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SECOND ORDER STABILITY FOR THE MONGE–AMPÈRE EQUATION AND STRONG SOBOLEV CONVERGENCE OF OPTIMAL TRANSPORT MAPS

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The aim of this note is to show that Alexandrov solutions of the Monge–Ampère equation, with right-hand side bounded away from zero and infinity, converge strongly in $W^{2,1}_{\mathrm{loc}}$ if their right-hand sides converge strongly in L^1_{loc}. As a corollary, we deduce strong $W^{1,1}_{\mathrm{loc}}$ stability of optimal transport maps.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. In [De Philippis and Figalli 2013], we showed that convex Alexandrov solutions of

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

with $0 < \lambda \le f \le \Lambda$, are $W^{2,1}_{\mathrm{loc}}(\Omega)$. More precisely, they were able to prove uniform interior $L \log L$-estimates for $D^2 u$. This result has also been improved in [De Philippis et al. 2013; Schmidt 2013], where it is actually shown that $u \in W^{2,\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma = \gamma(n, \lambda, \Lambda) > 1$: more precisely, for any $\Omega' \Subset \Omega$,

$$\int_{\Omega'} |D^2 u|^\gamma \le C(n, \lambda, \Lambda, \Omega, \Omega'). \tag{1-2}$$

A question which naturally arises in view of the previous results is the following: choose a sequence of functions f_k with $\lambda \le f_k \le \Lambda$ which converges to f strongly in $L^1_{\mathrm{loc}}(\Omega)$, and denote by u_k and u the solutions of (1-1) corresponding to f_k and f, respectively. By the convexity of u_k and u and the uniqueness of solutions to (1-1), it is immediately deduced that $u_k \to u$ uniformly, and $\nabla u_k \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. What can be said about the strong convergence of $D^2 u_k$? Due to the highly nonlinear character of the Monge–Ampère equation, this question is nontrivial. (Note that weak $W^{2,1}_{\mathrm{loc}}$ convergence is immediate by compactness, even under the weaker assumption that f_k converges to f weakly in $L^1_{\mathrm{loc}}(\Omega)$.)

The aim of this short note is to prove that strong convergence holds. Our main result is the following:

Theorem 1.1. *Let $\Omega_k \subset \mathbb{R}^n$ be a family of convex domains, and let $u_k : \Omega_k \to \mathbb{R}$ be convex Alexandrov solutions of*

$$\begin{cases} \det D^2 u_k = f_k & \text{in } \Omega_k, \\ u_k = 0 & \text{on } \partial\Omega_k, \end{cases} \tag{1-3}$$

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with $0 < \lambda \le f_k \le \Lambda$. *Assume that* Ω_k *converges to some convex domain* Ω *in the Hausdorff distance*, *and* $f_k \chi_{\Omega_k}$ *converges to* f *in* $L^1_{\mathrm{loc}}(\Omega)$. *Then*, *if* u *denotes the unique Alexandrov solution of*

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for any $\Omega' \Subset \Omega$, *we have*

$$\|u_k - u\|_{W^{2,1}(\Omega')} \to 0 \quad \text{as } k \to \infty. \tag{1-4}$$

(*Obviously*, *since the functions* u_k *are uniformly bounded in* $W^{2,\gamma}(\Omega')$, *this gives strong convergence in* $W^{2,\gamma'}(\Omega')$ *for any* $\gamma' < \gamma$.)

As a consequence, we can prove the following stability result for optimal transport maps:

Theorem 1.2. *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be two bounded domains with* Ω_2 *convex*, *and let* f_k, g_k *be a family of probability densities such that* $0 < \lambda \le f_k, g_k \le \Lambda$ *inside* Ω_1 *and* Ω_2, *respectively. Assume that* $f_k \to f$ *in* $L^1(\Omega_1)$ *and* $g_k \to g$ *in* $L^1(\Omega_2)$, *and let* $T_k : \Omega_1 \to \Omega_2$ (*resp.* $T : \Omega_1 \to \Omega_2$) *be the* (*unique*) *optimal transport map for the quadratic cost sending* f_k *onto* g_k (*resp.* f *onto* g). *Then* $T_k \to T$ *in* $W^{1,\gamma'}_{\mathrm{loc}}(\Omega_1)$ *for some* $\gamma' > 1$.

We point out that, in order to prove (1-4) and the local $W^{1,1}$ stability of optimal transport maps, the interior $L \log L$-estimates from [De Philippis and Figalli 2013] are sufficient. Indeed, the $W^{2,\gamma}$-estimates are used just to improve the convergence from $W^{2,1}_{\mathrm{loc}}$ to $W^{2,\gamma'}_{\mathrm{loc}}$ with $\gamma' < \gamma$.

This paper is organized as follows: in the next section, we collect some notation and preliminary results. Then in Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2.

2. Notation and preliminaries

Given a convex function $u : \Omega \to \mathbb{R}$, we define its *Monge–Ampère measure* as

$$\mu_u(E) := |\partial u(E)| \quad \text{for all } E \subset \Omega \text{ Borel}$$

(see [Gutiérrez 2001, Theorem 1.1.13]), where

$$\partial u(E) := \bigcup_{x \in E} \partial u(x).$$

Here $\partial u(x)$ is the subdifferential of u at x, and $|F|$ denotes the Lebesgue measure of a set F. In case $u \in C^{1,1}_{\mathrm{loc}}$, by the area formula [Evans and Gariepy 1992, Paragraph 3.3], the following representation holds:

$$\mu_u = \det D^2 u \, dx.$$

The main property of the Monge–Ampère measure we are going to use is the following (see [Gutiérrez 2001, Lemmas 1.2.2 and 1.2.3]):

Proposition 2.1. *Let* $u_k : \Omega \to \mathbb{R}$ *be a sequence of convex functions converging locally uniformly to* u. *Then the associated Monge–Ampère measures* μ_{u_k} *converge to* μ_u *in duality with the space of continuous* *functions compactly supported in* Ω*. In particular,*

$$\mu_u(A) \le \liminf_{k\to\infty} \mu_{u_k}(A)$$

for any open set $A \subset \Omega$*.*

Given a Radon measure ν on $\mathbb{R}^n$ and a bounded convex domain $\Omega \subset \mathbb{R}^n$, we say that a convex function $u : \Omega \to \mathbb{R}$ is an *Alexandrov solution* of the Monge–Ampère equation

$$\det D^2 u = \nu \quad \text{in } \Omega$$

if $\mu_u(E) = \nu(E)$ for every Borel set $E \subset \Omega$.

If $v : \overline{\Omega} \to \mathbb{R}$ is a continuous function, we define its *convex envelope inside* Ω as

$$\Gamma_v(x) := \sup\{\ell(x) : \ell \le v \text{ in } \Omega,\ \ \ell \text{ affine}\}. \tag{2-1}$$

In case Ω is a convex domain and $v \in C^2(\Omega)$, it is easily seen that

$$D^2 v(x) \ge 0 \quad \text{for every } x \in \{v = \Gamma_v\} \cap \Omega \tag{2-2}$$

in the sense of symmetric matrices. Moreover, the following inequality between measures holds in Ω:

$$\mu_{\Gamma_v} \le \det D^2 v \mathbf{1}_{\{v=\Gamma_v\}}\, dx \tag{2-3}$$

(here $\mathbf{1}_E$ is the characteristic function of a set E).[1]

We recall that a continuous function v is said to be *twice differentiable* at x if there exists a (unique) vector $\nabla v(x)$ and a (unique) symmetric matrix $\nabla^2 v(x)$ such that

$$v(y) = v(x) + \nabla v(x)\cdot(y-x) + \tfrac{1}{2}\nabla^2 v(x)[y-x,\, y-x] + o(|y-x|^2).$$

In case v is twice differentiable at some point $x_0 \in \{v = \Gamma_v\}$, it is immediate to check that

$$\nabla^2 v(x_0) \ge 0. \tag{2-5}$$

[1] To see this, let us first recall that by [Gutiérrez 2001, Lemma 6.6.2], if $x_0 \in \Omega \setminus \{\Gamma_v = v\}$ and $a \in \partial\Gamma_v(x_0)$, then the convex set

$$\{x \in \Omega : \Gamma_v(x) = a\cdot(x-x_0) + \Gamma_v(x_0)\}$$

is nonempty and contains more than one point. In particular,

$$\partial\Gamma_v\big(\Omega \setminus \{\Gamma_v = v\}\big) \subset \{p \in \mathbb{R}^n : \text{there exist distinct } x, y \in \Omega \text{ such that } p \in \partial\Gamma_v(x) \cap \partial\Gamma_v(y)\}.$$

This last set is contained in the set of nondifferentiability of the convex conjugate of Γ_v, so it has zero Lebesgue measure (see [Gutiérrez 2001, Lemma 1.1.12]), and hence

$$\big|\partial\Gamma_v\big(\Omega \setminus \{\Gamma_v = v\}\big)\big| = 0. \tag{2-4}$$

Moreover, since $v \in C^1(\Omega)$, for any $x \in \{\Gamma_v = v\} \cap \Omega$, we have $\partial\Gamma_v(x) = \{\nabla v(x)\}$. Thus, using (2-4) and (2-2), for any open set $A \Subset \Omega$, we have

$$\mu_{\Gamma_v}(A) = \big|\partial\Gamma_v\big(A \cap \{\Gamma_v = v\}\big)\big| = \big|\nabla v\big(A \cap \{\Gamma_v = v\}\big)\big| \le \int_{A\cap\{\Gamma_v=v\}} |\det D^2 v| = \int_{A\cap\{\Gamma_v=v\}} \det D^2 v,$$

as desired. (The inequality above follows from the area formula in [Evans and Gariepy 1992, Paragraph 3.3.2] applied to the C^1 map ∇v.)

By the Alexandrov theorem, any convex function is twice differentiable almost everywhere (see, for instance, [Evans and Gariepy 1992, Paragraph 6.4]). In particular, (2-5) holds almost everywhere on $\{v = \Gamma_v\}$ whenever v is the difference of two convex functions.

Finally we recall that, in case $v \in W^{2,1}_{\mathrm{loc}}$, the pointwise Hessian of v coincides almost everywhere with its distributional Hessian [Evans and Gariepy 1992, Sections 6.3 and 6.4]. Since in the sequel we are going to deal with $W^{2,1}_{\mathrm{loc}}$ convex functions, we will use D^2u to denote both the pointwise and the distributional Hessian.

3. Proof of Theorem 1.1

We are going to use the following result:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $u, v : \overline{\Omega} \to \mathbb{R}$ be two continuous strictly convex functions such that $\mu_u = f\,dx$ and $\mu_v = g\,dx$, with $f, g \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\mu_{\Gamma_{u-v}} \le \big(f^{1/n} - g^{1/n}\big)^n \mathbf{1}_{\{u-v=\Gamma_{u-v}\}}\,dx. \tag{3-1}$$

Proof. In case u, v are of class C^2 inside Ω, by (2-2) we have

$$0 \le D^2u(x) - D^2v(x) \quad \text{for every } x \in \{u - v = \Gamma_{u-v}\},$$

so using the monotonicity and the concavity of the function $\det^{1/n}$ on the cone of nonnegative symmetric matrices, we get

$$0 \le \det(D^2u - D^2v) \le \big((\det D^2u)^{1/n} - (\det D^2v)^{1/n}\big)^n \quad \text{on} \quad \{u - v = \Gamma_{u-v}\},$$

which, combined with (2-3), gives the desired result.

Now, for the general case, we consider a sequence of smooth uniformly convex domains Ω_k increasing to Ω and two sequences of smooth functions f_k and g_k converging respectively to f and g in $L^1_{\mathrm{loc}}(\Omega)$, and we solve

$$\begin{cases} \det D^2u_k = f_k & \text{in } \Omega_k, \\ u_k = u * \rho_k & \text{on } \partial\Omega_k, \end{cases} \qquad \begin{cases} \det D^2v_k = g_k & \text{in } \Omega_k, \\ v_k = v * \rho_k & \text{on } \partial\Omega_k, \end{cases}$$

where ρ_k is a smooth sequence of convolution kernels. In this way, both u_k and v_k are smooth on $\overline{\Omega}_k$ [Gilbarg and Trudinger 2001, Theorem 17.23], and $\|u_k - u\|_{L^\infty(\Omega_k)} + \|v_k - v\|_{L^\infty(\Omega_k)} \to 0$ as $k \to \infty$.[2] Hence, $\Gamma_{u_k - v_k}$ also converges locally uniformly to Γ_{u-v}. Moreover, it follows easily from the definition of a contact set that

$$\limsup_{k\to\infty} \mathbf{1}_{\{u_k - v_k = \Gamma_{u_k - v_k}\}} \le \mathbf{1}_{\{u-v=\Gamma_{u-v}\}}. \tag{3-2}$$

We now observe that the previous step applied to u_k and v_k gives

$$\mu_{\Gamma_{u_k-v_k}} \le \big((\det D^2u_k)^{1/n} - (\det D^2v_k)^{1/n}\big)^n \mathbf{1}_{\{u_k-v_k=\Gamma_{u_k-v_k}\}}\,dx.$$

Thus, letting $k \to \infty$ and taking into account Proposition 2.1 and (3-2), we obtain (3-1). □

[2] Indeed, it is easy to see that u_k and v_k converge uniformly to u and v, respectively, both on $\partial\Omega_k$ and in any compact subdomain of Ω. Then, using for instance a contradiction argument, one exploits the convexity of u_k (resp. v_k) and Ω_k and the uniform continuity of u (resp. v) to show that the convergence is actually uniform on the whole Ω_k.

Proof of Theorem 1.1. The L^1_{loc} convergence of u_k (resp. ∇u_k) to u (resp. ∇u) is easy and standard, so we focus on the convergence of the second derivatives.

Without loss of generality, we can assume that Ω' is convex, and that $\Omega' \Subset \Omega_k$ (since $\Omega_k \to \Omega$ in the Hausdorff distance, this is always true for k sufficiently large). Fix $\varepsilon \in (0,1)$, let $\Gamma_{u-(1-\varepsilon)u_k}$ be the convex envelope of $u-(1-\varepsilon)u_k$ inside Ω' (see (2-1)), and define

$$A_k^\varepsilon := \bigl\{x \in \Omega' : u(x) - (1-\varepsilon)u_k(x) = \Gamma_{u-(1-\varepsilon)u_k}(x)\bigr\}.$$

Since $u_k \to u$ locally uniformly, $\Gamma_{u-(1-\varepsilon)u_k}$ converges uniformly to $\Gamma_{\varepsilon u} = \varepsilon u$ (as u is convex) inside Ω'. Hence, by applying Proposition 2.1 and (3-1) to u and $(1-\varepsilon)u_k$ inside Ω', we get that

$$\varepsilon^n \int_{\Omega'} f = \mu_{\Gamma_{\varepsilon u}}(\Omega') \le \liminf_{k\to\infty} \mu_{\Gamma_{u-(1-\varepsilon)u_k}}(\Omega') \le \liminf_{k\to\infty} \int_{\Omega'\cap A_k^\varepsilon} \bigl(f^{1/n} - (1-\varepsilon) f_k^{1/n}\bigr)^n.$$

We now observe that, since f_k converges to f in $L^1_{\text{loc}}(\Omega)$, we have

$$\left| \int_{\Omega'\cap A_k^\varepsilon} \bigl(f^{1/n} - (1-\varepsilon) f_k^{1/n}\bigr)^n - \int_{\Omega'\cap A_k^\varepsilon} \varepsilon^n f \right| \le \int_{\Omega'} \left| \bigl(f^{1/n} - (1-\varepsilon) f_k^{1/n}\bigr)^n - \varepsilon^n f \right| \to 0$$

as $k \to \infty$. Hence, combining the two estimates above, we immediately get

$$\int_{\Omega'} f \le \liminf_{k\to\infty} \int_{\Omega'\cap A_k^\varepsilon} f,$$

or equivalently,

$$\limsup_{k\to\infty} \int_{\Omega'\setminus A_k^\varepsilon} f = 0.$$

Since $f \ge \lambda$ inside Ω (as a consequence of the fact that $f_k \ge \lambda$ inside Ω_k), this gives

$$\lim_{k\to\infty} |\Omega' \setminus A_k^\varepsilon| = 0 \quad \text{for all} \quad \varepsilon \in (0,1). \tag{3-3}$$

We now recall that, by the results in [Caffarelli 1990; De Philippis and Figalli 2013; De Philippis et al. 2013; Schmidt 2013], both u and $(1-\varepsilon)u_k$ are strictly convex and belong to $W^{2,1}(\Omega')$. Hence we can apply (2-5) to deduce that

$$D^2 u - (1-\varepsilon) D^2 u_k \ge 0 \quad \text{almost everywhere on } A_k^\varepsilon.$$

In particular, by (3-3),

$$\bigl|\Omega' \setminus \{D^2 u \ge (1-\varepsilon) D^2 u_k\}\bigr| \to 0 \quad \text{as} \quad k \to \infty.$$

By a similar argument (exchanging the roles of u and u_k),

$$\bigl|\Omega' \setminus \{D^2 u_k \ge (1-\varepsilon) D^2 u\}\bigr| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, if we set $B_k^\varepsilon := \{x \in \Omega' : (1-\varepsilon) D^2 u_k \le D^2 u \le (1/(1-\varepsilon)) D^2 u_k\}$, we have

$$\lim_{k\to\infty} |\Omega' \setminus B_k^\varepsilon| = 0 \quad \text{for all} \quad \varepsilon \in (0,1).$$

Moreover, by (1-2) applied to both u_k and u, we have[3]

$$\begin{aligned}\int_{\Omega'}|D^2u-D^2u_k| &= \int_{\Omega'\cap B_k^\varepsilon}|D^2u-D^2u_k|+\int_{\Omega'\setminus B_k^\varepsilon}|D^2u-D^2u_k|\\ &\le \frac{\varepsilon}{1-\varepsilon}\int_{\Omega'}|D^2u|+\|D^2u-D^2u_k\|_{L^\gamma(\Omega')}|\Omega'\setminus B_k^\varepsilon|^{1-1/\gamma}\\ &\le C\biggl(\frac{\varepsilon}{1-\varepsilon}+|\Omega'\setminus B_k^\varepsilon|^{1-1/\gamma}\biggr).\end{aligned}$$

Hence, first letting $k\to\infty$ and then sending $\varepsilon\to 0$, we obtain the desired result. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we will need the following lemma (note that for the next result we do not need to assume the convexity of the target domain):

Lemma 4.1. *Let $\Omega_1,\Omega_2\subset\mathbb{R}^n$ be two bounded domains, and let f_k, g_k be a family of probability densities such that $0<\lambda\le f_k, g_k\le\Lambda$ inside Ω_1 and Ω_2, respectively. Assume that $f_k\to f$ in $L^1(\Omega_1)$ and $g_k\to g$ in $L^1(\Omega_2)$, and let $T_k:\Omega_1\to\Omega_2$ (resp. $T:\Omega_1\to\Omega_2$) be the (unique) optimal transport map for the quadratic cost sending f_k onto g_k (resp. f onto g). Then*

$$\frac{f_k}{g_k\circ T_k}\to\frac{f}{g\circ T}\quad \text{in } L^1(\Omega_1).$$

Proof. By stability of optimal transport maps (see, for instance, [Villani 2009, Corollary 5.23]) and the fact that $f_k\ge\lambda$ (and so $f\ge\lambda$), we know that $T_k\to T$ in measure (with respect to Lebesgue) inside Ω.

We claim that $g\circ T_k\to g\circ T$ in $L^1(\Omega_1)$. Indeed, this is obvious if g is uniformly continuous (by the convergence in measure of T_k to T). In the general case, we choose $g_\eta\in C(\overline{\Omega}_2)$ such that $\|g-g_\eta\|_{L^1(\Omega_2)}\le\eta$, and we observe that (recall that $f_k, f\ge\lambda$, $g_k, g\le\Lambda$, and that by the definition of transport maps, we have $T_\# f_k=g_k$, $T_\# f=g$)

$$\begin{aligned}\int_{\Omega_1}|g\circ T_k-g\circ T| &\le \int_{\Omega_1}|g_\eta\circ T_k-g_\eta\circ T|+\int_{\Omega_1}|g_\eta\circ T_k-g\circ T_k|\frac{f_k}{\lambda}+\int_{\Omega_1}|g_\eta\circ T-g\circ T|\frac{f}{\lambda}\\ &= \int_{\Omega_1}|g_\eta\circ T_k-g_\eta\circ T|+\int_{\Omega_2}|g_\eta-g|\frac{g_k}{\lambda}+\int_{\Omega_2}|g_\eta-g|\frac{g}{\lambda}\\ &\le \int_{\Omega_1}|g_\eta\circ T_k-g_\eta\circ T|+2\frac{\Lambda}{\lambda}\,\eta.\end{aligned}$$

Thus

$$\limsup_{k\to\infty}\int_{\Omega_1}|g\circ T_k-g\circ T|\le 2\frac{\Lambda}{\lambda}\,\eta,$$

and the claim follows by the arbitrariness of η.

[3] If instead of (1-2) we only had uniform $L\log L$ a priori estimates, in place of Hölder's inequality we could apply the elementary inequality $t\le\delta t\log(2+t)+e^{1/\delta}$ with $t=|D^2u-D^2u_k|$ inside $\Omega'\setminus B_k^\varepsilon$, and we would first let $k\to\infty$ and then send $\delta,\varepsilon\to 0$.

Since

$$\begin{aligned}\int_{\Omega_1} |g_k \circ T_k - g \circ T| &\le \int_{\Omega_1} |g_k \circ T_k - g \circ T_k| \frac{f_k}{\lambda} + \int_{\Omega_1} |g \circ T_k - g \circ T| \\ &= \int_{\Omega_2} |g_k - g| \frac{g_k}{\lambda} + \int_{\Omega_1} |g \circ T_k - g \circ T| \\ &\le \frac{\Lambda}{\lambda} \|g_k - g\|_{L^1(\Omega_2)} + \int_{\Omega_1} |g \circ T_k - g \circ T|,\end{aligned}$$

from the claim above we immediately deduce that also $g_k \circ T_k \to g \circ T$ in $L^1(\Omega_1)$.

Finally, since $g_k, g \ge \lambda$ and $f \le \Lambda$,

$$\begin{aligned}\int_{\Omega_1} \left| \frac{f_k}{g_k \circ T_k} - \frac{f}{g \circ T} \right| &\le \int_{\Omega_1} \left| \frac{f_k - f}{g_k \circ T_k} \right| + \int_{\Omega_1} f \left| \frac{1}{g_k \circ T_k} - \frac{1}{g \circ T} \right| \\ &\le \frac{1}{\lambda} \| f_k - f \|_{L^1(\Omega_1)} + \Lambda \int_{\Omega_1} \frac{|g_k \circ T_k - g \circ T|}{g_k \circ T_k \, g \circ T} \\ &\le \frac{1}{\lambda} \| f_k - f \|_{L^1(\Omega_1)} + \frac{\Lambda}{\lambda^2} \| g_k \circ T_k - g \circ T \|_{L^1(\Omega_1)},\end{aligned}$$

from which the desired result follows. ☐

Proof of Theorem 1.2. Since T_k are uniformly bounded in $W^{1,\gamma}(\Omega_1')$ for any $\Omega_1' \Subset \Omega$, it suffices to prove that $T_k \to T$ in $W^{1,1}_{\mathrm{loc}}(\Omega_1)$.

Fix $x_0 \in \Omega_1$ and $r > 0$ such that $B_r(x_0) \subset \Omega_1$. By compactness, it suffices to show that there is an open neighborhood $\mathcal{U}_{x_0}$ of x_0 such that $\mathcal{U}_{x_0} \subset B_r(x_0)$ and

$$\int_{\mathcal{U}_{x_0}} |T_k - T| + |\nabla T_k - \nabla T| \to 0.$$

It is well known [Caffarelli 1992] that T_k (resp. T) can be written as ∇u_k (resp. ∇u) for some strictly convex function $u_k : B_r(x_0) \to \mathbb{R}$ (resp. $u : B_r(x_0) \to \mathbb{R}$). Moreover, up to subtracting a constant from u_k (which will not change the transport map T_k), one may assume that $u_k(x_0) = u(x_0)$ for all $k \in \mathbb{N}$.

Since the functions $T_k = \nabla u_k$ are bounded (as they take values in the bounded set Ω_2), by classical stability of optimal maps (see for instance [Villani 2009, Corollary 5.23]) we get that $\nabla u_k \to \nabla u$ in $L^1_{\mathrm{loc}}(B_r(x_0))$. (Actually, if one uses [Caffarelli 1992], ∇u_k are locally uniformly Hölder maps, so they converge locally uniformly to ∇u.) Hence, to conclude the proof we only need to prove the convergence of $D^2 u_k$ to $D^2 u$ in a neighborhood of x_0.

To this aim, we observe that, by strict convexity of u, we can find a linear function $\ell(z) = a \cdot z + b$ such that the open convex set $Z := \{z : u(z) < u(x_0) + \ell(z)\}$ is nonempty and compactly supported inside $B_{r/2}(x_0)$. Hence, by the uniform convergence of u_k to u (which follows from the L^1_{loc} convergence of the gradients, the convexity of u_k and u, and the fact that $u_k(x_0) = u(x_0)$), and the fact that ∇u is transversal to ℓ on ∂Z, we get that $Z_k := \{z : u_k(z) < u_k(x_0) + \ell(z)\}$ are nonempty convex sets which converge in the Hausdorff distance to Z.

Moreover, by [Caffarelli 1992], the maps $v_k := u_k - \ell$ solve in the Alexandrov sense

$$\begin{cases} \det D^2 v_k = \dfrac{f_k}{g_k \circ T_k} & \text{in } Z_k, \\ v_k = 0 & \text{on } \partial Z_k \end{cases}$$

(here we used that the Monge–Ampère measures associated to v_k and u_k are the same). Therefore, thanks to Lemma 4.1, we can apply Theorem 1.1 to deduce that $D^2 u_k \to D^2 u$ in any relatively compact subset of Z, which concludes the proof. □

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