dx \, dt. \tag{9-12}
\]

By a use of the Cauchy–Schwarz inequality, (9-12) is estimated by
\[
\sum_{K_1} K_1^{-(2s-1)/2} \| u \|_{Y_s} \left( \sum_{K} |v^*_K|^2 \right)^{1/2} \left( \sum_{K_0} K_0 |u^*_K|^2 \right)^{1/2} \left( \sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2}
\cdot \left( \sum_{K_3 \leq K_1} K_3^2 |u_{K_3}|^2 \right)^{1/2} \left( \sum_{K_3 \leq K_1} \sum_{M} M^{2s} |g_{K_3, M}|^2 \right)^{1/2} \, dx \, dt. \tag{9-13}
\]

Using Hölder’s inequality, we then bound it further by
\[
\sum_{K_1} K_1^{-(2s-1)/2} \| u \|_{Y_s} \left( \sum_{K} |v^*_K|^2 \right)^{1/2} \left( \sum_{K_0} K_0 |u^*_K|^2 \right)^{1/2} \left( \sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2}
\cdot \left( \sum_{K_3 \leq K_1} K_3^{2s} |u_{K_3}|^2 \right)^{1/2} \left( \sum_{K_3 \leq K_1} \sum_{M} M^{2s} |g_{K_3, M}|^2 \right)^{1/2} \, dx \, dt, \tag{9-14}
\]
which is majorized by
\[
\sum_{K_1} K_1^{-(2s-1)/2} \| u \|_{Y_s}^4 \sum_{K_3 \leq K_1} K_3^{-s} \left( \sum_{M} M^{2s} |g_{K_3, M}|^2 \right)^{1/2} \leq \sum_{K_1} K_1^{-(2s-1)/2} \| u \|_{Y_s}^4 \sum_{K_3 \leq K_1} K_3^{-s} \| \partial_x^s g_{K_3} \|_\infty. \tag{9-15}
\]

From the definition of \( g_{K_3} \), we have
\[
\partial_x g_{K_3}(x, t) = O(M K_3) \| u \|_{Y_s} = O(M K_3) \| \phi \|_{H^s}. \tag{9-15}
\]

Hence, for \( s < 1 \),
\[
\| \partial_x^s g_{K_3} \|_\infty \leq C M K_3^s \| \phi \|_{H^s}. \tag{9-16}
\]
Since \( s > 1/2 \), we then have
\[
(9-14) \leq C \mathcal{M} \| \phi \|_{H^s} \sum_{K_1} K_1^{-\frac{(2s-1)}{2}+\epsilon} \| u \|_{Y_s}^4 \leq C \mathcal{M} \| \phi \|_{H^s} \| u \|_{Y_s}^4.
\] (9-17)

This completes our discussion of Case (8-11).

### 10. Proof of case (8-12)

In this case, it suffices to consider the following subcases:

\[
\begin{align*}
K &\leq 2^{10} K_2, \quad (10-1) \\
K &\leq 2^{10} M, \quad (10-2) \\
K > 2^9(K_2 + M) &\quad \text{and} \quad K_3 \geq K_0^{1/2}, \quad (10-3) \\
K > 2^9(K_2 + M), K_3 &\leq K_0^{1/2}, \quad \text{and} \quad M \geq 2^{-10} K_0^{2/3}, \quad (10-4) \\
K > 2^9(K_2 + M), K_3 &\leq K_0^{1/2}, \quad \text{and} \quad M < 2^{-10} K_0^{2/3}. \quad (10-5)
\end{align*}
\]

The first two cases can be handled in exactly the same way as cases (9-1) and (9-2).

For case (10-3), observe that (8-12) and (10-3) imply
\[
K \leq C K_1 \quad (10-6)
\]
and
\[
K_0^{1/2} \leq K_2^{1/2} K_3^{1/2}. \quad (10-7)
\]

Hence (8-10) is bounded by
\[
\int \sum_{K_1} \sum_{K \leq CK_1} K^s v_K^s |u_{K_1}| \sum_{K_0 \geq K_2 \geq K_3} K_0 u_{K_0}^s |u_{K_2}| \| u_{K_3} \| \| G_{K_3} \|_\infty \, dx \, dt. \quad (10-8)
\]

Applying Hölder’s inequality, we estimate (10-8) by
\[
C \mathcal{M} \left( \int \sum_K \left| v_K^s \right|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \prod_{j=0,2,3} \left( \sum_{K_j} K_j^{1+\epsilon} |u_{K_j}|^2 \right)^{1/2} \, dx \, dt. \quad (10-9)
\]

One more use of Hölder’s inequality yields that (10-8) is bounded by
\[
C \mathcal{M} \left\| \left( \sum_K \left| v_K^s \right|^2 \right)^{1/2} \right\|_4 \left\| \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \right\|_4 \prod_{j=0,2,3} \left\| \left( \sum_{K_j} K_j^{1+\epsilon} |u_{K_j}|^2 \right)^{1/2} \right\|_6.
\]

Hence we obtain
\[
(10-8) \leq C \mathcal{M} \| u \|_{Y_s}^4. \quad (10-10)
\]

This finishes the proof of (10-3).
For case (10-4), we estimate (8-10) by
\[
\sum_{K_2, K_3} \int \sum_{K_1}^{K \leq CK_1} \sum_{K_0} K^x v_K^* |u_{K_1}| \sum_{K_0} K_0 |u_{K_0}^*| |u_{K_2}| |u_{K_3}| \sum_{M \geq CK_0^{2/3}} |g_{K_3, M}| \, dx \, dt, \tag{10-11}
\]
which is dominated by
\[
C \sum_{K_2, K_3} \int \left( \sum_K |v_K^*|^2 \right)^{1/2} \left( \sum_{K_1} K^{2x} |u_{K_1}|^2 \right)^{1/2} |u_{K_2}| |u_{K_3}| \cdot \left( \sum_{K_0} K_0 |u_{K_0}^*|^2 \right)^{1/2} \left( \sum_M M^{3/2} |g_{K_3, M}|^2 \right)^{1/2} \, dx \, dt. \tag{10-12}
\]
By Hölder’s inequality with $L^4$ norms for the first two functions in the integrand, $L^{6+}$ norms for the next three functions, and an $L^p$ norm (very large $p$) for the last one, (10-12) is dominated by
\[
C \|u\|_{Y_x} \sum_{K_2, K_3} \|u_{K_2}\|_{6+} \|u_{K_3}\|_{6+} \left\| \left( \sum_{K_0} K_0 |u_{K_0}^*|^2 \right)^{1/2} \right\|_{6+} \| \partial_t^{3/4} G_{K_3}\|_{\infty}, \tag{10-13}
\]
Applying (9-16), we estimate (10-12) by
\[
C M \|\phi\|_{H_t} \|u\|_{Y_x}^2 \prod_{j=2}^3 \sum_{K_j} K_j^{3/8} |u_{K_j}|_{6+} \leq C M \|\phi\|_{H_t} \|u\|_{Y_x}^2 \prod_{j=2}^3 \sum_{K_j} K_j^{3/8} |u_{K_j}|_{0+} \leq C M \|\phi\|_{H_t} \|u\|_{Y_x}^4,
\]
as desired. This completes the discussion of (10-4).

We now turn to case (10-5). In this case, we have
\[
|n_0 + n_1| + 2K_2 + M \geq |n| \geq K/2 \geq 2^8 (K_2 + M), \tag{10-14}
\]
which implies
\[
|n_0 + n_1| \geq 2^5 (K_2 + M). \tag{10-15}
\]
Notice that
\[
(n_0 + n_1 + n_2 + n_3 + m)^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3 = 3(n_0 + n_1)(n_0 + n_2 + n_3 + m)(n_1 + n_2 + n_3 + m) + (n_2 + n_3 + m)^3 - n_2^3 - n_3^3 - m^3. \tag{10-16}
\]
From (10-15), (10-16), and (10-5), we obtain
\[
|n^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3| \geq C (K_2 + M) K_0 K_1 \geq C K_0 K_1 \geq C K_0^2. \tag{10-17}
\]
Hence one of the following four statements must be true:
\[
|\lambda - n^3| \geq K_0^2, \tag{10-18}
|\lambda - \lambda_1 - \lambda_2 - \lambda_3 - \mu| - n_0^3 \geq K_0^2, \tag{10-19}
\]
there exists an $i \in \{1, 2, 3\}$ such that $|\lambda_i - n_i^3| \geq K_0^2$, \tag{10-20}
\[
|\mu| \geq K_0^2. \tag{10-21}
\]
For case (10-18), we set
\[ \tilde{v}(x, t) = (\hat{u}1_{|\lambda - n^3| \geq K_0^2})^\vee(x, t). \] (10-22)

We then estimate (8-10) by
\[ \sum_{K_2, K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} |\partial_x u_{K_0}| \sum_{K_1 \leq C K_1} K^s \tilde{v}_K^* |u_{K_1}| \, dx \, dt. \] (10-23)

This is clearly bounded by
\[ C \mathcal{M} \|u\|_Y^2 \sum_{K_0} \int K_0 |u_{K_0}^*| \left( \sum_K |\tilde{v}_K^*|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \, dx \, dt. \] (10-24)

Using the Cauchy–Schwarz inequality, we bound (10-24) by
\[ C \mathcal{M} \|u\|_Y^2 \int \left( \sum_{K_0} K_0^\delta |u_{K_0}^*|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2-s} \sum_K |\tilde{v}_K^*|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \, dx \, dt. \] (10-25)

By Hölder’s inequality, (10-25) is majorized by
\[ C \mathcal{M} \|u\|_Y^2 \left\| \left( \sum_{K_0} K_0^\delta |u_{K_0}^*|^2 \right)^{1/2} \right\|_4 \left\| \left( \sum_{K_0} K_0^{2-s} \sum_K |\tilde{v}_K^*|^2 \right)^{1/2} \right\|_2 \left\| \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \right\|_4, \]

which is controlled by
\[ C \mathcal{M} \|u\|_Y^2 \left\| \partial_x^\delta u \right\|_4 \left( \sum_{K_0} K_0^{2-s} \|\tilde{v}\|_2^2 \right)^{1/2} \leq C \mathcal{M} \|u\|_Y^2 \left\| \partial_x^\delta u \right\|_4 \sum_{K_0} K_0^{-s/2} \leq C \mathcal{M} \|u\|_Y^4. \] (10-26)

This finishes the proof of case (10-18).

For case (10-19), let \( \tilde{u} \) be defined by
\[ \tilde{u} = (\hat{u}1_{|\lambda - n^3| \geq K_0^2})^\vee. \] (10-27)

Then (8-10) can be estimated by
\[ \sum_{K_2, K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} |\partial_x \tilde{u}_{K_0}| \sum_{K_1 \leq C K_1} K^s v_K^* |u_{K_1}| \, dx \, dt. \] (10-28)

By Schur’s test and Hölder’s inequality, we control (10-28) by
\[ \sum_{K_2, K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \|\partial_x \tilde{u}_{K_0}\|_2 \left\| \left( \sum_K |v_K|^2 \right)^{1/2} \right\|_4 \left\| \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \right\|_4, \] (10-29)

which is bounded by
\[ C \mathcal{M} \|u\|_Y^3 \sum_{K_0} \|u_{K_0}\|_{0.1/2} \leq C \mathcal{M} \|u\|_Y^4. \] (10-30)

This completes the proof of case (10-19).
For case (10-20), if \( j = 1 \), we dominate (8-10) by
\[
\sum_{K_2, K_3} \| u_{K_2} \|_\infty \| u_{K_3} \|_\infty \| G_{K_3} \|_\infty \sum_{K_0} \int |\partial_t u_{K_0}| \sum_{K_1} \sum_{K \leq C K_1} K^s v^*_K |\tilde{u}_{K_1}| \, dx \, dt. \tag{10-31}
\]
As we did in case (10-19), we bound (10-31) by
\[
C \mathcal{M} \| u \|_{Y_s}^2 \sum_{K_0} \| \partial_t u_{K_0} \|_4 \| v \|_4 \left\| \left( \sum_{K_1} K^{2s} |\tilde{u}_{K_1}|^2 \right) \right\|_2. \tag{10-32}
\]
This can be further controlled by
\[
C \mathcal{M} \| u \|_{Y_s}^3 \sum_{K_0} \frac{1}{K_0} \| \partial_t u_{K_0} \|_4 \| v \|_4 \leq C \mathcal{M} \| u \|_{Y_s}^3 \sum_{K_0} \frac{1}{K_0} \| u_{K_0} \|_{1, 1/3} \leq C \mathcal{M} \| u \|_{Y_s}^4, \tag{10-33}
\]
as desired.

We now consider \( j = 2 \) or \( j = 3 \). Without loss of generality, assume \( j = 2 \). In this case, we estimate (8-10) by
\[
\sum_{K_3} \| u_{K_3} \| \| G_{K_3} \|_\infty \sum_{K_0} \int |\partial_t u_{K_0}| \sum_{K_1} \sum_{K \leq C K_1} K^s v^*_K |u_{K_1}| \sum_{K_2 \leq C K_0} |\tilde{u}_{K_2}| \, dx \, dt, \tag{10-34}
\]
which is bounded by
\[
C \mathcal{M} \| u \|_{Y_s} \sum_{K_0} \| \partial_t u_{K_0} \|_{\infty} \sum_{K_2 \leq K_0} \| \tilde{u}_{K_2} \|_2 \| v \|_4 \left\| \left( \sum_{K_1} K^{2s} |u_{K_1}|^2 \right) \right\|_4. \tag{10-35}
\]
Notice that
\[
\sum_{K_0} \| \partial_t u_{K_0} \|_{\infty} \sum_{K_2 \leq K_0} \| \tilde{u}_{K_2} \|_2 \leq C \sum_{K_0} \frac{1}{K_0} \| \partial_t u_{K_0} \|_{\infty} \| u \|_{Y_s}
\]
\[
\leq C \sum_n \int |\hat{u}(n, \lambda)| \, d\lambda \| u \|_{Y_s} \leq C \| u \|_{Y_s}^2. \tag{10-36}
\]
Hence (10-34) is dominated by
\[
C \mathcal{M} \| u \|_{Y_s}^4. \tag{10-37}
\]
This completes case (10-20).

We now turn to the most difficult case, (10-21) in case (8-12). We should decompose \( G_{K_3} \), with respect to the \( t \)-variable, into Littlewood–Paley multipliers in the same spirit as before. More precisely, for any dyadic number \( L \), let \( Q_L \) be
\[
Q_L u(x, t) = \int \psi_L(\tau) u(x, t - \tau) \, d\tau. \tag{10-36}
\]
Here the Fourier transform of \( \psi_L \) is a bump function supported on \([-2L, 2L]\) and \( \hat{\psi}(x) = 1 \) if \( x \in [-L, L] \). Let
\[
\Pi_L u = Q_L u - Q_{L/2} u. \tag{10-37}
\]
Then $\Pi_L u$ gives a Littlewood–Paley multiplier with respect to the time variable $t$. Using this multiplier, we represent

$$u_K = \sum_L u_{K,L}. \quad (10-38)$$

Here $u_{K,L} = \Pi_L (u_K)$. We decompose $G_{K_3}$ as

$$G_{K_3} = C + \sum_L (F_3(Q_L P_{4K_3} u, \ldots, Q_L P_{K_3/8} u) - F_3(Q_{L/2} P_{4K_3} u, \ldots, Q_{L/2} P_{K_3/8} u)) = C + \sum_{j=4,2,1} H_{K_3,L} u_{jK_3,L}, \quad (10-39)$$

where $H_{K_3,L}$ is given by

$$H_{K_3,L} = F_4(Q_{\ell L} P_{4K_3} u, \ldots, Q_{\ell L} P_{K_3/8} u; \ell = 1, \frac{1}{2}). \quad (10-40)$$

Let $\mathcal{M}_1$ be defined by

$$\mathcal{M}_1 = \sup \{|D^\alpha F_4(u_1, \ldots, u_{12})| : u_j \text{ satisfies } \|u_j\|_{Y_2} \leq C_0 \|\phi\|_{H^\varphi} \text{ for all } j = 1, \ldots, 12; \alpha\}. \quad (10-41)$$

Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{12}}^{\alpha_{12}}$ and $\alpha$ is taken over all tuples $(\alpha_1, \ldots, \alpha_{12}) \in (\mathbb{N} \cup \{0\})^{12}$ with $\sum |\alpha_j| \leq 1$. $\mathcal{M}_1$ is a real number because $F_4 \in C^1$.

In order to finish the proof, we need to consider a further three subcases:

$$L \leq 2^{10} K_3^3, \quad (10-42)$$

$$2^{10} K_3^3 < L \leq 2^{-5} K_0^2, \quad (10-43)$$

$$L > 2^{-5} K_0^2. \quad (10-44)$$

For the contribution of $(10-42)$, we set

$$h_{K_0,jK_3,L} = (H_{K_3,L} u_{jK_3,L} 1_{|\mu| \geq K_0^2})^\vee. \quad (10-45)$$

Here $j = 4, 2, 1, 1/2, 1/4, 1/8$. From the definition of $H_{K_3,L}$, we get

$$\|h_{K_0,jK_3,L}\|_4 \leq C \mathcal{M}_1 \|\phi\|_{H^\varphi} \frac{L}{K_0^2} \|u_{jK_3,L}\|_4. \quad (10-46)$$

Then $(8-10)$ is bounded by

$$\sum_{K_2} \|u_{K_2}\|_{\infty} \sum_{K_0} \int K_0 u_{K_0}^* \sum_{K_3 \leq CK_0^{1/2}} \|u_{K_3}\|_{\infty} \sum_{L \leq CK_3^3} \|h_{K_0,jK_3,L}\| \sum_{K_1 \leq CK_1} \sum_{K \leq CK_1} K^4 v_K^* |u_{K_1}| \, dx \, dt, \quad (10-47)$$

which is majorized by

$$\sum_{K_2} \|u_{K_2}\|_{\infty} \sum_{K_0} \sum_{K_3 \leq CK_0^{1/2}} \|u_{K_3}\|_{\infty} \int K_0 u_{K_0}^* \sum_{L \leq CK_3^3} |h_{K_0,jK_3,L}| \left(\sum_K |v_K^*|^2\right)^{1/2} \left(\sum_K K_1^{2s} |u_{K_1}|^2\right)^{1/2} \, dx \, dt. \quad (10-48)$$
Using Hölder’s inequality with $L^4$ norms for the four functions in the integrand, we estimate (10-48) as follows:

$$C \mathcal{M}_1 \| \phi \|_{H^s} \| u \|_Y^2 \sum_{K_0} K_0 \| u_{K_0} \|_4 \sum_{K_3 \leq K_0^{1/2}} \| u_{K_3} \|_\infty \sum_{L \leq CK_3^3} \frac{L}{K_0^2} \| u_{jK_3,L} \|_4 \leq C \mathcal{M}_1 \| \phi \|_{H^s} \| u \|_Y^3 \sum_{K_0} K_0^{1/2} \| u_{K_0} \|_{0,1/3} \leq C \mathcal{M}_1 \| \phi \|_{H^s} \| u \|_Y^4.$$  

(10-49)

This finishes case (10-42).

For the contribution of (10-43), we bound (8-10) by

$$\sum_{K_2} \| u_{K_2} \|_\infty \sum_{K_3} \| u_{K_3} \|_\infty \int \sum_{K_0} \| \partial_x u_{K_0} \| \sum_{2^{10}K_3^3 < L \leq 2^{-10}K_0^3} \| h_{K_0,jK_3,L} \| \sum_{K_1} \sum_{K \leq CK_1} K^5 v_K^* |u_{K_1}| \, dx \, dt, \quad (10-50)$$

which is dominated by

$$C \| u \|_Y \sum_{K_3} \| u_{K_3} \|_\infty \sum_{\Delta \leq 2^{-10} \Delta \text{dyadic}} \int \sum_{K_0} \| \partial_x u_{K_0} \| \sum_{2^{10}K_3^3 < L \leq (\Delta/2)K_0^3} \| h_{K_0,jK_3,L} \| \sum_{K_1} \sum_{K} K^5 v_K^* |u_{K_1}| \, dx \, dt \cdot \left( \sum_{K} |v_K^*|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \, dx \, dt. \quad (10-51)$$

By the Cauchy–Schwarz inequality, we further estimate (10-51) by

$$C \| u \|_Y \sum_{K_3} \| u_{K_3} \|_\infty \sum_{\Delta \leq 2^{-10} \Delta \text{dyadic}} \Delta^{-1/2} \int \sum_{K_0} \| \partial_x u_{K_0} \| \frac{K_0}{K_0} \left( \sum_{2^{10}K_3^3 < L \leq (\Delta/2)K_0^3} \| h_{K_0,jK_3,L} \|^2 \right)^{1/2} \left( \sum_{K} |v_K^*|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \, dx \, dt. \quad (10-52)$$

Applying Hölder’s inequality with an $L^\infty$ norm for the first function in the integrand, an $L^2$ norm for the second, and $L^4$ norms for the last two functions, we then majorize (10-52) by

$$C \| u \|_Y^2 \sum_{K_3} \| u_{K_3} \|_\infty \sum_{\Delta \leq 2^{-10} \Delta \text{dyadic}} \Delta^{-1/2} \sum_{K_0} \| \partial_x u_{K_0} \| \infty \left( \sum_{2^{10}K_3^3 < L \leq (\Delta/2)K_0^3} \| h_{K_0,jK_3,L} \|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \equiv 2. \quad (10-53)$$

Notice that if $L \sim \Delta K_0^2$,

$$\| h_{K_0,jK_3,L} \|_2 \leq C \mathcal{M}_1 \| \phi \|_{H^s} \Delta \| u_{jK_3,L} \|_2. \quad (10-54)$$
Thus we have
\[
\left\| \left( \sum_{2^{10}K_3 \leq L} L|h_{K_0,jK_3,L}|^2 \right)^{1/2} \right\|_2 \leq C\mathcal{M}_1 \| \phi \|_{H^s} \Delta \left( \sum_{2^{10}K_3 \leq L} L\|u_{jK_3,L}\|^2 \right)^{1/2} 
\leq C\mathcal{M}_1 \| \phi \|_{H^s} \Delta \|u_{jK_3}\|_{0,1/2} 
\leq C\mathcal{M}_1 \| \phi \|_{H^s}^2 \Delta. 
\tag{10-55}
\]

From (10-55), (10-53) is bounded by
\[
C\mathcal{M}_1 \| \phi \|_{H^s}^2 \|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\| \Delta \sum_{K_0} \Delta^{1/2} \sum_{\Delta \leq 2^{-10}} \Delta^{1/2} \sum_{K_0} \sum_{K \leq CK_1} \|\partial_x u_{K_0}\|_K, 
\tag{10-56}
\]
which is clearly majorized by
\[
C\mathcal{M}_1 \| \phi \|_{H^s}^2 \|u\|_{Y_s}^4. 
\tag{10-57}
\]

This finishes case (10-43).

For the contribution of (10-44), we estimate (8-10) by
\[
\sum_{K_2} \|u_{K_2}\| \sum_{K_3} \|u_{K_3}\| \int \sum_{K_0} |\partial_x u_{K_0}| \sum_{L \leq 2^{-5}K_0} |h_{K_0,jK_3,L}| \sum_{K_1} \sum_{K \leq CK_1} K^s v^*_K |u_{K_1}| \, dx \, dt, 
\tag{10-58}
\]
which is bounded by
\[
\sum_{K_2} \|u_{K_2}\| \sum_{K_3} \|u_{K_3}\| \int \left( \sum_{K_0} \frac{|\partial_x u_{K_0}|^2}{K_0^2} \right)^{1/2} \cdot \left( \sum_{L \leq 2^{-5}K_0^2} L|h_{K_0,jK_3,L}| \right) \left( \sum_{K} |v^*_K|^2 \right)^{1/2} \left( \sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \, dx \, dt. 
\tag{10-59}
\]
Applying Hölder’s inequality, we further have
\[
(10-59) \leq C\mathcal{M}_1 \|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\| \sum_{K_0} \frac{|\partial_x u_{K_0}|}{K_0} \left( \sum_{L \leq 2^{-5}K_0^2} L\|u_{jK_3,L}\|^2 \right)^{1/2} 
\leq C\mathcal{M}_1 \|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\| \sum_{K_0} \frac{|\partial_x u_{K_0}|}{K_0} \|u_{jK_3}\|_{0.1/2}. 
\tag{10-60}
\]
This is clearly majorized by
\[
C\mathcal{M}_1 \| \phi \|_{H^s} \|u\|_{Y_s}^4. 
\tag{10-61}
\]
Hence we complete case (10-44).
11. Proof of case (8-13)

In this case, it suffices to consider the following subcases:

\[ M \geq 2^{-10}K_0^{2/3}, \quad (11-1) \]
\[ M < 2^{-10}K_0^{2/3} \text{ and } K_2^2K_3 \geq 2^{-10}K_0^2, \quad (11-2) \]
\[ M < 2^{-10}K_0^{2/3} \text{ and } K_2^2M \geq 2^{-10}K_0^2, \quad (11-3) \]
\[ M < 2^{-10}K_0^{2/3}, \quad K_2^2K_3 < 2^{-10}K_0^2 \text{ and } K_2^2M < 2^{-10}K_0^2. \quad (11-4) \]

For case (11-1), notice that we have

\[ K \leq CM^{3/2}. \quad (11-5) \]

Hence we estimate (8-10) by

\[
\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M K^s v_K^* \sum_{K_0 \leq CM^{3/2}} K_0 u_{K_0}^* |g_{K_3,M}| \, dx \, dt, \quad (11-6)
\]

which is bounded by

\[
\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \left( \sum_M M^{3/2(1-s)} |g_{K_3,M}| \right) \sum_{K \leq CM^{3/2}} K^s v_K^* \left( \sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \right)^{1/2} \, dx \, dt, \quad (11-7)
\]

since \( 1/2 < s < 1 \). Applying Schur’s test, we estimate (11-7) by

\[
\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \left( \sum_M M^3 |g_{K_3,M}|^2 \right)^{1/2} \left( \sum_K |v_K^*|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \right)^{1/2} \, dx \, dt. \quad (11-8)
\]

By Hölder’s inequality and \( s > 1/2 \), (11-8) is majorized by

\[
\sum_{K_1 \geq K_2 \geq K_3} \| \partial_x^{3/2} G_{K_3} \|_\infty \left( \prod_{j=1}^3 \| u_{K_j} \|_{6+} \right) \left( \sum_K |v_K|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \right)^{1/2} \leq CM\left( \| \phi \|_{H^1} + \| \phi \|_{H^2}^2 \right) \| u \|_{Y_{4+}} \sum_{K_1 \geq K_2 \geq K_3} K_3^{3/2} \prod_{j=1}^3 \| u_{K_j} \|_{6+} \]
\[
\leq C\| \phi \|_{H^1} + \| \phi \|_{H^2}^2 \| u \|_{Y_{4+}} \prod_{j=1}^3 \sum_{K_j} K_j^{1/2} \| u_{K_j} \|_{0+1/2} \]
\[
\leq C\| \phi \|_{H^1} + \| \phi \|_{H^2}^2 \| u \|_{Y_{4+}}^4. \quad (11-9)
\]

This finishes case (11-1).

For case (11-2), observe that, in this case,

\[ K_0 \leq CK_1^{1/2}K_2^{1/2}K_3^{1/2}. \quad (11-10) \]
We estimate (8-10) by

\[
\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}| |u_{K_2}| |u_{K_3}| \sum_{K \leq C K_0} K^s v_K^* \sum_{K_0 \leq C (K_1 K_2 K_3)^{1/2}} K_0 u_{K_0}^* \| G_{K_3} \|_\infty dx \, dt,
\]

(11-11)

which is bounded by

\[
C \mathcal{M} \int \left( \sum_K |v_K^*|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \right)^{1/2} \prod_{j=1}^3 K_j^{1/2} |u_{K_j}| \, dx \, dt.
\]

(11-12)

Using Hölder’s inequality with \(L^4\) norms for the first two functions and \(L^6\) norms for the last three functions in the integrand, we obtain

\[
C \mathcal{M} \|u\|_{Y_2} \prod_{j=1}^3 \|K_j^{1/2} |u_{K_j}|\|_6 \leq C \mathcal{M} \|u\|^4_{Y_2}.
\]

(11-13)

This completes case (11-2).

For case (11-3) we have

\[
K_0 \leq C K_1^{1/2} K_2^{1/2} M^{1/2}.
\]

(11-14)

Hence we dominate (8-10) by

\[
\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}| |u_{K_2}| |u_{K_3}| \sum_M |g_{K_3, M}| \sum_{K \leq C K_0} K^s v_K^* \sum_{K_0 \leq C (K_1 K_2 M)^{1/2}} K_0 u_{K_0}^* \, dx \, dt,
\]

(11-15)

which is bounded by

\[
C \sum_{K_3} \int \left( \sum_K |v_K^*|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \right)^{1/2} |u_{K_3}| \cdot \left( \sum_M |g_{K_3, M}|^2 \right)^{1/2} \prod_{j=1}^2 K_j^{1/2} |u_{K_j}| \, dx \, dt.
\]

(11-16)

Using Hölder’s inequality with \(L^4\) norms for the first two functions, \(L^6\) norms for the third, an \(L^p\) norm with \(p\) very large for the fourth, and \(L^{6+}\) for the last two functions in the integrand, we obtain

\[
C \|u\|_{Y_2} \prod_{j=1}^2 \|K_j^{1/2} |u_{K_j}|\|_6 \sum_{K_3} \|u_{K_3}\|_6 \|\partial_x^{1/2} G_{K_3}\|_\infty.
\]

(11-17)

Clearly (11-17) is dominated by

\[
C \mathcal{M} \|\phi\|_{H^s} \|u\|^3_{Y_2} K_3^{1/2} \|u_{K_3}\|_6 \leq C \mathcal{M} \|\phi\|_{H^s} \|u\|^4_{Y_2}.
\]

(11-18)

Hence case (11-3) is done.

For case (11-4) we observe that

\[
M^2 K_2 \leq 2^{-10} K_0^2.
\]

(11-19)
In fact, if (11-19) does not hold, then, from (11-4),
\[ M^2K_2 > 2^{-10}K_0^2 > K_2^2M. \]
Thus \( M > K_2 \), which immediately yields
\[ M^3 > M^2K_2 > 2^{-10}K_0^2, \]
contradicting \( M < 2^{-10}K_0^{2/3} \). Hence (11-19) must be true. From (11-19), \( K_2^2K_3 + K_2^2M < 2^{-9}K_0^2 \), we get
\[ |(n_2 + n_3 + m)^3 - n_2^3 - n_3^3 - m^3| \leq 2^{-5}K_0^2. \]  
(11-20)
Since \( n_1 + n_2 + n_3 + m \neq 0 \), from (8-13), (11-4), and (11-20), the crucial arithmetic observation (10-16) yields
\[ |n^3 - n_0^3 - n_2^3 - n_3^3 - m^3| \geq 2K_0^2. \]  
(11-21)
Hence one of the following statements must be true:
\[ |\lambda - n_3| \geq K_0^2, \]  
(11-22)
\[ |(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - \mu) - n_0^3| \geq K_0^2, \]  
(11-23)
there exists an \( i \in \{1, 2, 3\} \) such that \( |\lambda_i - n_i| \geq K_0^2, \) \( |\mu| \geq K_0^2. \)  
(11-24)
(11-25)
For case (11-22), we estimate (8-10) by
\[ \sum_{K_1, K_2, K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |u_{K_0}^*|^2 \sum_{K \geq C_{K_0}} \partial^s_x u_K \| \, dx \, dt. \]  
(11-26)
Then the Cauchy–Schwarz inequality yields
\[ C\mathcal{M}\|u\|_{Y_3}^3 \left( \sum_{K_0} K_0^{2s} \sum_{K \leq C_{K_0}} \partial^s_x \bar{u}_K \right)^{1/2} \left( \sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \right)^{1/2} \leq C\mathcal{M}\|u\|_{Y_3}^4 \left( \sum_{K_0} K_0^{2s} \sum_{K \leq C_{K_0}} \partial^s_x \bar{u}_K \right)^{1/2} \leq C\mathcal{M}\|u\|_{Y_3}^4. \]  
(11-27)
This finishes the proof of case (11-22).
For case (11-23), (8-10) can be estimated by
\[ \sum_{K_1, K_2, K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |u_{K_0}^*|^2 \sum_{K \leq C_{K_0}} \partial^s_x \bar{u}_K \| \, dx \, dt. \]  
(11-28)
By Schur’s test and Hölder’s inequality, we control (11-28) by
\[ C\mathcal{M}\|u\|_{Y_3}^3 \left( \sum_K |v_K^*|^2 \right)^{1/2} \left( \sum_{K_0} K_0^{2s+2} |\bar{u}_{K_0}^*|^2 \right)^{1/2}, \]  
(11-29)
which is clearly bounded by
\[ C \mathfrak{M} \| u \|_{Y_1}^3 \left( \sum_{K_0} K_0^{2s} \| u_{K_0} \|_{0,1/2}^2 \right)^{1/2} \leq C \mathfrak{M} \| u \|_{Y_1}^4. \] (11-30)

This completes the proof of case (11-23).

For case (11-24), without loss of generality, assume \( j = 1 \). We then dominate (8-10) by
\[ \sum_{K_2, K_3} \| u_{K_2} \|_{\infty} \| u_{K_3} \|_{\infty} \| G_{K_3} \|_{\infty} \sum_{K_1} \sum_{K_0} \int K_0 |u_{K_0}^*| |\tilde{u}_{K_1}| K \leq C K_0 \] (11-31)

By Hölder’s inequality, we bound (11-31) by
\[ \sum_{K_2, K_3} \| u_{K_2} \|_{\infty} \| u_{K_3} \|_{\infty} \| G_{K_3} \|_{\infty} \sum_{K_1} \| u_{K_1} \|_{0,1/2} \sum_{K_0} \sum_{K \leq C K_0} K^s |u_{K_0}| \| v_K \|_4. \] (11-32)

By Schur’s test, we dominate (11-32) by
\[ C \mathfrak{M} \| u \|_{Y_1}^2 \sum_{K_1} \| u_{K_1} \|_{0,1/2} \left( \sum_{K_0} K_0^{2s} \| u_{K_0} \|_{0,1/2}^2 \right)^{1/2} \left( \sum_K \| v_K \|_4^2 \right)^{1/2} \leq C \mathfrak{M} \| u \|_{Y_1}^3 \left( \sum_{K_0} K_0^{2s} \| u_{K_0} \|_{0,1/3}^2 \right)^{1/2} \left( \sum_K \| v_K \|_{0,1/3}^2 \right)^{1/2} \leq C \mathfrak{M} \| u \|_{Y_1}^4. \] (11-33)

Hence case (11-24) is done.

In order to finish the proof, as is done in (10-36), we need to consider three further subcases:
\[ L \leq 2^{10} K_3^3, \] (11-34)
\[ 2^{10} K_3^3 < L \leq 2^{-5} K_0^2, \] (11-35)
\[ L > 2^{-5} K_0^2. \] (11-36)

For the contribution of (11-34), notice that
\[ \| h_{K_0,jK_3,L} \|_6 \leq C \mathfrak{M}_1 \| \phi \|_{H^s} \frac{L}{K_0^2} \| u_{jK_3,L} \|_6. \] (11-37)

Here \( h_{K_0,jK_3,L} \) is defined as in (10-45). In this case we also have \( K_3 \leq K_0^{2/3} \), from
\[ K_3^2 K_3 \leq 2^{-10} K_0^2. \]

Then (8-10) is bounded by
\[ \int \sum_{K_0} K_0 u_{K_0}^* \sum_{K \leq C K_0} K^s v_K^* \sum_{K_1 \leq C K_2 \leq K_3} u_{K_1} |u_{K_2}||u_{K_3}| \sum_{L \leq C K_3} |h_{K_0,jK_3,L}| \, dx \, dt. \] (11-38)
Write (11-38) as
\[
\sum_{\Delta \text{dyadic}} \int \sum_{K_0} K_0 u_{K_0}^* \sum_{K \leq C K_0} K^s v_K^* \sum_{K_1 \geq K_2 \geq K_3} K_1^{1/2} K_2^{1/2} K_3^{1/2} \sum_{\Delta \leq 1} |u_{K_1}| |u_{K_2}| |u_{K_3}| \sum_{L \leq C K_3^3} |h_{K_0,jK_3,L}| \, dx \, dt. 
\]  
(11-39)

Observe that if \( \Delta K_0^{2/3} / 2 < K_3 \leq \Delta K_0^{2/3} \), we have
\[
K_0 \leq \Delta^{-3/2} K_1^{1/2} K_2^{1/2} K_3^{1/2}. 
\]  
(11-40)

Hence
\[
C \|u\|_{Y_s} \sum_{K_0} \sum_{K \leq K_0} K^s \sum_{K_1,K_2} K_1^{1/2} K_2^{1/2} \sum_{\Delta \leq 1} \Delta^{-3/2} \sum_{K_3 \sim \Delta K_0^{2/3}} K_3^{1/2} \int u_{K_0}^* v_K^* |u_{K_1}| |u_{K_2}| \sum_{L \leq C K_3^3} |h_{K_0,jK_3,L}| \, dx \, dt. 
\]  
(11-41)

Applying Hölder’s inequality with \( L^4 \) norms for first two functions and \( L^6 \) for the last three, and then using (11-37), we get
\[
C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s} \sum_{K_0} \sum_{K \leq K_0} K^s \sum_{K_1,K_2} K_1^{1/2} K_2^{1/2} \sum_{\Delta \leq 1} \Delta^{-3/2} \sum_{K_3 \sim \Delta K_0^{2/3}} L_{K_3} \sum_{L \leq C \Delta K_0^{3/2}} \|u_{K_0}\|_4 \|v_K^*\|_4 
\cdot \sum_{K_1} K_1^{1/2} \|u_{K_1}\|_{0+,1/2} \sum_{K_2} K_2^{1/2} \|u_{K_2}\|_{0+,1/2} \sum_{K_3} K_3^{1/2} \|u_{K_3,L}\|_{0+,1/2} 
\leq C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^3 \sum_{K_0} \sum_{K \leq C K_0} K^s \sum_{\Delta \leq 1} \Delta^{3/2} \sum_{K_3 \sim \Delta K_0^{2/3}} L_{K_3} \sum_{L \leq C \Delta K_0^{3/2}} \|u_{K_0}\|_4 \|v_K\|_4 
\leq C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^3 \left( \sum_{K_0} \sum_{K \leq C K_0} K_0^{2s} \|u_{K_0}\|_{0,1/3}^2 \right)^{1/2} \left( \sum_{K} \|v_K\|_{0,1/3}^2 \right)^{1/2} 
\leq C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^4. 
\]  
(11-42)

which is bounded by
\[
C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s} \sum_{K_0} \sum_{K \leq K_0} K^s \sum_{\Delta \leq 1} \sum_{L \leq C \Delta K_0^{3/2}} L_{K_3} \sum_{K} \|v_K\|_4 \|u_{K_0}\|_4 
\cdot \sum_{K_1} K_1^{1/2} \|u_{K_1}\|_{0+,1/2} \sum_{K_2} K_2^{1/2} \|u_{K_2}\|_{0+,1/2} \sum_{K_3} K_3^{1/2} \|u_{K_3,L}\|_{0+,1/2} 
\leq C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^3 \sum_{K_0} \sum_{K \leq C K_0} K^s \sum_{\Delta \leq 1} \Delta^{3/2} \sum_{K_3 \sim \Delta K_0^{2/3}} L_{K_3} \sum_{L \leq C \Delta K_0^{3/2}} \|u_{K_0}\|_4 \|v_K\|_4 
\leq C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^3 \left( \sum_{K_0} \sum_{K \leq C K_0} K_0^{2s} \|u_{K_0}\|_{0,1/3}^2 \right)^{1/2} \left( \sum_{K} \|v_K\|_{0,1/3}^2 \right)^{1/2} 
\leq C \mathcal{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^4. 
\]  
(11-43)

This completes case (11-34).

For the contribution of (11-35), we bound (8-10) by
\[
\sum_{K_1} \|u_{K_1}\|_\infty \sum_{K_2} \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \int \sum_{K_0} \sum_{K \leq C K_0} K^s v_K^* K_0 u_{K_0}^* \sum_{2^{10} K_0^2 < L \leq 2^{-5} K_0^2} |h_{K_0,jK_3,L}| \, dx \, dt. 
\]  
(11-44)
which is dominated by

\[
C \|u\|_{Y_2}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\Delta \leq 2^{-5}} \sum_{K_0} \sum_{K \leq CK_0} K^s \int K_0 u_{K_0}^* v_K^* \sum_{2^{10} K_3^3 < L}^{(\Delta/2) K_0^3 < L \leq \Delta K_0^2} |h_{K_0, jK_3, L}| \, dx \, dt. \tag{11-45}
\]

Using the Cauchy–Schwarz inequality, we further estimate (11-45) by

\[
C \|u\|_{Y_2}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\Delta \leq 2^{-5}} \sum_{K_0} \sum_{K \leq CK_0} K^s \int u_{K_0}^* v_K^* \left( \sum_{2^{10} K_3^3 < L}^{(\Delta/2) K_0^3 < L \leq \Delta K_0^2} L |h_{K_0, jK_3, L}|^2 \right)^{1/2} \, dx \, dt. \tag{11-46}
\]

Employing Hölder’s inequality with \(L^4\) norms for the first two functions and an \(L^2\) for the last one, we bound (11-46) by

\[
C \|u\|_{Y_2}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\Delta \leq 2^{-10}} \sum_{K_0} \sum_{K \leq CK_0} K^s \|u_{K_0}\|_4 \|v_K\|_4 \left( \sum_{2^{10} K_3^3 < L}^{(\Delta/2) K_0^3 < L \leq \Delta K_0^2} L |h_{K_0, jK_3, L}|^2 \right)^{1/2} \|2\). \tag{11-47}
\]

From (10-55), (11-47) is majorized by

\[
C\mathcal{M}_1 \|\phi\|_{H^3}^2 \|u\|_{Y_2}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\Delta \leq 2^{-10}} \sum_{K_0} \sum_{K \leq CK_0} K^s \|u_{K_0}\|_4 \|v_K\|_4 \leq C\mathcal{M}_1 \|\phi\|_{H^3}^2 \|u\|_{Y_2}^3 \left( \sum_{K_0} K_0^{2s} \|u_{K_0}\|_0^2 \right)^{1/2} \left( \sum_K \|v_K\|_0^{1/2} \right)^{1/2} \leq C\mathcal{M}_1 \|\phi\|_{H^3}^2 \|u\|_{Y_2}^4. \tag{11-48}
\]

This finishes the proof for case (11-35).

For the contribution of (11-36), we estimate (8-10) by

\[
\sum_{K_1, K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \int K_0 u_{K_0}^* \sum_{L > 2^{-5} K_0^2} |h_{K_0, jK_3, L}| \sum_{K \leq CK_0} K^s v_K^* \, dx \, dt. \tag{11-49}
\]

By the Cauchy–Schwarz inequality, (11-49) is bounded by

\[
\sum_{K_1, K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \sum_{K_0} \sum_{K \leq CK_0} K^s \int v_K^* u_{K_0}^* \left( \sum_{L > 2^{-10} K_0^2} L |h_{K_0, jK_3, L}|^2 \right)^{1/2} \, dx \, dt. \tag{11-50}
\]

Employing Hölder’s inequality with \(L^4\) norms for the first two functions and an \(L^2\) for the last one, we
dominate (11-50) by

\[ C M_1 \| u \|_Y^2 \sum_{L \geq 2^5 K_0^2} \| u_{K_0} \|_{\infty} \sum_{K_0 \leq C K_0} K^s \| u_{K_0} \|_4 \| v_{K_0} \|_4 \left( \sum_{L > 2^5 K_0^2} L \| u_{j K_0 L} \|_2 \right)^{1/2} \leq C M_1 \| u \|_Y^2 \sum_{K_3} \| u_{K_3} \|_{\infty} \sum_{K_0 \leq C K_0} K^s \| u_{K_0} \|_{0,1/3} \| v_{K_0} \|_{0,1/3} \| u \|_{0,1/2} \leq C M_1 \| \phi \|_{H^s} \| u \|_Y^4. \]  

Hence we complete case (11-36).

References


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