MAGNETIC INTERPRETATION OF THE NODAL DEFECT ON GRAPHS
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Yves Colin de Verdière

We present a natural proof of a recent and surprising result of Gregory Berkolaiko interpreting the Courant nodal defect as a Morse index. This proof is inspired by a nice paper of Miroslav Fiedler published in 1975.

1. Introduction

The “nodal defect” of an eigenfunction of a Schrödinger operator is closely related to the difference between the upper bound on the number of nodal domains given by Courant’s theorem and the number of nodal domains. Berkolaiko [2013] has proved a nice formula for the nodal defect of an eigenfunction of a Schrödinger operator on a finite graph in terms of the Morse index of the corresponding eigenvalue as a function of a magnetic deformation of the operator. His proof remains mysterious and rather indirect. In order to get a better understanding in view of possible generalizations, it is desirable to have a more direct approach. This is what we do here, with a proof inspired by [Fiedler 1975].

After reviewing our notations, we state the main result, as well as a reinterpretation in terms of Hessians of a determinant, and give an informal description of the proof in Section 3. The proof itself is implemented in Sections 4 and 5 with an alternative view provided in Appendix A. The continuous Schrödinger operator on a circle was considered in the preprint version of this paper [Colin de Verdière 2012]. The case of quantum graphs, i.e., graphs as 1-dimensional simplicial complexes, is worked out in [Berkolaiko and Weyand 2012].

2. Notation

Let $G = (X, E)$ be a finite connected graph, where $X$ is the set of vertices and $E$ the set of unoriented edges. We denote by $\{x, y\}$ the edge linking the vertices $x$ and $y$. We denote by $\tilde{E}$ the set of oriented edges and by $[x, y]$ the edge from $x$ to $y$; the set $\tilde{E}$ is a 2-fold cover of $E$. A 1-form $\alpha$ on $G$ is a map $\tilde{E} \to \mathbb{R}$ such that $\alpha([y, x]) = -\alpha([x, y])$ for all $\{x, y\} \in E$. We denote by $\Omega^1(G)$ the vector space of dimension $\#E$ of 1-forms on $G$. The operator $d : \mathbb{R}^X \to \Omega^1(G)$ is defined by $df([x, y]) = f(y) - f(x)$. If $Q$ is a nondegenerate, not necessarily positive, quadratic form on $\Omega^1(G)$, we denote by $d^*$ the adjoint of $d$, where $\mathbb{R}^X$ carries the canonical Euclidean structure and $\Omega^1(G)$ is equipped with the symmetric inner product $\hat{Q}$ associated to $Q$. We have dim $\ker d^* = \beta$, where $\beta = 1 + \#E - \#X$ is the dimension

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of the space of cycles of $G$. We will show later that, in our context, we have the Hodge decomposition \( \Omega^1(G) = d\mathbb{R}^X \oplus \ker d^* \), where both spaces are \( \hat{Q} \)-orthogonal.

Following [Colin de Verdière 1998], we denote by \( \mathcal{C}_G \) the set of \( X \times X \) real symmetric matrices \( H \) which satisfy \( h_{x,y} < 0 \) if \( \{x, y\} \in E \) and \( h_{x,y} = 0 \) if \( \{x, y\} \notin E \) and \( x \neq y \). Note that the diagonal entries of \( H \) are arbitrary. An element \( H \) of \( \mathcal{C}_G \) is called a **Schrödinger operator** on the graph \( G \). It will be useful to write the quadratic form associated to \( H \) as

\[
q_1(f) = -\sum_{\{x,y\}\in E} h_{x,y}(f(x) - f(y))^2 + \sum_{x\in X} V_x f(x)^2,
\]

with \( V_x = h_{x,x} + \sum_{y\sim x} h_{x,y} \). A **magnetic field** on \( G \) is a map \( B : \vec{E} \to U(1) \) defined by \( B([x, y]) = e^{i\alpha_{x,y}} \), where \([x, y] \mapsto \alpha_{x,y} \) is a 1-form on \( G \). We denote by \( \mathcal{B}_G = e^{i\Omega^1(G)} \) the manifold of magnetic fields on \( G \). The magnetic Schrödinger operator \( H_B \) associated to \( H \in \mathcal{C}_G \) and \( B = e^{i\alpha} \) is defined by the quadratic form

\[
q_B(f) = -\frac{1}{2} \sum_{\{x,y\}\in \vec{E}} h_{x,y}|f(x) - e^{i\alpha_{x,y}} f(y)|^2 + \sum_{x\in X} V_x |f(x)|^2
\]

associated to a Hermitian form on \( \mathbb{C}^X \). More explicitly, if \( f \in \mathbb{C}^X \),

\[
H f(x) = h_{x,x} f(x) + \sum_{y\sim x} h_{x,y} e^{i\alpha_{x,y}} f(y).
\]

We fix \( H \) and we denote by

\[
\lambda_1(B) \leq \lambda_2(B) \leq \cdots \leq \lambda_n(B) \leq \cdots \leq \lambda_{\#X}(B)
\]

the eigenvalues of \( H_B \). It will be important to notice that \( \lambda_n(\vec{B}) = \lambda_n(B) \). Moreover, we have a gauge invariance: the operators \( H_B \) and \( H_{B'} \) with \( \alpha' = \alpha + df \) for some \( f \in \mathbb{R}^X \) are unitarily equivalent. Hence they have the same eigenvalues. This implies that, if \( \Omega^1(G) = d\mathbb{R}^X \oplus \ker d^* \) (this is not always the case because \( Q \) is not positive), it is enough to consider 1-forms in the subspace \( \ker d^* \) of \( \Omega^1(G) \) when studying the map \( \Lambda_n : B \to \lambda_n(B) \). This holds in particular for investigations concerning the Hessian and the Morse index.

### 3. Statement of Berkolaiko’s magnetic theorem

Before stating the main result, we recall:

**Definition 1.** The **Morse index** \( j(q) \in \mathbb{N} \cup \{+\infty\} \) of a quadratic form \( q \) on a real vector space \( E \) is defined by \( j(q) = \sup_F \dim F \), where \( F \) is a subspace of \( E \) such that \( q_{|F \backslash 0} \) is less than 0. The **nullity** of \( q \) is the dimension of the kernel of \( q \).

The **Morse index** of a smooth real-valued function \( f \) defined on a smooth manifold \( M \) at a **critical point** \( x_0 \in M \) (i.e., a point satisfying \( df(x_0) = 0 \)) is the Morse index of the Hessian of \( f \), which is a canonically defined quadratic form on the tangent space \( T_{x_0}M \). The critical point \( x_0 \) is called **nondegenerate** if the previous Hessian is nondegenerate. The **nullity** of the critical point \( x_0 \) of \( f \) is the nullity of the Hessian of \( f \) at the point \( x_0 \).
The aim of this note is to prove the following nice results due to Berkolaiko [2008; 2013]:

**Theorem 1.** Let $G = (X, E)$ be a finite connected graph and $\beta$ the dimension of the space of cycles of $G$. We suppose that the $n$-th eigenvalue $\lambda_n$ of $H \in \mathcal{O}_G$ is simple. We assume moreover that an associated nonzero eigenfunction $\phi_n$ satisfies $\phi_n(x) \neq 0$ for all $x \in X$. Then, the number $\nu$ of edges along which $\phi_n$ changes sign satisfies $n - 1 \leq \nu \leq n - 1 + \beta$.

Moreover $\Lambda_n : B \to \lambda_n(B)$ is smooth at $B \equiv 1$ which is a critical point of $\Lambda_n$ and the nodal defect, $\delta_n = \nu - (n - 1)$, is the Morse index of $\Lambda_n$ at that point. If $M$ is the manifold of dimension $\beta$ of magnetic fields on $G$ modulo the gauge transforms, the function $[B] \rightarrow \lambda_n(B)$ has $[B = 1]$ as a nondegenerate critical point.

**Remark 1.** The previous results can be extended by replacing the critical point $B \equiv 1$ by $B_{x,y} = \pm 1$ for all edges $\{x, y\} \in E$. The number $\nu$ is then the number of edges $\{x, y\} \in E$ satisfying $B_{x,y}\phi_n(x)\phi_n(y) < 0$ where $\phi_n$ is the corresponding eigenfunction.

**Remark 2.** The assumptions on $H$ are satisfied for $H$ in an open dense subset of $\mathcal{O}_G$.

The upper bound of $\nu$ in the first part of Theorem 1 is related to the Courant nodal theorem (see [Courant and Hilbert 1953, Section VI.6]) as follows: a nodal domain on a graph for the eigenfunction $\phi_n$ is a connected component of the subgraph $G'$ of $G$ obtained by removing the edges along which $\phi_n$ changes sign. Denoting by $\mu$ the number of nodal domains of $\phi_n$, the Courant theorem for graphs (see [Colin de Verdière 1998, Theorem 2.4]) asserts that $\mu \leq n$; using the Euler formula for the graph $G'$ and because $\mu = \nu_0(G')$, the number of connected components of the graph $G'$, we get also a lower bound (see [Berkolaiko 2008]):

**Corollary 1.** Under the assumptions of Theorem 1, we have $n - \beta \leq \mu \leq n$.

**Example 3.1** (bipartite graphs). Let $G = (V, E)$ be a bipartite graph: $V = Y \cup Z$ and all edges have one vertex in $Y$ and the other in $Z$. Let $U$ be the involution on $\mathbb{R}^V$ given by $Uf(x) = -f(x)$ if $x \in Y$ and $Uf(x) = f(x)$ if $x \in Z$ and let $B$ be a magnetic field. Then $UH_BU = -H'_B$, with $H' \in \mathcal{O}_G$, so that $\lambda_{|V|}(H_B) = -\lambda_1(H'_B)$. And hence it follows from the diamagnetic inequality that $B \rightarrow \lambda_{|V|}(H_B)$ has a maximum at $B \equiv 1$. And hence the Morse index of the Hessian of $B \rightarrow \lambda_{|V|}(H_B)$ at $B \equiv 1$ is the dimension of the manifold of magnetic fields, namely $\beta$. On the other hand the first eigenfunction $\phi_1$ of $H'$ is everywhere greater than 0 and the number of sign changes of $U\phi_1$ is $|E|$. So Berkolaiko’s formula for $\lambda_{|V|}$ gives $(|V| - 1) + \beta = |E|$. This is the Euler formula.

**Theorem 1** can be reinterpreted as follows:

**Theorem 2.** Under the assumptions as in Theorem 1, consider the functional $D_n : B \mapsto \det(H_B - \lambda_n(1))$. Then $B \equiv 1$ is a nondegenerate critical point of $D_n$ whose Morse index is $\delta_n$ if $n$ is odd and $\beta - \delta_n$ if $n$ is even.

**Proof.** Under the assumptions of the theorem we have

$$\det(H_B - \lambda_n(1)) = (\lambda_n(B) - \lambda_n(1)) \det'(H_B - \lambda_n(1))$$
where \( \det'(H_B) = F(B) \) is the product of the eigenvalues \( \lambda_j - \lambda_n(1) \) for \( j \neq n \). The following lemma is easy to check by direct computations of the second derivatives:

**Lemma 1.** Let \( F = fG \) where \( F, f, G \) are smooth real valued functions defined near a point \( x_0 \) on a smooth manifold. Let us assume that \( f(x_0) = 0 \) and \( f'(x_0) = 0 \); then the Hessian of \( F \) at the point \( x_0 \) is \( G(x_0) \) times the Hessian of \( f \) at \( x_0 \).

From the lemma, we get that the Hessian of \( D_n \) at \( B = 1 \) is \( F(1) \) times the Hessian of \( \Lambda_n \). We have \((-1)^{n-1} F(1) > 0 \). The conclusion follows. \( \square \)

There is a formula for the characteristic polynomial of a magnetic Laplacian on graphs due to Robin Forman [1993] and reproved by Richard Kenyon [2012] and Yurii Burman [2012]. Using the gauge change \( f \rightarrow f \phi_n \) as in [Colin de Verdière 1998] gives a Laplace type operator whose entries can be of any sign. Forman’s formula extends to that case and it would be nice to relate Berkolaiko’s formula to Forman’s formula.

**Important warning:** Without loss of generality, we can and will assume in the rest of this note that \( \lambda_n = \Lambda_n(1) = 0 \). This implies that the Morse index of \( q_1 \) is \( n - 1 \).

In the course of the proof we will use a special choice of gauge in which we can compute the Hessian explicitly. More precisely, according to the classical perturbation formulae,

\[
\ddot{\lambda} = (\phi, \ddot{H}\phi) + 2(\dot{H}\phi, \dot{\phi}),
\]

where we assumed that \( \lambda \) is at a critical point: \( \dot{\lambda} = 0 \). The first term is easy to calculate explicitly; for perturbation in the direction of the 1-form \( \omega \) it is

\[
Q(\omega) = \frac{1}{2} \sum_{[x,y]} a_{x,y} \omega([x, y])^2 \quad \text{with} \quad a_{x,y} = -h_{x,y}\phi_n(x)\phi_n(y) = a_{y,x}.
\]

Considered as a quadratic form in \( \omega \), \( Q \) is already in the diagonal form. Its index is clearly the number of negative values among \( \{-h_{x,y}\phi_n(x)\phi_n(y)\} \), or, in other words, the number \( \nu \) of edges where \( \phi_n \) changes sign!

We will present an explicit choice of gauge in which the second term vanishes. The condition for this is \( \dot{H}\phi = 0 \) which, after explicit calculation, can be interpreted as \( \omega \in \ker d^* \), where \( d^* \) is the conjugate of \( d \) with respect to the inner product induced by (2).

Finally, we observe that the index of \( Q(\omega) \) has been computed to be \( \nu \) in the whole of \( \Omega^1(G) \), whereas we should be restricting ourselves to our chosen gauge, \( \omega \in \ker d^* \). We will show that this restriction reduces the index precisely by \( n - 1 \). Indeed, the splitting \( \Omega^1(G) = d\mathbb{R}^X \oplus \ker d^* \) is orthogonal with respect to the form \( Q \); therefore

\[
\text{ind}(Q) = \text{ind}(Q|_{d\mathbb{R}^X}) + \text{ind}(Q|_{\ker d^*}).
\]

We establish that \( \text{ind}(Q|_{d\mathbb{R}^X}) = n - 1 \) by relating the form \( Q \) on \( d\mathbb{R}^X \) to the quadratic form \( q_1 \) around the point \( \phi_n \).
4. The quadratic form $Q$

**Lemma 2.** The set of forms $f \rightarrow (f(x) - f(y))^2$ where $\{x, y\} \in \mathcal{P}_2(X)$, the set of subsets with two elements of $X$, and $f \rightarrow f(x)^2$ with $x \in X$ is a basis of the set of quadratic forms on $\mathbb{R}^X$.

**Definition 2.** A quadratic form $q$ on $\mathbb{R}^X$ is said of Laplace type if for all $f \in \mathbb{R}^X$, $\hat{q}(1, f) \equiv 0$ where $\hat{q}$ is the symmetric bilinear form associated to $q$.

**Lemma 3.** The set of forms $f \rightarrow (f(x) - f(y))^2$, $\{x, y\} \in \mathcal{P}_2(X)$ is a basis of the space of quadratic forms of Laplace type.

The form $\tilde{q}_1 : f \rightarrow q_1(\phi_n f)$, where $\phi_n f$ is the pointwise product of $\phi_n$ and $f$, is of Laplace type because

$$\hat{q}_1(1, g) = \langle H\phi_n|\phi_n g \rangle = \langle 0|\phi_n g \rangle.$$  

Hence $\tilde{q}_1(1, g) = 0$.

Moreover, $\tilde{q}_1(f) = Q(df)$. Indeed, because of Lemma 3, it is enough to compare the coefficients of the basis forms $f \rightarrow (f(x) - f(y))^2$. The form $f \rightarrow Q(df)$ is already expanded in this basis. To find the coefficient for the form $f \rightarrow \tilde{q}_1(f)$, we observe that (because we know it is of Laplace type) the coefficient in question is minus the coefficient in front of the term $f(x)f(y)$, divided by two. This evaluates to $a_{x,y}$ (see (2)).

In fact, we will need to use $\hat{Q}(df, dg) = \langle (H\phi_n f)|\phi_n g \rangle$.

**Lemma 4.** The Morse index of $Q|_{d\mathbb{R}^X}$ is equal to $n - 1$.

It is a general fact that the Morse index of the quadratic form $f \rightarrow Q(Af)$ is the same as the Morse index of the restriction of $Q$ to the image of $A$. Hence, the Morse index of $Q|_{d\mathbb{R}^X}$ is the Morse index of $\tilde{q}_1$ on $\mathbb{R}^X$. Because $f \rightarrow \phi_n f$ is a linear isomorphism, this index is equal to the index of $q_1$ by the Sylvester theorem. Since $\lambda_n = 0$, the index of $q_1$ is $n - 1$ by elementary spectral theory.

**Lemma 5.** Let us denote by $d^*$ the adjoint of $d$ where $\mathbb{R}^X$ is equipped with the canonical Euclidean structure and $\Omega^1(G)$ with the inner product associated to $Q$. The space $\Omega^1(G)$ splits as

$$\Omega^1(G) = d\mathbb{R}^X \oplus \ker d^*$$

(Hodge type splitting), and this decomposition is $Q$-orthogonal.

More explicitly $d^*$ is given by

$$d^*\omega(x) = \sum_{y \sim x} a_{x,y} \omega([y, x]).$$

If $\omega = df$ satisfies $d^*\omega = 0$, we have $d^*df = 0$. Hence $\hat{Q}(df, dg) = 0$ for all $g$ and $\langle (H\phi_n f)|\phi_n g \rangle = 0$. Because $\lambda_n$ is of multiplicity 1, this implies that $f$ is constant and hence $df = 0$. So $d\mathbb{R}^X \cap \ker d^* = \{0\}$ and the conclusions follow.

At this point, we know that the nodal defect is the Morse index of the restriction of $Q$ to the space $\ker d^*$ of dimension $\beta$. The first part of Theorem 1 follows.
5. The magnetic Hessian

We need one more fact to complete the proof: to identify the Hessian of $\Lambda_n$ on $e^{i\text{ker}d^*}$ at $B \equiv 1$ with the restriction of $Q$ to $\text{ker}d^*$.

Let us denote by $S \subset \mathbb{C}^X$ the set of unit vectors $f$ normalized so that $f(x_0)$ is real and $f(x_0) > 0$ where $x_0$ is chosen in $X$.

**Lemma 6.** The point $B \equiv 1$ is a critical point of $\Lambda_n$. If $\phi_n(B) \in S$ is the eigenfunction of $H_B$ corresponding to the eigenvalue $\lambda_n(B)$, the differential of $B \rightarrow \phi_n(B)$ vanishes at $B \equiv 1$ on $\text{ker}d^*$.

The first property comes from the fact that $\Lambda_n(B) = \Lambda_n(B)$. We can compute, for any variation $e^{i\alpha}$, $t$ close to 0, of $B \equiv 1$, that $\dot{H}_B \phi_n + H \dot{\phi}_n = 0$. The condition $d^*\alpha = 0$ can be written as

$$\sum_{y \sim x} h_{x,y} \phi_n(y)\alpha_{x,y} = 0 \quad \text{for all } x \in X.$$  

From (1), this is equivalent to $\dot{H}_B \phi_n = 0$. Hence $H(\phi_n) = 0$ and $\dot{\phi}_n = c\phi_n$ since $\lambda_n$ is simple. From the normalization $\|\phi_n(B)\| = 1$, we get $c \in i\mathbb{R}$ and, since $\dot{\phi}_n(x_0) \in \mathbb{R}$, the number $c$ is real. We deduce that $\dot{\phi}_n = 0$.

**Lemma 7.** The function $F : S \times e^{i\text{ker}d^*} \rightarrow \mathbb{R}$ defined by $F(f, e^{i\alpha}) = \langle H_{e^{i\alpha}} f | f \rangle$ admits $(\phi_n, 0)$ as a critical point and the Hessian of $(\Lambda_n)_{\text{ker}d^*}$ at the point $B \equiv 1$ is the form $Q$.

The differential of $F$ with respect to $f$ vanishes because $f$ is an eigenfunction of $H$. The differential with respect to $\text{ker}d^*$ vanishes, because $F(f, e^{i\alpha}) = F(f, e^{-i\alpha})$. The Hessian of $F$ at $(\phi_n, 0)$ is well defined. Because the differential at $B \equiv 1$ of $B \rightarrow \phi_n(B)$ vanishes on $e^{i\text{ker}d^*}$, the Hessians of $\Lambda_n : B \rightarrow F(\phi_n(B), B)$ and $M_n : B \rightarrow F(\phi_n(1), B)$ agree. A simple calculation of the Hessian of $M_n$ gives the result:

$$M_n(e^{i\alpha}) = -\frac{1}{2} \sum_{[x,y] \in \mathcal{E}} h_{x,y} |\phi_n(x) - e^{i\alpha_{x,y}} \phi_n(y)|^2 + \sum_{x \in X} V_x |\phi_n(x)|^2$$

$$= -\sum_{[x,y] \in \mathcal{E}} h_{x,y} (\phi_n(x)^2 + \phi_n(y)^2 - 2 \cos \alpha_{x,y} \phi_n(x)\phi_n(y)) + \sum_{x \in X} V_x |\phi_n(x)|^2.$$  

Computing the second derivative with respect to $\alpha$ at $\alpha = 0$ gives Hessian($M_n$) = $Q(\alpha)$.

**Appendix A: A pedestrian approach to the calculus of the Hessian of $\Lambda_n$ in Section 5**

We will derive a direct approach to the calculus of the second derivative of an eigenvalue which could be used directly in the proof of Lemma 7. Let $t \rightarrow A(t)$ be a $C^2$ curve defined near $t = 0$ in the space of Hermitian matrices on a finite-dimensional Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$. Let us assume that $\lambda(0)$ is an eigenvalue of $A(0)$ of multiplicity one with a normalized eigenvector $\phi(0)$. Then, for $t$ close to 0, $A(t)$ has a simple eigenvalue $\lambda(t)$ of multiplicity one which is a $C^2$ function of $t$. We can choose an associated eigenfunction $\phi(t)$ which is $C^2$ with respect to $t$. The following assertions give the values of the first and second derivatives of $\lambda(t)$ at $t = 0$:
Appendix A.

If $\lambda'(0) = 0$, we have

$$\lambda''(0) = \langle A''(0)\phi(0)\vert \phi(0)\rangle + 2\langle \phi'(0)\vert A'(0)\phi(0)\rangle,$$

where $\phi'(0)$ is any solution of $(A(0) - \lambda(0))\phi'(0) = -A(0)\phi(0)$.

In particular, if $A'(0)\phi(0) = 0$,

$$\lambda''(0) = \langle A''(0)\phi(0)\vert \phi(0)\rangle.$$

Proof. We start with $(A(t) - \lambda(t))\phi(t) = 0$ where $\phi(t)$ is an eigenfunction of $A(t)$ which depends in a $C^2$ way on $t$. Taking the first derivative, we get

$$(A'(t) - \lambda'(t))\phi(t) + (A(t) - \lambda(t))\phi'(t) = 0. \quad (3)$$

Putting $t = 0$ and taking the scalar product with $\phi(0)$, we get the formula for $\lambda'(0)$. Similarly, the $t$-derivative of (3) is

$$(A''(t) - \lambda''(t))\phi(t) + 2(A'(t) - \lambda'(t))\phi'(t) + (A(t) - \lambda(t))\phi''(t) = 0. \quad (4)$$

Putting $t = 0$, taking the scalar product with $\phi(0)$ and using $\lambda'(0) = 0$, we get the result. \qed

We can apply this to $A(t) := H_{\mu\alpha}$ with $\alpha \in \ker d^*$ in order to get the Hessian of $\Lambda_n$ in Section 5. The condition $A'(0)\phi(0) = 0$ is exactly $d^*\alpha = 0!$

Appendix B: The case where the eigenfunction vanishes at some vertex

In this appendix, we take $H \in \mathcal{C}_G$ and assume that $\lambda_n = 0$ is nondegenerate eigenvalue of $H$ with a normalized eigenfunction $\phi$. We have:

**Proposition 2.** Let us assume that, for all vertices $x$ satisfying $\phi(x) = 0$, there exists a vertex $y \sim x$ so that $\phi(y) \neq 0$. Then, for any $\psi \in \mathbb{R}^X$ orthogonal to $\phi$, there exists a smooth deformation $H_t \in \mathcal{C}_G$ of $H$ so that $\dot{\phi} = \psi$.

It is enough to check that the space of $\dot{H}\phi$ is $\mathbb{R}^X$ and to use the first variation formulae given in Appendix A.

**Theorem 3.** Let us assume that the function $\phi$ vanishes at the unique vertex $x_0$. Then, the nullity of the Hessian of the “magnetic variation” of $H$ is at least $|n_+ - n_-|$ where $n_\pm$ is the number of vertices $x \sim x_0$ so that $\pm\phi(x) > 0$.

Proof. Choose a smooth variation $H_t$ of $H$ so that $\dot{\phi}(x_0) = 1$. Let $\nu$ be the number of sign changes of $\phi$ away from $x_0$. Then, for $t > 0$ small enough, the number of sign changes of $\phi_t$ is $\nu + n_-$ while, for $t < 0$ small enough, it is $\nu + n_+$. We see from Theorem 1 that the magnetic Morse index is $\nu + n_- - (n - 1)$ for $t > 0$ and $\nu + n_+ - (n - 1)$. The discontinuity of the Morse index at $t = 0$ is $|n_+ - n_-|$. This gives the lower bound on the nullity. \qed

**Corollary 2.** If $|n_+ - n_-| > \beta$, the eigenvalue 0 is degenerate.
Let us remark that this lower bound is not always sharp. In the following example, we have $n_+ = n_-$, $\beta = 2$ and the nullity of the Hessian is 2.

**Example B.1.** The graph $G$ is made of 2 cycles of length 3 with a common vertex. The matrix of $H$ is chosen as follows:

$$[H] = -\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \end{pmatrix}.$$  

Using the fact that the graph has a symmetry of order 2 exchanging the 2 cycles, one can split $\mathbb{R}^X$ and the matrix $H$ into the even and odd parts. This allows us to check that $\lambda_4 = 0$ is nondegenerate. In order to compute the magnetic Hessian, we check that it is possible to build a decomposition $\Omega^1(G) = d\mathbb{R}^X \oplus K$ which is $Q$-orthogonal and with $K \subset \ker d^*$. It is then easy to check that the magnetic Hessian evaluated on $K$ vanishes.

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**References**


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YVES COLIN DE VERDIÈRE: yves.colin-de-verdiere@ujf-grenoble.fr

Unité Mixte de Recherche CNRS-UJF 5582, Institut Fourier, BP 74, 38402 Saint Martin d’Hères cedex, France

http://www-fourier.ujf-grenoble.fr/~ycolver/
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<td><a href="mailto:vaugan.f.jones@vanderbilt.edu">vaugan.f.jones@vanderbilt.edu</a></td>
</tr>
<tr>
<td>Herbert Koch</td>
<td>Universität Bonn, Germany</td>
<td><a href="mailto:koch@math.uni-bonn.de">koch@math.uni-bonn.de</a></td>
</tr>
<tr>
<td>Izabella Laba</td>
<td>University of British Columbia, Canada</td>
<td><a href="mailto:ilaba@math.ubc.ca">ilaba@math.ubc.ca</a></td>
</tr>
<tr>
<td>Gilles Lebeau</td>
<td>Université de Nice Sophia Antipolis, France</td>
<td><a href="mailto:lebeau@unice.fr">lebeau@unice.fr</a></td>
</tr>
<tr>
<td>László Lempert</td>
<td>Purdue University, USA</td>
<td><a href="mailto:lempert@math.purdue.edu">lempert@math.purdue.edu</a></td>
</tr>
<tr>
<td>Richard B. Melrose</td>
<td>Massachusetts Institute of Technology, USA</td>
<td><a href="mailto:rbm@math.mit.edu">rbm@math.mit.edu</a></td>
</tr>
<tr>
<td>Frank Merle</td>
<td>Université de Cergy-Pontoise, France</td>
<td><a href="mailto:Frank.Merle@u-cergy.fr">Frank.Merle@u-cergy.fr</a></td>
</tr>
<tr>
<td>William Minicozzi II</td>
<td>Johns Hopkins University, USA</td>
<td><a href="mailto:minicozz@math.jhu.edu">minicozz@math.jhu.edu</a></td>
</tr>
<tr>
<td>Werner Müller</td>
<td>Universität Bonn, Germany</td>
<td><a href="mailto:mueller@math.uni-bonn.de">mueller@math.uni-bonn.de</a></td>
</tr>
<tr>
<td>Yuval Peres</td>
<td>University of California, Berkeley, USA</td>
<td><a href="mailto:peres@stat.berkeley.edu">peres@stat.berkeley.edu</a></td>
</tr>
<tr>
<td>Gilles Pisier</td>
<td>Texas A&amp;M University, and Paris 6</td>
<td><a href="mailto:pisier@math.tamu.edu">pisier@math.tamu.edu</a></td>
</tr>
<tr>
<td>Tristan Rivièr</td>
<td>ETH, Switzerland</td>
<td><a href="mailto:riviere@math.ethz.ch">riviere@math.ethz.ch</a></td>
</tr>
<tr>
<td>Igor Rodnianski</td>
<td>Princeton University, USA</td>
<td><a href="mailto:irod@math.princeton.edu">irod@math.princeton.edu</a></td>
</tr>
<tr>
<td>Wilhelm Schlag</td>
<td>University of Chicago, USA</td>
<td><a href="mailto:schlag@math.uchicago.edu">schlag@math.uchicago.edu</a></td>
</tr>
<tr>
<td>Sylvia Serfaty</td>
<td>New York University, USA</td>
<td><a href="mailto:serfaty@cims.nyu.edu">serfaty@cims.nyu.edu</a></td>
</tr>
<tr>
<td>Yum-Tong Siu</td>
<td>Harvard University, USA</td>
<td><a href="mailto:siu@math.harvard.edu">siu@math.harvard.edu</a></td>
</tr>
<tr>
<td>Terence Tao</td>
<td>University of California, Los Angeles, USA</td>
<td><a href="mailto:tao@math.ucla.edu">tao@math.ucla.edu</a></td>
</tr>
<tr>
<td>Michael E. Taylor</td>
<td>Univ. of North Carolina, Chapel Hill, USA</td>
<td><a href="mailto:met@math.unc.edu">met@math.unc.edu</a></td>
</tr>
<tr>
<td>Gunther Uhlmann</td>
<td>University of Washington, USA</td>
<td><a href="mailto:gunther@math.washington.edu">gunther@math.washington.edu</a></td>
</tr>
<tr>
<td>András Vasy</td>
<td>Stanford University, USA</td>
<td><a href="mailto:andras@math.stanford.edu">andras@math.stanford.edu</a></td>
</tr>
<tr>
<td>Dan Virgil Voiculescu</td>
<td>University of California, Berkeley, USA</td>
<td><a href="mailto:dvv@math.berkeley.edu">dvv@math.berkeley.edu</a></td>
</tr>
<tr>
<td>Steven Zelditch</td>
<td>Northwestern University, USA</td>
<td><a href="mailto:zelditch@math.northwestern.edu">zelditch@math.northwestern.edu</a></td>
</tr>
</tbody>
</table>

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