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MAGNETIC INTERPRETATION OF THE NODAL DEFECT ON GRAPHS

YVES COLIN DE VERDIÈRE

We present a natural proof of a recent and surprising result of Gregory Berkolaiko interpreting the Courant nodal defect as a Morse index. This proof is inspired by a nice paper of Miroslav Fiedler published in 1975.

1. Introduction

The "nodal defect" of an eigenfunction of a Schrödinger operator is closely related to the difference between the upper bound on the number of nodal domains given by Courant's theorem and the number of nodal domains. Berkolaiko [2013] has proved a nice formula for the nodal defect of an eigenfunction of a Schrödinger operator on a finite graph in terms of the Morse index of the corresponding eigenvalue as a function of a magnetic deformation of the operator. His proof remains mysterious and rather indirect. In order to get a better understanding in view of possible generalizations, it is desirable to have a more direct approach. This is what we do here, with a proof inspired by [Fiedler 1975].

After reviewing our notations, we state the main result, as well as a reinterpretation in terms of Hessians of a determinant, and give an informal description of the proof in Section 3. The proof itself is implemented in Sections 4 and 5 with an alternative view provided in Appendix A. The continuous Schrödinger operator on a circle was considered in the preprint version of this paper [Colin de Verdière 2012]. The case of quantum graphs, i.e., graphs as 1-dimensional simplicial complexes, is worked out in [Berkolaiko and Weyand 2012].

2. Notation

Let $G = (X, E)$ be a finite connected graph, where X is the set of vertices and E the set of unoriented edges. We denote by $\{x, y\}$ the edge linking the vertices x and y. We denote by $\vec{E}$ the set of oriented edges and by $[x, y]$ the edge from x to y; the set $\vec{E}$ is a 2-fold cover of E. A 1-form α on G is a map $\vec{E} \to \mathbb{R}$ such that $\alpha([y, x]) = -\alpha([x, y])$ for all $\{x, y\} \in E$. We denote by $\Omega^1(G)$ the vector space of dimension $\#E$ of 1-forms on G. The operator $d : \mathbb{R}^X \to \Omega^1(G)$ is defined by $df([x, y]) = f(y) - f(x)$. If Q is a nondegenerate, not necessarily positive, quadratic form on $\Omega^1(G)$, we denote by $d^\star$ the adjoint of d, where $\mathbb{R}^X$ carries the canonical Euclidean structure and $\Omega^1(G)$ is equipped with the symmetric inner product $\hat{Q}$ associated to Q. We have $\dim \ker d^\star = \beta$, where $\beta = 1 + \#E - \#X$ is the dimension

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of the space of cycles of G. We will show later that, in our context, we have the Hodge decomposition $\Omega^1(G) = d\mathbb{R}^X \oplus \ker d^\star$, where both spaces are $\hat{Q}$-orthogonal.

Following [Colin de Verdière 1998], we denote by $\mathbb{O}_G$ the set of $X \times X$ real symmetric matrices H which satisfy $h_{x,y} < 0$ if $\{x, y\} \in E$ and $h_{x,y} = 0$ if $\{x, y\} \notin E$ and $x \neq y$. Note that the diagonal entries of H are arbitrary. An element H of $\mathbb{O}_G$ is called a *Schrödinger operator* on the graph G. It will be useful to write the quadratic form associated to H as

$$q_1(f) = -\sum_{\{x,y\}\in E} h_{x,y}(f(x) - f(y))^2 + \sum_{x\in X} V_x f(x)^2,$$

with $V_x = h_{x,x} + \sum_{y\sim x} h_{x,y}$. A *magnetic field* on G is a map $B : \vec{E} \to U(1)$ defined by $B([x, y]) = e^{i\alpha_{x,y}}$, where $[x, y] \mapsto \alpha_{x,y}$ is a 1-form on G. We denote by $\mathcal{B}_G = e^{i\Omega^1(G)}$ the manifold of magnetic fields on G. The magnetic Schrödinger operator H_B associated to $H \in \mathbb{O}_G$ and $B = e^{i\alpha}$ is defined by the quadratic form

$$q_B(f) = -\frac{1}{2} \sum_{[x,y]\in\vec{E}} h_{x,y} |f(x) - e^{i\alpha_{x,y}} f(y)|^2 + \sum_{x\in X} V_x |f(x)|^2$$

associated to a Hermitian form on $\mathbb{C}^X$. More explicitly, if $f \in \mathbb{C}^X$,

$$Hf(x) = h_{x,x} f(x) + \sum_{y\sim x} h_{x,y} e^{i\alpha_{x,y}} f(y). \tag{1}$$

We fix H and we denote by

$$\lambda_1(B) \leq \lambda_2(B) \leq \cdots \leq \lambda_n(B) \leq \cdots \leq \lambda_{\#X}(B)$$

the eigenvalues of H_B. It will be important to notice that $\lambda_n(\bar{B}) = \lambda_n(B)$. Moreover, we have a gauge invariance: the operators H_B and $H_{B'}$ with $\alpha' = \alpha + df$ for some $f \in \mathbb{R}^X$ are unitarily equivalent. Hence they have the same eigenvalues. This implies that, if $\Omega^1(G) = d\mathbb{R}^X \oplus \ker d^\star$ (this is not always the case because Q is not positive), it is enough to consider 1-forms in the subspace $\ker d^\star$ of $\Omega^1(G)$ when studying the map $\Lambda_n : B \to \lambda_n(B)$. This holds in particular for investigations concerning the Hessian and the Morse index.

3. Statement of Berkolaiko's magnetic theorem

Before stating the main result, we recall:

Definition 1. The *Morse index* $j(q) \in \mathbb{N} \cup \{+\infty\}$ of a quadratic form q on a real vector space E is defined by $j(q) = \sup_F \dim F$, where F is a subspace of E such that $q_{|F\setminus 0}$ is less than 0. The *nullity* of q is the dimension of the kernel of q.

The *Morse index* of a smooth real-valued function f defined on a smooth manifold M at a *critical point* $x_0 \in M$ (i.e., a point satisfying $df(x_0) = 0$) is the Morse index of the Hessian of f, which is a canonically defined quadratic form on the tangent space $T_{x_0}M$. The critical point x_0 is called *nondegenerate* if the previous Hessian is nondegenerate. The *nullity* of the critical point x_0 of f is the nullity of the Hessian of f at the point x_0.

The aim of this note is to prove the following nice results due to Berkolaiko [2008; 2013]:

Theorem 1. *Let $G = (X, E)$ be a finite connected graph and β the dimension of the space of cycles of G. We suppose that the n-th eigenvalue λ_n of $H \in \mathcal{O}_G$ is simple. We assume moreover that an associated nonzero eigenfunction ϕ_n satisfies $\phi_n(x) \neq 0$ for all $x \in X$. Then, the number ν of edges along which ϕ_n changes sign satisfies $n - 1 \leq \nu \leq n - 1 + \beta$.*

Moreover $\Lambda_n : B \to \lambda_n(B)$ is smooth at $B \equiv 1$ which is a critical point of Λ_n and the nodal defect, *$\delta_n = \nu - (n-1)$, is the Morse index of Λ_n at that point. If M is the manifold of dimension β of magnetic fields on G modulo the gauge transforms, the function $[B] \to \Lambda_n(B)$ has $[B = 1]$ as a nondegenerate critical point.*

Remark 1. The previous results can be extended by replacing the critical point $B \equiv 1$ by $B_{x,y} = \pm 1$ for all edges $\{x, y\} \in E$. The number ν is then the number of edges $\{x, y\} \in E$ satisfying $B_{x,y}\phi_n(x)\phi_n(y) < 0$ where ϕ_n is the corresponding eigenfunction.

Remark 2. The assumptions on H are satisfied for H in an open dense subset of $\mathcal{O}_G$.

The upper bound of ν in the first part of Theorem 1 is related to the Courant nodal theorem (see [Courant and Hilbert 1953, Section VI.6]) as follows: a nodal domain on a graph for the eigenfunction ϕ_n is a connected component of the subgraph G' of G obtained by removing the edges along which ϕ_n changes sign. Denoting by μ the number of nodal domains of ϕ_n, the Courant theorem for graphs (see [Colin de Verdière 1998, Theorem 2.4]) asserts that $\mu \leq n$; using the Euler formula for the graph G' and because $\mu = b_0(G')$, the number of connected components of the graph G', we get also a lower bound (see [Berkolaiko 2008]):

Corollary 1. *Under the assumptions of Theorem 1, we have $n - \beta \leq \mu \leq n$.*

Example 3.1 (bipartite graphs). Let $G = (V, E)$ be a bipartite graph: $V = Y \cup Z$ and all edges have one vertex in Y and the other in Z. Let U be the involution on $\mathbb{R}^V$ given by $Uf(x) = -f(x)$ if $x \in Y$ and $Uf(x) = f(x)$ if $x \in Z$ and let B be a magnetic field. Then $UH_BU = -H'_B$ with $H' \in O_G$, so that $\lambda_{|V|}(H_B) = -\lambda_1(H'_B)$. And hence it follows from the diamagnetic inequality that $B \to \lambda_{|V|}(H_B)$ has a maximum at $B \equiv 1$. And hence the Morse index of the Hessian of $B \to \lambda_{|V|}(H_B)$ at $B \equiv 1$ is the dimension of the manifold of magnetic fields, namely β. On the other hand the first eigenfunction ϕ_1 of H' is everywhere greater than 0 and the number of sign changes of $U\phi_1$ is $|E|$. So Berkolaiko's formula for $\lambda_{|V|}$ gives $(|V| - 1) + \beta = |E|$. This is the Euler formula.

Theorem 1 can be reinterpreted as follows:

Theorem 2. *Under the assumptions as in Theorem 1, consider the functional $D_n : B \mapsto \det(H_B - \lambda_n(1))$. Then $B \equiv 1$ is a nondegenerate critical point of D_n whose Morse index is δ_n if n is odd and $\beta - \delta_n$ if n is even.*

Proof. Under the assumptions of the theorem we have

$$\det(H_B - \lambda_n(1)) = (\lambda_n(B) - \lambda_n(1)) \det{}'(H_B - \lambda_n(1))$$

where $\det'(H_B) = F(B)$ is the product of the eigenvalues $\lambda_j - \lambda_n(1)$ for $j \neq n$. The following lemma is easy to check by direct computations of the second derivatives:

Lemma 1. *Let $F = fG$ where F, f, G are smooth real valued functions defined near a point x_0 on a smooth manifold. Let us assume that $f(x_0) = 0$ and $f'(x_0) = 0$; then the Hessian of F at the point x_0 is $G(x_0)$ times the Hessian of f at x_0.*

From the lemma, we get that the Hessian of D_n at $B \equiv 1$ is $F(1)$ times the Hessian of Λ_n. We have $(-1)^{n-1}F(1) > 0$. The conclusion follows. □

There is a formula for the characteristic polynomial of a magnetic Laplacian on graphs due to Robin Forman [1993] and reproved by Richard Kenyon [2012] and Yurii Burman [2012]. Using the gauge change $f \to f\phi_n$ as in [Colin de Verdière 1998] gives a Laplace type operator whose entries can be of any sign. Forman's formula extends to that case and it would be nice to relate Berkolaiko's formula to Forman's formula.

Important warning: Without loss of generality, we can and *will* assume in the rest of this note that $\lambda_n = \Lambda_n(1) = 0$. This implies that the Morse index of q_1 is $n-1$.

In the course of the proof we will use a special choice of gauge in which we can compute the Hessian explicitly. More precisely, according to the classical perturbation formulae,

$$\ddot{\lambda} = (\phi, \ddot{H}\phi) + 2(\dot{H}\phi, \dot{\phi}),$$

where we assumed that λ is at a critical point: $\dot{\lambda} = 0$. The first term is easy to calculate explicitly; for perturbation in the direction of the 1-form ω it is

$$Q(\omega) = \frac{1}{2} \sum_{\vec{E}} a_{x,y}\omega([x, y])^2 \quad \text{with } a_{x,y} = -h_{x,y}\phi_n(x)\phi_n(y) = a_{y,x}. \tag{2}$$

Considered as a quadratic form in ω, Q is already in the diagonal form. Its index is clearly the number of negative values among $\{-h_{x,y}\phi_n(x)\phi_n(y)\}$, or, in other words, the number ν of edges where ϕ_n changes sign!

We will present an explicit choice of gauge in which the second term vanishes. The condition for this is $\dot{H}\phi = 0$ which, after explicit calculation, can be interpreted as $\omega \in \ker d^\star$, where $d^\star$ is the conjugate of d with respect to the inner product induced by (2).

Finally, we observe that the index of $Q(\omega)$ has been computed to be ν in the whole of $\Omega^1(G)$, whereas we should be restricting ourselves to our chosen gauge, $\omega \in \ker d^\star$. We will show that this restriction reduces the index precisely by $n-1$. Indeed, the splitting $\Omega^1(G) = d\mathbb{R}^X \oplus \ker d^\star$ is orthogonal with respect to the form Q; therefore

$$\operatorname{ind}(Q) = \operatorname{ind}(Q|_{d\mathbb{R}^X}) + \operatorname{ind}(Q|_{\ker d^\star}).$$

We establish that $\operatorname{ind}(Q|_{d\mathbb{R}^X}) = n-1$ by relating the form Q on $d\mathbb{R}^X$ to the quadratic form q_1 around the point ϕ_n.

4. The quadratic form Q

Lemma 2. *The set of forms $f \to (f(x) - f(y))^2$ where $\{x, y\} \in \mathcal{P}_2(X)$, the set of subsets with two elements of X, and $f \to f(x)^2$ with $x \in X$ is a basis of the set of quadratic forms on $\mathbb{R}^X$.*

Definition 2. A quadratic form q on $\mathbb{R}^X$ is said of Laplace type if for all $f \in \mathbb{R}^X$, $\hat{q}(1, f) \equiv 0$ where $\hat{q}$ is the symmetric bilinear form associated to q.

Lemma 3. *The set of forms $f \to (f(x) - f(y))^2$, $\{x, y\} \in \mathcal{P}_2(X)$ is a basis of the space of quadratic forms of Laplace type.*

The form $\tilde{q}_1 : f \to q_1(\phi_n f)$, where $\phi_n f$ is the pointwise product of ϕ_n and f, is of Laplace type because

$$\widehat{\tilde{q}_1}(1, g) = \langle H\phi_n|\phi_n g\rangle = \langle 0|\phi_n g\rangle.$$

Hence $\widehat{\tilde{q}_1}(1, g) = 0$.

Moreover, $\tilde{q}_1(f) = Q(df)$. Indeed, because of Lemma 3, it is enough to compare the coefficients of the basis forms $f \to (f(x) - f(y))^2$. The form $f \to Q(df)$ is already expanded in this basis. To find the coefficient for the form $f \to \tilde{q}_1(f)$, we observe that (because we know it is of Laplace type) the coefficient in question is minus the coefficient in front of the term $f(x)f(y)$, divided by two. This evaluates to $a_{x,y}$ (see (2)).

In fact, we will need to use $\hat{Q}(df, dg) = \langle H(\phi_n f)|\phi_n g\rangle$.

Lemma 4. *The Morse index of $Q_{|d\mathbb{R}^X}$ is equal to $n - 1$.*

It is a general fact that the Morse index of the quadratic form $f \to Q(Af)$ is the same as the Morse index of the restriction of Q to the image of A. Hence, the Morse index of $Q_{|d\mathbb{R}^X}$ is the Morse index of $\tilde{q}_1$ on $\mathbb{R}^X$. Because $f \to \phi_n f$ is a linear isomorphism, this index is equal to the index of q_1 by the Sylvester theorem. Since $\lambda_n = 0$, the index of q_1 is $n - 1$ by elementary spectral theory.

Lemma 5. *Let us denote by $d^\star$ the adjoint of d where $\mathbb{R}^X$ is equipped with the canonical Euclidean structure and $\Omega^1(G)$ with the inner product associated to Q. The space $\Omega^1(G)$ splits as*

$$\Omega^1(G) = d\mathbb{R}^X \oplus \ker d^\star$$

(*Hodge type splitting*), *and this decomposition is Q-orthogonal.*

More explicitly $d^\star$ is given by

$$d^\star\omega(x) = \sum_{y\sim x} a_{x,y}\omega([y, x]).$$

If $\omega = df$ satisfies $d^\star\omega = 0$, we have $d^\star df = 0$. Hence $\hat{Q}(df, dg) = 0$ for all g and $\langle H(\phi_n f)|\phi_n g\rangle\rangle = 0$. Because λ_n is of multiplicity 1, this implies that f is constant and hence $df = 0$. So $d\mathbb{R}^X \cap \ker d^\star = \{0\}$ and the conclusions follow.

At this point, we know that the nodal defect is the Morse index of the restriction of Q to the space $\ker d^\star$ of dimension β. The first part of Theorem 1 follows.

5. The magnetic Hessian

We need one more fact to complete the proof: to identify the Hessian of Λ_n on $e^{i\ker d^\star}$ at $B\equiv 1$ with the restriction of Q to $\ker d^\star$.

Let us denote by $S\subset\mathbb{C}^X$ the set of unit vectors f normalized so that $f(x_0)$ is real and $f(x_0)>0$ where x_0 is chosen in X.

Lemma 6. *The point $B\equiv 1$ is a critical point of Λ_n. If $\phi_n(B)\in S$ is the eigenfunction of H_B corresponding to the eigenvalue $\lambda_n(B)$, the differential of $B\to\phi_n(B)$ vanishes at $B\equiv 1$ on* $\ker d^\star$.

The first property comes from the fact that $\Lambda_n(\bar{B})=\Lambda_n(B)$. We can compute, for any variation $e^{it\alpha}$, t close to 0, of $B\equiv 1$, that $\dot{H}_B\phi_n+H\dot{\phi}_n=0$. The condition $d^\star\alpha=0$ can be written as

$$\sum_{y\sim x} h_{x,y}\phi_n(y)\alpha_{x,y}=0 \quad \text{for all } x\in X.$$

From (1), this is equivalent to $\dot{H}_B\phi_n=0$. Hence $H(\dot{\phi}_n)=0$ and $\dot{\phi}_n=c\phi_n$ since λ_n is simple. From the normalization $\|\phi_n(B)\|=1$, we get $c\in i\mathbb{R}$ and, since $\dot{\phi}_n(x_0)\in\mathbb{R}$, the number c is real. We deduce that $\dot{\phi}_n=0$.

Lemma 7. *The function $F:S\times e^{i\ker d^\star}\to\mathbb{R}$ defined by $F(f,e^{i\alpha})=\langle H_{e^{i\alpha}}f\,|\,f\rangle$ admits $(\phi_n,0)$ as a critical point and the Hessian of $(\Lambda_n)_{|e^{i\ker d^\star}}$ at the point $B\equiv 1$ is the form Q.*

The differential of F with respect to f vanishes because f is an eigenfunction of H. The differential with respect to $\ker d^\star$ vanishes, because $F(f,e^{i\alpha})=F(f,e^{-i\alpha})$. The Hessian of F at $(\phi_n,0)$ is well defined. Because the differential at $B=1$ of $B\to\phi_n(B)$ vanishes on $e^{i\ker d^\star}$, the Hessians of $\Lambda_n:B\to F(\phi_n(B),B)$ and $M_n:B\to F(\phi_n(1),B)$ agree. A simple calculation of the Hessian of M_n gives the result:

$$\begin{aligned} M_n(e^{i\alpha}) &= -\frac{1}{2}\sum_{[x,y]\in\vec{E}} h_{x,y}|\phi_n(x)-e^{i\alpha_{x,y}}\phi_n(y)|^2+\sum_{x\in X}V_x|\phi_n(x)|^2 \\ &= -\sum_{[x,y]\in E} h_{x,y}\big(\phi_n(x)^2+\phi_n(y)^2-2\cos\alpha_{x,y}\phi_n(x)\phi_n(y)\big)+\sum_{x\in X}V_x|\phi_n(x)|^2. \end{aligned}$$

Computing the second derivative with respect to α at $\alpha=0$ gives $\mathrm{Hessian}(M_n)=Q(\alpha)$.

Appendix A: A pedestrian approach to the calculus of the Hessian of Λ_n in Section 5

We will derive a direct approach to the calculus of the second derivative of an eigenvalue which could be used directly in the proof of Lemma 7. Let $t\to A(t)$ be a C^2 curve defined near $t=0$ in the space of Hermitian matrices on a finite-dimensional Hilbert space $(\mathcal{H},\langle\cdot|\cdot\rangle)$. Let us assume that $\lambda(0)$ is an eigenvalue of $A(0)$ of multiplicity one with a normalized eigenvector $\phi(0)$. Then, for t close to 0, $A(t)$ has a simple eigenvalue $\lambda(t)$ of multiplicity one which is a C^2 function of t. We can choose an associated eigenfunction $\phi(t)$ which is C^2 with respect to t. The following assertions give the values of the first and second derivatives of $\lambda(t)$ at $t=0$:

Proposition 1. *Under the previous assumptions, we have*

$$\lambda'(0) = \langle A'(0)\phi(0)|\phi(0)\rangle.$$

If $\lambda'(0) = 0$, *we have*

$$\lambda''(0) = \langle A''(0)\phi(0)|\phi(0)\rangle + 2\langle\phi'(0)|A'(0)\phi(0)\rangle,$$

where $\phi'(0)$ *is any solution of* $(A(0) - \lambda(0))\phi'(0) = -A'(0)\phi(0)$.

In particular, if $A'(0)\phi(0) = 0$,

$$\lambda''(0) = \langle A''(0)\phi(0)|\phi(0)\rangle.$$

Proof. We start with $(A(t) - \lambda(t))\phi(t) = 0$ where $\phi(t)$ is an eigenfunction of $A(t)$ which depends in a C^2 way on t. Taking the first derivative, we get

$$(A'(t) - \lambda'(t))\phi(t) + (A(t) - \lambda(t))\phi'(t) = 0. \tag{3}$$

Putting $t = 0$ and taking the scalar product with $\phi(0)$, we get the formula for $\lambda'(0)$. Similarly, the t-derivative of (3) is

$$(A''(t) - \lambda''(t))\phi(t) + 2(A'(t) - \lambda'(t))\phi'(t) + (A(t) - \lambda(t))\phi''(t) = 0. \tag{4}$$

Putting $t = 0$, taking the scalar product with $\phi(0)$ and using $\lambda'(0) = 0$, we get the result. □

We can apply this to $A(t) := H_{e^{it\alpha}}$ with $\alpha \in \ker d^\star$ in order to get the Hessian of Λ_n in Section 5. The condition $A'(0)\phi(0) = 0$ is exactly $d^\star\alpha = 0$!

Appendix B: The case where the eigenfunction vanishes at some vertex

In this appendix, we take $H \in \mathcal{O}_G$ and assume that $\lambda_n = 0$ is nondegenerate eigenvalue of H with a normalized eigenfunction ϕ. We have:

Proposition 2. *Let us assume that, for all vertices x satisfying $\phi(x) = 0$, there exists a vertex $y \sim x$ so that $\phi(y) \neq 0$. Then, for any $\psi \in \mathbb{R}^X$ orthogonal to ϕ, there exists a smooth deformation $H_t \in \mathcal{O}_G$ of H so that $\dot{\phi} = \psi$.*

It is enough to check that the space of $\dot{H}\phi$ is $\mathbb{R}^X$ and to use the first variation formulae given in Appendix A.

Theorem 3. *Let us assume that the function ϕ vanishes at the unique vertex x_0. Then, the nullity of the Hessian of the "magnetic variation" of H is at least $|n_+ - n_-|$ where $n_\pm$ is the number of vertices $x \sim x_0$ so that $\pm\phi(x) > 0$.*

Proof. Choose a smooth variation H_t of H so that $\dot{\phi}(x_0) = 1$. Let ν be the number of sign changes of ϕ away from x_0. Then, for $t > 0$ small enough, the number of sign changes of ϕ_t is $\nu + n_-$ while, for $t < 0$ small enough, it is $\nu + n_+$. We see from Theorem 1 that the magnetic Morse index is $\nu + n_- - (n-1)$ for $t > 0$ and $\nu + n_+ - (n-1)$. The discontinuity of the Morse index at $t = 0$ is $|n_+ - n_-|$. This gives the lower bound on the nullity. □

Corollary 2. *If* $|n_+ - n_-| > \beta$, *the eigenvalue* 0 *is degenerate.*

Let us remark that this lower bound is not always sharp. In the following example, we have $n_+ = n_-$, $\beta = 2$ and the nullity of the Hessian is 2.

Example B.1. The graph G is made of 2 cycles of length 3 with a common vertex. The matrix of H is chosen as follows:

$$[H] = -\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Using the fact that the graph has a symmetry of order 2 exchanging the 2 cycles, one can split $\mathbb{R}^X$ and the matrix H into the even and odd parts. This allows us to check that $\lambda_4 = 0$ is nondegenerate. In order to compute the magnetic Hessian, we check that it is possible to build a decomposition $\Omega^1(G) = d\mathbb{R}^X \oplus K$ which is Q-orthogonal and with $K \subset \ker d^\star$. It is then easy to check that the magnetic Hessian evaluated on K vanishes.

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