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SHARP MODULUS OF CONTINUITY FOR PARABOLIC EQUATIONS ON MANIFOLDS AND LOWER BOUNDS FOR THE FIRST EIGENVALUE

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We derive sharp estimates on the modulus of continuity for solutions of the heat equation on a compact Riemannian manifold with a Ricci curvature bound, in terms of initial oscillation and elapsed time. As an application, we give an easy proof of the optimal lower bound on the first eigenvalue of the Laplacian on such a manifold as a function of diameter.

1. Introductory comments

In our previous papers [Andrews and Clutterbuck 2009a; 2009b] we proved sharp bounds on the modulus of continuity of solutions of various parabolic boundary value problems on domains in Euclidean space. In this paper, our aim is to extend these estimates to parabolic equations on manifolds. Precisely, let (M, g) be a compact Riemannian manifold with induced distance function d, diameter

$$\sup\{d(x, y) : x, y \in M\} = D,$$

and lower Ricci curvature bound $\mathrm{Ric}(v, v) \geq (n-1)\kappa g(v, v)$. Let $a : T^*M \to \mathrm{Sym}_2(T^*M)$ be a parallel equivariant map (so that $a(S^*\omega)(S^*\mu, S^*\nu) = a(\omega)(\mu, \nu)$ for any ω, μ, ν in T_x^*M and $S \in O(T_xM)$, while $\nabla\,(a(\omega)(\mu, \nu)) = 0$ whenever $\nabla\omega = \nabla\mu = \nabla\nu = 0$). Then we consider solutions to the parabolic equation

$$\frac{\partial u}{\partial t} = a^{ij}(Du)\nabla_i\nabla_j u. \tag{1}$$

Our assumptions imply that the coefficients a^{ij} have the form

$$a(Du)(\xi, \xi) = \alpha(|Du|)\frac{(Du \cdot \xi)^2}{|Du|^2} + \beta(|Du|)\left(|\xi|^2 - \frac{(Du \cdot \xi)^2}{|Du|^2}\right) \tag{2}$$

for some smooth positive functions α and β. Of particular interest are the cases of the heat equation (with $\alpha = \beta = 1$) and the p-laplacian heat flows (with $\alpha = (p-1)|Du|^{p-2}$ and $\beta = |Du|^{p-2}$). Here we are principally concerned with the case of manifolds without boundary, but can also allow M to have a nontrivial convex boundary (in which case we impose Neumann boundary conditions $D_\nu u = 0$). Our main aim is to provide the following estimates on the modulus of continuity of solutions in terms of the initial oscillation, elapsed time, κ, and D.

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Theorem 1 (modulus of continuity estimate). *Let (M, g) be a compact Riemannian manifold (possibly with smooth, uniformly locally convex boundary) with diameter D and Ricci curvature bound* $\mathrm{Ric} \geq (n-1)\kappa g$ *for some constant $\kappa \in \mathbb{R}$. Let $u : M \times [0, T) \to \mathbb{R}$ be a smooth solution to (1) with Neumann boundary conditions if $\partial M \neq \varnothing$. Suppose that*

- *$u(\cdot, 0)$ has a smooth modulus of continuity $\varphi_0 : [0, D/2] \to \mathbb{R}$ with $\varphi_0(0) = 0$ and $\varphi_0' \geq 0$;*
- *$\varphi : [0, D/2] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies*
  - (i) *$\varphi(z, 0) = \varphi_0(z)$ for each $z \in [0, D/2]$;*
  - (ii) *$\partial\varphi/\partial t \geq \alpha(\varphi')\varphi'' - (n-1)\boldsymbol{T}_\kappa\beta(\varphi')\varphi'$;*
  - (iii) *$\varphi' \geq 0$ on $[0, D/2] \times \mathbb{R}_+$.*

Then $\varphi(\cdot, t)$ is a modulus of continuity for $u(\cdot, t)$ for each $t \in [0, T)$:

$$|u(x, t) - u(y, t)| \leq 2\varphi\left(\frac{d(x, y)}{2}, t\right).$$

Here we use the notation

$$\boldsymbol{C}_\kappa(\tau) = \begin{cases} \cos\sqrt{\kappa}\tau, & \kappa > 0, \\ 1, & \kappa = 0, \\ \cosh\sqrt{-\kappa}\tau, & \kappa < 0 \end{cases} \quad \text{and} \quad \boldsymbol{S}_\kappa(\tau) = \begin{cases} (1/\sqrt{\kappa})\sin\sqrt{\kappa}\tau, & \kappa > 0, \\ \tau, & \kappa = 0, \\ (1/\sqrt{-\kappa})\sinh\sqrt{-\kappa}\tau, & \kappa < 0, \end{cases} \tag{3}$$

and

$$\boldsymbol{T}_\kappa(s) := \kappa\frac{\boldsymbol{S}_\kappa(s)}{\boldsymbol{C}_\kappa(s)} = \begin{cases} \sqrt{\kappa}\tan\left(\sqrt{\kappa}s\right), & \kappa > 0, \\ 0, & \kappa = 0, \\ -\sqrt{-\kappa}\tanh\left(\sqrt{-\kappa}s\right), & \kappa < 0. \end{cases}$$

These estimates are sharp, holding exactly for certain symmetric solutions on particular warped product spaces. The modulus of continuity estimates also imply sharp gradient bounds which hold in the same situation. The central ingredient in our argument is a comparison result for the second derivatives of the distance function (Theorem 3) which is a close relative of the well-known Laplacian comparison theorem. We remark that the assumption of smoothness can be weakened: for example, in the case of the p-laplacian heat flow, we do not expect solutions to be smooth near spatial critical points, but nevertheless solutions are smooth at other points, and this is sufficient for our argument.

As an immediate application of the modulus of continuity estimates, we provide a new proof of the optimal lower bound on the smallest positive eigenvalue of the Laplacian in terms of D and κ. Precisely, if we define

$$\lambda_1(M, g) = \inf\left\{\int_M |Du|_g^2\, d\mathrm{Vol}(g) : \int_M u^2\, d\mathrm{Vol}(g) = 1, \int_M u\, d\mathrm{Vol}(g) = 0\right\}$$

and

$$\lambda_1(D, \kappa, n) = \inf\{\lambda_1(M, g) : \dim(M) = n, \mathrm{diam}(M) \leq D, \mathrm{Ric} \geq (n-1)\kappa g\},$$

then we characterize $\lambda_1(D, \kappa)$ precisely as the first eigenvalue of a certain one-dimensional Sturm–Liouville problem.

Theorem 2 (lower bound on the first eigenvalue). *Let μ be the first eigenvalue of the Sturm–Liouville problem*

$$\begin{gathered}\frac{1}{\boldsymbol{C}_\kappa^{n-1}}\left(\Phi'\boldsymbol{C}_\kappa^{n-1}\right)' + \mu\Phi = 0 \quad \text{on } [-D/2, D/2],\\ \Phi'(\pm D/2) = 0.\end{gathered} \tag{4}$$

Then $\lambda_1(D, \kappa, n) = \mu$.

Previous results in this direction include those derived from gradient estimates in [Li 1979; Li and Yau 1980], with the sharp result for nonnegative Ricci curvature first proved in [Zhong and Yang 1984]. The complete result as stated above is implicit in [Kröger 1992, Theorem 2]. Chen and Wang [1994] used stochastic methods to prove an apparently equivalent result. The result appears to have been first explicitly stated in the form above in by Bakry and Qian [2000, Theorem 14], who also used gradient estimate methods. Our contribution is the rather simple proof using the long-time behavior of the heat equation (a method which was also central in our work on the fundamental gap conjecture [Andrews and Clutterbuck 2011], and which has also been employed successfully in [Ni 2013]), which seems considerably easier than the previously available arguments. In particular, the complications arising in previous works from possible asymmetry of the first eigenfunction are avoided in our argument. A similar argument proving the sharp lower bound for λ_1 on a Bakry–Emery manifold may be found in [Andrews and Ni 2012].

The estimate in Theorem 2 is sharp (that is, we obtain an equality and not just an inequality), since, for a given diameter D and Ricci curvature bound κ, we can construct a sequence of manifolds satisfying these bounds on which the first eigenvalue approaches μ_1; see the remarks after Corollary 1 in [Kröger 1992]. We include a discussion of these examples in Section 5, since the examples required for our purposes are a simpler subset of those constructed in [Kröger 1992]. We also include in Section 6 a discussion of the implications for a conjectured inequality of Li.

2. A comparison theorem for the second derivatives of distance

Theorem 3. *Let (M, g) be a complete connected Riemannian manifold with a lower Ricci curvature bound* $\mathrm{Ric} \geq (n-1)\kappa g$, *and let φ be a smooth function with $\varphi' \geq 0$. Then on $(M \times M) \setminus \{(x, x) : x \in M\}$ the function $v(x, y) = 2\varphi(d(x, y)/2)$ is a viscosity supersolution of*

$$\mathscr{L}[\nabla^2 v, \nabla v] = 2[\alpha(\varphi')\varphi'' - (n-1)\boldsymbol{T}_\kappa\beta(\varphi')\varphi']|_{d/2},$$

where

$$\mathscr{L}[B, \omega] = \inf\left\{ \operatorname{tr}(AB) : \begin{array}{l} A \in \mathrm{Sym}_2(T^*_{x,y}(M \times M)),\\ A \geq 0,\\ A|_{T^*_x M} = a(\omega|_{T_x M}),\\ A|_{T^*_y M} = a(\omega|_{T_y M}) \end{array} \right\}$$

for any $B \in \mathrm{Sym}_2(T_{x,y}(M \times M)$ and $\omega \in T^*_{(x,y)}(M \times M)$.

Proof. By approximation it suffices to consider the case where φ' is strictly positive. Let x and y be fixed, with $y \neq x$ and $d = d(x, y)$, and let $\gamma : [-d/2, d/2] \to M$ be a minimizing geodesic from x to y (that is, with $\gamma(-d/2) = x$ and $\gamma(d/2) = y$) parametrized by arc length. Choose an orthonormal basis $\{E_i\}_{1\le i\le n}$ for T_xM with $E_n = \gamma'(-d/2)$. Use parallel transport along γ to produce an orthonormal basis $\{E_i(s)\}_{1\le i\le n}$ for $T_{\gamma(s)}M$ with $E_n(s) = \gamma'(s)$ for each $s \in [-d/2, d/2]$. Let $\{E_*^i\}_{1\le i\le n}$ be the dual basis for $T^*_{\gamma(s)}M$.

To prove the theorem, consider any smooth function ψ defined on a neighborhood of (x, y) in $M \times M$ such that $\psi \le v$ and $\psi(x, y) = v(x, y)$. We must prove that

$$\mathcal{L}[\nabla^2\psi, \nabla\psi]|_{(x,y)} \le 2[\alpha(\varphi')\varphi'' - (n-1)\beta(\varphi')\varphi'\boldsymbol{T}_\kappa]|_{d(x,y)/2}.$$

By definition of $\mathcal{L}$, it suffices to find a nonnegative $A \in \mathrm{Sym}_2(T^*_{x,y}(M\times M))$ such that $A|_{T_xM} = a(\nabla\psi|_{T_xM})$ and $A|_{T_yM} = a(\nabla\psi|_{T_yM})$, with $\mathrm{tr}(AD^2\psi) \le 2[\alpha(\varphi')\varphi'' - (n-1)\beta(\varphi')\varphi'\boldsymbol{T}_\kappa]|_{d/2}$.

Before choosing this, we observe that $\nabla\psi$ is determined by d and φ: We have $\psi \le 2\varphi \circ d/2$ with equality at (x, y). In particular, we have (since φ is nondecreasing)

$$\psi(\gamma(s), \gamma(t)) \le 2\varphi(d(\gamma(s), \gamma(t))/2) \le 2\varphi(L[\gamma|_{[s,t]}]/2) \le 2\varphi(|t-s|/2)$$

for all $s \neq t$, with equality when $t = d/2$ and $s = -d/2$. This gives $\nabla\psi(E_n, 0) = -\varphi'(d/2)$ and $\nabla\psi(0, E_n) = \varphi'(d/2)$. To identify the remaining components of $\nabla\psi$, we define

$$\gamma_i^y(r, s) = \exp_{\gamma(s)}(r(1/2 + s/d)E_i(s))$$

for $1 \le i \le n-1$. Then we have

$$\psi(x, \exp_y(rE_i)) \le 2\varphi(L[\gamma_i^y(r, \cdot)]/2)$$

with equality at $r = 0$. The right-hand side is a smooth function of r with derivative zero, from which it follows that $\nabla\psi(0, E_i) = 0$. Similarly, we have $\nabla\psi(E_i, 0) = 0$ for $i = 1, \dots, n-1$. Therefore we have

$$\nabla\psi|_{(x,y)} = \varphi'(d(x, y)/2)(-E_*^n, E_*^n).$$

In particular, by (2), we have

$$a(\nabla\psi|_{T_xM}) = \alpha(\varphi')E_n \otimes E_n + \beta(\varphi')\sum_{i=1}^{n-1} E_i \otimes E_i,$$

and similarly for y.

Now we choose A as follows:

$$A = \alpha(\varphi')(E_n, -E_n) \otimes (E_n, -E_n) + \beta(\varphi')\sum_{i=1}^{n-1}(E_i, E_i) \otimes (E_i, E_i). \tag{5}$$

This is manifestly nonnegative, and agrees with a on T_xM and T_yM as required. This choice gives

$$\mathrm{tr}(A\nabla^2\psi) = \alpha(\varphi)\nabla^2\psi((E_n, -E_n), (E_n, -E_n)) + \beta(\varphi')\sum_{i=1}^{n-1}\nabla^2\psi((E_i, E_i), (E_i, E_i)). \tag{6}$$

For each $i \in \{1, \dots, n-1\}$ let $\gamma_i : (-\varepsilon, \varepsilon) \times [-d/2, d/2] \to M$ be any smooth one-parameter family of curves with $\gamma_i(r, \pm d/2) = \exp_{\gamma(\pm d/2)}(r E_i(\pm d/2))$ for $i = 1, \dots, n-1$, and $\gamma_i(0, s) = \gamma(s)$. Then $d(\exp_x(r E_i), \exp_y(r E_i)) \le L[\gamma_i(r, \cdot)]$, and hence

$$\begin{aligned}\psi(\exp_x(r E_i), \exp_y(r E_i)) &\le v(\exp_x(r E_i), \exp_y(r E_i)) \\ &= 2\varphi\left(\frac{d(\exp_x(r E_i), \exp_y(r E_i))}{2}\right) \\ &\le 2\varphi\left(\frac{L[\gamma_i(r, \cdot)]}{2}\right),\end{aligned}$$

since φ is nondecreasing. Since the functions on the left and the right are both smooth functions of r and equality holds for $r = 0$, it follows that

$$\nabla^2\psi((E_i, E_i), (E_i, E_i)) \le 2\frac{d^2}{dr^2}\left(\varphi\left(\frac{L[\gamma_i(r, \cdot)]}{2}\right)\right)\Big|_{r=0}. \tag{7}$$

Similarly, since $d - 2r = L[\gamma|_{[-d/2+r, d/2-r]}] \ge d(\gamma(-d/2 + r), \gamma(d/2 - r))$, we have

$$\nabla^2\psi(E_n, -E_n), (E_n, -E_n)) \le 2\frac{d^2}{dr^2}\left(\varphi\left(\frac{d}{2} - r\right)\right)\Big|_{r=0} = 2\varphi''\left(\frac{d}{2}\right). \tag{8}$$

Now we make a careful choice of the curves $\gamma_i(r, \cdot)$, motivated by the situation in the model space, in order to get a useful result on the right-hand side in inequality (7): To begin with, if $K > 0$, we assume that $d < \pi/\sqrt{K}$ (we will return to deal with the equality case later). We choose

$$\gamma_i(r, s) = \exp_{\gamma(s)}\left(\frac{r \boldsymbol{C}_\kappa(s) E_i}{\boldsymbol{C}_\kappa(d/2)}\right),$$

where $\boldsymbol{C}_\kappa$ is given by (3). Now we proceed to compute the right-hand side of (7): Denoting s derivatives of γ_i by γ' and r derivatives by $\dot\gamma$, we find

$$\begin{aligned}\frac{d}{dr}\left(L[\gamma_i(r, \cdot)]\right) &= \frac{d}{dr}\left(\int_{-d/2}^{d/2} \|\gamma'(r, s)\|\, ds\right) \\ &= \int_{-d/2}^{d/2} \frac{\langle \gamma', \nabla_r \gamma' \rangle}{\|\gamma'\|}\, ds.\end{aligned}$$

In particular this gives zero when $r = 0$. Differentiating again, we obtain (using $\|\gamma'(0, s)\| = 1$ and the expression $\dot\gamma(0, s) = (\boldsymbol{C}_\kappa(s)/\boldsymbol{C}_\kappa(d/2)) E_i$)

$$\frac{d^2}{dr^2}(L[\gamma_i(r, \cdot)])|_{r=0} = \int_{-d/2}^{d/2} \|\nabla_r \gamma'\|^2 - \langle \gamma', \nabla_r \gamma' \rangle^2 + \langle \gamma', \nabla_r \nabla_r \gamma' \rangle\, ds.$$

Now we observe that

$$\nabla_r \gamma' = \nabla_s \dot\gamma = \nabla_s\left(\frac{\boldsymbol{C}_\kappa(s)}{\boldsymbol{C}_\kappa(d/2)} E_i\right) = \frac{\boldsymbol{C}_\kappa'(s)}{\boldsymbol{C}_\kappa(d/2)} E_i,$$

while

$$\nabla_r\nabla_r\gamma' = \nabla_r\nabla_s\dot\gamma = \nabla_s\nabla_r\dot\gamma - R(\dot\gamma,\gamma')\dot\gamma = -\frac{\boldsymbol{C}_\kappa(s)^2}{\boldsymbol{C}_\kappa(d/2)^2}R(E_i,E_n)E_i,$$

since by the definition of $\gamma_i(r,s)$ we have $\nabla_r\dot\gamma = 0$. This gives

$$\frac{d^2}{dr^2}(L[\gamma_i(r,\cdot)])|_{r=0} = \frac{1}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}\{\boldsymbol{C}_\kappa'(s)^2 - \boldsymbol{C}_\kappa(s)^2R(E_i,E_n,E_i,E_n)\}\,ds.$$

Summing over i from 1 to $n-1$ gives

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{d^2}{dr^2}(L[\gamma_i(r,\cdot)])|_{r=0} &= \frac{1}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}\left\{(n-1)\boldsymbol{C}_\kappa'(s)^2 - \boldsymbol{C}_\kappa(s)^2\sum_{i=1}^{n-1}R(E_i,E_n,E_i,E_n)\right\}ds\\ &= \frac{1}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}\{(n-1)\boldsymbol{C}_\kappa'(s)^2 - \boldsymbol{C}_\kappa(s)^2\,\mathrm{Ric}(E_n,E_n)\}\,ds\\ &\le \frac{n-1}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}\{\boldsymbol{C}_\kappa'(s)^2 - \kappa\boldsymbol{C}_\kappa(s)^2\}\,ds.\end{aligned}$$

In the case $\kappa = 0$, the integral is zero; in the case $\kappa < 0$, or the case $\kappa > 0$ with $d < \pi/\sqrt{\kappa}$, we have

$$\begin{aligned}\frac{1}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}\{\boldsymbol{C}_\kappa'(s)^2 - \kappa\boldsymbol{C}_\kappa(s)^2\}\,ds &= \frac{1}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}(-\kappa\boldsymbol{S}_\kappa\boldsymbol{C}_\kappa' - \kappa\boldsymbol{S}_\kappa'\boldsymbol{C}_\kappa)\,ds\\ &= -\frac{\kappa}{\boldsymbol{C}_\kappa(d/2)^2}\int_{-d/2}^{d/2}(\boldsymbol{C}_\kappa\boldsymbol{S}_\kappa)'\,ds\\ &= -\frac{2\kappa\boldsymbol{C}_\kappa(d/2)\boldsymbol{S}_\kappa(d/2)}{\boldsymbol{C}_\kappa(d/2)^2}\\ &= -2\boldsymbol{T}_\kappa(d/2).\end{aligned}$$

Finally, we have

$$\frac{d}{dr}\left(\varphi\left(\frac{L[\gamma_i(r,\cdot)]}{2}\right)\right)\Big|_{r=0} = \varphi'\frac{d}{dr}\left(\frac{L[\gamma_i(r,\cdot)]}{2}\right)\Big|_{r=0} = 0,$$

and so

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{d^2}{dr^2}\left(\varphi\left(\frac{L[\gamma_i(r,\cdot)]}{2}\right)\right)\Big|_{r=0} &= \sum_{i=1}^{n-1}\left(\varphi'\frac{d^2}{dr^2}\left(\frac{L[\gamma_i(r,\cdot)]}{2}\right)\Big|_{r=0} + \varphi''\left(\frac{d}{dr}\left(\frac{L[\gamma_i(r,\cdot)]}{2}\right)\Big|_{r=0}\right)^2\right)\\ &\le -(n-1)\varphi'\boldsymbol{T}_\kappa|_{d/2}.\end{aligned}$$

Now, using the inequalities (7) and (8), we have from (6) that

$$\mathscr{L}[\nabla^2\psi,\nabla\psi] \le \mathrm{trace}(A\nabla^2\psi) \le 2[\alpha(\varphi')\varphi'' - (n-1)\beta(\varphi')\varphi'\boldsymbol{T}_\kappa]|_{d/2}, \tag{9}$$

as required.

In the case $d = \pi/\sqrt{K}$, we instead choose $\gamma_i(r,s) = \exp_{\gamma(s)}(r\boldsymbol{C}_{\kappa'}(s)E_i/(\boldsymbol{C}_{\kappa'}(d/2)))$, for arbitrary $\kappa' < \kappa$. Then the computation above gives

$$\sum_{i=1}^{n-1} \nabla^2\psi((E_i, E_i), (E_i, E_i)) \le -2(n-1)\varphi'\boldsymbol{T}_\kappa.$$

Since the right-hand side approaches $-\infty$ as κ' increases to κ, we have a contradiction to the assumption that ψ is smooth. Hence no such ψ exists and there is nothing to prove. □

3. Estimate on the modulus of continuity for solutions of heat equations

In this section we prove Theorem 1, which extends the oscillation estimate from domains in $\mathbb{R}^n$ to compact Riemannian manifolds. The estimate is analogous to [Andrews and Clutterbuck 2009b, Theorem 4.1], the modulus of continuity estimate for the Neumann problem on a convex Euclidean domain.

Proof of Theorem 1. Recall that (M, g) is a compact Riemannian manifold, possibly with boundary (in which case we assume that the boundary is locally convex). Define an evolving quantity, Z, on the product manifold $M \times M \times [0, \infty)$:

$$Z(x, y, t) := u(y, t) - u(x, t) - 2\varphi(d(x, y)/2, t) - \epsilon(1 + t)$$

for small $\epsilon > 0$.

We have assumed that φ is a modulus of continuity for u at $t = 0$, and so $Z(\cdot, \cdot, 0) \le -\epsilon < 0$. Note also that Z is continuous on $M \times M \times [0, \infty)$, and $Z(x, x, t) = -\varepsilon(1 + t) < 0$ for each $x \in M$ and $t \in [0, T)$. It follows that if Z ever becomes positive, there exists a first time $t_0 > 0$ and points $x_0 \ne y_0$ in M such that $Z(x_0, y_0, t_0) = 0$. There are two possibilities: either both x_0 and y_0 are in the interior of M, or at least one of them (say x_0) lies in the boundary ∂M.

We deal with the first case first: Clearly $Z(x, y, t) \le 0$ for all $x, y \in M$ and $t \in [0, t_0]$. In particular, if we let $v(x, y) = 2\varphi(d(x, y)/2, t_0)$ and $\psi(x, y) = u(y, t_0) - u(x, t_0) - \varepsilon(1 + t_0)$, then

$$\psi(x, y) \le v(x, y)$$

for all $x, y \in M$, while $\psi(x_0, y_0) = v(x_0, y_0)$. Since ψ is smooth, by Theorem 3 we have

$$\mathcal{L}[\nabla^2\psi, \nabla\psi] \le 2[\alpha(\varphi')\varphi'' - (n-1)\boldsymbol{T}_\kappa\beta(\varphi')\varphi']|_{d(x_0,y_0)/2}.$$

Now we observe that since the mixed partial derivatives of $\nabla^2\psi$ all vanish, we have for any admissible A in the definition of $\mathcal{L}$ that

$$\operatorname{tr}(A\nabla^2\psi) = (a(Du)^{ij}\nabla_i\nabla_j u)|_{(y_0,t_0)} - (a(Du)^{ij}\nabla_i\nabla_j u)|_{(x_0,t_0)},$$

and therefore

$$\mathcal{L}[\nabla^2\psi, \nabla\psi] = (a(Du)^{ij}\nabla_i\nabla_j u)|_{(y_0,t_0)} - (a(Du)^{ij}\nabla_i\nabla_j u)|_{(x_0,t_0)}.$$

It follows that

$$a(Du)^{ij}\nabla_i\nabla_j u|_{(y_0,t_0)} - a(Du)^{ij}\nabla_i\nabla_j u|_{(x_0,t_0)} \le 2[\alpha(\varphi')\varphi'' - (n-1)\boldsymbol{T}_\kappa\beta(\varphi')\varphi']|_{d(x_0,y_0)/2}. \tag{10}$$

We also know that the time derivative of Z is nonnegative at (x_0, y_0, t_0), since $Z(x_0, y_0, t) \le 0$ for $t < t_0$:

$$\frac{\partial Z}{\partial t}|_{(x_0,y_0,t_0)} = a(Du)^{ij}\nabla_i\nabla_j u|_{(y_0,t_0)} - a(Du)^{ij}\nabla_i\nabla_j u|_{(x_0,t_0)} - 2\frac{\partial \varphi}{\partial t} - \varepsilon \ge 0. \tag{11}$$

Combining the inequalities (10) and (11), we obtain

$$\frac{\partial \varphi}{\partial t} < \alpha(\varphi')\varphi'' - (n-1)\boldsymbol{T}_\kappa\beta(\varphi')\varphi',$$

where all terms are evaluated at the point $d(x_0, y_0)/2$. This contradicts assumption (ii) in Theorem 1.

Now we consider the second case, where $x_0 \in \partial M$. Under the assumption that ∂M is convex, there exists [Bartolo et al. 2002] a length-minimizing geodesic $\gamma : [0, d] \to M$ from x_0 to y_0, such that $\gamma(s)$ is in the interior of M for $0 < s < d$ and $\gamma'(0) \cdot \nu(x_0) > 0$, where $\nu(x_0)$ is the inward-pointing unit normal to ∂M at x_0. We compute

$$\frac{d}{ds}Z(\exp_{x_0}(s\nu(x_0)), y_0, t_0) = -\nabla_{\nu(x_0)}u - \varphi'(d/2)\nabla d(\nu(x_0), 0) = \varphi'(d/2)\gamma'(0)\cdot\nu(x_0) \ge 0.$$

In particular, $Z(\exp_{x_0}(s\nu(x_0)), y_0, t_0) > 0$ for all small positive s, contradicting the fact that $Z(x, y, t_0) \le 0$ for all $x, y \in M$.

Therefore Z remains negative for all $(x, y) \in M$ and $t \in [0, T)$. Letting ε approach zero proves the theorem. □

4. The eigenvalue lower bound

Now we provide the proof of the sharp lower bound on the first eigenvalue (Theorem 2), which follows very easily from the modulus of continuity estimate from Theorem 1.

Proposition 4. *For M and u as in Theorem 1 applied to the heat equation ($\alpha \equiv \beta \equiv 1$ in (2)), we have the oscillation estimate*

$$|u(y, t) - u(x, t)| \le Ce^{-\mu t},$$

where C depends on the modulus of continuity of $u(\cdot, 0)$, and μ is the smallest positive eigenvalue of the Sturm–Liouville equation

$$\begin{gathered}\Phi'' - (n-1)\boldsymbol{T}_\kappa\Phi' + \mu\Phi = \frac{1}{\boldsymbol{C}_\kappa^{n-1}}(\Phi'\boldsymbol{C}_\kappa^{n-1})' + \mu\Phi = 0 \quad \text{on } [-D/2, D/2],\\ \Phi'(\pm D/2) = 0.\end{gathered} \tag{12}$$

Proof. The eigenfunction-eigenvalue pair (Φ, μ) is defined as follows: For any $\sigma \in \mathbb{R}$ we define $\Phi_\sigma(x)$ to be the solution of the initial value problem

$$\begin{aligned}\Phi_\sigma'' - (n-1)\boldsymbol{T}_\kappa\Phi_\sigma' + \sigma\Phi_\sigma &= 0;\\ \Phi_\sigma(0) &= 0;\\ \Phi_\sigma'(0) &= 1.\end{aligned}$$

Then $\mu = \sup\{\sigma : x \in [-D/2, D/2] \Longrightarrow \Phi'_\sigma(x) > 0\}$. In particular, for $\sigma < \mu$ the function Φ_σ is strictly increasing on $[-D/2, D/2]$, and $\Phi_\sigma(x)$ is decreasing in σ and converges smoothly to $\Phi(x) = \Phi_\mu(x)$ as σ approaches μ for $x \in (0, D/2]$ and $0 < \sigma < \mu$.

Now we apply Theorem 1: Since Φ is smooth, has positive derivative at $x = 0$ and is positive for $x \in (0, D/2]$, there exists $C > 0$ such that $C\Phi$ is a modulus of continuity for $u(\,.\,, 0)$. Then, for each $\sigma \in (0, \mu)$, $\varphi_0 = C\Phi_\sigma$ is also a modulus of continuity for $u(\,.\,, 0)$, with $\varphi_0(0) = 0$ and $\varphi'_0 > 0$. Defining $\varphi(x, t) = C\Phi_\sigma(x)\mathrm{e}^{-\sigma t}$, all the conditions of Theorem 1 are satisfied, and we deduce that $\varphi(\,.\,, t)$ is a modulus of continuity for $u(\,.\,, t)$, each $t \geq 0$. Letting σ approach μ, we deduce that $C\Phi\mathrm{e}^{-\mu t}$ is also a modulus of continuity. That is, for all x, y and $t \geq 0$,

$$|u(y, t) - u(x, t)| \leq C\mathrm{e}^{-\mu t}\Phi\left(\frac{d(x, y)}{2}\right) \leq C \sup \Phi\mathrm{e}^{-\mu t}. \qquad \square$$

Proof of Theorem 2. Observe that if (φ, λ) is the first eigenfunction-eigenvalue pair, then $u(x, t) = e^{-\lambda t}\varphi(x)$ satisfies the heat equation on M for all $t > 0$. From Proposition 4, we have $|u(y, t) - u(x, t)| \leq Ce^{-\mu t}$, and so $|\varphi(y) - \varphi(x)| \leq Ce^{-(\mu-\lambda)t}$ for all $x, y \in M$ and $t > 0$. Since φ is nonconstant, letting $t \to \infty$ implies that $\mu - \lambda \leq 0$. $\square$

5. Sharpness of the estimates

In the previous section we proved that $\lambda_1(D, \kappa, n) \geq \mu$. To complete the proof of Theorem 2, we must prove that $\lambda_1(D, \kappa, n) \leq \mu$. To do this, we construct examples of Riemannian manifolds with given diameter bounds and Ricci curvature lower bounds such that the first eigenvalue is as close as desired to μ. The construction is similar to that given in [Kröger 1992; Bakry and Qian 2000], but we include it here because the construction also produces examples proving that the modulus of continuity estimates of Theorem 1 are sharp.

Fix κ and D, and let $M = S^{n-1} \times [-D/2, D/2]$ with the metric

$$g = ds^2 + a\boldsymbol{C}^2_\kappa(s)\bar{g},$$

where $\bar{g}$ is the standard metric on S^{n-1}, and $a > 0$. The Ricci curvatures of this metric are given by

$$\begin{aligned}
&\mathrm{Ric}(\partial_s, \partial_s) = (n-1)\kappa;\\
&\mathrm{Ric}(\partial_s, v) = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{for } v \in TS^{n-1};\\
&\mathrm{Ric}(v, v) = \left((n-1)\kappa + (n-2)\frac{1/a - \kappa}{\boldsymbol{C}^2_\kappa}\right)|v|^2 \quad \text{for } v \in TS^{n-1}.
\end{aligned}$$

In particular, the lower Ricci curvature bound $\mathrm{Ric} \geq (n-1)\kappa$ is satisfied for any a if $\kappa \leq 0$ and for $a \leq 1/\kappa$ if $\kappa > 0$.

To demonstrate the sharpness of the modulus of continuity estimate in Theorem 1, we construct solutions of (1) on M which satisfy the conditions of Theorem 1 and satisfy the conclusion with equality for positive times: Let $\varphi_0 : [0, D/2]$ be as given in Theorem 1, and extend by odd reflection to $[-D/2, D/2]$ and

define φ to be the solution of the initial-boundary value problem

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= \alpha(\varphi')\varphi'' + (n-1)\boldsymbol{T}_\kappa\beta(\varphi')\varphi';\\ \varphi(x,0) &= \varphi_0(x);\\ \varphi'(\pm D/2, t) &= 0.\end{aligned}$$

Now define $u(z,s,t) = \varphi(s,t)$ for $s \in [-D/2, D/2]$, $z \in S^{n-1}$, and $t \geq 0$. Then a direct calculation shows that u is a solution of (1) on M. If φ_0 is concave on $[0, D/2]$, we have $|\varphi_0(a) - \varphi_0(b)| \leq 2\varphi_0(|b-1|/2)$ for all a and b in $[-D/2, D/2]$. For our choice of φ, this also remains true for positive times. Note also that for any $w, z \in S^{n-1}$ and $a, b \in [-D/2, D/2]$ we have $d((w,a),(z,b)) \geq |b-a|$. Therefore we have

$$|u(w,a,t) - u(z,b,t)| = |\varphi(a,t) - \varphi(b,t)| \leq 2\varphi\left(\frac{|b-a|}{2}, t\right) \leq 2\varphi\left(\frac{d((w,a),(z,b))}{2}, t\right),$$

so that $\varphi(\,.\,,t)$ is a modulus of continuity for $u(\,.\,,t)$ as claimed. Furthermore, this holds with equality whenever $w = z$ and $b = -a$, so there is no smaller modulus of continuity and the estimate is sharp.

Now we proceed to the sharpness of the eigenvalue estimate. On the manifold constructed above, we have an explicit eigenfunction of the Laplacian, given by $\varphi(z,s) = \Phi(s)$, where Φ is the first eigenfunction of the one-dimensional Sturm–Liouville problem given in Proposition 4. That is, we have $\lambda_1(M,g) \leq \mu$. In this example we have the required Ricci curvature lower bound, and the diameter approaches D as $a \to 0$. Since μ depends continuously on D, the result follows.

A slightly more involved construction shows that the bound is sharp even in the smaller class of manifolds without boundary. This is achieved by smoothly attaching spherical caps to the ends of the above examples; see the similar construction in [Andrews and Ni 2012, Section 2].

6. Implications for the "Li conjecture"

In this section we mention some implications of the sharp eigenvalue estimate and a conjecture attributed to Peter Li. The result of Lichnerowicz [1958] is that $\lambda_1 \geq n\kappa$ whenever $\mathrm{Ric} \geq (n-1)\kappa g_{ij}$ (so that, by the Bonnet–Myers estimate, $D \leq \pi/\sqrt{\kappa}$). An estimate from [Zhong and Yang 1984] gives $\lambda_1 \geq \pi^2/D^2$ for $\mathrm{Ric} \geq 0$. Both of these are sharp, and the latter estimate should also be sharp as $D \to 0$ for any lower Ricci curvature bound. Interpolating linearly (in κ) between these estimates, we obtain Li's conjecture:

$$\lambda_1 \geq \frac{\pi^2}{D^2} + (n-1)\kappa.$$

By construction this holds precisely at the endpoints $\kappa \to 0$ and $\kappa \to \pi^2/D^2$.

Several previous attempts to prove such inequalities have been made, particularly towards proving inequalities of the form $\lambda_1 \geq \pi^2/D^2 + a\kappa$ for some constant a, which are linear in κ and have the correct limit as $\kappa \to 0$. These include works of DaGang Yang [1999], Jun Ling [2006] and Ling and Lu [2010], the latter showing that $\alpha = \frac{34}{100}$ holds. These are all superseded by the result of Shi and Zhang [2007] which proves $\lambda_1 \geq \sup_{s\in(0,1)}\{4s(1-s)\pi^2/D^2 + (n-1)s\kappa\}$, so in particular $\lambda_1 \geq \pi^2/D^2 + ((n-1)/2)\kappa$ by taking $s = \frac{1}{2}$.

We remark here that the inequality with $a = (n-1)/2$ is the best possible of this kind, and in particular the Li conjecture is false. This can be seen by computing an asymptotic expansion for the sharp lower bound μ given by Theorem 2. For fixed $D = \pi$ we perturb about $\kappa = 0$ (as in [Andrews and Ni 2012, Section 4]), obtaining

$$\mu = 1 + \frac{(n-1)}{2}\kappa + O(\kappa^2).$$

By scaling, this amounts to the estimate

$$\mu = \frac{\pi^2}{D^2} + \frac{(n-1)}{2}\kappa + O(\kappa D^2).$$

Since the lower bound $\lambda_1 \geq \mu$ is sharp, this shows that the inequality $\lambda_1 \geq \pi^2/D^2 + a\kappa$ is false for any $a > (n-1)/2$, and in particular for $a = n-1$.

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